INVESTIGATION OF ADIABATIC MODES AND SUPER CHARGES USING SOFT THEOREMS AND SYMMETRIES IN ASYMPTOTICALLY FLAT SPACETIMES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
PHYSICS

JULY 2023

Approval of the thesis:

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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ABSTRACT

INVESTIGATION OF ADIABATIC MODES AND SUPER CHARGES USING SOFT THEOREMS AND SYMMETRIES IN ASYMPTOTICALLY FLAT SPACETIMES

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July 2023, 85 pages

This thesis aims to investigate and review the connection between adiabatic modes, Ward identities and Weinberg's soft theorem in the context of asymptotic symmetries and the Bondi-Metzner-Sachs group. Adiabatic modes are low-energy excitations related to the spontaneous breaking of continuous symmetries, while Ward identities are constraints on physical quantities arising from the existence of certain symmetries. Weinberg's soft theorem, on the other hand, relates the behaviour of low-energy particles in the presence of spontaneously broken symmetries to the symmetries themselves. The main focus will be the examination of how Weinberg's soft theorem can be produced by the use of the symmetries of the Bondi-Metzner-Sachs group, including the supertranslations and superrotations, to gain insights into the behaviour of low-energy particles in gravity.

Keywords: soft theorem, supertranslation, superrotation, asymptotic spacetime, memory effect

ÖZ

ADİYABATİK MODLARIN VE SÜPER YÜKLERİN SOFT TEOREMLER VE ASİMPTOTİK DÜZ UZAYZAMAN SİMETRİLERİ KULLANILARAK İNCELENMESİ

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Temmuz 2023 , 85 sayfa

Bu tezin hedefi asimptotik simetriler ve Bondi-Metzner-Sachs grubu bağlamında adiyabatik modlar, Ward Özdeşlikleri ve Weinberg'in soft teoremi arasındaki ilişkiyi incelemektir. Adiyabatik modlar kendiliğinden simetri kırılması ile ilişkili düşük enerji uyarılmaları iken, Ward Özdeşlikleri bazı simetrilerin varlığından dolayı oluşan fiziksel büyüklüklerin üzerindeki kısıtlamalardır. Öte yandan, Weinberg'in soft teoremi, kendiliğinden simetri kırılmasının varlığındaki düşük enerjili parçacıkların davranışlarını simetrilerin kendileriyle ilişkilendirir. Kütleçekimdeki düşük enerjili parçacıkların davranışları üzerine bir yaklaşım elde edebilmek için tezin odak noktası süperötelemeleri ve süperdönmeleri içeren Bondi-Metzner-Sachs grubunun simetrilerini kullanarak Weinberg'in soft teoremini türetmek olacaktır.

Anahtar Kelimeler: soft teorem, süperöteleme, süperdönme, asimptotik uzayzaman, hafıza etkisi

To my beloved companion Datça

ACKNOWLEDGMENTS

This thesis is a product of a long solitary but delightful scientific adventure. Looking back to my thesis work, with the freedom that I had, I can't keep myself away from imagining a bird soaring over a boundless ocean by itself, with no weight, exhilarated by the countless dazzling waves.

I would like to thank my advisor, Bayram Tekin, for introducing me to such amazing ideas in quantum cosmology, which later turned out to be the essential parts of this thesis, and hopefully to future projects. I would also like to show my gratitude for his warm personality and wonderful lectures which kept the scientific passion blazing in me throughout my undergraduate and graduate years. For their detailed suggestions and corrections, I would like thank to Emine Şeyma Kutluk, Bayram Tekin and Arev Pelin Sümer. Finally, I would like to thank my family for their endless support and encouragement to keep my dreams alive and for being there for me when life is grey.

TABLE OF CONTENTS

LIST OF FIGURES

FIGURES

LIST OF ABBREVIATIONS

AdS	Anti de Sitter
BMS	Bondi-Metzner-Sachs
CFT	Conformal Field Theory
DOF	Degrees of Freedom
EM	Electromagnetic
FLRW	Friedmann–Lemaître–Robertson–Walker
GR	General Relativity
NG	Nambu-Goldstone
QED	Quantum Electrodynamics
QFT	Quantum Field Theory

CHAPTER 1

INTRODUCTION

Adiabatic modes can be viewed as large diffeomorphisms that can be represented (locally) by physical perturbations with long wavelengths (Mirbabayi & Simonović, 2016). One can define them in electrodynamics and gravity in an asymptotically flat spacetime. These are residual gauge transformations, which after local gauge fixing stay unfixed. They also give rise to soft theorems which can be originated from Ward identities (Ward, 1950) of spontaneously broken asymptotic symmetry groups. It is also crucial to know that to be able to talk about a physical observable, the asymptotic Ward identity must be expressible as the limit of a conservation law in terms of quantities at a finite distance. However, adiabatic modes grow with radius r and hence have a different r-dependence compared to both radiation and the large gauge transformations of asymptotic conservation laws. It is also well known that there is a strong connection between adiabatic modes and soft theorems even though they seem like very distinct fields.

There are several reasons for employing soft theorems in the study of gravity. One reason is that they provide a way to assess the consistency of perturbative quantum gravity. In perturbative quantum gravity, scattering amplitudes are calculated using Feynman diagrams. These diagrams are constructed using perturbation theory, which involves expanding the scattering amplitudes in a series of terms, each multiplied by a small coupling constant. However, this expansion is not always clear cut, making it challenging to determine the behaviour of scattering amplitudes at low energies. Soft theorems offer a way to test the consistency of perturbative quantum gravity by relating the scattering amplitudes of low-energy particles to the symmetries of the asymptotic region of spacetime. If the scattering amplitudes of these particles do not satisfy the constraints imposed by the soft theorems, it indicates that the perturbative expansion is not well-defined and the theory is not consistent. In addition to providing a way to test the consistency of perturbative quantum gravity, soft theorems also have important implications for one's understanding of the structure of QFT in the presence of gravity. These theorems allow one to connect the behaviour of low-energy particles to the symmetries of the asymptotic region of spacetime, which can help with the understanding of the role of these symmetries in the structure of the universe. Equipped with this, it is easy to see that in cosmology, for each adiabatic mode, there is a locally conserved current and by using the conservation of this associated current one can also derive the Weinberg soft theorem (Weinberg, 2003) corresponding to each current. Weinberg's soft theorem connects the scattering amplitudes of particles with infinitesimal energies to the symmetries of the asymptotic region of spacetime. This theorem had a vital part in the comprehension of the QFT in the presence of gravity and has prompted the development of new approaches to perturbative quantum gravity. It will be evident in the upcoming chapters of this thesis that there are indeed infinitely many adiabatic modes but elegantly they resolve into the same leading soft theorem.

This study will also shed light on the role of symmetry in the description of gravity and its possible connection to the emergence of Goldstone bosons and soft modes (Goldstone, 1961; Goldstone et al., 1962). Soft modes, also known as Nambu-Goldstone modes, are low-energy excitations that arise in structures which contains spontaneously broken global symmetries. In the context of asymptotic symmetry and the BMS group (Bondi et al., 1962), these modes can be understood as the low-energy excitations associated with the spontaneously broken symmetries generated by supertranslations and superrotations. The relationship between soft modes and Weinberg's soft theorem, which connects the scattering amplitudes of particles with arbitrarily low energies to the symmetries of the asymptotic region of spacetime, can be understood by considering the role of these symmetries in the behaviour of low-energy particles.

1.1 Intuitive Picture for the BMS group and Supertranslations

Asymptotic symmetries and their associated group, the BMS group, have long been a focus of study in theoretical physics. Today, it is known that these symmetries pertain to asymptotically flat spacetimes and are generated by supertranslations and superrotations, which maintain the asymptotic nature of the spacetime metric.

The BMS group, also as a subgroup involves the Poincare' group. Hence, the BMS group has been demonstrated to be significant in the structure of the asymptotic region of spacetime and has been utilized in formulating a consistent theory of scattering in gravity.

For any asymptotically flat system, Bondi coordinates exist in a neighbourhood of $\mathcal{I}^+$(the future boundary for null geodesics) and any two Bondi coordinate systems are related by some BMS transformation, indicating that the BMS group entails all possible gauge transformations required to discuss the limits of asymptotically flat spacetime (Boyle, 2016).

The short discussion here will be built on inertial emitters in Minkowski space. So one has translations of both time and space. Generalizing these, the structure of supertranslations will be constructed. Starting with the time translations, consider an emitter $\mathcal{Y}$ with the proper time $\tau_{\mathcal{Y}}$ with the assigned retarded time $u_{\mathcal{Y}} = \tau_{\mathcal{Y}}$ and a similar construction for the emitter $\mathcal{P}$ follows as $u_{\mathcal{P}} = \tau_{\mathcal{P}}$ (see Figure 1.1).

Now, if the emitters' time scales are related by a simple time translation such that $\tau_{\mathcal{P}} = \tau_{\mathcal{Y}} - \delta t$, there exists a relation between the retarded times as

$$u_{\mathcal{P}} = u_{\mathcal{Y}} - \delta t. \tag{1.1}$$

In terms of spherical coordinates, this can also be written as

$$u \to u + \alpha^{0,0} Y_{0,0}(\theta, \phi), \tag{1.2}$$

where $\alpha^{0,0}$ is a constant. The change in the retarded-time coordinate does not depend

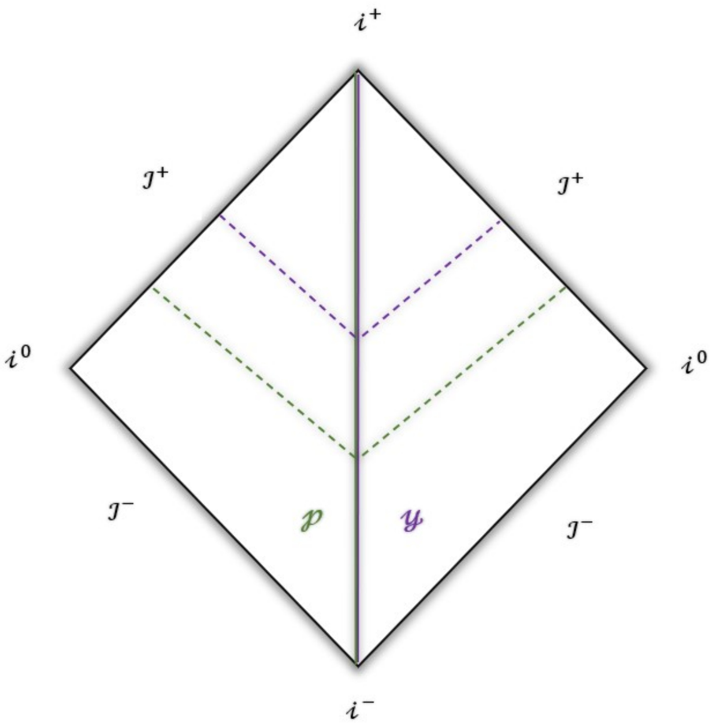

Figure 1.1: $\mathcal{Y}$ and $\mathcal{P}$ represent the same emitter with the same spatial coordinates but for time coordinates they have different origins. The two null cones (dashed lines) correspond to the two origins of the time coordinate.

on the direction (l=0), in other words, it is isotropic. The same argument is not valid for space translations and generalizing this idea will be the essential part to grasp the structure of supertranslations.

Now considering the space translations, let the emitter $\mathcal{P}$ be displaced from the emitter $\mathcal{Y}$ by δx, but let them be stationary with respect to each other. The null cone $\mathcal{N}_{\mathcal{P}}$ originates from the origin of $\mathcal{P}$ and intersects $\mathcal{I}^+$ at two points. These same points are on null rays from two separate null cones of $\mathcal{Y}$.

