ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

ON THE HYPERSURFACES WITH NON-DIAGONALIZABLE SHAPE OPERATOR IN MINKOWSKI SPACES

M.Sc. THESIS

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

MİNKOWSKİ UZAYLARINDA KÖŞEGENLEŞTİRİLEMEYEN ŞEKİL OPERATÖRÜNE SAHİP HİPERYÜZEYLER ÜZERİNE

YÜKSEK LİSANS TEZİ

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Nilgün ÜNSAL, a M.Sc. student of ITU Graduate School student ID 509181215, successfully defended the thesis entitled "ON THE HYPERSURFACES WITH NON-DIAGONALIZABLE SHAPE OPERATOR IN MINKOWSKI SPACES", which he/she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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To my family,

FOREWORD

I started my master's degree at Istanbul Technical University in 2019 and gained very important experiences during this time. My dear advisor Assoc. Prof. Dr. Nurettin Cenk Turgay shared his knowledge and experience with me in every subject and contributed greatly to the development of my academic skills and has always been with me throughout the time we worked together. I would like to thank my dear advisor Assoc. Prof. Dr. Nurettin Cenk Turgay for his efforts and contributions to the emergence of this thesis.

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June 2022 Nilgün ÜNSAL

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ABBREVIATIONS

SYMBOLS

M	:	Hypersurface
N	:	Unit Normal Vector
S	:	Shape Operator of Hypersurface
h	:	Second Fundamental Form of Hypersurface
H	:	Mean Curvature
K	:	Gauss–Kronecker Curvature
$\mathbb{E}_1^n$	:	n-dimensional Minkowski Space
∇	:	Levi-Civita Connection of Hypersurface
$\tilde{\nabla}$	:	Levi-Civita Connection of Minkowski Space
$\overline{\nabla}$	:	Van Der Waerden-Bortolotti Connection
$\nabla^{\perp}$	:	Normal Connection
w_{ij}	:	Connection Form

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ON THE HYPERSURFACES WITH NON-DIAGONALIZABLE SHAPE OPERATOR IN MINKOWSKI SPACES

SUMMARY

Let M be an oriented hypersurface of the Minkowski space $\mathbb{E}_1^{n+1}$. One of the most important extrinsic object of M is its shape operator S defined by the Wiengarten formula

$$SX = -\tilde{\nabla}_X N,$$

where N is the unit normal vector field to M whenever $X \in TM$.

The shape operator can be used to determine how the tangent plane and its normal move in all directions. Note that S is a self adjoint endomorphism in TM. Therefore, it is diagonalizable when M is Riemannian. However, if M is Lorentzian, then its shape operator can be non-diagonalizable. In this case, the shape operator S has four canonical forms. These canonical forms are written with respect to either an orthonormal basis or a pseudo-orthonormal basis. If the basis is orthonormal, then it is called a orthonormal frame field. An orthonormal frame of vector fields in a neighborhood of any point in M is a basis $\{E_1, \dots, E_n\}$ such that

$$(E_1, E_1) = -1, \quad (E_1, E_i) = 0, \quad (E_i, E_j) = \delta_{ij}$$

for $2 \leq i,\ j \leq n$. On the other hand if the basis is pseudo-orthonormal, then it is called a pseudo-orthonormal frame field. A pseudo-orthonormal frame of vector fields in a neighborhood of any point in M is a basis $\{X, Y, E_1, \dots, E_{n-2}\}$ such that

$$(X, X) = (Y, Y) = 0, \quad (X, E_i) = (Y, E_i) = 0, \quad (X, Y) = -1$$

and

$$(E_i, E_j) = \delta_{ij}$$

for $1 \leq i,\ j \leq n-2$.

The eigenvalues and eigenvectors of S are called the principal curvatures and principal directions of M, respectively. If the shape operator S is diagonalizable and M has constant principal curvatures, then the hypersurface M is said to be isoparametric. If S is non-diagonalizable and its minimal polynomial is constant, then M is said to be isoparametric.

In this thesis, we study isoparametric hypersurfaces with non-diagonalizable shape operator in Minkowski space $\mathbb{E}_1^4$. This thesis consists of five sections. First section is introduction. In the second section, we give some basic concepts on Lorentzian inner product and also some basic facts on hypersurfaces of $\mathbb{E}_1^{n+1}$. In the third section, a theorem about isoparametric hypersurfaces is given. We note that these theorems are proved by Magid in 1985. We prove these theorems by using another method. In fact, this theorem implies that there is only four classes of isoparametric hypersurface using

the Codazzi and Gauss equations in $\mathbb{E}^4_1$. Then, we give another theorem which proves that there is no isoparametric hypersurface in $\mathbb{E}^4_1$ with complex principal curvatures.

In the fourth section, we construct a family of hypersurfaces with non-diagonalizable shape operator in $\mathbb{E}^5_1$. We obtain the shape operator, the mean curvature, Gauss-Kronecker curvature and Levi-Civita connection of this hypersurface. Then, we give the necessary and sufficient condition for this hypersurface to be minimal with a theorem.

MİNKOWSKİ UZAYLARINDA KÖŞEGENLEŞTİRİLEMEYEN ŞEKİL OPERATÖRÜNE SAHİP HİPERYÜZEYLER ÜZERİNE

ÖZET

Metrik tensörü

$$\tilde{g} = dx_1^2 + dx_2^2 + \cdots + dx_n^2 - dx_{n+1}^2$$

ile verilen $(n+1)$ boyutlu $\mathbb{E}_1^{n+1}$ Minkowski uzayının bir M hiperyüzeyi göz önüne alınsın ve N ile M hiperyüzeyinin birim normal vektörü gösterilsin. M alt manifoldu üzerinde tanımlı en önemli dışsal nesnelerden biri,

$$SX = -\tilde{\nabla}_X N$$

Wiengarten formülü ile tanımlanan S şekil operatörüdür. Burada, $\tilde{\nabla}$ ile Minkowski uzayının Levi-Civita konneksiyonu gösterilmiştir.

Şekil operatörü, teğet düzlemin ve normalinin her yöne nasıl hareket ettiğini belirlemek için kullanılır. S şekil operatörünün TM'de kendine eş bir endomorfizmdir. Bu nedenle, M hiperyüzeyi Riemannian olduğunda köşegenleştirilebilir. Ancak, M Lorentzian ise, şekil operatörü her zaman köşegenleştirilemez. Bu durumda, S şekil operatörünün dört kanonik formu vardır. Bu kanonik formlar, ortonormal veya sözde-ortonormal baza göre yazılabilir.

M hiperyüzeyinin teğet uzayının bir $\{E_1, \ldots, E_n\}$ baz alanına $2 \leq i,\ j \leq n$ olmak üzere

$$(E_1, E_1) = -1, \quad (E_1, E_i) = 0, \quad (E_i, E_j) = \delta_{ij}$$

şeklinde ise ortonormaldir denir. Diğer taraftan, M hiperyüzeyinin teğet uzayının bir $\{X, Y, E_1, \ldots, E_{n-2}\}$ çatı alanına $1 \leq i,\ j \leq n-2$ olmak üzere

$$(X, X) = (Y, Y) = 0, \quad (X, E_i) = (Y, E_i) = 0, \quad (X, Y) = -1$$

ve

$$(E_i, E_j) = \delta_{ij}$$

denklemlerinin sağlanması durumunda ise sözde ortonormaldir denir.

S şekil operatörünün özdeğerleri ve özvektörleri, sırasıyla asal eğrilikler ve asal doğrultular olarak adlandırılır. S şekil operatörü köşegenleştirilebilirse ve M hiperyüzeyi sabit asal eğriliklere sahipse, o zaman M hiperyüzeyinin izoparametrik olduğu söylenir. Ayrıca, S şekil operatörü köşegenleştirilemezse ve minimal polinomu sabitse, o zaman M hiperyüzeyine izoparametriktir denir.

Eğer S şekil operatörü köşegenleştirilemez ise üç kanonik durumdan birine indirgenebilir. Eğer S şekil operatörünün geometrik katlılığı 1, cebirsel katlılığı ise 2 veya 3 olan bir özdeğeri varsa, matris temsili, sırasıyla,

$$S = \begin{pmatrix} \lambda & 1 & \\ 0 & \lambda & \\ & & D_{n-2} \end{pmatrix}$$

veya

$$S = \begin{pmatrix} \lambda & 0 & 1 & \\ 0 & \lambda & 0 & \\ 0 & -1 & \lambda & \\ & & & D_{n-3} \end{pmatrix}.$$

olacak şekilde sözde ortonormal bir baz alanı mevcuttur ki burada D_k ile k-boyutlu diagonal bir matris gösterilmiştir. Diğer taraftan, eğer S şekil operatörünün kompleks eşlenik bir özdeğer çifti varsa, matris temsili

$$S = \begin{pmatrix} \lambda & -\nu & \\ \nu & \lambda & \\ & & D_{n-2} \end{pmatrix}.$$

olacak şekilde ortonormal bir baz alanı mevcuttur.

Minkowski uzaylarındaki izoparametrik hiperyüzeyler, şekil operatörü köşegenleştirilebilir olduğu durumda 1981 yılında Nomizu; köşegenleştirilemez olduğu durumda ise 1985 yılında Magid tarafından incelenmiştir. Magid'in elde ettiği sonuçlar özel olarak $n = 3$ durumu göz önüne alındığında $\mathbb{E}_1^4$ Minkowski uzayındaki bir M izoparametrik hiperyüzeyinin dört hiperyüzey ailesinden birinden olduğu görülür. Eğer M hiperyüzeyinin şekil operatörünün minimal polinomu $P(\lambda) = \lambda^2$ veya $P(\lambda) = (\lambda - a)^2\lambda$ ise yerel parametrizasyonu uygun seçilmiş bir $x(s)$ ışıksal eğrisi için

$$f(s,u,w) = x(s) + uY(s) + wW(s)$$

şeklindedir. Diğer taraftan, $P(\lambda) = \lambda^2(\lambda - a)$ veya $P(\lambda) = (\lambda - a)^2$ olması durumunda ise M hiperyüzeyinin yerel parametrizasyonu

$$f(s,u,z) = x(s) + uY(s) + zZ(s) + \left(\frac{1}{a} - \sqrt{\frac{1}{a^2} - z^2}\right) C(s)$$

olur.

Yarı-Riemann uzay formlarındaki başka bir önemli hiperyüzey ailesi de rotasyonel hiperyüzeylerdir. Her rotasyonel hiperyüzeyin λ ve ν şeklinde sadece iki tane ayrık asal eğriliği vardır ve bu asal eğrilikler arasında

$$\lambda = f(\nu)$$

şeklinde fonksiyonel bir ilişki vardır.

Bu tez çalışmasında, $\mathbb{E}_1^4$ Minkowski uzayında köşegenleştirilemeyen şekil operatörüne sahip izoparametrik hiperyüzeyleri incelenmiştir. Bu tez beş bölümden oluşmaktadır. Tezin ilk bölümünde, konunun tarihsel gelişimi anlatılmıştır. Tezin ikinci bölümünde, diğer bölümlerde kullanılacak olan notasyon ve Minkowski uzaylarının hiperyüzeyleri hakkında temel bilgiler verilmiştir.

Tezin üçüncü bölümünde, 1985 yılında Magid tarafından elde edilmiş izoparametrik hiperyüzeylere ait bazı teoremler farklı bir yöntem kullanılarak yeniden ispat edilmiştir. Bu teoremlerden ilkinde minimal polinomu

$$P(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)$$

olan bir izoparametrik hiperyüzey için

$$\lambda_1\lambda_2 = 0$$

olduğu, Codazzi ve Gauss denklemleri kullanılarak gösterilmiştir. Diğer teoremde ise $\mathbb{E}_1^4$ Minkowski uzayında

$$\lambda \pm ir$$

kompleks özdeğerlere sahip izoparametrik bir hiperyüzeyin olmadığı gösterilmiştir.

