ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

LIE SYMMETRIES AND EXACT SOLUTIONS OF BENNEY–ROSKES/ZAKHAROV–RUBENCHIK SYSTEM

M.Sc. THESIS

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Department of Mathematics Engineering

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JANUARY 2023

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BENNEY–ROSKES/ZAKHAROV–RUBENCHICK SİSTEMİNİN LIE SİMETRİLERİ VE TAM ÇÖZÜMLERİ

YÜKSEK LİSANS TEZİ

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To my family,

FOREWORD

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ABBREVIATIONS

ODE : Ordinary Differantial Equation
PDE : Partial Differantial Equation
KDD : Kısmi Diferansiyel Denklem
BR : Benney–Roskes
ZR : Zakharov–Rubenchik
BR/ZR : Benney–Roskes/Zakharov–Rubenchik
DS : Davey–Stewartson
KMV : Kac–Moody–Virasoro
2D : Two-dimensional
3D : Three-dimensional

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LIE SYMMETRIES AND EXACT SOLUTIONS OF BENNEY–ROSKES/ZAKHAROV–RUBENCHIK SYSTEM

SUMMARY

Many physical phenomena in our lives are modeled using ordinary differential equations (ODE) and partial differential equations (PDE). Unlike PDEs, ODEs can be solved using more familiar and straightforward techniques. Partial differential equations are widely utilized in scientific fields that place on mathematics, such as physics and engineering. A wide range of partial differential equation types has been derived as a result of the diversity of the sources. Many methods have been developed to deal with the resulting individual equations. One of these methods used to solve PDEs is Lie symmetry analysis. Lie groups and Lie algebras are useful tools for reducing the number of independent variables in a PDE by using the reduction method. Lie's method leads to group-invariant solutions and conservation laws for partial differential equations. PDEs can be classified into equivalence classes and new solutions can be derived from existing ones by taking advantage of their symmetry. The first step in the method is finding the determining equations for the system's symmetries. By solving the determining equations, the vector field that generates the transformation group of the equation is obtained, which is called the infinitesimal generator of the symmetry group. In other words, we find the infinitesimal generators of the transformation groups, which will leave the solution of the system invariant. From this generator, the Lie algebra structure of the system emerges. However, applying Lie group methods to systems of equations takes a lot of time and effort. Even solving elementary differential equations is prone to mistakes if we do it with a pen and paper. All of that has changed thanks to the accessibility of computer algebra systems like Mathematica and Maple. Some of the calculations in this thesis were done using these programs.

This thesis investigates the Lie symmetry algebra of the Benney–Roskes/Zakharov–Rubenchik (BR/ZR) system and presents exact solutions to this system of equations. BR/ZR system includes the well-known Davey-Stewartson (DS) system and Zakharov system in the limiting case. Although the first appearance of the BR system dates back a few decades, it is seen that the research on qualitative and numerical analysis of the system finds place in the recent literature. As this literature lacks the results on Lie symmetries and solitary-type analytic solutions of the system, it has been this work's main purpose to fulfill this gap.

In Chapter 1, the problem statement of the thesis and the literature review of the problem are given.

In Chapter 2, the fundamental definitions and notations for the Lie symmetry analysis of differential equations are provided .

In Chapter 3, (2+1)-dimensional BR/ZR system and in Chapter 4, (3+1) BR/ZR system are investigated by the tools of Lie group analysis.

The symmetry algebra of the (2+1)-dimensional BR/ZR system is obtained as an infinite dimensional Lie algebra. We found that the symmetry algebra is as not rich as the symmetry algebra of the DS system, which is one of the integrable equations in (2+1) dimensions. We succeeded in finding solutions in the forms of a line soliton and hyperbolic type.

We also discovered the Lie symmetry algebra of the (3+1) BR/ZR system. The invariance algebra of the system turns out to be infinite-dimensional. Concentrating on traveling solutions, we found wave components of sech − tanh type, which proceed as line solitons and kinks in two-dimensional cross-sections in space.

With this thesis, we have added original results to the literature on group-theoretical properties and exact solutions of the BR/ZR system. We believe that these results will serve as a source for future numerical and qualitative studies on this system.

BENNEY–ROSKES/ZAKHAROV–RUBENCHICK SİSTEMİNİN LIE SİMETRİLERİ VE TAM ÇÖZÜMLERİ

ÖZET

Bu tezde, Benney–Roskes/Zakharov– Rubenchik denklem sisteminin Lie simetri cebiri belirlenmiş ve denklem sisteminin tam çözümleri bulunmuştur.

Hayatımızdaki birçok fiziksel olayın matematiksel modellenmesi, adi diferansiyel denklemler ve kısmi diferansiyel denklemler (KDD) kullanılarak yapılır. Fizik ve mühendislik gibi matematik odaklı bilimsel disiplinlerde KDD'ler yaygın olarak kullanılır. Adi diferansiyel denklemlere çözüm bulmak daha bilindik ve sistematik yöntemlere dayanırken kısmi diferansiyel denklemler için durum farklıdır. KDD'lere çözüm bulmak için birçok yöntem vardır ve bu yöntemlerden birisi de ilk olarak Sophus Lie'nin çalışmalarıyla litaretüre kazandırılan Lie simetri metodudur. Lie yöntemi ile, diferansiyel denklemin türü ve simetrilerine bağlı olarak, bir diferansiyel denklemin mertebesi düşürülebilir veya bağımsız değişken sayısı azaltılabilir. Yeterince zengin bir simetri cebiri varsa, belli koşullar altında kısmi bir diferansiyel denklem bir adi diferansiyel denkleme dönüşebilir. Lie yöntemi sadece bir diferansiyel denklemin çözüm bulmamıza değil aynı zamanda bilinen bir çözümden başka bir çözüme ulaşmamızı da sağlar. KDD'lerin simetrilerinden yararlanarak KDD'ler denklik sınıflarına ayrılabilir. KDD'lere uygulanan Lie yöntemi ile, grup değişmez çözümlere ve koruma yasalarına ulaşılabilir. Bu yöntemin temelinde etkileri bir denklemin çözüm uzayını değişmez bırakan dönüşüm grupları vardır. Yöntemdeki ilk adım, sistemin simetrileri için belirleyici denklemleri bulmaktır. Belirleyici denklemler çözülerek dönüşüm gruplarının oluşturduğu vektör alanları belirlenir. Bu vektör alanları, sistemin çözümünü değişmez bırakacak olan dönüşüm gruplarının sonsuz küçüklükteki üreteçleridir. Bu üreteçten sistemin Lie cebiri yapısı ortaya çıkar. Ancak bu yöntemi denklem sistemlerine uygulamak çok zaman ve çaba gerektirir. Mathematica ve Maple gibi bilgisayar paketlerinin erişilebilirliği sayesinde bu hesaplar daha sistemli bir şekilde yapılabilmektedir. Bu tezdeki bazı hesaplamalarda da bu programlardan yararlanılmıştır.

Bu tezin konusunu, yerçekimi etkisi altındaki su dalgaları bağlamında Benney ve Roskes (1969) tarafından türetilmiş olan Benney-Roskes (BR) denklem sistemi oluşturmaktadır. Benney-Roskes denklem sistemi, bağımsız olarak Zakharov ve Rubenchik tarafından düşük frekanslı akustik tip salınımlar ile spektral olarak dar küçük genlikli yüksek frekanslı dalga paketlerinin etkileşimini tanımlayan bir sistem olarak türetilen Zakharov-Rubenchik (ZR) sistemi ile denk bir denklem takımıdır. Birçok fiziksel durumda kısa ve uzun dalgaların etkileşimini modelleyen su dalgaları teorisinde Benney-Roskes olarak bilinen Zakharov-Rubenchik sistemi içerisinde zengin dinamik yapılar bulunmaktadır. Bu denklem sistemi çeşitli limitlerde lineer olmayan Schrödinger denklemi ile bir dalga denklemini birleştiren klasik (skaler) Zakharov sistemini ve lineer olmayan Schrödinger denklemi ile bir eliptik denklemi

birleştiren Davey-Stewartson (DS) sistemini içerir. Davey- Stewartson (1974) denklem sistemi sonlu derinlikte suda dalga paketinin yayılımını modelleyen, asimptotik yöntemler sonucu elde edilen bir denklem sistemidir. Bu denklem takımı, daha önce türetilmiş olan Benney-Roskes denklem sisteminin özel bir hali olarak karşımıza çıkar. BR sisteminin ilk ortaya çıkışı çok eskiye dayansa da sistem üzerinde nitel ve sayısal analizler üzerine yapılan araştırmaların son dönem literatüründe yer bulduğu görülmektedir. Literatürde sistemin Lie simetrileri ve analitik çözümleri ile ilgili sonuçlar bulunmadığından, bu çalışmanın asıl amacı bu boşluğu doldurmak olmuştur.

Birinci bölümde, tezin konusu olan problem verilmiş ve probleme ait literatür taraması verilmiştir.

İkinci bölümde, diferansiyel denklemlerin Lie simetri analizi için bilinmesi gereken temel tanımlar ve gösterimler verilmiştir.

Üçüncü bölümde, (2+1)-boyutlu BR/ZR sisteminin simetri cebiri incelenmiş ve simetri cebirinin sonsuz boyutlu bir Lie cebirine sahip olduğu bulunmuştur. Parametrelerden birinin özel bir değeri için sistem, (2+1)-boyutlu DS sistemi haline gelmektedir. DS sisteminin simetri cebirine ilişkin literatürde mevcut olan sonuçlar ile karşılaştırma yapılmıştır. (2+1)-boyutlu BR/ZR sistemine çizgi soliton, yumru tipi durağan (lump type stationary) çözüm ayrıca eliptik, hiperbolik ve trigonometrik tip periyodik çözümler şeklinde çözümler elde edildi.

Dördüncü bölümde, (3+1)-boyutlu BR/ZR sisteminin Lie simetri cebiri bulundu. Sistemin değişmezlik cebirinin sonsuz boyutlu olduğu keşfedildi. Hareket eden dalga çözümlerine odaklanarak, sech − tanh tipinde, uzayın iki boyutlu kesitlerinde çizgisel soliton ve kink soliton şeklinde ilerleyen dalga bileşenleri elde edilmiştir.

Bu tez çalışması ile litaretüre, BR/ZR sisteminin grup-teorik özellikleri ve tam çözümlerine ilişkin orjinal sonuçlar ekledik. Bu sonuçların bu sistemle ilgili ilerideki nümerik ve niteliksel çalışmalara kaynaklık edeceğine inanıyoruz.

1. INTRODUCTION

1.1 Purpose of Thesis

The aim of this thesis is to determine the Lie symmetry algebra and find exact solutions of the Benney–Roskes/Zakharov–Rubenchik system, considered in [1] in the form

$$\begin{aligned} i\psi_t &= -\varepsilon\psi_{zz} - a\Delta_\perp\psi + b|\psi|^2\psi + w(\rho + d\varphi_z)\psi, \\ \rho_t + c\rho_z &= -\Delta_\perp\varphi - \varphi_{zz} - d(|\psi|^2)_z, \\ \varphi_t + c\varphi_z &= -\frac{1}{m}\rho - |\psi|^2 \end{aligned} \tag{1.1.1}$$

in two and three space dimensions.

In system (1.1.1), $\psi(\vec{\mathbf{x}},t)$ denotes the complex-valued high frequency function, where $\vec{\mathbf{x}} = (x,z)$ if $n = 2$ and $\vec{\mathbf{x}} = (x,y,z)$ if $n = 3$, and $\varphi(\vec{\mathbf{x}},t)$ denotes the real-valued hydrodynamic potential function and $\rho(\vec{\mathbf{x}},t)$ is the density fluctuation. $\Delta_\perp = \partial_x^2$ in the case $n = 2$ or $\Delta_\perp = \partial_x^2 + \partial_y^2$ in the case $n = 3$. As for the parameters, m is the Mach number [2], ε is a dispersion constant, d is related to the Doppler shift and w measures the strength of the coupling to acoustic waves. The constants a and c depend on the group velocity and the parameter b measures the self-interaction of the carrying wave. We refer to [1,2] for the explicit expressions for these constants, and we consider them to be nonzero real numbers in our analysis.

The analysis in two space dimensions is performed in Chapter 3, and Chapter 4 deals with the three-dimensional case.

1.2 Literature Review

The Benney–Roskes system of equations (1.1.1) was derived in the context of gravity waves by Benney and Roskes in 1969 [3]. Independently of Benney–Roskes system, Zakharov and Rubenchik derived the Zakharov-Rubenchik system describing the interaction of a spectrally narrow small amplitude high-frequency wave packet with a low-frequency oscillation of acoustic type [4]. The Benney–Roskes system is identical to the Zakharov–Rubenchik system (more details [5]), of which derivation and physical background are in attendance in [6] with details.

The global well-posedness of the one-dimensional Zakharov–Rubenchik equation is proven in the space $H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$ by Oliveira [7]. Also in [7], the existence and the orbital stability of solitary wave solutions for the one-dimensional ZR system are proven. To simulate the one-dimensional ZR equation, Zhao and Li [8] suggested time-splitting Fourier pseudo-spectral (FP) approach, conservative Crank–Nicolson(CN) finite difference method (FDM) and non-conservative FDM. Martinez and Palacios investigated the decay properties for the solutions to the initial value problem of the 1-D ZR/BR system [9].

