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FEN BİLİMLERİ ENSTİTÜSÜ**

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FINITE NILPOTENT AND SOLUBLE GROUPS

MATEMATİK ANABİLİM DALI

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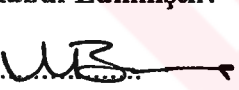
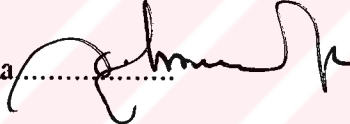

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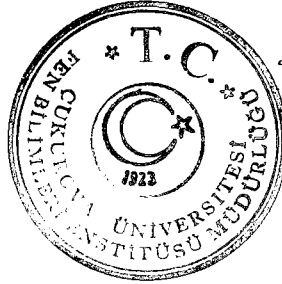
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ABSTRACT

MSc THESIS

FINITE NILPOTENT AND SOLUBLE GROUPS

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Group theory is not only one of the basic studies of Mathematics, but is practically applied in Physics and other important parts of Science. Finite groups have been studied for a long time and the classification of finite groups has been completed. With the help of this Thesis there will be a both theoretical and applied referance about finite nilpotent and soluble groups.

Keywords: Group, normal subgroup, group series, soluble, nilpotent.

ÖZ

YÜKSEK LİSANS TEZİ

SONLU NİLPOTENT VE ÇÖZÜLEBİLİR GRUPLAR

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Gruplar teorisi cebirin en temel çalışma alanlarından biri olmakla kalmayıp, matematiğin diğer dallarında ve matematik dışındaki (fizik v.b.) gibi bilim dallarında da önemli bir uygulama alanı bulmuştur. Sonlu gruplar uzun süreden beri üzerinde çalışılmış ve sonlu grupların büyük ölçüde sınıflandırılması tamamlanmıştır. Bu tezle birlikte sonlu nilpotent ve çözülebilir gruplar konusunda teorik ve uygulamalı bir kaynak ortaya çıkmış olacaktır.

Anahtar Kelimeler: Grup, normal altgrup, grup serileri, çözülebilir, nilpotent.

TEŐEKKÜR

Bu alıŐmayı yÖneten ve deęerli zamanını harcayarak yakın ilgi, yardım ve tecrübelerini esirgemeyen sayın hocam Prof. Dr. Melih BORAL'a, tÖm Matematik BÖlÖmÖ akademik personeline, aileme ve asistan arkadaşlarım Mine VARAT, Gonca GÖNGÖR ve Ali ÖZKURT'a teŐekkÖr ederim.



In the beginning of this thesis we will try to give the basic definitions and theorems about groups and also the proofs of some special theorems actually. Among the books and articles about group theory we will clarify the ones which are related with finite nilpotent and soluble groups and we will specify the problems which have to be especially solved in the same subject.

The first chapter will be composed of the basic definitions and theorems about finite nilpotent and soluble groups and in the second chapter; the solutions of the problems related with the important results in the first chapter will be given.

Specifically knowledge is presupposed of the notions of normal subgroups of a group, group series, soluble groups, derived subgroup of a group G , nilpotent groups, p -groups and Hall subgroups of a group. Also the selected problems about finite nilpotent and soluble groups will be helpful to understand the subject better and it will show how it's used practically in applied Mathematics.

In some books and articles 'Soluble Group' is called as 'Solvable Group', but they are the same in definition and properties. Thus it doesn't mean that they are different in their properties and structure.

1. FINITE NILPOTENT AND SOLUBLE GROUPS: MELIS MINISKER

We start with basic definitions on normal subgroups, factor groups and group series.

DEFINITION: Let G be a group. A subgroup $K \leq G$ is a normal subgroup, denoted by $K \triangleright G$, if $gKg^{-1} = K$ for every $g \in G$.

DEFINITION: Let K be a subgroup of G . $Kx = \{kx : k \in K\}$ is called a right coset of K . If $K \triangleright G$, then the cosets of K in G form a group called the factor group of G by K , denoted by G/K , of order $|G : K|$.

DEFINITION: Let G be a group. A normal series of G is a sequence of subgroups $G = G_0 \geq G_1 \geq G_2 \geq \dots \geq G_n = 1$ in which $G_{i+1} \triangleright G_i$ for all i . The factor groups of this series are the groups G_i/G_{i+1} for all $i = 0, 1, \dots, n-1$; the length of the normal series is the number of strict inclusions; that is the length is the number of nontrivial factor subgroups.

DEFINITION: A soluble series of a group G is a normal series all of whose factor groups are abelian. (G_i/G_{i+1} are abelian for all i). A group G is soluble if it has a soluble series.

THEOREM 1.1: Every subgroup of a soluble group is soluble.

PROOF: Let H be a subgroup of G . If $G = G_0 \geq G_1 \geq G_2 \geq \dots \geq G_n = 1$ is a soluble series, consider the series

$$H = H_0 \geq (H \cap G_1) \geq (H \cap G_2) \geq \dots \geq (H \cap G_n) = 1.$$

This is a normal series of H , for the second isomorphism theorem gives $H \cap G_{i+1} = (H \cap G_i) \cap G_{i+1} \triangleright H \cap G_i$ for all i . Now

$(H \cap G_i)/(H \cap G_{i+1}) \cong G_{i+1}(H \cap G_i)/G_{i+1} \leq G_i/G_{i+1}$; as G_i/G_{i+1} is abelian, so is any of its subgroups. Therefore H has a soluble series.

THEOREM 1.2: Every quotient of a soluble group is soluble.

THEOREM 1.3: If $H \triangleright G$ and if both H and G/H are soluble, then G is soluble.

PROOF: Suppose that H and G/H are soluble. Let $G/H \geq K_1^* \geq K_2^* \geq \dots \geq K_n^* = 1$ be a soluble series. By the correspondence theorem, we can construct a sequence looking like the beginning of a soluble series from G to H ; there are subgroups K_i (with $H \leq K_i$ and $K_i/H \cong K_i^*$) such that $G \geq K_1 \geq K_2 \geq \dots \geq K_n = H$, $K_{i+1} \triangleright K_i$, and $K_i/K_{i+1} (\cong K_i^*/K_{i+1}^*)$ is abelian. Since H is soluble, it has a soluble series;

$$H \geq H_1 \geq H_2 \geq \dots \geq H_m = 1.$$

If we combine these two sequences together at H , we obtain a soluble series for G .

DEFINITION: If H and K are groups, then their direct product, denoted by $H \times K$ is the group with elements all ordered pairs (h, k) , where $h \in H$ and $k \in K$, and with operation

$$(h, k)(h', k') = (hh', kk').$$

It's easy to check that $H \times K$ is a group; the identity is $(1, 1)$; the inverse of (h, k) is (h^{-1}, k^{-1}) .

COROLLARY: If H and K are soluble groups, then $H \times K$ is soluble.

PROOF: If $G = H \times K$, then $H \triangleright G$ and $G/H \cong K$.

DEFINITION: Let G be a group. G' is a subgroup of G defined as $G' = \{[x, y] = xyx^{-1}y^{-1} : x, y \in G\}$. G' is called the commutator subgroup of G .

DEFINITION: The higher commutator subgroups of G are defined inductively: $G^{(0)} = G$; $G^{(i+1)} = G^{(i)'}; that is $G^{(i+1)}$ is the commutator subgroup of $G^{(i)}$. The series $G = G^{(0)} \geq G^{(1)} \geq \dots$ is called the derived series of G .$

LEMMA: Let G be a soluble group and $G = G_0 \geq G_1 \geq G_2 \geq \dots G_n = 1$ be a soluble series of G . Then $G_i \geq G^{(i)}$ for all i .

THEOREM 1.4: A group G is soluble if and only if $G^{(n)} = 1$ for some n .

PROOF: Let $G = G_0 \geq G_1 \geq \dots \geq G_n = 1$ be a soluble series. By the above lemma, $G_i \geq G^{(i)}$ for all i . In particular, $1 = G_n \geq G^{(n)}$, and so $G^{(n)} = 1$.

Conversely if $G^{(n)} = 1$, then the derived series is a normal series; since it has abelian factor groups, it is a soluble series for G .

DEFINITION: A normal subgroup H of G is a minimal normal subgroup of

G if $H \neq 1$ and there's no normal subgroup K of G with $1 < K < H$.

DEFINITION: Let $H \leq G$. $C_G(H) = \{ x \in G: [x, h] = 1 \text{ for all } h \in H \}$ is the centralizer of H in G . We say that a group K centralizes H if $K \leq C_G(H)$; it's easy to see that K centralizes H if and only if $[H, K] = 1$.

If $x, y \in G$ and $[x, y] \in K$, where $K \triangleright G$, then x and y commute mod K ; that is, $xKyK = yKxK$ in G/K .

DEFINITION: We define characteristic subgroups $\gamma_i(G)$ of G by induction: $\gamma_1(G) = G$; $\gamma_{i+1}(G) = [\gamma_i(G), G]$. Notice that $\gamma_2(G) = [\gamma_1(G), G] = [G, G] = G' = G^{(1)}$. It's easy to check that $\gamma_{i+1}(G) \leq \gamma_i(G)$.

DEFINITION: The lower central series (or descending series) of G is the series $G = \gamma_1(G) \geq \gamma_2(G) \geq \dots$ (this need not be a normal series because it may not reach 1).

DEFINITION: The subgroups $\tau^i(G)$ of G are the characteristic subgroups of G defined by induction: $\tau^0(G) = 1$; $\tau^{i+1}(G)/\tau^i(G) = Z(G/\tau^i(G))$; that is, if $v_i: G \rightarrow G/\tau^i(G)$ is the natural map, then $\tau^{i+1}(G)$ is the inverse image of the center. Of course $\tau^1(G) = Z(G)$.

DEFINITION: The upper central series (or ascending central series) of G is $1 = \tau^0(G) \leq \tau^1(G) \leq \tau^2(G) \leq \dots$. When no confusion can occur, we may abbreviate $\tau^1(G)$ by τ^i and $\gamma_i(G)$ by γ_i .

THEOREM 1.5: If G is a group, then there is an integer c with $\tau^c(G) = G$

if and only if $\gamma_{c+1}(G) = 1$. Moreover in this case, $\gamma_{i+1}(G) \leq \tau^{c-i}(G)$ for all i .

DEFINITION: A group G is nilpotent if there's an integer c such that $\gamma_{c+1}(G) = 1$; the least such c is called the class of the nilpotent group G .

THEOREM 1.6: *i-*) Every nilpotent group G is solvable.

ii-) If $G \neq 1$ is nilpotent; then $Z(G) \neq 1$.

iii-) S_3 is a solvable group that is not nilpotent.

PROOF: *i-*) An easy induction shows that $G^{(i)} \leq \gamma_i(G)$ for all i . It follows that if $\gamma_{c+1}(G) = 1$, then $G^{(c+1)} = 1$; that is, if G is nilpotent (of class $\leq c$), then G is solvable. (with derived length $\leq c + 1$).

ii-) Assume that $G \neq 1$ is nilpotent of class c , so that $\gamma_{c+1}(G) = 1$ and $\gamma_c(G) \neq 1$.