The relationship between the retarded time coordinates for any point on S^2 is

$$u_{\mathcal{P}} = u_{\mathcal{Y}} + \delta x \cdot r. \tag{1.3}$$

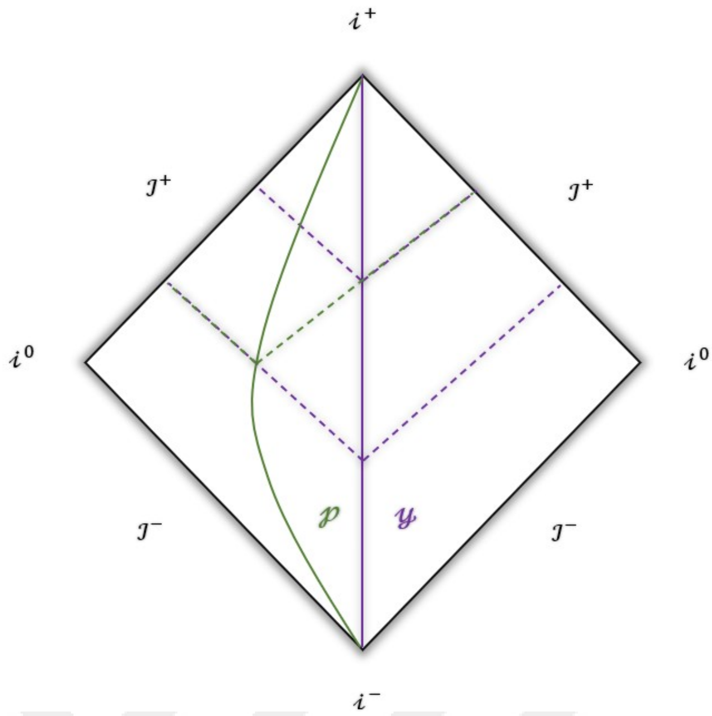

Figure 1.2: $\mathcal{Y}$ and $\mathcal{P}$ represent emitters that are displaced relatively. The two intersection points of the green dashed line and $\mathcal{I}^+$ are found on two separate null cones emitted by $\mathcal{Y}$.

In terms of spherical coordinates, this can also be written as

$$u \to u + \sum_{m=-1}^{1} \alpha^{1,m} Y_{1,m}(\theta, \phi). \tag{1.4}$$

Fusing these two transformation laws into a single law for general spacetime translations gives

$$u_{\mathcal{P}} = u_{\mathcal{Y}} - \sum_{l \in \{0,1\}} \sum_{m=-l}^{l} \alpha^{l,m} Y_{l,m}(\theta, \phi), \tag{1.5}$$

where

$$
\begin{aligned}
\alpha^{0,0} &= \sqrt{4\pi}\delta t, \\
\alpha^{1,-1} &= -\sqrt{\frac{2\pi}{3}}(\delta x + i\delta y), \\
\alpha^{1,0} &= -\sqrt{\frac{4\pi}{3}}\delta z, \\
\alpha^{1,1} &= -\sqrt{\frac{2\pi}{3}}(-\delta x + i\delta y).
\end{aligned} \tag{1.6}
$$

Here $(\delta x, \delta y, \delta z)$ is a spatial vector (Boyle, 2016), suggesting that the final generalization is the expansion of the range of the sum over l to all positive integers, while keeping the retarded time coordinate real. More specifically, the final generalization can be obtained by making a coordinate transformation such that

$$
u^{'} = u - \alpha(\theta, \phi), \tag{1.7}
$$

where α is any real-valued function on the S^2. Conventionally this transformation is referred to as supertranslations and they are also symmetries of the asymptotic metric. Obviously, it's not generally a symmetry of the metric in the interior of the spacetime, but something important happens once one gets to the $\mathcal{I}^+$. In the upcoming chapters, one will see that this behaviour at the null infinity gives rise to many other physically enticing ideas.

CHAPTER 2

ANALYSIS AND RESULTS

2.1 The Theory

2.1.1 The Symmetry

In field theory, symmetry transformations are transformations $\triangle\phi$ that leave the action invariant, or on the same footing, change the $\mathcal{L}$ by $\triangle\mathcal{L} = \partial_\mu F^\mu$. If one introduces two different states that digress by a symmetry transformation as the same physical state, then this symmetry is considered as a gauge symmetry. On the other hand, If $\phi^{'}_{sol}$ and ϕ_{sol} are physically distinguishable, the transformation is called a global symmetry where $\phi^{'}_{sol} = \phi_{sol} + \triangle\phi_{sol}$.

If the generator Q generates new solutions, it indicates that this generator satisfies the relation $[Q, H] = 0$ (one can think of this bracket as a Poisson bracket in classical theory and as a commutator in quantum theory), where H is the Hamiltonian. There exists a conserved current by the Noether theorem

$$J^\nu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\Delta\phi - F^\mu, \tag{2.1}$$

with $\partial_\mu J^\mu = 0$. Then the corresponding conserved charge is

$$Q = \int_\Sigma \sqrt{h} J^\mu n_\mu d^3x, \tag{2.2}$$

where n^μ is a timelike vector field which identifies the hypersurface Σ which one integrates over. Q generates transformations of the fields as

$$i[Q, \phi] = \Delta\phi. \tag{2.3}$$

2.1.2 Linearly Realized Symmetries

By quantizing the theory and turning Q and ϕ into operators one can see that the symmetry induced by Q is unbroken in the $|\Omega\rangle$ if and only if

$$\langle\Omega| [Q, \phi] |\Omega\rangle = 0. \tag{2.4}$$

When expanding $\Delta\phi$, one has to begin with a linear term in ϕ but it cannot consists of a constant term. Assuming that $\Delta\phi$ is precisely linear in ϕ, that is

$$\Delta\phi_a = D_{ab}\phi_b, \tag{2.5}$$

where D_{ab} is a group of matrices representing the symmetry group. All linearly realized symmetries take single-excited states into other single-excited states. That is, if $|0\rangle$ is the vacuum, $\phi_a(x)|0\rangle$ is mapped into

$$Q\phi_a(x)|0\rangle = [Q, \phi_a(x)]|0\rangle = -iD_{ab}\phi_b(x)|0\rangle. \tag{2.6}$$

The energy of the new excited state has to be the with the energy of the original one (since Q commutes with H). Q is said to be an internal symmetry, if D_{ab} has number (real or complex) entries, Q is said to be a spacetime symmetry if D_{ab} contains functions or derivatives of spacetime (Pajer, 2018).

2.1.3 Non-Linearly Realized Symmetries

In QFT one can define spontaneously broken symmetry as

$$\langle\Omega| [Q, \phi] |\Omega\rangle \neq 0. \tag{2.7}$$

It is possible to work with fields with vanishing expectation values by using a field redefinition such as $\phi \to \phi - \langle\phi\rangle$. Then in terms of these fields, the broken symmetry transformation has to involve a constant term. So, it is understood that a spontaneously broken symmetry has to be non-linearly realized

$$i[Q, \phi] = \Delta\phi = const + \mathcal{O}(\phi). \tag{2.8}$$

In other words, one can think about non-linearly realized transformation as a transformation that functions non-linearly on the solutions of the theory, meaning given the two solutions $\phi_{sol,1} = \phi_{sol,2}$ one finds $\Delta\phi_{sol,1} \neq \Delta\phi_{sol,2}$ (Pajer, 2018). Then it is obvious that there exists a degenerate vacuum since both states $|\alpha\rangle = U(\alpha)|\Omega\rangle$ and $|\Omega\rangle$ have the equal energy where $U(\alpha)$ is the unitary symmetry operator for the transformation parameter α.

2.2 Adiabatic Modes in Gravity

It is known that general relativity is invariant under infinitesimal diffeomorphisms when the background is Minkowski spacetime. Defining the canonically normalized metric fluctuation by

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{2.9}$$

where $\kappa^2 = 32\pi G$ one sees that under a diffeomorphism ξ^μ, it transforms non-linearly as

$$h_{\mu\nu} \to h_{\mu\nu} + \kappa^{-1}\partial_\mu\xi_\nu + \kappa^{-1}\partial_\nu\xi_\mu + \xi^\sigma\partial_\sigma h_{\mu\nu}. \tag{2.10}$$

The synchronous gauge $h_{00} = h_{0i} = 0$ can be preferred to fix local diffeomorphisms. Also the transverse time independent large diffeomorphisms $\partial_0\xi_i^T = \partial_i\xi_i^T$ preserve the synchronous gauge. These large diffeomorphisms generate a set of solutions with infinite wavelength when they act on the Minkowski vacuum.

To understand this better, one needs to investigate the linearized Einstein equations

$$\Box h_{\mu\nu} - \partial_\mu \partial_\sigma - \partial_\nu \partial_\sigma h^\sigma_\mu - \partial_\mu \partial_\nu h - \eta_{\mu\nu} \Box h + \eta_{\mu\nu} \partial_\sigma \partial_\rho h^{\sigma\rho} = \kappa/2 T_{\mu\nu}. \tag{2.11}$$

The $[0i]$ component acts as a constraint and when there are no sources it becomes

$$\partial_0 (\partial_i h_{kk} - \partial_k h_{ik}) = 0. \tag{2.12}$$

Any $h_{ij} = \partial_i \xi^T_j + \partial_j \xi^T_i$ with $\partial_0 \xi^T_i = 0$ satisfies this equation. But one needs to satisfy the continuity to finite frequency when $\partial_0 \xi^T_i \neq 0$ i.e. when the metric perturbation is changing with time (propagation of gravitational waves). For linearized gravity, the requirement of continuity to finite frequency translates to the time derivative of the perturbation approaching zero as the frequency of the perturbation approaches zero. To achieve this, it is required to assume a stronger condition, which is

$$\partial_i h_{kk} - \partial_k h_{ik} = 0. \tag{2.13}$$

This condition forces ξ^T_i to obey the relation

$$\nabla^2 \xi^T_i = 0. \tag{2.14}$$

These large diffeomorphisms can be organized as Taylor series such as

$$\xi^T_i = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \epsilon_{i i_0 i_1 \cdots i_n} x^{i_0} \cdots x^{i_n}. \tag{2.15}$$

One can derive the Noether current by varying the action once, finding the symmetry of the gauge fixed action, or directly from the equation of motion after linearizing it in $h_{\mu\nu}$. After linearizing the equation of motion it can be written as

$$\partial^\alpha H_{\alpha\mu\nu} = \frac{\kappa}{2} T_{\mu\nu}, \tag{2.16}$$

where $H_{\alpha\mu\nu} = \partial_\alpha h_{\mu\nu} + \eta_{\nu\mu} \partial^\beta h_{\alpha\beta} + \eta_{\nu\alpha} \partial_\mu h^\beta_\beta - (\alpha \leftrightarrow \mu)$.

However, in the presence of hard gravitons linearization in $h_{\mu\nu}$ is no more acceptable. But remembering that the final soft theorem only depends on asymptotic hard states, one can involve the energy momentum of hard gravitons in $T_{\mu\nu}$. By claiming that the divergence of the Noether current results in a projection of the given equation of motion onto the adiabatic mode, one finds

$$K^{\mu} = \partial_i \xi_j^T H^{\mu ij} - \frac{\kappa}{2} \xi_i^T T^{i\mu}. \tag{2.17}$$

2.2.1 Adiabatic Modes & Gauge Transformations

So far one has seen that broken symmetries are non-linearly realized since their field transformations involve constant terms. However, the fact that the nature of the realization is independent of the symmetry and it's dependency on the state of the theory is evident. Meaning that a symmetry can be linearly realized or non-linearly realized depending on the states.