Saut and Ponce [1] reduced the BR/ZR system to a nonlinear Schrödinger equation with nonlinear terms involving nonlocal terms and derivatives of the unknown and they studied the local well-posedness of the Cauchy problem. Luong et al. handle the Cauchy problem again for the two and three-dimensional Zakharov–Rubenchik system and they analyze its perturbation by a line soliton [10]. In [11], the Zakharov-Rubenchik system has the form

$$\begin{aligned}
&\psi_t - \sigma_3 \psi_x - i\delta \psi_{xx} - i\sigma_1 \Delta_\perp \psi + i\sigma_2 |\psi|^2 + W(\rho + D\phi_x)\psi = 0,\\
&\rho_t + \Delta\phi + D(|\psi|^2)_x = 0, \\
&\phi_t + \frac{1}{M^2}\rho + |\psi|^2 = 0
\end{aligned} \tag{1.2.1}$$

where $\psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, ρ, $\phi : \mathbb{R} \to \mathbb{R}^d$, $d = 2,3$ decribe the fast oscillating and, resp., acoustic type waves. Eq. (1.2.1) has the two conserved quantities, the L^2 norm

$$\int |\psi(x, y_\perp, t)|^2 = \int |\psi(x, y_\perp, 0)|^2,$$

where $y_\perp = y$ or (y,z), and, after the change of variable $x \to (x + \sigma_3 t, t)$, the Hamiltonian

$$\begin{aligned}
E(t) = \int_{\mathbb{R}^d} \Big(&\frac{\delta}{2}|\psi_x|^2 + \frac{\sigma_1}{2}|\nabla_\perp \psi|^2 + \frac{\sigma_2}{4}|\psi|^4 + \frac{W}{4M^2}\rho^2 + \frac{W}{4}|\nabla\phi|^4 + \frac{W}{2}(\rho + D\phi_x)|\psi|^2\Big) \\
= E(0) &
\end{aligned} \tag{1.2.2}$$

Quintero and Cordero studied the nonlinear orbital instability of ground state standing waves for the Benney–Roskes/Zakharov–Rubenchik system in $N = 2$ and $N = 3$ spatial directions [12]. Quintero also analyzed the stability and instability of standing waves for a generalized BR/ZR system in spatial dimensions $N = 2,3$ in another study [13]. A simplified version of the ’universal’ model of the BR/ZR system of equations turns into the Davey–Stewartson system [10]. Davey–Stewartson (DS) system was first derived by Davey and Stewartson in [14]. The DS system describes the propagation of two-dimensional water waves moving under the force of gravity in waters of finite depth. Ref. [15] contains a brief summary of results on the BR/ZR system and a detailed account of the limiting case DS systems. With similar simplification, the Zakharov system can also be obtained. The Zakharov system is introduced in [16] to describe the propagation of Langmuir waves in plasma. Ref. [17] includes a rigorous justification of the Zakharov limit of the ZR system and [18] presents the Schrödinger limit of the ZR system in one spatial dimension case for well-prepared initial data [10]. Refs. [18] and [19] are to be mentioned for the well-posedness results in the one-dimensional setting.

Djordjevic and Redekopp extended the study of Davey and Stewartson to include the effects of capillarity [20]. In [21], Ghidaglia and Saut studied the well-posedness of the Cauchy problem for the DS equation. In the elliptic-hyperbolic and hyperbolic-hyperbolic cases of the DS system, Linares showed in [22] that the initial value problem with small assumptions on the data is locally well-posed in weighted Sobolev space.

The symmetry algebra of the Davey–Stewartson system is identified by Champagne and Winternitz in [23] as an infinite-dimensional algebra of Kac–Moody–Virasoro (KMV) type, in the case when the system is integrable. Admitting a KMV algebra as the invariance algebra is a property that is seen in some other integrable equations in (2+1)-dimensions [24].

1.3 Hypothesis

We focus on determining the Lie algebra of Benney–Roskes/Zakharov–Rubenchik system of equations and find some exact solutions. The recent literature includes works that have taken into consideration this system from several approaches. The system has connections with the well-known widely studied Davey–Stewartson system of equations. From the group-theoretical point of view, the DS system of equations admits an infinite-dimensional Lie symmetry algebra of KMV type exactly in the integrable case, which is a feature shared by some other integrable equations in (2+1)-dimensions. Besides, for the DS system, there is also a vast amount of literature devoted to the search for exact solutions. The fact that the BR/ZR system includes the DS system in the limiting case motivated us to investigate the Lie invariance algebra of the BR/ZR system and also search for any type of exact solutions that might contribute to the available literature. Therefore in this thesis, we answer the following questions:

- What is the Lie symmetry algebra of the BR/ZR system in two and three space dimensions?
- How does two-dimensional (2D) BR/ZR algebra compare to 2D DS algebra?
- Can we obtain exact solutions of traveling type for 2D and three-dimensional (3D) BR/ZR system?

2. BASIC NOTATIONS AND DEFINITIONS OF LIE SYMMETRY ANALYSIS

In this Chapter, we give basic notions and definitions of Lie groups which are available in [25] and [26]. The content in this Chapter is not to be understood as original content submitted by the author of this thesis in fulfillment of this degree.

2.1 Lie Groups of Transformations

Definition 2.1.1 ***(Symmetry)*** *Symmetry can be defined as a transformation that leaves an object invariant. If a transformation satisfies the following conditions it is a symmetry .*
(S1) The transformation preserves the structure.
(S2) The transformation is a diffeomorphism.
(S3) The transformation maps the object to itself.

Definition 2.1.2 ***(One-Parameter Lie Group of Transformation)*** *Suppose that an object occupying a subset of R^N has an infinite set of symmetries.*

$$\mathscr{S}_\varepsilon : x^k \to \tilde{x}^k(x^1,...,x^N;\varepsilon), \qquad k=1,....,N \tag{2.1.1}$$

where ε is a real parameter, and that the following conditions are satisfied.
(LI) $\mathscr{S}_0$ is the trivial symmetry, so that $\tilde{x}^k = x^k$ when $\varepsilon = 0$.
(L2) $\mathscr{S}_\varepsilon$ is a symmetry for every ε in some neighbourhood of zero.
(L3) $\mathscr{S}_\delta\mathscr{S}_\varepsilon = \mathscr{S}_{\delta+\varepsilon}$ for every δ,ε sufficiently close to zero.
(L4) Each $\tilde{x}^k$ may be represented as a Taylor series in ε (in some neighbourhood of $\varepsilon = 0$), and therefore

$$\tilde{x}^k(x^1,...,x^N;\varepsilon) = x^k + \varepsilon\xi^k(x^1,...,x^N) + O(\varepsilon^2), \qquad k=1,....,N. \tag{2.1.2}$$

Then the set of symmetries $\mathscr{S}_\varepsilon$ is a *one-parameter local Lie group*. The term "local" (which we will usually omit hereafter) refers to the fact that the conditions need only apply in some neighborhood of $\varepsilon = 0$.

2.2 One-Parameter Groups of Point Transformations and Their Infinitesimal Generators in the Plane

Definition 2.2.1 ***(One-parameter group of point transformation)*** *A point transformation in the plane, where x is the independent variable and y is the dependent variable is defined as*

$$\tilde{x} = \tilde{x}(x,y), \qquad \tilde{y} = \tilde{y}(x,y). \tag{2.2.1}$$

It maps points (x,y) into points $(\tilde{x},\tilde{y})$. If we consider this in the frame of symmetries, we have to consider point transformations that depend on (at least) one arbitrary parameter ε,

$$\tilde{x} = \tilde{x}(x,y;\varepsilon), \qquad \tilde{y} = \tilde{y}(x,y;\varepsilon). \tag{2.2.2}$$

This transformation which also satisfies properties $L_1 - L_3$ is called a one-parameter group of point transformations in the plane.

Rotations are a straightforward example of a one-parameter group

$$(\tilde{x},\tilde{y}) = (xcos\varepsilon - ysin\varepsilon, xsin\varepsilon + ycos\varepsilon) \tag{2.2.3}$$

Another example is a translation

$$\tilde{x} = x, \qquad \tilde{y} = y + \varepsilon. \tag{2.2.4}$$

Definition 2.2.2 ***(Infinitesimal Generators)*** *We take an arbitrary point (x,y) and write*

$$\tilde{x}(x,y;\varepsilon) = x + \varepsilon\xi(x,y)... = x + \varepsilon Xx + ..., \tag{2.2.5}$$

$$\tilde{y}(x,y;\varepsilon) = y + \varepsilon\eta(x,y)... = x + \varepsilon Xy + ..., \tag{2.2.6}$$

where the function ξ and η are defined by

$$\xi(x,y) = \frac{\partial\tilde{x}}{\partial\varepsilon}|_{\varepsilon=0}, \qquad \eta(x,y) = \frac{\partial\tilde{y}}{\partial\varepsilon}|_{\varepsilon=0}, \tag{2.2.7}$$

and the operator X is given by

$$\mathbf{X}=\xi(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y} \tag{2.2.8}$$

The operator **X** *is called Infinitesimal Generator of the transformation.*

Let's illustrate it on (2.2.3). We have

$$\frac{\partial \tilde{x}}{\partial \varepsilon}|_{\varepsilon=0}=-y, \qquad \frac{\partial \tilde{y}}{\partial \varepsilon}|_{\varepsilon=0}=x \tag{2.2.9}$$

so that corresponding the infinitesimal generator is given by

$$\mathbf{X}=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y} \tag{2.2.10}$$

For a translation (2.2.4) we obtain,

$$\frac{\partial \tilde{x}}{\partial \varepsilon}|_{\varepsilon=0}=0, \qquad \frac{\partial \tilde{y}}{\partial \varepsilon}|_{\varepsilon=0}=1, \qquad X=\frac{\partial}{\partial y}. \tag{2.2.11}$$

From this point of view, we can think that we can find the finite transformation by integrating from a given generator. Let's we have

$$\mathbf{X}=3x\frac{\partial}{\partial x}+4y\frac{\partial}{\partial y} \tag{2.2.12}$$

so,

$$\xi(x,y)=\frac{\partial \tilde{x}}{\partial \varepsilon}=3\tilde{x}, \qquad \eta(x,y)=\frac{\partial \tilde{y}}{\partial \varepsilon}=4\tilde{y} \tag{2.2.13}$$

then we can integrate **X** with respect to $\varepsilon=0$. The solution with initial condition $\tilde{x}(0)=0$ and $\tilde{y}(0)=0$ is

$$\tilde{x}=e^{3\varepsilon}x, \qquad \tilde{y}=e^{4\varepsilon}y. \tag{2.2.14}$$

2.3 Extensions of Transformations and Their Generators in the Plane

Definition 2.3.1 ***(Prolongation of transformation)*** *If*

$$\mathbf{X}=\xi(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y} \tag{2.3.1}$$

is infinitesimal generators of transformation,then

$$\mathbf{X}^{(n)}=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\eta'\frac{\partial}{\partial y'}+\ldots+\eta^{(n)}\frac{\partial}{\partial y^{(n)}} \tag{2.3.2}$$

is its prolongation (extension) up to the nth derivative where $\eta^{(n)}(x,y,y',...,y^{(n)})$ *are defined by*

$$\eta^{(n)} = \frac{d^n}{dx^n}(\eta - y'\xi) + y^{(n+1)}\xi \tag{2.3.3}$$

Notice that $\eta^{(n)}$ *does not mean the nth derivative of* η *and here* $\frac{d^n}{dx^n}$ *denotes the total derivative.*

Definition 2.3.2 *The nth order differential equation given in the following form*

$$F(x,y,y',...,y^n) = 0 \tag{2.3.4}$$

is invariant under symmetry transformation $\tilde{x} = \tilde{x}(x,y)$ *and* $\tilde{y} = \tilde{y}(x,y)$ *if*

$$F(\tilde{x},\tilde{y},\tilde{y}',...,\tilde{y}^n) = 0. \tag{2.3.5}$$

This definition means that the existence of symmetry is unaffected by the variables chosen for the differential equation and its solutions. Also (2.3.5) has to be valid for all values of ε.

Definition 2.3.3 *An ordinary differential equation*

$$F(x,y,y',...,y^{(n)}) = 0 \tag{2.3.6}$$

admits a group of symmetries with generator **X** *if and only if*

$$\mathbf{X}^{(n)}F\big|_{F=0} = 0 \tag{2.3.7}$$

holds.

Making use of (2.3.6), we can find **X** infinitesimal generator of a given differential equation. We can give it step by step as follows.

- The prolongation $\mathbf{X}^{(n)}$ is applied to the given differential equation F. We call the equations resulting from this application as determining equations.
- The determining equations are solved.

- The solution of the determining equations gives us the coefficients of the infinitesimal generator ξ, η.

- Finally the infinitesimal generator **X** is found.

Definition 2.3.4 *A Lie algebra is a real vector space L together with a binary bracket operation*

$$[\cdot,\cdot] = L \times L \to L \tag{2.3.8}$$

such that any $X,Y,Z \in L$

(B1) *(Linearity)* $[aX_1 + bX_2, Y] = a[X_1,Y] + b[X_2,Y]$

(B2) *(Antisymmetry)* $[X,Y] = -[Y,X]$

(B3) *(Jacobi identity)* $[X,[Y,Z]] + [Y,[Z,X]] + [Z[X,Y]] = 0$

The bracket $[\cdot,\cdot]$ is called the Lie bracket which is defined as

$$[X,Y] = XY - YX. \tag{2.3.9}$$

2.4 Symmetries of PDEs

Suppose we have system

$$\mathscr{F}(x^n, u^\alpha, u_n^\alpha, u_{nm,...}^\alpha) = 0 \tag{2.4.1}$$

of PDE in the N independent variables x^n, the M dependent variables $u^\alpha(x^n)$, and their derivatives, denoted by

$$u_n^\alpha = \frac{\partial u^\alpha}{\partial x^n}, \qquad u_{nm}^\alpha = \frac{\partial^2 u^\alpha}{\partial x^n \partial x^m} \qquad etc. \tag{2.4.2}$$

Definition 2.4.1 *A one-parameter group of point transformation is a transformation between independent and dependent variables,*

$$\begin{aligned} &\tilde{x}^n = \tilde{x}^n(x^i, u^\beta; \varepsilon), && \tilde{u}^n = \tilde{u}^n(x^i, u^\beta; \varepsilon), \\ &n,i = 1,...,N, && \alpha,\beta = 1,...,M. \end{aligned} \tag{2.4.3}$$

The corresponding infinitesimal transformation

$$\begin{aligned} &\tilde{x}^n = x^n + \varepsilon\xi^n(x^i,u^\beta) + O(\varepsilon^2), \qquad \xi^n = \frac{\partial \tilde{x}^n}{\partial \varepsilon}|_{\varepsilon=0} \\ &\tilde{u}^\alpha = u^\alpha + \varepsilon\eta^\alpha(x^i,u^\beta) + O(\varepsilon^2), \qquad \eta^\alpha = \frac{\partial \tilde{u}^\alpha}{\partial \varepsilon}|_{\varepsilon=0} \end{aligned} \tag{2.4.4}$$

is generated by

$$\mathbf{X} = \xi^n(x^i, u^\beta)\frac{\partial}{\partial x^n} + \eta^\alpha(x^i, u^\beta)\frac{\partial}{\partial u^\alpha}. \tag{2.4.5}$$

We have to expand the finite point transformation (2.4.3) and its generator (2.4.5) to the derivatives in order to apply them to a partial differential equation. The generator (2.4.5) the extended to derivatives of arbitrary order is given by the next definition.

