ON THE DEPINNING TRANSITION OF THE DIRECTED POLYMER IN A RANDOM ENVIRONMENT WITH A DEFECT LINE

by

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Table of Contents

List of Figures

Abstract

We study the depinning transition of the $1+1$ dimensional directed polymer in a random environment with a defect line model. The monomer locations of the polymer is modeled by the space-time trajectory of a one dimensional simple symmetric random walk. Random environment is introduced by assigning each site of $\mathbb{Z}^2$ an independent and identically distributed normal random variable that interacts with the polymer when it visits that site. The defect line is incorporated to the model by having an additional constant potential u at the origin which gives a reward or penalty to the polymer as it visits the origin. There is a critical value of u above which polymer is pinned, placing a positive fraction of its monomers at zero with high probability.

Our first result is that the quenched free energy exists and the self-averaging holds, which implies that there is a nonrandom quenched critical point. To see the effect of disorder on the depinning transition, we compare the quenched free energy of the system as a function of u to the corresponding annealed system. Our main result is that the quenched and annealed free energies differ significantly only in a very small neighborhood of the critical point and we show that the size of this neighborhood scales as at most β as $\beta \to 0$, where β is the inverse temperature.

Chapter 1

Introduction

In chapter one, we introduce the polymer models and basic statistical mechanics concepts in this framework. We will review and summarize some of the main existing results, which we need to use in the following chapters, for random walk pinning models, general homogeneous and disordered pinning models.

In chapter two, after reviewing the relevant physics literature on the directed polymer in a random environment with a defect line model, we introduce our problem in mathematical framework. We show that self-averaging occurs, meaning that the quenched free energy and critical point are nonrandom, off a null set.

In chapter three, we present our main results and develop the necessary tools to prove them.

1.1 Random Walk Pinning Model

In this section, we introduce the one dimensional random walk pinning model and basic statistical mechanics concepts in this framework. Let P_x be the distribution of the one dimensional simple symmetric random walk, $S = \{S_j, j \geq 0\}$, on $\mathbb{Z}$ starting at the point x, and we use P for P_0. We consider $S = \{S_j, j \geq 0\}$ as a path of random walk and all probability measures will be defined on the path space. Let

$$L_N(\cdot) := L_N(S) = \sum_{j=1}^{N} 1_{S_j=0}$$

be the local time of random walk at 0 up to time N.

We introduce the following polymer (probability) measures on the path space in Boltzmann-Gibbs way as follows:

$$\frac{\mathrm{d}\mu_{N,u}}{\mathrm{d}P}(S) = \frac{1}{Z_{N,u}} e^{uL_N(S)} \tag{1.1.1}$$

where

$$Z_{N,u} = E^{P_0}(e^{uL_N(S)}) \tag{1.1.2}$$

is the normalizing constant and called the *partition function* of the system.

The exponential factor $H_{N,u}(S) := uL_N(S)$ is called the *Hamiltonian* of the system which is the energy functional and identifies the (negative) energy of each micro-state.

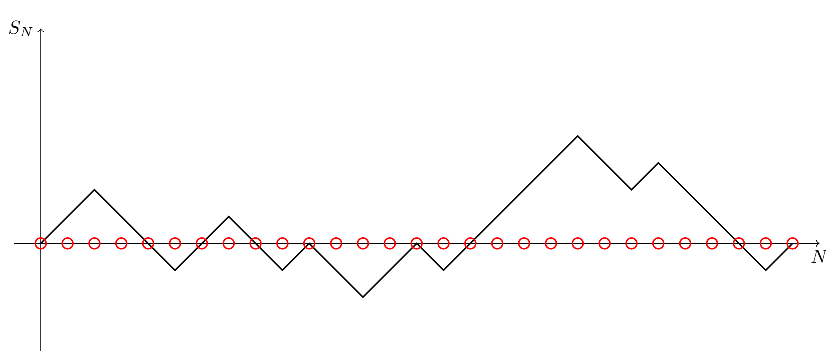

Figure 1.1: 1-D pinning model

One of the most fundamental result in statistical mechanics (based on the law of energy conservation and the basic postulate that all micro-states of the same energy level are equiprobable) is that when the system is in thermal equilibrium with its environment, the probability of finding the system in a given micro-state is given by Boltzmann-Gibbs distribution as we defined above.

Note that under the usual random walk measure P_0, each path of length N has equal probability but polymer measure $\mu_{N,u}(\cdot)$ favors paths which has more return to zero. Under the polymer measure $\mu_{N,u}(\cdot)$, random walk paths are weighted proportional to their returns to zero in the first N steps.

1.1.1 Partition Function and the Existence of the Free Energy

Definition 1.1.1. *A sequence $\{a_n\}_{n\geq 1}$ is called sub-additive if it satisfies the inequality*

$$a_{m+n} \leq a_m + a_n$$

for all $m, n \geq 1$.

For the proof of the following lemma, see the appendix 7 of [25].

Lemma 1.1.2 (Fekete's sub-additive lemma)**.** *For every sub-additive sequence $\{a_n\}_{n\geq 1}$, the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists and is equal to $\inf_{n\geq 1} \frac{a_n}{n} \in [-\infty, \infty)$.*

Lemma 1.1.3. *Let $a_n(u) = u + \log E^{P_0}(e^{uL_n})$. Then $\{a_n(u)\}_{n\geq 1}$ defines a sub-additive sequence for any $u \in \mathbb{R}$.*

Proof. By Markov property of SSRW, we have

$$\begin{aligned}
E^{P_0}(e^{uL_{n+k}}) &= \sum_x E^{P_0}(e^{uL_{n+k}} 1_{S_n=x}) \\
&= \sum_x E^{P_0}(e^{uL_n} 1_{S_n=x}) E^{P_x}(e^{uL_k}) \\
&\leq \sum_x E^{P_0}(e^{uL_n} 1_{S_n=x}) e^u E^{P_0}(e^{uL_k}) \\
&= e^u E^{P_0}(e^{uL_n}) E^{P_0}(e^{uL_k})
\end{aligned}$$

Multiplying both sides by e^u, and then taking the log of both sides, we get

$$a_{n+k}(u) \leq a_n(u) + a_k(u).$$

□

By Lemma 1.1.2,

$$\lim_{n\to\infty} \frac{a_n(u)}{n}$$

exists for all $u \in \mathbb{R}$.

The limit

$$\mathrm{F}(u) = \lim_{n\to\infty} \frac{\log E^{P_0}(e^{uL_n})}{n}$$

is called the *free energy* of the system.

Observe that for $u \leq 0$, $E^{P_0}(e^{uL_n}) \leq 1$. Therefore,

$$\mathrm{F}(u) = \lim_{n\to\infty} \frac{\log E^{P_0}(e^{uL_n})}{n} \quad \leq 0$$

and also for any $u \in \mathbb{R}$,

$$\begin{aligned} E^{P_0}(e^{uL_n}) &\geq E^{P_0}(e^{uL_n}1_{\tau>n}) \\ E^{P_0}(e^{uL_n}) &\geq P_0(\tau > n) \end{aligned}$$

Therefore,

$$\mathrm{F}(u) = \lim_{n\to\infty} \frac{\log E^{P_0}(e^{uL_n})}{n} \geq 0$$

since $P_0(\tau > n) \sim Cn^{-1/2}$ as $n \to \infty$, where $\tau = \inf\{n \geq 1 : S_n = 0\}$ is the first return time of random walk to the origin.

Recall that for two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$, $a_n \sim b_n$ means that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

The following lemma gives an exponential order lower bound in terms of free energy for the partition function of the system of size N.

Lemma 1.1.4. *For all* $N \geq 1$*,*

$$E^{P_0}(e^{uL_N}) \geq e^{-u} e^{N\mathrm{F}(u)}.$$

Proof. Let $a_n(u) = u + \log E^{P_0}(e^{uL_n})$.

By sub-additivity

$$\frac{a_N(u)}{N} \geq \inf_{n\geq 1} \frac{a_n(u)}{n} = \lim_{n\to\infty} \frac{a_n(u)}{n} = \mathrm{F}(u), \quad \forall N \geq 1;$$

Therefore

$$u + \log E^{P_0}(e^{uL_N}) \geq N\mathrm{F}(u)$$

By taking the exponential of both sides, we get the result. □

A similar upper bound is proved in [3].

Lemma 1.1.5. *There exists a $K > 0$ such that*

$$\forall j \geq 1, \quad E^{P_0}(e^{uL_{jM}}) \leq Kje^j$$

where $M = \frac{1}{\mathrm{F}(u)}$ is the correlation length.

Here are some properties of the free energy function $\mathrm{F}(\cdot)$. For the proofs, see [25] and [27] by Giacomin.

a) $\mathrm{F}(u)$ is 0 on $(-\infty, 0]$ and strictly increasing and positive on $(0, \infty)$.

b) $\mathrm{F}(u)$ is a convex function.

c) $\mathrm{F}(u)$ is real analytic except at the origin.

d) $\mathrm{F}(u) \sim c_1 u^2$, as $u \to 0^+$ for a positive constant c_1.

The estimate in d) shows that $\mathrm{F}(u)$ is not C^2 at the origin but by convexity, it is C^1. In a standard statistical mechanics terminology this means that the system undergoes a *second order phase transition,* in the sense that the non-analyticity of the free energy comes from a singularity in the second derivative of the free energy [25].

Note that

$$\begin{aligned}
\mathrm{F}'(u) &= \lim_{n\to\infty} \frac{1}{N}\frac{\mathrm{d}}{\mathrm{du}} \log E^{P_0}(e^{uL_N}) \\
&= \lim_{n\to\infty} \frac{1}{N}\frac{E^{P_0}(L_N e^{uL_N})}{E^{P_0}(e^{uL_N})} \\
&= \lim_{n\to\infty} E_{\mu_{N,u}}\left(\frac{L_N}{N}\right)
\end{aligned}$$

where $E_{\mu_{N,u}}$ is the expectation with respect to the polymer measure $\mu_{N,u}$. We see that there is a drastic change in the system as u passes from non-positive to positive regime.

Indeed, $\mathrm{F}'(u)$ is the expected fraction of the visits to zero by random walk under the polymer measure and it passes from zero to a positive value. This is an example of transition from *delocalized* to *localized* phase.

We define

$$u_c = \sup\{u : \mathrm{F}(u) = 0\}$$

as the *critical point* of the system. The parameter space is split into two regions:

$$\mathcal{L} := \{u : u > u_c\}$$

is called the *localized phase* and

$$\mathcal{D} := \{u : u \leq u_c\}$$

is called the *delocalized phase.*

From properties of F(·), we see that for 1-dimensional random walk pinning model $u_c = 0$.

1.1.2 Variational Form of the Free Energy

Let's recall some facts on SSRW on $\mathbb{Z}^d$:

Let

$$\tau = \inf\{n \geq 1 : S_n = 0\}$$

be the first return time of random walk to the origin. We can also consider it as an excursion length. Then, as $n \to \infty$, we have:

$$\begin{aligned}
d = 1, \quad & P(S_n = 0) \sim \frac{1}{\sqrt{2\pi n}}, & P(\tau = n) \sim \frac{K_1}{n^{3/2}} \\
d = 2, \quad & P(S_n = 0) \sim \frac{A_2}{n}, & P(\tau = n) \sim \frac{K_2}{n(\log n)^2} \\
d \geq 3, \quad & P(S_n = 0) \sim \frac{A_d}{n^{d/2}}, & P(\tau = n) \sim \frac{K_d}{n^{d/2}}
\end{aligned}$$

Note that, in all cases $P(\tau = n) \sim n^{-c}\phi(n)$ as $n \to \infty$, where $c \geq 1$, and $\phi(n)$ a slowly varying function, that is,

$$\frac{\phi(bn)}{\phi(n)} \to 1 \quad \text{as} \quad n \to \infty, \quad \text{for all } b > 0.$$

Note that

$$L_N \geq k \iff \tau_k \leq N$$

where

$$\tau_1 = \tau, \tag{1.1.3}$$

$$\tau_k = \inf\{i > \tau_{k-1} : S_i = 0\}, \quad k \geq 2 \tag{1.1.4}$$

time of the k^{th} return. Therefore, for $a > 0$,

$$\begin{aligned} P(L_{ak} \geq k) &= P(\tau_k \leq ak) \\ &= P(\tau_k \leq ak) \\ &\leq e^{\alpha ak} E e^{-\alpha \tau_k} \\ &= e^{\alpha ak} (E e^{-\alpha \tau})^k \\ &= e^{-k(-\log M_\tau(-\alpha) - \alpha a)} \end{aligned}$$

where $M_\tau(t) = Ee^{t\tau}$ is the moment generating function of τ. Since the last inequality true for all $\alpha > 0$, we have

$$P(\tau_k \leq ak) \leq e^{-k \sup_{\alpha>0}(-\log M_\tau(-\alpha) - \alpha a)}$$

Let's define

$$I_\tau(a) = \sup_{\alpha>0}(-\log M_\tau(-\alpha) - \alpha a)$$

which is called *the large deviation rate function* related to τ.

By Large Deviation Theory,

$$\lim_{k\to\infty}\frac{1}{k}\log P(\tau_k\leq ak)=-I_\tau(a)$$

and also

$$\lim_{\epsilon\to0^+}\lim_{k\to\infty}\frac{1}{k}\log P(\frac{\tau_k}{k}\in(a-\epsilon,a+\epsilon))=-I_\tau(a)$$

In a more general context, we have the following lemma. For the proof and more on Large Deviation Theory, see [18].

Lemma 1.1.6. *For a recurrent Markov Chain*

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\log P(L_N\geq\delta N)&=\delta\lim_{N\to\infty}\frac{1}{\delta N}\log P(\tau_{\delta N}\leq\delta^{-1}\delta N)\\&=-\delta I_\tau(\delta^{-1})\end{aligned}$$

Also,

$$\lim_{\epsilon\to0^+}\lim_{N\to\infty}\frac{1}{N}\log P(\frac{L_N}{N}\in(\delta-\epsilon,\delta+\epsilon))=-\delta I_\tau(\delta^{-1})\tag{1.1.5}$$

The following lemma gives a variational formula for the free energy of the model which was proved in [2].

Lemma 1.1.7.

$$\mathrm{F}(u)=\sup_{\delta\in(0,1)}(u\delta-\delta I_\tau(\delta^{-1}))$$

Proof. Let $k > 1$ be a large integer,

$$\begin{aligned}
\mathrm{F}(u) &= \lim_{N\to\infty} \frac{1}{N} \log E^{P_0}(e^{uL_N(S)}) \\
&= \max_{0\le j\le k-1} \lim_{N\to\infty} \frac{1}{N} \log E^{P_0}(e^{uL_N(S)} 1_{\frac{L_N}{N}\in(\frac{j}{k},\frac{j+1}{k}]}) \\
&\le \max_{0\le j\le k-1} \lim_{N\to\infty} \frac{1}{N} \log(e^{u(j+1)N/k} P(\frac{L_N}{N} \ge \frac{j}{k})) \\
&\le \max_{0\le j\le k-1} \lim_{N\to\infty} \frac{1}{N} \log(e^{u(j+1)N/k} e^{-\frac{j}{k} I_\tau(k/j)N}) \\
&\le \sup_{\delta\in(0,1)} (u\delta - \delta I_\tau(\delta^{-1})) + \frac{u}{k}
\end{aligned}$$

If we let k go to infinity, we have

$$\mathrm{F}(u) \le \sup_{\delta\in(0,1)} (u\delta - \delta I_\tau(\delta^{-1}))$$

The reverse inequality is proved in a similar way. □

1.2 General Homogeneous Pinning Model

Polymer pinning models describe the interaction between a directed polymer and a one dimensional defect line. In absence of interaction, the polymer spatial configuration is modeled by $(n, X_n)_{n\ge0}$, where $(X_n)_{n\ge0}$ is a Markov Chain in a state space Σ with a fixed element which we call 0, with law $\mathbf{P}$. The defect line is $\mathbb{N} \times \{0\}$ and each time the polymer touches the line, that is $X_n = 0$, it gets an energy reward or penalty which is either positive or negative. The interaction between polymer and the defect line depends only on the law of the return time to 0 of the Markov chain. For this reason, the general pinning model involves a renewal sequence $\tau = \{\tau_0, \tau_1, \dots\}$ where $\tau_0 = 0$ and $\{\tau_i - \tau_{i-1}\}_{i\geq 1}$ are independent identically distributed positive integer valued random variables. If the polymer is modeled by a Markov chain $(X_n)_{n\geq 0}$, then $\tau = \{\tau_0, \tau_1, \dots\}$ is the set of return times of $(X_n)_{n\geq 0}$ to 0 with the assumption $\tau_0 = 0$, and L_N is $|\tau \cap (0, N]|$. One of the reasons polymer pinning models are studied extensively by physicists and mathematicians is that they show a so called *localization-delocalization* phase transition.

The most interesting cases for polymer pinning models involve a renewal sequence with power law, more precisely,

$$K(n) := \mathbf{P}(\tau_1 = n) = \frac{L(n)}{n^{1+\alpha}} \tag{1.2.1}$$

where $\alpha \geq 0$ and L is a slowly varying function.

