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GENELLEŞTİRİLMİŞ KESİRLİ İNTEGRALLER YARDIMIYLA İKİ DEĞİŞKENLİ FONKSİYONLAR İÇİN YENİ EŞİTSİZLİKLER

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DOKTORA TEZİ
MATEMATİK ANABİLİM DALI

DANIŞMAN
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SİMGELER

F'	F fonksiyonunun 1. Dereceden Türevi
F''	F fonksiyonunun 2. Dereceden Türevi
$J^{\varphi}_{\kappa_1+}F(\varkappa)$	Riemann-Liouville Sol Kesirli İntegral
$J^{\varphi}_{\kappa_2-}F(\varkappa)$	Riemann-Liouville Sağ Kesirli İntegral
$\mathbf{J}^{\varphi}_{\kappa_1+}F(\varkappa)$	Hadamard Sol Kesirli İntegral
$\mathbf{J}^{\varphi}_{\kappa_2-}F(\varkappa)$	Hadamard Sağ Kesirli İntegral
$\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa)$	Genelleştirilmiş Sol Kesirli İntegral
$\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa)$	Genelleştirilmiş Sağ Kesirli İntegral
$L_1[\kappa_1,\kappa_2]$	1. Dereceden (κ_1,κ_2) Aralığında İntegrallenebilen Fonksiyonlar Kümesi
$\mathbb{R}$	Reel Sayılar
$\mathbb{R}^n$	n-boyutlu Öklid Uzayı
Γ	Gamma Fonksiyonu

ÖZET

GENELLEŞTİRİLMİŞ KESİRLİ İNTEGRALLER YARDIMIYLA İKİ DEĞİŞKENLİ FONKSİYONLAR İÇİN YENİ EŞİTSİZLİKLER

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Bu tez altı bölümden oluşmaktadır. İlk bölüm tezle ilgili literatürdeki çalışmaları içermektedir. İkinci bölümde, bu tezde kullanılacak bazı temel tanım ve teoremler, bazı temel eşitsizlikler ve kesirli integral tanımları sunulmuştur. L_∞ ve L_p uzaylarına ait fonksiyonlar kullanılarak, iki değişkenli fonksiyonlar için genelleştirilmiş kesirli integralleri içeren bazı Trapezoid ve Ostrowski tipli eşitsizler sırasıyla üçüncü ve dördüncü bölümde ispatlanmıştır. Beşinci bölümde, sınırlı kısmi türevli fonksiyonlar yardımıyla genelleştirilmiş kesirli integraller için yeni Trapezoid ve Ostrowski tipli eşitsizlikler elde edilmiştir. Son olarak altıncı bölümde, tezde yapılan çalışmalar özetlenmiş ve sonraki çalışmalar için bazı öneriler verilmiştir.

Anahtar sözcükler: Kesirli integraller, Ostrowski tipli eşitsizlik, Trapezoid tipli eşitsizlik.

ABSTRACT

NEW INEQUALITIES FOR TWO VARIABLES FUNCTIONS VIA GENERALIZED FRACTIONAL INTEGRALS

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Graduate School, Department of Mathematics
Doctoral Thesis
Supervisor: Doç. Dr. Hüseyin BUDAK
February 2023, 81 pages

This thesis consists of six chapters. The first part includes the studies in the literature related to the thesis. In the second chapter, some basic definitions and theorems to be used in this thesis, some basic inequalities and definitions of fractional integral are presented. Using functions of the spaces L_∞ and L_p, some Trapezoid and Ostrowski type inequalities including generalized fractional integrals for functions of two variables are proved in the third and fourth sections, respectively. In the fifth chapter, new Trapezoid and Ostrowski type inequalities are obtained for generalized fractional integrals with the help of bounded partially derivative functions. Finally, in the sixth chapter, the studies carried out in the thesis are summarized and some suggestions for further studies are given.

Keywords: Fractional integrals, Ostrowski type inequality, Trapezoid type inequality.

EXTENDED ABSTRACT

NEW INEQUALITIES FOR TWO VARIABLES FUNCTIONS VIA GENERALIZED FRACTIONAL INTEGRALS

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1. INTRODUCTION

The fractional calculations started when the question "the derivative meaning of the $n-$ order integer could also be when n is not an integer" was raised. This question was asked by L'Hopital in 1695. In his letter one day, Leibniz asked the question of what the outcome would be when L'Hopital $n = 1/2$, and Leibniz replied, "a day like a paradox will come as a useful result." The n derivative of the $F(\varkappa) = \varkappa$ function is expressed by Leibniz as $d^n\varkappa/d\varkappa^n$. This is about Euler, Laplace, Fourier, Lacroix, Abel, Riemann and Liouville, many of the leading mathematicians have noticed. Several mathematicians, such as Riemann, Grunwald, Weyl [1], have identified fractional derivative operators using various approaches.

Later in the years, an article by Liouville laid the foundations of fractional derivative and fractional integral concepts. The main source of this idea is the concept of fractional derivative and fractional integral, "Are derivatives and internal products for integer only?" question [2].

In light of the latest developments in mathematics, the fractional calculation is in mathematical modeling of mechanical and electrical properties of objects, fluid theory, control theory, electrical circuits, electro-analytical chemistry, it is very popular in many areas, such as heat conduction, computational analysis and engineering.

The first detailed study of the theory of inequality began to enter literature in early 1900. Today, the theory of inequality is systematically studied, and the work on this topic is increasing every day. The presence of integrated inequalities that help to determine the limits for unknown functions is very important for the differential theory.

One of the basic mathematical inequalities is found in the literature as "Ostrowski inequality". Ostrowski inequality has statistical, probability and optimization theory, stochastic, knowledge and integrated operator theory applications. S. In 2001, on the Dragomir side, Ostrowski contributed by expanding its inequality.

Since the last 20 years, the field of inequality has been a remarkable development. Many research articles have been published on inequalities called Cebysev, Gruss, Trapezoid, Ostrowski, Hadamard and Jensen in particular. Some of the research and monographies published over the last few years have been an important part of the progress in the field of inequality.

The purpose of this thesis is to obtain Trapezoid and Ostrowski type inequalities for two variable functions with the help of fractional integrals and to obtain some generalized fractional Trapezoid and Ostrowski type inequalities for bounded partially derivative functions.

2. MATERIAL AND METHODS

First, the definition of convexity and characteristics of the convex function that ensure Hermite-Hadamard inequality are given. Some important inequalities such as Holder inequality and Minkowski inequality are also presented. Later, we recall the definitions Riemann-Liouville fractional integrals, Hadamard fractional integrals, and generalized fractional integrals for functions of both single and two variables. Moreover, we give some Ostrowski and trapezoid type inequalities including these fractional integrals.

3. RESULTS AND DISCUSSIONS

The main results of this thesis are given in sections 3, 4 and 5. In the Sections 3 and 4, Trapezoid and Ostrowski type inequalities are obtained for functions of two variable with the help of generalized fractional integrals. The results to be presented in both sections have been in the form of generalizations of previously existing studies in the literature.

Section 5 is presented under the three titles. In the first part, some Trapezoid-type inequalities are created for generalized fractional integrals by using an equality obtained for twice partial derivative functions. The second part presents some Ostrowski type inequalities for bounded partially derivative functions. In the third part, some new trapezoid-type inequalities are established for generalized fractional integrals using the results in the previous section.

4. CONCLUSION AND OUTLOOK

In this thesis, we present several trapezoid and Ostrowski type inequalities including generalized fractional integrals for functions of two variables. In the future studies, the inequalities obtained in this thesis can also be proved for different types fractional integrals.In addition, by using the technique used in this thesis, one can obtain some important inequalities such as Simpson, Bullen, Newton etc.

1. GİRİŞ

1695 yılında L'Hospital tarafından Leibniz'e $\dfrac{d^n}{d\varkappa^n}$ notasyonunun $n = \dfrac{1}{2}$ için ne anlama geldiği sorusuyla ilk kez kesirli türev ve integral kavramı ortaya çıkmıştır. Kesirli türev ve integral alışılmış tamsayı yöntemlerine göre keyfi mertebeli türev ve integraller uygulamalarda çok daha doğru sonuçlar vermektedir. Günümüzde kesirli türev ve integralleri içeren problemler için büyük çapta yaklaşım yöntemleri geliştirilmiştir. Hızlı değişimi tam mertebeli türev ve integral yerine kesirli türev ve integraller ile ifade ve temsil etmek daha uygun olmaktadır.

Leibniz ve L'Hospital den sonra Euler 1730 da, J.L. Lagrange ise 1772 yılında, 1812 yılında P.S. Laplace, 1819 yılında S.F. Lacroix, J.B.J. Fourier 1822 de kesirli analiz üzerinde çalışmalar yapmışlardır. 1823 yılında N.H. Abel kesirli operatörleri kullanmıştır. J. Liouville 1832 yılında kesirli analiz tanımlarını teorik problemlere uygulayarak ilk önemli çalışmayı yapmıştır. 1867 yılında A.K. Grünwald, G.F.B. Riemann 1892 de, A.V. Letnikov, 1868 ile 1872 yılları arasında kesirli analize katkıda bulunmuşlardır. 1900 ve 1970 yılları arasında H.H. Hardy, S. Sambo, H. Weyl [1], M. Riezs, S. Blair gibi matematikçiler konuya katkılarını sunmuşlardır. 1970 den sonra J. Spanier, K.B. Oldham, B. Ross, K. Nishimoto, R. Bagley, K.S. Miller, M. Caputo, S.G. Samko, A.A. Kilbas, O.I. Marichev çalışmalar ortaya koymuşlardır [2], [3]. Kesirli analiz, kuantum mekaniği, difüzyon, elektrik devreleri, sıvı analizi, akışlar teorisi gibi bir çok konuda kullanılmaktadır [4]-[7].

Eşitsizlik ile ilgili çalışmalar uygulamalı matematiğin gelişmesinde önemli yer tutmaktadır. Bu kavram 1900 lü yıllardan beri neredeyse tüm alanlarda kullanılmıştır. Mühendislikte optimizasyon, mukavemet, haritalarda bir yerin tahmin edilmesinde, nümerik integrasyonda hata hesaplarında kullanılmaktadır. Konveks fonksiyon eş zamanlı olarak bulunup, tanımı ve eşitsizlik olarak görülmüştür. Çoğu eşitsizlikler konveks fonksiyon ile elde edilmiştir. 1883'te Hermite'nin oluşturduğu eşitsizlik daha sonra Hadamard tarafından ele alınarak aşağıdaki Hermite-Hadamard eşitsizliği elde edilmiştir.

$\mathcal{F}: I \subset \mathbb{R} \to \mathbb{R}$ fonksiyonu konveks ise $a, b \in I$ olmak üzere $a < b$ için,

$$\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)d\varkappa \leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}, \tag{1.1}$$

eşitsizliği elde edilir.

Hermite-Hadamard eşitsizliğinin sağ tarafı olan,

$$\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2} - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)d\varkappa,$$

farkı için Dragomir ve Agarwal [8], Yamuk tipli (Trapezoid),

Hermite-Hadamard eşitsizliğinin sol tarafı olan,

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)d\varkappa - \mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right),$$

farkı için Kırmacı [9], Orta-Nokta tipli (midpoint) için üst sınırlar elde etmişlerdir. Sarıkaya ve ark. [10] tarafından kesirli integraller için (1.1) eşitsizliklerini genelleştirdi ve yazarlar ayrıca bazı karşılık gelen trapezoid tipli eşitsizlikleri kanıtlamıştır. İqbal ve ark. [11]' de konveks fonksiyonlar için bazı kesirli orta nokta tipli eşitsizlikler sunmuşlardır.

Diğer bir eşitsizlik ise adını A.M. Ostrowski den alan 1938 yılında literatüre girmiş olan Ostrowski eşitsizliğidir. Bu eşitsizlik aşağıdaki şekilde ifade edilir.

$\mathcal{F}: [\kappa_1, \kappa_2] \to \mathbb{R}$, (κ_1, κ_2) üzerinde türevlenebilir bir fonksiyon olsun. (κ_1, κ_2) aralığında $\mathcal{F}': (\kappa_1, \kappa_2) \to \mathbb{R}$ de sınırlı başka bir ifade ile $\|\mathcal{F}'\|_\infty = \sup\limits_{t \in (\kappa_1, \kappa_2)} |\mathcal{F}'(t)| < +\infty$, $\forall \varkappa \in [\kappa_1, \kappa_2]$ olmak üzere,

$$\left|\mathcal{F}(\varkappa) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(t)dt\right| \leq \left[\frac{1}{4} + \frac{\left(\varkappa - \frac{\kappa_1+\kappa_2}{2}\right)^2}{(\kappa_2-\kappa_1)^2}\right](\kappa_2-\kappa_1)\left\|\mathcal{F}'\right\|_\infty,$$

eşitsizliği elde edilir [12]. Burada $\frac{1}{4}$ sabiti mümkün olan en iyi sabittir. Ostrowski eşitsizliği fonksiyonun herhangi bir $\varkappa \in [\kappa_1, \kappa_2]$ noktasının görüntüsü ile

$$\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\eta) d\eta,$$

integral ortalama arasındaki fark için bir üst sınır vermiştir.

Ostrowski tipli eşitsizliklerin birçok varyasyonu, konveks fonksiyonlar, sınırlı fonksiyonlar, sınırlı varyasyonlu fonksiyonlar (teorik) gibi çeşitli fonksiyon sınıfları için incelenmiştir. Özellikle, konvekslik teorisi, saf ve uygulamalı matematiğin farklı dallarından çok sayıda problemi çözmenin etkili ve güçlü bir yolu olduğundan, birçok bilim adamı konveks fonksiyonlar için Ostrowski eşitsizliğine yönelmiştir. Örneğin, Alomari ve ark. [13] tarafından ξ-konveks fonksiyonlar için bazı Ostrowski tipli eşitsizlikler kanıtlamışlardır. Ayrıca, diğer konvekslik türleri için Ostrowski tipli eşitsizlikler birçok farklı alanda incelenmiştir [14]-[17]. [18]' de ilk olarak ξ-konveks fonksiyonlar için Ostrowski eşitsizliğinin Riemann-Liouville kesirli versiyonunu elde etmiştir. Buna ek olarak, birçok araştırmacı k-Riemann-Liouville kesirli integraller, local kesirli integraller, Raina kesirli integraller gibi belirli kesirli integral operatörleri için Ostrowski tipli eşitsizlikler elde etmişlerdir. Bu alanda bir çok çalışma yapılmıştır [19]-[32].

Diğer bir konuda çok değişkenli fonksiyonların uygulama alanları tek değişkenli fonksiyonlara göre daha fazladır. Güncel hayat problemleri birden fazla değişken içerir. Basıncın hesaplanması, cisimlerin hacimleri gibi, bağımsız değişken sayısı birden çok ise tek değişkenli fonksiyonlar kullanılmaz. Çok değişkenli fonksiyonları sorunsuz bir şekilde hesaplamak için çok katlı integrallere ihtiyaç duyulmuştur.

Çok katlı integralleri ilk olarak Newton kullanmıştır. 18. yüzyılda Leonhard Euler sınırlı bölgede incelemeler yapmıştır. Joseph Lous Lagrange 1775 yılında çok katlı integralleri kullanmıştır. 19. yüzyılda ise çok katlı integraller yaygın olarak kullanılmıştır. Bu konuda en temel eşitsizlikler Ostrowski ve Hermite-Hadamard tipli eşitsizliklerin iki değişkenli versiyonlarıdır. Bu iki değişkenli konveks fonksiyonlar için çift katlı Riemann integralleri ve çift katlı Riemann Liouville kesirli integralleri içeren Ostrowski eşitsizlikleri sırasıyla [33] ve [34] makalelerinde sunulmuştur. Öte yandan Dragomir, [35]' de iki değişkenli konveks fonksiyonlar için Hermite-Hadamard eşitsizliklerini kanıtlamıştır. İki değişkenli konveks fonksiyonlar için orta nokta ve trapezoid tipli eşitsizlikler sırasıyla [36] ve [37] makalelerinde kanıtlamışlardır. Ayrıca Sarıkaya, [38]' de iki değişkenli fonksiyonlar için kesirli Hermite-Hadamard ve kesirli trapezoid tipli eşitsizlikleri elde etmiştir. Tunç ve ark. [39]'de iki değişkenli konveks fonksiyonlar için bazı kesirli orta nokta tipli eşitsizlikleri sunmuşlardır. Diğer benzer eşitsizlikler [40]-[45] referanslardan görülebilir.

Bu tezde asıl amaç, genelleştirilmiş kesirli integraller yardımıyla iki değişkenli fonksiyonlar için Trapezoid ve Ostrowski tipli eşitsizlerin elde edilmesi ve sınırlı kısmi türevli fonksiyonlar için bazı genelleştirilmiş kesirli Trapezoid ve Ostrowski tipli eşitsizliklerin elde edilmesidir.

Bu tezin geri kalan kısmı aşağıdaki gibi düzenlenmiştir.

Kesirli türev ve integral kavramının ortaya çıkışı, beraberinde getirdikleri, kavramların tarihsel gelişimleri tezin giriş kısmını oluşturmaktadır.

Tezin ikimci bölümünde, temel tanım ve teoremler, bazı temel eşitsizlikler ve kesirli integrallere yer verilmiştir.

Tezin üçüncü ve dördüncü bölümünde, kesirli integraller yardımıyla iki değişkenli fonksiyonlar için Trapezoid ve Ostrowski tipli eşitsizler elde edilmiştir.

Tezin beşinci bölümünde ise sınırlı kısmi türevli fonksiyonlar için bazı genelleştirilmiş kesirli Trapezoid ve Ostrowski tipli eşitsizlikler elde edilmiştir.

Tezin altıncı bölümünde tezden elde edilen sonuçlar ve öneriler ifade edilmiştir.

2. TEMEL KAVRAMLAR

Bu bölümde, temel tanım ve teoremlere yer verilecektir.

2.1. TEMEL TANIM VE TEOREMLER

Tanım 2.1. $\varkappa > 0$ olmak üzere,

$$\Gamma(\varkappa) = \int_0^\infty e^{-\eta} \eta^{\varkappa-1} d\eta,$$

şeklinde tanımlanan fonksiyona Gamma fonksiyonu denir.

Tanım 2.2. $\digamma(\varkappa)$ fonksiyonu $[\kappa_1, \kappa_2]$ aralığında tanımlı olsun. Her $\varkappa \in [\kappa_1, \kappa_2]$ için $|\digamma(\varkappa)| \leq M$ olacak şekilde bir $M > 0$ sayısı varsa, $\digamma(\varkappa)$ fonksiyonu $[\kappa_1, \kappa_2]$ aralığında sınırlıdır denir.

Tanım 2.3. $A \subset \mathbb{R}$, $\digamma : A \to \mathbb{R}$ bir fonksiyon ve $\kappa_1 \in A$ olsun. $\forall \varepsilon > 0$ sayısı için $|\varkappa - \kappa_1| < \delta$ olduğunda $|\digamma(\varkappa) - \digamma(\kappa_1)| < \varepsilon$ olacak şekilde öyle bir $\delta > 0$ sayısı varsa $\digamma$ fonksiyonuna $\kappa_1 \in A$ noktasında süreklidir denir.

Tanım 2.4. $A \subset \mathbb{R}$, $\digamma : A \to \mathbb{R}$ bir fonksiyon olsun. $\forall \varepsilon > 0$ sayısı için $|\varkappa - \kappa_1| < \delta$ eşitsizliğini sağlayan $\forall \varkappa, \kappa_1 \in A$ için $|\digamma(\varkappa) - \digamma(\kappa_1)| < \varepsilon$ olacak şekilde öyle bir $\delta > 0$ sayısı varsa $\digamma$ fonksiyonu A üzerinde düzgün süreklidir denir.

Tanım 2.5. $\digamma : I \to \mathbb{R}$, I içerisindeki örtüşmeyen aralıkların sonlu ailesi $\{[l_i, m_i]\,; 1 \leq i \leq n\}$ olmak üzere, bu aralıkların uzunlukları toplamı,

$$\sum_{i=1}^{n} |m_i - l_i| < \delta \quad \text{ise} \quad \sum_{i=1}^{n} |\digamma(m_i) - \digamma(l_i)| < \varepsilon, \tag{2.1}$$

olacak şekilde $\delta > 0$ varsa $\digamma$ fonksiyonu I üzerinde mutlak süreklidir.

(i) (2.1) ifadesinde $n = 1$ olarak alındığında,

$$|m_1 - l_1| < \delta \quad \text{ise} \quad |\digamma(m_1) - \digamma(l_1)| < \varepsilon,$$

olacak şekilde $\delta > 0$ varsa $\digamma$ fonksiyonu I üzerinde düzgün süreklidir. Böylece mutlak sürekli her fonksiyon düzgün süreklidir. Ayrıca, $\digamma$ ve g mutlak sürekli fonksiyonlar ise $|\digamma|$, $\digamma + g$, $\digamma - g$ ve $\digamma.g$ ifadelerindeki fonksiyonlarda mutlak süreklidir.

(*ii*) (2.1) ifadesinde $n = 1$ ve l_i ler sabit olarak seçilirse,

$$|m_1 - \kappa_1| < \delta \quad \text{ise} \quad |\digamma(m_1) - \digamma(\kappa_1)| < \varepsilon,$$

olacak şekilde $\delta > 0$ varsa $\digamma$ fonksiyonu I üzerinde süreklidir. Böylece düzgün sürekli her fonksiyon süreklidir.

(*iii*) $|\digamma(\kappa_2) - \digamma(\kappa_1)| < k.|\kappa_2 - \kappa_1|$ olacak şekilde $k \in \mathbb{R}$ varsa $\digamma$ fonksiyonunda k sabitine bağlı Lipschitz süreklidir denir.

$\digamma$ mutlak sürekli bir fonksiyon ise bu fonksiyon sınırlı varyasyonlu fonksiyondur. $\digamma$ in türev fonksiyonu hemen hemen her yerde vardır.

Tanım 2.6. $\digamma(\varkappa)$ fonksiyonu I aralığında tanımlı bir fonksiyon olsun. $\varkappa_1 < \varkappa_2$ şartını sağlayan $\forall \varkappa_1, \varkappa_2 \in I$ için,

(*i*) $\digamma(\varkappa_1) < \digamma(\varkappa_2)$ ise $\digamma$ fonksiyonu I üzerinde artandır.

(*ii*) $\digamma(\varkappa_1) \leq \digamma(\varkappa_2)$ ise $\digamma$ fonksiyonu I üzerinde azalmayandır.

(*iii*) $\digamma(\varkappa_1) > \digamma(\varkappa_2)$ ise $\digamma$ fonksiyonu I üzerinde azalandır.

(*iv*) $\digamma(\varkappa_1) \geq \digamma(\varkappa_2)$ ise $\digamma$ fonksiyonu I üzerinde artmayandır.

Teorem 2.7. I, $\mathbb{R}$ de bir aralık olmak üzere $\digamma$, I üzerinde sürekli ve I° üzerinde diferansiyellenebilir bir fonksiyon olsun, bu durumda $\forall \varkappa \in I^{\circ}$ için,

(*i*) $\digamma'(\varkappa) > 0$ ise $\digamma$ fonksiyonu I üzerinde artandır.

(*ii*) $\digamma'(\varkappa) \geq 0$ ise $\digamma$ fonksiyonu I üzerinde azalmayandır.

(*iii*) $\digamma'(\varkappa) < 0$ ise $\digamma$ fonksiyonu I üzerinde azalandır.

(*iv*) $\digamma'(\varkappa) \leq 0)$ ise $\digamma$ fonksiyonu I üzerinde artmayandır.

Teorem 2.8. fonksiyonu (κ_1, κ_2) aralığında diferensiyellenebilir bir fonksiyon olsun. Bu durumda $\digamma$ fonksiyonunun konveks (kesin konveks) olması için gerek ve yeter şart $\digamma'$ nin artan (kesin artan) olmasıdır.

Teorem 2.9. $\digamma$ fonksiyonunun I açık aralığında ikinci türevi mevcutsa, $\digamma$ fonksiyonunun bu aralık üzerinde konveks olması için gerek ve yeter şart $\forall\varkappa \in I$ için,

$$\digamma''(\varkappa) \geq 0,$$

olmasıdır.

Tanım 2.10. $\digamma : I \to \mathbb{R}$ fonksiyonu $\varkappa, \gamma \in I$ ve $\eta \in [0,1]$ için,

$$\digamma(\eta\varkappa + (1-\eta)\gamma) \leq \eta\digamma(\varkappa) + (1-\eta)\digamma(\gamma),$$

eşitsizliğini sağlıyorsa bu fonksiyona konveks fonksiyon denir. Eşitsizlik yön değiştirirse $\digamma$ fonksiyonuna konkav fonksiyon denir.

2.1.1. Konveks Fonksiyonların özellikleri

(a_1) Kapalı aralıkta tanımlı konveks fonksiyon sınırlıdır.

(a_2) $\digamma : I \to \mathbb{R}$ konveks fonksiyon ise I° de herhangi bir $[\kappa_1, \kappa_2]$ aralığında Lipschitz şartını sağlar. Bu nedenle $\digamma$ fonksiyonu $[\kappa_1, \kappa_2]$ aralığında mutlak sürekli ve I° de süreklidir.

(a_3) $\digamma : I \to \mathbb{R}$ konveks fonksiyon ise I° de $\digamma'_-(\varkappa)$ ve $\digamma'_+(\varkappa)$ vardır ve artandır.

(a_4) $\digamma : I \to \mathbb{R}$ fonksiyonu I açık aralığında konveks ise, sayılabilir bir E kümesi haricinde $\digamma'$ var ve süreklidir.

(a_5) $\digamma_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, 3, ..., k$ fonksiyonları konveks olsun. Bu durumda $e_j > 0$ olmak üzere,

$$\digamma(\varkappa) = \sum_{j=1}^{k} e_j \digamma_j(k),$$

fonksiyonu da konvekstir.

(a_6) $g : \mathbb{R} \to \mathbb{R}$ azalmayan ve konveks fonksiyon ayrıca $h : \mathbb{R}^n \to \mathbb{R}$ konveks olsun. Bu durumda $\digamma : \mathbb{R}^n \to \mathbb{R}$, $\digamma(\varkappa) = (g \circ h)(\varkappa)$ olarak tanımlanan $\digamma$ bileşke fonksiyonuda konvekstir.

(a_7) $g:\mathbb{R}^m \to \mathbb{R}$ konveks ve $h:\mathbb{R}^n \to \mathbb{R}$ fonksiyonu ise $h(\varkappa) = A\varkappa + B$ formunda konveks olmak üzere $F(\varkappa) = g(h(\varkappa))$ fonksiyonuda konvekstir.

Tanım 2.11. $F(\varkappa,\gamma)$ fonksiyonu $T = \{(\varkappa,\gamma); -\infty < \kappa_1 < \kappa_2 < \infty,\ -\infty < \kappa_3 < \kappa_4 < \infty\}$ bölgesinde sürekli bir fonksiyon olmak üzere,

$$\int_{\kappa_1}^{\kappa_2}\left(\int_{\kappa_1}^{\varkappa} F(\varkappa,\gamma)d\gamma\right)d\varkappa = \int_{\kappa_1}^{\kappa_2}\left(\int_{\gamma}^{\kappa_2} F(\varkappa,\gamma)d\varkappa\right)d\gamma,$$

eşitliğine Dirichlet Formülü denir.[2]

Tanım 2.12. $T = \{(\varkappa,\gamma); \kappa_1 \leq \varkappa \leq \kappa_2,\ \kappa_3 \leq \gamma \leq \kappa_4\}$ ve $F : T \to \mathbb{R}$ fonksiyonu sürekli olsun. Bu durumda,

$$\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,\gamma)d\gamma d\varkappa = \int_{\kappa_3}^{\kappa_4}\int_{\kappa_1}^{\kappa_2} F(\varkappa,\gamma)d\varkappa d\gamma,$$

şeklindeki eşitliğe Fubini Teoremi denir.

Tanım 2.13. $\kappa_1,\kappa_2 \in \mathbb{R}$ olmak üzere, $L_1(\kappa_1,\kappa_2)$ uzayı

$$\|F\|_1 = \int_{\kappa_1}^{\kappa_2} |F(\varkappa)|d\varkappa < \infty,$$

normu ile verilen fonksiyonların uzayıdır.

Tanım 2.14. $1 \leq p \leq \infty$ olmak üzere $\mathbb{R}$ üzerinde,

$$L_p = L_p(\mathbb{R}) = \left\{F : \|F\|_p = \left(\int_{-\infty}^{\infty} |F(\varkappa)|^p\right)^{1/p} < \infty,\right\},$$

şeklinde verilen F fonksiyonların uzayına $L_p$uzayı denir.

Tanım 2.15. $\varkappa,\gamma \in \mathbb{R}$ olmak üzere,

$$|\varkappa+\gamma| \leq |\varkappa| + |\gamma|,$$

şeklinde eşitsizliğe Üçgen Eşitsizlği denir.

Tanım 2.16. $\digamma$ fonksiyonu $[\kappa_1, \kappa_2]$ aralığında sürekli ve reel değerli bir fonksiyon olsun. Bu durumda,

$$\left|\int_{\kappa_1}^{\kappa_2} \digamma(\varkappa)d\varkappa\right| \leq \int_{\kappa_1}^{\kappa_2} |\digamma(\varkappa)|d\varkappa,$$

eşitsizliğine integraller için Üçgen Eşitsizliği denir.

Teorem 2.17. $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)$ ve $\rho = (\rho_1, \rho_2, ..., \rho_n)$ reel veya kompleks sayıların iki n-lisi olsun. Bu takdirde,

$$\frac{1}{p} + \frac{1}{q} = 1,$$

olmak üzere,

(a) $p > 1$ ise,

$$\sum_{k=1}^{n} |\varphi_k \rho_k| \leq \left(\sum_{k=1}^{n} |\varphi_k|^p\right)^{1/p} \left(\sum_{n=1}^{n} |\rho_k|^q\right)^{1/q},$$

(b) $p < 0$ veya $q < 0$ ise,

$$\sum_{k=1}^{n} |\varphi_k \rho_k| \geq \left(\sum_{k=1}^{n} |\varphi_k|^p\right)^{1/p} \left(\sum_{n=1}^{n} |\rho_k|^q\right)^{1/q},$$

eşitsizliklerine Ayrık Hölder Eşitsizliği denir [46].

