

$\mathcal{H}^\infty$ DESIGN FOR AN X-29 AIRCRAFT

by

Oguzhan Cifdaloz



A Thesis Presented in Partial Fulfillment
of the Requirements for the Degree
Master of Science

ARIZONA STATE UNIVERSITY

May 2002

$\mathcal{H}^\infty$ DESIGN FOR AN X-29 AIRCRAFT

by

Oguzhan Cifdaloz

has been approved

May 2002

APPROVED:

_____, Chair

Supervisory Committee

ACCEPTED:

Department Chair

Dean, Graduate College

ABSTRACT

This thesis addresses $\mathcal{H}^\infty$ mixed sensitivity control system design for an X-29 aircraft. The X-29 is a fighter aircraft with a unique forward-swept wing design that trades off stability for fuel efficiency and improved maneuverability. Instability results because the aircraft center of gravity (c.g.) lies aft of the center of pressure (c.p.). This results in a negative static margin and an unstable short period mode. While the instability results in increased maneuverability, the instability is too fast for any human pilot to control. An automatic control system is therefore essential. In this thesis, control of both the longitudinal and lateral dynamics are examined.

The X-29 has multiple control surfaces. The longitudinal dynamics are controlled by three surfaces: the strake flap (situated at the rear of the aircraft), symmetric flap (situated on the trailing edge of the wing), and the canard (situated in front of the wing). In principle, these three surfaces may be used to control three independent quantities (e.g., speed, pitch angle, angle of attack). This is in contrast to conventional fighters, which typically have an elevator-type surface available for longitudinal control. The lateral dynamics are controlled by two surfaces: the differential flap (situated on the trailing edge of the wing) and the rudder (situated at the rear of the aircraft).

The $\mathcal{H}^\infty$ /bilinear transformation design method was applied to various single-input single-output (SISO) unstable plants in order to develop insight into the selection of weighting functions. The impacts of unmodeled high-frequency dynamics on closed loop system robustness were considered. High-frequency dynamics were then modeled to improve performance and understand how the nominal design changes. A similar approach was applied to various X-29 models.

For the longitudinal dynamics, two models were considered: (1) a model using pitch feedback, (2) a model using pitch and speed feedback.

$\mathcal{H}^\infty$ designs were developed for each. For the lateral dynamics, designs were developed using roll and sideslip angles as feedback variables. For each case, the impacts of actuator dynamics were considered.

ACKNOWLEDGMENTS

My deep thanks must commence with my wife, Defne Burduloğlu Çifdalöz, who encouraged me to continue my higher education in such a far country, even though she would sacrifice many things just to be with me.

The core of this work comes from the tutoring and guidance of my advisor, Professor Armando Antonio Rodriguez. My greatest debt is to him, who shared his knowledge, thoughts and precious time with me so generously. I feel very fortunate to be his student.

I can not repay the debt I owe to my father, Turhan, my mother Filiz, and my brothers Gülhan, Metehan, Ali İhsan and Emir Doğan Çifdalöz for their continuous support, unfailing love and trust.



TABLE OF CONTENTS

	Page
LIST OF TABLES	vii
LIST OF FIGURES	viii
CHAPTER	
1 Introduction	1
2 Model Analysis	3
2.1 X-29 Aircraft Longitudinal Dynamics	3
2.2 X-29 Aircraft Lateral Dynamics	18
3 Simple SISO Examples	30
3.1 Selection of Weighting Functions for Design Shaping	30
3.2 SISO Unstable Plant with High Frequency Dynamics	35
3.3 SISO Unstable Plant with High Frequency Dynamics: Impact of Sensitivity Weighting Design Parameters	45
3.4 SISO Unstable Plant with High Frequency Dynamics: Impact of Control Sensitivity Weighting Design Parameters	56
3.5 SISO Unstable Plant with High Frequency Dynamics: Impact of Complementary Sen- sitivity Weighting Design Parameters	65
3.6 SISO Unstable Plant with High Frequency Dynamics: Use of Bilinear Transformation	72
3.7 SISO Unstable Plant with Time Delay	80
4 $\mathcal{H}^\infty$ Mixed Sensitivity Control System Design: X-29 Lateral Dynamics	87
5 $\mathcal{H}^\infty$ Control System Design: X-29 Longitudinal Dynamics	110
5.0.1 Design 1: Pitch Attitude Feedback	110
5.0.2 Design 2: Speed and Pitch Attitude Feedback	128
6 Summary and Directions for Future Research	149

CHAPTER	Page
REFERENCES	150
APPENDIX	
A Matlab Macros	153



LIST OF TABLES

Table	Page
2.1 X-29 Aircraft Characteristics	4
2.2 X-29 Aircraft Longitudinal Control Surface Limits	6
2.3 Poles for X-29 Longitudinal Dynamics (Powered Approach)	6
2.4 Eigenvalues and Eigenvectors of Longitudinal Dynamics of X-29 (Powered Approach)	7
2.5 X-29 Aircraft Lateral Control Surface Limits	19
2.6 Poles for X-29 Lateral Dynamics (Powered Approach)	20
2.7 Eigenvalues and Eigenvectors for X-29 Lateral Dynamics (Powered Approach)	20
3.1 Weighting Functions and Design Parameters	35

LIST OF FIGURES

Figure		Page
2.1	X-29 Forward-Swept Wing Aircraft	3
2.2	Visualization of Poles for X-29 Longitudinal Dynamics (Powered Approach)	6
2.3	Visualization of X-29 Aircraft Phugoid Mode ($x_o = Re(x_1)$)	9
2.4	Visualization of X-29 Aircraft Phugoid Mode ($x_o = Im(x_1)$)	9
2.5	Visualization of X-29 Aircraft Damping Mode ($x_o = x_3$)	10
2.6	Visualization of X-29 Aircraft Pitch-Up Instability Mode ($x_o = x_4$)	11
2.7	Frequency Response - δ_c to v	13
2.8	Frequency Response - δ_c to α	13
2.9	Frequency Response - δ_c to θ	14
2.10	Frequency Response - δ_{sf} to v	14
2.11	Frequency Response - δ_{sf} to α	14
2.12	Frequency Response - δ_{sf} to θ	15
2.13	Frequency Response - δ_{stf} to v	15
2.14	Frequency Response - δ_{stf} to α	15
2.15	Frequency Response - δ_{stf} to θ	16
2.16	X-29 Aircraft Singular Values MIMO Frequency Response (Longitudinal Dynamics)	16
2.17	X-29 SVD at DC for Longitudinal Dynamics - $v_1 \rightarrow \sigma_1 u_1 = G(j0)v_1$	17
2.18	X-29 SVD at DC for Longitudinal Dynamics - $v_2 \rightarrow \sigma_2 u_2 = G(j0)v_2$	17
2.19	X-29 SVD at DC for Longitudinal Dynamics - $v_3 \rightarrow \sigma_3 u_3 = G(j0)v_3$	18
2.20	Visualization of Poles for X-29 Lateral Dynamics (Powered Approach)	20
2.21	Visualization of X-29 Aircraft Dutch Roll Mode ($x_o = Re(x_1)$)	21
2.22	Visualization of X-29 Aircraft Dutch Roll Mode ($x_o = Im(x_1)$)	21
2.23	Visualization of X-29 Aircraft Roll Subsidence Mode ($x_o = x_3$)	22
2.24	Visualization of X-29 Aircraft Spiral Divergence Mode ($x_o = x_4$)	24
2.25	Frequency Response - δ_{df} to ϕ	25
2.26	Frequency Response - δ_{df} to β	26

Figure	Page
2.27 Frequency Response - δ_r to ϕ	26
2.28 Frequency Response - δ_r to β	27
2.29 X-29 Aircraft Singular Values MIMO Frequency Response (Lateral Dynamics)	27
2.30 X-29 SVD at DC for Lateral Dynamics - $v_1 \rightarrow \sigma_1 u_1 = G(j0)v_1$	28
2.31 X-29 SVD at DC for Lateral Dynamics - $v_2 \rightarrow \sigma_2 u_2 = G(j0)v_2$	29
3.1 Plant Frequency Responses - $P = \frac{1}{s-1}$ (plant 1), $P = \frac{1}{s-1} \left[\frac{10}{s+10} \right]^2$ (plant 2)	38
3.2 Inverse of Weighting Functions - W_1^{-1} , W_2^{-1} , W_3^{-1}	39
3.3 Controller Frequency Responses - K	39
3.4 Open Loop Frequency Responses - $L = PK$	39
3.5 Sensitivity Frequency Responses - $S = \frac{1}{1+PK}$	40
3.6 Complementary Sensitivity Frequency Responses - $T = \frac{PK}{1+PK}$	41
3.7 Input Disturbance to Output Frequency Responses - $T_{d_{iy}} = SP$	42
3.8 Reference to Control Frequency Responses - $T_{ru} = KS$	42
3.9 Exogenous Signals to Regulated Variables - $\sigma_i(T_{wz})$	43
3.10 Output Response to Step Reference Command (without Pre-filter)	44
3.11 Control Response to Step Reference Command (without Pre-filter)	44
3.12 Output Response to Step Reference Command (with Pre-filter)	45
3.13 Control Response to Step Reference Command (with Pre-filter)	45
3.14 Dependence of $ S(j\epsilon\omega_b) $ on ϵ	46
3.15 Dependence of 3 dB Closed Loop (T) Bandwidth on ϵ	46
3.16 Dependence of Sensitivity Gain Crossover Frequency on ϵ	47
3.17 Dependence of Open Loop Gain Crossover on ϵ	47
3.18 Dependence of $\ T_{wz}\ _{\mathcal{H}^\infty}$ on ϵ	47
3.19 Dependence of Gain Margins on ϵ	48
3.20 Dependence of Phase Margin on ϵ	48
3.21 Dependence of Overshoot on ϵ (Step Command, No Prefilter)	49
3.22 Dependence of Settling Time on ϵ (Step Command, No Prefilter)	49
3.23 Dependence of 3 dB Closed Loop (T) Bandwidth on ω_b	50

Figure	Page
3.24 Dependence of Sensitivity Gain Crossover Frequency on ω_b	50
3.25 Dependence of Open Loop Gain Crossover on ω_b	50
3.26 Dependence of $ S(j\omega_b) $ on ω_b	51
3.27 Dependence of $\ T_{wz}\ _{\mathcal{H}^\infty}$ on ω_b	51
3.28 Dependence of Gain Margins on ω_b	52
3.29 Dependence of Phase Margin on ω_b	52
3.30 Dependence of Overshoot on ω_b (without Prefilter)	52
3.31 Dependence of Settling Time on ω_b (without Prefilter)	53
3.32 Dependence of Step Responses on ω_b for Design 1 (without Prefilter)	53
3.33 Dependence of Step Responses ω_b for Design 1 ⁺ (without pre-filter)	54
3.34 Dependence of Step Responses on ω_b for Design 2 (without Prefilter)	54
3.35 Dependence of 3 dB Closed Loop (T) Bandwidth on M_s	55
3.36 Dependence of Sensitivity Gain Crossover on M_s	55
3.37 Dependence of Open Loop Gain Crossover on M_s	55
3.38 Dependence of $\ T_{wz}\ _{\mathcal{H}^\infty}$ on M_s	56
3.39 Dependence of Gain Margins on M_s	56
3.40 Dependence of Phase Margin on M_s	56
3.41 Dependence of Overshoot on M_s (without Prefilter)	57
3.42 Dependence of Settling Time on M_s (without Prefilter)	57
3.43 Dependence of $\ T_{ru}\ _{\mathcal{H}^\infty}$ on M_u	58
3.44 Dependence of 3 dB Closed Loop (T) Bandwidth on M_u	58
3.45 Dependence of Sensitivity Gain Crossover Frequency on M_u	58
3.46 Dependence of Open Loop Gain Crossover on M_u	59
3.47 Dependence of $\ T_{wz}\ _{\mathcal{H}^\infty}$ on M_u	60
3.48 Dependence of Gain Margins on M_u	60
3.49 Dependence of Phase Margin on M_u	61
3.50 Dependence of Overshoot on M_u (without Prefilter)	61
3.51 Dependence of Settling Time on M_u (without Prefilter)	61
3.52 Dependence of $\ T_{ru}\ _{\mathcal{H}^\infty}$ on ω_{bu}	62

Figure	Page
3.53 Dependence of T_{ru} Gain Crossover on ω_{bu}	62
3.54 Dependence of 3 dB Closed Loop (T) Bandwidth on ω_{bu}	63
3.55 Dependence of Sensitivity Gain Crossover on ω_{bu}	63
3.56 Dependence of Open Loop Gain Crossover on ω_{bu}	63
3.57 Dependence of $\ T_{wz}\ _{\mathcal{H}^\infty}$ on ω_{bu}	64
3.58 Dependence of Gain Margins on ω_{bu}	64
3.59 Dependence of Phase Margin on ω_{bu}	64
3.60 Dependence of Overshoot on ω_{bu}	65
3.61 Dependence of Settling Time on ω_{bu}	65
3.62 Dependence of 3 dB Closed Loop (T) Bandwidth on ω_{bc}	66
3.63 Dependence of Sensitivity Gain Crossover on ω_{bc}	66
3.64 Dependence of Open Loop Gain Crossover on ω_{bc}	67
3.65 Dependence of $\ T_{wz}\ _{\mathcal{H}^\infty}$ on ω_{bc}	67
3.66 Dependence of Gain Margins on ω_{bc}	68
3.67 Dependence of Phase Margin on ω_{bc}	68
3.68 Dependence of Overshoot on ω_{bc} (without Prefilter)	68
3.69 Dependence of Settling Time on ω_{bc} (without Prefilter)	69
3.70 Dependence of 3 dB Closed Loop (T) Bandwidth on M_y	69
3.71 Dependence of Sensitivity Gain Crossover Frequency on M_y	70
3.72 Dependence of Open Loop Gain Crossover on M_y	70
3.73 Dependence of $\ T_{wz}\ _{\mathcal{H}^\infty}$ on M_y	70
3.74 Dependence of Gain Margins on M_y	71
3.75 Dependence of Phase Margin on M_y	71
3.76 Dependence of Overshoot on M_y (without Prefilter)	72
3.77 Dependence of Settling Time on M_y (without Prefilter)	72
3.78 Dependence of 3 dB Closed Loop (T) Bandwidth on p_1	76
3.79 Dependence of Sensitivity Gain Crossover on p_1	77
3.80 Dependence of Open Loop Gain Crossover on p_1	77
3.81 Dependence of $\ T_{wz}\ _{\mathcal{H}^\infty}$ on p_1	77

Figure	Page
3.82 Dependence of Gain Margins on p_1	78
3.83 Dependence of Phase Margin on p_1	78
3.84 Dependence of Overshoot on p_1	79
3.85 Dependence of Settling Time on p_1	79
3.86 Step Responses for Design 1 - Dependence on p_1	79
3.87 Step Responses for Design 1 ⁺ - Dependence on p_1	80
3.88 Step Responses for Design 2 - Dependence on p_1	80
3.89 Plant Frequency Response - $P = \frac{1}{s-1}$, $P = \frac{1}{s-1} \left[\frac{20-s}{s+20} \right]$	82
3.90 Controller Frequency Response	82
3.91 Open Loop Frequency Response - $L = PK$	83
3.92 Sensitivity Frequency Response - $S = \frac{1}{1+PK}$	83
3.93 Complementary Sensitivity Frequency Response - $T = \frac{PK}{1+PK}$	83
3.94 Input Disturbance to Output Frequency Response - $T_{d_{iy}} = SP$	84
3.95 Reference to Control Frequency Response: $T_{ru} = KS$	84
3.96 Exogenous Signal to Regulated Signals - $\frac{1}{\gamma}\sigma_i(T_{wz})$	84
3.97 Output Response to Step Reference Command (without Pre-Filter)	85
3.98 Control Response to Step Reference Command (without Pre-Filter)	85
3.99 Output Response to Step Reference Command (with Pre-filter)	86
3.100 Control Response to Step Reference Command (with Pre-filter)	86
4.1 Singular Values - TITO X-29 Lateral Dynamics (with/without Actuator Dynamics)	89
4.2 Weighting Functions for Mixed-Sensitivity $\mathcal{H}^\infty$ Optimization	90
4.3 Compensator Singular Values	100
4.4 Open Loop Singular Values at Plant Output - $L_o = PK$	101
4.5 Open Loop Singular Values at Plant Input - $L_i = KP$	101
4.6 Sensitivity Singular Values at Plant Output - $S_o = [I + PK]^{-1}$	102
4.7 Sensitivity Singular Values at Plant Input - $S_i = [I + KP]^{-1}$	103
4.8 Complementary Sensitivity Singular Values at Plant Output - $T_o = PK[I + PK]^{-1}$	103
4.9 Complementary Sensitivity Singular Values at Plant Input - $T_i = KP[I + KP]^{-1}$	104

Figure	Page
4.10 Closed Loop Singular Values - T_{ry}	105
4.11 Input Disturbance to Output Singular Values - T_{d_iy}	105
4.12 Reference to Control Singular Values - T_{ru}	106
4.13 Response to Step Reference Command - Outputs y	107
4.14 Response to Step Reference Command - Controls u	107
4.15 Response to ϕ Doublet Pulse Command - Outputs y	108
4.16 Response to ϕ Doublet Pulse Command - Controls u	108
4.17 Response to β Doublet Pulse Command - Outputs	109
4.18 Response to β Doublet Pulse Command - Controls	109
5.1 X-29 Singular Values	112
5.2 Weighting Functions for Mixed-Sensitivity $\mathcal{H}^\infty$ Optimization	113
5.3 Compensator Singular Values	122
5.4 Open Loop Singular Values at Plant Output - $L_o = PK$	123
5.5 Open Loop Singular Values at Plant Input - $L_i = KP$	123
5.6 Sensitivity Singular Values at Plant Output - $S_o = [I + PK]^{-1}$	124
5.7 Frequency Response: $S_i = [I + KP]^{-1}$	125
5.8 Complementary Sensitivity Singular Values at Plant Output - $T_o = PK[I + PK]^{-1}$	125
5.9 Closed Loop Singular Values - T_{ry}	126
5.10 Complementary Sensitivity Singular Values at Plant Input - $T_i = KP[I + KP]^{-1}$	126
5.11 Input Disturbance to Output Singular Values - T_{d_iy}	127
5.12 Reference to Control Singular Values - T_{ru}	128
5.13 Response to Step Reference Command - Outputs	129
5.14 Response to Step Reference Command - Controls	129
5.15 Response to θ Doublet Pulse Command - Outputs	130
5.16 Response to θ Doublet Pulse Command - Controls	130
5.17 X-29 Singular Values	132
5.18 Weighting Functions for Mixed-Sensitivity $\mathcal{H}^\infty$ Optimization	133
5.19 Compensator Singular Values	141

Figure	Page
5.20 Open Loop Singular Values at Plant Output - $L_o = PK$	142
5.21 Open Loop Singular Values at Plant Input - $L_i = KP$	142
5.22 Sensitivity Singular Values at Plant Output - $S_o = [I + PK]^{-1}$	143
5.23 Frequency Response: $S_i = [I + KP]^{-1}$	144
5.24 Complementary Sensitivity Singular Values at Plant Output - $T_o = PK[I + PK]^{-1}$	144
5.25 Closed Loop Singular Values - T_{ry}	145
5.26 Complementary Sensitivity Singular Values at Plant Input - $T_i = KP[I + KP]^{-1}$	146
5.27 Input Disturbance to Output Singular Values - $T_{d_{iy}}$	146
5.28 Response to Step Reference Command - Outputs	147
5.29 Response to Step Reference Command - Outputs	148
5.30 Response to Step Reference Command - Controls	148

CHAPTER 1

Introduction

Contributions of Work. This thesis shows how $\mathcal{H}^\infty$ Mixed Sensitivity Optimization may be used to systematically design a flight control system for a high performance aircraft. Three control system designs were developed for the X-29 aircraft – two longitudinal and one lateral. These will serve as benchmarks for future MIMO studies. This includes development of a gain scheduled flight control system.

$\mathcal{H}^\infty$ controller design method. The $\mathcal{H}^\infty$ theory provides a direct, reliable procedure for synthesizing MIMO controllers which satisfy singular value loop shaping specifications. In this thesis, the $\mathcal{H}^\infty$ design method is used to design controllers for the selected aircraft system.

Bilinear Transform. It was found that simple bilinear transform can be extremely useful when used with robust control synthesis techniques. All the left half plane poles are canceled when $\mathcal{H}^\infty$ design methodology is used. If a plant has lightly damped stable poles, cancelation of these poles results in unacceptable designs. We use bilinear transformation to shift such lightly damped poles into the right half plane in order to prevent them from being canceled. Combining the $\mathcal{H}^\infty$ method and the bilinear transform is a powerful way of controlling closed-loop system performance. The bilinear transform has been applied for designing controllers of both the lateral and longitudinal dynamics of the X-29 aircraft in this thesis.

Outline. The rest of this thesis is organized as follows. Chapter 2 describes the mathematical models of the X-29 aircraft for both longitudinal and lateral dynamics. Chapter 3 describes the application of $\mathcal{H}^\infty$ /bilinear transformation design method to various single-input single-output (SISO) unstable plants in order to develop insight into the selection of weighting functions and the impact of unmodeled high-frequency dynamics on closed loop system robustness. High-frequency dynamics were then modeled to improve performance and understand how the nominal design changes. Chapter 4 describes an $\mathcal{H}^\infty$ Mixed Sensitivity Control System Design for the X-29 Lateral Dynamics. Chapter 5 includes for $\mathcal{H}^\infty$ Mixed Sensitivity Control System Designs for the X-29 Longitudinal Dynamics. Two models were considered: (1) a model using pitch feedback, (2) a model using pitch and speed feedback. In this design, bank angle and

sideslip angle were used as feedback variables. Chapter 6 summarizes the thesis and presents directions for future research.



CHAPTER 2

Model Analysis

This chapter describes the mathematical models for the longitudinal and lateral dynamics of X-29 Aircraft discussed in this thesis.

2.1 X-29 Aircraft Longitudinal Dynamics

In this section, we examine the longitudinal dynamics for an X-29 forward-swept wing aircraft [1], [9]. The X-29 is shown in Figure 2.1.

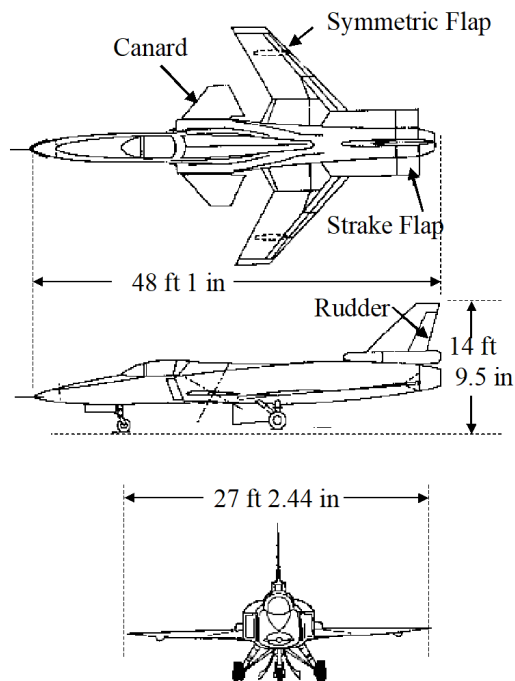


Figure 2.1: X-29 Forward-Swept Wing Aircraft

Background. The X-29 program was initiated in 1977 by DARPA (Defense Advanced Research Projects Agency) in cooperation with Grumman Aerospace Corporation. The goal was to develop a fast (transonic), highly maneuverable, aerodynamically efficient fighter. To achieve speed and high maneuverability, designers at Grumman implemented a forward-swept wing design. Graphite-epoxy composite materials were used in the wing skin rather than metal in order to provide strength and reduce weight. The increased stiffness provided by composites helps with wing bending and torsional mode problems associated with forward-swept designs.

Instability: Need for a Flight Control System. While the forward-swept design offers reduced drag in comparison with comparable aft-swept wing designs, the forward-swept wing results in a center of pressure (c.p.) which lies in front of the center of gravity (c.g.). This, in turn, results in a natural *pitch-up instability* that is much too fast for a pilot to manage without the assistance of a control system. The absence of a tail and a forward situated canard contribute to the instability. Because a pilot simply could not be able to respond quickly enough to stabilize and control this natural instability, a flight control system exploiting computer technology is essential.

Aircraft Characteristics. Characteristics of the X-29 are shown in Table 2.1 [1]:

Power source	Single F404-GE-400 engine
Maximum thrust	approximately 16,000 lbs
Vehicle empty weight	14,000 lbs
Fuel capacity	4,000 lbs
Length	48.1 ft
Height	14.8 ft
Wing span	27.2 ft
Wing area	185 ft ²
Wing leading-edge forward sweep	29.3 deg
Mean aerodynamic chord	7.2 ft
Canard area	37 ft ²
Roll moment of inertia	$I_{xx} = 4,597 \text{ lb ft s}^2$
Pitch moment of inertia	$I_{yy} = 52,805 \text{ lb ft s}^2$
Yaw moment of inertia	$I_{zz} = 55,754 \text{ lb ft s}^2$
Cross moment of inertia	$I_{xz} = 2,468 \text{ lb ft s}^2$

Table 2.1: X-29 Aircraft Characteristics

Longitudinal Dynamics: Powered Approach. We examine the aircraft during a straight and level powered approach [1, page 19]. The precise flight condition is as follows:

- speed of Mach 0.258 (160 kts equivalent airspeed), 284 ft/sec (193.6 mph) true velocity
- altitude of 4,000 ft
- angle of attack of 8.48° for a 1g trim
- control surfaces trimmed as follows:
 - canard at -6.55°

- symmetric flap at 20.15°
- strake flap at -10.93°
- weight of 14,777 lbs.

This flight condition is appropriate for examining performance during take-off or landing. A state space model for the longitudinal dynamics at the above flight condition is as follows [1, pp. 19, 21]:

$$\dot{x} = Ax + Bu \quad (2.1)$$

$$y = Cx + Du \quad (2.2)$$

$$A = \begin{bmatrix} -0.0425 & -0.1399 & -0.0079 & -0.5613 \\ -0.0452 & -0.5482 & 0.9893 & 0.0000 \\ 0.0224 & 3.6040 & -0.2221 & 0.0000 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix} \quad (2.3)$$

$$B = \begin{bmatrix} -0.0343 & -0.0922 & 0.0048 \\ -0.0610 & -0.0766 & -0.0242 \\ 1.5666 & -0.3201 & -0.7655 \\ 0 & 0 & 0 \end{bmatrix} \quad (2.4)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2.5)$$

$$u = \begin{bmatrix} \delta_c & - & \text{canard} & (deg) \\ \delta_{sf} & - & \text{symmetric flap} & (deg) \\ \delta_{stf} & - & \text{strake flap} & (deg) \end{bmatrix} \quad (2.6)$$

$$x = \begin{bmatrix} v & - & \text{speed} & (ft/sec) \\ \alpha & - & \text{angle of attack} & (deg) \\ q & - & \text{pitch rate} & (deg/sec) \\ \theta & - & \text{pitch angle} & (deg) \end{bmatrix} \quad (2.7)$$

$$y = \begin{bmatrix} v & - & \text{speed} & (ft/sec) \\ \alpha & - & \text{angle of attack} & (deg) \\ \theta & - & \text{pitch angle} & (deg) \end{bmatrix} \quad (2.8)$$

The aircraft's three control surfaces are as follows:

- canard - forward situated control surface,
- symmetric flap - situated on the wings,
- strake flap - rearward situated control surface.

These surfaces, in principle, may be used to control three independent variables. Control surface limits are as follows:

Control Surface	Position limit, deg	Rate limit, deg/sec
Canard	-60, 30	± 100
Symmetric Flap	-10, 25	± 70
Strake Flap	-30, 30	± 30

Table 2.2: X-29 Aircraft Longitudinal Control Surface Limits

Comment 2.1.1 (Linear Model: Perturbation Step Sizes)

The above linear model was developed from a six-degree-of-freedom nonlinear model with a full envelop aerodynamic database obtained from wind-tunnel test results. The following perturbation step sizes about trim were used to obtain the model: $\pm 1\%$ speed of sound for speed, $\pm 2^\circ$ for angle of attack, and $\pm 1^\circ$ for the remaining states and control surfaces. Flight data was not used to update the model. ■

Modal Analysis: Natural Dynamics. In this section, the natural modes for the X-29's longitudinal dynamics are examined. The associated system poles (eigenvalues of A) are listed in Table 2.1 and may be visualized as shown in Figure 2.2. These poles define three distinct modes. Each mode is now described. To do so, we refer to Table 2.1 which contains each of the eigenvalues (i.e. system poles) and associated (right) eigenvectors.

Pole	Damping	Frequency (rad/s)	Mode Name
$-0.0219 \pm j0.1544$	0.1402	0.1560	Phugoid Mode
-2.2879	1	2.2879	Damping Mode
1.5189	-1	1.5189	Pitch-Up Instability

Table 2.3: Poles for X-29 Longitudinal Dynamics (Powered Approach)

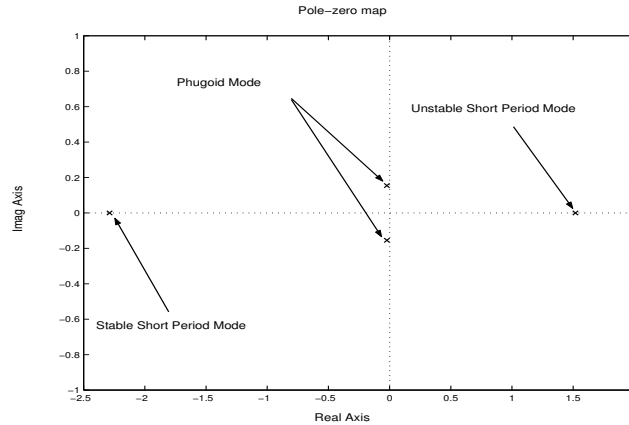


Figure 2.2: Visualization of Poles for X-29 Longitudinal Dynamics (Powered Approach)

- *Phugoid Mode* ($\lambda_{1,2}$). The phugoid mode [7, pp. 179, 182] is associated with complex conjugate left half plane poles at $\lambda_{1,2} = -0.0219 \pm j0.1544$. The damping ratio is $\zeta = 0.1402$, the undamped natural frequency is $\omega_n = 0.1560 \text{ rad/sec}$. To examine this mode, the real and imaginary parts of x_1 are

Eigenvalues	Eigenvectors
Phugoid Mode $\lambda_{1,2} = -0.0219 \pm j0.1544$	$x_{1,2} = \begin{bmatrix} -0.9060 \pm j0.3252 \\ 0.0029 \mp j0.0029 \\ -0.0394 \pm j0.0137 \\ 0.1227 \pm j0.2379 \end{bmatrix} = \begin{bmatrix} 0.9626e^{\pm j160.3^\circ} \\ 0.0041e^{\mp j45^\circ} \\ 0.0417e^{\pm j160.83^\circ} \\ 0.2677e^{\pm j62.72^\circ} \end{bmatrix}$
Damping Mode $\lambda_3 = -2.2879$	$x_3 = \begin{bmatrix} -0.1140 \\ -0.4614 \\ 0.8062 \\ -0.3524 \end{bmatrix}$
Pitch-Up Instability $\lambda_4 = 1.5189$	$x_4 = \begin{bmatrix} 0.2156 \\ -0.3665 \\ -0.7560 \\ -0.4977 \end{bmatrix}$

Table 2.4: Eigenvalues and Eigenvectors of Longitudinal Dynamics of X-29 (Powered Approach)

selected separately as initial conditions (see Table 2.1). More specifically, if

$$x_o = k_1 \operatorname{Re}\{e^{\lambda_1 t} x_1\} + k_2 \operatorname{Im}\{e^{\lambda_1 t} x_1\} = k_1 \begin{bmatrix} -0.9060 \\ 0.0029 \\ -0.0394 \\ 0.1227 \end{bmatrix} + k_2 \begin{bmatrix} \pm 0.3252 \\ \mp 0.0029 \\ \pm 0.0137 \\ \pm 0.2379 \end{bmatrix} \quad (2.9)$$

where k_1 and k_2 are arbitrary real constants, then

$$x(t) = k_1 \operatorname{Re}\{e^{\lambda_1 t} x_1\} + k_2 \operatorname{Im}\{e^{\lambda_1 t} x_1\} \quad (2.10)$$

$$= k_1 \begin{bmatrix} 0.9626e^{-0.0219t} \cos(0.1544t \pm 160.3^\circ) \\ 0.0041e^{-0.0219t} \cos(0.1544t \mp 45^\circ) \\ 0.0417e^{-0.0219t} \cos(0.1544t \pm 160.83^\circ) \\ 0.2677e^{-0.0219t} \cos(0.1544t \pm 62.72^\circ) \end{bmatrix} \quad (2.11)$$

$$+ k_2 \begin{bmatrix} 0.9626e^{-0.0219t} \sin(0.1544t \mp 160.3^\circ) \\ 0.0041e^{-0.0219t} \sin(0.1544t \pm 45^\circ) \\ 0.0417e^{-0.0219t} \sin(0.1544t \mp 160.83^\circ) \\ 0.2677e^{-0.0219t} \sin(0.1544t \mp 62.72^\circ) \end{bmatrix}. \quad (2.12)$$

Examination of the polar form of x_1 shows that

- v and q are nearly in phase
- v leads θ by $160.3^\circ - 62.72^\circ = 97.58^\circ$
- θ leads α by $62.72^\circ - (-45)^\circ = 107.72^\circ$
- v leads α by $160.3^\circ - (-45)^\circ = 205.3^\circ$

Figure 2.3 contains the time response which results when the initial condition is $x_o = \operatorname{Re}(x_1)$ (i.e. $k_1 =$

2, $k_2 = 0$). Figure 2.4 contains the time response which results when $x_o = Im(x_1)$ (i.e. $k_1 = 0, k_2 = 2$). Examination of the eigenvector x_1 (or x_2) shows that this mode is associated with a nearly constant angle of attack; i.e. the variable α associated with the linear model does not change much. This implies that the pitch and flight path angles are approximately equal. The eigenvector x_1 (or x_2) also shows that the mode is primarily associated with the aircraft's speed. This mode is associated with the exchange of potential and kinetic energy. As the pitch angle increases, the aircraft climbs, its speed decreases, and it gains potential energy. As the aircraft continues to climb, its pitch angle levels off, the speed reduces to zero. As the pitch angle then decreases, the speed goes negative, and potential energy is converted into kinetic energy. The increased kinetic energy results in additional lift which causes the aircraft to climb.

- This qualitative behavior is consistent with the fact that

$$\dot{v} \approx -0.5613 \theta. \quad (2.13)$$

From this, we have

$$V \approx -\frac{0.5613}{s} \Theta. \quad (2.14)$$

Since $s = -0.0219 + j0.1544 = 0.1544e^{j90^\circ}$, this approximate relationship yields

$$\frac{V}{\Theta} \approx e^{j180^\circ} \frac{0.5613}{0.1559e^{j98.1^\circ}} = 3.6e^{j90^\circ}. \quad (2.15)$$

This relationship is an approximation to what the first and fourth entries of x_1 yield, namely:

$$\frac{V}{\Theta} = \frac{0.9626e^{j160.3^\circ}}{0.2677e^{j62.72^\circ}} = 3.5958e^{j97.58^\circ}. \quad (2.16)$$

This stable lightly damped mode has a time constant of $\tau = 45.7 \text{ sec}$ and hence takes approximately $t_s = 5\tau = 228.62 \text{ sec} = 3.8 \text{ min}$ to settle. With the aircraft traveling at 193.6 mph (or 3.23 miles per minute), this mode takes approximately 12.3 miles to decay.

- *Short Period Mode* (λ_3 and λ_4). Commercial aircraft typically exhibit a lightly damped sinusoidal exponential short period mode. The mode is associated with a constant speed. The X-29's forward swept wing results in a center of pressure (c.p.) that lies in front of the center of gravity (c.g.). This results in a stable damping mode ($\lambda_3 = -2.2879$) and a pitch-up instability ($\lambda_4 = 1.5188$). Each mode is now described.

- *Damping Mode.* Associated with the damping mode is a stable pole at $\lambda_3 = -2.2879$. This mode has a time constant of $\tau = 0.4371 \text{ sec}$ and a settling time of $t_s = 5\tau = 2.1854 \text{ sec}$. With the aircraft traveling at 284 ft/sec, this mode decays in approximately 620.7 ft or 0.12 mile. This

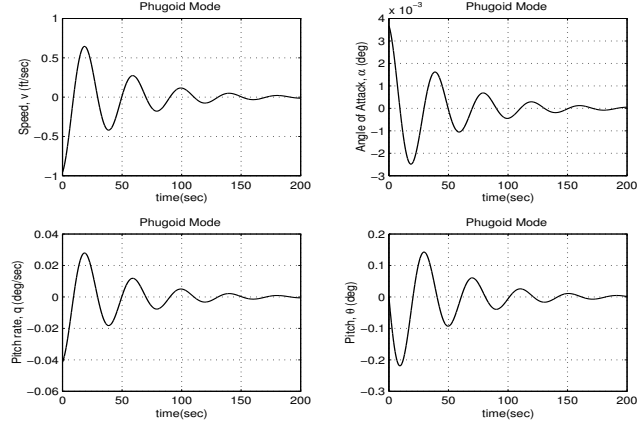


Figure 2.3: Visualization of X-29 Aircraft Phugoid Mode ($x_o = Re(x_1)$)

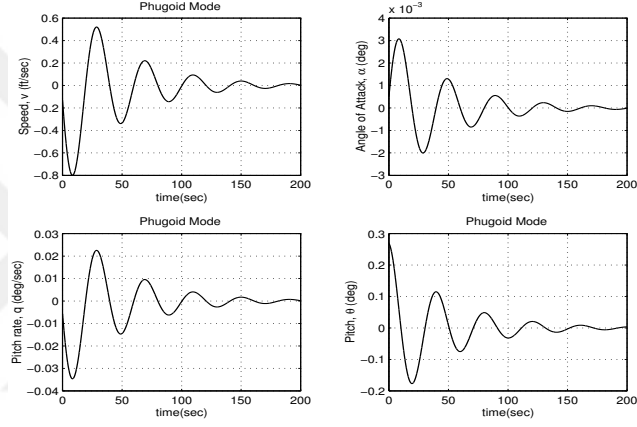


Figure 2.4: Visualization of X-29 Aircraft Phugoid Mode ($x_o = Im(x_1)$)

mode is primarily associated with pitch rate and rotational aerodynamic damping. To examine this mode, x_3 is selected as an initial condition. That is, if

$$x_o = \begin{bmatrix} -0.1140 \\ -0.4614 \\ 0.8062 \\ -0.3524 \end{bmatrix} \quad (2.17)$$

then

$$x(t) = x_3 e^{\lambda_3 t} = \begin{bmatrix} -0.1140 \\ -0.4614 \\ 0.8062 \\ -0.3524 \end{bmatrix} e^{-2.2879t}. \quad (2.18)$$

The aircraft is initially pitched downward and is pitching upward. After approximately $t_s = 5\tau = 2.1854 \text{ sec}$, aerodynamic damping causes the aircraft to return to its equilibrium flight

condition. Figure 2.5 contains the time response which results when $x_o = x_3$. For this mode, speed deviations are small - relatively speaking.

- *Pitch-Up Instability.* Associated with the pitch-up instability is an unstable pole at $\lambda_4 = 1.5188$. This mode has a time to double of $t_{double} = 1.6586 \text{ sec}$. With the aircraft traveling at 284 ft/sec, this mode doubles in approximately 471 ft or 0.09 mile. To examine this mode, x_4 is selected as an initial condition. That is, if

$$x_o = x_4 = \begin{bmatrix} 0.2156 \\ -0.3665 \\ -0.7560 \\ -0.4977 \end{bmatrix} \quad (2.19)$$

then

$$x(t) = x_4 e^{\lambda_4 t} = \begin{bmatrix} 0.2156 \\ -0.3665 \\ -0.7560 \\ -0.4977 \end{bmatrix} e^{1.5188t}. \quad (2.20)$$

With the above choice, the aircraft pitches downward and speeds up. Figure 2.6 contains the time response which results when $x_o = x_4$. For this mode, speed deviations are small - relatively speaking. The mode is primarily associated with pitch rate. The pitch-up instability is specifically attributable to the fact that the c.g. lies aft of the c.p. a distance of about 35% of the aircraft's mean aerodynamic chord (the wing's average width) at low dynamic pressure subsonic flight conditions. This is in contrast to 7% for an F-16A fighter. This mode makes the X-29 aircraft very maneuverable, but too unstable for a pilot to control open loop. A stabilizing feedback control system is therefore essential to reduce pilot workload.

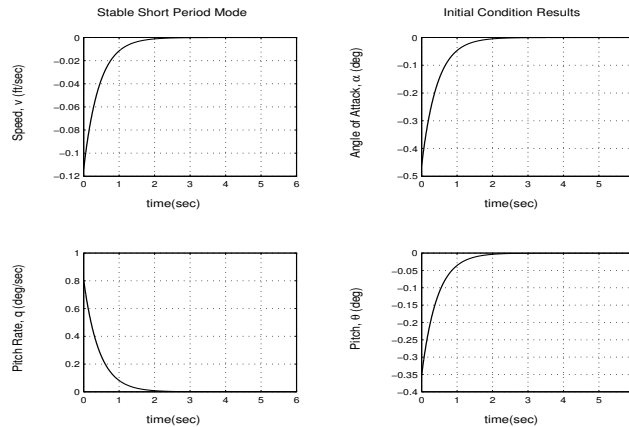


Figure 2.5: Visualization of X-29 Aircraft Damping Mode ($x_o = x_3$)

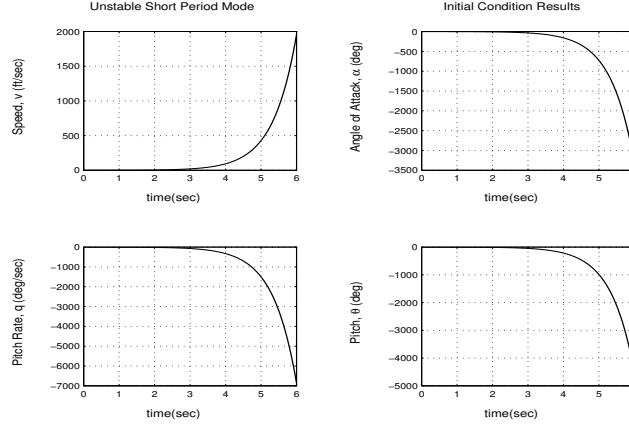


Figure 2.6: Visualization of X-29 Aircraft Pitch-Up Instability Mode ($x_o = x_4$)

Transmission Zeros. The three-input three-output transfer function matrix from u to y contains no transmission zeros. When one or two output variables are considered, then the system is nonsquare and it possesses zeros at every point s in the complex plane.

Transfer Function Matrix. The system transfer function matrix from u to y is given by:

$$G(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} G_{\delta_c v} & G_{\delta_{sf} v} & G_{\delta_{stf} v} \\ G_{\delta_c \alpha} & G_{\delta_{sf} \alpha} & G_{\delta_{stf} \alpha} \\ G_{\delta_c \theta} & G_{\delta_{sf} \theta} & G_{\delta_{stf} \theta} \end{bmatrix} \quad (2.21)$$

where

$$G_{\delta_c v} = \frac{-0.0343(s + 0.3679)(s^2 + 0.5152s + 28.42)}{(s + 2.288)(s - 1.519)(s^2 + 0.04375s + 0.02433)} \quad (2.22)$$

$$G_{\delta_c \alpha} = \frac{-0.061025(s - 25.2)(s^2 + 0.04356s + 0.02535)}{(s + 2.288)(s - 1.519)(s^2 + 0.04375s + 0.02433)} \quad (2.23)$$

$$G_{\delta_c \theta} = \frac{1.5666(s + 0.4151)(s + 0.03477)}{(s + 2.288)(s - 1.519)(s^2 + 0.04375s + 0.02433)} \quad (2.24)$$

$$G_{\delta_{sf} v} = \frac{-0.09217(s - 2.37)(s + 2.539)(s + 0.4571)}{(s + 2.288)(s - 1.519)(s^2 + 0.04375s + 0.02433)} \quad (2.25)$$

$$G_{\delta_{sf} \alpha} = \frac{-0.07661(s + 4.304)(s^2 + 0.0404s + 0.02756)}{(s + 2.288)(s - 1.519)(s^2 + 0.04375s + 0.02433)} \quad (2.26)$$

$$G_{\delta_{sf} \theta} = \frac{-0.32014(s + 1.453)(s + 0.006553)}{(s + 2.288)(s - 1.519)(s^2 + 0.04375s + 0.02433)} \quad (2.27)$$

$$G_{\delta_{stf} v} = \frac{0.004827(s + 0.5491)(s^2 + 2.178s + 107.3)}{(s + 2.288)(s - 1.519)(s^2 + 0.04375s + 0.02433)} \quad (2.28)$$

$$G_{\delta_{stf}\alpha} = \frac{-0.024175(s + 31.56)(s^2 + 0.04197s + 0.02586)}{(s + 2.288)(s - 1.519)(s^2 + 0.04375s + 0.02433)} \quad (2.29)$$

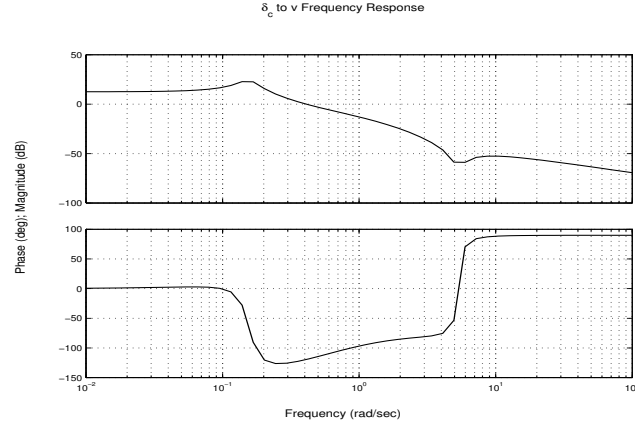
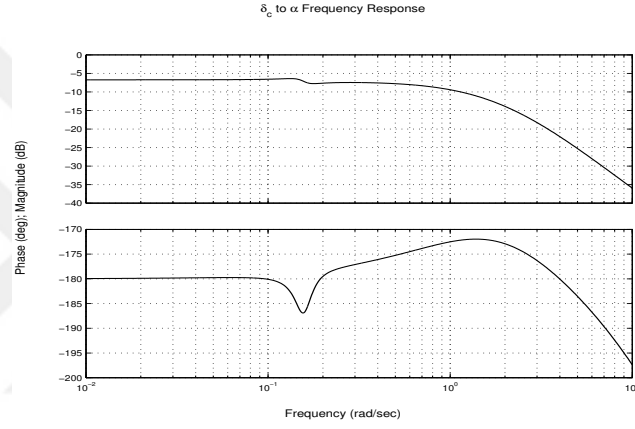
$$G_{\delta_{stf}\theta} = \frac{-0.76553(s + 0.6706)(s + 0.0338)}{(s + 2.288)(s - 1.519)(s^2 + 0.04375s + 0.02433)} \quad (2.30)$$

The individual transfer functions show the presence of the plant poles at $s = -0.0219 \pm 0.1544i$ (phugoid mode), $s = -2.2879$ (damping mode), and $s = 1.5189$ (pitch-up instability). The transfer function $G_{\delta_{sf}v}$ associated with the symmetric flap shows the presence of a right half plane zero at $s = 2.37$. This right half plane (non-minimum phase) zero, although not a transmission zero, along with multivariable coupling makes the unstable X-29 aircraft difficult to control.

Frequency Responses. The frequency responses (Bode plots) for each of the nine system transfer functions is given in Figures 2.7-2.15.

- The impact of the canard is seen in Figures 2.7-2.9.
- The impact of the symmetric flap is seen in Figures 2.10-2.12.
- The impact of the symmetric flap is seen in Figures 2.13-2.15.
- The lightly damped phugoid mode is seen to impact the speed transfer functions the most. It is hardly seen in the angle of attack transfer functions. This is because the phugoid mode is associated with a constant angle of attack.
- A constant control surface deflection (any surface) is seen to impact speed much more than angle of attack or pitch angle.
- All variables (v, α, θ) are impacted most by the canard. The canard impacts speed the most.
- All variables (v, α, θ) are impacted least by the symmetric flap. This is consistent with the fact that the symmetric flap is held constant (at trim: 20.15° trailing edge down) during a powered approach.
- A constant control surface deflection (any surface) is seen to impact pitch angle the least. Analysis at dc shows that a positive constant:
 1. canard deflection produces a positive speed, negative angle of attack, and a negative pitch angle
 2. symmetric flap deflection produces a negative speed, positive angle of attack, and a positive pitch angle
 3. strake flap deflection produces a negative speed, positive angle of attack, and a positive pitch angle

MIMO Frequency Response : Singular Values The singular values (MIMO frequency response) for the transfer function matrix relating the controls u to the output variables y are plotted in Figure 2.16. The figure shows, as expected, the peaking due to the lightly damped phugoid mode ($\omega = 0.1560 \text{ rad/sec}, \zeta = 0.1402$).

Figure 2.7: Frequency Response - δ_c to v Figure 2.8: Frequency Response - δ_c to α

DC Gain Analysis: Singular Value Decomposition. Analyzing the X-29 aircraft model at DC yields the following matrix of DC gains:

$$\begin{bmatrix} v \\ \alpha \\ \theta \end{bmatrix} = \begin{bmatrix} 4.2412 & -2.9980 & -3.3644 \\ -0.4610 & 0.1075 & 0.2333 \\ -0.2674 & 0.0361 & 0.2052 \end{bmatrix} \begin{bmatrix} \delta_c \\ \delta_{sf} \\ \delta_{stf} \end{bmatrix} \quad (2.31)$$

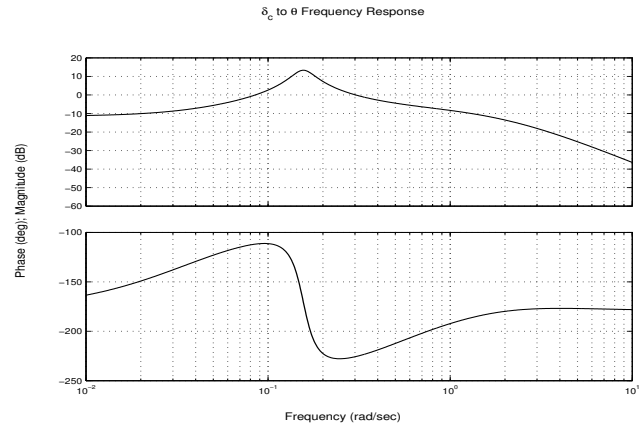
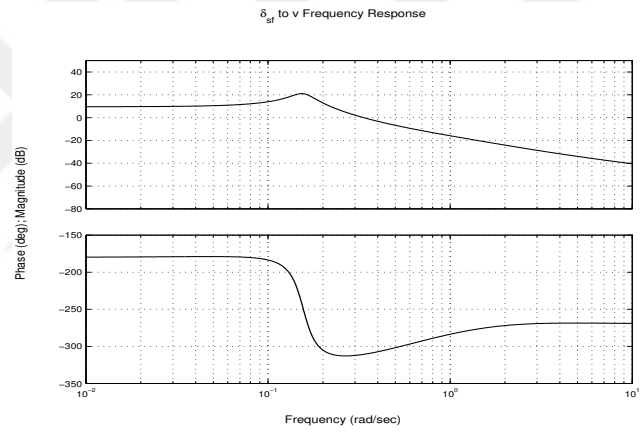
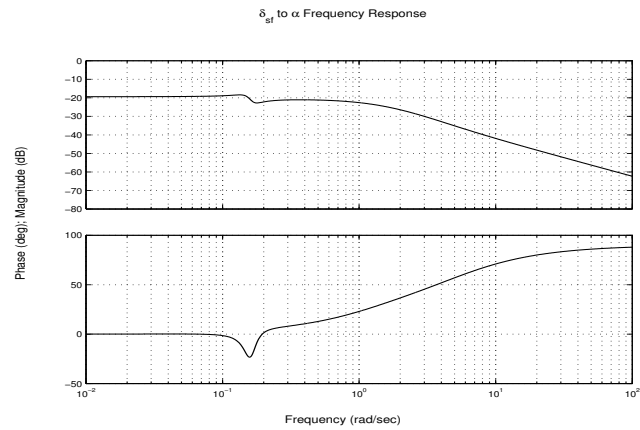
A singular value decomposition at DC yields the following:

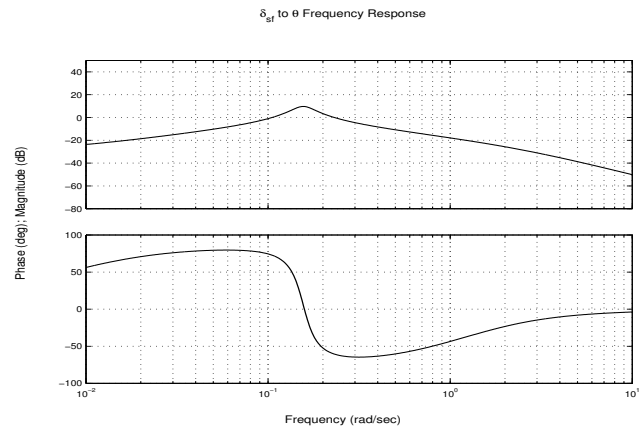
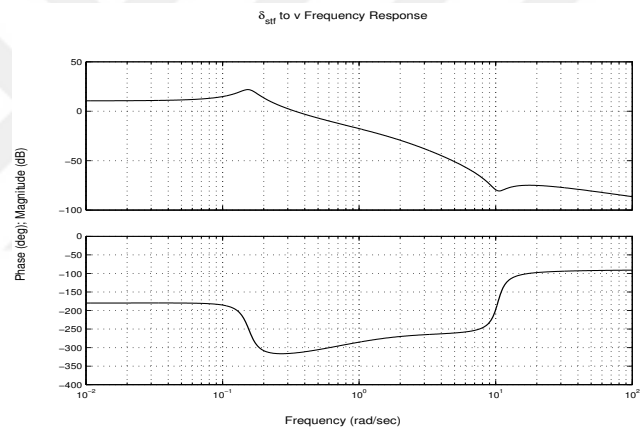
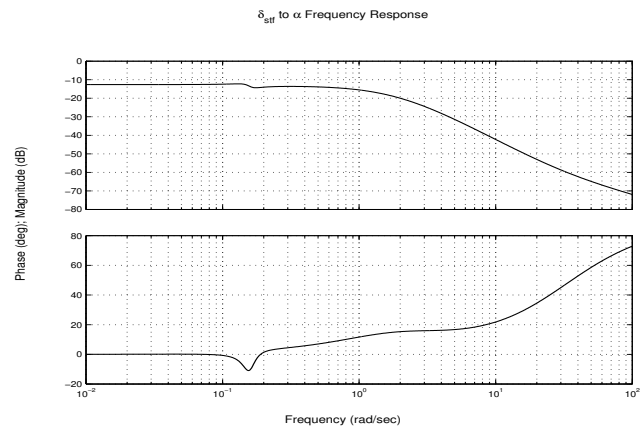
$$G(j0) = C(-A)^{-1}B + D \quad (2.32)$$

$$= U\Sigma V^T \quad (2.33)$$

$$= \begin{bmatrix} 4.2412 & -2.9980 & -3.3644 \\ -0.4610 & 0.1075 & 0.2333 \\ -0.2674 & 0.0361 & 0.2052 \end{bmatrix} \quad (2.34)$$

$$U = \begin{bmatrix} 0.9955 & 0.0942 & -0.0032 \\ -0.0797 & 0.8234 & -0.5619 \\ -0.0503 & 0.5596 & 0.8272 \end{bmatrix} \quad (2.35)$$

Figure 2.9: Frequency Response - δ_c to θ Figure 2.10: Frequency Response - δ_{sf} to v Figure 2.11: Frequency Response - δ_{sf} to α

Figure 2.12: Frequency Response - δ_{sf} to θ Figure 2.13: Frequency Response - δ_{stf} to v Figure 2.14: Frequency Response - δ_{stf} to α

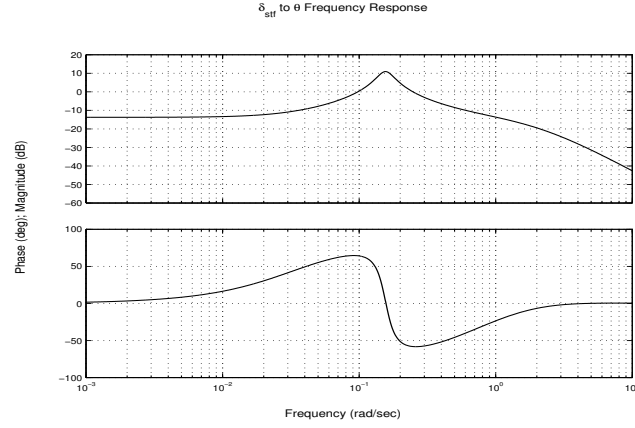
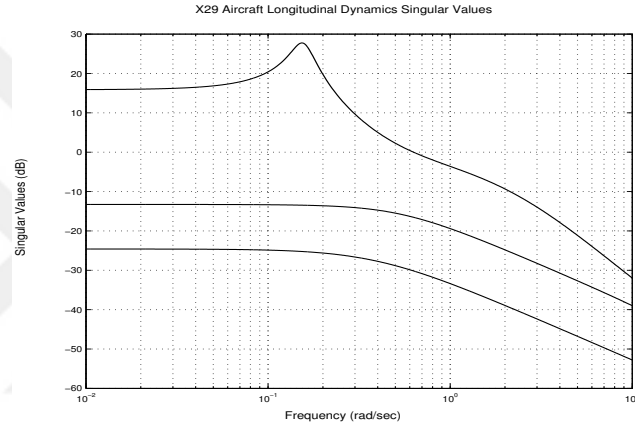
Figure 2.15: Frequency Response - δ_{stf} to θ 

Figure 2.16: X-29 Aircraft Singular Values MIMO Frequency Response (Longitudinal Dynamics)

$$\Sigma = \begin{bmatrix} 6.2160 & 0 & 0 \\ 0 & 0.2171 & 0 \\ 0 & 0 & 0.0589 \end{bmatrix} \quad (2.36)$$

$$V = \begin{bmatrix} 0.6874 & -0.5975 & 0.4130 \\ -0.4818 & -0.8005 & -0.3563 \\ -0.5435 & -0.0459 & 0.8382 \end{bmatrix} \quad (2.37)$$

From this singular value decomposition, several conclusions may be drawn steady state input-output coupling - assuming, of course, that a stabilizing controller is in place.

- Examination of the first columns of V , Σ , and U shows that canard and strake flap combinations (canard mainly) have the greatest impact on the speed. This may be visualized as shown in Figure 2.17. The analysis suggests that the canard should be used as the primary control for maintaining steady speed perturbations from equilibrium.
- Examination of the second columns of V , Σ , and U shows that symmetric flap and canard combinations (symmetric flap mainly) have the greatest impact on the angle of attack and pitch (mainly angle of attack). This may be visualized as shown in Figure 2.18. The analysis suggests that the symmetric

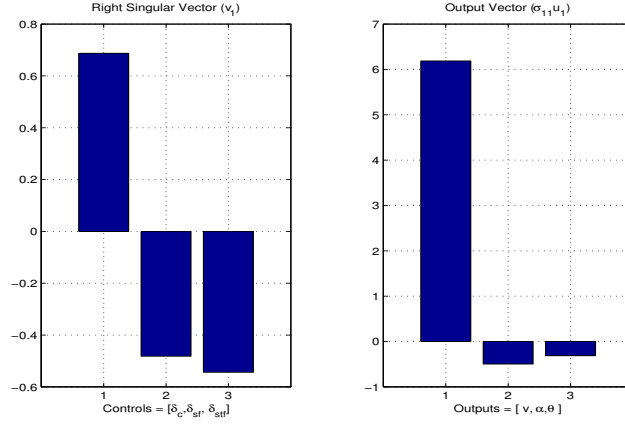


Figure 2.17: X-29 SVD at DC for Longitudinal Dynamics - $v_1 \rightarrow \sigma_1 u_1 = G(j0)v_1$

flap should be used as the primary control for maintaining steady angle of attack perturbations from equilibrium.

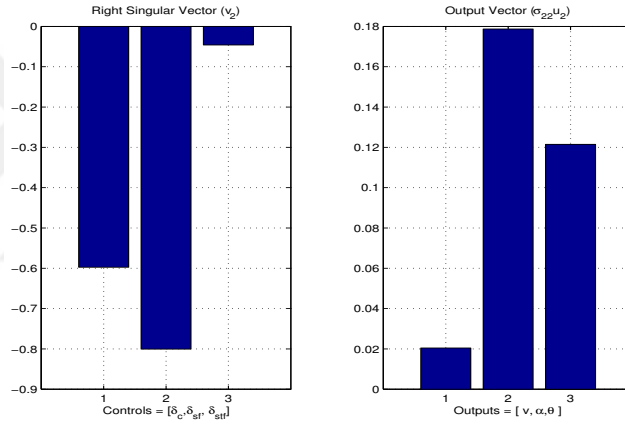


Figure 2.18: X-29 SVD at DC for Longitudinal Dynamics - $v_2 \rightarrow \sigma_2 u_2 = G(j0)v_2$

- Examination of the third columns of V , Σ , and U shows that strake flap and canard combinations (strake flap mainly) have the greatest impact on the pitch and angle of attack (mainly pitch angle). This may be visualized as shown in Figure 2.19. The analysis suggests that the strake flap should be used as the primary control for maintaining steady pitch angle perturbations from equilibrium.

Actuator Dynamics. Associated with each of the three control surfaces is an actuator. A dynamical model for each is now presented:

$$H_{canard} = 0.885 \left[\frac{20.2}{s + 20.2} \right] \left[\frac{144.9}{s + 144.9} \right] \left[\frac{(71.4)^2}{s^2 + 2(0.736)(71.4)s + (71.4)^2} \right] \quad (2.38)$$

$$H_{sym\ flap} = \left[\frac{20.2}{s + 20.2} \right] \left[\frac{144.9}{s + 144.9} \right] \left[\frac{(71.4)^2}{s^2 + 2(0.736)(71.4)s + (71.4)^2} \right] \quad (2.39)$$

$$H_{strake\ flap} = \left[\frac{50}{s + 50} \right] \left[\frac{100}{s + 100} \right] \left[\frac{(325)^2}{s^2 + 2(0.7)(325)s + (325)^2} \right] \quad (2.40)$$

$$(2.41)$$

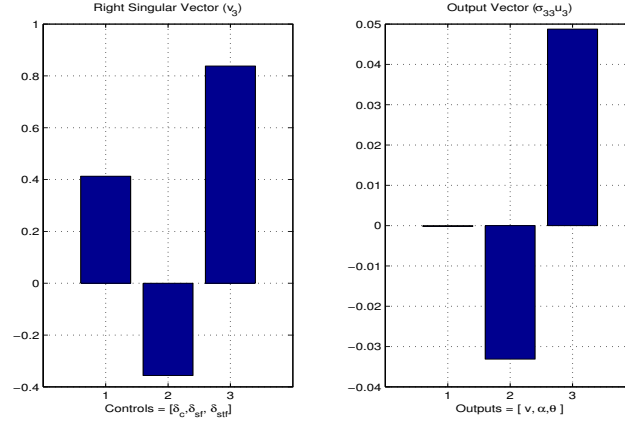


Figure 2.19: X-29 SVD at DC for Longitudinal Dynamics - $v_3 \rightarrow \sigma_3 u_3 = G(j0)v_3$

Sensor Dynamics. The aircraft uses a pitch rate gyro. Its dynamics are as follows:

$$H_{pitch\ rate\ gyro} = \frac{(137)^2}{s^2 + 2(0.704)(137)s + (137)^2}. \quad (2.42)$$

2.2 X-29 Aircraft Lateral Dynamics

Lateral Dynamics: Powered Approach. We examine the aircraft during a straight and level powered approach [1, page 19]. The precise flight condition is as follows:

- speed of Mach 0.258 (160 kts equivalent airspeed), 284 ft/sec (193.6 mph) true velocity
- altitude of 4,000 ft
- weight of 14,777 lbs.