In cosmology there are large classes of non-linearly realized symmetries which have vital part in dynamical gravity scenarios, they will be discussed in the upcoming chapters. These symmetries are continuously connected to physical perturbations and they form a subset of large diffeomorphisms. And most importantly when acting on an unperturbed FLRW spacetime, newly obtained solutions, which are the well-known adiabatic modes, are generated by them.

Most of the time, while working on cosmology one has a non-linear, exact solution that expresses an isotropic & homogeneous background, when it is expanded in a small perturbation, it takes the form

$$g_{\mu\nu}(x,t) = \bar{g}_{\mu\nu}(t) + h_{\mu\nu}(x,t). \tag{2.18}$$

In order to obtain conclusive results, splitting the metric and the matter parts to vectors, scalars and tensors because of the rotational invariance of the background is well-known (It is more accurate to say that into scalar, transverse vectors and transverse traceless tensor parts). These parts coincide with the representation of the orthogonal group $SO(2)$, which has the lowest possible dimension. One can recognize these parts as the cosmological analog of single particle states since they are the irreducible representations of the Poincare' symmetry group. Continuing with the parametrization of the metric

$$ds^2 = -(1+Z)dt^2+2a(\partial_i Y+M_i)dtdx^i+a^2[(1+K)\delta_{ij}+\partial_i\partial_j L+2\partial_{(i}F_{j)}+\gamma_{ij}]dx^i dx^j, \tag{2.19}$$

where $\{Z, Y, K, L\}$ are four scalars, $\{M_i, F_i\}$ two vectors and γ_{ij} is a tensor (Weinberg, 2008). γ_{ij} satisfies the relation

$$\gamma_{ii} = \partial_i \gamma_{ij} = \partial_i F_i = \partial_i M_i = 0. \tag{2.20}$$

Here one assumes the source to be a single perfect fluid, with the energy momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{2.21}$$

with p the pressure, ρ the energy density and $u_\mu u^\mu = -1$ being the normalised four-velocity. For the multiple fluids case, one can make generalizations (Pajer, 2018). Another important side note is the assumption that the fluid has vanishing anisotropic stresses, but this premise will be relaxed later. The fluid velocity is broken down into a scalar δu and vector δu^i_V as

$$u_\mu = (u_0, u_1), \qquad u_i = \partial_i \delta u + \delta u_i^V, \qquad \partial_i \delta u_i^V = 0, \tag{2.22}$$

and logically it follows as $\delta u_i^V = 0$ (in order to have gauge invariant potential flow which simplifies the analysis). Change of coordinates by the covariance of the GR gives

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x), \tag{2.23}$$

as the symmetries of the theory. This is suitably denoted with regards to the well known gauge transformations in cosmological perturbation theory (Pajer, 2018). To linear order in ϵ and the perturbations, the gauge transformations of these perturbations are

$$\begin{aligned}\Delta h_{ij} &= 2a^2 H\delta_{ij}\epsilon_0 - (\partial_j\epsilon_i + \partial_i\epsilon_j),\\ \Delta h_{0i} &= -\dot{\epsilon}_i - \partial_i\epsilon_0 + 2H\epsilon_i,\\ \Delta h_{00} &= -2\dot{\epsilon}_0,\\ \Delta u_i &= -\partial_i\epsilon_0,\\ \frac{\Delta\delta\rho}{\dot{\rho}} &= \frac{\Delta\delta\rho}{\dot{p}} = \epsilon_0.\end{aligned} \tag{2.24}$$

Once again, one can see that the states that differ by a gauge transformation are physically identical. It is straightforward to find some gauge invariant variables in some suitable gauge to guarantee that these solutions are physically separate results of the theory instead of being gauge transformations for an arbitrary single solution. Consider small gauge transformations to fix the gauge, which are defined by their behaviour at spatial infinity

$$\lim_{|x|\to\infty} \epsilon^\mu(t,x) = 0. \tag{2.25}$$

Then the Scalar-Vector-Tensor components of the metric and $T_{\mu\nu}$ transform as

$$\begin{aligned}\Delta K &= 2H\epsilon_0, & \Delta L &= -\frac{2}{a^2}\epsilon^S,\\ \Delta F_i &= -\frac{1}{a^2}\epsilon_i^V, & \Delta Z &= 2\dot{\epsilon}_0,\\ \Delta Y &= \frac{1}{a}(-\epsilon_0 - \dot{\epsilon}^S + 2H\epsilon^S), & \Delta M_i &= \frac{1}{a}(-\dot{\epsilon}_i^V + 2H\epsilon_i^V),\\ \Delta\delta\rho &= \dot{\bar{\rho}}\epsilon_0, & \Delta\delta p &= \dot{\bar{p}}\epsilon_0,\\ \Delta\delta u &= -\epsilon_0, & \Delta\pi^S &= \Delta\pi_i^V = \Delta\pi_{ij}^T = \Delta\delta u_i^V = 0,\end{aligned} \tag{2.26}$$

where the Scalar-Vector-Tensor splitting of the gauge parameter is specified as $\epsilon^\mu = \{\epsilon^0, \partial^i\epsilon^S + \epsilon_V^i\}$ (Pajer, 2018). Building gauge invariant perturbations now is uncomplicated and there are many viable options. The most convenient variables are curvature perturbations on comoving ($\mathcal{R}$) and constant density (ζ) hypersurfaces, respectively given as

$$\begin{aligned} \mathcal{R} &\equiv \frac{K}{2} + H\delta u, \\ \zeta &\equiv \frac{K}{2} - H\frac{\delta\rho}{\dot{\rho}}. \end{aligned} \tag{2.27}$$

For these variables, the gauge invariance is satisfied only to linear order, and one must add additional terms for second and higher-order cases.

In order to understand the most inclusive scalar and tensor adiabatic modes, one can make a simple four step pathway. It starts with fixing the small gauge by choosing the comoving gauge. After choosing this gauge condition, finding the residual large diffeomorphisms that respect it comes next. Now as the third step, one needs to solve the Einstein equation non-trivially, in other words, finding the subset of large diffeomorphisms that continue to finite momentum. And as the final step, to produce adiabatic modes one needs to act on the unperturbed FLRW metric with the newly discovered diffeomorphisms.

Using the transformations of Scalar-Vector-Tensor components of $T_{\mu\nu}$

$$L^{'} = L + \Delta L = 0, \qquad \delta u^{'} = \delta u + \delta\delta u = 0, \qquad F_i + \Delta F_i = 0. \tag{2.28}$$

This designates small diffeomorphisms entirely (now one can solve for small ϵ^{μ} uniquely).

In this gauge, the value of curvature perturbations on comoving hypersurface turns out to be

$$\mathcal{R} \mid_{comoving} \equiv \mathcal{R} = \frac{K}{2}. \tag{2.29}$$

By obtaining the gauge transformations of these perturbations, it is evident that acting on the unperturbed FLRW background with a large gauge transformation generates the following perturbations (Pajer & Jazayeri, 2018; Simonovic et al., 2014)

$$
\begin{aligned}
\mathcal{R} &= H\epsilon_0 - \frac{1}{3a^2}\partial_k\epsilon_k, & N_1 &= \dot{\epsilon}_0, \\
N_i &= -\partial_i\epsilon_0 + 2H\epsilon_i - \dot{\epsilon}_i, & \frac{\delta\rho}{\dot{\rho}} &= \epsilon_0, \\
\gamma_{ij} &= -2\partial_{<i}\epsilon^j_{>}, & \delta u_i &= -\partial_i\epsilon_0.
\end{aligned}
\tag{2.30}
$$

Where $< ... >$ is an indication for the symmetric traceless part

$$
T_{<ij>} \equiv \frac{1}{2}(T_{ij} + T_{ji}) - \frac{1}{3}T_{kk}\delta_{ij}. \tag{2.31}
$$

These perturbations are solutions of the equations of motion for any large ϵ, of the form

$$
\epsilon^\mu = \sum_n a^\mu_{i_1 i_2 i_3 \ldots i_n}(t) x^{i_1} x^{i_2} \cdots x^{i_n}. \tag{2.32}
$$

In Fourier space, the above expression is just a sum of derivatives of $\delta^3(k)$ since it is non-vanishing only at $k = 0$. Because of this property, this solution is called the zero momentum solution (Pajer, 2018). But one can see that because of the change of coordinates these zero momentum solutions are just FLRW in unfamiliar coordinates. Now by extending these solutions to the finite momentum, it is possible to interpret them as the $k \to 0$ limit of some perturbations in the comoving gauge. To achieve this, one needs to impose

$$
\partial_i\gamma_{ij} = \gamma_{ii} = 0, \qquad \nabla^2\epsilon_i = -\frac{1}{3}\partial_i\partial_k\epsilon_k, \tag{2.33}
$$

which is the general way of representing a transverse traceless tensor. This specifically signifies

$$
\nabla^2\partial_i\epsilon^i = 0. \tag{2.34}
$$

This result is still not satisfying. A quick computation reveals that in the linear order the off-diagonal and the ij parts of the Einstein equations take the form

$$
\begin{aligned}
k_i k_j(N_1 + \mathcal{R} + \dot{\psi} + H\psi) &= 0, \\
k_j(\dot{N}_i^V + HN_i^V) &= 0, \\
k_i(HN_1 - \dot{\mathcal{R}}) &= 0,
\end{aligned}
\tag{2.35}
$$

where $N_i = \partial_i \psi + N_i^V$. While these equations are directly satisfied at $k = 0$, they are not satisfied in general at k. To extend them to finite momentum one needs additional requirements to be satisfied

$$
\begin{aligned}
(N_1 + \mathcal{R} + \dot{\psi} + H\psi) &\stackrel{?}{=} 0, \\
(\dot{N}_i^V + HN_i^V) &\stackrel{?}{=} 0, \\
k_i(HN_1 - \dot{\mathcal{R}}) &\stackrel{?}{=} 0.
\end{aligned}
\tag{2.36}
$$

Using the perturbation (2.30) $\left(R = H\epsilon_0 - \frac{1}{3a^2}\partial_k \epsilon_k\right)$ one finds

$$
\epsilon_0 = \frac{1}{3\dot{H}}\partial_k \dot{\epsilon}^k \Rightarrow \nabla^2 \epsilon_0 = 0. \tag{2.37}
$$

Integrating the first constraint in (2.36), one arrives at

$$
\psi = -\epsilon + \frac{1}{3a}\int^t dt^{'} a(t^{'})\partial_k \epsilon^k. \tag{2.38}
$$

Using (2.34) and (2.37) it turns out that $\nabla^2\psi = 0$ (Pajer, 2018). Comparing this with the perturbation that was found earlier (2.30) ($N_i = -\partial_i \epsilon_0 + 2H\epsilon_i - \dot{\epsilon}_i$) one finds

$$
\partial_i \psi = N_i = -\partial_i \epsilon_0 + 2H\partial_i \epsilon_i - \partial_i \dot{\epsilon}_i, \tag{2.39}
$$

and the solution for this is

$$
\epsilon^i(t, x) = \bar{\epsilon}^i(x) - \partial_i \partial_k \bar{\epsilon}^k \int^t \frac{dt^{'}}{3a(t^{'})^3} \int^{t^{'}} dt^{''} a(t^{''}). \tag{2.40}
$$

Based on the perturbations that were found earlier, these diffeomorphisms generate the solution

$$
\begin{aligned}
\mathcal{R} &= -\frac{1}{3}\partial_k \bar{\epsilon}^k, \\
\psi &= \frac{1}{3a}\partial_k \bar{\epsilon}^k \int^t dt^{'} a(t^{'}), \\
\gamma_{ij} &= -2\partial_{<i}\bar{\epsilon}^{j>} + 2\partial_i\partial_j\partial_k\bar{\epsilon}^k \int^t \frac{dt^{'}}{3a(t^{'})^3}\int^{t^{'}} dt^{''} a(t^{''}).
\end{aligned} \tag{2.41}
$$

Now to discuss the results for only the leading adiabatic mode, consider the following diffeomorphism

$$
\epsilon^{\mu} = \{0, w_{ij}x^j\}. \tag{2.42}
$$

According to the gauge transformations of the perturbations that were separated into space and time components, the diagonal part of this diffeomorphism (w_{ii}) generates the constant curvature mode when acting on the unperturbed FLRW background

$$
\text{Scalar curvature mode} : \mathcal{R} = -\frac{w_{ii}}{3}, \qquad \psi = \frac{w_{ii}}{3a}\int a(t^{'})dt^{'}. \tag{2.43}
$$

The anti-symmetric part $w_{[ij]}$ is just a rotation and since FLRW is rotationally invariant, it does not generate any perturbation. And lastly, the adiabatic tensor mode is given by term $w_{<ij>}$ as

$$
\text{Tensor adiabatic mode} : \gamma_{ij} = -2w_{<ij>}. \tag{2.44}
$$

This derivation proves Weinberg's famous discovery that there always exist a constant scalar and constant tensor modes independent of the expansion phases of the universe and its constituents of it (Weinberg, 2003). The scalar adiabatic mode equation must be the solution on large scales if the structure under consideration has a single active scalar DOF (like single field inflation) (Weinberg, 2008).