Teorem 2.18. $p > 1$ ve $\frac{1}{p} + \frac{1}{q} = 1$ olsun. $\digamma$ ve g $[\kappa_1, \kappa_2]$ aralığında tanımlı reel fonksiyonlar $|\digamma|^p$ ve $|g|^q$, $[\kappa_1, \kappa_2]$ aralığında integrallenebilir fonksiyonlar ise,

$$\int_{\kappa_1}^{\kappa_2} |\digamma(\varkappa)g(\varkappa)|d\varkappa \leq \left(\int_{\kappa_1}^{\kappa_2} |\digamma(\varkappa)|^p d\varkappa\right)^{1/p} \left(\int_{\kappa_1}^{\kappa_2} |g(\varkappa)|^q d\varkappa\right)^{1/q},$$

eşitsizliğine integraller için Hölder Eşitsizliği denir.

Teorem 2.19. $1 \leq p \leq \infty$ ve $\delta_1 = [\kappa_1, \kappa_2]$, $\delta_2 = [\kappa_1, \kappa_2]$ olmak üzere δ_1 ve δ_2 kümeleri üzerinde ölçülebilir $F(\varkappa, \gamma)$ fonksiyonu için,

$$\left\{ \int_{\delta_1} \left(\int_{\delta_2} F(\varkappa, \gamma) d\varkappa \right)^p d\gamma \right\}^{1/p} \leq \int_{\delta_2} \left(\int_{\delta_1} |F(\varkappa, \gamma)|^p d\gamma \right)^{1/p} d\varkappa,$$

şeklindeki eşitsizliğe integraller için Minkowski Eşitsizliği denir.

2.2. KESİRLİ İNTEGRALLER

Tanım 2.20. $F \in L_1[\kappa_1, \kappa_2]$ olsun. $\varphi > 0$ olmak üzere $J^{\varphi}_{\kappa_1+}F$ ve $J^{\varphi}_{\kappa_2-}F$ Riemann - Liouville kesirli integralleri sırasıyla,

$$J^{\varphi}_{\kappa_1+}F(\varkappa) = \frac{1}{\Gamma(\varphi)} \int_{\kappa_1}^{\varkappa} (\varkappa - \eta)^{\varphi-1} F(\eta) d\eta, \quad \varkappa > \kappa_1,$$

ve

$$J^{\varphi}_{\kappa_2-}F(\varkappa) = \frac{1}{\Gamma(\varphi)} \int_{\varkappa}^{\kappa_2} (\eta - \varkappa)^{\varphi-1} F(\eta) d\eta, \quad \varkappa < \kappa_2,$$

şeklinde tanımlanır. Burada $\Gamma(\varphi)$ Gamma fonksiyonu ve $J^{0}_{\kappa_1+}F(\varkappa) = J^{0}_{\kappa_2-}F(\varkappa) = F(\varkappa)$ dır [2].

Tanım 2.21. $F \in L_1([\kappa_1, \kappa_2])$ olsun. $\varphi > 0$ ile $\kappa_1 \geq 0$ olmak üzere $\mathbf{J}^{\varphi}_{\kappa_1+}F$ ve $\mathbf{J}^{\varphi}_{\kappa_2-}F$ Hadamard kesirli integralleri sırasıyla,

$$\mathbf{J}^{\varphi}_{\kappa_1+}F(\varkappa) = \frac{1}{\Gamma(\varphi)} \int_{\kappa_1}^{\varkappa} \left(\ln \frac{\varkappa}{\eta} \right)^{\varphi-1} \frac{F(\eta)}{\eta} d\eta, \quad \varkappa > \kappa_1,$$

ve

$$\mathbf{J}^{\varphi}_{\kappa_2-}F(\varkappa) = \frac{1}{\Gamma(\varphi)} \int_{\varkappa}^{\kappa_2} \left(\ln \frac{\eta}{\varkappa} \right)^{\varphi-1} \frac{F(\eta)}{\eta} d\eta, \quad \varkappa < \kappa_2,$$

şeklinde tanımlanır [2, 3].

Tanım 2.22. g, $(\kappa_1, \kappa_2]$ üzerinde integrallenebilen fonksiyon, $g : [\kappa_1, \kappa_2] \to \mathbb{R}$ monoton artan bir fonksiyon olsun. $g'(\varkappa)$ de (κ_1, κ_2) üzerinde sürekli bir fonksiyon $\varphi > 0$ ile $\kappa_1 \geq 0$

olmak üzere $(\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa))$ ve $(\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa))$ kesirli integralleri sırasıyla,

$$\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa) = \frac{1}{\Gamma(\varphi)} \int_{\kappa_1}^{\varkappa} \frac{g'(\eta)F(\eta)}{[g(\varkappa)-g(\eta)]^{1-\varphi}} d\eta, \quad \varkappa > \kappa_1, \tag{2.2}$$

ve

$$\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa) = \frac{1}{\Gamma(\varphi)} \int_{\varkappa}^{\kappa_2} \frac{g'(\eta)F(\eta)}{[g(\eta)-g(\varkappa)]^{1-\varphi}} d\eta, \quad \varkappa < \kappa_2, \tag{2.3}$$

şeklinde ifade edilir. Bu integrallere sırasıyla sağ ve sol genelleştirilmiş kesirli integralleri denir [2].

Tanım 2.22 ü kullanılarak, bazı özel seçimlerle iyi bilinen kesirli integraller elde edilebilir. Örneğin;

1. $g(\eta) = \eta$ olarak alındığında (2.2) ve (2.3) operatörleri sırasıyla $J^{\varphi}_{\kappa_1+}F(\varkappa)$ ve $J^{\varphi}_{\kappa_2-}F(\varkappa)$ Riemann-Liouville kesirli integrallerine indirgenir.
2. $g(\eta) = \ln\eta$ alındığında (2.2) ve (2.3) operatörleri sırasıyla $\mathbf{J}^{\varphi}_{\kappa_1+}F(\varkappa)$, $\mathbf{J}^{\varphi}_{\kappa_2-}F(\varkappa)$ Hadamard kesirli integrallerine indirgenir.

İki değişkenli bir fonksiyonun Riemann - Liouville kesirli integralleri şu şekilde verilebilir.

Tanım 2.23. $F \in L_1(\Delta = [\kappa_1, \kappa_2] \times [\kappa_3, \kappa_4])$ olsun. $\varphi, \rho > 0$ olmak üzere $J^{\varphi,\rho}_{\kappa_1+,\kappa_3+}F$, $J^{\varphi,\rho}_{\kappa_1+,\kappa_4-}F$, $J^{\varphi,\rho}_{\kappa_2-,\kappa_3+}F$ ve $J^{\varphi,\rho}_{\kappa_2-,\kappa_4-}F$ Riemann-Liouville kesirli integralleri sırasıyla,

$$J^{\varphi,\rho}_{\kappa_1+,\kappa_3+}F(\varkappa,\gamma) = \frac{1}{\Gamma(\varphi)\Gamma(\rho)} \int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma} (\varkappa-\eta)^{\varphi-1}(\gamma-\xi)^{\rho-1}F(\eta,\xi)d\xi d\eta, \quad \varkappa > \kappa_1,\ \gamma > \kappa_3,$$

$$J^{\varphi,\rho}_{\kappa_1+,\kappa_4-}F(\varkappa,\gamma) = \frac{1}{\Gamma(\varphi)\Gamma(\rho)} \int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4} (\varkappa-\eta)^{\varphi-1}(\xi-\gamma)^{\rho-1}F(\eta,\xi)d\xi d\eta, \quad \varkappa > \kappa_1,\ \gamma < \kappa_4,$$

$$J^{\varphi,\rho}_{\kappa_2-,\kappa_3+}F(\varkappa,\gamma) = \frac{1}{\Gamma(\varphi)\Gamma(\rho)} \int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma} (\eta-\varkappa)^{\varphi-1}(\gamma-\xi)^{\rho-1}F(\eta,\xi)d\xi d\eta, \quad \varkappa < \kappa_2,\ \gamma > \kappa_3,$$

ve

$$J^{\varphi,\rho}_{\kappa_2-,\kappa_4-}F(\varkappa,\gamma)=\frac{1}{\Gamma(\varphi)\Gamma(\rho)}\int\limits_{\varkappa}^{\kappa_2}\int\limits_{\gamma}^{\kappa_4}(\eta-\varkappa)^{\varphi-1}(\gamma-\xi)^{\rho-1}F(\eta,\xi)d\xi d\eta,\quad \varkappa<\kappa_2,\ \gamma<\kappa_4,$$

şeklinde tanımlanır.

İki değişkenli bir fonksiyonun Hadamart kesirli integralleri şu şekilde verilebilir.

Tanım 2.24. $F\in L_1(\Delta=[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4])$ olsun. $\varphi,\rho>0$ ile $\kappa_1,\kappa_3\geq 0$ olmak üzere $\mathbf{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+}F$, $\mathbf{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-}F$, $\mathbf{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+}F$ ve $\mathbf{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-}F$ Hadamard kesirli integralleri sırasıyla, $\varkappa>\kappa_1$, $\gamma>\kappa_3$ için

$$\mathbf{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+}F(\varkappa,\gamma)=\frac{1}{\Gamma(\varphi)\Gamma(\rho)}\int\limits_{\kappa_1}^{\varkappa}\int\limits_{\kappa_3}^{\gamma}\left(\ln\frac{\varkappa}{\eta}\right)^{\varphi-1}\left(\ln\frac{\gamma}{\xi}\right)^{\rho-1}\frac{f(\eta,\xi)}{\eta\xi}d\xi d\eta,$$

$\varkappa>\kappa_1$, $\gamma<\kappa_4$ için

$$\mathbf{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-}F(\varkappa,\gamma)=\frac{1}{\Gamma(\varphi)\Gamma(\rho)}\int\limits_{\kappa_1}^{\varkappa}\int\limits_{\gamma}^{\kappa_4}\left(\ln\frac{\varkappa}{\eta}\right)^{\varphi-1}\left(\ln\frac{\xi}{\gamma}\right)^{\rho-1}\frac{f(\eta,\xi)}{\eta\xi}d\xi d\eta,$$

$\varkappa<\kappa_2$, $\gamma>\kappa_3$ için

$$\mathbf{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+}F(\varkappa,\gamma)=\frac{1}{\Gamma(\varphi)\Gamma(\rho)}\int\limits_{\varkappa}^{\kappa_2}\int\limits_{\kappa_3}^{\gamma}\left(\ln\frac{\eta}{\varkappa}\right)^{\varphi-1}\left(\ln\frac{\gamma}{\xi}\right)^{\rho-1}\frac{F(\eta,\xi)}{\eta\xi}d\xi d\eta,$$

$\varkappa<\kappa_2$ $\gamma<\kappa_4$ için

$$\mathbf{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-}F(\varkappa,\gamma)=\frac{1}{\Gamma(\varphi)\Gamma(\rho)}\int\limits_{\varkappa}^{\kappa_2}\int\limits_{\gamma}^{\kappa_4}\left(\ln\frac{\eta}{\varkappa}\right)^{\varphi-1}\left(\ln\frac{\gamma}{\xi}\right)^{\rho-1}\frac{F(\eta,\xi)}{\eta\xi}d\xi d\eta,$$

şeklinde tanımlanır.

Genelleştirilmiş kesirli integral operatörleri aşağıdaki gibi verilebilir.

Tanım 2.25. $g:[\kappa_1,\kappa_2]\to\mathbb{R}$ fonksiyonu $(\kappa_1,\kappa_2]$ üzerinde monoton artan fonksiyon, $g'(\varkappa)$ de (κ_1,κ_2) üzerinde sürekli bir fonksiyon ve $w:[\kappa_3,\kappa_4]\to\mathbb{R}$ fonksiyonu $(\kappa_3,\kappa_4]$ üzerinde monoton artan fonksiyon, $w'(\gamma)$ de (κ_3,κ_4) üzerinde sürekli bir fonksiyon olsun.

$\varphi, \rho > 0$ için $F \in L_1(\Delta)$ ise iki değişkenli fonksiyonlar için genelleştirilmiş kesirli integral operatörleri,

$\varkappa > \kappa_1$, $\gamma > \kappa_3$ için

$$\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+;g,w}F(\varkappa,\gamma) = \frac{1}{\Gamma(\varphi)\Gamma(\rho)} \int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma} \frac{g'(\eta)}{[g(\varkappa)-g(\eta)]^{1-\varphi}} \frac{w'(\xi)}{[w(\gamma)-w(\xi)]^{1-\rho}} F(\eta,\xi)d\xi d\eta, \tag{2.4}$$

$\varkappa > \kappa_1$, $\gamma < \kappa_4$ için

$$\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-;g,w}F(\varkappa,\gamma) = \frac{1}{\Gamma(\varphi)\Gamma(\rho)} \int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4} \frac{g'(\eta)}{[g(\varkappa)-g(\eta)]^{1-\varphi}} \frac{w'(\xi)}{[w(\xi)-w(\gamma)]^{1-\rho}} F(\eta,\xi)d\xi d\eta, \tag{2.5}$$

$\varkappa < \kappa_2$, $\gamma > \kappa_3$ için

$$\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+;g,w}F(\varkappa,\gamma) = \frac{1}{\Gamma(\varphi)\Gamma(\rho)} \int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma} \frac{g'(\eta)}{[g(\eta)-g(\varkappa)]^{1-\varphi}} \frac{w'(\xi)}{[w(\gamma)-w(\xi)]^{1-\rho}} F(\eta,\xi)d\xi d\eta, \tag{2.6}$$

$\varkappa < \kappa_2$, $\gamma < \kappa_4$ için

$$\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-;g,w}F(\varkappa,\gamma) = \frac{1}{\Gamma(\varphi)\Gamma(\rho)} \int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4} \frac{g'(\eta)}{[g(\eta)-g(\varkappa)]^{1-\varphi}} \frac{w'(\xi)}{[w(\xi)-w(\gamma)]^{1-\rho}} F(\eta,\xi)d\xi d\eta, \tag{2.7}$$

olarak tanımlanır.

Tanım 2.25 kullanılarak, bazı özel seçimlerle iyi bilinen kesirli integraller elde edilebilir. Örneğin;

1. $g(\eta) = \eta$ ve $w(\xi) = \xi$ olarak alındığında (2.4), (2.5), (2.6) ve (2.7) operatörleri sırasıyla $J^{\varphi,\rho}_{\kappa_1+,\kappa_3+}F$, $J^{\varphi,\rho}_{\kappa_1+,\kappa_4-}F$, $J^{\varphi,\rho}_{\kappa_2-,\kappa_3+}F$ ve $J^{\varphi,\rho}_{\kappa_2-,\kappa_4-}F$ Riemann-Liouville kesirli integrallerine indirgenir.
2. $g(\eta) = \ln\eta$ ve $w(\xi) = \ln\xi$ alındığında (2.4), (2.5), (2.6) ve (2.7) operatörleri sırasıyla $\mathbf{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+}F$, $\mathbf{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-}F$, $\mathbf{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+}F$ ve $\mathbf{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-}F$ Hadamard kesirli integrallerine indirgenir.

2.3. TEK DEĞİŞKENLİ FONKSİYONLAR İÇİN BAZI TEOREMLER

2.3.1. Hermite-Hadamard Eşitsizliği

Teorem 2.26. $\digamma : I \subseteq \mathbb{R} \to \mathbb{R}$ konveks bir fonksiyon olmak üzere, her $\kappa_1, \kappa_2 \in I^{\circ}$ için,

$$\digamma\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \digamma(\varkappa)d\varkappa \leq \frac{\digamma(\kappa_1)+\digamma(\kappa_2)}{2},$$

şeklindeki çift taraflı eşitsizlik Hermite-Hadamard eşitsizliğidir [47].

Teorem 2.27. $\digamma : [\kappa_1, \kappa_2] \to \mathbb{R}$ pozitif bir fonksiyon ve $0 \leq \kappa_1 < \kappa_2$, $\digamma(\varkappa) \in L[\kappa_1, \kappa_2]$ olsun. Ayrıca $\digamma$ fonksiyonu $[\kappa_1, \kappa_2]$ aralığında konveks bir fonksiyon olmak üzere $\varphi > 0$ için

$$\digamma\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{\Gamma(\kappa_1+1)}{2(\kappa_2-\kappa_1)^{\varphi}}\left[\mathbf{J}^{\varphi}_{\kappa_1+}\digamma(\kappa_2)+\mathbf{J}^{\varphi}_{\kappa_2-}\digamma(\kappa_1)\right] \leq \frac{\digamma(\kappa_1)+\digamma(\kappa_2)}{2},$$

eşitsizliği sağlanır. Bu eşitsizlik literatürde iyi bilinen Kesirli Hermite-Hadamard eşitsizliğidir [10].

Tanım 2.28. $\digamma : [\kappa_1, \kappa_2] \to \mathbb{R}$ fonksiyonu $0 \leq \kappa_1 < \kappa_2 < \infty$ olsun. $\digamma \in L_{\infty}(\kappa_1, \kappa_2)$ için $\mathcal{J}^{\varphi}_{\kappa_1+;g}\digamma(\varkappa)$ ve $\mathcal{J}^{\varphi}_{\kappa_2-;g}\digamma(\varkappa)$ tanımlı olsun. Aşağıdaki,

$$\tilde{\digamma}(\varkappa) = \digamma(\kappa_1+\kappa_2-\varkappa) \quad \text{ve} \quad F(\varkappa) = \digamma(\varkappa)+\tilde{\digamma}(\varkappa),$$

eşitlikleri tanımlansın.

Teorem 2.29. $\varphi > 0$ olsun. $\digamma$, $[\kappa_1, \kappa_2]$ aralığında konveks bir fonksiyon ise,

$$\digamma\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{\Gamma(\kappa_1+1)}{4\left[g(\kappa_2)-g(\kappa_1)\right]^{\varphi}}\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\kappa_2)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\kappa_1)\right] \leq \frac{\digamma(\kappa_1)+\digamma(\kappa_2)}{2},$$

şeklindeki eşitsizlik sağlanır [48].

2.3.2. Midpoint Eşitsizliği

Teorem 2.30. $F:[\kappa_1,\kappa_2]\subseteq\mathbb{R}\to\mathbb{R}$ fonksiyonu (κ_1,κ_2) aralığında diferansiyellenebilir bir fonksiyon ve $|F'|$, $[\kappa_1,\kappa_2]$ aralığında konveks olmak üzere,

$$\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(\varkappa)d\varkappa-F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\leq\frac{(\kappa_2-\kappa_1)}{8}\left(|F'(\kappa_1)|+|F'(\kappa_2)|\right),$$

eşitsizliği sağlanır. Bu eşitsizlik literatürde orta nokta eşitsizliği olarak bilinir [9].

Teorem 2.31. $F:[\kappa_1,\kappa_2]\to\mathbb{R}$ fonksiyonu $\kappa_1<\kappa_2$ için (κ_1,κ_2) üzerinde diferansiyellenebilir olsun. $|F'|$, $[\kappa_1,\kappa_2]$ aralığında konveks ise aşağıdaki kesirli integral eşitsizliği sağlanır[49],

$$\begin{aligned}&\left|\frac{\Gamma(\kappa_1+1)}{2(\kappa_2-\kappa_1)^{\varphi}}\left[J^{\varphi}_{\kappa_1+}F(\kappa_2)+J^{\varphi}_{\kappa_2-}F(\kappa_1)\right]-F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\\&\leq\frac{(\kappa_2-\kappa_1)}{4(\varphi+1)}\left(\varphi-1+\frac{1}{2^{\varphi-1}}\right)\left(|F'(\kappa_1)|+|F'(\kappa_2)|\right).\end{aligned}$$

2.3.3. Trapezoid Eşitsizliği

Teorem 2.32. $F:[\kappa_1,\kappa_2]\subseteq\mathbb{R}\to\mathbb{R}$ fonksiyonu (κ_1,κ_2) aralığında diferansiyellenebilir bir fonksiyon ve $|F'|$, $[\kappa_1,\kappa_2]$ aralığında konveks olmak üzere,

$$\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(\eta)d\eta\right|\leq\frac{(\kappa_2-\kappa_1)}{8}\left(|F'(\kappa_1)|+|F'(\kappa_2)|\right),$$

eşitsizliği sağlanır. Bu eşitsizlik literatürde trapezoid eşitsizliği olarak bilinir [50].

Teorem 2.33. $F:[\kappa_1,\kappa_2]\to\mathbb{R}$ fonksiyonu (κ_1,κ_2) üzerinde diferansiyellenebilir olsun. $|F'|$, $[\kappa_1,\kappa_2]$ aralığında konveks ise

$$\begin{aligned}&\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{\Gamma(\kappa_1+1)}{2(\kappa_2-\kappa_1)^{\varphi}}\left[J^{\varphi}_{\kappa_1+}F(\kappa_2)+J^{\varphi}_{\kappa_2-}F(\kappa_1)\right]\right|\\&\leq\frac{(\kappa_2-\kappa_1)}{2(\varphi+1)}\left(1-\frac{1}{2^{\varphi}}\right)[F'(\kappa_1)+F'(\kappa_2)],\end{aligned}$$

kesirli integral eşitsizliği sağlanır [10].

Teorem 2.34. $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ fonksiyonu (κ_1, κ_2) üzerinde diferansiyellenebilir olsun. Ayrıca $\varkappa, \gamma \in [\kappa_1, \kappa_2]$ için,

$$\mathcal{L}_g^{\varphi}(\varkappa,\gamma) = \int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} |\varkappa - u||g(\gamma) - g(u)|^{\varphi} du - \int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2} |\varkappa - u||g(\gamma) - g(u)|^{\varphi} du,$$

ve

$$\mathcal{J}_g^{\varphi}(\kappa_1,\kappa_2) = \mathcal{L}_g^{\varphi}(\kappa_2,\kappa_2) + \mathcal{L}_g^{\varphi}(\kappa_1,\kappa_2) - \mathcal{L}_g^{\varphi}(\kappa_2,\kappa_1) - \mathcal{L}_g^{\varphi}(\kappa_1,\kappa_1),$$

$|F'|$, $[\kappa_1, \kappa_2]$ aralığında konveks olsun.

$$\begin{aligned}
&\left| \frac{F(\kappa_1)+F(\kappa_2)}{2} - \frac{\Gamma(\kappa_1+1)}{4[g(\kappa_2)-g(\kappa_1)]^{\varphi}} \left[\mathcal{J}_{\kappa_1+;g}^{\varphi} F(\kappa_2) + \mathcal{J}_{\kappa_2-;g}^{\varphi} F(\kappa_1) \right] \right| \\
&\leq \frac{\mathcal{J}_g^{\varphi}(\kappa_1,\kappa_2)}{4[g(\kappa_2)-g(\kappa_1)]^{\varphi}(\kappa_2-\kappa_1)} \left(|F'(\kappa_1)| + |F'(\kappa_2)| \right),
\end{aligned}$$

eşitsizliği sağlanır [48].

2.3.4. Ostrowski Eşitsizliği

Teorem 2.35. $F : [\kappa_1, \kappa_2] \subseteq \mathbb{R} \to \mathbb{R}$, (κ_1, κ_2) üzerinde diferansiyellenebilir bir fonksiyon ve $F' \in L[\kappa_1, \kappa_2]$ olsun. Her $\varkappa \in [\kappa_1, \kappa_2]$ için $|F'(\varkappa)| \leq M$ ise,

$$\begin{aligned}
\left| F(\varkappa) - \frac{1}{\kappa_2-\kappa_1} \int_{\kappa_1}^{\kappa_2} F(\eta) d\eta \right| &\leq \frac{M}{\kappa_2-\kappa_1} \left[\frac{(\varkappa-\kappa_1)^2 + (\kappa_2-\varkappa)^2}{2} \right] \\
&= M(\kappa_2-\kappa_1) \left[\frac{1}{4} - \frac{\left(\varkappa - \frac{\kappa_1+\kappa_2}{2}\right)^2}{(\kappa_2-\kappa_1)^2} \right],
\end{aligned}$$

eşitsizliği sağlanır. Buradaki $\frac{1}{4}$ katsayısı bu şartlar altındaki en iyi katsayıdır. Bu eşitsizlik Ostrowski Eşitsizliği olarak bilinir [12].

Teorem 2.36. $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ fonksiyonu (κ_1, κ_2) aralığında türevlenebilir ve $F' \in L[\kappa_1, \kappa_2]$ olsun. Her $\varkappa \in [\kappa_1, \kappa_2]$ için $|F'(\varkappa)| \leq M$ ve $\varphi > 0$, $|F'|$, $[\kappa_1, \kappa_2]$ aralığında konveks ise,

$$\left| \left(\frac{(\varkappa-\kappa_1)^{\varphi} + (\kappa_2-\varkappa)^{\varphi}}{\kappa_2-\kappa_1} \right) F(\varkappa) - \frac{\Gamma(\varphi+1)}{\kappa_2-\kappa_1} \left[J_{\varkappa+}^{\varphi} F(\kappa_2) + J_{\varkappa-}^{\varphi} F(\kappa_1) \right] \right|$$

$$\leq \frac{M}{\kappa_2-\kappa_1}\left[\frac{(\varkappa-\kappa_1)^{\varphi+1}+(\kappa_2-\varkappa)^{\varphi+1}}{\varphi+1}\right],$$

kesirli integral eşitsizliği sağlanır[18] .

Teorem 2.37. $\mathcal{F}:[\kappa_1,\kappa_2]\subseteq\mathbb{R}\to\mathbb{R}$, (κ_1,κ_2) üzerinde diferansiyellenebilir bir fonksiyon ve $\varkappa\in[\kappa_1,\kappa_2]$ için $|\mathcal{F}'(\varkappa)|\leq M$, g monoton artan bir fonksiyon ve $g'(\varkappa)\geq 1$ olsun. $\varphi,\rho>0$ olmak üzere,

$$\begin{aligned}&\left|((g(\kappa_2)-g(\varkappa))^{\rho}+(g(\varkappa)-g(\kappa_1))^{\varphi})\mathcal{F}(\varkappa)-\left(\Gamma(\rho+1)\mathcal{J}^{\varphi}_{\kappa_2-;g}\mathcal{F}(\varkappa)+\Gamma(\varphi+1)\mathcal{J}^{\varphi}_{\kappa_1+;g}\mathcal{F}(\varkappa)\right)\right|\\&\leq M\left(\frac{\varphi}{\varphi+1}(g(\kappa_2)-g(\varkappa))^{\rho+1}+\frac{\rho}{\rho+1}(g(\varkappa)-g(\kappa_1))^{\varphi+1}\right),\end{aligned}$$

eşitsizliği sağlanır [19].

2.4. İKİ DEĞİŞKENLİ FONKSİYONLAR İÇİN BAZI TEOREMLER

Şimdi, $\mathbb{R}^2$ de $\kappa_1<\kappa_2$ ve $\kappa_3<\kappa_4$ olan bir $\Delta=[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4]$, aralığı verilsin.

$\mathcal{F}:\Delta\to\mathbb{R}$, $\forall(\varkappa,u),(\gamma,w)\in\Delta$ ve $\eta\in[0,1]$ için,

$$\mathcal{F}(\eta\varkappa+(1-\eta)\gamma,\eta u+(1-\eta)w)\leq\eta\mathcal{F}(\varkappa,u)+(1-\eta)\mathcal{F}(\gamma,w)),$$

eşitsizliği sağlanıyorsa $\mathcal{F}$'ye Δ üzerinde bir konveks fonksiyon denir.

Tanım 2.38. $\Delta=[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4]$ ve $\mathcal{F}:\Delta\to\mathbb{R}$ fonksiyonu verilsin. $\eta,\xi\in[0,1]$ ve $(\varkappa,u),(\gamma,w)\in\Delta$ için,

$$\begin{aligned}\mathcal{F}(\eta\varkappa+(1-\eta)\gamma,\xi u+(1-\xi)w)\leq{}&\eta\xi\mathcal{F}(\varkappa,u)+\xi(1-\eta)\mathcal{F}(\gamma,u)+\eta(1-\xi)\mathcal{F}(\varkappa,w)\\&+(1-\eta)(1-\xi)\mathcal{F}(\gamma,w),\end{aligned}$$

eşitsizliği sağlanıyor ise $\mathcal{F}$'ye Δ üzerinde koordinatlarda konveks fonksiyon denir [35].

$$\mathcal{F}_{\gamma}:[\kappa_1,\kappa_2]\to\mathbb{R},\quad \mathcal{F}_{\gamma}(u)=\mathcal{F}(u,\gamma),$$

ve

$$\mathcal{F}_{\varkappa}:[\kappa_3,\kappa_4]\to\mathbb{R},\quad \mathcal{F}_{\varkappa}(v)=\mathcal{F}(\varkappa,v),$$

ifadeleri $\forall \varkappa \in [\kappa_1, \kappa_2]$ ve $\gamma \in [\kappa_3, \kappa_4]$ için konvekstir [35].

2.4.1. Hermite-Hadamard Eşitsizliği

Teorem 2.39. $F : \Delta \to \mathbb{R}$, Δ üzerinde koordinatlarda konveks fonksiyon olmak üzere,

$$
\begin{aligned}
F\left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F\left(\varkappa, \frac{\kappa_3+\kappa_4}{2}\right) d\varkappa + \frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4} F\left(\frac{\kappa_1+\kappa_2}{2}, \gamma\right) d\gamma\right] \\
&\leq \frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,\gamma)\, d\gamma\, d\varkappa \\
&\leq \frac{1}{4}\left[\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa,\kappa_3)\, d\varkappa + \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa,\kappa_4)\, d\varkappa \right.\\
&\left. + \frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4} F(\kappa_1,\gamma)\, d\gamma + \frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4} F(\kappa_2,\gamma)\, d\gamma\right] \\
&\leq \frac{F(\kappa_1,\kappa_3)+F(\kappa_1,\kappa_4)+F(\kappa_2,\kappa_3)+F(\kappa_2,\kappa_4)}{4},
\end{aligned}
$$

eşitsizliği sağlanır [35].