This flight condition is appropriate for examining performance during take-off or landing. A state space model for the lateral dynamics at the above flight condition is as follows [1, pp. 19, 21]:

$$\dot{x} = Ax + Bu \quad (2.43)$$

$$y = Cx + Du \quad (2.44)$$

where

$$A = \begin{bmatrix} -0.1850 & 0.1475 & -0.9825 & 0.1120 \\ -3.4670 & -1.7100 & 0.9029 & 0.0000 \\ 1.1740 & -0.0825 & -0.1826 & -0.0000 \\ 0 & 1.0000 & 0.1492 & 0 \end{bmatrix} \quad (2.45)$$

$$B = \begin{bmatrix} -0.0256 & 0.0230 \\ 21.2869 & 3.1446 \\ 1.5202 & -0.7741 \\ 0 & 0 \end{bmatrix} \quad (2.46)$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (2.47)$$

$$u = \begin{bmatrix} \delta_{df} - \text{differential flap (deg)} \\ \delta_r - \text{rudder (deg)} \end{bmatrix} \quad (2.48)$$

$$x = \begin{bmatrix} \beta - \text{side slip angle (deg)} \\ p - \text{roll rate (deg/sec)} \\ r - \text{yaw rate (deg/sec)} \\ \phi - \text{roll angle (deg)} \end{bmatrix} \quad (2.49)$$

$$y = \begin{bmatrix} \phi - \text{roll angle (deg)} \\ \beta - \text{side slip angle (deg)} \end{bmatrix} \quad (2.50)$$

The aircraft's two control surfaces are as follows:

- differential flap - situated on wing surface,
- rudder - situated on the tail,

These surfaces, in principle, may be used to control two independent variables. Control surface limits are as follows:

Control Surface	Position limit, deg	Rate limit, deg/sec
Differential Flap	-17.5, 17.5	± 70
Symmetric Flap	-30, 30	± 125

Table 2.5: X-29 Aircraft Lateral Control Surface Limits

Comment 2.2.1 (Linear Model: Perturbation Step Sizes)

The above linear model was developed from a six-degree-of-freedom nonlinear model with a full envelop aerodynamic database obtained from wind-tunnel test results. The following perturbation step sizes about trim were used to obtain the model: $\pm 1\%$ speed of sound for speed, $\pm 2^\circ$ for angle of attack, and $\pm 1^\circ$ for the remaining states and control surfaces. Flight data was not used to update the model. ■

Modal Analysis: Natural Dynamics In this section, the natural modes of the X-29's lateral dynamics of the X-29 aircraft are examined. The associated system poles (eigenvalues of A) are listed in Table 2.2 and may be visualized as shown in Figure 2.20. These poles define three distinct modes. Each mode is now described. To do so, we refer to Table 2.2 which contains each of the eigenvalues (i.e. system poles) and associated (right) eigenvectors.

- *Dutch Roll Mode* ($\lambda_{1,2}$). The dutch roll mode is associated with complex conjugate left half plane poles at $\lambda_{1,2} = -0.2455 \pm j1.2703$. The associated damping ratio ζ is 0.1898. The undamped natural frequency is $\omega_n = 1.2938 \text{ rad/sec}$. This mode may be objectionable due to light damping. (Passengers sitting in the rear of a commercial aircraft may sense the side to side yaw-roll motion associated with

Pole	Damping	Frequency(rad/s)	
$-0.2455 \pm j1.2703$	0.1898	1.2938	Dutch roll mode
-1.6183	1	1.6183	Roll subsidence mode
0.0318	-1	0.0318	Spiral divergence mode

Table 2.6: Poles for X-29 Lateral Dynamics (Powered Approach)

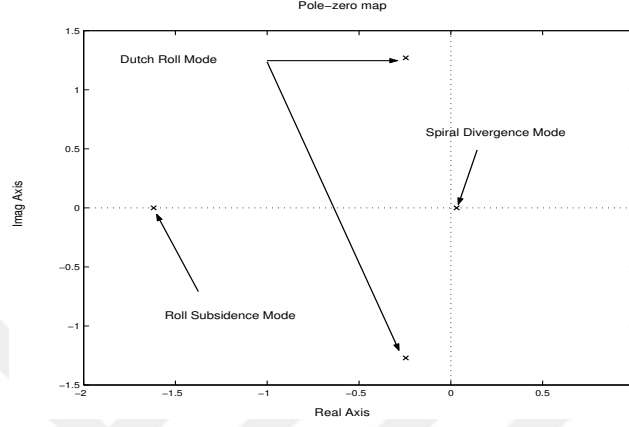


Figure 2.20: Visualization of Poles for X-29 Lateral Dynamics (Powered Approach)

Eigenvalues	Eigenvectors
Dutch Roll Mode $\lambda_{1,2} = -0.2455 \pm j1.2703$	$x_{1,2} = \begin{bmatrix} -0.2565 \pm j0.2524. \\ 0.2241 \mp j0.6455. \\ 0.2869 \pm j0.2374. \\ -0.5021 \mp j0.1131. \end{bmatrix} = \begin{bmatrix} 0.3598e^{\pm j135.46^\circ} \\ 0.6833e^{\mp j70.85^\circ} \\ 0.3724e^{\pm j39.60^\circ} \\ 0.5147e^{\mp j167.31^\circ} \end{bmatrix}$
Roll Subsidence Mode $\lambda_3 = -1.6183$	$x_3 = \begin{bmatrix} 0.0080. \\ -0.8471. \\ -0.0552. \\ 0.5285. \end{bmatrix}$
Spiral Divergence Mode $\lambda_4 = 0.0318$	$x_4 = \begin{bmatrix} 0.0213. \\ 0.0151. \\ 0.1108. \\ 0.9935. \end{bmatrix}$

Table 2.7: Eigenvalues and Eigenvectors for X-29 Lateral Dynamics (Powered Approach)

this mode.) To examine this mode, the real and imaginary parts of x_1 may be selected separately as initial conditions (see Table 2.2). More specifically, if

$$x_o = k_1 \operatorname{Re}\{ e^{\lambda_1 t} x_1 \} + k_2 \operatorname{Im}\{ e^{\lambda_1 t} x_1 \} = k_1 \begin{bmatrix} -0.2565 \\ 0.2241 \\ 0.2869 \\ -0.5021 \end{bmatrix} + k_2 \begin{bmatrix} \pm 0.2524 \\ \mp 0.6455 \\ \pm 0.2374 \\ \mp 0.1131 \end{bmatrix} \quad (2.51)$$

where k_1 and k_2 are arbitrary real constants, then

$$x(t) = k_1 \operatorname{Re}\{e^{\lambda_1 t} x_1\} + k_2 \operatorname{Im}\{e^{\lambda_1 t} x_1\} \quad (2.52)$$

$$= k_1 \begin{bmatrix} 0.3598e^{-0.2455t} \cos(1.2703t \pm 135.46^\circ) \\ 0.6833e^{-0.2455t} \cos(1.2703t \mp 70.85^\circ) \\ 0.3724e^{-0.2455t} \cos(1.2703t \pm 39.60^\circ) \\ 0.5147e^{-0.2455t} \cos(1.2703t \mp 167.31^\circ) \end{bmatrix} \quad (2.53)$$

$$+ k_2 \begin{bmatrix} 0.3598e^{-0.2455t} \sin(1.2703t \pm 135.46^\circ) \\ 0.6833e^{-0.2455t} \sin(1.2703t \mp 70.85^\circ) \\ 0.3724e^{-0.2455t} \sin(1.2703t \pm 39.60^\circ) \\ 0.5147e^{-0.2455t} \sin(1.2703t \mp 167.31^\circ) \end{bmatrix}. \quad (2.54)$$

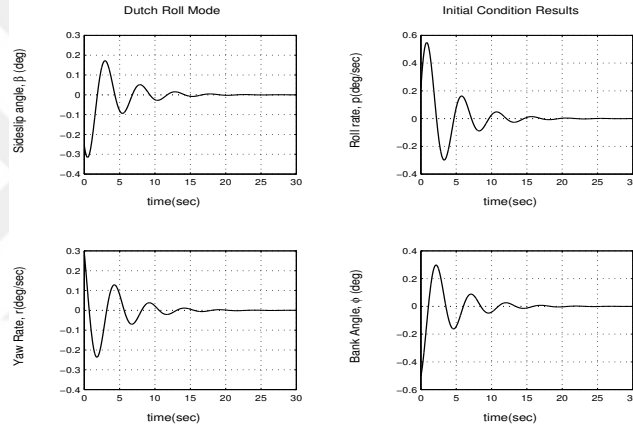


Figure 2.21: Visualization of X-29 Aircraft Dutch Roll Mode ($x_o = \operatorname{Re}(x_1)$)

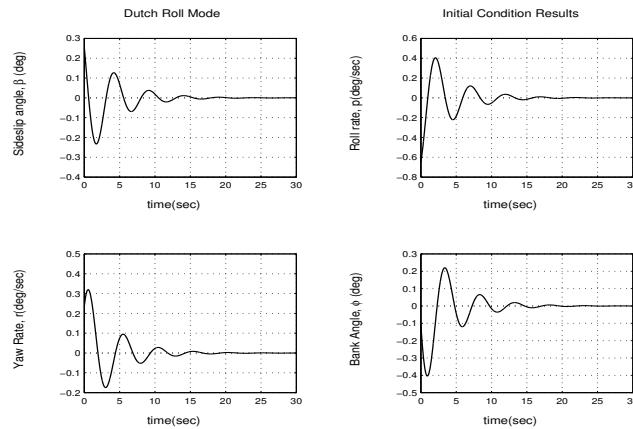


Figure 2.22: Visualization of X-29 Aircraft Dutch Roll Mode ($x_o = \operatorname{Im}(x_1)$)

This stable lightly damped mode has a time constant of $\tau = 4.07 \text{ sec}$ and hence takes approximately $t_s = 5\tau = 20.36 \text{ sec}$ to settle. Analysis of the eigenvector x_1 shows that

- The mode is primarily associated with p and ϕ , but does significantly impact r and β . The mode is characteristic of a ice skater skating in a canal. Such motion couples side to side motion

with banking.

- p leads ϕ by 96.46°
- ϕ leads β by 57.23°
- β leads r by 95.86°

- *Roll Subsidence Mode* (λ_3). The roll subsidence mode is associated with roll damping. This mode corresponds to a stable pole at $\lambda_3 = -1.6183$. Figure 2.23 shows the time response which is associated with the roll subsidence mode. To examine this mode, x_3 is selected as the initial condition. That is, if

$$x_o = \begin{bmatrix} 0.0080 \\ -0.8471 \\ -0.0552 \\ 0.5285 \end{bmatrix} \quad (2.55)$$

then

$$x(t) = x_3 e^{\lambda_3 t} = \begin{bmatrix} 0.0080 \\ -0.8471 \\ -0.0552 \\ 0.5285 \end{bmatrix} e^{-1.6183t}. \quad (2.56)$$

As expected, this roll damping mode is primarily associated with p and ϕ . Figure 2.23 contains the time response which results when $x_o = x_3$. The mode has a time constant of $\tau = 0.618$ sec and a settling time of $5\tau = 3.09$ sec.

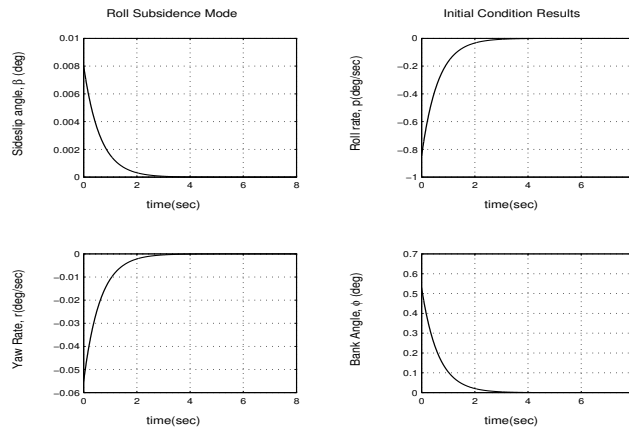


Figure 2.23: Visualization of X-29 Aircraft Roll Subsidence Mode ($x_o = x_3$)

- *Spiral Divergence Mode* (λ_4). The spiral divergence mode is associated with the natural tendency of the aircraft to spiral downward - rolling and yawing. Associated with this mode is the unstable pole at $\lambda_4 = 0.0318$. It has a time to double of $t_{double} = \frac{\ln 2}{0.0318} = 21.8$ sec. This mode typically

does not concern pilots because its time to double is large and pilots have little trouble controlling it. The mode can cause problems for pilots during periods of low visibility which force pilots to rely on instrumentation rather than on external visual indicators. During such periods pilots not trained for instrumented flight may lose orientation, pull back on the stick (trying to increase lift and altitude), and tightening the radius of the spiral - often referred to as the *graveyard spiral*.

Comment 2.2.2 (John F. Kennedy Jr. Piper Saratoga Accident)

On July 17, 1999, John F. Kennedy Jr.'s single engine Piper Saratoga aircraft spiraled into the ocean during a night flight from New Jersey to Martha's Vineyard in Massachusetts. JFK Jr. - not trained for instrumented (poor visibility) flight - lost orientation and piloted his aircraft into a ever tightening graveyard spiral. Radar data showed that the plane plunged 1100 feet in 14 sec (4714 feet per minute, 53.6 miles/hr). A nominal rate of descent would be about 500 to 700 feet per minute (5.68-7.96 miles/hour). The Piper Saratoga aircraft generally cannot handle a descent faster than 1500 feet per minute (17.1 miles/hour). 40 hours of flight time are typically required to become a licensed private pilot. Kennedy had 92 hours. Pilots trained for instrumented flight require additional hours. Instrumented flight training teaches a pilot to rely on the following six basic flight indicators:

- *altimeter*
- *vertical speed indicator*
- *artificial horizon indicator (provides roll-pitch information)*
- *heading indicator*
- *air speed indicator*
- *engine rpm indicator*

■

To examine the X-29's spiral divergence mode, x_4 is selected as the initial condition. That is, if

$$x_o = \begin{bmatrix} 0.0213 \\ 0.0151 \\ 0.1108 \\ 0.9935 \end{bmatrix} \quad (2.57)$$

then

$$x(t) = x_4 e^{\lambda_4 t} = \begin{bmatrix} 0.0213 \\ 0.0151 \\ 0.1108 \\ 0.9935 \end{bmatrix} e^{0.0318t}. \quad (2.58)$$

Examination of x_4 , as expected, shows that this mode is primarily associated with r and ϕ . Figure 2.24 contains the time response which results when $x_o = x_4$.

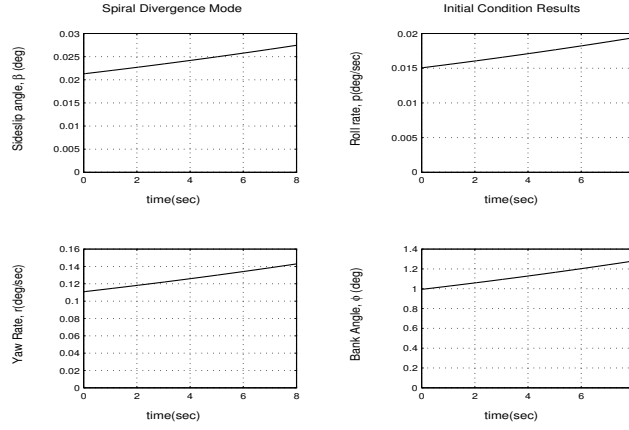


Figure 2.24: Visualization of X-29 Aircraft Spiral Divergence Mode ($x_o = x_4$)

Transmission Zeros. The two-input two-output transfer function matrix from u to y contains no transmission zeros.

Transfer Function Matrix The system transfer function matrix from u to y is given by:

$$G(s) = C(sI - A)^{-1}B = \begin{bmatrix} G_{\delta_{df}\phi} & G_{\delta_r\phi} \\ G_{\delta_{df}\beta} & G_{\delta_r\beta} \end{bmatrix} \quad (2.59)$$

where

$$G_{\delta_{df}\phi} = \frac{21.5138(s^2 + 0.4392s + 1.458)}{(s + 1.618)(s - 0.0318)(s^2 + 0.4911s + 1.674)} \quad (2.60)$$

$$G_{\delta_{df}\beta} = \frac{-0.025613(s - 63.82)(s + 1.108)(s + 0.3329)}{(s + 1.618)(s - 0.0318)(s^2 + 0.4911s + 1.674)} \quad (2.61)$$

$$G_{\delta_r\phi} = \frac{3.0291(s^2 + 0.04081s + 0.318)}{(s + 1.618)(s - 0.0318)(s^2 + 0.4911s + 1.674)} \quad (2.62)$$

$$G_{\delta_r\beta} = \frac{0.023035(s + 53.52)(s + 1.551)(s - 0.02114)}{(s + 1.618)(s - 0.0318)(s^2 + 0.4911s + 1.674)}. \quad (2.63)$$

Each transfer function exhibits the Dutch roll mode, the roll subsidence mode, and the spiral divergence mode. Salient characteristics of each transfer function are as follows:

- $G_{\delta_{df}\phi}$ is hardly impacted by the Dutch roll mode because of a near pole-zero cancelation. The mode is much more pronounced in the other system transfer functions. Root locus ideas suggest that a simple gain could be used to stabilize this transfer function. DC analysis implies that a negative differential flap deflection is required to hold a positive roll angle - if the rudder is not used.
- $G_{\delta_{df}\beta}$ exhibits a zero near 0.33 rad/sec. The presence of this low frequency zero (an approximate differentiator) suggests that rapid differential flap deflections will result in a considerable transient side slip. A relatively large right half plane zero near 64 has little impact on the aircraft's (low

frequency) dynamics. DC analysis implies that a negative differential flap deflection is required to hold a positive side slip angle - if the rudder is not used.

- $G_{\delta_r \phi}$ exhibits lightly damped zeros (a slight notch) near the Dutch roll frequency. The presence of these zeros (approximate differentiators) suggests that rapid rudder deflections will result in a considerable transient roll. DC analysis implies that a negative rudder deflection is required to hold a positive roll angle - if the differential flap is not used.
- $G_{\delta_r \beta}$ contains a near right half plane pole-zero cancelation near the spiral mode. This suggests that β may be difficult to control via δ_r alone. δ_{df} may be particularly helpful during transients. The fact that the right half plane zero is closer to the origin than the right half plane spiral divergence pole is of particular concern. A near left half plane pole-zero cancelation in $G_{\delta_r \beta}$ suggests that the roll subsidence mode has relatively little impact on this transfer function. DC analysis implies that a positive rudder deflection is required to hold a positive side slip angle - if the differential flap is not used.

Frequency Responses. The frequency responses (Bode plots) for the four system transfer functions (δ_{df} to ϕ , δ_{df} to β , δ_r to ϕ , δ_r to β), are given in Figures 2.25-2.28 respectively.

- Figure 2.25 shows that δ_{df} impacts ϕ considerably at dc. The figure shows that the Dutch roll mode hardly impacts $G_{\delta_{df} \phi}$ because of a near pole-zero cancelation. The mode is much more pronounced in the other system transfer functions. The figure indicates a natural roll bandwidth of about 4 rad/sec. Because of the above near pole-zero cancelation, the frequency response is dominated by roll subsidence and spiral divergence dynamics.

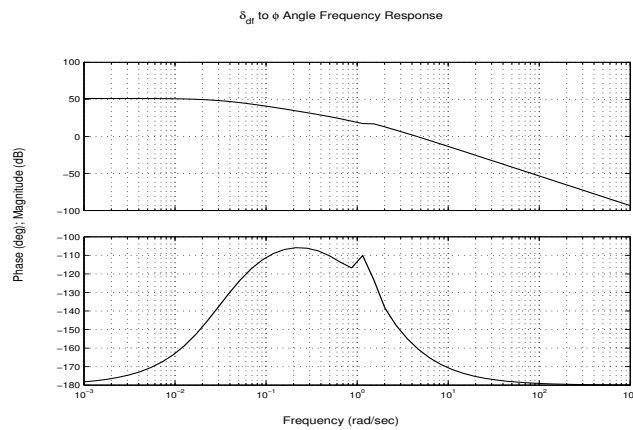


Figure 2.25: Frequency Response - δ_{df} to ϕ

- Figure 2.26 shows that δ_{df} also impacts β considerably at dc, but not as much as ϕ .

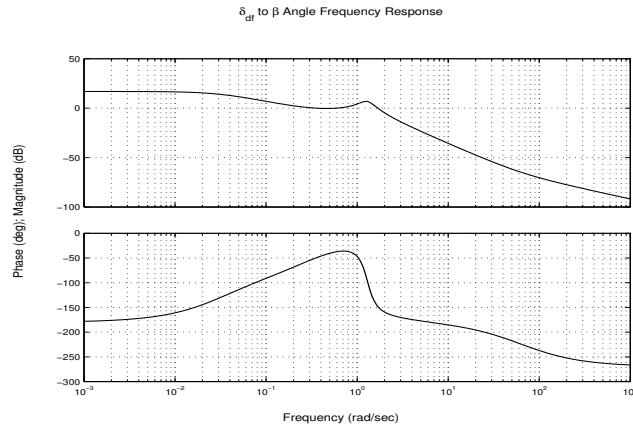


Figure 2.26: Frequency Response - δ_{df} to β

- Figure 2.27 shows a slight notch just below the Dutch roll mode frequency. This results in three unity gain crossover frequencies. The figure also shows that δ_r significantly impacts ϕ at dc.

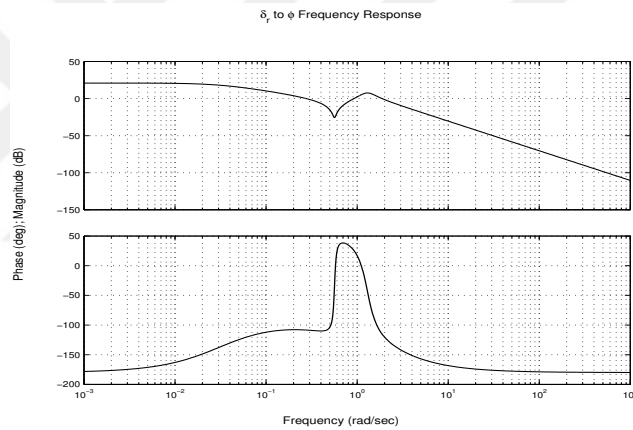
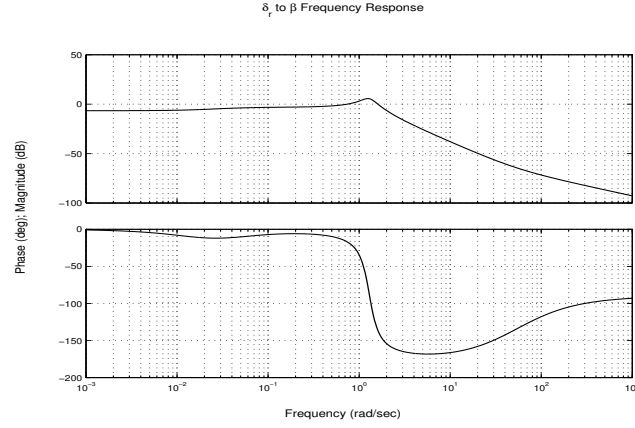


Figure 2.27: Frequency Response - δ_r to ϕ

- The near right half plane pole-zero cancelation in $G_{\delta_r, \beta}$ manifests itself in Figure 2.28 in the form of some (albeit little) lag at low frequencies. Over and above the near right half plane pole-zero cancelation, this transfer function is dominated by the Dutch roll dynamics.

Figure 2.28: Frequency Response - δ_r to β

MIMO Frequency Response: Singular Values. The singular values (MIMO frequency response) for the transfer function matrix relating the controls (δ_{df}, δ_r) to the output variables (ϕ, β) are plotted in Figure 2.29.

- At low frequencies, the maximum singular value is associated primarily with δ_{df} and ϕ .
- At low frequencies, the minimum singular value is associated primarily with δ_r and β .

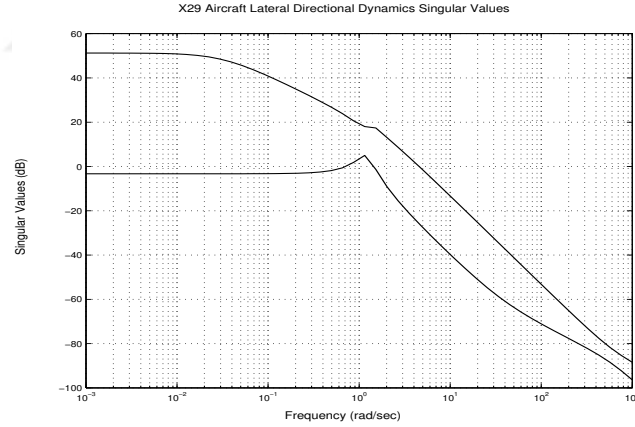


Figure 2.29: X-29 Aircraft Singular Values MIMO Frequency Response (Lateral Dynamics)

DC Gain Analysis (MIMO Coupling): Singular Value Decomposition. Analyzing the X-29 aircraft lateral model at DC yields the following matrix of DC gains:

$$\begin{bmatrix} \phi \\ \beta \end{bmatrix} = \begin{bmatrix} -364.2160 & -11.1837 \\ -7.0021 & 0.4693 \end{bmatrix} \begin{bmatrix} \delta_{df} \\ \delta_r \end{bmatrix} \quad (2.64)$$

A singular value decomposition of the plant at DC yields the following results:

$$G(j0) = C(-A)^{-1}B = U\Sigma V^T \quad (2.65)$$

where

$$U = \begin{bmatrix} -0.9998 & -0.0192 \\ -0.0192 & 0.9998 \end{bmatrix} \quad (2.66)$$

$$\Sigma = \begin{bmatrix} 364.4546 & 0 \\ 0 & 0.6839 \end{bmatrix} \quad (2.67)$$

$$V = \begin{bmatrix} 0.9995 & -0.0307 \\ 0.0307 & 0.9995 \end{bmatrix} \quad (2.68)$$

and

$$G(j0) = \begin{bmatrix} -364.2160 & -11.1837 \\ -7.0021 & 0.4693 \end{bmatrix}. \quad (2.69)$$

- Examination of the first columns of V , Σ , and U shows that differential flap has the greatest impact on roll. This may be visualized as shown in Figure 2.30. The analysis suggests that the differential flap should be used as the primary control for maintaining steady state roll perturbations from equilibrium.
- Examination of the second columns of V , Σ , and U shows that rudder has the greatest impact on the sideslip angle. This may be visualized as shown in Figure 2.31. The analysis suggests that the rudder should be used as the primary control for maintaining steady state sideslip perturbations from equilibrium.

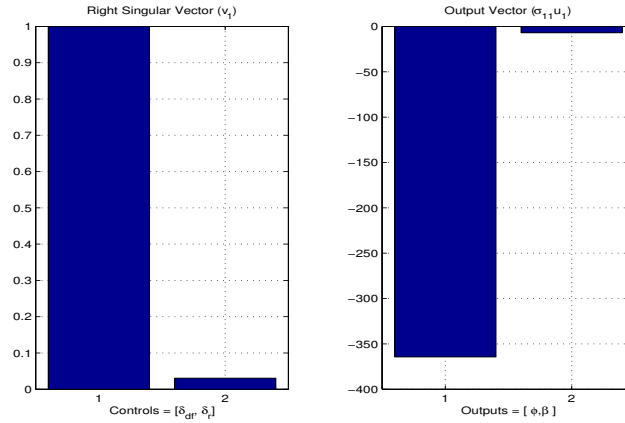


Figure 2.30: X-29 SVD at DC for Lateral Dynamics - $v_1 \rightarrow \sigma_1 u_1 = G(j0)v_1$

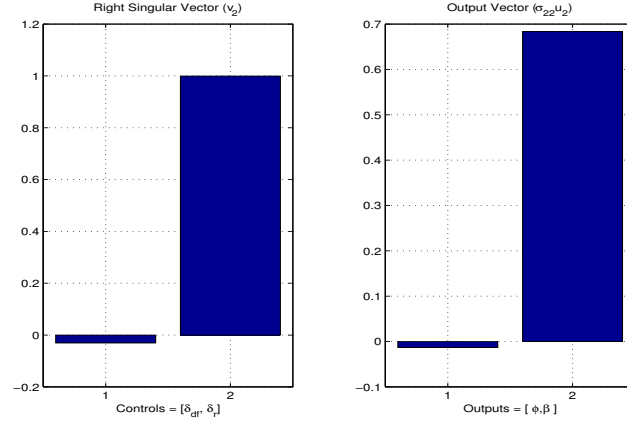


Figure 2.31: X-29 SVD at DC for Lateral Dynamics - $v_2 \rightarrow \sigma_2 u_2 = G(j0)v_2$

Actuator Dynamics. Associated with each of the two control surfaces is an actuator. A dynamical model for each is now presented:

$$H_{dif\ flap} = \left[\frac{(54.1)^2}{s^2 + 2(1.53)(54.1)s + 54.1^2} \right] \left[\frac{(71.4)^2}{s^2 + 2(0.736)(71.4)s + 71.4^2} \right] \quad (2.70)$$

$$H_{rudder} = \left[\frac{(54.1)^2}{s^2 + 2(1.53)(54.1)s + 54.1^2} \right] \left[\frac{(71.4)^2}{s^2 + 2(0.736)(71.4)s + 71.4^2} \right]. \quad (2.71)$$

Sensor Dynamics. The aircraft relies on roll and yaw rate gyros. The dynamics for each gyro are as follows:

$$H_{roll\ rate\ gyro} = \frac{(157)^2}{s^2 + 2(0.701)(157) + (157)^2} \quad (2.72)$$

$$H_{yaw\ rate\ gyro} = \frac{(137)^2}{s^2 + 2(0.704)(137) + (137)^2}. \quad (2.73)$$

CHAPTER 3

Simple SISO Examples

In this section, we consider control system design for SISO plants. Control system designs are obtained as solutions to weighted $\mathcal{H}^\infty$ mixed-sensitivity optimization problems possessing the form:

- *Weighted $\mathcal{H}^\infty$ Mixed-Sensitivity Optimization Problem:* Find a stabilizing compensator K such that

$$\|T_{wz}\|_{\mathcal{H}^\infty} = \left\| \begin{bmatrix} W_1 S \\ W_2 K S \\ W_3 T \end{bmatrix} \right\|_{\mathcal{H}^\infty} < \gamma \quad (3.1)$$

where W_1 , W_2 and W_3 are weighting functions and $\gamma > 0$ is a parameter to be minimized.

3.1 Selection of Weighting Functions for Design Shaping

General rules (guidelines) for selecting the weighting functions W_1 , W_2 and W_3 are now developed [3].

- **Sensitivity Weighting.** In many applications, the weighting function W_1 on the sensitivity S is selected to have the form

$$W_1(s) = \frac{1}{M_s} \left[\frac{s + M_s \omega_b}{s + \epsilon \omega_b} \right]. \quad (3.2)$$

where $\epsilon > 0$, $\omega_b > 0$ and $M_s > 0$ are design parameters used to shape the sensitivity function. Typically we have $0 < \epsilon \ll 1 < M_s < \infty$.

Using Bode approximation ideas, it follows that

$$W_1 \approx \begin{cases} \frac{1}{\epsilon} & 0 & \leq |s| < \epsilon \omega_b \\ \frac{\omega_b}{s} & \epsilon \omega_b < |s| < M_s \omega_b \\ \frac{1}{M_s} & M_s \omega_b < |s| < \infty \end{cases} \quad (3.3)$$

Noting that $|W_1 S| < \gamma$ for all frequencies, it follows that $|S| < \gamma |W_1^{-1}|$. Given this, we have the following (approximate) relationship for the sensitivity:

$$|S| < \gamma \begin{cases} \frac{1}{\epsilon} & 0 & \leq & |s| & \ll & \epsilon \omega_b \\ \frac{\omega_b}{s} & \epsilon \omega_b & \ll & |s| & \ll & M_s \omega_b \\ M_s & M_s \omega_b & \ll & |s| & < & \infty \end{cases} \quad (3.4)$$

Sensitivity Design Parameter Interpretations. From this, we have the following interpretations for the design parameters ϵ , ω_b , M_s , and γ :

1. *Low Frequency Sensitivity Reduction.*

ϵ is an upperbound for the desired low frequency sensitivity reduction;
this sensitivity reduction is to take place for $\omega \leq \epsilon \omega_b$;

2. *Sensitivity Unity Gain Crossover Frequency.*

ω_b is an upperbound for the desired sensitivity unity gain crossover frequency;

3. *Sensitivity High Frequency Upper Bound.*

M_s is an upper bound for the sensitivity at frequencies $\omega \geq M_s \omega_b$.

By making γ smaller, the above upper bounds are improved; i.e. the effective sensitivity reduction is reduced, the effective sensitivity gain crossover frequency is increased, and the effective upper bound for the sensitivity at low frequencies is reduced. We therefore wish to make γ as small as possible.

Steeper Sensitivity. If a steeper sensitivity is required near ω_b , then one might select W_1 to have the form

$$W_1(s) = \left(\frac{1}{\sqrt[k_1]{M_s}} \left[\frac{s + \sqrt[k_1]{M_s} \omega_b}{s + \sqrt[k_1]{\epsilon} \omega_b} \right] \right)^{k_1} \quad (3.5)$$

where k_1 is selected to be a sufficiently large integer. Near ω_b , this higher order weighting function satisfies $W_1 \approx \left(\frac{\omega_b}{s} \right)^{k_1}$.

- **Control Sensitivity Weighting.** The weighting function W_2 on KS imposes a penalty on the controls that result. In this sense, W_2 is a control weighting function. In many applications, W_2 is selected to be a constant:

$$W_2(s) = M_u. \quad (3.6)$$

Since $|W_2 KS| < \gamma$ for all frequencies, it follows that

$$|KS| < \gamma \frac{1}{M_u}. \quad (3.7)$$

Given this, it follows that

we increase (decrease) M_u in order to reduce (increase) the peak of $|KS|$.

Typically, $0 < M_u < 1$ so that $1 < \frac{1}{M_u} < \infty$ represents a nominal desired upper bound on $|KS|$. Given this, the parameter M_u is henceforth referred to as the *control sensitivity peaking parameter* or *control penalty parameter*. By making γ small, we make the peak even smaller than $\frac{1}{M_u}$. This is typically very desirable because it translates into less control action.

Dynamic Control Sensitivity Weighting. For some applications, it is important to control the bandwidth associated with KS as well as the peak. For such application, the following dynamic weighting function may be more appropriate:

$$W_2(s) = \frac{1}{\epsilon} \left[\frac{s + \omega_{b_u} M_u}{s + \frac{\omega_{b_u}}{\epsilon}} \right] \quad (3.8)$$

where $M_u > 0$, $\omega_{b_u} > 0$, and $\epsilon > 0$ are design parameters used to shape the control sensitivity $|KS|$. The important parameters are M_u and ω_{b_u} . Typically, $0 < \epsilon < M_u < 1$. Here, $\epsilon > 0$ is typically very small - non-zero so that W_2 is proper.

Using Bode approximation ideas, it follows that

$$W_2 \approx \begin{cases} M_u & 0 & \leq |s| << \omega_{b_u} M_u \\ \frac{s}{\omega_{b_u}} & \omega_{b_u} M_u << |s| << \frac{\omega_{b_u}}{\epsilon} \\ \frac{1}{\epsilon} & \frac{\omega_{b_u}}{\epsilon} << |s| < \infty \end{cases} \quad (3.9)$$

Noting that $|W_2 KS| < \gamma$ for all frequencies, it follows that $|KS| < \gamma |W_2^{-1}|$. Given this, we have the following (approximate) relationship for the control sensitivity:

$$|KS| < \gamma \begin{cases} \frac{1}{M_u} & 0 & \leq |s| << \omega_{b_u} M_u \\ \frac{\omega_{b_u}}{s} & \omega_{b_u} M_u << |s| << \frac{\omega_{b_u}}{\epsilon} \\ \epsilon & \frac{\omega_{b_u}}{\epsilon} << |s| < \infty \end{cases} \quad (3.10)$$

Control Sensitivity Design Parameter Interpretations. From this, we have the following interpretations for the design parameters M_u , ω_{b_u} , ϵ , and γ :

1. *Bound on Peak Control Sensitivity.*

$\frac{1}{M_u}$ is an upperbound for the desired peak control sensitivity $|KS|$ at low frequencies;

2. *Control Sensitivity Bandwidth and Roll-Off.*

ω_{b_u} is an upperbound for the desired control sensitivity $|KS|$ bandwidth; i.e. high frequency unity gain crossover frequency;

3. Control Sensitivity High Frequency Upper Bound.

ϵ is an upper bound for the control sensitivity $|KS|$ at frequencies $\omega \geq \frac{\omega_{b_u}}{\epsilon}$.

By making γ smaller, the above upper bounds are all made smaller. We therefore wish to make γ as small as possible.

Steeper Control Sensitivity. If a steeper control sensitivity $|KS|$ is required near ω_{b_u} , then one might select W_2 to have the form

$$W_2(s) = \left(\frac{1}{k_2 \sqrt[k_2]{\epsilon}} \left[\frac{s + \omega_{b_u} \sqrt[k_2]{M_u}}{s + \frac{\omega_{b_u}}{k_2 \sqrt[k_2]{\epsilon}}} \right] \right)^{k_2} \quad (3.11)$$

where k_2 is selected to be a sufficiently large integer. Near ω_{b_u} , this higher order weighting function satisfies $W_2 \approx \left(\frac{s}{\omega_{b_u}} \right)^{k_2}$.

- **Complementary Sensitivity Weighting.** In many applications, the weighting function W_3 on the complementary sensitivity T is selected to have the following form:

$$W_3(s) = \frac{s + \frac{\omega_{b_c}}{M_y}}{\omega_{b_c}} \quad (3.12)$$

where $M_y > 0$ and $\omega_{b_c} > 0$ are design parameters used to shape the $|T|$. (Note that if the denominator is replaced by $\epsilon s + \omega_{b_c}$ with $\epsilon > 0$ small, then the function W_3 becomes proper. This is sometimes required.)

Using Bode approximation ideas, it follows that

$$W_3 \approx \begin{cases} \frac{1}{M_y} & 0 \leq |s| < \frac{\omega_{b_c}}{M_y} \\ \frac{s}{\omega_{b_c}} & \frac{\omega_{b_c}}{M_y} \ll |s| < \infty \end{cases} \quad (3.13)$$

Noting that $|W_3 T| < \gamma$ for all frequencies, it follows that $|T| < \gamma |W_3^{-1}|$. Given this, we have the following (approximate) relationship for the complementary sensitivity:

$$|T| < \begin{cases} M_y & 0 \leq |s| < \frac{\omega_{b_c}}{M_y} \\ \frac{\omega_{b_c}}{s} & \frac{\omega_{b_c}}{M_y} \ll |s| < \infty \end{cases} \quad (3.14)$$

Complementary Sensitivity Design Parameter Interpretations. From this, we have the following interpretations for the design parameters M_y , ω_{b_c} , and γ :

1. *Bound on Peak Complementary Sensitivity.*

M_y is an upperbound for the desired peak complementary sensitivity $|T|$ at low frequencies;

2. *Complementary Sensitivity Bandwidth and Roll-Off.*

ω_{b_c} is an upperbound for the desired complementary sensitivity $|T|$ bandwidth; i.e. high frequency unity gain crossover frequency.

Typically, we have $1 < M_y \ll \infty$. By making γ smaller, the above upper bounds are made smaller. We therefore wish to make γ as small as possible.

Steeper Complementary Sensitivity. If a steeper complementary sensitivity $|T|$ is required near ω_{b_c} , then one might select W_3 to have the form

$$W_3(s) = \left(\frac{1}{k_3 \sqrt[k_3]{\epsilon}} \left[\frac{s + \frac{\omega_{b_u}}{k_3 \sqrt[k_3]{M_u}}}{s + \frac{\omega_{b_c}}{k_3 \sqrt[k_3]{\epsilon}}} \right] \right)^{k_3} \quad (3.15)$$

where k_3 is selected to be a sufficiently large integer and $\epsilon \geq 0$ is introduced to achieve properness. Near ω_{b_c} , this higher order weighting function satisfies $W_3 \approx \left(\frac{s}{\omega_{b_c}} \right)^{k_3}$.

Significance of γ Vis-a-Vis Design Specifications. One might use MATLAB's "hinftot" command to find the minimum γ and a near-optimal controller K . (Caution: The command uses $\frac{1}{\gamma}$ instead of γ .) Given this, there are three cases worth noting.

- *Non-demanding Design Specifications.* A γ less than unity, implies that

$$|S(j\omega)| < |W_1^{-1}(j\omega)| \quad (3.16)$$

$$|K(j\omega)S(j\omega)| < |W_2^{-1}(j\omega)| \quad (3.17)$$

$$|T(j\omega)| < |W_3^{-1}(j\omega)| \quad (3.18)$$

for all ω . This suggests that weighting functions (design specifications) reflecting not very demanding (i.e. relaxed) were used. In such a case, one might then try to tighten the specifications by selecting more aggressive design parameters..

- *Demanding Design Specifications.* A γ greater than unity, suggests that weighting functions reflecting very demanding (i.e. aggressive) design specifications were used. In such a case, one might then try to loosen the specifications by selecting less aggressive design parameters.
- *Unrealistic Design Specifications.* If a controller does not exist (i.e. any one of the necessary and sufficient conditions for existence are violated), then this implies that the weighting functions (and the associated design specifications) used were unrealistic; i.e. far too aggressive. As a general rule,

the design problem becomes more difficult as low and high frequency
design specifications (reflected in the weighting functions) overlap.

Why is this? Low frequency specifications typically call for a minimum bandwidth. High frequency specifications typically call for a maximum bandwidth. If the specifications are too demanding, a

very high order (possibly infinite-dimensional) controller may be required to achieve the overlapping specifications. In such a case, a near optimal controller possessing dimension equal to that of the generalized plant may not exist.

3.2 SISO Unstable Plant with High Frequency Dynamics

The following example considers an unstable SISO plant with high frequency dynamics.

Example 3.2.1 ($\mathcal{H}^\infty$ Design for First Order Unstable Plant with High Frequency Dynamics)

In this example, we consider an unstable plant $P = \frac{1}{s-1} \left[\frac{10}{s+10} \right]^2$. The plant has a first order instability at $s = 1$ and high frequency second order dynamics. To design a control system for this plant, we consider two designs:

- the first is based on a nominal first order plant model $P = \frac{1}{s-1}$,
- the second is based on the full order plant model $P = \frac{1}{s-1} \left[\frac{10}{s+10} \right]^2$.

Each design was obtained by solving a standard weighted $\mathcal{H}^\infty$ mixed-sensitivity control problem with weighting functions and design parameters as specified in Table 3.1. It should be noted that little time was spent tuning the parameters.

Table 3.1: Weighting Functions and Design Parameters

W_1	W_2	W_3
$M_s = 10$ $\omega_b = 1$ $\epsilon = 0.01$	$M_u = 0.01$	$M_y = 10$ $\omega_{bc} = 30$
$W_1^{-1} = \frac{s+\epsilon\omega_b}{\frac{s}{M_s}+\omega_b}$ $= \frac{s+0.01}{0.1s+1}$	$W_2^{-1} = \frac{1}{M_u}$ $= 100$	$W_3^{-1} = \frac{\omega_{bc}}{s+\frac{\omega_{bc}}{M_y}}$ $= \frac{30}{s+3}$

Design 1 - Neglect High Frequency Dynamics in Design Process. The first design was obtained using the nominal (low frequency) plant model

$$P = \frac{1}{s-1}. \quad (3.19)$$

This low frequency model neglects the high frequency dynamics $\left[\frac{10}{s+10} \right]^2$. This approach, of neglecting dynamics, is widely used in practice to maintain a handle on mathematical tractability and reduce computational requirements. The resulting controller - obtained using MATLAB's "hinftol" command with an

optimality tolerance of 0.0001 - is as follows:

$$K = \frac{3393773.861(s + 0.6534)}{(s + 0.01)(s + 5.108 \times 10^5)} \approx \frac{6.6436(s + 0.6534)}{(s + 0.01)}. \quad (3.20)$$

Note that K is second order - the order of P plus the order of W_1 . W_3 does not contribute to the order of K . This is because W_3 is improper and when it is lumped into our generalized plant G , it does not increase the order of G . W_2 , being a constant, also does not contribute to the order of K .

The controller provides high gain at low frequencies - for good low frequency command following and good low frequency disturbance attenuation. K essentially contains an integrator; i.e. a pole near $s \approx 0$. The zero in K provides lead needed to ensure closed loop stability. This follows from classical root locus thinking. For this design, the downward gain margin is $\downarrow GM \approx 0.1491$ (or -16.5 dB) and the upward gain margin is $\uparrow GM = \infty$. The associated minimum γ was found to be

$$\gamma = 0.2519. \quad (3.21)$$

Since $\gamma < 1$, it follows that our specifications (as reflected in W_1 , W_2 and W_3 were not very demanding.

The resulting complementary sensitivity is given by

$$T = \frac{3385956.0265(s + 0.6534)}{(s + 4.74)(s + 0.9138)(s + 5.108 \times 10^5)} \approx \frac{6.6284(s + 0.6534)}{(s + 4.74)(s + 0.9138)}. \quad (3.22)$$

The closed loop system is stable with well damped poles. A dominant closed loop pole is situated at $s = -0.9138$. The associated time constant is $\tau = 1.09$ sec. The associated settling time is $t_s = 5\tau = 5.47$ sec.

Design 1⁺: Include High Frequency Dynamics After Design Process. When the above controller K is evaluated using the full order plant

$$P = \frac{1}{s - 1} \left[\frac{10}{(s + 10)} \right]^2, \quad (3.23)$$

we refer to the design as Design 1⁺. The resulting complementary sensitivity is given by

$$T = \frac{338595602.6502(s + 0.6534)}{(s + 16.11)(s + 0.8489)(s + 5.108 \times 10^5)(s^2 + 2.055s + 31.68)} \quad (3.24)$$

$$\approx \frac{662.8291(s + 0.6534)}{(s + 16.11)(s + 0.8489)(s^2 + 2.055s + 31.68)}. \quad (3.25)$$

Note that Design 1⁺ exhibits lightly damped closed loop poles near $s \approx -1.0275 \pm j5.5339$ ($\zeta = 0.183$, $\omega_n = 5.63$). Because this is not desirable, it provides motivation for the next design.

Design 2: Include High Frequency Dynamics in Design Process. The second design was obtained

using the full order plant model

$$P = \frac{1}{s-1} \left[\frac{10}{(s+10)} \right]^2. \quad (3.26)$$

The resulting controller - obtained using MATLAB's "hinftopt" command with an optimality tolerance of 0.0001 - is as follows:

$$K = \frac{6769870.7704(s+0.6094)(s+10)^2}{(s+2.128 \times 10^5)(s+0.01)(s^2+37.93s+614.2)} \quad (3.27)$$

$$\approx \frac{31.8094(s+0.6094)(s+10)^2}{(s+0.01)(s^2+37.93s+614.2)} \quad (3.28)$$

$$\approx \frac{5.179(s+0.6094)}{(s+0.01)} \left[\frac{s+10}{10} \right]^2 \left[\frac{614.2}{s^2+37.93s+614.2} \right] \quad (3.29)$$

Note that K is fourth order - the order of P plus the order of W_1 . This design is essentially the first design (see pole near $s = -0.01$ and zero near $s = -0.6$) with additional lead that inverts the plant's two high frequency poles at $s = -10$. Poles at $s \approx -18.97 \pm j15.95$ and another high frequency pole ensure that the controller is strictly proper; i.e. degree of denominator is greater than degree of numerator.

- **Caution: Inversion of Plant Dynamics.** One must be very careful when using the $\mathcal{H}^\infty$ design methodology or any other design methodology that may invert plant dynamics (e.g. LQG/LTR). In the above discussion, for example, one sees how Design 2 inverts the plant's high frequency dynamics. In general, an $\mathcal{H}^\infty$ controller - if permitted - may try to invert plant poles that are near the imaginary axis! This is typically highly undesirable. When plant poles near the imaginary axis are present, one can use a bilinear transformation method to prevent undesirable inversions by the resulting compensator. The use of the bilinear transformation for this purpose will be explored in a subsequent example (see Example 3.6.1, page 72).

The associated minimum γ was found to be

$$\gamma = 0.3334. \quad (3.30)$$

Since $\gamma < 1$, it follows that our weighing functions (and the associated design specifications) were not very aggressive. The resulting complementary sensitivity is given by

$$T = \frac{-2.2287 \times 10^{-7}(s - 3.028 \times 10^{15})(s + 0.6095)}{(s + 2.128 \times 10^5)(s + 5.504)(s + 0.9362)(s^2 + 30.5s + 375.1)} \quad (3.31)$$

$$\approx \frac{3171.2673(s + 0.6095)}{(s + 5.504)(s + 0.9362)(s^2 + 30.5s + 375.1)} \quad (3.32)$$

$$\approx \frac{8.4545(s + 0.6095)}{(s + 5.504)(s + 0.9362)} \left[\frac{375.1}{s^2 + 30.5s + 375.1} \right]. \quad (3.33)$$

This shows that the resulting T is very similar to that obtained in Design 1 (see Equation 3.22).

Frequency Response Data. Frequency response data for each design is now presented.

- **Plant.** The plant frequency responses for the two plant models used - $P = \frac{1}{s-1}$ for Design 1 and $P = \frac{1}{s-1} \left[\frac{10}{s+10} \right]^2$ for Designs 1⁺ and 2 - are provided in Figure 3.1. Both frequency responses are similar at low frequencies with a 3 dB frequency near 1 rad/sec. The first response exhibits a high frequency roll-off of -20 dB/dec (one pole roll-off). The second response exhibits a high frequency roll-off of -60 dB/dec (three pole roll-off).

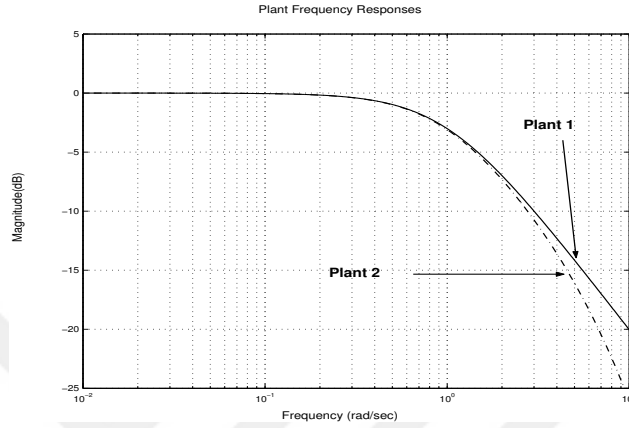


Figure 3.1: Plant Frequency Responses - $P = \frac{1}{s-1}$ (plant 1), $P = \frac{1}{s-1} \left[\frac{10}{s+10} \right]^2$ (plant 2)

- **Weighting Functions.** The inverse of each weighting function W_1, W_2, W_3 may be visualized as shown in Figure 3.2. Each provides a bound for a different closed loop transfer function:

$$|S| < \gamma |W_1|^{-1} \quad |KS| < \gamma |W_2|^{-1} \quad |T| < \gamma |W_3|^{-1}. \quad (3.34)$$

Note that $|W_1|^{-1}$ and $|W_3|^{-1}$ look like desirable sensitivity and complementary sensitivity magnitude responses. This justifies our selection:

$$W_1(s) = \frac{1}{M_s} \left[\frac{s + M_s \omega_b}{s + \epsilon \omega_b} \right] = \frac{1}{10} \left[\frac{s + 10 (1)}{s + (0.01) (1)} \right] = 0.1 \left[\frac{s + 10}{s + 0.01} \right] \quad (3.35)$$

$$W_3(s) = \frac{s + \frac{\omega_{bc}}{M_y}}{\omega_{bc}} = \frac{s + \frac{30}{10}}{30} = \frac{s + 3}{30}. \quad (3.36)$$

- **Controller.** The corresponding controller frequency responses are provided in Figure 3.3. Each design exhibits high gain at low frequencies (i.e. near integral action). Both exhibit lead near 0.6 rad/sec (due to a zero). Design 2 exhibits additional lead near 10 rad/sec - the additional lead inverting the plant's high frequency dynamics. Note: In practice, one must be careful when inverting high frequency dynamics. Such dynamics are typically uncertain and the resulting design may be sensitive to the associated modeling errors.
- **Open Loop.** The open loop frequency responses for Designs 1, 1⁺, and 2 are provided in Figure 3.4. Design 1 exhibits the highest gain crossover frequency - followed by Designs 2 and 1⁺. At low frequencies, the open loop gain of each design is high (due to near-integral action in controller). This

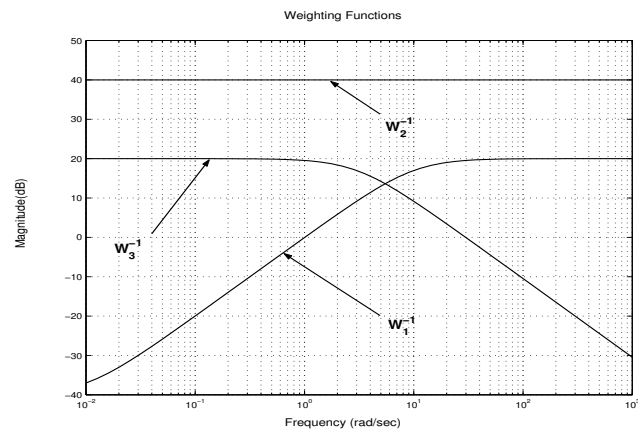


Figure 3.2: Inverse of Weighting Functions - W_1^{-1} , W_2^{-1} , W_3^{-1}

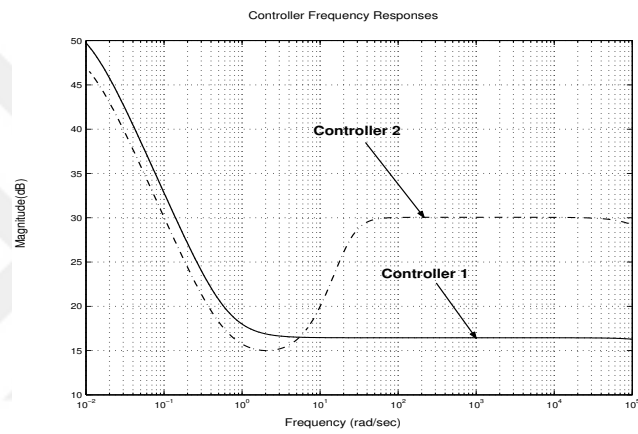


Figure 3.3: Controller Frequency Responses - K

helps provide good low frequency command following and good low frequency disturbance attenuation. The gain crossover frequency ω_g for all designs lies between 5 and 6 rad/sec.

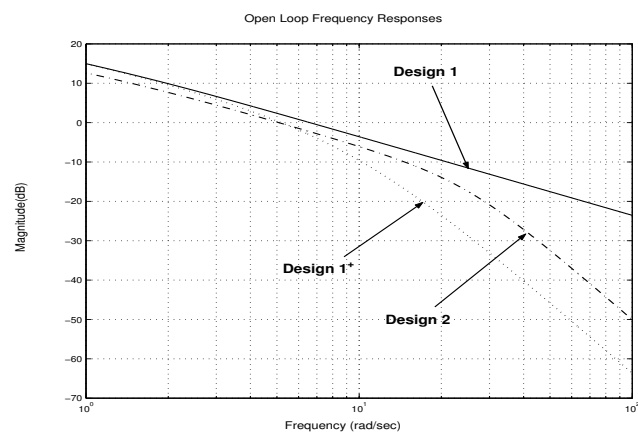


Figure 3.4: Open Loop Frequency Responses - $L = PK$

- Sensitivity. The sensitivity frequency responses for Designs 1, 1⁺, and 2 are provided in Figure 3.5.

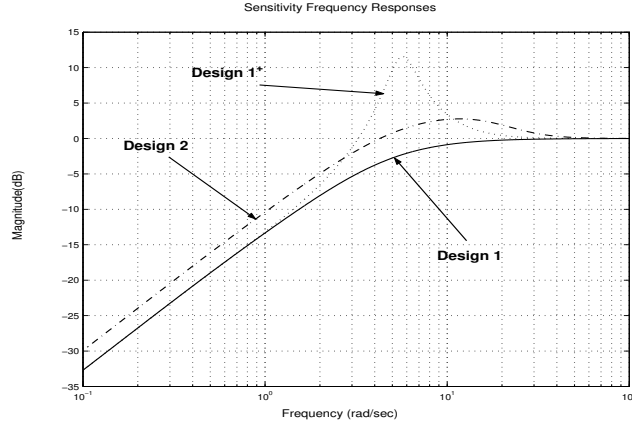


Figure 3.5: Sensitivity Frequency Responses - $S = \frac{1}{1+PK}$

Each sensitivity exhibits a low frequency slope of 20 dB/dec. This is because of the near integral action requested by W_1^{-1} at low frequencies. This helps provide good low frequency command following and good low frequency disturbance attenuation. Design 1 exhibits a sensitivity with no peaking. Design 2 exhibits some peaking. Design 1⁺, which did not consider the plant's high frequency dynamics, exhibits the most sensitivity peaking. The large sensitivity peaking is due to the presence of lightly damped closed loop poles near $s \approx -1.0275 \pm j5.5339$ ($\omega_n = 5.6$ rad/sec, $\zeta = 0.18$).

- Sensitivity Peaking and Guaranteed Stability Margins. Typically, we would like the peak sensitivity to lie below $\alpha = 2$ (i.e. 6 dB). Associated with such an upper bound are the following guaranteed margins:

$$\uparrow GM > \frac{\alpha}{\alpha - 1} \quad \downarrow GM < \frac{\alpha}{\alpha + 1} \quad |PM| > 2 \sin^{-1} \left(\frac{1}{2\alpha} \right). \quad (3.37)$$

It should be noted that these bounds may be very conservative, i.e. the actual margins may be much greater than suggested by the bound.

At $\omega = 0.1$, the sensitivity for design 1 satisfies $|S| < \gamma|W_1^{-1}| = 0.2519(0.1) = 0.02519$ (-31.98 dB). For design 2, the sensitivity for design 2 satisfies $|S| < \gamma|W_1^{-1}| = 0.3334(0.1) = 0.03334$ (-29.54 dB). We see that for each design minimizing γ has resulted in additional sensitivity reduction beyond the 0.1 (-20 dB) “requested” by $W_1^{-1} = 10 \left[\frac{s+0.01}{s+10} \right]$ at $\omega = 0.1$ rad/sec.

At $\omega = 10$ rad/sec, we have $|W_1| \approx \frac{10}{\sqrt{2}} = 7.071$ (16.9 dB). (A very non-demanding specification.) At this frequency, the sensitivity for design 1 satisfies $|S| < \gamma|W_1^{-1}| = 0.2519(7.071) = 1.7812$ (5.01 dB). The sensitivity actually lies below 0 dB - well below the upper bound. At this frequency, the sensitivity for design 2 satisfies $|S| < \gamma|W_1^{-1}| = 0.3334(7.071) = 2.3575$ (7.45 dB). The sensitivity actually lies below 4 dB - well below the upper bound.

- Complementary Sensitivity. The complementary sensitivity frequency responses for Designs 1, 1^+ , and 2 are provided in Figure 3.6.

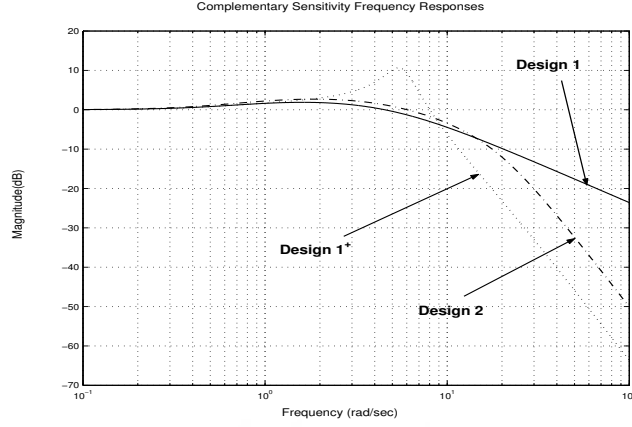


Figure 3.6: Complementary Sensitivity Frequency Responses - $T = \frac{PK}{1+PK}$

All complementary sensitivities are near unity (0 dB) at low frequencies. This is because of the near integral action requested by W_1^{-1} at low frequencies. Design 1 exhibits a complementary sensitivity with little peaking. Design 2 exhibits more peaking. Design 1^+ , which did not consider the plant's high frequency dynamics in the design process, exhibits the most complementary sensitivity peaking. The large complementary sensitivity peaking is due to the presence of lightly damped closed loop poles near $s \approx -1.0275 \pm j5.5339$ ($\omega_n = 5.6$ rad/sec, $\zeta = 0.18$).

- Complementary Sensitivity Peaking and Guaranteed Stability Margins. Typically, we would like the peak complementary sensitivity to lie below $\beta = 2$ (i.e. 6 dB). Associated with such an upper bound are the following guaranteed margins:

$$\uparrow GM > \frac{\beta}{\beta - 1} \quad \downarrow GM < \frac{\beta}{\beta + 1} \quad |PM| > 2 \sin^{-1} \left(\frac{1}{2\beta} \right). \quad (3.38)$$

It should be noted that these bounds may be very conservative, i.e. the actual margins may be much greater than suggested by the bound.

- Input Disturbance to Output. The impact of input disturbances (i.e. additive disturbances modeled at the plant input) on the plant output for Designs 1, 1^+ , and 2 may be ascertained from the frequency responses in Figure 3.7 - for $T_{d,y} = \frac{P}{1+PK} = SP$.

While each design shows that input disturbances d_i will be attenuated at all frequencies, the impact of the lightly damped closed loop pole associated with Design 1^+ is visible in the form of additional peaking near 5.6 rad/sec.

- Note that our mixed sensitivity optimization did not directly address $T_{d,y}$. We could have ad-

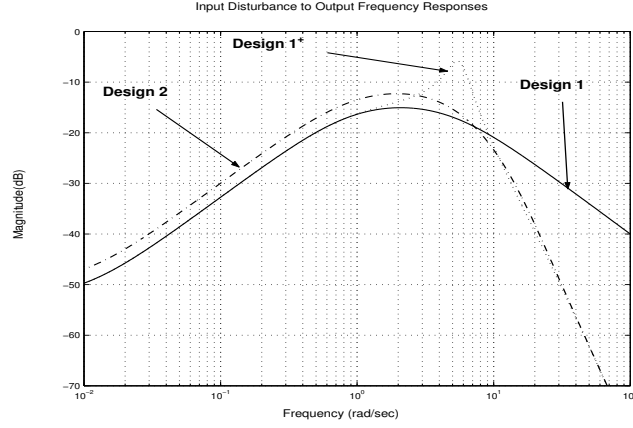


Figure 3.7: Input Disturbance to Output Frequency Responses - $T_{d_{iy}} = SP$

dressed $T_{d_{iy}}$ directly in the design process by solving the following optimization problem:

$$K = \arg \left\{ \min_K \gamma \mid \|T_{wz}(K)\|_{\mathcal{H}^\infty} = \left\| \begin{bmatrix} W_1 S \\ W_2 K S \\ W_3 T \\ W_4 S P \end{bmatrix} \right\|_{\mathcal{H}^\infty} < \gamma \right\} \quad (3.39)$$

where W_4 is an stable weighting function selected to shape $T_{d_{iy}} = SP$.

- **Reference to Control.** *The impact of reference commands on the controls for Designs 1, 1^+ , and 2 may be ascertained from the frequency responses in Figure 3.8 for $T_{ru} = \frac{K}{1+PK} = KS$. The figure shows that Design 1 amplifies high frequency reference commands the least. Design 2 amplifies high frequency reference commands the most. This is because of the additional lead in Design 2's controller - lead that inverts the plant's high frequency dynamics. Design 1^+ is characterized by undesirable peaking near 5.6 rad/sec.*

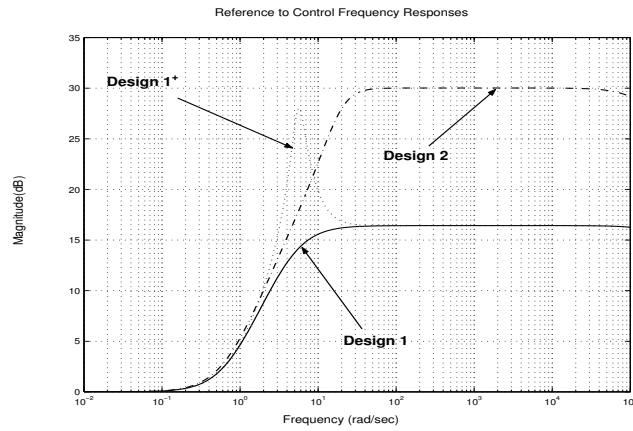


Figure 3.8: Reference to Control Frequency Responses - $T_{ru} = KS$

- **Optimization Norm Objective.** *It is interesting to examine the γ -scaled singular values of the transfer*

function matrix from the exogenous signal to the regulated signals:

$$\frac{1}{\gamma} T_{wz} \stackrel{\text{def}}{=} \frac{1}{\gamma} \begin{bmatrix} W_1 S \\ W_2 K S \\ W_3 T \end{bmatrix}. \quad (3.40)$$

This is done in Figure 3.9 for Designs 1, 1⁺, and 2. The figure shows that the singular values for Designs 1 and 2 are flat for most frequencies - rolling off at high frequencies. This implies that $\frac{1}{\gamma} T_{wz}$ is unitary at low frequencies; i.e.

$$T_{wz}^H(j\omega) T_{wz}(j\omega) = \gamma^2 I \text{ at low frequencies}$$

for Designs 1 and 2. Finally, note that Design 1⁺ exhibits undesirable peaking near 5.6 rad/sec.