Tezin dördüncü bölümünde, $\mathbb{E}_1^5$ Minkowski uzayında köşegenleştirilemeyen şekil operatörüne sahip bir hiperyüzey ailesi oluşturulmuştur. Bu hiperyüzeyin asal eğrilikleri arasında aynı rotasyonel hiperyüzeylerde olduğu gibi $\lambda = f(v)$ şeklinde fonksiyonel bir ilişki bulunmaktadır. Bu hiperyüzeyin şekil operatörü, ortalama eğriliği, Gauss-Kronecker eğriliği ve Levi-Civita konneksiyonu elde edilmiştir. Daha sonra bu hiperyüzeyin minimal olması için gerek ve yeter koşul bir teorem ile verilmiştir.

1. INTRODUCTION

One of the most important intrinsic objects of a hypersurface of a pseudo-Riemannian space form is its shape operator. The eigenvalues of the shape operator are called principal curvatures of the hypersurface. A lot of notions have been presented by considering the form of principal curvatures of hypersurfaces. Perhaps one of the simplest classes of hypersurfaces are isoparametric.

By the definition of Cartan, a hypersurface of a Riemannian space form is said to be isoparametric if all of its principal curvatures are constant. In [1], Cartan obtained that the number r of distinct principal curvatures of a hypersurface of Euclidean space $\mathbb{E}^{n+1}$ and hyperbolic space are at most two. Moreover, if $r = 2$, one of the principal curvatures must be 0. He demonstrated that this is not true on the sphere $\mathbb{S}^{n+1}$.

Note that the shape operator of a Riemannian space form is always diagonalizable. On the other hand, when hypersurfaces of Lorentzian space forms were considered, it was proved that the shape operator is not always diagonalizable. Therefore, geometers presented the definition of isoparametric hypersurfaces in a different way. In [2], Nomizu described isoparametric hypersurfaces in Lorentzian space forms. He extended the fundamental identity in Cartan's theory regarding the constant principal curvatures of the isoparametric hypersurface to a space-like hypersurface case in any of the standard Lorentzian spaces. Nomizu proved a theorem about an isoparametric hypersurface with more than two distinct constant principal curvatures and considered hypersurfaces of Minkowski spaces with diagonalizable shape operator. In this case, isoparametric hypersurface can be defined by exactly the same way did for hypersurfaces of Riemannian space forms. However, in [3], Magid considered the other classes of hypersurfaces. He defined the hypersurface as isoparametric if the shape operator is non-diagonalizable and the minimal polynomial of the shape operator is constant. Magid showed that if a hypersurface is isoparametric in the Minkowski space $\mathbb{E}^{n+1}_1$ and there are two distinct principal curvatures, one of them must be zero. Hereby, it was shown that in this case the hypersurface can have at most one non-zero true principal curvature. In the same work, by classifying the four canonical forms of the shape operator, Lorentzian isoparametric hypersurfaces were locally classified.

In [4], Liang Xiao use a result of Magid to conclude that there are four types of local isoparametric hypersurfaces in the anti-de sitter space $\mathbb{H}_1^{n+1}$. He showed that a Lorentzian isoparametric hypersurface has at most a pair of conjugate complex and two real principal curvatures. He obtained the local classification of Lorentzian isoparametric hypersurfaces in $\mathbb{H}_1^{n+1}$ and their properties.

On the other hand, hypersurfaces with non-constant principal curvatures also have caught interest of many geometers so far. Note that one of the important classes of submanifolds is rotational hypersurfaces. It is well known that a rotational hypersurface has only two principal curvatures λ and ν. They are related by a functional equation, that is $\lambda = f(\nu)$. (See for example [5–7]).

This thesis consists of five section. In the second section of the thesis, concepts and notations that will be used in the rest of the thesis will be shown. Also, some basic calculations will be given in detail.

In Section 3, the isoparamteric hypersurfaces of $\mathbb{E}_1^4$ were considered. Some theorems about isoparametric hypersurfaces with non-diagonalizable shape operator will be proved. We show that there is only four classes of the hypersurface in $\mathbb{E}_1^4$. We give another theorem which proves that the hypersurface, given the shape operator, can not be isoparametric in $\mathbb{E}_1^4$ with complex principal curvatures.

In Section 4, a new family of hypersurfaces will be constructed. Some calculations for this hypersurface and the curvatures of this hypersurface will be obtained. In addition, a theorem will be given about the necessary condition for this hypersurface to be minimal.

In the last section of the thesis, a brief summary of what has been done in the thesis will be given and some problems that can be studied in the future will be mentioned.

2. PRELIMINARIES

In this section, some important notions and informations that we will use in the rest of the thesis will be given. The definitions below are taken from [8].

Definition 2.1. *The Lorentz-Minkowski space is the metric space* $\mathbb{E}_1^{n+1} = \left(\mathbb{R}^{n+1}, \langle , \rangle\right)$, *where the metric* $\langle , \rangle$ *is the canonical Lorentzian metric defined by*

$$\langle u, v\rangle = u_1v_1 + u_2v_2 + \cdots + u_nv_n - u_{n+1}v_{n+1}, \tag{2.1}$$

and we put $u = (u_1, u_2, \ldots, u_{n+1})$, $v = (v_1, v_2, \ldots, v_{n+1})$.

The Lorentzian metric is a non-degenerate metric of index 1. We also use Minkowski space and Minkowski metric terminology to denote space and metric, respectively.

Definition 2.2. *A vector* $v \in \mathbb{E}_1^n$ *is said*

(1) spacelike if $\langle v, v\rangle > 0$ *or* $v = 0$,

(2) timelike if $\langle v, v\rangle < 0$,

(3) lightlike if $\langle v, v\rangle = 0$ *and* $v \neq 0$.

We are going to use the following propositions later:

Proposition 2.3. *Two lightlike vectors* $u, v \in \mathbb{E}_1^n$ *are linearly dependent if and only* $\langle u, v\rangle = 0$.

Proposition 2.4. *Let* U *be a subspace of* $\mathbb{E}_1^n$, *then* U *is Lorentzian if and only if* $U^{\perp}$ *is Riemannian.*

2.1 Lorentzian Inner Product Spaces

In this subsection a brief summary of basic facts on inner product spaces are given. Note that a non-degenerated inner product space is called Lorentzian if its index is 1.

Definition 2.5. *Let* $(U, \langle , \rangle)$ *be a non-degenerate inner product space an endomorphism* $L : U \to U$ *is said to be self-adjoint if* $\langle Lu, v\rangle = \langle u, Lv\rangle$.

If $(U, \langle , \rangle)$ is a Lorentzian inner product space and $\{\tilde{e}_1, \tilde{e}_2\}$ is an orthonormal base of U, then the matrix representation of L has the form

$$L = \begin{pmatrix} L_{11} & L_{12} \\ -L_{12} & L_{22} \end{pmatrix} \qquad \text{w.r.t } \{\tilde{e}_1, \tilde{e}_2\} \tag{2.2}$$

for some $L_{11}, L_{12}, L_{22} \in \mathbb{R}$.

Because of (2.2), $\lambda \in \mathbb{R}$ is an eigenvalue of L if and only if

$$\lambda^2 - (L_{11} + L_{22})\lambda + L_{11}L_{22} + {L_{12}}^2 = 0. \tag{2.3}$$

First, we are going to prove the following proposition. Note that a base $\{X, Y\}$ is said to be pseudo-orthonormal if

$$\langle X, X\rangle = \langle Y, Y\rangle = 0, \quad \langle X, Y\rangle = -1. \tag{2.4}$$

Proposition 2.6. *Let U be Lorentzian,* $\dim U = 2$ *and $L = U \to U$ be self-adjoint. Assume that L is not proportional to identity operator. If L has only one real eigenvalue, then there exists a pseudo-orthonormal base $\{X, Y\}$ such that*

$$L = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \textit{w.r.t } \{X, Y\}. \tag{2.5}$$

Proof. Because of (2.3), if L has only one real eigenvalue, then we have

$$\frac{L_{11} - L_{22}}{2L_{12}} = \pm 1.$$

Therefore, by replacing e_2 with $-e_2$, one may assume that

$$L = \begin{pmatrix} 2a & a-b \\ b-a & 2b \end{pmatrix} \qquad \text{w.r.t } \{\tilde{e}_1, \tilde{e}_2\} \tag{2.6}$$

for some a, b and we have $\lambda = a + b$. Note that we have $a \neq b$ because of hypothesis. In this case, because of (2.6) we have

$$Le_1 = 2a\tilde{e}_1 + (b-a)\tilde{e}_2, \tag{2.7a}$$

$$Le_2 = (a-b)\tilde{e}_1 + 2b\tilde{e}_2. \tag{2.7b}$$

If we define a pseudo-orthonormal base $\tilde{X}, \tilde{Y}$ by $\tilde{X} = \frac{1}{\sqrt{2}}(\tilde{e}_1 - \tilde{e}_2)$ and $\tilde{Y} = \frac{1}{\sqrt{2}}(\tilde{e}_1 + \tilde{e}_2)$, then (2.7) implies

$$L\tilde{X} = \frac{a+b}{\sqrt{2}}\tilde{e}_1 - \frac{a+b}{\sqrt{2}}\tilde{e}_2 = \lambda\tilde{X}, \tag{2.8a}$$

$$L\tilde{Y} = \frac{3a-b}{\sqrt{2}}\tilde{e}_1 + \frac{-a+3b}{\sqrt{2}}\tilde{e}_2 = c\tilde{X} + \lambda\tilde{Y}, \tag{2.8b}$$

where $c \neq 0$ is a constant.

Now, if we define X, Y by $X = \phi\tilde{X}$ and $Y = \frac{1}{\phi}\tilde{Y}$, then (2.8) implies

$$LX = \lambda\tilde{X},$$
$$LY = \frac{c}{\phi^2}X + \lambda Y.$$

Therefore, by choosing ϕ properly and replacing X, Y with $-X, -Y$ if necessary, one can obtain (2.5). □

Proposition 2.7. *Let U be Lorentzian,* $\dim U = 2$ *and $L = U \to U$ is self-adjoint. If L has a complex eigenvalue, then there exists an orthonormal base $\{e_1, e_2\}$ such that*

$$L = \begin{pmatrix} \lambda & -\nu \\ \nu & \lambda \end{pmatrix} \qquad w.r.t\ \{e_1, e_2\} \tag{2.9}$$

for some $\lambda, \nu \in \mathbb{R}$, $\nu \neq 0$.