Teorem 2.40. $F : \Delta \to \mathbb{R}$, Δ üzerinde koordinatlarda konveks fonksiyon ve $F \in L_1(\Delta)$ olmak üzere,

$$
\begin{aligned}
F\left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2}\right) &\leq \frac{\Gamma(\varphi+1)}{4(\kappa_2-\kappa_1)^{\varphi}}\left[J^{\varphi}_{\kappa_1+}F\left(\kappa_2, \frac{\kappa_3+\kappa_4}{2}\right) + J^{\varphi}_{\kappa_2-}F\left(\kappa_1, \frac{\kappa_3+\kappa_4}{2}\right)\right] \\
&+ \frac{\Gamma(\rho+1)}{4(\kappa_4-\kappa_3)^{\rho}}\left[J^{\rho}_{\kappa_3+}F\left(\frac{\kappa_1+\kappa_2}{2}, \kappa_4\right) + J^{\rho}_{\kappa_4-}F\left(\frac{\kappa_1+\kappa_2}{2}, \kappa_3\right)\right] \\
&\leq \frac{\Gamma(\varphi+1)\Gamma(\rho+1)}{4(\kappa_2-\kappa_1)^{\varphi}(\kappa_4-\kappa_3)^{\rho}}[J^{\varphi,\rho}_{\kappa_1+,\kappa_3+}F(\kappa_2,\kappa_4) + J^{\varphi,\rho}_{\kappa_1+,\kappa_4-}F(\kappa_2,\kappa_3) \\
&+ J^{\varphi,\rho}_{\kappa_2-,\kappa_3+}F(\kappa_1,\kappa_4) + J^{\varphi,\rho}_{\kappa_2-,\kappa_4-}F(\kappa_1,\kappa_3)] \\
&\leq \frac{\Gamma(\varphi+1)}{4(\kappa_2-\kappa_1)^{\varphi}}[J^{\varphi}_{\kappa_1+}F(\kappa_2,\kappa_3) + J^{\varphi}_{\kappa_1+}F(\kappa_2,\kappa_4) + J^{\varphi}_{\kappa_2-}F(\kappa_1,\kappa_3) \\
&+ J^{\varphi}_{\kappa_2-}F(\kappa_1,\kappa_4)] + \frac{\Gamma(\rho+1)}{4(\kappa_4-\kappa_3)^{\rho}}[J^{\rho}_{\kappa_3+}F(\kappa_1,\kappa_4) + J^{\rho}_{\kappa_3+}F(\kappa_2,\kappa_4) \\
&+ J^{\rho}_{\kappa_4-}F(\kappa_1,\kappa_3) + J^{\rho}_{\kappa_4-}F(\kappa_2,\kappa_3)] \\
&\leq \frac{F(\kappa_1,\kappa_3)+F(\kappa_1,\kappa_4)+F(\kappa_2,\kappa_3)+F(\kappa_2,\kappa_4)}{4},
\end{aligned}
$$

sağlanır [38].

$\digamma : \Delta \to \mathbb{R}$ olsun. $(\varkappa, \gamma) \in [\kappa_1, \kappa_2] \times [\kappa_3, \kappa_4]$ için,

$$\begin{aligned}
\tilde{\digamma}_1(\varkappa,\gamma) &= \digamma(\kappa_1+\kappa_2-\varkappa,\gamma),\\
\tilde{\digamma}_2(\varkappa,\gamma) &= \digamma(\varkappa,\kappa_3+\kappa_4-\gamma),\\
\tilde{\digamma}_3(\varkappa,\gamma) &= \digamma(\kappa_1+\kappa_2-\varkappa,\kappa_3+\kappa_4-\gamma),\\
G(\varkappa,\gamma) &= \digamma(\varkappa,\gamma)+\tilde{\digamma}_2(\varkappa,\gamma),\\
H(\varkappa,\gamma) &= \digamma(\varkappa,\gamma)+\tilde{\digamma}_1(\varkappa,\gamma),\\
K(\varkappa,\gamma) &= \tilde{\digamma}_1(\varkappa,\gamma)+\tilde{\digamma}_3(\varkappa,\gamma),\\
L(\varkappa,\gamma) &= \tilde{\digamma}_2(\varkappa,\gamma)+\tilde{\digamma}_3(\varkappa,\gamma),\\
F(\varkappa,\gamma) &= \tilde{\digamma}_1(\varkappa,\gamma)+\tilde{\digamma}_2(\varkappa,\gamma)+\tilde{\digamma}_3(\varkappa,\gamma)+\digamma(\varkappa,\gamma)\\
&= \frac{G(\varkappa,\gamma)+H(\varkappa,\gamma)+K(\varkappa,\gamma)+L(\varkappa,\gamma)}{2},
\end{aligned}$$

olsun.

Teorem 2.41. $g : [\kappa_1, \kappa_2] \to \mathbb{R}$ fonksiyonu $(\kappa_1, \kappa_2]$ üzerinde monoton artan fonksiyon, $g'(\varkappa)$ de (κ_1, κ_2) üzerinde sürekli bir fonksiyon ve $w : [\kappa_3, \kappa_4] \to \mathbb{R}$ fonksiyonu $(\kappa_3, \kappa_4]$ üzerinde monoton artan fonksiyon, $w'(\gamma)$ de (κ_3, κ_4) üzerinde sürekli bir fonksiyon olsun. $\digamma : \Delta \to \mathbb{R}$, Δ üzerinde bir konveks fonksiyon ve $\varphi, \rho > 0$ için,

$$\begin{aligned}
\digamma\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right) &\le \frac{\Gamma(\varphi+1)}{8\,[g(\kappa_2)-g(\kappa_1)]^{\varphi}}\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}H\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)+\mathcal{J}^{\varphi}_{\kappa_2-;g}H\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)\right]\\
&\quad+\frac{\Gamma(\rho+1)}{8\,(w(\kappa_4)-w(\kappa_3))^{\rho}}\left[\mathcal{J}^{\rho}_{\kappa_3+;w}G\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)+\mathcal{J}^{\rho}_{\kappa_4-;w}G\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)\right]\\
&\le \frac{\Gamma(\varphi+1)\Gamma(\rho+1)}{16\,[g(\kappa_2)-g(\kappa_1)]^{\varphi}\,[w(\kappa_4)-w(\kappa_3)]^{\rho}}[\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+;g,w}\digamma(\kappa_2,\kappa_4)\\
&\quad+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-;g,w}\digamma(\kappa_2,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+;g,w}\digamma(\kappa_1,\kappa_4)+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-;g,w}\digamma(\kappa_1,\kappa_3)]\\
&\le \frac{\Gamma(\varphi+1)}{16\,[g(\kappa_2)-g(\kappa_1)]^{\varphi}}[\mathcal{J}^{\varphi}_{\kappa_1+;g}H(\kappa_2,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\kappa_1+;g}\digamma(\kappa_2,\kappa_3)\\
&\quad+\mathcal{J}^{\varphi}_{\kappa_2-,\kappa_3+;g,w}H(\kappa_2,\kappa_4)+\mathcal{J}^{\varphi}_{\kappa_2-;g}H(\kappa_1,\kappa_3)+\mathcal{J}^{\varphi}_{\kappa_2-;g}H(\kappa_1,\kappa_4)]\\
&\quad+\frac{\Gamma(\rho+1)}{16\,[w(\kappa_4)-w(\kappa_3)]^{\rho}}[\mathcal{J}^{\rho}_{\kappa_3+;w}G(\kappa_1,\kappa_4)\\
&\quad+\mathcal{J}^{\rho}_{\kappa_3+;w}G(\kappa_2,\kappa_4)+\mathcal{J}^{\rho}_{\kappa_4-;w}G(\kappa_1,\kappa_3)+\mathcal{J}^{\rho}_{\kappa_4-;w}G(\kappa_2,\kappa_3)]\\
&\le \frac{\digamma(\kappa_1,\kappa_3)+\digamma(\kappa_1,\kappa_4)+\digamma(\kappa_2,\kappa_3)+\digamma(\kappa_2,\kappa_4)}{4},
\end{aligned}$$

şeklindeki Hermite-Hadamard tipli eşitsizlik elde edilir [51].

2.4.2. Midpoint Eşitsizliği

Teorem 2.42. $\digamma : \Delta \to \mathbb{R}$ fonksiyonu $\kappa_1 < \kappa_2$, $\kappa_3 < \kappa_4$ olmak üzere Δ üzerinde kısmi diferansiyellenebilir olsun. $\left|\frac{\partial^2 \digamma}{\partial\xi\partial\eta}\right|$, Δ üzerinde koordinatlarda konveks ise,

$$B = \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} \digamma\left(\varkappa, \frac{\kappa_3+\kappa_4}{2}\right) d\varkappa + \frac{1}{\kappa_4 - \kappa_3}\int_{\kappa_3}^{\kappa_4} \digamma\left(\frac{\kappa_1+\kappa_2}{2}, \gamma\right) d\gamma,$$

olmak üzere,

$$\left|\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} d\gamma d\varkappa + \digamma\left(\frac{a+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2}\right) - B\right|$$
$$\leq \frac{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}{16}$$
$$\left[\frac{\left|\frac{\partial^2 \digamma}{\partial\xi\partial\eta}\digamma(\kappa_1,\kappa_3)\right| + \left|\frac{\partial^2 \digamma}{\partial\xi\partial\eta}\digamma(\kappa_1,\kappa_4)\right| + \left|\frac{\partial^2 \digamma}{\partial\xi\partial\eta}\digamma(\kappa_2,\kappa_3)\right| + \left|\frac{\partial^2 \digamma}{\partial\xi\partial\eta}\digamma(\kappa_2,\kappa_4)\right|}{4}\right],$$

eşitsizliği sağlanır [36].

Teorem 2.43. $\digamma : \Delta \to \mathbb{R}$ fonksiyonu $\kappa_1 < \kappa_2$, $\kappa_3 < \kappa_4$ olmak üzere Δ üzerinde kısmi diferansiyellenebilir olsun. $\left|\frac{\partial^2 \digamma}{\partial\xi\partial\eta}\right|$, Δ üzerinde koordinatlarda konveks ise,

$$\left|\digamma\left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2}\right) - \left[\frac{\Gamma(\varphi+1)}{4(\kappa_2-\kappa_1)^\varphi}\left[J^\varphi_{\kappa_1+}\digamma\left(\kappa_2, \frac{\kappa_3+\kappa_4}{2}\right) + J^\varphi_{\kappa_2-}\digamma\left(\kappa_1, \frac{\kappa_3+\kappa_4}{2}\right)\right]\right.\right.$$
$$\left.\frac{\Gamma(\rho+1)}{4(\kappa_4-\kappa_3)^\rho}\left[J^\rho_{\kappa_3+}\digamma\left(\frac{\kappa_1+\kappa_2}{2}, \kappa_4\right) + J^\rho_{\kappa_4-}\digamma\left(\frac{\kappa_1+\kappa_2}{2}, \kappa_3\right)\right]\right]$$
$$+\frac{\Gamma(\varphi+1)\Gamma(\rho+1)}{4(\kappa_2-\kappa_1)^\varphi(\kappa_4-\kappa_3)^\rho}[J^{\varphi,\rho}_{\kappa_1+,\kappa_3+}\digamma(\kappa_2,\kappa_4) + J^{\varphi,\rho}_{\kappa_1+,\kappa_4-}\digamma(\kappa_2,\kappa_3)$$
$$\left.+J^{\varphi,\rho}_{\kappa_2-,\kappa_3+}\digamma(\kappa_1,\kappa_4) + J^{\varphi,\rho}_{\kappa_2-,\kappa_4-}\digamma(\kappa_1,\kappa_3)]\right|$$
$$\leq \frac{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}{4}\left(\frac{1}{2} + \frac{1-2^\varphi}{2^{\varphi(\varphi+1)}}\right)\left(\frac{1-2^\rho}{2^{\rho(\rho+1)}}\right)$$
$$\times\left(\left|\frac{\partial^2 \digamma}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right| + \left|\frac{\partial^2 \digamma}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right| + \left|\frac{\partial^2 \digamma}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right| + \left|\frac{\partial^2 \digamma}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|\right),$$

eşitsizliği sağlanır [39].

2.4.3. Trapezoid Eşitsizliği

Teorem 2.44. $\digamma : \Delta \to \mathbb{R}$ fonksiyonu $\kappa_1 < \kappa_2$, $\kappa_3 < \kappa_4$ olmak üzere Δ üzerinde kısmi diferansiyellenebilir olsun. $\left|\frac{\partial^2 \digamma}{\partial \xi \partial \eta}\right|$, Δ üzerinde koordinatlarda konveks ise,

$$C = \frac{1}{2}\left[\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}[\digamma(\varkappa,\kappa_3)+\digamma(\varkappa,\kappa_4)]d\varkappa + \frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}[\digamma(\kappa_1,\gamma)+\digamma(\kappa_2,\gamma)]d\gamma\right],$$

olmak üzere,

$$\begin{aligned}
&\left|\frac{\digamma(\kappa_1,\kappa_3)+\digamma(\kappa_1,\kappa_4)+\digamma(\kappa_2,\kappa_3)+\digamma(\kappa_2,\kappa_4)}{4}\right.\\
&\left.+\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\digamma(\varkappa,\gamma)d\gamma d\varkappa - C\right|\\
&\leq \frac{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}{16}\\
&\times\left[\frac{\left|\frac{\partial^2 \digamma}{\partial \xi \partial \eta}(\kappa_1,\kappa_3)\right|+\left|\frac{\partial^2 \digamma}{\partial \xi \partial \eta}(\kappa_1,\kappa_4)\right|+\left|\frac{\partial^2 \digamma}{\partial \xi \partial \eta}(\kappa_2,\kappa_3)\right|+\left|\frac{\partial^2 \digamma}{\partial \xi \partial \eta}(\kappa_2,\kappa_4)\right|}{4}\right],
\end{aligned}$$

eşitsizliği sağlanır [37].

Teorem 2.45. $\digamma : \Delta \to \mathbb{R}$ fonksiyonu $\kappa_1 < \kappa_2$, $\kappa_3 < \kappa_4$ olmak üzere Δ üzerinde kısmi diferansiyellenebilir olsun. $\left|\frac{\partial^2 \digamma}{\partial \xi \partial \eta}\right|$, Δ üzerinde koordinatlarda konveks ise,

$$\begin{aligned}
D = {}&\frac{\Gamma(\rho+1)}{4(\kappa_4-\kappa_3)^{\rho}}[J^{\rho}_{\kappa_3+}\digamma(\kappa_1,\kappa_4)+J^{\rho}_{\kappa_3+}\digamma(\kappa_2,\kappa_4)+J^{\rho}_{\kappa_4-}\digamma(\kappa_1,\kappa_3)+J^{\rho}_{\kappa_4-}\digamma(\kappa_2,\kappa_3)]\\
&+\frac{\Gamma(\varphi+1)}{4(\kappa_2-\kappa_1)^{\varphi}}[J^{\varphi}_{\kappa_1+}\digamma(\kappa_2,\kappa_3)+J^{\varphi}_{\kappa_1+}\digamma(\kappa_2,\kappa_4)+J^{\varphi}_{\kappa_2-}\digamma(\kappa_1,\kappa_3)+J^{\varphi}_{\kappa_2-}\digamma(\kappa_1,\kappa_4)],
\end{aligned}$$

olmak üzere,

$$\begin{aligned}
&\left|\frac{\digamma(\kappa_1,\kappa_3)+\digamma(\kappa_1,\kappa_4)+\digamma(\kappa_2,\kappa_3)+\digamma(\kappa_2,\kappa_4)}{4}\right.\\
&+\frac{\Gamma(\varphi+1)\Gamma(\rho+1)}{4(\kappa_2-\kappa_1)^{\varphi}(\kappa_4-\kappa_3)^{\rho}}[J^{\varphi,\rho}_{\kappa_1+,\kappa_3+}\digamma(\kappa_2,\kappa_4)+J^{\varphi,\rho}_{\kappa_1+,\kappa_4-}\digamma(\kappa_2,\kappa_3)
\end{aligned}$$

$$
+J^{\varphi,\rho}_{\kappa_2-,\kappa_3+}F(\kappa_1,\kappa_4)+J^{\varphi,\rho}_{\kappa_2-,\kappa_4-}F(\kappa_1,\kappa_3)]-D\Bigg|
$$

$$
\leq \frac{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}{4(\varphi+1)(\rho+1)}
$$

$$
\times\left(\left|\frac{\partial^2 F}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|\right),
$$

eşitsizliği sağlanır [38].

2.4.4. Ostrowski Eşitsizliği

Teorem 2.46. $F:\Delta\to\mathbb{R}$ fonksiyonu $\kappa_1<\kappa_2$, $\kappa_3<\kappa_4$ olmak üzere Δ üzerinde kısmi diferansiyellenebilir olsun. $\left|\frac{\partial^2 F}{\partial\xi\partial\eta}\right|$, Δ üzerinde koordinatlarda konveks ve $(\varkappa,\gamma)\in\Delta$ için $\left|\frac{\partial^2 F}{\partial\xi\partial\eta}F(\varkappa,\gamma)\right|\leq M$ olsun. $\forall(\varkappa,\gamma)\in\Delta$ için,

$$
E=\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}F(\varkappa,v)dv+\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(u,\gamma)du,
$$

olmak üzere,

$$
\left|F(\varkappa,\gamma)+\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}F(u,v)dvdu-E\right|
$$

$$
\leq M\left[\frac{(\varkappa-\kappa_1)^2+(\kappa_2-\varkappa)^2}{2(\kappa_2-\kappa_1)}\right]\left[\frac{(\gamma-\kappa_3)^2+(\kappa_4-\gamma)^2}{2(\kappa_4-\kappa_3)}\right],
$$

eşitsizliği sağlanır [33].

Teorem 2.47. $F:\Delta\to\mathbb{R}$ fonksiyonu $\kappa_1<\kappa_2$, $\kappa_3<\kappa_4$ olmak üzere Δ üzerinde kısmi diferansiyellenebilir olsun. $\left|\frac{\partial^2 F}{\partial\xi\partial\eta}\right|$, Δ üzerinde ikili koordinatlarda konveks ve $(\varkappa,\gamma)\in\Delta$ için $\left|\frac{\partial^2 F}{\partial\xi\partial\eta}F(\varkappa,\gamma)\right|\leq M$ varsa $\forall(\varkappa,\gamma)\in\Delta$ ve $\varphi,\rho>0$ için,

$$
N=\frac{\Gamma(\varphi+1)\Gamma(\rho+1)}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\left[J^{\varphi,\rho}_{\varkappa-,\gamma-}F(\kappa_1,\kappa_3)+J^{\varphi,\rho}_{\varkappa-,\gamma+}F(\kappa_1,\kappa_4)+J^{\varphi,\rho}_{\varkappa+,\gamma-}F(\kappa_2,\kappa_3)\right.
$$

$$
\left.+J^{\varphi,\rho}_{\varkappa+,\gamma+}F(\kappa_2,\kappa_4)\right]-\frac{\left[(\varkappa-\kappa_1)^{\varphi}+(\kappa_2-\varkappa)^{\varphi}\right]\Gamma(\rho+1)}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\left[J^{\rho}_{\gamma-}F(\varkappa,\kappa_3)+J^{\rho}_{\gamma+}F(\varkappa,\kappa_4)\right]
$$

$$
-\frac{\left[(\gamma-\kappa_3)^{\rho}+(\kappa_4-\gamma)^{\rho}\right]\Gamma(\varphi+1)}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\left[J^{\rho}_{\varkappa-}F(\kappa_1,\gamma)+J^{\rho}_{\varkappa+}F(\kappa_2,\gamma)\right],
$$

olmak üzere,

$$\left|\frac{\left[(\kappa_2-\varkappa)^{\varphi}+(\varkappa-\kappa_1)^{\varphi}\right]\left[(\kappa_4-\gamma)^{\rho}+(\gamma-\kappa_3)^{\rho}\right]}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}F(\varkappa,\gamma)+N\right|$$
$$\leq M\frac{(\varphi\rho+2\varphi+2\rho+4)\left[(\kappa_2-\varkappa)^{\varphi+1}+(\varkappa-\kappa_1)^{\varphi+1}\right]\left[(\kappa_4-\gamma)^{\rho+1}+(\gamma-\kappa_3)^{\rho+1}\right]}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)(\varphi+1)(\varphi+2)(\rho+1)(\rho+2)},$$

eşitsizliği sağlanır [52].

3. GENELLEŞTİRİLMİŞ KESİRLİ İNTEGRALLER YARDIMIYLA İKİ DEĞİŞKENLİ FONKSİYONLAR İÇİN TRAPEZOID TİPLİ EŞİTSİZLİKLER

Bu bölümde genelleştirilmiş kesirli integralleri içeren iki değişkenli fonksiyonlar için Trapezoid tipli integral eşitsizlikleri oluşturulacaktır. Bu bölümde sunulacak sonuçlar daha önce literatürde var olan çalışmaların genellemeleri olacaktır.

Not 3.1. Bu tez çalışması boyunca $F_{\eta\xi}$ ile $\frac{\partial^2 F}{\partial\eta\partial\xi}$ kısmi türevi gösterilecektir. Δ da $\kappa_1 < \kappa_2$ ve $\kappa_3 < \kappa_4$ olan bir $\Delta = [\kappa_1, \kappa_2] \times [\kappa_3, \kappa_4]$ olarak ifade edilecektir. Ayrıca, $g : [\kappa_1, \kappa_2] \to \mathbb{R}$ fonksiyonu $(\kappa_1, \kappa_2]$ üzerinde monoton artan fonksiyon, $g'(\varkappa)$ de (κ_1, κ_2) üzerinde sürekli bir fonksiyon ve $w : [\kappa_3, \kappa_4] \to \mathbb{R}$ fonksiyonu $(\kappa_3, \kappa_4]$ üzerinde monoton bir fonksiyon, $w'(\gamma)$ de (κ_3, κ_4) üzerinde sürekli bir fonksiyon olarak alınacaktır.

İlk olarak sıklıkla kullanılacak olan aşağıdaki fonksiyonlar tanımlansın
$(\varkappa, \gamma) \in \Delta$ için,

$$M_g^{\varphi}(\kappa_1, \varkappa) = \frac{[g(\varkappa) - g(\kappa_1)]^{\varphi}}{\Gamma(\varphi + 1)}, \quad N_w^{\rho}(\kappa_3, \gamma) = \frac{[w(\gamma) - w(\kappa_3)]^{\rho}}{\Gamma(\rho + 1)},$$

ve

$$M_g^{\varphi}(\kappa_2, \varkappa) = \frac{[g(\kappa_2) - g(\varkappa)]^{\varphi}}{\Gamma(\varphi + 1)}, \quad N_w^{\rho}(\kappa_4, \gamma) = \frac{[w(\kappa_4) - w(\gamma)]^{\rho}}{\Gamma(\rho + 1)}.$$

Ek olarak $g(\eta) = \ln \eta$, $\eta \in [\kappa_1, \kappa_2]$ ve $w(\xi) = \ln \xi$, $\xi \in [\kappa_3, \kappa_4]$ olmak üzere seçilirse,

$$M_{\ln}^{\varphi}(\kappa_1, \varkappa) = \frac{\left[\ln \frac{\varkappa}{\kappa_1}\right]^{\varphi}}{\Gamma(\varphi + 1)}, \quad N_{\ln}^{\rho}(\kappa_3, \gamma) = \frac{\left[\ln \frac{\gamma}{\kappa_3}\right]^{\rho}}{\Gamma(\rho + 1)},$$

ve

$$M_{\ln}^{\varphi}(\kappa_2, \varkappa) = \frac{\left[\ln \frac{\kappa_2}{\varkappa}\right]^{\varphi}}{\Gamma(\varphi + 1)}, \quad N_{\ln}^{\rho}(\kappa_4, \gamma) = \frac{\left[\ln \frac{\kappa_4}{\gamma}\right]^{\rho}}{\Gamma(\rho + 1)},$$

eşitsizlikleri elde edilir. Ayrıca, $(\varkappa,\gamma)\in\Delta$ için,

$$\begin{aligned} A(g,w) &= M_g^{\varphi}(\kappa_1,\varkappa)N_w^{\rho}(\kappa_3,\gamma)\digamma(\kappa_1,\kappa_3)+M_g^{\varphi}(\kappa_1,\varkappa)N_w^{\rho}(\kappa_4,\gamma)\digamma(\kappa_1,\kappa_4) \\ &+M_g^{\varphi}(\kappa_2,\varkappa)N_w^{\rho}(\kappa_3,\gamma)\digamma(\kappa_2,\kappa_3)+M_g^{\varphi}(\kappa_2,\varkappa)N_w^{\rho}(\kappa_4,\gamma)\digamma(\kappa_2,\kappa_4), \end{aligned}$$

olsun.

Şimdi, daha sonra sıklıkla kullanılacak olan aşağıdaki özdeşlik ispatlanacaktır.

Yardımcı Teorem 3.2. $\digamma:\Delta\to\mathbb{R}$ fonksiyonu Δ° üzerinde iki kez kısmi türevlenebilir olsun. $\varphi,\rho>0$ için $\digamma_{\eta\xi}\in L(\Delta)$ ise

$$I_1=\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\varkappa)-g(\eta))^{\varphi}(w(\gamma)-w(\xi))^{\rho}\digamma_{\eta\xi}(\eta,\xi)d\xi d\eta,$$

$$I_2=\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}(g(\varkappa)-g(\eta))^{\varphi}(w(\xi)-w(\gamma))^{\rho}\digamma_{\eta\xi}(\eta,\xi)d\xi d\eta,$$

$$I_3=\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}(g(\eta)-g(\varkappa))^{\varphi}(w(\gamma)-w(\xi))^{\rho}\digamma_{\eta\xi}(\eta,\xi)d\xi d\eta,$$

$$I_4=\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}(g(\eta)-g(\varkappa))^{\varphi}(w(\xi)-w(\gamma))^{\rho}\digamma_{\eta\xi}(\eta,\xi)d\xi d\eta,$$

olmak üzere,

$$\begin{aligned} &A(g,w)-N_w^{\rho}(\kappa_3,\gamma)\left[\mathcal{J}_{\kappa_1+;g}^{\varphi}\digamma(\varkappa,\kappa_3)+\mathcal{J}_{\kappa_2-;g}^{\varphi}\digamma(\varkappa,\kappa_3)\right] \\ &-N_w^{\rho}(\kappa_4,\gamma)\left[\mathcal{J}_{\kappa_1+;g}^{\varphi}\digamma(\varkappa,\kappa_4)+\mathcal{J}_{\kappa_2-;g}^{\varphi}\digamma(\varkappa,\kappa_4)\right] \\ &-M_g^{\varphi}(\kappa_1,\varkappa)\left[\mathcal{J}_{\kappa_3+;w}^{\rho}\digamma(\kappa_1,\gamma)+\mathcal{J}_{\kappa_4-;w}^{\rho}\digamma(\kappa_1,\gamma)\right] \\ &-M_g^{\varphi}(\kappa_2,\varkappa)\left[\mathcal{J}_{\kappa_3+;w}^{\rho}\digamma(b,\gamma)+\mathcal{J}_{\kappa_4-;w}^{\rho}\digamma(\kappa_2,\gamma)\right] \end{aligned}$$

$$+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+;g,w}F(\varkappa,\gamma)+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-;g,w}F(\varkappa,\gamma)$$

$$+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+;g,w}F(\varkappa,\gamma)+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-;g,w}F(\varkappa,\gamma)$$

$$=\frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)}\left[I_1-I_2-I_3+I_4\right],$$

elde edilir.

