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**ON SOME BEST PROXIMITY POINT RESULTS FOR
MULTIVALUED MAPPINGS**

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ON SOME BEST PROXIMITY POINT RESULTS FOR MULTIVALUED
MAPPINGS

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May 2023

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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ABSTRACT

ON SOME BEST PROXIMITY POINT RESULTS FOR MULTIVALUED MAPPINGS

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Master of Science in Mathematics

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May 2023

This thesis is divided into four chapters. The first chapter discusses the motivation for the thesis and provides some background information on fixed point theory and best proximity point theory. In the second chapter, we recall the fundamentals definitions and notations, and some theorems related to our results about fixed point and best proximity point. We obtain our theorems in chapter three. The concepts of KW -type Ψ -contraction mapping inspired by the approaches of Wardowski and Klim-Wardowski have been presented. Then, we investigate the existence of a best proximity point for such mappings by considering a new family which is larger than the family of functions that is often used in fixed point results for multivalued mappings. Also, to demonstrate to effectiveness of our result we give a comparative example in which similar results in the literature cannot be applied. We provide our findings and suggestions in the fourth chapter.

2023, 38 pages

Keywords: Fixed point, Best proximity point, Multivalued mapping, Ψ -contraction, Complete metric space

ÖZET

KÜME DEĞERLİ DÖNÜŞÜMLER İÇİN BAZI EN İYİ YAKINLIK NOKTASI SONUÇLARI ÜZERİNE

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Matematik, Yüksek Lisans

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Bu tez dört bölüme ayrılmıştır. İlk bölümde, tezin motivasyonunu ve sabit nokta teorisi ve en iyi yakınlık noktası teorisi hakkında bazı arka plan bilgileri verildi. İkinci bölümde, sabit nokta ve en iyi yakınlık noktası ile ilgili temel tanımları ve notasyonları ve sonuçlarımızla ilgili bazı teoremleri hatırlıyoruz. Sonuçlarımızı üçüncü bölümde sunuyoruz. İlk önce Wardowski ve Klim-Wardowski'nin yaklaşımlarından ilham alarak KW tipi Ψ -büzülme eşleme kavramlarını tanıtıyoruz. Ardından, çok değerli eşlemeler için sabit nokta sonuçlarında sıklıkla kullanılan fonksiyon ailesinden daha büyük yeni bir aileyi dikkate alarak bu tür eşlemeler için en iyi yakınlık noktasının varlığını araştırıyoruz. Ayrıca, sonuçlarımızın etkinliğini göstermek için literatürdeki benzer sonuçların uygulanamadığı karşılaştırmalı bir örnek veriyoruz. Bulgu ve önerilerimizi dördüncü bölümde sunuyoruz.

2023, 38 sayfa

Anahtar Kelimeler: Sabit nokta, En iyi yakınlık noktası, Küme değerli dönüşüm, Ψ -büzülme, Tam metrik uzay

PREFACE AND KNOWLEDGEMENTS

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LIST OF SYMBOLS

\geq	Greater than or equal
\leq	Less than or equal
\subseteq	Subset or equals
\mathbb{N}	The set of natural numbers
\mathbb{Q}	The set of rational
\mathbb{R}	The set of all real numbers



LIST OF ABBREVIATIONS

<i>itf</i>	Infimum
<i>maš</i>	Maximum
<i>mit</i>	Minimum
<i>sup</i>	Supremum



1. INTRODUCTION

The area of mathematics known as fixed point theory is fascinating. It combines geometry and analysis topology. One of the most active fields of nonlinear analytic research is fixed point theory. It can be used in a variety of disciplines, including computer science, economics, and many more. The Banach contraction principle, commonly recognized as the most significant in this direction, was proposed by the Polish mathematician Banach (1922) as follows:

Let $\varphi: (\mathfrak{S}, \rho) \rightarrow (\mathfrak{S}, \rho)$ be a mapping where (\mathfrak{S}, ρ) is a complete metric space. If there is k in $[0,1)$ satisfying

$$\rho(\varphi\check{s}, \varphi\check{u}) \leq k\rho(\check{s}, \check{u})$$

for all $\check{s}, \check{u} \in \mathfrak{S}$, then φ has a unique fixed point in \mathfrak{S} .

This theory not only provides a method to find the fixed points of these mappings but also ensures the existence and uniqueness of the fixed points of self-mappings. With the help of this method, one may effectively address a wide range of issues in the fields of dynamic programming, game theory, ordinary differential equations, partial differential equations, and integral equations. The Banach contraction principle is later applied to many different ways (Abbas *et al.* 2019, Cosentin *et al.* 2015, Feng and Liu 2006, Klim and Wardowski 2007, Mizoguchi and Takahashi 1989, Muhammed and Kumam 2019, Nadler 1969, O'Regan 2019, Reich 1972a, Reich 1972b, Vetro 2015). One of them has been obtained by Wardowski (2012). He presented a new class of functions, denoted by \mathcal{F} , and proved the following result:

Let $\varphi: (\mathfrak{S}, \rho) \rightarrow (\mathfrak{S}, \rho)$ be a mapping where (\mathfrak{S}, ρ) is a complete metric space and $\Psi \in \mathcal{F}$. If there is $\tau > 0$ satisfying

$$\tau + \Psi(\rho(\varphi\check{s}, \varphi\check{u})) \leq \Psi(\rho(\check{s}, \check{u}))$$

for all $\check{s}, \check{u} \in \mathfrak{S}$, then φ has a unique fixed point in \mathfrak{S} .

There are other findings in the literature that are connected to Wardowski's result (Altun *et al.* 2015, Altun *et al.* 2016, Aydi *et al.* 2017, Durmaz *et al.* 2016, Piri and Kumam 2014, Sahin *et al.* 2019, Secelean 2013).

Nadler (1969) has obtained one of the generalizations of Banach's result via Pompei-Hausdorff metric. Hence, first, he has presented the following result for multivalued mappings.

Let $\varphi: (\mathfrak{S}, \rho) \rightarrow \mathcal{CA}(\mathfrak{S})$ be a multivalued mapping where (\mathfrak{S}, ρ) is a complete metric space. If there is k in $[0,1)$ satisfying

$$H(\varphi\check{s}, \varphi\check{u}) \leq k\rho(\check{s}, \check{u})$$

for all $\check{s}, \check{u} \in \mathfrak{S}$, then φ has a unique fixed point in \mathfrak{S} .

Then, Feng and Liu (2006) proved a fixed point result which is a generalization of Nadler's result without Pompei-Hausdorff metric. After that, Klim and Wardowski (2007) presented a result by inspring Feng-Liu and Mizoguichi-Takahashi.

Recently, the idea of the best proximity point was introduced by Basha and Veeramani (1977). The best proximity point theory investigates the circumstances under which the optimization problem represented by the expression $if_{\check{s} \in \Gamma} \rho(\check{s}, \varphi\check{s})$ has a solution. Assume that Γ and Λ are non-empty subsets of \mathfrak{S} , $\varphi: \Gamma \rightarrow \Lambda$ is a mapping, and (\mathfrak{S}, ρ) is a metric space. The best proximity point of the mapping φ is $a \in \Gamma$ if the statement $\rho(a, \varphi a) = \rho(\Gamma, \Lambda)$ holds. Numerous writers have studied this problem since every best proximity point result becomes a fixed point result when $\Gamma = \Lambda$ (Abkar and Gabeleh 2013, Ahmadi *et al.* 2021, Altun *et al.* 2020, Aslantas *et al.* 2021, Aslantas 2021, Aslantas 2022, Sahin *et al.* 2020, Sahin 2021, Sahin 2022).

This thesis is divided into four chapters. The first chapter discusses the motivation for the thesis and provides some background information on fixed point theory and best proximity point theory. In the second chapter, we recall the fundamental definitions and notations, and some theorems related to our results about fixed point theory and best proximity point. We obtain our theorems in chapter three. The concepts of KW -type Ψ -contraction mapping inspired by the approaches of Wardowski and Klim-Wardowski have been presented. Then, we investigate the existence of a best proximity point for such mappings by considering a new family which is larger than the family of functions that is often used in fixed point results for multivalued mappings. Also, to demonstrate the effectiveness of our result we give a comparative example in which similar results in the literature cannot be applied. We provide our findings and suggestions in the fourth chapter.



2. PRELIMINARIES

In this section, we recall some definitions and theorems related to our main section.

2.1 Metric Spaces

Definition 2.1. Let \mathfrak{S} be a nonempty set and $\rho: \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}$ a mapping with the following properties:

- (i) $\rho(\check{s}, \check{u}) = 0$ if and only if $\check{u} = \check{s}$.
- (ii) $\rho(\check{s}, \check{u}) = \rho(\check{u}, \check{s})$ for all $\check{s}, \check{u} \in \mathfrak{S}$.
- (iii) $\rho(\check{s}, \check{u}) + \rho(\check{u}, z) \geq \rho(\check{s}, z)$ for all $\check{s}, \check{u}, z \in \mathfrak{S}$.