1. By theorem 1.5, $1 \neq \gamma_c(G) \leq \tau^1(G) = Z(G)$.

iii-) The group $G = S_3$ is soluble and $Z(S_3) = 1$.

THEOREM 1.7: Every subgroup H of a nilpotent group G is nilpotent. Moreover if G is nilpotent of class c , then H is nilpotent of class $\leq c$.

PROOF: It is easily proved by induction that $H \leq G$ implies $\gamma_i(H) \leq \gamma_i(G)$ for all i . Therefore $\gamma_{c+1}(G) = 1$ forces $\gamma_{c+1}(H) = 1$.

THEOREM 1.8: Let G be a nilpotent group and let the class of G be c . Then for every subgroup H of G , there's a series of length c from H to G . In particular, every subgroup of G is subnormal in G .

DEFINITION: Let G be a group and let $K \leq G$. K is called the maximal subgroup of G if there's no subgroup H of G such that $K < H < G$.

DEFINITION: Let G be a group and let $H \leq G$. Then $N_G(H) = \{g \in G : gHg^{-1} = H\}$ is called the normalizer of H in G .

DEFINITION: Let G be a group. The intersection of all maximal subgroups of G is a subgroup of G which is called the Frattini subgroup of G and is shown as $\Phi(G)$.

THEOREM 1.9: The following statements are equivalent:

- i-) G is nilpotent.
- ii-) Every subgroup of G is subnormal in G .
- iii-) Whenever $H < G$, $H < N_G(H)$.
- iv-) Every maximal subgroup of G is normal in G .
- v-) $G' \leq \Phi(G)$.
- vi-) Every Sylow subgroup of G is normal in G .
- vii-) G is a direct product of groups of prime power orders.

PROOF: (i) \Rightarrow (ii): This is a special case of the theorem 1.8.

(ii) \Rightarrow (iii): Let $H < G$. By (ii) there's a series from H to G . Hence there's a proper series from H to G , say $H = H_0 \triangleright H_1 \triangleright H_2 \triangleright H_3 \triangleright \dots \triangleright H_n = G$. Since $H < G$, $n > 0$, then $H < H_1 \leq N_G(H)$.

(iii) \Rightarrow (iv): Let M be a maximal subgroup of G . By (iii) $M < N_G(M) \leq G$. Now the maximality of M implies that $N_G(M) = G$. Thus $M \triangleright G$. Before the proof of (iv) \Rightarrow (v) we'll give Theorem 1.10.

THEOREM 1.10: Let M be a maximal subgroup of G and let $M \triangleright G$. Then G/M has prime order.

(iv) \Rightarrow (v): Let M be a maximal subgroup of G . By (iv) $M \triangleright G$. Then by the Theorem 1.10 G/M is cyclic of prime order. The rest of the proof is left to the reader.

In the proofs of other equivalences we'll use the following theorems.

THEOREM 1.11: For any group G , the derived group G' is the unique smallest normal subgroup K of G such that G/K is abelian.

THEOREM 1.12: Let K be a normal subgroup of G . Then every subgroup of G/K is of the form H/K , where $K \leq H \leq G$. Moreover $\frac{H}{K} \triangleright \frac{G}{K} \iff H \triangleright G$, and if so, $\frac{G/K}{H/K} \cong G/H$.

THEOREM 1.13: Let G be a finite group and P a Sylow p -subgroup of G . Then for every subgroup H of G which contains $N_G(P)$, $N_G(H) = H$.

THEOREM 1.14: Let G be finite group and let p_1, p_2, \dots, p_s be the distinct prime divisors of $|G|$, where s is a positive integer. Suppose that, for each $i = 1, 2, 3, \dots, s$, G has a normal Sylow p_i -subgroup P_i . Then $G =$

$D_r(P_1 P_2 P_3 \dots P_s)$ and G is nilpotent.

(v) \Rightarrow (iv): Let M be a maximal subgroup of G . Then (v) implies that $G' \leq M$. Hence M / G' is a subgroup of the abelian group G / G' and so $\frac{M}{G'} \triangleright \frac{G}{G'}$. Hence by the Theorem 1.12, $M \triangleright G$.

(iv) \Rightarrow (vi): Let P be a Sylow p -subgroup of G . Suppose that $N_G(P) < G$. Then there's a maximal subgroup M of G which contains $N_G(P)$. But then by the Theorem 1.13 $N_G(P) = M$. So as a result $N_G(P) = M$, this is a contradiction to (iv). Hence (iv) implies that $N_G(P) = G$, that is, $P \triangleright G$.

(vi) \Rightarrow (vii): Let the distinct prime divisors of $|G|$ be $p_1, p_2, p_3, \dots, p_s$, where s is a positive integer. By (vi) G has a normal Sylow p_i -subgroup P_i for each $i = 1, 2, 3, \dots, s$. Then, by the Theorem 1.14, $G = D_r(P_1 P_2 P_3 \dots P_s)$.

(vii) \Rightarrow (i): This follows by the Theorem 1.15 (i), and repeated application of Theorem 1.15 (ii).

THEOREM 1.15 i-): Let G be a finite p -group. Then G is nilpotent.

ii-) Suppose that $G = H \times K$. If H and K are both nilpotent then G is nilpotent.

iii-) If H and K are both soluble then G is soluble.

DEFINITIONS: i-) A factor H / K of a series of G is said to be a central factor of G if K is normal in G and $H / K \leq Z(G / K)$.

ii-) G is said to be nilpotent if it has a series all of whose factors are

central factors of G . Such a series is called a central series.

iii-) G is said to be soluble (or solvable) if it has a series all of whose factors are abelian. We shall call such a series an abelian series. Groups which are not soluble are said to be insoluble.

LEMMA 1.16: Let $K \triangleright G$. Then $K \leq \Phi(G) \Leftrightarrow$ There's no proper subgroup H of G such that $HK = G$.

PROOF: \Rightarrow Suppose first that $K \leq \Phi(G)$. Let $H < G$. Then there's a maximal subgroup M of G such that $H \leq M < G$. Since $K \leq \Phi(G)$, $K \leq M$. Hence $HK \leq M < G$. Thus there's no proper subgroup H of G such that $HK = G$.

\Leftarrow Suppose K is not a subgroup of $\Phi(G)$. Then $G \neq 1$ and, by definition of $\Phi(G)$, there's a maximal subgroup M of G such that K is not a subgroup of M . Then $M < MK \leq G$. The maximality of M implies that $MK = G$. Thus in this case M is a proper subgroup of G such that $MK = G$. This is a contradiction to the hypothesis. So $K \leq \Phi(G)$.

THEOREM 1.17: $\Phi(G)$ is nilpotent.

PROOF: Let P be any Sylow subgroup of $\Phi(G)$. Since $\Phi(G) \triangleright G$, Frattini's Lemma shows that $G = N_G(P) \cdot \Phi(G)$. Hence by Lemma 1.16 $N_G(P) = G$. Thus $P \triangleright G$, and so $P \triangleright \Phi(G)$. Now it follows from Theorem 1.9 that $\Phi(G)$ is nilpotent.

LEMMA 1.18 (FRATTINI'S LEMMA): If K is a finite normal subgroup of G and P is a sylow p -subgroup of K then $G = N_G(P) \cdot K$.

COROLLARY 1.19: $\Phi(G) \leq F(G)$.

The nilpotent radical of a finite group G is denoted by $F(G)$ and called the Fitting subgroup of G .

LEMMA 1.20: Let $H \leq G$ and $K \triangleright G$.

i-) If $K \leq \Phi(H)$ then $K \leq \Phi(G)$.

ii-) $\Phi(K) \leq \Phi(G)$.

PROOF: *i-*) Suppose that K is not a subgroup of $\Phi(G)$. Then by Lemma 1.16 there's a proper subgroup J of G such that $JK = G$. Assume that $K \leq \Phi(H)$. Then $K \leq H \leq G = JK$ so that, by Dedekind's rule (In other words let A, B, C be subgroups of G such that $B \leq A$. Then $A \cap (BC) = B(A \cap C)$) $H = (H \cap J).K$.

By Lemma 1.16 the assumption that $K \leq \Phi(H)$ implies that $H \cap J = H$. Then $K \leq H \leq J$ and therefore $G = JK = J < G$ (a contradiction). Hence if K is not a subgroup of $\Phi(G)$, it follows that K is not a subgroup of $\Phi(H)$.

The proof of the second part is left to the reader.

LEMMA 1.21: Let $K \triangleright G$. Then

i-) $\frac{\Phi(G)K}{K} \leq \Phi\left(\frac{G}{K}\right)$.

ii-) If $K \leq \Phi(G)$ then $\frac{\Phi(G)}{K} = \Phi\left(\frac{G}{K}\right)$.

PROOF: It's clear from the following theorem that every maximal subgroup of G/K is of the form, M/K , where M is a maximal subgroup of

G containing K . Moreover, for every such M , M/K is a maximal subgroup of G/K . The statements (i) and (ii) follow.

THEOREM 1.22: Let $K \triangleright G$. Then every subgroup of G/K is of the form H/K , where $K \leq H \leq G$. Moreover, $\frac{H}{K} \triangleright \frac{G}{K} \Leftrightarrow H \triangleright G$, and if so, $\frac{\frac{G}{K}}{\frac{H}{K}} \cong \frac{G}{H}$.

LEMMA 1.23: Let $H \triangleright G$, $K \triangleright G$ and $G = H \times K$. And π, ρ are projections of G onto H and K . Let $L \leq G$.

i-) $(H \cap L) \triangleright L\pi \leq H$, $(K \cap L) \triangleright L\rho \leq K$ and $\frac{L\pi}{H \cap L} \cong \frac{L\rho}{K \cap L}$.

ii-) $L = (H \cap L) \times (K \cap L) \Leftrightarrow L\pi = H \cap L$. (Or if and only if $L\rho = K \cap L$).

THEOREM 1.24: Let H and K be normal subgroups of G such that $G = H \times K$ and let π be the projection of G onto H . Let $L \leq G$ and let $J = (H \cap L) \times (K \cap L)$, then $J \triangleright L$ and $\frac{L}{J} \cong \frac{L\pi}{H \cap L}$.

DEFINITION: Let X be a set of generators of G . We say that X is a minimal set of generators of G , if for every proper subset Y of X , $\langle Y \rangle$ is a proper subgroup of G . However, even for a finite abelian group there can be minimal sets of generators containing different numbers of elements. For instance let G be a cyclic group of order 6, say $G = \langle x \rangle$. Then $\{x\}$ is a minimal set of generators of G . But also $G = \langle x^2, x^3 \rangle$ and, because $\langle x^2 \rangle < G$ and $\langle x^3 \rangle < G$, $\{x^2, x^3\}$ is a minimal set of generators of G .

What we are going to prove is that if G is a p -group then any two minimal sets of generators of G do contain the same number of elements.