İspat. Kısmi integrasyon yardımıyla,

$$I_1=\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\varkappa)-g(\eta))^{\varphi}(w(\gamma)-w(\xi))^{\rho}F_{\eta\xi}(\eta,\xi)d\xi d\eta \tag{3.1}$$

$$=\int_{\kappa_1}^{\varkappa}(g(\varkappa)-g(\eta))^{\varphi}\left[(w(\gamma)-w(\xi))^{\rho}F_{\eta}(\eta,\xi)\Big|_{\kappa_3}^{\gamma}\right.$$

$$\left.+\rho\int_{\kappa_3}^{\gamma}(w(\gamma)-w(\xi))^{\rho-1}w'(\xi)F_{\eta\xi}(\eta,\xi)d\xi\right]d\eta$$

$$=-\int_{\kappa_1}^{\varkappa}(g(\varkappa)-g(\eta))^{\varphi}(w(\gamma)-w(\kappa_3))^{\rho}F_{\eta}(\eta,\kappa_3)d\eta$$

$$+\rho\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\varkappa)-g(\eta))^{\varphi}(w(\gamma)-w(\xi))^{\rho-1}w'(\xi)F_{\eta}(\eta,\xi)d\xi d\eta$$

$$=-(w(\gamma)-w(\kappa_3))^{\rho}$$

$$\times\left[(g(\varkappa)-g(\eta))^{\varphi}F(\eta,\kappa_3)\Big|_{\kappa_1}^{\varkappa}\right.$$

$$\left.+\varphi\int_{\kappa_1}^{\varkappa}(g(\varkappa)-g(\eta))^{\varphi-1}g'(\eta)F(\eta,\kappa_3)d\eta\right]$$

$$+\rho\int_{\kappa_3}^{\gamma}(w(\gamma)-w(\xi))^{\rho-1}w'(\xi)$$

$$\times\left[(g(\varkappa)-g(\eta))^{\varphi}F(\eta,\xi)\Big|_{\kappa_1}^{\varkappa}\right.$$

$$\left.+\varphi\int_{\kappa_1}^{\varkappa}(g(\varkappa)-g(\eta))^{\varphi-1}g'(\eta)F(\eta,\xi)d\eta\right]d\xi$$

$$=(w(\gamma)-w(\kappa_3))^{\rho}(g(\varkappa)-g(\kappa_1))^{\varphi}F(\kappa_1,\kappa_3)$$

$$-\Gamma(\varphi+1)(w(\gamma)-w(\kappa_3))^{\rho}\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_3)$$

$$-\Gamma(\rho+1)(g(\varkappa)-g(\kappa_1))^{\varphi}\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_1,\gamma)$$

$$+\Gamma(\varphi+1)\Gamma(\rho+1)\mathcal{J}^{\varphi,\rho}_{\kappa_1+\kappa_3+;g,w}F(\varkappa,\gamma),$$

elde edilir. Benzer şekilde,

$$I_2=-(g(\varkappa)-g(\kappa_1))^{\varphi}(w(\kappa_4)-w(\gamma))^{\rho}F(\kappa_1,\kappa_4) \tag{3.2}$$

$$+\Gamma(\varphi+1)(w(\kappa_4)-w(\gamma))^{\rho}\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_4)$$

$$+\Gamma(\rho+1)(g(\varkappa)-g(\kappa_1))^{\varphi}\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_1,\gamma)$$

$$-\Gamma(\varphi+1)\Gamma(\rho+1)\mathcal{J}^{\varphi,\rho}_{\kappa_1+\kappa_4-;g,w}F(\varkappa,\gamma),$$

$$I_3=-(g(\kappa_2)-g(\varkappa))^{\varphi}(w(\gamma)-w(\kappa_3))^{\rho}F(\kappa_2,\kappa_3) \tag{3.3}$$

$$+\Gamma(\varphi+1)(w(\gamma)-w(\kappa_3))^{\rho}\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_3)$$

$$+\Gamma(\rho+1)(g(\kappa_2)-g(\varkappa))^{\varphi}\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_2,\gamma)$$

$$-\Gamma(\varphi+1)\Gamma(\rho+1)\mathcal{J}^{\varphi,\rho}_{\kappa_2-\kappa_3+;g,w}F(\varkappa,\gamma),$$

ve

$$I_4=(g(\kappa_2)-g(\varkappa))^{\varphi}(w(\kappa_4)-w(\gamma))^{\rho}F(\kappa_2,\kappa_4) \tag{3.4}$$

$$-\Gamma(\varphi+1)(w(\kappa_4)-w(\gamma))^{\rho}\mathcal{J}^{\varphi}_{\kappa_3+;g}F(\varkappa,\kappa_4)$$

$$-\Gamma(\rho+1)(g(\kappa_2)-g(\varkappa))^{\varphi}\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_2,\gamma)$$

$$+\Gamma(\varphi+1)\Gamma(\rho+1)\mathcal{J}^{\varphi,\rho}_{\kappa_2-\kappa_4-;g,w}F(\varkappa,\gamma),$$

eşitlikleri vardır. (3.1)–(3.4) eşitliklerinden,

$$A(g,w)-N^{\rho}_{w}(\kappa_3,\gamma)\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_3)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_3)\right]$$

$$-N^{\rho}_{w}(\kappa_4,\gamma)\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_4)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_4)\right]$$

$$-M^{\varphi}_{g}(\kappa_1,\varkappa)\left[\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_1,\gamma)+\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_1,\gamma)\right]$$

$$-M^{\varphi}_{g}(\kappa_2,\varkappa)\left[\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_2,\gamma)+\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_2,\gamma)\right]$$

$$=\frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)}\left[I_1-I_2-I_3+I_4\right],$$

elde edilir ve ispat tamamlanır. □

Teorem 3.3. Lemma 3.2 şartları altında $\frac{F_{\eta\xi}}{g'w'}\in L_{\infty}(\Delta)$ ise genelleştirilmiş kesirli integraller için aşağıdaki Trapezoid tipli eşitsizlik elde edilir.

$$\Big|A(g,w)-N^{\rho}_{w}(\kappa_3,\gamma)\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_3)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_3)\right]$$

$$-N^{\rho}_{w}(\kappa_4,\gamma)\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_4)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_4)\right]$$

$$-M^{\varphi}_{g}(a,\varkappa)\left[\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_1,\gamma)+\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_1,\gamma)\right]$$

$$-M_g^{\varphi}(\kappa_2,\varkappa)\left[\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_2,\gamma)+\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_2,\gamma)\right]$$

$$+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+;g,w}F(\varkappa,\gamma)+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-;g,w}F(\varkappa,\gamma)$$

$$+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+;g,w}F(\varkappa,\gamma)+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-;g,w}F(\varkappa,\gamma)\Big|$$

$$\leq\left[M_g^{\varphi+1}(\kappa_1,\varkappa)+M_g^{\varphi+1}(\kappa_2,\varkappa)\right]$$

$$\times\left[N_w^{\rho+1}(\kappa_3,\gamma)+N_w^{\rho+1}(\kappa_4,\gamma)\right]\left\|\frac{F_{\eta\xi}}{g'w'}\right\|_{\Delta,\infty}.$$

Burada,

$$\left\|\frac{F_{\eta\xi}}{g'w'}\right\|_{\Delta,\infty}=\sup_{(\eta,\xi)\in\Delta}\left|\frac{F_{\eta\xi}(\eta,\xi)}{g'(\eta)w'(\xi)}\right|,$$

dır.

İspat. Lemma 3.2 kullanılarak

$$\Big|A(g,w)-N_w^{\rho}(\kappa_3,\gamma)\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_3)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_3)\right] \tag{3.5}$$

$$-N_w^{\rho}(\kappa_4,\gamma)\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_4)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_4)\right]$$

$$-M_g^{\varphi}(\kappa_1,\varkappa)\left[\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_1,\gamma)+\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_1,\gamma)\right]$$

$$-M_g^{\varphi}(\kappa_2,\varkappa)\left[\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_2,\gamma)+\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_2,\gamma)\right]$$

$$+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+;g,w}F(\varkappa,\gamma)+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-;g,w}F(\varkappa,\gamma)$$

$$+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+;g,w}F(\varkappa,\gamma)+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-;g,w}F(\varkappa,\gamma)\Big|$$

$$\leq\frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)}\left[|I_1|+|I_2|+|I_3|+|I_4|\right],$$

elde edilir. Teorem 3.3 varsayımlarıyla,

$$
\begin{aligned}
|I_1| &= \left| \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (g(\varkappa)-g(t))^{\varphi} (w(\gamma)-w(\xi))^{\rho} F_{\eta\xi}(\eta,\xi) d\xi d\eta \right| \qquad (3.6) \\
&\leq \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (g(\varkappa)-g(\eta))^{\varphi} (w(\gamma)-w(\xi))^{\rho} \left| F_{\eta\xi}(\eta,\xi) \right| d\xi d\eta \\
&= \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (g(\varkappa)-g(\eta))^{\varphi} (w(\gamma)-w(\xi))^{\rho} \\
&\times \left| \frac{F_{\eta\xi}(\eta,\xi)}{g'(\eta)w'(\xi)} \right| g'(\eta)w'(\xi) d\xi d\eta \\
&\leq \left\| \frac{F_{\eta\xi}}{g'w'} \right\|_{[\kappa_1,\varkappa]\times[\kappa_3,\gamma],\infty} \\
&\times \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (g(\varkappa)-g(\eta))^{\varphi} (w(\gamma)-w(\xi))^{\rho} g'(\eta)w'(\xi) d\xi d\eta \\
&= \left\| \frac{F_{\eta\xi}}{g'w'} \right\|_{[\kappa_1,\varkappa]\times[\kappa_3,\gamma],\infty} \\
&\times \frac{(g(\varkappa)-g(\kappa_1))^{\varphi+1}(w(\gamma)-w(\kappa_3))^{\rho+1}}{(\varphi+1)(\rho+1)},
\end{aligned}
$$

eşitsizliği bulunur. Benzer şekilde,

$$
\begin{aligned}
|I_2| &= \left| \int_{\kappa_1}^{\varkappa} \int_{\gamma}^{\kappa_4} (g(\varkappa)-g(\eta))^{\varphi} (w(\xi)-w(\gamma))^{\rho} F_{\eta\xi}(\eta,\xi) d\xi d\eta \right| \qquad (3.7) \\
&\leq \left\| \frac{F_{\eta\xi}}{g'w'} \right\|_{[a,\varkappa]\times[\gamma,\kappa_4],\infty} \\
&\times \frac{(g(\varkappa)-g(\kappa_1))^{\varphi+1}(w(\kappa_4)-w(\gamma))^{\rho+1}}{(\varphi+1)(\rho+1)},
\end{aligned}
$$

$$
\begin{aligned}
|I_3| &= \left| \int_{\varkappa}^{\kappa_2} \int_{\kappa_3}^{\gamma} (g(\kappa_2)-g(t))^{\varphi} (w(\xi)-w(\kappa_3))^{\rho} F_{\eta\xi}(\eta,\xi) d\xi d\eta \right| \qquad (3.8) \\
&\leq \left\| \frac{F_{\eta\xi}}{g'w'} \right\|_{[\varkappa,\kappa_2]\times[\kappa_3,\gamma],\infty} \\
&\times \frac{(g(\kappa_2)-g(\varkappa))^{\varphi+1}(w(\gamma)-w(\kappa_3))^{\rho+1}}{(\varphi+1)(\rho+1)},
\end{aligned}
$$

ve

$$|I_4| = \left| \int_{\varkappa}^{\kappa_2} \int_{\gamma}^{\kappa_4} (g(\kappa_2)-g(\eta))^{\varphi} (w(\kappa_4)-w(\xi))^{\rho} F_{\eta\xi}(\eta,\xi) d\xi d\eta \right| \tag{3.9}$$
$$\leq \left\| \frac{F_{\eta\xi}}{g'w'} \right\|_{[\varkappa,\kappa_2]\times[\gamma,\kappa_4],\infty}$$
$$\times \frac{(g(\kappa_2)-g(\varkappa))^{\varphi+1}(w(\kappa_4)-w(\gamma))^{\rho+1}}{(\varphi+1)(\rho+1)},$$

eşitsizlikleri vardır. (3.6)-(3.9) eşitsizliklerini (3.5) de kullanarak,

$$\left|A(g,w) - N_w^{\rho}(\kappa_3,\gamma)\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_3) + \mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_3)\right]\right.$$

$$-N_w^{\rho}(\kappa_4,\gamma)\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_4) + \mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_4)\right]$$

$$-M_g^{\varphi}(\kappa_1,\varkappa)\left[\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_1,\gamma) + \mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_1,\gamma)\right]$$

$$-M_g^{\varphi}(\kappa_2,\varkappa)\left[\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_2,\gamma) + \mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_2,\gamma)\right]$$

$$+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+;g,w}F(\varkappa,\gamma) + \mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-;g,w}F(\varkappa,\gamma)$$

$$\left.+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+;g,w}F(\varkappa,\gamma) + \mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-;g,w}F(\varkappa,\gamma)\right|$$

$$\leq \frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)}$$
$$\times \left[\left\| \frac{F_{\eta\xi}}{g'w'} \right\|_{[\kappa_1,\varkappa]\times[\kappa_3,\gamma],\infty} \right.$$
$$\times \frac{(g(\varkappa)-g(\kappa_1))^{\varphi+1}(w(\gamma)-w(\kappa_3))^{\rho+1}}{(\varphi+1)(\rho+1)}$$
$$+ \left\| \frac{F_{\eta\xi}}{g'w'} \right\|_{[\kappa_1,\varkappa]\times[\gamma,\kappa_4],\infty}$$
$$\times \frac{(g(\varkappa)-g(\kappa_1))^{\varphi+1}(w(\kappa_4)-w(\gamma))^{\rho+1}}{(\varphi+1)(\rho+1)}$$
$$+ \left\| \frac{F_{\eta\xi}}{g'w'} \right\|_{[\varkappa,\kappa_2]\times[\kappa_3,\gamma],\infty}$$

$$\times \frac{(g(\kappa_2)-g(\varkappa))^{\varphi+1}(w(\gamma)-w(\kappa_3))^{\rho+1}}{(\varphi+1)(\rho+1)}$$

$$\left\| \frac{F_{\eta\xi}}{g'w'} \right\|_{[\varkappa,\kappa_2]\times[\gamma,\kappa_4],\infty} \Bigg]$$

$$\times \frac{(g(\kappa_2)-g(\varkappa))^{\varphi+1}(w(\kappa_4)-w(\gamma))^{\rho+1}}{(\varphi+1)(\rho+1)}$$

$$\leq \left[M_g^{\varphi+1}(\kappa_1,\varkappa)+M_g^{\rho+1}(\kappa_2,\varkappa)\right]$$

$$\times \left[N_g^{\varphi+1}(\kappa_3,\gamma)+N_g^{\rho+1}(\kappa_4,\gamma)\right] \left\| \frac{F_{\eta\xi}}{g'w'} \right\|_{[\kappa_1,\kappa_2]\times[\kappa_2,\kappa_4],\infty},$$

istenilen sonuç elde edilir. Böylece kanıt tamamlanır. □

Özellik 3.4. Theorem 3.3 de $g(\eta)=\eta$, $\eta\in[\kappa_1,\kappa_2]$ ve $w(\xi)=\xi$, $\xi\in[\kappa_3,\kappa_4]$ şeklinde seçilirse Riemann-Liouville kesirli integralleri için aşağıdaki Trapezoid tipli eşitsizlik elde edilir;

$$\Bigg| A(\eta,\xi)-\frac{(\gamma-\kappa_3)^{\rho}}{\Gamma(\rho+1)}\left[J_{\kappa_1+}^{\varphi}F(\varkappa,\kappa_3)+J_{\kappa_2-}^{\varphi}F(\varkappa,\kappa_3)\right] \tag{3.10}$$

$$-\frac{(\kappa_4-\gamma)^{\rho}}{\Gamma(\rho+1)}\left[J_{\kappa_1+}^{\varphi}F(\varkappa,\kappa_4)+J_{\kappa_2-}^{\varphi}F(\varkappa,\kappa_4)\right]$$

$$-\frac{(\varkappa-\kappa_1)^{\varphi}}{\Gamma(\varphi+1)}\left[J_{\kappa_3+}^{\rho}F(\kappa_1,\gamma)+J_{\kappa_4-}^{\rho}F(\kappa_1,\gamma)\right]$$

$$-\frac{(\kappa_2-\varkappa)^{\varphi}}{\Gamma(\varphi+1)}\left[J_{\kappa_3+}^{\rho}F(\kappa_2,\gamma)+J_{\kappa_4-}^{\rho}F(\kappa_2,\gamma)\right]$$

$$+J_{\kappa_1+,\kappa_3+}^{\varphi,\rho}F(\varkappa,\gamma)+J_{\kappa_1+,\kappa_4-}^{\varphi,\rho}F(\varkappa,\gamma)$$

$$+J_{\kappa_2-,\kappa_3+}^{\varphi,\rho}F(\varkappa,\gamma)+J_{\kappa_2-,\kappa_4-}^{\varphi,\rho}F(\varkappa,\gamma)\Bigg|$$

$$\leq \frac{1}{\Gamma(\varphi+2)\Gamma(\rho+2)}\left[(\varkappa-\kappa_1)^{\varphi+1}+(\kappa_2-\varkappa)^{\varphi+1}\right]$$

$$\times\left[(\gamma-\kappa_3)^{\rho+1}+(\kappa_4-\gamma)^{\rho+1}\right]\left\|F_{\eta\xi}\right\|_{\Delta,\infty}.$$

Not 3.5. Sonuç 3.4 de $\varphi = \rho = 1$ alınırsa aşağıdaki eşitsizlik elde edilir,

$$\begin{aligned}
&|(\varkappa-\kappa_1)(\gamma-\kappa_3)F(\kappa_1,\kappa_3)+(\varkappa-\kappa_1)(\kappa_4-\gamma)F(\kappa_1,\kappa_4)\\
&+(\kappa_2-\varkappa)(\gamma-\kappa_3)F(\kappa_2,\kappa_3)+(\kappa_2-\varkappa)(\kappa_4-\gamma)F(\kappa_2,\kappa_4)\\
&-(\gamma-\kappa_3)\int_{\kappa_1}^{\kappa_2}F(\eta,\kappa_3)d\eta-(\kappa_4-\gamma)\int_{\kappa_1}^{\kappa_2}F(\eta,\kappa_4)d\eta\\
&-(\varkappa-\kappa_1)\int_{\kappa_3}^{\kappa_4}F(\kappa_1,\xi)d\xi-(\kappa_2-\varkappa)\int_{\kappa_3}^{\kappa_4}F(\kappa_2,\xi)d\xi\\
&+\left.\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}F(\eta,\xi)\,d\eta d\xi\right|\\
&\leq\frac{\left[(\varkappa-\kappa_1)^2+(\kappa_2-\varkappa)^2\right]\left[(\gamma-\kappa_3)^2+(\kappa_4-\gamma)^2\right]}{4}\left\|F_{\eta\xi}\right\|_{\Delta,\infty}.
\end{aligned}$$

Özellik 3.6. Theorem 3.3 de $a,c>0$ olmak üzere $g(\eta)=\ln\eta$, $\eta\in[\kappa_1,\kappa_2]$ ve $w(\xi)=\ln\xi$, $\xi\in[\kappa_3,\kappa_4]$ şeklinde seçilirse Hadamard kesirli integralleri için aşağıdaki Trapezoid tipli eşitsizlik elde edilir,

$$\begin{aligned}
&\left|A(\ln,\ln)-N_{\ln}^{\rho}(\kappa_3,\gamma)\left[\mathbf{J}_{\kappa_1+}^{\varphi}F(\varkappa,\kappa_3)+\mathbf{J}_{\kappa_2-}^{\varphi}F(\varkappa,\kappa_3)\right]\right.\\
&-N_{\ln}^{\rho}(\kappa_4,\gamma)\left[\mathbf{J}_{\kappa_1+}^{\varphi}F(\varkappa,\kappa_4)+\mathbf{J}_{\kappa_2-}^{\varphi}F(\varkappa,\kappa_4)\right]\\
&-M_{\ln}^{\varphi}(\kappa_1,\varkappa)\left[\mathbf{J}_{\kappa_3+}^{\rho}F(\kappa_1,\gamma)+\mathbf{J}_{\kappa_4-}^{\rho}F(\kappa_1,\gamma)\right]\\
&-M_{\ln}^{\varphi}(\kappa_2,\varkappa)\left[\mathbf{J}_{\kappa_3+}^{\rho}F(\kappa_2,\gamma)+\mathbf{J}_{\kappa_4-}^{\rho}F(\kappa_2,\gamma)\right]\\
&+\mathbf{J}_{\kappa_1+,\kappa_3+}^{\varphi,\rho}F(\varkappa,\gamma)+\mathbf{J}_{\kappa_1+,\kappa_4-}^{\varphi,\rho}F(\varkappa,\gamma)\\
&\left.+\mathbf{J}_{\kappa_2-,\kappa_3+}^{\varphi,\rho}F(\varkappa,\gamma)+\mathbf{J}_{\kappa_2-,\kappa_4-}^{\varphi,\rho}F(\varkappa,\gamma)\right|
\end{aligned}$$

$$\leq \left[M_{\ln}^{\varphi+1}(\kappa_1,\varkappa)+M_{\ln}^{\rho+1}(\kappa_2,\varkappa)\right]$$

$$\times\left[N_{\ln}^{\varphi+1}(\kappa_3,\gamma)+N_{\ln}^{\rho+1}(\kappa_4,\gamma)\right]\left\|\frac{F_{\eta\xi}}{g'w'}\right\|_{[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4],\infty}.$$

Teorem 3.7. Lemma 3.2 şartları altında $\frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\in L_p(\Delta)$, $\frac{1}{p}+\frac{1}{q}=1$ ve

$$\left\|\frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\right\|_{\Delta,p}=\left(\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\left|\frac{F_{\eta\xi}(\eta,\xi)}{(g'(\eta)w'(\xi))^{\frac{1}{q}}}\right|^p d\xi d\eta\right)^{\frac{1}{p}}<+\infty,$$

olsun. Bu durumda genelleştirilmiş kısmi integral yardımıyla aşağıdaki trapezoid tipli eşitsizlik elde edilir.

$$\Big|A(g,w)-N_w^{\rho}(\kappa_3,\gamma)\left[\mathcal{J}_{\kappa_1+;g}^{\varphi}F(\varkappa,\kappa_3)+\mathcal{J}_{\kappa_2-;g}^{\varphi}F(\varkappa,\kappa_3)\right]$$

$$-N_w^{\rho}(\kappa_4,\gamma)\left[\mathcal{J}_{\kappa_1+;g}^{\varphi}F(\varkappa,\kappa_4)+\mathcal{J}_{\kappa_2-;g}^{\varphi}F(\varkappa,\kappa_4)\right]$$

$$-M_g^{\varphi}(\kappa_1,\varkappa)\left[\mathcal{J}_{\kappa_3+;w}^{\rho}F(\kappa_1,\gamma)+\mathcal{J}_{\kappa_4-;w}^{\rho}F(\kappa_1,\gamma)\right]$$

$$-M_g^{\varphi}(\kappa_2,\varkappa)\left[\mathcal{J}_{\kappa_3+;w}^{\rho}F(\kappa_2,\gamma)+\mathcal{J}_{\kappa_4-;w}^{\rho}F(\kappa_2,\gamma)\right]$$

$$+\mathcal{J}_{\kappa_1+,\kappa_3+;g,w}^{\varphi,\rho}F(\varkappa,\gamma)+\mathcal{J}_{\kappa_1+,\kappa_4-;g,w}^{\varphi,\rho}F(\varkappa,\gamma)$$

$$+\mathcal{J}_{\kappa_2-,\kappa_3+;g,w}^{\varphi,\rho}F(\varkappa,\gamma)+\mathcal{J}_{\kappa_2-,\kappa_4-;g,w}^{\varphi,\rho}F(\varkappa,\gamma)\Big|$$

$$\leq\frac{1}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}\Gamma(\varphi+1)\Gamma(\rho+1)}$$

$$\times\left\|\frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\right\|_{[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4],p}$$

$$\times\left[(g(\varkappa)-g(\kappa_1))^{\varphi+\frac{1}{q}}+(g(\kappa_2)-g(\varkappa))^{\varphi+\frac{1}{q}}\right]$$

$$\times\left[(w(\gamma)-w(\kappa_3))^{\rho+\frac{1}{q}}+(w(\kappa_4)-w(\gamma))^{\rho+\frac{1}{q}}\right].$$

İspat. Hölder eşitsizliği ve mutlak değerin özellikleri kullanılarak,

$$\begin{aligned}
|I_1| &\leq \int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\varkappa)-g(\eta))^{\varphi}(w(\gamma)-w(\xi))^{\rho}\left|F_{\eta\xi}(\eta,\xi)\right|d\xi d\eta \qquad (3.11)\\
&= \int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\varkappa)-g(\eta))^{\varphi}(w(\gamma)-w(\xi))^{\rho}\\
&\times\left|\frac{F_{\eta\xi}(\eta,\xi)}{g'(\eta)w'(\xi)}\right|(g'(\eta))^{\frac{1}{p}+\frac{1}{q}}(w'(\xi))^{\frac{1}{p}+\frac{1}{q}}d\xi d\eta\\
&\leq \left(\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}\left|\frac{F_{\eta\xi}(\eta,\xi)}{g'(\eta)w'(\xi)}\right|^{p}g'(\eta)w'(\xi)d\xi d\eta\right)^{\frac{1}{p}}\\
&\times\left(\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\varkappa)-g(\eta))^{\varphi q}(w(\gamma)-w(\xi))^{\rho q}g'(\eta)w'(\xi)d\xi d\eta\right)^{\frac{1}{q}}\\
&\leq \left\|\frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\right\|_{[\kappa_1,\varkappa]\times[\kappa_3,\gamma],p}\\
&\times\left(\int_{\kappa_1}^{\varkappa}(g(\varkappa)-g(t))^{\varphi q}g'(\eta)d\eta\right)^{\frac{1}{q}}\\
&\times\left(\int_{\kappa_3}^{\gamma}(w(\gamma)-w(\xi))^{\rho q}w'(\xi)d\xi\right)^{\frac{1}{q}}\\
&= \left(\frac{(g(\varkappa)-g(\kappa_1))^{\varphi q+1}}{\varphi q+1}\right)^{\frac{1}{q}}\\
&\times\left(\frac{(w(\gamma)-w(\kappa_3))^{\rho q+1}}{\rho q+1}\right)^{\frac{1}{q}}\left\|\frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\right\|_{[\kappa_1,\varkappa]\times[\kappa_3,\gamma],p}\\
&= \left\|\frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\right\|_{[\kappa_1,\varkappa]\times[\kappa_3,\gamma],p}\\
&\times\frac{(g(\varkappa)-g(\kappa_1))^{\varphi+\frac{1}{q}}(w(\gamma)-w(\kappa_3))^{\rho+\frac{1}{q}}}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}},
\end{aligned}$$

eşitsizliği bulunur. Benzer şekilde,

$$|I_2| \leq \left\| \frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}} \right\|_{[\kappa_1,\varkappa]\times[\gamma,\kappa_4],p} \times \frac{(g(\varkappa)-g(\kappa_1))^{\varphi+\frac{1}{q}}(w(\kappa_4)-w(\gamma))^{\rho+\frac{1}{q}}}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}}, \tag{3.12}$$

$$|I_3| \leq \left\| \frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}} \right\|_{[\varkappa,\kappa_2]\times[\kappa_3,\gamma],p} \times \frac{(g(\kappa_2)-g(\varkappa))^{\varphi+\frac{1}{q}}(w(\gamma)-w(\kappa_3))^{\rho+\frac{1}{q}}}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}}, \tag{3.13}$$

ve

$$|I_4| \leq \left\| \frac{F_{\eta}}{(g'w')^{\frac{1}{q}}} \right\|_{[\varkappa,\kappa_2]\times[\gamma,\kappa_4],p} \times \frac{(g(\kappa_2)-g(\varkappa))^{\varphi+\frac{1}{q}}(w(\kappa_4)-w(\gamma))^{\rho+\frac{1}{q}}}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}}, \tag{3.14}$$

olduğu açıktır. (3.11)–(3.14) eşitsizliklerini (3.5) de kullanarak,

$$\Big|A(g,w) - N_w^{\rho}(\kappa_3,\gamma)\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_3) + \mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_3)\right]$$

$$-N_w^{\rho}(\kappa_4,\gamma)\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_4) + \mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_4)\right]$$

$$-M_g^{\varphi}(\kappa_1,\varkappa)\left[\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_1,\gamma) + \mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_1,\gamma)\right]$$

$$-M_g^{\varphi}(\kappa_2,\varkappa)\left[\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_2,\gamma) + \mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_2,\gamma)\right]$$

$$+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+;g,w}F(\varkappa,\gamma) + \mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-;g,w}F(\varkappa,\gamma)$$

$$+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+;g,w}F(\varkappa,\gamma) + \mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-;g,w}F(\varkappa,\gamma)\Big|$$

$$\leq \frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)}$$

$$\times \Bigg[\left\| \frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}} \right\|_{[\kappa_1,\varkappa]\times[\kappa_3,\gamma],p}$$

$$\times \frac{(g(\varkappa)-g(\kappa_1))^{\varphi+\frac{1}{q}}(w(\gamma)-w(\kappa_3))^{\rho+\frac{1}{q}}}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}}$$

$$+ \left\| \frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}} \right\|_{[\kappa_1,\varkappa]\times[\gamma,\kappa_4],p}$$

$$\times \frac{(g(\varkappa)-g(\kappa_1))^{\varphi+\frac{1}{q}}(w(\kappa_4)-w(\gamma))^{\rho+\frac{1}{q}}}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}}$$

$$+ \left\| \frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}} \right\|_{[\varkappa,\kappa_2]\times[\kappa_3,\gamma],p}$$

$$\times \frac{(g(\kappa_2)-g(\varkappa))^{\varphi+\frac{1}{q}}(w(\gamma)-w(\kappa_3))^{\rho+\frac{1}{q}}}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}}$$

$$+ \left\| \frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}} \right\|_{[\varkappa,\kappa_2]\times[\gamma,\kappa_4],p} \Bigg]$$

$$\times \frac{(g(\kappa_2)-g(\varkappa))^{\varphi+\frac{1}{q}}(w(\kappa_4)-w(\gamma))^{\rho+\frac{1}{q}}}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}}$$

$$\leq \frac{1}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}\Gamma(\varphi+1)\Gamma(\rho+1)}$$

$$\times \left\| \frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}} \right\|_{[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4],p}$$

$$\times \left[(g(\varkappa)-g(\kappa_1))^{\varphi+\frac{1}{q}} + (g(\kappa_2)-g(\varkappa))^{\varphi+\frac{1}{q}} \right]$$

$$\times \left[(w(\gamma)-w(\kappa_3))^{\rho+\frac{1}{q}} + (w(\kappa_4)-w(\gamma))^{\rho+\frac{1}{q}} \right],$$

bulunur. Böylece ispat tamamlanır. □

Özellik 3.8. Theorem 3.7 de $g(\eta)=\eta$, $\eta\in[\kappa_1,\kappa_2]$ ve $w(\xi)=\xi$, $\xi\in[\kappa_3,\kappa_4]$ şeklinde seçilirse Riemann–Liouville kesirli integralleri için aşağıdaki Trapezoid tipli eşitsizlik elde edilir,

$$\Bigg| A(\eta,\xi) - \frac{(\gamma-\kappa_3)^{\rho}}{\Gamma(\rho+1)} \left[J^{\varphi}_{\kappa_1+}F(\varkappa,\kappa_3) + J^{\varphi}_{\kappa_2-}F(\varkappa,\kappa_3) \right]$$

$$-\frac{(\kappa_4-\gamma)^{\rho}}{\Gamma(\rho+1)}\left[J^{\varphi}_{\kappa_1+}F(\varkappa,\kappa_4)+J^{\varphi}_{\kappa_2-}F(\varkappa,\kappa_4)\right]$$

$$-\frac{(\varkappa-\kappa_1)^{\varphi}}{\Gamma(\varphi+1)}\left[J^{\rho}_{\kappa_3+}F(\kappa_1,\gamma)+J^{\rho}_{\kappa_4-}F(\kappa_1,\gamma)\right]$$

$$-\frac{(\kappa_2-\varkappa)^{\varphi}}{\Gamma(\varphi+1)}\left[J^{\rho}_{\kappa_3+}F(\kappa_2,\gamma)+J^{\rho}_{\kappa_4-}F(\kappa_2,\gamma)\right]$$

$$+J^{\varphi,\rho}_{\kappa_1+,\kappa_3+}F(\varkappa,\gamma)+J^{\varphi,\rho}_{\kappa_1+,\kappa_4-}F(\varkappa,\gamma)$$

$$+J^{\varphi,\rho}_{\kappa_2-,\kappa_3+}F(\varkappa,\gamma)+J^{\varphi,\rho}_{\kappa_2-,\kappa_4-}F(\varkappa,\gamma)\Big|$$

$$\leq\frac{\left[(\varkappa-\kappa_1)^{\varphi+\frac{1}{q}}+(\kappa_2-\varkappa)^{\varphi+\frac{1}{q}}\right]}{\Gamma(\varphi+1)\Gamma(\rho+1)}$$

$$\times\frac{\left[(\gamma-\kappa_3)^{\rho+\frac{1}{q}}+(\kappa_4-\gamma)^{\rho+\frac{1}{q}}\right]}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}}\left\|F_{\eta\xi}\right\|_{[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4],p}.$$