Thus, we called ρ is a metric on \mathfrak{S} and (\mathfrak{S}, ρ) is a metric space.

Example 2.2. i) The function $\rho: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ given by, for all $\check{s}, \check{u} \in \mathbb{R}$

$$\rho(\check{s}, \check{u}) = |\check{s} - \check{u}|$$

is a metric on \mathbb{R} . This metric is known as usual metric.

ii) The function $\rho: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$\rho((\check{s}_1, \check{s}_2), (\check{u}_1, \check{u}_2)) = \sqrt{(\check{s}_1 - \check{u}_1)^2 + (\check{s}_2 - \check{u}_2)^2}$$

is a metric on \mathbb{R}^2 called the euclidean metric on \mathbb{R}^2 .

iii) Let \mathfrak{S} be a non-empty set and ρ the function from $\mathfrak{S} \times \mathfrak{S}$ into \mathbb{R} defined by

$$\rho(\check{s}, \check{u}) = \begin{cases} 0, & \text{if } \check{s} = \check{u} \\ 1, & \text{if } \check{s} \neq \check{u} \end{cases}.$$

Then ρ is a metric on \mathfrak{S} and is called the discrete metric.

Definition 2.3. The set

$$A_r(\check{s}_0) = \{\check{s} \in \mathfrak{S} : \rho(\check{s}, \check{s}_0) < r\}$$

in a metric space (\mathfrak{S}, ρ) is an open ball in \mathfrak{S} with radius $r > 0$ centered \check{s}_0 and the set

$$\overline{A}_r(\check{s}_0) = \{\check{s} \in \mathfrak{S} : \rho(\check{s}, \check{s}_0) \leq r\}$$

is a closed ball in \mathfrak{S} with radius $r > 0$ centered \check{s}_0

Definition 2.4. Let (\mathfrak{S}, ρ) be a metric space and $U \subseteq \mathfrak{S}$.

i) If for every $\check{s} \in U$ there is a number $\varepsilon > 0$ such that $A_r(\check{s}) \subseteq U$, then the set U is called open set.

ii) If the set $\mathfrak{S} - U$ is open, the set U is called a closed set.

Theorem 2.5. Let (\mathfrak{S}, ρ) be a metric space. Then, $A_r(\check{s}_0)$ is an open set.

Theorem 2.6. Let (\mathfrak{S}, ρ) be a metric space. Then, $\overline{A}_r(\check{s}_0)$ is a closed set.

Definition 2.7. Let $\{\check{s}_t\}$ be a sequence in a metric space (\mathfrak{S}, ρ) . We say that

i) the sequence $\{\check{s}_t\}$ converges to $\check{s} \in \mathfrak{S}$ (denoted by $\check{s}_t \rightarrow \check{s}$) if, for any $\varepsilon > 0$, there is natural number K such that $\rho(\check{s}, \check{s}_t) < \varepsilon$ for all $t > K$.

ii) the sequence $\{\check{s}_t\}$ is a Cauchy sequence if, for any $\varepsilon > 0$, there is a natural number K such that $\rho(\check{s}_m, \check{s}_t) < \varepsilon$ for all $m, t > K$.

iii) the sequence $\{\check{s}_t\}$ is bounded if there exists $M > 0$ such that $\rho(\check{s}_s, \check{s}_t) \leq M$ for all $s, t \in \mathbb{N}$.

Theorem 2.8. i) The limit of a convergent sequence in any metric space (\mathfrak{S}, ρ) is unique.

ii) Every convergent sequence in a metric space (\mathfrak{S}, ρ) is a Cauchy sequence.

Example 2.9. i) Let (\mathbb{R}, ρ) be a usual metric space. Then, (\mathbb{R}, ρ) is complete metric space.

ii) Let (\mathfrak{S}, ρ) be a discrete metric space. Then, (\mathfrak{S}, ρ) is complete metric space.

iii) Let $\mathcal{C}([a, \delta])$ be the set of all continuous function and $\rho: \mathcal{C}([a, \delta]) \times \mathcal{C}([a, \delta]) \rightarrow \mathbb{R}$ be a function defined by

$$\rho(f, g) = \sup \{|f(\check{s}) - g(\check{s})|: \check{s} \in [0, 1]\}.$$

Then, $(\mathcal{C}([a, \delta]), \rho)$ is a complete metric space.

iv) Let (\mathbb{Q}, ρ) be a usual metric space. Then, (\mathbb{Q}, ρ) is not complete metric space.

Definition 2.10. Let $(\mathfrak{S}, \rho_{\mathfrak{S}})$ and (Y, ρ_Y) be two metric spaces and $v \in \mathfrak{S}$. We say that a function $h: \mathfrak{S} \rightarrow Y$ is continuous at v if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\rho_Y(h\check{s}, hv) < \varepsilon \text{ whenever } \rho_{\mathfrak{S}}(\check{s}, v) < \delta$$

Equivalently, a function $h: \mathfrak{X} \rightarrow Y$ is continuous at v if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$h\left(\Lambda_{\rho_{\mathfrak{X}}}(v, \delta)\right) \subseteq \Lambda_{\rho_Y}(hv, \varepsilon).$$

Theorem 2.11. Let $(\mathfrak{X}, \rho_{\mathfrak{X}})$ and (Y, ρ_Y) be two metric spaces and $v \in \mathfrak{X}$. A function $h: \mathfrak{X} \rightarrow Y$ is said to be continuous at $v \in \mathfrak{X}$, if and only if $h(\check{s}_t) \rightarrow h(v)$ in Y , whenever $\check{s}_t \rightarrow v$ in \mathfrak{X} , for every sequence $\{\check{s}_t\}$ in \mathfrak{X} .

Definition 2.12. Let $h: \mathfrak{X} \rightarrow Y$ be a function and $\check{s}_0 \in \mathfrak{X}$ where (\mathfrak{X}, ρ) is a metric space. If for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$h(\check{s}_0) - \varepsilon < h(\check{s})$$

for all $\check{s} \in \Lambda(\check{s}_0, \delta) \cap W$, then the function h is called lower semicontinuous function. Similarly, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$h(\check{s}_0) < h(\check{s}) + \varepsilon$$

for all $\check{s} \in \Lambda(\check{s}_0, \delta) \cap W$, then the function h is called upper semi-continuous function.

Theorem 2.13. Let (\mathfrak{X}, ρ) be a metric space, $h: \mathfrak{X} \rightarrow Y$ be a function and $\check{s}_0 \in \mathfrak{X}$. For every sequence $\{\check{s}_t\}$ converging to \check{s}_0 ,

$$\liminf_{\check{s} \rightarrow \check{s}_0} h(\check{s}) \geq h(\check{s}_0)$$

if and only if the function h is lower semi-continuous function. Similarly, if for every sequence $\{\check{s}_t\}$ converging to \check{s}_0 ,

$$\liminf_{\check{s} \rightarrow \infty} h(\check{s}_t) \leq h(\check{s}_0)$$

if and only if the function h is upper semi-continuous function.

2.2 Fixed Point Theory

In this section, we provide some fundamental definitions and theorems about fixed points of the mapping H on a metric space (\mathfrak{S}, ρ) , together with the prerequisites for their existence and uniqueness.

Definition 2.14. Let $H: \mathfrak{S} \rightarrow \mathfrak{S}$ be a mapping. Then, a point \check{s} is said to be a fixed point of H if it satisfies $H\check{s} = \check{s}$.

Now, we give the following example

Example 2.15. i) Let $H: \mathbb{R} \rightarrow [0,1]$ be a mapping given as $H\check{s} = \sin \check{s}$. Then, it has a fixed point which is 0, but $(\pi/2)$ is not fixed of φ .

ii) Let $H: (\mathfrak{S}, \rho) \rightarrow (\mathfrak{S}, \rho)$ be a mapping given as $H\check{s} = \check{s} + a$ where $0 \neq a \in \mathbb{R}$. Then, H does not have a fixed point.

iii) Let $H: (\mathfrak{S}, \rho) \rightarrow (\mathfrak{S}, \rho)$ be a mapping given as $H\check{s} = \check{s}$. The mapping has infinite fixed points.

Fixed point theory in full metric spaces was first studied by Banach in 1922.

Definition 2.16. Let (\mathfrak{S}, ρ) be a metric space and $H: \mathfrak{S} \rightarrow \mathfrak{S}$ be a mapping. The mapping is called contraction mapping if there exists q in $[0,1]$ such that

$$\rho(H\check{s}, H\check{u}) \leq q\rho(\check{s}, \check{u})$$

for all $\check{s}, \check{u} \in \mathfrak{S}$.