The perturbations which emerged very soon after the big bang are called primordial perturbations, and impressively, it is the scalar adiabatic mode that produces each and every cosmological perturbation one has ever observed in the cosmos. However, the tensor adiabatic mode is yet to be observed. There are ongoing experimental efforts to detect it using the CMB data.

2.3 Asymptotic Expansions

2.3.1 Soft Theorems

Adiabatic modes and soft theorems are concepts that are closely related in the context of gravitational scattering amplitudes. In the case of gravity, the soft behavior of amplitudes is related to the presence of Goldstone bosons (2.8) associated with large diffeomorphisms, and leads to relationships between low-energy and high-energy scattering amplitudes.

Soft theorems describe the behaviour of massless particles when they become soft, or have zero energy. These theorems are general features of Feynman diagrams and scattering amplitudes that preserve the consistency of quantum field theory while allowing for the production of an infinite number of soft particles in any physical process.

Specifically, these theorems state that the amplitude with the added gauge boson can be written as a product of a soft factor and the initial amplitude without the additional boson. This behaviour is observed when the momentum of the included gauge boson is choose to be soft (Pasterski, 2019).

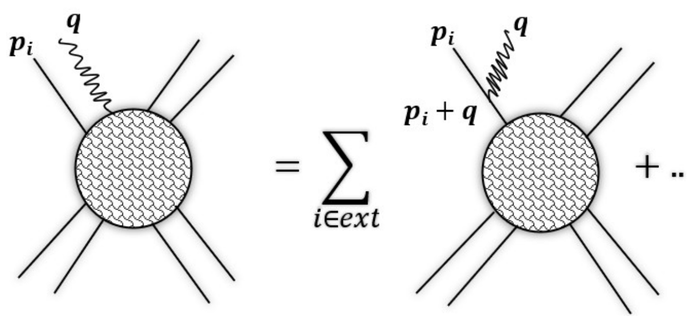

Figure 2.1: With an addition of an extra boson with its energy going to zero, expansion of the amplitude can be illustrated like this diagrammatically.

In the study of gravitational physics, the term "infrared" refers to the behaviour of low-energy particles in the presence of spontaneously broken symmetries. These symmetries, which include those described by soft theorems, can affect the behaviour of low-energy particles in a predictable manner. The gravitational memory effect, a non-linear phenomenon involving the permanent displacement of objects due to the passage of a gravitational wave, has also been studied in the context of low-energy particle behaviour. These two are related through their connection to momentum space poles in scattering amplitudes. The Fourier transform of a pole in frequency space is a step function in time, which can be recognized as a domain wall connecting two unequal vacua associated by an asymptotic symmetry (Strominger, 2018). As such, the memory effect directly evaluates the action of asymptotic symmetries. Every symmetry has a corresponding Ward identity that relates the scattering amplitudes of symmetry-related states.

2.3.2 Asymptotic Expansion for Electrodynamics

In terms of soft theorems and symmetries, electrodynamics have strong similarities to gravity. In both of their analysis, one deals with the Cauchy data at $\mathcal{I}^{\pm}$, matching conditions, infinitely many conservation laws, large gauge transformations and so on. So electrodynamics is a solid point to start the investigation to construct the exact nature of gravity.

In retarded coordinates $(r, u, z, \overline{z})$, the Minkowski line element is

$$ds^2 = -du^2 - dudr + 2r^2\gamma_{z\overline{z}}dzd\overline{z}. \tag{2.45}$$

These coordinates are suitable in the neighbourhood of $\mathcal{I}^+$. Where r is the radial coordinate, $u = t - r$ is the retarded time coordinate and z is a complex coordinate on the unit sphere with metric

$$\gamma_{z\overline{z}} = \frac{2}{(1 + z\overline{z})^2}. \tag{2.46}$$

Keeping $(u, z, \overline{z})$ fixed and taking the limit $r \to \infty$, one moves out along the null line to $\mathcal{I}^+$. The standard Minkowski metric is

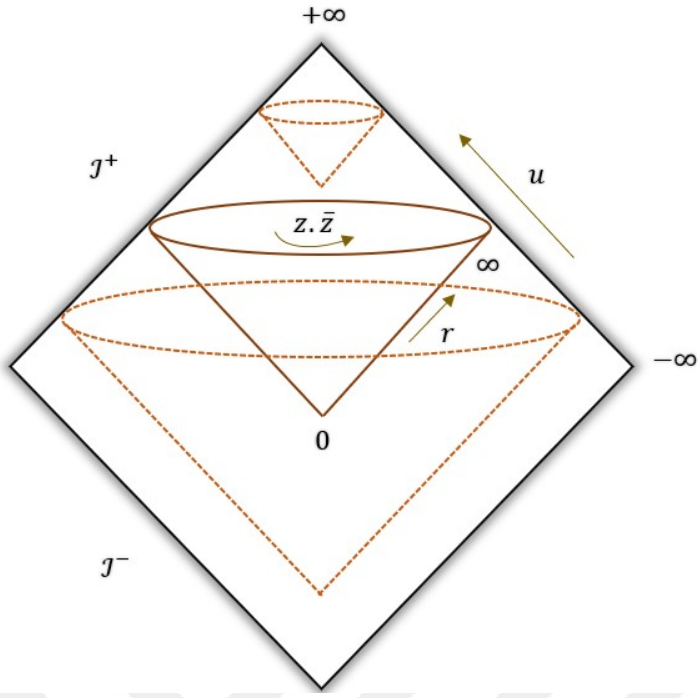

Figure 2.2: Future null infinity is parameterized by u and $(z, \bar{z})$ in the retarded Bondi coordinates.

$$ds^2 = -dt^2 + (d\vec{x})^2. \tag{2.47}$$

This metric is related to the original one by the coordinate transformations

$$(\vec{x})^2 = r^2, \qquad t = u - r, \qquad x^1 + ix^2 = \frac{2rz}{1+z\bar{z}}, \qquad x^3 = r\frac{1-z\bar{z}}{1+z\bar{z}}, \tag{2.48}$$

here the domain of the z is the entire complex plane; $z = 0$ is the north pole, $z = \infty$ is the south pole, $z\bar{z}$ is the equator and $z \to -1/\bar{z}$ is the antipodal map. Near $\mathcal{I}^+$ this coordinate system is optimal since the considered quantities fall-off near $\mathcal{I}^+$, so fields can be expanded in powers of r^{-1}.

On the other hand, these coordinates can not be used near $\mathcal{I}^-$, because $u = -\infty$ there. Advanced coordinates must be introduced to work in a neighbourhood of $\mathcal{I}^-$. The advanced line element is

$$ds^2 = -d\nu^2 + d\nu dr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}. \tag{2.49}$$

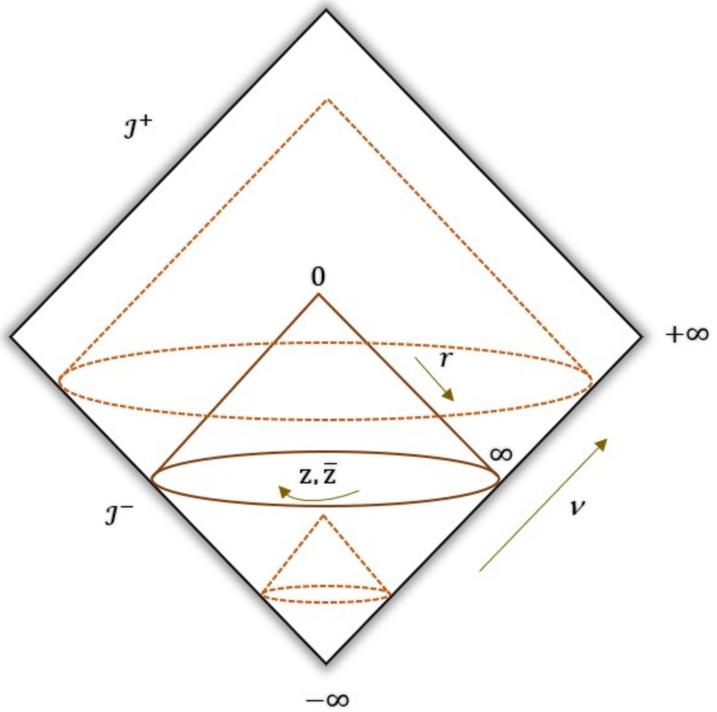

Figure 2.3: Past null infinity is parameterized by ν and $(z, \bar{z})$ in the advanced Bondi coordinates.

The above metric can be acquired with the aid of coordinate transformations such as

$$(\vec{x})^2 = r^2, \qquad t = \nu - r, \qquad x^1 + ix^2 = \frac{-2rz}{1+z\bar{z}}, \qquad x^3 = -r\frac{1-z\bar{z}}{1+z\bar{z}}. \tag{2.50}$$

In the last two terms, there are minus signs which signify the role of the antipodal map. This relation indicates that z in the advanced coordinates is the antipodal point on the sphere to the z in the retarded coordinates. Now the next step is expanding around $\mathcal{I}^+$. Given a field, one can reformulate it as a sum of expansions. If the field under consideration is the z-component of the vector potential, the sum takes the form

$$A_z(u, r, z, \bar{z}) = \sum_{n=0}^{\infty} \frac{A_z^{(n)}(u, r, z, \bar{z})}{r^n}, \tag{2.51}$$

where the coefficients depend only on the $(u, r, z, \bar{z})$ which parameterize $\mathcal{I}^+$.

The superscript (n) denotes the order in the expansion about $r = \infty$. By the antipodal mapping condition, one can define the matching condition as

$$F^{(2)}_{(ru)}(z, \bar{z}) \mid_{\mathcal{I}^+_-} = F^{(2)}_{(r\nu)}(z, \bar{z}) \mid_{\mathcal{I}^-_+}, \tag{2.52}$$

where $F^{(2)}_{(ru)}$ is the $\frac{1}{r^2}$ term in the expansion of the ru-component of the field strength around $\mathcal{I}^+$. Again, evaluating it at $\mathcal{I}^+_-$ using the antipodal matching gives (by taking $u = -\infty$)

$$F^{(2)}_{(ru)}(z, \bar{z}) \mid_{\mathcal{I}^+_-} = F^{(2)}_{(ru)}(-\infty, z, \bar{z}). \tag{2.53}$$

And it can be seen that the z values on $\mathcal{I}^+$ are antipodally correlated to those on $\mathcal{I}^-$.