Not 3.9. Sonuç 3.8 de $\varphi=\rho=1$ alınırsa aşağıdaki eşitsizlik elde edilir,

$$|(\varkappa-\kappa_1)(\gamma-\kappa_3)F(\kappa_1,\kappa_3)+(\varkappa-\kappa_1)(\kappa_4-\gamma)F(\kappa_1,\kappa_4)$$

$$+(\kappa_2-\varkappa)(\gamma-\kappa_3)F(\kappa_2,\kappa_3)+(\kappa_2-\varkappa)(\kappa_4-\gamma)F(\kappa_2,\kappa_4)$$

$$-(\gamma-\kappa_3)\int_{\kappa_1}^{\kappa_2}F(\eta,\kappa_3)d\eta-(\kappa_4-\gamma)\int_{\kappa_1}^{\kappa_2}F(\eta,\kappa_4)d\eta$$

$$-(\varkappa-\kappa_1)\int_{\kappa_3}^{\kappa_4}F(\kappa_1,\xi)d\xi-(\kappa_2-\varkappa)\int_{\kappa_3}^{\kappa_4}F(\kappa_2,\xi)d\xi$$

$$+\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}F(\eta,\xi)\,d\eta d\xi\Bigg|$$

$$\leq\frac{\left[(\varkappa-\kappa_1)^{1+\frac{1}{q}}+(\kappa_2-\varkappa)^{1+\frac{1}{q}}\right]\left[(\gamma-\kappa_3)^{1+\frac{1}{q}}+(\kappa_4-\gamma)^{1+\frac{1}{q}}\right]}{(q+1)^{\frac{2}{q}}}\left\|F_{\eta\xi}\right\|_{\Delta,p}.$$

Özellik 3.10. Theorem 3.7 de $\kappa_1, \kappa_3 > 0$ olmak üzere $g(\eta) = \ln \eta$, $\eta \in [\kappa_1, \kappa_2]$ ve $w(\xi) = \ln \xi$, $\xi \in [\kappa_3, \kappa_4]$ şeklinde seçilirse Hadamard kesirli integralleri için aşağıdaki Trapezoid tipli eşitsizlik elde edilir;

$$
\begin{aligned}
&\Big| A(\ln, \ln) - N_{\ln}^{\rho}(\kappa_3, \gamma)\left[\mathbf{J}_{\kappa_1+}^{\varphi} F(\varkappa, \kappa_3) + \mathbf{J}_{\kappa_2-}^{\varphi} F(\varkappa, \kappa_3)\right] \\
&- N_{\ln}^{\rho}(\kappa_4, \gamma)\left[\mathbf{J}_{\kappa_1+}^{\varphi} F(\varkappa, \kappa_4) + \mathbf{J}_{\kappa_2-}^{\varphi} F(\varkappa, \kappa_4)\right] \\
&- M_{\ln}^{\varphi}(\kappa_1, \varkappa)\left[\mathbf{J}_{\kappa_3+}^{\rho} F(\kappa_1, \gamma) + \mathbf{J}_{\kappa_4-}^{\rho} F(\kappa_1, \gamma)\right] \\
&- M_{\ln}^{\varphi}(\kappa_2, \varkappa)\left[\mathbf{J}_{\kappa_3+}^{\rho} F(\kappa_2, \gamma) + \mathbf{J}_{\kappa_4-}^{\rho} F(\kappa_2, \gamma)\right] \\
&+ \mathbf{J}_{\kappa_1+,\kappa_3+}^{\varphi,\rho} F(\varkappa, \gamma) + \mathbf{J}_{\kappa_1+,\kappa_4-}^{\varphi,\rho} F(\varkappa, \gamma) \\
&+ \mathbf{J}_{\kappa_2-,\kappa_3+}^{\varphi,\rho} F(\varkappa, \gamma) + \mathbf{J}_{\kappa_2-,\kappa_4-}^{\varphi,\rho} F(\varkappa, \gamma) \Big| \\
&\leq \frac{\left[(\ln \frac{\varkappa}{a})^{\varphi+\frac{1}{q}} + (\ln \frac{\kappa_2}{\varkappa})^{\varphi+\frac{1}{q}}\right]\left[(\ln \frac{\gamma}{\kappa_3})^{\rho+\frac{1}{q}} + (\ln \frac{\kappa_4}{\gamma})^{\rho+\frac{1}{q}}\right]}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}\Gamma(\varphi+1)\gamma(\rho+1)} \\
&\times \left\| \frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}} \right\|_{[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4],p}.
\end{aligned}
$$

Not 3.11. Theorem 3.3 ve Theorem 3.7 de $\varkappa = \frac{\kappa_1+\kappa_2}{2}$ ve $\gamma = \frac{\kappa_3+\kappa_4}{2}$ olarak seçilirse genelleştirilmiş kesirli integraller için bazı yeni Midpoint tipli eşitsizlikler elde edilebilir [53].

4. GENELLEŞTİRİLMİŞ KESİRLİ İNTEGRALLER YARDIMIYLA İKİ DEĞİŞKENLİ FONKSİYONLAR İÇİN OSTROWSKİ TİPLİ EŞİTSİZLİKLER

Bu bölümde genelleştirilmiş kesirli integralleri içeren iki değişkenli fonksiyonlar için Ostrowski tipli integral eşitsizlikleri oluşturulacaktır. Bu bölümde sunulacak sonuçlar daha önce literatürde var olan çalışmaların genellemeleri olacaktır.

İlk olarak sıklıkla kullanılacak olan aşağıdaki fonksiyonlar tanımlansın.
$(\varkappa,\gamma)\in\Delta$ için,

$$M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)=\frac{[g(\varkappa)-g(\kappa_1)]^{\varphi}+[g(\kappa_2)-g(\varkappa)]^{\varphi}}{\Gamma(\varphi+1)},$$

ve

$$N_w^{\rho}(\kappa_3,\kappa_4;y)=\frac{[w(\gamma)-w(\kappa_3)]^{\rho}+[w(\kappa_4)-w(\gamma)]^{\rho}}{\Gamma(\rho+1)},$$

olsun. Ayrıca, ve $N_w^{\rho}(\kappa_3,\kappa_4)$,

$$M_g^{\varphi}(\kappa_1,\kappa_2)=M_g^{\varphi}(\kappa_1,\kappa_2;\kappa_1)=M_g^{\varphi}(\kappa_1,\kappa_2;\kappa_2)=\frac{[g(\kappa_2)-g(\kappa_1)]^{\varphi}}{\Gamma(\varphi+1)},$$

ve

$$N_w^{\rho}(\kappa_3,\kappa_4)=N_w^{\rho}(\kappa_3,\kappa_4;\kappa_3)=N_w^{\rho}(\kappa_3,\kappa_4;\kappa_4)=\frac{[w(\kappa_4)-w(\kappa_3)]^{\rho}}{\Gamma(\rho+1)},$$

olarak tanımlansın. Burada $g(\eta)=\ln\eta$, $\eta\in[\kappa_1,\kappa_2]$ ve $w(\xi)=\ln\xi$, $\xi\in[\kappa_3,\kappa_4]$ olarak seçilirse,

$$M_{\ln}^{\varphi}(\kappa_1,\kappa_2;\varkappa)=\frac{\left[\ln\frac{\varkappa}{\kappa_1}\right]^{\varphi}+\left[\ln\frac{\kappa_2}{\varkappa}\right]^{\varphi}}{\Gamma(\varphi+1)},\quad N_{\ln}^{\rho}(\kappa_3,\kappa_4;\gamma)=\frac{\left[\ln\frac{\gamma}{\kappa_3}\right]^{\rho}+\left[\ln\frac{\kappa_4}{\gamma}\right]^{\rho}}{\Gamma(\rho+1)},$$

ve

$$M_{\ln}^{\varphi}(\kappa_1,\kappa_2)=\frac{\left[\ln\frac{\kappa_2}{\kappa_1}\right]^{\varphi}}{\Gamma(\varphi+1)},\quad N_{\ln}^{\rho}(\kappa_3,\kappa_4)=\frac{\left[\ln\frac{\kappa_4}{\kappa_3}\right]^{\rho}}{\Gamma(\rho+1)},$$

elde edilir.

Yardımcı Teorem 4.1. $\mathcal{F}:\Delta\to\mathbb{R}$ fonksiyonu Δ° üzerinde iki kez kısmi türevlenebilir olsun. $\varphi,\rho>0$ için $\mathcal{F}_{\eta\xi}\in L(\Delta)$ ise

$$I_1=\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\eta)-g(\kappa_1))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}\mathcal{F}_{\eta\xi}(\eta,\xi)d\xi d\eta,$$

$$I_2=\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}(g(\eta)-g(\kappa_1))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}\mathcal{F}_{\eta\xi}(\eta,\xi)d\xi d\eta,$$

$$I_3=\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}(g(\kappa_2)-g(\eta))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}\mathcal{F}_{\eta\xi}(\eta,\xi)d\xi d\eta,$$

ve

$$I_4=\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}(g(\kappa_2)-g(\eta))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}\mathcal{F}_{\eta\xi}(\eta,\xi)d\xi d\eta,$$

olmak üzere,

$$\begin{aligned}
&\mathcal{F}(\varkappa,\gamma)-\frac{1}{M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)}\left[\mathcal{J}^{\varphi}_{\varkappa^-;g}\mathcal{F}(\kappa_1,\gamma)+\mathcal{J}^{\varphi}_{\varkappa^+;g}\mathcal{F}(\kappa_2,\gamma)\right]\\
&-\frac{1}{N_w^{\rho}(\kappa_3,\kappa_4;\gamma)}\left[\mathcal{J}^{\rho}_{\gamma^-;w}\mathcal{F}(\varkappa,\kappa_3)+\mathcal{J}^{\rho}_{\gamma^+;w}\mathcal{F}(\varkappa,\kappa_4)\right]\\
&+\frac{1}{M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)N_w^{\rho}(\kappa_3,\kappa_4;\gamma)}\\
&\times\left[\mathcal{J}^{\varphi,\rho}_{\varkappa^-\gamma^-;g,w}\mathcal{F}(\kappa_1,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\varkappa^-\gamma^+;g,w}\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{J}^{\varphi,\rho}_{\varkappa^+\gamma^-;g,w}\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\varkappa^+\gamma^+;g,w}\mathcal{F}(\kappa_2,\kappa_4)\right]\\
&=\frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)N_w^{\rho}(\kappa_3,\kappa_4;\gamma)}\left[I_1-I_2-I_3+I_4\right],
\end{aligned}$$

eşitliği vardır.

İspat. Kısmi integral yardımıyla,

$$\begin{aligned}
I_1&=\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\eta)-g(\kappa_1))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}\mathcal{F}_{\eta\xi}(\eta,\xi)d\xi d\eta \qquad (4.1)\\
&=\int_{\kappa_1}^{\varkappa}(g(\eta)-g(\kappa_1))^{\varphi}\\
&\times\left[(w(\xi)-w(\kappa_3))^{\rho}\mathcal{F}_{\eta}(\eta,\xi)\Big|_{\kappa_3}^{\gamma}-\rho\int_{\kappa_3}^{\gamma}(w(\xi)-w(\kappa_3))^{\rho-1}w'(\xi)\mathcal{F}_{\eta}(\eta,\xi)d\xi\right]d\eta
\end{aligned}$$

$$= \int_{\kappa_1}^{\varkappa} (g(\eta) - g(\kappa_1))^{\varphi} (w(\gamma) - w(\kappa_3))^{\rho} \digamma_{\eta}(\eta, \gamma) d\eta$$

$$- \rho \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (g(\eta) - g(\kappa_1))^{\varphi} (w(\xi) - w(\kappa_3))^{\rho-1} w'(\xi) \digamma_{\eta}(\eta, \xi) d\xi d\eta$$

$$= (w(\gamma) - w(\kappa_3))^{\rho}$$

$$\times \left[(g(\eta) - g(\kappa_1))^{\varphi} \digamma(\eta, \gamma) - \varphi \int_{\kappa_1}^{\varkappa} (g(\eta) - g(\kappa_1))^{\varphi-1} g'(\eta) \digamma(\eta, \gamma) d\eta \right]$$

$$- \rho \int_{\kappa_3}^{\gamma} (w(\xi) - w(\kappa_3))^{\rho-1} w'(\xi)$$

$$\times \left[(g(\eta) - g(\kappa_1))^{\varphi} \digamma(\eta, \xi) \Big|_{\kappa_1}^{\varkappa} - \varphi \int_{\kappa_1}^{\varkappa} (g(\eta) - g(\kappa_1))^{\varphi-1} g'(\eta) \digamma(\eta, \xi) d\eta \right] d\xi$$

$$= (w(\gamma) - w(\kappa_3))^{\rho} (g(\varkappa) - g(\kappa_1))^{\varphi} \digamma(\varkappa, \gamma) - \Gamma(\varphi + 1)(w(\gamma) - w(\kappa_3))^{\rho} \mathcal{J}^{\varphi}_{\varkappa^-;g} \digamma(\kappa_1, \gamma)$$

$$- \Gamma(\rho + 1)(g(\varkappa) - g(\kappa_1))^{\varphi} \mathcal{J}^{\rho}_{\gamma^-;w} \digamma(\varkappa, \kappa_3) + \Gamma(\varphi + 1)\Gamma(\rho + 1) \mathcal{J}^{\varphi,\rho}_{\varkappa^-\gamma^-;g,w} \digamma(\kappa_1, \kappa_3),$$

elde edilir. Benzer şekilde,

$$I_2 = -(g(\varkappa) - g(\kappa_1))^{\varphi} (w(\kappa_4) - w(\gamma))^{\rho} \digamma(\varkappa, \gamma) \tag{4.2}$$

$$+ \Gamma(\varphi + 1)(w(\kappa_4) - w(\gamma))^{\rho} \mathcal{J}^{\varphi}_{\varkappa^-;g} \digamma(\kappa_1, \gamma)$$

$$+ \Gamma(\rho + 1)(g(\varkappa) - g(\kappa_1))^{\varphi} \mathcal{J}^{\rho}_{\gamma^+;w} \digamma(\varkappa, \kappa_4)$$

$$- \Gamma(\varphi + 1)\Gamma(\rho + 1) \mathcal{J}^{\varphi,\rho}_{\varkappa^-\gamma^+;g,w} \digamma(\kappa_1, \kappa_4),$$

$$I_3 = -(g(\kappa_2) - g(\varkappa))^{\varphi} (w(\gamma) - w(\kappa_3))^{\rho} \digamma(\varkappa, \gamma) \tag{4.3}$$

$$+ \Gamma(\varphi + 1)(w(\gamma) - w(\kappa_3))^{\rho} \mathcal{J}^{\varphi}_{\varkappa^+;g} \digamma(\kappa_2, \gamma)$$

$$+ \Gamma(\rho + 1)(g(\kappa_2) - g(\varkappa))^{\varphi} \mathcal{J}^{\rho}_{\gamma^-;w} \digamma(\varkappa, \kappa_3)$$

$$-\Gamma(\varphi+1)\Gamma(\rho+1)\mathcal{J}^{\varphi,\rho}_{\varkappa^+\gamma^-;g,w}F(\kappa_2,\kappa_3),$$

ve

$$I_4=(g(\kappa_2)-g(\varkappa))^{\varphi}(w(\kappa_4)-w(\gamma))^{\rho}F(\varkappa,\gamma) \tag{4.4}$$

$$-\Gamma(\varphi+1)(w(\kappa_4)-w(\gamma))^{\rho}\mathcal{J}^{\varphi}_{\varkappa^+;g}F(\kappa_2,\gamma)$$

$$-\Gamma(\rho+1)(g(\kappa_2)-g(\varkappa))^{\varphi}\mathcal{J}^{\rho}_{\gamma^+;w}F(\varkappa,\kappa_4)$$

$$+\Gamma(\varphi+1)\Gamma(\rho+1)\mathcal{J}^{\varphi,\rho}_{\varkappa^+\gamma^+;g,w}F(\kappa_2,\kappa_4),$$

eşitlikleri vardır. (4.1)–(4.4) eşitlikleri yardımıyla,

$$I_1-I_2-I_3+I_4=\Gamma(\varphi+1)\Gamma(\rho+1)M^{\varphi}_g(\kappa_1,\kappa_2;\varkappa)N^{\rho}_w(\kappa_3,\kappa_4;\gamma)F(\varkappa,\gamma)$$

$$-\Gamma(\varphi+1)\Gamma(\rho+1)N^{\rho}_w(\kappa_3,\kappa_4;\gamma)\left[\mathcal{J}^{\varphi}_{\varkappa^-;g}F(\kappa_1,\gamma)+\mathcal{J}^{\varphi}_{\varkappa^+;g}F(\kappa_2,\gamma)\right]$$

$$-\Gamma(\varphi+1)\Gamma(\rho+1)M^{\varphi}_g(\kappa_1,\kappa_2;\varkappa)\left[\mathcal{J}^{\rho}_{\gamma^-;w}F(\varkappa,\kappa_3)+\mathcal{J}^{\rho}_{\gamma^+;w}F(\kappa_2,\gamma)\right]$$

$$+\Gamma(\varphi+1)\Gamma(\rho+1)$$

$$\times\Big[\mathcal{J}^{\varphi,\rho}_{\varkappa^-\gamma^-;g,w}F(\kappa_1,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\varkappa^-\gamma^+;g,w}F(\kappa_1,\kappa_4)$$

$$+\mathcal{J}^{\varphi,\rho}_{\varkappa^+\gamma^-;g,w}F(\kappa_2,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\varkappa^+\gamma^+;g,w}F(\kappa_2,\kappa_4)\Big],$$

elde edilir. □

Teorem 4.2. Lemma 4.1 in şartları altında $\frac{F_{\eta\xi}}{g'w'} \in L_{\infty}(\Delta)$ olsun. Bu durumda,

$$\left\| \frac{F_{\eta\xi}}{g'w'} \right\|_{\Delta,\infty} = \sup_{(\eta,\xi)\in\Delta} \left| \frac{F_{\eta\xi}(\eta,\xi)}{g'(\eta)w'(\xi)} \right|,$$

olmak üzere genelleştirilmiş kesirli integraller için aşağıdaki Ostrowski tipli eşitsizlik sağlanır.

$$\Bigg| F(\varkappa,\gamma) - \frac{1}{M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)} \left[\mathcal{J}_{\varkappa^-;g}^{\varphi} F(\kappa_1,\gamma) + \mathcal{J}_{\varkappa^+;g}^{\varphi} F(\kappa_2,\gamma) \right]$$
$$- \frac{1}{N_w^{\rho}(\kappa_3,\kappa_4;\gamma)} \left[\mathcal{J}_{\gamma^-;w}^{\rho} F(\varkappa,\kappa_3) + \mathcal{J}_{\gamma^+;w}^{\rho} F(\varkappa,\kappa_4) \right] + \frac{1}{M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa) N_w^{\rho}(\kappa_3,\kappa_4;\gamma)}$$
$$\times \left[\mathcal{J}_{\varkappa^-\gamma^-;g,w}^{\varphi,\rho} F(\kappa_1,\kappa_3) + \mathcal{J}_{\varkappa^-\gamma^+;g,w}^{\varphi,\rho} F(\kappa_1,\kappa_4) + \mathcal{J}_{\varkappa^+\gamma^-;g,w}^{\varphi,\rho} F(\kappa_2,\kappa_3) + \mathcal{J}_{\varkappa^+\gamma^+;g,w}^{\varphi,\rho} F(\kappa_2,\kappa_4) \right] \Bigg|$$
$$\leq \frac{M_g^{\varphi+1}(\kappa_1,\kappa_2;\varkappa) N_w^{\rho+1}(\kappa_3,\kappa_4;\gamma)}{M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa) N_w^{\rho}(\kappa_3,\kappa_4;\gamma)} \left\| \frac{F_{\eta\xi}}{g'w'} \right\|_{\Delta,\infty}.$$

İspat. Lemma 4.1 kullanılarak,

$$\Bigg| F(\varkappa,\gamma) - \frac{1}{M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)} \left[\mathcal{J}_{\varkappa^-;g}^{\varphi} F(\kappa_1,\gamma) + \mathcal{J}_{\varkappa^+;g}^{\varphi} F(\kappa_2,\gamma) \right] \tag{4.5}$$
$$- \frac{1}{N_w^{\rho}(\kappa_3,\kappa_4;\gamma)} \left[\mathcal{J}_{\gamma^-;w}^{\rho} F(\varkappa,\kappa_3) + \mathcal{J}_{\gamma^+;w}^{\rho} F(\varkappa,\kappa_4) \right] + \frac{1}{M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa) N_w^{\rho}(\kappa_3,\kappa_4;\gamma)}$$
$$\times \left[\mathcal{J}_{\varkappa^-\gamma^-;g,w}^{\varphi,\rho} F(\kappa_1,\kappa_3) + \mathcal{J}_{\varkappa^-\gamma^+;g,w}^{\varphi,\rho} F(\kappa_1,\kappa_4) + \mathcal{J}_{\varkappa^+\gamma^-;g,w}^{\varphi,\rho} F(\kappa_2,\kappa_3) + \mathcal{J}_{\varkappa^+\gamma^+;g,w}^{\varphi,\rho} F(\kappa_2,\kappa_4) \right] \Bigg|$$
$$\leq \frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1) M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa) N_w^{\rho}(\kappa_3,\kappa_4;\gamma)} \left[|I_1| + |I_2| + |I_3| + |I_4| \right],$$

elde edilir. Burada,

$$\begin{aligned} |I_1| &= \left| \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (g(\eta)-g(\kappa_1))^{\varphi} (w(\xi)-w(\kappa_3))^{\rho} F_{\eta\xi}(\eta,\xi) d\xi d\eta \right| \qquad (4.6) \\ &\leq \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (g(\eta)-g(\kappa_1))^{\varphi} (w(\xi)-w(\kappa_3))^{\rho} \left| F_{\eta\xi}(\eta,\xi) \right| d\xi d\eta \\ &= \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (g(\eta)-g(\kappa_1))^{\varphi} (w(\xi)-w(\kappa_3))^{\rho} \left| \frac{F_{\eta\xi}(\eta,\xi)}{g'(\eta)w'(\xi)} \right| g'(\eta)w'(\xi) d\xi d\eta \\ &\leq \left\| \frac{F_{\eta\xi}}{g'w'} \right\|_{[\kappa_1,\varkappa]\times[\kappa_3,\gamma],\infty} \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (g(\eta)-g(\kappa_1))^{\varphi} (w(\xi)-w(\kappa_3))^{\rho} g'(\eta)w'(\xi) d\xi d\eta \end{aligned}$$

$$= \frac{(g(\varkappa)-g(\kappa_1))^{\varphi+1}(w(\gamma)-w(\kappa_3))^{\rho+1}}{(\varphi+1)(\rho+1)}\left\|\frac{\digamma_{\eta\xi}}{g'w'}\right\|_{[\kappa_1,\varkappa]\times[\kappa_3,\gamma],\infty},$$

ve

$$\begin{aligned} |I_2| &= \left|\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}(g(\eta)-g(\kappa_1))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}\digamma_{\eta\xi}(\eta,\xi)d\xi d\eta\right| \qquad (4.7)\\ &\leq \int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}(g(\eta)-g(\kappa_1))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}\left|\digamma_{\eta\xi}(\eta,\xi)\right|d\xi d\eta\\ &= \int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}(g(\eta)-g(\kappa_1))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}\left|\digamma_{\eta\xi}(\eta,\xi)\right|d\xi d\eta\\ &= \int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}(g(\eta)-g(a))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}\left|\frac{\digamma_{\eta\xi}(\eta,\xi)}{g'(\eta)w'(\xi)}\right|g'(\eta)w'(\xi)d\xi d\eta\\ &\leq \left\|\frac{\digamma_{\eta\xi}}{g'w'}\right\|_{[\kappa_1,\varkappa]\times[\gamma,\kappa_4],\infty}\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}(g(\eta)-g(\kappa_1))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}g'(\eta)w'(\xi)d\xi d\eta\\ &= \frac{(g(\varkappa)-g(\kappa_1))^{\varphi+1}(w(\kappa_4)-w(\gamma))^{\rho+1}}{(\varphi+1)(\rho+1)}\left\|\frac{\digamma_{\eta\xi}}{g'w'}\right\|_{[\kappa_1,\varkappa]\times[\gamma,\kappa_4],\infty}, \end{aligned}$$

elde edilir. Benzer şekilde,

$$\begin{aligned} |I_3| &= \left|\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}(g(\kappa_2)-g(\eta))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}\digamma_{\eta\xi}(\eta,\xi)d\xi d\eta\right| \qquad (4.8)\\ &\leq \frac{(g(\kappa_2)-g(\varkappa))^{\varphi+1}(w(\gamma)-w(\kappa_3))^{\rho+1}}{(\varphi+1)(\rho+1)}\left\|\frac{\digamma_{\eta\xi}}{g'w'}\right\|_{[\varkappa,\kappa_2]\times[\kappa_3,\gamma],\infty}, \end{aligned}$$

ve

$$\begin{aligned} |I_4| &= \left|\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}(g(\kappa_2)-g(\eta))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}\digamma_{\eta\xi}(\eta,\xi)d\xi d\eta\right| \qquad (4.9)\\ &\leq \frac{(g(\kappa_2)-g(\varkappa))^{\varphi+1}(w(\kappa_4)-w(\gamma))^{\rho+1}}{(\varphi+1)(\rho+1)}\left\|\frac{\digamma_{\eta\xi}}{g'w'}\right\|_{[\varkappa,\kappa_2]\times[\gamma,\kappa_4],\infty}, \end{aligned}$$

olduğu açıktır. (4.6)-(4.9) eşitsizlikleri (4.5) de yazılırsa,

$$\left|\digamma(\varkappa,\gamma)-\frac{1}{M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)}\left[\mathcal{J}^{\varphi}_{\varkappa^-;g}\digamma(\kappa_1,\gamma)+\mathcal{J}^{\varphi}_{\varkappa^+;g}\digamma(\kappa_2,\gamma)\right]\right.$$

$$-\frac{1}{N_w^{\rho}(\kappa_3,\kappa_4;\gamma)}\left[\mathcal{J}^{\rho}_{\gamma^-;w}F(\varkappa,\kappa_3)+\mathcal{J}^{\rho}_{\gamma^+;w}F(\varkappa,\kappa_4)\right]+\frac{1}{M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)N_w^{\rho}(\kappa_3,\kappa_4;\gamma)}$$

$$\times\left[\mathcal{J}^{\varphi,\rho}_{\varkappa^-\gamma^-;g,w}F(\kappa_1,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\varkappa^-\gamma^+;g,w}F(\kappa_1,\kappa_4)+\mathcal{J}^{\varphi,\rho}_{\varkappa^+\gamma^-;g,w}F(\kappa_2,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\varkappa^+\gamma^+;g,w}F(\kappa_2,\kappa_4)\right]\Bigg|$$
$$\le\frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)N_w^{\rho}(\kappa_3,\kappa_4;\gamma)}$$
$$\times\Bigg[\frac{(g(\varkappa)-g(\kappa_1))^{\varphi+1}(w(\gamma)-w(\kappa_3))^{\rho+1}}{(\varphi+1)(\rho+1)}\left\|\frac{F_{\eta\xi}}{g'w'}\right\|_{[\kappa_1,\varkappa]\times[\kappa_3,\gamma],\infty}$$
$$+\frac{(g(\varkappa)-g(\kappa_1))^{\varphi+1}(w(\kappa_4)-w(\gamma))^{\rho+1}}{(\varphi+1)(\rho+1)}\left\|\frac{F_{\eta\xi}}{g'w'}\right\|_{[\kappa_1,\varkappa]\times[\gamma,\kappa_4],\infty}$$
$$+\frac{(g(\kappa_2)-g(\varkappa))^{\varphi+1}(w(\gamma)-w(\kappa_3))^{\rho+1}}{(\varphi+1)(\rho+1)}\left\|\frac{F_{\eta\xi}}{g'w'}\right\|_{[\varkappa,\kappa_2]\times[\kappa_3,\gamma],\infty}$$
$$\frac{(g(\kappa_2)-g(\varkappa))^{\varphi+1}(w(\kappa_4)-w(\gamma))^{\rho+1}}{(\varphi+1)(\rho+1)}\left\|\frac{F_{\eta\xi}}{g'w'}\right\|_{[\varkappa,\kappa_2]\times[\gamma,\kappa_4],\infty}\Bigg]$$

$$\le\frac{1}{\Gamma(\varphi+2)\Gamma(\rho+2)M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)N_w^{\rho}(\kappa_3,\kappa_4;\gamma)}\left((g(\varkappa)-g(\kappa_1))^{\varphi+1}+(g(\kappa_2)-g(\varkappa))^{\varphi+1}\right)$$

$$\times\left((w(\gamma)-w(\kappa_3))^{\rho+1}+(w(\kappa_4)-w(\gamma))^{\rho+1}\right)\left\|\frac{F_{\eta\xi}}{g'w'}\right\|_{[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4],\infty},$$

elde edilir ve ispat tamamlanır. □

Özellik 4.3. Theorem 4.2 de $g(\eta)=\eta$, $\eta\in[\kappa_1,\kappa_2]$ ve $w(\xi)=\xi$, $\xi\in[\kappa_3,\kappa_4]$ şeklinde seçilirse Riemann–Liouville kesirli integralleri için aşağıdaki Ostrowski tipli eşitsizlik elde edilir;

$$\Bigg|F(\varkappa,\gamma)-\frac{\Gamma(\varphi+1)}{(\varkappa-\kappa_1)^{\varphi}+(\kappa_2-\varkappa)^{\varphi}}\left[J^{\varphi}_{\varkappa^-}F(\kappa_1,\gamma)+J^{\varphi}_{\varkappa^+}F(\kappa_2,\gamma)\right] \tag{4.10}$$
$$-\frac{\Gamma(\rho+1)}{(\gamma-\kappa_3)^{\rho}+(\kappa_4-\gamma)^{\rho}}\left[J^{\rho}_{\gamma^-}F(\varkappa,\kappa_3)+J^{\rho}_{\gamma^+}F(\varkappa,\kappa_4)\right]$$
$$+\frac{\Gamma(\varphi+1)\Gamma(\rho+1)}{\left[(\varkappa-\kappa_1)^{\varphi}+(\kappa_2-\varkappa)^{\varphi}\right]\left[(\gamma-\kappa_3)^{\rho}+(\kappa_4-\gamma)^{\rho}\right]}$$
$$\times\left[J^{\varphi,\rho}_{\varkappa^-\gamma^-}F(\kappa_1,\kappa_3)+J^{\varphi,\rho}_{\varkappa^-\gamma^+}F(\kappa_1,\kappa_4)+J^{\varphi,\rho}_{\varkappa^+\gamma^-}F(\kappa_2,\kappa_3)+J^{\varphi,\rho}_{\varkappa^+\gamma^+}F(\kappa_2,\kappa_4)\right]\Bigg|$$
$$\le\frac{\left[(\varkappa-\kappa_1)^{\varphi+1}+(\kappa_2-\varkappa)^{\varphi+1}\right]\left[(\gamma-\kappa_3)^{\rho+1}+(\kappa_4-\gamma)^{\rho+1}\right]}{(\varphi+1)(\rho+1)\left[(\varkappa-\kappa_1)^{\varphi}+(\kappa_2-\varkappa)^{\varphi}\right]\left[(\gamma-\kappa_3)^{\rho}+(\kappa_4-\gamma)^{\rho}\right]}\left\|F_{\eta\xi}\right\|_{\Delta,\infty}.$$

Not 4.4. Sonuç 4.3 de $\varphi=\rho=1$ alınırsa (4.10) eşitsizliği, [54, Theorem 2.1] de oluşturulan eşitsizliğe indirgenir.