Proof. Because of (2.3), if L has complex eigenvalue, then we have

$$-1 < \frac{L_{11} - L_{22}}{2L_{12}} < 1.$$

Now, for a $\theta \in \mathbb{R}$ one may define an orthonormal base $\{e_1, e_2\}$ by

$$e_1 = \cosh\theta e_1 + \sinh\theta e_2, \quad e_2 = -\sinh\theta e_1 + \cosh\theta e_2$$

to get

$$Le_1 = \left(\cosh^2\theta L_{11} - \sinh^2\theta L_{22} - \sinh 2\theta L_{12}\right) e_1 - \nu e_2, \tag{2.10a}$$

$$Le_2 = \nu e_1 + \left(-\sinh^2\theta L_{11} + \cosh^2\theta L_{22} + \sinh 2\theta L_{12}\right) e_2 \tag{2.10b}$$

from (2.2), where $\nu \in \mathbb{R}$. Note that if $\theta = \tanh^{-1}\left(\frac{L_{11}-L_{22}}{2L_{12}}\right)$, then we have

$$\lambda = \cosh^2\theta L_{11} - \sinh^2\theta L_{22} - \sinh 2\theta L_{12} = -\sinh^2\theta L_{11} + \cosh^2\theta L_{22} + \sinh 2\theta L_{12}.$$

In this case, (2.10) gives (2.9). □

Proposition 2.8. *Let U be Lorentzian,* $\dim U = 2$ *and $L = U \to U$ is self-adjoint. If L has two distinct real eigenvalues, then there exists an orthonormal base $\{e_1, e_2\}$ such that*

$$L = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad w.r.t\ \{e_1, e_2\} \tag{2.11}$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

Proof. Because of (2.3), if L has two distinct real eigenvalues, then we have

$$\frac{L_{11}-L_{22}}{2L_{12}} > 1$$

or

$$\frac{L_{11}-L_{22}}{2L_{12}} < -1.$$

Now, for a $\theta \in \mathbb{R}$ one may define an orthonormal base $\{e_1, e_2\}$ by

$$e_1 = \cosh\theta e_1 + \sinh\theta e_2, \quad e_2 = -\sinh\theta e_1 + \cosh\theta e_2$$

to get

$$Le_1 = \left(\cosh^2\theta L_{11} - \sinh^2\theta L_{22} - \sinh 2\theta L_{12}\right) e_1, \tag{2.12a}$$

$$Le_2 = \left(-\sinh^2\theta L_{11} + \cosh^2\theta L_{22} + \sinh 2\theta L_{12}\right) e_2 \tag{2.12b}$$

from (2.2), where $\lambda_1, \lambda_2 \in \mathbb{R}$. Note that if $\theta = \coth^{-1}\left(\frac{L_{11}-L_{22}}{2L_{12}}\right)$, then we have

$$\begin{aligned} \lambda_1 &= \cosh^2\theta L_{11} - \sinh^2\theta L_{22} - \sinh 2\theta L_{12}, \\ \lambda_2 &= -\sinh^2\theta L_{11} + \cosh^2\theta L_{22} + \sinh 2\theta L_{12}. \end{aligned}$$

In this case, (2.12) gives (2.11). □

Theorem 2.9. *Let U be Lorentzian,* $\dim U = n \geq 3$. *A self-adjoint operator $L = U \to U$ can be put into one of the following four forms:*

Case 1.

$$L = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \qquad w.r.t\ \{e_1, e_2, \ldots, e_n\}. \tag{2.13}$$

Case 2.

$$L = \begin{pmatrix} \lambda & 1 & & & & \\ 0 & \lambda & & & & \\ & & \lambda_1 & & & \\ & & & \lambda_2 & & \\ & & & & \ddots & \\ & & & & & \lambda_{n-2} \end{pmatrix} \qquad w.r.t\ \{X, Y, e_3, e_4, \ldots, e_n\}. \tag{2.14}$$

Case 3.

$$L = \begin{pmatrix} \lambda & -\nu & & & & \\ \nu & \lambda & & & & \\ & & \lambda_1 & & & \\ & & & \lambda_2 & & \\ & & & & \ddots & \\ & & & & & \lambda_{n-2} \end{pmatrix} \qquad w.r.t \ \{e_1, e_2, \dots, e_n\}. \tag{2.15}$$

Case 4.

$$L = \begin{pmatrix} \lambda & 0 & 1 & & & & \\ 0 & \lambda & 0 & & & & \\ 0 & -1 & \lambda & & & & \\ & & & \lambda_1 & & & \\ & & & & \lambda_2 & & \\ & & & & & \ddots & \\ & & & & & & \lambda_{n-3} \end{pmatrix} \qquad w.r.t \ \{X, Y, e_3, e_4, \dots, e_n\}. \tag{2.16}$$

Proof. The theorem is going to be proved by induction.

Case $n = 3$. First assume that L has a time-like or space-like eigenvector e_3 which implies $Le_3 = \lambda_3 e_3$ for some $\lambda_3 \in \mathbb{R}$. In this case by Proposition 2.4, $V = (\operatorname{span} e_3)^{\perp}$ is either time-like or space-like subject to being space-like or time-like of e_3, respectively. Consider $\tilde{L} = L|_V$.

If e_3 is time-like, then V is space-like. Therefore $\tilde{L}$ is diagonalizable, i.e., there exist $e_1, e_2 \in V \subset U$ such that

$$\tilde{L}e_i = Le_i = \lambda_i e_i, \qquad i = 1, 2$$

for some λ_1, λ_2. Therefore, we have the Case I of the theorem.

If e_3 is space-like, then V is Lorentzian. Therefore, $\tilde{L}$ has one of the matrix representations given in Proposition 2.6, Proposition 2.7 and Proposition 2.8. Consequently, we have Case I, II or III of the theorem.

Now, L is assumed to have only null eigenvectors. If L has two distinct eigenvalues λ_1, λ_2, then, the corresponding eigenvectors e_1, e_2 are perpendicular to each other, i.e., $\langle e_1, e_2 \rangle = 0$. In this case, e_1 and e_2 are linearly dependent. However, this is impossible.

On the other hand, if L has two null eigenvectors v_1, v_2 corresponding to an eigenvalue λ_1, then $v_1 + v_2$ is also an eigenvalue corresponding to λ_1. However, in this case $v_1 + v_2$ is not null and we have Case I, II or III of the theorem.

Finally, assume that L has only one distinct eigenvalue λ with only one corresponding linearly independent eigenvector u_1, u_2, u_3 such that

$$Lu_1 = \lambda u_1, \quad Lu_2 = u_1 + \lambda u_2, \quad Lu_3 = u_2 + \lambda u_3.$$

Since L is self adjoint, we have

$$\langle Lu_1, u_2 \rangle = \langle Lu_2, u_1 \rangle$$

$$\langle Lu_1, u_3 \rangle = \langle Lu_3, u_1 \rangle$$

$$\langle Lu_2, u_3 \rangle = \langle Lu_3, u_2 \rangle$$

which give

$$\langle u_1, u_1 \rangle = \langle u_1, u_2 \rangle = 0, \quad \langle u_1, u_3 \rangle = \langle u_2, u_2 \rangle.$$

Put

$$\langle u_1, u_3 \rangle = \langle u_2, u_2 \rangle = a, \quad \langle u_2, u_3 \rangle = b, \quad \langle u_3, u_3 \rangle = c.$$

We can choose A, B_1, B_2, B_3, C_1, C_2, C_3 so that $X = Au_1$, $Y = B_1u_1 + B_2u_2 + B_3u_3$, $e_3 = C_1u_1 + C_2u_2 + C_3u_3$ satisfy $LX = \lambda X$, $LY = -e_3 + \lambda Y$, $Le_3 = X + \lambda e_3$ which gives

Case 4. of the theorem for $n = 3$.

Case $n > 3$. Assume that the theorem is true for the case $n = k \geq 3$.

If L has only one distinct eigenvalue λ with only one corresponding linearly independent eigenvector, then there exists a linearly independent set $\{u_1, u_2, \ldots, u_{k+1}\}$ such that

$$Lu_1 = \lambda u_1, \quad Lu_2 = u_1 + \lambda u_2, \quad Lu_3 = u_2 + \lambda u_3, \ldots, \quad Lu_{k+1} = u_k + \lambda u_{k+1}.$$

In this case, since L is self-adjoint, the equations

$$\langle Lu_i, u_j \rangle = \langle Lu_j, u_i \rangle, \qquad i, j = 1, 2, \ldots, k+1 \tag{2.17}$$

are satisfied. The equation (2.17) for $i = 1$, $j = 2$; $i = 1$, $j = 3$; $i = 1$, $j = 4$ and $i = 2$, $j = 3$ imply

$$\langle u_1, u_1 \rangle = \langle u_1, u_2 \rangle = \langle u_2, u_2 \rangle = 0.$$

However, because of Proposition 2.3, u_1, u_2 are linearly dependent which yields a contradiction.

Now, if L has at least two distinct eigenvalues λ_1, λ_{k+1} or only one distinct eigenvalue with two linearly independent eigenvectors, then similar to the proof for case $n = 3$, L has a space-like or time-like eigenvector u_{k+1}. Put $V = (\operatorname{span} e_{k+1})^{\perp}$. Then, V has dimension k and it is either space-like or Lorentzian. By the assumption, $\tilde{L} = L|_V$ has one of the matrix representations given in the theorem. Consequently, proof is completed for the case $n = k+1$. □

2.2 Basic Facts on Hypersurfaces of $\mathbb{E}_1^{n+1}$

In this subsection, we will give some basic informations about hypersurfaces of Minkowski space. These informations will be written using [9], [10], and [11].

Definition 2.10. *An n-dimensional manifold is a topological space with a neighborhood of each point homeomorphic to an open subset of n-dimensional Euclidean space* $\mathbb{R}^n$.

Definition 2.11. *A manifold M is a submanifold of a manifold N provided:*

(1) M is a topological subspace of N.

(2) The inclusion map $j : M \subset N$ *is smooth and at each point* $p \in M$ *its differential map* dj *is one-to-one.*

Definition 2.12. *An n-dimensional submanifold M of an* $(n+1)$*-dimensional manifold N is called as a hypersurface of N.*

Let M be an oriented hypersurface in $(n+1)$-dimensional Minkowski Space $\mathbb{E}_1^{n+1}$. We denote the Levi-Civita connections of $\tilde{M}$ and M by $\tilde{\nabla}$ and ∇ respectively. Then, the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.18}$$

$$\tilde{\nabla}_X N = -S(X) \tag{2.19}$$

for any vector fields X and Y tangent to M, where h and S are the second fundamental form and the shape operator of M, respectively, and N is the unit normal vector field associated with the orientation of M.

The Gauss and Codazzi equations are given, respectively, by

$$R(X,Y,Z,W) = \langle h(Y,Z), h(X,W)\rangle - \langle h(X,Z), h(Y,W)\rangle, \tag{2.20}$$

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z) \tag{2.21}$$

for any X, Y, Z and W tangent to M where the curvature tensor R and $\overline{\nabla}h$ are defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.22}$$

$$(\overline{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{2.23}$$

for any X, Y and Z tangent to M, where $\overline{\nabla}$ is the Van der Waerden-Bortolotti connection and $\nabla^{\perp}$ is the normal connection of M.

Definition 2.13. *Let M be a hypersurface of $\mathbb{E}^{n+1}_1$ with the shape operator S for $n \geq 3$.*

(a) S is diagonalizable. If $Se_i = \lambda_i e_i$, then λ_i and e_i are called principal curvature and principal direction of M.

(b) S is non-diagonalizable. The functions $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n$ appearing on Theorem 2.9 are called principal curvatures of M.

Definition 2.14. *The mean curvature of M is defined by*

$$H = \frac{1}{n} \operatorname{tr}(S). \tag{2.24}$$

Definition 2.15. *A hypersurface M of $\mathbb{E}^{n+1}_1$ is said to be minimal if its mean curvature vanishes identically i.e.,*

$$H = 0.$$

Definition 2.16. *The Gauss-Kronecker curvature of M is defined by*

$$K = \det(S). \tag{2.25}$$

Let $n = 3$, i.e., M^3 be a hypersurface of $\mathbb{E}^4_1$. A base field $\{e_1, e_2, e_3\}$ is said to be orthonormal if

$$\langle e_1, e_1 \rangle = -1, \quad \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = 1, \quad \langle e_i, e_j \rangle = 0$$

when $i \neq j$.