Definition 2.4.2 *If* (2.4.5) *is the infinitesimal generator of the transformation* (2.4.3)*, then*

$$X^{(n)} = \xi^n\frac{\partial}{\partial x^n} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \eta^\alpha_n\frac{\partial}{\partial u^\alpha_n} + \eta^\alpha_{nm}\frac{\partial}{\partial u^\alpha_{nm}} + \dots \tag{2.4.6}$$

where

$$\eta^\alpha_{nm} = \frac{D\eta^\alpha_n}{Dx^m} - u^\alpha_{ni}\xi^i\frac{D\xi^i}{Dx^m} \tag{2.4.7}$$

is its prolongation (extension).

Definition 2.4.3 *The partial differential equation*

$$\mathscr{F}(x^n, u^\alpha, u^\alpha_n, u^\alpha_{nm,\dots}) = 0 \tag{2.4.8}$$

admits the symmetry transformation generated by (2.4.6) *if and only if*

$$\mathbf{X}^{(n)}\mathscr{F}\big|_{\mathscr{F}=0} = 0 \tag{2.4.9}$$

where $\mathbf{X}^{(n)}$ *is prolongation of* (2.4.5)*, that is,* (2.4.6).

In this Chapter, the sources used were [25] and [26]. For further reference on Lie group analysis of differential equations, we can mention the classical monographs [27] and [28].

3. 2D BENNEY–ROSKES/ZAKHAROV–RUBENCHIK SYSTEM

Benney–Roskes system was derived in the context of water waves [3]. Let us consider the Benney–Roskes (BR) system of [3], also appearing in [1],

$$\begin{aligned}
&\frac{\partial b_1}{\partial T} - i\frac{\varepsilon}{2}\Big\{\omega_{11}\frac{\partial^2 b_1}{\partial X^2} + \omega_{22}\frac{\partial^2 b_1}{\partial Z^2}\Big\} + i\varepsilon b_1\Big\{k\frac{\partial a_0}{\partial X} + \frac{1}{2g\omega}(g^2k^2-\omega^4)b_0\Big\} + i\varepsilon|b_1|^2 b_1 = 0,\\
&\frac{\partial a_0}{\partial T} - c_g\frac{\partial a_0}{\partial X} + gb_0 + \frac{1}{\omega^2}(g^2k^2-\omega^4)|b_1|^2 = 0,\\
&\frac{\partial b_0}{\partial T} - c_g\frac{\partial b_0}{\partial X} + h\Big(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}\Big)a_0 + \frac{2gk}{\omega}\frac{\partial}{\partial X}|b_1|^2 = 0.
\end{aligned} \tag{3.0.1}$$

Let us replace $b_1 \to \psi$, $a_0 \to \varphi$, $b_0 \to \rho$, $T \to t$, $X \to z$, $Z \to x$, which gives

$$\begin{aligned}
&i\psi_t + \frac{\varepsilon}{2}\omega_{22}\psi_{xx} + \frac{\varepsilon}{2}\omega_{11}\psi_{zz} = \varepsilon|\psi|^2\psi + \frac{\varepsilon}{2g\omega}(g^2k^2-\omega^4)\rho\psi + \varepsilon k\varphi_z\psi,\\
&\rho_t - c_g\rho_z + h(\varphi_{xx}+\varphi_{zz}) + \frac{2gk}{\omega}(|\psi|^2)_z = 0,\\
&\varphi_t - c_g\varphi_z + g\rho + \frac{1}{\omega^2}(g^2k^2-\omega^4)|\psi|^2 = 0.
\end{aligned} \tag{3.0.2}$$

Hence it is seen that the Benney–Roskes system is identical to the Zakharov–Rubenchik (ZR) system, also given in [1], as

$$\begin{aligned}
&i\psi_t = -\varepsilon\psi_{zz} - a\Delta_\perp\psi + b|\psi|^2\psi + w(\rho + d\varphi_z)\psi,\\
&\rho_t + c\rho_z = -\Delta_\perp\varphi - \varphi_{zz} - d(|\psi|^2)_z,\\
&\varphi_t + c\varphi_z = -\frac{1}{m}\rho - |\psi|^2,
\end{aligned} \tag{3.0.3}$$

in the spatial domain $\mathbb{R}^n$, $n = 2,3$, describing the interaction of a spectrally narrow small amplitude high-frequency wave packet with a low-frequency oscillation of acoustic type. For a detailed account of the system we can refer to [1]. Using the third equation of (3.0.3), we substitute

$$\rho = -m(\varphi_t + c\varphi_z + |\psi|^2) \tag{3.0.4}$$

to the first and second equations. We get

$$\begin{aligned}
&i\psi_t + a\Delta_\perp\psi + \varepsilon\psi_{zz} = (b - wm)|\psi|^2\psi - w\psi\Big\{m\frac{\partial}{\partial t} + (mc-d)\frac{\partial}{\partial z}\Big\}\varphi,\\
&\Delta_\perp\varphi + (1-mc^2)\varphi_{zz} - m\varphi_{tt} - 2mc\varphi_{zt} = \Big\{m\frac{\partial}{\partial t} + (mc-d)\frac{\partial}{\partial z}\Big\}|\psi|^2.
\end{aligned} \tag{3.0.5}$$

For simplicity, we call the system of equations (3.0.5) 2D BR/ZR system if $n = 2$ while 3D BR/ZR system if $n = 3$. In this Chapter, we deal with the case $n = 2$ only and investigate the Lie symmetry algebra of the 2D BR/ZR system and present exact solutions.

3.1 The Symmetry Algebra of 2D BR/ZR System

In order to determine the Lie symmetries of the BR/ZR system of equations (3.0.5), we apply the well-known algorithm first introduced by Sophus Lie [29]–[31]. The analysis we will do here is standard.
When we consider the system (3.0.5) for $n = 2$, replacing $z \to y$ and $\varepsilon \to \varepsilon_0$, it becomes

$$\begin{aligned} &i\psi_t + a\psi_{xx} + \varepsilon_0\psi_{yy} = (b - wm)|\psi|^2\psi - w\psi\Big\{m\frac{\partial}{\partial t} + (mc - d)\frac{\partial}{\partial y}\Big\}\varphi, \\ &\varphi_{xx} + (1 - mc^2)\varphi_{yy} - m\varphi_{tt} - 2mc\varphi_{yt} = \Big\{m\frac{\partial}{\partial t} + (mc - d)\frac{\partial}{\partial y}\Big\}|\psi|^2. \end{aligned} \tag{3.1.1}$$

We would like to point out here that a, b, c are considered as nonzero real numbers and ε_0, d, m, w as positive. Clearly, when $m = 0$, this system reduces to (2+1)-dimensional Davey–Stewartson equations.

In the analysis below, we first find the symmetry algebra of the system (3.1.1). After that, since the derivatives on the right hand sides of (3.1.1) are in the same direction, we make a transformation which will put (3.1.1) to a somewhat simpler form and determine the symmetry algebra in this second form. Lastly, we give the symmetry algebra of the DS system in again two cases and we conclude this subsection with a comparison of these results.

3.1.1 BR/ZR Symmetry Algebra

Separating $\psi = u + iv$, we get

$$\begin{aligned} u_t + av_{xx} + \varepsilon_0 v_{yy} &= (b - wm)(u^2 + v^2)v - vw\Big(m\varphi_t + (mc - d)\varphi_y\Big), \\ -v_t + au_{xx} + \varepsilon_0 u_{yy} &= (b - wm)(u^2 + v^2)u - uw\Big(m\varphi_t + (mc - d)\varphi_y\Big), \\ \varphi_{xx} + (1 - mc^2)\varphi_{yy} - m\varphi_{tt} - 2mc\varphi_{yt} &= m(2uu_t + 2vv_t) + (mc - d)(2uu_y + 2vv_y). \end{aligned} \tag{3.1.2}$$

the Lie symmetry algebra BR/ZR system (3.1.1) is generated by the infinitesimal

$$\mathscr{V} = \tau_0\partial_t + \xi_0\partial_x + \eta_0\partial_y - \chi(t)(v\partial_u - u\partial_v) + \Big[\frac{1}{mw}\chi(t) + P(x,\zeta)\Big]\partial_\varphi \tag{3.1.3}$$

where τ_0, ξ_0, η_0 and

$$\zeta = my - (mc - d)t, \tag{3.1.4}$$

with $\chi(t)$ and $P(x,\zeta)$ satisfying

$$P_{xx} + m(m - d^2)P_{\zeta\zeta} = \frac{1}{w}\ddot{\chi}(t). \tag{3.1.5}$$

The right hand side of this equation depends only on t, and its left hand side depends on x and ζ. Therefore, it must be equal to a constant, say, c_1.

$$P_{xx} + m(m - d^2)P_{\zeta\zeta} = \frac{1}{w}\ddot{\chi}(t) = c_1. \tag{3.1.6}$$

Then, we have $\chi(t) = \frac{c_1 w}{2}t^2 + s_1 t + s_0$. For the solution to $P_{xx} + m(m - d^2)P_{\zeta\zeta} = c_1$, let $P(x,\zeta) = \bar{P}(x,\zeta) + \frac{c_1}{2}x^2$. Then $\bar{P}$ satisfies

$$\bar{P}_{xx} + m(m - d^2)\bar{P}_{\zeta\zeta} = 0. \tag{3.1.7}$$

(A) Suppose $m < d^2$. Set $\alpha_0^2 = m(d^2 - m)$. Then

$$\bar{P}_{xx} - \alpha_0^2\bar{P}_{\zeta\zeta} = 0 \tag{3.1.8}$$

is integrated to

$$\bar{P} = F(\alpha_0 x - \zeta) + G(\alpha_0 x + \zeta) \tag{3.1.9}$$

and hence

$$P(x,\zeta) = F(\alpha_0 x - \zeta) + G(\alpha_0 x + \zeta) + \frac{c_1}{2}x^2. \tag{3.1.10}$$

We see that the Lie algebra is spanned by

$$\begin{aligned}
&\mathscr{X}_1 = \partial_t, \quad \mathscr{X}_2 = \partial_x, \quad \mathscr{X}_3 = \partial_y,\\
&\mathscr{X}_4 = -v\partial_u + u\partial_v + \frac{1}{mw}\partial_\varphi,\\
&\mathscr{X}_5 = t(-v\partial_u + u\partial_v) + \frac{t}{mw}\partial_\varphi,\\
&\mathscr{X}_6 = t^2(-v\partial_u + u\partial_v) + (\frac{t^2}{mw} + \frac{x^2}{w})\partial_\varphi,\\
&\mathscr{P}_{\mathscr{A}}(F) = F\Big(\sqrt{m(d^2 - m)}\,x - my + (mc - d)t\Big)\partial_\varphi,\\
&\mathscr{Q}_{\mathscr{A}}(G) = G\Big(\sqrt{m(d^2 - m)}\,x + my - (mc - d)t\Big)\partial_\varphi
\end{aligned} \tag{3.1.11}$$

where $F(s)$ and $G(s)$ are arbitrary single-variable functions. The nonzero commutations are

$$\begin{aligned}
&[\mathscr{X}_1, \mathscr{Q}_{\mathscr{A}}(G)] = (d-mc)\mathscr{Q}_{\mathscr{A}}(G'), && [\mathscr{X}_1, \mathscr{X}_6] = 2\mathscr{X}_5,\\
&[\mathscr{X}_1, \mathscr{P}_{\mathscr{A}}(F)] = (mc-d)\mathscr{P}_{\mathscr{A}}(F'), && [\mathscr{X}_1, \mathscr{X}_5] = \mathscr{X}_4,\\
&[\mathscr{X}_2, \mathscr{P}_{\mathscr{A}}(F)] = \sqrt{m(d^2-m)}\,\mathscr{P}_{\mathscr{A}}(F'), && [\mathscr{X}_3, \mathscr{P}_{\mathscr{A}}(\mathscr{F})] = -m\mathscr{P}_{\mathscr{A}}(F'),\\
&[\mathscr{X}_2, \mathscr{Q}_{\mathscr{A}}(G)] = \sqrt{m(d^2-m)}\,\mathscr{Q}_{\mathscr{A}}(G'), && [\mathscr{X}_3, \mathscr{Q}_{\mathscr{A}}(G)] = m\mathscr{P}_{\mathscr{A}}(G'),\\
&[\mathscr{X}_2, \mathscr{X}_6] = \frac{1}{w\sqrt{m(d^2-m)}}(\mathscr{P}_{\mathscr{A}}|_{F(s)=s} + \mathscr{Q}_{\mathscr{A}}|_{G(s)=s}).
\end{aligned} \tag{3.1.12}$$