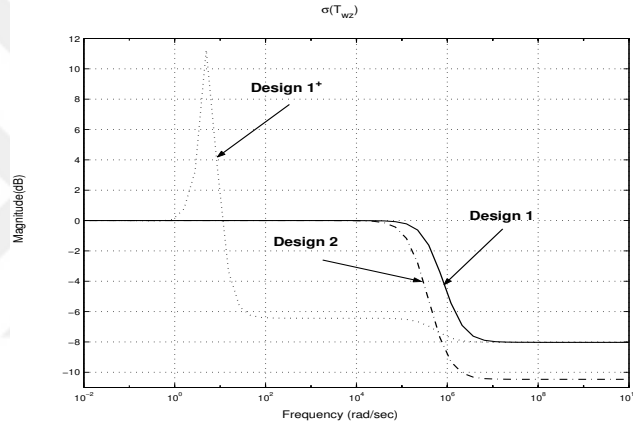


Figure 3.9: Exogenous Signals to Regulated Variables - $\sigma_i(T_{wz})$

Time Response Data: Step Command Following. We now address step command following for each design. This is done without a pre-filter and with a pre-filter.

Step Command Following With No Pre-Filter. Figure 3.10 contains the output response to a step reference command for each design. Design 1 exhibits 20% overshoot. This overshoot is primarily due to a zero in K (Design 1) at $s = -0.6534$ since the associated closed loop poles are at desirable locations: $s = -0.9138, -4.74, -5.108 \times 10^5$. Design 2 exhibits 30% overshoot. This overshoot is primarily due to a zero in K (Design 2) at $s = -0.6059$ since the associated closed loop poles are at desirable locations: $s = -0.9362, -5.504, -15.25 \pm j11.9389, -2.128 \times 10^5$. Design 1⁺ exhibits a large overshoot due to the presence of lightly damped closed loop poles. Figure 3.11 contains the corresponding controls for each design. As expected from the zeros in each controller and from the associated $|KS|$ plots, the controls are large. The time responses in these Figures 3.10-3.11 suggest that a reference command pre-filter is necessary.

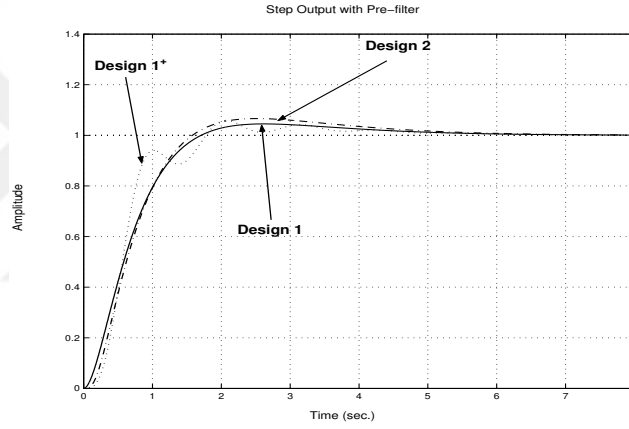


Figure 3.10: Output Response to Step Reference Command (without Pre-filter)

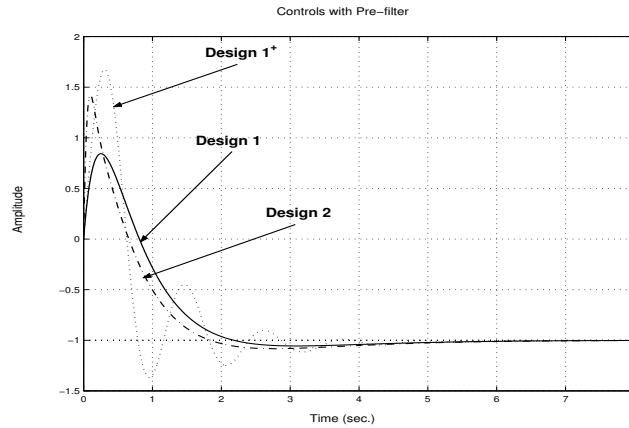


Figure 3.11: Control Response to Step Reference Command (without Pre-filter)

Step Command Following With Pre-Filter. In order to compensate for the derivative action in each design, we developed a pre-filter for each design. The pre-filter was selected to be $W(s) = \frac{2z}{s+2z}$ where $z = z_1 =$

0.6534 for Design 1 and $z = z_2 = 0.6095$ for Design 2. Figure 3.12 contains the output response to a filtered step reference command for each design. Designs 1 and 2 exhibit little overshoot. Design 1⁺ exhibits a large overshoot due to the presence of lightly damped closed loop poles.

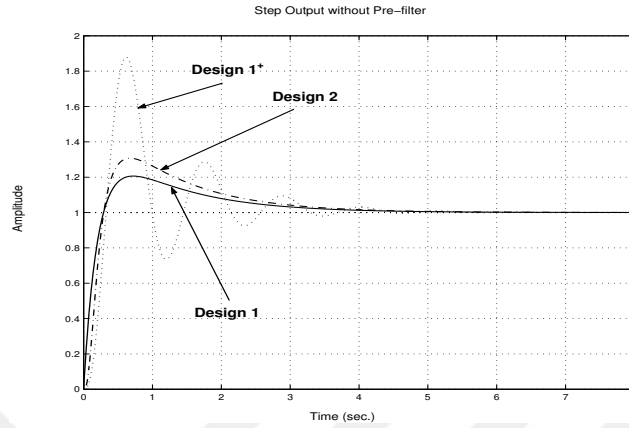


Figure 3.12: Output Response to Step Reference Command (with Pre-filter)

Figure 3.13 contains the corresponding controls. As expected, the corresponding controls are much smaller. Design 1 exhibits the least control action. Design 2 exhibits slightly more control action. Design 1⁺ exhibits the most control action due to the presence of lightly damped closed loop poles.

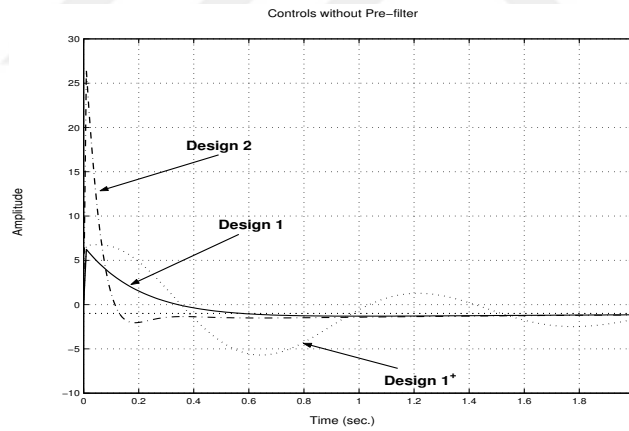


Figure 3.13: Control Response to Step Reference Command (with Pre-filter)

■

3.3 SISO Unstable Plant with High Frequency Dynamics: Impact of Sensitivity Weighting Design Parameters

The following example shows how the sensitivity weighting function design parameters may be used to influence a design.

Example 3.3.1 (Impact of Sensitivity Weighting Function Design Parameters)

The impact of key design parameters associated with the sensitivity weighting function W_1 is now examined.

Impact of Sensitivity Reduction Parameter ϵ . The impact of the sensitivity reduction parameter ϵ on the designs may be visualized as shown in Figures 3.14-3.22. The nominal parameter value is $\epsilon = 0.01$.

- Sensitivity Reduction. The impact of ϵ on sensitivity reduction at $\omega = \epsilon\omega_b$ is seen in Figure 3.14. The figure demonstrates the purpose of ϵ . As ϵ is reduced, the sensitivity reduction that occurs at $\omega = \epsilon\omega_b$ increases; i.e. $|S(j\epsilon\omega_b)|$ decreases with decreasing ϵ .

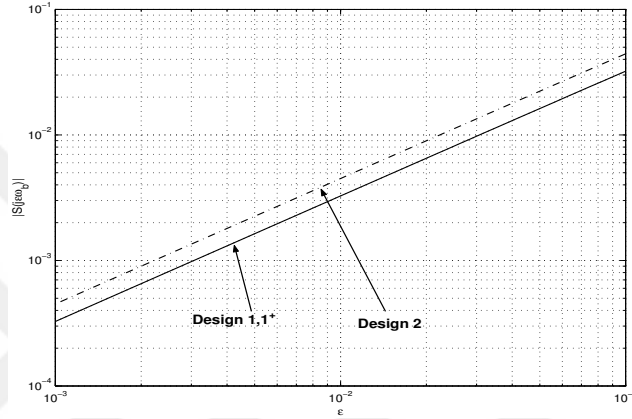


Figure 3.14: Dependence of $|S(j\epsilon\omega_b)|$ on ϵ

- Bandwidth. Figures 3.15-3.17 show that for small ϵ , the parameter has little impact on “bandwidth” - open loop or closed loop.

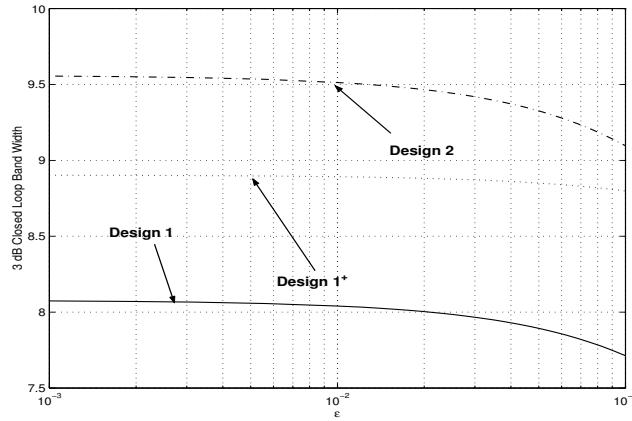


Figure 3.15: Dependence of 3 dB Closed Loop (T) Bandwidth on ϵ

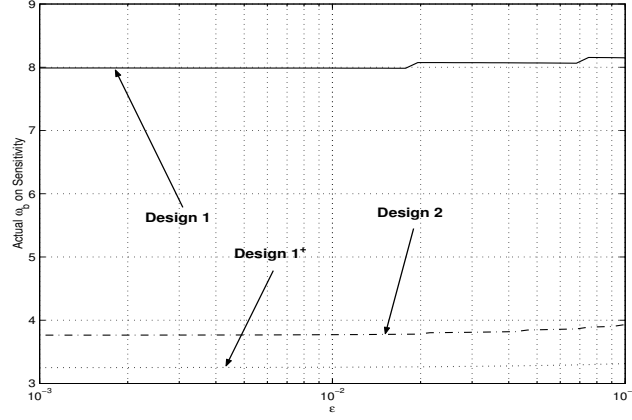


Figure 3.16: Dependence of Sensitivity Gain Crossover Frequency on ϵ

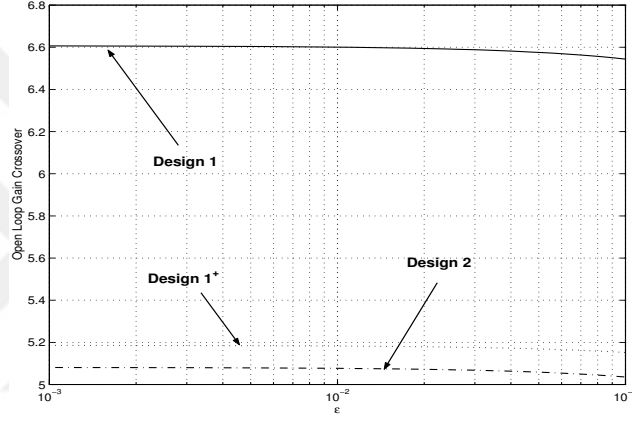


Figure 3.17: Dependence of Open Loop Gain Crossover on ϵ

- Optimization Norm Objective. Figure 3.18 shows that the sensitivity reduction parameter ϵ has little impact on $\|T_{wz}\|_{\mathcal{H}^\infty}$. The figure shows that $\|T_{wz}\|_{\mathcal{H}^\infty}$ remains relatively constant for $\epsilon \in [0.001, 0.1]$

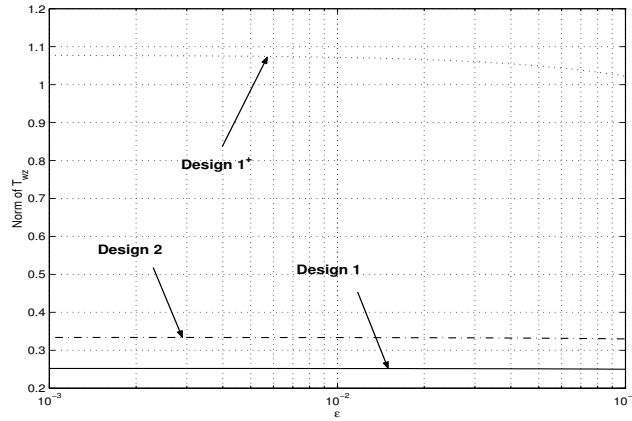


Figure 3.18: Dependence of $\|T_{wz}\|_{\mathcal{H}^\infty}$ on ϵ

- Stability Margins. Figure 3.19 show that the sensitivity reduction parameter ϵ has little impact on stability margins; i.e. upward gain margin, downward gain margin, and phase margin remain relatively

constant for $\epsilon \in [0.001, 0.1]$. For small $\epsilon > 0$, we have $W_1 \approx \frac{1}{M_s} \left[\frac{s+M_s\omega_b}{s} \right]$ which dictates the loop margins.

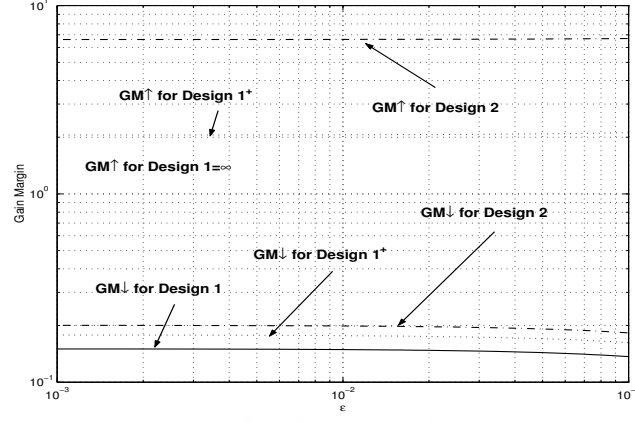


Figure 3.19: Dependence of Gain Margins on ϵ

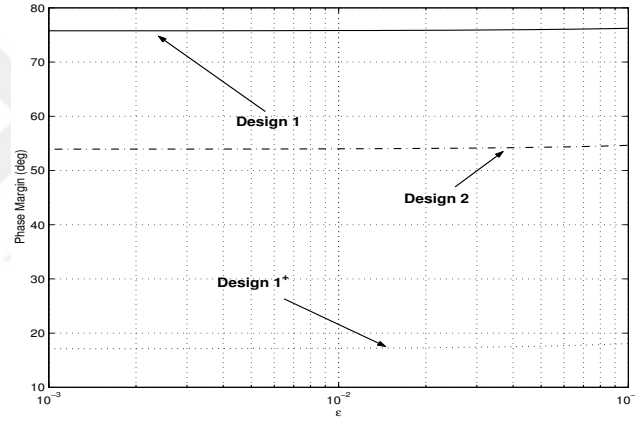


Figure 3.20: Dependence of Phase Margin on ϵ

- **Overshoot and Settling Time.** Figures 3.21 and 3.22 show that the sensitivity reduction parameter ϵ has little impact on the overshoot and settling time that results from a step reference command (with no prefilter). The figures show that both decrease for large ϵ . For large ϵ , we have $W_1 \approx \frac{1}{M_s} \left[\frac{s + M_s \omega_b}{\epsilon \omega_b} \right]$ which penalizes high frequencies more.

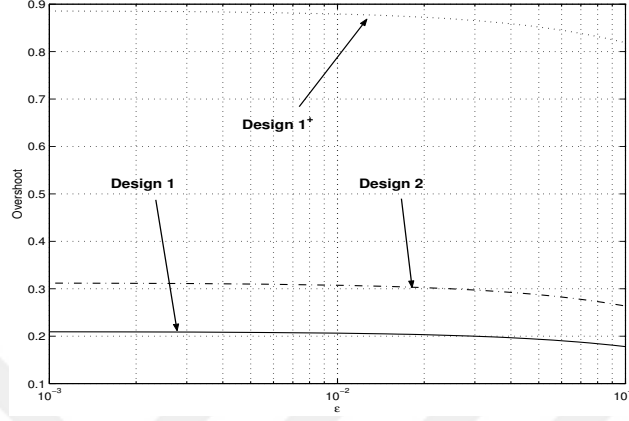


Figure 3.21: Dependence of Overshoot on ϵ (Step Command, No Prefilter)

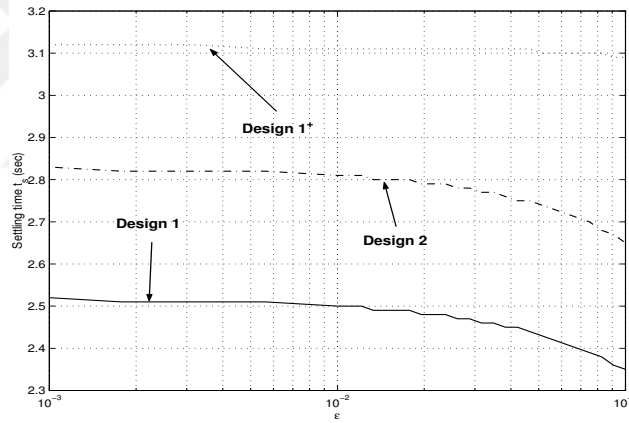


Figure 3.22: Dependence of Settling Time on ϵ (Step Command, No Prefilter)

Impact of Sensitivity Gain Crossover Parameter ω_b . The nominal value of the “bandwidth design parameter” ω_b is $\omega_b = 1$ rad/sec. Its impact on various quantities is now examined.

- **3 dB Closed Loop Bandwidth Associated with T .** The impact of ω_b on the 3 dB closed loop bandwidth associated with the complementary sensitivity T is seen in Figure 3.23. As desired, the 3 dB bandwidth increases with increasing ω_b .
- **Sensitivity Gain Crossover Frequency.** The impact of ω_b on the sensitivity gain crossover frequency - that is, frequency at which $|S| = 1$ - is seen in Figure 3.24. As desired, the sensitivity gain crossover frequency increases with increasing ω_b . Because of Figures 3.23-3.24, it makes sense to refer to ω_b as a bandwidth design parameter.
- **Open Loop Gain Crossover Frequency.** The impact of ω_b on the open loop gain crossover frequency

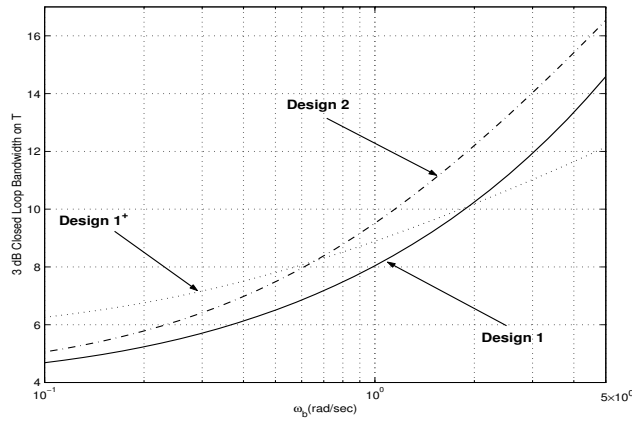


Figure 3.23: Dependence of 3 dB Closed Loop (T) Bandwidth on ω_b

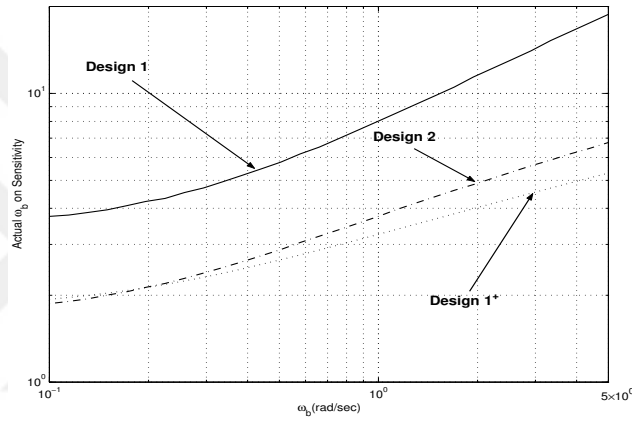


Figure 3.24: Dependence of Sensitivity Gain Crossover Frequency on ω_b

is seen in Figure 3.25. As desired, the open loop gain crossover frequency increases with increasing ω_b .

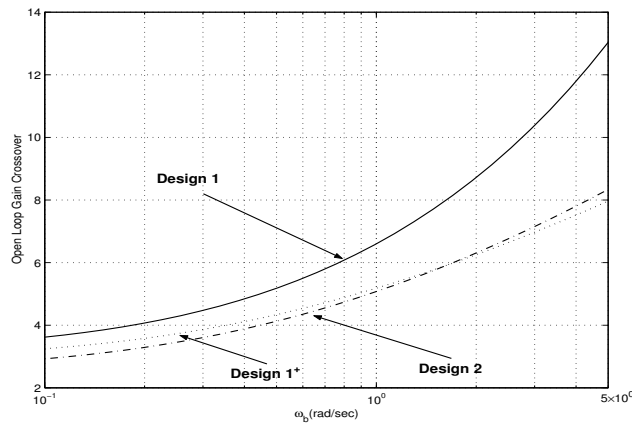
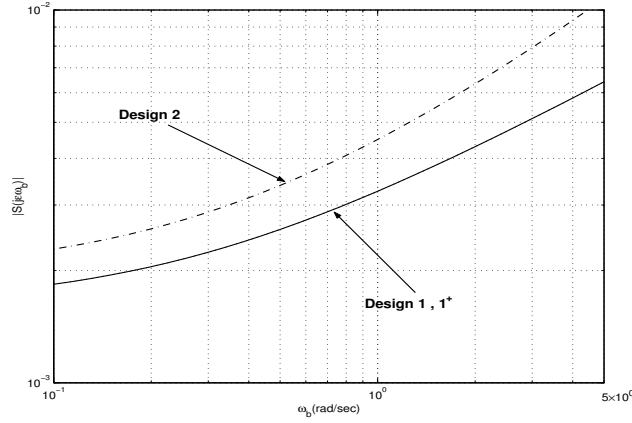
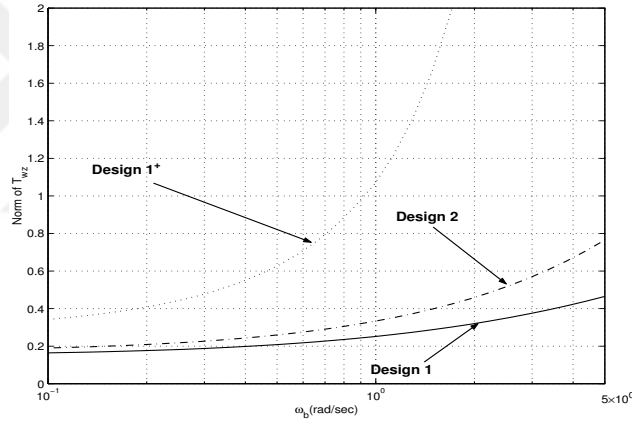


Figure 3.25: Dependence of Open Loop Gain Crossover on ω_b

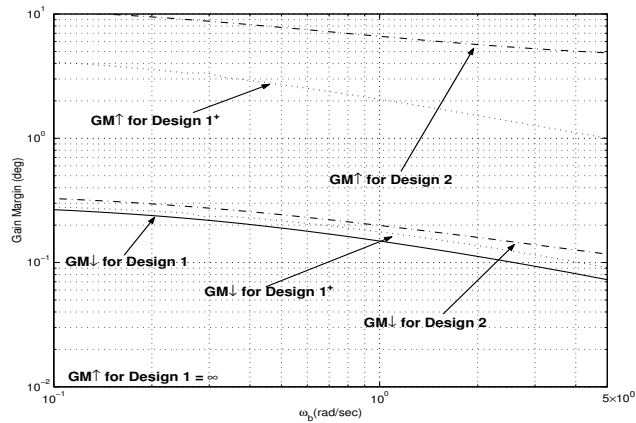
- Sensitivity Reduction. The dependence of $S(j\epsilon\omega_b)$ on ω_b is seen in Figure 3.26. As ω_b increases, the sensitivity reduction $|S(j\epsilon\omega_b)|$ worsens; i.e. $|S(j\epsilon\omega_b)|$ increases with ω_b .

Figure 3.26: Dependence of $|S(j\omega_b)|$ on ω_b

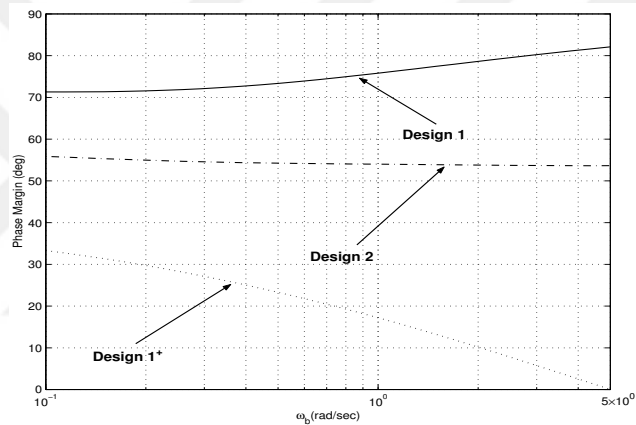
- **Optimization Norm Objective.** Figure 3.27 shows that as ω_b increases, $\|T_{wz}\|_{\mathcal{H}^\infty}$ increases. For design 1^+ , increasing ω_b too much results in closed loop instability and a $\|T_{wz}\|_{\mathcal{H}^\infty}$ that approaches infinity.

Figure 3.27: Dependence of $\|T_{wz}\|_{\mathcal{H}^\infty}$ on ω_b

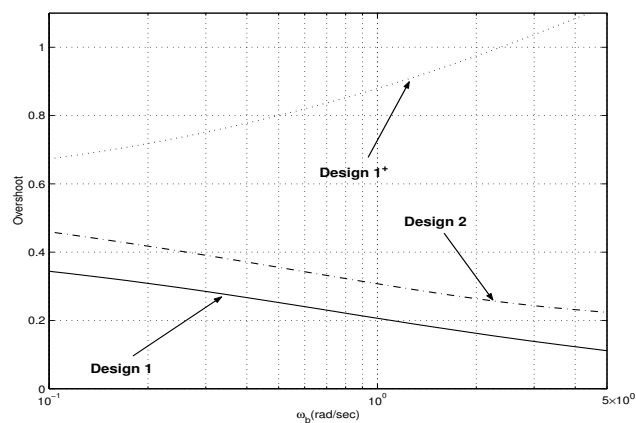
- **Gain Margin.** The dependence of gain margin on ω_b is seen in Figure 3.28. The upward gain margin for Design 1 is infinite for any ω_b . This follows from classical root locus thinking. As ω_b increases, the upward gain margin for Design 1^+ is reduced. Although not shown in the figure, Design 1^+ goes unstable if ω_b is increased too much. This is because the plant's high frequency poles are not taken into consideration in the design process. Because of this, a high bandwidth (i.e. high gain) design based on a first order model results in an unstable closed loop system. For Design 2, the upward gain margin decreases and then levels off. In contrast, the downward gain margins for each design improves (i.e. decreases) with increasing ω_b . One must therefore tradeoff upward gain margin for downward gain margin as ω_b increases.
- **Phase Margin.** The dependence of phase margin on ω_b is seen in Figure 3.29. For Design 1, the phase margin increases with increasing ω_b . For Design 2, the phase margin remains almost flat - decreasing slightly with increasing ω_b . For Design 1^+ , the phase margin decreases with increasing ω_b .

Figure 3.28: Dependence of Gain Margins on ω_b

Although not shown in the figure, Design 1⁺ goes unstable if ω_b is increased too much.

Figure 3.29: Dependence of Phase Margin on ω_b

- Overshoot to Step Reference Command. Figure 3.30 shows that as ω_b increases, the overshoot to a step reference command decreases for Designs 1 and 2 and increases for Design 1⁺. Although not shown in the figure, Design 1⁺ goes unstable if ω_b is increased too much.

Figure 3.30: Dependence of Overshoot on ω_b (without Prefilter)

- Settling Time for Step Reference Command. Figure 3.31 shows that as ω_b increases, the settling time (1% rule) to a step reference command decreases for Designs 1 and 2. For Design 1⁺, the settling time increases significantly if ω_b is increased above a certain value. This is because Design 1⁺ goes unstable if ω_b is increased too much.

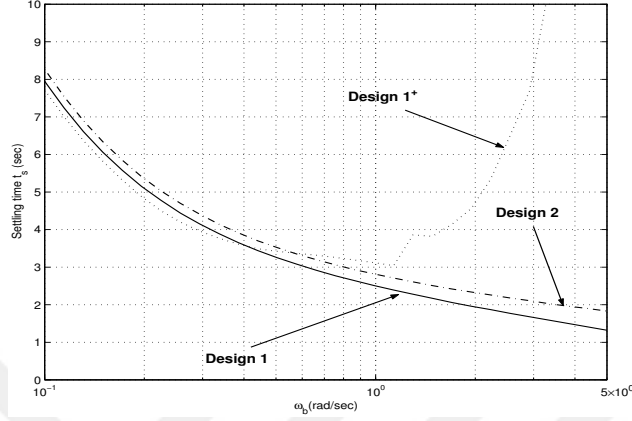


Figure 3.31: Dependence of Settling Time on ω_b (without Prefilter)

- Response to Step Reference Command: Design 1. Figure 3.32 clearly shows that for Design 1, the overshoot and settling time decrease as ω_b is increased. This is consistent with Figures 3.30-3.31. The overshoot in the step response is primarily due to the low frequency zero in the compensator. For small ω_b , the compensator zero is near the origin - in the left half plane. As ω_b increases the zero moves toward the left in the left half plane.

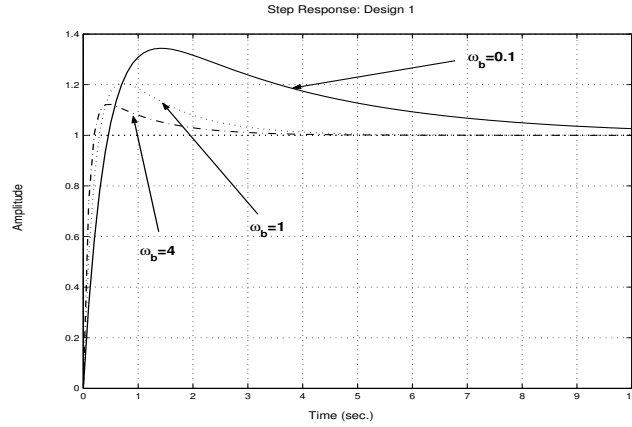


Figure 3.32: Dependence of Step Responses on ω_b for Design 1 (without Prefilter)

- Response to Step Reference Command: Design 1⁺. Figure 3.33 shows that the overshoot for Design 1⁺ increases as ω_b is increased. See Figure 3.30. The figure also shows that settling time decreases and then increases as ω_b is increased. This is consistent with Figure 3.31. If ω_b is increased too much, Design 1⁺ goes unstable.
- Response to Step Reference Command: Design 2. Figure 3.34 shows that for Design 2, the overshoot and settling time decrease as ω_b is increased. The overshoot in the step response is primarily due to

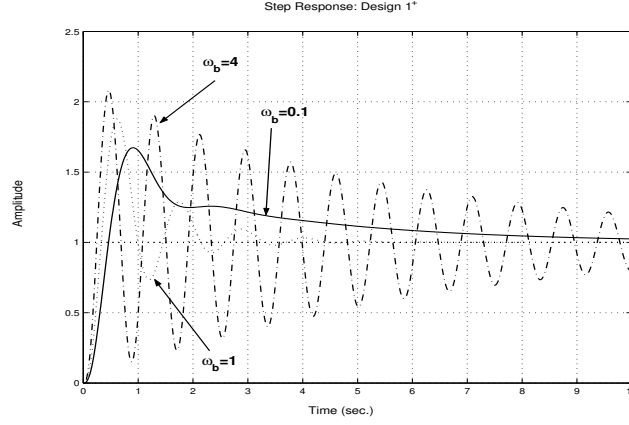


Figure 3.33: Dependence of Step Responses ω_b for Design 1⁺ (without pre-filter)

the low frequency zero in the compensator. For small ω_b , the zero is near the origin - in the left half plane. As ω_b increases the zero moves toward the left in the left half plane.

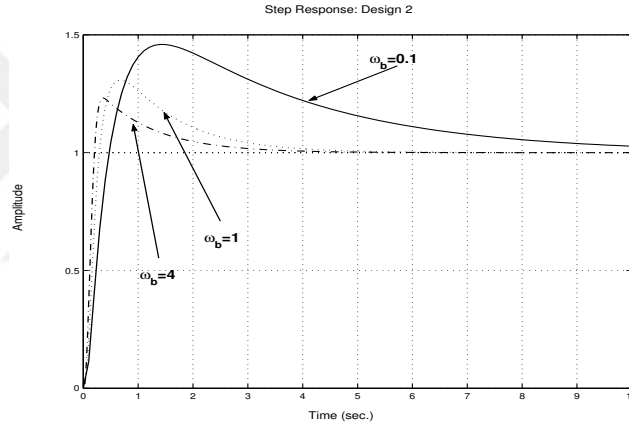


Figure 3.34: Dependence of Step Responses on ω_b for Design 2 (without Prefilter)

Impact of Sensitivity High Frequency Bound Parameter M_s . The impact of the high frequency sensitivity bound parameter M_s on each of the designs may be visualized as shown in Figures 3.35-3.42.

- **Bandwidth.** As M_s is increased, the bandwidth of Designs 1 and 2 increase - leveling off for large M_s . This is seen in Figures 3.35-3.37.
- **Optimization Norm Objective.** For large M_s , the parameter has little impact on $\|T_{wz}\|_{\mathcal{H}^\infty}$. This is seen in Figure 3.38.
- **Stability Margins.** For large M_s , the parameter has little impact on stability margins. This is seen in Figure 3.39.
- **Overshoot and Settling Time (without Prefilter).** For Designs 1 and 2, the overshoot and settling time to a step reference command (without prefilter) both decrease and level off as M_s increases. For Design 1⁺, the overshoot increase and levels off. The settling time decreases and levels off.

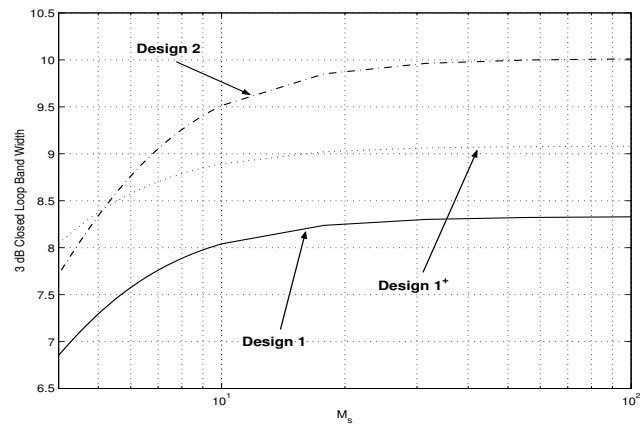


Figure 3.35: Dependence of 3 dB Closed Loop (T) Bandwidth on M_s

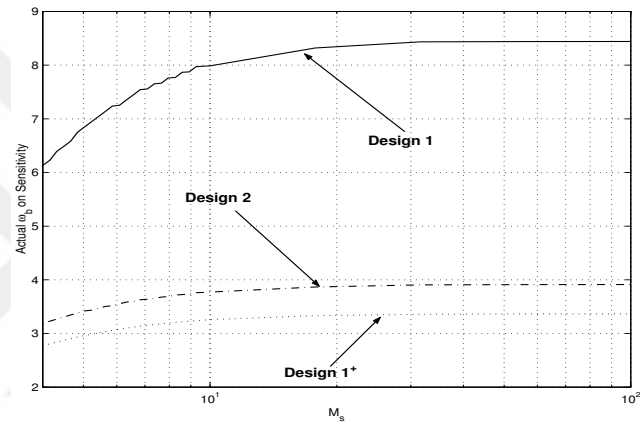


Figure 3.36: Dependence of Sensitivity Gain Crossover on M_s

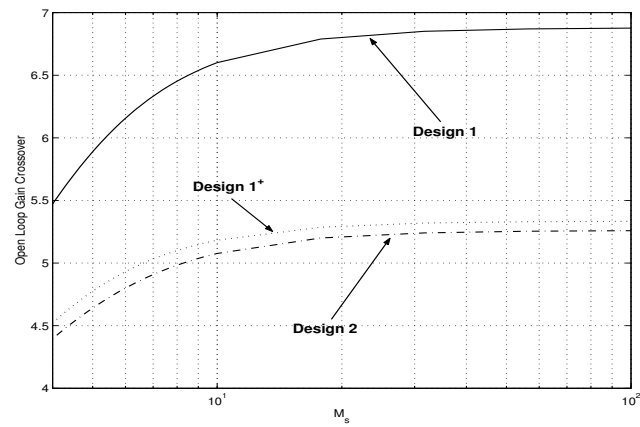
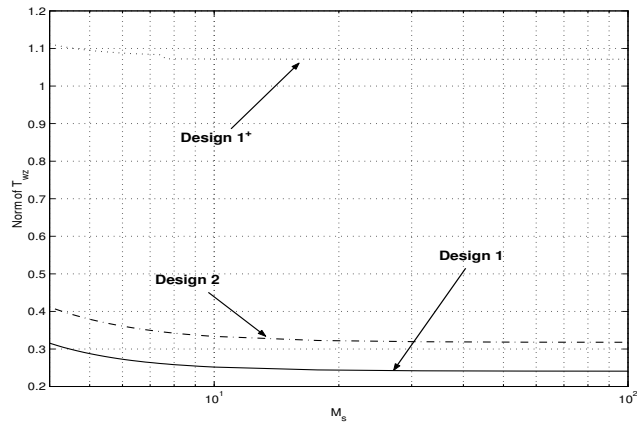
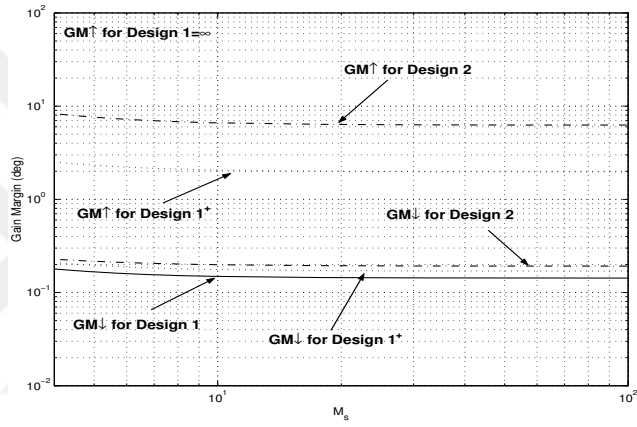
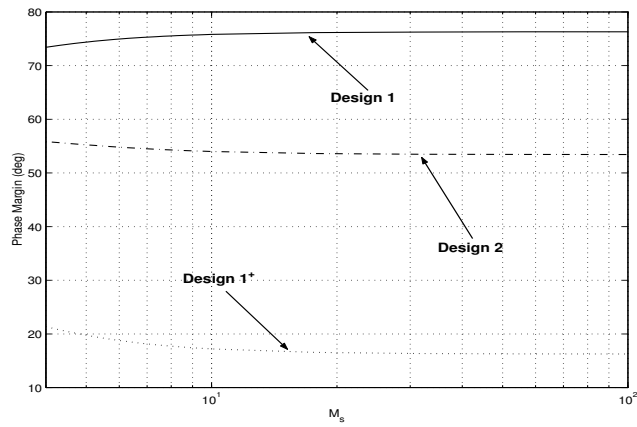


Figure 3.37: Dependence of Open Loop Gain Crossover on M_s

Figure 3.38: Dependence of $\|T_{wz}\|_{\mathcal{H}^\infty}$ on M_s Figure 3.39: Dependence of Gain Margins on M_s Figure 3.40: Dependence of Phase Margin on M_s

3.4 SISO Unstable Plant with High Frequency Dynamics: Impact of Control Sensitivity Weighting Design Parameters

The following example shows how the control sensitivity weighting function design parameters may be used to influence a design.

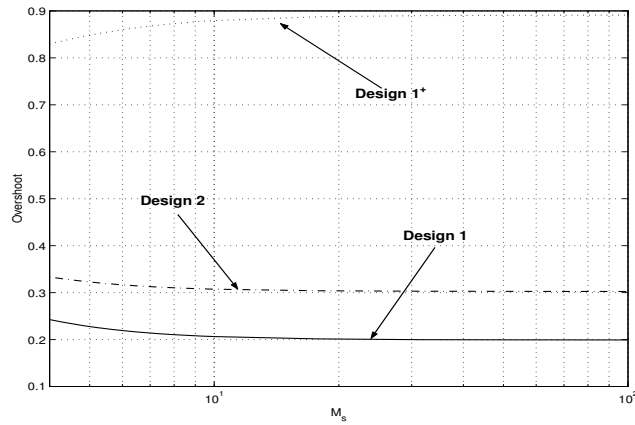


Figure 3.41: Dependence of Overshoot on M_s (without Prefilter)

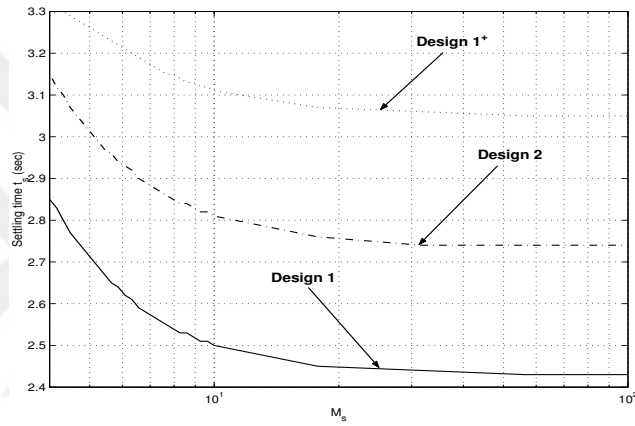


Figure 3.42: Dependence of Settling Time on M_s (without Prefilter)

Example 3.4.1 (Impact of Control Sensitivity Weighting Function Design Parameters)

The impact of key design parameters associated with the control sensitivity weighting function W_2 is now examined.

Impact of Control Sensitivity Peaking Parameter M_u . The impact of the low frequency control sensitivity peaking parameter M_u on each of the designs may be visualized as shown in Figures 3.43-3.51.

- **Peak Control Sensitivity.** While the dependence of M_u on the peak control sensitivity $T_{ru} = KS$ is much more pronounced for Design 2, the general trend is that the peak of KS decreases with increasing M_u (as expected). By increasing M_u , we penalize control action more. This forces the peak of KS to decrease. The decrease is achieved by rolling KS or K off at a lower frequency. As such K contributes less lead at high frequencies.
- **Bandwidth.** Figures 3.44-3.46 show that, generally, the bandwidth decreases with increasing M_u .

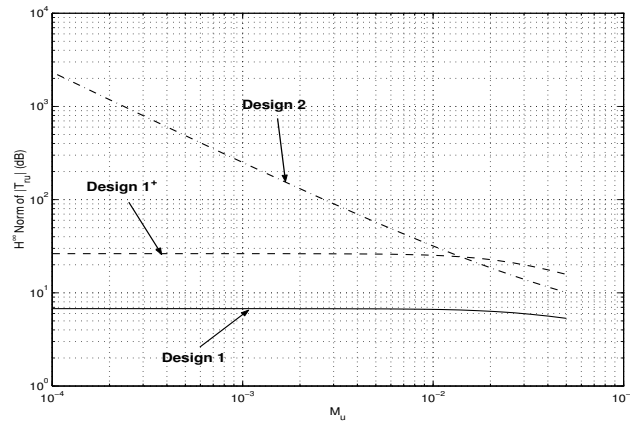


Figure 3.43: Dependence of $\|T_{ru}\|_{\mathcal{H}^\infty}$ on M_u

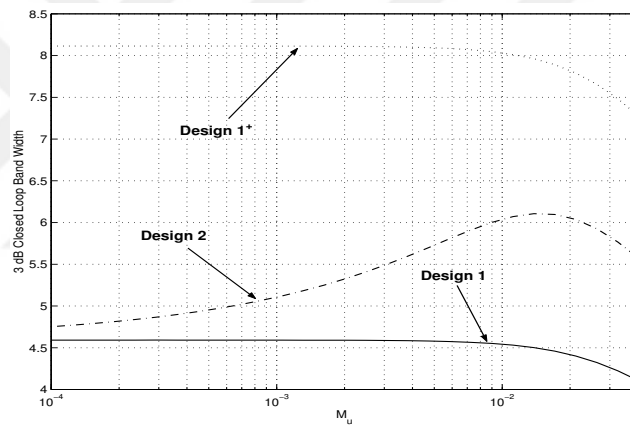


Figure 3.44: Dependence of 3 dB Closed Loop (T) Bandwidth on M_u

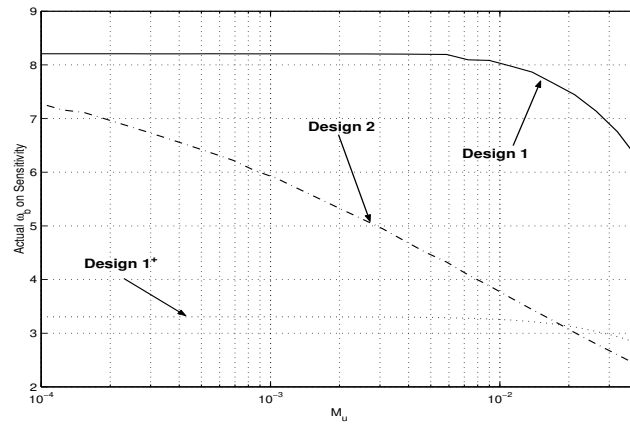


Figure 3.45: Dependence of Sensitivity Gain Crossover Frequency on M_u

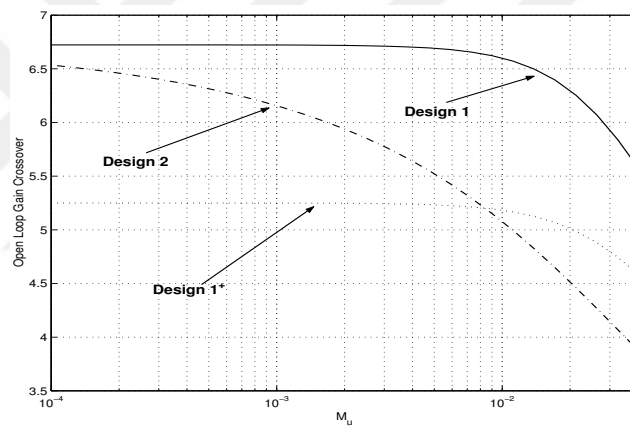


Figure 3.46: Dependence of Open Loop Gain Crossover on M_u

- **Optimization Norm Objective.** As M_u increases, the peak of KS gets smaller. This results in less control action or less bandwidth. Less bandwidth (gain) translates into a larger S and T . As a result, $\|T_{wz}\|_{\mathcal{H}^\infty}$ increases. This is seen in Figure 3.47 for Designs 1, 1^+ , and 2..

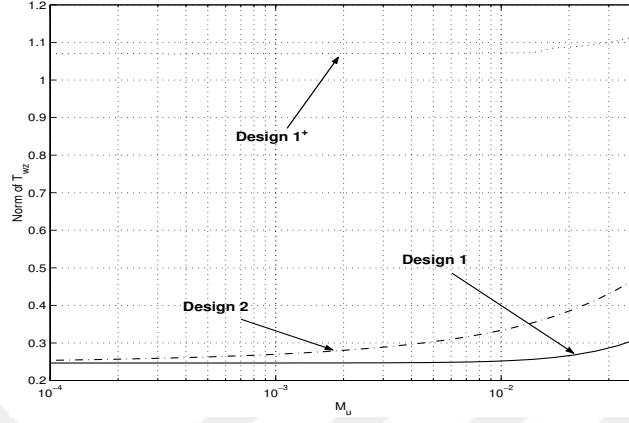


Figure 3.47: Dependence of $\|T_{wz}\|_{\mathcal{H}^\infty}$ on M_u

- **Stability Margins.** The dependence of gain and phase margins on M_u may be visualized as shown in Figures 3.48-3.49. For Design 1, the upward gain margin is infinite. For each design the downward gain margin is finite. For Design 2, the upward gain margin gets worse as M_u increases. For both designs, the downward gain margin and phase margin gets worse as M_u increases. This is expected, since increasing M_u generally reduces the lead provided by the compensator (see Figure 3.43). It also reduces the gain crossover frequency (see Figure 3.46). A consequence of this is reduced phase margin as illustrated in Figure 3.49.

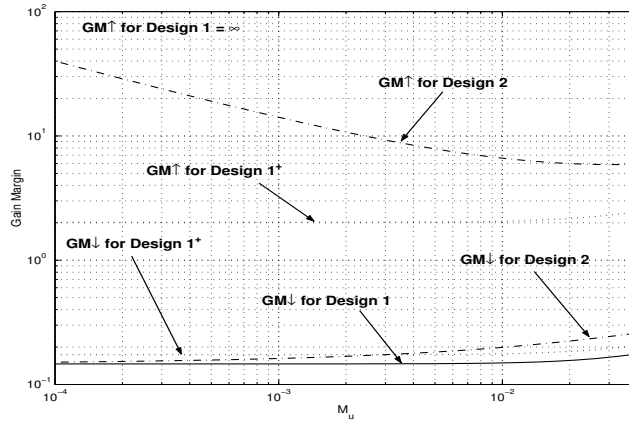
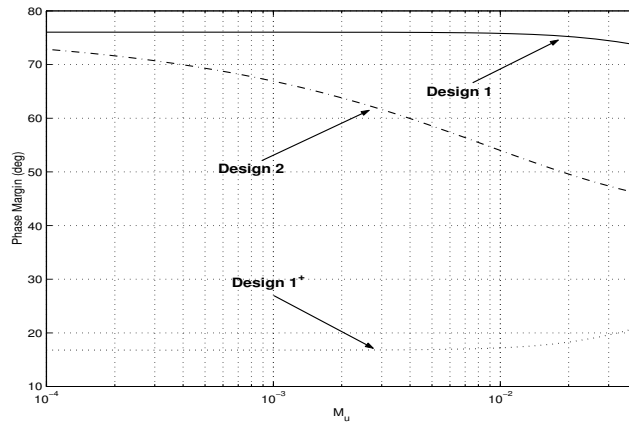


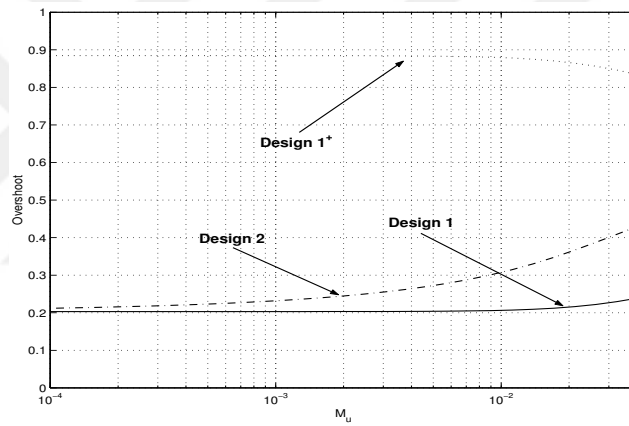
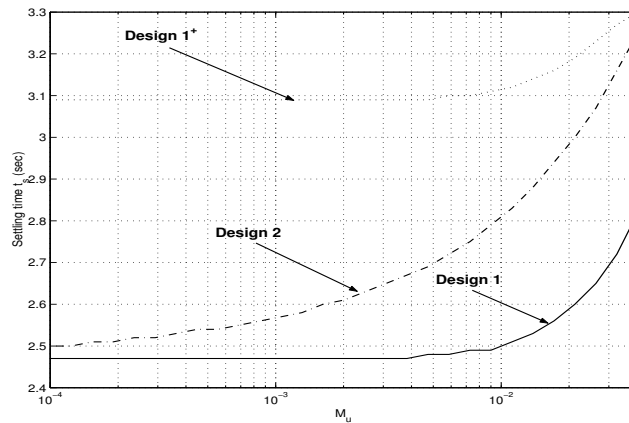
Figure 3.48: Dependence of Gain Margins on M_u

- **Overshoot and Settling Time.** As M_u increases, the overshoot associated with a step reference command (without prefilter) increases for Designs 1 and 2. This is seen in Figures 3.50. Increased overshoot - with increasing M_u - may be associated with degraded stability margins that results from M_u increasing (see Figures 3.48-3.49).

As M_u increases, the settling time associated with a step reference command (without prefilter) de-

Figure 3.49: Dependence of Phase Margin on M_u

creases for Designs 1 and 2. This is seen in Figures 3.51. Decreased settling time - with increasing M_u - may be associated with the lower bandwidth that results from M_u increasing (see Figure 3.44).

Figure 3.50: Dependence of Overshoot on M_u (without Prefilter)Figure 3.51: Dependence of Settling Time on M_u (without Prefilter)

It should be noted that when a dynamic control sensitivity weighting $W_2 = \frac{1}{\epsilon} \left[\frac{s + \omega_{bu} M_u}{s + \frac{\omega_{bu}}{\epsilon}} \right]$ is used, the patterns observed above for the parameter M_u (i.e. for a static weighting $W_2 = M_u$) will be very similar.

Dynamic Weighting W_2 : Impact of Control Sensitivity Bandwidth Parameter ω_{bu} . Sometimes, it is essential to use a dynamic weighting $W_2 = \frac{1}{\epsilon} \left[\frac{s + \frac{\omega_{bu}}{M_u}}{s + \frac{\omega_{bu}}{\epsilon}} \right]$ where $M_u > 0$, $\omega_{bu} > 0$, and $\epsilon > 0$ are design parameters used to shape the control sensitivity $|KS|$. Here $\epsilon > 0$ is typically very small - non-zero so that W_2 is proper. Here, we chose $\epsilon = 0.01$, $M_u = 100$, and we vary ω_{bu} . The impact of ω_{bu} can be visualized as shown in Figures 3.52-3.61.

- **Control Sensitivity Peak.** For Designs 1 and 2, $\|T_{ru}\|_{\mathcal{H}^\infty} = \|KS\|_{\mathcal{H}^\infty}$ increases and then levels off as ω_{bu} increases. This is seen in Figure 3.52. The leveling off for larger values of ω_{bu} is associated with the fact that $W_2 = \frac{1}{\epsilon} \left[\frac{s + \frac{\omega_{bu}}{M_u}}{s + \frac{\omega_{bu}}{\epsilon}} \right] \approx \frac{1}{M_u}$ is independent of ω_{bu} for ω_{bu} large. For Design 1⁺, $\|T_{ru}\|_{\mathcal{H}^\infty} = \|KS\|_{\mathcal{H}^\infty}$ increases, then decreases, and then levels off as ω_{bu} increases.

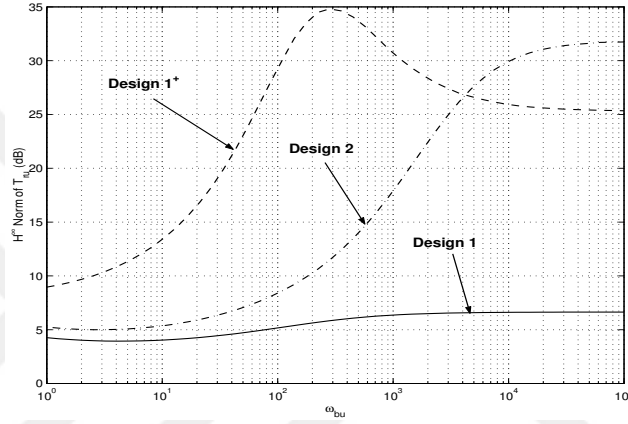


Figure 3.52: Dependence of $\|T_{ru}\|_{\mathcal{H}^\infty}$ on ω_{bu}

- **Bandwidth.** Figure 3.53 shows that as ω_{bu} increases, the (high frequency) unity gain crossover frequency for $T_{ru} = KS$ increases for all designs. This behavior is observed for the complementary sensitivity 3 dB frequency for Design 1⁺ (see Figure 3.54). For Designs 1 and 2, the complementary sensitivity 3 dB frequency increases, then decreases, and then levels off as ω_{bu} increases (see Figure 3.54). For all designs, the sensitivity and open loop unity gain crossover frequencies increase with ω_{bu} (see Figures 3.55-3.56).

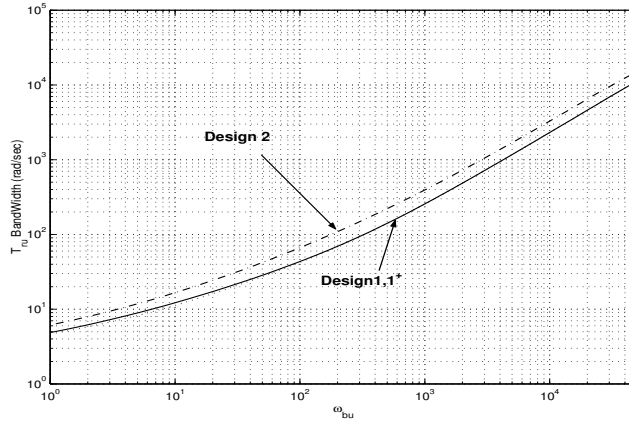


Figure 3.53: Dependence of T_{ru} Gain Crossover on ω_{bu}

- **Optimization Norm Objective.** Figure 3.57 shows the $\|T_{wz}\|_{\mathcal{H}^\infty}$ decreases with ω_{bu} for all designs.

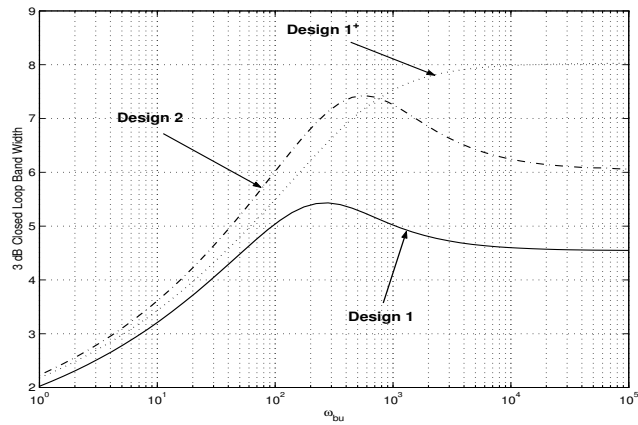


Figure 3.54: Dependence of 3 dB Closed Loop (T) Bandwidth on ω_{bu}

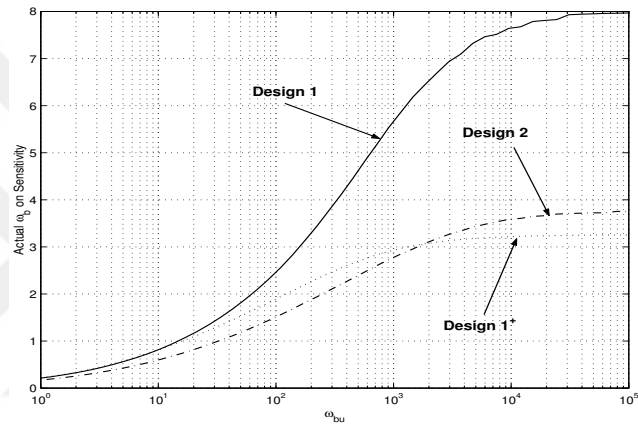


Figure 3.55: Dependence of Sensitivity Gain Crossover on ω_{bu}

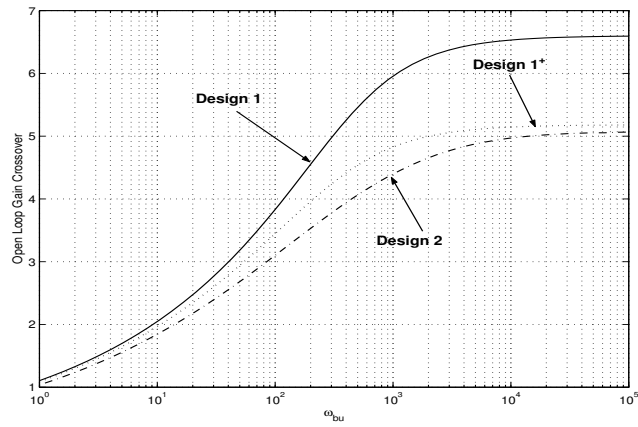


Figure 3.56: Dependence of Open Loop Gain Crossover on ω_{bu}

- **Stability Margins.** Figure 3.58 shows that for all designs, the downward gain margins improves and then levels off as ω_{bu} is increased. The upward gain margin for Design 1 is infinite. For the other designs, the margin worsens, then improves, and then levels off as ω_{bu} is increased. For all designs, the phase margin generally increases with ω_{bu} . The phase margin for Design 1⁺ experiences a slight dip.

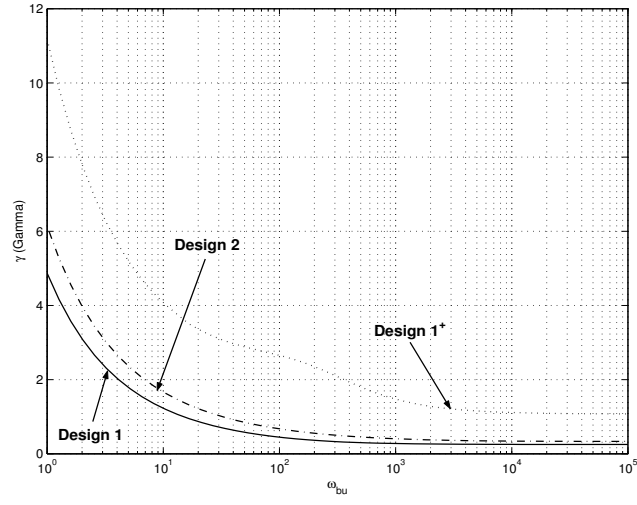


Figure 3.57: Dependence of $\|T_{wz}\|_{\mathcal{H}^\infty}$ on ω_{bu}

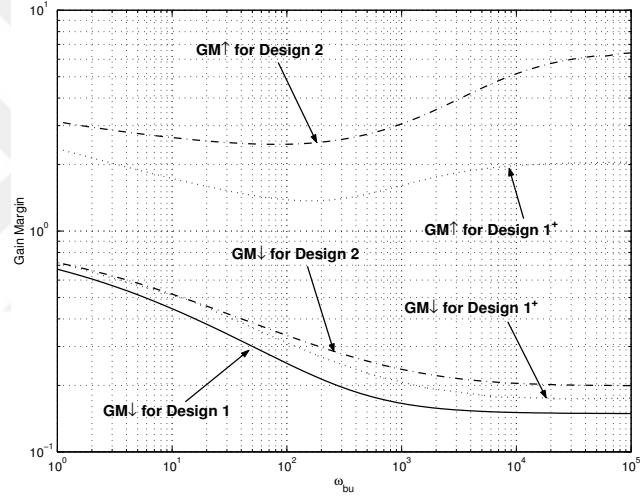


Figure 3.58: Dependence of Gain Margins on ω_{bu}

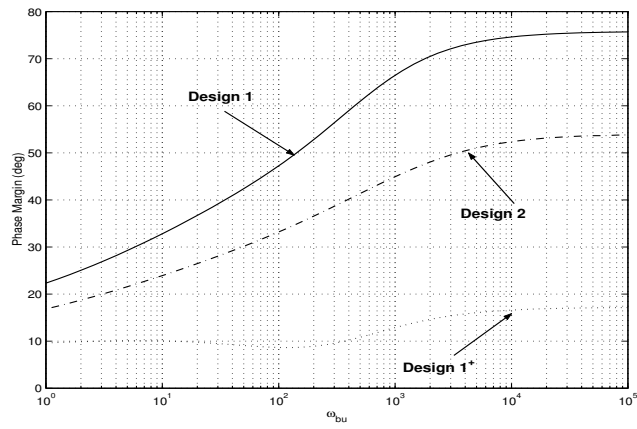


Figure 3.59: Dependence of Phase Margin on ω_{bu}

- **Overshoot and Settling Time.** Figures 3.60-3.61 shows that for Designs 1 and 2, the overshoot and settling time decrease with increasing ω_{bu} . While the overshoot for Design 1⁺ decreases, the settling

time exhibits a more complex (albeit generally decreasing) pattern.