Özellik 4.5. $\kappa_1, \kappa_3 > 0$ olmak üzere $g(\eta) = \ln\eta$, $\eta \in [\kappa_1, \kappa_2]$ ve $w(\xi) = \ln\xi$, $\xi \in [\kappa_3, \kappa_4]$ olarak seçilirse Theorem 4.2 de Hadamard kesirli integralleri için aşağıdaki Ostrowski tipli eşitsizliği,

$$\begin{aligned}
&\Bigg| \digamma(\varkappa,\gamma) - \frac{1}{M_{\ln}^{\varphi}(\kappa_1,\kappa_2;\varkappa)} \left[\mathbf{J}^{\varphi}_{\varkappa^-;g}\digamma(\kappa_1,\gamma) + \mathbf{J}^{\varphi}_{\varkappa^+;g}\digamma(\kappa_2,\gamma) \right] \\
&- \frac{1}{N_{\ln}^{\rho}(\kappa_3,\kappa_4;\gamma)} \left[\mathbf{J}^{\rho}_{\gamma^-;w}\digamma(\varkappa,\kappa_3) + \mathbf{J}^{\rho}_{\gamma^+;w}\digamma(\varkappa,\kappa_4) \right] + \frac{1}{M_{\ln}^{\varphi}(\kappa_1,\kappa_2;\varkappa) N_{\ln}^{\rho}(\kappa_3,\kappa_4;\gamma)} \\
&\times \left[\mathbf{J}^{\varphi,\rho}_{\varkappa^-\gamma^-;g,w}\digamma(\kappa_1,\kappa_3) + \mathbf{J}^{\varphi,\rho}_{\varkappa^-\gamma^+;g,w}\digamma(\kappa_1,\kappa_4) + \mathbf{J}^{\varphi,\rho}_{\varkappa^+\gamma^-;g,w}\digamma(\kappa_2,\kappa_3) + \mathbf{J}^{\varphi,\rho}_{\varkappa^+\gamma^+;g,w}\digamma(\kappa_2,\kappa_4) \right] \Bigg| \\
&\leq \frac{M_{\ln}^{\varphi+1}(\kappa_1,\kappa_2;\varkappa) N_{\ln}^{\rho+1}(\kappa_3,\kappa_4;\gamma)}{M_{\ln}^{\varphi}(\kappa_1,\kappa_2;\varkappa) N_{\ln}^{\rho}(\kappa_3,\kappa_4;\gamma)} \left\| \frac{\digamma_{\eta\xi}}{g'w'} \right\|_{\Delta,\infty},
\end{aligned}$$

elde edilir.

Teorem 4.6. Lemma 4.1 in şartları altında $\frac{\digamma_{\eta\xi}}{g'w'} \in L_p(\Delta)$ olsun. $\frac{1}{p} + \frac{1}{q} = 1$ için,

$$\|\digamma\|_{\Delta,p} = \left(\int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} |\digamma(\eta,\xi)|^p \, d\xi \, d\eta \right)^{\frac{1}{p}} < +\infty,$$

olmak üzere genelleştirilmiş kesirli integraller için,

$$\begin{aligned}
&\Bigg| \digamma(\varkappa,\gamma) - \frac{1}{M_{g}^{\varphi}(\kappa_1,\kappa_2;\varkappa)} \left[\mathcal{J}^{\varphi}_{\varkappa^-;g}\digamma(\kappa_1,\gamma) + \mathcal{J}^{\varphi}_{\varkappa^+;g}\digamma(\kappa_2,\gamma) \right] \\
&- \frac{1}{N_{w}^{\rho}(\kappa_3,\kappa_4;\gamma)} \left[\mathcal{J}^{\rho}_{\gamma^-;w}\digamma(\varkappa,\kappa_3) + \mathcal{J}^{\rho}_{\gamma^+;w}\digamma(\varkappa,\kappa_4) \right] + \frac{1}{M_{g}^{\varphi}(\kappa_1,\kappa_2;\varkappa) N_{w}^{\rho}(\kappa_3,\kappa_4;\gamma)} \\
&\times \left[\mathcal{J}^{\varphi,\rho}_{\varkappa^-\gamma^-;g,w}\digamma(\kappa_1,\kappa_3) + \mathcal{J}^{\varphi,\rho}_{\varkappa^-\gamma^+;g,w}\digamma(\kappa_1,\kappa_4) + \mathcal{J}^{\varphi,\rho}_{\varkappa^+\gamma^-;g,w}\digamma(\kappa_2,\kappa_3) + \mathcal{J}^{\varphi,\rho}_{\varkappa^+\gamma^+;g,w}\digamma(\kappa_2,\kappa_4) \right] \Bigg| \\
&\leq \frac{\left[(g(\varkappa) - g(\kappa_1))^{\varphi+\frac{1}{q}} + (g(\kappa_2) - g(\varkappa))^{\varphi+\frac{1}{q}} \right] \left[(w(\gamma) - w(\kappa_3))^{\rho+\frac{1}{q}} + (w(\kappa_4) - w(\gamma))^{\rho+\frac{1}{q}} \right]}{(\varphi q+1)^{\frac{1}{q}} (\rho q+1)^{\frac{1}{q}} \Gamma(\varphi+1)\Gamma(\rho+1) M_{g}^{\varphi}(\kappa_1,\kappa_2;\varkappa) N_{w}^{\rho}(\kappa_3,\kappa_4;\gamma)} \\
&\times \left\| \frac{\digamma_{\eta\xi}}{(g'w')^{\frac{1}{q}}} \right\|_{[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4],p},
\end{aligned}$$

Ostrowski tipli eşitsizliği sağlanır.

İspat. Hölder eşitsizliği ve mutlak değerin özellikleri yardımıyla,

$$|I_1| \leq \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (g(\eta) - g(\kappa_1))^{\varphi} (w(\xi) - w(\kappa_3))^{\rho} \left| \digamma_{\eta\xi}(\eta,\xi) \right| d\xi \, d\eta \tag{4.11}$$

$$\leq \int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\eta)-g(\kappa_1))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}\left|\frac{\digamma_{\eta\xi}(\eta,\xi)}{g'(\eta)w'(\xi)}\right|(g'(\eta))^{\frac{1}{p}+\frac{1}{q}}(w'(\xi))^{\frac{1}{p}+\frac{1}{q}}d\xi d\eta$$

$$\leq \left(\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}\left|\frac{\digamma_{\eta\xi}(\eta,\xi)}{g'(\eta)w'(\xi)}\right|^{p}g'(\eta)w'(\xi)d\xi d\eta\right)^{\frac{1}{p}}$$

$$\times\left(\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\eta)-g(\kappa_1))^{\varphi q}(w(\xi)-w(\kappa_3))^{\rho q}g'(\eta)w'(\xi)d\xi d\eta\right)^{\frac{1}{q}}$$

$$\leq \left\|\frac{\digamma_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\right\|_{[\kappa_1,\varkappa]\times[\kappa_3,\gamma],p}\left(\int_{\kappa_1}^{\varkappa}(g(\eta)-g(\kappa_1))^{\varphi q}g'(\eta)d\eta\right)^{\frac{1}{q}}$$

$$\times\left(\int_{\kappa_3}^{\gamma}(w(\xi)-w(\kappa_3))^{\rho q}w'(\xi)d\xi\right)^{\frac{1}{q}}$$

$$=\left(\frac{(g(\varkappa)-g(\kappa_1))^{\varphi q+1}}{\varphi q+1}\right)^{\frac{1}{q}}\left(\frac{(w(\gamma)-w(\kappa_3))^{\rho q+1}}{\rho q+1}\right)^{\frac{1}{q}}\left\|\frac{\digamma_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\right\|_{[\kappa_1,\varkappa]\times[\kappa_3,\gamma],p}$$

$$=\frac{(g(\varkappa)-g(\kappa_1))^{\varphi+\frac{1}{q}}(w(\gamma)-w(\kappa_3))^{\rho+\frac{1}{q}}}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}}\left\|\frac{\digamma_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\right\|_{[\kappa_1,\varkappa]\times[\kappa_3,\gamma],p},$$

bulunur. Benzer şekilde aşağıdaki eşitsizlikler yazılabilir;

$$|I_2|\leq \frac{(g(\varkappa)-g(\kappa_1))^{\varphi+\frac{1}{q}}(w(\kappa_4)-w(\gamma))^{\rho+\frac{1}{q}}}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}}\left\|\frac{\digamma_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\right\|_{[\kappa_1,\varkappa]\times[\gamma,\kappa_4],p}, \tag{4.12}$$

$$|I_3|\leq \frac{(g(\kappa_2)-g(\varkappa))^{\varphi+\frac{1}{q}}(w(\gamma)-w(\kappa_3))^{\rho+\frac{1}{q}}}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}}\left\|\frac{\digamma_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\right\|_{[\varkappa,\kappa_2]\times[\kappa_3,\gamma],p}, \tag{4.13}$$

ve

$$|I_4|\leq \frac{(g(\kappa_2)-g(\varkappa))^{\varphi+\frac{1}{q}}(w(\kappa_4)-w(\gamma))^{\rho+\frac{1}{q}}}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}}\left\|\frac{\digamma_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\right\|_{[\varkappa,\kappa_2]\times[\gamma,\kappa_4],p}. \tag{4.14}$$

(4.11)–(4.14) eşitsizlikleri (4.5) de yazılırsa,

$$\Bigg|\digamma(\varkappa,\gamma)-\frac{1}{M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)}\left[\mathcal{J}^{\varphi}_{\varkappa^-;g}\digamma(\kappa_1,\gamma)+\mathcal{J}^{\varphi}_{\varkappa^+;g}\digamma(\kappa_2,\gamma)\right]$$

$$-\frac{1}{N_w^{\rho}(\kappa_3,\kappa_4;\gamma)}\left[\mathcal{J}^{\rho}_{\gamma^-;w}\digamma(\varkappa,\kappa_3)+\mathcal{J}^{\rho}_{\gamma^+;w}\digamma(\kappa_2,\gamma)\right]+\frac{1}{M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)N_w^{\rho}(\kappa_3,\kappa_4;\gamma)}$$

$$\times\left[\mathcal{J}^{\varphi,\rho}_{\varkappa^-\gamma^-;g,w}F(\kappa_1,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\varkappa^-\gamma^+;g,w}F(\kappa_1,\kappa_4)+\mathcal{J}^{\varphi,\rho}_{\varkappa^+\gamma^-;g,w}F(\kappa_2,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\varkappa^+\gamma^+;g,w}F(\kappa_2,\kappa_4)\right]\Bigg|$$

$$\leq\frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)M^{\varphi}_{g}(\kappa_1,\kappa_2;\varkappa)N^{\rho}_{w}(\kappa_3,\kappa_4;\gamma)}$$

$$\times\left[\frac{(g(\varkappa)-g(\kappa_1))^{\varphi+\frac{1}{q}}(w(\gamma)-w(\kappa_3))^{\rho+\frac{1}{q}}}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}}\left\|\frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\right\|_{[\kappa_1,\varkappa]\times[\kappa_3,\gamma],p}\right.$$

$$+\frac{(g(\varkappa)-g(\kappa_1))^{\varphi+\frac{1}{q}}(w(\kappa_4)-w(\gamma))^{\rho+\frac{1}{q}}}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}}\left\|\frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\right\|_{[\kappa_1,\varkappa]\times[\gamma,\kappa_4],p}$$

$$+\frac{(g(\kappa_2)-g(\varkappa))^{\varphi+\frac{1}{q}}(w(\gamma)-w(\kappa_3))^{\rho+\frac{1}{q}}}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}}\left\|\frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\right\|_{[\varkappa,\kappa_2]\times[\kappa_3,\gamma],p}$$

$$\left.+\frac{(g(\kappa_2)-g(\varkappa))^{\varphi+\frac{1}{q}}(w(\kappa_4)-w(\gamma))^{\rho+\frac{1}{q}}}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}}\left\|\frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\right\|_{[\varkappa,\kappa_2]\times[\gamma,\kappa_4],p}\right]$$

$$\leq\frac{1}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}\Gamma(\varphi+1)\Gamma(\rho+1)M^{\varphi}_{g}(\kappa_1,\kappa_2;\varkappa)N^{\rho}_{w}(\kappa_3,\kappa_4;\gamma)}\left\|\frac{F_{\eta\xi}}{(g'w')^{\frac{1}{q}}}\right\|_{[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4],p}$$

$$\times\left[(g(\varkappa)-g(\kappa_1))^{\varphi+\frac{1}{q}}+(g(\kappa_2)-g(\varkappa))^{\varphi+\frac{1}{q}}\right]\left[(w(\gamma)-w(\kappa_3))^{\rho+\frac{1}{q}}+(w(\kappa_4)-w(\gamma))^{\rho+\frac{1}{q}}\right],$$

elde edilir. Böylece ispat tamamlanır. □

Özellik 4.7. Theorem 4.6 de $g(\eta)=\eta$, $\eta\in[\kappa_1,\kappa_2]$ ve $w(\xi)=\xi$, $\xi\in[\kappa_3,\kappa_4]$ şeklinde seçilirse Riemann-Liouville kesirli integralleri için aşağıdaki Ostrowski tipli eşitsizlik elde edilir;

$$\Bigg|F(\varkappa,\gamma)-\frac{\Gamma(\varphi+1)}{(\varkappa-\kappa_1)^{\varphi}+(\kappa_2-\varkappa)^{\varphi}}\left[J^{\varphi}_{\varkappa^-}F(\kappa_1,\gamma)+J^{\varphi}_{\varkappa^+}F(\kappa_2,\gamma)\right] \tag{4.15}$$

$$-\frac{\Gamma(\rho+1)}{(\gamma-\kappa_3)^{\rho}+(\kappa_4-\gamma)^{\rho}}\left[J^{\rho}_{\gamma^-}F(\varkappa,\kappa_3)+J^{\rho}_{\gamma^+}F(\varkappa,\kappa_4)\right]$$

$$+\frac{\Gamma(\varphi+1)\Gamma(\rho+1)}{\left[(\varkappa-\kappa_1)^{\varphi}+(\kappa_2-\varkappa)^{\varphi}\right]\left[(\gamma-\kappa_3)^{\rho}+(\kappa_4-\gamma)^{\rho}\right]}$$

$$\times\left[J^{\varphi,\rho}_{\varkappa^-\gamma^-}F(\kappa_1,\kappa_3)+J^{\varphi,\rho}_{\varkappa^-\gamma^+}F(\kappa_1,\kappa_4)+J^{\varphi,\rho}_{\varkappa^+\gamma^-}F(\kappa_2,\kappa_3)+J^{\varphi,\rho}_{\varkappa^+\gamma^+}F(\kappa_2,\kappa_4)\right]\Bigg|$$

$$\leq\frac{\left[(\varkappa-\kappa_1)^{\varphi+\frac{1}{q}}+(\kappa_2-\varkappa)^{\varphi+\frac{1}{q}}\right]\left[(\gamma-\kappa_3)^{\rho+\frac{1}{q}}+(\kappa_4-\gamma)^{\rho+\frac{1}{q}}\right]}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}\left[(\varkappa-\kappa_1)^{\varphi}+(\kappa_2-\varkappa)^{\varphi}\right]\left[(\gamma-\kappa_3)^{\rho}+(\kappa_4-\gamma)^{\rho}\right]}\left\|F_{\eta\xi}\right\|_{\Delta,p}.$$

Not 4.8. Sonuç 4.7 de $\varphi = \rho = 1$ şeklinde alınırsa,

$$\left| \digamma(\varkappa,\gamma) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \digamma(\eta,\gamma)d\eta - \frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4} \digamma(\varkappa,\xi)d\xi \right.$$

$$\left. + \frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} \digamma(\eta,\xi)d\xi d\eta \right|$$

$$\leq \frac{\left[(\varkappa-\kappa_1)^{1+\frac{1}{q}} + (\kappa_2-\varkappa)^{1+\frac{1}{q}}\right]\left[(\gamma-\kappa_3)^{1+\frac{1}{q}} + (\kappa_4-\gamma)^{1+\frac{1}{q}}\right]}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)(q+1)^{\frac{2}{q}}} \left\| \digamma_{\eta\xi} \right\|_{\Delta,p},$$

bulunur.

Özellik 4.9. Theorem 4.6 de $\kappa_1, \kappa_3 > 0$ olmak üzere $g(\eta) = \ln\eta$, $\eta \in [\kappa_1, \kappa_2]$ ve $w(\xi) = \ln\xi$, $\xi \in [\kappa_3, \kappa_4]$ şeklinde seçilirse,

$$h(\eta,\xi) = \frac{1}{\xi\eta},$$

olmak üzere,

$$\left| \digamma(\varkappa,\gamma) - \frac{1}{M_{\ln}^{\varphi}(\kappa_1,\kappa_2;\varkappa)}\left[\mathbf{J}_{\varkappa^-;g}^{\varphi}\digamma(\kappa_1,\gamma) + \mathbf{J}_{\varkappa^+;g}^{\varphi}\digamma(\kappa_2,\gamma)\right] \right.$$

$$-\frac{1}{N_{\ln}^{\rho}(\kappa_3,\kappa_4;\gamma)}\left[\mathbf{J}_{\gamma^-;w}^{\rho}\digamma(\varkappa,\kappa_3) + \mathbf{J}_{\gamma^+;w}^{\rho}\digamma(\varkappa,\kappa_4)\right] + \frac{1}{M_{\ln}^{\varphi}(\kappa_1,\kappa_2;\varkappa)N_{\ln}^{\rho}(\kappa_3,\kappa_4;\gamma)}$$

$$\left. \times\left[\mathbf{J}_{\varkappa^-\gamma^-;g,w}^{\varphi,\rho}\digamma(\kappa_1,\kappa_3) + \mathbf{J}_{\varkappa^-\gamma^+;g,w}^{\varphi,\rho}\digamma(\kappa_1,\kappa_4) + \mathbf{J}_{\varkappa^+\gamma^-;g,w}^{\varphi,\rho}\digamma(\kappa_2,\kappa_3) + \mathbf{J}_{\varkappa^+\gamma^+;g,w}^{\varphi,\rho}\digamma(\kappa_2,\kappa_4)\right] \right|$$

$$\leq \frac{\left[(\ln\frac{\varkappa}{\kappa_1})^{\varphi+\frac{1}{q}} + (\ln\frac{\kappa_2}{\varkappa})^{\varphi+\frac{1}{q}}\right]\left[(\ln\frac{\gamma}{\kappa_3})^{\rho+\frac{1}{q}} + (\ln\frac{\kappa_4}{\gamma})^{\rho+\frac{1}{q}}\right]}{(\varphi q+1)^{\frac{1}{q}}(\rho q+1)^{\frac{1}{q}}\Gamma(\varphi+1)\Gamma(\rho+1)M_{\ln}^{\varphi}(\kappa_1,\kappa_2;\varkappa)N_{\ln}^{\rho}(\kappa_3,\kappa_4;\gamma)} \left\| \frac{\digamma_{\eta\xi}}{(h)^{\frac{1}{q}}} \right\|_{[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4],p},$$

Hadamard kesirli integralleri için Ostrowski tipli eşitsizlik elde edilir.

Not 4.10. Theorem 4.2 ve Theorem 4.6 de $\varkappa = \frac{\kappa_1+\kappa_2}{2}$ ve $\gamma = \frac{\kappa_3+\kappa_4}{2}$ olarak seçilirse genelleştirilmiş kesirli integraller için bazı yeni Midpoint tipli eşitsizlikler elde edilebilir.

5. SINIRLI KISMİ TÜREVLİ FONKSİYONLAR İÇİN BAZI GENELLEŞTİRİLMİŞ KESİRLİ TRAPEZOID VE OSTROWSKI TİPLİ EŞİTSİZLİKLER

Bu bölüm üç başlık altında sunulacaktır. Birinci bölümde, elde edilen bir özdeşliği iki kez kısmi türevlenebilir fonksiyonlar için kullanarak, genelleştirilmiş kesirli integraller için bazı Trapezoid tipli eşitsizlikler oluşturulacaktır. İkinci bölümde, iki kez kısmi türevlenebilir fonksiyonlar için bir eşitlik ve sınırlı kısmi türevli fonksiyonlar için bazı Ostrowski tipli eşitsizlikler sunulacaktır. Ayrıca ana sonuçlarımızın özel durumlarını da verilecektir. Üçüncü bölümde ise önceki bölümdeki sonuçlar kullanılarak genelleştirilmiş kesirli integraller için bazı yeni trapezoid tipli eşitsizlikler oluşturulacaktır.

5.1. TRAPEZOID TİPLİ EŞİTSİZLİKLER

Teorem 5.1. $F \in L_1(\Delta)$ sürekli fonksiyon ve $m_1(\varkappa,\gamma)$, $M_1(\varkappa,\gamma)$, $m_2(\varkappa,\gamma)$, $M_2(\varkappa,\gamma)$, $m_3(\varkappa,\gamma)$, $M_3(\varkappa,\gamma)$, $m_4(\varkappa,\gamma)$, $M_4(\varkappa,\gamma)$ reel sayılar olmak üzere,

$$m_1(\varkappa,\gamma) \leq F_{\eta\xi}(\eta,\xi) \leq M_1(\varkappa,\gamma), \quad (\eta,\xi) \in (\kappa_1,\varkappa) \times (\kappa_3,\gamma), \tag{5.1}$$

$$m_2(\varkappa,\gamma) \leq F_{\eta\xi}(\eta,\xi) \leq M_2(\varkappa,\gamma), \quad (\eta,\xi) \in (\kappa_1,\varkappa) \times (\gamma,\kappa_4),$$

$$m_3(\varkappa,\gamma) \leq F_{\eta\xi}(\eta,\xi) \leq M_3(\varkappa,\gamma), \quad (\eta,\xi) \in (\varkappa,\kappa_2) \times (\kappa_3,\gamma),$$

$$m_4(\varkappa,\gamma) \leq F_{\eta\xi}(\eta,\xi) \leq M_4(\varkappa,\gamma), \quad (\eta,\xi) \in (\varkappa,\kappa_4) \times (\gamma,\kappa_4),$$

olsun. Bu durumda, Lemma 3.2 göz önüne alınarak,

$$\frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)}\left[m_1(\varkappa,\gamma)\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}[g(\varkappa)-g(\eta)]^{\varphi}[w(\gamma)-w(\xi)]^{\rho}\,d\xi d\eta\right.$$

$$-M_2(\varkappa,\gamma)\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}[g(\varkappa)-g(\eta)]^{\varphi}[w(\xi)-w(\gamma)]^{\rho}\,d\xi d\eta$$

$$-M_3(\varkappa,\gamma)\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}[g(\eta)-g(\varkappa)]^{\varphi}[w(\gamma)-w(\xi)]^{\rho}\,d\xi d\eta$$

$$\left.+m_4(\varkappa,\gamma)\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}[g(\eta)-g(\varkappa)]^{\varphi}[w(\xi)-w(\gamma)]^{\rho}\,d\xi d\eta\right]$$

$$\leq A(g,w)-N_w^{\rho}(\kappa_3,\gamma)\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_3)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_3)\right]$$

$$-N_w^{\rho}(\kappa_4,\gamma)\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_4)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_4)\right]$$

$$-M_g^{\varphi}(\kappa_1,\varkappa)\left[\mathcal{J}^{\rho}_{c+;w}F(\kappa_1,\gamma)+\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_1,\gamma)\right]$$

$$-M_g^{\varphi}(\kappa_2,\varkappa)\left[\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_2,\gamma)+\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_2,\gamma)\right]$$

$$+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+;g,w}F(\varkappa,\gamma)+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-;g,w}F(\varkappa,\gamma)$$

$$+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+;g,w}F(\varkappa,\gamma)+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-;g,w}F(\varkappa,\gamma)$$

$$\leq\frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)}\left[M_1(\varkappa,\gamma)\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}[g(\varkappa)-g(\eta)]^{\varphi}[w(\gamma)-w(s)]^{\rho}\,d\xi d\eta\right.$$

$$-m_2(\varkappa,\gamma)\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}[g(\varkappa)-g(\eta)]^{\varphi}[w(\xi)-w(\gamma)]^{\rho}\,d\xi d\eta$$

$$-m_3(\varkappa,\gamma)\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}[g(\eta)-g(\varkappa)]^{\varphi}[w(\gamma)-w(\xi)]^{\rho}\,d\xi d\eta$$

$$\left.+M_4(\varkappa,\gamma)\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}[g(\eta)-g(\varkappa)]^{\varphi}[w(\xi)-w(\gamma)]^{\rho}\,d\xi d\eta\right],$$

eşitsizliği gerçeklenir.