Theorem 2.17. Let $H: (\mathfrak{S}, \rho) \rightarrow (\mathfrak{S}, \rho)$ be a mapping where (\mathfrak{S}, ρ) is a complete metric space. Then, if H is a contraction mapping, then H has a unique fixed point.

The Banach fixed point theorem is an existence and uniqueness theorem for the fixed points of particular mappings. The proof will show us that it also gives us a useful method for steadily improving fixed point approximations. We begin by choosing an arbitrary \check{s}_0 from a given set, and then compute a sequence recursively by letting

$$\check{s}_{t+1} = \varphi\check{s}_t \text{ for all } t \in \mathbb{N}$$

We call this process an iteration. Such iteration methods are widely employed in applied mathematics, and Banach's Fixed Point Theorem is frequently applied to ensure the convergence of the scheme and the uniqueness of the solution. Basics of Banach contraction mapping is one of the unique analysis findings. It is commonly used as the basis for metric fixed point theory. The broad variety of mathematics and fields it.

Nadler (1969) derived one of the generalizations of Banach's findings using the Hausdorff metric. First, it has been found that multivalued mappings have a fixed point result. We will now go through a few terms and concepts related to multivalued mappings.

$$P(\mathfrak{S}) = \{U \subseteq \mathfrak{S}: U \text{ is totemptũ su}\delta\text{set}\}$$

$$C(\mathfrak{S}) = \{U \subseteq \mathfrak{S}: U \text{ is totemptũ } \mu\text{losep su}\delta\text{set}\}$$

$$CA(\mathfrak{S}) = \{U \subseteq \mathfrak{S}: U \text{ is totemptũ } \mu\text{losep atp } \delta\text{outpep su}\delta\text{set}\}$$

Definition 2.18. Let (\mathfrak{S}, ρ) be a metric space and $U, V \in CA(\mathfrak{S})$. A function $H: CA(\mathfrak{S}) \times CA(\mathfrak{S}) \rightarrow \mathbb{R}$ defined by

$$H(U, V) = \max\{\sup_{\check{s} \in U} \rho(\check{s}, V), \sup_{\check{u} \in V} \rho(U, \check{u})\}$$

where $\rho(\check{s}, V) = \inf_{\check{u} \in V} \rho(\check{s}, \check{u})$ is called Pompeiu-Hausdorff metric on $\mathcal{CA}(\mathfrak{S})$.

Theorem 2.19. Let (\mathfrak{S}, ρ) be a complete metric space and $\varphi: \mathfrak{S} \rightarrow \mathcal{CA}(\mathfrak{S})$ be a mapping. If φ is a multivalued contraction mapping, that is, there exists $k \in [0, 1)$ such that

$$H(\varphi\check{s}, \varphi\check{u}) \leq k\rho(\check{s}, \check{u})$$

for all $\check{s}, \check{u} \in \mathfrak{S}$, then φ has a unique fixed point.

Later, taking $\mathcal{C}(\mathfrak{S})$ instead of $\mathcal{CA}(\mathfrak{S})$ in Theorem 2 Feng and Liu (2006) obtained a generalization of Nadler's result as follows:

Theorem 2.20. Let $\varphi: \mathfrak{S} \rightarrow \mathcal{C}(\mathfrak{S})$ be a multivalued mapping where (\mathfrak{S}, ρ) is a complete metric space. Assume that $\check{s} \rightarrow \rho(\check{s}, \varphi\check{s})$ is lower semicontinuous and for all $\check{s} \in \mathfrak{S}$ there exist $\check{u} \in I_{\delta}^{\check{s}} = \{\check{u} \in \varphi\check{s} : \delta\rho(\check{s}, \check{u}) \leq \rho(\check{s}, \varphi\check{s})\}$ such that

$$\rho(\check{u}, \varphi\check{u}) \leq \mu\rho(\check{s}, \check{u})$$

for some $\delta, \mu \in [0, 1]$ with $\mu < \delta$. Then, φ has a fixed point in \mathfrak{S} .

Then, Klim and Wardowski (2007) introduced a new contraction for multivalued mappings by considering the ideas of Feng–Liu and Mizoguchi–Takahashi. Hence, they proved the following nice result for such contractive mappings.

Theorem 2.21. Let (\mathfrak{S}, ρ) be a complete metric space and $\varphi: \mathfrak{S} \rightarrow \mathcal{C}(\mathfrak{S})$ be multivalued mapping. Assume that $\check{s} \rightarrow \rho(\check{s}, \varphi\check{s})$ is lower semicontinuous and there exist $\delta \in (0, 1)$ and $\varphi: [0, \infty) \rightarrow [0, \delta)$ such that for all $s \geq 0$

$$\limsup_{t \rightarrow s^+} \varphi(t) < \delta \quad (1.1)$$

and for all $\check{s} \in \mathfrak{S}$ there exists $\check{u} \in I_\delta^{\check{s}}$ satisfying

$$\rho(\check{u}, \varphi\check{u}) \leq \varphi(\rho(\check{s}, \check{u}))\rho(\check{s}, \check{u}).$$

Then, φ has a fixed point in \mathfrak{S} .

On the other hand, Wardowski (2012) presented the following concept called Ψ -contraction, which includes many contractions in the literature, including the Banach contraction principle.

Definition 2.22. Let $\Psi : (0, \infty) \rightarrow \mathbb{R}$ be a mapping with the below axioms :

(Ψ_1) is strictly increasing,

(Ψ_2) for each sequence $\{s_t\}$ with the positive real numbers,

$$\lim_{t \rightarrow \infty} s_t = 0 \Leftrightarrow \lim_{t \rightarrow \infty} \mathcal{F}(s_t) = -\infty,$$

(Ψ_3) there is $k \in (0,1)$ with $\lim_{t \rightarrow 0} t^k \mathcal{F}(t) = 0$.

We will denote the class of functions Ψ satisfying the conditions in Definition 2.31 with \mathcal{F} . Then, Wardowski (2012) presented the definition of Ψ -contraction mapping and proved the following result.

Definition 2.23. Let (\mathfrak{S}, ρ) be a complete metric space, $\varphi: \mathfrak{S} \rightarrow \mathfrak{S}$ be a mapping and $\Psi \in \mathcal{F}$. If there exists $\tau > 0$ such that

$$\tau + \Psi(\rho(\varphi\check{s}, \varphi\check{u})) \leq \Psi(\rho(\check{s}, \check{u}))$$

for all $\check{s}, \check{u} \in \mathfrak{S}$ with $\rho(\varphi\check{s}, \varphi\check{u}) > 0$, then φ is called a Ψ -contraction mapping.

Theorem 2.24. Let (\mathfrak{S}, ρ) is a complete metric space. φ has a unique fixed point in \mathfrak{S} if $\varphi: (\mathfrak{S}, \rho) \rightarrow (\mathfrak{S}, \rho)$ is a Ψ -contraction mapping.

Recently, Secelean (2013) presented the following conditions which are equivalent to the (Ψ_2) , but easy condition.

$$(\Psi'_2) \inf \Psi = -\infty$$

or

$$(\Psi''_2) \text{ there exists the sequence } \{\lambda_t\} \subseteq (0, \infty) \text{ such that } \lim_{t \rightarrow \infty} \Psi(\lambda_t) = -\infty.$$

Then, Secelean (2013) proved the following important lemma.

Lemma 2.25. Let $\Psi: (0, \infty) \rightarrow \mathbb{R}$ be an increasing function and $\{\lambda_t\} \subseteq (0, \infty)$ be a sequence. Then, the following statements are true.

- (i) if $\lim_{t \rightarrow \infty} \Psi(\lambda_t) = -\infty$, then $\lim_{t \rightarrow \infty} \lambda_t = 0$.
- (ii) if $\inf \Psi = -\infty$ and $\lim_{t \rightarrow \infty} \lambda_t = 0$, then $\lim_{t \rightarrow \infty} \Psi(\lambda_t) = -\infty$.

We will denote the set of all functions $\Psi: (0, \infty) \rightarrow \mathbb{R}$ satisfying (Ψ_1) , (Ψ'_2) and (Ψ_3) by \mathcal{F}' .

Recently, taking into account the idea of F-contraction for multivalued mappings, multivalued F-contraction mappings were presented by Altun *et al.* (2015).

Definition 2.26. Let $\varphi: \mathfrak{S} \rightarrow \mathcal{CA}(\mathfrak{S})$ be a multivalued mapping on a metric space (\mathfrak{S}, ρ) if there is $\tau > 0$ and $\Psi \in \mathcal{F}$, such that

$$\tau + \Psi(H(\varphi\check{s}, \varphi\check{u})) \leq \Psi(\rho(\check{s}, \check{u}))$$

For all $\check{s}, \check{u} \in \mathfrak{S}$ with $H(\varphi\check{s}, \varphi\check{u}) > 0$, then is called a multivalued F-contraction mappings.