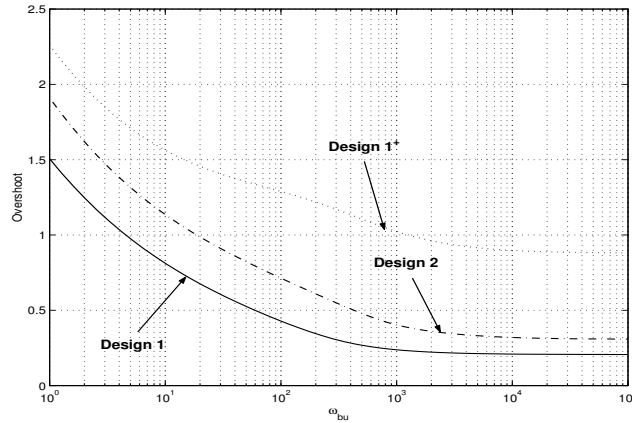


Figure 3.60: Dependence of Overshoot on ω_{bu}

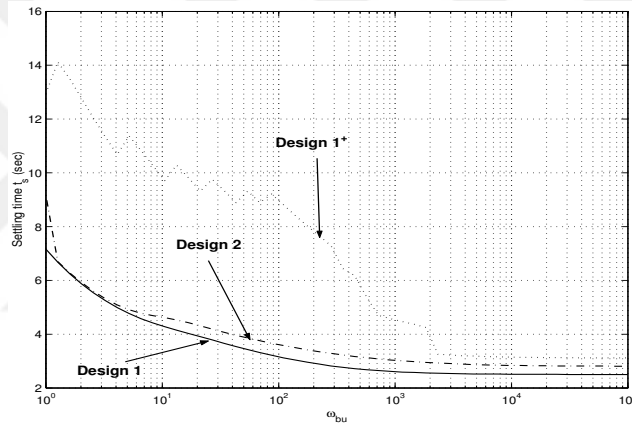


Figure 3.61: Dependence of Settling Time on ω_{bu}

■

3.5 SISO Unstable Plant with High Frequency Dynamics: Impact of Complementary Sensitivity Weighting Design Parameters

The following example shows how the complementary sensitivity weighting function design parameters may be used to influence a design.

Example 3.5.1 (Impact of Complementary Sensitivity Weighting Function Design Parameters) *The impact of key design parameters associated with the complementary sensitivity weighting function W_3 is now examined.*

Impact of Complementary Sensitivity Bandwidth Parameter ω_{bc} . The impact of the complementary sensitivity bandwidth parameter ω_{bc} on each of the designs may be visualized as shown in Figures 3.62-3.69.

- **Bandwidth.** As ω_{bc} is increased, the bandwidth of Designs 1 and 2 increase - leveling off for large ω_{bc} . This is seen in Figures 3.62-3.64. While the same trend holds for Design 1⁺, instability will result if ω_{bc} is increased too much. (This was also true for the sensitivity bandwidth parameter ω_b . It was not true for the control sensitivity bandwidth parameter ω_{bu} .) Figure 3.66, for example, show that the upward gain margin and phase margin for Design 1⁺ worsen as ω_{bc} increases.

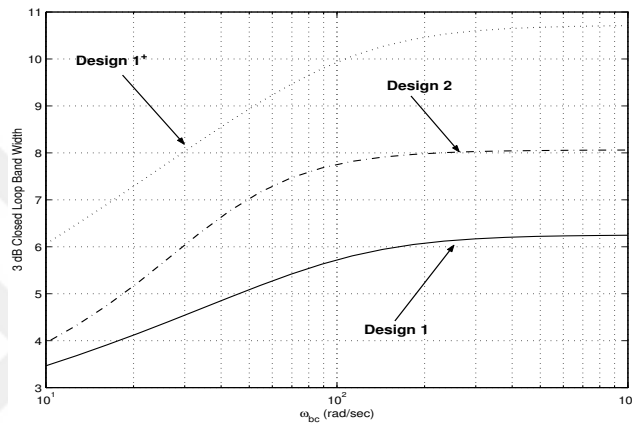


Figure 3.62: Dependence of 3 dB Closed Loop (T) Bandwidth on ω_{bc}

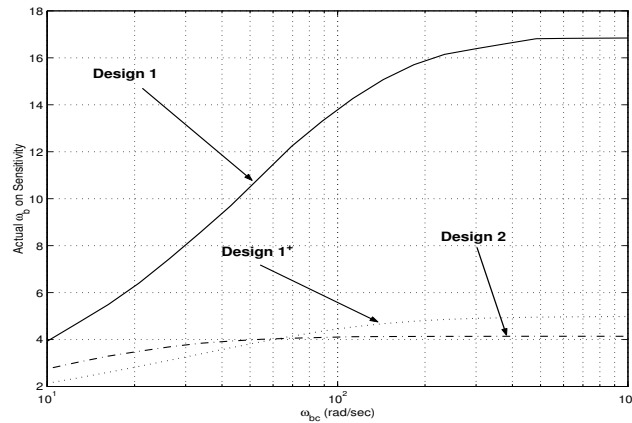


Figure 3.63: Dependence of Sensitivity Gain Crossover on ω_{bc}

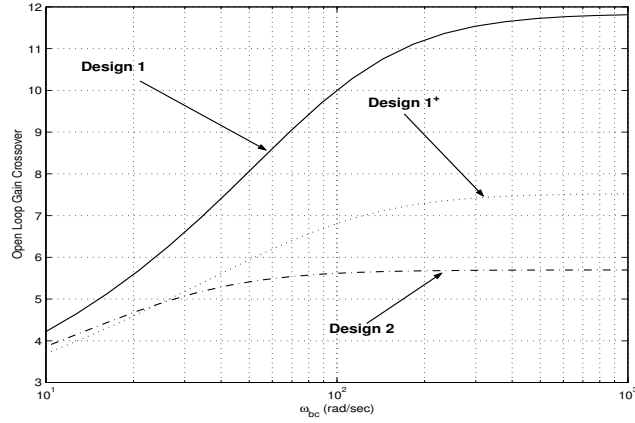


Figure 3.64: Dependence of Open Loop Gain Crossover on ω_{bc}

- Optimization Norm Objective. Figure 3.65 shows that for Designs 1 and 2, $\|T_{wz}\|_{\mathcal{H}^\infty}$ decreases with ω_{bc} . For Design 1⁺, $\|T_{wz}\|_{\mathcal{H}^\infty}$ dips slightly and then increases with ω_{bc} .

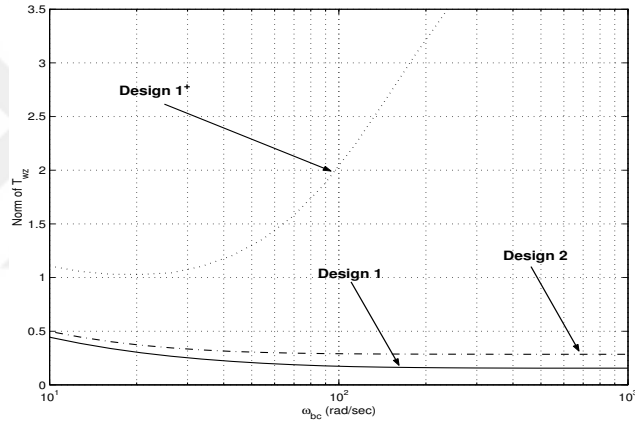
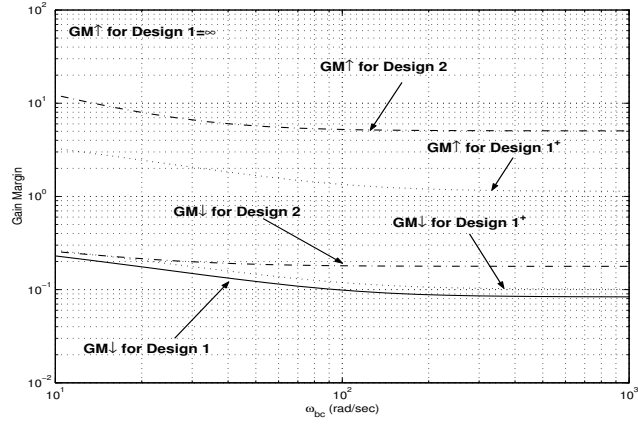
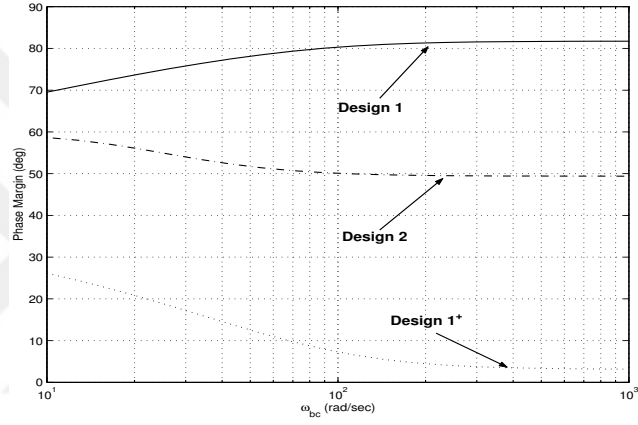
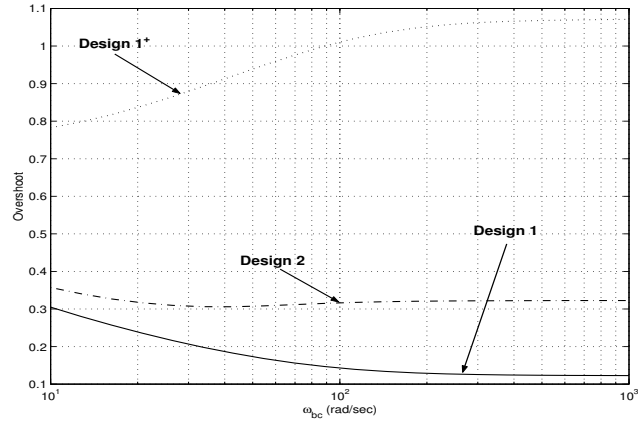


Figure 3.65: Dependence of $\|T_{wz}\|_{\mathcal{H}^\infty}$ on ω_{bc}

- Stability Margins. Figure 3.66 shows that the downward gain margins decrease with ω_{bc} - leveling off for large ω_{bc} . The upward gain margin for Design 1 is infinite. The upward gain margins for Design 2 decreases with ω_{bc} - leveling off for large ω_{bc} . The upward gain margins for Design 1⁺ decreases with ω_{bc} - approaching unity for large ω_{bc} . This implies that Design 1⁺ goes unstable for large ω_{bc} . The figure also shows that the phase margin for Design 1 increases with ω_{bc} - leveling off for large ω_{bc} . The phase margin for Designs 1⁺ and 2 decreases with ω_{bc} - leveling off for large ω_{bc} . For Design 1⁺, the phase margin gets small for large ω_{bc} .
- Overshoot and Settling Time. Consider the response to step reference command (without prefilter). The overshoot and settling time - for Design 1⁺ - increases significantly if ω_{bc} is increased too much. This is seen in Figures 3.68-3.69. For Design 1, the overshoot decreases with ω_{bc} - leveling off for large ω_{bc} . For Design 2, it dips slightly, then rises, then levels off for large ω_{bc} . The settling time decreases with ω_{bc} for both Designs 1 and 2.

Impact of Complementary Sensitivity Bound Parameter M_y . The impact of the low frequency

Figure 3.66: Dependence of Gain Margins on ω_{bc} Figure 3.67: Dependence of Phase Margin on ω_{bc} Figure 3.68: Dependence of Overshoot on ω_{bc} (without Prefilter)

complementary sensitivity gain parameter M_y on each of the designs may be visualized as shown in Figures 3.70-3.77.

- Bandwidth. Figures 3.70-3.72 show that bandwidth is inversely with M_y ; i.e. as M_y decreases (increases), the bandwidth increases (decreases).

As M_y is increased, the bandwidth of Designs 1 and 2 decrease - leveling off for large M_y . For large

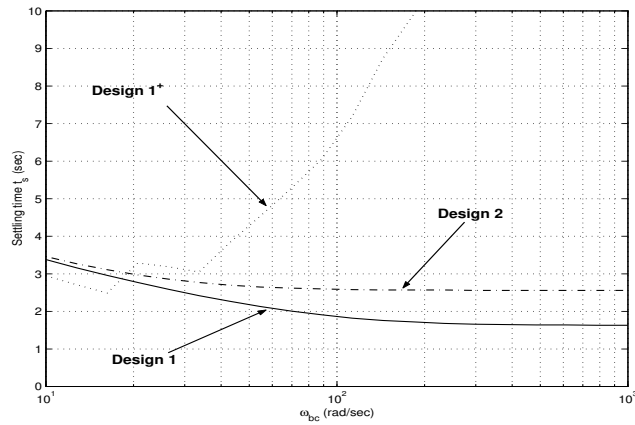


Figure 3.69: Dependence of Settling Time on ω_{bc} (without Prefilter)

M_y , the parameter is essentially insignificant.

In these figures, no data is provided for $M_y < 2$. This is because for $M_y < 2$, the resulting controller for Design 1 has so high a bandwidth that Design 1+ (which includes high frequency dynamics after the design process) is unstable. This is corroborated by a unity upward gain margin and zero phase margin for Design 1+ in Figure 3.74 as M_y approaches 2 from above.

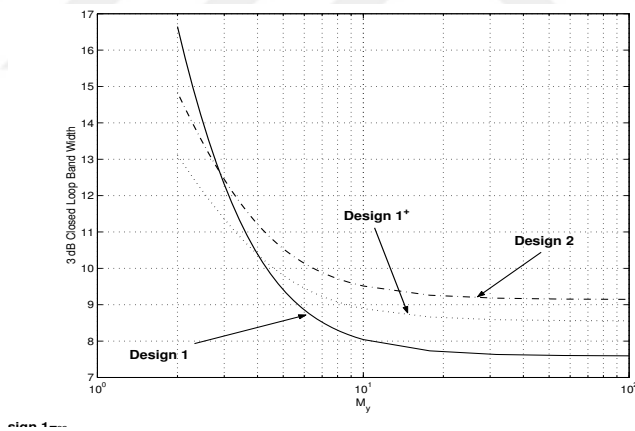


Figure 3.70: Dependence of 3 dB Closed Loop (T) Bandwidth on M_y

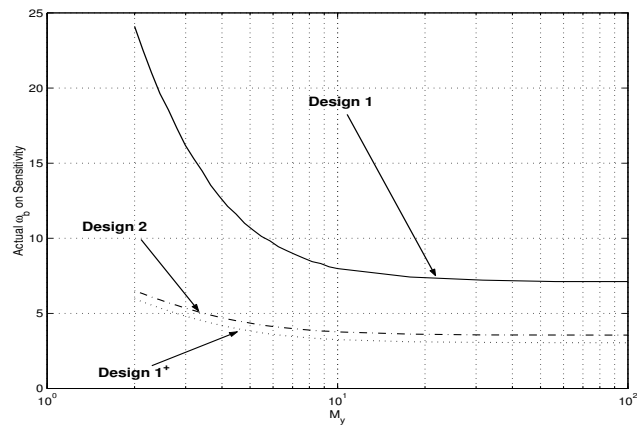


Figure 3.71: Dependence of Sensitivity Gain Crossover Frequency on M_y

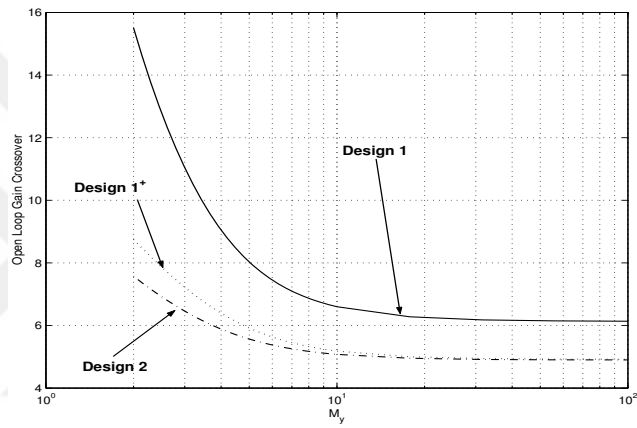


Figure 3.72: Dependence of Open Loop Gain Crossover on M_y

- Optimization Norm Objective. As M_y increases, $\|T_{wz}\|_{\mathcal{H}^\infty}$ decreases (for all designs) - leveling off for large M_y . As M_y decreases, $\|T_{wz}\|_{\mathcal{H}^\infty}$. For Design 1⁺, the norm increases significantly - approaching ∞ as M_y approaches 2 from above.

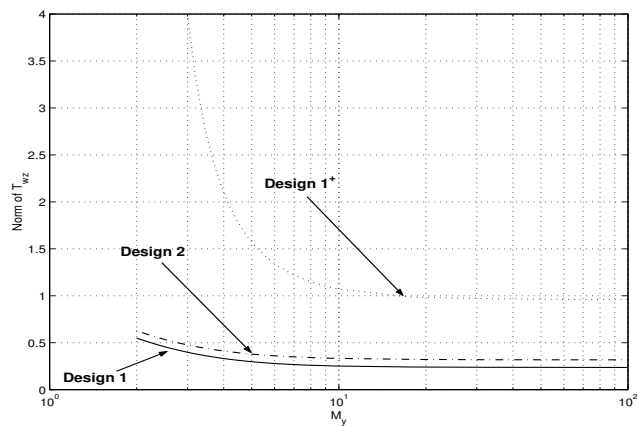


Figure 3.73: Dependence of $\|T_{wz}\|_{\mathcal{H}^\infty}$ on M_y

- Stability Margins. Figure 3.74 shows that all downward gain margins worsen as M_y increases -

leveling off for large M_y . This is consistent with the fact that as M_y increases, the bandwidth (gain) decreases. The upward gain margin for Design 1 is infinite. As M_y increases, the upward gain margins for Designs 1^+ and 2 increase - leveling off for large M_y . As M_y increases, the phase margins for Designs 1 and 2 decrease - leveling off for large M_y . As M_y increases, the phase margin for Design 1^+ increases - leveling off for large M_y . As M_y approaches 2 from above, the phase margin for Design 1^+ decreases toward zero.

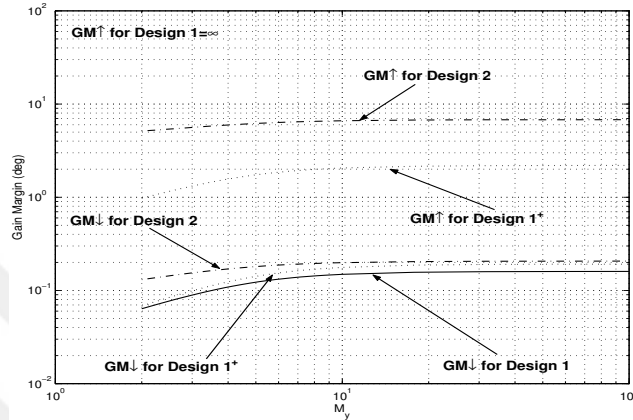


Figure 3.74: Dependence of Gain Margins on M_y

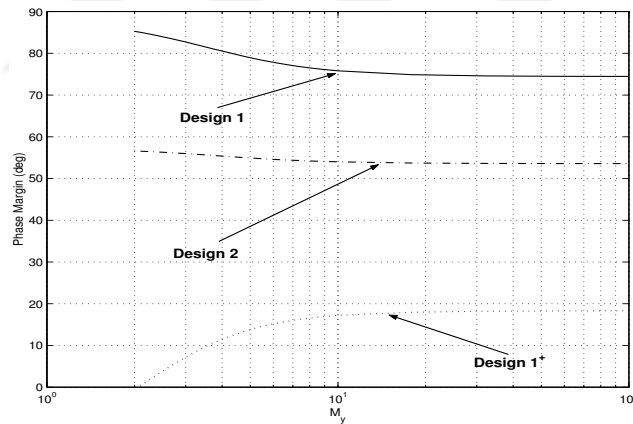


Figure 3.75: Dependence of Phase Margin on M_y

- **Overshoot and Settling Time.** Figure 3.76 shows that for Designs 1 and 2, the overshoot to a step reference command (without prefilter) decreases with decreasing M_y . It increases with M_y - leveling off for large M_y . For Design 1^+ , the overshoot increases significantly as M_y decreases toward 2. The same holds for Design 1^+ settling time. The settling time for Design 1 decreases as M_y decreases toward 2, while that for Design 2 increases.

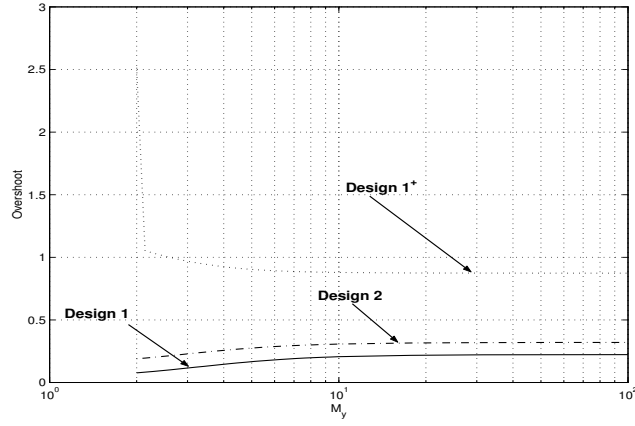


Figure 3.76: Dependence of Overshoot on M_y (without Prefilter)

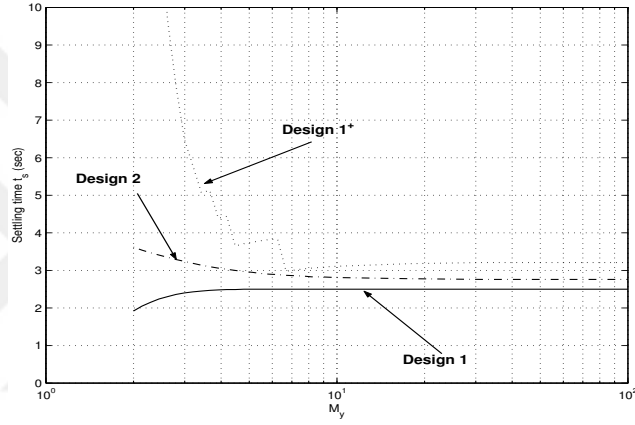


Figure 3.77: Dependence of Settling Time on M_y (without Prefilter)

3.6 SISO Unstable Plant with High Frequency Dynamics: Use of Bilinear Transformation

The following example shows how the bilinear transformation may be used to influence $\mathcal{H}^\infty$ control system designs.

Example 3.6.1 ($\mathcal{H}^\infty$ Design for First Order Unstable Plant: Use of Bilinear Transformation)

In this example, we consider weighted $\mathcal{H}^\infty$ mixed-sensitivity designs for the unstable plant $P = \frac{1}{s-1}$ considered in Example 3.2.1. The weighting functions W_1 , W_2 , and W_3 used in this example are identical to those used in Example 3.2.1 (see page 35). The purpose of this example is to show how the bilinear transformation

$$s = \frac{\hat{s} + p_1}{\frac{\hat{s}}{p_2} + 1} \quad \hat{s} = \frac{s - p_1}{1 - \frac{s}{p_2}} \quad (3.41)$$

where $p_1, p_2 \in \mathcal{R}$, may be used to influence the final control system design.

Bilinear Transformation Parameters Used. *The bilinear transformation parameters were selected as follows:*

$$p_1 = -1.5 \quad (3.42)$$

$$p_2 = -10^{20}. \quad (3.43)$$

This selection yields

$$s = \frac{\hat{s} + p_1}{\frac{\hat{s}}{p_2} + 1} = \frac{\hat{s} - 1.5}{\frac{\hat{s}}{-10^{20}} + 1} \approx \hat{s} - 1.5 \quad (\text{inverse transformation}) \quad (3.44)$$

$$\hat{s} \approx s + 1.5. \quad (\text{transformation}) \quad (3.45)$$

This selection of parameters results in a rightward shifting transformation $\hat{s} \approx s + 1.5$ and a leftward shifting transformation $s \approx \hat{s} - 1.5$. The selection thus shifts all plant poles and zeros 1.5 units to the right. The transformed plant is combined with the weighting functions to form the generalized plant G . (Note that the transformation is NOT applied to the weighting functions!) Designs are based on the resulting generalized plant G . After a design $\hat{K}$ is obtained, the inverse transform is applied to $\hat{K}$ - shifting its poles and zeros in the $\hat{s}$ plane toward the left to get poles and zeros in the s -plane. Since $\hat{K}$ internally stabilized $\hat{P}$, the resulting controller K is guaranteed to stabilize the original plant P .

Design 1. *The first design is obtained using*

$$P = \frac{1}{s - 1} \quad (3.46)$$

as the plant model. The design process is initiated by transforming the generalized plant into the $\hat{s}$ -plane using the above bilinear transformation. After this is done, a design is obtained. The resulting $\hat{s}$ -plane design is then transformed back into the s -plane. The resulting minimum γ (optimality tolerance of 0.0001) was found to be

$$\gamma = 0.3133. \quad (3.47)$$

The resulting s -domain controller is given by

$$K = \frac{2780879.8595(s + 2.494)}{(s + 1.51)(s + 3.26 \times 10^5)} \approx \frac{8.5303(s + 2.494)}{(s + 1.51)}. \quad (3.48)$$

The associated complementary sensitivity is given by

$$T = \frac{2583506.3856(s + 2.494)}{(s + 5.34)(s + 3.701)(s + 3.26 \times 10^5)} \approx \frac{7.9249(s + 2.494)}{(s + 5.34)(s + 3.701)}. \quad (3.49)$$

Design 1⁺. When the above control system K is evaluated with the higher order model

$$P = \frac{1}{s-1} \left[\frac{10}{s+10} \right]^2, \quad (3.50)$$

we refer to the design as Design 1⁺. The associated complementary sensitivity is given by

$$T = \frac{258350638.5636(s+2.494)}{(s+16.71)(s+2.789)(s+3.26 \times 10^5)(s^2+1.009s+42.4)} \quad (3.51)$$

$$\approx \frac{792.4866(s+2.494)}{(s+16.71)(s+2.789)(s^2+1.009s+42.4)} \quad (3.52)$$

$$\approx \frac{18.6907(s+2.494)}{(s+16.71)(s+2.789)} \left[\frac{42.4}{s^2+1.009s+42.4} \right]. \quad (3.53)$$

Design 2. The second design is obtained using the full order plant model

$$P = \frac{1}{s-1} \left[\frac{10}{s+10} \right]^2. \quad (3.54)$$

The resulting minimum γ was found to be

$$\gamma = 0.4541. \quad (3.55)$$

The resulting controller is given by

$$K = \frac{7815798.2793(s+2.354)(s+10)^2}{(s+1.764 \times 10^5)(s+1.51)(s^2+41.52s+727)} \quad (3.56)$$

$$\approx \frac{44.3072(s+2.354)(s+10)^2}{(s+1.51)(s^2+41.52s+727)} \quad (3.57)$$

$$\approx \frac{6.0945(s+2.354)}{(s+1.51)} \left[\frac{s+10}{10} \right]^2 \left[\frac{727}{s^2+41.52s+727} \right]. \quad (3.58)$$

The associated complementary sensitivity is given by

$$T = \frac{699314887.3915(s+2.354)}{(s+1.764 \times 10^5)(s+6.001)(s+3.817)(s^2+32.22s+407.4)} \quad (3.59)$$

$$\approx \frac{3964.3701(s+2.354)}{(s+6.001)(s+3.817)(s^2+32.22s+407.4)} \quad (3.60)$$

$$\approx \frac{9.7309(s+2.354)}{(s+6.001)(s+3.817)} \left[\frac{407.4}{s^2+32.22s+407.4} \right]. \quad (3.61)$$

■

Comment 3.6.1 (Use of Bilinear Transformation and Shaping of S , KS , and T)

When the bilinear transformation is applied to our $\mathcal{H}^\infty$ mixed sensitivity problem, we obtain

$$\|T_{wz}\|_{\mathcal{H}^\infty} = \left\| \begin{bmatrix} W_1 \hat{S} \\ W_2 \hat{K} \hat{S} \\ W_3 \hat{T} \end{bmatrix} \right\|_{\mathcal{H}^\infty} < \gamma \quad (3.62)$$

where $\hat{S} = [I + \hat{P}\hat{K}]$ is the transformed sensitivity, $\hat{K}\hat{S} = \hat{K}[I + \hat{P}\hat{K}]$ is the transformed control sensitivity, and $\hat{T} = I - \hat{S} = \hat{P}\hat{K}[I + \hat{P}\hat{K}]$ is the transformed complementary sensitivity. This implies that

$$|\hat{S}| < \gamma |W_1^{-1}| \quad |\hat{K}\hat{S}| < \gamma |W_2^{-1}| \quad |\hat{T}| < \gamma |W_3^{-1}| \quad (3.63)$$

on the imaginary axis in the $\hat{s}$ plane. These very useful inequalities - the key to shaping closed loop frequency responses - do NOT hold for S , KS , and T ! Transforming back into the s -plane yields the following inequalities:

$$|S| < \gamma |\hat{W}_1^{-1}| \quad |KS| < \gamma |\hat{W}_2^{-1}| \quad |T| < \gamma |\hat{W}_3^{-1}| \quad (3.64)$$

on the imaginary axis in the s plane, where

$$\hat{W}_i(s) = W_i \left(\frac{s - p_1}{1 - \frac{s}{p_2}} \right) \quad (3.65)$$

for $i = 1, 2, 3$. These inequalities are not very insightful for frequency response shaping! A consequence of this is the following:

Although using the bilinear transformation preserves stability,
our ability to systematically shape S , KS , and T is lost!

Trial and error tuning - carefully observing tradeoffs - is often required when the bilinear transformation is used.

Effective Shaping Functions. Since our W_i have the form

$$W_i(\hat{s}) = k \begin{bmatrix} \hat{s} + a \\ \hat{s} + b \end{bmatrix} \quad (3.66)$$

we obtain

$$\hat{W}_i(s) = k \begin{bmatrix} \hat{s} + a \\ \hat{s} + b \end{bmatrix}_{\hat{s} = \frac{s - p_1}{1 - \frac{s}{p_2}}} \quad (3.67)$$

$$= k \begin{bmatrix} p_2 - a \\ p_2 - b \end{bmatrix} \begin{bmatrix} s + \frac{a - p_1}{1 - \frac{a}{p_2}} \\ \frac{b - p_1}{s + \frac{b - p_1}{1 - \frac{b}{p_2}}} \end{bmatrix}. \quad (3.68)$$

Application to our $\mathcal{H}^\infty$ mixed sensitivity weighting functions yields

$$\hat{W}_1(s) = \frac{1}{M_s} \left[\frac{p_2 - M_s \omega_b}{p_2 - \epsilon \omega_b} \right] \left[\frac{s + \frac{M_s \omega_b - p_1}{1 - \frac{M_s \omega_b}{p_2}}}{s + \frac{\epsilon \omega_b - p_1}{1 - \frac{M_s \omega_b}{p_2}}} \right] \quad (3.69)$$

$$\hat{W}_2(s) = \frac{1}{\epsilon} \left[\frac{p_2 - M_u \omega_{bu}}{p_2 - \frac{\omega_{bu}}{\epsilon}} \right] \left[\frac{s + \frac{M_u \omega_{bu} - p_1}{1 - \frac{M_u \omega_{bu}}{p_2}}}{s + \frac{\frac{\omega_{bu}}{\epsilon} - p_1}{1 - \frac{\epsilon}{p_2}}} \right] \quad (3.70)$$

$$\hat{W}_3(s) = \frac{1}{\omega_{bc}} \left[\frac{p_2 - \frac{\omega_{bc}}{M_y}}{p_2 - s} \right] \left[\frac{s + \frac{\frac{\omega_{bc}}{M_y} - p_1}{1 - \frac{M_y}{p_2}}}{p_2 - s} \right]. \quad (3.71)$$

When p_2 is a large (negative) quantity, the above simplify significantly. ■

The following example shows how the bilinear transformation “shift parameter” p_1 - a time constant parameter - may be used to control the bandwidth of the resulting design.

Example 3.6.2 (Impact of Bilinear Transformation Shift Parameter p_1 .)

In this example, we examine the impact of the bilinear transformation shift parameter p_1 . Nominally, we have $p_1 = -1.5$ and $p_2 = -10^{20}$ as in Example 3.6.1.

Figures 3.78- 3.88 shows the impact of the bilinear transformation on the resulting controller and closed loop system properties as the transformation parameter p_1 is varied between -5.0 and 0 . $p_1 = 0$ corresponds to the design obtained in Example 3.2.1.

- Bandwidth. As p_1 is made more negative, the bandwidth increases. This is seen in Figure 3.78-3.80.

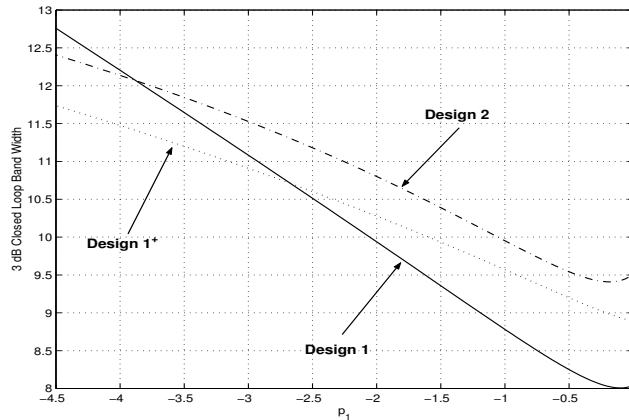
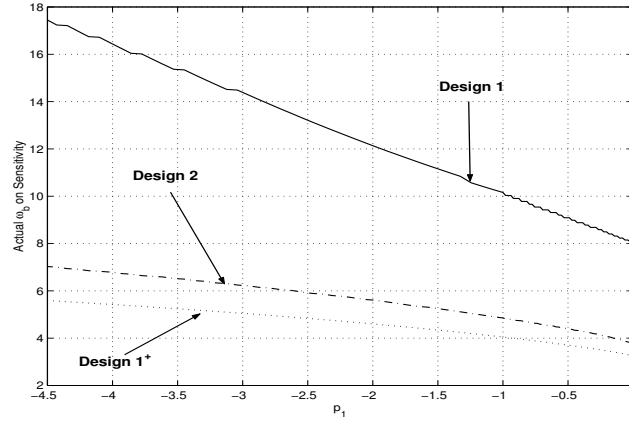
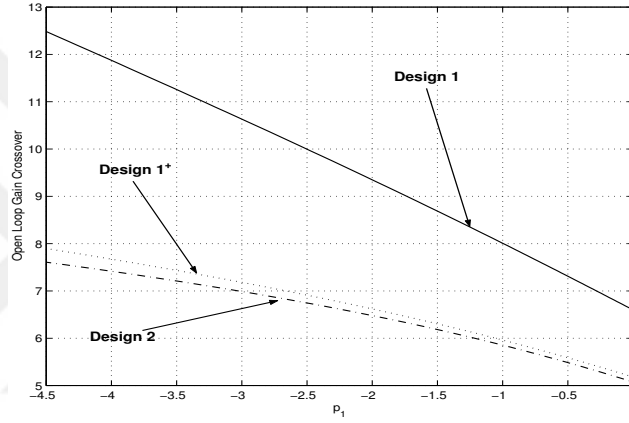
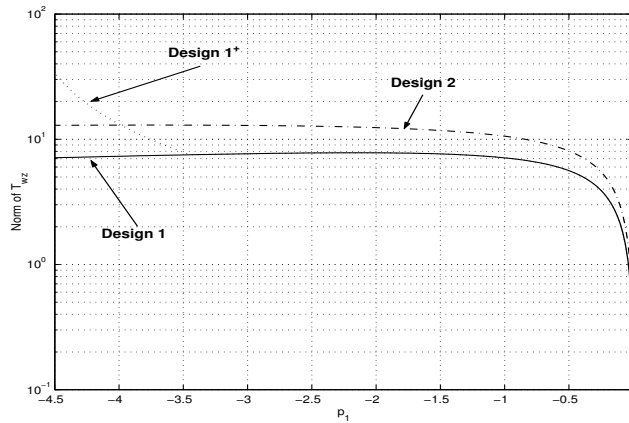


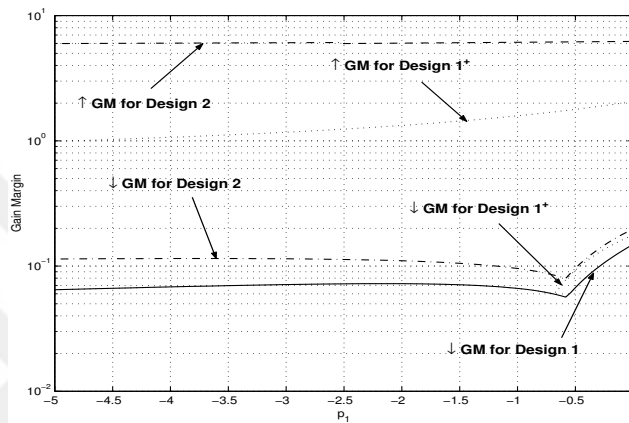
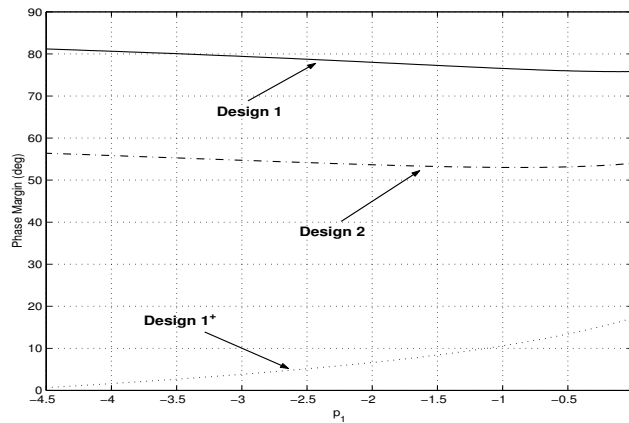
Figure 3.78: Dependence of 3 dB Closed Loop (T) Bandwidth on p_1

Figure 3.79: Dependence of Sensitivity Gain Crossover on p_1 Figure 3.80: Dependence of Open Loop Gain Crossover on p_1

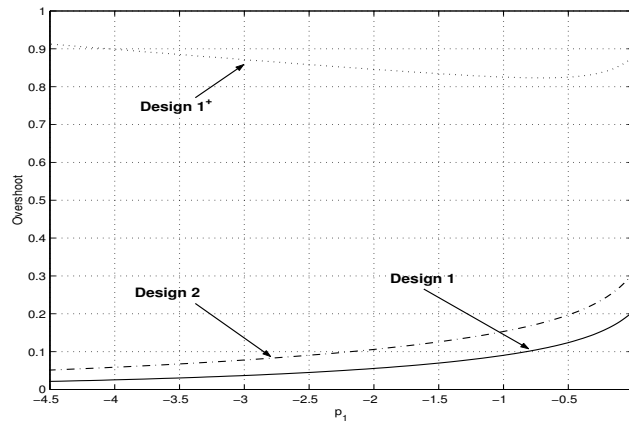
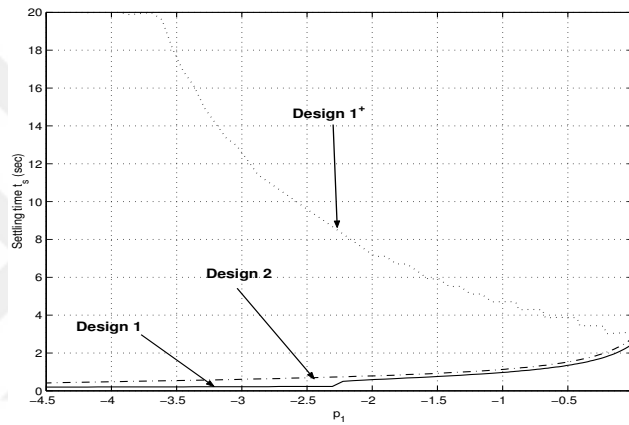
- Optimization Norm Objective. Figure 3.81 shows that as p_1 becomes more negative, $\|T_{wz}\|_{\mathcal{H}^\infty}$ increases - leveling off as p_1 becomes very negative for Designs 1 and 2. For Design 1⁺, $\|T_{wz}\|_{\mathcal{H}^\infty}$ increases significantly as p_1 becomes very negative. This is because the design becomes unstable. This is seen in Figure 3.82 which shows the upward gain margin approaching unity and the phase margin approaching zero as p_1 becomes very negative (for Design 1⁺).

Figure 3.81: Dependence of $\|T_{wz}\|_{\mathcal{H}^\infty}$ on p_1

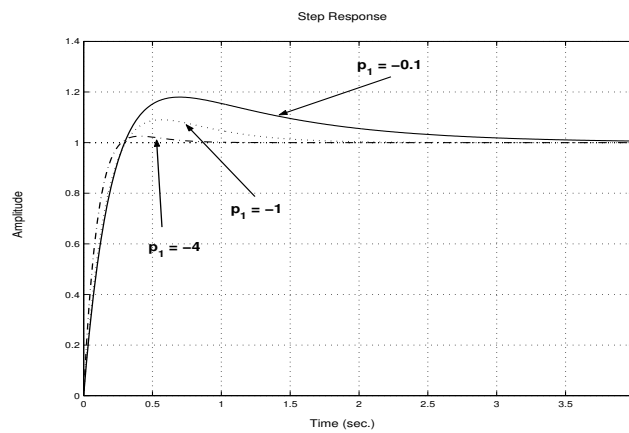
- **Stability Margins.** Figure 3.82 shows that downward gain margins decreasing, increasing, and then leveling off as p_1 becomes more negative. The upward gain margin for Design 1 is infinite. The upward gain margin for Design 2 decreases as p_1 becomes more negative. This is consistent with an increasing gain crossover frequency. The upward gain margin for Design 1⁺ decreases toward unity as p_1 becomes more negative; i.e. Design 1⁺ becomes unstable as p_1 becomes more negative. The phase margins for Designs 1 and 2 increase (slowly) as p_1 becomes more negative. The phase margin for Designs 1⁺ decreases toward zero as p_1 becomes more negative; i.e. Design 1⁺ becomes unstable as p_1 becomes more negative.

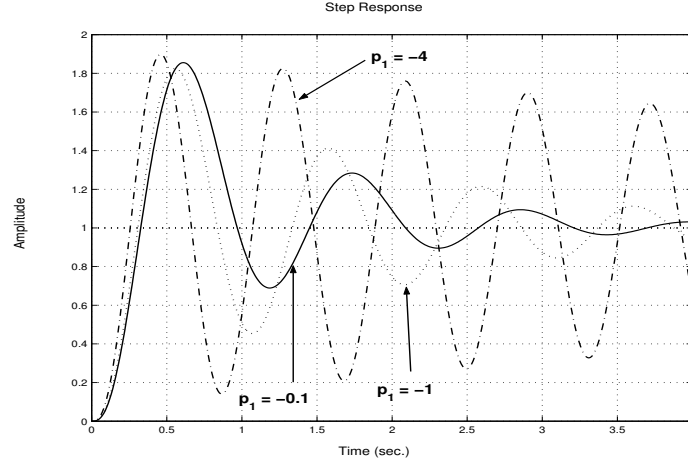
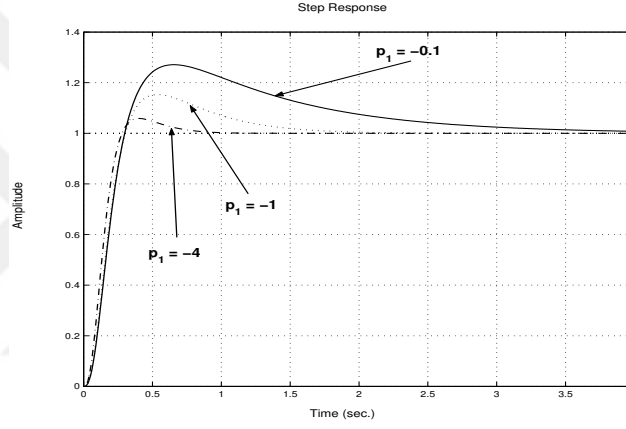
Figure 3.82: Dependence of Gain Margins on p_1 Figure 3.83: Dependence of Phase Margin on p_1

- **Overshoot and Settling Time.** As p_1 is made more negative, the overshoot to a step reference command decreases for Design 1. This is seen in Figures 3.84 and 3.87. Figures 3.84 and 3.87 show that as p_1 is made more and more negative, the overshoot for Design 1⁺ decreases and then increases. If p_1 is made sufficiently negative, Design 1⁺ goes unstable. As p_1 is made more negative, the overshoot to a step reference command decreases for Design 2. This is seen in Figures 3.84 and 3.88. As p_1 is made more negative, the settling time for Designs 1 and 2 decreases, while that for Design 1⁺ increases.
- **Step Responses.** Figure 3.87 shows the overshoot and settling time for Design 1 decreasing as p_1 is

Figure 3.84: Dependence of Overshoot on p_1 Figure 3.85: Dependence of Settling Time on p_1

made more negative. It also shows that the overshoot for Design 1⁺ decreases and then increases. This is consistent with Figure 3.84. Figure 3.88 shows that the overshoot and settling time for Design 2 decreases as p_1 is made more negative. This is consistent with Figures 3.84 and 3.85.

Figure 3.86: Step Responses for Design 1 - Dependence on p_1

Figure 3.87: Step Responses for Design 1⁺ - Dependence on p_1 Figure 3.88: Step Responses for Design 2 - Dependence on p_1

3.7 SISO Unstable Plant with Time Delay

The following example considers $\mathcal{H}^\infty$ control system design for an unstable plant with a time delay. Its purpose is to provide another illustrative example.

Example 3.7.1 ($\mathcal{H}^\infty$ Design for First Order Unstable Plant with a Time Delay)

In this example, we consider a plant

$$P = \frac{1}{s-1} e^{-s\Delta} \quad (3.72)$$

where $\Delta = 0.1$ represents a time delay. The time delay is approximated as follows:

$$e^{-\Delta s} = \frac{e^{-\frac{\Delta}{2}s}}{e^{\frac{\Delta}{2}s}} \approx \frac{1 - \frac{\Delta}{2}s}{1 + \frac{\Delta}{2}s} = \frac{\frac{2}{\Delta} - s}{s + \frac{2}{\Delta}} = \frac{20 - s}{s + 20}. \quad (3.73)$$

This approximation to the delay is a first order Pade' approximation. Two designs are obtained as in Example 3.2.1. The weighting functions W_1 , W_2 , and W_3 used in Example 3.2.1 will be used in this example.

Design 1. The first design is obtained using the first order plant model

$$P = \frac{1}{s-1}. \quad (3.74)$$

This model contains no information about the time delay. The resulting minimum γ (with an optimality tolerance of $\text{tol} = 0.0001$) is

$$\gamma = 0.2519. \quad (3.75)$$

The resulting controller is given by

$$K = \frac{3393773.861(s+0.6534)}{(s+0.01)(s+5.108 \times 10^5)} \approx \frac{6.644(s+0.6534)}{(s+0.01)}. \quad (3.76)$$

Corresponding to this controller is the following complementary sensitivity:

$$T = \frac{3385956.0265(s+0.6534)}{(s+4.74)(s+0.9138)(s+5.108 \times 10^5)} \approx \frac{6.6287(s+0.6534)}{(s+4.74)(s+0.9138)}. \quad (3.77)$$

Design 1⁺. When the above controller K is evaluated with the higher order plant model (containing a first order approximation to the time delay)

$$P = \frac{1}{s-1} \left[\frac{20-s}{s+20} \right] \quad (3.78)$$

the design is referred to as Design 1⁺. The resulting complementary sensitivity is given by

$$T = \frac{3385956.0265(s+0.6534)(20-s)}{(s+5.108 \times 10^5)(s+0.8782)(s^2+11.49s+98.63)} \quad (3.79)$$

$$\approx \frac{6.6287(s+0.6534)(20-s)}{(s+0.8782)(s^2+11.49s+98.63)}. \quad (3.80)$$

Design 2. The second design is obtained using

$$P = \frac{1}{s-1} \left[\frac{20-s}{s+20} \right]. \quad (3.81)$$

This model contains a first order Pade' approximation to the time delay. The resulting minimum γ is

$$\gamma = 0.3516. \quad (3.82)$$

The resulting controller is given by

$$K = \frac{272170037.3032(s+0.6008)(s+20)}{(s+39.06)(s+0.01)(s+2.81 \times 10^7)} \approx \frac{9.6858(s+0.6008)(s+20)}{(s+39.06)(s+0.01)}. \quad (3.83)$$

Corresponding to this controller is the following complementary sensitivity:

$$T = \frac{271256483.6998(20-s)(s+0.6008)}{(s+21.78)(s+5.667)(s+0.9398)} \quad (3.84)$$

$$\approx \frac{9.6533(20-s)(s+0.6008)}{(s+21.78)(s+5.667)(s+0.9398)} \quad (3.85)$$

$$\approx \frac{8.8640(s+0.6008)}{(s+5.667)(s+0.9398)} \left[\frac{20-s}{20} \right] \left[\frac{21.78}{s+21.78} \right]. \quad (3.86)$$

Frequency Responses. Figures 3.89-3.96 contain relevant frequency responses. It should be noted that right half plane (non-minimum phase) zero at $s = 20$ limits the achievable loop gain (bandwidth).

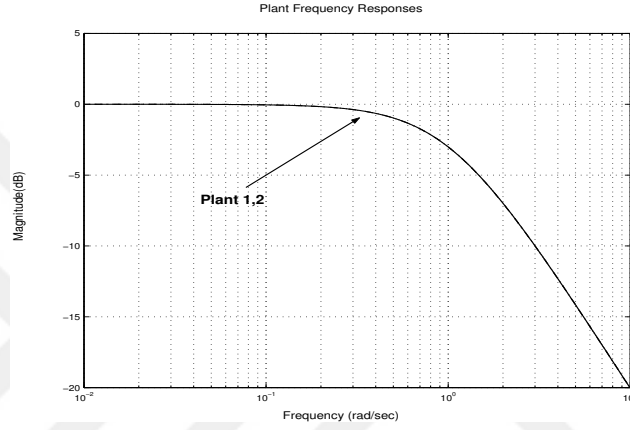


Figure 3.89: Plant Frequency Response - $P = \frac{1}{s-1}$, $P = \frac{1}{s-1} \left[\frac{20-s}{s+20} \right]$

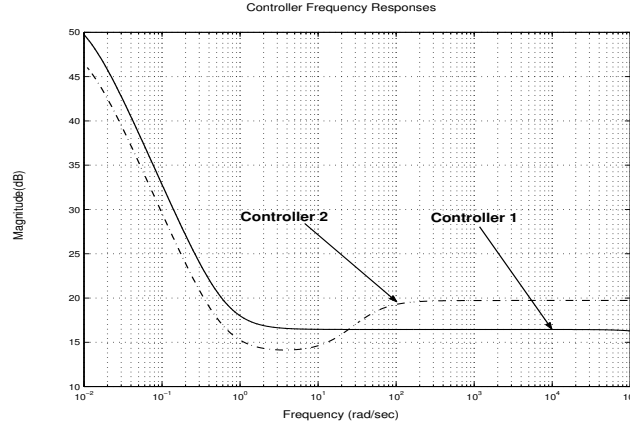


Figure 3.90: Controller Frequency Response

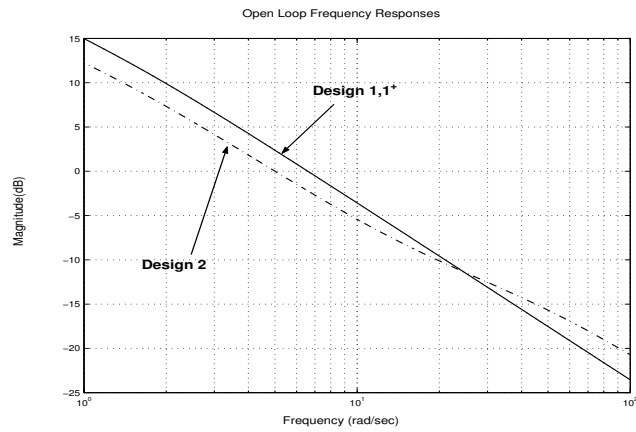


Figure 3.91: Open Loop Frequency Response - $L = PK$

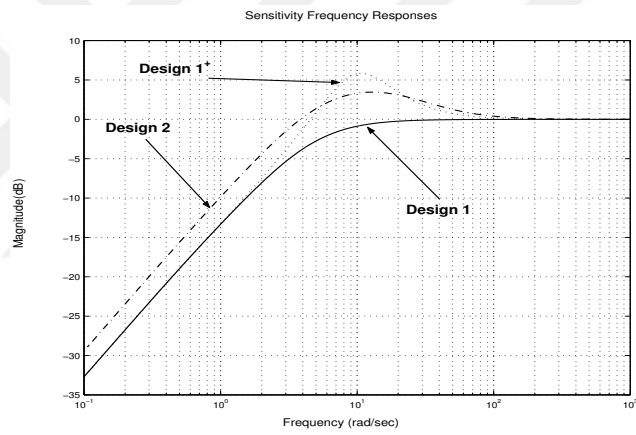


Figure 3.92: Sensitivity Frequency Response - $S = \frac{1}{1+PK}$

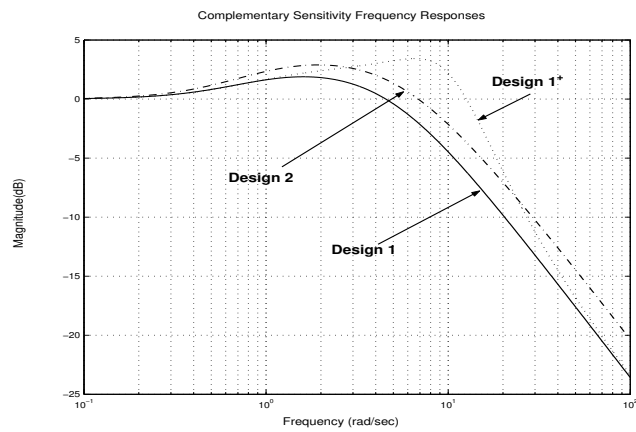


Figure 3.93: Complementary Sensitivity Frequency Response - $T = \frac{PK}{1+PK}$

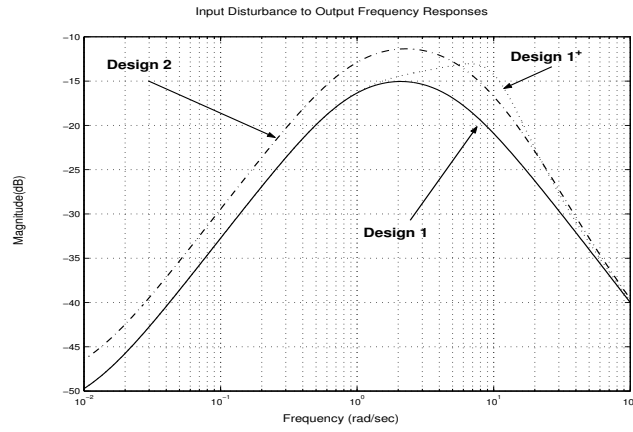


Figure 3.94: Input Disturbance to Output Frequency Response - $T_{dy} = SP$

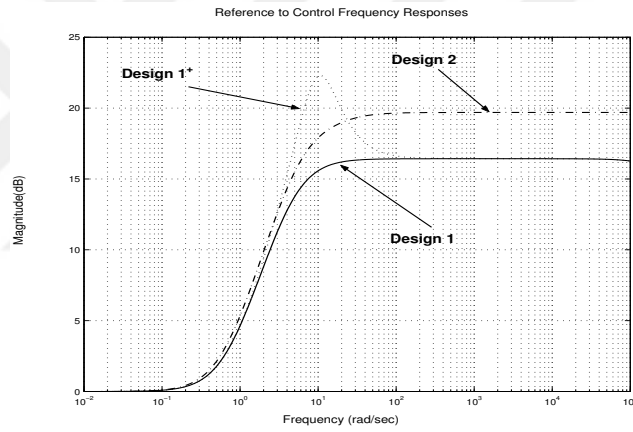


Figure 3.95: Reference to Control Frequency Response: $T_{ru} = KS$

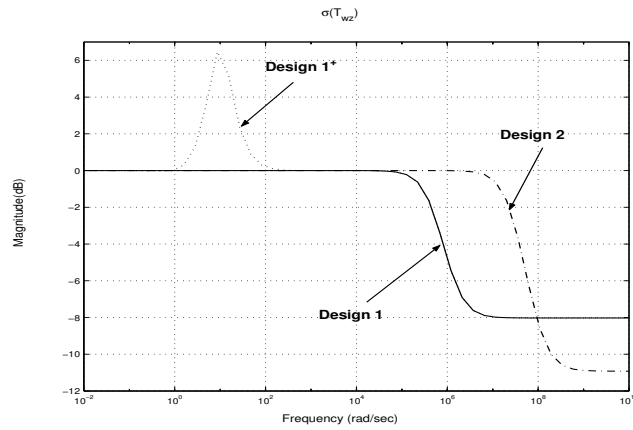


Figure 3.96: Exogenous Signal to Regulated Signals - $\frac{1}{\gamma} \sigma_i(T_{wz})$

Time Response Data: Step Command Following. We now address step command following for each design. This is done without a pre-filter and with a pre-filter.

Step Command Following (Without Pre-Filter). Figure 3.97 contains the output response to a step reference command for each design (without pre-filter). Designs 1 and 2 exhibit significant overshoot because of zeros present in their compensators K . The associated controls are plotted in Figure 3.100. Design 1 exhibits the least control action. Design 2 exhibits slightly more control action. Design 1⁺ exhibits the most control action due to the presence of lightly damped closed loop poles.

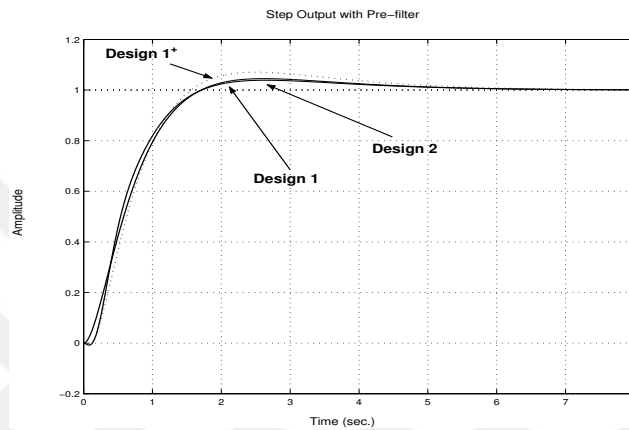


Figure 3.97: Output Response to Step Reference Command (without Pre-Filter)

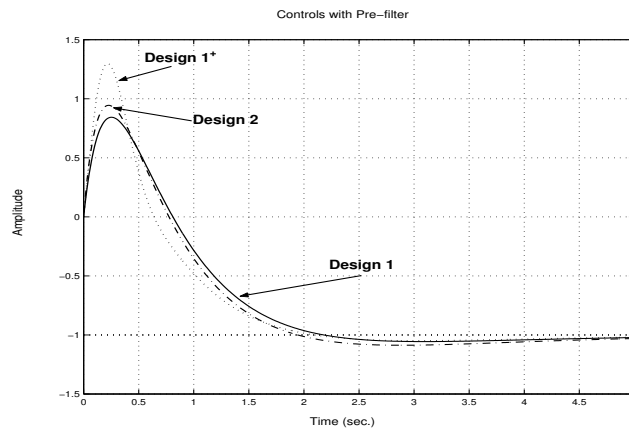


Figure 3.98: Control Response to Step Reference Command (without Pre-Filter)

Step Command Following (With Pre-Filter). Figure 3.97 contains the output response to a filtered step reference command for each design. The prefilter for each design was selected as follows:

$$W(s) = \frac{2z}{s + 2z} \quad (3.87)$$

where z was selected to be the magnitude of the compensator low frequency zero for each Design; i.e. $z = z_1 = 0.6534$ for Design 1 and $z = z_2 = 0.6008$ for Design 2. It is these zeros that are responsible for the overshoot seen in Figure 3.97. Figure 3.99 shows that Designs 1 and 2 exhibit little overshoot. Design

1^+ exhibits a large overshoot due to the presence of lightly damped closed loop poles. Figure 3.100 contains the corresponding controls.

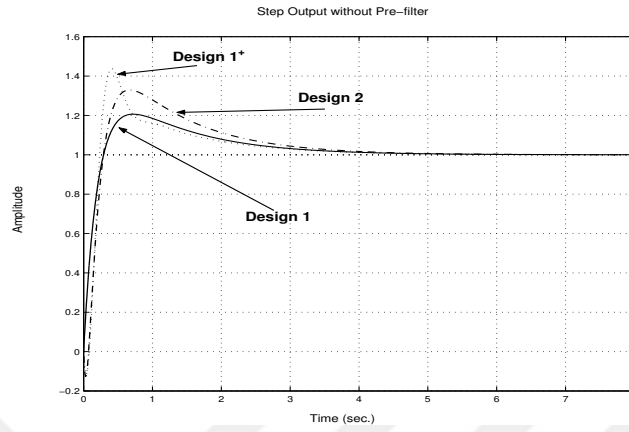


Figure 3.99: Output Response to Step Reference Command (with Pre-filter)

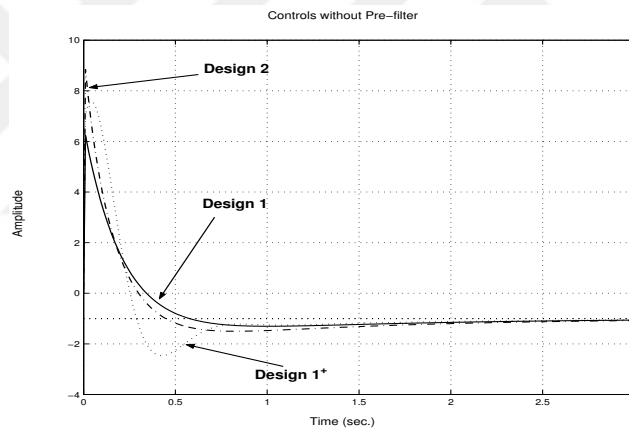


Figure 3.100: Control Response to Step Reference Command (with Pre-filter)

CHAPTER 4

$\mathcal{H}^\infty$ Mixed Sensitivity Control System Design: X-29 Lateral Dynamics

In this section, we consider the design of a control system for the lateral (yaw-roll) dynamics of a forward swept wing X-29 aircraft. A two-input two-output (TITO) model is used to develop a weighted mixed-sensitivity $\mathcal{H}^\infty$ control system design.

Model for X-29 Lateral Dynamics. The TITO model for the X-29 lateral dynamics (powered approach, Mach 0.258, 4,000 ft altitude, 14,777 lbs) is as follows [1]:

$$\dot{x} = Ax + Bu \quad (4.1)$$

$$y = Cx + Du \quad (4.2)$$

where

$$A = \begin{bmatrix} -0.1850 & 0.1475 & -0.9825 & 0.1120 \\ -3.4670 & -1.7100 & 0.9029 & 0.0000 \\ 1.1740 & -0.0825 & -0.1826 & -0.0000 \\ 0 & 1.0000 & 0.1492 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -0.0256 & 0.0230 \\ 21.2869 & 3.1446 \\ 1.5202 & -0.7741 \\ 0 & 0 \end{bmatrix} \quad (4.3)$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (4.4)$$

$$u = \begin{bmatrix} \delta_{df} - \text{differential flap} \quad (deg) \\ \delta_r - \text{rudder} \quad (deg) \end{bmatrix} \quad (4.5)$$

$$x = \begin{bmatrix} \beta - \text{side slip angle} \quad (deg) \\ p - \text{roll rate} \quad (deg/sec) \\ r - \text{yaw rate} \quad (deg/sec) \\ \phi - \text{roll angle} \quad (deg) \end{bmatrix} \quad (4.6)$$

$$y = \begin{bmatrix} \phi & - & \text{roll angle} & (deg) \\ \beta & - & \text{side slip angle} & (deg) \end{bmatrix} \quad (4.7)$$

The aircraft is characterized by a lightly damped stable dutch roll mode ($s = -0.2455 \pm j1.2703$), a stable roll subsidence mode ($s = -1.6183$), and an unstable spiral divergence mode ($s = 0.0318$). Fundamentally, the differential flap is used to control roll while the rudder is used to control yaw or side slip.