İspat. (5.1) şartları kullanılarak,

$$m_1(\varkappa,\gamma)\int\limits_{\kappa_1}^{\varkappa}\int\limits_{\kappa_3}^{\gamma}(g(\varkappa)-g(\eta))^{\varphi}(w(\gamma)-w(\xi))^{\rho}d\xi d\eta$$
$$\leq\int\limits_{\kappa_1}^{\varkappa}\int\limits_{\kappa_3}^{\gamma}(g(\varkappa)-g(\eta))^{\varphi}(w(\gamma)-w(\xi))^{\rho}F_{\eta\xi}(\eta,\xi)d\xi d\eta$$
$$\leq M_1(\varkappa,\gamma)\int\limits_{\kappa_1}^{\varkappa}\int\limits_{\kappa_3}^{\gamma}(g(\varkappa)-g(\eta))^{\varphi}(w(\gamma)-w(\xi))^{\rho}d\xi d\eta,$$

$$m_2(\varkappa,\gamma)\int\limits_{\kappa_1}^{\varkappa}\int\limits_{\gamma}^{\kappa_4}(g(\varkappa)-g(\eta))^{\varphi}(w(\xi)-w(\gamma))^{\rho}d\xi d\eta$$
$$\leq\int\limits_{\kappa_1}^{\varkappa}\int\limits_{\gamma}^{\kappa_4}(g(\varkappa)-g(\eta))^{\varphi}(w(\xi)-w(\gamma))^{\rho}F_{\eta\xi}(\eta,\xi)d\xi d\eta$$
$$\leq M_2(\varkappa,\gamma)\int\limits_{\kappa_1}^{\varkappa}\int\limits_{\gamma}^{\kappa_4}(g(\varkappa)-g(\eta))^{\varphi}(w(\xi)-w(\gamma))^{\rho}d\xi d\eta,$$

$$m_3(\varkappa,\gamma)\int\limits_{\varkappa}^{\kappa_2}\int\limits_{\kappa_3}^{\gamma}(g(\eta)-g(\varkappa))^{\varphi}(w(\gamma)-w(\xi))^{\rho}d\xi d\eta$$
$$\leq\int\limits_{\varkappa}^{\kappa_2}\int\limits_{\kappa_3}^{\gamma}(g(\eta)-g(\varkappa))^{\varphi}(w(\gamma)-w(\xi))^{\rho}F_{\eta\xi}(\eta,\xi)d\xi d\eta$$
$$\leq M_3(\varkappa,\gamma)\int\limits_{\varkappa}^{\kappa_2}\int\limits_{\kappa_3}^{\gamma}(g(\eta)-g(\varkappa))^{\varphi}(w(\gamma)-w(\xi))^{\rho}d\xi d\eta,$$

ve

$$m_4(\varkappa,\gamma)\int\limits_{\varkappa}^{\kappa_2}\int\limits_{\gamma}^{\kappa_4}(g(\eta)-g(\varkappa))^{\varphi}(w(\xi)-w(\gamma))^{\rho}d\xi d\eta$$
$$\leq\int\limits_{\varkappa}^{\kappa_2}\int\limits_{\gamma}^{\kappa_4}(g(\eta)-g(\varkappa))^{\varphi}(w(\xi)-w(\gamma))^{\rho}F_{\eta\xi}(\eta,\xi)d\xi d\eta$$
$$\leq M_4(\varkappa,\gamma)\int\limits_{\varkappa}^{\kappa_2}\int\limits_{\gamma}^{\kappa_4}(g(\eta)-g(\varkappa))^{\varphi}(w(\xi)-w(\gamma))^{\rho}d\xi d\eta,$$

eşitsizlikleri yazılabilir. Bu eşitsizlikler sonucunda,

$$
\begin{aligned}
& m_1(\varkappa,\gamma)\int\limits_{\kappa_1}^{\varkappa}\int\limits_{\kappa_3}^{\gamma}[g(\varkappa)-g(\eta)]^{\varphi}[w(\gamma)-w(\xi)]^{\rho}\,d\xi d\eta \\
& -M_2(\varkappa,\gamma)\int\limits_{\kappa_1}^{\varkappa}\int\limits_{\gamma}^{\kappa_4}[g(\varkappa)-g(\eta)]^{\varphi}[w(\xi)-w(\gamma)]^{\rho}\,d\xi d\eta \\
& -M_3(\varkappa,\gamma)\int\limits_{\varkappa}^{\kappa_2}\int\limits_{\kappa_3}^{\gamma}[g(\eta)-g(\varkappa)]^{\varphi}[w(\gamma)-w(\xi)]^{\rho}\,d\xi d\eta \\
& +m_4(\varkappa,\gamma)\int\limits_{\varkappa}^{\kappa_2}\int\limits_{\gamma}^{\kappa_4}[g(\eta)-g(\varkappa)]^{\varphi}[w(\xi)-w(\gamma)]^{\rho}\,d\xi d\eta \\
& \leq I_1-I_2-I_3+I_4 \\
& \leq M_1(\varkappa,\gamma)\int\limits_{\kappa_1}^{\varkappa}\int\limits_{\kappa_3}^{\gamma}[g(\varkappa)-g(\eta)]^{\varphi}[w(\gamma)-w(\xi)]^{\rho}\,d\xi d\eta \\
& -m_2(\varkappa,\gamma)\int\limits_{\kappa_1}^{\varkappa}\int\limits_{\gamma}^{\kappa_4}[g(\varkappa)-g(\eta)]^{\varphi}[w(\xi)-w(\gamma)]^{\rho}\,d\xi d\eta \\
& -m_3(\varkappa,\gamma)\int\limits_{\varkappa}^{\kappa_2}\int\limits_{\kappa_3}^{\gamma}[g(\eta)-g(\varkappa)]^{\varphi}[w(\gamma)-w(\xi)]^{\rho}\,d\xi d\eta \\
& +M_4(\varkappa,\gamma)\int\limits_{\varkappa}^{\kappa_2}\int\limits_{\gamma}^{\kappa_4}[g(\eta)-g(\varkappa)]^{\varphi}[w(\xi)-w(\gamma)]^{\rho}\,d\xi d\eta,
\end{aligned}
$$

olup, istenilen sonuç elde edilir. □

Özellik 5.2. Theorem 5.1 ün şartlarını sağlasın. m,M birer sabit olmak üzere $m \leq M$ için,

$$m \leq \digamma_{\eta\xi} \leq M, \quad \forall(\eta,\xi)\in\Delta,$$

olsun. Bu takdirde,

$$
\begin{aligned}
& \frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)}\Bigg[m\int\limits_{\kappa_1}^{\varkappa}\int\limits_{\kappa_3}^{\gamma}[g(\varkappa)-g(\eta)]^{\varphi}[w(\gamma)-w(\xi)]^{\rho}\,d\xi d\eta \\
& -M\int\limits_{\kappa_1}^{\varkappa}\int\limits_{\gamma}^{\kappa_4}[g(\varkappa)-g(\eta)]^{\varphi}[w(\xi)-w(\gamma)]^{\rho}\,d\xi d\eta
\end{aligned}
$$

$$-M\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}[g(\eta)-g(\varkappa)]^{\varphi}[w(\gamma)-w(\xi)]^{\rho}\,d\xi d\eta$$

$$+m\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}[g(\eta)-g(\varkappa)]^{\varphi}[w(\xi)-w(\gamma)]^{\rho}\,d\xi d\eta\Bigg]$$

$$\leq A(g,w)-N_w^{\rho}(\kappa_3,\gamma)\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_3)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_3)\right]$$

$$-N_w^{\rho}(\kappa_4,\gamma)\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_4)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_4)\right]$$

$$-M_g^{\varphi}(\kappa_1,\varkappa)\left[\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_1,\gamma)+\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_1,\gamma)\right]$$

$$-M_g^{\varphi}(\kappa_2,\varkappa)\left[\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_2,\gamma)+\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_2,\gamma)\right]$$

$$+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+;g,w}F(\varkappa,\gamma)+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-;g,w}F(\varkappa,\gamma)$$

$$+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+;g,w}F(\varkappa,\gamma)+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-;g,w}F(\varkappa,\gamma)$$

$$\leq\frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)}\Bigg[M\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}[g(\varkappa)-g(\eta)]^{\varphi}[w(\gamma)-w(\xi)]^{\rho}\,d\xi d\eta$$

$$-m\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}[g(\varkappa)-g(\eta)]^{\varphi}[w(\xi)-w(\gamma)]^{\rho}\,d\xi d\eta$$

$$-m\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}[g(\eta)-g(\varkappa)]^{\varphi}[w(\gamma)-w(\xi)]^{\rho}\,d\xi d\eta$$

$$+M\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}[g(\eta)-g(\varkappa)]^{\varphi}[w(\xi)-w(\gamma)]^{\rho}\,d\xi d\eta\Bigg],$$

eşitsizliği gerçeklenir.

Özellik 5.3. Theorem 5.1 ün şartlarını sağlasın. m,M birer sabit olmak üzere $m\leq M$ için,

$$m\leq F_{\eta\xi}\leq M,\quad\forall(\eta,\xi)\in\Delta,$$

olsun. Bu durumda,

$$\Psi_g(\kappa_1,\kappa_2;\varkappa)=\begin{cases}\left[\frac{\kappa_2-\kappa_1}{2}+\left|\varkappa-\frac{\kappa_1+\kappa_2}{2}\right|\right]\left[(g(\varkappa)-g(\kappa_1))^{\varphi}+(g(\kappa_2)-g(\varkappa))^{\varphi}\right]\\ \\ \left[(\varkappa-\kappa_1)^p+(\kappa_2-\varkappa)^p\right]^{\frac{1}{p}}\left[(g(\varkappa)-g(\kappa_1))^{\varphi q}+(g(\kappa_2)-g(\varkappa))^{\varphi q}\right]^{\frac{1}{q}},\frac{1}{p}+\frac{1}{q}=1\\ \\ (\kappa_2-\kappa_1)\left[\frac{g(\kappa_2)-g(\kappa_1)}{2}+\left|g(\varkappa)-\frac{g(\kappa_2)+g(\kappa_1)}{2}\right|\right]^{\varphi}\end{cases}$$

ve

$$\Upsilon_w(\kappa_3,\kappa_4;\gamma)=\begin{cases}\left[\frac{\kappa_4-\kappa_3}{2}+\left|\gamma-\frac{\kappa_3+\kappa_4}{2}\right|\right]\left[(w(\gamma)-w(\kappa_3))^{\rho}+(w(\kappa_4)-w(\gamma))^{\rho}\right]\\ \\ \left[(\gamma-\kappa_3)^p+(\kappa_4-\gamma)^p\right]^{\frac{1}{p}}\ \left[(w(\gamma)-w(\kappa_3))^{\rho q}+(w(\kappa_4)-w(\gamma))^{\rho q}\right]^{\frac{1}{q}},\frac{1}{p}+\frac{1}{q}=1\\ \\ (\kappa_4-\kappa_3)\left[\frac{w(\kappa_4)-w(\kappa_3}{2}+\left|w(\gamma)-\frac{w(\kappa_4)+w(\kappa_3)}{2}\right|\right]^{\rho}\end{cases}$$

olmak üzere,

$$\left|A(g,w)-N_w^{\rho}(\kappa_3,\gamma)\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_3)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_3)\right]\right.$$

$$-N_w^{\rho}(\kappa_4,\gamma)\left[\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\varkappa,\kappa_4)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\varkappa,\kappa_4)\right]$$

$$-M_g^{\varphi}(\kappa_1,\varkappa)\left[\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_1,\gamma)+\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_1,\gamma)\right]$$

$$-M_g^{\varphi}(\kappa_2,\varkappa)\left[\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_2,\gamma)+\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_2,\gamma)\right]$$

$$+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+;g,w}F(\varkappa,\gamma)+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-;g,w}F(\varkappa,\gamma)$$

$$+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+;g,w}F(\varkappa,\gamma)+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-;g,w}F(\varkappa,\gamma)$$

$$-\frac{M+m}{2\Gamma(\varphi+1)\Gamma(\rho+1)}$$

$$\times \left[\int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} [g(\varkappa) - g(\eta)]^{\varphi} [w(\gamma) - w(\xi)]^{\rho} \, d\xi d\eta \right.$$

$$- \int_{\kappa_1}^{\varkappa} \int_{\gamma}^{\kappa_4} [g(\varkappa) - g(\eta)]^{\varphi} [w(\xi) - w(\gamma)]^{\rho} \, d\xi d\eta$$

$$- \int_{\varkappa}^{\kappa_2} \int_{\kappa_3}^{\gamma} [g(\eta) - g(\varkappa)]^{\varphi} [w(\gamma) - w(\xi)]^{\rho} \, d\xi d\eta$$

$$\left. + \int_{\varkappa}^{\kappa_2} \int_{\gamma}^{\kappa_4} [g(\eta) - g(\varkappa)]^{\varphi} [w(\xi) - w(\gamma)]^{\rho} \, d\xi d\eta \right] \Bigg|$$

$$\leq \frac{M - m}{2\Gamma(\varphi + 1)\Gamma(\rho + 1)} \Psi_g(\kappa_1, \kappa_2; \varkappa) \Upsilon_w(\kappa_3, \kappa_4; \gamma),$$

eşitsizliği sağlanır.

İspat. Sonuç 5.2 den

$$\Big| A(g,w) - N_w^{\rho}(\kappa_3, \gamma) \left[\mathcal{J}_{\kappa_1+;g}^{\varphi} F(\varkappa, \kappa_3) + \mathcal{J}_{\kappa_2-;g}^{\varphi} F(\varkappa, \kappa_3) \right]$$

$$- N_w^{\rho}(\kappa_4, \gamma) \left[\mathcal{J}_{\kappa_1+;g}^{\varphi} F(\varkappa, \kappa_4) + \mathcal{J}_{\kappa_2-;g}^{\varphi} F(\varkappa, \kappa_4) \right]$$

$$- M_g^{\varphi}(\kappa_1, \varkappa) \left[\mathcal{J}_{\kappa_3+;w}^{\rho} F(\kappa_1, \gamma) + \mathcal{J}_{\kappa_4-;w}^{\rho} F(\kappa_1, \gamma) \right]$$

$$- M_g^{\varphi}(\kappa_2, \varkappa) \left[\mathcal{J}_{\kappa_3+;w}^{\rho} F(\kappa_2, \gamma) + \mathcal{J}_{\kappa_4-;w}^{\rho} F(\kappa_2, \gamma) \right]$$

$$+ \mathcal{J}_{\kappa_1+,\kappa_3+;g,w}^{\varphi,\rho} F(\varkappa, \gamma) + \mathcal{J}_{\kappa_1+,\kappa_4-;g,w}^{\varphi,\rho} F(\varkappa, \gamma)$$

$$+ \mathcal{J}_{\kappa_2-,\kappa_3+;g,w}^{\varphi,\rho} F(\varkappa, \gamma) + \mathcal{J}_{\kappa_2-,\kappa_4-;g,w}^{\varphi,\rho} F(\varkappa, \gamma)$$

$$- \frac{M + m}{2\Gamma(\varphi + 1)\Gamma(\rho + 1)} \left[\int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} [g(\varkappa) - g(\eta)]^{\varphi} [w(\gamma) - w(\xi)]^{\rho} \, d\xi d\eta \right.$$

$$- \int_{\kappa_1}^{\varkappa} \int_{\gamma}^{\kappa_4} [g(\varkappa) - g(\eta)]^{\varphi} [w(\xi) - w(\gamma)]^{\rho} \, d\xi d\eta$$

$$-\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}[g(\eta)-g(\varkappa)]^{\varphi}[w(\gamma)-w(\xi)]^{\rho}\,d\xi d\eta$$

$$\left.\left.+\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}[g(\eta)-g(\varkappa)]^{\varphi}[w(\xi)-w(\gamma)]^{\rho}\,d\xi d\eta\right]\right|$$

$$\leq \frac{M-m}{2\Gamma(\varphi+1)\Gamma(\rho+1)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}|g(\varkappa)-g(\eta)|^{\varphi}|w(\gamma)-w(\xi)|^{\rho}\,d\xi d\eta$$

$$=\frac{M-m}{2\Gamma(\varphi+1)\Gamma(\rho+1)}\left(\int_{\kappa_1}^{\kappa_2}|g(\varkappa)-g(\eta)|^{\varphi}\,d\eta\right)\left(\int_{\kappa_3}^{\kappa_4}|w(\gamma)-w(\xi)|^{\rho}\,d\xi\right),$$

elde edilir. g fonksiyonu (κ_1,κ_2) üzerinde artan bir fonksiyon olduğundan ayrık Hölder eşitsizliği yardımıyla,

$$\int_{\kappa_1}^{\kappa_2}|g(\varkappa)-g(\eta)|^{\varphi}\,d\eta$$

$$=\int_{\kappa_1}^{\varkappa}[g(\varkappa)-g(\eta)]^{\varphi}\,d\eta+\int_{\varkappa}^{\kappa_2}[g(\eta)-g(\varkappa)]^{\varphi}\,d\eta$$

$$\leq(\varkappa-\kappa_1)(g(\varkappa)-g(\kappa_1))^{\varphi}+(\kappa_2-\varkappa)(g(\kappa_2)-g(\varkappa))^{\varphi}$$

$$\leq\begin{cases}\left[\frac{\kappa_2-\kappa_1}{2}+\left|\varkappa-\frac{\kappa_1+\kappa_2}{2}\right|\right]\left[(g(\varkappa)-g(\kappa_1))^{\varphi}+(g(\kappa_2)-g(\varkappa))^{\varphi}\right]\\[2ex] \left[(\varkappa-\kappa_1)^{p}+(\kappa_2-\varkappa)^{p}\right]^{\frac{1}{p}}\left[(g(\varkappa)-g(\kappa_1))^{\varphi q}+(g(\kappa_2)-g(\varkappa))^{\varphi q}\right]^{\frac{1}{q}}\ ,\ \frac{1}{p}+\frac{1}{q}=1\\[2ex] (\kappa_2-\kappa_1)\left[\frac{g(\kappa_2)-g(\kappa_1)}{2}+\left|g(\varkappa)-\frac{g(\kappa_2)+g(\kappa_1)}{2}\right|\right]^{\alpha}\end{cases}$$

$$:\ =\Psi_g(\kappa_1,\kappa_2;\varkappa),$$

olduğu kolayca görülebilir. Benzer şekilde w fonksiyonu (κ_3,κ_4) üzerinde artan bir fonksiyon olmak üzere,

$$\int_{\kappa_3}^{\kappa_4}|w(\gamma)-w(\xi)|^{\rho}\,d\xi\leq\Upsilon_w(\kappa_3,\kappa_4;\gamma),$$

elde edilir ve ispat tamamlanır. $\square$

Not 5.4. Sonuç 5.3 te $g(\eta)=\eta,\ \eta\in[\kappa_1,\kappa_2],\ \ w(\xi)=\xi,\ \xi\in[\kappa_3,\kappa_4]$, $\varkappa=\frac{\kappa_1+\kappa_2}{2}$ ve $\gamma=\frac{\kappa_3+\kappa_4}{2}$ alınırsa,

$$\begin{aligned}
&\Bigg|\frac{F(\kappa_1,\kappa_3)+F(\kappa_1,\kappa_4)+F(\kappa_2,\kappa_3)+F(\kappa_2,\kappa_4)}{4}\\
&-\frac{2^{\varphi-2}\Gamma(\varphi+1)}{(\kappa_2-\kappa_1)^{\varphi}}\left[J^{\varphi}_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+J^{\varphi}_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)\right.\\
&\left.+J^{\varphi}_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)+J^{\varphi}_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\right]\\
&-\frac{2^{\rho-2}\Gamma(\rho+1)}{(\kappa_4-\kappa_3)^{\rho}}\left[J^{\rho}_{\kappa_3+}F\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+J^{\rho}_{\kappa_4-}F\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)\right.\\
&\left.+J^{\rho}_{\kappa_3+}F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)+J^{\rho}_{\kappa_4-}F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\right]\\
&+\frac{2^{\varphi+\rho-2}\Gamma(\varphi+1)\Gamma(\rho+1)}{(\kappa_2-\kappa_1)^{\varphi}(\kappa_4-\kappa_3)^{\rho}}\\
&\times\left[J^{\varphi,\rho}_{\kappa_1+,\kappa_3+}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+J^{\varphi,\rho}_{\kappa_1+,\kappa_4-}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\right.\\
&\left.+J^{\varphi,\rho}_{\kappa_2-,\kappa_3+}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+J^{\varphi,\rho}_{\kappa_2-,\kappa_4-}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\right]\Bigg|\\
&\le\frac{M-m}{8}(\kappa_2-\kappa_1)(\kappa_4-\kappa_3),
\end{aligned}$$

elde edilir.

5.2. OSTROWSKİ TİPLİ EŞİTSİZLİKLER

Yardımcı Teorem 5.5. $F:\Delta\to\mathbb{R}$ fonksiyonu Δ° üzerinde iki kez kısmi türevlenebilir olsun. $\varphi,\rho>0$, $F_{\eta\xi}\in L(\Delta)$ için,

$$I_5=\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\eta)-g(\kappa_1))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}F_{\eta\xi}(\eta,\xi)d\xi d\eta,$$

$$I_6=\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}(g(\eta)-g(\kappa_1))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}\digamma_{\eta\xi}(\eta,\xi)d\xi d\eta,$$

$$I_7=\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}(g(\kappa_2)-g(\eta))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}\digamma_{\eta\xi}(\eta,\xi)d\xi d\eta$$

$$I_8=\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}(g(\kappa_2)-g(\eta))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}\digamma_{\eta\xi}(\eta,\xi)d\xi d\eta.$$

olmak üzere,

$$M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)N_w^{\rho}(\kappa_3,\kappa_4;\gamma)\digamma(\varkappa,\gamma)$$

$$-N_w^{\rho}(\kappa_3,\kappa_4;\gamma)\left[\mathcal{J}^{\varphi}_{\varkappa-;g}\digamma(\kappa_1,\gamma)+\mathcal{J}^{\varphi}_{\varkappa+;g}\digamma(\kappa_2,\gamma)\right]$$

$$-M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)\left[\mathcal{J}^{\rho}_{\gamma-;w}\digamma(\varkappa,\kappa_3)+\mathcal{J}^{\rho}_{\gamma+;w}\digamma(\varkappa,\kappa_4)\right]$$

$$+\left[\mathcal{J}^{\varphi,\rho}_{\varkappa-,\gamma-;g,w}\digamma(\kappa_1,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\varkappa-,\gamma+;g,w}\digamma(\kappa_1,\kappa_4)\right.$$

$$\left.+\mathcal{J}^{\varphi,\rho}_{\varkappa+,\gamma-;g,w}\digamma(\kappa_2,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\varkappa+,\gamma+;g,w}\digamma(\kappa_2,\kappa_4)\right]$$

$$=\frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)}\left[I_5-I_6-I_7+I_8\right]$$

eşitliği elde edilir.

İspat. Kısmi integral yardımıyla

$$I_5=\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\eta)-g(\kappa_1))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}\digamma_{\eta,\xi}(\eta,\xi)d\xi d\eta \tag{5.2}$$

$$=\int_{\kappa_1}^{\varkappa}(g(\eta)-g(\kappa_1))^{\varphi}$$

$$\times\left[(w(\xi)-w(\kappa_3))^{\rho}F_{\eta}(\eta,\xi)\Big|_{\kappa_3}^{\gamma}-\rho\int_{\kappa_3}^{\gamma}(w(\xi)-w(\kappa_3))^{\rho-1}w'(\xi)F_{\eta}(\eta,\xi)d\xi\right]d\eta$$

$$=\int_{\kappa_1}^{\varkappa}(g(\eta)-g(\kappa_1))^{\varphi}(w(\gamma)-w(\kappa_3))^{\rho}F_{\eta}(\eta,\gamma)d\eta$$

$$-\rho\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\eta)-g(\kappa_1))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho-1}w'(\xi)F_{\eta}(\eta,\xi)d\xi d\eta$$

$$=(w(\gamma)-w(\kappa_3))^{\rho}\left[(g(\eta)-g(\kappa_1))^{\varphi}F(\eta,\gamma)-\varphi\int_{\kappa_1}^{\varkappa}(g(\eta)-g(\kappa_1))^{\varphi-1}g'(\eta)F(\eta,\gamma)d\eta\right]$$

$$-\rho\int_{\kappa_3}^{\gamma}(w(\xi)-w(\kappa_3))^{\rho-1}w'(\xi)$$

$$\times\left[(g(\eta)-g(\kappa_1))^{\varphi}F(\eta,\xi)\Big|_{\kappa_1}^{\varkappa}-\varphi\int_{\kappa_1}^{\varkappa}(g(\eta)-g(\kappa_1))^{\varphi-1}g'(\eta)F(\eta,\xi)d\eta\right]d\xi$$

$$=(w(\gamma)-w(\kappa_3))^{\rho}(g(\varkappa)-g(\kappa_1))^{\varphi}F(\varkappa,\gamma)-\Gamma(\varphi+1)(w(\gamma)-w(\kappa_3))^{\rho}\mathcal{J}^{\varphi}_{\varkappa-;g}F(\kappa_1,\gamma)$$

$$-\Gamma(\rho+1)(g(\varkappa)-g(\kappa_1))^{\varphi}\mathcal{J}^{\rho}_{\gamma-;w}F(\varkappa,\kappa_3)+\Gamma(\varphi+1)\Gamma(\rho+1)\mathcal{J}^{\varphi,\rho}_{\varkappa-,\gamma-;g,w}F(\kappa_1,\kappa_3),$$

bulunur. Benzer şekilde,

$$I_6=-(g(\varkappa)-g(\kappa_1))^{\varphi}(w(\kappa_4)-w(\gamma))^{\rho}F(\varkappa,\gamma) \quad (5.3)$$

$$+\Gamma(\varphi+1)(w(\kappa_4)-w(\gamma))^{\rho}\mathcal{J}^{\varphi}_{\varkappa-;g}F(\kappa_1,\gamma)$$

$$+\Gamma(\rho+1)(g(\varkappa)-g(\kappa_1))^{\varphi}\mathcal{J}^{\rho}_{\gamma+;w}F(\varkappa,\kappa_4)$$

$$-\Gamma(\varphi+1)\Gamma(\rho+1)\mathcal{J}^{\varphi,\rho}_{\varkappa-,\gamma+;g,w}F(\kappa_1,\kappa_4),$$

$$I_7 = -(g(\kappa_2)-g(\varkappa))^{\varphi}(w(\gamma)-w(\kappa_3))^{\rho}F(\varkappa,\gamma) \tag{5.4}$$

$$+\Gamma(\varphi+1)(w(\gamma)-w(\kappa_3))^{\rho}\mathcal{J}^{\varphi}_{\varkappa+;g}F(\kappa_2,\gamma)$$

$$+\Gamma(\rho+1)(g(\kappa_2)-g(\varkappa))^{\varphi}\mathcal{J}^{\rho}_{\gamma-;w}F(\varkappa,\kappa_3)$$

$$-\Gamma(\varphi+1)\Gamma(\rho+1)\mathcal{J}^{\varphi,\rho}_{\varkappa+,\gamma-;g,w}F(\kappa_2,\kappa_3),$$

ve

$$I_8 = (g(\kappa_2)-g(\varkappa))^{\varphi}(w(\kappa_4)-w(\gamma))^{\rho}F(\varkappa,\gamma) \tag{5.5}$$

$$-\Gamma(\varphi+1)(w(\kappa_4)-w(\gamma))^{\rho}\mathcal{J}^{\varphi}_{\varkappa+;g}F(\kappa_2,\gamma)$$

$$-\Gamma(\rho+1)(g(\kappa_2)-g(\varkappa))^{\varphi}\mathcal{J}^{\rho}_{\gamma+;w}F(\varkappa,\kappa_4)$$

$$+\Gamma(\varphi+1)\Gamma(\rho+1)\mathcal{J}^{\varphi,\rho}_{\varkappa+,\gamma+;g,w}F(\kappa_2,\kappa_4),$$

olduğu açıktır. (5.2)-(5.5) eşitlikleriyle istenen sonuç elde edilir. □

Özellik 5.6. $\varkappa = \frac{\kappa_1+\kappa_2}{2}$ ve $\gamma = \frac{\kappa_3+\kappa_4}{2}$ alınırsa Lemma 5.5,

$$M^{\varphi}_{g}\left(\kappa_1,\kappa_2;\frac{a+b}{2}\right)N^{\rho}_{w}\left(\kappa_3,\kappa_4;\frac{\kappa_3+\kappa_4}{2}\right)F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)$$

$$-N^{\rho}_{w}(\kappa_3,\kappa_4;\frac{\kappa_3+\kappa_4}{2})\left[\mathcal{J}^{\varphi}_{\frac{\kappa_1+\kappa_2}{2}-;g}F\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+\mathcal{J}^{\varphi}_{\frac{\kappa_1+\kappa_2}{2}+;g}F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\right]$$

$$-M^{\varphi}_{g}\left(\kappa_1,\kappa_2;\frac{\kappa_1+\kappa_2}{2}\right)\left[\mathcal{J}^{\rho}_{\frac{\kappa_3+\kappa_4}{2}-;w}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+\mathcal{J}^{\rho}_{\frac{\kappa_3+\kappa_4}{2}+;w}F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\right]$$

$$+\mathcal{J}^{\varphi,\rho}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}-g,w}F(\kappa_1,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}+;g,w}F(\kappa_1,\kappa_4)$$

$$+\mathcal{J}^{\varphi,\rho}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}-;g,w}F(\kappa_2,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}+;g,w}F(\kappa_2,\kappa_4)$$

$$=\frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)}\left[\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}(g(\eta)-g(\kappa_1))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}F_{\eta\xi}(\eta,\xi)d\xi d\eta\right.$$

$$-\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\int_{\frac{\kappa_3+\kappa_4}{2}}^{\kappa_4}(g(\eta)-g(\kappa_1))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}F_{\eta\xi}(\eta,\xi)d\xi d\eta$$

$$-\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}(g(\kappa_2)-g(\eta))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}F_{\eta\xi}(\eta,\xi)d\xi d\eta$$

$$\left.+\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\int_{\frac{\kappa_3+\kappa_4}{2}}^{\kappa_4}(g(\kappa_2)-g(\eta))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}F_{\eta\xi}(\eta,\xi)d\xi d\eta\right],$$

eşitliğine dönüşür.

Teorem 5.7. Theorem 5.1 şartları sağlandığını varsayalım. Bu durumda,

$$\frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)}\left[m_1(\varkappa,\gamma)\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\eta)-g(\kappa_1))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}d\xi d\eta\right.$$

$$-M_2(\varkappa,\gamma)\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}(g(\eta)-g(\kappa_1))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}d\xi d\eta$$

$$-M_3(\varkappa,\gamma)\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}(g(\kappa_2)-g(\eta))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}d\xi d\eta$$

$$\left.+m_4(\varkappa,\gamma)\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}(g(\kappa_2)-g(\eta))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}d\xi d\eta\right]$$

$$\leq M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)N_w^{\rho}(\kappa_3,\kappa_4;\gamma)F(\varkappa,\gamma)$$

$$-N_w^{\rho}(\kappa_3,\kappa_4;\gamma)\left[\mathcal{J}_{\varkappa-;g}^{\varphi}\digamma(\kappa_1,\gamma)+\mathcal{J}_{\varkappa+;g}^{\varphi}\digamma(\kappa_2,\gamma)\right]$$

$$-M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)\left[\mathcal{J}_{\gamma-;w}^{\rho}\digamma(\varkappa,\kappa_3)+\mathcal{J}_{\gamma+;w}^{\rho}\digamma(\kappa_2,\gamma)\right]$$

$$+\mathcal{J}_{\varkappa-,\gamma-;g,w}^{\varphi,\rho}\digamma(\kappa_1,\kappa_3)+\mathcal{J}_{\varkappa-,\gamma+;g,w}^{\varphi,\rho}\digamma(\kappa_1,\kappa_4)$$

$$+\mathcal{J}_{\varkappa+,\gamma-;g,w}^{\varphi,\rho}\digamma(\kappa_2,\kappa_3)+\mathcal{J}_{\varkappa+,\gamma+;g,w}^{\varphi,\rho}\digamma(\kappa_2,\kappa_4)$$

$$\leq\frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)}\left[M_1(\varkappa,\gamma)\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\eta)-g(\kappa_1))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}d\xi d\eta\right.$$

$$-m_2(\varkappa,\gamma)\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}(g(\eta)-g(\kappa_1))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}d\xi d\eta$$

$$-m_3(\varkappa,\gamma)\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}(g(\kappa_2)-g(\eta))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}d\xi d\eta$$

$$\left.+M_4(\varkappa,\gamma)\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}(g(\kappa_2)-g(\eta))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}d\xi d\eta\right],$$

eşitsizliği sağlanır.