Then, they proved that a compact valued $(K(\mathfrak{S}))$ multivalued F-contraction mapping φ on al complete metric space (\mathfrak{S}, ρ) has a fixed point \check{s} In addition , they investigated whether the same result is valid if the mapping φ is $CA(\mathfrak{S})$ valued and proved a fixed-point result for the new type of mappings by assuming the following property in addition to $(\Psi_1) - (\Psi_3)$.

(Ψ_4) For all $\Gamma \subseteq (0, \infty)$ with $\inf \Gamma > 0$, we have $\Psi(\inf \Gamma) = \inf \Psi(\Gamma)$.

We denote the class of functions $\Psi \in \mathcal{F}$ having (Ψ_4) by \mathcal{F}_* throughout this study.

2.3 Best Proximity Point Theory

It has become quite popular recently since each result in the best proximity point theory converts a result in fixed point theory into a specific example.

Let $\varphi : \Gamma \rightarrow \Lambda$ be a mapping where (\mathfrak{S}, ρ) is a metric spcae and $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{S}$. If $\rho(\Gamma, \Lambda) > 0$, then the mapping φ cannot have fixed point. In this sense, it makes sense to investigate the presence of a point $\check{s} \in \Gamma$ such that $\rho(\check{s}, \varphi\check{s}) = \rho(\Gamma, \Lambda)$ which is called best proximity point of φ (Basha and Veeramani 1977).

Now, we present some notations, definitions and theorems about best proximity point theory. Consider the following sets

$$U_0 = \{\check{s} \in U : \rho(\check{s}, \check{u}) = \rho(U, V) \text{ for some } \check{u} \in V\}$$

and

$$V^0 = \{\check{s} \in V: \rho(\check{s}, \check{u}) = \rho(U, V) \text{ for some } \check{u} \in U\}.$$

The following definition is a version of contraction mapping in the best proximity point theory.

Definition 2.27. Let (\mathfrak{X}, ρ) be a metric space and $\emptyset \neq U, V \subseteq \mathfrak{X}$. A mapping $H: U \rightarrow V$ is said to be proximal contraction if there exists q in $[0,1)$ such that

$$\left. \begin{array}{l} \rho(\check{s}_1, Hu_1) = \rho(U, V) \\ \rho(\check{s}_2, Hu_2) = \rho(U, V) \end{array} \right\} \Rightarrow \rho(\check{s}_1, \check{s}_2) \leq q\rho(u_1, u_2)$$

for all $\check{s}_1, \check{s}_2, u_1, u_2 \in U$.

After that, Raj (2013) gave the concept of P -Property and proved a best proximity point result.

Definition 2.28. Let (\mathfrak{X}, ρ) be a metric space and Γ, Λ be nonempty subsets of \mathfrak{X} . The pair (Γ, Λ) is said to have the P -Property if and only if

$$\left. \begin{array}{l} \rho(\check{s}_1, \check{u}_1) = \rho(\Gamma, \Lambda) \\ \rho(\check{s}_2, \check{u}_2) = \rho(\Gamma, \Lambda) \end{array} \right\} \Rightarrow \rho(\check{s}_1, \check{s}_2) = \rho(\check{u}_1, \check{u}_2)$$

for all $\check{s}_1, \check{s}_2 \in \Gamma$ and $\check{u}_1, \check{u}_2 \in \Lambda$.

Sahin (2022) introduced the following definition which is a generalization of P -Property.

Definition 2.29. Let Γ and Λ be non-empty subsets of a metric space (\mathfrak{X}, ρ) . Then, if it holds

$$\left. \begin{array}{l} \rho(\check{s}_1, \check{u}_1) = \rho(\Gamma, \Lambda) \\ \rho(\check{s}_2, \check{u}_2) = \rho(\Gamma, \Lambda) \end{array} \right\} \rho(\check{s}_1, \check{s}_2) = \rho(\check{u}_1, \check{u}_2)$$

for all $\check{s}_1, \check{s}_2 \in \Gamma$ with $\check{s}_1 \neq \check{s}_2$ and $\check{u}_1, \check{u}_2 \in \Lambda$ with $\check{u}_1 \neq \check{u}_2$, then the pair (Γ, Λ) is said to have a modified P-Property.

The pair (Γ, Λ) has the modified P-property if the pair (Γ, Λ) possesses the P-property. The opposite, though, might not always be true. Let $\mathfrak{S} = \mathbb{R}$ have the standard metric d , in fact. Take into account the subsets $\Gamma = \{0, 4\}$ and $\Lambda = \{2\}$ respectively. The result is $\rho(\Gamma, \Lambda) = 1$. Despite having the modified P-Property, the pair does not have the P-Property, as can be shown. If we take $\check{s}_1 = 0, \check{s}_2 = 4$ and $\check{u}_1 = 2 = \check{u}_2$, then even though $\rho(\check{s}_1, \check{u}_1) = \rho(\Gamma, \Lambda)$ and $\rho(\check{s}_2, \check{u}_2) = \rho(\Gamma, \Lambda)$, we have

$$\rho(\check{s}_1, \check{s}_2) = 2 \neq 0 = \rho(\check{u}_1, \check{u}_2).$$

Very recently, Sahin (2022) introduced generalized multivalued F-contraction mapping inspired by the ideas of Feng-Liu and Wardowski, and then proved the following result for such mappings.

Theorem 2.30. Let $\mu, \Lambda \subseteq \mathfrak{S}$ be closed, having a modified P-Property, where (\mathfrak{S}, ρ) is a complete metric space and $\Gamma_0 \neq \emptyset$. suppose that $\varphi: \Gamma \times \Lambda \rightarrow C(\Lambda)$ is a multivalued mapping and $f(\check{s}, \check{u}) = \rho(\check{u}, \varphi\check{s})$ is lower semi-continuous on $\Gamma \times \Lambda$. If there are $\tau > 0$ and $\Psi \in \mathcal{F}_*$ such that for all $\check{s} \in \Gamma_*$ with $\rho(\check{s}, \varphi\check{s}) > \rho(\Gamma, \Lambda)$ and $\check{u} \in \varphi\check{s}$, there exists $z \in \Gamma_0$ satisfying

$$\rho(\check{u}, z) = \rho(\Gamma, \Lambda)$$

and

$$\tau + \Psi(\rho(\check{u}, \varphi z)) \leq \Psi(\rho(\check{s}, z)),$$

Thus, the main result for $CB(\mathfrak{S})$ valued mappings in (Altun *et al.* 2015) has been generalized in an interesting way. Note that one needs the family \mathcal{F}_* to show the existence of a fixed point for $CB(\mathfrak{S})$ valued mappings. However, Aslantas (2021) presented another approach by a new family, which includes \mathcal{F}_* .

Now, let $\varphi : \Gamma \rightarrow \rho(\Lambda)$ be a mapping defined on a metric space (\mathfrak{S}, ρ) , where $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{S}$. Define a family by

$${}^{\rho}\mathcal{F}_{**} = \{\Psi \in \mathcal{F}' : \Psi_{\sigma, \varphi}^{\check{s}, \check{u}} \neq \emptyset \text{ for all } (\check{s}, \check{u}) \in \rho_{\Lambda}^{\Gamma} \text{ and } \sigma \geq 0\},$$

where

$$\Psi_{\sigma, \varphi}^{\check{s}, \check{u}} = \{z \in \varphi\check{s} : \Psi(\rho(\check{u}, z)) \leq \Psi(\rho(\check{u}, \varphi\check{s})) + \sigma\}$$

and

$$\rho_{\Lambda}^{\Gamma} = \{(\check{s}, \check{u}) \in \Gamma \times \Lambda : \rho(\check{s}, \check{u}) = \rho(\Gamma, \Lambda) \text{ and } \rho(\check{u}, \varphi\check{s}) > 0\}.$$

Then, it can be shown that \mathcal{F}_* is a proper subset of ${}^{\rho}\mathcal{F}_{**}$. Indeed, let $\mathfrak{S} = \{0\} \cup [1, \infty)$ and (\mathfrak{S}, ρ) be usual metric space. Consider the subsets $\Lambda = [1, \infty)$ and $\Gamma = \{0\}$. Let consider the mappings $\varphi : \Gamma \rightarrow \mathcal{C}(\Lambda)$ and $\Psi : (0, \infty) \rightarrow \mathbb{R}$ by $\varphi 0 = [1, \infty)$ and

$$\Psi\alpha = \begin{cases} \ln\alpha & , \quad \alpha \leq 1 \\ 2\alpha & , \quad \alpha > 1 \end{cases}$$

respectively. Then, we have $\Psi \notin \mathcal{F}_*$. But, $\Psi \in {}^{\rho}\mathcal{F}_{**}$, to see this, let $\check{s} \in \Gamma, \check{u} \in \varphi\check{s}$ and $\sigma \geq 0$. Then, we have

$$\Psi_{\sigma, \varphi}^{0, \check{u}} = \{z \in [1, \infty): \Psi(\rho(\check{u}, z)) \leq \Psi(\rho(0, \varphi 0)) + \sigma\}$$

$$= \{z \in [1, \infty): \Psi(|\check{u} - z|) \leq \Psi(1) + \sigma\}.$$

In this case, if we choose $z = \check{u} + \frac{1}{2}$, then $z \in \Psi_{\sigma, \varphi}^{0, \check{u}}$, and so $\Psi \in {}_{\sigma}^{\rho}\mathcal{F}_{**}$.