Actuator Dynamics. Associated with each control signal are actuator dynamics and an actuator command input:

$$\delta_{df} = H_{dif\ flap} \delta_{df_c} \quad (4.8)$$

$$\delta_r = H_{rudder} \delta_{r_c}. \quad (4.9)$$

$$H_{dif\ flap} = \left[\frac{(54.1)^2}{s^2 + 2(1.53)(54.1)s + (54.1)^2} \right] \left[\frac{(71.4)^2}{s^2 + 2(0.736)(71.4)s + (71.4)^2} \right] \quad (4.10)$$

$$H_{rudder} = \left[\frac{(54.1)^2}{s^2 + 2(1.53)(54.1)s + (54.1)^2} \right] \left[\frac{(71.4)^2}{s^2 + 2(0.736)(71.4)s + (71.4)^2} \right] \quad (4.11)$$

Such dynamics become very important in evaluating aircraft performance when the closed loop system possesses a “high bandwidth.” Each actuator has a lower undamped natural frequency of $\omega_n = 54.1$ rad/sec. We therefore expect that when the closed loop system exhibits a bandwidth greater than ~ 5 rad/sec (settling time ~ 1 sec), we expect problems. We expect significant problems particularly if the actuator dynamics have not been accounted for in the design of the flight control system. These ideas will be illustrated in what follows.

Open Loop Bandwidth Design Specification. Based on the above actuator dynamics, we sought an open loop bandwidth of about 5 rad/sec - about one decade below our actuator dynamics.

Plant Singular Values. The plant singular values - with and without actuator dynamics - are plotted in Figure 4.1. The peak near 1.29 rad/sec is due to the aircraft’s lightly damped dutch roll mode.

Figure 4.1 shows that the plant singular values are widely spread at low frequencies. An svd at dc reveals that

- the maximum singular value at dc is primarily associated with differential flap and roll
- the minimum singular value at dc (below 0 dB) is primarily associated with rudder and side slip.

The wide spread at low frequencies suggests that the resulting controller will have to significantly compensate for the low plant gain in the rudder channel. Given this, we anticipate that significant rudder activity will be required in order to achieve a loop with desirable low frequency command following and disturbance attenuation properties (e.g. $\sigma_{min} [PK] > 20dB$ at low frequencies).

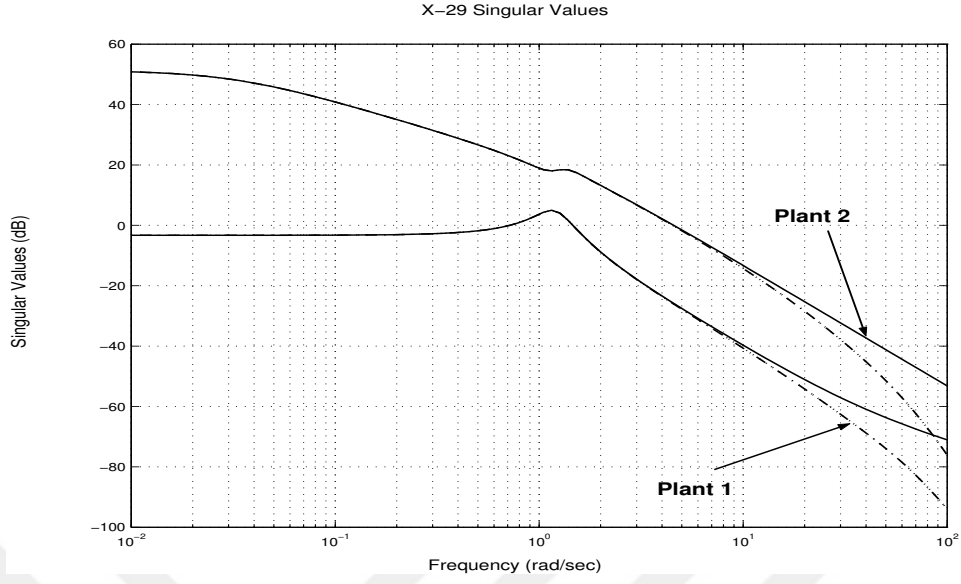


Figure 4.1: Singular Values - TITO X-29 Lateral Dynamics (with/without Actuator Dynamics)

General Design Specifications. We seek a design that significantly improves the handling qualities of our lightly damped aircraft.

$\mathcal{H}^\infty$ Mixed-Sensitivity Weighting Function Parameters. $\mathcal{H}^\infty$ mixed-sensitivity weighting function parameters were selected as follows:

$$\omega_{b_1} = 1.35 \quad \omega_{b_2} = 0.90 \quad (4.12)$$

$$M_{s_1} = 10 \quad M_{s_2} = 10 \quad (4.13)$$

$$\omega_{bu_1} = 4000 \quad \omega_{bu_2} = 250 \quad (4.14)$$

$$M_{u_1} = 0.001 \quad M_{u_2} = 0.001 \quad (4.15)$$

$$\omega_{bc_1} = 45 \quad \omega_{bc_2} = 45 \quad (4.16)$$

$$M_{y_1} = 100 \quad M_{y_2} = 100 \quad (4.17)$$

$$\epsilon = 0.001 \quad (4.18)$$

Weighting Functions. Weighting functions were selected as follows:

$$W_1^{-1} = \begin{bmatrix} \left(\frac{s/\sqrt{M_{s_1}} + \omega_{b_1}}{s + \omega_{b_1} \sqrt{\epsilon}} \right)^2 & 0 \\ 0 & \left(\frac{s/\sqrt{M_{s_2}} + \omega_{b_2}}{s + \omega_{b_2} \sqrt{\epsilon}} \right)^2 \end{bmatrix}^{-1} \quad (4.19)$$

$$= \begin{bmatrix} \left(\frac{s+0.0427}{0.316s+1.35} \right)^2 & 0 \\ 0 & \left(\frac{s+0.0285}{0.316s+0.9} \right)^2 \end{bmatrix} \quad (4.20)$$

$$W_2^{-1} = \begin{bmatrix} \frac{s+\omega_{bu1} M_{u1}}{\varepsilon s+\omega_{bu1}} & 0 \\ 0 & \frac{s+\omega_{bu2} M_{u2}}{\varepsilon s+\omega_{bu2}} \end{bmatrix}^{-1} \quad (4.21)$$

$$= \begin{bmatrix} \frac{0.001s+4000}{s+4} & 0 \\ 0 & \frac{0.001s+250}{s+0.25} \end{bmatrix} \quad (4.22)$$

$$W_3^{-1} = \begin{bmatrix} \left(\frac{s+\omega_{bc1}/\sqrt{M_{y1}}}{\sqrt{\varepsilon}s+\omega_{bc1}} \right)^2 & 0 \\ 0 & \left(\frac{s+\omega_{bc2}/\sqrt{M_{y2}}}{\sqrt{\varepsilon}s+\omega_{bc2}} \right)^2 \end{bmatrix}^{-1} \quad (4.23)$$

$$= \begin{bmatrix} \left(\frac{0.0316s+45}{s+4.5} \right)^2 & 0 \\ 0 & \left(\frac{0.0316s+45}{s+4.5} \right)^2 \end{bmatrix} \quad (4.24)$$

The above weighting functions may be visualized as shown in Figure 4.2. Reasons behind their selection are given below.

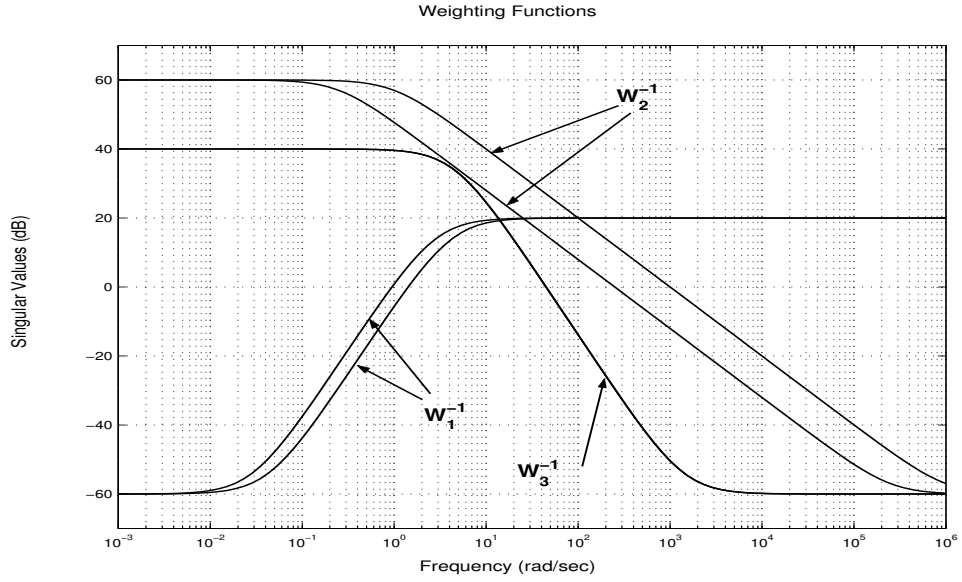


Figure 4.2: Weighting Functions for Mixed-Sensitivity $\mathcal{H}^\infty$ Optimization

Closed Loop Poles: A Bad Design. The closed loop poles that result - from a straight forward $\mathcal{H}^\infty$ mixed-sensitivity optimization - are as follows:

Poles	Damping	Frequency(rad/sec)
$-1.42e + 003 \pm 2.91e + 001i$	$1.00e + 000$	$1.42e + 003$
$-1.42e + 003$	$1.00e + 000$	$1.42e + 003$
$-1.42e + 003$	$1.00e + 000$	$1.42e + 003$
$-3.26e + 002$	$1.00e + 000$	$3.26e + 002$
$-4.28e + 001$	$1.00e + 000$	$4.28e + 001$
$-1.08e + 001 \pm 1.13e + 001i$	$6.89e - 001$	$1.56e + 001$
$-2.07e + 000 \pm 3.11e + 000i$	$5.54e - 001$	$3.74e + 000$
$-4.26e + 000 \pm 9.16e - 002i$	$1.00e + 000$	$4.26e + 000$
$-2.46e - 001 \pm 1.27e + 000i$	$1.90e - 001$	$1.29e + 000$
$-2.38e + 000 \pm 5.26e - 001i$	$9.76e - 001$	$2.43e + 000$
$-3.18e - 002$	$1.00e + 000$	$3.18e - 002$
$-1.62e + 000$	$1.00e + 000$	$1.62e + 000$

The resulting minimum γ is

$$\gamma = 0.3636. \quad (4.25)$$

Because the design results in very lightly damped closed loop poles ($\zeta = 0.19$), the design is unacceptable. This occurred because straight-forward $\mathcal{H}^\infty$ mixed-sensitivity optimization results in a controller that cancels the plant's lightly damped dutch roll mode (see Table 2.2, page 20). To prevent this undesirable plant pole inversion, we use the bilinear transformation.

Bilinear Transformation Parameters. The bilinear transformation (BLT) parameters selected were as follows:

$$p_1 = -1.2 \quad (4.26)$$

$$p_2 = -10^{20}. \quad (4.27)$$

This selection results in

$$s = \frac{\hat{s} + p_1}{\frac{\hat{s}}{p_2} + 1} = \frac{\hat{s} - 1.2}{\frac{\hat{s}}{-10^{20}} + 1} \approx \hat{s} - 1.2 \quad (\text{transform}) \quad (4.28)$$

$$\tilde{s} \approx s + 1.2 \quad (\text{inverse transform}) \quad (4.29)$$

Reasons for this selection are provided below. The transform performs a rightward (“destabilizing”) shift. The inverse transform performs a leftward (“stabilizing”) shift.

Design 1: Actuator Dynamics Neglected. Our first design neglected the aircraft's actuator dynamics. The design procedure for Design 1 was as follows:

- The plant was transformed using the above bilinear transformation. The transformation moves all plant poles and zeros to the right 1.2 units - making the plant look more unstable. The transformation specifically moves the lightly damped (troublesome) dutch roll mode into the right half plane - thus preventing an undesirable cancelation.
- A generalized plant G was formed using the transformed plant $\hat{P}$ and the original weighting functions W_1 , W_2 , and W_3 .
- A design $\hat{K}$ was obtained using the generalized plant (including the transformed plant and weighting functions).
- The resulting controller was then transformed back to get the final control system design K .

Additional details are given below. The Robust Control Toolbox's "hinftol" command was used to obtain the design. An optimality tolerance of 0.0001 was used. The resulting minimum γ is

$$\gamma = 0.7619. \quad (4.30)$$

Since $\gamma < 1$, this implies that our design specification (reflected within our weighting functions) were not too aggressive.

Comment 4.0.1 (Selection of Weighting Functions for X-29 Lateral Control System Design)

We now provide some rational for the selection of the weighting functions (W_1, W_2, W_3) and the associated design parameters.

- Selection of Bilinear Transformation Shifting Parameter p_1 : Closed Loop Damping. *The nominal plant P is transformed via the bilinear transformation to prevent its lightly damped dutch roll poles from being canceled by the controller. We selected $p_2 = -10^{20}$ to be very large so that the bilinear transformation becomes a mere rightward shifting transformation by $-p_1$ units.*
 - *Since all left half plane plant poles are canceled when the $\mathcal{H}^\infty$ design methodology is used, the shifting parameter p_1 must be chosen such that all plant poles which must NOT be canceled by the resulting controller are moved into the right half plane after applying the transformation.*

This suggests that a lower bound on the magnitude $|p_1|$ must be enforced in order to shift the desired poles into the right half plane. More specifically, p_1 has to be smaller than the minimum of the real parts of the plant poles to be moved into the right half plane. For our problem, the dutch roll mode dictates that $p_1 < -0.2455$.

*We selected $p_1 = -1.2$ to be on the order of the imaginary parts
of the dutch roll mode ($s = -0.2455 \pm j1.2703$).*

This selection is based on the eigen-structure of Hamiltonian systems and our desire to achieve a damping of $\zeta = 0.7$. The detailed reasoning is as follows:

- This selection of p_1 moves the dutch roll mode into the right half plane to $\hat{s} = -0.2455 \pm j1.2703 + 1.2 = 0.9545 \pm j1.2703$. The associated open loop Hamiltonian will have an eigenvalue at the reflection of this location; i.e. at $\hat{s} = -0.9545 \pm j1.2703$. The $\mathcal{H}^\infty$ design method places a zero at this left half plane location. When shifted back, the final controller ends up with a zero at $s = -0.9545 \pm j1.2703 - 1.2 = -2.1455 \pm j1.2703$ ($\zeta = 0.8605$). The zeros listed above have a zero at this location - thus supporting our argument. It is this zero at $s = -2.1455 \pm j1.2703$ ($\zeta = 0.8605$) that draws the dutch roll mode to a location that has desirable damping; i.e. $\zeta > 0.7$.
- Impact of Shifting Parameter p_1 . Decreasing p_1 (i.e. making it more negative), increases the resulting bandwidth (open and closed loop). As the bandwidth increases, more control action results for typical test commands (e.g. step commands). This suggests that p_1 should not be made too negative.
- Loss of Integral Action: Accommodating Steady State Error to Step Command. The transformed plant $\hat{P}$ and original weighting functions are used in the design process. Since the resulting design $\hat{K}$ is transformed back, near-integrators associated with the transformed controller $\hat{K}$ (due to W_1 selection) are shifted into the left half plane by $|p_1| = 1.2$ units. This, however, destroys the low frequency command following properties of the design $\hat{K}$. When a step reference command is issued to the resulting closed loop system (with K), a steady state error e_{ss} results. To address this, we incorporated a constant matrix into a reference command pre-filter design (described below). The matrix was selected to be $T_o(0)^{-1}$ - the inverse of the complementary sensitivity transfer function matrix $T_o = PK[I + PK]^{-1}$. While this resulted in an acceptable design - with desirable step command and low frequency command following - it must be noted that in practice, such an approach may be too sensitive with respect to low frequency uncertainty within the feedback loop.
- Loss of Closed Loop Frequency Response Shaping Inequalities. The standard $\mathcal{H}^\infty$ mixed sensitivity inequality $\|\hat{T}_{wz}\|_{\mathcal{H}^\infty} < \gamma$ holds for the closed loop $\hat{T}_{wz}$ formed with the transformed plant $\hat{P}$, the transformed controller $\hat{K}$, and the original weighting functions W_1, W_2, W_3 . The inequality gives rise to response shaping inequalities for $\hat{S}, \hat{K}\hat{S}$, and $\hat{T}$. These are not very useful.
The inequality also holds for the closed loop T_{wz} formed with P, K , and the back transformed weighting functions (i.e. weighting functions that are more stable). Given this, it follows that
 - When the BLT is used, the inequality $\|T_{wz}\|_{\mathcal{H}^\infty} < \gamma$ does NOT hold for the closed loop system T_{wz} consisting of P, K , and the weighting functions W_1, W_2, W_3 .
 - A consequence of this is the loss in the physical significance (meaning) of the weighting functions W_1, W_2, W_3 .
- Sensitivity Weighting. A second order weighting function was selected for W_1 in order to increase the slope of the sensitivity at low frequencies, $\omega < \omega_{b_i}$. This was desired in order to achieve a $T_{d_{iy}} = SP$ which is smaller at low frequencies. A first order weighting function proved to be inadequate - resulting in a $T_{d_{iy}} = SP$ which was well above 0 dB at low frequencies.
 - Initially, we started with $\omega_{b_1} = \omega_{b_2} = 1$. ω_{b_1} was increased in order to reduce bank angle overshoot. ω_{b_2} was decreased in order to slow down the response to a side slip step command.

- Since $\omega_{b_1} > \omega_{b_2}$, it follows that we are more aggressive with respect to bank angle errors than side slip errors.
- As the sensitivity bandwidth parameters ω_{b_i} are increased (decreased), the closed loop system bandwidth increases (decreases) (e.g. gain crossover frequency for the sensitivity, complementary sensitivity transfer function 3 dB bandwidth). Increasing and/or decreasing the ω_{b_i} too much results in an unacceptable design.
- Control Sensitivity Weighting. Initially, we selected W_2 to be a constant (i.e. $W_2 = M_u$). This selection resulted in large controls and did not permit us to adequately control the roll off of the control sensitivity KS . To remedy this, we selected the above dynamic W_2 . By so doing, we were able to shape the high frequency behavior of KS ; e.g. its associated high frequency gain crossover frequencies.
 - Initially, we started with $\omega_{b_{u1}} = \omega_{b_{u2}} = 500$ rad/sec - two decades above our desired open loop gain crossover of 5 rad/sec (about one decade below our actuator dynamics at 54.1 rad/sec).
 - $\omega_{b_{u1}}$ was increased in order to encourage more differential flap activity.
 - $\omega_{b_{u2}}$ was decreased in order to encourage less rudder activity.

Since $\omega_{b_{u1}} > \omega_{b_{u2}}$, it follows that we are more aggressive with respect to differential flap action than rudder action.

- Complementary Sensitivity Weighting. A second order weighting function was selected for W_3 in order to control the bandwidth associated with the complementary sensitivity transfer function matrix T .
 - Initially, we started with $\omega_{c1} = \omega_{c2} = 50$ rad/sec - one decade above our desired gain crossover frequency of 5 rad/sec.
 - Both were decreased simultaneously to reduce bandwidth, overshoot, and improve damping.

■

Closed Loop Poles for Design 1: A Good Design. The closed loop poles that result from our BLT (bilinear transformation) approach are as follows:

Poles	Damping	Frequency(rad/sec)
$-5.85e + 002$	$1.00e + 000$	$5.85e + 002$
$-1.42e + 003 \pm 2.91e + 001i$	$1.00e + 000$	$1.42e + 003$
$-1.43e + 003$	$1.00e + 000$	$1.43e + 003$
$-1.42e + 003$	$1.00e + 000$	$1.42e + 003$
$-4.39e + 001$	$1.00e + 000$	$4.39e + 001$
$-1.14e + 001 \pm 1.06e + 001i$	$7.32e - 001$	$1.56e + 001$
$-2.92e + 000 \pm 2.81e + 000i$	$7.20e - 001$	$4.05e + 000$
$-5.46e + 000 \pm 9.21e - 002i$	$1.00e + 000$	$5.46e + 000$
$-2.15e + 000 \pm 1.27e + 000i$	$8.61e - 001$	$2.50e + 000$
$-1.62e + 000$	$1.00e + 000$	$1.62e + 000$
$-3.42e + 000 \pm 5.60e - 001i$	$9.87e - 001$	$3.47e + 000$
$-2.43e + 000$	$1.00e + 000$	$2.43e + 000$

We observe that all closed loop poles for design 1 exhibit a damping $\zeta > 0.7$ - as desired.

Closed Loop Zeros for Design 1. The resulting closed loop transmission zeros (associated with the resulting complementary sensitivity transfer function matrix) are as follows:

Zeros	Damping	Frequency(rad/sec)
$-4.00e + 006$	$1.00e + 000$	$4.00e + 006$
$-2.50e + 005$	$1.00e + 000$	$2.50e + 005$
$-3.75e + 001$	$1.00e + 000$	$3.75e + 001$
$-1.42e + 003 \pm 1.45e - 003i$	$1.00e + 000$	$1.42e + 003$
$-1.42e + 003$	$1.00e + 000$	$1.42e + 003$
$-1.42e + 003$	$1.00e + 000$	$1.42e + 003$
$-1.38e + 000 \pm 1.15e + 000i$	$7.70e - 001$	$1.80e + 000$
$-2.00e + 000 \pm 7.06e - 001i$	$9.43e - 001$	$2.12e + 000$
$-1.83e + 000$	$1.00e + 000$	$1.83e + 000$
$-1.62e + 000$	$1.00e + 000$	$1.62e + 000$

One low frequency closed loop zero (also a compensator zero) which will significantly increase the overshoot resulting from step reference commands is situated at $s = -1.83$. Because of this zero, a reference command pre-filter will be essential. The design of this pre-filter for Design 1 will be based on the location of this zero. This is discussed below. (Note that the zeros at $s = -2 \pm j0.706$, -1.62 will not impact overshoot much since they have nearby poles.)

Minimum γ for Design 1. The resulting minimum γ for Design 2 is

$$\gamma = 0.7619. \quad (4.31)$$

The design was found using the Robust Control Toolbox's "hinftol" command - with an optimality tolerance of 0.0001.

Design #1⁺: Actuator Dynamics Included After Design Process.

When the actuator dynamics are inserted into the feedback loop, we obtain what we call Design 1⁺. This design did not consider the actuator dynamics in the design process. Given this, and given our high bandwidth of ~ 5 rad/sec, we anticipate trouble with this design.

Closed Loop Poles for Design #1⁺. The closed loop poles associated with Design 1⁺ are as follows:

Poles	Damping	Frequency(rad/sec)
$-1.42e + 003 \pm 2.91e + 001i$	$1.00e + 000$	$1.42e + 003$
$-1.43e + 003$	$1.00e + 000$	$1.43e + 003$
$-1.42e + 003$	$1.00e + 000$	$1.42e + 003$
$-5.85e + 002$	$1.00e + 000$	$5.85e + 002$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-5.27e + 001 \pm 4.85e + 001i$	$7.36e - 001$	$7.17e + 001$
$-5.25e + 001 \pm 4.84e + 001i$	$7.35e - 001$	$7.14e + 001$
$-5.10e + 001$	$1.00e + 000$	$5.10e + 001$
$-2.05e + 001 \pm 2.19e + 001i$	$6.84e - 001$	$3.00e + 001$
$-1.92e + 001$	$1.00e + 000$	$1.92e + 001$
$-9.60e + 000$	$1.00e + 000$	$9.60e + 000$
$-1.19e + 000 \pm 6.95e + 000i$	$1.69e - 001$	$7.05e + 000$
$-1.30e + 000 \pm 3.42e + 000i$	$3.55e - 001$	$3.66e + 000$
$-1.79e + 000 \pm 1.15e + 000i$	$8.41e - 001$	$2.13e + 000$
$-2.57e + 000 \pm 5.07e - 001i$	$9.81e - 001$	$2.62e + 000$
$-1.62e + 000$	$1.00e + 000$	$1.62e + 000$
$-2.18e + 000$	$1.00e + 000$	$2.18e + 000$

As expected, we have a problem. One of the closed loop poles has a damping of 0.355 and undamped natural frequency of $\omega_n = 3.66$ rad/sec. If this closed loop pole were situated at a very high frequency, its residue matrix would be small, and it would not bother us so much. This, however, is not the case.

Closed Loop Zeros for Design 1⁺. The actuator dynamics do not introduce additional zeros. Given this, the closed loop transmission zeros associated with Design 1⁺ are identical to those associated with Design 1.

Reference Command Pre-Filter for Designs 1 and 1⁺. The following reference command pre-filter was designed on the basis of controller zeros. Designs 1 and 1⁺ have the same controller. The controller has

a zero at $s = -1.83$. Based on this, we developed the following reference command pre-filter for Designs 1 and 1⁺:

$$W_o = [T_{ryo}(0)]^{-1} \begin{bmatrix} \frac{3}{s+3} & 0 \\ 0 & \frac{1.9}{s+1.9} \end{bmatrix} \quad (4.32)$$

$$= \begin{bmatrix} 1.0025 & -0.0631 \\ 0.0005 & 0.8874 \end{bmatrix}^{-1} \begin{bmatrix} \frac{3}{s+3} & 0 \\ 0 & \frac{1.9}{s+1.9} \end{bmatrix} \quad (4.33)$$

$$= \begin{bmatrix} \frac{2.9925}{s+3} & \frac{0.1348}{s+1.9} \\ \frac{-0.0018}{s+3} & \frac{2.1410}{s+1.9} \end{bmatrix} \quad (4.34)$$

where $T_{ryo} = PK[I + PK]^{-1}$ is the nominal complementary sensitivity for Design 1. Note that its dc gain suggests very little coupling between bank angle side slip angle at low frequencies. Also note that our pre-filter has a low pass filter in each reference command channel. Initially, we started with a low pass filter $\frac{z}{s+z}$ in each channel with $z = 1.9$ - slightly larger than the compensator zero. We then increased the bank angle filter parameter to 3; i.e. until the overshoot to a step reference bank angle command was on the border of acceptability. Further increase would result in additional bank angle overshoot.

Design 2: Actuator Dynamics Included In Design Process. Because Design 1⁺ does not adequately address our actuator dynamics, we developed another design - design 2. This design includes the actuator dynamics in the initial plant model P . After application of the BLT, the resulting $\hat{P}$ is combined with W_1 , W_2 , and W_3 to form the generalized plant G . A design $\hat{K}$ is then obtained and transformed back to get the final design K .

Closed Loop Poles for Design 2. The closed loop poles associated with Design 2 are as follows:

Poles	Damping	Frequency(rad/sec)
$-5.10e + 003$	$1.00e + 000$	$5.10e + 003$
$-1.42e + 003 \pm 1.26e - 004i$	$1.00e + 000$	$1.42e + 003$
$-1.42e + 003 \pm 1.21e - 004i$	$1.00e + 000$	$1.42e + 003$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-5.22e + 001 \pm 4.79e + 001i$	$7.37e - 001$	$7.09e + 001$
$-2.41e + 001 \pm 1.89e + 001i$	$7.87e - 001$	$3.06e + 001$
$-5.25e + 001 \pm 4.84e + 001i$	$7.35e - 001$	$7.14e + 001$
$-1.15e + 001 \pm 1.11e + 001i$	$7.18e - 001$	$1.60e + 001$
$-2.98e + 000 \pm 2.97e + 000i$	$7.08e - 001$	$4.21e + 000$
$-1.62e + 000$	$1.00e + 000$	$1.62e + 000$
$-2.15e + 000 \pm 1.27e + 000i$	$8.61e - 001$	$2.50e + 000$
$-2.43e + 000$	$1.00e + 000$	$2.43e + 000$
$-3.50e + 000 \pm 5.47e - 001i$	$9.88e - 001$	$3.54e + 000$
$-5.46e + 000 \pm 9.30e - 002i$	$1.00e + 000$	$5.46e + 000$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-2.01e + 001$	$1.00e + 000$	$2.01e + 001$
$-5.25e + 001 \pm 4.84e + 001i$	$7.35e - 001$	$7.14e + 001$
$-5.25e + 001 \pm 4.84e + 001i$	$7.35e - 001$	$7.14e + 001$
$-2.01e + 001$	$1.00e + 000$	$2.01e + 001$
$-2.01e + 001$	$1.00e + 000$	$2.01e + 001$

As expected, all closed loop poles associated with design 2 are at desirable locations - with damping $\zeta > 0.7$.

Closed Loop Zeros for Design 2. The closed loop transmission zeros associated with Design 2 are as follows:

Zeros	Damping	Frequency(rad/sec)
$3.30e + 007$	$-1.00e + 000$	$3.30e + 007$
$-3.54e + 006$	$1.00e + 000$	$3.54e + 006$
$-2.50e + 005$	$1.00e + 000$	$2.50e + 005$
$-1.42e + 003$	$1.00e + 000$	$1.42e + 003$
$-1.42e + 003 \pm 1.47e - 002i$	$1.00e + 000$	$1.42e + 003$
$-1.42e + 003$	$1.00e + 000$	$1.42e + 003$
$-1.45e + 002 \pm 3.04e - 008i$	$1.00e + 000$	$1.45e + 002$
$-5.25e + 001 \pm 4.84e + 001i$	$7.35e - 001$	$7.14e + 001$
$-5.25e + 001 \pm 4.84e + 001i$	$7.35e - 001$	$7.14e + 001$
$-3.75e + 001$	$1.00e + 000$	$3.75e + 001$
$-1.33e + 000 \pm 1.14e + 000i$	$7.60e - 001$	$1.75e + 000$
$-1.90e + 000 \pm 6.76e - 001i$	$9.42e - 001$	$2.02e + 000$
$-1.79e + 000$	$1.00e + 000$	$1.79e + 000$
$-1.62e + 000$	$1.00e + 000$	$1.62e + 000$
$-2.01e + 001 \pm 2.16e - 006i$	$1.00e + 000$	$2.01e + 001$

As for Design 1, one low frequency closed loop zero (also a compensator zero) which will significantly increase the overshoot resulting from step reference commands is situated at $s = -1.79$. Because of this zero, a reference command pre-filter will be essential. The design of this pre-filter for Design 2 will be based on the location of this zero. This is discussed below. (Note that the zeros at $s = -1.9 \pm j0.0676$, -1.62 will not impact overshoot much since they have nearby poles.)

Minimum γ for Design 2. The resulting minimum γ for Design 2 is

$$\gamma = 0.8828. \quad (4.35)$$

The design was found using the Robust Control Toolbox's "hinftopt" command - with an optimality tolerance of 0.0001.

Reference Command Pre-Filter for Design 2. The following reference command pre-filter was designed on the basis of Design 2's controller zeros. Design 2's controller has a zero at $s = -1.79$. The following pre-filter was developed like that developed for Designs 1 and 1⁺ above:

$$W_o = [T_{ry_o}(0)]^{-1} \begin{bmatrix} \frac{3}{s+3} & 0 \\ 0 & \frac{1.9}{s+1.9} \end{bmatrix} \quad (4.36)$$

$$= \begin{bmatrix} 1.0033 & -0.0849 \\ 0.0007 & 0.8699 \end{bmatrix}^{-1} \begin{bmatrix} \frac{3}{s+3} & 0 \\ 0 & \frac{1.9}{s+1.9} \end{bmatrix} \quad (4.37)$$

$$= \begin{bmatrix} \frac{2.9899}{s+3} & \frac{0.1848}{s+1.9} \\ \frac{-0.0023}{s+3} & \frac{2.1841}{s+1.9} \end{bmatrix} \quad (4.38)$$

Frequency Response Data.

- *Compensator Singular Values.* The compensator singular values are plotted in Figure 4.3. Because we used the bilinear transformation, the integral behavior “requested” by W_1 has been “shifted away.” We therefore see that for frequencies below 1.2 rad/sec, the compensator singular values are constant - with the minimum singular value being very low! (BLT parameter $p_1 = -1.2$). This will result in steady state errors to step reference commands for ϕ and β . At higher frequencies, the singular values roll off to ensure that high frequency sensor noise n in e is attenuated.

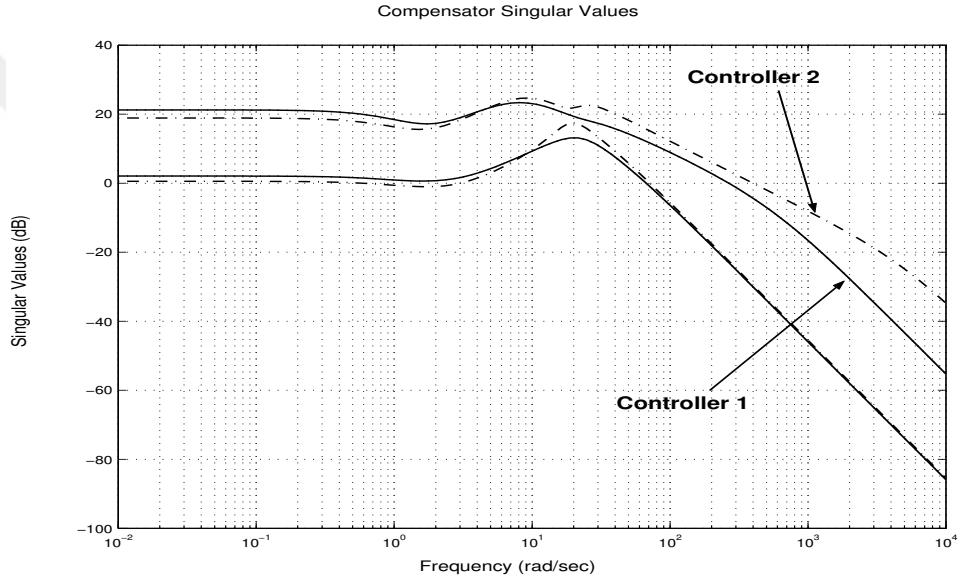


Figure 4.3: Compensator Singular Values

- *Open Loop Singular Values at Plant Output.* Breaking the loop at e or y yields the open loop transfer function matrix $L_o = PK$. Its singular values are plotted in Figure 4.4. The singular values are constant at low frequencies because K is constant at low frequencies - a consequence of the BLT. The plot suggests that (to some extent) low frequency reference commands r will be followed, low frequency output disturbances d_o will be attenuated, and high frequency sensor noise n will be attenuated. More precisely, reference commands r with frequency content below 1 rad/sec should be followed to within about 15 dB (5.6234); that is, with a steady state error of about 15.1% ($\frac{1}{1+5.6234} = \frac{1}{6.6234}$). Similarly, output disturbances d_o with frequency content below 1 rad/sec should be attenuated by the same amount. Sensor noise n with frequency content above 6 rad/sec (approximate unity gain high frequency crossover for Design 2) will be attenuated.
- *Open Loop Singular Values at Plant Input.* Breaking the loop at u yields the open loop transfer function matrix $L_i = KP$. The singular value plot for that is given in Figure 4.5. L_i therefore suggests that low frequency input disturbances will be attenuated only very little. This could be improved

- at the expense of more control action. It should be noted that to properly assess the disturbance attenuation properties of a design, one must properly model the anticipated disturbances. Stochastic Dryden spectrum (low pass filter) models, for example, are typically used to model disturbances in the atmosphere.

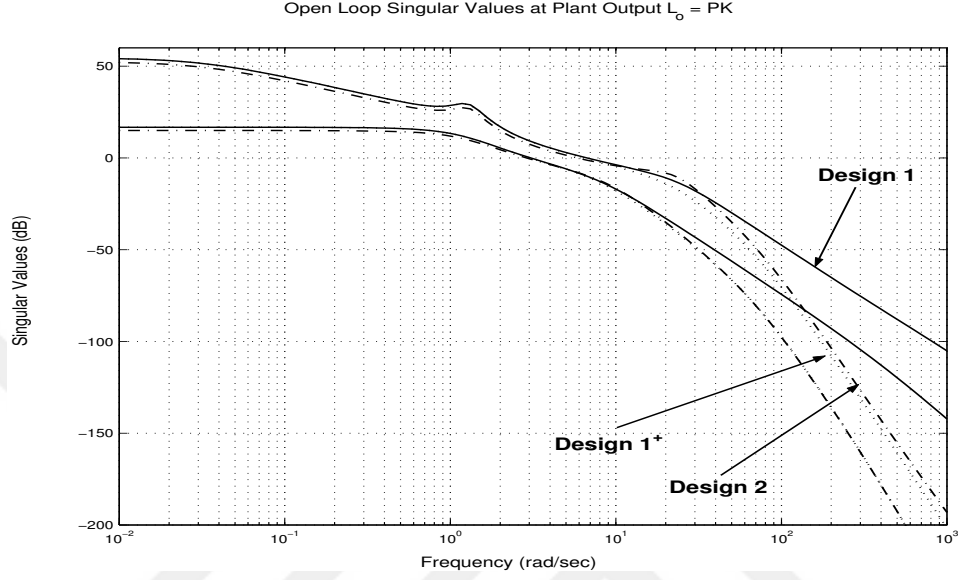


Figure 4.4: Open Loop Singular Values at Plant Output - $L_o = PK$

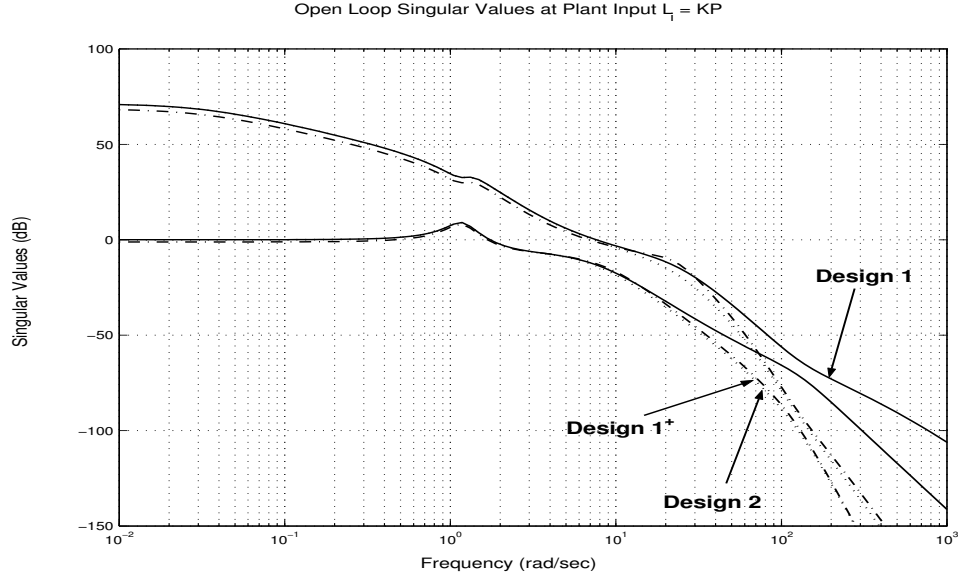


Figure 4.5: Open Loop Singular Values at Plant Input - $L_i = KP$

- *Sensitivity Singular Values at Plant Output.* The sensitivity transfer function matrix associated with e is given by:

$$S_o = [I + PK]^{-1} = [I + L_o]^{-1}. \quad (4.39)$$

The singular values of S_o are plotted in Figure 4.6. The plot suggests that (to some extent) low frequency reference commands will be followed and low frequency output disturbances will be attenuated. More precisely, reference commands r with frequency content below 0.3 rad/sec should be followed to within about 16 dB (6.3096); that is, with a steady state error of about 15.85% ($\frac{1}{6.3096}$). Similarly, output disturbances d_o with frequency content below 0.3 rad/sec should be attenuated by this amount. The plot also shows acceptable peaking for Designs 1 and 2. design 1⁺ (as expected) exhibits undesirable peaking.

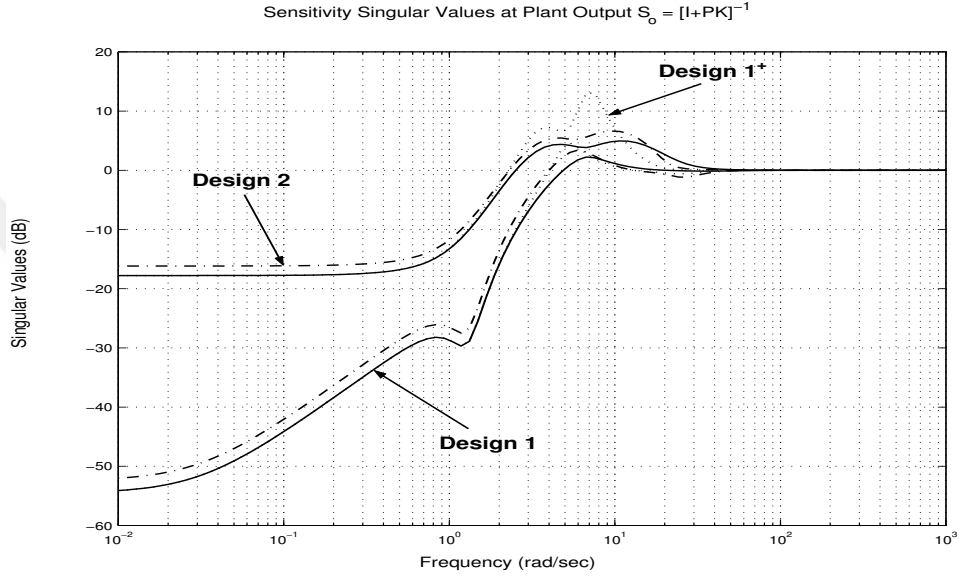


Figure 4.6: Sensitivity Singular Values at Plant Output - $S_o = [I + PK]^{-1}$

- *Sensitivity Singular Values at Plant Input.* The sensitivity transfer function matrix associated with controls u is given by:

$$S_i = [I + KP]^{-1} = [I + L_i]^{-1}. \quad (4.40)$$

This is also the transfer function matrix from input disturbances d_i to plant inputs u_p . The singular values of S_i are plotted in Figure 4.7. The plot indicates that one singular value is small at low frequencies while the other is near unity (~ -1 dB for Designs 1 and 2) at frequencies below 0.3. We also see that the worse case input disturbance occurs at about 3.3 rad/sec where S_i exhibits a peak of about 11.7 dB (3.85). An svd at this frequency reveals that the worse case disturbance is associated with a differential flap - rudder disturbance that primarily impacts the rudder. As stated earlier, disturbances must be properly modeled in order to judiciously assess the disturbance attenuation properties of a design.

- *Complementary Sensitivity Singular Values at Plant Output.* The complementary sensitivity transfer function matrix associated with the output y or error $e = r - y$ is given by:

$$T_o = I - S_o = PK[I + PK]^{-1}. \quad (4.41)$$

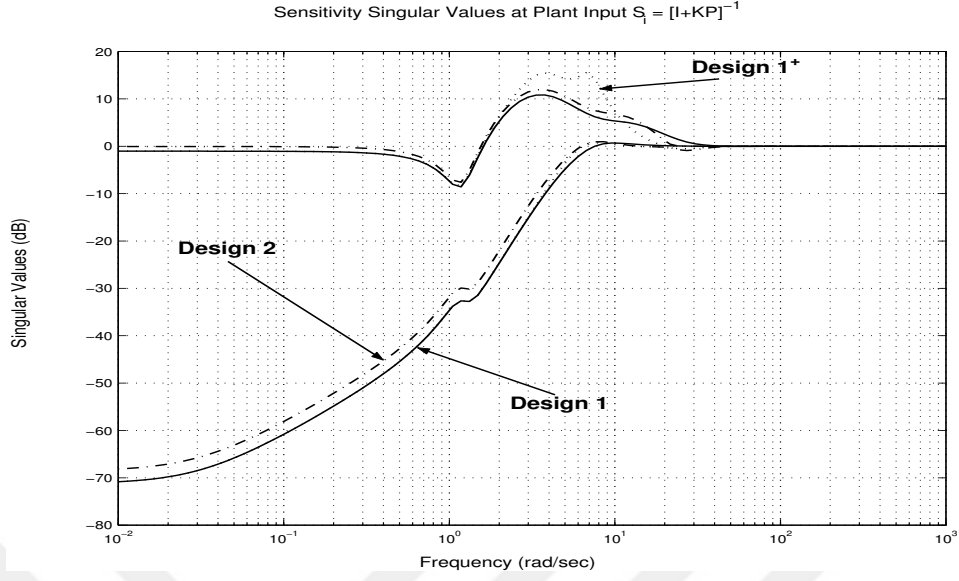


Figure 4.7: Sensitivity Singular Values at Plant Input - $S_i = [I + KP]^{-1}$

The singular values of T_o are plotted in Figure 4.8. These singular values are important for assessing the impact of a unfiltered reference command r on the output y . The bump in our figure suggests that a pre-filter is essential to prevent excessive overshoot to step reference commands. Since the bump starts to grow around 2 rad/sec (near a compensator zero), the command pre-filter was selected to be as in (4.34) for Design 1 and as in (4.38) for Design 2. The plot shows that low frequency reference commands will be followed (to some extent). The singular value for S_o better illustrate the reference command following properties of a design. The plot also shows that sensor noise with frequency content above 10 rad/sec will be attenuated. The peaking in T_o (for Designs 1 and 2) is acceptable.

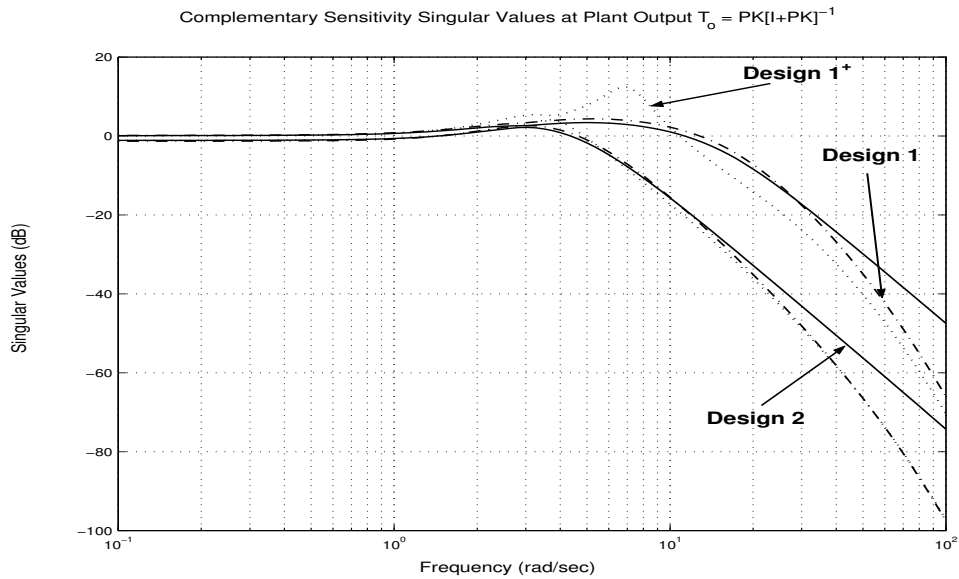


Figure 4.8: Complementary Sensitivity Singular Values at Plant Output - $T_o = PK[I + PK]^{-1}$

- *Complementary Sensitivity Singular Values at Plant Input.* The singular values of $T_i = KP[I + KP]^{-1}$

are plotted in Figure 4.9. T_i is the transfer function matrix from d_i to u . This plot indicates that low frequency input disturbances with frequency content below 1 rad/sec will be amplified by a factor of about 1.475 (3.38 dB). This is not desirable. One must reflect upon the comments made earlier with respect to disturbance modeling as it relates to disturbance attenuation assessment.

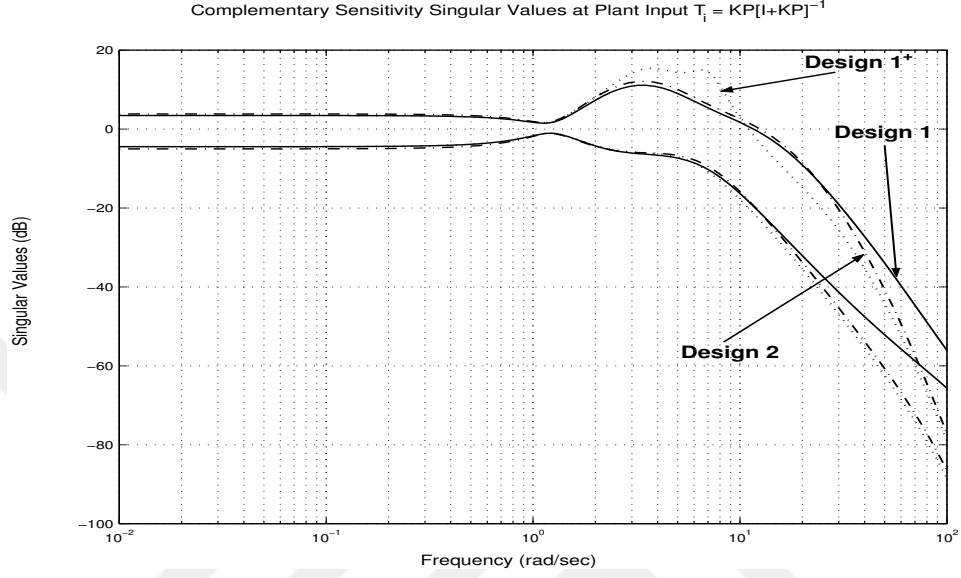


Figure 4.9: Complementary Sensitivity Singular Values at Plant Input - $T_i = KP[I + KP]^{-1}$

- *Closed Loop Singular Values: Reference to Output.* The transfer function matrix from reference commands r to the outputs ϕ and β is

$$T_{ry} = [I + PK]^{-1}PKW_o \quad (4.42)$$

where W_o refers to the reference command pre-filters designed earlier. The singular values of T_{ry} are plotted in Figure 4.10. The plot suggests that low frequency reference commands will be followed in the steady state and that little overshoot will result during the transient.

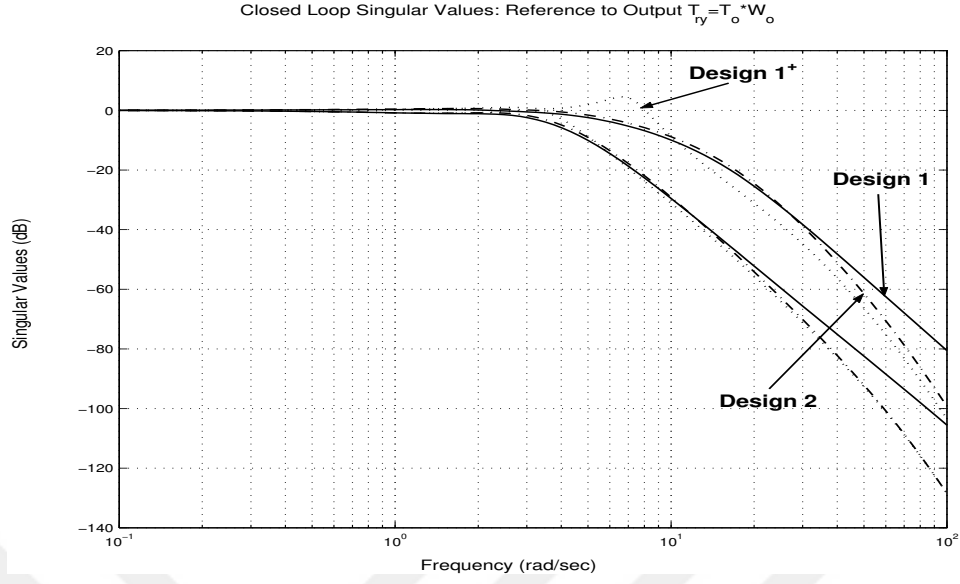
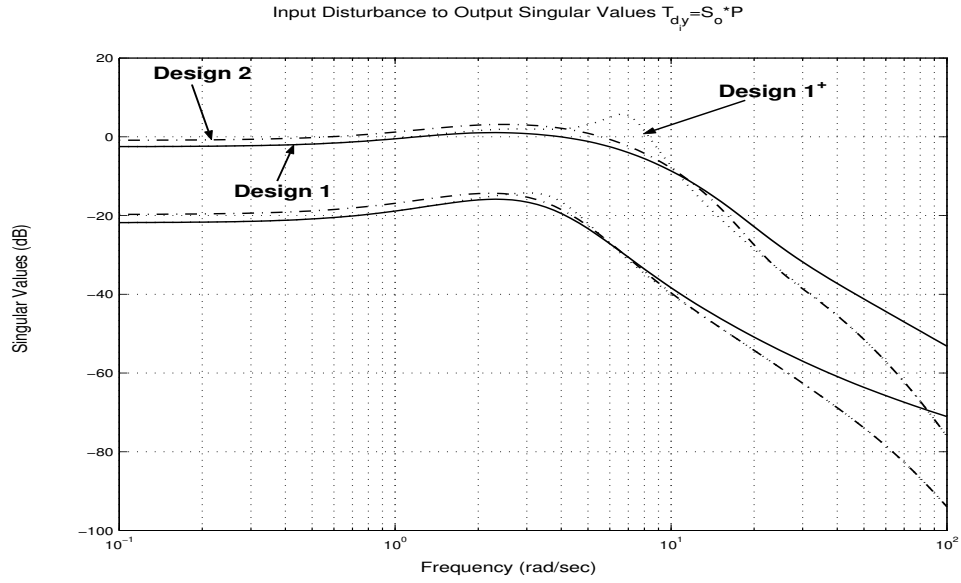
- *Input Disturbance to Output Singular Values.* To assess the impact of input disturbances d_i on the output y , one considers the transfer function matrix from d_i to y :

$$T_{d_i y} = [I + PK]^{-1}P. \quad (4.43)$$

Its singular values are plotted in Figure 4.11. The plot suggests that low frequency input disturbances will be attenuated only very little. High frequency input disturbances will be attenuated. This is expected since such high frequency input disturbances are directly attenuated by the plant.

- *Reference to Control Singular Values.* To assess the impact of reference commands on control action, one considers the transfer function matrix from r to u :

$$T_{ru} = [I + KP]^{-1}KW = K[I + PK]^{-1}W. \quad (4.44)$$

Figure 4.10: Closed Loop Singular Values - T_{ry} Figure 4.11: Input Disturbance to Output Singular Values - T_{diy}

Its singular values are plotted in Figure 4.12.

The small maximum singular value at low frequencies suggests that low frequency reference commands will not require too much steady state control action. The large peak does suggest that control transients may be large. An svd at $\omega 6 \text{ rad/sec}$ yields a maximum singular value

$$\sigma_1 = 8.1244 \quad (18.1958 \text{ dB}) \quad (4.45)$$

a right singular vector

$$v_1 = \begin{bmatrix} 0.1998 & -0.3986 - j0.8951 \end{bmatrix}^T = \begin{bmatrix} 0.1998 & 0.9798 e^{-j114^\circ} \end{bmatrix}^T \quad (4.46)$$

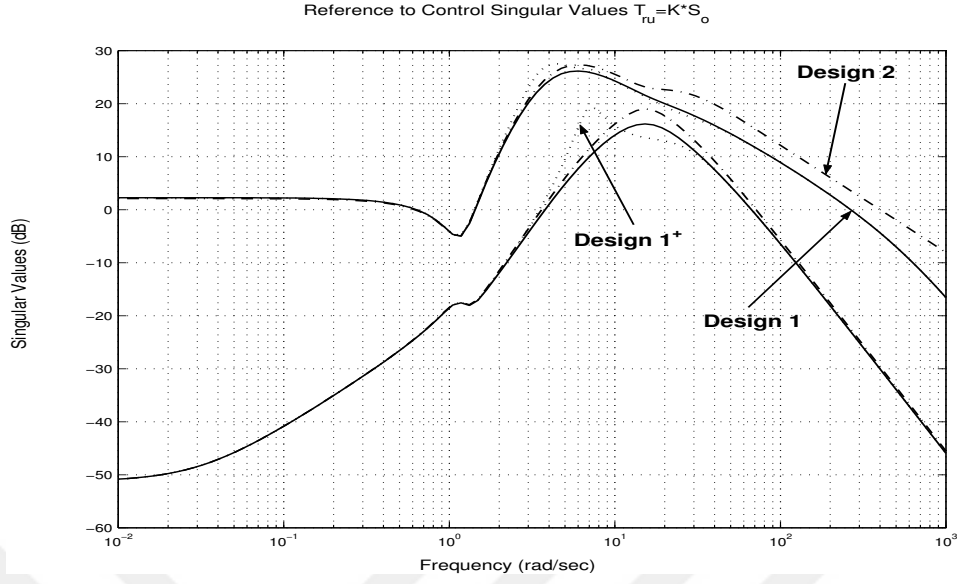


Figure 4.12: Reference to Control Singular Values - T_{ru}

and a left singular vector

$$u_1 = \begin{bmatrix} 0.1354 + j0.0760 & -0.9403 - j0.3030 \end{bmatrix}^T = \begin{bmatrix} 0.1553 e^{j30^\circ} & 0.9879 e^{-j162^\circ} \end{bmatrix}^T \quad (4.47)$$

This svd information implies that if a reference command

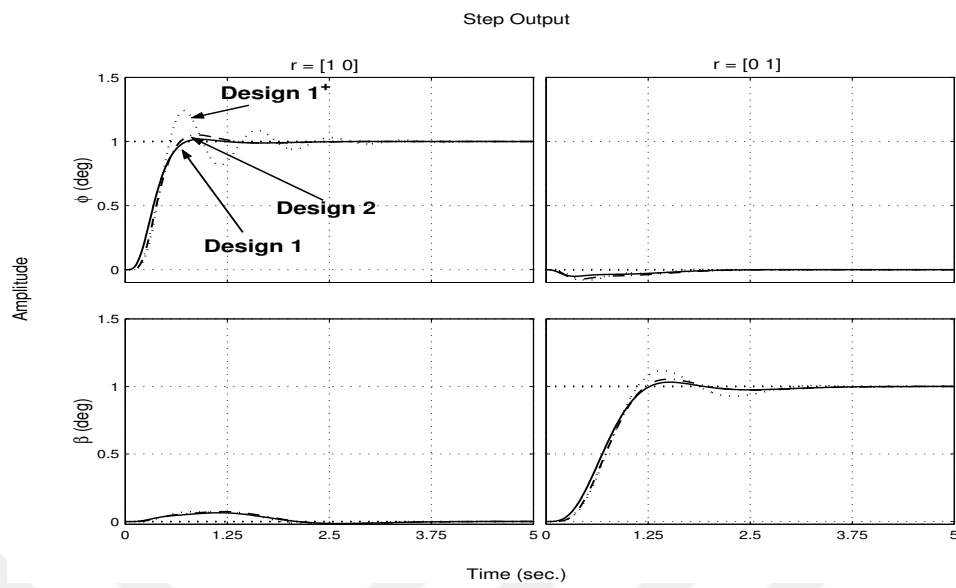
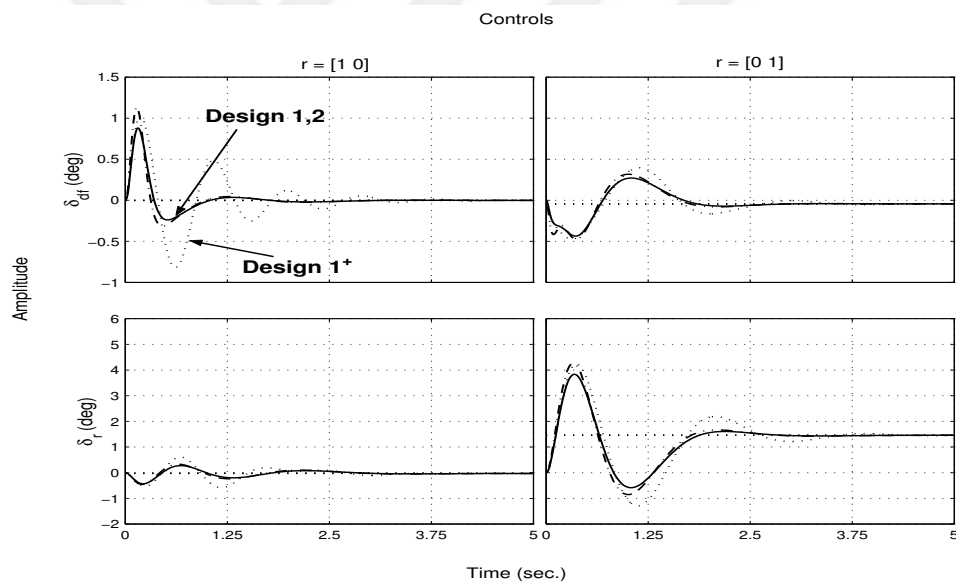
$$r(t) = \begin{bmatrix} 0.1998 \sin 6t & 0.9798 \sin(6t - 114^\circ) \end{bmatrix}^T \quad (4.48)$$

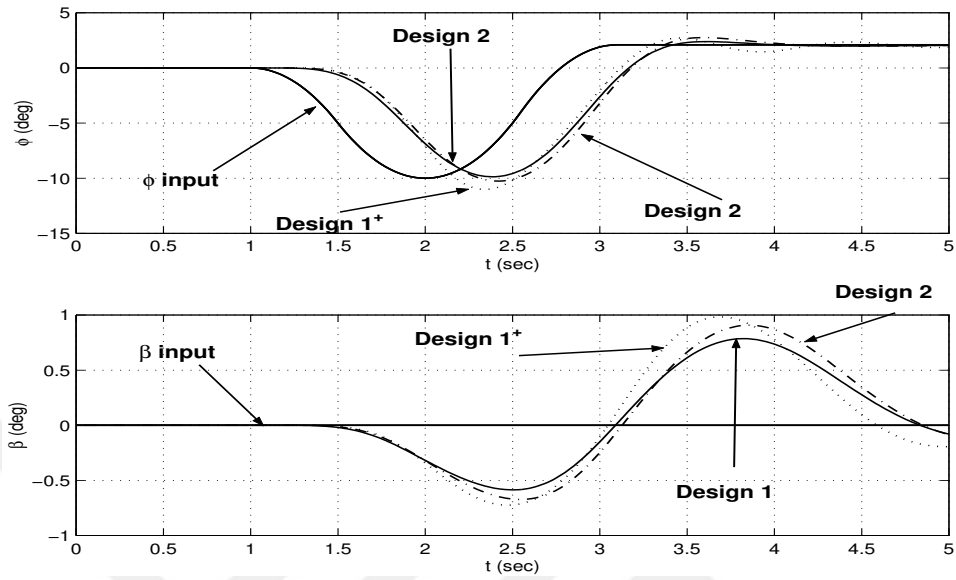
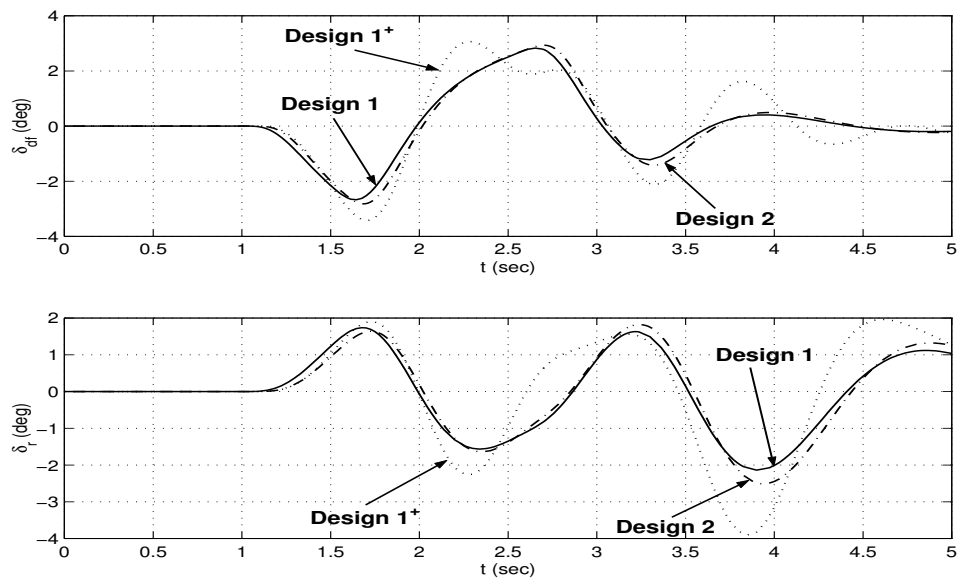
is issued to the system, then the resulting steady state control will be

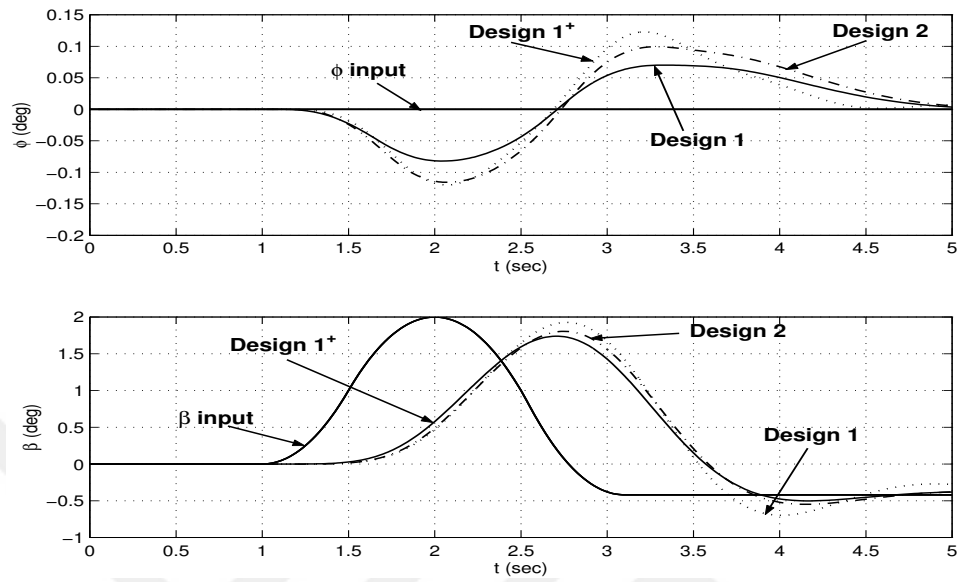
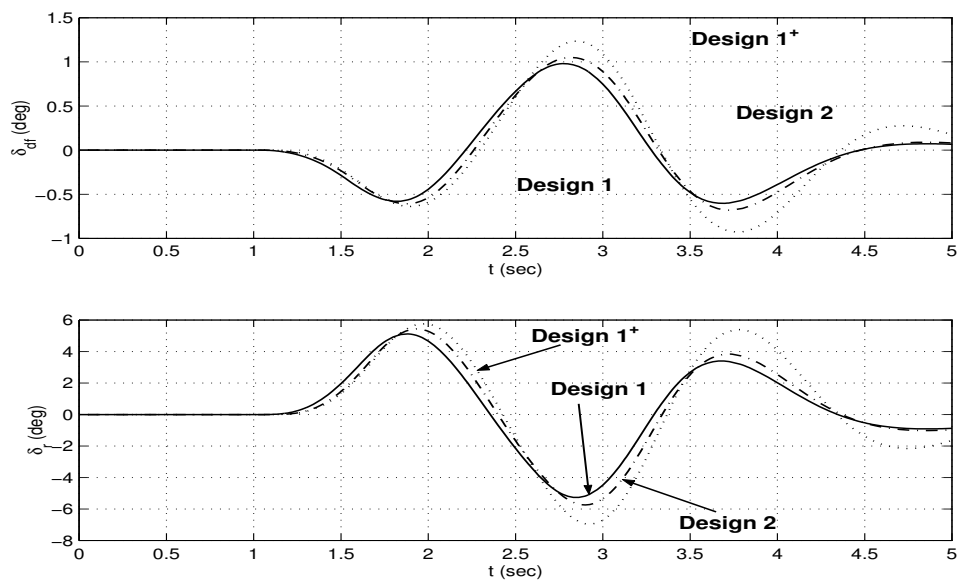
$$u(t) = 8.1244 \begin{bmatrix} 0.1553 \sin(6t + 30^\circ) & 0.9879 \sin(6t - 162^\circ) \end{bmatrix}^T. \quad (4.49)$$

Time Response Data: Reference Command Following.