2.3.3 An Infinity of Conserved Charges

On Minkowski spacetime, take any function ϵ restricted by the boundary condition

$$\epsilon(z, \bar{z}) \mid_{\mathcal{I}^+_-} = \epsilon(z, \bar{z}) \mid_{\mathcal{I}^-_+} . \tag{2.54}$$

Bear in mind that $\epsilon(z, \bar{z})$ is not smooth in the neighbourhood of spatial infinity, rather it is antipodally identified. Continuing with defining past and future charges one gets

$$\mathcal{Q}^-_\epsilon = \frac{1}{e^2} \int_{\mathcal{I}^-_+} \epsilon * F, \qquad \mathcal{Q}^+_\epsilon = \frac{1}{e^2} \int_{\mathcal{I}^+_-} \epsilon * F. \tag{2.55}$$

It immediately follows that, from the matching condition, one for every function ϵ there exist infinite number of conservation laws, in any theory involving electromagnetism. And by the antipodal mapping, the relation between the future and the past charges is

$$\mathcal{Q}^+_\epsilon = \mathcal{Q}^-_\epsilon . \tag{2.56}$$

For example, ϵ could be a spherical harmonic, so that one has a conservation law for every value of the angular momentum. Now the future charges can be written as (by the use of explicit coordinate representations)

$$\mathcal{Q}_{\epsilon}^{+} = \frac{1}{e^2}\int_{\mathcal{I}_{-}^{+}} \epsilon * F = \frac{1}{e^2}\int_{\mathcal{I}_{-}^{+}} d^2z\gamma_{z\bar{z}}\epsilon F_{ru}^{(2)}. \tag{2.57}$$

There is a r^2 in the Hodge dual, which cancels the $1/r^2$ that involved in the $F_{ru}^{(2)}$ term, enabling an integral that is finite for the limit $r \to \infty$. Using the relation (2.54) one can write the past charge as

$$\mathcal{Q}_{\epsilon}^{-} = \frac{1}{e^2}\int_{\mathcal{I}_{+}^{-}} d^2z\gamma_{z\bar{z}}\epsilon F_{r\nu}^{(2)}. \tag{2.58}$$

This needs to be integrated by parts and to do this using the constraint equations on the null surfaces $\mathcal{I}^{\pm}$ is optimal. The constraint equation near $\mathcal{I}^{+}$ has an expansion in powers of $\frac{1}{r}$, with the leading term

$$\partial_u F_{ru}^{(2)} + D^z F_{uz}^{(0)} + D^{\bar{z}} F_{u\bar{z}}^{(0)} + e^2 j_u^{(2)} = 0. \tag{2.59}$$

D_z is the covariant derivative defined on the $\mathcal{S}^2$ with the metric $\gamma_{z\bar{z}}$ and $D^z = \gamma^{z\bar{z}}D_{\bar{z}}$. Choosing $\partial_u\epsilon \mid_{\mathcal{I}^+} = 0$ and using the constraint equation, followed by integrating the boundary expression for $\mathcal{Q}_{\epsilon}^{+}$ by parts gives two terms

$$\mathcal{Q}_{\epsilon}^{+} = \frac{1}{e^2}\int_{\mathcal{I}^{+}} dud^2z(\partial_z\epsilon F_{u\bar{z}}^{(0)} + \partial_{\bar{z}}\epsilon F_{uz}^{(0)}) + \int_{\mathcal{I}^{+}} dud^2z\epsilon\gamma_{z\bar{z}} j_u^{(2)}, \tag{2.60}$$

where the first term of the integral is called a "soft charge" (will be denoted as $\mathcal{Q}_S^{+}$) which is linear in the EM field, and the second term of the integral is "hard charge" (will be denoted as $\mathcal{Q}_H^{+}$), which is mostly linear in the charge current. By "soft charge" one indicates entities with zero energy, while by "hard charge" one indicates energetic excitation. $\mathcal{Q}_S^{+}$ vanishes if ϵ is constant and through future null infinity, $\mathcal{Q}_{\epsilon}^{+}$ becomes the total charge flux. For the case where ϵ is not equal to a constant, the hard term is non-vanishing and null infinity piercing charges are weighted by an angle dependent arbitrary function.

Now looking at the soft charge, it is clear that it involves a term in the form

$$\int_{-\infty}^{\infty} du F_{uz}^{(0)} \equiv N_z, \tag{2.61}$$

convoluted with $\partial_{\bar{z}}\epsilon$ and integrated over the sphere. This is the $w \to 0$ limit of

$$\int_{-\infty}^{\infty} du F_{uz}^{(0)} e^{iwu}, \tag{2.62}$$

which is a nonzero energy and frequency w Fourier component of the EM field. When it is considered as a quantum operator, it creates and annihilates outgoing photons of energy w. Similarly, in the current case, one deals with an expression which is the $w \to 0$ limit. And following the same argument, one concludes that this term creates and annihilates soft particles with zero energy, making the use of the name "soft theorem" more comprehensible.

These outgoing photons have polarization $\partial_{\bar{z}}\epsilon$. The soft photon mode ($N_z$) is linked to gauge transformations at $\mathcal{I}^+_\pm$, it will be clear if one considers the curl

$$\begin{aligned}\partial_{\bar{z}} N_z - \partial_z N_z &= \int_{-\infty}^{\infty} du [\partial_{\bar{z}} F_{uz}^{(0)} - \partial_z F_{u\bar{z}}^{(0)}] \\ &= -\int_{-\infty}^{\infty} du \partial_u F_{z\bar{z}}^{(0)} = -F_{z\bar{z}} \mid_{\mathcal{I}^+_-}^{\mathcal{I}^+_+},\end{aligned} \tag{2.63}$$

where the Bianchi identity is used in the second line. Assuming that $F_{z\bar{z}} \mid_{\mathcal{I}^+_\pm} = 0$ (long range magnetic fields asymptotic states not allowed) then the curl vanishes. The next step is to define the real scalar N as

$$N_z \equiv e^2 \partial_z N. \tag{2.64}$$

2.3.4 Large Gauge Symmetry

The commutator action of the future and past charges ($\mathcal{Q}^+_\epsilon$, $\mathcal{Q}^-_\epsilon$) has a term involving the matter field and a linear term involving $\partial_z N$ and $\partial_{\bar{z}} N$ (known as the soft photon terms). However, $A_z^{(0)}$ and the soft terms do not commute

$$[\mathcal{Q}_\epsilon^+, A_z^{(0)}(u, z, \overline{z})] = i\partial_z \epsilon(z, \overline{z}), \tag{2.65}$$

for the past charge, calculating on $\mathcal{I}^-$ yields

$$[\mathcal{Q}_\epsilon^-, A_z^{(0)}(\nu, z, \overline{z})] = i\partial_z \epsilon(z, \overline{z}). \tag{2.66}$$

This concludes that, in the canonical formalism, an infinite number of symmetries are generated by $\mathcal{Q}_\epsilon^+$ and they are all gauge transformations with the parameter ϵ. By further analysis, one can see that $\mathcal{Q}_\epsilon^+$ transforms A_z by a large gauge transformation which is non-trivial even when the ϵ is an arbitrary function (Strominger, 2018). This large gauge transformation also does not vanish at infinity. Checking the gauge parameter, it becomes an angle dependent (but u-independent) function at $\mathcal{I}^+$. Using the antipodal argument, it is natural for $\mathcal{Q}_\epsilon^-$ to generate transformations in which the gauge parameter approaches to the antipodally modified angle dependent function at $\mathcal{I}^-$. An important side note to consider is, $A_z = 0$ is not invariant under these symmetries, which indicates that one is working with spontaneously broken symmetries and an infinitely degenerate vacuum.

So far it is proven that the commutator of the gauge field itself and the soft charge is a large gauge transformation. For the case which involves matter, $\mathcal{Q}_\epsilon^+$ has to be checked if it appropriately produces the gauge transformations on the matter fields. And as expected, it will be the contribution of the hard term.

2.3.5 Ward Identity

Ward Identities basically give relations between the quantum scattering amplitudes. They are also used to convey the dynamical outcomes of the commutation of the conserved charges with Hamiltonian or equivalently the S-matrix. Since the S-matrix and the Hamiltonian related as

$$S \sim \exp(iHT), \tag{2.67}$$

for $T \to \infty$. Quantum scattering amplitudes can be written as

$$\langle out| \, S \, |in\rangle \, . \tag{2.68}$$

Following this, charge conservation takes the form

$$\langle out| \, (\mathcal{Q}_\epsilon^+ S - S \mathcal{Q}_\epsilon^-) \, |in\rangle = 0. \tag{2.69}$$

Use of the matching condition (2.56) $\mathcal{Q}_\epsilon^+$ is equal to $\mathcal{Q}_\epsilon^-$, however, $\mathcal{Q}_\epsilon^-$ is used when acting on in states while $\mathcal{Q}_\epsilon^+$ is used while acting on out states (T. M. He, 2018). Using the well-known method to generate finite symmetry, one exponentiates the charge, and the charge conservation equation can be read as the following statement. Starting with an in state A which develops to B an out state, a large gauge transformed in state A develops to a large gauge transformed out state B. The action of $\mathcal{Q}_\epsilon^-$ on the in state (A) can be written as

$$\mathcal{Q}_\epsilon^- \, |in\rangle = \underbrace{-2 \int d^2 z \partial_{\bar{z}} \epsilon \partial_z N^-(z, \bar{z}) \, |in\rangle}_{\text{soft}} + \underbrace{\sum_{k=1}^{m} \mathcal{Q}_k^{in} \epsilon(z_k^{in}, \bar{z}_k^{in}) \, |in\rangle}_{\text{hard}}, \tag{2.70}$$

where $N^-(z, \bar{z})$ denotes the incoming soft photon field on $\mathcal{I}^-$. Here it is accepted that using the m hard particles that are coming in at points on the CS^2 indicated by z_k^{in}, one can construct the in state. The first term is the contribution of the soft charge, and the second term is the contribution of the hard charge.

Likewise, the action of the future charge ($\mathcal{Q}_\epsilon^+$) on the B turns out to be

$$\langle out| \, \mathcal{Q}_\epsilon^+ = \underbrace{2 \int d^2 z \partial_z \partial_{\bar{z}} \epsilon \, \langle out| \, N(z, \bar{z})}_{\text{soft}} + \underbrace{\sum_{k=1}^{n} \mathcal{Q}_k^{out} \epsilon(z_k^{out}, \bar{z}_k^{out}) \, \langle out| \, .}_{\text{hard}} \tag{2.71}$$

Combining these results, the Ward Identity becomes

$$2\int d^2z\partial_z\partial_{\bar{z}}\epsilon \langle out|\, N(z,\bar{z})S - SN^{-}(z,\bar{z})\,|in\rangle$$
$$= \left[\sum_{k=1}^{m} Q_k^{in}\epsilon(z_k^{in},\bar{z}_k^{in}) - \sum_{k=1}^{n} Q_k^{out}\epsilon(z_k^{out},\bar{z}_k^{out})\right]\langle out|\, S\,|in\rangle\,. \quad (2.72)$$

From this equation, one can deduce that corresponding to every function ϵ on the sphere there is one Ward Identity. So in total, there is an infinite number of them. These Ward Identities link any S-matrix element between incoming and outgoing states times the term in the brackets, to the same S-matrix element with the insertion of particular soft photon modes.