THEOREM 1.25: (BURNSIDE'S BASIS THEOREM) Let G be a non-trivial

p -group. $\bar{G} = G / \Phi(G)$ and $|\bar{G}| = p^d$. For each $x \in G$, let $\bar{x} = x\Phi(G) \in \bar{G}$, and for each non empty subset X of G , let $\bar{X} = \{\bar{x} : x \in X\} \subseteq \bar{G}$. If X is a minimal set of generators of G then \bar{X} is a base of \bar{G} and $|X| = d$.

Conversely, if X is a subset of G such that $|X| = d$ and \bar{X} is a base of \bar{G} then X is a minimal set of generators of G .

PROOF: Since $G \neq 1$, $d > 0$. (Let G be a p -group. Then $\Phi(G)$ is the unique smallest normal subgroup of G such that $\frac{G}{\Phi(G)}$ is elementary abelian.) By the theorem in parenthesis \bar{G} is elementary abelian and therefore can be viewed as a vector space over Z_p . Since $|\bar{G}| = p^d$, the dimension of this vector space is d . Let $\emptyset \subset X \subseteq G$. If $\langle X \rangle = G$ then $\langle \bar{X} \rangle = \bar{G}$. Then regarded as a set of vectors of the vector space \bar{G} , \bar{X} certainly spans \bar{G} .

Suppose conversely that \bar{X} is a set of vectors spanning \bar{G} . Then by definition of the vector space structure on \bar{G} , \bar{X} is a set of generators of elementary abelian group \bar{G} . Let $\langle X \rangle = H \leq G$. Then $\bar{G} = \langle \bar{X} \rangle = \frac{H\Phi(G)}{\Phi(G)}$. Hence $G = H\Phi(G)$. Now it follows by Lemma 1.16 that $H = G$.

COROLLARY 1.26: The only soluble simple groups are the groups of prime power orders.

DEFINITION: Let G be a group. The largest normal p -subgroup of G is called p -radical of G and is denoted by $O_p(G)$.

THEOREM 1.27: Suppose that $O_p(G) \neq 1$. Let $L = O_p(G)$ and $|L| = p^n$. Suppose also that G has a maximal subgroup M such that $M_G = 1$. Then
i-) L is elementary abelian.

ii-) M is a complement to L in G ; in particular $|G : M| = p^n$.

iii-) L is minimal normal in G , and

iv-) $C_G(L) = 1$; hence L is the unique minimal normal subgroup of G .

Suppose further that there's a prime q such that $O_q(M) \neq 1$. Then for any such q :

v-) $p^n \equiv 1 \pmod{q}$; in particular, $q \neq p$, and

vi-) Every complement to L in G is conjugate to M .

PROOF: *i-*) Since M is a maximal subgroup of G , $\Phi(G) \leq M$, and since $\Phi(G) \triangleright G$ it follows that $\Phi(G) \leq M_G = 1$. $L \triangleright G$, $\Phi(G) \leq \Phi(L) = 1$. Thus $\Phi(L) = 1$. As $\Phi(G) \leq M$, and since $\Phi(G) \triangleright G$, a p -group, and since $\Phi(L) = 1$, it follows that L is elementary abelian.

ii-) Since $1 \neq L \triangleright G$ and $M_G = 1$, L is not a subgroup of M . Therefore $M < ML \leq G$ so the maximality of M implies that $ML = G$. Since $L \triangleright G$, $M \cap L \triangleright M$ and since, by (i), L is abelian, $M \cap L \triangleright L$. Hence $M \cap L \triangleright ML = G$, so that $M \cap L \leq M_G = 1$. Thus $ML = G$ and $M \cap L = 1$, that is, M is a complement to L in G . In particular, $|G : M| = |L| = p^n$.

iii-) Let $1 < K \triangleright G$ with $K \leq L$. Then $M \leq MK \leq G$. Since $M_G = 1$, K is not a subgroup of M so $MK \neq M$. Therefore, by the maximality of M , $MK = G$.

Moreover $M \cap K \leq M \cap L = 1$, by (ii). Hence M is a complement to K in G , and so $|K| = |G : M| = |L|$, by (ii). Since $K \leq L$, it follows that $K = L$. Thus L is minimal normal in G .

iv-) By (i) and (ii), $L \leq C_G(L) \leq G = ML$. Therefore, by Dedekind's rule, $C_G(L) = (M \cap C_G(L)).L$. (For every $L \leq G$, $C_G(L) \triangleright N_G(L)$ and $N_G(L) / C_G(L)$ can be embedded in $\text{Aut}H$.) By this theorem $C_G(L) \triangleright G$. Therefore $M \cap C_G(L) \triangleright M$. So L centralizes and normalizes $M \cap C_G(L)$. Hence $M \cap C_G(L) \triangleright ML = G$. Since $M_G = 1$, it follows that $M \cap C_G(L) = 1$. Thus $C_G(L) = L$.

Suppose that G has a minimal normal subgroup $N \neq L$. Then $L \cap N \triangleright G$ and, by (iii), N is not a subgroup of L , so that $L \cap N < N$. Since N is minimal normal in G , it follows that $L \cap N = 1$. Then $[L, N] = 1$. (This is for the reason that $[H, K] \leq H \cap K$.) Hence $N \leq C_G(L) = L$. This is a contradiction. Therefore L is the unique minimal normal subgroup of G .

v-) Since $Q \triangleright M$ and $L \triangleright G$, $QL \triangleright ML = G$. (Suppose that $B \triangleright A \leq G$ and $C \leq G$. Then $(B \cap C) \triangleright (A \cap C)$ and $(A \cap C) / (B \cap C) \cong B(A \cap C) / B$. If also $C \triangleright G$ then $BC \triangleright AC$.) By the theorem given in parenthesis, also, by (ii), $Q \cap L = 1$ and so $|QL| = |Q||L|$. Since $Q \neq 1$ and $L = O_p(G)$, it follows that $q \neq p$. Thus Q is a Sylow q -subgroup of QL .

Now $1 < Q \triangleright M$ and, since $M_G = 1$, Q is not a normal subgroup of G . Therefore the maximality of M implies that $N_G(Q) = M$. Hence $N_{QL}(M) = (QL) \cap M = Q(L \cap M) = Q$, by Dedekind's rule and (ii). Thus $|L| = |QL : Q| = |QL : N_{QL}(Q)| \equiv 1 \pmod{q}$, by Sylow's theorem; that is, $p^n \equiv 1 \pmod{q}$.

vi-) Let M^* be any complement to L in G , and let $Q^* = O_q(M^*)$. The natural homomorphism v of G onto G/L maps Q to QL/L and Q^* to Q^*L/L .

Since, by (ii), M is a complement to L in G , the restriction of v to M is an isomorphism of M onto G/L . Therefore, v must map $Q = O_q(M)$ to $O_q(\frac{G}{L})$. Similarly, v must map Q^* to $O_q(\frac{G}{L})$. Thus $\frac{QL}{L} = O_q(\frac{G}{L}) = \frac{Q^*L}{L}$, so

that $QL = Q^*L$. Since by (v), $q \neq p$, it follows that Q and Q^* are Sylow q -subgroups of QL . Therefore, by Sylow's theorem, $Q^* = Q^x$ for some $x \in QL$. (We must define $Q^x = \{xqx^{-1} : q \in Q\}$). As in (v), $M = N_G(Q)$. Since $Q^* \triangleright M^*$; it follows that $M^* \leq N_G(Q^*) = N_G(Q^x) = M^x$. (If U is a non-empty subset of G and $g \in G$. Then $N_G(U^g) = N_G(U)^g$). Because M and M^* are complements to L in G , $|M^*| = \left| \frac{G}{L} \right| = |M| = |M^x|$.

Hence $M^* = M^x$. (M_G = The largest normal subgroup of G in M).

COROLLARY: Suppose that G is a finite group such that $F(G)$ is abelian. Then $F(G)$ is the unique maximal abelian normal subgroup of G . Moreover, if G is soluble then $F(G)$ is a maximal abelian subgroup of G .

COROLLARY 1.28: Suppose that G is finite and that p is the smallest prime divisor of $|G|$. If H is a subgroup of index p in G then $H \triangleright G$.

DEFINITION: For every $n \in \mathbb{N}$ the groups S_n are defined as the groups of permutations from the set $\{1, 2, 3, \dots, n\}$ to itself.

DEFINITIONS: w always denotes a set of prime numbers.

i-) A positive integer n is said to be a w -number if every prime divisor of n belongs to w . Note that we do not require that every prime in w divides n ; for instance 6 is a $\{2, 3, 5\}$ -number. By convention, 1 is a w -number for every set w of primes. (and if $w = \emptyset$, 1 is the only w -number). If $w = \{p\}$, the w -numbers are powers of p : $1, p, p^2, p^3, \dots$.

ii-) Let G be a finite group. We say that G is a w -group if $|G|$ is a w -number. Thus, for example, S_3, S_4, S_5 are all $\{2, 3, 5\}$ groups. If $w = \{p\}$, a

w -group is called a p -group.

COROLLARY: If G is a finite w -group then all subgroups and all quotient groups of G are w -groups.

COROLLARY: Let G be a finite group. Then G has a unique largest normal w -subgroup, which is denoted by $O_w(G)$ and called the w -radical of G .

(Here $O_w(G)$ is largest in the sense that it contains every normal w -subgroup of G).

THEOREM 1.29: (P.HALL) Suppose for every maximal subgroup M of G , $|G : M|$ is either a prime or a square of a prime. Then G is soluble.

PROOF: We argue by induction on $|G|$. We may suppose $G \neq 1$. Let K be a minimal normal subgroup of G . Any maximal subgroup of G/K is of the form M/K , where M is a maximal subgroup of G . Hence every maximal subgroup of G/K has index either a prime or the square of a prime. Therefore by the induction hypothesis, G/K is soluble.

Let p be the largest prime divisor of $|K|$ and let P be a Sylow p -subgroup of K . If $N_G(P) = G$ then, since K is minimal normal in G , $K = P$. Thus K is soluble. Now suppose that $N_G(P) < G$. Then there's a maximal subgroup M of G such that $N_G(P) \leq M$. $G = N_G(P).K$. Also $M \cap K \triangleright M$ and since $P \leq M \cap K \leq K$, P is a Sylow p -subgroup of $M \cap K$ and $M = N_M(P).(M \cap K)$. Now it follows by the following theorem and Sylow's theorem, that

$$|G : N_G(P)| = |K : N_K(P)| \equiv 1 \pmod{p} \dots\dots\dots(i)$$

and $|M : N_M(P)| = |M \cap K : N_{M \cap K}(P)| \equiv 1 \pmod{p} \dots\dots\dots(ii).$

(THEOREM: Let $H \leq K$ and $K \triangleright G$. Then $H \cap K \triangleright H$ and $\frac{H}{H \cap K} \cong \frac{HK}{K}$.)

Since $N_G(P) \leq M \leq G = N_G(P).K$, $G = MK$, and therefore

$$|G : M| = |K : M \cap K| \dots \dots \dots (iii).$$

Moreover $N_M(P) = M \cap N_G(P) = N_G(P)$ and $|G : N_G(P)| = |G : M| |M : N_G(P)|$.