- *Unit Step Reference Command Following.* The responses to unit step reference commands for ϕ and β are plotted in Figure 4.13. As expected, ϕ and β follow the commands very well - with a little overshoot and a settling time of about 1 sec. The corresponding controls are plotted in Figure 4.14. They are acceptable in size.
- *Smooth Pulse Reference Command Following: Bank Angle.* The response to a smooth (doublet) pulse reference command for ϕ is plotted in Figure 4.15. The corresponding controls are plotted in Figure 4.16. They are acceptable in size [1, pp. 50–51, 21].
- *Smooth Pulse Reference Command Following: Side Slip Angle.* The response to a smooth (doublet) pulse reference command for β is plotted in Figure 4.17. The corresponding controls are plotted in Figure 4.18. They are acceptable in size [1, pp. 53–54, 21].

Figure 4.13: Response to Step Reference Command - Outputs y Figure 4.14: Response to Step Reference Command - Controls u

Figure 4.15: Response to ϕ Doublet Pulse Command - Outputs y Figure 4.16: Response to ϕ Doublet Pulse Command - Controls u

Figure 4.17: Response to β Doublet Pulse Command - OutputsFigure 4.18: Response to β Doublet Pulse Command - Controls

CHAPTER 5

$\mathcal{H}^\infty$ Control System Design: X-29 Longitudinal Dynamics

In this section, we develop two $\mathcal{H}^\infty$ mixed-sensitivity control designs for the X-29 aircraft's longitudinal dynamics. The first is a pitch attitude control system - exploiting only pitch feedback. The second design is a speed-pitch control system in which both speed and pitch measurements are fed back.

5.0.1 Design 1: Pitch Attitude Feedback

In this section, we consider the design of a control system for the longitudinal dynamics of a forward swept wing X-29 aircraft. A three-input one-output model is used to develop a weighted $\mathcal{H}^\infty$ mixed-sensitivity design.

Model for X-29 Longitudinal Dynamics. A model for the X-29 pitch dynamics is as follows:

$$\dot{x} = Ax + Bu \quad (5.1)$$

$$y = Cx + Du \quad (5.2)$$

where

$$A = \begin{bmatrix} -0.0425 & -0.1399 & -0.0079 & -0.5613 \\ -0.0452 & -0.5482 & 0.9893 & 0.0000 \\ 0.0224 & 3.6040 & -0.2221 & 0.0000 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix} \quad (5.3)$$

$$B = \begin{bmatrix} -0.0343 & -0.0922 & 0.0048 \\ -0.0610 & -0.0766 & -0.0242 \\ 1.5666 & -0.3201 & -0.7655 \\ 0 & 0 & 0 \end{bmatrix} \quad (5.4)$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \quad (5.5)$$

$$D = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad (5.6)$$

$$u = \begin{bmatrix} \delta_c & - & \text{canard} & (deg) \\ \delta_{sf} & - & \text{flap} & (deg) \\ \delta_{stf} & - & \text{strake flap} & (deg) \end{bmatrix} \quad (5.7)$$

$$x = \begin{bmatrix} v & - & \text{speed} & (ft/sec) \\ \alpha & - & \text{angle of attack} & (deg) \\ q & - & \text{pitch rate} & (deg/sec) \\ \theta & - & \text{pitch angle} & (deg) \end{bmatrix} \quad (5.8)$$

$$y = \begin{bmatrix} \theta & - & \text{pitch angle} & (deg) \end{bmatrix} \quad (5.9)$$

The aircraft is characterized by a lightly damped stable phugoid mode, a damping mode, and a pitch-up instability. The canard, symmetric flap, and strake flap are used to control the pitch angle. (In principle, these three surfaces can be used to control three variables independently. From the beginning, the X-29 was designed as a research vehicle.)

Actuator Dynamics. Associated with each of the three control surfaces is an actuator. A dynamical model for each is now presented:

$$H_{canard} = 0.885 \left[\frac{20.2}{s + 20.2} \right] \left[\frac{144.9}{s + 144.9} \right] \left[\frac{(71.4)^2}{s^2 + 2(0.736)(71.4)s + (71.4)^2} \right] \quad (5.10)$$

$$H_{sym\ flap} = \left[\frac{20.2}{s + 20.2} \right] \left[\frac{144.9}{s + 144.9} \right] \left[\frac{(71.4)^2}{s^2 + 2(0.736)(71.4)s + (71.4)^2} \right] \quad (5.11)$$

$$H_{strake\ flap} = \left[\frac{50}{s + 50} \right] \left[\frac{100}{s + 100} \right] \left[\frac{(325)^2}{s^2 + 2(0.7)(325)s + (325)^2} \right]. \quad (5.12)$$

Such dynamics become very important in evaluating aircraft performance when the closed loop system possesses a “high bandwidth.” Each actuator has a low frequency pole situated at $s = -20.2$ rad/sec. We therefore expect that when the closed loop system exhibits a bandwidth greater than ~ 2 rad/sec, we expect problems. We expect significant problems particularly if the actuator dynamics have not been accounted for in the design of the flight control system. These ideas will be illustrated in what follows.

Open Loop Bandwidth Design Specification. Based on the above actuator dynamics, we sought an open loop bandwidth of about 2 rad/sec - about one decade below our actuator dynamics.

Plant Singular Values. The plant singular values - with and without actuator dynamics - are plotted in Figure 5.1. The peak near 0.156 rad/sec is due to the aircraft's lightly damped phugoid mode.

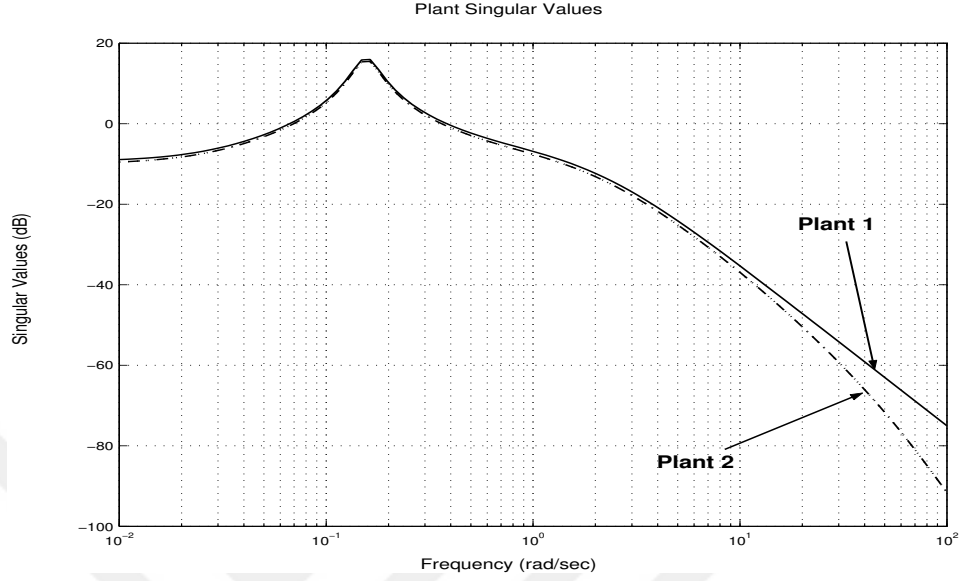


Figure 5.1: X-29 Singular Values

Weighting Function Parameters. Weighting function parameters were selected as follows:

$$\omega_{b1} = 1 \quad (5.13)$$

$$M_{s1} = 10 \quad (5.14)$$

$$\omega_{bu1} = 833 \quad \omega_{bu2} = 200 \quad \omega_{bu3} = 666 \quad (5.15)$$

$$M_{u1} = 0.01 \quad M_{u2} = 0.01 \quad M_{u3} = 0.01 \quad (5.16)$$

$$\omega_{bc1} = 10 \quad (5.17)$$

$$M_{y1} = 1000 \quad (5.18)$$

$$\epsilon = 0.001 \quad (5.19)$$

Weighting Functions. Weighting functions were selected as follows:

$$W_1^{-1} = \left[\frac{s/M_{s1} + \omega_{b1}}{s + \omega_{b1}\epsilon} \right]^{-1} = \left[\frac{s + 0.001}{0.1s + 1} \right] \quad (5.20)$$

$$W_2^{-1} = \begin{bmatrix} \frac{s + \omega_{bu1}M_{u1}}{\epsilon s + \omega_{bu1}} & 0 & 0 \\ 0 & \frac{s + \omega_{bu2}M_{u2}}{\epsilon s + \omega_{bu2}} & 0 \\ 0 & 0 & \frac{s + \omega_{bu3}M_{u3}}{\epsilon s + \omega_{bu3}} \end{bmatrix}^{-1} \quad (5.21)$$

$$= \begin{bmatrix} \frac{0.001s + 833}{s + 8.333} & 0 & 0 \\ 0 & \frac{0.001s + 200}{s + 2} & 0 \\ 0 & 0 & \frac{0.001s + 666}{s + 6.666} \end{bmatrix} \quad (5.22)$$

$$W_3^{-1} = \left[\left(\frac{s + \omega_{bc1}/\sqrt{M_{y1}}}{\sqrt{\epsilon}s + \omega_{bc1}} \right)^2 \right]^{-1} = \left[\left(\frac{0.0316s + 10}{s + 0.3162} \right)^2 \right] \quad (5.23)$$

The above weighting functions may be visualized as shown in Figure 5.2.

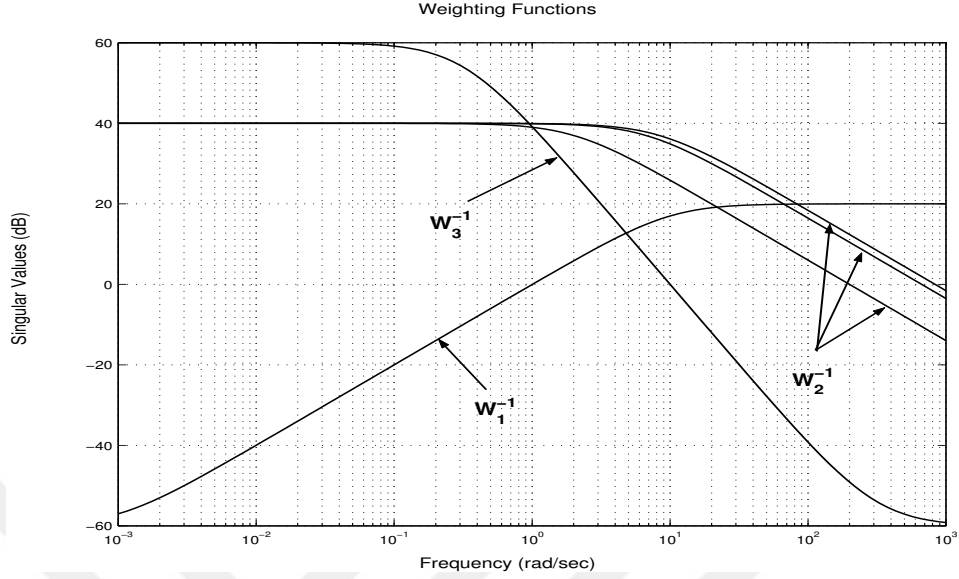


Figure 5.2: Weighting Functions for Mixed-Sensitivity $\mathcal{H}^\infty$ Optimization

Closed Loop Poles: A Bad Design. The closed loop poles that result - from a straight forward $\mathcal{H}^\infty$ mixed-sensitivity optimization - are as follows:

Poles	Damping	Frequency(rad/sec)
$-3.39e + 005$	$1.00e + 000$	$3.39e + 005$
$-3.23e + 002$	$1.00e + 000$	$3.23e + 002$
$-3.09e + 002$	$1.00e + 000$	$3.09e + 002$
$-1.63e + 001$	$1.00e + 000$	$1.63e + 001$
$-6.89e + 000$	$1.00e + 000$	$6.89e + 000$
$-3.42e + 000 \pm 3.01e + 000i$	$7.51e - 001$	$4.56e + 000$
$-2.29e + 000$	$1.00e + 000$	$2.29e + 000$
$-2.19e - 002 \pm 1.54e - 001i$	$1.40e - 001$	$1.56e - 001$
$-3.01e - 002$	$1.00e + 000$	$3.01e - 002$
$-1.51e + 000$	$1.00e + 000$	$1.51e + 000$
$-2.02e + 000$	$1.00e + 000$	$2.02e + 000$
$-5.41e - 001$	$1.00e + 000$	$5.41e - 001$

The resulting minimum γ is

$$\gamma = 0.6148 \quad (5.24)$$

Because the design results in very lightly damped closed loop poles ($\zeta = 0.14$), the design is unacceptable. This occurred because a straight-forward $\mathcal{H}^\infty$ mixed-sensitivity optimization results in a controller that cancels the plant's lightly damped phugoid mode. To prevent this undesirable plant pole inversion,

we use the bilinear transformation.

Bilinear Transformation Parameters. The bilinear transformation (BLT) parameters selected were as follows:

$$p_1 = -0.10 \quad (5.25)$$

$$p_2 = -10^{20}. \quad (5.26)$$

This selection results in

$$s = \frac{\tilde{s} + p_1}{\frac{\tilde{s}}{p_2} + 1} = \frac{\tilde{s} - 0.10}{\frac{\tilde{s}}{-10^{20}} + 1} \approx \tilde{s} - 0.10 \quad (\text{inverse transform}) \quad (5.27)$$

$$\tilde{s} \approx s + 0.10 \quad (\text{transform}) \quad (5.28)$$

Reasons for this selection are provided below. The transform performs a rightward (“destabilizing”) shift. The inverse transform performs a leftward (“stabilizing”) shift.

Design 1: Actuator Dynamics Neglected In Design Process. Our first design neglected the aircraft’s actuator dynamics. The design procedure for Design 1 was as follows:

- The plant was transformed using the above bilinear transformation. The transformation moves all plant poles and zeros to the right 0.1 units - making the plant look more unstable. The transformation specifically moves the lightly damped (troublesome) phugoid mode into the right half plane - thus preventing an undesirable cancelation.
- A generalized plant G was formed using the transformed plant $\hat{P}$ and the original weighting functions W_1 , W_2 , and W_3 .
- A design $\hat{K}$ was obtained using the generalized plant (including the transformed plant and weighting functions).
- The resulting controller was then transformed back to get the final control system design K .

Additional details are given below. The Robust Control Toolbox’s “hinftol” command was used to obtain the design. An optimality tolerance of 0.0001 was used. The resulting minimum γ is

$$\gamma = 0.6703. \quad (5.29)$$

Since $\gamma < 1$, this implies that our design specification (reflected within our weighting functions) were not too aggressive.

Comment 5.0.1 (Selection of Weighting Functions for X-29 Longitudinal Control System Design)

We now provide some rationale for the selection of the weighting functions (W_1, W_2, W_3) and the associated design parameters.

- Selection of Bilinear Transformation Shifting Parameter p_1 : Closed Loop Damping. *The nominal plant P is transformed via the bilinear transformation to prevent its lightly damped phugoid poles from being canceled by the controller. We selected $p_2 = -10^{20}$ to be very large so that the bilinear transformation becomes a mere rightward shifting transformation by $-p_1$ units.*

- *Since all left half plane plant poles are canceled when the $\mathcal{H}^\infty$ design methodology is used, the shifting parameter p_1 must be chosen such that all plant poles which must NOT be canceled by the resulting controller are moved into the right half plane after applying the transformation.*

This suggests that a lower bound on the magnitude $|p_1|$ must be enforced in order to shift the desired poles into the right half plane. More specifically, p_1 has to be smaller than the minimum of the real parts of the plant poles to be moved into the right half plane. For our problem, the phugoid mode dictates that $p_1 < -0.0219$.

*We selected $p_1 = -0.1$ to be on the order of the imaginary parts
of the phugoid mode ($s = -0.0219 \pm j0.1544$).*

This selection is based on the eigen-structure of Hamiltonian systems and our desire to achieve a damping of $\zeta = 0.7$. The detailed reasoning is as follows:

- *This selection of p_1 moves the phugoid mode into the right half plane to $\hat{s} = -0.0219 \pm j0.1544 + 0.1 = 0.0781 \pm j0.1544$. The associated open loop Hamiltonian will have an eigenvalue at the reflection of this location; i.e. at $\hat{s} = -0.0781 \pm j0.1544$. The $\mathcal{H}^\infty$ design method places a zero at this left half plane location. When shifted back, the final controller ends up with a zero at $s = -0.0781 \pm j0.1544 - 0.1 = -0.1781 \pm j0.1544$ ($\zeta = 0.7560$). The zeros listed above have a zero at this location - thus supporting our argument. It is this zero at $-0.1781 \pm j0.1544$ ($\zeta = 0.7560$) that draws the phugoid mode to a location that has desirable damping; i.e. $\zeta > 0.7$.*
- Impact of Shifting Parameter p_1 . *Decreasing p_1 (i.e. making it more negative), increases the resulting bandwidth (open and closed loop). As the bandwidth increases, more control action results for typical test commands (e.g. step commands). This suggests that p_1 should not be made too negative.*
- Loss of Integral Action: Accommodating Steady State Error to Step Command. *The transformed plant $\hat{P}$ and original weighting functions are used in the design process. Since the resulting design $\hat{K}$ is transformed back, near-integrators associated with the transformed controller $\hat{K}$ (due to W_1 selection) are shifted into the left half plane by $|p_1| = 0.1$ units. This, however, destroys the low frequency command following properties of the design $\hat{K}$. When a step reference command is issued to the resulting closed loop system (with K), a steady state error e_{ss} results. To address this, we incorporated a constant matrix into a reference command pre-filter design (described below). The matrix was selected to be $T_o(0)^{-1}$ - the inverse of the complementary sensitivity transfer function matrix $T_o = PK[I + PK]^{-1}$. While this resulted in an acceptable design - with desirable step command and low frequency command following - it must be noted that in practice, such an approach may be too sensitive with respect to low frequency uncertainty within the feedback loop.*

- **Loss of Closed Loop Frequency Response Shaping Inequalities.** *The standard $\mathcal{H}^\infty$ mixed sensitivity inequality $\|\hat{T}_{wz}\|_{\mathcal{H}^\infty} < \gamma$ holds for the closed loop $\hat{T}_{wz}$ formed with the transformed plant $\hat{P}$, the transformed controller $\hat{K}$, and the original weighting functions W_1, W_2, W_3 . The inequality gives rise to response shaping inequalities for $\hat{S}, \hat{K}\hat{S}$, and $\hat{T}$. These are not very useful.*

The inequality also holds for the closed loop T_{wz} formed with P, K , and the back transformed weighting functions (i.e. weighting functions that are more stable). Given this, it follows that

- *When the BLT is used, the inequality $\|T_{wz}\|_{\mathcal{H}^\infty} < \gamma$ does NOT hold for the closed loop system T_{wz} consisting of P, K , and the weighting functions W_1, W_2, W_3 .*
- *A consequence of this is the loss in the physical significance (meaning) of the weighting functions W_1, W_2, W_3 .*
- **Sensitivity Weighting.** *A first order weighting function was selected for W_1 . ω_{b_1} was picked to be 1 rad/sec.*
As the sensitivity bandwidth parameters ω_{b_i} are increased (decreased), the closed loop system bandwidth increases (decreases) (e.g. gain crossover frequency for the sensitivity, complementary sensitivity transfer function 3 dB bandwidth). Increasing and/or decreasing the ω_{b_i} too much results in an unacceptable design.
- **Control Sensitivity Weighting.** *Initially, we selected W_2 to be a constant (i.e. $W_2 = M_u$). This selection resulted in large controls and did not permit us to control the adequately control the roll off of the control sensitivity KS . To remedy this, we selected the above dynamic W_2 . By so doing, we were able to shape the high frequency behavior of KS ; e.g. its associated high frequency gain crossover frequencies.*

- *Initially, we started with $\omega_{b_{u1}} = \omega_{b_{u2}} = 250$ rad/sec - about two decades above our desired open loop gain crossover of 2 rad/sec (about one decade below our actuator dynamics at 20.2 rad/sec).*
- *$\omega_{b_{u1}}$ was increased in order to encourage more canard activity.*
- *$\omega_{b_{u2}}$ was decreased in order to encourage less symmetric flap activity.*
- *$\omega_{b_{u3}}$ was increased in order to encourage more strake flap activity.*

Since $\omega_{b_{u1}} > \omega_{b_{u3}} > \omega_{b_{u2}}$, it follows that we are more aggressive with respect to symmetric flap action than strake flap action and, we are more aggressive with respect to strake flap action than canard action.

- **Complementary Sensitivity Weighting.** *A second order weighting function was selected for W_3 in order to control the bandwidth associated with the complementary sensitivity transfer function matrix T .*
 - *Initially, we started with $\omega_{bc} = 20$ rad/sec - one decade above our desired gain crossover frequency of 2 rad/sec.*
 - *ω_{bc} was decreased to reduce bandwidth, overshoot, and improve damping.*

■

Closed Loop Poles for Design 1: A Good Design. The closed loop poles that result from our BLT (bilinear transformation) approach are as follows:

Poles	Damping	Frequency(rad/sec)
$-4.85e + 004$	$1.00e + 000$	$4.85e + 004$
$-3.23e + 002$	$1.00e + 000$	$3.23e + 002$
$-3.09e + 002$	$1.00e + 000$	$3.09e + 002$
$-1.64e + 001$	$1.00e + 000$	$1.64e + 001$
$-6.99e + 000$	$1.00e + 000$	$6.99e + 000$
$-3.41e + 000 \pm 2.98e + 000i$	$7.53e - 001$	$4.53e + 000$
$-1.71e + 000$	$1.00e + 000$	$1.71e + 000$
$-1.78e - 001 \pm 1.54e - 001i$	$7.56e - 001$	$2.36e - 001$
$-1.85e - 001$	$1.00e + 000$	$1.85e - 001$
$-2.29e + 000$	$1.00e + 000$	$2.29e + 000$
$-5.46e - 001$	$1.00e + 000$	$5.46e - 001$
$-2.12e + 000$	$1.00e + 000$	$2.12e + 000$

We observe that all closed loop poles for Design 1 exhibit a damping $\zeta > 0.7$ - as desired.

Closed Loop Zeros for Design 1. The resulting closed loop (complementary sensitivity) transmission zeros are as follows:

Zeros	Damping	Frequency(rad/sec)
$-8.00e + 005$	$1.00e + 000$	$8.00e + 005$
$-3.16e + 002$	$1.00e + 000$	$3.16e + 002$
$-3.16e + 002$	$1.00e + 000$	$3.16e + 002$
$-7.01e + 000$	$1.00e + 000$	$7.01e + 000$
$-2.29e + 000$	$1.00e + 000$	$2.29e + 000$
$-2.12e + 000$	$1.00e + 000$	$2.12e + 000$
$-1.93e - 001 \pm 1.81e - 001i$	$7.30e - 001$	$2.65e - 001$
$-4.92e - 001$	$1.00e + 000$	$4.92e - 001$
$-5.51e - 001$	$1.00e + 000$	$5.51e - 001$
$-1.86e - 001$	$1.00e + 000$	$1.86e - 001$

One low frequency closed loop zero (also a compensator zero) which will significantly increase the overshoot resulting from step reference commands is situated at $s = -0.492$. Because of this zero, a reference command pre-filter will be essential. The design of this pre-filter for Design 1 will be based on the location of this zero. This is discussed below.

Minimum γ for Design 1. The resulting minimum γ for Design 1 is

$$\gamma = 0.6703. \quad (5.30)$$

The design was found using the Robust Control Toolbox's "hinftol" command - with an optimality tolerance of 0.0001.

Design 1⁺: Actuator Dynamics Included After Design Process.

When the actuator dynamics are inserted into the feedback loop, we obtain what we call Design 1⁺. This design did not consider the actuator dynamics in the design process. Given this, and given our high bandwidth of ~ 2 rad/sec, we anticipate trouble with this design.

Closed Loop Poles for Design 1⁺. The closed loop poles associated with Design 1⁺ are as follows:

Poles	Damping	Frequency(rad/sec)
$-4.85e + 004$	$1.00e + 000$	$4.85e + 004$
$-2.28e + 002 \pm 2.32e + 002i$	$7.00e - 001$	$3.25e + 002$
$-3.23e + 002$	$1.00e + 000$	$3.23e + 002$
$-3.09e + 002$	$1.00e + 000$	$3.09e + 002$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-1.00e + 002$	$1.00e + 000$	$1.00e + 002$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-4.98e + 001$	$1.00e + 000$	$4.98e + 001$
$-5.26e + 001 \pm 4.83e + 001i$	$7.37e - 001$	$7.14e + 001$
$-2.06e + 001 \pm 8.53e + 000i$	$9.24e - 001$	$2.23e + 001$
$-6.90e + 000$	$1.00e + 000$	$6.90e + 000$
$-1.27e + 000 \pm 3.44e + 000i$	$3.45e - 001$	$3.67e + 000$
$-1.08e - 001 \pm 1.47e - 001i$	$5.91e - 001$	$1.82e - 001$
$-2.91e - 001$	$1.00e + 000$	$2.91e - 001$
$-5.78e - 001$	$1.00e + 000$	$5.78e - 001$
$-2.35e + 000$	$1.00e + 000$	$2.35e + 000$
$-1.60e + 000$	$1.00e + 000$	$1.60e + 000$
$-2.06e + 000$	$1.00e + 000$	$2.06e + 000$
$-5.26e + 001 \pm 4.83e + 001i$	$7.36e - 001$	$7.14e + 001$
$-2.02e + 001$	$1.00e + 000$	$2.02e + 001$

As expected, we have a problem. One of the closed loop poles has a damping of 0.345 and undamped natural frequency of $\omega_n = 3.67$ rad/sec. If this closed loop pole were situated at a very high frequency, its residue matrix would be small, and it would not bother us so much. This, however, is not the case.

Closed Loop Zeros for Design 1⁺. The actuator dynamics do not introduce additional zeros into the loop. Given this, the closed loop (complementary sensitivity) transmission zeros associated with Design 1⁺ are identical to those associated with Design 1.

Reference Command Pre-Filter for Designs 1 and 1⁺. Designs 1 and 1⁺ have the same controller. The controller has a zero at $s = -0.492$. Based on this, we developed the following reference command pre-filter for Designs 1 and 1⁺:

$$W_o = [T_{ry_o}(0)]^{-1} \left[\frac{0.9}{s+0.9} \right] \quad (5.31)$$

$$= \left[1.0242 \right]^{-1} \left[\frac{0.9}{s+0.9} \right] \quad (5.32)$$

$$= \left[\frac{0.8787}{s+0.9} \right] \quad (5.33)$$

where $T_{ry_o} = PK[I + PK]^{-1}$ is the nominal complementary sensitivity for Design 1. Also note that our pre-filter has a low pass filter which filters pitch attitude reference commands. The low pass filter $\frac{z}{s+z}$ was selected with $z = 0.9$ - about 2 times larger than the compensator zero at $s = -0.492$.

Design 2: Actuator Dynamics Included In Design Process. Because Design 1⁺ does not adequately address our actuator dynamics, we developed another design - Design 2. This design includes the actuator dynamics in the initial plant model P . After application of the BLT, the resulting $\hat{P}$ is combined with W_1 , W_2 , and W_3 to form the generalized plant G . A design $\hat{K}$ is then obtained and transformed back to get the final design K .

Closed Loop Poles for Design 2. The closed loop poles associated with Design 2 are as follows:

Poles	Damping	Frequency(rad/sec)
$-1.46e + 004$	$1.00e + 000$	$1.46e + 004$
$-3.16e + 002$	$1.00e + 000$	$3.16e + 002$
$-3.16e + 002$	$1.00e + 000$	$3.16e + 002$
$-2.28e + 002 \pm 2.32e + 002i$	$7.00e - 001$	$3.25e + 002$
$-2.28e + 002 \pm 2.32e + 002i$	$7.00e - 001$	$3.25e + 002$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-1.00e + 002$	$1.00e + 000$	$1.00e + 002$
$-5.25e + 001 \pm 4.83e + 001i$	$7.36e - 001$	$7.14e + 001$
$-4.96e + 001$	$1.00e + 000$	$4.96e + 001$
$-1.68e + 001 \pm 5.43e + 000i$	$9.51e - 001$	$1.76e + 001$
$-7.01e + 000$	$1.00e + 000$	$7.01e + 000$
$-3.45e + 000 \pm 3.19e + 000i$	$7.34e - 001$	$4.70e + 000$
$-1.78e - 001 \pm 1.54e - 001i$	$7.56e - 001$	$2.36e - 001$
$-1.82e - 001$	$1.00e + 000$	$1.82e - 001$
$-5.69e - 001$	$1.00e + 000$	$5.69e - 001$
$-1.71e + 000$	$1.00e + 000$	$1.71e + 000$
$-2.12e + 000$	$1.00e + 000$	$2.12e + 000$
$-2.29e + 000$	$1.00e + 000$	$2.29e + 000$
$-1.00e + 002$	$1.00e + 000$	$1.00e + 002$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-5.00e + 001$	$1.00e + 000$	$5.00e + 001$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-5.26e + 001 \pm 4.83e + 001i$	$7.36e - 001$	$7.14e + 001$
$-5.26e + 001 \pm 4.83e + 001i$	$7.36e - 001$	$7.14e + 001$
$-5.26e + 001 \pm 4.83e + 001i$	$7.36e - 001$	$7.14e + 001$
$-2.02e + 001$	$1.00e + 000$	$2.02e + 001$
$-2.02e + 001$	$1.00e + 000$	$2.02e + 001$
$-2.02e + 001$	$1.00e + 000$	$2.02e + 001$

As expected, all closed loop poles associated with Design 2 are at desirable locations - with damping $\zeta > 0.7$.

Closed Loop Zeros for Design 2. The resulting closed loop (complementary sensitivity) transmission zeros are as follows:

Zeros	Damping	Frequency(rad/sec)
$1.29e + 012$	$-1.00e + 000$	$1.29e + 012$
$-6.77e + 005$	$1.00e + 000$	$6.77e + 005$
$-2.28e + 002 \pm 2.32e + 002i$	$7.00e - 001$	$3.25e + 002$
$-3.17e + 002$	$1.00e + 000$	$3.17e + 002$
$-3.16e + 002$	$1.00e + 000$	$3.16e + 002$
$-6.52e + 001 \pm 8.40e + 001i$	$6.13e - 001$	$1.06e + 002$
$-1.30e + 002$	$1.00e + 000$	$1.30e + 002$
$-1.00e + 002$	$1.00e + 000$	$1.00e + 002$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-5.26e + 001 \pm 4.83e + 001i$	$7.36e - 001$	$7.14e + 001$
$-5.26e + 001 \pm 4.83e + 001i$	$7.36e - 001$	$7.14e + 001$
$-5.26e + 001 \pm 4.83e + 001i$	$7.36e - 001$	$7.14e + 001$
$-5.00e + 001$	$1.00e + 000$	$5.00e + 001$
$-4.01e + 001$	$1.00e + 000$	$4.01e + 001$
$-7.03e + 000$	$1.00e + 000$	$7.03e + 000$
$-2.29e + 000$	$1.00e + 000$	$2.29e + 000$
$-2.12e + 000$	$1.00e + 000$	$2.12e + 000$
$-5.71e - 001$	$1.00e + 000$	$5.71e - 001$
$-4.26e - 001$	$1.00e + 000$	$4.26e - 001$
$-1.93e - 001 \pm 1.91e - 001i$	$7.12e - 001$	$2.72e - 001$
$-2.02e + 001$	$1.00e + 000$	$2.02e + 001$
$-2.02e + 001$	$1.00e + 000$	$2.02e + 001$
$-1.84e - 001$	$1.00e + 000$	$1.84e - 001$
$-2.02e + 001$	$1.00e + 000$	$2.02e + 001$

Minimum γ for Design 2. The resulting minimum γ for Design 2 is

$$\gamma = 0.8209. \quad (5.34)$$

The design was found using the Robust Control Toolbox's "hinftol" command - with an optimality tolerance of 0.0001.

Reference Command Pre-Filter for Design 2. The following reference command pre-filter was designed on the basis of Design 2's controller zeros. Design 2's controller has a zero at $s = -0.524$. The

following pre-filter was developed like that developed for Designs 1 and 1⁺ above:

$$W_o = [T_{ry_o}(0)]^{-1} \left[\frac{0.9}{s+0.9} \right] \quad (5.35)$$

$$= \left[1.0301 \right]^{-1} \left[\frac{0.9}{s+0.9} \right] \quad (5.36)$$

$$= \left[\frac{0.8737}{s+0.9} \right] \quad (5.37)$$

Frequency Response Data.

- *Compensator Singular Values.* The compensator singular values are plotted in Figure 5.3. Because we used the bilinear transformation, the integral behavior “requested” by W_1 has been “shifted away.” We therefore see that for frequencies below 0.1 rad/sec, the compensator singular value is constant - with the minimum singular value being very low! (BLT parameter $p_1 = -0.1$). This will result in steady state errors to step reference command for θ . At higher frequencies, the singular values roll off to ensure that high frequency sensor noise n in e is attenuated.

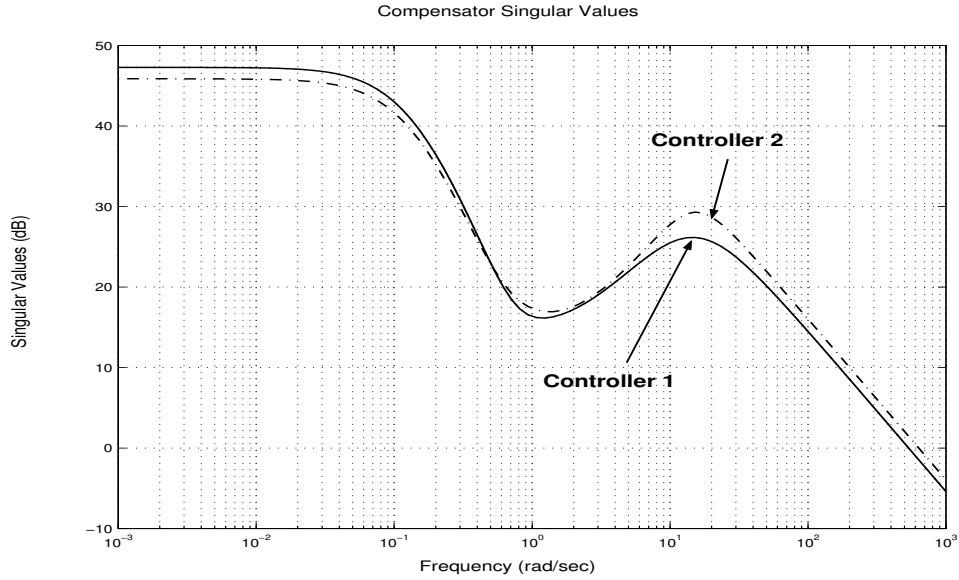


Figure 5.3: Compensator Singular Values

- *Open Loop Singular Values at Plant Output.* Breaking the loop at e or y yields the open loop transfer function matrix $L_o = PK$. Its singular value is plotted in Figure 5.4. The singular values are constant at low frequencies because K is constant at low frequencies - a consequence of the BLT. The plot suggests that low frequency reference commands r will be followed, low frequency output disturbances d_o will be attenuated, and high frequency sensor noise n will be attenuated. More precisely, reference commands r with frequency content below 0.3 rad/sec should be followed to within about 20 dB; that is, with a steady state error of about 10%. Similarly, output disturbances d_o with frequency content below 0.3 rad/sec should be attenuated by approximately 20 dB. Sensor noise n with frequency content above 4 rad/sec will be attenuated.
- *Open Loop Singular Values at Plant Input.* Breaking the loop at u yields the open loop transfer

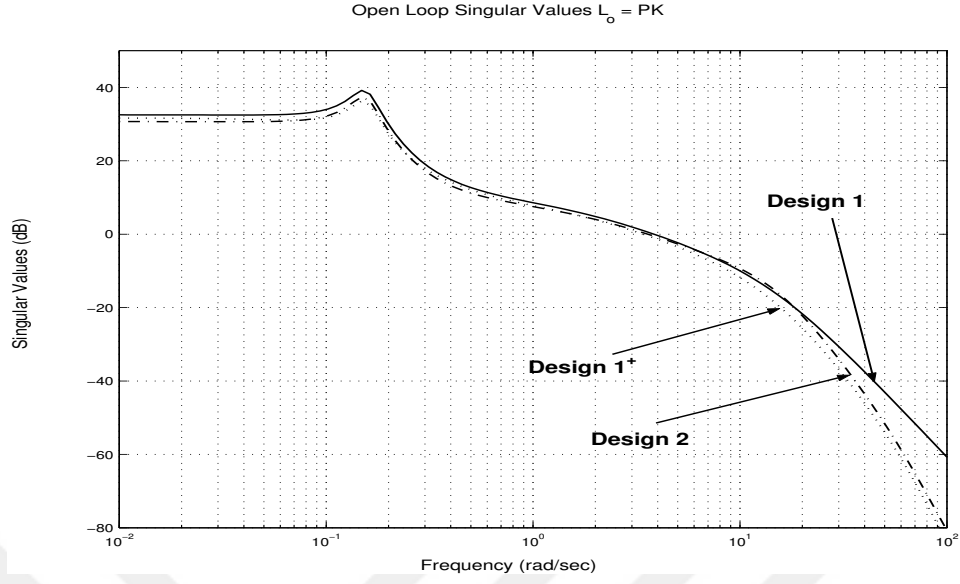


Figure 5.4: Open Loop Singular Values at Plant Output - $L_o = PK$

function matrix $L_i = KP$. The singular values of L_i are plotted in Figure 5.5. The singular value plot for L_i suggests that input disturbances d_i with frequency content below 0.5 rad/sec will be attenuated by about 20 dB - resulting in a u_p having about one tenth the amplitude of the disturbance.

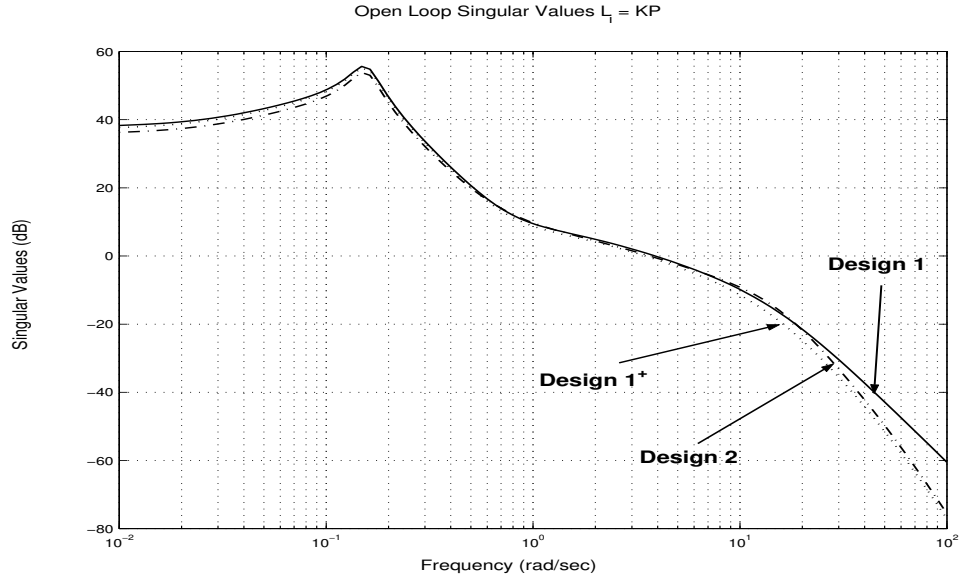


Figure 5.5: Open Loop Singular Values at Plant Input - $L_i = KP$

- *Sensitivity Singular Values at Plant Output.* The sensitivity transfer function matrix associated with e is given by:

$$S_o = [I + PK]^{-1} = [I + L_o]^{-1}. \quad (5.38)$$

The singular values of S_o are plotted in Figure 5.6. The plot suggests that low frequency reference

command will be followed and low frequency output disturbances will be attenuated. More precisely, reference commands r with frequency content below 0.3 rad/sec should be followed to within about 20 dB; that is, with a steady state error of about 10%. Similarly, output disturbances d_o with frequency content below 0.3 rad/sec should be attenuated by approximately 20 dB. The plot also shows acceptable peaking for Designs 1 and 2. Design 1⁺ (as expected) exhibits undesirable peaking.

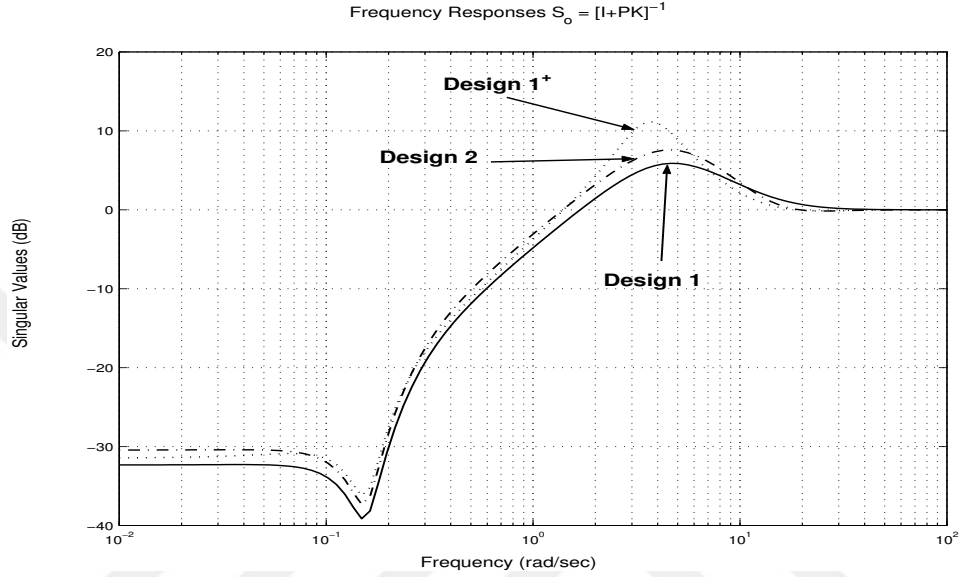


Figure 5.6: Sensitivity Singular Values at Plant Output - $S_o = [I + PK]^{-1}$

- *Sensitivity Singular Values at Plant Input.* The sensitivity transfer function matrix associated with u is given by:

$$S_i = [I + KP]^{-1} = [I + L_i]^{-1}. \quad (5.39)$$

This is also the transfer function matrix from d_i to u_p . The singular values of S_i are plotted in Figure 5.7. The plot indicates that one singular value is small at low frequencies while the others are large. This is expected for this 3-input 3-output transfer function matrix from d_i to u_p because only one variable (i.e. pitch attitude) is being feedback (e.g. PK is SISO). In order to be able to control impact of all three input disturbances, the feedback loop must have at least three feedback variables (for example pitch attitude, speed and angle of attack).

- *Complementary Sensitivity Singular Values at Plant Output.* The complementary sensitivity transfer function matrix associated with the output or e is given by:

$$T_o = I - S_o = PK[I + PK]^{-1}. \quad (5.40)$$

The singular values of T_o are plotted in Figure 5.8. These singular values are important for assessing the impact of a unfiltered reference command r on the output y . The bump in our figure suggests that a prefilter is essential to prevent excessive overshoot to step reference commands. Since the

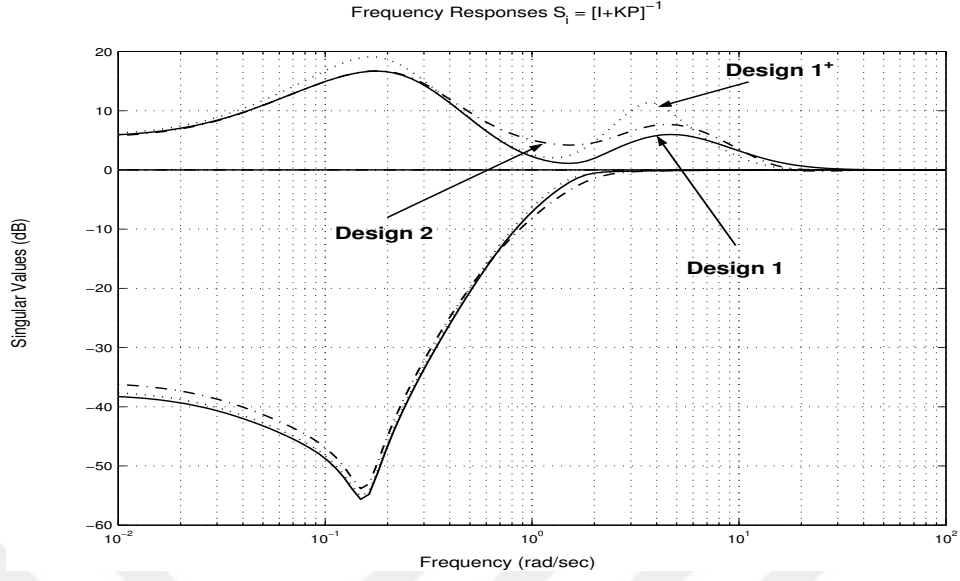


Figure 5.7: Frequency Response: $S_i = [I + KP]^{-1}$

bump starts to grow around 0.5 rad/sec (a compensator zero), the command pre-filter was selected to be $\frac{0.9}{s+0.9}$. The plot shows that low frequency reference commands will be followed. The singular value for S_o shows this a little better. The plot also shows that sensor noise with frequency content above 6 rad/sec will be attenuated. Finally, just as S_o provides us with stability robustness margins, so does T_o .

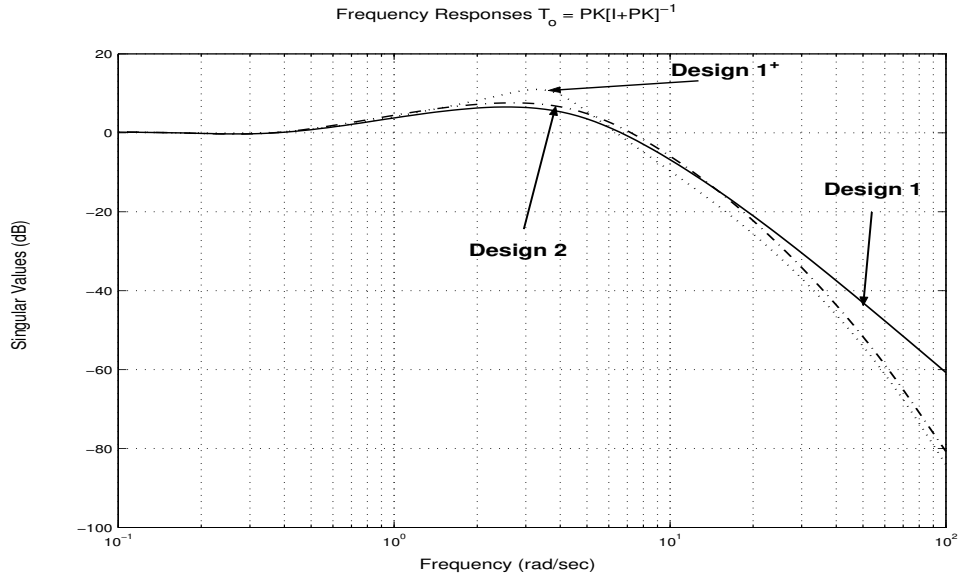


Figure 5.8: Complementary Sensitivity Singular Values at Plant Output - $T_o = PK[I + PK]^{-1}$

- *Closed Loop Singular Values: Reference to Output.* The transfer function matrix from reference commands r to θ is

$$T_{ry} = [I + PK]^{-1} PKW_o \quad (5.41)$$

where W_o refers to the reference command pre-filters designed earlier. Its singular values are plotted in Figure 5.9. The plot suggests that low frequency reference commands will be followed in the steady state and that little overshoot will result during the transient.

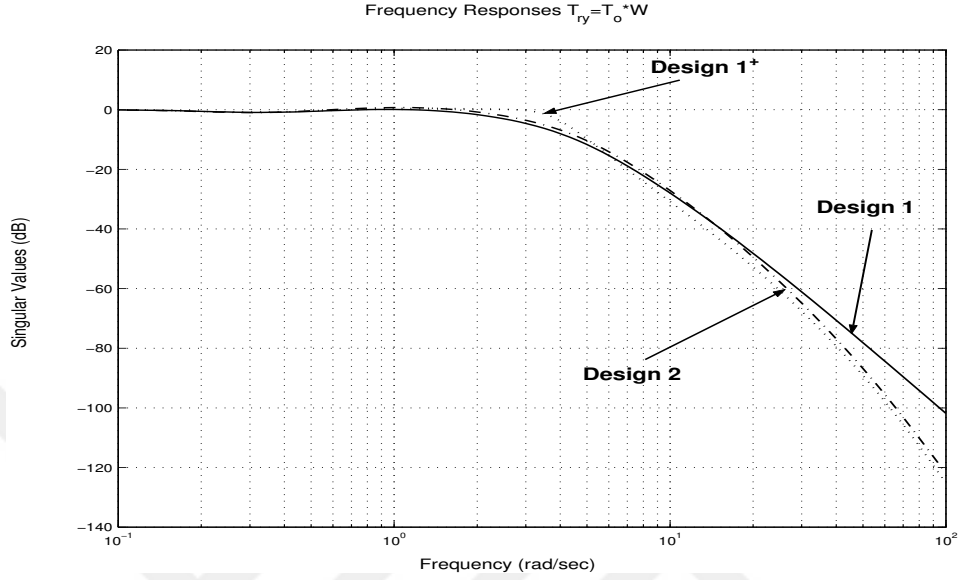


Figure 5.9: Closed Loop Singular Values - T_{ry}

- *Complementary Sensitivity Singular Values at Plant Input.* Figure 5.10 The singular values of T_i are plotted in Figure 5.10. T_i is the transfer function matrix from d_i to u . This plot indicates that low frequency input disturbances with frequency content below 1 rad/sec will be amplified. The amplification in the worst case will be about 20 dB.

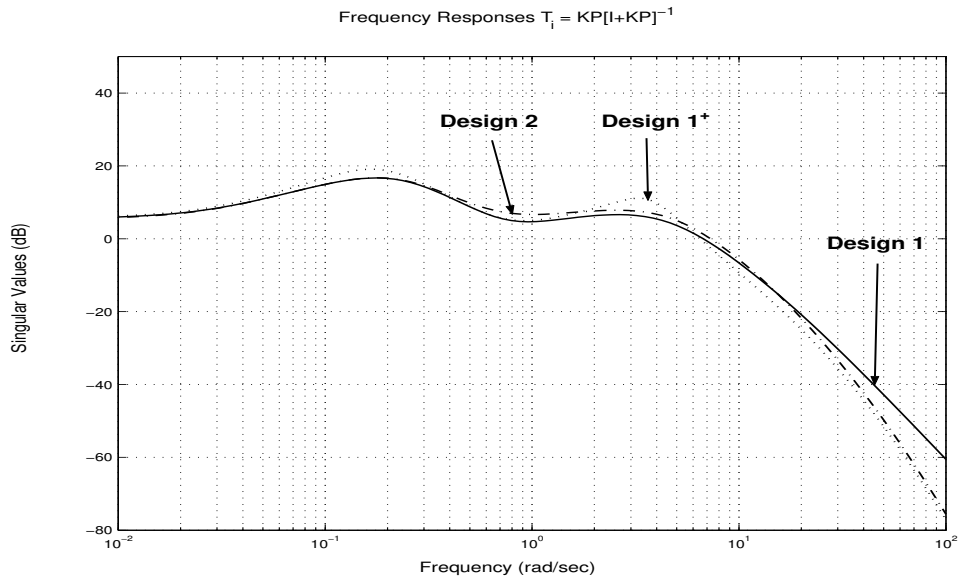


Figure 5.10: Complementary Sensitivity Singular Values at Plant Input - $T_i = KP[I + KP]^{-1}$

- *Input Disturbance to Output Singular Values.* To assess the impact of input disturbances d_i on the

output y , one considers the transfer function matrix from d_i to y :

$$T_{d_i y} = [I + PK]^{-1}P. \quad (5.42)$$

Its singular values are plotted in Figure 5.11. The plot suggests that input disturbances with frequency content below 0.2 rad/sec will be attenuated by 20 dB. High frequency input disturbances will also be attenuated. This is expected since such high frequency input disturbances are directly attenuated by the plant. The plot also shows that some input disturbances with frequency of 2 rad/sec will be attenuated by 10 dB.

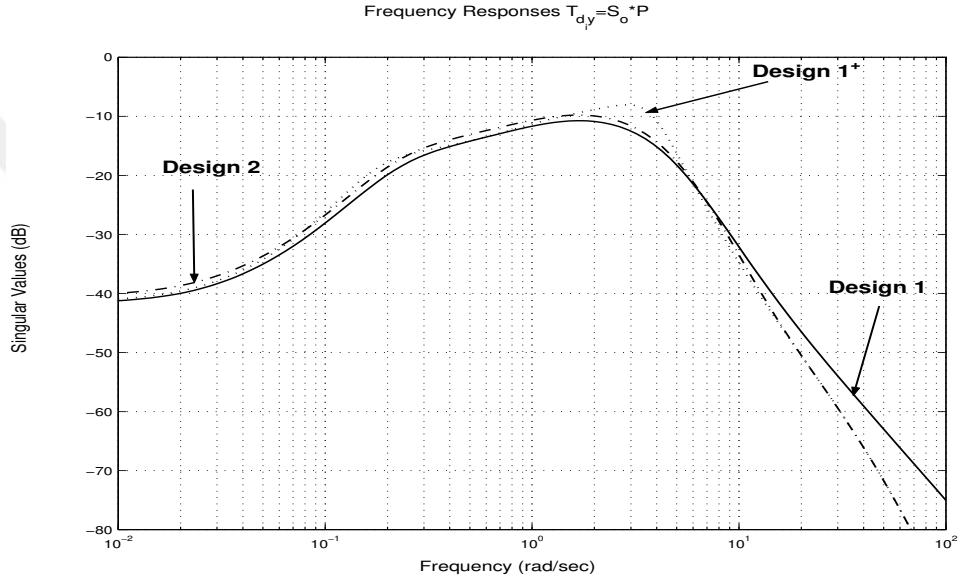


Figure 5.11: Input Disturbance to Output Singular Values - $T_{d_i y}$

- *Reference to Control Singular Values.* To assess the impact of reference commands on control action, one considers the transfer function matrix from r to u :

$$T_{ru} = [I + KP]^{-1}KW = K[I + PK]^{-1}W. \quad (5.43)$$

Its singular values are plotted in Figure 5.12.

An svd at dc yields a maximum singular value

$$\sigma_1 = 5.4635 \quad (14.7495 \text{ dB}) \quad (5.44)$$

a right singular vector

$$v_1 = \begin{bmatrix} -0.5618 & -0.7726 & 0.2956 \end{bmatrix}^T \quad (5.45)$$

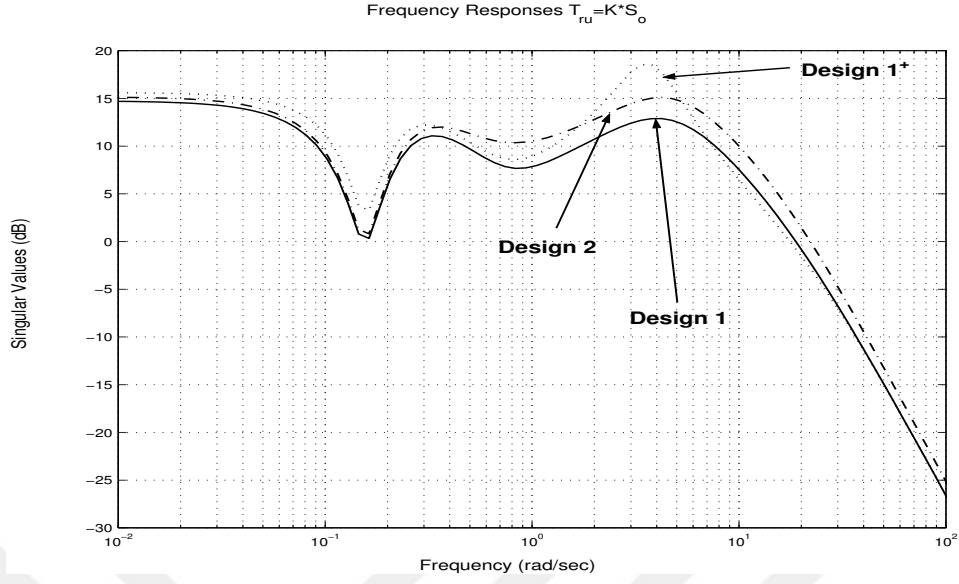


Figure 5.12: Reference to Control Singular Values - T_{ru}

and a left singular vector

$$u_1 = 1 \quad (5.46)$$

This svd implies that if a step reference command acts on the system, then the resulting steady state control output will be

$$u(t) = 5.4635 \begin{bmatrix} -0.5618 & -0.7726 & 0.2956 \end{bmatrix}^T. \quad (5.47)$$

- *Step Reference Command Following.* The response to a unit step θ command is plotted in Figure 5.13. Corresponding speed, v and angle of attack, α can also be seen on the plot. As expected, θ follows the step command well - with a little overshoot. The corresponding controls are plotted in Figure 5.14. They are acceptable in size.
- *Smooth Pulse Reference Command Following.* The response to a smooth pulse reference command θ command is plotted in Figure 5.15. The corresponding controls are plotted in Figure 5.16. They are acceptable in size [1, pp. 48-49, 21].

5.0.2 Design 2: Speed and Pitch Attitude Feedback

In this section, we consider the design of a control system for the longitudinal dynamics of a forward swept wing X-29 aircraft. A three-input two-output model is used to develop a weighted $\mathcal{H}^\infty$ mixed-sensitivity design.