If we put $w_{ij}(e_k) = \langle \nabla_{e_k} e_i, e_j \rangle$, then we get

$$\nabla_{e_i} e_1 = w_{12}(e_i) e_2 + w_{13}(e_i) e_3, \tag{2.26a}$$

$$\nabla_{e_i} e_2 = w_{12}(e_i) e_1 + w_{23}(e_i) e_3, \tag{2.26b}$$

$$\nabla_{e_i} e_3 = w_{13}(e_i) e_1 - w_{23}(e_i) e_2. \tag{2.26c}$$

On the other hand $\{e_1, e_2, e_3\}$ is said to be a pseudo-orthonormal base field if

$$\langle e_1, e_2 \rangle = -1, \quad \langle e_3, e_3 \rangle = 1, \quad \langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_1, e_3 \rangle = \langle e_2, e_3 \rangle = 0.$$

In this case if we put $w_{ij}(e_k) = \langle \nabla_{e_k} e_i, e_j \rangle$, then we get

$$\nabla_{e_i} e_1 = -w_{12}(e_i)e_1 + w_{13}(e_i)e_3, \tag{2.27a}$$

$$\nabla_{e_i} e_2 = w_{12}(e_i)e_2 + w_{23}(e_i)e_3, \tag{2.27b}$$

$$\nabla_{e_i} e_3 = w_{23}(e_i)e_1 + w_{13}(e_i)e_2. \tag{2.27c}$$

3. ISOPARAMETRIC HYPERSURFACES

3.1 Shape Operators of Isoparametric Hypersurfaces

In this section we are going to consider isoparametric hypersurfaces in the Minkowski space $\mathbb{E}^4_1$.

Before we start to our summary, we would like to recall the definiton of isoparametric hypersurfaces.

Definition 3.1. [3] If the shape operator S is diagonalizable and S has constant principal curvatures, a hypersurface is said to be isoparametric. If S is not diagonalizable and the minimal polynomial of the shape operator is constant, define a hypersurface to be isoparametric.

In the following theorem which proves that there is only four classes of hypersurface in $\mathbb{E}^4_1$.

Theorem 3.2. *Let M be a hypersurface in $\mathbb{E}^4_1$ with the shape operator S given by*

$$S = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{3.1}$$

with respect to a pseudo-orthonormal base. If λ_1 and λ_3 are constants and $(\lambda_1 - \lambda_3) \neq 0$, then $\lambda_1\lambda_3 = 0$.

Proof. Let M be a hypersurface in $\mathbb{E}^4_1$ with the shape operator S given by (3.1) for some distinct constants λ_1, λ_3. Note that because of (3.1), we have

$$h(e_1, e_1) = 0, \tag{3.2}$$

$$h(e_1, e_2) = -\lambda_1, \tag{3.3}$$

$$h(e_2, e_2) = -1, \tag{3.4}$$

$$h(e_3, e_3) = \lambda_3. \tag{3.5}$$

First, we compute both sides of the Codazzi Equation (2.21) for $X = Y = e_1$ and $Z = e_2$.

$$\begin{aligned} e_1(h(e_1,e_1)) - h(\nabla_{e_1}e_1,e_2) - h(e_1,\nabla_{e_1}e_2) &= e_1(-\lambda_1) \\ &\quad +h(w_{12}(e_1)e_1 - w_{13}(e_1)e_3,e_2) \\ &\quad -h(e_1,w_{12}(e_1)e_2 + w_{23}(e_1)e_3) \\ &= e_1(-\lambda_1) - w_{12}(e_1)(\lambda_1) + w_{12}(e_1)(\lambda_1) \\ &= e_1(-\lambda_1) \end{aligned}$$

Therefore, we have

$$e_1(h(e_1,e_1)) - h(\nabla_{e_1}e_1,e_2) - h(e_1,\nabla_{e_1}e_2) = e_1(-\lambda_1). \tag{3.6}$$

On the other hand, we also have

$$\begin{aligned} e_2(h(e_1,e_1)) - 2h(\nabla_{e_2}e_1,e_1) &= e_2(0) - 2h(-w_{12}(e_2)e_1 + w_{13}(e_2)e_3,e_1) \\ &= e_2(0) \\ &= 0. \end{aligned} \tag{3.7}$$

By combining (3.6) and (3.7), we obtain

$$e_1(\lambda_1) = 0. \tag{3.8}$$

Next, we deal with the Codazzi Equation (2.21) for $Y = e_1$ and $X = Z = e_2$.

$$\begin{aligned} e_2(h(e_1,e_2)) - h(\nabla_{e_2}e_1,e_2) - h(e_1,\nabla_{e_2}e_2) &= e_2(-\lambda_1) \\ &\quad +h(w_{12}(e_2)e_1 - w_{13}(e_2)e_3,e_2) \\ &\quad -h(e_1,w_{12}(e_2)e_2 + w_{23}(e_2)e_3) \\ &= e_2(-\lambda_1). \end{aligned}$$

Thus, we have

$$e_2(h(e_1,e_2)) - h(\nabla_{e_2}e_1,e_2) - h(e_1,\nabla_{e_2}e_2) = e_2(-\lambda_1). \tag{3.9}$$

On the other hand, we also have

$$\begin{aligned} e_1(h(e_2,e_2)) - 2h(\nabla_{e_1}e_2,e_2) &= e_1(-1) - 2h(w_{12}(e_2)e_2 + w_{23}(e_2)e_3,e_2) \\ &= 2w_{12}(e_2). \end{aligned} \tag{3.10}$$

By combining (3.9) and (3.10), we obtain

$$e_2(-\lambda_1) = 2w_{12}(e_2). \tag{3.11}$$

Since λ_1, λ_3 are constants, we have

$$w_{12}(e_2) = 0. \tag{3.12}$$

We compute both sides of the Codazzi Equation (2.21) for $X = Y = e_1$ and $Z = e_3$.

$$\begin{aligned} e_1(h(e_1,e_3)) - h(\nabla_{e_1}e_1, e_3) - h(e_1, \nabla_{e_1}e_3) &= e_1(0) + h(w_{12}(e_1)e_1 - w_{13}(e_1)e_3, e_3) \\ &\quad -h[e_1, (w_{23}(e_1)e_1 + w_{13}(e_1)e_2)] \\ &= -w_{13}(e_1)(\lambda_3) - w_{13}(e_1)(-\lambda_1) \\ &= w_{13}(e_1)[\lambda_1 - \lambda_3] \end{aligned}$$

So, we have

$$e_1(h(e_1,e_3)) - h(\nabla_{e_1}e_1, e_3) - h(e_1, \nabla_{e_1}e_3) = w_{13}(e_1)(\lambda_1 - \lambda_3). \tag{3.13}$$

On the other hand, we also have

$$e_3(h(e_1,e_1)) - 2h(\nabla_{e_3}e_1, e_1) = e_3(0) - 2h(-w_{12}(e_3)e_1 + w_{13}(e_3)e_3, e_1) = 0. \tag{3.14}$$

By combining (3.13) and (3.14), we obtain

$$w_{13}(e_1)(\lambda_1 - \lambda_3) = 0. \tag{3.15}$$

Since λ_1, λ_3 are constants and $\lambda_1 \neq \lambda_3$ than $(\lambda_1 - \lambda_3) \neq 0$, it follows

$$w_{13}(e_1) = 0. \tag{3.16}$$

Now, consider the equation (2.21) for $Y = e_1$ and $X = Z = e_3$.

Similar to (3.11) we obtain

$$w_{13}(e_1)(\lambda_1 - \lambda_3) = e_1(\lambda_3). \tag{3.17}$$

Because λ_1, λ_3 are constants and $\lambda_1 \neq \lambda_3$, (3.17) implies

$$w_{13}(e_1) = 0. \tag{3.18}$$

We also deal with the Codazzi Equation (2.21) for $X = Y = e_2$ and $Z = e_3$.

$$\begin{aligned} e_2(h(e_2,e_3)) - h(\nabla_{e_2}e_2,e_3) - h(e_2,\nabla_{e_2}e_3) &= e_2(0) - h(w_{12}(e_2)e_2 + w_{23}(e_2)e_3, e_3) \\ &\quad -h(e_2, w_{23}(e_2)e_1 + w_{13}(e_2)e_2) \\ &= -w_{23}(e_2)(\lambda_3) - w_{23}(e_2)(-\lambda_1) \\ &\quad -w_{13}(e_2)(-1) \\ &= w_{23}(e_2)(\lambda_1 - \lambda_3) + w_{13}(e_2) \end{aligned}$$

Therefore, we get

$$e_2(h(e_2,e_3)) - h(\nabla_{e_2}e_2,e_3) - h(e_2,\nabla_{e_2}e_3) = w_{23}(e_2)(\lambda_1 - \lambda_3) + w_{13}(e_2). \quad (3.19)$$

On the other hand, we also have

$$\begin{aligned} e_3(h(e_2,e_2)) - 2h(\nabla_{e_3}e_2,e_2) &= e_3(-1) - 2h(w_{12}(e_3)e_2 + w_{23}(e_3)e_3, e_2) \\ &= e_3(-1) - 2(w_{12}(e_3)(-1)) \\ &= 2(w_{12}(e_3)). \end{aligned}$$

Thus, we have

$$e_3(h(e_2,e_2)) - 2h(\nabla_{e_3}e_2,e_2) = 2(w_{12}(e_3)). \quad (3.20)$$

By combining (3.19) and (3.20), we obtain

$$w_{23}(e_2)(\lambda_1 - \lambda_3) + w_{13}(e_2) = 2(w_{12}(e_3)). \quad (3.21)$$

Next, we consider the Codazzi Equation (2.21) for $Y = e_2$ and $X = Z = e_3$.

$$\begin{aligned} e_3(h(e_2,e_3)) - h(\nabla_{e_3}e_2,e_3) - h(e_2,\nabla_{e_3}e_3) &= e_3(0) - h(w_{12}(e_3)e_2 + w_{23}(e_3)e_3, e_3) \\ &\quad -h(e_2, w_{23}(e_3)e_1 + w_{13}(e_3)e_2) \\ &= -w_{23}(e_3)(\lambda_3) - w_{23}(e_3)(-\lambda_1) \\ &\quad -w_{13}(e_3)(-1) \\ &= w_{23}(e_3)(\lambda_1 - \lambda_3) \end{aligned}$$

So, we have

$$e_3(h(e_2,e_3)) - h(\nabla_{e_3}e_2,e_3) - h(e_2,\nabla_{e_3}e_3) = w_{23}(e_3)(\lambda_1 - \lambda_3). \quad (3.22)$$

On the other hand, we also have

$$\begin{aligned} e_2(h(e_3,e_3)) - 2h(\nabla_{e_2}e_3,e_3) &= e_2(\lambda_3) - 2h(w_{23}(e_2)e_1 + w_{13}(e_2)e_2, e_3) \\ &= e_2(\lambda_3). \end{aligned} \tag{3.23}$$

By combining (3.22) and (3.23), we obtain

$$w_{23}(e_3)(\lambda_1 - \lambda_3) = e_2(\lambda_3). \tag{3.24}$$

Because λ_1, λ_3 are constants and $\lambda_1 \neq \lambda_3$ than $(\lambda_1 - \lambda_3) \neq 0$, we get

$$w_{23}(e_3) = 0. \tag{3.25}$$

Next, we evaluate each sides of the Codazzi Equation (2.21) for $X = e_1$, $Y = e_2$ and $Z = e_3$.

$$\begin{aligned} e_1(h(e_2,e_3)) - h(\nabla_{e_1}e_2,e_3) - h(e_2,\nabla_{e_1}e_3) &= e_1(0) - h(w_{12}(e_1)e_2 + w_{23}(e_1)e_3, e_3) \\ &\quad - h(e_2, w_{23}(e_1)e_1 + w_{13}(e_1)e_2) \\ &= w_{23}(e_1)(\lambda_1 - \lambda_3) + w_{13}(e_1) \\ &= w_{23}(e_1)(\lambda_1 - \lambda_3) \end{aligned}$$

Therefore, we have

$$e_1(h(e_2,e_3)) - h(\nabla_{e_1}e_2,e_3) - h(e_2,\nabla_{e_1}e_3) = w_{23}(e_1)(\lambda_1 - \lambda_3). \tag{3.26}$$

On the other hand, we also have

$$\begin{aligned} e_2(h(e_1,e_3)) - h(\nabla_{e_2}e_1,e_3) - h(e_1,\nabla_{e_2}e_3) &= e_2(0) + h(w_{12}(e_2)e_1 - w_{13}(e_2)e_3, e_3) \\ &\quad - h(e_1, w_{23}(e_2)e_1 + w_{13}(e_2)e_2) \\ &= w_{13}(e_2)(\lambda_1 - \lambda_3). \end{aligned}$$