(B) Suppose $m > d^2$. $\bar{P}(x,\zeta)$ satisfies

$$\bar{P}_{xx} + m(m-d^2)\bar{P}_{\zeta\zeta} = 0. \tag{3.1.13}$$

Let us scale $\hat{\zeta} = \dfrac{\zeta}{\sqrt{m(m-d^2)}}$. Then we have

$$\bar{P}_{xx} + \bar{P}_{\hat{\zeta}\hat{\zeta}} = 0. \tag{3.1.14}$$

Then

$$\bar{P} = H(x,\hat{\zeta}), \tag{3.1.15}$$

where H is any harmonic function. Therefore we find that

$$P(x,\zeta) = H\Big(x, \frac{\zeta}{\sqrt{m(m-d^2)}}\Big) + \frac{c_1}{2}x^2. \tag{3.1.16}$$

Similarly, we obtain the previous generators $\mathscr{X}_i$, $i = 1,...,6$ and

$$\mathscr{P}_{\mathscr{B}}(H) = H(x,\sigma)\partial_{\varphi}, \qquad \sigma = \frac{my-(mc-d)t}{\sqrt{m(m-d^2)}} \tag{3.1.17}$$

where H is an arbitrary harmonic function. The nonzero commutation relations are

$$\begin{aligned}
&[\mathscr{X}_1, \mathscr{X}_5] = \mathscr{X}_4, && [\mathscr{X}_1, \mathscr{X}_6] = 2\mathscr{X}_5,\\
&[\mathscr{X}_2, \mathscr{X}_6] = \mathscr{P}_{\mathscr{B}}(H)|_{H(x,\sigma)=2x/w}, && [\mathscr{X}_1, \mathscr{P}_{\mathscr{B}}(H)] = \frac{d-mc}{\sqrt{m(m-d^2)}}\mathscr{P}_{\mathscr{B}}\Big(H_{\sigma}(x,\sigma)\Big),\\
&[\mathscr{X}_2, \mathscr{P}_{\mathscr{B}}(H)] = \mathscr{P}_{\mathscr{B}}\Big(H_x(x,\sigma)\Big), && [\mathscr{X}_3, \mathscr{P}_{\mathscr{B}}(H)] = \sqrt{\frac{m}{m-d^2}}\,\mathscr{P}_{\mathscr{B}}\Big(H_{\sigma}(x,\sigma)\Big).
\end{aligned} \tag{3.1.18}$$

(C) Suppose $m = d^2$. Then solving $P_{xx}(x,\zeta) = c_1$ yields

$$P(x,\zeta) = \frac{c_1}{2}x^2 + R(\zeta)x + S(\zeta), \tag{3.1.19}$$

where R, S are arbitrary functions. In addition to $\mathscr{X}_i$, $i = 1,...,6$, we obtain the generators

$$\begin{aligned}\mathscr{P}_{\mathscr{C}}(R) &= xR\big(dy-(cd-1)t\big)\partial_\varphi,\\ \mathscr{Q}_{\mathscr{C}}(S) &= S\big(dy-(cd-1)t\big)\partial_\varphi\end{aligned} \tag{3.1.20}$$

where $R(s)$ and $S(s)$ are arbitrary functions of a single variable. We present the nonzero commutation relations as follows.

$$\begin{aligned}&[\mathscr{X}_1,\mathscr{X}_5] = \mathscr{X}_4, && [\mathscr{X}_1,\mathscr{X}_6] = 2\mathscr{X}_5,\\ &[\mathscr{X}_1,\mathscr{P}_{\mathscr{C}}(R)] = (1-cd)\mathscr{P}_{\mathscr{C}}(R'), && [\mathscr{X}_2,\mathscr{P}_{\mathscr{C}}(R)] = \mathscr{Q}_{\mathscr{C}}(R),\\ &[\mathscr{X}_3,\mathscr{P}_{\mathscr{C}}(R)] = d\mathscr{P}_{\mathscr{C}}(R'), && [\mathscr{X}_1,\mathscr{Q}_{\mathscr{C}}(S)] = (1-cd)\mathscr{Q}_{\mathscr{C}}(S')\\ &[\mathscr{X}_3,\mathscr{Q}_{\mathscr{C}}(S)] = d\mathscr{Q}_{\mathscr{C}}(S'), && [\mathscr{X}_2,\mathscr{X}_6] = \mathscr{P}_{\mathscr{C}}(R)|_{R(s)=2/w}.\end{aligned} \tag{3.1.21}$$

Proposition 3.1.1 *The invariance algebra of the BR/ZR system* (3.1.1) *is infinite-dimensional and has the following bases.*

- ∘ $m < d^2$: $\mathscr{L}_{\mathscr{A}} = \{\mathscr{X}_i, \mathscr{P}_{\mathscr{A}}(F), \mathscr{Q}_{\mathscr{A}}(G)\}$, $i = 1,\dots,6$,
- ∘ $m > d^2$: $\mathscr{L}_{\mathscr{B}} = \{\mathscr{X}_i, \mathscr{P}_{\mathscr{B}}(H)\}$, $i = 1,\dots,6$,
- ∘ $m = d^2$: $\mathscr{L}_{\mathscr{C}} = \{\mathscr{X}_i, \mathscr{P}_{\mathscr{C}}(R), \mathscr{Q}_{\mathscr{C}}(S)\}$, $i = 1,\dots,6$.

3.1.2 BR/ZR in the Second Form

Notice that the directional derivatives on the right hand side of Eqs. (3.1.1) are in the same direction. This gives us the chance to simplify the right hand sides of these equations by a rotational transformation. We keep the variables (x,ψ,φ) as they are and do the transformation $(y,t) \to (\xi,\eta)$ with

$$t = k_1\xi + k_2\eta, \qquad y = \ell_1\xi + \ell_2\eta. \tag{3.1.22}$$

with the condition $\kappa_0 = k_1\ell_2 - \ell_1 k_2 \neq 0$. When we choose $k_1 = m$, $\ell_1 = mc - d$, the mentioned directional derivative becomes

$$m\frac{\partial}{\partial t} + (mc-d)\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi}. \tag{3.1.23}$$

With this transformation, we write (3.1.1) as

$$\begin{aligned}i\Big(\frac{-\ell_1}{\kappa_0}\Big)\psi_\eta + a\psi_{xx} + \frac{\varepsilon_0 k_2^2}{\kappa_0^2}\psi_{\xi\xi} + \frac{\varepsilon_0 k_1^2}{\kappa_0^2}\psi_{\eta\eta} - \frac{2\varepsilon_0 k_1 k_2}{\kappa_0^2}\psi_{\xi\eta}&\\ + i\Big(\frac{\ell_2}{\kappa_0}\Big)\psi_\xi = (b-wm)|\psi^2|\psi - w\psi\varphi_\xi.&\end{aligned} \tag{3.1.24}$$

This way, the term $\varphi_{\xi\eta}$ appearing on the left hand side of (3.1.1) is eliminated if we choose $k_2 = d\sqrt{m}$, $\ell_2 = \sqrt{m}(cd-1)$ and we obtain

$$\varphi_{xx} + \frac{1}{d^2-m}(\varphi_{\xi\xi} - \varphi_{\eta\eta}) = (|\psi|^2)_\xi \tag{3.1.25}$$

with the condition $\kappa = \sqrt{m}(d^2-m) \neq 0$. Differentiating (3.1.25) with respect to ξ once and replacing $\varphi_\xi = \phi$, we consider (3.1.24) and (3.1.25) in the form

$$ia_1\psi_t + a\psi_{xx} + a_2\psi_{yy} + a_3\psi_{tt} + a_4\psi_{ty} + ia_5\psi_y = (b - wm)|\psi^2|\psi - w\psi\phi, \tag{3.1.26}$$

$$\phi_{xx} + \frac{1}{d^2-m}(\phi_{yy} - \phi_{tt}) = (|\psi|^2)_{yy} \tag{3.1.27}$$

with

$$\begin{aligned} a_1 &= \frac{d-mc}{(d^2-m)\sqrt{m}}, \quad a_2 = \frac{\varepsilon_0 d^2}{(d^2-m)^2}, \quad a_3 = \frac{\varepsilon_0 m}{(d^2-m)^2}, \\ a_4 &= -\frac{2\varepsilon_0 d\sqrt{m}}{(d^2-m)^2}, \quad a_5 = \frac{dc-1}{d^2-m}. \end{aligned} \tag{3.1.28}$$

The symmetries are generated by the vector fields

$$\begin{aligned} &\bar{\mathscr{X}}_1 = \partial_t, \quad \bar{\mathscr{X}}_2 = \partial_x, \quad \bar{\mathscr{X}}_3 = \partial_y, \quad \bar{\mathscr{X}}_4 = -v\partial_u + u\partial_v, \\ &\bar{\mathscr{X}}_5 = (dt + \sqrt{m}y)(-v\partial_u + u\partial_v) + \frac{1}{w\sqrt{m}}\partial_\phi, \\ &\bar{\mathscr{X}}_6 = (dt + \sqrt{m}y)^2(-v\partial_u + u\partial_v) + \frac{2(dt+\sqrt{m}y)}{w\sqrt{m}}\partial_\phi \end{aligned} \tag{3.1.29}$$

with the nonzero commutations being

$$\begin{aligned} &[\bar{\mathscr{X}}_1, \bar{\mathscr{X}}_5] = d\bar{\mathscr{X}}_4, \qquad [\bar{\mathscr{X}}_1, \bar{\mathscr{X}}_6] = 2d\bar{\mathscr{X}}_5, \\ &[\bar{\mathscr{X}}_3, \bar{\mathscr{X}}_5] = \sqrt{m}\bar{\mathscr{X}}_4, \quad [\bar{\mathscr{X}}_3, \bar{\mathscr{X}}_6] = 2\sqrt{m}\bar{\mathscr{X}}_5. \end{aligned} \tag{3.1.30}$$

Proposition 3.1.2 *The BR/ZR system* (3.1.26)-(3.1.27) *admits a six-dimensional Lie algebra with the basis* $\{\bar{\mathscr{X}}_i\}$, $i = 1, \ldots, 6$.

3.1.3 DS Symmetry Algebra

It is clear that in the case of $m = 0$ BR/ZR system (3.1.1) turns into (2+1) DS system. Here, firstly, the Davey–Stewartson system of equations in the literature and its Lie algebra is given for the reader to make comparisons. Then, the Lie algebra of the DS system is obtained from the BR/ZR system is found and compared with the DS system in the literature. Following the work [23], we consider the Davey–Stewartson system in the form

$$\begin{aligned} i\psi_t + \psi_{xx} + \varepsilon_1 \psi_{yy} &= \varepsilon_2 |\psi|^2 \psi + \psi \phi, \\ \phi_{xx} + \delta_1 \phi_{yy} &= \delta_2 (|\psi|^2)_{yy}. \end{aligned} \tag{3.1.31}$$

This system was first derived in [14], appearing as their equation (2.24a,b). Considering $\psi = u + iv$, $\varepsilon_1 = \mp 1$, $\varepsilon_2 = \mp 1$, the Lie symmetry algebra of the DS system (3.1.31) is reported in [23] to be generated by

$$\bar{\mathscr{V}} = \bar{\mathscr{X}}(f) + \bar{\mathscr{Y}}(g) + \bar{\mathscr{Z}}(h) + \bar{\mathscr{W}}(m) \tag{3.1.32}$$

with

$$\begin{aligned} \bar{\mathscr{X}}(f) &= f(t)\partial_t + \frac{\dot{f}(t)}{2}(x\partial_x + y\partial_y - u\partial_u - v\partial_v - 2\phi\partial_\phi) \\ &\quad - \frac{x^2 + \varepsilon_1 y^2}{8}\ddot{f}(t)(v\partial_u - u\partial_v) - \frac{\dddot{f}(t)}{8}(x^2 + \varepsilon_1 y^2)\partial_\phi, \\ \bar{\mathscr{Y}}(g) &= g(t)\partial_x - \frac{x}{2}\dot{g}(t)(v\partial_u - u\partial_v) - \frac{x}{2}\ddot{g}(t)\partial_\phi, \\ \bar{\mathscr{Z}}(h) &= h(t)\partial_y - \frac{\varepsilon_1 y}{2}\dot{h}(t)(v\partial_u - u\partial_v) - \frac{\varepsilon_1 y}{2}\ddot{h}(t)\partial_\phi, \\ \bar{\mathscr{W}}(m) &= m(t)(v\partial_u - u\partial_v) + \dot{m}(t)\partial_\phi. \end{aligned} \tag{3.1.33}$$

where $g(t)$, $h(t)$, $m(t)$ are arbitrary functions and $f(t)$ is stated to satisfy

$$f(t) = \begin{cases} \text{arbitrary}, & \delta_1 = -\varepsilon_1 = \mp 1, \\ at^2 + bt + c, & \delta_1 \neq -\varepsilon_1. \end{cases} \tag{3.1.34}$$

One of the main results of [23] is that, exactly in the integrable case $\delta_1 = -\varepsilon_1$ of the DS system, the symmetry algebra of the DS system (3.1.31) is an infinite-dimensional Lie algebra of Kac–Moody–Virasoro type, generated by vector fields in (3.1.33) depending on four arbitrary functions of time. The algebra has the Levi decomposition $\bar{\mathscr{X}}(f) \uplus \{\bar{\mathscr{Y}}(g), \bar{\mathscr{Z}}(h), \bar{\mathscr{W}}(m)\}$ [23]. For the structural information on KMV algebras and further examples of PDEs enjoying the KMV algebra as the invariance algebra we can refer to [24].