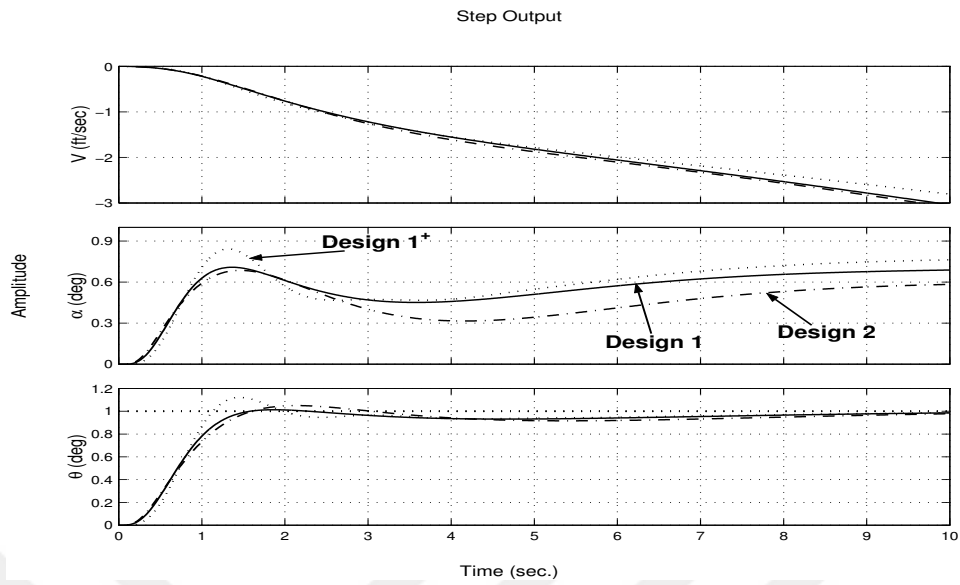


Figure 5.13: Response to Step Reference Command - Outputs

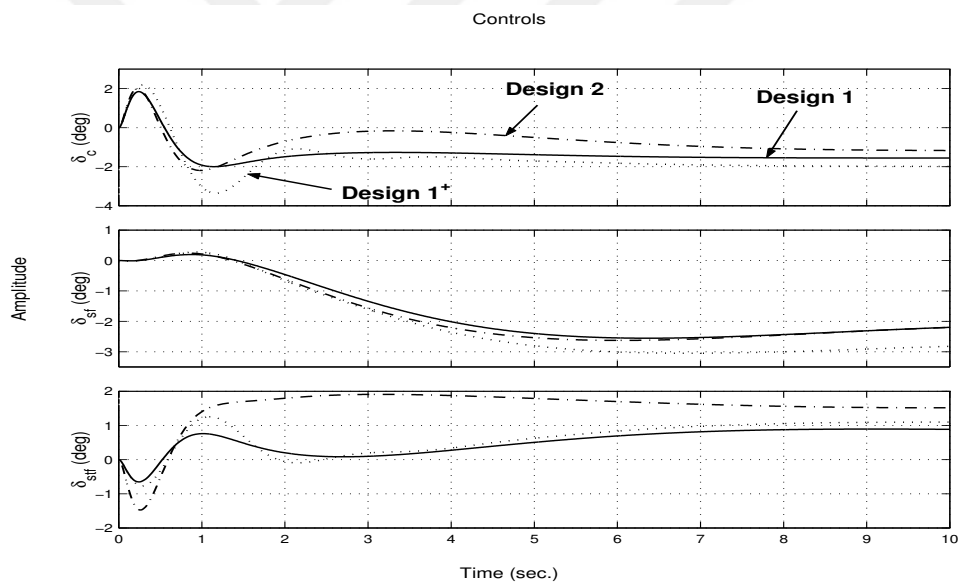
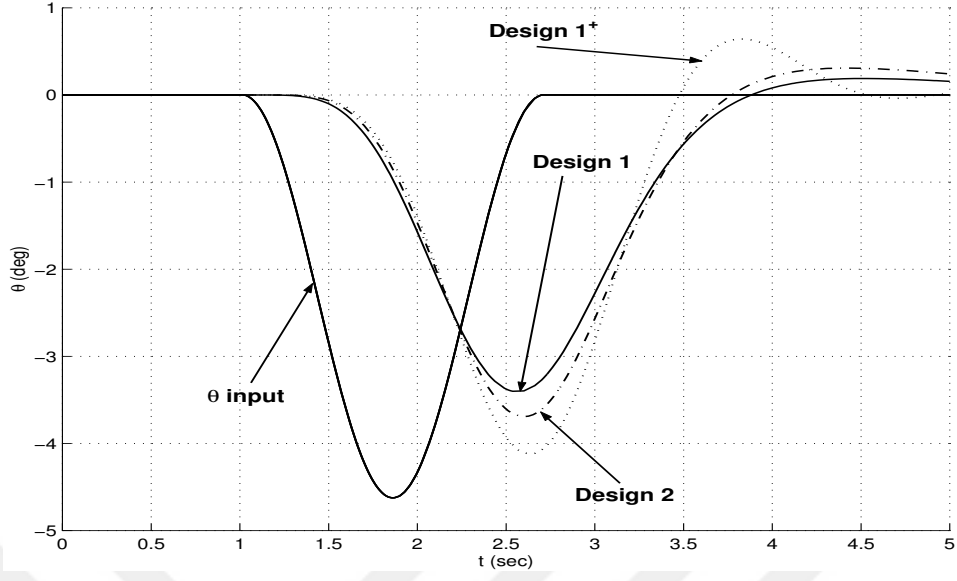
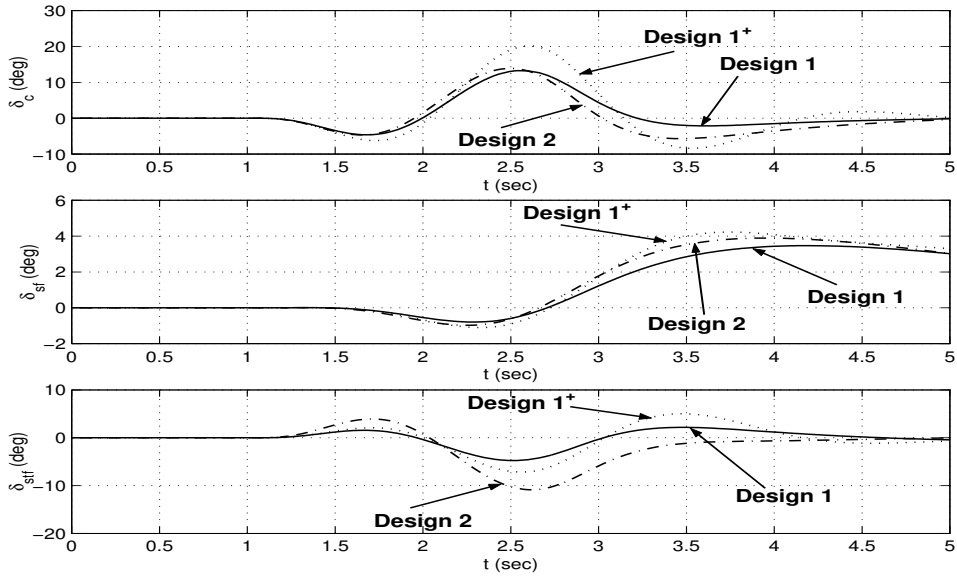


Figure 5.14: Response to Step Reference Command - Controls

Model for X-29 Longitudinal Dynamics. A model for the X-29 pitch dynamics is as follows:

$$\dot{x} = Ax + Bu \quad (5.48)$$

$$y = Cx + Du \quad (5.49)$$

Figure 5.15: Response to θ Doublet Pulse Command - OutputsFigure 5.16: Response to θ Doublet Pulse Command - Controls

where

$$A = \begin{bmatrix} -0.0425 & -0.1399 & -0.0079 & -0.5613 \\ -0.0452 & -0.5482 & 0.9893 & 0.0000 \\ 0.0224 & 3.6040 & -0.2221 & 0.0000 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix} \quad (5.50)$$

$$B = \begin{bmatrix} -0.0343 & -0.0922 & 0.0048 \\ -0.0610 & -0.0766 & -0.0242 \\ 1.5666 & -0.3201 & -0.7655 \\ 0 & 0 & 0 \end{bmatrix} \quad (5.51)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5.52)$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5.53)$$

$$u = \begin{bmatrix} \delta_c - \text{canard} & (deg) \\ \delta_{sf} - \text{flap} & (deg) \\ \delta_{stf} - \text{strake flap} & (deg) \end{bmatrix} \quad (5.54)$$

$$x = \begin{bmatrix} v - \text{speed} & (ft/sec) \\ \alpha - \text{angle of attack} & (deg) \\ q - \text{pitch rate} & (deg/sec) \\ \theta - \text{pitch angle} & (deg) \end{bmatrix} \quad (5.55)$$

$$y = \begin{bmatrix} v - \text{speed} & (ft/sec) \\ \theta - \text{pitch angle} & (deg) \end{bmatrix} \quad (5.56)$$

The aircraft is characterized by a lightly damped stable phugoid mode, a damping mode, and a pitch-up instability. The canard, symmetric flap and strake flap are used to control the speed and the pitch angle.

Actuator Dynamics. Associated with each of the three control surfaces is an actuator. A dynamical model for each is now presented:

$$H_{canard} = 0.885 \left[\frac{20.2}{s + 20.2} \right] \left[\frac{144.9}{s + 144.9} \right] \left[\frac{(71.4)^2}{s^2 + 2(0.736)(71.4)s + (71.4)^2} \right] \quad (5.57)$$

$$H_{sym\ flap} = \left[\frac{20.2}{s + 20.2} \right] \left[\frac{144.9}{s + 144.9} \right] \left[\frac{(71.4)^2}{s^2 + 2(0.736)(71.4)s + (71.4)^2} \right] \quad (5.58)$$

$$H_{strake\ flap} = \left[\frac{50}{s + 50} \right] \left[\frac{100}{s + 100} \right] \left[\frac{(325)^2}{s^2 + 2(0.7)(325)s + (325)^2} \right]. \quad (5.59)$$

Such dynamics become very important in evaluating aircraft performance when the closed loop system possesses a “high bandwidth.” Each actuator has a lower undamped natural frequency of $\omega_n = 20.2$ rad/sec. We therefore expect that when the closed loop system exhibits a bandwidth greater than ~ 2 rad/sec, we expect problems. We expect significant problems particularly if the actuator dynamics have not been accounted for in the design of the flight control system. (Note that the canard actuator has a dc gain of 0.885, i.e. near unity). These ideas will be illustrated in what follows.

Open Loop Bandwidth Design Specification. Based on the above actuator dynamics, we sought an open loop bandwidth of about 2 rad/sec - about one decade below our actuator dynamics.

Plant Singular Values. The plant singular values - with and without actuator dynamics - are plotted in Figure 5.17. The peak near 0.156 rad/sec is due to the aircraft’s lightly damped phugoid mode.

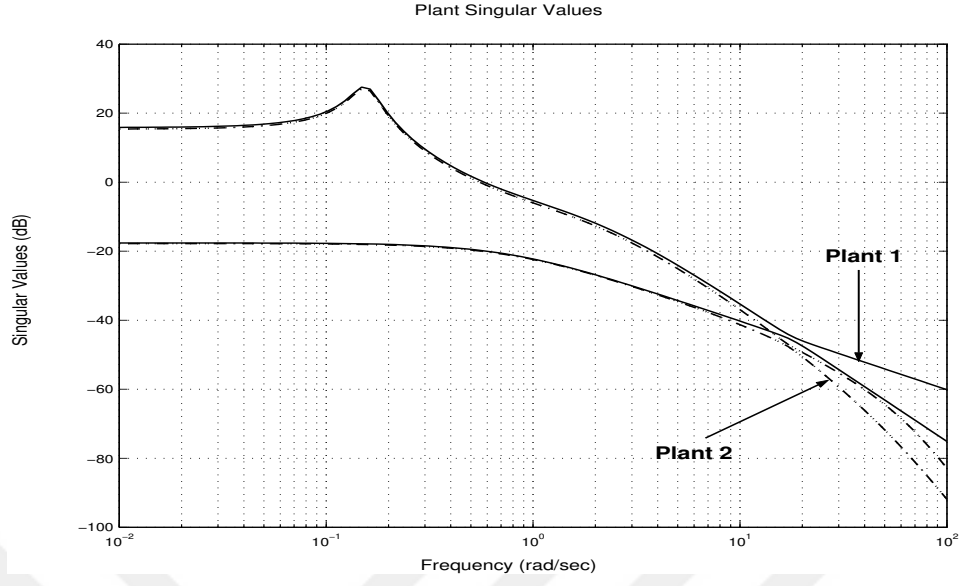


Figure 5.17: X-29 Singular Values

Weighting Function Parameters. Weighting function parameters were selected as follows:

$$\omega_{b_1} = 0.3 \quad \omega_{b_2} = 0.6 \quad (5.60)$$

$$M_{s_1} = 10 \quad M_{s_2} = 10 \quad (5.61)$$

$$\omega_{bu_1} = 300 \quad \omega_{bu_2} = 200 \quad \omega_{bu_3} = 500 \quad (5.62)$$

$$M_{u_1} = 0.01 \quad M_{u_2} = 0.01 \quad M_{u_3} = 0.01 \quad (5.63)$$

$$\omega_{bc_1} = 10 \quad \omega_{bc_2} = 10 \quad (5.64)$$

$$M_{y_1} = 100 \quad M_{y_2} = 100 \quad (5.65)$$

$$\epsilon = 0.01 \quad (5.66)$$

Weighting Functions. Weighting functions were selected as follows:

$$W_1^{-1} = \begin{bmatrix} \frac{s/M_{s_1} + \omega_{b_1}}{s + \omega_{b_1}\epsilon} & 0 \\ 0 & \frac{s/M_{s_2} + \omega_{b_2}}{s + \omega_{b_2}\epsilon} \end{bmatrix}^{-1} \quad (5.67)$$

$$= \begin{bmatrix} \frac{s+0.003}{0.1s+0.3} & 0 \\ 0 & \frac{s+0.006}{0.1s+0.6} \end{bmatrix} \quad (5.68)$$

$$W_2^{-1} = \begin{bmatrix} \frac{s + \omega_{bu_1} M_{u_1}}{\epsilon s + \omega_{bu_1}} & 0 & 0 \\ 0 & \frac{s + \omega_{bu_2} M_{u_2}}{\epsilon s + \omega_{bu_2}} & 0 \\ 0 & 0 & \frac{s + \omega_{bu_3} M_{u_3}}{\epsilon s + \omega_{bu_3}} \end{bmatrix}^{-1} \quad (5.69)$$

$$= \begin{bmatrix} \frac{0.01s+300}{s+3} & 0 & 0 \\ 0 & \frac{0.01s+200}{s+2} & 0 \\ 0 & 0 & \frac{0.01s+500}{s+5} \end{bmatrix} \quad (5.70)$$

$$W_3^{-1} = \begin{bmatrix} \left(\frac{s + \omega_{bc1}/\sqrt{M_{y1}}}{\sqrt{\varepsilon}s + \omega_{bc1}} \right)^2 & 0 \\ 0 & \left(\frac{s + \omega_{bc2}/\sqrt{M_{y2}}}{\sqrt{\varepsilon}s + \omega_{bc2}} \right)^2 \end{bmatrix}^{-1} \quad (5.71)$$

$$= \begin{bmatrix} \left(\frac{0.1s+10}{s+1} \right)^2 & 0 \\ 0 & \left(\frac{0.1s+10}{s+1} \right)^2 \end{bmatrix} \quad (5.72)$$

The above weighting functions may be visualized as shown in Figure 5.18.

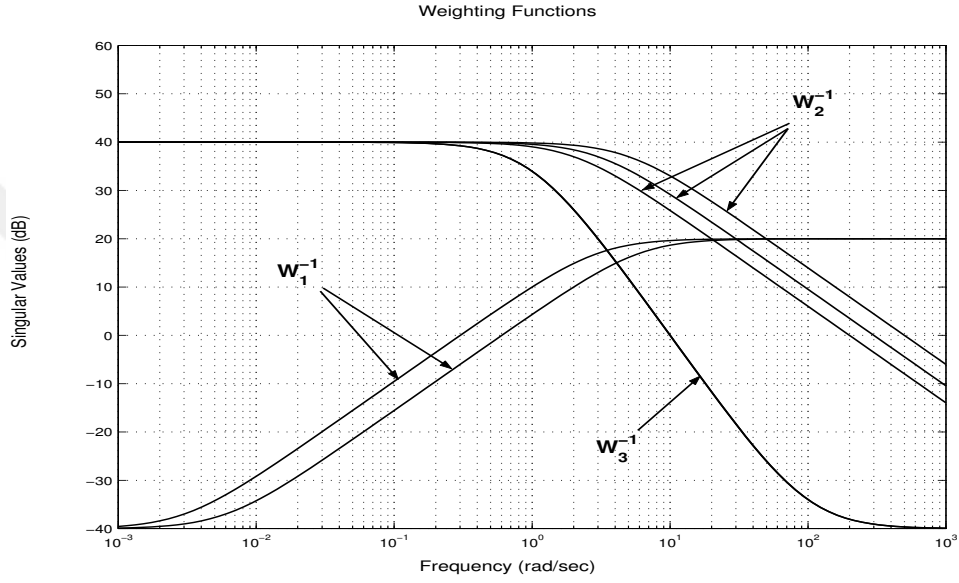


Figure 5.18: Weighting Functions for Mixed-Sensitivity $\mathcal{H}^\infty$ Optimization

Closed Loop Poles: A Bad Design. The closed loop poles that result from a straight forward $\mathcal{H}^\infty$ mixed-sensitivity optimization are as follows:

Poles	Damping	Frequency(rad/sec)
$-2.88e + 006$	$1.00e + 000$	$2.88e + 006$
$-1.01e + 002 \pm 1.05e + 001i$	$9.95e - 001$	$1.01e + 002$
$-1.03e + 002$	$1.00e + 000$	$1.03e + 002$
$-9.70e + 001$	$1.00e + 000$	$9.70e + 001$
$-7.46e + 000$	$1.00e + 000$	$7.46e + 000$
$-2.74e + 000 \pm 2.53e + 000i$	$7.35e - 001$	$3.73e + 000$
$-4.23e + 000$	$1.00e + 000$	$4.23e + 000$
$-1.40e + 000 \pm 1.29e + 000i$	$7.35e - 001$	$1.91e + 000$
$-2.19e - 002 \pm 1.54e - 001i$	$1.40e - 001$	$1.56e - 001$
$-3.99e - 001$	$1.00e + 000$	$3.99e - 001$
$-1.51e + 000$	$1.00e + 000$	$1.51e + 000$
$-2.29e + 000$	$1.00e + 000$	$2.29e + 000$
$-2.31e + 000$	$1.00e + 000$	$2.31e + 000$

The resulting minimum γ is

$$\gamma = 0.5728 \quad (5.73)$$

Because the design results in very lightly damped closed loop poles ($\zeta = 0.14$), the design is unacceptable. This occurred because a straight-forward $\mathcal{H}^\infty$ mixed-sensitivity optimization results in a controller that cancels the plant's lightly damped phugoid mode. To prevent this undesirable plant pole inversion, we use the bilinear transformation.

Bilinear Transformation Parameters. The bilinear transformation parameters selected were as follows:

$$p_1 = -0.15 \quad (5.74)$$

$$p_2 = -10^{20}. \quad (5.75)$$

This selection results in

$$s = \frac{\tilde{s} + p_1}{\frac{\tilde{s}}{p_2} + 1} = \frac{\tilde{s} - 0.15}{\frac{\tilde{s}}{-10^{20}} + 1} \approx \tilde{s} - 0.15 \quad (\text{transform}) \quad (5.76)$$

$$\tilde{s} \approx s + 0.15 \quad (\text{inverse transform}) \quad (5.77)$$

Reasons for this selection are provided below. The transform performs a rightward (“destabilizing”) shift. The inverse transform performs a leftward (“stabilizing”) shift.

Design Procedure. The same design procedure that was described in the previous section was applied.

Comment 5.0.2 (Selection of Weighting Functions and Design Parameters)

- Sensitivity Weighting. *A second order weighting function was selected for W_1 .*
 - *Initially, we started with $\omega_{b_1} = \omega_{b_2} = 1$. ω_{b_1} and ω_{b_2} were decreased in order to reduce overshoot and to slow down the response to a step command.*
 - *As the sensitivity bandwidth parameters ω_{b_i} are increased (decreased), the closed loop system bandwidth increases (decreases) (e.g. gain crossover frequency for the sensitivity, complementary sensitivity transfer function 3 dB bandwidth). Increasing and/or decreasing the ω_{b_i} too much results in an unacceptable design.*
- Control Sensitivity Weighting. *Initially, we selected W_2 to be a constant (i.e. $W_2 = M_u$). This selection resulted in large controls and did not permit us to control the adequately control the roll off of the control sensitivity KS . To remedy this, we selected the above dynamic W_2 . By so doing, we were able to shape the high frequency behavior of KS ; e.g. its associated high frequency gain crossover frequencies.*
 - *Initially, we started with $\omega_{b_{u1}} = \omega_{b_{u2}} = 250$ rad/sec - about two decades above our desired open loop gain crossover of 2 rad/sec (about one decade below our actuator dynamics at 20.2 rad/sec).*

- $\omega_{b_{u1}}$ was increased in order to encourage more canard activity.
- $\omega_{b_{u2}}$ was decreased in order to encourage less symmetric flap activity.
- $\omega_{b_{u3}}$ was increased in order to encourage more strake flap activity.

Since $\omega_{b_{u3}} > \omega_{b_{u1}} > \omega_{b_{u2}}$, it follows that we are more aggressive with symmetric flap action than canard and, we are more aggressive with canard than strake flap.

- **Complementary Sensitivity Weighting.** A second order weighting function was selected for W_3 in order to control the bandwidth associated with the complementary sensitivity transfer function matrix T .
 - Initially, we started with $\omega_{bc1} = \omega_{bc2} = 20$ rad/sec - one decade above our desired gain crossover frequency of 2 rad/sec.
 - Both were decreased simultaneously to reduce bandwidth, overshoot, and improve damping.

■

Closed Loop Poles for Design 1: A Good Design. The closed loop poles that result from our BLT (bilinear transformation) approach are as follows:

Poles	Damping	Frequency(rad/sec)
$-7.08e + 005$	$1.00e + 000$	$7.08e + 005$
$-1.01e + 002 \pm 1.05e + 001i$	$9.95e - 001$	$1.01e + 002$
$-1.03e + 002$	$1.00e + 000$	$1.03e + 002$
$-9.72e + 001$	$1.00e + 000$	$9.72e + 001$
$-7.61e + 000$	$1.00e + 000$	$7.61e + 000$
$-2.83e + 000 \pm 2.51e + 000i$	$7.48e - 001$	$3.78e + 000$
$-4.37e + 000$	$1.00e + 000$	$4.37e + 000$
$-1.57e + 000 \pm 1.29e + 000i$	$7.73e - 001$	$2.03e + 000$
$-1.80e + 000$	$1.00e + 000$	$1.80e + 000$
$-2.78e - 001 \pm 1.54e - 001i$	$8.74e - 001$	$3.18e - 001$
$-2.48e + 000$	$1.00e + 000$	$2.48e + 000$
$-4.06e - 001$	$1.00e + 000$	$4.06e - 001$
$-2.29e + 000$	$1.00e + 000$	$2.29e + 000$

We observe that all closed loop poles for Design 1 exhibit a damping $\zeta > 0.7$ - as desired.

Closed Loop Zeros for Design 1. The resulting closed loop (complementary sensitivity) transmission zeros for Design 1 are as follows:

Zeros	Damping	Frequency(rad/sec)
$-3.51e + 004$	$1.00e + 000$	$3.51e + 004$
$-2.03e + 004$	$1.00e + 000$	$2.03e + 004$
$-4.43e + 000$	$1.00e + 000$	$4.43e + 000$
$-2.29e + 000$	$1.00e + 000$	$2.29e + 000$
$-2.62e - 001 \pm 1.64e - 001i$	$8.47e - 001$	$3.10e - 001$
$-4.47e - 001$	$1.00e + 000$	$4.47e - 001$
$-4.05e - 001$	$1.00e + 000$	$4.05e - 001$
$-1.00e + 002 \pm 2.96e - 005i$	$1.00e + 000$	$1.00e + 002$
$-1.00e + 002$	$1.00e + 000$	$1.00e + 002$
$-1.00e + 002$	$1.00e + 000$	$1.00e + 002$

One low frequency closed loop zero (also a compensator zero) which will significantly increase the overshoot resulting from step reference commands is situated at $s = -0.447$. Because of this zero, a reference command pre-filter will be essential. The design of this pre-filter for Design 1 will be based on the location of this zero. This is discussed below.

Minimum γ for Design 1. The resulting minimum γ for Design 2 is

$$\gamma = 0.6320. \quad (5.78)$$

The design was found using the Robust Control Toolbox's "hinftol" command - with an optimality tolerance of 0.0001.

Design #1⁺: Actuator Dynamics Included After Design Process.

When the actuator dynamics are inserted into the feedback loop, we obtain what we call Design 1⁺. This design did not consider the actuator dynamics in the design process. Given this, and given our high bandwidth of ~ 2 rad/sec, we anticipate trouble with this design.

Closed Loop Poles for Design 1⁺. The closed loop poles associated with Design 1⁺ are as follows:

Poles	Damping	Frequency(rad/sec)
$-7.08e + 005$	$1.00e + 000$	$7.08e + 005$
$-2.28e + 002 \pm 2.32e + 002i$	$7.00e - 001$	$3.25e + 002$
$-1.45e + 002 \pm 1.85e - 004i$	$1.00e + 000$	$1.45e + 002$
$-1.01e + 002 \pm 1.05e + 001i$	$9.95e - 001$	$1.01e + 002$
$-1.03e + 002$	$1.00e + 000$	$1.03e + 002$
$-9.72e + 001$	$1.00e + 000$	$9.72e + 001$
$-1.00e + 002$	$1.00e + 000$	$1.00e + 002$
$-5.25e + 001 \pm 4.83e + 001i$	$7.36e - 001$	$7.14e + 001$
$-5.26e + 001 \pm 4.83e + 001i$	$7.36e - 001$	$7.14e + 001$
$-4.98e + 001$	$1.00e + 000$	$4.98e + 001$
$-1.77e + 001$	$1.00e + 000$	$1.77e + 001$
$-2.01e + 001$	$1.00e + 000$	$2.01e + 001$
$-1.35e + 001$	$1.00e + 000$	$1.35e + 001$
$-1.17e + 000 \pm 2.79e + 000i$	$3.86e - 001$	$3.02e + 000$
$-4.65e + 000$	$1.00e + 000$	$4.65e + 000$
$-3.36e + 000$	$1.00e + 000$	$3.36e + 000$
$-1.35e + 000 \pm 1.16e + 000i$	$7.58e - 001$	$1.78e + 000$
$-1.97e + 000$	$1.00e + 000$	$1.97e + 000$
$-1.50e + 000$	$1.00e + 000$	$1.50e + 000$
$-2.99e - 001 \pm 1.87e - 001i$	$8.48e - 001$	$3.53e - 001$
$-4.06e - 001$	$1.00e + 000$	$4.06e - 001$

As expected, we have a problem. One of the closed loop poles has a damping of 0.386 and undamped natural frequency of $\omega_n = 3.02$ rad/sec. If this closed loop pole were situated at a very high frequency, its residue matrix would be small, and it would not bother us so much. This, however, is not the case.

Closed Loop Zeros for Design 1⁺. The actuator dynamics do not introduce additional zeros into the loop. Given this, the closed loop transmission zeros associated with Design 1⁺ are identical to those associated with Design 1.

Reference Command Pre-Filter for Designs 1 and 1⁺. The following reference command pre-filter was designed on the basis of controller zeros. Designs 1 and 1⁺ have the same controller. The controller has a zero at $s = -0.492$. Based on this, we developed the following reference command pre-filter for Designs 1 and 1⁺:

$$W_o = [T_{ry_o}(0)]^{-1} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{0.9}{s+0.9} \end{bmatrix} \quad (5.79)$$

$$= \begin{bmatrix} 0.9864 & -0.1771 \\ -0.0309 & 1.0488 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{0.9}{s+0.9} \end{bmatrix} \quad (5.80)$$

$$= \begin{bmatrix} \frac{1.0192}{s+1} & \frac{0.1549}{s+0.9} \\ \frac{0.0300}{s+1} & \frac{0.8627}{s+0.9} \end{bmatrix} \quad (5.81)$$

where $T_{ryo} = PK[I + PK]^{-1}$ is the nominal complementary sensitivity for Design 1. Also note that our pre-filter has a low pass filter in each reference command channel. Initially, we started with a low pass filter $\frac{z}{s+z}$ in each channel with $z = 0.9$ - about 2 times larger than the compensator zero. The bandwidth was opened up a bit for the filter in the first (i.e. speed) reference command channel.

Design 2: Actuator Dynamics Included In Design Process. Because Design 1⁺ does not adequately address our actuator dynamics, we developed another design - Design 2. This design includes the actuator dynamics in the initial plant model P . After application of the BLT, the resulting $\hat{P}$ is combined with W_1 , W_2 , and W_3 to form the generalized plant G . A design $\hat{K}$ is then obtained and transformed back to get the final design K .

Closed Loop Poles for Design 2. The closed loop poles associated with Design 2 are as follows:

Poles	Damping	Frequency(rad/sec)
$-2.40e + 005$	$1.00e + 000$	$2.40e + 005$
$-2.27e + 002 \pm 2.32e + 002i$	$7.00e - 001$	$3.25e + 002$
$-2.28e + 002 \pm 2.32e + 002i$	$7.00e - 001$	$3.25e + 002$
$-9.87e + 001 \pm 3.19e + 000i$	$9.99e - 001$	$9.87e + 001$
$-1.03e + 002$	$1.00e + 000$	$1.03e + 002$
$-1.02e + 002$	$1.00e + 000$	$1.02e + 002$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-9.84e + 001$	$1.00e + 000$	$9.84e + 001$
$-1.45e + 002$	$1.00e + 000$	$1.45e + 002$
$-5.25e + 001 \pm 4.83e + 001i$	$7.36e - 001$	$7.14e + 001$
$-5.26e + 001 \pm 4.83e + 001i$	$7.36e - 001$	$7.14e + 001$
$-1.00e + 002$	$1.00e + 000$	$1.00e + 002$
$-4.97e + 001$	$1.00e + 000$	$4.97e + 001$
$-7.55e + 000$	$1.00e + 000$	$7.55e + 000$
$-2.83e + 000 \pm 2.62e + 000i$	$7.34e - 001$	$3.86e + 000$
$-4.29e + 000$	$1.00e + 000$	$4.29e + 000$
$-1.49e + 000 \pm 1.30e + 000i$	$7.53e - 001$	$1.98e + 000$
$-2.78e - 001 \pm 1.54e - 001i$	$8.74e - 001$	$3.18e - 001$
$-4.15e - 001$	$1.00e + 000$	$4.15e - 001$
$-1.81e + 000$	$1.00e + 000$	$1.81e + 000$
$-2.45e + 000$	$1.00e + 000$	$2.45e + 000$
$-5.00e + 001$	$1.00e + 000$	$5.00e + 001$
$-2.29e + 000$	$1.00e + 000$	$2.29e + 000$
$-1.97e + 001$	$1.00e + 000$	$1.97e + 001$
$-2.07e + 001$	$1.00e + 000$	$2.07e + 001$
$-1.45e + 002 \pm 8.19e - 013i$	$1.00e + 000$	$1.45e + 002$
$-5.26e + 001 \pm 4.83e + 001i$	$7.36e - 001$	$7.14e + 001$
$-5.26e + 001 \pm 4.83e + 001i$	$7.36e - 001$	$7.14e + 001$
$-2.02e + 001$	$1.00e + 000$	$2.02e + 001$
$-2.02e + 001$	$1.00e + 000$	$2.02e + 001$

As expected, all closed loop poles associated with Design 2 are at desirable locations - with damping $\zeta > 0.7$.

Closed Loop Zeros for Design 2. The resulting closed loop (complementary sensitivity) transmission zeros for Design 2 are as follows:

Zeros	Damping	Frequency(rad/sec)
$-4.91e + 004$	$1.00e + 000$	$4.91e + 004$
$-2.05e + 004$	$1.00e + 000$	$2.05e + 004$
$-2.28e + 002 \pm 2.32e + 002i$	$7.00e - 001$	$3.25e + 002$
$-5.42e + 001 \pm 6.07e + 001i$	$6.66e - 001$	$8.14e + 001$
$-1.40e + 002$	$1.00e + 000$	$1.40e + 002$
$-5.00e + 001$	$1.00e + 000$	$5.00e + 001$
$-1.45e + 002 \pm 6.64e - 013i$	$1.00e + 000$	$1.45e + 002$
$-3.06e + 001$	$1.00e + 000$	$3.06e + 001$
$-4.37e + 000$	$1.00e + 000$	$4.37e + 000$
$-2.29e + 000$	$1.00e + 000$	$2.29e + 000$
$-2.60e - 001 \pm 1.70e - 001i$	$8.37e - 001$	$3.11e - 001$
$-4.27e - 001$	$1.00e + 000$	$4.27e - 001$
$-3.93e - 001$	$1.00e + 000$	$3.93e - 001$
$-5.26e + 001 \pm 4.83e + 001i$	$7.36e - 001$	$7.14e + 001$
$-5.26e + 001 \pm 4.83e + 001i$	$7.36e - 001$	$7.14e + 001$
$-1.00e + 002$	$1.00e + 000$	$1.00e + 002$
$-1.00e + 002 \pm 1.06e - 004i$	$1.00e + 000$	$1.00e + 002$
$-1.00e + 002$	$1.00e + 000$	$1.00e + 002$
$-1.00e + 002$	$1.00e + 000$	$1.00e + 002$
$-2.02e + 001 \pm 4.32e - 013i$	$1.00e + 000$	$2.02e + 001$

One low frequency closed loop zero (also a compensator zero) which will significantly increase the overshoot resulting from step reference commands is situated at $s = -0.393$. Because of this zero, a reference command pre-filter will be essential. The design of this pre-filter for Design 2 will be based on the location of this zero. This is discussed below.

Minimum γ for Design 2. The resulting minimum γ for Design 2 is

$$\gamma = 0.7430. \quad (5.82)$$

The design was found using the Robust Control Toolbox's "hinftopt" command - with an optimality tolerance of 0.0001.

Reference Command Pre-Filter for Design 2. The following reference command pre-filter was designed on the basis of Design 2's controller zeros. Design 2's controller has a zero at $s = -0.393$. Given

this, we selected the following pre-filter for Designs 2:

$$W_o = [T_{ry_o}(0)]^{-1} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{0.9}{s+0.9} \end{bmatrix} \quad (5.83)$$

$$= \begin{bmatrix} 0.9873 & -0.1983 \\ -0.0355 & 1.0592 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{0.9}{s+0.9} \end{bmatrix} \quad (5.84)$$

$$= \begin{bmatrix} \frac{1.0197}{s+1} & \frac{0.1718}{s+0.9} \\ \frac{0.0342}{s+1} & \frac{0.8555}{s+0.9} \end{bmatrix} \quad (5.85)$$

Frequency Response Data.

- *Compensator Singular Values.* The compensator singular values are plotted in Figure 5.19. Because we used the bilinear transformation, the integral behavior “requested” by W_1 has been “shifted away.” We therefore see that for frequencies below 0.1 rad/sec, the compensator singular value is constant - with the minimum singular value being very low! (BLT parameter $p_1 = -0.15$). This will result in steady state errors to step reference commands for θ and v . At higher frequencies, the singular values roll off to ensure that high frequency sensor noise n in e is attenuated.

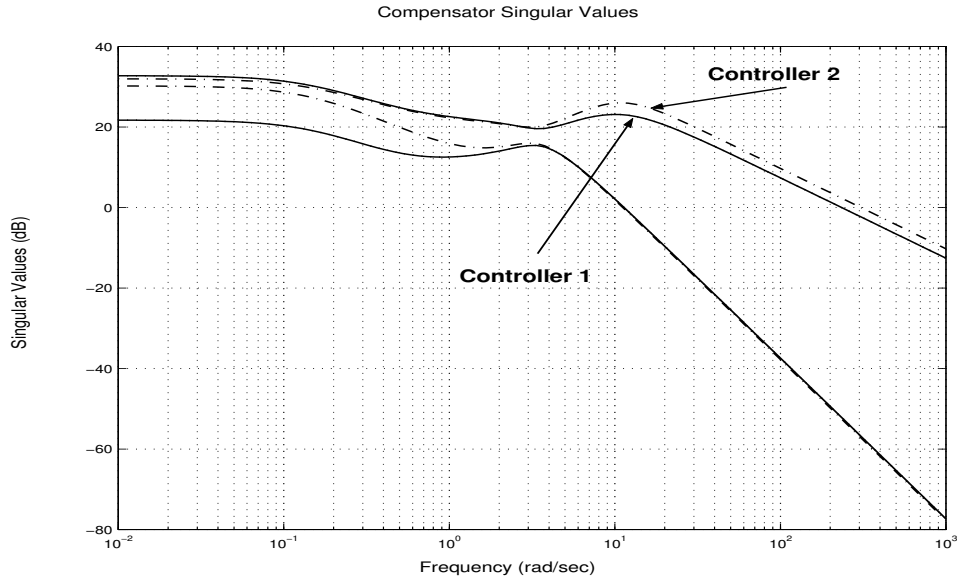


Figure 5.19: Compensator Singular Values

- *Open Loop Singular Values at Plant Output.* Breaking the loop at e or y yields the open loop transfer function matrix $L_o = PK$. Its singular value is plotted in Figure 5.20. The singular values are constant at low frequencies because K is constant at low frequencies - a consequence of the BLT. The plot suggests that low frequency reference commands r will be followed, low frequency output disturbances d_o will be attenuated, and high frequency sensor noise n will be attenuated. More precisely, reference commands r with frequency content below 0.1 rad/sec should be followed to within about 15 dB; that is, with a steady state error of about 17%. Similarly, output disturbances

d_o with frequency content below 0.1 rad/sec should be attenuated by approximately 15 dB. Sensor noise n with frequency content above 3 rad/sec will be attenuated.

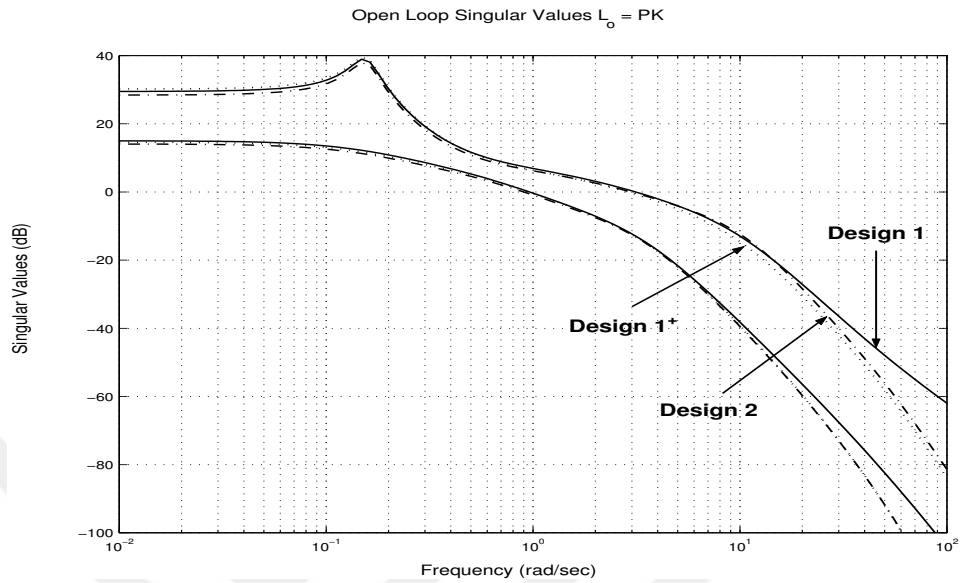


Figure 5.20: Open Loop Singular Values at Plant Output - $L_o = PK$

- *Open Loop Singular Values at Plant Input.* Breaking the loop at u yields the open loop transfer function matrix $L_i = KP$. The singular values of L_i are plotted in Figure 5.21. The singular value plot for L_i therefore suggests that input disturbances d_i with frequency content below 0.1 rad/sec will be attenuated only by about 3 dB.

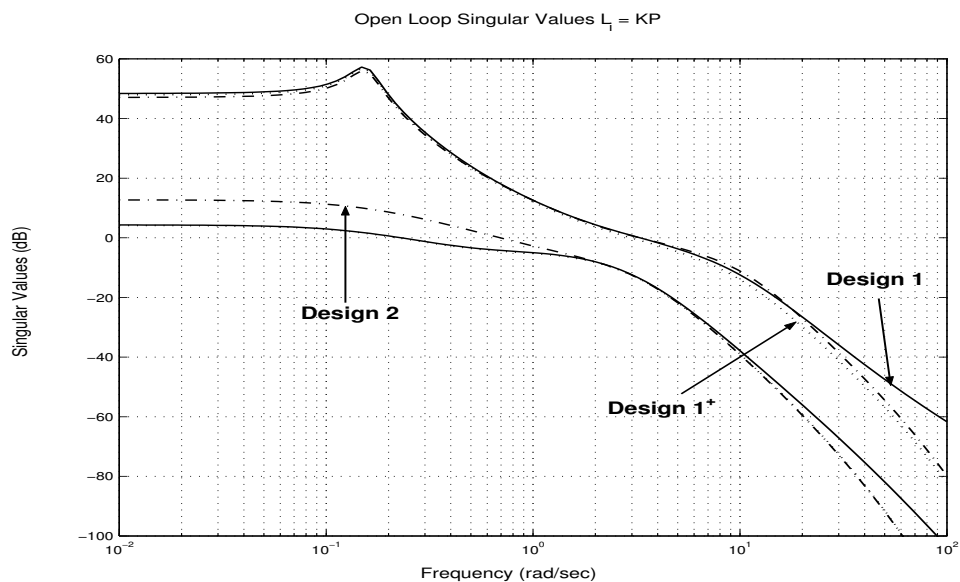


Figure 5.21: Open Loop Singular Values at Plant Input - $L_i = KP$

- *Sensitivity Singular Values at Plant Output.* The sensitivity transfer function matrix associated with

e is given by:

$$S_o = [I + PK]^{-1} = [I + L_o]^{-1}. \quad (5.86)$$

The singular values of S_o are plotted in Figure 5.22. The plot suggests that low frequency reference command will be followed and low frequency output disturbances will be attenuated. More precisely, reference commands r with frequency content below 0.1 rad/sec should be followed to within about 14 dB; that is, with a steady state error of about 20%. Similarly, output disturbances d_o with frequency content below 0.1 rad/sec should be attenuated by approximately 14 dB.

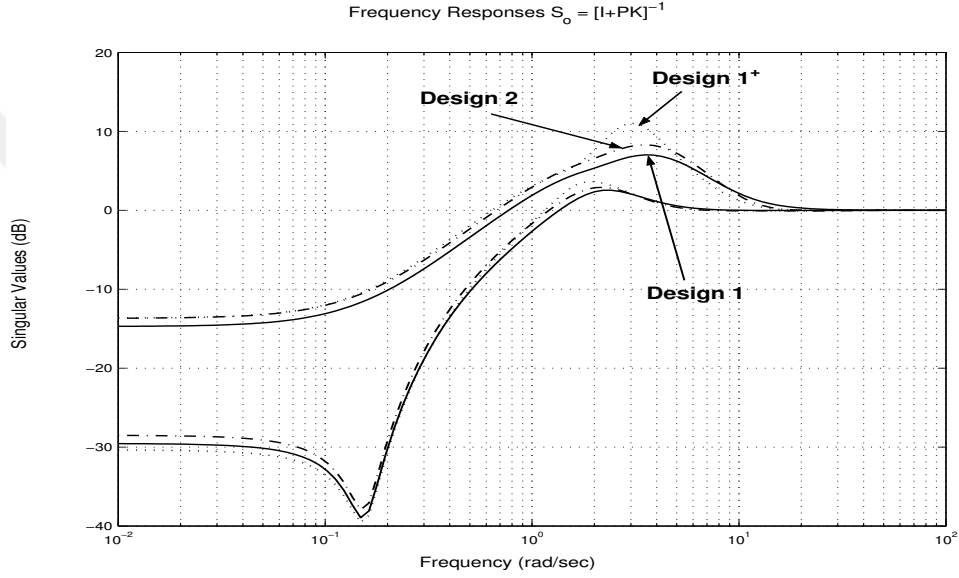


Figure 5.22: Sensitivity Singular Values at Plant Output - $S_o = [I + PK]^{-1}$

- *Sensitivity Singular Values at Plant Input.* The sensitivity transfer function matrix associated with u is given by:

$$S_i = [I + KP]^{-1} = [I + L_i]^{-1}. \quad (5.87)$$

This is also the transfer function matrix from d_i to u_p . The singular values of S_i are plotted in Figure 5.23. The plot indicates that two singular values are small at low frequencies while the other is large. This is expected for this 3-input 3-output transfer function matrix from d_i to u_p because only two variables (i.e. speed and pitch attitude) are being fed back (e.g. PK is TITO). In order to be able to control impact of all three input disturbances, the feedback loop must have at least three feedback variables (for example pitch attitude, speed and angle of attack).

- *Complementary Sensitivity Singular Values at Plant Output.* The complementary sensitivity transfer function matrix associated with the output or e is given by:

$$T_o = I - S_o = PK[I + PK]^{-1}. \quad (5.88)$$

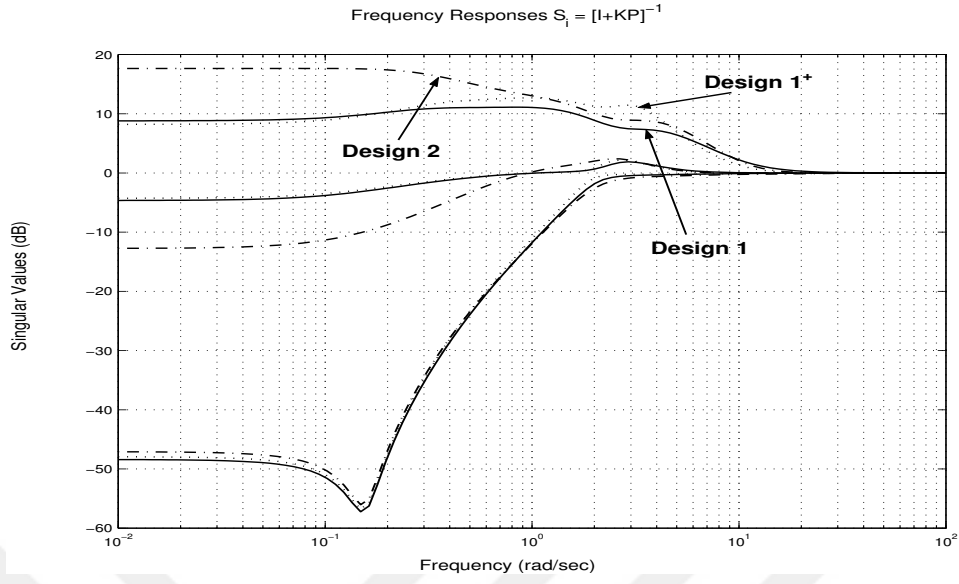


Figure 5.23: Frequency Response: $S_i = [I + KP]^{-1}$

The singular values of T_o are plotted in Figure 5.24. These singular values are important for assessing the impact of a unfiltered reference command r on the output y . The bump in our figure suggests that a prefilter is essential to prevent excessive overshoot to step reference commands. Since the bump starts to grow around 0.5 rad/sec (a compensator zero), the command pre-filter was selected to be as in (5.81) for Design 1 and (5.85) for design 2. The plot shows that low frequency reference commands will be followed. The singular value for S_o shows this a little better. The plot also shows that sensor noise with frequency content above 6 rad/sec will be attenuated.

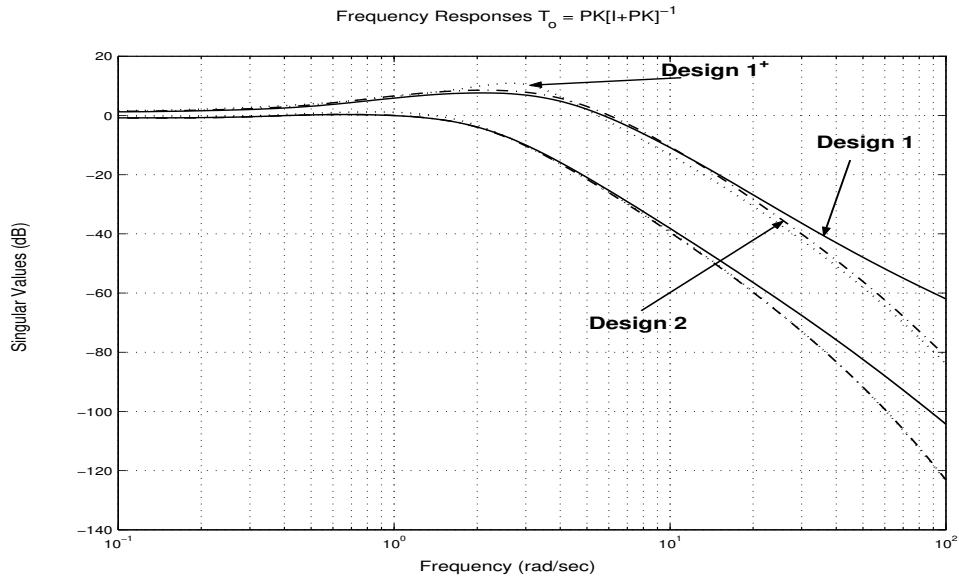


Figure 5.24: Complementary Sensitivity Singular Values at Plant Output - $T_o = PK[I + PK]^{-1}$

- *Closed Loop Singular Values: Reference to Output.* The transfer function matrix from reference

commands r to v and θ are

$$T_{ry} = [I + PK]^{-1}PKW. \quad (5.89)$$

where W_o refers to the reference command pre-filters designed earlier. Its singular values are plotted in Figure 5.25. The plot suggests that low frequency reference commands will be followed in the steady state and that little overshoot will result during the transient.

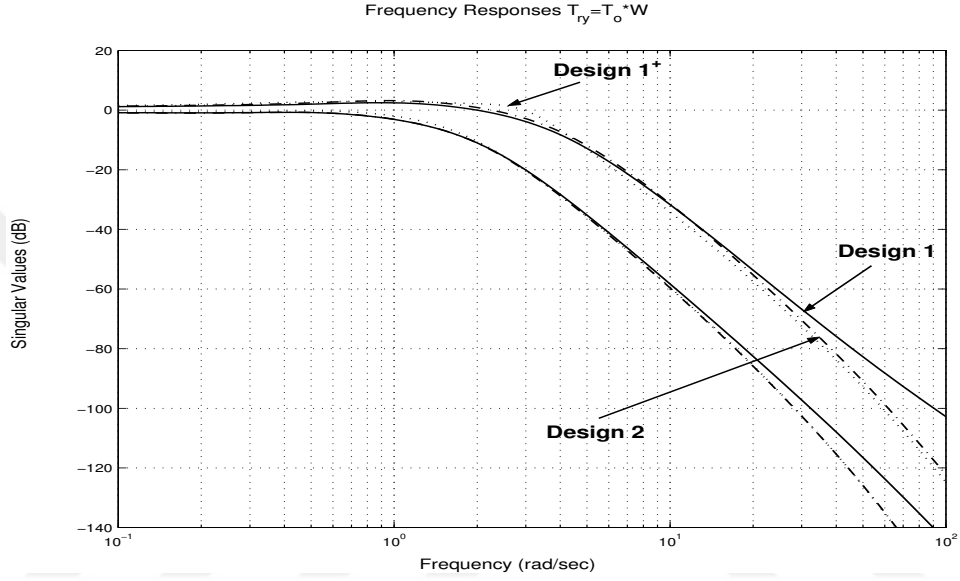


Figure 5.25: Closed Loop Singular Values - T_{ry}

- *Complementary Sensitivity Singular Values at Plant Input.* The singular values of T_i are plotted in Figure 5.26. T_i is the transfer function matrix from d_i to u . This plot indicates that low frequency input disturbances with frequency content below 6 rad/sec will be amplified by a factor of 3.16 (10 dB) for Design 1 and by a factor of 10 (20dB) for Design 2.
- *Input Disturbance to Output Singular Values.* To assess the impact of input disturbances d_i on the output y , one considers the transfer function matrix from d_i to y :

$$T_{d_iy} = [I + PK]^{-1}P. \quad (5.90)$$

Its singular values are plotted in Figure 5.27. The plot suggests that input disturbances with frequency content below 0.2 rad/sec will be attenuated by 10 dB. High frequency input disturbances will also be attenuated. This is expected since such high frequency input disturbances are directly attenuated by the plant. The plot also shows that, in the worst case, input disturbances with frequency of 2 rad/sec will be attenuated by 8 dB (by a factor of 2.512).

- *Reference to Control Singular Values.* To assess the impact of reference commands on control action,

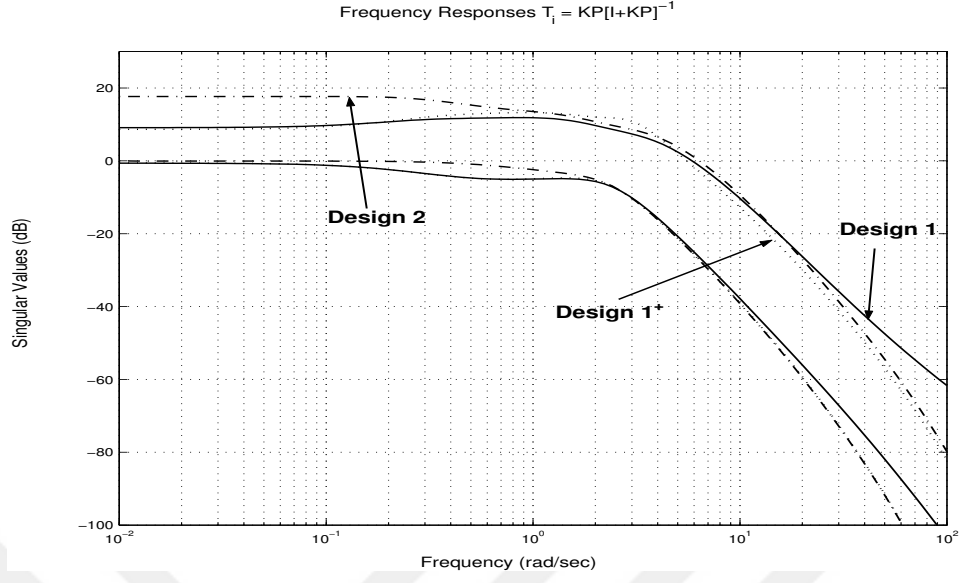


Figure 5.26: Complementary Sensitivity Singular Values at Plant Input - $T_i = KP[I + KP]^{-1}$

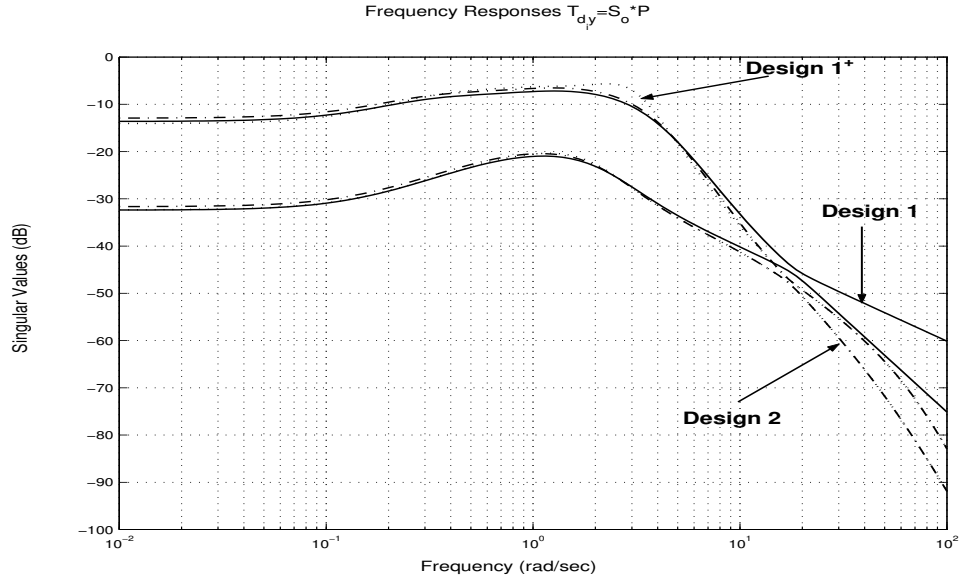


Figure 5.27: Input Disturbance to Output Singular Values - $T_{d_i y}$

one considers the transfer function matrix from r to u :

$$T_{ru} = [I + KP]^{-1}KW = K[I + PK]^{-1}W. \quad (5.91)$$

Its singular values are plotted in Figure 5.28. The plot suggests that issued reference commands r will be attenuated to produce the necessary steady state control u . An svd at $s = j1 \text{ rad/sec}$ yields a maximum singular value

$$\sigma_1 = 11.4163 \quad (21.1505 \text{ dB}) \quad (5.92)$$

a right singular vector

$$v_1 = \begin{bmatrix} 0.6082 & 0.4114 + j0.6789 \end{bmatrix}^T \quad (5.93)$$

$$= \begin{bmatrix} 0.6082 & 0.7938 e^{j58.78^\circ} \end{bmatrix}^T \quad (5.94)$$

and a left singular vector

$$u_1 = \begin{bmatrix} -0.3060 + j0.1822 & -0.8562 + j0.3694 & 0.0466 - j0.0374 \end{bmatrix}^T \quad (5.95)$$

$$= \begin{bmatrix} 0.3561 e^{j149^\circ} & 0.9325 e^{j156.66^\circ} & 0.0596 e^{-j38.75^\circ} \end{bmatrix}^T. \quad (5.96)$$

This svd implies that if a reference command

$$r(t) = \begin{bmatrix} 0.6082 \sin t & 0.7938 \sin(t + 58.78^\circ) \end{bmatrix}^T \quad (5.97)$$

acts on the system, then the resulting steady state control output will be

$$u(t) = 11.413 \begin{bmatrix} 0.3561 \sin(t + 149^\circ) & 0.9325 \sin(t + 156.66^\circ) & 0.0596 \sin(t - 38.75^\circ) \end{bmatrix}^T \quad (5.98)$$

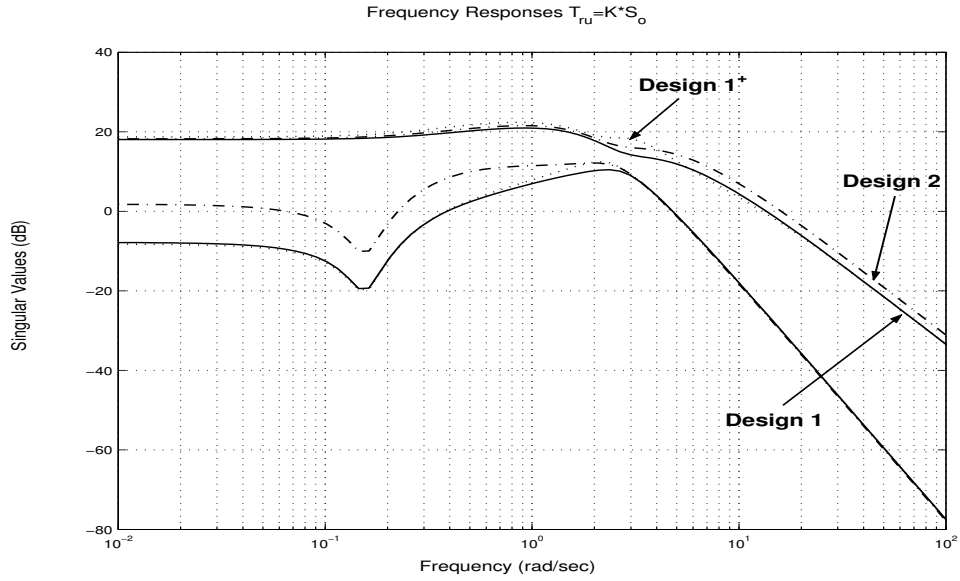


Figure 5.28: Response to Step Reference Command - Outputs

Time Response Data: Reference Command Following.

- *Step Reference Command Following.* The responses to unit step v and θ commands are plotted in Figure 5.29. The corresponding angle of attack, α can also be seen in the figure. As expected, v and θ follow the step command well - with a little overshoot. The corresponding controls are plotted in Figure 5.30. For a unit step v command, they are acceptable in size. For a unit step θ command, controls have relatively large steady state values. This is expected because increasing the

pitch attitude while maintaining speed (zero speed command) simultaneously is not natural for an aircraft.

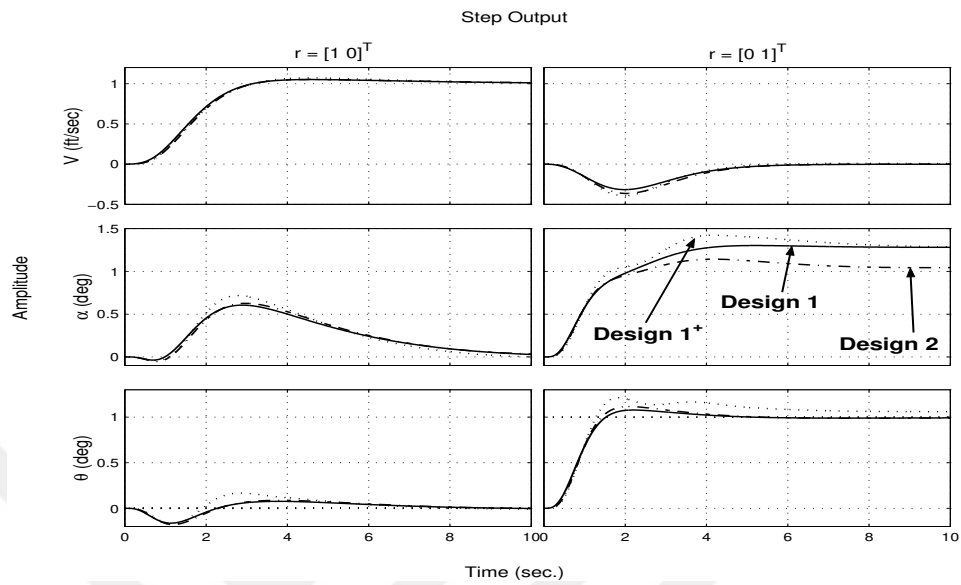


Figure 5.29: Response to Step Reference Command - Outputs

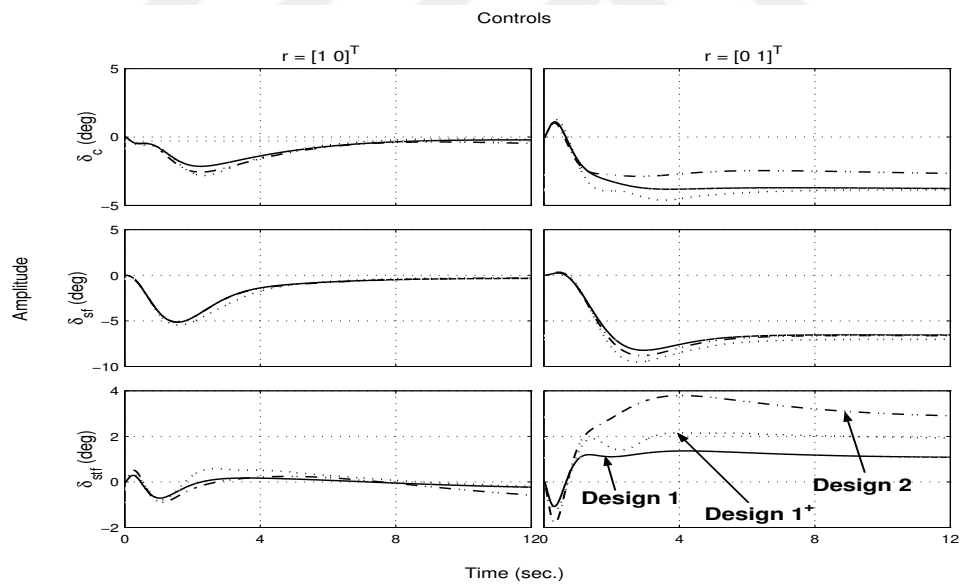


Figure 5.30: Response to Step Reference Command - Controls

CHAPTER 6

Summary and Directions for Future Research

This thesis described how $\mathcal{H}^\infty$ Mixed Sensitivity Optimization may be used to design a flight control system for a high performance aircraft. Three control system designs were developed for the X-29 aircraft – two longitudinal and one lateral. These will serve as benchmarks for future MIMO studies. This includes development of a gain scheduled flight control system.

The $\mathcal{H}^\infty$ /bilinear transformation design method was applied to various single-input single-output (SISO) unstable plants in order to develop insight into the selection of weighting functions. The impact of unmodeled high-frequency dynamics on closed loop system robustness were considered. High-frequency dynamics were then modeled to improve performance and understand how the nominal design changes.

For the longitudinal dynamics, two models were considered: (1) a model using pitch feedback, (2) a model using pitch and speed feedback.

$\mathcal{H}^\infty$ designs were developed for each. For the lateral dynamics, designs were developed using roll and sideslip angles as feedback variables. For each case, the impact of actuator dynamics were considered.

REFERENCES

- [1] J.T. Bosworth, "Linearized Aerodynamic and Control Law Models of the X-29A Airplane and Comparison with Flight Data," *NASA Dryden Flight Research Facility*, NASA Technical Memorandum 4356, Edwards, CA, 1992
- [2] J.H. Blakelock, "Automatic Control of Aircraft and Missiles," 2nd Edition, *Wiley Interscience*, 1991.
- [3] K. Zhou, J.C. Doyle, "Essentials of Robust Control," *Prentice Hall*, 1998.
- [4] R. Chiang and M. Safonov, "Robust Control Toolbox Version 2," *MathWorks Inc.*, 1998.
- [5] R.Y. Chiang, M.G. Safonov, " $\mathcal{H}^\infty$ Synthesis Using a Bilinear Pole Shifting Transform," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 5, Sep-Oct, pp. 1111-1117, 1992.
- [6] Robert C. Nelson, "Flight Stability and Automatic Control," *WCB McGraw-Hill*, 2nd Edition, 1998.
- [7] B. Etkin, "Dynamics of Flight: Stability and Control," 2nd Edition, *John Wiley and Sons*, New York, 1982.
- [8] K. Zhou, J.C. Doyle, K. and Glover, "Robust and Optimal Control," *Prentice Hall*, 1996.
- [9] Kaplan, G., "The X-29: Is it coming or going?," *IEEE Spectrum*, pp. 54-60 June, 1985.
- [10] R.A. Hyde, " H^∞ Aerospace Control Design," *Springer*, 1995.
- [11] W.L. Rogers, D.J. Collins, "X-29 $\mathcal{H}^\infty$ Control Synthesis," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 4, Jul-Aug, pp. 962-967, 1992.
- [12] J.F. Magni, S. Bennani, Jan Terlouw, "Robust Flight Control: A Design Challenge," *Springer*, pp. 64,81,300,464, 1996.
- [13] R.A. Hyde, K. Glover, "Development of Robust Flight Control Law for a VSTOL Aircraft," *IFAC Automatic Control in Aerospace*, No. 12, Ottobrunn, Germany September 7-11, pp. 97-102, 1992.
- [14] John T. Bosworth, Robert Clarke, John J. Burken, "X-29 Flight Control System: Lesson Learned," *NASA Dryden Flight Research Facility*, NASA Technical Memorandum 4598, Edwards, CA, 1994.
- [15] A.B. Whitaker, J. Chin, "X-29 Digital Flight Control System Design," *Active Control Systems Review, Evaluation and Projections*, AGARD-CP-384, 1984, pp. 6.1-6.13.
- [16] G. Spacht, "The Forward Swept Wing: A Unique Design Challenge," presented at the *1980 AIAA Aircraft Systems Meeting*, AIAA-80-1885., Anaheim, CA, August 4-6, 1980.
- [17] J. Gera, J.T. Bosworth, "Dynamic Stability and Handling Qualities Tests of a Highly Augmented, Statically Unstable Airplane," *NASA Dryden Flight Research Facility*, NASA Technical Memorandum 88297, Edwards, CA, 1987.
- [18] J. Gera, "Dynamics and Controls Flight Testing of the X-29A Airplane," *NASA Dryden Flight Research Facility*, NASA Technical Memorandum 86803, Edwards, CA, 1986.
- [19] Wilma W. Quinn, "Multivariable Control of a Forward Swept Wing Aircraft," *M.S. Thesis, MIT*, 1986.