Thus, we have

$$e_2(h(e_1,e_3)) - h(\nabla_{e_2}e_1,e_3) - h(e_1,\nabla_{e_2}e_3) = w_{13}(e_2)(\lambda_1 - \lambda_3). \tag{3.27}$$

On the other hand, we also have

$$\begin{aligned} e_3(h(e_1,e_2)) - h(\nabla_{e_3}e_1,e_2) - h(e_1,\nabla_{e_3}e_2) &= e_3(-\lambda_1) \\ &\quad + h(w_{12}(e_3)e_1 - w_{13}(e_3)e_3, e_2) \\ &\quad - h(e_1, w_{12}(e_3)e_2 + w_{23}(e_3)e_3) \\ &= e_3(-\lambda_1). \end{aligned}$$

So, we have

$$e_3(h(e_1,e_2)) - h(\nabla_{e_3}e_1,e_2) - h(e_1,\nabla_{e_3}e_2) \quad = e_3(-\lambda_1). \tag{3.28}$$

By combining (3.26) and (3.27) and (3.28), we obtain

$$w_{23}(e_1)(\lambda_1-\lambda_3) = w_{13}(e_2)(\lambda_1-\lambda_3) = e_3(-\lambda_1). \tag{3.29}$$

Because λ_1, λ_3 are constants and $\lambda_1 \neq \lambda_3$ than $(\lambda_1-\lambda_3)\neq 0$, we get

$$w_{23}(e_1) = w_{13}(e_2) = 0. \tag{3.30}$$

Because of $w_{12}(e_2) = w_{13}(e_1) = w_{13}(e_2) = w_{13}(e_3) = w_{23}(e_1) = w_{23}(e_3) = 0$, the equation (2.27) turns into

$$\nabla_{e_1}e_1 = -w_{12}(e_1)e_1, \quad \nabla_{e_2}e_1 = -w_{12}(e_2)e_1 = 0, \quad \nabla_{e_3}e_1 = -w_{12}(e_3)e_1, \tag{3.31a}$$

$$\nabla_{e_1}e_2 = w_{12}(e_1)e_2, \quad \nabla_{e_2}e_2 = w_{23}(e_2)e_3, \quad \nabla_{e_3}e_2 = w_{12}(e_3)e_2, \tag{3.31b}$$

$$\nabla_{e_1}e_3 = 0, \quad \nabla_{e_2}e_3 = w_{23}(e_2)e_1, \quad \nabla_{e_3}e_3 = 0. \tag{3.31c}$$

Next, we use the Gauss Equation (2.20) for $X=e_1$, $Y=W=e_3$, $Z=e_2$ and substitute (3.2)-(3.5) to get

$$R(e_1,e_3,e_2,e_3) = \lambda_1\lambda_3. \tag{3.32}$$

First we compute the left hand side of (3.32)

$$\begin{aligned} R(e_1,e_3,e_2,e_3) &= \langle \nabla_{e_1}\nabla_{e_3}e_2 - \nabla_{e_3}\nabla_{e_1}e_2 - \nabla_{[e_1,e_3]}e_2, e_3\rangle \\ &= \langle \nabla_{e_1}[w_{12}(e_3)e_2 + w_{23}(e_3)e_3] - \nabla_{e_3}[w_{12}(e_1)e_2 + w_{23}(e_1)e_3] \\ &\quad -\nabla_{\nabla_{e_1}e_3 - \nabla_{e_3}e_1}e_2, e_3\rangle \\ &= \langle -w_{12}(e_1)\nabla_{e_3}e_2, e_3\rangle \\ &= \langle -w_{12}(e_1)[w_{12}(e_3)e_2], e_3\rangle \\ &= 0 \end{aligned}$$

which gives

$$R(e_1,e_3,e_2,e_3) \quad = \quad 0. \tag{3.33}$$

By combining (3.32) with (3.33), we get

$$0 = \lambda_1\lambda_3. \tag{3.34}$$

Hence the proof is completed.

□

In the following theorem which proves that there is no isoparametric hypersurface in $\mathbb{E}^4_1$ with complex principal curvatures.

Theorem 3.3. *Let M be a hypersurface in $\mathbb{E}^4_1$ with the shape operator S given by*

$$S = \begin{pmatrix} \lambda_1 & \nu & 0 \\ -\nu & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{3.35}$$

with respect to an orthonormal base. Then, M can not be isoparametric.

Proof. Let M be a hypersurface in $\mathbb{E}^4_1$ with the shape operator S given by (3.35) for some constants $\lambda_1, \lambda_3, \nu$. Note that because of (3.35), we have

$$\nu \neq 0, \tag{3.36}$$

$$h(e_1, e_1) = -\lambda_1, \tag{3.37}$$

$$h(e_1, e_2) = -\nu, \tag{3.38}$$

$$h(e_2, e_2) = \lambda_1, \tag{3.39}$$

$$h(e_3, e_3) = \lambda_3. \tag{3.40}$$

First, we compute both sides of the Codazzi Equation (2.21) for $X = Y = e_1$ and $Z = e_2$.

$$\begin{aligned} e_1(h(e_1,e_2)) - h(\nabla_{e_1}e_1, e_2) - h(e_1, \nabla_{e_1}e_2) &= e_1(-\nu) \\ &\quad -h(w_{12}(e_1)e_2 + w_{13}(e_1)e_3, e_2) \\ &\quad -h(e_1, w_{12}(e_1)e_1 + w_{23}(e_1)e_3) \\ &= e_1(-\nu) - w_{12}(e_1)(\lambda_1) + w_{12}(e_1)(\lambda_1) \\ &= e_1(-\nu) \end{aligned}$$

Therefore, we have

$$e_1(h(e_1,e_2)) - h(\nabla_{e_1}e_1,e_2) - h(e_1,\nabla_{e_1}e_2) = e_1(-\nu). \tag{3.41}$$

On the other hand, we also have

$$\begin{aligned} e_2(h(e_1,e_1)) - 2h(\nabla_{e_2}e_1,e_1) &= e_2(-\lambda_1) - 2h(w_{12}(e_2)e_2 + w_{13}(e_2)e_3,e_1) \\ &= e_2(-\lambda_1) - 2w_{12}(e_2)(-\nu) \\ &= e_2(-\lambda_1) + 2w_{12}(e_2)\nu. \end{aligned}$$

So, we have

$$e_2(h(e_1,e_1)) - 2h(\nabla_{e_2}e_1,e_1) = e_2(-\lambda_1) + 2w_{12}(e_2)\nu. \tag{3.42}$$

By combining (3.41) and (3.42), we obtain

$$e_1(-\nu) = e_2(-\lambda_1) + 2w_{12}(e_2)\nu. \tag{3.43}$$

Since λ_1 and ν are constants, we have

$$w_{12}(e_2) = 0. \tag{3.44}$$

Next, we deal with the Codazzi Equation (2.21) for $X = Z = e_2$ and $Y = e_1$.

$$\begin{aligned} e_2(h(e_1,e_2)) - h(\nabla_{e_2}e_1,e_2) - h(e_1,\nabla_{e_2}e_2) &= e_2(-\nu) \\ &\quad -h(w_{12}(e_2)e_2 + w_{13}(e_2)e_3,e_2) \\ &\quad -h(e_1,w_{12}(e_2)e_1 + w_{23}(e_2)e_3) \\ &= e_2(-\nu) - w_{12}(e_2)(\lambda_1) + w_{12}(e_2)(\lambda_1) \\ &= e_2(-\nu). \end{aligned}$$

Thus, we have

$$e_2(h(e_1,e_2)) - h(\nabla_{e_2}e_1,e_2) - h(e_1,\nabla_{e_2}e_2) = e_2(-\nu). \tag{3.45}$$

On the other hand, we also have

$$\begin{aligned} e_1(h(e_2,e_2)) - 2h(\nabla_{e_1}e_2,e_2) &= e_1(\lambda_1) - 2h(w_{12}(e_1)e_1 + w_{23}(e_1)e_3,e_2) \\ &= e_1(\lambda_1) - 2w_{12}(e_1)(-\nu). \end{aligned}$$

So, we get

$$e_1(h(e_2,e_2)) - 2h(\nabla_{e_1}e_2,e_2) = e_1(\lambda_1) - 2w_{12}(e_1)(-\nu). \tag{3.46}$$

By combining (3.45) and (3.46), we obtain

$$e_2(-\nu) = e_1(\lambda_1) + 2w_{12}(e_1)\nu. \tag{3.47}$$

Since λ_1 and ν are constants, we have

$$w_{12}(e_1) = 0. \tag{3.48}$$

We also compute both sides of the Codazzi Equation (2.21) for $X = Y = e_1$ and $Z = e_3$.

$$\begin{aligned} e_1(h(e_1,e_3)) - h(\nabla_{e_1}e_1,e_3) - h(e_1,\nabla_{e_1}e_3) &= e_1(0) - h(w_{12}(e_1)e_2 + w_{13}(e_1)e_3,e_3) \\ &\quad -h(e_1,w_{13}(e_1)e_1 - w_{23}(e_1)e_2) \\ &= -w_{13}(e_1)(\lambda_3) - w_{13}(e_1)(-\lambda_1) \\ &\quad +w_{23}(e_1)(-\nu) \\ &= w_{13}(e_1)(\lambda_1-\lambda_3) + w_{23}(e_1)(-\nu) \end{aligned}$$

So, we have

$$\begin{aligned} e_1(h(e_1,e_3)) - h(\nabla_{e_1}e_1,e_3) - h(e_1,\nabla_{e_1}e_3) &= w_{13}(e_1)(\lambda_1-\lambda_3) \\ &\quad +w_{23}(e_1)(-\nu). \end{aligned} \tag{3.49}$$

On the other hand, we also have

$$\begin{aligned} e_3(h(e_1,e_1)) - 2h(\nabla_{e_3}e_1,e_1) &= e_3(-\lambda_1) - 2h(-w_{12}(e_3)e_2 + w_{13}(e_3)e_3,e_1) \\ &= e_3(-\lambda_1) - 2w_{12}(e_3)(-\nu). \end{aligned}$$

So, we get

$$e_3(h(e_1,e_1)) - 2h(\nabla_{e_3}e_1,e_1) = e_3(-\lambda_1) - 2w_{12}(e_3)(-\nu). \tag{3.50}$$

By combining (3.49) and (3.50), we obtain

$$w_{13}(e_1)(\lambda_1-\lambda_3) + w_{23}(e_1)(-\nu) = e_3(-\lambda_1) - 2w_{12}(e_3)(-\nu). \tag{3.51}$$

Since λ_1, λ_3 and ν are constants, it follows

$$w_{13}(e_1)(\lambda_1 - \lambda_3) + w_{23}(e_1)(-\nu) = 2w_{12}(e_3)\nu. \tag{3.52}$$

Now, consider the equation (2.21) for $X = Z = e_3$ and $Y = e_1$.