Therefore we deduce from (i) and (ii) that

$$|G : M| \equiv 1 \pmod{p} \dots \dots \dots (iv).$$

By hypothesis there's a prime q such that

$$|G : M| \text{ is either } q \text{ or } q^2 \dots \dots \dots (v).$$

Certainly $q \neq p$ and, by (iii), q divides $|K|$. Therefore, by the choice of p , it follows that $q < p$. This implies from (iv) and (v) that $|G : M| = q^2 \equiv 1 \pmod{p}$. This last congruence is possible with $q < p$ only if $p = 3$ and $q = 2$. Thus $|G : M| = 4$. Now (iii) shows that K has a proper subgroup of index 4. Therefore K has a proper normal subgroup L such that K/L can be embedded in S_4 , which is soluble. Then K/L is soluble and so K/L has a non-trivial abelian quotient group. Hence K has a non trivial abelian quotient group, so that $K' < K$. Since K' is characteristic in K , $K' \triangleright G$. Because K is minimal normal in G , it follows that $K' = 1$. Thus K is abelian. Now in any case both K and G/K are soluble. Therefore G is soluble. This completes the induction argument.

DEFINITION: For any set w of prime numbers, we denote by w' the set of all primes which do not belong to w .

Let $H \leq G$. Then H is said to be a Hall w -subgroup of G if $|H|$ is

a w -number and $|G:H|$ is a w' -number. Note also that if G has a Hall w -subgroup H then $|H|$ is determined by $|G|$; namely, $|H|$ must be the largest w -number which divides $|G|$.

THEOREM 1.30: Suppose that G is soluble and let w be any set of primes. Then

i-) G has a Hall w -subgroup.

ii-) If H is a Hall w -subgroup of G and V is any w -subgroup of G then $V \leq H^g$ for some $g \in G$. In particular, the Hall w -subgroups of G form a single conjugacy class of subgroups of G .

PROOF: The theorem is clear if $|G|=1$. Suppose that $|G|>1$. Since all subgroups and quotient groups of G are soluble, the induction hypothesis implies that the theorem is true for every proper subgroups of G and for every quotient of G by a non-trivial normal subgroup. Let $R = O_w(G)$ and suppose first that $R \neq 1$. Then by the induction hypothesis, G/R has a Hall w -subgroup H/R , where $H \leq G$. Then $|H/R| = \frac{|H|}{|R|}$, which is a w -number, and $|G:H| = \frac{|G/R|}{|R/R|} = \frac{|G/R|}{1}$ which is a w' -number. Thus H is a Hall w -subgroup of G . Now let V be any w -subgroup of G . Then VR/R is a w -subgroup of G/R and so, by the induction hypothesis, $\frac{VR}{R} \leq \left(\frac{H}{R}\right)^{Rg} = \frac{H^g}{R}$ for some $g \in G$. Then $V \leq VR \leq H^g$. This completes the argument in this case. Case (ii) is left to the reader.

DEFINITION: Let G a non-trivial group. Suppose that $|G| = p_1^{m_1} p_2^{m_2} \dots p_s^{m_s}$ where s, m_1, m_2, \dots, m_s , are positive integers and p_1, p_2, \dots, p_s distinct primes. Let $S = \{1, 2, \dots, s\}$. Then the possible orders of non-trivial Hall subgroups of G are

the $2^s - 1$ distinct numbers $\prod_{j \in T} p_j^{m_j}$, where T ranges over the $2^s - 1$ distinct non-empty subsets of S .

DEFINITION: Let G be a finite group. A complement to a Sylow p -subgroup of G is called a p -complement of G . Note that a subgroup H of G is a p -complement of $G \iff |G : H|$ is a power of p and p does not divide $|H|$. In particular a p -complement of G is a Hall subgroup of G . If $|G| = p_1^{m_1} p_2^{m_2} \dots p_s^{m_s}$ then any set $Q = \{H_1, H_2, \dots, H_s\}$, with H_i a p_i -complement of G for each $i = 1, 2, \dots, s$ will be called a complement system of G .

THEOREM 1.31: *i-*) Let $G = H \times K$. ($H \triangleright G, K \triangleright G$).

$\pi : H \times K \mapsto H, \rho : H \times K \mapsto K$ are projections.

Let $L \leq G$. L is a subdirect product of H and K if $L\pi = H$ and $L\rho = K$.

ii-) Suppose that G is finite and that $\left(\left|\frac{H}{H'}\right|, \left|\frac{K}{K'}\right|\right) = 1$. Then no proper normal subgroup of G is a subdirect product of H and K .

THEOREM 1.32: Let M be a maximal subgroup of G . Assume that P is a Sylow p -subgroup of M and P is not a normal subgroup of G . Then $N_G(P) = M$ and P is a Sylow p -subgroup of G .

DEFINITION: Let P be a Sylow p -subgroup of G . If $P \leq Z(N_G(P))$ then G is called p -nilpotent.

COROLLARY: For each non-empty subset U of G ,

$|\text{the conjugacy class of } U \text{ in } G| = |G : N_G(U)|$. (That is, $|G : N_G(U)|$ is the number of distinct conjugates of U in G .) $\text{Stab}_G(U) = \{g \in G : g^{-1}Ug = U\}$ is

called the normalizer of U in G and is denoted by $N_G(U)$.)

THEOREM 1.33: Let K be a normal Hall subgroup of the finite group G , and suppose either K or G/K is soluble. Then the complements to K in G form a single conjugacy class of subgroups of G .

COROLLARY: Let G be a finite group and let $H, K \leq G$ with

$$(|G : H|, |G : K|) = 1.$$

Then $|G : H \cap K| = |G : H| \cdot |G : K|$ and $G = HK$.

COROLLARY 1.34: Let G be a non-trivial soluble group. Then the system normalizers of G form a single conjugacy class of nilpotent subgroups of G .

PROOF: Let $\rho = \{H_1, H_2, \dots, H_s\}$ be a complement system of G . The corresponding system normalizer of G is the subgroup $L = N_G(\rho) = \bigcap_{i=1}^s N_G(H_i)$. For any element $g \in G$, $L^g = \bigcap_{i=1}^s N_G[H_i]^g = \bigcap_{i=1}^s N_G((H^G)_i) = N_G(\rho^g)$, the system normalizer corresponding to the complement system ρ^g . This remark, shows that the system normalizers of G form a single conjugacy class of subgroups of G .

Suppose that for each $i = 1, 2, \dots, s$ H_i is a p_i -complement of G . If $s = 1$ then G is a p_1 -group, $H_1 = 1$ and $L = N_G(H_1) = G$, which in this case is nilpotent. Suppose now that $s > 1$ and for each $i = 1, 2, \dots, s$, let $P_i = \bigcap_{i \neq j} H_j$. P_i is a Sylow p_i -subgroup of G . Moreover $L = N_G(\rho) \leq N_G(P_i)$ for each i . Since $P_i \triangleright N_G(P_i)$, $L \cap P_i \triangleright L$; and it's easy to check that $L \cap P_i$ is a Sylow p_i -subgroup of L . This shows that every Sylow subgroup of L is normal in L , because p_1, p_2, \dots, p_s are the only prime divisors of $|G|$. Hence, L is nilpotent.

LEMMA 1.35: Suppose that G has three soluble groups H_1, H_2, H_3 , and suppose that $(|G:H_i|, |G:H_j|) = 1$ whenever $i, j \in \{1, 2, 3\}$ with $i \neq j$. Then G is soluble.

PROOF: We argue by induction on $|G|$. Since $(|G:H_1|, |G:H_2|) = 1$ then $G = H_1H_2$. Since H_2 is soluble, we may suppose $H_1 \neq 1$. Let L_1 be a minimal normal subgroup of H_1 . Then, since H_1 is soluble, L_1 is an elementary abelian p -group for some prime p . By hypothesis, p does not divide both $|G:H_2|$ and $|G:H_3|$. We suppose, without loss of generality, that p does not divide $|G:H_2|$. Let $J = H_1 \cap H_2$. Then $|G:H_2| = \frac{|H_1H_2|}{|H_2|} = |H_1:J|$. Thus p does not divide $|H_1:J|$. Since $J \leq H_1$ and $L_1 \triangleright H_1$, $J \leq JL_1 \leq H_1$. Hence p does not divide $|JL_1:J| = |L_1:J \cap L_1|$ and since L_1 is a p -group $|L_1:J \cap L_1|$ is a power of p . Therefore $|JL_1:J| = 1$ that is, $L_1 \leq J$. Now let $L = L_1^G$, the normal closure of L_1 in G . Then $L = \langle x^g : x \in L_1, g \in G \rangle = \langle x^{h_1h_2} : x \in L_1, h_1 \in H_1, h_2 \in H_2 \rangle$. $L = \langle x^{h_1h_2} : x \in L_1, h_1 \in H_1, h_2 \in H_2 \rangle \leq H_2$. Since $L_1 \triangleright H_1$ and $L_1 \leq J \leq H_2$. Because H_2 is soluble, L is soluble. Moreover $L < L_1 \leq L \triangleleft G$. We may now apply the induction hypothesis to G/L . For each $i = 1, 2, 3$ $\frac{H_iL}{L} \leq \frac{G}{L}$ and $\frac{H_iL}{L} \cong \frac{H_i}{H_i \cap L}$, this is soluble, since H_i is soluble. Whenever $i \neq j$ $(|\frac{G}{L}:\frac{H_iL}{L}|, |\frac{G}{L}:\frac{H_jL}{L}|) = (|G:H_iL|, |G:H_jL|)$. This number divides $(|G:H_i|, |G:H_j|)$ and is equal to 1. Hence by the induction hypothesis, G/L is soluble.

Finally, since L and G/L are both soluble, it follows that G is soluble. This completes the induction argument.

THEOREM 1.36: G has a normal subgroup of index $p \iff p$ divides $|\frac{G}{G'}|$.

THEOREM 1.37: Suppose that $|G| = p^m q^n$, where p and q are distinct primes

and m and n are non-negative integers. Then G is soluble.

REMARK: The following three statements are equivalent:

i-) G is soluble.

ii-) G has a Hall w -subgroup for every set w of primes.

iii-) G has a p -complement for every prime divisor p of $|G|$.

COROLLARY: G acts on itself by conjugation.

COROLLARY (THE CLASS EQUATION): If G is a finite group with k distinct conjugacy classes of elements, and if x_1, x_2, \dots, x_k are elements of G , one from each of these k classes, then $|G| = \sum_{i=1}^k |G : C_G(x_i)|$.

COROLLARY: G acts on $\varphi(G)$ by conjugation.

THEOREM 1.38: Let K be a normal Hall subgroup of the finite group G . Then G splits over K .

THEOREM 1.39: $F(G)/\Phi(G)$ is abelian.

DEFINITION: The product of all minimal normal subgroups of a group G is a group shown as $S(G)$.

DEFINITION: Let A be an abelian subgroup of G . If there's no abelian subgroup M of G such that $A < M < G$ then A is called maximal abelian subgroup of G .

THEOREM 1.40: Let A be an abelian subgroup of G . Then A is a maximal abelian subgroup of $G \iff C_G(A) = A$.

THEOREM 1.41: A simple group G can not be expressed in the form $G = HK$ with $H < G$, $K < G$, H abelian and $H \cap K \neq \{1\}$.