Recall that a function $L : (0, \infty) \to (0, \infty)$ is called *slowly varying* at infinity if

$$\lim_{x\to\infty} \frac{L(rx)}{L(x)} = 1$$

for every $r > 0$. A slowly varying function diverges or vanishes at infinity slower than any polynomial.

Recall that a renewal sequence is called *recurrent* if $\sum_{n\geq 1} K(n) = 1$ and *transient* if $\sum_{n\geq 1} K(n) < 1$.

In pinning model literature, it is usually assumed that the renewal sequence τ is recurrent and hence it contains infinitely many points $\mathbf{P}$-a.s..

Observe that in the case $\alpha = 0$, the assumption $\sum_{n\geq 1} K(n) = 1$ implies that $L(\cdot)$ tends to zero as n approaches to infinity.

Note also that

$$\mathbf{E}(\tau_1) = \sum_{n\geq 1} nK(n)$$

is finite for $\alpha > 1$ and infinite for $\alpha < 1$. Therefore, τ occupies a positive fraction of $\mathbb{N}$, and called *positive recurrent* if $\mathbf{E}(\tau_1) < \infty$; whereas the density of τ in $\mathbb{N}$ is zero and it is called *null recurrent* if $\tau_1 < \infty$ a.s. and $\mathbf{E}(\tau_1) = +\infty$.

As we have seen in the previous section, asymptotically this model covers random walk pinning models in any dimension.

The polymer measure is defined in Boltzmann-Gibbs way by

$$\frac{\mathrm{d}\mathbf{P}_{N,u}}{\mathrm{d}\mathbf{P}}(\tau) = \frac{1}{Z_{N,u}} e^{u|\tau\cap(0,N]|} 1_{N\in\tau} \tag{1.2.2}$$

where $Z_{N,u} = \mathbf{E}\left(e^{u|\tau\cap(0,N]|} 1_{N\in\tau}\right)$ is called the partition function of the constrained system.

Remark 1.2.1. *The constraint* $1_{N\in\tau}$ *in the definition of the polymer measure is boundary condition. The free model is defined in the following way:*

$$\frac{\mathrm{d}\mathbf{P}^f_{N,u}}{\mathrm{d}\mathbf{P}}(\tau) = \frac{1}{Z^f_{N,u}} e^{u|\tau\cap(0,N]|} \tag{1.2.3}$$

where $Z^f_{N,u} = \mathbf{E}\left(e^{u|\tau\cap(0,N]|}\right)$. *The constrained model 1.2.2 is more manageable but for most of the results there is no difference between the models. See [25].*

Remark 1.2.2. *Note that the assumption that the renewal sequence is recurrent is harmless because it only changes the exponential factor:*

Let $\Sigma = \sum_{n\geq 1} K(n) < 1$, *then define* $\hat{\Sigma} := K(n)/\Sigma$. *It is easy to see that* $\hat{\Sigma}$ *is recurrent and that the polymer measure and partition function of the model with* $K(\cdot)$ *is the same as the model with* $\hat{\Sigma}$ *provided that* u *is replaced by* $u + \log \Sigma$.

By sub-additivity argument, it is easy to show that the following limit exists:

$$\mathrm{F}(u) = \lim_{N\to\infty} \frac{1}{N} \log \mathbf{E}\left(e^{u|\tau\cap(0,N]|} 1_{N\in\tau}\right) \tag{1.2.4}$$

The function $\mathrm{F}(\cdot)$ is called the *free energy* of the system.

Let

$$l_N = \frac{|\tau \cap (0, N]|}{N}$$

Note that

$$\begin{aligned}\mathrm{F}'(u) &= \lim_{N\to\infty} \log \frac{1}{N} \frac{\mathrm{d}}{\mathrm{d}u} \log \mathbf{E}\left(e^{u|\tau\cap(0,N]|} 1_{N\in\tau}\right) \\ &= \lim_{N\to\infty} \mathbf{E}_{N,u}\left(l_N\right)\end{aligned}$$

$\mathrm{F}'(u)$ is the expected fraction of the visits to zero by the polymer under the polymer measure in the limit as N tends to infinity.

We define

$$u_c = \sup\{u : \mathrm{F}(u) = 0\}$$

The critical point u_c is either zero or positive which depends on whether the renewal is *recurrent*, or *transient* as we see in Theorem 1.2.3.

For the proof of the following theorem which summarizes the main results in this section, see [25] and [27].

Theorem 1.2.3. *For homogeneous pinning model ,*

$$u_c = -\log \sum_{n\geq 1} K(n) \tag{1.2.5}$$

and as $u \searrow u_c$

$$\mathrm{F}(u) \sim (u - u_c)^{\max(1/\alpha,1)} \hat{L}(1/(u-u_c)), \tag{1.2.6}$$

where $\hat{L}$ *is a slowly varying function associated with* L*. Therefore, the transition is of* k^{th} *order if* $\alpha \in (1/k, 1/(k-1))$*. The order of the transition for* $\alpha = 1/k, k \geq 1$*, is either* k^{th} *or* $(k+1)^{th}$ *order, and this depends on the slowly varying function* $L(n)$ *that defines* $K(n)$*.*

In statistical physics literature, the *specific heat exponent* ν is introduced as

$$\nu = 2 - \lim_{u \searrow u_c} \frac{\log \mathrm{F}(u)}{\log(u-u_c)}$$

provided the limit exists. From Theorem 1.2.3, we see that

$$\nu = 2 - \max(1/\alpha, 1). \tag{1.2.7}$$

Note that $\nu > 0$ as soon as $\alpha > 1/2$.

In the next section, we will see that when the disorder is present, the exponent α being greater than 1/2 or less than 1/2 will be important for the disorder relevance.

1.3 Disordered Pinning Model

In previous section, we have seen that the phase transition of the homogeneous pinning model can be of any order, from first order to infinite, depending on the distribution $K(\cdot)$ of the renewal sequence τ. In this section, we summarize the results on the effect of disorder on the pinning/depinning transition.

The main questions of physical interest are

- whether for given β, u the polymer is "pinned", meaning that it places a positive fraction of its monomers at 0 for large N with high probability.
- the location and nature of the depinning transition as β and u vary
- the effect of disorder, as seen by comparing the transition to the annealed case

Let $\omega = \{\omega_i\}_{i\geq 0}$ be a collection of i.i.d. random variables with law $\mathbb{P}$, which plays the role of quenched randomness in the system. We assume that

$$\mathbb{E}(\omega_1) = 0, \text{ and } \mathbb{E}(\omega_1^2) = 1$$

where $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$.

In this disordered pinning model, the Hamiltonian of the system and the *quenched polymer measure* is defined to be in the following form:

$$H_{N,\beta}(\tau) = \sum_{j=1}^{N} (u + \beta\omega_j)\delta_j$$

$$\frac{d\mu_N^{\beta,u,q}}{d\mathbf{P}}(\tau) = \frac{1}{Z_N^{\beta,u,q}} e^{H_{N,\beta}(\tau)} 1_{\delta_N}$$

where $\delta_j := 1_{j \in \tau}, u \in \mathbb{R}, \beta \geq 0$. Superscript q denotes the quenched randomness and the normalizing sum is called the quenched partition function:

$$Z_N^{\beta,u,q} = \mathbf{E}\left(e^{H_{N,\beta}(\tau)} 1_{\delta_N}\right)$$

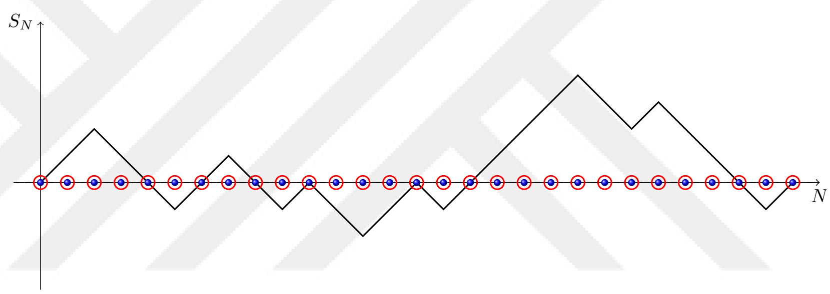

Figure 1.2: 1-D disordered pinning model: disorder is only on the defect line.
$\circ$: potential. $\bullet$: disorder.

There have been a great progress in the mathematical understanding of the role of the quenched randomness in the polymer models in last ten years. In particular, most of the physicists' predictions on the effect of disorder on pinning/depinning transition, the location of the critical points, and the path properties of the polymer in the two thermodynamic phases are justified rigorously, see [3], [19], [24] and [26].

Expectation with respect to the polymer measure will be denoted by $\mathbf{E}_N^{\beta,u,q}(\cdot)$

The following theorem shows that the thermodynamic limit of free energy exists and does not depend on the realization of the randomness ω :

Theorem 1.3.1. *If* $\mathbb{E}(|\omega_1|) < \infty$, *then the limit*

$$F^q(\beta, u) := \lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta,u,q} = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left(\log Z_N^{\beta,u,q}\right)$$

exists for every $\beta \geq 0, u \in \mathbb{R}$ *and it is* $\mathbb{P}$*-a.s. constant.*

For the proof see [25].

$F^q(\beta, u)$ is convex in (β, u), non-decreasing, continuous everywhere and differentiable almost everywhere in u as a consequence of convexity.

1.3.1 Localization-Delocalization Transition

Observe that

$$F(\beta, u) \geq \frac{1}{N} \mathbb{E} \log \mathbf{E}\left(e^{\sum_{j=1}^{N}(u+\beta\omega_j)\delta_n} 1_{\tau_1=N}\right) = \frac{u}{N} + \frac{1}{N} \log K(N)$$

so this simple argument shows that

$$F(\beta, u) \geq 0.$$

Since $F(\beta, u)$ is non-decreasing for a given β, the localization/delocalization critical point is defined to be

$$u_c^q(\beta) := \sup\{u : F^q(\beta, u) = 0\} \tag{1.3.1}$$

and the the function $\beta \to u_c^q(\beta)$ is called the *critical line.*

The parameter space (β, u) is split into two regions:

$$\mathcal{L} := \{(\beta, u) : \beta \geq 0, u > u_c^q(\beta)\}$$

is called the *localized phase* and

$$\mathcal{D} := \{(\beta, u) : \beta \geq 0, u \leq u_c^q(\beta)\}$$

is called the *delocalized phase.*

The critical point $u_c(\beta)$ is strictly decreasing as a function of β and since $u_c(\beta)$ is a concave function,

$$u_c^q(\beta) \to -\infty \text{ as } \beta \to \infty.$$

This means that even the disorder on average is repulsive, the defect line can pin the polymer. See [2] and [25].

The *annealed free energy* and the *annealed critical point* are defined in the following way:

$$F^a(\beta, u) := \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} Z_N^{\beta,u,q} \tag{1.3.2}$$

$$u_c^a(\beta) := \sup\{u : F^a(\beta, u) = 0\} \tag{1.3.3}$$

It is an easy application of Jensen's inequality that

$$F^q(\beta, u) \leq F^a(\beta, u) \tag{1.3.4}$$

therefore

$$u_c^a(\beta) \leq u_c^q(\beta) \text{ for all } \beta \geq 0. \tag{1.3.5}$$

Note also that

$$u_c^a(\beta) = u_c^q(0) - \log M(\beta) \tag{1.3.6}$$

where $M(\beta) = \mathbb{E}(e^{\beta\omega_1})$, in particular $\log M(\beta) = \frac{\beta^2}{2}$ for the Gaussian disorder.

Thanks to a series of papers [3, 4, 5, 19, 23, 24, 26, 28, 47, 55, 56] by Alexander, Zygouras and Derrida, Giacomin, Lacoin, Toninelli the effect of disorder on the depinning transition is well understood in most of the cases under Gaussian disorder. We can summarize their main results as follows:

(i) for every $\alpha > 0$, $u_c^a(\beta) \neq u_c^q(\beta)$ if β is large.

(ii) for every $\alpha > 1/2$ and $\beta > 0$, $u_c^a(\beta) \neq u_c^q(\beta)$.

(iii) for every $\alpha < 1/2$, $u_c^a(\beta) = u_c^q(\beta)$ if β is small.

(iv) for $\alpha = 0$ and $\beta > 0$, $u_c^a(\beta) = u_c^q(\beta)$.

(v) for $\alpha = 1/2$ and $\sum_{n\geq 1} \frac{1}{nL(n)^2} < \infty$, $u_c^a(\beta) = u_c^q(\beta)$ if β is small.

In particular, they showed that

(vi) for $1/2 < \alpha < 1$, there exists a constant c and slowly varying function $\tilde{L}$ associated with L such that for all small β

$$c^{-1}\beta^{\frac{2\alpha}{2\alpha-1}}\tilde{L}(\frac{1}{\beta}) < u_c^q(\beta) - u_c^a(\beta) < c\beta^{\frac{2\alpha}{2\alpha-1}}\tilde{L}(\frac{1}{\beta})$$

(vii) for $\alpha = 1$,

$$c^{-1}\beta^2 \tilde{L}(\frac{1}{\beta}) < u_c^q(\beta) - u_c^a(\beta)$$

(viii) for $\alpha > 1$,

$$c^{-1}\beta^2 < u_c^q(\beta) - u_c^a(\beta) < c\beta^2$$

The case $\alpha = 1/2$ is marginal and not fully understood. It is believed that, see [**?**]

$$\text{for every } \beta > 0,\ u_c^a(\beta) < u_c^q(\beta) \ \text{ as long as } \sum_{n \geq 1} \frac{1}{nL(n)^2} = \infty.$$

For the proofs of statements (i) and (ii), see [3], [19] and [25].

Statements (iii) and (v) are proved by using second moment method by Alexander [3] and by using replica-coupling approach by Toninelli [56] for Gaussian disorder; and then by using a Martingale approach by Lacoin for general disorder case [47].

Statements (vi) and (vii) are proved in [2], [3], [4], [19].

Chapter 2

Directed Polymers in a Random Environment with a Defect Line

2.1 Physical Motivation

The directed polymer in a random environment is a model in the theory of disordered systems. The $1+1$ dimensional version of the model first appeared in statistical physics literature by Huse and Henley [36] in 1985 as a model for the interface in two-dimensional Ising models with random exchange interaction. This was followed by other physics papers by Huse, Henley and Fisher [37], Kardar [45], Kardar and Zhang [44]. Since then it has been used to describe a variety of phenomena: formation of magnetic domains in spin-glasses [36], vortex lines in superconductors [50], turbulence in viscous incompressible fluids (Burger turbulence) [10] and KPZ equation [43].

The statistical mechanics of an elastic object in a random environment has been the focus of many studies in the theoretical physics literature for more than a two decades [36], [44], [52]. In particular, understanding the competition between extended and point defects and their role on the pinning phenomena has been an important quest for physicists working on high-temperature superconductors. In physics literature, an early study of pinning/depinning transition of directed polymer in random environment(point defect) with a defect line(extended defect) was done by Kardar [42], who investigated numerically the pinning of $1+1$ dimensional directed polymer in a random environment to the defect line. The polymer was found to depin from the line defect, if the pinning potential of the defect line is smaller than a certain threshold value. Critical behavior associated with the depinning transition was later investigated by Zapatocky and Halpin-Healy [57]. The results of Kardar, and Zapatocky and Halpin-Healy were challenged by the work of Tang and Lyuksyutov [54], who argued that a directed polymer is always localized, although weakly, to the defect line in $1+1$ dimensions, and claim that the depinning transition only exists above $1+1$ dimensions. Their conclusion was supported by Balents and Kardar [7], who performed numerical simulations and developed a functional renormalization group analysis. On the other hand, analysis of directed polymer itself by Kolomeisky and Strately [46], have led to a different conclusion when different renormalization group Ansatz were used. In a later work by Hwa and Natterman [39], a systematic analysis of the competition in pinning between point and line defects for a directed polymer was investigated. By using the known results for the directed polymer in a random environment in $1+1$ dimension in the absence of defect line, they constructed a renormalization group analysis directly in $1+1$ dimension, which is the critical dimension for this problem. Their results support the conclusion of Tang and Lyuksyutov, that the polymer is always pinned at and below $1+1$ dimensions. All these physics results need to be justified rigorously and our work is toward understanding the depining transition of directed polymer in a random environment with a defect line.

2.2 Mathematical Formulation of the Problem

2.2.1 DPRE model

The d-dimensional integer lattice version can be formulated as follows: Let P be the distribution of the simple symmetric random walk $S = \{S_j, j \geq 0\}$ on $\mathbb{Z}^d$ starting at the origin, $d \geq 1$. We shall represent the polymer chain as a graph $\{(j, S_j)\}_{j=1}^n$ in $\mathbb{N} \times \mathbb{Z}^d$, so that the polymer lives in $(1 + d)$ dimensional discrete lattice and stretches in the direction of the first coordinate. Each point $(j, S_j) \in \mathbb{N} \times \mathbb{Z}^d$ stands for the position of the j-th monomer in this picture.

The random environment is described by $V = \{v(i, x) : i \geq 1, x \in \mathbb{Z}^d\}$ which is an independent and identically distributed collection of random variables defined on a probability space $(\Xi, \mathcal{G}, Q)$ such that $E^Q[\exp(\beta v(i, x))] < \infty$, for all $\beta \in \mathbb{R}$. Random environment represents the *bulk disorder* or impurities in the system. In this dissertation, we assume that the disorder distribution is Gaussian with zero mean and unit variance.