- [20] Michael G. Gilbert, "Dynamic Modeling and Active Control of Aeroelastic Aircraft," *M.S. Thesis, Purdue University*, 1982.
- [21] Richard Y. Chiang, Michael G. Safonov, "MATLAB Robust Control Toolbox User's Guide," *MathWorks, Inc*, 1998.
- [22] J.E. Bauer, D.B. Crawford, D. Andrisani, J. Gera, "Real Time Comparison of X-29A Flight Data and Simulation Data," presented at the *1987 AIAA 25th Aerospace Sciences Meeting*, AIAA-87-0344, Reno, NV, January 12-15, 1987.
- [23] J.T. Bosworth, J.C. West, "Real Time Open Loop Frequency Response Analysis of Flight Test Data," presented at the *1986 AIAA SFTE 3rd Flight Testing Conference*, AIAA-86-9738, Las Vegas, NV, April 2-4, 1986.
- [24] Kenneth W. Iliff, "X-29 Lateral-Directional Stability and Control Derivatives Extracted From High-Angle-of-Attack Flight Data," *NASA Dryden Flight Research Facility*, NASA Technical Paper 3664, Edwards, CA, 1996.
- [25] N.J. Krone, Jr., "Forward Swept Wing Flight Demonstrator," presented at the *1980 AIAA Aircraft Systems and Technology Meeting*, AIAA-80-1882., Anaheim, CA, August 4-6, 1980.
- [26] Robert Sims, "X-29A Aircraft Structural Loads Flight Testing," *NASA Dryden Flight Research Facility*, NASA Technical Memorandum 101715, Edwards, CA, 1989.
- [27] T.W. Putnam, "X-29A Flight Research Program," *NASA Dryden Flight Research Facility*, NASA Technical Memorandum 86025, Edwards, CA, 1984.
- [28] Walter J. Sefic, Cleo M. Maxwell, "X-29A Technology Demonstrator Flight Test Program Overview," *NASA Dryden Flight Research Facility*, NASA Technical Memorandum 86809, Edwards, CA, 1986.
- [29] John W. Hicks, Neil W. Matheny, "Preliminary Flight Assessment of the X-29A Advanced Technology Demonstrator," *NASA Dryden Flight Research Facility*, NASA Technical Memorandum 100407, Edwards, CA, 1987.
- [30] Gary A. Trippensee, David P. Lux, "X-29A Forward-Swept-wing Research Program Status," *NASA Dryden Flight Research Facility*, NASA Technical Memorandum 100413, Edwards, CA, 1987.
- [31] John W. Hicks, James M. Cooper, Walter J. Sefic, "Flight Test Techniques for the X-29A Aircraft," *NASA Dryden Flight Research Facility*, NASA Technical Memorandum 88289, Edwards, CA, 1987.
- [32] Michael W. Kehoe, Lisa J. Bjarke, "An In-Flight Interaction of the X-29A Canard and Flight Control System," *NASA Dryden Flight Research Facility*, NASA Technical Memorandum 101718, Edwards, CA, 1990.
- [33] Vince Chacon, David McBride, "Operational Viewpoint of the X-29A Digital Flight Control System," *NASA Dryden Flight Research Facility*, NASA Technical Memorandum 100434, Edwards, CA, 1988.
- [34] J.E. Bauer, Robert Clarke, John J. Burken, "Flight Test of the X-29A at High Angle of Attack: Flight Dynamics and Controls," *NASA Dryden Flight Research Facility*, NASA Technical Memorandum 100434, Edwards, CA, 1995.
- [35] E.H.J. Pallet, S. Coyle, "Automatic Flight control," *BSP Professional Books*, 1993.
- [36] Donal McLean, "Automatic Flight Control Systems," *Prentice Hall*, 1990.
- [37] M. Vukobratovic, R. Stojic, "Modern Aircraft Flight Control," *Springer*, 1985.
- [38] Scott A. Speigle, "Navigation and Control Technologies for Unmanned Systems 2," *SPIE-The International Society for Optical Engineering*, 1997.
- [39] Malcolm J. Abzug, E. Eugene Larrabee, "Airplane Stability and Control," *Cambridge University Press*, 1997.
- [40] Gene F. Franklin, J. David Powell, Abbas Emami-Naeini, "Feedback Control Of Dynamic Systems," *Prentice Hall Addison-Wesley Publishing Company*, 1994.

- [41] S.P. Boyd and C.H. Barratt, "Linear Controller Design: Limits of Performance," *Prentice Hall*, Englewood Cliffs, NJ, 1991.
- [42] J.J. D'Azzo and C.H. Houpis, "Linear Control System Analysis & Design," *McGraw Hill*, 3rd Edition, New York, 1988.
- [43] R.C. Dorf, "Modern Control Systems," *Addison-Wesley*, 6th Edition, Reading, MA, 1992.
- [44] G. Franklin, J.D. Powell, A. Emami-Naeini, "Feedback Control of Dynamic Systems," *Addison-Wesley*, 3rd Edition, Reading, MA, 1994.
- [45] B.C. Kuo, "Automatic Control Systems," *Prentice Hall*, Sixth Edition, Upper Saddle River, N.J., , 1991.
- [46] K. Ogata, "Modern Control Engineering," *Prentice-Hall*, Third Edition, Upper Saddle River, N.J., 1997.
- [47] C.E. Rohrs, J.L. Melsa, and D.G. Schultz, "Linear Control Systems," *McGraw Hill*, 1993.
- [48] P.M. DeRusso, R.J. Roy, C.M. Close, A.A. Desrochers, "State Variables for Engineers," *John Wiley and Sons*, 2nd Edition, 1998.
- [49] Vidyasagar, M., "Nonlinear Systems Analysis," *Prentice Hall*, 2nd Edition, 1993.
- [50] C. Borghesani, "Controls Tutor Using MATLAB," *PWS*, 1996.
- [51] D. Etter, "Introduction to MATLAB for Engineers and Scientists," *Prentice Hall*, 1996.
- [52] D.K. Frederick and J.H. Chow, "Feedback Control Problems Using MATLAB and the Control Systems Toolbox," *PWS*, 1995.
- [53] A. Grace, "Optimization Toolbox for use with Matlab," *The MathworksInc.*, Natick, MA, 1990.
- [54] Duane Hanselman, Bruce Littlefield, "MATLAB User's Guide," *Prentice Hall*, Upper Saddle River, NJ, 1997.
- [55] *Control System Toolbox Version 4*, MathWorks Inc, 1996.
- [56] L. Ljung, "System Identification Toolbox: User Guide," *MathWorks Inc.*, 1991.
- [57] *MATRIX-X User's Guide*, Integrated Systems Inc., Donation to ASU, 1998.
- [58] *MATLAB Based Books*, MathWorks Inc., Natick, Massachusetts, Spring 1996, <http://www.mathworks.com>, info@mathworks.com.
- [59] The Mathworks, " MATLAB & SIMULINK," "<http://www.mathworks.com/>."

APPENDIX



APPENDIX A

Matlab Macros



Macro: siso.m

```

1  clc
2  close all
3  clear all
4
5  %***** PLANT *****
6  High_Frequency_Dynamics=tf([10],[1 10])*tf([10],[1 10]);
7  %High_Frequency_Dynamics=zpk([20],[-20],[-1]);
8  P=tf([1],[1 -1]);
9  %*****
10 for plant=1:3
11   if plant == 1
12     P=tf([1],[1 -1]);
13     shape = 'k-';
14   elseif plant == 2
15     P=tf([1],[1 -1]);
16     shape = 'b:.';
17   elseif plant == 3
18     P=tf([1],[1 -1])*High_Frequency_Dynamics;
19     shape = 'r-.';
20   end
21
22   zpk(P);
23   [Ap,Bp,Cp,Dp]=ssdata(P);
24
25   %***** Weighting Functions *****
26   Eps=0.01;
27   Gamma=1;
28
29   Ms=10;
30   Wb=1;
31   Wbc=30;
32   My=10;
33   Wbu=5000;
34   %Mu=100;
35   Mu=0.01;
36   %***** WEIGHTING FUNCTIONS *****
37   numW1=[1/Ms Wb];denW1=[1 Eps*Wb];
38   numW2=[0 Mu];denW2=[0 1];
39   %numW2=[1 Wbu/Mu];denW2=[Eps Wbu];
40   numW3=[1 Wbc/My];denW3=[0 Wbc];
41
42
43   W1_tf=tf(numW1,denW1);
44   W2_tf=tf(numW2,denW2);
45   W3_tf=tf(numW3,denW3);
46
47   W1=[numW1;denW1];

```

```

48 W2=[numW2;denW2];
49 W3=[numW3;denW3];
50 %*****
51
52 %***** DESIGN *****
53 Plant_Tree = mksys(Ap,Bp,Cp,Dp);
54 [Augmented_Tree]=augtf(Plant_Tree,W1,W2,W3);
55 %[A,B1,B2,C1,C2,D11,D12,D21,D22]=branch(Augmented_Tree);
56 %System_to_be_optimized= mksys(A,B1,B2,C1,C2,D11,D12,D21,D22,'tss');
57 [Gamma,ss_cp,ss_cl,hinfo] = hinft(Augmented_Tree,[1:3],[0.0001 1 0]);
58 Gamma
59 1/Gamma
60
61 [Abk,Bbk,Cbk,Dbk] = branch(ss_cp);
62 K=ss(Abk,Bbk,Cbk,Dbk);
63
64 K_Poles=pole(K);
65 K_Zeros=zero(K);
66
67 %***** PRE-FILTER*****
68 H=1;
69 Lo=series(K,P);
70 To=feedback(Lo,1);
71 Wo=inv(dcgain(To));
72 a=0.75;
73
74
75 %*****
76 if plant == 2
77     P=tf([1],[1 -1])*High_Frequency_Dynamics;
78 end
79
80 Lo=series(K,P);
81 So=feedback(1,Lo);
82 To=feedback(Lo,1);
83 Try=series(Wo,To);
84 Tru=series(So,K)*Wo;
85 zpk(P)
86 zpk(K)
87 zpk(Try)
88 damp(pole(Try))
89 damp(tzero(Try))
90
91
92 pause
93 figure(1)
94 grid on;hold on
95 sigma(P,logspace(-2,1,1000),shape);

```

```

96 ylabel('Magnitude(dB)')
97 title('Plant Frequency Responses')
98
99 figure(2)
100 grid on;hold on
101 sigma(K,logspace(-2,5,1000),shape);
102 ylabel('Magnitude(dB)')
103 title('Controller Frequency Responses')
104
105 figure(3)
106 grid on;hold on
107 sigma(Lo,logspace(-0,2,1000),shape);
108 ylabel('Magnitude(dB)')
109 title('Open Loop Frequency Responses')
110
111 figure(4)
112 grid on;hold on
113 sigma(So,logspace(-1,2,1000),shape);
114 ylabel('Magnitude(dB)')
115 title('Sensitivity Frequency Responses')
116
117 figure(5)
118 grid on;hold on
119 sigma(To*Wo,logspace(-1,2,1000),shape);
120 ylabel('Magnitude(dB)')
121 title('Complementary Sensitivity Frequency Responses')
122
123 figure(6)
124 grid on;hold on
125 Tdiy=So*P;
126 sigma(Tdiy,logspace(-2,2,1000),shape)
127 ylabel('Magnitude(dB)')
128 title('Input Disturbance to Output Frequency Responses')
129
130 figure(7)
131 grid on;hold on
132 sigma(Tru,logspace(-2,5,1000),shape)
133 ylabel('Magnitude(dB)')
134 title('Reference to Control Frequency Responses')
135
136 figure(8)
137 grid on;hold on
138 step(Try*tf([a],[1 a]), 0:0.01:8,shape);
139 title('Step Output with Pre-filter')
140 figure(81)
141 grid on;hold on
142 step(Try, 0:0.01:8, shape);
143 title('Step Output without Pre-filter')

```

```

144
145 figure(9)
146 grid on;hold on
147 step(Tru*tf([a],[1 a]), 0:0.01:8,shape);
148 title('Controls with Pre-filter')
149 figure(91)
150 grid on;hold on
151 step(Tru, 0:0.01:8,shape);
152 title('Controls without Pre-filter')
153
154 [Atcl,Btcl,Ctcl,Dtcl] = branch(ss_cl);
155 Ty1u1=ss(Atcl,Btcl,Ctcl,Dtcl);
156
157 H_inf=[Gamma*W1_tf*tf(So);
158         Gamma*W2_tf*K*tf(So);
159         Gamma*W3_tf*tf(To)];
160
161 figure(10)
162 grid on;hold on
163 sigma(H_inf,logspace(-2,10),shape);
164 ylabel('Magnitude(dB)')
165 title('\sigma(T_{wz})')
166
167
168 end
169
170 figure(11)
171 grid on;
172 sigma(inv(W3_tf),logspace(-2,3));
173 hold on
174 sigma(inv(W1_tf),logspace(-2,3));
175 hold on
176 sigma(inv(W2_tf),logspace(-2,3));
177 ylabel('Magnitude(dB)')
178 title('Weighting Functions')
179
180
181 return
182 figure(1)
183 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\siso_p', 'eps')
184
185 figure(2)
186 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\siso_k', 'eps')
187
188 figure(3)
189 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\siso_pk', 'eps')
190
191 figure(4)

```

```

192 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\siso_so', 'eps')
193
194 figure(5)
195 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\siso_to', 'eps')
196
197 figure(6)
198 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\siso_tdiy', 'eps')
199
200 figure(7)
201 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\siso_tru', 'eps')
202
203 figure(8)
204 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\siso_step', 'eps')
205
206 figure(81)
207 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\siso_step_pre', 'eps')
208
209 figure(9)
210 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\siso_controls', 'eps')
211
212 figure(91)
213 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\siso_controls_pre', 'eps')
214
215 figure(10)
216 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\siso_hinf', 'eps')
217
218 figure(11)
219 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\siso_wf', 'eps')

```

<i>Macro: siso_bil.m</i>

```

1 clc
2 close all
3 set(0,'DefaultLineLineWidth',1)
4
5 %***** PLANT *****
6 High_Frequency_Dynamics=tf([10],[1 10])*tf([10],[1 10]);
7 %High_Frequency_Dynamics=zpk([20],[-20],[-1]);
8 %*****
9 for plant=1:3
10 if plant == 1
11     P=tf([1],[1 -1]);
12     shape = 'b-';
13 elseif plant == 2
14     P=tf([1],[1 -1]);
15     shape = 'b:.';

```

```

16 elseif plant == 3
17     P=tf([1],[1 -1])*High_Frequency_Dynamics;
18     shape = 'r-.';
19 end
20
21 zpk(P);
22 [Ap,Bp,Cp,Dp]=ssdata(P);
23
24 %***** Bilinear Transformation *****
25 p2 = -1e20; p1=-1.5;
26 [At,Bt,Ct,Dt]=bilin(Ap,Bp,Cp,Dp,1,'Sft_jw',[p2 p1]);
27
28 %***** Weighting Functions *****
29 Eps=0.01;
30 Gamma=1;
31
32 Ms=10;
33 Wb=1;
34 Mu=0.01;
35 Wbc=30;
36 My=10;
37
38 %***** WEIGHTING FUNCTIONS *****
39 numW1=[1/Ms Wb];denW1=[1 Eps*Wb];
40 numW2=[0 Mu];denW2=[0 1];
41 numW3=[1 Wbc/My];denW3=[0 Wbc];
42
43
44 W1_tf=tf(numW1,denW1);
45 W2_tf=tf(numW2,denW2);
46 W3_tf=tf(numW3,denW3);
47
48 W1=[Gamma*numW1;denW1];
49 W2=[numW2;denW2];
50 W3=[numW3;denW3];
51 %*****
52
53 %***** DESIGN *****
54 Plant_Tree = mksys(At,Bt,Ct,Dt);
55 [Augmented_Tree]=augtf(Plant_Tree,W1,W2,W3);
56 [A,B1,B2,C1,C2,D11,D12,D21,D22]=branch(Augmented_Tree);
57 System_to_be_optimized= mksys(A,B1,B2,C1,C2,D11,D12,D21,D22,'tss');
58 [Gamma,ss_cp,ss_cl,hinfo] = hinftopt(Augmented_Tree,[1:3],[0.0001 1 0]);
59 Gamma
60 1/Gamma
61
62 W1=[Gamma*numW1;denW1];
63 W2=[Gamma*numW2;denW2];

```

```

64 W3=[Gamma*numW3;denW3];
65
66 [Augmented_Tree]=augtf(Plant_Tree,W1,W2,W3);
67 [ss_cp,ss_cl,hinfo] = hinf(Augmented_Tree,0);
68
69 [Abk,Bbk,Cbk,Dbk] = branch(ss_cp);
70
71 [Atk,Btk,Ctk,Dtk]=bilin(Abk,Bbk,Cbk,Dbk,-1,'Sft_jw',[p2 p1]);
72 K=ss(Atk,Btk,Ctk,Dtk);
73 %*****
74
75 K_Poles=pole(K);
76 K_Zeros=zero(K);
77
78 %***** PRE-FILTER*****
79 H=1;
80 Lo=series(K,P);
81 To=feedback(Lo,1);
82 Wo=inv(dcgain(To));
83 a=min(abs(K_Zeros));
84 %Wo=Wo*tf([3*a],[1 3*a]);
85
86
87 %*****
88 if plant == 2
89     P=tf([1],[1 -1])*High_Frequency_Dynamics;
90 end
91
92 Lo=series(K,P);
93 So=feedback(1,Lo);
94 To=feedback(Lo,1);
95 Try=series(Wo,To);
96 Tru=series(So,K)*Wo;
97 zpk(K)
98 zpk(Try)
99 %damp(pole(Try))
100 %damp(tzero(Try))
101 pause
102
103 figure(1)
104 grid on;hold on
105 sigma(P,logspace(-2,1,1000),shape);
106 title('Frequency Responses: P')
107
108 figure(2)
109 grid on;hold on
110 sigma(K,logspace(-2,5,1000),shape);
111 title('Frequency Responses: K')

```

```

112
113 figure(3)
114 grid on;hold on
115 sigma(Lo,logspace(-1,2,1000),shape);
116 title('Frequency Responses: PK')
117
118 figure(4)
119 grid on;hold on
120 sigma(So,logspace(-1,2,1000),shape);
121 title('Frequency Responses: S_o=[I+PK]^{-1}')
122
123 figure(5)
124 grid on;hold on
125 sigma(To*Wo,logspace(-1,2,1000),shape)
126 title('Frequency Responses T_o = PK[I+PK]^{-1}')
127
128 figure(6)
129 grid on;hold on
130 Tdiy=So*P;
131 sigma(Tdiy,logspace(-2,2,1000),shape)
132 title('Frequency Responses T_{d-iy}')
133
134 figure(7)
135 grid on;hold on
136 sigma(Tru,logspace(-2,5,1000),shape)
137 title('Frequency Responses T_{r-u}')
138
139 figure(8)
140 grid on;hold on
141 if plant~=2
142     hold on
143     step(Try*tf([3*a],[1 3*a]), 0:0.01:8);
144 else
145     step(Try,[0:0.01:8]);
146 end
147 title('Step Output')
148
149 figure(9)
150 grid on;hold on
151 if plant~=2
152     hold on
153     step(Tru*tf([3*a],[1 3*a]), 0:0.01:8);
154 else
155     step(Tru, 0:0.01:8);
156 end
157 title('Controls')
158
159 [Atcl,Btcl,Ctcl,Dtcl] = branch(ss_cl);

```

```

160 Ty1u1=ss(Atcl,Btcl,Ctcl,Dtcl);
161
162 H_inf=[Gamma*W1_tf*tf(So);
163         Gamma*W2_tf*K*tf(So);
164         Gamma*W3_tf*tf(To)];
165
166 figure(10)
167 grid on;hold on
168 sigma(H_inf,logspace(-2,6),shape);
169 title('\sigma(T_{zw})')
170
171
172 end
173
174 figure(11)
175 grid on;
176 sigma(inv(W3_tf),logspace(-2,3));
177 hold on
178 sigma(inv(W1_tf),logspace(-2,3));
179 hold on
180 sigma(inv(W2_tf),logspace(-2,3));
181 title('Weighting Functions')
182
183
184 return
185 figure(1)
186 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisobil_p', 'eps')
187
188 figure(2)
189 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisobil_k', 'eps')
190
191 figure(3)
192 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisobil_pk', 'eps')
193
194 figure(4)
195 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisobil_so', 'eps')
196
197 figure(5)
198 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisobil_to', 'eps')
199
200 figure(6)
201 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisobil_tdiy', 'eps')
202
203 figure(7)
204 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisobil_tru', 'eps')
205
206 figure(8)
207 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisobil_step', 'eps')

```

```

208
209 figure(9)
210 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisobil_controls', 'eps')
211
212 figure(10)
213 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisobil_hinf', 'eps')
214
215 figure(11)
216 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisobil_wf', 'eps')

```

Macro: siso_delay.m

```

1 clc
2 clear all
3 close all
4
5 set(0,'DefaultLineLineWidth',1)
6
7 %***** PLANT *****
8 %High_Frequency_Dynamics=tf([10],[1 10])*tf([10],[1 10]);
9 High_Frequency_Dynamics=zpk([20],[-20],[-1]);
10 P=tf([1],[1 -1]);
11 %*****
12 for plant=1:3
13 if plant == 1
14     P=tf([1],[1 -1]);
15     shape = 'b-';
16
17 elseif plant == 2
18     P=tf([1],[1 -1]);
19     shape = 'b:.';
20 elseif plant == 3
21     P=tf([1],[1 -1])*High_Frequency_Dynamics;
22     shape = 'r-.';
23 end
24
25 zpk(P);
26 [Ap,Bp,Cp,Dp]=ssdata(P);
27
28 %***** Bilinear Transformation *****
29 p2 = -1e20; p1=0.0;
30 [At,Bt,Ct,Dt]=bilin(Ap,Bp,Cp,Dp,1,'Sft_jw',[p2 p1]);
31
32 %***** Weighting Functions *****
33 Eps=0.01;
34 Gamma=1;

```

```

35
36 Ms=10;
37 Wb=1;
38 Mu=0.01;
39 Wbc=30;
40 My=10;
41
42 %***** WEIGHTING FUNCTIONS *****
43 numW1=[1/Ms Wb];denW1=[1 Eps*Wb];
44 numW2=[0 Mu];denW2=[0 1];
45 numW3=[1 Wbc/My];denW3=[0 Wbc];
46
47
48 W1_tf=tf(numW1,denW1);
49 W2_tf=tf(numW2,denW2);
50 W3_tf=tf(numW3,denW3);
51
52 W1=[numW1;denW1];
53 W2=[numW2;denW2];
54 W3=[numW3;denW3];
55 %*****
56
57 %***** DESIGN *****
58 Plant_Tree = mksys(At,Bt,Ct,Dt);
59 [Augmented_Tree]=augtf(Plant_Tree,W1,W2,W3);
60 [A,B1,B2,C1,C2,D11,D12,D21,D22]=branch(Augmented_Tree);
61 System_to_be_optimized= mksys(A,B1,B2,C1,C2,D11,D12,D21,D22,'tss');
62 [Gamma,ss_cp,ss_cl,hinfo] = hinftopt(Augmented_Tree,[1:3],[0.0001 1 0]);
63 Gamma
64 1/Gamma
65
66 %W1=[Gamma*numW1;denW1];
67 %W2=[Gamma*numW2;denW2];
68 %W3=[Gamma*numW3;denW3];
69
70 %[Augmented_Tree]=augtf(Plant_Tree,W1,W2,W3);
71 %[ss_cp,ss_cl,hinfo] = hinf(Augmented_Tree,0);
72
73 [Abk,Bbk,Cbk,Dbk] = branch(ss_cp);
74 K=ss(Abk,Bbk,Cbk,Dbk);
75
76 [Atk,Btk,Ctk,Dtk]=bilin(Abk,Bbk,Cbk,Dbk,-1,'Sft_jw',[p2 p1]);
77 K=ss(Atk,Btk,Ctk,Dtk);
78 %*****
79
80 K_Poles=pole(K);
81 K_Zeros=zero(K);
82

```

```

83 %***** PRE-FILTER*****
84 H=1;
85 Lo=series(K,P);
86 To=feedback(Lo,1);
87 Wo=inv(dcgain(To));
88 a=min(abs(K_Zeros));
89 %Wo=Wo*tf([3*a],[1 3*a]);
90
91
92 %*****
93 if plant == 2
94     P=tf([1],[1 -1])*High_Frequency_Dynamics;
95 end
96
97 Lo=series(K,P);
98 So=feedback(1,Lo);
99 To=feedback(Lo,1);
100 Try=series(Wo,To);
101 Tru=series(So,K)*Wo;
102 zpk(K)
103 zpk(Try)
104 %damp(pole(Try))
105 %damp(tzero(Try))
106 pause
107
108 figure(1)
109 grid on;hold on
110 sigma(P,logspace(-2,1,1000),shape);
111 ylabel('Magnitude(dB)')
112 title('Plant Frequency Responses')
113
114 figure(2)
115 grid on;hold on
116 sigma(K,logspace(-2,5,1000),shape);
117 ylabel('Magnitude(dB)')
118 title('Controller Frequency Responses')
119
120 figure(3)
121 grid on;hold on
122 sigma(Lo,logspace(-0,2,1000),shape);
123 ylabel('Magnitude(dB)')
124 title('Open Loop Frequency Responses')
125
126 figure(4)
127 grid on;hold on
128 sigma(So,logspace(-1,3,1000),shape);
129 ylabel('Magnitude(dB)')
130 title('Sensitivity Frequency Responses')

```

```

131
132 figure(5)
133 grid on;hold on
134 sigma(To*Wo,logspace(-1,2,1000),shape);
135 ylabel('Magnitude(dB)')
136 title('Complementary Sensitivity Frequency Responses')
137
138 figure(6)
139 grid on;hold on
140 Tdiy=So*P;
141 sigma(Tdiy,logspace(-2,2,1000),shape)
142 ylabel('Magnitude(dB)')
143 title('Input Disturbance to Output Frequency Responses')
144
145 figure(7)
146 grid on;hold on
147 sigma(Tru,logspace(-2,5,1000),shape)
148 ylabel('Magnitude(dB)')
149 title('Reference to Control Frequency Responses')
150
151 figure(8)
152 grid on;hold on
153 step(Try*tf([2*a],[1 2*a]), 0:0.01:8);
154 title('Step Output with Pre-filter')
155 figure(81)
156 grid on;hold on
157 step(Try, 0:0.01:8);
158 title('Step Output without Pre-filter')
159
160 figure(9)
161 grid on;hold on
162 step(Tru*tf([2*a],[1 2*a]), 0:0.01:8);
163 title('Controls with Pre-filter')
164 figure(91)
165 grid on;hold on
166 step(Tru, 0:0.01:8);
167 title('Controls without Pre-filter')
168
169 [Atcl,Btcl,Ctcl,Dtcl] = branch(ss_cl);
170 Ty1u1=ss(Atcl,Btcl,Ctcl,Dtcl);
171
172 H_inf=[Gamma*W1_tf*tf(So);
173        Gamma*W2_tf*K*tf(So);
174        Gamma*W3_tf*tf(To)];
175
176 figure(10)
177 grid on;hold on
178 sigma(H_inf,logspace(-2,10),shape);

```

```

179 ylabel('Magnitude(dB)')
180 title('\sigma(T_{wz})')
181
182
183 end
184
185 figure(11)
186 grid on;
187 sigma(inv(W3_tf),logspace(-2,3));
188 hold on
189 sigma(inv(W1_tf),logspace(-2,3));
190 hold on
191 sigma(inv(W2_tf),logspace(-2,3));
192 ylabel('Magnitude(dB)')
193 title('Weighting Functions')
194
195
196
197 return
198 figure(1)
199 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisodelay_p', 'eps')
200
201 figure(2)
202 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisodelay_k', 'eps')
203
204 figure(3)
205 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisodelay_pk', 'eps')
206
207 figure(4)
208 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisodelay_so', 'eps')
209
210 figure(5)
211 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisodelay_to', 'eps')
212
213 figure(6)
214 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisodelay_tdiy', 'eps')
215
216 figure(7)
217 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisodelay_tru', 'eps')
218
219 figure(8)
220 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisodelay_step', 'eps')
221
222 figure(81)
223 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisodelay_step_pre', 'eps')
224
225 figure(9)
226 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisodelay_controls', 'eps')

```

```

227
228 figure(91)
229 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisodelay_controls_pre', 'eps')
230
231 figure(10)
232 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisodelay_hinf', 'eps')
233
234 figure(11)
235 saveas(gcf, 'D:\PCTeX32\Thesis\Eps\sisodelay_wf', 'eps')

```

Macro: hinf_palat.m

```

1 %close all
2 clear all
3 clc
4 DesignIndex = 3;
5 SaveFigures = 0;
6 set(0,'DefaultLineLineWidth',1)
7
8 if SaveFigures >= 1
9     Ilk = 1;
10    Son = 3;
11 else
12     Ilk = DesignIndex;
13     Son = DesignIndex;
14 end
15
16 for Design = Ilk:Son
17
18 if Design == 1
19     shape='b-';
20 elseif Design == 2
21     shape='b:.';
22 elseif Design == 3
23     shape='b-.';
24 end
25
26
27 disp('Lateral-Directional state space matrices for AR-PA mode at')
28 disp('M = 0.258 and h = 4000 ft.')
```

29	Ao=[-0.1850E+00 0.1475E+00 -0.9825E+00 0.1120E+00;
30	-0.3467E+01 -0.1710E+01 0.9029E+00 -0.5843E-06;
31	0.1174E+01 -0.8248E-01 -0.1826E+00 -0.4428E-07;
32	0.0000E+00 0.1000E+01 0.1492E+00 0.0000E+00];
33	
34	Bo=[-0.4470E-03 0.4020E-03;

```

35     0.3715E+00 0.5488E-01;
36     0.2653E-01 -0.1351E-01;
37     0.0000E+00 0.0000E+00];
38
39 Co=[0.5730E+02 0.0000E+00 0.0000E+00 0.0000E+00;
40     0.0000E+00 0.5730E+02 0.0000E+00 0.0000E+00;
41     0.0000E+00 0.0000E+00 0.5730E+02 0.0000E+00;
42     0.0000E+00 0.0000E+00 0.0000E+00 0.5730E+02];
43
44 Do=zeros(4,2);
45
46 Sx=[57.3 0 0 0;
47     0 57.3 0 0;
48     0 0 57.3 0;
49     0 0 0 57.3];
50
51 Ao = Sx*Ao*inv(Sx);
52 Bo = Sx*Bo;
53 Co = Co*inv(Sx);
54 Do = Do;
55 Po=ss(Ao,Bo,Co,Do);
56
57 Ap = Ao;
58 Bp = Bo;
59 Cp = [Co(4,:); Co(1,:)];
60 Dp = zeros(2,2);
61 P=ss(Ap,Bp,Cp,Dp);
62 Actuator_Dynamics =
63 [tf([54.1^2],[1 2*1.53*54.1 54.1^2])*tf([71.4^2],[1 2*0.735*71.4 71.4^2]) 0;
64 0 tf([54.1^2],[1 2*1.53*54.1 54.1^2])*tf([71.4^2],[1 2*0.735*71.4 71.4^2])];
65
66 if Design == 3
67     P=ss(Ap,Bp,Cp,Dp);
68     P=P*Actuator_Dynamics;
69     [Ap,Bp,Cp,Dp]=ssdata(P);
70 end
71
72 [Aact,Bact,Cact,Dact]=ssdata(Actuator_Dynamics);
73 Act=ss(Aact,Bact,Cact,Dact);
74
75
76 p2 = -1e20; p1 = -1.2;
77 [At,Bt,Ct,Dt]=bilinear(P.a,P.b,P.c,P.d,1,'Sft_jw',[p2 p1]);
78 P_transformed=ss(At,Bt,Ct,Dt);
79
80
81 Eps = 0.001;
82 Ms1 = 10;          wb1 = 1.35;

```

```

83 Ms2 = 10;          wb2 = 0.90;          q = 2; p = 2;
84 W1 = [tf([1/(Ms1^(1/q)) wb1], [1 wb1*(Eps^(1/q))])^q 0;
85         0 tf([1/(Ms2^(1/p)) wb2], [1 wb2*(Eps^(1/p))])^p];
86 [aw1,bw1,cw1,dw1]=ssdata(W1);
87
88 wbu1 = 4000; Mu1 = 1000;
89 wbu2 = 250; Mu2 = 1000;
90 W2 = [tf([1 wbu1/Mu1], [Eps wbu1]) 0;
91         0 tf([1 wbu2/Mu2], [Eps wbu2])];
92 [aw2,bw2,cw2,dw2]=ssdata(W2);
93
94 My1 = 100;wbc1= 45;
95 My2 = 100;wbc2= 45; v = 2; w = 2;
96 W3 = [tf([1 wbc1/(My1^(1/v))], [(Eps^(1/v)) wbc1])^v 0;
97         0 tf([1 wbc2/(My2^(1/w))], [(Eps^(1/w)) wbc2])^w];
98 [aw3,bw3,cw3,dw3]=ssdata(W3);
99
100 %***** hinf *****
101 ss_W1=ss(aw1,bw1,cw1,dw1);
102 ss_W2=ss(aw2,bw2,cw2,dw2);
103 ss_W3=ss(aw3,bw3,cw3,dw3);
104
105 Gamma=1;
106 [Augmented_Tree]=augss(P_transformed,ss_W1,ss_W2,ss_W3);
107 [Gamma,ss_cp,ss_cl,hinfo] = hinfopt(Augmented_Tree,[1:6],[0.01 2 0]);
108 Gamma
109 %1/Gamma
110 [Augmented_Tree]=augss(P_transformed,Gamma*ss_W1,Gamma*ss_W2,Gamma*ss_W3);
111 %[ss_cp,ss_cl,hinfo,tss_k]=hinf(Augmented_Tree,1);
112
113
114 [Acp,Bcp,Ccp,Dcp] = branch(ss_cp);
115 [Atk,Btk,Ctk,Dtk]=bilinear(Acp,Bcp,Ccp,Dcp,-1,'Sft_jw',[p2 p1]);
116
117 K=ss(Atk,Btk,Ctk,Dtk);
118
119 %***** PRE DC FILTER *****
120 H=eye(2);
121 Wo=eye(2);
122 a=3.0;
123 b=1.9;
124 Lo=series(K,P);To = feedback(Lo,H);
125 Wo=Wo*inv(dcgain(To));
126 PreFilter=tf([a], [1 a]) 0;
127         0 tf([b], [1 b]);
128 Wo=Wo*PreFilter;
129 [Woa,Wob,Woc,Wod]=ssdata(Wo);
130 W=ss(Woa,Wob,Woc,Wod);

```

```

131 %***** design check *****
132 if Design == 2
133     P=ss(Ap,Bp,Cp,Dp);
134     P=P*Actuator_Dynamics;
135 end
136 Lo=series(K,P);Li=series(P,K);
137 So=feedback(eye(2),Lo);Si=feedback(eye(2),Li);
138 To=feedback(Lo,eye(2));Ti=feedback(Li,eye(2));
139
140 save evallat Po K Act W
141
142
143 damp(pole(To))
144 damp(tzero(To))
145
146 return
147
148 figure(1)
149 hold on;grid on;
150 sigma(P,logspace(-2,2,100),shape);
151 title('TITO X-29 Singular Values')
152
153 figure(2)
154 hold on;grid on;
155 sigma(K,logspace(-2,4,100),shape);
156 title('Compensator Singular Values')
157
158 figure(3)
159 hold on;grid on;
160 sigma(Lo,logspace(-2,3,100),shape);
161 title('Open Loop Singular Values at Plant Output L_o = PK')
162
163 figure(4)
164 hold on;grid on;
165 sigma(Li,logspace(-2,3,100),shape);
166 title('Open Loop Singular Values at Plant Input L_i = KP')
167
168 figure(5)
169 hold on;grid on;
170 sigma(So,logspace(-2,3,100),shape)
171 title('Sensitivity Singular Values at Plant Output S_o = [I+PK]^{-1}')
172
173 figure(6)
174 hold on;grid on;
175 sigma(Si,logspace(-2,3,100),shape)
176 title('Sensitivity Singular Values at Plant Input S_i = [I+KP]^{-1}')
177
178 figure(7)

```

```

179 hold on;grid on;
180 sigma(To,logspace(-1,2,100),shape)
181 title('Complementary Sensitivity Singular Values at Plant Output T_o = PK[I+PK]^{-1}')
182
183 figure(8)
184 hold on;grid on;
185 sigma(Ti,logspace(-2,2,100),shape)
186 title('Complementary Sensitivity Singular Values at Plant Input T_i = KP[I+KP]^{-1}')
187
188 figure(9)
189 hold on;grid on;
190 Try=series(Wo,To);
191 sigma(Try,logspace(-1,2,100),shape)
192 title('Closed Loop Singular Values: Reference to Output T_r_y=T_o*W_o')
193
194 figure(10)
195 hold on;grid on;
196 Tdiy=series(P,So);
197 sigma(Tdiy,logspace(-1,2,100),shape)
198 title('Input Disturbance to Output Singular Values T_{d_iy}=S_o*P')
199
200 figure(11)
201 hold on;grid on;
202 Tru=series(So,K);
203 %sigma(Tru*Wo,logspace(-2,3,100),shape)
204 sigma(Tru,logspace(-2,3,100),shape)
205 title('Reference to Control Singular Values T_r_u=K*S_o')
206
207 figure(12)
208 hold on;grid on;
209 step(Try,[0:0.01:5],shape);
210 title('Step Output')
211
212 figure(13)
213 hold on;grid on;
214 step(Tru*Wo, 0:0.01:5, shape);
215 title('Controls')
216
217
218 [Y,T]=step(Try,[0:0.01:5]);
219
220 figure(15)
221 hold on
222 subplot(2,1,1)
223 plot(T,Y(:,1,1),shape)
224 ylabel('\phi (deg)')
225 xlabel('t (sec)')
226 title('Response to \phi Step Reference Command - Outputs')

```

```

227 grid on
228 hold on
229 subplot(2,1,2)
230 plot(T,Y(:,2,1),shape)
231 ylabel('\beta (deg)')
232 xlabel('t (sec)')
233 grid on
234
235 figure(16)
236 hold on
237 subplot(2,1,1)
238 plot(T,Y(:,1,2),shape)
239 ylabel('\phi (deg)')
240 xlabel('t (sec)')
241 title('Response to \beta Step Reference Command - Outputs')
242 grid on
243 hold on
244 subplot(2,1,2)
245 plot(T,Y(:,2,2),shape)
246 ylabel('\beta (deg)')
247 xlabel('t (sec)')
248 grid on
249
250 [Y,T]=step(Tru*Wo, 0:0.01:5);
251
252 figure(17)
253 hold on
254 subplot(2,1,1)
255 plot(T,Y(:,1,1),shape)
256 ylabel('\delta_{df} (deg)')
257 xlabel('t (sec)')
258 title('Response to \phi Step Reference Command - Controls')
259 grid on
260 hold on
261 subplot(2,1,2)
262 plot(T,Y(:,2,1),shape)
263 ylabel('\delta_r (deg)')
264 xlabel('t (sec)')
265 grid on
266
267 figure(18)
268 hold on
269 subplot(2,1,1)
270 plot(T,Y(:,1,2),shape)
271 ylabel('\delta_{df} (deg)')
272 xlabel('t (sec)')
273 title('Response to \beta Step Reference Command - Controls')
274 grid on

```

```

275 hold on
276 subplot(2,1,2)
277 plot(T,Y(:,2,2),shape)
278 ylabel('\delta_r (deg)')
279 xlabel('t (sec)')
280 grid on
281
282 display('Press any key')
283 pause
284 end
285
286 if SaveFigures == 2
287
288 figure(1)
289 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_p', 'eps')
290
291 figure(2)
292 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_k', 'eps')
293
294 figure(3)
295 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_pk', 'eps')
296
297 figure(4)
298 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_kp', 'eps')
299
300 figure(5)
301 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_so', 'eps')
302
303 figure(6)
304 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_si', 'eps')
305
306 figure(7)
307 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_to', 'eps')
308
309 figure(8)
310 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_ti', 'eps')
311
312 figure(9)
313 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_try', 'eps')
314
315 figure(10)
316 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_tdiy', 'eps')
317
318 figure(11)
319 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_tru', 'eps')
320
321 figure(12)
322 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_step', 'eps')

```

```

323
324 figure(13)
325 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_controls', 'eps')
326
327 figure(14)
328 grid on
329 sigma(inv(W1),logspace(-3,6,100))
330 hold on
331 sigma(inv(W2),logspace(-3,6,100))
332 hold on
333 sigma(inv(W3),logspace(-3,6,100))
334 title('Weighting Functions')
335
336 figure(14)
337 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_wf', 'eps')
338
339 figure(15)
340 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_outputs_step_phi', 'eps')
341
342 figure(16)
343 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_outputs_step_beta', 'eps')
344
345 figure(17)
346 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_controls_step_phi', 'eps')
347
348 figure(18)
349 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_controls_step_beta', 'eps')
350
351 figure(14)
352 grid on
353 sigma(inv(W1),logspace(-3,5,100))
354 hold on
355 sigma(inv(W2),logspace(-3,5,100))
356 hold on
357 sigma(inv(W3),logspace(-3,5,100))
358 title('Weighting Functions')
359
360 figure(14)
361 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalat_wf', 'eps')
362
363 end

```

Macro: hinf_palon1.m

1

2 clear all

```

3  clc
4  DesignIndex = 1;
5  SaveFigures = 0;
6  set(0,'DefaultLineLineWidth',1)
7
8  if SaveFigures > 0
9      Ilk = 1;
10     Son = 3;
11 else
12     Ilk = DesignIndex;
13     Son = DesignIndex;
14 end
15
16 for Design = Ilk:Son
17
18 if Design == 1
19     shape='b-';
20 elseif Design == 2
21     shape='b:.';
22 elseif Design == 3
23     shape='b-.';
24 end
25
26
27 %***** Plant Description *****
28 disp('Longitudinal state space matrices for AR-PA mode at')
29 disp('M = 0.258 and h = 4000 ft.')
30 Ao=[-0.4251E-01 -0.8019E+01 -0.4538E+00 -0.3216E+02;
31     -0.7889E-03 -0.5482E+00  0.9893E+00  0.1812E-09;
32     0.3910E-03  0.3604E+01 -0.2221E+00  0.6564E-08;
33     0.0000E+00  0.0000E+00  0.1000E+01  0.0000E+00;];
34
35 Bo=[-0.3430E-01 -0.9217E-01 0.4827E-02;
36     -0.1065E-02 -0.1337E-02 -0.4219E-03;
37     0.2734E-01 -0.5587E-02 -0.1336E-01;
38     0.0000E+00  0.0000E+00  0.0000E+00;];
39
40 Co=[0.1000E+01 0.0000E+00 0.0000E+00 0.0000E+00;
41     0.0000E+00 0.5730E+02 0.0000E+00 0.0000E+00;
42     0.0000E+00 0.0000E+00 0.5730E+02 0.0000E+00;
43     0.0000E+00 0.0000E+00 0.0000E+00 0.5730E+02];
44
45 Do=[0.0000E+00 0.0000E+00 0.0000E+00;
46     0.0000E+00 0.0000E+00 0.0000E+00;
47     0.0000E+00 0.0000E+00 0.0000E+00;
48     0.0000E+00 0.0000E+00 0.0000E+00];
49
50

```

```

51 Sx=[1 0 0 0;
52     0 57.3 0 0;
53     0 0 57.3 0;
54     0 0 0 57.3];
55
56
57 Ao = Sx*Ao*inv(Sx);
58 Bo = Sx*Bo;
59 Co = Co*inv(Sx);
60 Do = Do;
61 Po=ss(Ao,Bo,Co,Do);
62
63
64 Ap = Ao;
65 Bp = Bo;
66 Cp = [Co(4,:)];
67 Dp   = zeros(1,3);
68 P=ss(Ap,Bp,Cp,Dp);
69
70
71
72 Canard_Actuator = tf([0.885*20.2],[1 20.2])*tf([71.4^2],[1 2*0.736*71.4 71.4^2])*tf([144.9],[1 144.9]);
73 Strake_Actuator = tf([20.2],[1 20.2])*tf([71.4^2],[1 2*0.736*71.4 71.4^2])*tf([144.9],[1 144.9]);
74 SymFlap_Actuator = tf([50],[1 50])*tf([325^2],[1 2*0.70*325 325^2])*tf([100],[1 100]);
75
76 Actuator_Dynamics = [Canard_Actuator 0 0;
77                      0 Strake_Actuator 0;
78                      0 0 SymFlap_Actuator];
79
80 [Aact,Bact,Cact,Dact]=ssdata(Actuator_Dynamics);
81 Act=ss(Aact,Bact,Cact,Dact);
82
83 if Design == 3
84     P=ss(Ap,Bp,Cp,Dp);
85     P=P*Actuator_Dynamics;
86     [Ap,Bp,Cp,Dp]=ssdata(P);
87 end
88
89 %***** Bilinear Transformation *****
90 p2 = -1e20; p1 = -0.10;
91 [At,Bt,Ct,Dt]=bilinear(P.a,P.b,P.c,P.d,1,'Sft_jw',[p2 p1]);
92 P_transformed=ss(At,Bt,Ct,Dt);
93
94 %***** Weight function parameters We,Wu,Wy *****
95 Eps = 0.001;
96 Ms1 = 10;wb1 = 1;
97 W1 = [tf([1/(Ms1)^(1/1) wb1], [1 wb1*(Eps)^(1/1)])^1];
98 [aw1,bw1,cw1,dw1]=ssdata(W1);

```

```

99
100 wbu1 = 1000/1.2; Mu1 = 100;
101 wbu2 = 200; Mu2 = 100;
102 wbu3 = 1000/1.5; Mu3 = 100;
103
104 W2 = [tf([1 wbu1/Mu1], [Eps wbu1]) 0 0;
105         0 tf([1 wbu2/Mu2], [Eps wbu2]) 0;
106         0 0 tf([1 wbu3/Mu3], [Eps wbu3])];
107 [aw2,bw2,cw2,dw2]=ssdata(W2);
108
109 My1 = 1000;wbc1= 10;
110 W3 = [tf([1 wbc1/(My1^(1/2))], [(Eps^(1/2)) wbc1])^2];
111 [aw3,bw3,cw3,dw3]=ssdata(W3);
112
113 %***** hinf *****
114 ss_W1=ss(aw1,bw1,cw1,dw1);
115 ss_W2=ss(aw2,bw2,cw2,dw2);
116 ss_W3=ss(aw3,bw3,cw3,dw3);
117
118 Gamma=1;
119 [Augmented_Tree]=augss(P_transformed,ss_W1,ss_W2,ss_W3);
120 [Gamma,ss_cp,ss_cl,hinfo] = hinft(Augmented_Tree,[1:5],[0.0001 1 0]);
121 Gamma
122
123 [Augmented_Tree]=augss(P_transformed,Gamma*ss_W1,Gamma*ss_W2,Gamma*ss_W3);
124 [ss_cp,ss_cl,hinfo,tss_k]=hinf(Augmented_Tree,1);
125
126
127 [Acp,Bcp,Ccp,Dcp] = branch(ss_cp);
128 [Atk,Btk,Ctk,Dtk]=bilinear(Acp,Bcp,Ccp,Dcp,-1,'Sft_jw',[p2 p1]);
129 K=ss(Atk,Btk,Ctk,Dtk);
130
131 %***** PRE DC FILTER *****
132 H=eye(1);
133 Wo=eye(1);
134 a=0.9;
135 Lo=series(K,P);To = feedback(Lo,H);
136 Wo=Wo*inv(dcgain(To));
137 PreFilter=tf([a], [1 a]);
138 Wo=Wo*PreFilter;
139 [Woa,Wob,Woc,Wod]=ssdata(Wo);
140 W=ss(Woa,Wob,Woc,Wod);
141 %***** design check *****
142 if Design == 2
143     P=ss(Ap,Bp,Cp,Dp);
144     P=P*Actuator_Dynamics;
145 end
146 Lo=series(K,P);Li=series(P,K);

```

```

147 So=feedback(eye(1),Lo);Si=feedback(eye(3),Li);
148 To=feedback(Lo,eye(1));Ti=feedback(Li,eye(3));
149
150 save evallon Po K Act W
151
152 damp(pole(To))
153 damp(tzero(To))
154
155 %H_inf=[W1*So;
156 %      W2*K*So;
157 %      W3*To];
158 %sigma(H_inf,logspace(-10,10,100))
159 return
160
161 figure(1)
162 hold on;grid on;
163 sigma(P,logspace(-2,2,100),shape);
164 title('Plant Singular Values')
165
166 figure(2)
167 hold on;grid on;
168 sigma(K,logspace(-1,3,100),shape);
169 title('Compensator Singular Values')
170
171 figure(3)
172 hold on;grid on;
173 sigma(Lo,logspace(-2,2,100),shape);
174 title('Open Loop Singular Values L_o = PK')
175
176 figure(4)
177 hold on;grid on;
178 sigma(Li,logspace(-2,2,100),shape);
179 title('Open Loop Singular Values L_i = KP')
180
181 figure(5)
182 hold on;grid on;
183 sigma(So,logspace(-2,2,100),shape)
184 title('Frequency Responses S_o = [I+PK]^{-1}')
185
186 figure(6)
187 hold on;grid on;
188 sigma(Si,logspace(-2,2,100),shape)
189 title('Frequency Responses S_i = [I+KP]^{-1}')
190
191 figure(7)
192 hold on;grid on;
193 sigma(To,logspace(-1,2,100),shape)
194 title('Frequency Responses T_o = PK[I+PK]^{-1}')

```

```

195
196 figure(8)
197 hold on;grid on;
198 sigma(Ti,logspace(-2,2,100),shape)
199 title('Frequency Responses T_i = KP[I+KP]^{-1}')
```

200

```

201 figure(9)
202 hold on;grid on;
203 Try=series(Wo,To);
204 sigma(Try,logspace(-1,2,100),shape)
205 title('Frequency Responses T_r_y=T_o*W')
```

206

```

207 figure(10)
208 hold on;grid on;
209 Tdiy=series(P,So);
210 sigma(Tdiy,logspace(-2,2,100),shape)
211 title('Frequency Responses T_{d_iy}=S_o*P')
```

212

```

213 figure(11)
214 hold on;grid on;
215 Tru=series(So,K);
216 sigma(Tru*Wo,logspace(-2,3,100),shape)
217 title('Frequency Responses T_r_u=K*S_o')
```

218

219

```

220 P1=ss(Ao,Bo,[Co(1,:);Co(2,:);Co(4,:)],zeros(3,3));
221
222 if Design == 2
223     P1=ss(Ao,Bo,[Co(1,:);Co(2,:);Co(4,:)],zeros(3,3));
224     P1=P1*Actuator_Dynamics;
225 end
226
227 Sys = feedback(P1*K,eye(1),[1],[3])*Wo;
```

228

```

229 figure(12)
230 hold on;grid on;
231 [Y,T]=step(Try,[0:0.01:10]);
232 %step(Try,[0:0.01:10],shape);
233 step(Sys,[0:0.01:10],shape);
234 title('Step Output')
```

235

```

236 figure(13)
237 hold on;grid on;
238 step(Tru*Wo, 0:0.01:10, shape);
239 title('Controls')
```

240

```

241 display('Press a key to continue')
242 pause
```

```

243 end
244
245 if SaveFigures == 2
246
247 figure(1)
248 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon1_p', 'eps')
249
250 figure(2)
251 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon1_k', 'eps')
252
253 figure(3)
254 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon1_pk', 'eps')
255
256 figure(4)
257 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon1_kp', 'eps')
258
259 figure(5)
260 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon1_so', 'eps')
261
262 figure(6)
263 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon1_si', 'eps')
264
265 figure(7)
266 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon1_to', 'eps')
267
268 figure(8)
269 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon1_ti', 'eps')
270
271 figure(9)
272 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon1_try', 'eps')
273
274 figure(10)
275 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon1_tdiy', 'eps')
276
277 figure(11)
278 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon1_tru', 'eps')
279
280 figure(12)
281 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon1_step', 'eps')
282
283 figure(13)
284 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon1_controls', 'eps')
285
286 figure(14)
287 grid on
288 sigma(inv(W1),logspace(-3,3,100))
289 hold on
290 sigma(inv(W2),logspace(-3,3,100))

```

```

291 hold on
292 sigma(inv(W3),logspace(-3,3,100))
293 title('Weighting Functions')
294
295 figure(14)
296 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon1_wf', 'eps')
297
298
299 end

```

<i>Macro: hinf_palon2.1.m</i>

```

1 close all
2 clear all
3 clc
4 DesignIndex = 3;
5 SaveFigures = 0;
6 set(0,'DefaultLineLineWidth',1)
7
8 if SaveFigures >= 1
9     Ilk = 1;
10    Son = 3;
11 else
12     Ilk = DesignIndex;
13     Son = DesignIndex;
14     close all
15 end
16
17 for Design = Ilk:Son
18
19 if Design == 1
20     shape='b-';
21 elseif Design == 2
22     shape='b:.';
23 elseif Design == 3
24     shape='b-.';
25 end
26
27 %***** Plant Description *****
28 disp('Longitudinal state space matrices for AR-PA mode at')
29 disp('M = 0.258 and h = 4000 ft.')
```

<pre> 30 Ac=[-0.4251E-01 -0.8019E+01 -0.4538E+00 -0.3216E+02; 31 -0.7889E-03 -0.5482E+00 0.9893E+00 0.1812E-09; 32 0.3910E-03 0.3604E+01 -0.2221E+00 0.6564E-08; 33 0.0000E+00 0.0000E+00 0.1000E+01 0.0000E+00;]; 34 </pre>

```

35 Bo=[-0.3430E-01 -0.9217E-01 0.4827E-02;
36     -0.1065E-02 -0.1337E-02 -0.4219E-03;
37     0.2734E-01 -0.5587E-02 -0.1336E-01;
38     0.0000E+00 0.0000E+00 0.0000E+00];
39
40 Co=[0.1000E+01 0.0000E+00 0.0000E+00 0.0000E+00;
41     0.0000E+00 0.5730E+02 0.0000E+00 0.0000E+00;
42     0.0000E+00 0.0000E+00 0.5730E+02 0.0000E+00;
43     0.0000E+00 0.0000E+00 0.0000E+00 0.5730E+02];
44
45 Do=[0.0000E+00 0.0000E+00 0.0000E+00;
46     0.0000E+00 0.0000E+00 0.0000E+00;
47     0.0000E+00 0.0000E+00 0.0000E+00;
48     0.0000E+00 0.0000E+00 0.0000E+00];
49
50
51 Sx=[1 0 0 0;
52     0 57.3 0 0;
53     0 0 57.3 0;
54     0 0 0 57.3];
55
56 Ao = Sx*Ao*inv(Sx);
57 Bo = Sx*Bo;
58 Co = Co*inv(Sx);
59 Do = Do;
60 Po=ss(Ao,Bo,Co,Do);
61
62 Ap = Ao;
63 Bp = Bo;
64 Cp = [Co(1,:); Co(4,:)];
65 Dp = zeros(2,3);
66 P=ss(Ap,Bp,Cp,Dp);
67
68 Actuator_Dynamics = ...
69 [tf([0.885*20.2],[1 20.2])*tf([71.4^2],[1 2*0.736*71.4 71.4^2])*tf([144.9],[1 144.9]) 0 0;
70 0 tf([20.2],[1 20.2])*tf([71.4^2],[1 2*0.736*71.4 71.4^2])*tf([144.9],[1 144.9]) 0;
71 0 0 tf([50],[1 50])*tf([325^2],[1 2*0.70*325 325^2])*tf([100],[1 100])];
72 [Aact,Bact,Cact,Dact]=ssdata(Actuator_Dynamics);
73 Act=ss(Aact,Bact,Cact,Dact);
74 if Design == 3
75     P=ss(Ap,Bp,Cp,Dp);
76     P=P*Actuator_Dynamics;
77     [Ap,Bp,Cp,Dp]=ssdata(P);
78 end
79
80
81 %***** Weight function parameters We,Wu,Wy *****
82 Eps = 0.01;

```

```

83 Ms1 = 10;wb1 = 1*0.3;
84 Ms2 = 10;wb2 = 2*0.3;
85 W1 = [tf([1/(Ms1) wb1], [1 wb1*(Eps)]) 0;
86         0 tf([1/(Ms2) wb2], [1 wb2*(Eps)])]);
87 [aw1,bw1,cw1,dw1]=ssdata(W1);
88
89 wbu1 = 300; Mu1 = 100;
90 wbu2 = 200; Mu2 = 100;
91 wbu3 = 500; Mu3 = 100;
92
93 W2 = [tf([1 wbu1/Mu1], [Eps wbu1]) 0 0;
94         0 tf([1 wbu2/Mu2], [Eps wbu2]) 0;
95         0 0 tf([1 wbu3/Mu3], [Eps wbu3])]);
96 [aw2,bw2,cw2,dw2]=ssdata(W2);
97
98 My1 = 100;wbc1= 10;
99 My2 = 100;wbc2= 10;
100 W3 = [tf([1 wbc1/(My1^(1/2))], [(Eps^(1/2)) wbc1])^2 0;
101         0 tf([1 wbc2/(My2^(1/2))], [(Eps^(1/2)) wbc2])^2];
102 [aw3,bw3,cw3,dw3]=ssdata(W3);
103
104 %***** Bilinear Transformation *****
105 p2 = -1e20; p1 = -0.15;
106 [At,Bt,Ct,Dt]=bilin(Ap,Bp,Cp,Dp,1,'Sft_jw',[p2 p1]);
107 P_transformed=ss(At,Bt,Ct,Dt);
108
109 %***** hinf *****
110 ss_W1=ss(aw1,bw1,cw1,dw1);
111 ss_W2=ss(aw2,bw2,cw2,dw2);
112 ss_W3=ss(aw3,bw3,cw3,dw3);
113
114 Gamma=1;
115 [Augmented_Tree]=augss(P_transformed,ss_W1,ss_W2,ss_W3);
116 [Gamma,ss_cp,ss_cl,hinfo] = hinfopt(Augmented_Tree,[1:7],[0.0001 1 0]);
117 Gamma
118
119 [Augmented_Tree]=augss(P_transformed,Gamma*ss_W1,Gamma*ss_W2,Gamma*ss_W3);
120 [ss_cp,ss_cl,hinfo,tss_k]=hinf(Augmented_Tree,1);
121
122
123 [Acp,Bcp,Ccp,Dcp] = branch(ss_cp);
124 [Atk,Btk,Ctk,Dtk]=bilin(Acp,Bcp,Ccp,Dcp,-1,'Sft_jw',[p2 p1]);
125 K=ss(Atk,Btk,Ctk,Dtk);
126
127 %***** PRE DC FILTER *****
128 H=eye(2);
129 Wo=eye(2);
130 a=1;

```

```

131 b=0.9;
132 Lo=series(K,P);To = feedback(Lo,H);
133 Wo=Wo*inv(dcgain(To));
134 PreFilter=tf([a], [1 a]) 0;
135           0 tf([b], [1 b]));
136 Wo=Wo*PreFilter;
137
138 %***** design check *****
139 if Design == 2
140     P=ss(Ap,Bp,Cp,Dp);
141     P=P*Actuator_Dynamics;
142 end
143 Lo=series(K,P);Li=series(P,K);
144 So=feedback(eye(2),Lo);Si=feedback(eye(3),Li);
145 To=feedback(Lo,eye(2));Ti=feedback(Li,eye(3));
146
147 damp(pole(To))
148 damp(tzero(To))
149
150 return
151
152 figure(1)
153 hold on;grid on;
154 sigma(P,logspace(-2,2,100),shape);
155 title('Plant Singular Values')
156
157 figure(2)
158 hold on;grid on;
159 sigma(K,logspace(-1,3,100),shape);
160 title('Compensator Singular Values')
161
162 figure(3)
163 hold on;grid on;
164 sigma(Lo,logspace(-2,2,100),shape);
165 title('Open Loop Singular Values L_o = PK')
166
167 figure(4)
168 hold on;grid on;
169 sigma(Li,logspace(-2,2,100),shape);
170 title('Open Loop Singular Values L_i = KP')
171
172 figure(5)
173 hold on;grid on;
174 sigma(So,logspace(-2,2,100),shape)
175 title('Frequency Responses S_o = [I+PK]^-1')
176
177 figure(6)
178 hold on;grid on;

```

```

179 sigma(Si,logspace(-2,2,100),shape)
180 title('Frequency Responses S_i = [I+KP]^-1')
181
182 figure(7)
183 hold on;grid on;
184 sigma(To,logspace(-1,2,100),shape)
185 title('Frequency Responses T_o = PK[I+PK]^-1')
186
187 figure(8)
188 hold on;grid on;
189 sigma(Ti,logspace(-2,2,100),shape)
190 title('Frequency Responses T_i = KP[I+KP]^-1')
191
192 figure(9)
193 hold on;grid on;
194 Try=series(Wo,To);
195 sigma(Try,logspace(-1,2,100),shape)
196 title('Frequency Responses T_r_y=T_o*W')
197
198 figure(10)
199 hold on;grid on;
200 Tdiy=series(P,So);
201 sigma(Tdiy,logspace(-2,2,100),shape)
202 title('Frequency Responses T_{d_iy}=S_o*P')
203
204 figure(11)
205 hold on;grid on;
206 Tru=series(So,K);
207 sigma(Tru*Wo,logspace(-2,3,100),shape)
208 title('Frequency Responses T_r_u=K*S_o')
209
210 P1=ss(Ao,Bo,[Co(1,:);Co(2,:);Co(4,:)],zeros(3,3));
211 if Design > 1
212     P1=ss(Ao,Bo,[Co(1,:);Co(2,:);Co(4,:)],zeros(3,3));
213     P1=P1*Actuator_Dynamics;
214 end
215 Sys = feedback(P1*K,eye(2),[1 2],[1 3])*Wo;
216
217 figure(12)
218 hold on;grid on;
219 [Y,T]=step(Try,[0:0.01:10]);
220 %step(Try,[0:0.01:10],shape);
221 step(Sys,[0:0.01:10],shape);
222 title('Step Output')
223
224 figure(13)
225 hold on;grid on;
226 step(Tru*Wo, 0:0.01:12, shape);

```

```

227 title('Controls')
228
229
230 t = 0:0.01:10;
231
232 u2 = ones(length(t),1);
233 u1 = -0.35*t;
234 u1 = u1';
235 u = [u1 u2];
236 hold on;grid on;
237 figure(15)
238 lsim(Try,u,t)
239 figure(16)
240 lsim(Tru*Wo,u,t)
241
242 display('Press any key to continue')
243 pause
244 end
245
246 if SaveFigures == 2
247
248 figure(1)
249 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon2_1_p', 'eps')
250
251 figure(2)
252 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon2_1_k', 'eps')
253
254 figure(3)
255 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon2_1_pk', 'eps')
256
257 figure(4)
258 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon2_1_kp', 'eps')
259
260 figure(5)
261 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon2_1_so', 'eps')
262
263 figure(6)
264 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon2_1_si', 'eps')
265
266 figure(7)
267 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon2_1_to', 'eps')
268
269 figure(8)
270 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon2_1_ti', 'eps')
271
272 figure(9)
273 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon2_1_try', 'eps')
274

```

```
275 figure(10)
276 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon2_1_tdiy', 'eps')
277
278 figure(11)
279 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon2_1_tru', 'eps')
280
281 figure(12)
282 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon2_1_step', 'eps')
283
284 figure(13)
285 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon2_1_controls', 'eps')
286
287 figure(14)
288 grid on
289 sigma(inv(W1),logspace(-3,3,100))
290 hold on
291 sigma(inv(W2),logspace(-3,3,100))
292 hold on
293 sigma(inv(W3),logspace(-3,3,100))
294 title('Weighting Functions')
295
296 figure(14)
297 saveas(gcf, 'D:\PCTeX32\Documents\Thesis\Eps\x29arpalon2_1_wf', 'eps')
298
299
300 end
```