In this section we solve certain selected problems on finite nilpotent and soluble groups. The problems are chosen from reference [1] Rose J., A Course On Group Theory. The lemmas and theorems of the previous section will be referred to as needed.

PROBLEM 1: Suppose that A is an abelian minimal normal subgroup of G . Then either $A \leq \Phi(G)$ or G splits over A .

SOLUTION: If $A \leq \Phi(G)$ then there's nothing to prove. If A is not a subgroup of $\Phi(G)$ then must show that G splits over A . $G \neq 1$ and by definition of $\Phi(G)$, there's a maximal subgroup M of G such that A is not a subgroup of M . Then $M < MA \leq G$, the maximality of M implies that $MA = G$. Thus in this case there's a proper subgroup M of G such that $MA = G$. And since A is an abelian minimal normal subgroup of G , we have $M \cap A = \{1\}$. So G splits over A .

PROBLEM 2: Let $x \in G$. Then x is said to be a non-generator of G , if whenever X is a set of generators of G , $X/\{x\}$ is also a set of generators of G . Prove that $\Phi(G)$ is the set of generators of G .

SOLUTION: We can say that x is a non-generator of G if $G = \langle x, X \rangle$ then $G = \langle X \rangle$. Now let x be a non-generator of G , and let M be a maximal subgroup of G . If $x \notin M$ then $G = \langle x, M \rangle = M$, it's a contradiction. Therefore $x \in M$, for all M , so $x \in \Phi(G)$. Conversely, if $z \in \Phi(G)$, assume that $G = \langle z, Y \rangle$. If $\langle Y \rangle \neq G$, then there's a maximal subgroup M with $\langle Y \rangle \leq M$. But $z \in M$, and so $G = \langle z, Y \rangle \leq M$, a contradiction. Therefore, z is a non-generator.

PROBLEM 3: Let $K \triangleright G$. A subgroup H of G is said to be a supplement to K in G if $HK = G$. In particular G itself is a supplement to K in G .

Let H be a minimal supplement to K in G ; that is let H be a supplement to K in G such that no proper subgroup of H is a supplement to K in G . Prove that then $H \cap K \leq \Phi(H)$.

SOLUTION: $K \triangleright G$. $HK = G$. Assume $H \cap K$ is not a subgroup of $\Phi(H)$. Then we'll now use Lemma 1.16. $H = (H \cap K).M$ and M is a proper subgroup of H . Then $G = HK = (H \cap K).M$ and M is a proper subgroup of H . Thus $G = HK = (H \cap K).M.K$. Since $M < H$, we have $(H \cap K).M < H$. This is a contradiction since H is a minimal supplement to K in G .

PROBLEM 4: Suppose that G is nilpotent. If $\frac{G}{G'}$ is cyclic then G is cyclic.

SOLUTION: We'll use Lemma 1.8. and Lemma 1.16. $G = G'.H$, and no such proper subgroup H of G occurs. Since $\frac{G}{G'}$ is cyclic, $\frac{G}{G'} = \langle yG' \rangle$. ($y \in G$).

Assume G is not cyclic, then $\langle y \rangle \neq G$, $\langle y \rangle < G$. We'll now show $G = G'\langle y \rangle$. Let $x \in G$. $xG' \in \frac{G}{G'} = \langle yG' \rangle$. $xG' = (yG')^n = y^n G'$. So $xy^{-n} \in G'$. $x = (xy^{-n}).y^n$. Since $xy^{-n} \in G'$ and $y^n \in \langle y \rangle$, then $x \in G'\langle y \rangle$. Thus $G = G'\langle y \rangle$. Therefore $\langle y \rangle = G$.

PROBLEM 5: If $\frac{G}{\Phi(G)}$ is a w -group then G is a w -group.

SOLUTION: We have $\left| \frac{G}{\Phi(G)} \right| = n$. So all the prime divisors of n belong to w . Assume G is not a w -group, then no prime divisors of $|G|$ belong to w .

If we say $|G| = k$, $\left(\left| \frac{G}{\Phi(G)} \right|, |\Phi(G)| \right) = \left(n, \frac{k}{n} \right) = 1$. So we'll apply Theorem 1.38. Since $\left(\left| \frac{G}{\Phi(G)} \right|, |\Phi(G)| \right) = 1$ then $\Phi(G)$ is a Hall-subgroup of G . So by 1.38 G

splits over $\Phi(G)$. We have $G = \Phi(G).H$ and $H \cap \Phi(G) = 1$. $H < G$, $\Phi(G) < G$. But if we use Lemma 1.16, $\Phi(G) \leq \Phi(G)$ so there's no proper subgroup H of G such that $G = \Phi(G).H$. We arrive a contradiction. Thus G is a w -group.

PROBLEM 6: If $\frac{K}{\Phi(G)}$ is a nilpotent normal subgroup of $\frac{G}{\Phi(G)}$ then K is nilpotent. Thus $F(\frac{G}{\Phi(G)}) = \frac{F(G)}{\Phi(G)}$. In particular, if $\frac{G}{\Phi(G)}$ is nilpotent then G is nilpotent.

SOLUTION: Let P be a Sylow p -subgroup of K , and P is also a Sylow p -subgroup of $P\Phi(G)$. If we use Frattini's Lemma 1.18 $G = N_G(P).P.\Phi(G)$. Then $N_G(P).P$ can't be a proper subgroup of G . $G = N_G(P).P$. Now we'll show $P \triangleright G$. Let $g \in G$ and $p \in P$, $g = ap_1$. $gpg^{-1} = ap_1pp_1^{-1}a^{-1} = ap_3a^{-1}$. Since $a \in N_G(P)$ we have $aPa^{-1} = P$. So $ap_3a^{-1} = p_4 \in P$. Then for all $g \in G$ and $p \in P$ we have $gpg^{-1} \in P$. So $P \triangleright G$, therefore $P \triangleright K$. This can be done for all Sylow p -subgroups of K . All Sylow p -subgroups of K are normal in K , therefore K is nilpotent.

If $\frac{G}{\Phi(G)}$ is nilpotent, then $\frac{G}{\Phi(G)}$ is a normal subgroup of $\frac{G}{\Phi(G)}$, so by the above argument G is nilpotent.

PROBLEM 7: *i-*) If A is an abelian subgroup of G such that $A \cap \Phi(G) = 1$ then G splits over A .

ii-) Suppose that $\Phi(G) = 1$. If $F(G) \neq 1$ then every non-trivial normal subgroup A of G contained in $F(G)$ is a direct product of abelian minimal normal subgroups of G and G splits over A . In particular, if G is a non-trivial soluble group such that $\Phi(G) = 1$ then $F(G) = S(G)$, the socle of G and G splits over $F(G)$.

SOLUTION: *i-*) Let H be a minimal complement to A in G then $H \cap A \leq \Phi(H)$. $A \triangleright G$. If we use Lemma 1.20 (i), $H \leq G$ and $K \triangleright G$. If $K \leq \Phi(H)$ then $K \leq \Phi(G)$. We have $G = AH$. Since $H \cap A \leq \Phi(H)$ then $H \cap A \leq \Phi(G)$. So $H \cap A \cap A \leq \Phi(G) \cap A = \{1\}$. Then $H \cap A = \{1\}$, then $G = AH$ and $A \cap H = \{1\}$, so G splits over A .

ii-) $\Phi(G) = 1$. By Theorem 1.39 $F(G)/\Phi(G)$ is abelian. Here

$$\frac{F(G)}{\Phi(G)} = \frac{F(G)}{1} = F(G) \text{ is abelian.}$$

Since $A \subset F(G)$, then A is abelian. A is normal in G . $A \cap \Phi(G) = A \cap \{1\} = 1$. By (i) G splits over A .

PROBLEM 8: Suppose that H and K are normal subgroups of G such that $G = H \times K$.

i-) For any maximal subgroup L of K , $H \times L$ is a maximal subgroup of G ; every maximal subgroup of G containing H is of the form $H \times L$, where L is a maximal subgroup of K .

ii-) If $K \neq 1$ then the intersection of all maximal subgroups of G containing H is $H \times \Phi(K)$.

iii-) Suppose that M is a subgroup of G which contains neither H nor K . Then M is a maximal subgroup of $G \iff M$ is a subdirect product of H and K and $\frac{H}{H \cap M}$ is simple.

iv-) Suppose that either H or K is soluble. Then any maximal subgroup

M of G which contains neither H nor K is necessarily normal in G . Moreover G has such a maximal subgroup $M \iff \left(\left| \frac{H}{H'} \right|, \left| \frac{K}{K'} \right| \right) > 1$.

SOLUTION: *i-*) Let $H \times L \leq M \times N \leq G = H \times K$. $H \times L \leq M \times N$, therefore $H \leq M$ and $L \leq N$. We'll show $H \leq M$. Let $x, y \in H$, $(x, 1) \in H \times L$ and $(y, 1) \in H \times L$. Since $H \times L \leq M \times N$ we have $(x, 1) \cdot (y, 1)^{-1} \in H \times L$. $(x, 1) \cdot (y^{-1}, 1) = (xy^{-1}, 1) \in H \times L$, then $xy^{-1} \in H$. So $H \leq M$. And $L \leq N$. Let $x, y \in L$, then $(1, x) \in H \times L$ and $(1, y) \in H \times L$. Since $H \times L \leq M \times N$ we have $(1, x) \cdot (1, y)^{-1} \in H \times L$. $(1, x) \cdot (1, y^{-1}) = (1, xy^{-1}) \in H \times L$, so $xy^{-1} \in L$. Thus $L \leq N$.

$H \times L \leq M \times N \leq H \times K = G$. So $L \leq N \leq K$ and $H \leq M \leq H$, $H = M$. $M \times N = H \times N$. Since $L \leq N \leq K$ and L is a maximal subgroup of K we have $N = K$ or $N = L$. If $N = K$ we have $M \times N = H \times N = H \times K = G$. If $N = L$ we have $M \times N = H \times N = H \times L$. Thus $M \times N = G$ or $M \times N = H \times L$. This shows that $H \times L$ is a maximal subgroup of G . Now let M be a maximal subgroup of G containing H . We'll show $M = H \times L$ where L is a maximal subgroup of G . We'll now apply Dedekind's rule. We have $H \leq M \leq G$. Then $M = H \times (M \cap K)$. Now we must show that $M \cap K$ is a maximal subgroup of K .

ii-) If A is a maximal subgroup of G containing H , then we have shown in *i-*) that $A = H \times L$ where L is a maximal subgroup of K . $\Phi(K) = \bigcap B$ (B is a maximal subgroup of K). We'll look at $\bigcap A$ ($H \subset A$, A is a maximal subgroup of G) $\bigcap A$ (A is a maximal subgroup of G , $H \subset A$) $= \bigcap H \times L = H \times \bigcap L = H \times \Phi(K)$. (L is a maximal subgroup of K).

iii-) \Leftarrow If M is a subdirect product of H and K , and if $M \leq L \leq G$, we'll show $L = G$ or $M = L$. Also we have $\frac{H}{H \cap M}$ is simple. By Lemma 1.23 we have $\frac{L\pi}{H \cap L} \cong \frac{L\rho}{K \cap L}$. $|\frac{L\pi}{H \cap L}| = \frac{|H|}{|H \cap L|} = \frac{|L\rho|}{|K \cap L|} = \frac{|K|}{|K \cap L|} \cdot |H| = \frac{|H \cap L| |K|}{|H \cap L|}$. If we use Theorem 1.24, $G = H \times K$. $\pi: H \times K \rightarrow H$ and $L \leq G$. $J = (H \cap L) \times (K \cap L)$, $J \triangleright L$ and $\frac{L}{J} \cong \frac{L\pi}{H \cap L \cdot (H \cap L) \times (K \cap L)} \cong \frac{L\rho}{(H \cap L) \times (K \cap L)} = \frac{|L\rho|}{|(H \cap L) \times (K \cap L)|} = \frac{|L\rho|}{|H \cap L|} = \frac{|H|}{|H \cap L|} \cdot |L| = |H| \cdot |K \cap L| = \frac{|H \cap L| |K|}{|K \cap L|} \cdot |K \cap L| \cdot |L| = |H \cap L| |K|$.