We define the Hamiltonian of the system which gives the energy of a polymer $\{(j, S_j)\}_{j=1}^n$ by

$$H_N(S) = \sum_{j=1}^{N} v(j, S_j)$$

and the *quenched polymer measure* is defined in the usual Boltzmann-Gibbs way by

$$\frac{d\mu_N^{\beta,q}}{dP}(S) = \frac{1}{Z_N^{\beta,q}} e^{\beta H_N(S)} \tag{2.2.1}$$

where $\beta > 0$ is the inverse temperature and

$$Z_N^{\beta,q} = E^P\left[e^{\beta H_N(S)}\right]$$

is the normalizing constant which is called *the quenched partition function.*

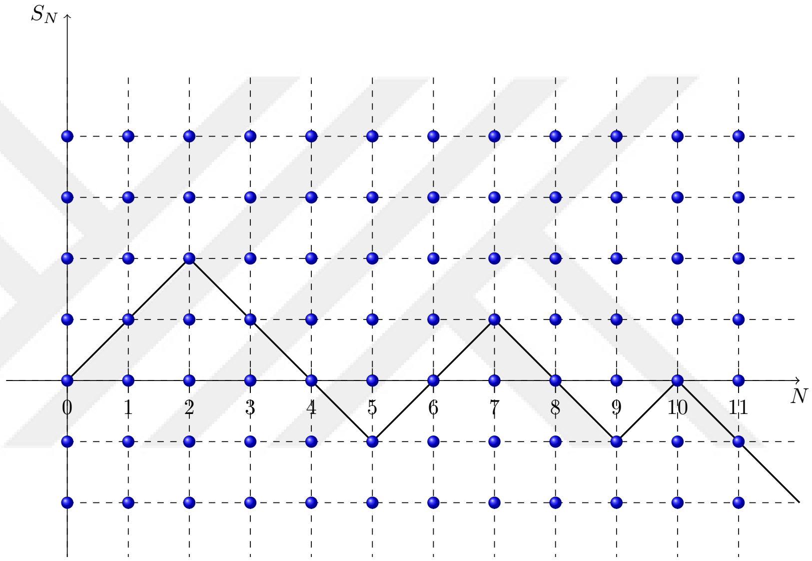

Figure 2.1: $1 + 1$ DPRE model

Note that quenched polymer measure and partition function are random quantities on the environment space $(\Xi, \mathcal{G}, Q)$, and the superscript q refers to this quenched randomness.

Note that when $\beta = 0$, the polymer measure 2.2.1 is the usual random walk measure and under the random walk measure, paths exhibit diffusive behavior.

The first rigorous mathematical work on directed polymers was done by Imbrie and Spencer [41] in 1988 and they proved that in dimension $d \geq 3$ with small enough β, the end point of the polymer scales as $n^{1/2}$, i.e. the polymer is diffusive. Bolthausen [9], by normalizing the partition function $W_n^{\beta,q} = Z_n^{\beta,q}/E^Q[Z_n^{\beta,q}]$ placed the polymer model in the framework of martingales, and observed that the almost sure limit of the rescaled partition function $W_\infty = \lim_{n\to\infty} W_n^{\beta,q}$ is subject to a dichotomy: there are only two possibilities for the positivity of the limit; $Q(W_\infty > 0) = 1$ or $Q(W_\infty = 0) = 1$ because the event $\{W_\infty = 0\}$ is a tail event. He also improved the result of Imbre and Spencer to a central limit theorem for the end point of the walk which means that in $d \geq 3$ entropy dominates at high enough temperature, namely the polymer behaves essentially as if it does not live in bulk disorder. In the literature [11, 15, 16], a natural manner for measuring the disorder due to the random environment is to call $Q(W_\infty > 0) = 1$ as *weak disorder* and $Q(W_\infty = 0) = 1$ as *strong disorder*. Comets and Yoshida [17], showed that there exists a critical value $\beta_c = \beta_c(d, v) \in [0, \infty]$ with $\beta_c = 0$, for $d = 1, 2$ and $0 < \beta_c \leq \infty$ for $d \geq 3$ such that $Q(W_\infty > 0) = 1$ If $\beta \in \{0\} \cup (0, \beta_c)$ and $Q(W_\infty = 0) = 1$ if $\beta > \beta_c$. The case $\beta_c = \infty$ can only occur if the environment random variable $v(i, x)$ is a.s. bounded.

2.2.2 DPRE with defect line model

The usual set up for directed polymer in a random environment which we described above doesn't contain a *defect line*. In some pinning models, disorder is present only at the defect line $\mathbb{N} \times \{0\}$, so there is no bulk disorder. In our model, we assumed that the polymer lives in *bulk disorder* and when it visits the origin, or the axis in space time, it encounters an additional deterministic potential, $u \in \mathbb{R}$, which plays the role of the strength of the *defect line*. To incorporate the effect of defect line to the model, we define the Hamiltonian of the system and the *quenched polymer measure* in the following way:

$$H_N^u(S) = \sum_{j=1}^{N}(v(j,S_j) + u1_{S_j=0}) \tag{2.2.2}$$

$$= H_N(S) + L_N(S), \tag{2.2.3}$$

$$\frac{d\mu_N^{\beta,u,q}}{dP}(S) = \frac{1}{Z_N^{\beta,u,q}} e^{\beta H_N^u(S)} \tag{2.2.4}$$

where $\beta > 0$ is the inverse temperature and

$$Z_N^{\beta,u,q} = E^P\left[e^{\beta H_N^u(S)}\right]$$

is the *the quenched partition function.* Here P is the distribution of the SSRW when $S_0 = 0$ a.s.. In this mathematical framework, the random environment V corresponds to "background point defects" and the potential u at the origin in space-time picture is the analog of "the extended defect" or "the defect line" in physics literature.

The *quenched free energy* is defined by

$$f^q(\beta, u) := \lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta,u,q}$$

where this limit is taken $Q-$ a.s.. The existence and non-randomness of this limit will be proved in the next section.

The effect of the quenched disorder on the phase transition is quantified by comparing the *quenched model* to the corresponding *annealed model*, which is obtained by averaging the quenched Boltzmann-Gibbs weight over the disorder to give the annealed Boltzmann-Gibbs weight, that is, the *annealed polymer measure* is defined by

$$\frac{d\mu_N^{\beta,u,a}}{dP}(S) = \frac{1}{Z_N^{\beta,u,a}} E^Q\left[e^{\beta H_N^u(S)}\right] \tag{2.2.5}$$

where

$$Z_N^{\beta,u,a} = E^P\left[E^Q\left[e^{\beta H_N^u(S)}\right]\right]$$

is the *the annealed partition function.* Note that by Fubini's theorem and the environment distribution being Gaussian,

$$E^P\left[E^Q\left[e^{\beta H_N^u(S)}\right]\right] = e^{\frac{\beta^2}{2}N} E^P\left[e^{\beta u L_N}\right].$$

The *annealed free energy* is defined by

$$\begin{aligned} f^a(\beta,u) &:= \lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta,u,a} \\ &= \lim_{N\to\infty} \frac{1}{N} \log\left(e^{\frac{\beta^2}{2}N} E^P[e^{\beta u L_N}]\right) \\ &= \frac{\beta^2}{2} + \mathrm{F}(\beta u), \end{aligned}$$

where $\mathrm{F}(\cdot)$ is the free energy of the 1-D SSRW pinning model.

By Jensen's inequality,

$$f^q(\beta,u) \le f^a(\beta,u) \text{ for all } \beta \ge 0 \text{ and } u \in \mathbb{R}. \tag{2.2.6}$$

We define the *quenched critical point* as

$$u_c^q(\beta) := \inf\{u > 0 : f^q(\beta, u) > f^q(\beta, 0)\}$$

and the *annealed critical point* as

$$u_c^a(\beta) := \inf\{u > 0 : f^a(\beta, u) > f^a(\beta, 0)\}.$$

Since the quenched free energy is nonrandom, $u_c^q(\beta)$ is constant Q-a.s.. Note that since the critical point u_c for 1-D SSRW random walk pinning model is zero, $u_c = 0$,

$$u_c^a(\beta) = 0 \text{ for all } \beta > 0.$$

Therefore, we have

$$u_c^a(\beta) \leq u_c^q(\beta) \text{ for all } \beta \geq 0. \tag{2.2.7}$$

2.3 Self-averaging of the Free Energy and Critical Point

In this section, we will first prove the existence of the quenched free energy for the constrained model and then by using concentration of measure property of Gaussian processes, we relate the quenched free energy of the constrained model with the quenched free energy of the unconstrained model. We will mostly follow the methods developed in the papers [11] and [12].

We will use the following Gaussian concentration lemma:

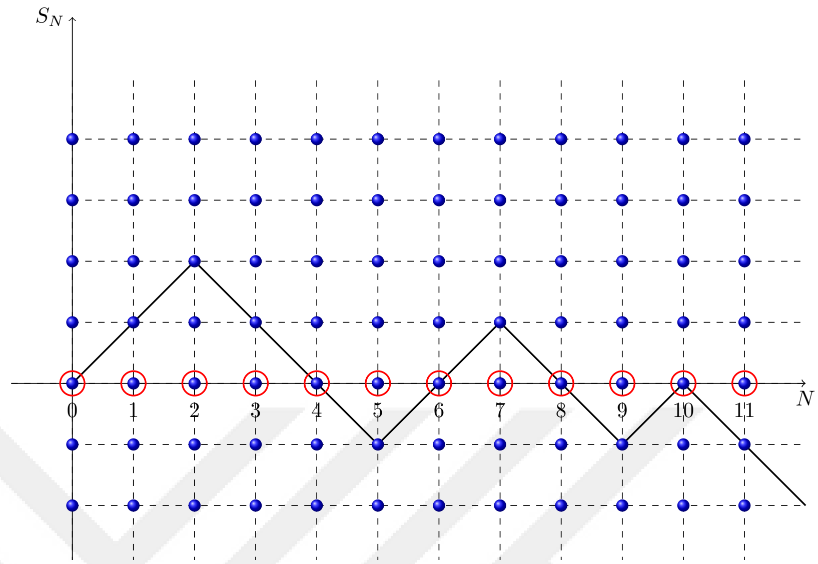

Figure 2.2: $1+1$ DPRE model with defect line. $\circ$: potential. $\bullet$: disorder.

Proposition 2.3.1. *Let $F : \mathbb{R}^N \to \mathbb{R}$ be a Lipschitz function with constant A, that is,*

$$|F(x) - F(y)| \leq A||x - y||, \quad x, y \in \mathbb{R}^N,$$

where $||\cdot||$ denotes the Euclidean norm on $\mathbb{R}^N$. Then, if $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_N)$ is a vector of i.i.d. standard normal random variables, we have

$$\mathbf{P}(|F(\mathbf{Z}) - \mathbf{E}(F(\mathbf{Z}))| > t) \leq \exp{(-\frac{t^2}{2A^2})}$$

for any $t > 0$.

For the proof of the above proposition, see [40].

For the sake of the notational convenience, we will not write β, u, q in the partition function in some places. Let's introduce some notations:

$$Z_N(x) = Z_N(x;V) := Z_{N,x}^{\beta,u,q} = E^{P_x}\left[e^{\sum_{j=1}^N \beta(v(j,S_j)+u1_{S_j=0})}\right]$$

$$Z_N(x,y) = Z_N(x,y;V) := E^{P_x}\left[e^{\sum_{j=1}^N \beta(v(j,S_j)+u1_{S_j=0})}1_{S_N=y}\right]$$

where P_x is the SSRW measure when $S_0 = x$ a.s.

Let θ_n be the time shift operator of order n on the environment V:

$$\text{for all } x \in \mathbb{Z}^d \text{ and } k,n \geq 1,\ (\theta_n v)(k,x) = v(k+n,x).$$

The proof of the following lemma is an easy application of the Markov property of SSRW:

Lemma 2.3.2. *For every $x,y \in \mathbb{Z}^d$ and every integer N, M, we have*

$$Z_{N+M}(x) = \sum_y Z_N(x,y;V)Z_M(y;\theta_N V) \tag{2.3.1}$$

$$Z_{N+M}(x,z) = \sum_y Z_N(x,y;V)Z_M(y,z;\theta_N V) \tag{2.3.2}$$

We use the notation $N \rightsquigarrow x$ to mean that there is a SSRW path from 0 to x in N steps.

The following is also standard.

Lemma 2.3.3. *For any $x \in \mathbb{Z}^d$ with $N \rightsquigarrow x$,*

$$E^{\mathcal{Q}}[\log Z_N(0,x;V)] \leq \frac{1}{2}E^{\mathcal{Q}}[\log Z_{2N}(0,0;V)].$$

Proof. From Lemma 2.3.2, for any $x \in \mathbb{Z}^d$ with $N \rightsquigarrow x$, we have

$$Z_{2N}(0,0;V) \geq Z_N(0,x;V)Z_N(x,0;\theta_N V) \tag{2.3.3}$$

By taking expectation and using the i.i.d. property of V, we get

$$E^{\mathcal{Q}}[\log Z_{2N}(0,0;V)] \geq E^{\mathcal{Q}}[\log Z_N(0,x;V)] + E^{\mathcal{Q}}[\log Z_N(x,0;\theta_N V)] \tag{2.3.4}$$

$$= 2E^{\mathcal{Q}}[\log Z_N(0,x;V)] \tag{2.3.5}$$

□

2.3.1 The Constrained Model

Due to periodicity of SSRW, we will assume that N, M are even integers in this subsection.

In the constrained model, polymer measure and the corresponding partition function are defined in the following way:

$$\frac{d\mu_N^{\beta,u,q,c}}{dP}(S) = \frac{1}{Z_N^{\beta,u,q,c}} e^{\sum_{j=1}^{N} \beta(v(j,S_j)+u1_{S_j=0})} 1_{S_N=0}$$

where $\beta > 0$ is the inverse temperature and

$$Z_N^{\beta,u,q,c} = E^P\left[e^{\sum_{j=1}^{N} \beta(v(j,S_j)+u1_{S_j=0})} 1_{S_N=0}\right]$$

is the *the quenched constrained partition function.*

By the Markov property of SSRW, we have

$$
\begin{aligned}
Z_{M+N}(0,0) &\geq E^{P_0}\left[e^{\sum_{j=1}^{M+N}\beta(v(j,S_j)+u1_{S_j=0})}1_{S_N=0}1_{S_{M+N}=0}\right] \\
\geq\ & E^{P_0}\left[e^{\sum_{j=1}^{N}\beta(v(j,S_j)+u1_{S_j=0})}1_{S_N=0}\right]E^{P_0}\left[e^{\sum_{j=1}^{M}\beta(\theta_N v(j,S_j)+u1_{S_j=0})}1_{S_M=0}\right] \\
\geq\ & Z_N(0,0;V)Z_M(0,0;\theta_N V)
\end{aligned}
$$

By taking logarithm and then expectation, we get

$$
E^{\mathcal{Q}}[\log Z_{N+M}(0,0;V)] \geq E^{\mathcal{Q}}[\log Z_N(0,0;V)] + E^{\mathcal{Q}}[\log Z_M(0,0;\theta_N V)] \tag{2.3.6}
$$

$$
= E^{\mathcal{Q}}[\log Z_N(0,0;V)] + E^{\mathcal{Q}}[\log Z_M(0,0;V)] \tag{2.3.7}
$$

Therefore, $\{E^{\mathcal{Q}}[\log Z_N(0,0;V)]\}_{N\geq 1}$ forms a super-additive sequence and hence by Sub-additive Lemma, the following limit exists:

$$
\lim_{N\to\infty}\frac{1}{N}E^{\mathcal{Q}}[\log Z_N(0,0;V)] = \sup_{N\geq 1}\frac{1}{N}E^{\mathcal{Q}}[\log Z_N(0,0;V)]
$$

Note that by using Jensen's inequality, Fubini's theorem and $L_N \leq N/2$, we have

$$
\begin{aligned}
E^{\mathcal{Q}}[\log Z_N(0,0;V)] &\leq \log E^{\mathcal{Q}}[Z_N(0,0;V)] \\
&\leq \log E^{P}[e^{\frac{\beta^2}{2}N+uL_N}] \\
&\leq \log\left(e^{\frac{\beta^2+u}{2}N}\right)
\end{aligned}
$$

Therefore, for all $N \geq 1$,

$$\frac{1}{N} E^{\mathcal{Q}}[\log Z_N(0,0;V)] \leq \frac{\beta^2 + u}{2} \tag{2.3.8}$$

Since the distribution of the random environment $V = \{v(i,x) : i \geq 1,\ x \in \mathbb{Z}^d\}$ is invariant under time shift operators θ_n for all $n \geq 1$, by Sub-additive Ergodic Theorem [21], the quenched free energy of the constrained model exists and $\mathcal{Q}$-a.s. constant:

$$f^{q,c}(\beta, u) = \lim_{N\to\infty} \frac{1}{N} \log Z_N(0,0;V) = \lim_{N\to\infty} \frac{1}{N} E^{\mathcal{Q}}[\log Z_N(0,0;V)].$$

Being almost sure constant is called the *self averaging property of* the free energy.

2.3.2 The Free Model

In this section we prove the existence of the free energy of the unconstrained model.