Similar to (3.47) we obtain

$$w_{13}(e_3)(\lambda_1 - \lambda_3) + w_{23}(e_3)(-\nu) = e_1(\lambda_3). \tag{3.53}$$

Because λ_1, λ_3 and ν are constants, (3.53) implies

$$w_{13}(e_3)(\lambda_1 - \lambda_3) + w_{23}(e_3)(-\nu) = 0. \tag{3.54}$$

We also deal with the Codazzi Equation (2.21) for $X = Y = e_2$ and $Z = e_3$.

$$\begin{aligned}
e_2(h(e_2,e_3)) - h(\nabla_{e_2}e_2,e_3) - h(e_2,\nabla_{e_2}e_3) &= e_2(0) - h(w_{12}(e_2)e_1 + w_{23}(e_2)e_3, e_3) \\
&\quad -h(e_2, w_{13}(e_2)e_1 - w_{23}(e_2)e_2) \\
&= -w_{23}(e_2)(\lambda_3) - w_{13}(e_2)(-\nu) \\
&\quad +w_{23}(e_2)(\lambda_1) \\
&= w_{23}(e_2)(\lambda_1 - \lambda_3) + w_{13}(e_2)\nu
\end{aligned}$$

Therefore, we get

$$e_2(h(e_2,e_3)) - h(\nabla_{e_2}e_2,e_3) - h(e_2,\nabla_{e_2}e_3) = w_{23}(e_2)(\lambda_1 - \lambda_3) + w_{13}(e_2)\nu. \tag{3.55}$$

On the other hand, we also have

$$\begin{aligned}
e_3(h(e_2,e_2)) - 2h(\nabla_{e_3}e_2,e_2) &= e_3(\lambda_1) - 2h(w_{12}(e_3)e_1 + w_{23}(e_3)e_3, e_2) \qquad (3.56)\\
&= e_3(\lambda_1) - 2(w_{12}(e_3)(-\nu)).
\end{aligned}$$

Thus, we have

$$e_3(h(e_2,e_2)) - 2h(\nabla_{e_3}e_2,e_2) = e_3(\lambda_1) - 2(w_{12}(e_3)(-\nu)). \tag{3.57}$$

Since λ_1, λ_3 and ν are constants, we get

$$e_3(\lambda_1) - 2(w_{12}(e_3)(-\nu)) = 2(w_{12}(e_3)\nu). \tag{3.58}$$

By combining (3.55) and (3.58), we obtain

$$w_{23}(e_2)(\lambda_1-\lambda_3)+w_{13}(e_2)\nu = 2w_{12}(e_3)\nu \tag{3.59}$$

Next, we consider the Codazzi Equation (2.21) for $X=Z=e_3$ and $Y=e_2$.

$$\begin{aligned} e_3(h(e_2,e_3))-h(\nabla_{e_3}e_2,e_3)-h(e_2,\nabla_{e_3}e_3) &= e_3(0)-h(w_{12}(e_3)e_1+w_{23}(e_3)e_3,e_3) \\ &\quad -h(e_2,w_{13}(e_3)e_1-w_{23}(e_3)e_2) \\ &= -w_{23}(e_3)(\lambda_3)-w_{13}(e_3)(-\nu) \\ &\quad +w_{23}(e_3)(\lambda_1) \\ &= w_{23}(e_3)(\lambda_1-\lambda_3)+w_{13}(e_3)\nu \end{aligned}$$

So, we have

$$e_3(h(e_2,e_3))-h(\nabla_{e_3}e_2,e_3)-h(e_2,\nabla_{e_3}e_3) = w_{23}(e_3)(\lambda_1-\lambda_3)+w_{13}(e_3)\nu. \tag{3.60}$$

On the other hand, we also have

$$\begin{aligned} e_2(h(e_3,e_3))-2h(\nabla_{e_2}e_3,e_3) &= e_2(\lambda_3)-2h(w_{13}(e_2)e_1-w_{23}(e_2)e_2,e_3) \quad (3.61) \\ &= e_2(\lambda_3) \end{aligned}$$

By combining (3.60) and (3.61), we obtain

$$w_{23}(e_3)(\lambda_1-\lambda_3)+w_{13}(e_3)\nu = e_2(\lambda_3). \tag{3.62}$$

Because λ_1,λ_3 and ν are constants, then we get

$$w_{23}(e_3)(\lambda_1-\lambda_3)+w_{13}(e_3)\nu = 0. \tag{3.63}$$

Next, we evaluate each sides of the Codazzi Equation (2.21) for $X=e_1$, $Y=e_2$ and $Z=e_3$.

$$\begin{aligned} e_1(h(e_2,e_3))-h(\nabla_{e_1}e_2,e_3)-h(e_2,\nabla_{e_1}e_3) &= e_1(0)-h(w_{12}(e_1)e_1+w_{23}(e_1)e_3,e_3) \\ &\quad -h(e_2,w_{13}(e_1)e_1-w_{23}(e_1)e_2) \\ &= w_{23}(e_1)(\lambda_1-\lambda_3)-w_{13}(e_1)(-\nu) \\ &= w_{23}(e_1)(\lambda_1-\lambda_3)+w_{13}(e_1)\nu \end{aligned}$$

Therefore, we have

$$e_1(h(e_2,e_3)) - h(\nabla_{e_1}e_2,e_3) - h(e_2,\nabla_{e_1}e_3) = w_{23}(e_1)(\lambda_1-\lambda_3) + w_{13}(e_1)\nu. \quad (3.64)$$

On the other hand, we also have

$$\begin{aligned} e_2(h(e_1,e_3)) - h(\nabla_{e_2}e_1,e_3) - h(e_1,\nabla_{e_2}e_3) &= e_2(0) - h(w_{12}(e_2)e_2 + w_{13}(e_2)e_3, e_3) \\ &\quad - h(e_1, w_{13}(e_2)e_1 - w_{23}(e_2)e_2) \\ &= w_{13}(e_2)(\lambda_1-\lambda_3) - w_{23}(e_2)\nu \end{aligned}$$

Thus, we have

$$e_2(h(e_1,e_3)) - h(\nabla_{e_2}e_1,e_3) - h(e_1,\nabla_{e_2}e_3) = w_{13}(e_2)(\lambda_1-\lambda_3) - w_{23}(e_2)\nu. \quad (3.65)$$

On the other hand, we also have

$$\begin{aligned} e_3(h(e_1,e_2)) - h(\nabla_{e_3}e_1,e_2) - h(e_1,\nabla_{e_3}e_2) &= e_3(-\nu) \\ &\quad - h(w_{12}(e_3)e_2 + w_{13}(e_3)e_3, e_2) \\ &\quad - h(e_1, w_{12}(e_3)e_1 + w_{23}(e_3)e_3) \\ &= e_3(-\nu). \end{aligned}$$

Since λ_1, λ_3 and ν are constants, we have

$$e_3(h(e_1,e_2)) - h(\nabla_{e_3}e_1,e_2) - h(e_1,\nabla_{e_3}e_2) = e_3(-\nu) = 0. \quad (3.66)$$

By combining (3.64) and (3.65) and (3.66), we obtain

$$w_{23}(e_1)(\lambda_1-\lambda_3) + w_{13}(e_1)\nu = w_{13}(e_2)(\lambda_1-\lambda_3) - w_{23}(e_2)\nu = 0. \quad (3.67)$$

Since, $w_{12}(e_1) = w_{12}(e_2) = 0$, the equation (2.26) turns into

$$\begin{aligned} \nabla_{e_1}e_1 &= w_{13}(e_1)e_3, \quad \nabla_{e_2}e_1 = w_{13}(e_2)e_3, && (3.68a) \\ \nabla_{e_3}e_1 &= w_{12}(e_3)e_2 + w_{13}(e_3)e_3, \\ \nabla_{e_1}e_2 &= w_{23}(e_1)e_3, \quad \nabla_{e_2}e_2 = w_{23}(e_2)e_3, && (3.68b) \\ \nabla_{e_3}e_2 &= w_{12}(e_3)e_1 + w_{23}(e_3)e_3, \\ \nabla_{e_1}e_3 &= w_{13}(e_1)e_1 - w_{23}(e_1)e_2, \quad \nabla_{e_2}e_3 = w_{13}(e_2)e_1 - w_{23}(e_2)e_2, && (3.68c) \\ \nabla_{e_3}e_3 &= w_{13}(e_3)e_1 - w_{23}(e_3)e_2. \end{aligned}$$

Furthermore, by combining (3.54), (3.59), (3.63) and (3.67), we get

$$A\begin{pmatrix} \lambda_1-\lambda_3 \\ \nu \end{pmatrix}=0, \tag{3.69}$$

where we put

$$A=\begin{pmatrix} w_{13}(e_1) & -w_{23}(e_1)-2w_{12}(e_3) \\ w_{23}(e_2) & w_{13}(e_2)-2w_{12}(e_3) \\ w_{23}(e_1) & w_{13}(e_1) \\ w_{13}(e_2) & -w_{23}(e_2) \end{pmatrix}. \tag{3.70}$$

Since $\nu\neq 0$, we have rank$A<2$. Therefore, there exists a function ξ such that

$$w_{13}(e_1) = \xi(-w_{23}(e_1)-2w_{12}(e_3)), \tag{3.71a}$$

$$w_{23}(e_2) = \xi(w_{13}(e_2)-2w_{12}(e_3)), \tag{3.71b}$$

$$w_{23}(e_1) = \xi w_{13}(e_1), \tag{3.71c}$$

$$w_{13}(e_2) = -\xi w_{23}(e_2), \tag{3.71d}$$

and we also have

$$det\begin{pmatrix} w_{13}(e_1) & -w_{23}(e_1)-2w_{12}(e_3) \\ w_{13}(e_2) & -w_{23}(e_2) \end{pmatrix} = 0, \tag{3.72a}$$

$$det\begin{pmatrix} w_{23}(e_2) & w_{13}(e_2)-2w_{12}(e_3) \\ w_{23}(e_1) & w_{13}(e_1) \end{pmatrix} = 0. \tag{3.72b}$$

Next, we use the Gauss Equation (2.20) for $X=Z=e_1$, $Y=W=e_2$ and substitute (3.36)-(3.40) to get

$$R(e_1,e_2,e_1,e_2)=\nu^2+\lambda_1^2. \tag{3.73}$$

By considering (3.68), we compute the left hand side of (3.73)

$$\begin{aligned} R(e_1,e_2,e_1,e_2) &= \langle\nabla_{e_1}\nabla_{e_2}e_1-\nabla_{e_2}\nabla_{e_1}e_1-\nabla_{[e_1,e_2]}e_1,e_2\rangle \\ &= \langle\nabla_{e_1}[w_{13}(e_2)e_3]-\nabla_{e_2}[w_{13}(e_1)e_3]-\nabla_{\nabla_{e_1}e_2-\nabla_{e_2}e_1}e_1,e_2\rangle \\ &= \langle -w_{13}(e_2)w_{23}(e_1)e_2+w_{13}(e_1)w_{23}(e_2)e_2-w_{23}(e_1)w_{12}(e_3)e_2 \\ &\quad +w_{13}(e_2)w_{12}(e_3)e_2,e_2\rangle \\ &= \lambda_1[-w_{13}(e_2)w_{23}(e_1)+w_{13}(e_1)w_{23}(e_2)] \\ &\quad -(\lambda_1)[(w_{23}(e_1)-w_{13}(e_2))w_{12}(e_3)]. \end{aligned}$$

So, we have

$$R(e_1,e_2,e_1,e_2) = \lambda_1[-w_{13}(e_2)w_{23}(e_1)+w_{13}(e_1)w_{23}(e_2)] \\ -(\lambda_1)[(w_{23}(e_1)-w_{13}(e_2))w_{12}(e_3)]. \tag{3.74}$$

We are going to show that $w_{12}(e_3)=0$.

Assume that $w_{12}(e_3)\neq 0$. In this case from (3.72), we get

$$2w_{12}(e_3)w_{13}(e_2)=2w_{12}(e_3)w_{23}(e_1) \tag{3.75}$$

which implies

$$w_{13}(e_2)=w_{23}(e_1). \tag{3.76}$$

Therefore, (3.71c) and (3.71d) imply

$$w_{13}(e_1)+w_{23}(e_2)=0. \tag{3.77}$$

However, if we sum (3.71a) with (3.71b) and use (3.76), (3.77), then we get

$$w_{12}(e_3)=0$$

which is a contradiction. Thus, $w_{12}(e_3)=0$ and (3.72a) implies

$$w_{13}(e_2)w_{23}(e_1)-w_{13}(e_1)w_{23}(e_2)=0. \tag{3.78}$$

Therefore, (3.74) turns into

$$R(e_1,e_2,e_1,e_2)=0.$$

Consequently, (3.73) implies

$$\nu=0$$

which yields a contradiction. Hence, the proof is completed.