\Rightarrow Let M be a maximal subgroup of G . We'll show $M\pi = H$ and $M\rho = K$. If we use Theorem 1.31, we must show $HM = G = KM$. We have $M < HM \leq G$. We have $M \neq HM$ because if $HM = M$ then $H \subset HM = M$ therefore $H \subset M$. But by the hypothesis H isn't a subset of M . So $M \neq HM$, $M < HM$. Since M is maximal in G , we have $HM = G$. By the same argument $M < KM = G$. So $HM = G = KM$. Thus $M\pi = H$ and $M\rho = K$. Now we'll show $\frac{H}{H \cap M}$ is simple. $M \triangleright G$. So if we apply the isomorphism theorem $\frac{H}{H \cap M} \cong \frac{HM}{M} = \frac{G}{M}$. We'll now show that $\frac{G}{M}$ is simple. Assume $\frac{G}{M}$ is not simple, there's a proper subgroup $\frac{A}{M}$ of $\frac{G}{M}$ such that $\frac{A}{M} \triangleright \frac{G}{M}$. Since $\frac{A}{M} < \frac{G}{M}$ we have $A < G$. $M \triangleright A$, $M \leq A < G$. Since M is maximal in G , $A = M$ or $A = G$. If $A = M$ then $\frac{A}{M} = \frac{M}{M} = M$. If $A = G$ then $\frac{G}{M} = \frac{A}{M}$. Thus $\frac{A}{M}$ is the only trivial subgroups of $\frac{G}{M}$. Hence $\frac{G}{M}$ is simple. Thus $\frac{H}{H \cap M}$ is simple.

v-) \Rightarrow Assume G has such a maximal subgroup M . We've shown that $M \triangleright G$. $M \neq G$. We'll show $(|\frac{H}{H'}|, |\frac{K}{K'}|) > 1$. Assume $(|\frac{H}{H'}|, |\frac{K}{K'}|) = 1$ then 1.31 (ii) says no proper normal subgroup of G is a direct product of H and K . But M is a direct product of H and K . (contradiction).

\Leftarrow Let $(|\frac{H}{H'}|, |\frac{K}{K'}|) > 1$. $(|\frac{H}{H'}|, |\frac{K}{K'}|) = a > 1$. $|\frac{G}{G'}| = \frac{|H \times K|}{|H' \times K'|} = \frac{|H|}{|H'|} \cdot \frac{|K|}{|K'|}$. Since $a \mid |\frac{H}{H'}|$ and $a \mid |\frac{K}{K'}|$, $a^2 \mid |\frac{H}{H'}| \cdot |\frac{K}{K'}| = |\frac{G}{G'}|$. $a > 1$ so $a^2 > 1$. Let us write a^2 as a

product of primes.

$a^2 = p_1 p_2 \dots p_k$. $p_1 \mid a^2$, $a^2 \mid \left| \frac{G}{G'} \right|$ and so $p_1 \mid \left| \frac{G}{G'} \right|$. By Theorem 1.36 G has a normal subgroup of index p_1 . $\left| \frac{G}{M} \right| = p_1$. ($M \triangleright G$). So M is maximal in G .

PROBLEM 9: Suppose that there's an insoluble finite group with an abelian maximal subgroup and let G be such a group of least possible order.

Let M be an abelian maximal subgroup of G .

i-) $M_G = 1$.

ii-) G' is the unique minimal normal subgroup of G .

iii-) M is a Hall subgroup of G .

iv-) M is a complement to G' in G .

v-) Every complement to G' in G is conjugate to M .

vi-) Every subgroup of G isomorphic to M is conjugate to M .

vii-) G is p nilpotent for every prime p dividing $|M|$.

viii-) Let q be a prime divisor of $|G'|$ and let Q be a sylow q -subgroup of G' . Then $N_G(Q)$ has a subgroup M^* isomorphic to M .

ix-) M^* is a maximal subgroup of G and $Q \triangleright G$.

SOLUTION: i-) $M_G = \bigcap g^{-1} M g$ ($g \in G$). $M_G \triangleright G$. And whenever $K \leq M \leq G$ with $K \triangleright G$, $K \leq M_G$. So M_G is the unique largest normal subgroup of G contained in M . Assume $M_G \neq 1$. We know $M_G \triangleright G$ and $M_G \leq M$. Since M is abelian, then M_G is abelian. Thus all subgroups of M_G are normal in M_G .

$M_G = M_0 \geq M_1 \geq M_2 \geq \dots \geq 1$. ($M_i \leq M_G$) $\frac{M_0}{M_1}, \frac{M_1}{M_2}, \frac{M_2}{M_3}, \dots$ are abelian. M_G is soluble. $\frac{M}{M_G}$ is also abelian. (Since M is abelian). $\frac{G}{M_G} \geq \frac{M}{M_G} \geq \left(\frac{M}{M_G} \right)_1 \geq \left(\frac{M}{M_G} \right)_2 \geq \dots \geq 1$. $\frac{M}{M_G}$ is abelian, by the above argument $\frac{M}{M_G}$ is soluble. Hence $\frac{G}{M_G}$ is soluble. Since $M_G \triangleright G$ and M_G is soluble, by Theorem 1.3 G is soluble. But

G must be insoluble. $M_G = 1$.

ii-) Let A be a minimal normal subgroup of G , we have $G' \cap A = G'$, so $G' \subseteq A$. $G' \leq A < G$. Since A is a minimal normal subgroup of G , we have $G' = A$. So G' is the unique minimal normal subgroup of G .

iii-) Since P is a Sylow p -subgroup of G then $|G| = p^m \cdot r$ and $|P| = p^m$. (p doesn't divide r). Let n be the number of distinct Sylow p -subgroups of G . Then $n = |G : N_G(P)|$, where P is any particular Sylow p -subgroup of G ; n divides r and $n \equiv 1 \pmod{p}$. Now we'll show $(|G : M|, |M|) = 1$. Since $M = N_G(P)$;

$$(|G : M|, |M|) = (|G : N_G(P)|, |N_G(P)|). \quad N_G(P) \triangleright G, \quad |G : N_G(P)| = \frac{|G|}{|N_G(P)|} = n.$$

Since $M = N_G(P)$ then M is a normal subgroup of G . Hence M is an abelian maximal normal subgroup of G . $(|G : M|, |M|) = \left(\frac{|G|}{|M|}, |M|\right) = 1$. (Because M is maximal in G).

iv-) We must show $G' \cap M = \{1\}$ and $G = MG'$. $M \leq MG' \leq G$. Since M is an abelian maximal subgroup of G , we have $MG' = M$ or $MG' = G$. If $MG' = M$ then $G' \leq MG' = M$. So G' is abelian. $G' \cap M = G'$. We know that $G' \neq 1$ since G is not abelian.

But we know that $G' \cap M = \{1\}$ (contradiction), so $MG' = G$.

v-) G' is normal in G . We'll now show that G' is a Hall subgroup of G . And since $\frac{G}{G'}$ is abelian, it's soluble. We can then apply 1.33. We'll show

$$(|G : G'|, |G'|) = 1.$$

$$\left(\frac{|G|}{|G'|}, |G'\right) = \left(\frac{|M||G'|}{|G'|}, |G'\right) = (|M|, |G'|) = a. \quad \text{We know that } (|M|, |G : M|) = 1.$$

And also $G = G' \cdot M$ and $G' \cap M = \{1\}$, so $|G| = |G'| \cdot |M|$. $(|M|, |G : M|) =$

$(|M|, \frac{|G'| \cdot |M|}{|M|}) = 1$. Hence $(|M|, |G'|) = 1$. $(|G : G'|, |G'|) = 1$, then G' is a Hall subgroup of G . If we apply 1.33, the complements to G' in G form a single conjugacy class of subgroups of G . Since M is a complement to G' in G , every complement to G' in G is conjugate to M .

vi-) Let A be a subgroup of G isomorphic to M . Then A is a complement to G' in G . Since the complements to G' in G form a single conjugacy class of subgroups of G and M is conjugate to the complements to G' in G , M is conjugate to A .

vii-) Since $p \parallel |M|$, we can say M has a Sylow p -subgroup. Assume P is a Sylow p -subgroup of M such that P is not a normal subgroup of G . And then P is a Sylow p -subgroup of G and $N_G(P) = M$. For this P we'll show that $P \leq Z(N_G(P))$.

Since $N_G(P) = M$, $Z(N_G(P)) = Z(M) = \{m \in M : \text{for all } x \in M \ mx = xm\}$. Since M is abelian, $Z(M) = M$. Therefore $Z(N_G(P)) = M$. We know that $P \leq M$, so $P \leq Z(N_G(P))$. Thus G is p -nilpotent.

viii-) $G' \triangleright G$. And Q is a Sylow q -subgroup of G' . $G' = N_G(Q).G'$. We also know that G' is a normal Hall subgroup of G . By Theorem 1.38 G splits over G' . $G = M^*.G'$. ($G' \cap M^* = \{1\}$). M^* is a complement to G' in G . M^* is conjugate to M , therefore is isomorphic to M . $M^* \leq N_G(Q)$. $M^* \cong M$.

ix-) Since $M^* \cong M$, M^* is a maximal subgroup of G . And $M^* < N_G(Q) \leq G$. M^* is a maximal subgroup of G , $N_G(Q) = G$. $N_G(Q) = \{g \in G : gQ = Qg\} = G$. So $Q \triangleright G$.

PROBLEM 10: Suppose that G has a non-trivial abelian normal subgroup. Then the following are equivalent:

- i*-) G has a maximal subgroup M such that $M_G = 1$.
- ii*-) $\Phi(G) = 1$ and G has a unique minimal normal subgroup.

SOLUTION: (*i*) \implies (*ii*) Assume G has a maximal subgroup M such that $M_G = 1$. $M_G = \bigcap g^{-1}Mg = \{1\}$ and $M_G \leq M$. M_G is the unique largest normal subgroup of G contained in M . $\Phi(G) = \bigcap A$ (A is maximal in G) $\leq M$. $\Phi(G) \triangleright G$ then $\Phi(G) \triangleright M$. M_G is the largest normal subgroup of G contained in M , $\Phi(G) \leq M_G = \{1\}$. Thus $\Phi(G) = 1$.