Let's consider the set $\Lambda = \{(i,x) : 1 \leq i \leq N, i \rightsquigarrow x\}$, where $i \rightsquigarrow x$ means that there is a SSRW path from 0 to x in i steps. Let $M = |\Lambda|$

By ordering the elements of the set Λ, we can use it as an index set for $\mathbb{R}^M$. For a given two SRRW paths γ, π of length N, let's define the following vectors v^γ, w^π in $\mathbb{R}^M$ as follows:

$$a^\gamma(i,x) = 1_{\gamma(i)=x}$$

We define a function $F : \mathbb{R}^M \to \mathbb{R}$ by

$$F(z) \quad = \quad \log E^P \left[e^{\sum_{i=1}^N \beta (z_{i,S_i} + u 1_{S_i=0})} \right]$$

$$= \log E^P \left[e^{\sum_{(i,x)\in\Lambda} \beta(z_{i,x} 1_{S_i=x} + u 1_{S_i=0})} \right]$$

$$= \log E^P \left[e^{\beta(a^S \cdot z + \sum_{(i,x)\in\Lambda} u 1_{S_i=0})} \right].$$

By the Cauchy-Schwartz's inequality, we get

$$|(a^S \cdot z + \sum_{(i,x)\in\Lambda} u 1_{S_i=0}) - (a^S \cdot z' + \sum_{(i,x)\in\Lambda} u 1_{S_i=0})| \le ||a^S|| \cdot ||z - z'|| \le \sqrt{N}||z - z'||$$

Therefore, F is a Lipschitz function with constant at most $\beta\sqrt{N}$.

By using the concentration result from Proposition 2.3.1, we have

$$E^{\mathcal{Q}}[e^{\frac{1}{\sqrt{N}}|\log Z_N(x) - E^{\mathcal{Q}} Z_N(x)|}] = \int_0^\infty \mathcal{Q}(|\log Z_N(x) - E^{\mathcal{Q}} Z_N(x)| \ge \sqrt{N}\log t)dt$$
$$\le 1 + \int_1^\infty e^{\frac{-(\log t)^2}{2\beta^2}} dt$$
$$= K(\beta) < \infty$$

Therefore, we have the following proposition:

Proposition 2.3.4. *a) For any* $t > 0$,

$$\mathcal{Q}(|\log Z_N^{\beta,u,q} - E^{\mathcal{Q}} Z_N^{\beta,u,q}| > t) \le e^{-\frac{t^2}{2N\beta^2}}$$

b) For any $t > 0$ *and* $N \rightsquigarrow x$,

$$\mathcal{Q}(|\log Z_N^{\beta,u,q}(x) - E^{\mathcal{Q}} Z_N^{\beta,u,q}(x)| > t) \le e^{-\frac{t^2}{2N\beta^2}}$$

c) There exists a constant $K = K(\beta) > 0$ such that

$$E^{\mathcal{Q}}[e^{\frac{1}{\sqrt{N}}|\log Z_N^{\beta,u,q}(x) - E^{\mathcal{Q}} Z_N^{\beta,u,q}(x)|}] \leq K, \quad N \geq 1, \; N \rightsquigarrow x$$

It is easy to see that the partition function of the unconstrained system of length N is bounded below by that of the constrained system of the same size, therefore

$$E^{\mathcal{Q}}[\log Z_N^{\beta,u,q}(0,0)] \leq E^{\mathcal{Q}}[\log Z_N^{\beta,u,q}].$$

We will now show that we can bound from above the expected value of the logarithm of the partition function of the system of length N by the expected value of the logarithm of the partition function of the constrained system of length $2N$ plus a term of order $o(N)$ as the system size tends to infinity.

Let $\epsilon = \frac{1}{\sqrt{N}}$ and consider,

$$\begin{aligned}
&E^{\mathcal{Q}}[\log Z_N] \leq \frac{1}{\epsilon} \log E^{\mathcal{Q}}[Z_N^{\epsilon}] \\
&= \frac{1}{\epsilon} \log E^{\mathcal{Q}} \left[\left(\sum_{x:N \rightsquigarrow x} Z_N(0,x) \right)^{\epsilon} \right] \\
&\leq \frac{1}{\epsilon} \log E^{\mathcal{Q}} \left[\sum_{x:N \rightsquigarrow x} Z_N(0,x)^{\epsilon} \right] \\
&= \frac{1}{\epsilon} \log E^{\mathcal{Q}} \left[\sum_{x:N \rightsquigarrow x} e^{\epsilon(\log Z_N(0,x) - E^{\mathcal{Q}}[\log Z_N(0,x)])} e^{\epsilon E^{\mathcal{Q}}[\log Z_N(0,x)]} \right] \\
&\leq \frac{1}{\epsilon} \log \left(e^{\frac{\epsilon}{2} E^{\mathcal{Q}}[\log Z_{2N}(0,0)]} E^{\mathcal{Q}} \left[\sum_{x:N \rightsquigarrow x} e^{\epsilon(\log Z_N(0,x) - E^{\mathcal{Q}}[\log Z_N(0,x)])} \right] \right) \\
&\leq \frac{1}{\epsilon} \log \left(e^{\frac{\epsilon}{2} E^{\mathcal{Q}}[\log Z_{2N}(0,0)]} \left(\sum_{x:N \rightsquigarrow x} E^{\mathcal{Q}} \left[e^{\epsilon(\log Z_N(0,x) - E^{\mathcal{Q}}[\log Z_N(0,x)])} \right] \right) \right)
\end{aligned}$$

$$\leq \frac{1}{\epsilon} \log \left(K(2N+1)^d e^{\frac{\epsilon}{2} E^{\mathcal{Q}}[\log Z_{2N}(0,0)]} \right)$$
$$\leq \frac{d}{\epsilon} \log \left(K(2N+1) \right) + \frac{1}{2} E^{\mathcal{Q}}[\log Z_{2N}(0,0)]$$

In the first inequality we used Jensen's inequality and in the second inequality we used the fact that for non-negative numbers $a_1, a_2, \ldots, a_n$ and $0 < \epsilon < 1$,

$$\left(\sum_{i=1}^{n} a_i \right)^{\epsilon} \leq \sum_{i=1}^{n} a_i^{\epsilon}.$$

In the third and fifth inequalities, we used Lemma 2.3.3 and part c) of Proposition 2.3.4, respectively.

Hence, we have

$$E^{\mathcal{Q}}[\log Z_N(0,0)] \leq E^{\mathcal{Q}}[\log Z_N] \leq d\sqrt{N} \log \left(K(2N+1) \right) + \frac{1}{2} E^{\mathcal{Q}}[\log Z_{2N}(0,0)]$$

Dividing each term by N and then letting $N \to \infty$, we get

$$f^{q,c}(\beta, u) \leq f^{q}(\beta, u) \leq f^{q,c}(\beta, u) \ \mathcal{Q} \text{ a.s.}$$

Therefore, the quenched free energy of the original model exists and equals to the quenched free energy of the constrained model:

Theorem 2.3.5. *For every $\beta > 0$ and $u \in \mathbb{R}$,*

$$f^{q}(\beta, u) = \lim_{N \to \infty} \frac{1}{N} \log Z_N^{\beta,u,q} = \lim_{N \to \infty} \frac{1}{N} E^{Q}[\log Z_N^{\beta,u,q,c}] \tag{2.3.9}$$

exists $\mathcal{Q}$-a.s.

Chapter 3

Main Results

3.1 Statement of the Main Results and Overview of the Proofs

In this chapter, we present our main results, and develop the necessary tools to prove them. We use coarse graining procedure to introduce a *coarse grained lattice* and a site percolation model on it by covering the first quadrant of the plane by rectangular boxes of side length proportional to the *annealed correlation length.* These rectangular boxes are associated with the vertices of the coarse grained lattice. Then by second moment method each site of the coarse grained lattice is identified as *open* if the disorder in the associated box is favorable for the polymer to stay in with a probability at least p, which can be made as close to 1 as we wish by tuning the parameters of the model, and otherwise the site is identified as *closed.* We then use recent results of [20] and [31] for Lipschitz percolation to identify an *infinite open path from the origin* for which the fraction of links which lie on the x-axis(defect line) can be made as close to 1 as we wish by letting the site percolation density be close to 1. Coarse graining procedure produces a k-*dependent percolation* model but by using stochastic domination [49], we can transfer the results of independent model to the dependent one.

Theorem 3.1.1. *Consider the* $1+1$ *dimensional directed polymer in a random environment with a defect line model which has the Hamiltonian defined as in (2.2.2). Suppose that* $V = \{v(i,x) : i \geq 1, x \in \mathbb{Z}^d\}$ *is a collection of independent and identically distributed standard normal random variables. Then, given* $0 < \epsilon < 1$, *there exists a* $K = K(\epsilon)$ *as follows:*

Provided that β *and* βu *are sufficiently small and* $u \geq K\beta$, *we have*

$$f^a(\beta, u) \geq f^q(\beta, u) \geq (1-\epsilon) f^a(\beta, u).$$

In particular, we have

$$u_c^a(\beta) \leq u_c^q(\beta) \leq u_c^a(\beta) + K(\epsilon)\beta.$$

Remark 3.1.2. *Note that* $u_c^a(\beta) = 0$ *for all* $\beta \geq 0$.

3.2 Some Preliminaries

In this section, we assume that C_L, c_s are two positive constants such that $C_L \geq 1 > 3\,c_s > 0$.

For two random walk paths $S^1 = (S^1_0, S^1_1, \cdots)$ and $S^2 = (S^2_0, S^2_1, \cdots)$, define

$$B_N(S^1, S^2) = \sum_{i=1}^{N} 1_{S^1_i = S^2_i} \tag{3.2.1}$$

as the overlap of two SSRW paths up to time N.

Recall also that

$$L_N := L_N(S) = \sum_{i=1}^{N} 1_{S_i=0}. \tag{3.2.2}$$

Lemma 3.2.1. *For any $N \geq 1$, $k \geq 1$, $u \geq 0$, and $x \in \mathbb{Z}$, we have*

$$E^{P_x} e^{uL_{kN}} \leq \left(E^{P_0} e^{u(L_N+1)}\right)^k$$

Proof. Note that for any $y \in \mathbb{Z}$, by Markov property of SSRW

$$E^{P_y} e^{uL_N} \leq E^{P_0} e^{u(L_N+1)}. \tag{3.2.3}$$

Therefore,

$$\begin{aligned} E^{P_x} e^{uL_{kN}} &= \sum_y E^{P_x}\left(e^{uL_{kN}} 1_{S_{(k-1)N}=y}\right) \\ &= \sum_y E^{P_x}\left(e^{uL_{(k-1)N}} 1_{S_{(k-1)N}=y}\right) E^{P_y}\left(e^{uL_N}\right) \end{aligned}$$

$$\leq \quad E^{P_0}\left(e^{u(L_N+1)}\right)E^{P_x}\left(e^{uL_{(k-1)N}}\right).$$

By iterating over k, we get

$$E^{P_x}e^{uL_{kN}} \leq \left(E^{P_0}e^{u(L_N+1)}\right)^k. \tag{3.2.4}$$

□

Note that for any $x \in \mathbb{Z}$,

$$E^{P_{x,x}^{\otimes 2}}e^{uB_N} = E^{P_{0,0}^{\otimes 2}}e^{uB_N} \tag{3.2.5}$$

and for $x \neq x'$,

$$E^{P_{x,x'}^{\otimes 2}}e^{uB_N} \leq E^{P_{0,0}^{\otimes 2}}e^{u(B_N+1)} \tag{3.2.6}$$

where $P_{x,x'}^{\otimes 2}$ is the product measure $P_x \otimes P_{x'}$.

Therefore, as a consequence of Lemma 3.2.1, we also have the following lemma:

Lemma 3.2.2. *For any $N \geq 1$, $k \geq 1, u \geq 0$, and $x, x' \in \mathbb{Z}$, we have*

$$E^{P_{x,x'}^{\otimes 2}}e^{uB_{kN}} \leq \left(E^{P_{0,0}^{\otimes 2}}e^{u(B_N+1)}\right)^k.$$

Lemma 3.2.3. *For any $N \geq 1, u \geq 0$, and $x \geq 1$,*

$$E^{P_{x+1}}e^{uL_N} \leq E^{P_x}e^{uL_N} \tag{3.2.7}$$

Proof. Let $\tau_x = \inf\{n \geq 1 : S_n = x\}$.

$$\begin{aligned} E^{P_{x+1}} e^{uL_N} &= \sum_{k=1}^{N} E^{P_{x+1}}\Big(e^{uL_N} 1_{\tau_x = k}\Big) + E^{P_{x+1}}\Big(e^{uL_N} 1_{\tau_x > N}\Big) \\ &= \sum_{k=1}^{N} E^{P_x}\Big(e^{uL_{N-k}}\Big) P_{x+1}(\tau_x = k) + P_{x+1}(\tau_x > N) \\ &\leq E^{P_x}\Big(e^{uL_N}\Big) P_{x+1}(\tau_x \leq N) + P_{x+1}(\tau_x > N) \\ &\leq E^{P_x} e^{uL_N}. \end{aligned}$$

□

Lemma 3.2.4. *Let $C_L \geq 1 > 3\,c_s > 0$ be two positive integers. Then*

$$\liminf_{N\to\infty} \inf_{|x| \leq c_s\sqrt{N}} P_x\Big(\max_{1\leq i\leq N} |S_i| \leq 3C_L\sqrt{N}, |S_N| \leq c_s\sqrt{N}\Big) > 0,$$
$$\liminf_{N\to\infty} \inf_{|x| \leq c_s\sqrt{N}} P_x\Big(\max_{1\leq i\leq N} |S_i| \leq 3C_L\sqrt{N}, |S_N - C_L\sqrt{N}| \leq c_s\sqrt{N}\Big) > 0$$
$$\liminf_{N\to\infty} \inf_{|x| \leq c_s\sqrt{N}} P_x\Big(\max_{1\leq i\leq N} |S_i| \leq 3C_L\sqrt{N}, |S_N + C_L\sqrt{N}| \leq c_s\sqrt{N}\Big) > 0$$

Proof. By Donsker invariance principle,

$$\lim_{N\to\infty} P_0\Big(\max_{1\leq i\leq N} |S_i| \leq 3C_L\sqrt{N}, |S_N| \leq c_s\sqrt{N}]\Big) = \mathbf{P}\Big(\max_{0\leq t\leq 1} |B_t| \leq 3C_L, |B_1| \leq c_s\Big)$$

where $(B_t)_{0\leq t\leq 1}$ is a 1-dimensional Brownian motion with $B_0 = 0$ $\mathbf{P}$-a.s..

Therefore, there exists an $N_0 = N_0(C_L, c_s) \geq 1$ such that for all $N \geq N_0$,

$$P_0\Big(\max_{1\leq i\leq N} |S_i| \leq 3C_L\sqrt{N}, |S_N| \leq c_s\sqrt{N}\Big) \geq \frac{1}{2}\mathbf{P}\Big(\max_{0\leq t\leq 1} |B_t| \leq 3C_L, |B_1| \leq c_s\Big)$$

Note that for $0 \le x \le c_s\sqrt{N}$,

$$\begin{aligned} & P_x\Big(\max_{1\le i\le N}|S_i| \le 3C_L\sqrt{N}, |S_N| \le c_s\sqrt{N}\Big) \\ \ge\ & P_x\Big(\max_{1\le i\le N}|S_i - x| \le 2C_L\sqrt{N}, x - c_s\sqrt{N} \le S_N \le x\Big) \\ =\ & P_0\Big(\max_{1\le i\le N}|S_i| \le 2C_L\sqrt{N}, -c_s\sqrt{N} \le S_N \le 0\Big) \end{aligned}$$

and for $-c_s\sqrt{N} \le x \le 0$,

$$\begin{aligned} & P_x\Big(\max_{1\le i\le N}|S_i| \le 3C_L\sqrt{N}, |S_N| \le c_s\sqrt{N}\Big) \\ \ge\ & P_0\Big(\max_{1\le i\le N}|S_i| \le 2C_L\sqrt{N}, 0 \le S_N \le c_s\sqrt{N}\Big) \\ =\ & P_0\Big(\max_{1\le i\le N}|S_i| \le 2C_L\sqrt{N}, -c_s\sqrt{N} \le S_N \le 0\Big) \end{aligned}$$

Due to invariance principle, there exists an $N_1 = N_1(C_L, c_s) \ge 1$ such that for all $N \ge N_1$

$$P_0\Big(\max_{1\le i\le N}|S_i| \le 2C_L\sqrt{N}, -c_s\sqrt{N} \le S_N \le 0\Big) \ge \frac{1}{2}\mathbf{P}\Big(\max_{0\le t\le 1}|B_t| \le 2C_L, -c_s \le B_1 \le 0\Big)$$

Hence, for all $N \ge N_1$ and for $|x| \le c_s\sqrt{N}$,

$$P_x\Big(\max_{1\le i\le N}|S_i| \le 3C_L\sqrt{N}, |S_N| \le c_s\sqrt{N}\Big) \ge \frac{1}{2}\mathbf{P}\Big(\max_{0\le t\le 1}|B_t| \le 2C_L, -c_s \le B_1 \le 0\Big).$$

This completes the proof of the first claim in the theorem.