We recall the next lemma, which is crucial for proving the connection between the prior results proven and our results.

Lemma 2.31. Let (Γ, ρ) and (Λ, ρ) be two metric spaces and $\varphi : \Gamma \rightarrow P(\Lambda)$ be an upper semicontinuous mapping. $f : \Gamma \times \Lambda \rightarrow \mathbb{R}$ defined by $f(\check{s}, \check{u}) = \rho(\check{u}, \varphi \check{s})$ is lower semicontinuous.

3. SOME RESULTS RELATED TO BEST PROXIMITY POINT THEORY

In this section, we obtain some best proximity point results for multivalued mappings on complete metric spaces. Now, we introduce the definition of KW -type Ψ -contraction mapping.

Definition 3.1. Let (\mathfrak{X}, ρ) be a metric space and $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{X}$. Assume that $\varphi : \Gamma \rightarrow \mathcal{C}(\Lambda)$ is a mapping and $\Psi \in \mathcal{F}$. Then, φ is called KW -type Ψ -contraction mapping if there exist $\sigma \geq 0$ and $\tau : (0, \infty) \rightarrow (\sigma, \infty)$ such that for all $s > 0$

$$\liminf_{t \rightarrow s^+} \tau(t) > \sigma \quad (3.1)$$

and for all $\check{s} \in \Gamma_0$ with $\rho(\check{s}, \varphi\check{s}) > \rho(\Gamma, \Lambda)$ and $\check{u} \in \varphi\check{s}$ there exists $z \in \Gamma_0$ satisfying

$$\rho(\check{u}, z) = \rho(\Gamma, \Lambda),$$

and

$$\tau(\rho(\check{s}, z)) + \Psi(\rho(\check{u}, \varphi z)) \leq \Psi(\rho(\check{s}, z)).$$

Theorem 3.2. Let (\mathfrak{X}, ρ) be a complete metric space, $\varphi : \Gamma \rightarrow \mathcal{C}(\Lambda)$ be a KW -type Ψ -contraction mapping where Γ, Λ are closed non-empty subsets closed having modified P -property, $\Gamma_0 \neq \emptyset$ and $\Psi \in {}^\rho\mathcal{F}_{**}$. Then, φ has a best proximity in Γ provided that $f(\check{s}, \check{u}) = \rho(\check{u}, \varphi\check{s})$ is lower semi-continuous on Δ where

$$\Delta = \{(\check{s}, \check{u}) \in \Gamma \times \Lambda : \rho(\check{s}, \check{u}) = \rho(\Gamma, \Lambda)\}.$$

Proof. Let \check{s}_0 be an arbitrary point in Γ_0 and choose $\check{u}_0 \in \varphi\check{s}_0$. If $\rho(\check{s}_0, \varphi\check{s}_0) = \rho(\Gamma, \Lambda)$, then the proof is completed.

Now, suppose that $\rho(\check{s}_0, \varphi\check{s}_0) > \rho(\Gamma, \Lambda)$. Since φ is a *KW*-type Ψ -contraction mapping, there exists $\check{s}_1 \in \Gamma_0$ such that

$$\rho(\check{u}_0, \check{s}_1) = \rho(\Gamma, \Lambda)$$

and

$$\tau(\rho(\check{s}_0, \check{s}_1)) + \Psi(\rho(\check{u}_0, \varphi\check{s}_1)) \leq \Psi(\rho(\check{s}_0, \check{s}_1)). \quad (3.2)$$

From Inequality (3.2), we have $\rho(\check{u}_0, \varphi\check{s}_1) > 0$ and $\rho(\check{s}_0, \check{s}_1) > 0$. Since $\Psi \in {}^\rho_{\varphi}\mathcal{F}_{**}$ we get $\Psi_{\sigma, \varphi}^{\check{s}_1, \check{u}_0} \neq \emptyset$. So, there exists $\check{u}_1 \in \varphi\check{s}_1$ such that

$$\Psi(\rho(\check{u}_0, \check{u}_1)) \leq \Psi(\rho(\check{u}_0, \varphi\check{s}_1)) + \sigma.$$

Again, if $\rho(\check{s}_1, \varphi\check{s}_1) = \rho(\Gamma, \Lambda)$, then \check{s}_1 is abest proximity point of φ , and so the proof is completed. Now, assume $\rho(\check{s}_1, \varphi\check{s}_1) > \rho(\Gamma, \Lambda)$. Since, φ is a *KW*-type Ψ -Contraction mapping, there exists $\check{s}_2 \in \Gamma_0$ such that

$$\rho(\check{u}_1, \check{s}_2) = \rho(\Gamma, \Lambda)$$

and

$$\tau(\rho(\check{s}_1, \check{s}_2)) + \Psi(\rho(\check{u}_1, \varphi\check{s}_2)) \leq \Psi(\rho(\check{s}_1, \check{s}_2)). \quad (3.3)$$

From Inequality (3.3) we have $\rho(\check{u}_1, \varphi\check{s}_2) > 0$ and $\rho(\check{s}_1, \check{s}_2) > 0$. Since $\Psi \in \mathcal{F}_{**}$, we get $\Psi_{\sigma, \varphi}^{\check{s}_2, \check{u}_1} \neq \emptyset$. So, there exists $\check{u}_2 \in \varphi\check{s}_2$ such that

$$\Psi(\rho(\check{u}_1, \check{u}_2)) \leq \Psi(\rho(\check{u}_1, \varphi\check{s}_2)) + \sigma.$$

By continuing in this manner, we may demonstrate that $\{\check{s}_t\}$ in Γ and $\{\check{u}_t\}$ in Λ with $\check{s}_t \neq \check{s}_{t+1}$ and $\check{u}_t \neq \check{u}_{t+1}$ such that (we can suppose $\rho(\check{s}_t, \varphi\check{s}_t) > \rho(\Gamma, \Lambda)$ for all $t \in \mathbb{N}$. Otherwise the proof is complete)

$$\rho(\check{u}_t, \check{s}_{t+1}) = \rho(\Gamma, \Lambda), \quad (3.4)$$

$$\tau(\rho(\check{s}_t, \check{s}_{t+1}) + \Psi(\rho(\check{u}_t, \varphi\check{s}_{t+1})) \leq \Psi(\rho(\check{s}_t, \check{s}_{t+1})) \quad (3.5)$$

and

$$\Psi(\rho(\check{u}_t, \check{u}_{t+1})) \leq \Psi(\rho(\check{u}_t, \varphi\check{s}_{t+1})) + \sigma. \quad (3.6)$$

for all $t \geq 0$. Further, since the pair (Γ, Λ) has the modified P-property, then from Inequality (3.4) we get

$$\rho(\check{s}_t, \check{s}_{t+1}) = \rho(\check{u}_{t-1}, \check{u}_t) \quad (3.7)$$

for all $t \geq 1$. Thus, using Inequalities (3.5) and (3.6) we obtain

$$\begin{aligned} \Psi(\rho(\check{s}_t, \check{s}_{t+1})) &= \Psi(\rho(\check{u}_{t-1}, \check{u}_t)) \\ &\leq \Psi(\rho(\check{u}_{t-1}, \varphi\check{s}_t)) + \sigma \\ &\leq \Psi(\rho(\check{u}_{t-1}, \check{s}_t)) + \sigma - \tau(\rho(\check{s}_{t-1}, \check{s}_t)) \\ &\leq \Psi(\rho(\check{s}_{t-1}, \check{s}_t)) \end{aligned} \quad (3.8)$$

for all $t \geq 1$. Hence, the sequence $\{\rho(\check{s}_t, \check{s}_{t+1})\}$ is decreasing. Then, $\{\rho(\check{s}_t, \check{s}_{t+1})\}$ converges to for some $u \in [0, \infty)$. Assueme that $u > 0$. Then, from hypothesis we obtain

$$\lim_{t \rightarrow \infty} itf\tau((\check{s}_t, \check{s}_{t+1})) > \sigma,$$

and so there is $\gamma > \sigma$ and $t_0 \in \mathbb{N}$ such that

$$\tau(\check{s}_t, \check{s}_{t+1}) > \gamma$$

for all $t \geq t_0$. Therefore, for all $t \geq t_0$,

$$\begin{aligned} \Psi(\rho(\check{s}_t, \check{s}_{t+1})) &= \Psi(\rho(\check{u}_{t-1}, \check{u}_t)) & (3.9) \\ &\leq \Psi(\rho(\check{s}_{t-1}, \check{s}_t)) + \sigma - \tau(\rho(\check{s}_{t-1}, \check{s}_t)) \\ &< \Psi(\rho(\check{s}_{t-1}, \check{s}_t)) + (\sigma - \gamma) \\ &\leq \Psi(\rho(\check{s}_{t-2}, \check{s}_{t-1})) + \sigma - \tau(\rho(\check{s}_{t-2}, \check{s}_{t-1})) + (\sigma - \gamma) \\ &< \Psi(\rho(\check{s}_{t-2}, \check{s}_{t-1})) + 2(\sigma - \gamma) \\ &\vdots \\ &\leq \Psi(\rho(\check{s}_{t_0}, \check{s}_{t_0+1})) + (t - t_0)(\sigma - \gamma). \end{aligned}$$