□

As a result of Theorems 3.2 and 3.3, there are four classes of isoparametric hypersurfaces in $\mathbb{E}^4_1$.

3.2 Examples of Isoparametric Hypersurfaces

Example 3.4. Let $x(s)$ be a null curve in $\mathbb{E}^4_1$ on which a pseudo-orthonormal frame field $\{X(s), Y(s), C(s), Z(s)\}$ exists such that

$$\langle X, X\rangle = \langle Y, Y\rangle = 0,\ \ \langle X, Y\rangle = -1,\ \ \langle C, C\rangle = \langle Z, Z\rangle = 1$$

and

$$\dot{x}(s) = X(s), \tag{3.79a}$$

$$\dot{C}(s) = -B(s)Y(s), \quad B \neq 0, \tag{3.79b}$$

$$\dot{Z}(s) = A_2(s)Y(s) \tag{3.79c}$$

for some functions $B(s)$ and $A_2(s)$. Then, we have

$$\dot{X}(s) = A_1(s)X(s) - B(s)C(s) + A_2(s)Z(s), \tag{3.80a}$$

$$\dot{Y}(s) = -A_1(s)Y(s). \tag{3.80b}$$

Consider the hypersurface M defined by

$$f(s,u,z) = x(s) + uY(s) + zZ(s) + \left(\frac{1}{a} - \sqrt{\frac{1}{a^2} - z^2}\right)C(s) \tag{3.81}$$

for a non-zero constant a. We obtain

$$E_1 = \frac{\partial}{\partial u} = Y(s), \tag{3.82a}$$

$$E_2 = \frac{\partial}{\partial s} = X(s) + Y(s)\left[\left(-\frac{1}{a} + \sqrt{\frac{1}{a^2} - z^2}\right)B(s) - uA_1(s) + zA_2(s)\right], \tag{3.82b}$$

$$E_3 = \frac{\partial}{\partial z} = \frac{zC(s)}{\sqrt{\frac{1}{a^2} - z^2}} + Z(s). \tag{3.82c}$$

By a direct computation, we observe that the vector fields

$$e_1 = E_1, \tag{3.83a}$$

$$e_2 = F_1E_1 + E_2, \tag{3.83b}$$

$$e_3 = \sqrt{1 - a^2z^2}E_3 \tag{3.83c}$$

form a pseudo-orthonormal base for the tangent bundle at M, where

$$F_1 = \frac{B(s) - \sqrt{1 - a^2z^2}B(s) + auA_1(s) - azA_2(s)}{a}.$$

By a further computation, we get

$$S = \begin{pmatrix} 0 & \xi & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} \tag{3.84}$$

for a function ξ. Hence, M is an isoparametric hypersurface and it's shape operator S has the minimal polynomial $\lambda^2(\lambda - a)$.

Example 3.5. Let $x(s)$ be a null curve in $\mathbb{E}^4_1$ on which a pseudo-orthonormal frame field $\{X(s), Y(s), C(s), W(s)\}$ exists such that

$$\langle X, X\rangle = \langle Y, Y\rangle = 0, \ \ \langle X, Y\rangle = -1, \ \ \langle C, C\rangle = \langle W, W\rangle = 1$$

and

$$\dot{x}(s) = X(s), \tag{3.85a}$$

$$\dot{C}(s) = -B(s)Y(s), \quad B \neq 0 \tag{3.85b}$$

for a function $B(s)$. Then, we have

$$\dot{X}(s) = -A_3(s)X(s) - B(s)C(s) + A_2(s)W(s), \tag{3.86a}$$

$$\dot{Y}(s) = A_1(s)W(s) + A_3(s)Y(s), \tag{3.86b}$$

$$\dot{W}(s) = A_1(s)X(s) + A_2(s)Y(s). \tag{3.86c}$$

Consider the hypersurface M defined by

$$f(s, u, w) = x(s) + uY(s) + wW(s). \tag{3.87}$$

We obtain

$$E_1 = \frac{\partial}{\partial u} = Y(s), \tag{3.88a}$$

$$E_2 = \frac{\partial}{\partial s} = uA_1(s)W(s) + X(s)[1 + wA_1(s)] + Y(s)[wA_2(s) + uA_3(s)], \tag{3.88b}$$

$$E_3 = \frac{\partial}{\partial w} = W(s). \tag{3.88c}$$

By a direct computation, we observe that the vector fields

$$e_1 = E_1, \tag{3.89a}$$

$$e_2 = F_1E_1 + F_2E_2 + F_3E_3, \tag{3.89b}$$

$$e_3 = E_3 \tag{3.89c}$$

form a pseudo-orthonormal base for the tangent bundle at M, where

$$F_1 = \frac{-wA_2(s) - uA_3(s)}{1 + wA_1(s)}, \;\; F_2 = \frac{1}{1 + wA_1(s)}, \;\; F_3 = \frac{-uA_1(s)}{1 + wA_1(s)}.$$

By a further computation, we get

$$S = \begin{pmatrix} 0 & \frac{B(s)}{1+wA_1(s)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.90}$$

Hence, M is an isoparametric hypersurface and it's shape operator S has the minimal polynomial λ^2.

Example 3.6. Let $x(s)$ be a null curve in $\mathbb{E}_1^4$ on which a pseudo-orthonormal frame field $\{X(s), Y(s), C(s), W(s)\}$ exists such that

$$\langle X, X\rangle = \langle Y, Y\rangle = 0, \;\; \langle X, Y\rangle = -1, \;\; \langle C, C\rangle = \langle W, W\rangle = 1$$

and

$$\dot{x}(s) = X(s), \tag{3.91a}$$

$$\dot{C}(s) = -aX(s) - B(s)Y(s), \quad B \neq 0, \tag{3.91b}$$

$$\dot{W}(s) = A_2(s)Y(s) \tag{3.91c}$$

for some functions $B(s)$, $A_2(s)$ and a non-zero constant a. Then, we have

$$\dot{X}(s) = A_1(s)X(s) - B(s)C(s) - A_2(s)W(s), \tag{3.92a}$$

$$\dot{Y}(s) = A_3(s)Y(s) - aC(s). \tag{3.92b}$$

Consider the hypersurface M defined by

$$f(s, u, w) = x(s) + uY(s) + wW(s). \tag{3.93}$$

We obtain

$$E_1 = \frac{\partial}{\partial u} = Y(s), \tag{3.94a}$$

$$E_2 = \frac{\partial}{\partial s} = -auC(s) + X(s) + Y(s)[wA_2(s) + uA_3(s)], \tag{3.94b}$$

$$E_3 = \frac{\partial}{\partial w} = W(s). \tag{3.94c}$$

By a direct computation, we observe that the vector fields

$$e_1 = E_1, \tag{3.95a}$$

$$e_2 = F_1 E_1 + E_2, \tag{3.95b}$$

$$e_3 = E_3 \tag{3.95c}$$

form a pseudo-orthonormal base for the tangent bundle at M, where

$$F_1 = \frac{a^2u^2 - 2wA_2(s) - 2uA_3(s)}{2}.$$

By a further computation, we get

$$S = \begin{pmatrix} -a & -B(s) + awA_2(s) & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.96}$$

for a non-zero constant a. Hence, M is an isoparametric hypersurface and it's shape operator S has the minimal polynomial $(\lambda - a)^2\lambda$.

Example 3.7. Let $x(s)$ be a null curve in $\mathbb{E}_1^4$ on which a pseudo-orthonormal frame field $\{X(s), Y(s), C(s), Z(s)\}$ exists such that

$$\langle X, X\rangle = \langle Y, Y\rangle = 0, \ \ \langle X, Y\rangle = -1, \ \ \langle C, C\rangle = \langle Z, Z\rangle = 1$$

and

$$\dot{x}(s) = X(s), \tag{3.97a}$$

$$\dot{C}(s) = -aX(s) - B(s)Y(s), \quad B \neq 0 \tag{3.97b}$$

for a function $B(s)$ and a non-zero constant a. Then, we have

$$\dot{X}(s) = A_1(s)X(s) - B(s)C(s) + A_2(s)Z(s), \tag{3.98a}$$

$$\dot{Y}(s) = A_3(s)Y(s) - aC(s) - A_2(s)Z(s), \tag{3.98b}$$

$$\dot{Z}(s) = A_4(s)X(s) + A_2(s)Y(s). \tag{3.98c}$$

Consider the hypersurface M defined by

$$f(s,u,z) = x(s) + uY(s) + zZ(s) + \left(\frac{1}{a} - \sqrt{\frac{1}{a^2} - z^2}\right)C(s) \tag{3.99}$$

for a non-zero constant a. We obtain

$$E_1 = \frac{\partial}{\partial u} = Y(s), \tag{3.100a}$$

$$E_2 = \frac{\partial}{\partial s} = -auC(s) - uA_2(s)Z(s) + Y(s)\left(\sqrt{\frac{1}{a^2} - z^2}B(s) - \frac{B(s)}{a} + uA_3(s) + zA_2(s)\right) \tag{3.100b}$$

$$+X(s)\left(a\sqrt{\frac{1}{a^2} - z^2} + zA_4(s)\right),$$

$$E_3 = \frac{\partial}{\partial z} = \frac{zC(s)}{\sqrt{\frac{1}{a^2} - z^2}} + Z(s). \tag{3.100c}$$

By a direct computation, we observe that the vector fields

$$e_1 = E_1, \tag{3.101a}$$

$$e_2 = \frac{F_1}{2a\left(a^2z^2 + z^2A_4(s)^2 - 1\right)^2}E_1 + F_2E_2 + F_3E_3, \tag{3.101b}$$

$$e_3 = \sqrt{1 - a^2z^2}E_3 \tag{3.101c}$$

form a pseudo-orthonormal base for the tangent bundle at M, where

$$\begin{aligned} F_1 \;=\; & a\Big(a^2u^2z^2A_2(s)^2\left(zA_4(s)\left(zA_4(s) - 2\sqrt{1-a^2z^2}\right) - a^2z^2 + 1\right) \\ & -2zA_2(s)\Big(zA_4(s)\Big(zA_4(s)\left(\left(a^2u^2-1\right)\sqrt{1-a^2z^2} + zA_4(s)\right) \\ & +\left(2a^2u^2+1\right)\left(a^2z^2-1\right)\Big) + \left(a^2u^2+1\right)\left(1-a^2z^2\right)^{3/2}\Big) + u\Big(a^2u\left(a^2z^2-1\right)F_4 \\ & +2A_3(s)\left(\sqrt{1-a^2z^2} - zA_4(s)\right)\left(a^2z^2 + z^2A_4(s)^2 - 1\right)\Big)\Big) \\ & -2B(s)\left(z\left(\sqrt{1-a^2z^2} - 1\right)A_4(s) + a^2z^2 + \sqrt{1-a^2z^2} - 1\right) \\ & \left(a^2z^2 + z^2A_4(s)^2 - 1\right), \end{aligned}$$

$F_4 = zA_4(s)\left(2\sqrt{1-a^2z^2} - zA_4(s)\right) + a^2z^2 - 1$, $F_2 = \dfrac{1}{\sqrt{1-a^2z^2} + zA_4(s)}$ and

$$F_3 = \frac{u\left(a^2z\left(z\sqrt{1-a^2z^2}A_4(s) + a^2z^2 - 1\right) + \left(a^2z^2-1\right)A_2(s)\left(\sqrt{1-a^2z^2} - zA_4(s)\right)\right)}{a^2z^2 + z^2A_4(s)^2 - 1}.$$

By a further computation, we get

$$S = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ 0 & -a & 0 \\ 0 & s_{32} & -a \end{pmatrix} \tag{3.102}$$

for some functions s_{11}, s_{12}, s_{13}, s_{32} and a non-zero constant a. Hence, M is an isoparametric hypersurface and it's shape operator S has the minimal polynomial $(\lambda - a)^2$.