For the last two claims, note that for $0 \le x \le c_s\sqrt{N}$,

$$\begin{aligned} & P_x\Big(\max_{1\le i\le N}|S_i| \le 3C_L\sqrt{N}, |S_N - C_L\sqrt{N}| \le c_s\sqrt{N}\Big) \\ \ge\ & P_x\Big(\max_{1\le i\le N}|S_i - x| \le 2C_L\sqrt{N}, x + (C_L - c_s)\sqrt{N} \le S_N \le x + C_L\sqrt{N}\Big) \\ =\ & P_0\Big(\max_{1\le i\le N}|S_i| \le 2C_L\sqrt{N}, (C_L - c_s)\sqrt{N} \le S_N \le C_L\sqrt{N}\Big) \end{aligned}$$

and for $-c_s\sqrt{N} \le x \le 0$,

$$\begin{aligned} & P_x\Big(\max_{1\le i\le N}|S_i| \le 3C_L\sqrt{N}, |S_N - C_L\sqrt{N}| \le c_s\sqrt{N}\Big) \\ \ge\ & P_0\Big(\max_{1\le i\le N}|S_i| \le 2C_L\sqrt{N}, C_L\sqrt{N} \le S_N \le (C_L + c_s)\sqrt{N}\Big) \end{aligned}$$

Again by invariance principle, there exists an $N_2 = N_2(C_L, c_s) \ge 1$ such that for all $N \ge N_2$ and for $|x| \le c_s\sqrt{N}$,

$$\begin{aligned} & P_x\Big(\max_{1\le i\le N}|S_i| \le 3C_L\sqrt{N}, |S_N - C_L\sqrt{N}| \le c_s\sqrt{N}\Big) \\ & \ge \frac{1}{2}\mathbf{P}\Big(\max_{0\le t\le 1}|B_t| \le 2C_L, C_L \le B_1 \le C_L + c_s\Big) \end{aligned}$$

By a similar argument, there exists an $N_3 = N_3(C_L, c_s) \ge 1$ such that for all $N \ge N_3$ and for $|x| \le c_s\sqrt{N}$,

$$P_x\Big(\max_{1\le i\le N}|S_i| \le 3C_L\sqrt{N}, |S_N + C_L\sqrt{N}| \le c_s\sqrt{N}\Big)$$

$$\geq \frac{1}{2}\mathbf{P}\Big(\max_{0\leq t\leq 1}|B_t| \leq 2C_L, -(C_L+c_s) \leq B_1 \leq -C_L+c_s\Big)$$

□

The main ideas for the proof of the following lemma is due to Prof. S.R.S. Varadhan. We are grateful to him.

Lemma 3.2.5. *Let $C_L \geq 1 > 3\,c_s > 0$. Then there exists a constant $0 < \epsilon_0 = \epsilon_0(C_L, c_s) < 1$, such that for all sufficiently large N and $|x| \leq c_s\sqrt{N}$,*

$$E^{P_x}\Big(e^{uL_N}1_{\Omega_N}\Big) \geq \epsilon_0 E^{P_x}\Big(e^{uL_N}\Big)$$

where $\Omega_N = \{S : |S_0| \leq c_s\sqrt{N}, \max_{1\leq i\leq N}|S_i| \leq C_L\sqrt{N}, |S_N| \leq c_s\sqrt{N}\}$

Proof. Let's define the following probability measure on the path space:

$$\mu_{N,x,u}(A) := \frac{E^{P_x}[e^{uL_N}1_A]}{E^{P_x}[e^{uL_N}]}.$$

Let $W(n,x) = E^{P_x}[e^{uL_n}] = E^{P_x}[e^{[V(S_1)+\cdots+V(S_n)]}]$ where $V(\cdot) = u1_0(\cdot)$.

The process with the distribution $\mu_{N,x,u}(\cdot)$ is a non-stationary Markov process with transition probabilities from x to $x \pm 1$ at time k given by $\pi(x,y,k,N,u)$ as follows:

$$\pi(x,y,k,N,u) = \begin{cases} \frac{e^{V(y)}}{2}\frac{W(N-k-1,y)}{W(N-k,x)} & \text{if } k < N \\ \frac{1}{2} & \text{if } k \geq N. \end{cases}$$

where $y = x \pm 1$. Note that, for $y = x \pm 1$ and $k < N$,

$$\begin{aligned}
\pi(x, y, k, N, u) &= \mu_{N,x,u}(S_{k+1} = y | S_k = x) \\
&= \frac{\mu_{N,x,u}(S_{k+1} = y, S_k = x)}{\mu_{N,x,u}(S_k = x)} \\
&= \frac{E^{P_x}[e^{uL_N} 1_{S_k=x} 1_{S_{k+1}=y}]}{E^{P_x}[e^{uL_N} 1_{S_k=x}]} \\
&= \frac{E^{P_x}[e^{uL_k} 1_{S_k=x}] E^{P_x}[e^{uL_{N-k}} 1_{S_1=y}]}{E^{P_x}[e^{uL_k} 1_{S_k} = x] E^{P_x}[e^{uL_{N-k}}]} \\
&= \frac{1}{2} \frac{E^{P_x}[e^{uL_k} 1_{S_k=x}] e^{V(y)} E^{P_y}[e^{uL_{N-k-1}}]}{E^{P_x}[e^{uL_k} 1_{S_k=x}] E^{P_x}[e^{uL_{N-k}}]} \\
&= \frac{e^{V(y)}}{2} \frac{W(N-k-1, y)}{W(N-k, x)}
\end{aligned}$$

and for $k \geq N$,

$$\pi(x, y, k, N, u) = P_x(S_1 = y) = \frac{1}{2}.$$

Note that

$$W(n+1, x) = \frac{1}{2} e^{V(x+1)} W(n, x+1) + \frac{1}{2} e^{V(x-1)} W(n, x-1)$$

and by Lemma 3.2.1 and Lemma 3.2.3, we have for $x \geq 1$,

$$W(n, x+1) \leq W(n, x) \text{ and } W(n, 1) \leq e^u W(n, 0),$$

and for $x \leq -1$,

$$W(n, x) \leq W(n, x+1) \text{ and } W(n, -1) \leq e^u W(n, 0).$$

Therefore, for $x \geq 1, k \geq 1$,

$$\pi(x, x-1, k, N, u) \geq \frac{1}{2},$$

and for $x \leq -1, k \geq 1$,

$$\pi(x, x+1, k, N, u) \geq \frac{1}{2}.$$

Hence, the $\mu_{N,x,u}$ chain S_N^u can be coupled to the P_x chain S_N in a such a way that

$$|S_n^u| \leq |S_n| \text{ for all } n \geq 1.$$

Therefore,

$$\mu_{N,x,u}(\Omega_N) \geq P_x(\Omega_N)$$

By Lemma 3.2.4, we have a lower bound for $P_x(\Omega_N)$. □

Lemma 3.2.6. *Let $0 < \epsilon < 1$ be given. Then, for sufficiently large N and $|x| \leq c_s\sqrt{N}$ with $x \neq 0$,*

$$E^{P_x}\left(e^{\beta u L_N}\right) \geq \frac{1}{2}\mathbf{P}(Z \geq \frac{c_s}{\sqrt{\epsilon}})e^{(1-\epsilon)N\mathrm{F}(\beta u)}$$

Proof. Let

$$\tau_x = \inf\{n \geq 1 : S_n = x\}.$$

For a given $0 < \epsilon < 1$, there exists an $N_0 = N_0(c_s, \epsilon)$ such that for all $N \geq N_0$ and for $0 < x \leq c_s\sqrt{N}$,

$$P_x(\tau_0 \leq \epsilon N) \quad = \quad P_0(\tau_x \leq \epsilon N)$$

$$\begin{aligned}
&\geq P_0(S_{\epsilon N} \geq c_s\sqrt{N}) \\
&= P_0(S_{\epsilon N} \geq \frac{c_s}{\sqrt{\epsilon}}\sqrt{\epsilon N}) \\
&\geq \frac{1}{2}\mathbf{P}(Z \geq \frac{c_s}{\sqrt{\epsilon}})
\end{aligned}$$

where Z denotes the standard normal random variable under the probability measure $\mathbf{P}$. Note that since 1-D SSRW is symmetric, the same bound would be true for $-c_s\sqrt{N} \leq x < 0$.

Therefore, for sufficiently large N and $|x| \leq c_s\sqrt{N}$,

$$\begin{aligned}
E^{P_x}\left(e^{\beta u L_N}\right) &\geq \sum_{k=x}^{\epsilon N} e^{\beta u} E^{P_0}\left(e^{\beta u L_{N-k}}\right) P_x(\tau_0 = k) \\
&\geq \sum_{k=x}^{\epsilon N} e^{\beta u} e^{-\beta u} e^{(N-k)\mathrm{F}(\beta u)} P_x(\tau_0 = k) \\
&\geq \sum_{k=x}^{\epsilon N} e^{(1-\epsilon)N\mathrm{F}(\beta u)} P_x(\tau_0 = k) \\
&= e^{(1-\epsilon)N\mathrm{F}(\beta u)} P_x(\tau_0 \leq \epsilon N) \\
&\geq \frac{1}{2}\mathbf{P}(Z \geq \frac{c_s}{\sqrt{\epsilon}}) e^{(1-\epsilon)N\mathrm{F}(\beta u)}
\end{aligned}$$

In the first inequality we used the Markov property of SSRW and in the second inequality we used Lemma 1.1.4. □

Remark 3.2.7. *By Lemma 1.1.4, we have*

$$E^{P_0}\left(e^{\beta u L_N}\right) \geq e^{-\beta u} e^{N\mathrm{F}(\beta u)}.$$

Therefore, for sufficiently small $\beta u > 0$,

$$E^{P_0}\left(e^{\beta u L_N}\right) \geq \frac{1}{2}\mathbf{P}(Z \geq \frac{c_s}{\sqrt{\epsilon}})e^{(1-\epsilon)N\mathrm{F}(\beta u)}.$$

We need to use some facts on the excursion length distribution of (p,q)-walks. First, a definition:

Definition 3.2.8. *Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables with $\mathbf{P}(X_1 = b) = \mathbf{P}(X_1 = -b) = p/2 \in (0, 1/2)$ and $\mathbf{P}(X_1 = 0) = q > 0$ where $p + q = 1$ and b is a positive integer.*

The partial sum process $S = (S_n)_{n\geq 0}$, where $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$, is called a (p,q)-walk.

Remark 3.2.9. *Let $\bar{S}_N = S_N^1 - S_N^2$, where S_N^1, S_N^2 are SSRWs. Note that $(\bar{S}_N)_{N\geq 1}$ is a $(1/2, 1/2)$-walk with $b = 2$, and $B_N(S^1, S^2) = L_N(\bar{S})$.*

Let

$$\tau = \inf\{n \geq 1 : S_n = 0\}.$$

For the proofs of the following propositions, see [25].

Proposition 3.2.10. *For any (p,q)-walk, $p \in (0,1)$, we have*

$$\mathbf{P}(\tau = n) \sim \sqrt{\frac{p}{2\pi}}n^{-3/2} \text{ as } n \to \infty.$$

Remark 3.2.11. *For* $(1,0)$*-walk, the corresponding asymptotic is the following:*

$$\mathbf{P}(\tau = 2n) \sim \sqrt{\frac{1}{4\pi}} n^{-3/2} \text{ as } n \to \infty.$$

3.2.1 The Annealed Correlation Length

Recall that the *annealed free energy* of the system is defined as

$$\begin{aligned} f^a(\beta, u) &:= \lim_{N\to\infty} \frac{1}{N} \log E^{P_0} E^Q \left(e^{\beta \sum_{j=1}^N (v(j,S_j) + u \mathbb{1}_{S_j=0})} \right) \\ &= \lim_{N\to\infty} \frac{1}{N} \log \left(e^{\frac{\beta^2}{2} N} E^{P_0} e^{\beta u L_N(S)} \right) \\ &= \frac{\beta^2}{2} + \mathrm{F}(\beta u) \end{aligned}$$

where $\mathrm{F}(\cdot)$ is the free energy of the 1-D SSRW pinning model which we reviewed in Chapter 1.

The *annealed correlation length* M will be defined as

$$M := M(\beta u) = \frac{c_1}{\mathrm{F}(\beta u)}$$

Note that $M \to \infty$ as $\beta u \to 0$ since $\mathrm{F}(u) \sim c_1 u^2$ as $u \to 0$.

Next proposition will be important when we deal with a system of length of multiple of one correlation length.

Proposition 3.2.12. *Let $0 < a < 1$ be given. Then there exists a constant $K = K(a) > 0$ such that for sufficiently small β and βu, and $u \geq K(a)\beta$, we have*

$$E^{P_{0,0}^{\otimes 2}}\left(e^{2\beta^2(B_M(S^1,S^2)+1)} - 1\right) \leq a.$$

We will prove Proposition 3.2.12 in the rest of this section.

Recall that a random variable X is said to have a *Geometric distribution* with parameter $p \in (0,1)$, if

$$\mathbf{P}(X = k) = (1-p)^{k-1}p$$

where $k = 1, 2, 3, \cdots$. Then $\mathbf{P}(X > k) = (1-p)^k$ and the moment generating function of X is given by the following formula:

$$\mathbf{E}(e^{tX}) = \frac{pe^t}{1-(1-p)e^t} \tag{3.2.8}$$

provided that $t < -\log(1-p)$.

Let E_i denote the length of the i^{th} excursion of $\bar{S} = S^1 - S^2$ from 0. Then

$$\begin{aligned} P(B_N + 1 > k) &\leq P(\max_{1\leq i\leq k} E_i \leq N) \\ &= (1 - P(E_1 > N))^k \quad \text{for all } N, k \geq 1. \end{aligned}$$

By Remark 3.2.11, $P(E_1 > N) \sim \sqrt{\frac{1}{\pi}}N^{-1/2}$ as $N \to \infty$, therefore for a fixed $0 < K_4 < 1$, there exists $N_4 = N_4(K_4)$ such that for all $N \geq N_4$

$$P(B_N + 1 > k) \leq (1 - K_4\sqrt{\frac{1}{\pi}}N^{-1/2})^k \quad \text{for all } k \geq 1. \tag{3.2.9}$$

Therefore, under the random walk measure P, $B_M + 1$ is stochastically dominated by a Geometric random variable G_M with parameter

$$p_M = K_4\sqrt{\frac{1}{\pi}}M^{-1/2}$$

for the annealed correlation length M being sufficiently large, which is equivalent to βu being sufficiently small.

Therefore, we will have for βu sufficiently small,

$$E^{P_{0,0}^{\otimes 2}}\left(e^{2\beta^2(B_M(S^1,S^2)+1)} - 1\right) \leq \frac{p_M e^{2\beta^2}}{1-(1-p_M)e^{2\beta^2}} - 1 \tag{3.2.10}$$

provided that $2\beta^2 < -\log(1-p_M)$, which is equivalent to the condition

$$p_M > 1 - e^{-2\beta^2}. \tag{3.2.11}$$

The condition 3.2.11 is necessary for the existence of the moment generating function of G_M for $2\beta^2$, and with the condition that $\beta u > 0$ be sufficiently small, it implies that $\beta > 0$ must be small as well.

From 3.2.10, for a given $0 < a < 1$, we would like to have

$$\frac{p_M e^{2\beta^2}}{1-(1-p_M)e^{2\beta^2}} - 1 \leq a \tag{3.2.12}$$

which is equivalent to

$$p_M \geq \frac{a+1}{a}(1-e^{-2\beta^2}) \tag{3.2.13}$$

Since $1 - e^{-2\beta^2} \leq 2\beta^2$ and $a < 1$, the condition

$$p_M \geq \frac{4\beta^2}{a} \tag{3.2.14}$$

will be sufficient for 3.2.11 and 3.2.12.

From part (d) of the properties of the free energy of the random walk pinning model on page 6, we have $M \sim (\beta u)^{-2}$ as $\beta u \to 0$. Therefore, for a fixed $0 < K_5 < 1$, we will have

$$(\beta u)^{-2} \geq K_5 M$$

for sufficiently small βu.

By using the definition of $p_M = K_4\sqrt{\frac{1}{\pi}}M^{-1/2}$, we get

$$p_M \quad \geq \quad K_4 K_5^{1/2}\sqrt{\frac{1}{\pi}}\beta u. \tag{3.2.15}$$

Therefore, for β and βu sufficiently small, to satisfy the condition 3.2.14,

$$K_4 K_5^{1/2}\sqrt{\frac{1}{\pi}}\beta u \quad \geq \quad \frac{4\beta^2}{a} \tag{3.2.16}$$

will be sufficient. This is equivalent to

$$u \geq K\beta \tag{3.2.17}$$

which will be a sufficient condition for 3.2.12 where $K := K(a) = 4\sqrt{\pi}/aK_4K_5^{1/2}$.

3.3 The Coarse Grained Lattice $\mathbb{L}_{CG}$

In this section, we introduce a coarse grained lattice

$$\mathbb{L}_{CG} := \{(I,J) \in \mathbb{Z}^2 : I \geq 0, 0 \leq J \leq I\}.$$

Let $C_L \geq 1 > 3\,c_s > 0$ be two positive integers and $N = k_0 M$ be a multiple of the annealed correlation length M such that N and $\sqrt{N}$ are integers. We use capital letters (I,J) for a site in the coarse grained lattice which corresponds to the segment

$$R(I,J) := \{(k,l) \in \mathbb{Z}^2 : k = IN,\ (JC_L - c_s)\sqrt{N} \leq l \leq (JC_L + c_s)\sqrt{N}\}$$

in the original lattice $\mathbb{Z}^2$.