PROBLEM 11: Suppose that G has an abelian minimal normal subgroup L such that $O_q(\frac{G}{L})$ is non-trivial for some prime q . Then the following two statements are equivalent:

- i*-) G has a maximal subgroup M such that $M_G = \{1\}$.
- ii*-) $C_G(L) = L$.

SOLUTION: (*i*) \implies (*ii*) Assume G has a maximal subgroup M such that $M_G = 1$. Then G has a unique minimal normal subgroup and it's L . $G = ML$. ($M \cap L = 1$). $C_G(L) = (M \cap C_G(L)).L$. $C_G(L) \triangleright G$, then $(M \cap C_G(L)) \triangleright M$. Therefore L centralizes and normalizes $M \cap C_G(L)$.

Hence $(M \cap C_G(L)) \triangleright ML = G$. Since $M_G = 1$, $M \cap C_G(L) = 1$ and therefore $C_G(L) = L$.

PROBLEM 12: Suppose that $O_p(G) \neq 1$, and that G has a maximal subgroup

M such that $M_G = 1$. Then

i-) $O_p(G) = F(G)$.

ii-) $O_p(G)$ is the unique maximal abelian normal subgroup of G and

iii-) $O_p(G)$ is a maximal abelian subgroup of G .

SOLUTION: *i-*) We'll use Theorem 1.27. We have $L = O_p(G)$ and

$$C_G(L) = C_G(O_p(G)) = O_p(G).$$

$S(G) = O_p(G)$. $F(G) \leq C_G(S(G)) = C_G(O_p(G)) = O_p(G)$. And $O_p(G) \leq F(G)$.
 $F(G) = O_p(G)$.

ii-) Since $F(G) = O_p(G) = L$, and $C_G(L) = L$ we have L is a maximal abelian subgroup of G . (This is the answer of *(iii)* also). Here we used Theorem 1.40. Here $F(G) = O_p(G)$ and $O_p(G)$ is abelian, so $F(G)$ is abelian. By the corollary after Theorem 1.27; $O_p(G) = F(G)$ is the unique maximal abelian normal subgroup of G .

PROBLEM 13: If G has 3 nilpotent subgroups H_1, H_2, H_3 such that $(|G : H_i|, |G : H_j|) = 1$ whenever $i, j \in \{1, 2, 3\}$ with $i \neq j$, then G is nilpotent. Moreover, if H_1, H_2 and H_3 are abelian then G is abelian.

SOLUTION: We'll use the corollary following Lemma 1.35. $(|G : H_1|, |G : H_2|) = 1$ then $G = H_1.H_2$. Since H_1 and H_2 are both nilpotent we have $H_1 \times H_2$ is also nilpotent. Let $H_1 \vee H_2$ be the smallest subgroup of G containing $H_1.H_2 = G$. Then $H_1 \vee H_2 = G$. Thus $H_1 \cap H_2 = \{1\}$ and for all $h_1 \in H_1$ and $h_2 \in H_2$ $h_1h_2 = h_2h_1$.

Let $\Phi : H_1 \times H_2 \longrightarrow G = H_1H_2$ $\Phi(h_1, h_2) = h_1h_2$. Φ is an isomorphism.

So $H_1 \times H_2 \cong H_1 H_2 = G$. Since H_1 and H_2 are nilpotent, $H_1 \times H_2$ is nilpotent, thus G is nilpotent. If H_1, H_2, H_3 are abelian, then by the same argument $H_1 \times H_2$ is abelian. Since $H_1 \times H_2 \cong G$, G is also abelian.

PROBLEM 14: *i*–) Suppose that G satisfies the converse of Lagrange's theorem, that is G has a subgroup of order m for every divisor m of $|G|$. Then G is soluble.

ii–) Let G be a soluble group of order $p_1^{m_1} p_2^{m_2} \dots p_s^{m_s}$, where s, m_1, m_2, \dots, m_s are positive integers and p_1, p_2, \dots, p_s are distinct primes. Let A be an abelian group of order $p_1^{m_1-1} p_2^{m_2-1} \dots p_s^{m_s-1}$. Then the group $G \times A$ satisfies the converse of Lagrange's theorem.

SOLUTION: *i*–) If we make the prime factorization of $|G|$, $|G| = p_1^{r_1} p_2^{r_2} \dots p_n^{r_n}$. Since G satisfies the converse of Lagrange's theorem, G has subgroups H_1, H_2, \dots, H_n such that $|H_1| = p_1^{r_1}, |H_2| = p_2^{r_2}, \dots, |H_n| = p_n^{r_n}$. $(p_i, p_j) = 1$. So $H_i \cap H_j = \{1\}$ for all $i, j \in \{1, 2, \dots, n\}$. Let $\Phi: H_1 \times H_2 \times \dots \times H_n \rightarrow G$ $\Phi(h_1, h_2, \dots, h_n) = h_1 h_2 \dots h_n$. Φ is an isomorphism.

$G \cong H_1 \times H_2 \times \dots \times H_n$. $|H_1| = p_1^{r_1}$, H_1 is a p_1 -group, so H_1 is a nilpotent group and therefore H_1 is soluble. By the same reason H_2, H_3, \dots, H_n are also soluble. Thus $H_1 \times H_2 \times \dots \times H_n$ is soluble. Then G is soluble.

The second part of the problem is left to the reader.

PROBLEM 15: Let G be a non-trivial group such that every subgroup of G satisfies the converse of Lagrange's theorem. Then G has a normal Sylow p -subgroup where p is the largest prime divisor of $|G|$.

SOLUTION: If G is a p -group, then $|G| = p^r$. Then the largest prime divisor of $|G|$ is p . The Sylow p -subgroup of G has degree p^r . So the Sylow p -subgroup of G is G and $G \triangleright G$. If G is not a p -group then $|G| = p_1^{r_1} p_2^{r_2} \dots p_n^{r_n}$

Let p_1 be the smallest prime divisor of $|G|$. Since $p_1^{r_1} p_2^{r_2} \dots p_n^{r_n} \mid |G|$ and G satisfies the converse of Lagrange's theorem G has a subgroup H such that $|H| = p_1^{r_1} p_2^{r_2} \dots p_n^{r_n} \cdot |G : H| = \frac{|G|}{|H|} = \frac{p_1^{r_1} p_2^{r_2} \dots p_n^{r_n}}{p_1^{r_1-1} p_2^{r_2} \dots p_n^{r_n}} = p_1$. So $H \leq G$ and it has index p_1 in G , p_1 is the smallest prime divisor of G . By Corollary 1.28, $H \triangleright G$. $H \leq G$, so H also satisfies the converse of Lagrange's theorem. Since $p_n^{r_n} \mid |H|$, H has a subgroup K of degree $p_n^{r_n}$. K is a Sylow p_n -subgroup of G . (p_n is the largest prime divisor of $|G|$.)

PROBLEM 16: The following three statements are equivalent:

i-) G is supersoluble.

ii-) Whenever $J < H \leq G$, with J a maximal subgroup of H , $|H : J|$ is a prime number.

iii-) Each subgroup of G satisfies the converse of Lagrange's theorem.

SOLUTION: (iii) \implies (i): Each subgroup of G satisfies the converse of Lagrange's theorem. Let $|G| = p_1^{r_1} p_2^{r_2} \dots p_n^{r_n}$. Let p_1 be the smallest prime divisor of $|G|$. Since $p_1^{r_1-1} p_2^{r_2} \dots p_n^{r_n} \mid |G|$, G has a subgroup H such that $|H| = p_1^{r_1-1} p_2^{r_2} \dots p_n^{r_n}$, so $|G : H| = p_1$. Then $H \triangleright G$. $|G : H| = \frac{|G|}{|H|} = p_1$. Then $\frac{G}{H}$ is cyclic. By the same argument $|H| = p_1^{r_1-1} p_2^{r_2} \dots p_n^{r_n}$ and the smallest prime divisor of $|H|$ is p_1 . Since $p_1^{r_1-1} p_2^{r_2} \dots p_n^{r_n} \mid |H|$, H has a subgroup H_1 such that $|H_1| = p_1^{r_1-2} p_2^{r_2} \dots p_n^{r_n}$. $|H : H_1| = p_1$. Thus $H_1 \triangleright H$ and $\frac{H}{H_1}$ is also cyclic. If we go on the same argument we obtain a series such that $G \geq H \geq H_1 \geq H_2 \geq \dots \geq 1$.

And $\frac{G}{H}$ is cyclic, $\frac{H}{H_1}$ is cyclic, $\frac{H_1}{H_2}$ is cyclic. $\frac{H_i}{H_{i+1}}$ is cyclic. Hence G is supersoluble.

(ii) \implies (i): Whenever $J < H \leq G$, with J a maximal subgroup of H , $|H : J|$ is a prime number. We'll show G is supersoluble. Let $H < G$ and $H \triangleright G$ and let H be a maximal subgroup of G . Then by ii-) $|G : H| = p$. $|G : H| = \frac{|G|}{|H|} = p$, so $\frac{G}{H}$ is cyclic. ($H < G \leq G$). Let $H_1 < H$ and $H_1 \triangleright H$, H_1 is maximal in H . $H_1 < H \leq G$, so $|H : H_1| = p_1 = |\frac{H}{H_1}|$. $\frac{H}{H_1}$ is cyclic. $G \geq H \geq H_1 \geq H_2 \geq \dots \geq 1$ and all factors are cyclic. G is supersoluble.

PROBLEM 17: Show by an example that an insoluble group G can have a subgroup of index p for every prime divisor p of $|G|$.

SOLUTION: We can take $G = A_5 \times Z_{60}$. A_5 is a non-abelian simple group and Z_{60} is a cyclic group.

$$|G| = |A_5 \times Z_{60}| = |A_5| \cdot |Z_{60}| = \frac{5!}{2} \cdot 60 = 60 \cdot 60 = 5^2 \cdot 2^4 \cdot 3^2.$$

Prime divisors of G are 5, 2 and 3. $5 \mid |G|$ and $5 \parallel |Z_{60}|$. Since Z_{60} is cyclic and $5^2 \cdot 3 \cdot 2^4 \mid 60$, Z_{60} has a subgroup H such that $|H| = 5^2 \cdot 2^4 \cdot 3$. So

$$|Z_{60} : H| = \frac{|Z_{60}|}{|H|} = \frac{5^2 \cdot 2^4 \cdot 3^2}{5^2 \cdot 2^4 \cdot 3} = 3. \quad A_5 \times H \leq A_5 \times Z_{60}.$$

$$|A_5 \times Z_{60} : A_5 \times H| = |A_5 : A_5| \cdot |Z_{60} : H| = 3.$$

$A_5 \times H \leq G$ and $|G : (A_5 \times H)| = 3$. For $p = 2$ we have again

$$5^2 \cdot 2^3 \cdot 3^2 \mid |Z_{60}| = 60.$$

Hence Z_{60} has a subgroup K such that $|K| = 5^2 \cdot 2^3 \cdot 3^2$.

$$|Z_{60} : K| = \frac{|Z_{60}|}{|K|} = \frac{5^2 \cdot 2^4 \cdot 3^2}{5^2 \cdot 2^3 \cdot 3^2} = 2. \quad A_5 \times K \leq G.$$

$$|G : (A_5 \times K)| = |A_5 \times Z_{60} : A_5 \times K| = |A_5 : A_5| \cdot |Z_{60} : K| = 1 \cdot 2 = 2.$$

For $p = 2$, $A_5 \times K \leq G$ such that $|G : A_5 \times K| = 2$. For $p = 5$, $5 \mid |Z_{60}| = 60$. $5 \cdot 2^4 \cdot 3^2 \mid |Z_{60}| = 60$. Again by the same reason Z_{60}

has a subgroup L such that $|L| = 5 \cdot 2^4 \cdot 3^2$. $|Z_{60} : L| = \frac{5^2 \cdot 2^4 \cdot 3^2}{5 \cdot 2^4 \cdot 3^2} = 5$. $A_5 \times L \leq G$.