İspat. (5.1) şartları yardımıyla,

$$m_1(\varkappa,\gamma)\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\eta)-g(\kappa_1))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}d\xi d\eta \tag{5.6}$$

$$\leq\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\eta)-g(\kappa_1))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}\digamma_{\eta\xi}(\eta,\xi)d\xi d\eta$$

$$\leq M_1(\varkappa,\gamma)\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}(g(\eta)-g(\kappa_1))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}d\xi d\eta,$$

$$m_2(\varkappa,\gamma)\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}(g(\eta)-g(\kappa_1))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}d\xi d\eta \tag{5.7}$$

$$\leq \int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}(g(\eta)-g(\kappa_1))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}\digamma_{\eta\xi}(\eta,\xi)d\xi d\eta$$

$$\leq M_2(\varkappa,\gamma)\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}(g(\eta)-g(\kappa_1))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}d\xi d\eta,$$

$$m_3(\varkappa,\gamma)\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}(g(\kappa_2)-g(\eta))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}d\xi d\eta \tag{5.8}$$

$$\leq \int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}(g(\kappa_2)-g(\eta))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}\digamma_{\eta\xi}(\eta,\xi)d\xi d\eta$$

$$\leq M_3(\varkappa,\gamma)\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}(g(\kappa_2)-g(\eta))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho}d\xi d\eta,$$

ve

$$m_4(\varkappa,\gamma)\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}(g(\kappa_2)-g(\eta))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}d\xi d\eta \tag{5.9}$$

$$\leq \int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}(g(\kappa_2)-g(\eta))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}\digamma_{\eta\xi}(\eta,\xi)d\xi d\eta$$

$$\leq M_4(\varkappa,\gamma)\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}(g(\kappa_2)-g(\eta))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho}d\xi d\eta,$$

elitsizlikleri yazılabilir. (5.6)-(5.9) eşitsizlikleri yardımıyla istenen sonuç elde edilir. □

Özellik 5.8. Theorem 5.7 un şartları altında m,M birer sabit olmak üzere $m \leq M$ için,

$$m \leq \digamma_{\eta\xi} \leq M,$$

olsun. Bu durumda,

$$\frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)}\left[m\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}[g(\eta)-g(\kappa_1)]^{\varphi}[w(\xi)-w(\kappa_3)]^{\rho}d\xi d\eta\right.$$

$$-M\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}[g(\eta)-g(\kappa_1)]^{\varphi}[w(\kappa_4)-w(\xi)]^{\rho}d\xi d\eta$$

$$-M\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}[g(\kappa_2)-g(\eta)]^{\varphi}[w(\xi)-w(\kappa_3)]^{\rho}\,d\xi d\eta$$

$$\left.+m\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}[g(\kappa_2)-g(\eta)]^{\varphi}[w(\kappa_4)-w(\xi)]^{\rho}\,d\xi d\eta\right]$$

$$\leq M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)N_w^{\rho}(\kappa_3,\kappa_4;\gamma)\digamma(\varkappa,\gamma)$$

$$-N_w^{\rho}(\kappa_3,\kappa_4;\gamma)\left[\mathcal{J}^{\varphi}_{\varkappa-;g}\digamma(\kappa_1,\gamma)+\mathcal{J}^{\varphi}_{\varkappa+;g}\digamma(\kappa_2,\gamma)\right]$$

$$-M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)\left[\mathcal{J}^{\rho}_{\gamma-;w}\digamma(\varkappa,\kappa_3)+\mathcal{J}^{\rho}_{\gamma+;w}\digamma(\varkappa,\kappa_4)\right]$$

$$+\mathcal{J}^{\varphi,\rho}_{\varkappa-,\gamma-;g,w}\digamma(\kappa_1,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\varkappa-,\gamma+;g,w}\digamma(\kappa_1,\kappa_4)$$

$$+\mathcal{J}^{\varphi,\rho}_{\varkappa+,\gamma-;g,w}\digamma(\kappa_2,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\varkappa+,\gamma+;g,w}\digamma(\kappa_2,\kappa_4)$$

$$\leq\frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)}\left[M\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}[g(\eta)-g(\kappa_1)]^{\varphi}[w(\xi)-w(\kappa_3)]^{\rho}\,d\xi d\eta\right.$$

$$-m\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}[g(\eta)-g(\kappa_1)]^{\varphi}[w(\kappa_4)-w(\xi)]^{\rho}\,d\xi d\eta$$

$$-m\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}[g(\kappa_2)-g(\eta)]^{\varphi}[w(\xi)-w(\kappa_3)]^{\rho}\,d\xi d\eta$$

$$\left.+M\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}[g(\kappa_2)-g(\eta)]^{\varphi}[w(\kappa_4)-w(\xi)]^{\rho}\,d\xi d\eta\right],$$

eşitsizlikleri elde edilir.

Özellik 5.9. Theorem 5.7 şartları altında m,M birer sabit olmak üzere $m\leq M$ için,

$$m\leq\digamma_{\eta\xi}\leq M,$$

olsun. Bu durumda,

$$\Big|M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)N_w^{\rho}(\kappa_3,\kappa_4;\gamma)\digamma(\varkappa,\gamma) \tag{5.10}$$

$$-N_w^{\rho}(\kappa_3,\kappa_4;\gamma)\left[\mathcal{J}_{\varkappa-;g}^{\varphi}\digamma(\kappa_1,\gamma)+\mathcal{J}_{\varkappa+;g}^{\varphi}\digamma(\kappa_2,\gamma)\right]$$

$$-M_g^{\varphi}(\kappa_1,\kappa_2;\varkappa)\left[\mathcal{J}_{\gamma-;w}^{\rho}\digamma(\varkappa,\kappa_3)+\mathcal{J}_{\gamma+;w}^{\rho}\digamma(\varkappa,\kappa_4)\right]$$

$$+\mathcal{J}_{\varkappa-,\gamma-;g,w}^{\varphi,\rho}\digamma(\kappa_1,\kappa_3)+\mathcal{J}_{\varkappa-,\gamma+;g,w}^{\varphi,\rho}\digamma(\kappa_1,\kappa_4)$$

$$+\mathcal{J}_{\varkappa+,\gamma-;g,w}^{\varphi,\rho}\digamma(\kappa_2,\kappa_3)+\mathcal{J}_{\varkappa+,\gamma+;g,w}^{\varphi,\rho}\digamma(\kappa_2,\kappa_4)$$

$$-\frac{M+m}{2\Gamma(\varphi+1)\Gamma(\rho+1)}\left[\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}[g(\eta)-g(\kappa_1)]^{\varphi}[w(\xi)-w(\kappa_3)]^{\rho}\,d\xi d\eta\right.$$

$$-\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}[g(\eta)-g(\kappa_1)]^{\varphi}[w(\kappa_4)-w(\xi)]^{\rho}\,d\xi d\eta$$

$$-\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}[g(\kappa_2)-g(\eta)]^{\varphi}[w(\xi)-w(\kappa_3)]^{\rho}\,d\xi d\eta$$

$$\left.+\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}[g(\kappa_2)-g(\eta)]^{\varphi}[w(\kappa_4)-w(\xi)]^{\rho}\,d\xi d\eta\right]\Bigg|$$

$$\leq\frac{M-m}{2\Gamma(\varphi+1)\Gamma(\rho+1)}\Psi_g(\kappa_1,\kappa_2;\varkappa)\Upsilon_w(\kappa_3,\kappa_4;\gamma),$$

eşitsizliği vardır. Burada Ψ_g ve Υ_w değerleri Sonuç 5.3 de olduğu gibi tanımlanır.

İspat. (5.10) eşitsizliği, Sonuç 5.3 e benzer şekilde kolayca ispatlanır. □

Not 5.10. Sonuç 5.9 de $g(\eta)=\eta$, $\eta\in[\kappa_1,\kappa_2]$, $w(\xi)=\xi$, $\xi\in[\kappa_3,\kappa_4]$, $\varkappa=\frac{\kappa_1+\kappa_2}{2}$ ve $\gamma=\frac{\kappa_3+\kappa_4}{2}$ alınırsa,

$$\left|\digamma\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)-\frac{2^{\varphi-1}\Gamma(\varphi+1)}{(\kappa_2-\kappa_1)^{\varphi}}\right.$$

$$\times\left[J^{\varphi}_{\frac{\kappa_1+\kappa_2}{2}-}F\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+J^{\varphi}_{\frac{\kappa_1+\kappa_2}{2}+}F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\right]$$

$$-\frac{2^{\rho-1}\Gamma(\rho+1)}{(\kappa_4-\kappa_3)^{\rho}}\left[J^{\rho}_{\frac{\kappa_3+\kappa_4}{2}-}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+J^{\rho}_{\frac{\kappa_3+\kappa_4}{2}+}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\right]$$

$$+\frac{2^{\varphi+\rho-2}\Gamma(\varphi+1)\Gamma(\rho+1)}{(\kappa_2-\kappa_1)^{\varphi}(\kappa_4-\kappa_3)^{\rho}}$$

$$\times\left[J^{\varphi,\rho}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}-}F(\kappa_1,\kappa_3)+J^{\varphi,\rho}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}+}F(\kappa_1,\kappa_4)\right.$$

$$\left.\left.+J^{\varphi,\rho}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}-}F(\kappa_2,\kappa_3)+J^{\varphi,\rho}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}+}F(\kappa_2,\kappa_4)\right]\right|$$

$$\leq\frac{M-m}{8}(\kappa_2-\kappa_1)(\kappa_4-\kappa_3),$$

eşitsizliği elde edilir.

5.3. TRAPEZOİD TİPLİ YENİ EŞİTSİZLİKLER

Bu bölümde, önceki bölümdeki sonuçları kullanarak genelleştirilmiş kesirli integraller için bazı yeni trapezoid tipli eşitsizlikler oluşturulacaktır.

Yardımcı Teorem 5.11. $F:\Delta\to\mathbb{R}$ fonksiyonu Δ° üzerinde iki kez kısmi türevlenebilir olsun. $\varphi,\rho>0$ için $F_{\eta\xi}\in L(\Delta)$ ise,

$$M^{\varphi}_{g}(\kappa_1,\kappa_2)N^{\rho}_{w}(\kappa_3,\kappa_4)\frac{F(\kappa_2,\kappa_4)+F(\kappa_2,\kappa_3)+F(\kappa_1,\kappa_4)+F(\kappa_1,\kappa_3)}{4} \tag{5.11}$$

$$-\frac{N^{\rho}_{w}(\kappa_3,\kappa_4)}{4}\left[\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\kappa_1,\kappa_4)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\kappa_1,\kappa_3)+\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\kappa_2,\kappa_4)+\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\kappa_2,\kappa_3)\right]$$

$$-\frac{M^{\varphi}_{g}(\kappa_1,\kappa_2)}{4}\left[\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_2,\kappa_3)+\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_2,\kappa_4)+\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_1,\kappa_3)+\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_1,\kappa_4)\right]$$

$$+\frac{1}{4}\left[\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-;g,w}F(\kappa_1,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+;g,w}F(\kappa_1,\kappa_4)\right.$$

$$\left.+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-;g,w}F(\kappa_2,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+;g,w}F(\kappa_2,\kappa_4)\right]$$

$$= \frac{1}{4\Gamma(\varphi+1)\Gamma(\rho+1)} \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_2} [(g(\kappa_2)-g(\eta))^{\varphi} - g(\eta)-g(\kappa_1))^{\varphi}]$$

$$\times [(w(\kappa_4)-w(\xi))^{\rho} - (w(\xi)-w(\kappa_3))^{\rho}] \mathcal{F}_{\eta\xi}(\eta,\xi) d\xi d\eta,$$

eşitliği sağlanır.

İspat. (5.2) de $\varkappa = \kappa_2$ ve $\gamma = \kappa_4$ alınırsa,

$$I_9 = \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_2} (g(\eta)-g(\kappa_1))^{\varphi} (w(\xi)-w(\kappa_3))^{\rho} \mathcal{F}_{\eta,\xi}(\eta,\xi) d\xi d\eta$$

$$\begin{aligned} &= (w(\kappa_4)-w(\kappa_3))^{\rho} (g(\kappa_2)-g(\kappa_1))^{\varphi} \mathcal{F}(\kappa_2,\kappa_4) \\ &- \Gamma(\varphi+1)(w(\kappa_4)-w(\kappa_3))^{\rho} \mathcal{J}^{\varphi}_{\kappa_2-;g} \mathcal{F}(\kappa_1,\kappa_4) \\ &- \Gamma(\rho+1)(g(\kappa_2)-g(\kappa_1))^{\varphi} \mathcal{J}^{\rho}_{\kappa_4-;w} \mathcal{F}(\kappa_2,\kappa_3) \\ &+ \Gamma(\varphi+1)\Gamma(\rho+1) \mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-;g,w} \mathcal{F}(\kappa_1,\kappa_3), \end{aligned} \tag{5.12}$$

(5.3) de $\varkappa = \kappa_2$ ve $\gamma = \kappa_3$ yazılırsa,

$$I_{10} = \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} (g(\eta)-g(\kappa_1))^{\varphi} (w(\kappa_4)-w(\xi))^{\rho} \mathcal{F}_{\eta\xi}(\eta,\xi) d\xi d\eta$$

$$\begin{aligned} &= -(g(\kappa_2)-g(\kappa_1))^{\varphi} (w(\kappa_4)-w(\kappa_3))^{\rho} \mathcal{F}(\kappa_2,\kappa_3) \\ &+ \Gamma(\varphi+1)(w(\kappa_4)-w(\kappa_3))^{\rho} \mathcal{J}^{\varphi}_{\kappa_2-;g} \mathcal{F}(\kappa_1,\kappa_3) \\ &+ \Gamma(\rho+1)(g(\kappa_2)-g(\kappa_1))^{\varphi} \mathcal{J}^{\rho}_{\kappa_3+;w} \mathcal{F}(\kappa_2,\kappa_4) \\ &- \Gamma(\varphi+1)\Gamma(\rho+1) \mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+;g,w} \mathcal{F}(\kappa_1,\kappa_4), \end{aligned} \tag{5.13}$$

(5.4) de $\varkappa = \kappa_1$ ve $\gamma = \kappa_4$ alınırsa,

$$
\begin{aligned}
I_{11} &= \int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} (g(\kappa_2)-g(\eta))^{\varphi}(w(\xi)-w(\kappa_3))^{\rho} F_{\eta\xi}(\eta,\xi)d\xi d\eta \\
&= -(g(\kappa_2)-g(\kappa_1))^{\varphi}(w(\kappa_4)-w(\kappa_3))^{\rho} F(\kappa_1,\kappa_4) \\
&+ \Gamma(\varphi+1)(w(\kappa_4)-w(\kappa_3))^{\rho} \mathcal{J}^{\varphi}_{\kappa_1+;g} F(\kappa_2,\kappa_4) \\
&+ \Gamma(\rho+1)(g(\kappa_2)-g(\kappa_1))^{\varphi} \mathcal{J}^{\rho}_{\kappa_4-;w} F(\kappa_1,\kappa_3) \\
&- \Gamma(\varphi+1)\Gamma(\rho+1)\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-;g,w} F(\kappa_2,\kappa_3),
\end{aligned} \tag{5.14}
$$

ve son olarak (5.5) de $\varkappa = \kappa_1$ ve $\gamma = \kappa_3$ yazılırsa,

$$
\begin{aligned}
I_{12} &= \int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} (g(\kappa_2)-g(\eta))^{\varphi}(w(\kappa_4)-w(\xi))^{\rho} F_{\eta\xi}(\eta,\xi)d\xi d\eta \\
&= (g(\kappa_2)-g(\kappa_1))^{\varphi}(w(\kappa_4)-w(\kappa_3))^{\rho} F(\kappa_1,\kappa_3) \\
&- \Gamma(\varphi+1)(w(\kappa_4)-w(\kappa_3))^{\rho} \mathcal{J}^{\varphi}_{\kappa_1+;g} F(\kappa_2,\kappa_3) \\
&- \Gamma(\rho+1)(g(\kappa_2)-g(\kappa_1))^{\varphi} \mathcal{J}^{\rho}_{\kappa_3+;w} F(\kappa_1,\kappa_4) \\
&+ \Gamma(\varphi+1)\Gamma(\rho+1)\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+;g,w} F(\kappa_2,\kappa_4),
\end{aligned} \tag{5.15}
$$

elde edilir. (5.12)-(5.15) yardımıyla,

$$
\begin{aligned}
&\frac{1}{4\Gamma(\varphi+1)\Gamma(\rho+1)}\left[I_9 - I_{10} - I_{11} + I_{12}\right] \\
&= M_g^{\varphi}(\kappa_1,\kappa_2)N_w^{\rho}(\kappa_3,\kappa_4)\frac{F(\kappa_2,\kappa_4)+F(\kappa_2,\kappa_3)+F(\kappa_1,\kappa_4)+F(\kappa_1,\kappa_3)}{4}
\end{aligned}
$$

$$-\frac{N_w^\rho(\kappa_3,\kappa_4)}{4}\left[\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\kappa_1,\kappa_4)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\kappa_1,\kappa_3)+\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\kappa_2,\kappa_4)+\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\kappa_2,\kappa_3)\right]$$

$$-\frac{M_g^\varphi(\kappa_1,\kappa_2)}{4}\left[\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_2,\kappa_3)+\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_2,\kappa_4)+\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_1,\kappa_3)+\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_1,\kappa_4)\right]$$

$$+\frac{1}{4}\left[\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-;g,w}F(\kappa_1,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+;g,w}F(\kappa_1,\kappa_4)\right.$$

$$\left.+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-;g,w}F(\kappa_2,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+;g,w}F(\kappa_2,\kappa_4)\right],$$

elde edilir. □

Teorem 5.12. Theorem 5.1 ün şartlarını sağlasın. m,M birer sabit olmak üzere $m\leq M$ için,

$$m\leq F_{\eta\xi}\leq M,$$

olsun. Bu durumda,

$$\left|M_g^\varphi(\kappa_1,\kappa_2)N_w^\rho(\kappa_3,\kappa_4)\frac{F(\kappa_2,\kappa_4)+F(\kappa_2,\kappa_3)+F(\kappa_1,\kappa_4)+F(\kappa_1,\kappa_3)}{4}\right. \tag{5.16}$$

$$-\frac{N_w^\rho(\kappa_3,\kappa_4)}{4}\left[\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\kappa_1,\kappa_4)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\kappa_1,\kappa_3)+\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\kappa_2,\kappa_4)+\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\kappa_2,\kappa_3)\right]$$

$$-\frac{M_g^\varphi(\kappa_1,\kappa_2)}{4}\left[\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_2,\kappa_3)+\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_2,\kappa_4)+\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_1,\kappa_3)+\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_1,\kappa_4)\right]$$

$$+\frac{1}{4}\left[\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-;g,w}F(\kappa_1,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+;g,w}F(\kappa_1,\kappa_4)\right.$$

$$\left.+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-;g,w}F(\kappa_2,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+;g,w}F(\kappa_2,\kappa_4)\right]$$

$$-\frac{m+M}{8\Gamma(\varphi+1)\Gamma(\rho+1)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\left[(g(\kappa_2)-g(\eta))^\varphi-(g(\eta)-g(\kappa_1))^\varphi\right]$$

$$\left.\times\left[(w(\kappa_4)-w(\xi))^\rho-(w(\xi)-w(\kappa_3))^\rho\right]d\xi d\eta\right|$$

$$\leq \frac{M-m}{8\Gamma(\varphi+1)\Gamma(\rho+1)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}|(g(\kappa_2)-g(\eta))^{\varphi}-(g(\eta)-g(\kappa_1))^{\varphi}|$$

$$\times\left|(w(\kappa_4)-w(\xi))^{\rho}-(w(\xi)-w(\kappa_3))^{\rho}\right|d\xi d\eta,$$

eşitsizliği gerçeklenir.

İspat. (5.11) eşitsizliğinden,

$$\frac{1}{4\Gamma(\varphi+1)\Gamma(\rho+1)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}[(g(\kappa_2)-g(\eta))^{\varphi}-(g(\eta)-g(\kappa_1))^{\varphi}] \tag{5.17}$$

$$\times\left[(w(\kappa_4)-w(\xi))^{\rho}-(w(\xi)-w(\kappa_3))^{\rho}\right]\left(\mathcal{F}_{\eta\xi}(\eta,\xi)-\frac{m+M}{2}\right)d\xi d\eta$$

$$=\frac{1}{\Gamma(\varphi+1)\Gamma(\rho+1)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}[(g(\kappa_2)-g(\eta))^{\varphi}-(g(\eta)-g(\kappa_1))^{\varphi}]$$

$$\times\left[(w(\kappa_4)-w(\xi))^{\rho}-(w(\xi)-w(\kappa_3))^{\rho}\right]\mathcal{F}_{\eta\xi}(\eta,\xi)d\xi d\eta$$

$$-\frac{m+M}{8\Gamma(\varphi+1)\Gamma(\rho+1)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}[(g(\kappa_2)-g(\eta))^{\varphi}-(g(\eta)-g(\kappa_1))^{\varphi}]$$

$$\times\left[(w(\kappa_4)-w(\xi))^{\rho}-(w(\xi)-w(\kappa_3))^{\rho}\right]d\xi d\eta$$

$$=M_g^{\varphi}(\kappa_1,\kappa_2)N_w^{\rho}(\kappa_3,\kappa_4)\frac{\mathcal{F}(\kappa_2,\kappa_4)+\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{F}(\kappa_1,\kappa_3)}{4}$$

$$-\frac{N_w^{\rho}(\kappa_3,\kappa_4)}{4}\left[\mathcal{J}_{\kappa_2-;g}^{\varphi}\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{J}_{\kappa_2-;g}^{\varphi}\mathcal{F}(\kappa_1,\kappa_3)+\mathcal{J}_{\kappa_1+;g}^{\varphi}\mathcal{F}(\kappa_2,\kappa_4)+\mathcal{J}_{\kappa_1+;g}^{\varphi}\mathcal{F}(\kappa_2,\kappa_3)\right]$$

$$-\frac{M_g^{\varphi}(\kappa_1,\kappa_2)}{4}\left[\mathcal{J}_{\kappa_4-;w}^{\rho}\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{J}_{\kappa_3+;w}^{\rho}\mathcal{F}(\kappa_2,\kappa_4)+\mathcal{J}_{\kappa_4-;w}^{\rho}\mathcal{F}(\kappa_1,\kappa_3)+\mathcal{J}_{\kappa_3+;w}^{\rho}\mathcal{F}(\kappa_1,\kappa_4)\right]$$

$$+\frac{1}{4}\left[\mathcal{J}_{\kappa_2-,\kappa_4-;g,w}^{\varphi,\rho}\mathcal{F}(\kappa_1,\kappa_3)+\mathcal{J}_{\kappa_2-,\kappa_3+;g,w}^{\varphi,\rho}\mathcal{F}(\kappa_1,\kappa_4)\right.$$

$$\left.+\mathcal{J}_{\kappa_1+,\kappa_4-;g,w}^{\varphi,\rho}\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{J}_{\kappa_1+,\kappa_3+;g,w}^{\varphi,\rho}\mathcal{F}(\kappa_2,\kappa_4)\right]$$

$$-\frac{m+M}{8\Gamma(\varphi+1)\Gamma(\rho+1)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}[(g(\kappa_2)-g(\eta))^{\varphi}-(g(\eta)-g(\kappa_1))^{\varphi}]$$

$$\times\left[(w(\kappa_4)-w(\xi))^{\rho}-(w(\xi)-w(\kappa_3))^{\rho}\right]d\xi d\eta,$$

yazılabilir. (5.17) eşitsizliğinin modülü alınırsa,

$$\left|M_g^{\varphi}(\kappa_1,\kappa_2)N_w^{\rho}(\kappa_3,\kappa_4)\frac{F(\kappa_2,\kappa_4)+F(\kappa_2,\kappa_3)+F(\kappa_1,\kappa_4)+F(\kappa_1,\kappa_3)}{4}\right.$$

$$-\frac{N_w^{\rho}(\kappa_3,\kappa_4)}{4}\left[\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\kappa_1,\kappa_4)+\mathcal{J}^{\varphi}_{\kappa_2-;g}F(\kappa_1,\kappa_3)+\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\kappa_2,\kappa_4)+\mathcal{J}^{\varphi}_{\kappa_1+;g}F(\kappa_2,\kappa_3)\right]$$

$$-\frac{M_g^{\varphi}(\kappa_1,\kappa_2)}{4}\left[\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_2,\kappa_3)+\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_2,\kappa_4)+\mathcal{J}^{\rho}_{\kappa_4-;w}F(\kappa_1,\kappa_3)+\mathcal{J}^{\rho}_{\kappa_3+;w}F(\kappa_1,\kappa_4)\right]$$

$$+\frac{1}{4}\left[\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_4-;g,w}F(\kappa_1,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\kappa_2-,\kappa_3+;g,w}F(\kappa_1,\kappa_4)\right.$$

$$\left.+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_4-;g,w}F(\kappa_2,\kappa_3)+\mathcal{J}^{\varphi,\rho}_{\kappa_1+,\kappa_3+;g,w}F(\kappa_2,\kappa_4)\right]$$

$$-\frac{m+M}{8\Gamma(\varphi+1)\Gamma(\rho+1)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}[(g(\kappa_2)-g(\eta))^{\varphi}-(g(\eta)-g(\kappa_1))^{\varphi}]$$

$$\left.\times\left[(w(\kappa_4)-w(\xi))^{\rho}-(w(\xi)-w(\kappa_3))^{\rho}\right]d\xi d\eta\right|$$

$$\leq \frac{1}{4\Gamma(\varphi+1)\Gamma(\rho+1)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}|(g(\kappa_2)-g(\eta))^{\varphi}-(g(\eta)-g(\kappa_1))^{\varphi}|$$

$$\times\left|(w(\kappa_4)-w(\xi))^{\rho}-(w(\xi)-w(\kappa_3))^{\rho}\right|\left|F_{\eta\xi}(\eta,\xi)-\frac{m+M}{2}\right|d\xi d\eta$$

$$\leq \frac{M-m}{8\Gamma(\varphi+1)\Gamma(\rho+1)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}|(g(\kappa_2)-g(\eta))^{\varphi}-(g(\eta)-g(\kappa_1))^{\varphi}|$$

$$\times\left|(w(\kappa_4)-w(\xi))^{\rho}-(w(\xi)-w(\kappa_3))^{\rho}\right|d\xi d\eta,$$

bulunur ve ispat tamamlanır. □

Not 5.13. Theorem 5.12 de $g(\eta)=\eta$, $\eta\in[\kappa_1,\kappa_2]$ ve $w(\xi)=\xi$, $\xi\in[\kappa_3,\kappa_4]$ seçilirse,

$$\Bigg|\frac{F(\kappa_2,\kappa_4)+F(\kappa_2,\kappa_3)+F(\kappa_1,\kappa_4)+F(\kappa_1,\kappa_3)}{4}$$

$$-\frac{\Gamma(\varphi+1)}{4(\kappa_2-\kappa_1)^{\varphi}}\left[J^{\varphi}_{\kappa_2-}F(\kappa_1,\kappa_4)+J^{\varphi}_{\kappa_2-}F(\kappa_1,\kappa_3)+J^{\varphi}_{\kappa_1+}F(\kappa_2,\kappa_4)+J^{\varphi}_{\kappa_1+}F(\kappa_2,\kappa_3)\right]$$

$$-\frac{\Gamma(\rho+1)}{4(\kappa_4-\kappa_3)^{\rho}}\left[J^{\rho}_{\kappa_4-}F(\kappa_2,\kappa_3)+J^{\rho}_{\kappa_3+}F(\kappa_2,\kappa_4)+J^{\rho}_{\kappa_4-}F(\kappa_1,\kappa_3)+J^{\rho}_{\kappa_3+}F(\kappa_1,\kappa_4)\right]$$

$$+\frac{\Gamma(\varphi+1)\Gamma(\rho+1)}{4(\kappa_2-\kappa_1)^{\varphi}(\kappa_4-\kappa_3)^{\rho}}$$

$$\times\left[J^{\varphi,\rho}_{\kappa_2-,\kappa_4-}F(\kappa_1,\kappa_3)+J^{\varphi,\rho}_{\kappa_2-,\kappa_3+}F(\kappa_1,\kappa_4)+J^{\varphi,\rho}_{\kappa_1+,\kappa_4-}F(\kappa_2,\kappa_3)+J^{\varphi,\rho}_{\kappa_1+,\kappa_3+}F(\kappa_2,\kappa_4)\right]\Bigg|$$

$$\leq\frac{(M-m)(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}{2(\varphi+1)(\rho+1)}\left(1-\frac{1}{2^{\varphi}}\right)\left(1-\frac{1}{2^{\rho}}\right),$$

elde edilir.

6. SONUÇLAR VE ÖNERİLER

Bu çalışmada ilk olarak genelleştirilmiş kesirli integraller yardımıyla L_∞ ve L_p uzaylarına ait fonksiyonlar için Trapezoid ve Ostrowski tipli eşitsizlikler elde edilmiştir. Ayrıca kısmi türevleri sınırlı olan fonksiyonlar için bazı Trapezoid ve Ostrowski tipli eşitsizlikler ispatlanmıştır. Diğer yandan, elde edilen Ostrowski eşitsizliklerin özel durumları olarak bazı Midpoint tipli eşitsizlikler verilmiştir. Daha sonra, genelleştirilmiş kesirli integralin özel seçimleri yardımıyla, Riemann-Liouville kesirli integralleri ve Hadamard kesirli integralleri için trapezoid ve Ostrowski tipili eşitsizlikler sunulmuştur.

Sonraki çalışmalarda, bu tezde elde edilen eşitsizlikler farklı kesirli integraller için de ispatlanabilir. Ayrıca bu tezde kullanılan ispat yöntemleri kullanılarak Simpson, Bullen, Newton vb. gibi önemli eşitsizlikler de elde edilebilir.

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ÖZGEÇMİŞ

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