We call $R(I,J)$ the *window*-(I,J) in $\mathbb{Z}^2$.

The *box* starting from the window-(I,J) is the following region in $\mathbb{Z}^2$:

$$B(I,J) := [IN, (I+1)N] \times [(JC_L - 3C_L)\sqrt{N}, (JC_L + 3C_L)\sqrt{N}]$$

We say that there is a *link* between sites (I,J) and (K,L) if

$$K = I + 1 \text{ and } |L - J| \leq 1.$$

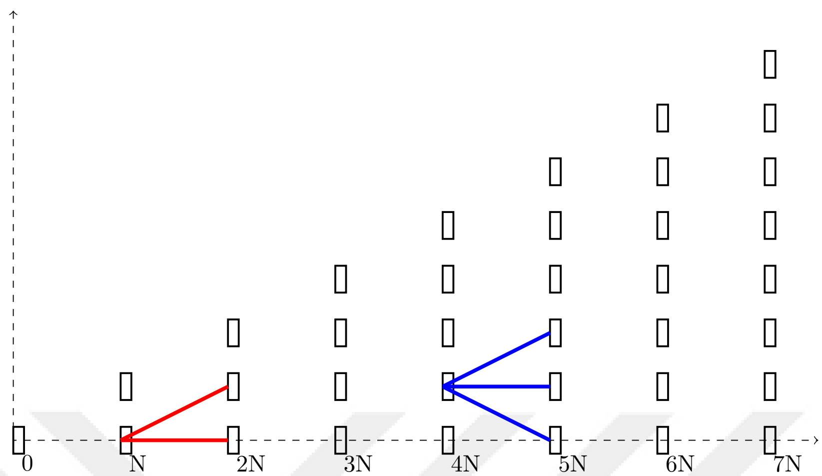

Figure 3.1: The Coarse Grained Lattice $\mathbb{L}_{CG}$.

A *path* $\Gamma_{(I,J)\to(K,L)}$ from site (I,J) to site (K,L) in the coarse grained lattice $\mathbb{L}_{CG}$ is a sequence of sites $(I_1,J_1)=(I,J),(I_2,J_2),\cdots,(I_N,J_N)=(K,L)$ such that

$$I_{i+1}=I_i+1,\ \text{and}\ |J_i-J_{i+1}|\leq 1,\ 1\leq i<N.$$

$\Gamma_{(I,J)\to(K,L)}(I_i)$ will denote the second coordinate J_i of a site (I_i,J_i) in the path $\Gamma_{(I,J)\to(K,L)}$.

We will use the notation $\Gamma_{(I,J)}$ for $\Gamma_{(0,0)\to(I,J)}$.

For a given two paths $\Gamma^1:=\Gamma^1_{(I,J)\to(K,L)}$ and $\Gamma^2:=\Gamma^2_{(I,J)\to(K,L)}$ from site (I,J) to site (K,L), we say that Γ^1 is closer to the x-axis than Γ^2 if

$$\Gamma^1(I_i)\leq\Gamma^2(I_i)\ \text{for each}\ I\leq I_i\leq K.$$

Assume that each site $(I,J) \in \mathbb{L}_{CG}$ is defined as *open* or *closed* in a well-defined way. Then we introduce the following definitions:

A path $\Gamma_{(I,J)\to(K,L)}$ from site (I,J) to site (K,L) is called *open* if its all sites are open.

A path $\Gamma_{(I,J)\to(K,L)}$ from site (I,J) to site (K,L) is called *maximal* if it has the maximum number of open sites among all paths from site (I,J) to site (K,L).

A path from site (I,J) to site (K,L) is called *optimal* if it is the maximal path which is closest to the x-axis. There is exactly one *optimal* path for given sites (I,J) and (K,L) and we denote it by $\Gamma^{\text{opt}}_{(I,J)\to(K,L)}$.

$\Gamma^{\infty}_{(I,J)}$ will denote a general infinite open path from the site (I,J).

An infinite open path from a site (I,J), which is closest to the x-axis among all such paths, is called the *infinite good path from the site* (I,J), and we denote it by $\Gamma^{\text{G},\infty}_{(I,J)}$. If there is an infinite good path from a site (I,J), then it must be unique.

$\Gamma^{\text{G},\infty}$ will denote the infinite good path from the site $(0,0)$.

For $0 \leq I \leq K$, $\Gamma^{\text{G},\infty}_{I\to K}$ will denote the finite segment of the path $\Gamma^{\text{G},\infty}$ between the sites with first coordinates I and K, respectively.

$\Gamma^{\text{G},\infty}_{I}$ will be used for $\Gamma^{\text{G},\infty}_{0\to I}$.

Note that if the site (I_0,J_0) is on the infinite good path from $(0,0)$, then $\Gamma^{\text{opt}}_{(I_0,J_0)}$ will be $\Gamma^{\text{G},\infty}_{I_0}$.

For a given path $\Gamma_{(I,J)}$ in $\mathbb{L}_{CG}$, we identify a subset of the SSRW paths of length IN in the following way:

$\Omega_{IN}^{\Gamma_{(I,J)}}$:=the set of SSRW paths of length IN from $(0,0)$ to the window-(I,J) which pass through all the windows $R(I',J')$ which correspond to sites (I',J') in the path $\Gamma_{(I,J)}$ and stay in the boxes $B(I',J')$ starting from the windows $R(I',J')$.

And for $0 \le I \le K$, $\Gamma_{I\to K}^{\mathrm{G},\infty}$, we define

$\Omega_{(K-I)N}^{\Gamma_{I\to K}^{\mathrm{G},\infty}}$:=the set of SSRW paths of length $(K-I)N$ from the window-$(I,\Gamma^{\mathrm{G},\infty}(I))$ to the window-$(K,\Gamma^{\mathrm{G},\infty}(K))$ which pass through all the windows $R(I',J')$ which correspond to the sites (I',J') in the path $\Gamma^{\mathrm{G},\infty}$ in between $(I,\Gamma^{\mathrm{G},\infty}(I))$ and $(K,\Gamma^{\mathrm{G},\infty}(K))$ and stay in the boxes $B(I',J')$ starting from the windows $R(I',J')$.

The corresponding partition function restricted to the subset $\Omega_{IN}^{\Gamma_{(I,J)}}$ of SSRW paths will be denoted by

$$Z_{IN}^{\beta,u,q}(\Omega_{IN}^{\Gamma_{(I,J)}}) := E^{P_0}\left(e^{\beta\sum_{i=1}^{IN}(v(i,S_i)+u1_{S_i=0})}1_{\Omega_{IN}^{\Gamma_{(I,J)}}}\right)$$

We define random probability measures on the window $R(I,J)$ associated with a given path $\Gamma_{(I,J)}$ in the following way:

For $x \in R(I,J)$,

$$\mu_{\Gamma_{(I,J)}}^{R(I,J)}(x) := \frac{Z_{IN}^{\beta,u,q}(\Omega_{IN}^{\Gamma_{(I,J)}}, S_{IN}=x)}{Z_{IN}^{\beta,u,q}(\Omega_{IN}^{\Gamma_{(I,J)}})}, \tag{3.3.1}$$

Let's consider the following subsets of SSRW paths:

$$\Omega_N^{\mathrm{up}} := \{(S_0,\cdots,S_N) : |S_0| \le c_s\sqrt{N}, |S_N - C_L\sqrt{N}| \le c_s\sqrt{N}, |S_i| \le 3C_L\sqrt{N}, 1 \le i \le N\},$$

$$\Omega_N^{\mathrm{forward}} := \{(S_0,\cdots,S_N) : |S_0| \le c_s\sqrt{N}, |S_N| \le c_s\sqrt{N}, |S_i| \le 3C_L\sqrt{N}, 1 \le i \le N\},$$

and

$$\Omega_N^{\text{down}} := \{(S_0, \cdots, S_N) : |S_0| \le c_s\sqrt{N}, |S_N + C_L\sqrt{N}| \le c_s\sqrt{N}, |S_i| \le 3C_L\sqrt{N}, 1 \le i \le N\}.$$

The sets $\Omega^{\text{up}}_{N,R(I,J)}, \Omega^{\text{forward}}_{N,R(I,J)}$ and $\Omega^{\text{down}}_{N,R(I,J)}$ will refer to the *up, forward, down* set of random walk paths which start at the window $R(I,J)$, stay in the box $B(I,J)$, and end up at the window $R(I+1, J+l),\ l = +1, 0, -1$, respectively.

Let $\nu^q_{(I,J)} := \mu^{R(I,J)}_{\Gamma^{\text{opt}}_{(I,J)}}$ be the random probability distribution on the window-(I,J) which comes from the optimal path $\Gamma^{\text{opt}}_{(I,J)}$, and let $\nu^q_{(0,0)} := \delta_0$ be the point mass at zero.

We define the partition functions over the up, forward, down SSRW paths between the window-(I,J) and the window-$(I+1, J+l)$, $I \ge 0,\ J \ge 1,\ l = +1, 0, -1$ with initial distribution $\nu^q_{(I,J)}$ as follows:

$$Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\text{g}}_{N,R(I,J)}) := E^{P_{\nu^q_{(I,J)}}}\left(e^{\beta\sum_{k=1}^N\left(v(IN+k, S_{IN+k}) + u1_{S_{IN+k}=0}\right)} 1_{\Omega^{\text{g}}_{N,R(I,J)}}\right),$$

where $\Omega^{\text{g}}_{N,R(I,J)}$ is $\Omega^{\text{up}}_{N,R(I,J)}, \Omega^{\text{forward}}_{N,R(I,J)}$ or $\Omega^{\text{down}}_{N,R(I,J)}$ for $l = +1, 0, -1$, respectively.

3.3.1 Assigning each site (I,J) in the coarse grained lattice $\mathbb{L}_{CG}$ as open or closed.

Let's consider the following filtrations:

$$\mathcal{F}_I := \sigma(\{v(i,x) : 1 \le i \le IN, x \in \mathbb{Z}\}),\ I \ge 1.$$

Note that the measures $\nu^q_{(I,J)}$ are $\mathcal{F}_I$-measurable for all $J \geq 0$. In the following, first we will identify two non-random constants U_{on} and U_{off} which will be lower bounds for partition functions associated to *on* and *off* axis coarse grained lattice links such that

$$U_{\text{on}} \leq \frac{1}{2} E^Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,0)}} (\Omega^{\text{forward}}_{N,R(I,0)})|\mathcal{F}_I\Big),\ Q-a.s. \text{ for each } I \geq 0,$$

and

$$U_{\text{off}} \leq \min\{\frac{1}{2} E^Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,0)}} (\Omega^{\text{up}}_{N,R(I,0)})|\mathcal{F}_I\Big), \frac{1}{2} E^Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}} (\Omega^{\text{g}}_{N,R(I,J)})|\mathcal{F}_I\Big)\}\ Q-a.s.$$

for each $I \geq 0, J \geq 1$, where $\Omega^{\text{g}}_{N,R(I,J)}$ is any one of $\Omega^{\text{up}}_{N,R(I,J)}, \Omega^{\text{forward}}_{N,R(I,J)}$ or $\Omega^{\text{down}}_{N,R(I,J)}$.

Note that for $I \geq 1$, by Lemma 3.2.5 and Lemma 3.2.6, for sufficiently large N, and sufficiently small βu, Q-a.s.

$$\begin{aligned}
& E^{\mathcal{Q}}\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,0)}} (\Omega^{\text{forward}}_{N,R(I,0)})|\mathcal{F}_I\Big) \\
= \quad & \sum_{x\in R(I,0)} \nu^q_{(I,0)}(x) E^{\mathcal{Q}}\Bigg(E^{P_x}\left[e^{\beta\sum_{k=1}^N \left(v(IN+k,S_{IN+k})+u1_{S_{IN+k}=0}\right)} 1_{\Omega^{\text{forward}}_{N,R(I,0)}}\right]\Bigg) \\
= \quad & \sum_{x\in R(I,0)} \nu^q_{(I,0)}(x) e^{\frac{\beta^2}{2}N} E^{P_x}\left[e^{\sum_{k=1}^N \beta u 1_{S_{IN+k}=0}} 1_{\Omega^{\text{forward}}_{N,R(I,0)}}\right] \\
\geq \quad & \sum_{x\in R(I,0)} \nu^q_{(I,0)}(x) e^{\frac{\beta^2}{2}N} \epsilon_0 \frac{1}{2}\mathbf{P}(Z \geq \frac{c_s}{\sqrt{\epsilon}}) e^{(1-\epsilon)N\mathrm{F}(\beta u)} \\
\geq \quad & \epsilon_0 \frac{1}{2}\mathbf{P}(Z \geq \frac{c_s}{\sqrt{\epsilon}}) e^{\frac{\beta^2}{2}(1-\epsilon)N} e^{(1-\epsilon)N\mathrm{F}(\beta \mathrm{u})} \\
= \quad & \epsilon_0 \frac{1}{2}\mathbf{P}(Z \geq \frac{c_s}{\sqrt{\epsilon}}) e^{(\frac{\beta^2}{2}+\mathrm{F}(\beta\mathrm{u}))(1-\epsilon)N} \\
= \quad & \epsilon_0 \frac{1}{2}\mathbf{P}(Z \geq \frac{c_s}{\sqrt{\epsilon}}) e^{f^a(\beta,u)(1-\epsilon)N}
\end{aligned}$$

Therefore, let's define

$$U_{\text{on}} := \epsilon_0 \frac{1}{4}\mathbf{P}(Z \geq \frac{c_s}{\sqrt{\epsilon}})e^{f^a(\beta,u)(1-\epsilon)N} \tag{3.3.2}$$

And for $I \geq 0, J \geq 1$, for sufficiently large N, we have Q-a.s.

$$\begin{aligned}
& E^{\mathcal{Q}}\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\text{g}}_{N,R(I,J)})|\mathcal{F}_I\Big) \\
= & \sum_{x\in R(I,J)} \nu^q_{(I,J)}(x)E^{\mathcal{Q}}\left(E^{P_x}\left[e^{\beta\sum_{k=1}^N\left(v(IN+k,S_{IN+k})+u1_{S_{IN+k}=0}\right)}1_{\Omega^{\text{g}}_{N,R(I,J)}}\right]\right) \\
\geq & \sum_{x\in R(I,J)} \nu^q_{(I,J)}(x)e^{\frac{\beta^2}{2}N}E^{P_x}\left[1_{\Omega^{\text{g}}_{N,R(I,J)}}\right] \\
\geq & \frac{1}{2}e^{\frac{\beta^2}{2}N}\mathbf{P}(A_{C_L,c_s})
\end{aligned}$$

where $A_{C_L,c_s} = \{\max_{0\leq t\leq 1}|B_t| \leq 2C_L, -(C_L+c_s) \leq B_1 \leq -C_L+c_s\}$.

Therefore, let's define

$$U_{\text{off}} := \frac{1}{4}e^{\frac{\beta^2}{2}N}\mathbf{P}(A_{C_L,c_s}). \tag{3.3.3}$$

We will now specify the rule which assigns each site (I,J) in the coarse grained lattice $\mathbb{L}_{CG}$ as *open* or *closed*.

The site $(I,J) = (0,0)$ is called *open* if

$$Z^{\beta,u,q}_N(\Omega^{\text{up}}_N) \geq U_{\text{off}} \text{ and } Z^{\beta,u,q}_N(\Omega^{\text{forward}}_N) \geq U_{\text{on}}.$$

Assume that all the sites (K,L), for $0 \leq K < I$ and $0 \leq L \leq K$ are defined as *open* or *closed*.

Then, conditionally on $\mathcal{F}_I$, the site $(I,0)$ is called *open* if

$$Z^{\beta,u,q}_{N,\nu^q_{(I,0)}}(\Omega^{\text{up}}_{N,R(I,0)}) \geq U_{\text{off}} \text{ and } Z^{\beta,u,q}_{N,\nu^q_{(I,0)}}(\Omega^{\text{forward}}_{N,R(I,0)}) \geq U_{\text{on}},$$

and the sites $(I,J), 0 < J \leq I$ are called *open* if

$$Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\text{up}}_{N,R(I,J)}) \geq U_{\text{off}},\ Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\text{forward}}_{N,R(I,J)}) \geq U_{\text{off}} \text{ and } Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\text{down}}_{N,R(I,J)}) \geq U_{\text{off}}.$$

Note that since the (random) probability measures $\nu^q_{(I,J)}$'s on the windows $R(I,J)$'s depend on the optimal path from $(0,0)$ to (I,J), they depend on the sites being open or closed defined in previous steps.

Let $\{X_{(I,J)} : (I,J) \in \mathbb{L}_{CG}\}$ be the indicator random variables such that

$$X_{(I,J)} = \begin{cases} 1 & \text{if the site } (I,J) \text{ is open,} \\ 0 & \text{if the site } (I,J) \text{ is closed.} \end{cases}$$

3.3.2 Second Moment Method and the Site Densities in $\mathbb{L}_{CG}$

The method which we describe now is generally called as *second moment method* in probability literature; for some applications, see [3], [48].