$|G : A_5 \times L| = |A_5 \times Z_{60} : A_5 \times L| = |A_5 : A_5| \cdot |Z_{60} : L| = 1 \cdot 5 = 5$. For $p = 5$, $A_5 \times L \leq G$ such that $|G : A_5 \times L| = 5$. Thus for all prime divisors p of $G = A_5 \times Z_{60}$, we can find a subgroup of index p .

PROBLEM 18: $G' \cap Z(G) \leq \Phi(G)$ for every group G .

SOLUTION: Denote $G' \cap Z(G)$ by D . If D is not a subgroup of $\Phi(G)$, there's a maximal subgroup M of G with D is not a subgroup of M . Therefore, $G = MD$, so that each $g \in G$ has a factorization $G = md$ with $m \in M$, $d \in D$.

Since $d \in Z(G)$, $gMg^{-1} = mdMd^{-1}m^{-1} = mMm^{-1} = M$, and so $M \triangleright G$. $\frac{G}{M}$ has prime order, hence is abelian. Therefore $G' \leq M$. But $D \leq G' \leq M$, contradicting D is not a subgroup of M .

In problem 19 we'll try to give the answer to the question if G has 2 subgroups A and B showing the same algebraic structure such that $(|G : A|, |G : B|) = 1$ then will G also contain the same property? In problem 19 A and B are abelian subgroups of G as we mentioned above.

PROBLEM 19: If G has abelian subgroups such that $(|G : A|, |G : B|) = 1$. Then G is soluble. This can be proved by the following argument.

Suppose the result false, and let G be an insoluble group of least possible order with abelian subgroups A, B such that $(|G : A|, |G : B|) = 1$. Then

i–) A and B are proper subgroups of G , and $G = AB$.

ii–) Let $1 < K \triangleleft G$. Then $\frac{G}{K}$ is soluble.

iii–) G is simple.

iv–) $A \cap B = 1$.

v–) A is a maximal subgroup of G .

vi–) Derive a contradiction by applying 9.

SOLUTION: *ii*–) $1 < K \triangleleft G$. If $\frac{G}{K}$ is insoluble, then $|\frac{G}{K}| = \frac{|G|}{|K|} < |G|$. Since G is an insoluble group of least possible order and $|\frac{G}{K}| < |G|$ we arrive a contradiction. So $\frac{G}{K}$ is soluble.

iii–) Let $K \triangleleft G$ and $K < G$. K is insoluble because if K is soluble then $\frac{G}{K}$ is soluble by (*ii*). So then G will be soluble. But G is assumed insoluble. Thus K is insoluble. But $|K| < |G|$ and G is insoluble of least possible order. Thus G has no non-trivial normal subgroup. G is simple.

iv–) We'll apply Theorem 1.41. $G = AB$. Since A is abelian, by Theorem 1.41 $A \cap B \neq \{1\}$ is wrong. Because if $A \cap B \neq \{1\}$, A is abelian so $A \cap B \triangleleft A$, $A \cap B \triangleleft B$. (B is also abelian.). Then $A \cap B \triangleleft AB$. Since $AB = G$ is simple and $A \cap B \neq \{1\}$, this idea is not true.

v–) Assume A is not a maximal subgroup of G , then we can find a $C < G$ such that $A < C < G = AB$. Since $A < C$ then $G = AB < CB \subseteq G$. So $G = CB$. $CB \leq G$, $CB = BC$. $G = BC$ and $A \cap B = \{1\} < B \cap C$, so $B \cap C \neq \{1\}$. But this is impossible by Theorem 1.41. So there's no proper subgroup C of G such that $A < C < G$. This shows A is a maximal subgroup of G .

vi–) We'll derive a contradiction by applying problem 9. Here A is an abelian maximal subgroup of G and G is an insoluble group of least possible order. Since we have solved problem 9 before we can directly write the following consequences:

i–) $A_G = 1$.

ii–) G' is the unique minimal normal subgroup of G .

iii–) A is a Hall subgroup of G .

iv–) G is p -nilpotent for every prime p dividing $|A|$.

v–) A is a complement to G' in G .

vi–) Every complement to G' in G is conjugate to A .

vii–) Every subgroup of G isomorphic to A is conjugate to A .

viii–) Let q be a prime divisor of $|G'|$ and let Q be a Sylow q -subgroup of G' . Then $N_G(Q)$ has a subgroup A^* isomorphic to A .

i × –) A^* is a maximal subgroup of G and $Q \triangleright G$.

× –) It follows G is soluble.

So we arrive a contradiction by assuming G insoluble.

PROBLEM 20: Show by an example that an insoluble group G can have soluble groups H_1 and H_2 such that $(|G : H_1|, |G : H_2|) = 1$.

SOLUTION: $G = A_5$. Since G is simple, for every $H < G$ $|G : H| \geq 5$ and therefore $|H| \leq 12$. And since $|G| = 60$, the possible orders of subgroups are 6, 10, 12. G has subgroups of order 12. For if we consider the natural action of G on the set $\{1, 2, 3, 4, 5\}$ then clearly $Stab_G(5) \cong A_4$. Let $H < G$ with $H \cong A_4$. Then H has a normal subgroup T of order 4 and it's unique. Because since

$H \cong A_4$ and $|A_4| = 12 = 2^2 \cdot 3$, the Sylow 2-subgroups of A_4 has degree 2^2 . And the number of sylow 2-subgroups of A_4 is $1+2k$ such that $1+2k \mid 12$. And we know that the only subgroup of A_4 which has degree 4 is $S = \{I, \sigma_2, \sigma_5, \sigma_8\}$. $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$, $\sigma_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$, $\sigma_8 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$. And $\sigma_2^2 = \sigma_2$, $\sigma_5^2 = \sigma_5$, $\sigma_8^2 = \sigma_8$. Then S is abelian. So S is the unique Sylow 2-subgroup of A_4 . Since it's unique, $S \triangleright A_4$. We have $H \cong A_4$, so $|T| = 4$ and $T \triangleright H$.

Now $H \leq N_G(T) < G$, since G is simple, $|H| = 12 \leq |N_G(T)| \leq 12$. Therefore $|N_G(T)| = 12$ means $N_G(T) = H$. And the number of Sylow 2-subgroups of G is equal to $|G : H| = 5$.

Since $H = N_G(T)$, we have $T \triangleright N_G(T) = H$. If we look at $\frac{N_G(T)}{T} = \{xT : x \in N_G(T)\}$; we can see easily that it's abelian.

Let $xT, yT \in N_G(T)$, then $(xT).(yT) = (Tx).(Ty) = Txy$. So $\frac{N_G(T)}{T}$ is abelian. And T is also abelian. $1 \triangleright T \triangleright N_G(T)$. $\frac{N_G(T)}{T}$ is abelian, $\frac{T}{1} = T$ is abelian. Thus $N_G(T) = H$ is soluble.

Let the numbers of Sylow 3-subgroups of and Sylow 5-subgroups of G be respectively n_3 and n_5 . Then n_3 divides $2^2 \times 5$ and $n_3 \equiv 1 \pmod{3}$, n_5 divides $2^2 \times 3$ and $n_5 \equiv 1 \pmod{5}$. Let U be any subgroup of G of order 3; V any subgroup of G of order 5, and $K = N_G(V)$. Then $|G : J| = n_3 = 10$ and $|G : K| = n_5 = 6$. Hence $|J| = 6$ and $|K| = 10$. $|V| = 5$, so V is cyclic since 5 is prime. Therefore V is abelian. $1 \triangleright V \triangleright N_G(V) = K$.

$\frac{N_G(V)}{V}$ is abelian. $\frac{V}{1} = V$ is abelian. Hence K is soluble. $|G : K| = 6$, $|G : H| = 5$ so $(|G : H|, |G : K|) = (5, 6) = 1$. So H and K are subgroups of A_5 which are soluble. A_5 is insoluble. $(|G : H|, |G : K|) = (|A_5 : H|, |A_5 : K|) = 1$. This is an example of such a group G .

PROBLEM 21: Let $G = GL_2(\mathbb{Z}_3)$ and $K = Z(G)$. Show that

- i*–) K is the unique maximal abelian normal subgroup of G ,
ii–) $K < F(G)$, and
iii–) K is not a maximal abelian subgroup of G .

SOLUTION: (*i*) Let $H = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in Z_3 \text{ with } ac \neq 0 \right\}$.

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} H = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : ac \neq 0 \right\}.$$

$H \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : ac \neq 0 \right\} = \left\{ \begin{pmatrix} a+b & a+b \\ c & c \end{pmatrix} : ac \neq 0 \right\}$. $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \in \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} H$, but $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \notin H \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. So $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} H \neq \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} H$. Thus H is not a normal subgroup of G . H is not abelian, because $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$. They are not equal. Now assume that L is an abelian normal subgroup of G such that $K < L < G$. Since $K < H < G$ and H is not abelian and normal in G we have two cases.

Case i–) $K < L < H < G$ or *Case ii*–) $K < H < L < G$. But in case *ii*–) since L is abelian, H must be abelian. But we have shown that H is not abelian. Thus we have only one condition: $K < L < H < G$. $|K| < |L| < |H|$, so $2 < |L| < 12$. So $|L| = 4$ or 6 . But $H_G = K$, here in this case $H_G = L$. So we arrive a contradiction. There's no abelian normal subgroup L of G such that $K < L < G$. So K is the maximal abelian normal subgroup of G .

Solution of (*ii*) is left to the reader.

iii–) We'll show K is not a maximal abelian subgroup of G . If $a = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$; $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = a^2$, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} = a^3$, $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = a^4$. Thus $a^4 = 1$.

$$\langle a \rangle = \left\{ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

ANSWERS TO SELECTED PROBLEMS MELIS MINISKER

Since $K = Z(G) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\} \langle a \rangle < GL_2(\mathbb{Z}_3)$ and $\langle a \rangle$ is abelian, hence K is not a maximal abelian subgroup of G .



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ÖZGEÇMİŞ

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