Taking limit as $t \rightarrow \infty$ in Inequality (3.9)

$$\lim_{t \rightarrow \infty} \Psi(\rho(\check{s}_t, \check{s}_{t+1})) = -\infty,$$

and so from (Ψ'_2) and Lemma 2.25 we get

$$u = \lim_{t \rightarrow \infty} \rho(\check{s}_t, \check{s}_{t+1}) = 0$$

which is a contradiction. Therefore, we get

$$\lim_{t \rightarrow \infty} \rho(\check{s}_t, \check{s}_{t+1}) = 0.$$

From the condition (Ψ_3) , there exists $k \in (0,1)$ such that

$$\lim_{n \rightarrow \infty} \rho(\check{s}_t, \check{s}_{t+1})^k \Psi(\rho(\check{s}_t, \check{s}_{t+1})) = 0.$$

From Inequality (3.9) we have

$$\left\{ \begin{array}{l} \rho(\check{s}_t, \check{s}_{t+1})^k \Psi(\rho(\check{s}_t, \check{s}_{t+1})) \\ -\rho(\check{s}_t, \check{s}_{t+1})^k \Psi(\rho(\check{s}_0, \check{s}_1)) \end{array} \right\} \leq \rho(\check{s}_t, \check{s}_{t+1})^k t(\sigma - \gamma) \leq 0$$

for all $t \geq t_0$. Hence, we get

$$t [\rho(\check{s}_t, \check{s}_{t+1})]^k \rightarrow 0 \text{ as } t \rightarrow \infty$$

Therefore, there exists $t_1 \in \mathbb{N}$ satisfying

$$t [\rho(\check{s}_t, \check{s}_{t+1})]^k \leq 1$$

for all $t \geq t_1$, which implies that

$$\rho(\check{s}_t, \check{s}_{t+1}) \leq \frac{1}{t^{1/k}}$$

for all $t \geq t_1$. Now, let $t, m \in \mathbb{N}$ with $m > t \geq t_0$. Hence, we get

$$\begin{aligned} \rho(\check{s}_t, \check{s}_m) &\leq \rho(\check{s}_t, \check{s}_{t+1}) + \rho(\check{s}_{t+1}, \check{s}_{t+2}) + \dots + \rho(\check{s}_{m-1}, \check{s}_m) \\ &\leq \frac{1}{t^{1/k}} + \frac{1}{(t+1)^{1/k}} + \dots + \frac{1}{(m-1)^{1/k}} \\ &\leq \sum_{i=t}^{m-1} \frac{1}{i^{1/k}} \\ &\leq \sum_{i=t}^{\infty} \frac{1}{i^{1/k}} \end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}} < \infty$, $\{\check{s}_t\}$ is a Cauchy sequence in Γ . From (2.7) $\{\check{u}_t\}$ is also a Cauchy sequence in Λ . since Γ and Λ are closed subsets of the complete metric space (\mathfrak{S}, ρ) ,

there exist $\check{s}^* \in \Gamma$ and $\check{u}^* \in \Lambda$ such that $\check{s}_t \rightarrow \check{s}^*$ and $\check{u}_t \rightarrow \check{u}^*$ as $t \rightarrow \infty$. Taking the limits as $t \rightarrow \infty$ in (2.4) we have

$$\rho(\check{s}^*, \check{u}^*) = \rho(\Gamma, \Lambda), \quad (3.10)$$

and so we get $(\check{s}^*, \check{u}^*) \in \Delta$. Moreover, from (2.8), (Ψ'_2) and lemma ???, we get

$$\lim_{t \rightarrow \infty} \Psi(\rho(\check{u}_t, \varphi\check{s}_{t+1})) = -\infty,$$

and so

$$\lim_{t \rightarrow \infty} \rho(\check{u}_t, \varphi\check{s}_{t+1}) = 0 \quad (3.11)$$

Now, since the function $f(\check{s}, \check{u}) = \rho(\check{u}, \varphi\check{s})$ is lower semi-continuous on Δ , $\check{s}_{t+1} \rightarrow \check{s}^*$ and $\check{u}_t \rightarrow \check{u}^*$ as $t \rightarrow \infty$, from Equation (3.11) we obtain

$$\rho(\check{u}^*, \varphi\check{s}^*) = f(\check{s}^*, \check{u}^*) \leq \liminf_{t \rightarrow \infty} f(\check{s}_{t+1}, \check{u}_t) = \lim_{t \rightarrow \infty} \rho(\check{u}_t, \varphi\check{s}_{t+1}) = 0,$$

and so $\check{u}^* \in \overline{\varphi\check{s}^*} = \varphi\check{s}^*$. Therefore, from Equation (3.10) we have

$$\rho(\Gamma, \Lambda) \leq \rho(\check{s}^*, \varphi\check{s}^*) \leq \rho(\check{s}^*, \check{u}^*) = \rho(\Gamma, \Lambda).$$

So, we get

$$d(x^*, Tx^*) = d(A, B),$$

that is, x^* is a best promixity point of T .

We obtain the generalized result below by using Theorem 3.2. The result generalizes the main result of Sahin (2022).

Corollary 3.3. Let $\varphi : \Gamma \rightarrow \mathcal{C}(\Lambda)$ be a mapping where (\mathfrak{S}, ρ) is a complete metric space and Γ, Λ are closed non-empty subsets closed having modified P -property. Assume that $\Gamma_0 \neq \emptyset$, $\tau > 0$ and $\Psi \in {}^\rho_{\varphi}\mathcal{F}_{**}$. Assume that for all $\check{s} \in \Gamma_0$ with $\rho(\check{s}, \varphi\check{s}) > \rho(\Gamma, \Lambda)$ and $\check{u} \in \varphi\check{s}$, there exist $z \in \Gamma_0$ such that

$$\rho(\check{u}, z) = \rho(\Gamma, \Lambda),$$

and

$$\tau + \Psi(\rho(\check{u}, \varphi z)) \leq \Psi(\rho(\check{s}, z)).$$

If $f(\check{s}, \check{u}) = \rho(\check{u}, \varphi\check{s})$ is lower semi-continuous on Δ , then φ has a best proximity point in Γ .

Proof. To show the existence by Theorem 3.2, it is enough to demonstrate that the contraction condition of Theorem 3.2 is satisfied. Now, let $\check{s} \in \Gamma_0$ with $\rho(\check{s}, \varphi\check{s}) > \rho(\Gamma, \Lambda)$ and $\check{u} \in \varphi\check{s}$ be arbitrary points. Then, we say that there exists $z \in \Gamma_0$ such that

$$\rho(\check{u}, z) = \rho(\Gamma, \Lambda),$$

and

$$\tau + \Psi(\rho(\check{u}, \varphi z)) \leq \Psi(\rho(\check{s}, z)). \quad (3.16)$$

On the other hand, since $\tau > 0$, there exist a real number σ such that $0 < \sigma < \tau$. Now, if we define the mapping $\tau' : (0, \infty) \rightarrow (\sigma, \infty)$ by $\tau'(t) = \tau$ for all $t > 0$. Then the Inequality (3.1) is satisfied. Also, from Inequality (3.16) we have

$$\tau'(\rho(\check{s}, z)) + \Psi(\rho(\check{u}, \varphi z)) \leq \Psi(\rho(\check{s}, z)),$$

and so from Theorem 3.2, the mapping φ has a best proximity point in Γ .

Theorem 3.2 is a proper generalization of Corollary 3.3, as demonstrated by the example that follows.