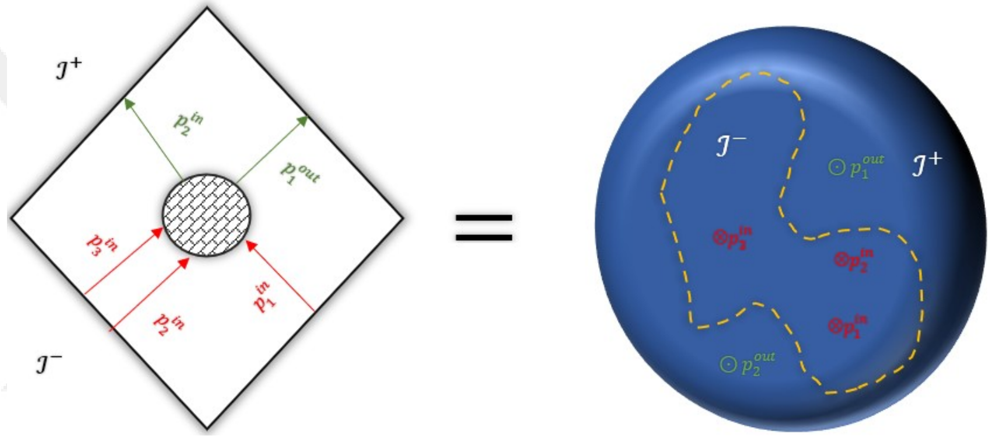

Figure 2.4: The S-matrix which is constructed firstly on a Minkowski space can be rewritten as a correlator on the CS^2. Massless outgoing and incoming particles are described by operators at the position where they penetrate the null infinity (Strominger, 2018).

So far by the use of advanced and retarded coordinates, particles have been characterized by the points at which they came in at null infinity also the conservation laws are derived from antipodal matching conditions. Now in order to prove that the Ward identity is a soft theorem one needs a couple more steps to follow.

In standard QFT, one engages in a foundation of plane waves. To utilize this notation, one needs to rewrite the Ward Identity equation in terms of a plane wave basis. This is possible by using the conventional mode expansion for A_z. It is not clear that the commutation relations $\left[\partial_u \hat{A}_z(u,z,\bar{z}), \hat{A}_{\hat{w}}(u',w,\bar{w})\right] = -\frac{ie^2}{2}\delta(u-u')\delta^2(z-w)$ and

$\left[\phi(z,\overline{z}), N(w,w)\right] = -\frac{i}{4\pi}\log|z-w|^2 + f(z,\overline{z}) + g(w,\overline{w})$ are equivalent to the conventional ones.

The important detail here is, they both arise from the covariant symplectic form which is independent of slicing. Now one needs to calculate it as opposed to the traditional use of plane wave basis, pushing the slice up to $\mathcal{I}^+$. These kinds of bases require the Cartesian coordinates for Minkowski space

$$ds^2 = -dt^2 + d\vec{x}.d\vec{x}. \tag{2.73}$$

Near $\mathcal{I}^+$, A_ν has the on-shell outgoing plane wave mode expansion

$$A_\nu(x) = e\sum_{\alpha=\pm}\int \frac{d^3q}{(2\pi)^3}\frac{1}{2w}\left[\epsilon_\nu^{*\alpha}(\vec{q})a_\alpha^{out}(\vec{q})e^{iq.x} + \epsilon_\nu^{\alpha}(\vec{q})a_\alpha^{out}(\vec{q})^\dagger e^{-iq.x}\right], \tag{2.74}$$

where $q^2 = 0$, the two polarization vectors satisfying a normalization condition $\epsilon_\alpha^\nu\epsilon_{\beta\nu}^* = \delta_{\alpha\beta}$ and

$$\left[a_\alpha^{out}(\vec{q}), a_\beta^{out}(\vec{q}\,')^\dagger\right] = \delta_{\alpha\beta}(2\pi)^3(2w_q)\delta^3(\vec{q}-\vec{q}\,'). \tag{2.75}$$

For the commutators of modes of the free EM field, this is the conventional formula.

Now with regards to the known creation and annihilation operators, one needs to rewrite the asymptotic quantities on $\mathcal{I}^+$. It is suitable to use retarded coordinates near $\mathcal{I}^+$. In these coordinates, the metric follows as

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}. \tag{2.76}$$

On the S^2, in the direction of the null vector, there exists a map from null vectors q^μ to points $(z,\overline{z})$, which takes the form

$$q^\mu = \frac{w}{1+z\overline{z}}(1+z\overline{z}, z+\overline{z}, -i(z-\overline{z}), 1-z\overline{z}) = (w, q^1, q^2, q^3). \tag{2.77}$$

Considering the field $A_z^{(0)}(u, r, z, \overline{z})$ at $\mathcal{I}^+$, it should be in the form

$$A_z^{(0)}(u, z, \overline{z}) = \lim_{r\to\infty} A_z(u, r, z, \overline{z}). \tag{2.78}$$

This form allows one to take all the q^μs in the mode expansion and re-express them in terms of points on the CS^2. Its already been established that in terms of creation and annihilation operators, one can write an expansion for $A_z(u, r, z, \overline{z})$. So, a natural deduction for $A_z^{(0)}(u, z, \overline{z})$ indicates that it should also have a similar expansion. Since $A_z^{(0)}(u, z, \overline{z})$ is an operator which is localized at the point $(z, \overline{z})$ on the CS^2, it is consistent for it to create and annihilate photons that penetrate at that point. In the case of a rotation about the point $(z, \overline{z})$, $A_z^{(0)}$ and $A_{\overline{z}}^{(0)}$ acts oppositely. They get opposite phases, consequently while $A_z^{(0)}$ creating one photon helicity and annihilating the other, $A_{\overline{z}}^{(0)}$ does the same thing oppositely.

By use of saddle point approximation (Saha, 2018), evaluation of $A_z^{(0)}(u, z, \overline{z})$ in a large-r yields to

$$A_z^{(0)}(u, z, \overline{z}) = -\frac{i}{8\pi^2}\frac{\sqrt{2}e}{1+z\overline{z}}\int_0^\infty dw[a_+^{out}(w\hat{x})e^{-iwu} - a_-^{out}(w\hat{x})^\dagger e^{iwu}], \tag{2.79}$$

where $\hat{x} = \hat{x}(z, \overline{z})$. $\hat{x}$ is a unit vector that points to $(z, \overline{z})$ on the S^2 and $w\hat{x}$,$\hat{x}$ is the three momentum involved in the creation and annihilation operators. It can be interpreted as, at the $\mathcal{I}^+$, it is the essential correlation between the out fields under $\frac{1}{r}$ expansion.

The related Ward identity involves $\partial_z N$. One has to define it in the zero momentum limit, to get its mode expansion. The zero momentum limit defined as

$$\partial_z N = \frac{1}{2e^2}\lim_{w\to 0^+}\int_{-\infty}^{\infty} du(e^{iwu} + e^{-iwu})F_{uz}^{(0)}. \tag{2.80}$$

With this, one guarantees that $\partial_z\partial_{\overline{z}}N$ is Hermitian. This would not be the case if the definition is made up only by one of the e^{iwu} or e^{-iwu}. Using the large-r saddle point approximation

$$\partial_z N = \frac{1}{8\pi e} \frac{\sqrt{2}}{1+z\bar{z}} \lim_{w\to 0^+} [wa_+^{out}(w\hat{x}) + wa_-^{out}(w\hat{x})^\dagger]. \tag{2.81}$$

There is a similar formula for $\partial_z N^-$

$$\partial_z N^- = \frac{1}{8\pi e} \frac{\sqrt{2}}{1+z\bar{z}} \lim_{w\to 0^+} [wa_+^{in}(w\hat{x}) + wa_-^{in}(w\hat{x})^\dagger]. \tag{2.82}$$

The Ward identity can now be expressed as

$$\begin{aligned} &\lim_{w\to 0} \left[w \langle out| \left(a_+^{out}(w\hat{x})S - Sa_-^{in}(w\hat{x})^\dagger\right) |in\rangle \right] \\ &\qquad = \sqrt{2}e(1+z\bar{z}) \left[\sum_{k=1}^{n} \frac{Q_k^{out}}{z - z_k^{out}} - \sum_{k=1}^{m} \frac{Q_k^{in}}{z - z_k^{in}} \right] \langle out| S |in\rangle , \end{aligned} \tag{2.83}$$

where ϵ is chosen to be $\epsilon = \frac{1}{z-w}$. This is a special case which mimics the Ward identity in 2-dimensional CFT. If ϵ is kept arbitrary, one gets a similar equation to (2.72). If one looks carefully, the resemblance between this expression and the standard soft photon theorem is clear. On the left hand side of the last expression, via explicit multiplication by w, a soft pole in the matrix element has been displayed. Looking at the right hand side of the expression for $z \to z_k$, there are also collinear poles.

2.4 Asymptotic Flat Spacetimes

To develop the BMS analysis, the asymptotic flat spacetimes have be to defined rigorously. Now following the conformal compactification condition proposed by Penrose in the 1960s one gets the definition:

Given a spacetime (M, g), it is called asymptotically simple if there exists a conformal embedding into a so called 'extended spacetime' $(\hat{M}, \hat{g})$. Extended spacetime can be considered as a manifold with a boundary that characterizes the points at infinity (Wheeler, 1970).

Now let (M, g) be a Lorentzian manifold. (M, g) is called spacetime if it is smooth, connected, 4-dim. and time orientable. The aim is to combine spacetimes that approach Minkowski spacetime at infinity. To achieve this, one constructs the boundary for the extended spacetime as

$$\mathcal{I} := i_+ \sqcup \mathcal{I}_+ \sqcup i_0 \sqcup \mathcal{I}_- \sqcup i_-, \tag{2.84}$$

where i_0 corresponds to spacelike infinity, i_+ and i_- to timelike infinity and $\mathcal{I}_+$ and $\mathcal{I}_-$ to lightlike infinity. It is related to the spacetime (M, g) via a conformal embedding, $\mathcal{C} : M \hookrightarrow \hat{M}$ (Geroch & Horowtiz, 1978; Prinz & Schmeding, 2022).

Let (M, g) be an oriented and casual spacetime. (M, g) is called an asymptotically simple spacetime, if it admits a conformal extension $(\hat{M}, \hat{g})$ indicating an embedding $\mathcal{C} : M \hookrightarrow \hat{M}$ and a smooth function $\xi \in C^\infty(\hat{M})$, such that;

(a) $\hat{M}$ is a manifold with interior $\mathcal{C}(M)$ and boundary $\mathcal{I}$; i.e., $\hat{M} \cong \mathcal{C}(M) \sqcup \mathcal{I}$.

(b)$\xi \mid_{\mathcal{C}(M)} > 0$, $\xi \mid_{\mathcal{I}} \equiv 0$ and $d\xi \mid_{\mathcal{I}} \neq 0$; additionally, $\mathcal{C}^* g \equiv \xi^2 \hat{g}$.

(c) Each null geodesic of $(\hat{M}, \hat{g})$ has two distinct endpoints on $\mathcal{I}$.

(d)$(R_{\mu\nu}) \mid_{\mathcal{C}^{-1}(\hat{O})} \equiv 0$, where $\hat{O} \subset \hat{M}$ is an open neighbourhood of $\mathcal{I} \subset \hat{M}$.