Before deriving the system (3.1.31), the first set of equations Davey and Stewartson derive in their work [14], which are the equations (2.14)-(2.15) of [14], is a system of the form

$$i\psi_t + \psi_{xx} + \varepsilon_1 \psi_{yy} = \varepsilon_2 |\psi|^2 \psi + \psi \varphi_y, \tag{3.1.35a}$$

$$\varphi_{xx} + \delta_1 \varphi_{yy} = \delta_2 (|\psi|^2)_y \tag{3.1.35b}$$

where $\delta_1, \delta_2, \varepsilon_1$ and ε_2 are real constants. Transition from (3.1.35) to (3.1.31) is straightforward by differentiating (3.1.35b) with respect to y and replacing $\varphi_y = \phi$

in both of the equations. Denoting $\psi = u + iv$ and assuming $\varepsilon_1 = \mp 1$, $\varepsilon_2 = \mp 1$, we determined that the Lie symmetry algebra of (3.1.35) is generated by the vector field

$$\mathscr{V} = \mathscr{X}(\tau) + \mathscr{Y}(\alpha) + \mathscr{Z}(\beta) + \mathscr{W}(\mu) + \mathscr{R}(\gamma) + \mathscr{S}(\delta) \tag{3.1.36}$$

with

$$\begin{aligned}
\mathscr{X}(\tau) &= \tau(t)\partial_t + \frac{\dot{\tau}(t)}{2}(x\partial_x + y\partial_y - u\partial_u - v\partial_v - \varphi\partial_\varphi) \\
&\quad - \frac{x^2+\varepsilon_1 y^2}{8}\ddot{\tau}(t)(v\partial_u - u\partial_v) - \frac{\dddot{\tau}(t)}{8}(x^2 y + \frac{\varepsilon_1}{3}y^3)\partial_\varphi, \\
\mathscr{Y}(\alpha) &= \alpha(t)\partial_x - \frac{x}{2}\dot{\alpha}(t)(v\partial_u - u\partial_v) - \frac{xy}{2}\ddot{\alpha}(t)\partial_\varphi, \\
\mathscr{Z}(\beta) &= \beta(t)\partial_y - \frac{y}{2\varepsilon_1}\dot{\beta}(t)(v\partial_u - u\partial_v) + \frac{\ddot{\beta}(t)}{4\varepsilon_1}(\delta_1 x^2 - y^2)\partial_\varphi, \\
\mathscr{W}(\mu) &= \mu(t)(v\partial_u - u\partial_v) + y\dot{\mu}(t)\partial_\varphi, \\
\mathscr{R}(\gamma) &= x\gamma(t)\partial_\varphi, \\
\mathscr{S}(\delta) &= \delta(t)\partial_\varphi.
\end{aligned} \tag{3.1.37}$$

where $\alpha(t)$, $\beta(t)$, $\mu(t)$, $\gamma(t)$, $\delta(t)$ are arbitrary functions whereas the constants ε_1, δ_1 and the function $\tau(t)$ must satisfy

$$(\varepsilon_1 + \delta_1)\,\dddot{\tau}(t) = 0. \tag{3.1.38}$$

Therefore we have

$$\tau(t) = \begin{cases} \text{arbitrary}, & \delta_1 = -\varepsilon_1 = \mp 1, \\ \tau_2 t^2 + \tau_1 t + \tau_0, & \delta_1 \neq -\varepsilon_1. \end{cases} \tag{3.1.39}$$

For the generators (3.1.37), the nonzero commutations are

$$\begin{aligned}
&[\mathscr{X}(\tau_1), \mathscr{X}(\tau_2)] = \mathscr{X}(\tau_1\dot{\tau}_2 - \dot{\tau}_1\tau_2), && [\mathscr{X}(\tau), \mathscr{Y}(\alpha)] = \mathscr{Y}(\tau\dot{\alpha} - \frac{1}{2}\dot{\tau}\alpha), \\
&[\mathscr{X}(\tau), \mathscr{Z}(\beta)] = \mathscr{Z}(\tau\dot{\beta} - \frac{1}{2}\dot{\tau}\beta), && [\mathscr{X}(\tau), \mathscr{W}(\mu)] = \mathscr{W}(\tau\dot{\mu}), \\
&[\mathscr{X}(\tau), \mathscr{R}(\gamma)] = \mathscr{R}(\tau\dot{\gamma} + \dot{\tau}\gamma), && [\mathscr{X}(\tau), \mathscr{S}(\delta)] = \mathscr{S}(\tau\dot{\delta} + \frac{1}{2}\dot{\tau}\delta), \\
&[\mathscr{Y}(\alpha_1), \mathscr{Y}(\alpha_2)] = -\frac{1}{2}\mathscr{W}(\alpha_1\dot{\alpha}_2 - \dot{\alpha}_1\alpha_2), && [\mathscr{Z}(\beta_1), \mathscr{Z}(\beta_2)] = -\frac{\varepsilon_1}{2}\mathscr{W}(\beta_1\dot{\beta}_2 - \dot{\beta}_1\beta_2), \\
&[\mathscr{Y}(\alpha), \mathscr{Z}(\beta)] = \frac{1}{2}\mathscr{R}(\ddot{\alpha}\beta + \delta_1\varepsilon_1\alpha\ddot{\beta}), && [\mathscr{Y}(\alpha), \mathscr{R}(\gamma)] = \mathscr{S}(\alpha\gamma), \\
&[\mathscr{Z}(\beta), \mathscr{W}(\mu)] = \mathscr{S}(\beta\dot{\mu}).
\end{aligned} \tag{3.1.40}$$

We can summarize this finding as follows .

Proposition 3.1.3 *When* $\delta_1 = -\varepsilon_1 = \mp 1$*, the symmetry algebra of the DS system* (3.1.35) *is an infinite-dimensional Kac–Moody–Virasoro algebra generated by the vector fields given in* (3.1.37)*, including 6 arbitrary functions of time. The algebra has the Levi decomposition*

$$\{\mathscr{X}(\tau)\} \uplus \{\mathscr{Y}(\alpha), \mathscr{Z}(\beta), \mathscr{W}(\mu), \mathscr{R}(\gamma), \mathscr{S}(\delta)\}. \tag{3.1.41}$$

Remark 3.1.1 *The Lie algebra of the DS system in* (3.1.31) *includes 4 arbitrary functions of time if* $\delta_1 = -\varepsilon_1$*, and 3 arbitrary functions of time if* $\delta_1 \neq -\varepsilon_1$*. The Lie algebra of the DS system in* (3.1.35) *includes 6 arbitrary functions of time if* $\delta_1 = -\varepsilon_1$*, and 5 arbitrary functions of time if* $\delta_1 \neq -\varepsilon_1$*; hence richer than that of* (3.1.31).

Remark 3.1.2 *The symmetry algebra of DS system is richer than the symmetry algebra of the BR/ZR system.*

3.2 Exact Solutions

A solution of the form

$$\psi = e^{i\theta}\Psi(\nu), \qquad \varphi = \Phi(\nu), \quad \nu = \nu_1 x + \nu_2 y + \nu_0 t, \quad \theta = \theta_1 x + \theta_2 y + \theta_0 t, \tag{3.2.1}$$

where ν_i, θ_i, $i = 0, 1, 2$ are real constants [32], reduces the BR/ZR system in (3.1.1) to the system of nonlinear ordinary differential equations

$$\begin{aligned} (a\nu_1^2 + \varepsilon_0 \nu_2^2)\Psi'' + i(2a\nu_1\theta_1 + 2\varepsilon_0\nu_2\theta_2 + \nu_0)\Psi' - (a\theta_1^2 + \varepsilon_0\theta_2^2 + \theta_0)\Psi \\ + (mw - b)\Psi^3 + w(m\nu_0 + mc\nu_2 - d\nu_2)\Psi\Phi' = 0, \end{aligned} \tag{3.2.2}$$

$$2(d\nu_2 - m\nu_0 - mc\nu_2)\Psi\Psi' + \left(\nu_1^2 + \nu_2^2 - m(\nu_0 + c\nu_2)^2\right)\Phi'' = 0 \tag{3.2.3}$$

which requires

$$2a\nu_1\theta_1 + 2\varepsilon_0\nu_2\theta_2 + \nu_0 = 0. \tag{3.2.4}$$

Integration of second equation one time and taking the constant of integration to be zero, we obtain

$$\Phi' = K\Psi^2, \qquad K = \frac{m\nu_0 + (mc - d)\nu_2}{\nu_1^2 + \nu_2^2 - m(\nu_0 + c\nu_2)^2}. \tag{3.2.5}$$

Substituting Φ' into first equation we obtain

$$\Psi'' = 2A\Psi^3 + B\Psi \tag{3.2.6}$$

where

$$A = \frac{b - mw - wK(m\nu_0 + (mc-d)\nu_2)}{2(a\nu_1^2 + \varepsilon_0\nu_2^2)}, \qquad B = \frac{a\theta_1^2 + \varepsilon_0\theta_2^2 + \theta_0}{a\nu_1^2 + \varepsilon_0\nu_2^2}. \tag{3.2.7}$$

Multiplying both sides of (3.2.8) by Ψ', we obtain

$$(\Psi')^2 = A\Psi^4 + B\Psi^2 + C \tag{3.2.8}$$

where C is an arbitrary constant.

Having obtained (3.2.8), we are going to produce several types of solutions. First of all, by a scaling $\Psi = \alpha_1 p$, $\nu = \beta_1 q$, (3.2.8) can be converted to a specific canonical form that yields elliptic functions. We first assume $\Delta = B^2 - 4AC > 0$ and present three different cases.

(i) When $A > 0$, $B < 0$, $C > 0$, for $\alpha_1^2 = C\beta_1^2$ and $\beta_1^2 = -\dfrac{B + \sqrt{B^2 - 4AC}}{2AC}$, (3.2.8) is transformed to

$$\left(\frac{dp}{dq}\right)^2 = (1-p^2)(1-\kappa^2p^2) \tag{3.2.9}$$

with $\kappa^2 = AC\beta_1^4$. We obtain

$$\Psi = \alpha_1 \mathrm{sn}\Big(\frac{1}{\beta_1}(\nu + \delta_0), \kappa\Big). \tag{3.2.10}$$

To illustrate this elliptic periodic solution at $t = 0$, we choose all the parameters in (3.1.1) and (3.2.1) equal to 1, except $\theta_0 = -5$, $\nu_0 = -4$, $C = 0.5$. Thus Figure 3.1 is the plot of

$$\Psi = 0.625953 \times \mathrm{sn}\Big(1.12965(x+y), 0.418871\Big). \tag{3.2.11}$$

(ii) In case $A < 0$, $C > 0$, with $\alpha_1^2 = -\dfrac{B + \sqrt{B^2 - 4AC}}{2A}$, $\kappa^2 = \dfrac{A\alpha_1^2}{2A\alpha_1^2 + B}$ and $\beta_1^2 = -\dfrac{\kappa^2}{A\alpha_1^2}$, (3.2.8) is transformed to

$$\left(\frac{dp}{dq}\right)^2 = (1-p^2)(\kappa'^2 + \kappa^2p^2) \tag{3.2.12}$$

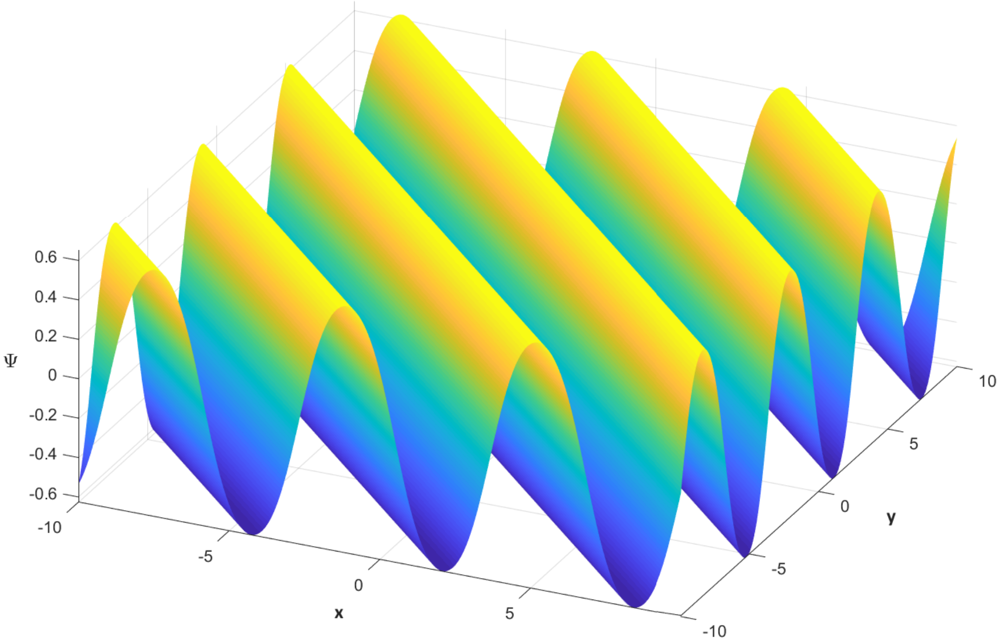


Figure 3.1 : The plot for Ψ of (3.2.10) for $t = 0$. The values of parameters are equal to 1 except $\theta_0 = -5$, $\nu_0 = -4$, $C = 0.5$.

with $\kappa'^2 = 1 - \kappa^2$. Then we get

$$\Psi = \alpha_1 \mathrm{cn}\Big(\frac{1}{\beta_1}(\nu + \delta_0), \kappa\Big). \tag{3.2.13}$$

(iii) When $A, B, C > 0$, for $\alpha_1^2 = C\beta_1^2$ and $\beta_1^2 = \dfrac{B - \sqrt{B^2 - 4AC}}{2AC}$, (3.2.8) is transformed to

$$\left(\frac{dp}{dq}\right)^2 = (1 + p^2)(1 + \kappa'^2 p^2) \tag{3.2.14}$$

with $\kappa'^2 = AC\beta_1^4$. We obtain

$$\Psi = \alpha_1 \mathrm{tn}\Big(\frac{1}{\beta_1}(\nu + \delta_0), \kappa\Big). \tag{3.2.15}$$

There are several possible types of elliptic solutions that can be obtained from (3.2.8) through a similar treatment yet not to be included here. Furthermore, one can also obtain trigonometric and hyperbolic type solutions, which can also be recovered as the limiting cases of the elliptic solutions family. In special, when $C = 0$ and $B \neq 0$ in (3.2.8), the equation reduces to

$$\frac{d\Psi}{\Psi\sqrt{A\Psi^2 + B}} = \varepsilon d\nu \tag{3.2.16}$$

where $\varepsilon = \mp 1$. We can evaluate this integral in the following three different cases.