Example 3.4. Let $\mathfrak{S} = \mathbb{R}^2$ and the function $\rho: \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}$ be the taxi-cab metric; that is; for all $\check{s} = (\check{s}_1, \check{s}_2), \check{u} = (\check{u}_1, \check{u}_2) \in \mathfrak{S}$

$$\rho(\check{s}, \check{u}) = |\check{s}_1 - \check{u}_1| + |\check{s}_2 - \check{u}_2|.$$

If we take the subsets

$$\Gamma = \left\{0, \frac{1}{t} : t \in \mathbb{N}\right\} \times \{0\}$$

and

$$\Lambda = \left\{0, \frac{1}{t} : t \in \mathbb{N}\right\} \times \{1\}$$

then, Γ and Λ are closed subsets. Moreover, $\rho(\Gamma, \Lambda) = 1$, $\Gamma_0 = \Gamma$, $\Lambda_0 = \Lambda$, and the pair (Γ, Λ) has the modified P -Property. Consider $\Psi: \mathbb{R}^+ \rightarrow \mathbb{R}$ and $\varphi: \Gamma \rightarrow C(\Lambda)$ as $\Psi(\alpha) = \ln \alpha$ and

$$\varphi^{\check{s}} = \begin{cases} \Lambda & , \quad \check{s} = (0,0) \\ \left\{ \left(\frac{1}{t+1}, 0 \right) \right\} & , \quad \check{s} = \left(\frac{1}{t}, 0 \right), t \in \mathbb{N} \end{cases}.$$

Choose $\sigma = 0$ and define the function $\tau: (0, \infty) \rightarrow (0, \infty)$ by

$$\tau(t) = \begin{cases} \ln\left(\frac{t+1}{t}\right), & t = \frac{1}{t(t+1)}, t \in \mathbb{N} \\ 1, & \text{otherwise} \end{cases}$$

Then, we have $\Psi \in {}^\rho\mathcal{F}_{**}$ and $\lim_{t \rightarrow s^+} \tau(t) = 1 > 0 = \sigma$ for all $s > 0$. We also have $f(\check{s}, \check{u}) = \rho(\check{u}, \varphi\check{s})$ is lower semi-continuous on Δ . Now we want to show that φ is a *KW*-type Ψ -contraction mapping. Let $\check{s} = \left(\frac{1}{t}, 0\right)$ for arbitrary $t \in \mathbb{N}$ and choose $\check{u} \in \left\{\left(\frac{1}{t+1}, 1\right)\right\} = \varphi\check{s}$. Then, it has to be $z = \left(\frac{1}{t+1}, 0\right)$, and so $\varphi z = \left\{\left(\frac{1}{t+2}, 1\right)\right\}$. In this case it is satisfied

$$\begin{aligned} \tau\left(\frac{1}{t(t+1)}\right) + \Psi\left(\frac{1}{(t+1)(t+2)}\right) &= \ln\left(\frac{t+1}{t}\right) + \ln\left(\frac{1}{(t+1)(t+2)}\right) \\ &= \ln\left(\frac{1}{t(t+2)}\right) \\ &\leq \ln\left(\frac{1}{t(t+1)}\right) \\ &= \Psi\left(\frac{1}{t(t+1)}\right) \end{aligned}$$

for all $t \in \mathbb{N}$. Hence, all conditions of Theorem 3.2 hold; so, Hence, φ has a best proximity point in Γ . However, Theorem 2.24 is not applied to this example because of the fact that the contraction condition of Corollary 3.3 is not satisfied. Assume the contrary, that is, there exist $\tau > 0$ such that for all $\check{s} \in \Gamma_0$ with $\rho(\check{s}, \varphi\check{s}) > \rho(\Gamma, \Lambda)$ and $\check{u} \in \varphi\check{s}$ there exists $z \in \Gamma_0$ such that

$$\rho(\check{u}, z) = \rho(\Gamma, \Lambda)$$

and

$$\tau + \Psi(\rho(\check{u}, \varphi z)) \leq \Psi(\rho(\check{s}, z)).$$

In this case, for $\check{s} = \left(\frac{1}{t}, 0\right)$, $t \in \mathbb{N}$ and choose $\check{u} \in \left\{\frac{1}{t+1}, 0\right\} = \varphi\check{s}$, Then, it is has to be $z = \left(\frac{1}{t+1}, 0\right)$, and so $\varphi z = \left\{\frac{1}{t+2}, 0\right\}$. In this case, it is satisfied

$$\tau + \Psi\left(\frac{1}{(t+1)(t+2)}\right) \leq \Psi\left(\frac{1}{t(t+1)}\right),$$

which implies that

$$\tau \leq \ln\left(\frac{t+2}{t}\right)$$

for all $t \in \mathbb{N}$. Taking the limit as $t \rightarrow \infty$, we have

$$\tau \leq 0$$

which is a contraction.

If we define function $\varphi : [0, \infty) \rightarrow [0, 1)$ by

$$\varphi(\ell) = \begin{cases} 0 & , \ell = 0 \\ \ell^{-\tau(\ell)} & , \ell > 0 \end{cases}$$

via the function τ in Theorem 3.2, then we have $\limsup_{\ell \rightarrow s^+} \varphi(\ell) < 1$ for all $s < 0$. Taking into account the function $\Psi: R^+ \rightarrow R$ by $\Psi(\alpha) = \ln \alpha$ in theorem 6, we also get, for all $\check{s} \in \Gamma_0$ and $\check{u} \in \varphi\check{s}$, there exists $z \in \Gamma_0$ satisfying

$$\rho(\check{u}, z) = \rho(\Gamma, \Lambda)$$

and

$$\rho(\tilde{u}, z) \leq \varphi(\rho(\check{s}, z))\rho(\check{s}, z).$$

Therefore, we can obtain the following best proximity point version of the fixed-point result of Klim and Wardowski with the help of Theorem 3.2.

Corollary 3.5. Let (\mathfrak{S}, ρ) be a complete metric space, $\varphi: \Gamma \rightarrow \mathcal{C}(\Lambda)$ be mapping where Γ, Λ are closed non-empty subsets having a modified p-property and $\Gamma_0 \neq \emptyset$. if there exist $\varphi: [0, \infty) \rightarrow [0, 1)$, such that, all $s > 0$

$$\lim_{\ell \rightarrow s^+} \sup \varphi(\ell) < 1 \tag{3.17}$$

and there exists $z \in \Gamma_0$ for all $\check{s} \in \Gamma_0$ and $\tilde{u} \in \varphi\check{s}$ satisfying

$$\rho(\tilde{u}, z) = \rho(\Gamma, \Lambda)$$

and

$$\rho(\tilde{u}, \varphi z) \leq \varphi(\rho(\check{s}, z))\rho(\check{s}, z),$$

and $f(\check{s}, \tilde{u}) = \rho(\tilde{u}, \varphi\check{s})$ is lower semi-continuous on Δ , then φ has a best proximity point in Γ .

The following theorem generalizing the finding of Altun *et al.* (2016) is reached when $\Gamma = \Lambda = \mathfrak{S}$ in Theorem 3.2.

Corollary 3.6. Let $\varphi: \mathfrak{S} \rightarrow \mathcal{C}(\mathfrak{S})$ be a multivalued mapping on a complete metric space (\mathfrak{S}, ρ) and $\Psi \in {}^{\rho}\mathcal{F}_{**}$. Assume that $f(\check{s}, \check{s}) = \rho(\check{s}, \varphi\check{s})$ is lower semi-continuous and there exist $\sigma > 0$ and $\tau: (0, \infty) \rightarrow (\sigma, 1)$, such that, for all $s > 0$

$$\liminf_{\ell \rightarrow s^+} \tau(\ell) > \sigma$$

and, for all $\check{s} \in \mathfrak{S}$ with $\rho(\check{s}, \varphi\check{s}) > 0$ and $\check{u} \in \varphi\check{s}$, it is satisfied

$$\tau(\rho(\check{s}, \check{u})) + \Psi(\rho(\check{u}, \varphi\check{u})) \leq \Psi(\rho(\check{s}, \check{u})).$$

Then, φ has a fixed point in \mathfrak{S} .



4. AN APPLICATION TO HOMOTOPY THEORY

Because homotopy theory and other disciplines of mathematics are closely related, there has been a recent upsurge in interest in this field. In light of this, numerous mathematicians have applied homotopy by drawing upon their fixed point findings (Abbas *et al.* 2019, Muhammed and Kumam 2019, O'Regan 2019). Taking into account this approach, We demonstrate how our top proximity point result applies to homotopy mappings. Now, we recall some basic concepts of this theory:

Definition 4.1. Let (\mathfrak{S}_1, τ_1) and (\mathfrak{S}_2, τ_2) be topological spaces, $\varphi, \Psi : \mathfrak{S}_1 \rightarrow \mathfrak{S}_2$ be continuous mappings. If there exists continuous function $H : \mathfrak{S}_1 \times [0,1] \rightarrow \mathfrak{S}_2$ such that $H(\check{s}, 0) = \varphi\check{s}$ and $H(\check{s}, 1) = \Psi\check{s}$ for all $\check{s} \in \mathfrak{S}_1$, then the mapping H is said to be homotopy.