There are a couple of important points one needs to check here. The first one is, letting (M, g) be an asymptotically simple and empty spacetime, which makes (M, g) parallelizable. The second one is, if (M, g) has a vanishing cosmological constant, then both $\mathcal{I}_\pm \supset \mathcal{I}$ components are homeomorphic to $\mathbb{R} \times \mathbb{S}^2$ (Ashtekar, 2014; Ashtekar et al., 2018).

As one moves far away, it is expected that the stress tensor falls off with a definite rate. So the question of how the metric approaches flatness is needed to be answered with the minimum number of assumptions utilized in order to have a reasonable definition.

In retarded coordinates near $\mathcal{I}^+$, flat Minkowski space is

$$ds^2 = -du^2 - 2dudr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z}. \tag{2.85}$$

The aim is, to find a metric which is asymptotically flat, but in the deep interior not completely equal to the flat metric. Suitable coordinates for this task are the Bondi coordinates $(u, r, z, \overline{z})$, shortened as $\Theta = (z, \overline{z})$. The most generic four dimensional metric in this gauge takes the form

$$ds^2 = -Udu^2 - 2e^{2\beta}dudr + g_{AB}(d\Theta^A + \frac{1}{2}U^A du)(d\Theta^B + \frac{1}{2}U^B du), \tag{2.86}$$

where

$$\partial_r det(\frac{g_{AB}}{r^2}) = 0. \tag{2.87}$$

Where r is the luminosity distance. By the conditions $g_{rr} = g_{rA} = 0$ together with $\partial_r det(\frac{g_{AB}}{r^2}) = 0$, the local diffeomorphism invariance is fixed. By this metric, one can locally express any geometry. At large r, requiring asymptotic flatness with the fixed $(u, z, \overline{z})$ is a step towards getting the fall-off conditions on the metric components. At this point, these fall-off conditions need to be chosen very carefully. They should be weak enough to allow all the compelling results but also strong enough to eliminate the unrealistic results.

Following the construction made by BMS (Bondi et al., 1962; Sachs, 1962), the metric is constrained to be

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + 2r^2\gamma_{z\overline{z}}dzd\overline{z} \\ &+ \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\overline{zz}}d\overline{z}^2 + D^2C_{zz}dudz + D^{\overline{z}}C_{\overline{zz}}dud\overline{z} \\ &+ \frac{1}{r}(\frac{4}{3}(N_z + u\partial_z m_B) - \frac{1}{4}\partial_z(C_{zz}C^{zz}))dudz + c.c. + ..., \end{aligned} \tag{2.88}$$

where D_z is the covariant derivative with respect to $\gamma_{z\overline{z}}$, C_{zz}, m_B and N_z depends on $(u, z, \overline{z})$ but not on r, and $c.c.$ refers to complex conjugate.

One can recognize the first line of the equation (2.88) as the flat Minkowski metric, this is true and additional terms are the leading corrections to the flat metric. Further subleading terms at large r are given in the ellipsis. One can associate Bondi coordinates with the Minkowski coordinates since to the leading order, the spacetime is flat near $\mathcal{I}^+$.

This metric is merely a geometric restriction to specify the class of spacetimes that is under investigation. m_B is known as the Bondi mass. In a general spacetime, it depends on the retarded time u and the angle $(z, \bar{z})$. And total Bondi mass as expected is the integral of m_B over S^2. N_z is the angular momentum characteristic (its integral over the S^2, contracted with the rotational vector field gives the total angular momentum). C_{zz} describes gravitational waves. It is transverse to $\mathcal{I}^+$. The Bondi news tensor is given by

$$N_{zz} = \partial_u C_{zz}. \tag{2.89}$$

One can consider it as the gravitational counterpart of the field strength $F_{uz} = \partial_u A_z$, and similarly, its square is harmonious with the energy flux over $\mathcal{I}^+$.

2.4.1 More Details on the BMS Coordinate Functions

Using the conformal automorphisms of the boundary $\mathcal{I}^+$ of M one can construct the BMS group. It is also useful to conceptualize $\mathbb{S}^2$ as the Riemann sphere with spherical coordinates $(\theta, \phi) \in [0, \pi[\times[0, 2\pi[$ by writing the boundary as $\mathcal{I}^+ = \mathbb{S}^2 \times \mathbb{R}$. To establish charts for the $\mathbb{S}^2$ as a complex 1-dim. manifold one can use the stereographic projections

$$st : \mathbb{S}^2 \setminus (0,0,1) \to \mathbb{C} \cong \mathbb{R}^2, \qquad st(\theta, \phi) := \zeta := e^{i\phi} cot(\frac{\theta}{2}). \tag{2.90}$$

Extending the stereographic projection to a diffeomorphism $\kappa : \mathbb{S}^2 \to \mathbb{C} \cup \{\infty\}$ via

$$\kappa(z) = \begin{cases} st(z) & \text{for} \quad z \in \mathbb{S}^2 \setminus \{(1,0,0)\} \\ \infty & \text{for} \quad else, \end{cases} \tag{2.91}$$

of $\mathbb{S}^2$ with the extended complex plane $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ (the Riemann sphere). The conventional coordinates are $z = (\zeta, \bar{\zeta})$.

Let (M, g) be an asymptotically simple spacetime with globally defined coordinate functions $x^\alpha : M \to \mathbb{R}^4$, denoted via $x^\alpha := (t, x, y, z)$. From these BMS coordinate function follows as $y^\alpha : M \to \mathbb{R} \times [0, \infty) \times \mathbb{S}^2$, denoted via $y^\alpha := (u, r, \theta, \phi)$. Transforming the (x, y, z) coordinates to (r, θ, ϕ) coordinates by the relations

$$r := \sqrt{x^2 + y^2 + z^2}, \qquad \theta := \arccos(\frac{z}{r}), \qquad \phi := \arctan(\frac{y}{x}), \tag{2.92}$$

and then combining the radial coordinate r with the timelike coordinate t to form the lightlike coordinate

$$u := t - r. \tag{2.93}$$

With such coordinates and by uniting the angles as $z^\alpha := (\theta, \phi)$, the metric turns out to be

$$\begin{aligned} g_{\mu\nu} dx^\mu \otimes dx^\nu = &-\frac{V}{r} e^{2\beta} du \otimes du - e^{2\beta}(du \otimes dr + dr \otimes du) \\ &+ r^2 h_{ab}(dz^a - U^a du) \otimes (dz^b - U^b du). \end{aligned} \tag{2.94}$$

where h_{ab} is the metric of the unit sphere

$$\begin{aligned} h_{ab} dz^a \otimes dz^b \equiv &\cosh(2\delta)(e^{2\gamma} d\theta \otimes d\theta + e^{-2\gamma} \sin^2(\theta) d\phi \otimes d\phi) \\ &+ \sin(\theta)\sin(2\delta)(d\theta \otimes \ d\phi + d\phi \otimes d\theta). \end{aligned} \tag{2.95}$$

The metric degrees of freedom are expressed via a vector field on the unit sphere $U \in \chi(S^2)$ and real functions on the spacetime $V, \beta, \gamma, \delta \in C^\infty(M, \mathbb{R})$. BMS coordinate functions can be considered as a diffeomorphism gauge fixing since with these coordinates, each degree of a globally hyperbolic Lorentzian metric is parameterized via a function (Prinz & Schmeding, 2022).

With the given coordinate functions, the (M, g) is now asymptotically flat, if for all $(u, r, \theta, \phi) \ni y^\alpha$ fixed. Equipped with these and by expanding the metric in a series of inverse powers of the radial coordinate r at null infinity, one gets the following fall-off conditions

$$\lim_{r\to+\infty} \frac{V}{r} = 1 + \mathcal{O}(1/r),$$
$$\lim_{r\to+\infty} h_{ab} = q_{ab} + \mathcal{O}(1/r),$$
$$\lim_{r\to+\infty} \beta = \mathcal{O}(1/r^3), \tag{2.96}$$
$$\lim_{r\to+\infty} U^\alpha = \mathcal{O}(1/r^3),$$

where $q_{ab} = \text{diag}(1, \sin^2(\theta))$. One way to interpret this is to see that at $\mathcal{I}$, the fall-off properties become the smoothness condition.

2.5 Supertranslations

In this section, the asymptotic symmetries of gravitational theories in asymptotically flat spacetimes will be discussed. It's already been established that these symmetries are generated by diffeomorphisms which preserve both the Bondi gauge and the boundary fall-off conditions. Historically it was envisioned that these symmetries generate the isometries of flat spacetime itself (the Poincare group) since the asymptotic region is considered almost flat. But what one ends with getting the so called BMS group, which is an infinite dimensional group (to be exact, it is a Fréchet Lie group, see (A.2)). And the well-known Poincare group is a subgroup of this new Lie group. Within this group, The four global translations are promoted to supertranslations that independently interact with each point on the CS^2. To generate these supertranslations, one can make a clarifying assumption which eliminates six Lorentz generators. Which is, one works only on the diffeomorphisms that have the large r fall-offs

$$\xi^u, \xi^r \sim \mathcal{O}(1), \qquad \xi^z, \xi^{\bar{z}} \sim \mathcal{O}(\frac{1}{r}). \tag{2.97}$$

This condition rule outs boosts and rotations that grow with r at infinity since the vector field is $\mathcal{O}(1)$ at large r in an orthonormal frame.

The Lie derivatives of the metric components at large r are then

$$\begin{aligned}
\mathcal{L}_\zeta g_{ur} &= -\partial_u \zeta^u + \mathcal{O}(\frac{1}{r}), \\
\mathcal{L}_\zeta g_{zr} &= r^2 \gamma_{z\bar{z}} \partial_r \zeta^{\bar{z}} - \partial_z \zeta^u + \mathcal{O}(\frac{1}{r}), \\
\mathcal{L}_\zeta g_{z\bar{z}} &= r\gamma_{z\bar{z}}[2\zeta^r + rD_z\zeta^z + rD_{\bar{z}}\zeta^{\bar{z}}] + \mathcal{O}(1), \\
\mathcal{L}_\zeta g_{uu} &= -2\partial_u \zeta^u - 2\partial_u \zeta^r + \mathcal{O}(\frac{1}{r}).
\end{aligned} \tag{2.98}$$

The next step is to ensure that Bondi gauge conditions and fall-offs are both preserved. One way of doing this is checking the infinitesimal BMS transformations that preserve the asymptotic form, which are given as

$$\begin{aligned}
u \to u - f, &\qquad r \to r - D^z D_z f, \\
z \to z + \frac{1}{r} D^z f, &\qquad \bar{z} \to \bar{z} + \frac{1}{r} D^{\bar{z}} f.
\end{aligned} \tag{2.99}$$

From these infinitesimal transformations, one can easily construct the desired vector field ζ at large r as

$$\zeta = -f\partial_u - D^z D_z f \partial_r + \frac{1}{r} D^z f \partial_z + \frac{1}{r} D^{\bar{z}} f \partial_{\bar{z}} + ... \tag{2.100}$$

where $f(z,\bar{z})$ is any function of $(z,\bar{z})$, this diffeomorphism preserves all the remaining conditions and the transformations generated by it are called supertranslations. As already mentioned, supertranslations are promoted versions of the four translations in Minkowski space. The geometry of spacetime is transformed into a new physically inequivalent geometry by supertranslations although they are diffeomorphisms.