(iv) In case $A > 0$ and $B > 0$,we obtain

$$\Psi(\nu) = -\varepsilon\sqrt{\frac{B}{A}}\,\operatorname{csch}\Big(\sqrt{B}(\nu+\delta_0)\Big), \tag{3.2.17}$$

$$\Phi(\nu) = -\frac{K\sqrt{B}}{A}\coth\Big(\sqrt{B}(\nu+\delta_0)\Big)+\Phi_0. \tag{3.2.18}$$

(v) When $A > 0$ and $B < 0$, we obtain

$$\Psi(\nu) = \sqrt{-\frac{B}{A}}\,\sec\Big(\sqrt{-B}(\nu+\delta_0)\Big), \tag{3.2.19}$$

$$\Phi(\nu) = \frac{K\sqrt{-B}}{A}\tan\Big(\sqrt{-B}(\nu+\delta_0)\Big)+\Phi_0. \tag{3.2.20}$$

(vi) If $A < 0$ and $B > 0$, we obtain

$$\Psi(\nu) = \sqrt{-\frac{B}{A}}\,\operatorname{sech}\Big(\sqrt{B}(\nu+\delta_0)\Big), \tag{3.2.21}$$

$$\Phi(\nu) = -\frac{K\sqrt{B}}{A}\tanh\Big(\sqrt{B}(\nu+\delta_0)\Big)+\Phi_0. \tag{3.2.22}$$

Remark 3.2.1 *For the (1+1)-dimensional case, [10] and [11] present an exact solution of a similar form to (vi) and in [11], there are also numerical investigations based on that solution. To the best of our knowledge, the solution* (3.2.21)-(3.2.22), *plotted in Figure 3.2, appear the first time in the literature for the (2+1)-dimensional BR/ZR system. We think this solution might also be useful for the kind of numerical investigations done for a stability analysis appearing in Chapter 6 of [11].*

To illustrate the solution in (3.2.21)-(3.2.22), we choose all the parameters in (3.1.1) and (3.2.1) equal to 1, except $w = 3$ and $\theta_2 = -3/2$. Therefore in Figure 3.2 we plot

$$\Psi = \sqrt{17}\,\operatorname{sech}\Big(\sqrt{\frac{17}{8}}(x+y)\Big), \quad \Phi = -\sqrt{34}\tanh\Big(\sqrt{\frac{17}{8}}(x+y)\Big). \tag{3.2.23}$$

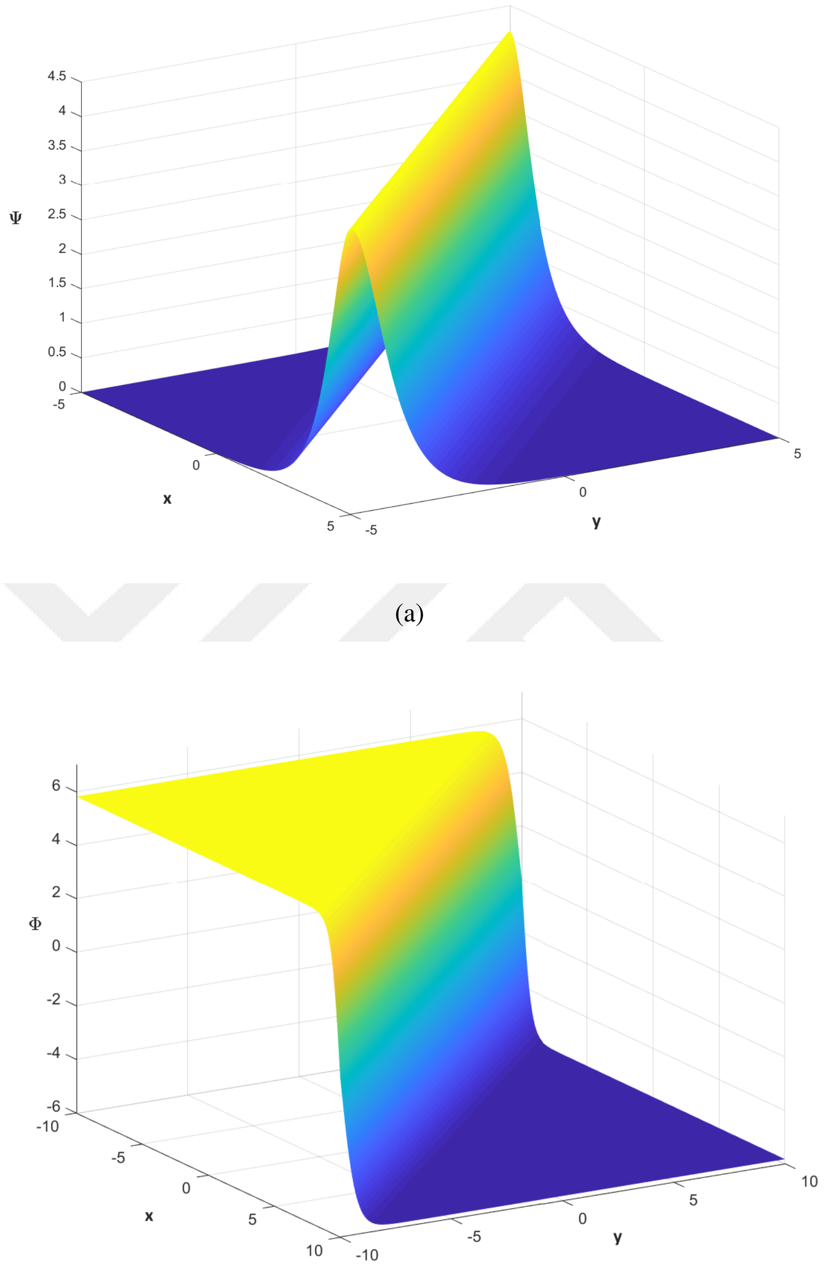
If $\Delta = B^2 - 4AC = 0$, then (3.2.8) takes the form

$$\frac{d\Psi}{2A\Psi^2+B} = \frac{\varepsilon}{2\sqrt{A}}\,d\nu \tag{3.2.24}$$

where $\varepsilon = \mp 1$.

(vii) If $B > 0$, we find

$$\Psi(\nu) = \varepsilon\sqrt{\frac{B}{2A}}\tan\big(\sqrt{\frac{B}{2}}(\nu+\delta_0)\big) \tag{3.2.25}$$



(a)

(b)

Figure 3.2 : Plots for (a) (3.2.21), (b) (3.2.22) for $t = 0$. The values of parameters are equal to 1 except $w = 3$ and $\theta_2 = -3/2$.

where A and C are positive, δ_0 is integration constant. We integrate (3.2.5) and find

$$\Phi(\nu) = \frac{K\sqrt{B}}{A\sqrt{2}} \tan\big(\sqrt{\frac{B}{2}}(\nu+\delta_0)\big) - \frac{BK}{2A}\ \nu + \Phi_0. \tag{3.2.26}$$

(viii) If $B < 0$, we obtain that

$$\Psi = -\varepsilon\sqrt{\frac{-B}{2A}} \tanh\Big[\sqrt{-\frac{B}{2}}(\nu+\delta_0)\Big], \tag{3.2.27}$$

$$\Phi = -\frac{K\sqrt{-B}}{A\sqrt{2}} \tanh\Big[\sqrt{-\frac{B}{2}}(\nu+\delta_0)\Big] - \frac{BK}{2A}\ \nu + \Phi_0. \tag{3.2.28}$$

3.2.1 A Lump Type Stationary Solution

Following the work [33], we suggest a solution of the form

$$\psi(x,y,t) = \frac{1}{\Omega(x,y)}, \qquad \varphi(x,y,t) = \lambda_3 \frac{\partial_y \Omega(x,y)}{\Omega(x,y)} \tag{3.2.29}$$

to (3.1.1), where $\Omega(x,y) = \lambda_1 x^2 + \lambda_2 y^2 + 1$. Under some restrictions for a and b, we determine $\lambda_1, \lambda_2, \lambda_3$ as

$$\lambda_1 = \frac{mw-b}{8a}, \qquad \lambda_2 = \frac{b-mw}{8\varepsilon_0}, \qquad \lambda_3 = \frac{4\varepsilon_0}{(mc-d)w} \tag{3.2.30}$$

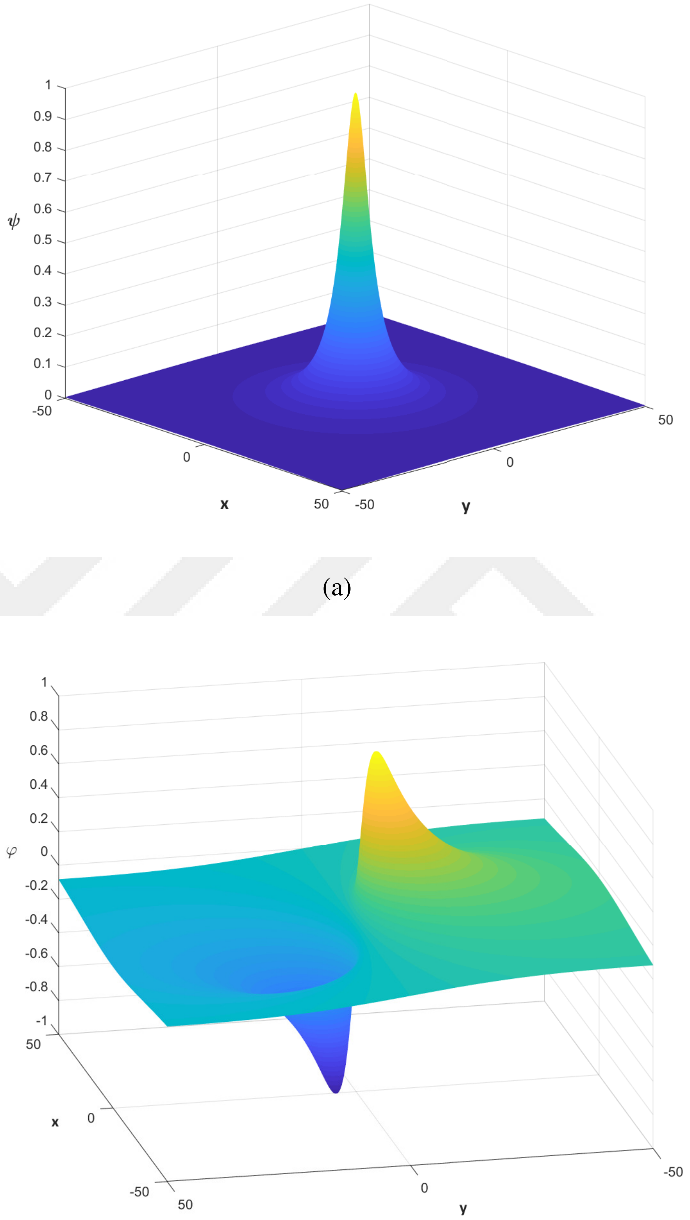
when

$$a = \frac{\varepsilon_0}{mc^2-1}, \qquad b = \frac{(c^2m^2-d^2-2m+2mcd)w}{2(mc^2-1)}. \tag{3.2.31}$$

The solution (3.2.29) can be used as an initial condition to obtain the development of a lump profile with numerical schemes. We plot this stationary solution for the values of the parameters $d = \varepsilon_0 = w = 1$ and $c = 1/4$, $m = 1/6$; that is, explicitly,

$$\psi(x,y,t) = \frac{1}{1+0.0574x^2+0.0580y^2}, \quad \varphi(x,y,t) = -\frac{0.4842y}{1+0.0574x^2+0.0580y^2} \tag{3.2.32}$$

is pictured in Figure 3.3.



(a)

(b)

Figure 3.3 : Plots of (3.2.32). The values of parameters are $d = \varepsilon_0 = w = 1$ and $c = 1/4$, $m = 1/6$.

4. 3D BENNEY–ROSKES/ZAKHAROV–RUBENCHIK SYSTEM

In Chapter 3, we investigated the Lie algebraic structure of the 2D BR/ZR system and presented some analytical solutions of periodic and (line) solitary type. A similar analysis can be performed in the 3D case. Therefore, in this Chapter, we investigate the Lie symmetry algebra of the 3D BR/ZR system and after that we employ the traveling wave ansatz, which converts the nonlinear PDEs into nonlinear ordinary differential equations to obtain exact solutions to the 3D BR/ZR system.