Let X be a random variable with finite mean and variance, and let $0 < \theta < 1$.

By the Chebyshev's Inequality, we have

$$\begin{aligned} P(X \leq (1-\theta)EX) &= P(X - EX \leq -\theta EX) \\ &\leq P(|X - EX| \geq \theta EX) \end{aligned}$$

$$\leq \frac{1}{\theta^2} \frac{Var(X)}{(EX)^2}$$

Similarly,

$$P(X \geq (1+\theta)EX) \leq \frac{1}{\theta^2} \frac{Var(X)}{(EX)^2}$$

Hence,

$$\begin{aligned} P((1-\theta)EX \leq X \leq (1+\theta)EX) &= P(X \leq (1+\theta)EX) - P(X \leq (1-\theta)EX) \\ &= 1 - P(X \geq (1+\theta)EX) - P(X \leq (1-\theta)EX) \\ &\geq 1 - \frac{2}{\theta^2} \frac{Var(X)}{(EX)^2} \\ &\geq 1 - \epsilon \end{aligned}$$

provided that

$$\frac{Var(X)}{(EX)^2} \leq \frac{\theta^2}{2}\epsilon. \tag{3.3.4}$$

Now, we will use the second moment method and control the probability of a site $(I, J) \in \mathbb{L}_{CG}$ being open with respect to the rule defined in the subsection 3.3.1.

Let's first consider a site $(I, 0)$ on the x-axis. We will use the fact that there are non-random constants U_{on} and U_{off} as we defined in 3.3.2 and 3.3.3 respectively, such that

$$U_{\text{on}} \leq \frac{1}{2} E^Q \left(Z^{\beta,u,q}_{N,\nu^q_{(I,0)}} (\Omega^{\text{forward}}_{N,R(I,0)}) | \mathcal{F}_I \right) \; Q - a.s.,$$

and Q-a.s.

$$U_{\rm off} \leq \min\{\frac{1}{2}E^Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,0)}}(\Omega^{\rm up}_{N,R(I,0)})|\mathcal{F}_I\Big), \frac{1}{2}E^Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\rm g}_{N,R(I,J)})|\mathcal{F}_I\Big)\}$$

where $\Omega^{\rm g}_{N,R(I,J)}$ is any one of $\Omega^{\rm up}_{N,R(I,J)}, \Omega^{\rm forward}_{N,R(I,J)}$ or $\Omega^{\rm down}_{N,R(I,J)}$.

Then, by the second moment method

$$Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,0)}}(\Omega^{\rm up}_{N,R(I,0)}) \leq U_{\rm off}|\mathcal{F}_I\Big) \leq 4\frac{Var_Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,0)}}(\Omega^{\rm up}_{N,R(I,0)})|\mathcal{F}_I\Big)}{\Big(E^Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,0)}}(\Omega^{\rm up}_{N,R(I,0)})\Big)|\mathcal{F}_I\Big)^2} \quad Q-a.s.$$

and similarly

$$Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,0)}}(\Omega^{\rm forward}_{N,R(I,0)}) \leq U_{\rm on}|\mathcal{F}_I\Big) \leq 4\frac{Var_Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,0)}}(\Omega^{\rm forward}_{N,R(I,0)})|\mathcal{F}_I\Big)}{\Big(E^Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,0)}}(\Omega^{\rm forward}_{N,R(I,0)})|\mathcal{F}_I\Big)\Big)^2} \quad Q-a.s.$$

Therefore, for the site $(I,0)$, Q-a.s.

$$\begin{aligned} Q(X_{(I,0)}=1|\mathcal{F}_I) &= 1-Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,0)}}(\Omega^{\rm up}_{N,R(I,0)}) \leq U_{\rm off} \text{ or } Z^{\beta,u,q}_{N,\nu^q_{(I,0)}}(\Omega^{\rm forward}_{N,R(I,0)}) \leq U_{\rm on}|\mathcal{F}_I\Big) \\ &\geq 1-\epsilon \end{aligned}$$

provided that Q-a.s.

$$\frac{Var_Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,0)}}(\Omega^{\rm forward}_{N,R(I,0)})|\mathcal{F}_I\Big)}{\Big(E^Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,0)}}(\Omega^{\rm forward}_{N,R(I,0)})|\mathcal{F}_I\Big)\Big)^2} \leq \frac{\epsilon}{8} \text{ and } \frac{Var_Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,0)}}(\Omega^{\rm up}_{N,R(I,0)})|\mathcal{F}_I\Big)}{\Big(E^Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,0)}}(\Omega^{\rm up}_{N,R(I,0)})|\mathcal{F}_I\Big)\Big)^2} \leq \frac{\epsilon}{8}.$$

Similarly, for a site (I,J) for $J \geq 1$, we will have Q-a.s.

$$Q(X_{(I,J)} = 1|\mathcal{F}_I) \geq 1-\epsilon$$

provided that Q-a.s.

$$\frac{Var_Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\text{up}}_{N,R(I,J)})|\mathcal{F}_I\Big)}{\Big(E^Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\text{up}}_{N,R(I,J)})|\mathcal{F}_I\Big)\Big)^2} \leq \frac{\epsilon}{12}, \quad \frac{Var_Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\text{forward}}_{N,R(I,J)})|\mathcal{F}_I\Big)}{\Big(E^Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\text{forward}}_{N,R(I,J)})|\mathcal{F}_I\Big)\Big)^2} \leq \frac{\epsilon}{12}$$

and

$$\frac{Var_Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\text{down}}_{N,R(I,J)})|\mathcal{F}_I\Big)}{\Big(E^Q\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\text{down}}_{N,R(I,J)})|\mathcal{F}_I\Big)\Big)^2} \leq \frac{\epsilon}{12}.$$

We will now control the ratio of the variance and the square of the mean for the partition function over the up, forward and down sets of SSRW paths.

Recall that $H_N(S) = \sum_{i=1}^{N} v(i, S_i)$.

Note that for given random walk paths S^1 and S^2, by using the fact that the distribution of the environment is Gaussian, we have

$$\begin{aligned}
& E^{\mathcal{Q}}\left(e^{\beta H_N(S^1)+\beta u L_N(S^1)} e^{\beta H_N(S^2)+\beta u L_N(S^2)}\right) \\
= \; & e^{\beta u L_N(S^1)} e^{\beta u L_N(S^2)} e^{\frac{(2\beta)^2}{2} B_N(S^1,S^2)} e^{\frac{\beta^2}{2} 2(N - B_N(S^1,S^2))} \\
= \; & e^{\beta u L_N(S^1)} e^{\beta u L_N(S^2)} e^{\beta^2 B_N(S^1,S^2)} e^{\beta^2 N}
\end{aligned}$$

Let $\Omega^{\text{g}}_{N,R(I,J)}$ denote any one of the *up*, *forward*, *down* sets of paths which starts from the window-(I,J).

Let's consider the following ratio:

$$T^{(I,J)}_{N,q} = T^{(I,J)}_{N,q}(\beta,u,k_0) := \frac{Var_{\mathcal{Q}}\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\mathrm{g}}_{N,R(I,J)})|\mathcal{F}_I\Big)}{\Big(E^{\mathcal{Q}}\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\mathrm{g}}_{N,R(I,J)})|\mathcal{F}_I\Big)\Big)^2}$$

$$= \frac{E^{\mathcal{Q}}\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\mathrm{g}}_{N,R(I,J)})^2|\mathcal{F}_I\Big) - \Big(E^{\mathcal{Q}}\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\mathrm{g}}_{N,R(I,J)})|\mathcal{F}_I\Big)\Big)^2}{\Big(E^{\mathcal{Q}}\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\mathrm{g}}_{N,R(I,J)})|\mathcal{F}_I\Big)\Big)^2}$$

First, we will deal with the numerator. Using the facts that $N = k_0M$, $(d-1)^2 \leq d^2-1$ for $d \geq 1$, the Cauchy-Schwartz inequality, and Lemma 3.2.1 and Lemma 3.2.2, we get Q-a.s.

$$\begin{aligned}
& E^{\mathcal{Q}}\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\mathrm{g}}_{N,R(I,J)})^2|\mathcal{F}_I\Big) - \Big(E^{\mathcal{Q}}\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\mathrm{g}}_{N,R(I,J)})|\mathcal{F}_I\Big)\Big)^2 \\
= \; & e^{\beta^2 N} \sum_{x,x' \in R(I,J)} \Big(\nu^q_{(I,J)}(x)\nu^q_{(I,J)}(x') \cdot \\
& \qquad E^{P^{\otimes 2}_{x,x'}}\Big(\Big(e^{\beta^2 B_N(S^1,S^2)} - 1\Big)e^{\beta u L_N(S^1)}e^{\beta u L_N(S^2)} 1_{\Omega^{\mathrm{g},\otimes 2}_{N,R(I,J)}}\Big)\Big) \\
\leq \; & e^{\beta^2 N} \sum_{x,x' \in R(I,J)} \nu^q_{(I,J)}(x)\nu^q_{(I,J)}(x') E^{P^{\otimes 2}_{x,x'}}\Big(\Big(e^{\beta^2 B_N(S^1,S^2)} - 1\Big)e^{\beta u L_N(S^1)}e^{\beta u L_N(S^2)}\Big) \\
\leq \; & e^{\beta^2 N} \sum_{x,x' \in R(I,J)} \Big[\nu^q_{(I,J)}(x)\nu^q_{(I,J)}(x')\Big(E^{P^{\otimes 2}_{x,x'}}\Big(e^{2\beta^2 B_N(S^1,S^2)} - 1\Big)\Big)^{1/2} \cdot \\
& \qquad \Big(E^{P_x} e^{2\beta u L_N(S^1)}\Big)^{1/2}\Big(E^{P_{x'}} e^{2\beta u L_N(S^2)}\Big)^{1/2}\Big] \\
\leq \; & e^{\beta^2 N}\Big(\Big(E^{P^{\otimes 2}_{0,0}} e^{2\beta^2(B_M(S^1,S^2)+1)}\Big)^{k_0} - 1\Big)^{1/2}\Big(E^{P_0} e^{2\beta u(L_M+1)}\Big)^{k_0} \\
= \; & e^{\beta^2 N}\Big(\Big(E^{P^{\otimes 2}_{0,0}}\Big(e^{2\beta^2(B_M(S^1,S^2)+1)} - 1\Big) + 1\Big)^{k_0} - 1\Big)^{1/2}\Big(E^{P_0} e^{2\beta u(L_M+1)}\Big)^{k_0}
\end{aligned}$$

and for the denominator, Q-a.s.

$$\begin{aligned}
& E^{\mathcal{Q}}\Big(Z^{\beta,u,q}_{N,\nu^q_{(I,J)}}(\Omega^{\mathrm{g}}_{N,R(I,J)})|\mathcal{F}_I\Big) \\
= & \sum_{x\in R(I,J)} \nu^q_{(I,J)}(x) E^{\mathcal{Q}}\left(E^{P_x}\left[e^{\beta\sum_{k=1}^N\left(v(IN+k,S_{IN+k})+u1_{S_{IN+k}=0}\right)} 1_{\Omega^{\mathrm{g}}_{N,R(I,J)}}\right]\right) \\
\geq & \sum_{x\in R(I,J)} \nu^q_{(I,J)}(x) e^{\frac{\beta^2}{2}N} E^{P_x}\left(1_{\Omega^{\mathrm{g}}_{N,R(I,J)}}\right) \\
\geq & e^{\frac{\beta^2}{2}N} K_6
\end{aligned}$$

where $K_6 = \frac{1}{2}\mathbf{P}(A_{C_L,c_s})$.

Note that by Lemma 1.1.5,

$$E^{P_0} e^{2\beta u L_M} \leq K e^{\mathrm{F}(2\beta u)M}$$

and $\mathrm{F}(2\beta u)M \to 4c_1$ as $\beta u \to 0$, since $\mathrm{F}(u) \sim c_1 u^2$ as $u \to 0$, and $M = c_1/\mathrm{F}(\beta u)$.

Therefore, there exist a $K_7 > 1$ such that

$$E^{P_0} e^{2\beta u(L_M+1)} \leq K_7 \text{ for sufficiently small } \beta u.$$

As a result, for any $(I,J) \in \mathbb{L}_{CG}$, we get Q-a.s.

$$\begin{aligned}
T^{(I,J)}_{N,q} &\leq \frac{1}{K_6^2}\left(E^{P_0} e^{2\beta u(L_M+1)}\right)^{k_0}\left(\left(E^{P^{\otimes 2}_{0,0}}\left[e^{2\beta^2(B_M(S^1,S^2)+1)} - 1\right] + 1\right)^{k_0} - 1\right)^{1/2} \\
&\leq K_6^{-2} K_7^{k_0}\left(\left(E^{P^{\otimes 2}_{0,0}}\left[e^{2\beta^2(B_M(S^1,S^2)+1)} - 1\right] + 1\right)^{k_0} - 1\right)^{1/2}
\end{aligned}$$

Hence, for any given $0 < \epsilon < 1$, we can choose a in Proposition 3.2.12 as follows:

$$a = \Big(\frac{K_6^4 \epsilon^2}{K_7^{2k_0}} + 1\Big)^{1/k_0} - 1. \tag{3.3.5}$$

Note that $0 < K_6 < 1$ and $K_7 > 1$, therefore for the above choice of a, a will be less than 1 for all $k_0 \geq 1$.

3.4 Lipschitz Percolation

Lipschitz percolation, the existence of open Lipschitz surfaces, was first introduced and studied in the papers [20] and [31]. In this section, we briefly summarize and adapt some of their results for dimension $d = 2$, to use in our context.

The site percolation model in $\mathbb{Z}^2$ is obtained by designating each site $x \in \mathbb{Z}^2$ *open* with probability p, otherwise *closed*, with different sites receiving independent states. The corresponding probability measure on the sample space $\Omega = \{0,1\}^{\mathbb{Z}^2}$ will be denoted by $\mathbb{P}_p$, and expectation by $\mathbb{E}_p$.

Let $\mathbb{Z}_0^+ = \{0, 1, 2, 3, \dots\}$.

A function $F : \mathbb{Z} \to \mathbb{Z}_0^+$ is called *Lipschitz* if

$$\text{for any } x, y \in \mathbb{Z} \text{ with } |x - y| = 1, \text{ we have } |F(x) - F(y)| \leq 1.$$

A Lipschitz function $F : \mathbb{Z} \to \mathbb{Z}_0^+$ is called *open* if for each $x \in \mathbb{Z}$, the site $(x, F(x)) \in \mathbb{Z}^2$ is open.

Remark 3.4.1. *In the original papers [20] and [31], it was assumed that* $F(\cdot) \geq 1$, *for our convenience we assume that* $F(\cdot) \geq 0$. *It does not change the results.*

Let $LIPF$ be the event that there exists an open Lipschitz function $F : \mathbb{Z} \to \mathbb{Z}_0^+$. The event $LIPF$ is invariant under the translation of $\mathbb{Z}^2$ by the unit vector $(1, 0)$. Therefore,

$\mathbb{P}_p(LIPF) = 0$ or 1. Since $LIPF$ is also an increasing event, there exists a $p_L \in [0, 1]$ such that

$$\mathbb{P}_p(LIPF) = \begin{cases} 0 & \text{if } p < p_L, \\ 1 & \text{if } p > p_L. \end{cases}$$

It was proved in [20] that $0 < p_L < 1$.

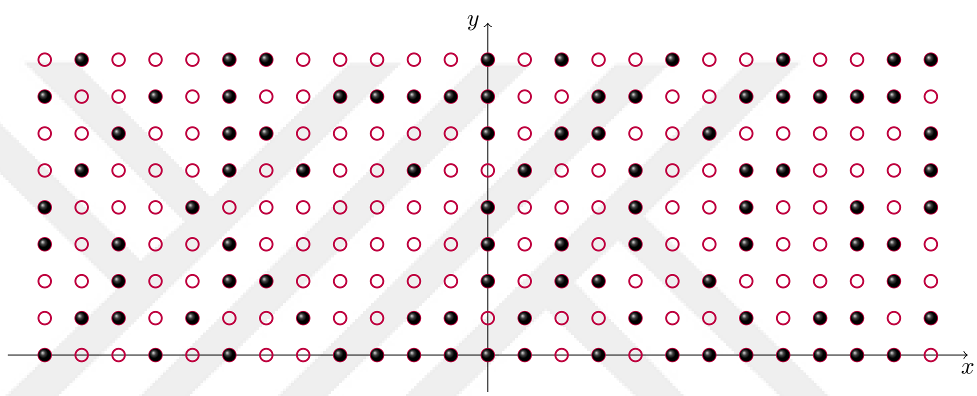

Figure 3.2: Site percolation on $\mathbb{Z}^2$. $\bullet$: open, $\circ$: closed

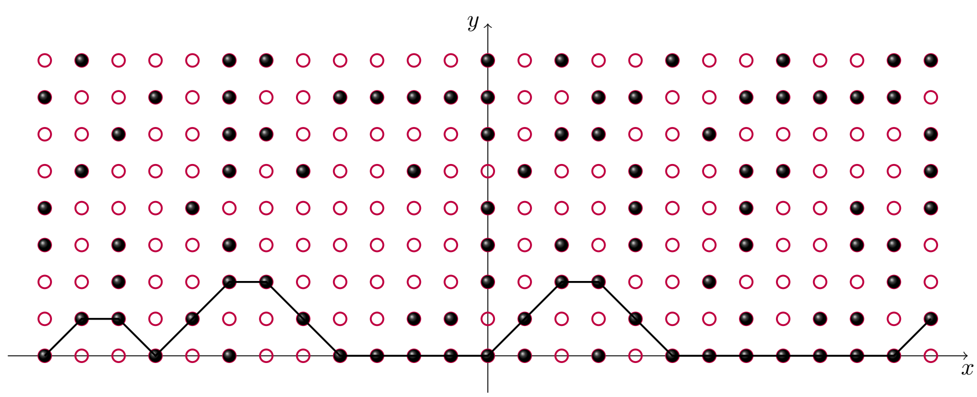

Figure 3.3: The lowest open Lipschitz function $F(\cdot)$

One of the main results of [20] is that the random field $(F(x) : x \in \mathbb{Z})$ is stationary and ergodic under each translation of $\mathbb{Z}$.