Example 3.8. Let $x(s)$ be a null curve in $\mathbb{E}^5_1$ on which a pseudo-orthonormal frame field $\{X(s), Y(s), Z(s), C(s), U(s)\}$ exists such that

$$\langle X, X \rangle = \langle Y, Y \rangle = 0, \ \langle X, Y \rangle = -1, \ \langle C, C \rangle = \langle Z, Z \rangle = \langle U, U \rangle = 1$$

and

$$\dot{x}(s) = X(s), \tag{3.103a}$$

$$\dot{C}(s) = B(s)Z(s), \quad B(s) \neq 0 \tag{3.103b}$$

for a function $B(s)$. Then, we have

$$\dot{X}(s) = A_1(s)X(s) + A_2(s)Z(s) + A_3(s)U(s), \tag{3.104a}$$

$$\dot{Y}(s) = -A_1(s)Y(s) + A_4(s)Z(s) + A_5(s)U(s), \tag{3.104b}$$

$$\dot{Z}(s) = A_4(s)X(s) + A_2(s)Y(s) - B(s)C(s) + A_6(s)U(s), \tag{3.104c}$$

$$\dot{U}(s) = A_5(s)X(s) + A_3(s)Y(s) - A_6(s)Z(s). \tag{3.104d}$$

Consider the hypersurface M defined by

$$f(s, u, z, w) = x(s) + uY(s) + zZ(s) + wU(s). \tag{3.105}$$

We obtain

$$E_1 = \frac{\partial}{\partial u} = Y(s), \tag{3.106a}$$

$$E_2 = \frac{\partial}{\partial s} = U(s)\left(uA_5(s) + zA_6(s)\right) + Y(s)\left(-uA_1(s) + wA_3(s) + zA_2(s)\right) \tag{3.106b}$$
$$+ Z(s)\left(uA_4(s) - wA_6(s)\right) + X(s)\left(wA_5(s) + zA_4(s) + 1\right) - zB(s)C(s),$$

$$E_3 = \frac{\partial}{\partial z} = Z(s), \tag{3.106c}$$

$$E_4 = \frac{\partial}{\partial w} = U(s). \tag{3.106d}$$

By a direct computation, we observe that the vector fields

$$e_1 = E_1, \tag{3.107a}$$

$$e_2 = F_1E_1 + F_2E_2 + F_3E_3 + F_4E_4, \tag{3.107b}$$

$$e_3 = E_3, \tag{3.107c}$$

$$e_4 = E_4 \tag{3.107d}$$

form a pseudo-orthonormal base for the tangent bundle at M, where

$$F_1 = \frac{2\,(wA_5(s) + zA_4(s) + 1)\,(uA_1(s) - wA_3(s) - zA_2(s)) + z^2B(s)^2}{2\,(wA_5(s) + zA_4(s) + 1)^2},$$

$$F_2 = \frac{1}{wA_5(s) + zA_4(s) + 1}, F_3 = \frac{wA_6(s) - uA_4(s)}{wA_5(s) + zA_4(s) + 1} \text{ and } F_4 = -\frac{uA_5(s) + zA_6(s)}{wA_5(s) + zA_4(s) + 1}.$$

We assume that $(1 + zA_4(s) + wA_5(s)) > 0$. Then we get

$$S = \begin{pmatrix} 0 & s_{12} & \frac{B(s)(wA_5(s)+1)}{(wA_5(s)+zA_4(s)+1)^2} & -\frac{zA_5(s)B(s)}{(wA_5(s)+zA_4(s)+1)^2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{B(s)(wA_5(s)+1)}{(wA_5(s)+zA_4(s)+1)^2} & 0 & 0 \\ 0 & \frac{zA_5(s)B(s)}{(wA_5(s)+zA_4(s)+1)^2} & 0 & 0 \end{pmatrix} \tag{3.108}$$

for a function s_{12}. Hence, M is an isoparametric hypersurface and it's shape operator S has the minimal polynomial λ^3.

4. HYPERSURFACES IN $\mathbb{E}^5_1$

In this section, we construct a family of hypersurfaces with non-diagonalizable shape operator in $\mathbb{E}^5_1$ with principal curvatures $\lambda_1, \lambda_3, \lambda_4$ satisfying

$$\lambda_3 = \phi_1(\lambda_1), \quad \lambda_4 = \phi_2(\lambda_1)$$

for some smooth functions ϕ_1, ϕ_2.

Here, $x(s)$ is a light-like curve in $\mathbb{E}^5_1$ and assume that there exists a pseudo-orthonormal frame field $\{X(s), Y(s), Z(s), C(s), U(s)\}$ along x such that

$$\langle X, X\rangle = \langle Y, Y\rangle = 0, \ \ \langle X, Y\rangle = -1, \ \ \langle C, C\rangle = \langle Z, Z\rangle = \langle U, U\rangle = 1$$

and

$$\dot{x}(s) = X(s), \tag{4.1a}$$

$$\dot{Z}(s) = A_1(s)X(s), \tag{4.1b}$$

$$\dot{U}(s) = A_2(s)X(s), \tag{4.1c}$$

$$\dot{C}(s) = A_3(s)X(s) + A_4(s)Y(s). \tag{4.1d}$$

for some smooth functions $A_1(s)$, $A_2(s)$, $A_3(s)$ and $A_4(s)$. Then, we have

$$\dot{X}(s) = A_5(s)X(s) + A_4(s)C(s), \tag{4.2a}$$

$$\dot{Y}(s) = -A_5(s)Y(s) + A_1(s)Z(s) + A_3(s)C(s) + A_2(s)U(s). \tag{4.2b}$$

for a smooth function $A_5(s)$.

Consider the hypersurface M defined by

$$f(s, u, z, w) = x(s) + uX(s) + wZ(s) + a(w + z)C(s) + zU(s) \tag{4.3}$$

for a smooth function a.

Note that we have

$$E_1 = \frac{\partial}{\partial u} = X(s), \tag{4.4a}$$

$$E_2 = \frac{\partial}{\partial s} = X(s)\left(A_3(s)a(w+z) + uA_5(s) + wA_1(s) + zA_2(s) + 1\right) \tag{4.4b}$$

$$+ A_4(s)Y(s)a(w+z) + uA_4(s)C(s),$$

$$E_3 = \frac{\partial}{\partial z} = C(s)a'(w+z) + U(s), \tag{4.4c}$$

$$E_4 = \frac{\partial}{\partial w} = C(s)a'(w+z) + Z(s). \tag{4.4d}$$

By a direct computation, we observe that the vector fields

$$e_1 = E_1, \tag{4.5a}$$

$$e_2 = B_1E_1 + B_2E_2, \tag{4.5b}$$

$$e_3 = C_1E_1 + C_2E_2 + C_3E_3, \tag{4.5c}$$

$$e_4 = D_1E_1 + D_2E_2 + D_3E_3 + D_4E_4 \tag{4.5d}$$

form a pseudo-orthonormal base for the tangent bundle at M, where

$$B_1 = \frac{A_4(s)u^2 - 2a(w+z)\left(A_3(s)a(w+z) + uA_5(s) + wA_1(s) + zA_2(s) + 1\right)}{2a(w+z)^2A_4(s)},$$

$$B_2 = \frac{1}{A_4(s)a(w+z)},\ \ C_1 = -\frac{ua'(w+z)}{a(w+z)\sqrt{a'(w+z)^2+1}},\ \ C_2 = 0,$$

$$C_3 = -\frac{1}{\sqrt{a'(w+z)^2+1}},\ D_1 = -\frac{ua'(w+z)}{a(w+z)\sqrt{2a'(w+z)^4+3a'(w+z)^2+1}},\ D_2 = 0,$$

$$D_3 = \frac{a'(w+z)^2}{\sqrt{2a'(w+z)^4+3a'(w+z)^2+1}},\ D_4 = -\frac{a'(w+z)^2+1}{\sqrt{2a'(w+z)^4+3a'(w+z)^2+1}}.$$

By a further computation, we get

$$S = \begin{pmatrix} \lambda_1 & s_{12} & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & -\frac{a''(w+z)}{\left(a'(w+z)^2+1\right)\sqrt{2a'(w+z)^2+1}} & s_{34} \\ 0 & 0 & s_{34} & -\frac{a''(w+z)}{\left(a'(w+z)^2+1\right)\left(2a'(w+z)^2+1\right)^{3/2}} \end{pmatrix} \tag{4.6}$$

where

$$\lambda_1 = \frac{1}{a(w+z)\sqrt{2a'(w+z)^2+1}},$$

$$s_{12} = -\frac{(A_1(s)+A_2(s))\,a(w+z)a'(w+z)+wA_1(s)+zA_2(s)+1}{A_4(s)a(w+z)^2\sqrt{2a'(w+z)^2+1}},$$

$$s_{34} = -\frac{\sqrt{2a'(w+z)^4+3a'(w+z)^2+1}\,a''(w+z)}{(a'(w+z)^2+1)^{3/2}\,(2a'(w+z)^2+1)^{3/2}}.$$

From (4.6), we have

$$\operatorname{tr}(S) = \frac{-2a(w+z)a''(w+z)+4a'(w+z)^2+2}{a(w+z)\,(2a'(w+z)^2+1)^{3/2}}, \tag{4.7}$$

$$\det(S) = 0. \tag{4.8}$$

By a further computation, from (4.7) we get

$$\lambda_3 = -\frac{a''(w+z)\left(2a'(w+z)^2+2\right)}{(a'(w+z)^2+1)\,(2a'(w+z)^2+1)^{3/2}}. \tag{4.9}$$

From (4.8), we have

$$\lambda_4 = 0. \tag{4.10}$$

Now, because of (4.8), M has zero Gauss-Kronecker curvature. On the other hand, (4.7) yields that M is minimal if and only if a is a smooth function satisfying

$$-2a(t)a''(t)+4a'(t)^2+2 = 0. \tag{4.11}$$

By multiplying both sides of this equation with $-\frac{a'(t)}{a(t)^5}$ we obtain

$$\frac{2a'(t)a''(t)}{a(t)^4} - \frac{4a'(t)^3}{a(t)^5} - \frac{2a'(t)}{a(t)^5} = 0. \tag{4.12}$$

By integrating this equation, we get

$$\frac{a'(t)^2}{a(t)^4} + \frac{1}{2a(t)^4} + c = 0. \tag{4.13}$$

for a constant c.

We obtain the following theorem by summing up all of the results.

Theorem 4.1. *The hypersurface M defined by* (4.3) *has non-diagonalizable shape operator and zero Gauss-Kronecker curvature. Furthermore, it is minimal if and only if* (4.13) *is satisfied.*

5. CONCLUSIONS AND RECOMMENDATIONS

In this thesis, hypersurfaces with non-diagonalizable shape operators in Minkowski space were studied. In Section 3, some theorems about isoparametric hypersurfaces were given. In Theorem 3.2, by using Codazzi and Gauss equations it was proved that the product of the principal curvatures satisfying the specified properties is zero for the isoparametric hypersurface with the shape operator given in (3.1). In this manner, we showed that there is only four classes of isoparametric hypersurface in $\mathbb{E}^4_1$. In Theorem 3.3, it was shown that the hypersurface with the shape operator given in (3.35) can not be isoparametric in $\mathbb{E}^4_1$.

In Section 4, we constructed a family of hypersurfaces with the non-diagonalizable shape operator in $\mathbb{E}^5_1$. We obtained the shape operator and mean curvature of this hypersurface by a further computation. Then, the Gauss-Kronecker curvature of this hypersurface was obtained. In Theorem 4.1, the necessary and sufficient condition for this hypersurface to be minimal was given.

In the future, several geometrical properties of family of hypersurfaces constructed in Section 4 can be studied. In particular the goemetrical properties of their position vectors and Gauss maps can be considered.

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