4.1 Lie Symmetry Algebra

When $n = 3$ in the system (3.0.5) becomes

$$\begin{aligned} &i\psi_t + a(\psi_{xx} + \psi_{yy}) + \varepsilon\psi_{zz} = (b - wm)|\psi|^2\psi - w\psi\left\{m\frac{\partial}{\partial t} + (mc-d)\frac{\partial}{\partial z}\right\}\varphi, \\ &\varphi_{xx} + \varphi_{yy} + (1 - mc^2)\varphi_{zz} - m\varphi_{tt} - 2mc\varphi_{zt} = \left\{m\frac{\partial}{\partial t} + (mc-d)\frac{\partial}{\partial z}\right\}|\psi|^2. \end{aligned} \tag{4.1.1}$$

Separating $\psi = u + iv$, we get

$$\begin{aligned} &u_t + a(v_{xx} + v_{yy}) + \varepsilon v_{zz} = (b - wm)(u^2 + v^2)v - vw\Big(m\varphi_t + (mc-d)\varphi_z\Big), \\ &-v_t + a(u_{xx} + u_{yy}) + \varepsilon u_{zz} = (b - wm)(u^2 + v^2)u - uw\Big(m\varphi_t + (mc-d)\varphi_z\Big), \\ &\varphi_{xx} + \varphi_{yy} + (1 - mc^2)\varphi_{zz} - m\varphi_{tt} - 2mc\varphi_{zt} = m(2uu_t + 2vv_t) + (mc-d)(2uu_z + 2vv_z). \end{aligned} \tag{4.1.2}$$

The Lie symmetry algebra of the BR/ZR system (4.1.1) is generated by the infinitesimal,

$$V = \tau_0\partial_t + (\xi_0 + \eta_0 y)\partial_x + (\nu_0 - \eta_0 x)\partial_y + \upsilon_0\partial_z + g(t)(u\partial_v - v\partial_u) + \left[\frac{g(t)}{mw} + J(x, y, \kappa)\right]\partial_\varphi \tag{4.1.3}$$

where τ_0, ξ_0, η_0, μ_0, ν_0, υ_0 are constants and

$$\kappa = mz - (mc - d)t \tag{4.1.4}$$

with $g(t)$ and $J(x, y, \kappa)$ satisfying

$$J_{xx} + J_{yy} + m(m - d^2)J_{\kappa\kappa} = \frac{1}{w}g''(t). \tag{4.1.5}$$

It is clearly seen that the right hand side of this equation depends only on t, and its left hand side depends on x, y and κ. Therefore, the sides of this equation are equal to a constant, say, c_0.

$$J_{xx}+J_{yy}+m(m-d^2)J_{\kappa\kappa}=\frac{1}{w}g''(t)=c_0. \tag{4.1.6}$$

Hence we have $g(t)=\frac{c_0 w}{2}t^2+g_1 t+g_0$, where g_0, g_1 are arbitrary constants. Setting $J(x,y,\kappa)=\Gamma(x,y,\kappa)+\frac{c_0}{2}x^2$ we have

$$\Gamma_{xx}+\Gamma_{yy}+m(m-d^2)\Gamma_{\kappa\kappa}=0. \tag{4.1.7}$$

When $m=d^2$, $\Gamma(x,y,\kappa)$ satisfies $\Gamma_{xx}+\Gamma_{yy}=0$. If $m-d^2>0$, (4.1.7) can be converted to the three-dimensional Laplace equation $\Gamma_{xx}+\Gamma_{yy}+\Gamma_{\tilde{\kappa}\tilde{\kappa}}=0$ by the scaling $\tilde{\kappa}=\kappa/\sqrt{m(m-d^2)}$. In case $m-d^2<0$, Eq. (4.1.7) is a (2+1)-dimensional wave equation and by the scaling $\tilde{\kappa}=\kappa/\sqrt{m(d^2-m)}$ it becomes $\Gamma_{\tilde{\kappa}\tilde{\kappa}}=\Gamma_{xx}+\Gamma_{yy}$.

We see that the Lie algebra is spanned by

$$\begin{aligned}
&\mathscr{X}_1=\partial_t, \quad \mathscr{X}_2=\partial_x, \quad \mathscr{X}_3=\partial_y, \quad \mathscr{X}_4=\partial_z,\\
&\mathscr{X}_5=y\partial_x-x\partial_y,\\
&\mathscr{X}_6=u\partial_v-v\partial_u+\frac{1}{mw}\partial_\varphi,\\
&\mathscr{X}_7=t(u\partial_v-v\partial_u)+\frac{t}{mw}\partial_\varphi,\\
&\mathscr{X}_8=t^2(u\partial_v-v\partial_u)+\Big(\frac{t^2}{mw}+\frac{x^2}{w}\Big)\partial_\varphi,\\
\mathscr{S}(\Gamma)&=\Gamma(x,y,\kappa)\partial_\varphi,
\end{aligned} \tag{4.1.8}$$

where $\Gamma(x,y,\kappa)$ is an arbitrary function satisfying Eq. (4.1.7).

The nonzero commutation relations are

$$\begin{array}{lll}
[\mathscr{X}_1,\mathscr{X}_7]=\mathscr{X}_6, & [\mathscr{X}_1,\mathscr{X}_8]=2\mathscr{X}_7, & [\mathscr{X}_3,\mathscr{X}_5]=\mathscr{X}_2,\\
[\mathscr{X}_2,\mathscr{X}_8]=\mathscr{S}(\Gamma)|_{\Gamma=2x/w}, & [\mathscr{X}_2,\mathscr{S}(\Gamma)]=\mathscr{S}(\Gamma_x), & [\mathscr{X}_3,\mathscr{S}(\Gamma)]=\mathscr{S}(\Gamma_y),\\
[\mathscr{X}_4,\mathscr{S}(\Gamma)]=m\mathscr{S}(\Gamma_\kappa), & [\mathscr{X}_5,\mathscr{X}_8]=\mathscr{S}(\Gamma)|_{\Gamma=2xy/w}, & [\mathscr{X}_5,\mathscr{X}_2]=\mathscr{X}_3,\\
[\mathscr{X}_5,\mathscr{S}(\Gamma)]=\mathscr{S}(y\Gamma_x-x\Gamma_y), & [\mathscr{X}_1,\mathscr{S}(\Gamma)]=(d-mc)\mathscr{S}(\Gamma_\kappa). &
\end{array} \tag{4.1.9}$$

Let us mention that if $\Gamma(x,y,\kappa)$ is an arbitrary function that satisfies (4.1.7), so are Γ_x, Γ_y, Γ_κ and $y\Gamma_x-x\Gamma_y$.

Proposition 4.1.1 *The Lie algebra of the (3+1)-dimensional Benney–Roskes/Zakharov–Rubenchik system* (4.1.1) *is infinite-dimensional. It is spanned by the vector fields* (4.1.8).

4.2 Traveling Wave Solutions

In order to obtain traveling wave solutions, we consider the three dimensional Abelian subalgebra

$$\mathscr{L}=\langle \mathscr{X}_2-e_1\mathscr{X}_1, \mathscr{X}_3-e_2\mathscr{X}_1, \mathscr{X}_4-e_3\mathscr{X}_1\rangle=\langle \partial_x-e_1\partial_t, \partial_y-e_2\partial_t, \partial_z-e_3\partial_t\rangle \tag{4.2.1}$$

of the full symmetry algebra of the equation, where $e_1, e_2, e_3 \in \mathbb{R}$. This subalgebra generates the 3-parameter Lie group of transformations

$$(x,y,z,t)\to(x+\varepsilon_1, y+\varepsilon_2, z+\varepsilon_3, t-e_1\varepsilon_1-e_2\varepsilon_2-e_3\varepsilon_3) \tag{4.2.2}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}$ are the group parameters, with the identity transformation in the dependent variables. The solutions of (4.1.1) which are invariant under this group will have the following traveling wave form

$$\psi=e^{iQ(\xi)}\Psi(\xi), \qquad \varphi=\Phi(\xi), \qquad \xi=e_1x+e_2y+e_3z+t, \tag{4.2.3}$$

where $\Psi(\xi)$ and $Q(\xi)$ are real functions. BR/ZR system reduces to the system of nonlinear ordinary differential equations

$$A\Psi''-A\Psi(Q')^2-\Psi Q'+wB\Psi\Phi'+(mw-b)\Psi^3=0, \tag{4.2.4}$$

$$A(\Psi Q''+2\Psi'Q')+\Psi'=0, \tag{4.2.5}$$

$$2B\Psi\Psi'-C\Phi''=0 \tag{4.2.6}$$

where

$$A=a(e_1^2+e_2^2)+\varepsilon e_3^2, \quad B=m+(mc-d)e_3, \quad C=e_1^2+e_2^2+e_3^2-m(1+ce_3)^2. \tag{4.2.7}$$

Assuming $A\neq 0$, we multiply (4.2.5) by Ψ and integrate to obtain

$$Q'=\frac{Q_0}{\Psi^2}-\frac{1}{2A} \tag{4.2.8}$$

where Q_0 is an arbitrary constant and Eq. (4.2.6) readily gives

$$\Phi'=\frac{B}{C}\Psi^2+\Phi_0 \tag{4.2.9}$$

with the arbitrary constant Φ_0. When Q' and Φ are substituted in (4.2.4), we see that Ψ satisfies

$$\Psi'' = \frac{1}{A}\Big(b - mw - w\frac{B^2}{C}\Big)\Psi^3 - \Big(\frac{1}{4A^2} + \Phi_0 w\frac{B}{A}\Big)\Psi + \frac{Q_0^2}{\Psi^3}. \tag{4.2.10}$$

Multiplying by Ψ', this can be integrated to

$$(\Psi')^2 = \frac{1}{2A}\Big(b - mw - w\frac{B^2}{C}\Big)\Psi^4 - \Big(\frac{1}{4A^2} + \Phi_0 w\frac{B}{A}\Big)\Psi^2 - \frac{Q_0^2}{\Psi^2} + \bar{K}. \tag{4.2.11}$$

where $\bar{K}$ is an arbitrary constant of integration. If we change the dependent variable by $\Psi = \sqrt{H}$, $H > 0$, we obtain

$$(H')^2 = \bar{A}H^3 + \bar{B}H^2 + 4\bar{K}H - 4Q_0^2 \tag{4.2.12}$$

with

$$\bar{A} = \frac{2}{A}\Big(b - mw - w\frac{B^2}{C}\Big), \qquad \bar{B} = -\frac{1}{A^2} - 4\Phi_0 w\frac{B}{A} \tag{4.2.13}$$

In general, the solutions of this equation can be obtained in terms of elliptic functions, which reduce to elemantary functions for the special cases of the parameters. These vast amount of solutions follow directly through the transformations available in [34]. However, we keep that analysis out of the scope of this thesis and concentrate on a rather simplified reduction.

We proceed with a slightly different ansatz than the form (4.2.3) of the group-invariant solutions and set

$$\psi = e^{i\tilde{Q}}\Psi(\xi), \qquad \varphi = \Phi(\xi), \qquad \tilde{Q} = q_1x + q_2y + q_3z + t, \qquad \xi = e_1x + e_2y + e_3z + t \tag{4.2.14}$$

in (4.1.1), where e_i, q_i, $i = 1,2,3$ are real constants. BR/ZR system reduces to the system of nonlinear ordinary differential equations

$$\begin{aligned}\big[a(e_1^2 + e_2^2) + \varepsilon e_3^2\big]\Psi'' - \big[a(q_1^2 + q_2^2) + \varepsilon q_3^2 + 1\big]\Psi + (mw - b)\Psi^3 &\\ + w[m + (mc - d)e_3]\Psi\Phi' = 0, &\qquad (4.2.15)\\ \big[2a(e_1q_1 + e_2q_2) + 2\varepsilon e_3q_3 + 1\big]\Psi' = 0, &\qquad (4.2.16)\\ 2\big[m + (mc - d)e_3\big]\Psi\Psi' - \big[e_1^2 + e_2^2 + e_3^2 - m(1 + ce_3)^2\big]\Phi'' = 0 &\qquad (4.2.17)\end{aligned}$$

which requires

$$2a(e_1q_1+e_2q_2)+2\varepsilon e_3q_3+1=0 \tag{4.2.18}$$

so that $\Psi' \neq 0$. Integrating (4.2.17) once, we obtain

$$\Phi' = \Phi_2\Psi^2+\Phi_1, \quad \Phi_2 = \frac{m+(mc-d)e_3}{e_1^2+e_2^2+e_3^2-m(1+ce_3)^2} \tag{4.2.19}$$

with the arbitrary constant Φ_1. Substituting Φ' into (4.2.15) we obtain

$$\Psi'' = 2M\Psi^3+L\Psi \tag{4.2.20}$$

where

$$M=\frac{b-mw-\Phi_2w(m+(mc-d)e_3)}{2(ae_1^2+ae_2^2+\varepsilon e_3^2)}, L=\frac{a(q_1^2+q_2^2)+\varepsilon q_3^2+1-\Phi_1w(m+(mc-d)e_3)}{a(e_1^2+e_2^2)+\varepsilon e_3^2}. \tag{4.2.21}$$

Multiplying both sides of the Eq. (4.2.20) by Ψ', we obtain

$$(\Psi')^2 = M\Psi^4+L\Psi^2+K \tag{4.2.22}$$

where K is an arbitrary constant.

As we stated for the treatment of (4.2.12), this equation can be integrated in terms of elliptic and elementary functions in various cases. Among them, we focus only on two cases, which we think will be interesting for the readers.

(A) When $K=0$ and $L\neq 0$ in (4.2.22), the equation reduces to

$$\frac{d\Psi}{\Psi\sqrt{M\Psi^2+L}} = \varepsilon d\xi \tag{4.2.23}$$

where $\varepsilon = \mp 1$. One can evaluate this integral in three different cases. Concentrating on the situation $M<0$ and $L>0$, upon integrating we get

$$\Psi(\xi) = \sqrt{-\frac{L}{M}}\,\mathrm{sech}\Big(\sqrt{L}(\xi+\delta_0)\Big), \tag{4.2.24}$$

$$\Phi(\xi) = -\frac{\Phi_2\sqrt{L}}{M}\tanh\Big(\sqrt{L}(\xi+\delta_0)\Big)+\Phi_1\xi+\Phi_0 \tag{4.2.25}$$

where Φ_1 and Φ_0 are arbitrary.

(B) When $\Delta = L^2 - 4MK = 0$, Eq.(4.2.22) takes the form

$$\frac{d\Psi}{2M\Psi^2 + L} = \frac{\varepsilon}{2\sqrt{M}}\, d\xi \tag{4.2.26}$$

where $\varepsilon = \mp 1$. If $L < 0$, we obtain

$$\Psi = -\varepsilon\sqrt{\frac{-L}{2M}}\tanh\left[\sqrt{-\frac{L}{2}}(\xi + \delta_0)\right], \tag{4.2.27}$$

$$\Phi = -\frac{\Phi_2\sqrt{-L}}{M\sqrt{2}}\tanh\left[\sqrt{-\frac{L}{2}}(\xi + \delta_0)\right] + \left(\Phi_1 - \frac{\Phi_2 L}{2M}\right)\xi + \Phi_0. \tag{4.2.28}$$

Let us remark that as Φ_1 is arbitrary, choosing $\Phi_1 = 0$ for (A) provides a sech-tanh couple of solution, whereas the choice $\Phi_1 = \Phi_2 L/(2M)$ gives a tanh-tanh couple for case (B), which therefore are bounded overall the 3D physical domain. Let us mention that we also obtained these hyperbolic profiles for the (2+1)-dimensional BR/ZR system in [5].