For any family $\mathcal{F}$ of Lipschitz functions, the lowest function

$$\bar{F}(x) = \inf\{F(x) : F \in \mathcal{F}\}$$

is also Lipschitz. If there exist an open Lipschitz function, then there exists a *lowest open Lipschitz function*, and it will be again denoted by F.

Theorem 3.4.2 (Theorem 2, [31])**.** *Let F be the lowest open Lipschitz function. For $p > p_L$, there exists $\alpha = \alpha(p) > 0$ such that*

$$\mathbb{P}_p(F(0) > n) \leq e^{-\alpha(n+1)}, \; n > 0.$$

Remark 3.4.3. *By Theorem 3.4.2, we can conclude that with positive probability there exists an infinite good path starting from $(0,0)$ in the coarse grained lattice $\mathbb{L}_{CG}$, if each site $(I,J) \in \mathbb{L}_{CG}$ is assigned independently open with probability $p > p_L$ or closed otherwise.*

Note that since the law of $F(x)$ is the same for all $x \in \mathbb{Z}^2$, the choice of the origin in the above theorem is arbitrary.

Let S be the set of all $x \in \mathbb{Z}$ for which $F(x) > 0$. Recall that by definition we assumed that $F \geq 0$.

Let S_0 be the vertex set of component containing 0 in the sub-graph of the nearest neighbor lattice of $\mathbb{Z}$ induced by S. We define $S_0 = \emptyset$ if $0 \notin S$.

The following theorem is a particular case of Theorem 3 of [31] for $d = 2$.

Theorem 3.4.4. *There exists $p'_L < 1$ such that for $p > p'_L$*

$$\exp(-\lambda n) \le \mathbb{P}_p(|S_0| > n) \le \exp(-\gamma n), \;\; n \ge 1,$$

where $\lambda = \lambda(p)$ and $\gamma = \gamma(p)$ are positive and finite.

Since $F(\cdot)$ is stationary and ergodic, by Ergodic Theorem

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} 1_{(F(i-1)=F(i)=0)} = \mathbb{P}_p(F(0) = F(1) = 0), \quad \mathbb{P}_p - \text{a.s.}$$

Let $l_s \in \mathbb{Z}$ such that $F(l_s) = 0$ and $F(l_s + 1) \neq 0$, and let

$$l_f = \inf\{l > l_s + 1 : F(l) = 0\}.$$

We call the sequence of vertices $(l_s, F(l_s)), (l_1, F(l_1), (l_2, F(l_2). \cdots, (l_f, F(l_f))$ a *detour from* $(l_s, 0)$ *to* $(l_f, 0)$, where $l_1 = l_s + 1$ and $l_{i+1} = l_i + 1$.

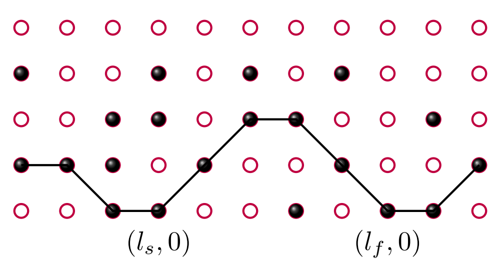

Figure 3.4: A detour

Note that

$$\begin{aligned}
&\mathbb{P}_p(F(0) = F(1) = 0) \\
= \quad & 1 - \mathbb{P}_p(\text{the link between } (0,0) \text{ and } (1,0) \text{ is not a part of the graph of } F(\cdot))
\end{aligned}$$

For $k, l \geq 0$, let $D_{-k,l}$ be the event that there is a detour from $(-k, 0)$ to $(l, 0)$.

For a given $\epsilon > 0$ and $0 < p' < 1$, we can choose l_0, l_1 sufficiently large so that $\exp(-\gamma(p')(l_0 + l_1 + 1)) < \epsilon/2$.

For $p > p'$ and close to 1, we can make $(l_0 + 1)l_1(l_0 + l_1 + 1)(1 - p) < \epsilon/2$. Since $\gamma(p)$ is increasing in p, by choosing p close to 1, we get

$$\begin{aligned}
\mathbb{P}_p \quad & (\text{the link between } (0,0) \text{ and } (1,0) \text{ is not in } F(\cdot)) \\
\leq \quad & \sum_{k \geq 0, l \geq 1} \mathbb{P}_p(D_{-k,l}) \\
\leq \quad & \sum_{\substack{0 \leq k \leq l_0 \\ 1 \leq l \leq l_1}} \mathbb{P}_p(D_{-k,l}) + \mathbb{P}_p(\text{there is a detour over } (0,0) \text{ of length} > l_0 + l_1 + 1) \\
\leq \quad & (l_0 + 1)l_1(l_0 + l_1)(1 - p) + \exp(-\gamma(p)(l_0 + l_1 + 1)) \\
\leq \quad & \epsilon
\end{aligned}$$

In the third inequality, we used the fact that there is a *detour* from $(-k, 0)$ to $(l, 0)$ implies that at least one of the sites in between must be closed. We also used Theorem 3.4.4 in the same inequality.

3.5 Stochastic Domination

Definition 3.5.1. *A collection of random variables* $(X_s)_{s\in\mathbb{Z}}$ *is called* k*-dependent if for each pair of subsets* $A, B \subset \mathbb{Z}$ *such that* $|a-b| > k$ *for each* $a \in A, b \in B$*, the families of random variables* $(X_s)_{s\in A}$ *and* $(X_s)_{s\in B}$ *are independent.*

The next theorem is a particular case of the main theorem of the paper [49] by Liggett, Schonmann and Stacey.

Theorem 3.5.2. *Let* $(X_s)_{s\in\mathbb{Z}}$ *be a collection of* 0-1 *valued* k*-dependent random variables, and suppose that there exists a* $p \in (0,1)$ *such that for each* $s \in \mathbb{Z}$

$$\mathbf{P}(X_s = 1) \geq p.$$

Then if

$$p > p_{SD}(k) = 1 - \frac{k^k}{(k+1)^{k+1}},$$

then $(X_s)_{s\in\mathbb{Z}}$ *is dominated from below by a product random field with density* $0 < \rho(p) < 1$*. Furthermore,* $\rho(p) \to 1$ *as* $p \to 1$*.*

Let's consider the following filtrations:

$$\mathcal{F}_I := \sigma(\{v(i,x) : 1 \leq i \leq IN, x \in \mathbb{Z}\}), \;\; I \geq 1,$$

and let $\mathbb{L}^I_{CG} := \{(I,J) \in \mathbb{L}_{CG} : 0 \leq J \leq I\}$.

Note that for each $I \geq 1$, conditionally on $\mathcal{F}_I$

$$\{X_{(I,J)} : (I, J) \in \mathbb{L}^I_{CG}\}$$

is a 3-dependent collection of random variables, and from section 3.3.2, we know that for a given $\epsilon > 0$, for sufficiently small $\beta u > 0$ and $\beta > 0$ with $u \geq K(\epsilon)\beta$, we have

$$Q(X_{(I,J)} = 1|\mathcal{F}_I) \geq 1 - \epsilon \quad Q - a.s. \text{ for each } I \geq 1, J \geq 0.$$

Therefore, by Theorem 3.5.2 there exists a collection of independent and identically distributed 0-1 valued random variables $\{Y_{(I,J)} : (I, J) \in \mathbb{L}_{CG}\}$ with $Q(Y_{(I,J)} = 1) = \rho(p)$ and

$$Q(X_{(I,J)} \geq Y_{(I,J)}|\mathcal{F}_I) = 1 \;\; Q - a.s. \tag{3.5.1}$$

where $p = p(\beta u)$, and $\rho(p) \to 1$ as $p \to 1$. Note also that p can be made greater than $1 - \epsilon$ for sufficiently small $\beta u > 0$ and $\beta > 0$ with $u \geq K(\epsilon)\beta$.

By taking the expectation of both sides with respect to Q in 3.5.1, we get

$$Q(X_{(I,J)} \geq Y_{(I,J)}) = 1$$

for all $(I, J) \in \mathbb{L}_{CG}$.

3.6 Final Steps

In section 3.3.2, we showed that for any $\epsilon \in (0,1)$, the site density p_{CG} of $\mathbb{L}_{CG}$ can be made greater than $1-\epsilon$ for sufficiently small $\beta u > 0$ and $\beta > 0$ with $u \geq K(\epsilon)\beta$. Therefore, we can assume that $p_{CG} > \max(p_L, p'_L, p_{SD}(3))$ and close to 1.

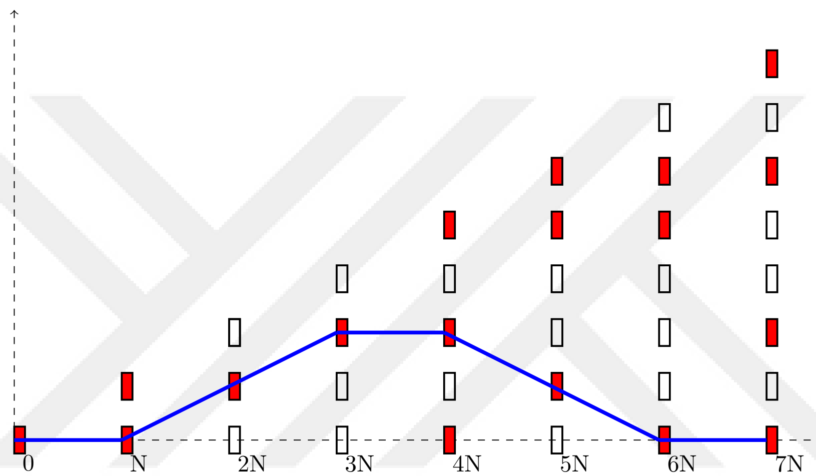

Figure 3.5: The infinite good path $\Gamma^{G,\infty}$ from the site $(0,0)$. Dark rectangles denote open sites and white ones denote closed sites in the coarse grained lattice $\mathbb{L}_{CG}$.

By Remark 3.4.3, with positive probability the infinite good path $\Gamma^{G,\infty}$ from the site $(0,0)$ exists.

Let

$$I_L := \text{the number of links of } \Gamma_L^{G,\infty} \text{ on the } x-\text{axis}.$$

Then from section 3.4, we know that the following limit exists Q-a.s. and constant:

$$\alpha = \alpha(\beta u) := \lim_{L\to\infty} \frac{I_L}{L}.$$

Furthermore, $\alpha \to 1$ as $p_{CG} \to 1$ thanks to the argument at the end of the section 3.4.

Recall that

$$U_{\text{off}} = \frac{1}{4} e^{\frac{\beta^2}{2} N} \mathbf{P}(A_{C_L, c_s}) = \Theta_1 e^{\frac{\beta^2}{2} N}$$

where $\Theta_1 = \frac{1}{4}\mathbf{P}(A_{C_L,c_s})$, and

$$U_{\text{on}} = \epsilon_0 \frac{1}{4} \mathbf{P}(Z \geq \frac{c_s}{\sqrt{\epsilon}}) e^{f^a(\beta,u)(1-\epsilon)N} = \Theta_2 e^{f^a(\beta,u)(1-\epsilon)N}$$

where $\Theta_2 = \epsilon_0 \frac{1}{4} \mathbf{P}(Z \geq \frac{c_s}{\sqrt{\epsilon}})$, and Z is a standard normal random variable.

For a standard normal random variable Z, the tail behavior is

$$\mathbf{P}(Z > x) \sim \frac{1}{\sqrt{2\pi} x} e^{-x^2/2} \text{ as } x \to \infty.$$

Therefore, we have

$$\Theta_2 \sim \frac{\epsilon_0}{4\sqrt{2\pi} c_s} \sqrt{\epsilon}\, e^{-c_s^2/2\epsilon} \text{ as } \epsilon \to 0. \tag{3.6.1}$$

Let's define $\Theta_3 = -(\alpha \log \Theta_2 + (1-\alpha)\log \Theta_1)$.

Note that $\Theta_3 > 0$, since $\alpha, \Theta_1, \Theta_2 \in (0,1)$; and

$$\Theta_3 = O(1/\epsilon) \text{ as } \epsilon \to 0. \tag{3.6.2}$$

Let $\Omega_{LN}^q := \Omega_{LN}^{\Gamma_L^{\mathrm{G},\infty}}$, where q denotes the quenched randomness of the environment.

Note that

$$\frac{1}{LN}\log Z_{LN}^{\beta,u,q} \geq \frac{1}{LN}\log Z_{LN}^{\beta,u,q}(\Omega_{LN}^{q})$$

and

$$Z_{LN}^{\beta,u,q}(\Omega_{LN}^{q}) = \prod_{I=1}^{L} \frac{Z_{IN}^{\beta,u,q}(\Omega_{IN}^{q})}{Z_{(I-1)N}^{\beta,u,q}(\Omega_{(I-1)N}^{q})} \tag{3.6.3}$$

$$= \prod_{I=1}^{L} Z_{N,\nu^{q}_{(I-1,\Gamma^{\mathrm{G},\infty}(I-1))}}^{\beta,u,q}(\Omega_{N}^{\Gamma^{\mathrm{G},\infty}_{I-1\to I}}) \tag{3.6.4}$$

where $Z_0^{\beta,u,q} := 1$.

Remark 3.6.1. *For a given realization of the environment, when we designate each site (I,J) in the coarse grained lattice $\mathbb{L}_{CG}$ as open or closed, we used random probability measures $\nu^{q}_{(I,J)} := \mu^{R(I,J)}_{\Gamma^{opt}_{(I,J)}}$ on the window-(I,J) which comes from the optimal path $\Gamma^{opt}_{(I,J)}$ as we defined in section 3.3. In that case, since the optimal path for a site on the infinite good path will be the finite part of that infinite good path, the random probability measures in the product 3.6.3 will be the same measures as we used initially to define sites as open or closed.*

Since there will be an infinite good path from $(0,0) \in \mathbb{L}_{CG}$ with positive probability, there exists a $c > 0$ such that for all $L \geq 1$,

$$Q\Big(\frac{1}{LN}\log Z_{LN}^{\beta,u,q} \geq \frac{1}{LN}\log U_{\mathrm{on}}^{I_L} U_{\mathrm{off}}^{L-I_L}\Big) \geq c$$

By using the fact that the quenched free energy $f^q(\cdot,\cdot)$ has self-averaging property, $N = Mk_0$, $M = c_1/\mathrm{F}(\beta u)$ and $f^a(\beta,u) = \mathrm{F}(\beta u) + \beta^2/2$ and letting $L \to \infty$, we get Q-a.s.

$$\begin{aligned}
f^q(\beta,u) &\geq \alpha\frac{1}{N}\log U_{\text{on}} + (1-\alpha)\frac{1}{N}\log U_{\text{off}} \\
&= \alpha\frac{1}{N}\log\left(\Theta_2 e^{f^a(\beta,u)(1-\epsilon)N}\right) + (1-\alpha)\frac{1}{N}\log\left(\Theta_1 e^{\frac{\beta^2}{2}N}\right) \\
&= \alpha(1-\epsilon)f^a(\beta,u) - \frac{1}{N}\Theta_3 + (1-\alpha)\frac{\beta^2}{2} \\
&= \alpha(1-\epsilon)f^a(\beta,u) - (\mathrm{F}(\beta u) + \frac{\beta^2}{2} - \frac{\beta^2}{2})\frac{1}{c_1k_0}\Theta_3 + (1-\alpha)\frac{\beta^2}{2} \\
&= \alpha(1-\epsilon)f^a(\beta,u) - f^a(\beta,u)\frac{1}{c_1k_0}\Theta_3 + \frac{\beta^2}{2}\left(1-\alpha+\frac{1}{c_1k_0}\Theta_3\right)
\end{aligned}$$

For sufficiently small $\beta > 0$ and $\beta u > 0$ with $u \geq K(\epsilon)\beta$, we can make $\alpha \geq 1-\epsilon$; and by choosing k_0 of order $\lfloor \epsilon^{-2} \rfloor$, the second term can be made greater than $-\epsilon f^a(\beta,u)$, and the third term stays bounded and positive since $\Theta_3 = O(1/\epsilon)$ as $\epsilon \to 0$. Therefore, we get

$$f^q(\beta,u) \geq (1-3\epsilon)f^a(\beta,u).$$

This completes the proof of Theorem 3.1.1.

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