The following definition was given in (Vetro 2015).

Definition 4.2. Let Γ be non-empty subset of a metric space (\mathfrak{S}, ρ) and $H : \Gamma \times [0,1] \rightarrow C(\Gamma)$ be a mapping. H is called closed multivalued mapping,

if $G(H)$ is a closed subset of $(\Gamma \times [0,1] \times \Gamma, \rho^*)$, then

We modified Definition 4.2 as follows.

Definition 4.3. Let $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{S}$ where (\mathfrak{S}, ρ) is a metric space, and $H : \Gamma \times [0,1] \rightarrow C(\Lambda)$ be a multivalued mapping. Then, H is said to be a ρ -closed multivalued mapping if

$$G_\rho(H) = \{(\check{s}, \beta, \check{u}) : \check{s} \in \Gamma, \beta \in [0,1] \text{ and } \rho(\check{u}, H(\check{s}, \beta)) = \rho(\Gamma, \Lambda)\}.$$

is a closed subset of $(\Gamma \times [0,1] \times \Lambda, \rho^*)$ where

$$G(H) = \{(\check{s}, \beta, \check{u}) : \check{s} \in \mathfrak{S}, \beta \in [0, 1] \text{ and } \check{u} \in H(\check{s}, \beta)\}$$

and

$$\rho^* \left((\check{s}_1, \beta_1, \check{u}_1), (\check{s}_2, \beta_2, \check{u}_2) \right) = \rho(\check{s}_1, \check{s}_2) + |\beta_1 - \beta_2| + \rho(\check{u}_1, \check{u}_2)$$

for all $(\check{s}_1, \beta_1, \check{u}_1) + (\check{s}_2, \beta_2, \check{u}_2) \in \Gamma \times [0, 1] \times \Gamma$.

Notice, if we take $\Gamma = \Lambda$ in Definition 4.3, then Definition 4.3 turns to Definition 4.2.

The following result is the main result of this section

Theorem 4.4. Let $\emptyset \neq \Gamma, \Lambda, \subseteq \mathfrak{S}$ where (\mathfrak{S}, ρ) is a complete metric space and Γ, Λ are closed, $\emptyset \neq U \subseteq \Gamma$ and $\Gamma_0 \neq \emptyset$. Suppose that the pair (Γ, Λ) has the modified P -Property. $H : \Gamma \times [0, 1] \rightarrow \mathcal{C}(\Lambda)$ is a closed multivalued mapping and $\Psi \in_{H(\cdot, \gamma)} \mathcal{F}_{**}^\rho$ for all $\gamma \in [0, 1]$. Assume that the following conditions are satisfied:

- i) $\rho(\check{s}, H(\check{s}, \beta)) > \rho(\Gamma, \Lambda)$ for all $\check{s} \in \Gamma \setminus U$ and $\beta \in [0, 1]$,
- ii) there exist $\sigma \geq 0$ and $\tau : (0, \infty) \rightarrow (\sigma, \infty)$ such that for all $s > 0$

$$\liminf_{t \rightarrow s^+} \tau(t) > \sigma$$

and for all $\check{s} \in \Gamma$ and $\beta \in [0, 1]$ with $\rho(\check{s}, H(\check{s}, \beta)) > \rho(\Gamma, \Lambda)$ and $\check{u} \in H(\check{s}, \beta)$

there exist $z \in U$ satisfying

$$\rho(\check{u}, z) = \rho(\Gamma, \Lambda)$$

and $\tau(\rho(\check{s}, z)) + \Psi(\rho(\check{u}, H(z, \beta))) \leq \Psi(\rho(\check{s}, z))$,

- (iii) for all $\beta \in (0, 1)$ the function $f : \Gamma \times \Lambda \rightarrow [0, \infty)$ defined by $f(\check{s}, \check{u}) =$

$\rho(\check{u}, H(\check{s}, \beta))$ is lower somecontinuous on Δ , where

$$\Delta = \{(\check{s}, \check{u}) \in \Gamma \times \Lambda : \rho(\check{s}, \check{u}) = \rho(\Gamma, \Lambda)\}.$$

(iv) if $\rho(\check{s}, H(\check{s}, \beta)) = \rho(\Gamma, \Lambda)$ for some $\check{s} \in \Gamma$ and $\beta \in [0, 1]$ then $H(\check{s}, \beta)$ is singleton.

If $H(\cdot, 0)$ has a best proximity point in Γ , then $H(\cdot, 1)$ has a best proximity point in Γ .

Proof. Consider the following set

$$K = \{(\beta, \check{s}) : \rho(\check{s}, H(\check{s}, \beta)) = \rho(\Gamma, \Lambda)\} \subseteq [0, 1] \check{U}.$$

Since there is a point \check{s} in U such that $\rho(\check{s}, \psi(\check{s}, 0)) = \rho(\Gamma, \Lambda)$, we get $H \neq \emptyset$. Now, consider the following partial order on H

$$(\gamma, \check{s}) \preceq (\mu, \check{u}) \Leftrightarrow \gamma \leq \mu \text{ and } \rho(\check{s}, \check{u}) \leq \mu - \gamma.$$

Assume that K is a totally ordered subset of H and $\gamma^* = \sup\{\gamma : (\gamma, \check{s}) \in K\}$. Consider $\{(\gamma_t, \check{s}_t)\} \subseteq K$ with $(\gamma_t, \check{s}_t) \preceq (\gamma_{t+1}, \check{s}_{t+1})$ for all $t \in N$ and $\gamma_t \rightarrow \gamma^*$ as $t \rightarrow \infty$. So, we get

$$\rho(\check{s}_t, \check{s}_m) \leq \gamma_m - \gamma_t,$$

for all $t, m \in N$ with $m > t$. Therefore, we have $\{\check{s}_t\} \subseteq U$ is a Cauchy sequence. There exists $\check{s}^* \in \Gamma$, such that $\check{s}_t \rightarrow \check{s}^*$ as $t \rightarrow \infty$, since Γ is a closed subset of complete metric space (Λ, ρ) . In addition, we have $(\check{s}_t, \gamma_t, \check{s}_t) \in G_\rho(\psi)$ and $(\check{s}_t, \gamma_t, \check{s}_t) \xrightarrow{\rho^*} (\check{s}^*, \gamma^*, \check{s}^*)$ as $t \rightarrow \infty$. Since ψ is a closed multivalued mapping, we get

$$\rho(\check{s}^*, \psi(\check{s}^*, \gamma^*)) = \rho(\Gamma, \Lambda).$$

From (i), we have $\check{s}^* \in \mathbf{U}$, and so $(\gamma^*, \check{s}^*) \in H$. We have $(\gamma, \check{s}) \preceq (\gamma^*, \check{s}^*)$ for all $(\gamma, \check{s}) \in K$, since K is totally ordered. Hence, K has an upper bound (γ^*, \check{s}^*) . Therefore, we conclude H has a maximal element $(\gamma_0, \check{s}_0) \in H$ using the Zorn Lemma. We claim that $\gamma_0 = 1$. If we suppose the contrary, then there is γ with $\gamma_0 < \gamma < 1$. Let $R = \gamma - \gamma_0 > 0$. Hence, considering (ii) and (iii), we say that $f: \bar{\Lambda}(\check{s}_0, M) \times \Lambda \rightarrow [0, \infty)$ defined by $f(\check{s}, \check{u}) = \rho(\check{u}, \psi(\check{s}, \gamma))$ is lower semi-continuous on Δ and the mapping $\psi(\cdot, \gamma): \bar{\Lambda}(\check{s}_0, M) \rightarrow \mathcal{C}(\Lambda)$ is a KW -type Ψ -contraction mapping. Therefore, we obtain that $\psi(\cdot, \gamma)$ has a best proximity point \check{s}_γ in $\bar{\Lambda}(\check{s}_0, M)$ using Theorem 6. From (i), $\check{s}_\gamma \in U$, and so, $(\gamma, \check{s}_\gamma) \in H$, which contradicts that (γ_0, \check{s}_0) is a maximal element of H . So, $\gamma_0 = 1$ and $\psi(\cdot, 1)$ has a best proximity point \check{s}_0 in Γ .

5. CONCLUSIONS

In this thesis, we sought to combine the approaches of Wardowski and Klim–Wardowski. Hence, we first introduced a new type of F-contraction, named KW-type F-contraction mapping. Then, we investigated the conditions of presence of a best proximity point for these mappings by considering a new family ${}^{\rho}_H\mathcal{F}_{**}$ which was larger than the family of functions \mathcal{F}_* . Moreover, we provided an example where our results can be applied, but to which the results in the literature cannot be applied. Finally, considering a new trend towards homotopy theory, we present an application for homotopic mappings.



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