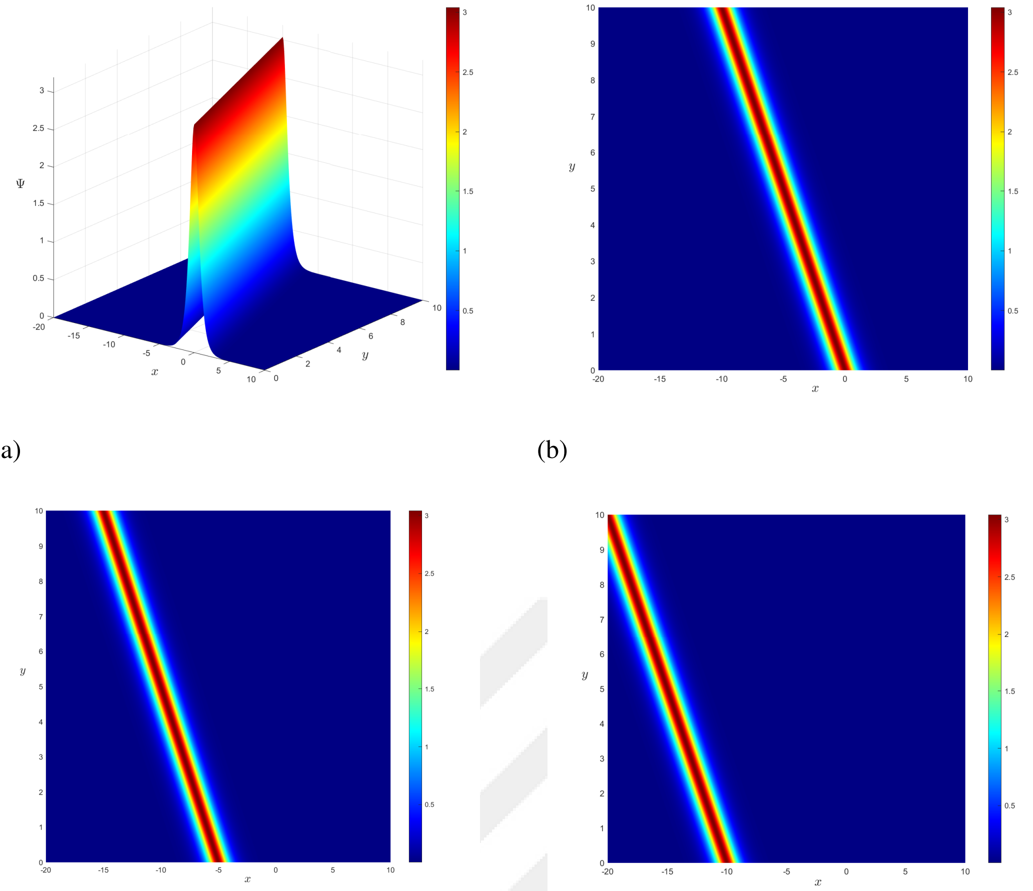
In the case $\Delta = L^2 - 4MK \neq 0$, integration of (4.2.22) yields exact solutions of elliptic type, which are periodic functions. Results in this direction can be found in [5]. Among this family of analytical solutions to the (3+1)-dimensional BR/ZR system, we would like to illustrate the couple (ψ, φ) that is obtained through (4.2.24) and (4.2.25). We choose the related constants as $a = b = c = \varepsilon = 1$, $w = d = 2$, $m = 1/2$, $e_1 = e_2 = e_3 = 1$, $q_1 = q_2 = 1$, $q_3 = -5/2$, $\delta_0 = \Phi_0 = \Phi_1 = 0$. We find $\Phi_2 = -1$, $M = -1/3$, $L = 37/12$. Therefore, explicitly, we plot

$$\Psi = \frac{\sqrt{37}}{2}\,\text{sech}\left[\sqrt{\frac{37}{12}}(x+y+z+t)\right], \tag{4.2.29}$$

$$\Phi = -\frac{\sqrt{111}}{2}\,\tanh\left[\sqrt{\frac{37}{12}}(x+y+z+t)\right]. \tag{4.2.30}$$

In Figure 4.1(a), we plot the magnitude Ψ of the ψ component of the modulation in the BR/ZR system (4.1.1); that is, we give the line soliton picture obtained when drawing (4.2.29) on the plane $z = 0$ when $t = 0$. Figures 4.1(b,c,d) are the contour plots of (4.2.29) on the plane $z = 0$ when $t = 0, 5, 10$, respectively, which well demonstrate the propagation of the line soliton.

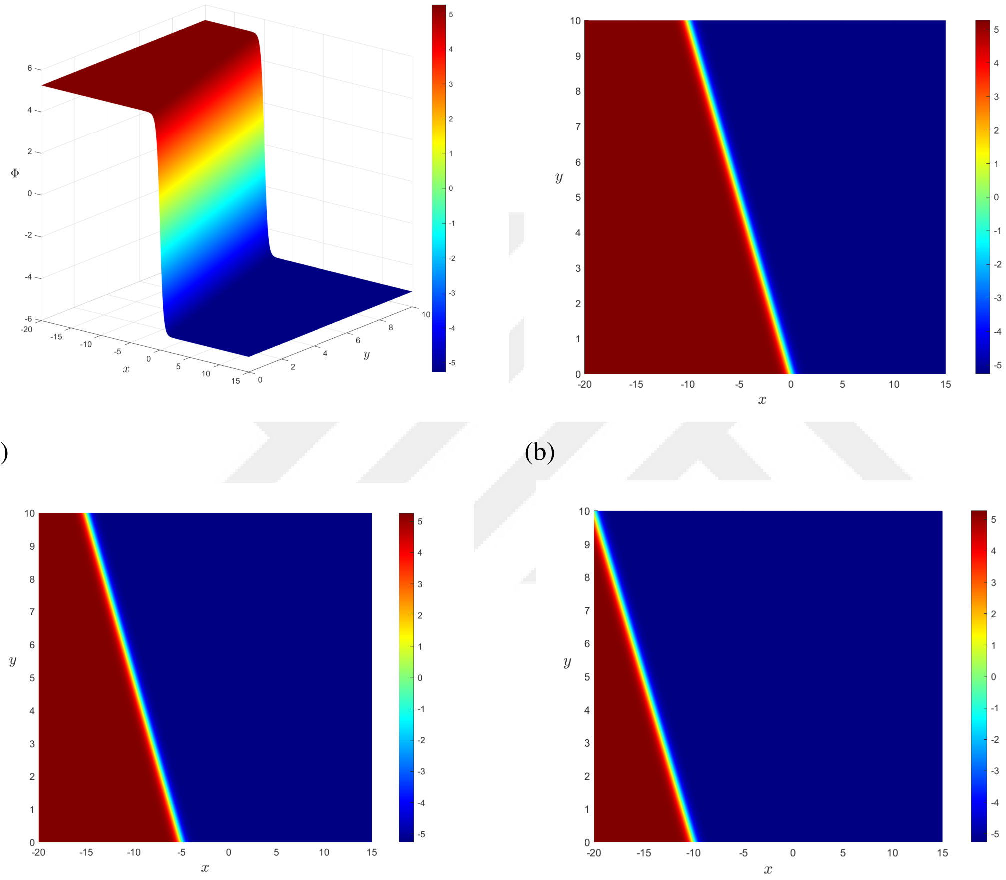
In Figure 4.2 we plot the φ component of the motion in the BR/ZR system. Φ of (4.2.30) is pictured in Figure 4.2(a) on the plane $z = 0$ for $t = 0$. Similarly, we provide contour plots of (4.2.30) on $z = 0$ for $t = 0, 5, 10$ in Figures 4.2(a,b,c).



(a) (b) (c) (d)

Figure 4.1 : Solution (4.2.29) on the plane $z = 0$; (a) $t = 0$, (b) $t = 0$, (c) $t = 5$, (d) $t = 10$.

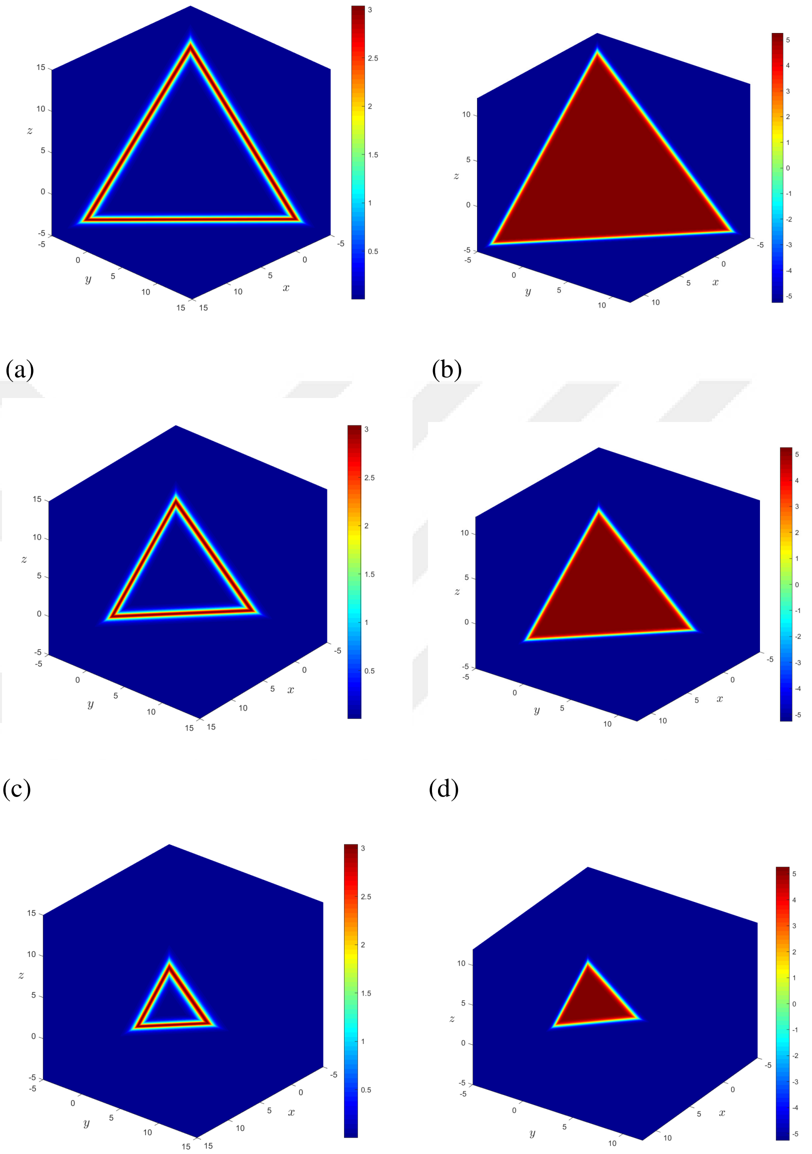
Clearly, Figures 4.1 and 4.2 illustrate the dynamics only on the plane $z = 0$. In the 3-dimensional spatial domain, to give a better illustration of the ongoing dynamics in the two-component modulation (ψ, φ) in the BR/ZR system, we picture the evolution of the solutions in (4.2.29) and (4.2.30) on the planes $x = -5$, $y = -5$ and $z = -5$ as contour plots. In Figures 4.3(a,c,e), we see the planes $x = -5$, $y = -5$ and $z = -5$ each of which contains a line soliton. Respectively, for the times $t = 0, 5, 10$, the separate figures in Figure 4.3(a,c,e) illustrate the propagation of the line soliton obtained by plotting Ψ of (4.2.29). The line solitons on the different planes are of the same shape due to the choice $e_1 = e_2 = e_3 = 1$. Similarly, on the same planes, the progress of Φ of (4.2.30) is seen as a kink profile in Figures 4.3(b,d,f).



(a) (b)

(c) (d)

Figure 4.2 : Solution (4.2.30) on the plane $z = 0$; (a) $t = 0$, (b) $t = 0$, (c) $t = 5$, (d) $t = 10$.



(a) (b)

(c) (d)

(e) (f)

Figure 4.3 : On the planes $x = -5$, $y = -5$ and $z = -5$, the solution (4.2.29) when (a) $t = 0$, (c) $t = 5$, (e) $t = 10$ and the solution (4.2.30) when (b) $t = 0$, (d) $t = 5$, (f) $t = 10$.

5. CONCLUSIONS

5.1 Conclusions

Through our analysis, in Chapter 3, the symmetry algebra of the 2D BR/ZR system is identified and compared with that of the Davey–Stewartson system. By a traveling wave ansatz, we obtained several exact solutions in terms of trigonometric, hyperbolic and elliptic functions. Besides the periodic the exact solutions, to our knowledge, we have obtained the result of a line soliton for the first time in the literature. We also obtained a lump-type stationary solution.

In Chapter 4, we determined the symmetry algebra of 3D BR/ZR system. It turns out to be infinite-dimensional. We succeeded in finding solutions in line soliton and hyperbolic type wave forms. We discovered wave components of the sech – tanh type that move as kinks and line solitons in two-dimensional cross-sections of space by focusing on traveling solutions.

We believe these solutions will draw the attentions of the community and they might support and motivate further research on the BR/ZR system.

5.2 Future Discussions

In this thesis, the Lie algebra of the BR system has been defined, but no study has been done on the reduction of the system. Group invariant solutions may provide further room for the exploration of exact solutions. The analytical solutions we provide are physically interesting. They can well serve as test solutions for numerical schemes developed for this class of equations. Further, they provide analytical initial profiles for numerical efforts that can be performed for the BR/ZR system. Therefore, further works may be planned based on the respects in available this thesis.

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