By evaluating the Lie derivative of the respective component of the metric and extracting the respective coefficient in the large r expansion, the action of supertranslations on the $\mathcal{I}^+$ data m_B, C_{zz} and N_{zz} can be determined

$$\begin{aligned}
\mathcal{L}_f N_{zz} &= f\partial_u N_{zz}, \\
\mathcal{L}_f m_B &= f\partial_u m_B + \frac{1}{4}[N^{zz} D_z^2 f + 2D_z N^{zz} D_z f + c.c.], \\
\mathcal{L}_f C_{zz} &= f\partial_u C_{zz} - 2D_z^2 f.
\end{aligned} \tag{2.101}$$

If one supertranslates flat Minkowski spacetime described by $m_B = N_{zz} = C_{zz} = 0$, these Lie derivative equations indicate that the supertranslated spacetime will still have $m_B = 0$ and $N_{zz} = 0$ and a vanishing Riemann tensor. It was expected since diffeomorphisms cannot create gravitational waves or change the physical mass squared. But the important point here is, for the supertranslated spacetime $C_{zz} \neq 0$, the vanishing of the curvature requires

$$C_{zz} = -D_z^2 C, \tag{2.102}$$

for some function $C(z, \overline{z})$. Under a supertranslation

$$\mathcal{L}_f C = f, \tag{2.103}$$

indicating that this function is the Goldstone boson, which can be considered as the product of the spontaneously broken supertranslation invariance. It parameterizes the physically distinct vacua.

It is possible to obtain a larger BMS^+ group (semi-direct product of supertranslations with Lorentz transformations on $\mathcal{I}^+$) by dropping the overly restrictive fall-offs (2.97) on ζ. Conventionally, for angular momentum, there is no BMS^+ invariant definition. But one should note that, for any classical vacuum, there exists a unique Poincare subgroup of BMS^+ under which it is invariant. So one can say that in flat space, it is always possible to find an unbroken Poincare subgroup of BMS^+.

Assume that the geometry is ruled by the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T^M_{\mu\nu}. \tag{2.104}$$

Since the structure under consideration is the null infinity, one can assume that $T^M_{\mu\nu}$ represents the massless modes. Inserting the metric and expanding in large r, the leading uu-component of Einstein's equations turns out to be

$$\partial_u m_B = \frac{1}{4}[D_z^2 N^{zz} + D_{\overline{z}}^2 N^{\overline{zz}}] - T_{uu}, \tag{2.105}$$

where T_{uu} is found as

$$T_{uu} = \frac{1}{4} N_{zz} N^{zz} + 4\pi G \lim_{r \to +\infty} [r^2 T^M_{uu}]. \tag{2.106}$$

The leading data at $\mathcal{I}^+$ is constrained by these two equations. Also, there exists an extra constraint involving N_z from the uz component of the Einstein equation.

Conventionally it is assumed that near the past and future boundaries of $\mathcal{I}^+$, $\mathcal{I}^+_-$ and $\mathcal{I}^+_+$, the news falls off faster than $\frac{1}{|u|}$. These asymptotic boundary conditions were shown to satisfied in a bounded neighbourhood of flat space. The spacetimes under investigation here obey this asymptotic behaviour but in the deep interior, it is not necessary to assume that they are nearly flat spaces.

The news dictates C_{zz} up to an integration function by integrating (2.89). Furthermore, the vanishing of the Weyl tensor at $\mathcal{I}^+_-$ calls for $C_{zz} \mid_{\mathcal{I}^+_-} = -2D_z^2 C \mid_{\mathcal{I}^+_-}$. So the integration function can be chosen as $C \mid_{\mathcal{I}^+_-}$. Given the initial data at $\mathcal{I}^+_-$ and the news tensor, on $\mathcal{I}^+$ the quantities m_B and N_z can be obtained by the integration of the constraints. Consequently, the Cauchy data consists of

$$\{N_{zz}(u, z, \bar{z}), C(z, \bar{z}) \mid_{\mathcal{I}^+_-}, m_B(z, \bar{z}) \mid_{\mathcal{I}^+_-}\}. \tag{2.107}$$

At higher orders in $\frac{1}{r}$, more data are required, along with $N_z \mid_{\mathcal{I}^+_-}$. This is going to play a crucial role in superrotations. Using the advanced Bondi coordinates $(\nu, r, z, \bar{z})$ one can make a similar deduction for $\mathcal{I}^-$. The metric expansion is given as

$$ds^2 = -d\nu^2 + 2d\nu dr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z} + \frac{2m_B}{r} d\nu^2 + rC_{zz} dz^2 + rC_{\bar{z}\bar{z}} d\bar{z}^2 + ..., \tag{2.108}$$

here m_B and C_{zz} depend on $(\nu, z, \bar{z})$. Supertranslations act on $\mathcal{I}^-$ as

$$\mathcal{L}_f N_{zz} = f \partial_\nu N_{zz}, \qquad \mathcal{L}_f C_{zz} = f \partial_\nu C_{zz} + 2D_z^2 f. \tag{2.109}$$

On $\mathcal{I}^-$, it can be expanded with boosts and rotations to obtain the action of BMS^-. The constraint equation follows as

$$\partial_\nu m_B = \frac{1}{4}(D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}) + T_{\nu\nu}, \quad T_{\nu\nu} = \frac{1}{4}N_{zz}N^{zz} + 4\pi G \lim_{r\to+\infty}[r^2 T^M_{\nu\nu}]. \tag{2.110}$$

Defining $C_{zz} \mid_{\mathcal{I}^-_+} = 2D_z^2 C \mid_{\mathcal{I}^-_+}$, the related Cauchy data is given as

$$\{N_{zz}(\nu, z, \bar{z}), C(z,\bar{z}) \mid_{\mathcal{I}^-_+}, m_B(z,\bar{z}) \mid_{\mathcal{I}^-_+}\}. \tag{2.111}$$

2.5.1 Gravitational Scattering Problem

The aim of the scattering problem is to obtain the map between the Cauchy data on $\mathcal{I}^-$ and the Cauchy data on $\mathcal{I}^+$. The tools one has so far up to this point are not enough to determine this map properly, right now one can determine the data on $\mathcal{I}^+$ at most up to a supertranslation. To make a better sense of the scattering problem in GR, by using the constraints one needs to choose a future BMS$^+$ frame, and establish the initial values for integrating N_z and m_B along $\mathcal{I}^+$. The choice of BMS$^+$ frame is dictated by the Lorentz invariant (also CPT) matching conditions

$$C(z,\bar{z}) \mid_{\mathcal{I}^+_-} = C(z,\bar{z}) \mid_{\mathcal{I}^-_+}, \qquad m_B(z,\bar{z}) \mid_{\mathcal{I}^+_-} = m_B(z,\bar{z}) \mid_{\mathcal{I}^-_+}. \tag{2.112}$$

These matching conditions destroy the joined BMS$^+\times$BMS$^-$ action $\mathcal{I}^+$ and $\mathcal{I}^-$ down to the diagonal subgroup that preserves these conditions

$$f(z,\bar{z}) \mid_{\mathcal{I}^+_-} = f(z,\bar{z}) \mid_{\mathcal{I}^-_+}. \tag{2.113}$$

Now the BMS$^+$ frame is fixed in regards of the BMS$^-$ frame. The symmetry of the gravitational scattering is the diagonal subgroup generated by this condition. Having a single solution for the scattering, infinitely many more solutions are generated by the group. Near spatial infinity, the conditions for $C(z,\bar{z})$ and $m_B(z,\bar{z})$ antipodally equates past and future fields (Strominger, 2014).

By use of Weinberg's soft graviton theorem the matching conditions for these two were established to be essential to all orders in standard weak field perturbation theory. So these conclusions push one to investigate if these matching conditions can be used to define the scattering problem for the cases where near the spatial infinity, the fields are weak.

2.5.2 Conserved Charges

Up until this point, it's been hinted that for every point on the CS^2, there exists a matching condition, which results in a conserved charge. So, one expects an infinite number of conserved charges when there are infinitely many matching conditions. Sticking to the same construction which has been done in the gauge theory to generate charges, one finds the supertranslation charges as

$$\begin{aligned} \mathcal{Q}_f^+ &= \frac{1}{4\pi G}\int_{\mathcal{I}_-^+} d^2z\gamma_{z\bar{z}} f m_B, \\ \mathcal{Q}_f^- &= \frac{1}{4\pi G}\int_{\mathcal{I}_+^-} d^2z\gamma_{z\bar{z}} f m_B. \end{aligned} \tag{2.114}$$

The conservation law presents itself right away from the matching conditions

$$\mathcal{Q}_f^+ = \mathcal{Q}_f^-. \tag{2.115}$$

Using the constraint equation and integrating by parts

$$\begin{aligned} \mathcal{Q}_f^+ &= \frac{1}{4\pi G}\int_{\mathcal{I}^+} dud^2z\gamma_{z\bar{z}} f[T_{uu} - \frac{1}{4}(D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}})], \\ \mathcal{Q}_f^- &= \frac{1}{4\pi G}\int_{\mathcal{I}^-} d\nu d^2z\gamma_{z\bar{z}} f[T_{\nu\nu} + \frac{1}{4}(D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}})], \end{aligned} \tag{2.116}$$

where $m_B \to 0$ in the far future. One interesting choice is, $f(z,\bar{z}) = \delta^2(z-w)$. With this choice one obtains (2.115), meaning that on $\mathcal{I}^+$, the integrated energy flux at a point w is equal to the integrated energy flux at the antipodal point w on $\mathcal{I}^-$ as

$$\int_{\mathcal{I}^+} du\gamma_{z\bar{z}}[T_{uu} - \frac{1}{4}(D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}})] = \int_{\mathcal{I}^-} d\nu\gamma_{z\bar{z}}\big[T_{\nu\nu} + \frac{1}{4}(D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}})\big]. \tag{2.117}$$

Additional to the stress tensor term, there is another term which is linear in the gravitational field and is also a total u derivative in the local energy equation. This extra term can be interpreted as the contribution to the local energy coming from soft graviton. From this, one can conclude that energy is conserved at every angle.

In the quantum theory, conserved charges commute with the S-matrix

$$\mathcal{Q}_f^+ S - S\mathcal{Q}_f^- = 0. \tag{2.118}$$

To form a Ward Identity from this statement, one can put it in between the in and out states

$$\langle out|\, a_{\pm} S\, |in\rangle = \sqrt{8\pi G} \sum_k \frac{\epsilon^{\pm\mu\nu} p_{k\mu} p_{k\nu}}{q.p_k} \langle out|\, S\, |in\rangle\,, \tag{2.119}$$

where $a_{\pm}$ annihilates a helicity $\pm$ graviton. This is same as the Weinberg's soft graviton theorem (T. He et al., 2015).

2.6 From Momentum to Asymptotic Position Space

Weinberg's soft graviton theorem is conveyed as momentum eigenmodes of field operators. But the supertranslation Ward identity which was obtained in the last chapter is expressed in terms of the integrated field operator P_z. One has to transform the field operator between these two bases to examine them with respect to each other.

For a massless particle with spatial momentum centred around $\vec{p}$ becomes localized at late times and large r near the point

$$\vec{p} = w\hat{x} \equiv w\frac{\vec{x}}{r} = \frac{w}{1 + z\bar{z}}(z + \bar{z}, -iz + i\bar{z}, 1 - z\bar{z}), \tag{2.120}$$

where $\vec{p}.\vec{p} = w^2$. So that the momentum of massless particles can be distinguished by $(w, z, \bar{z})$ or p^{μ}. By the use of the mode expansion, the gravitational field can be estimated as

$$h_{\mu\nu}^{out}(x) = \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2w_q} [\epsilon_{\mu\nu}^{\alpha *}(\vec{q}) a_\alpha^{out}(\vec{q}) e^{iq.x} + \epsilon_{\mu\nu}^{\alpha}(\vec{q}) a_\alpha^{out}(\vec{q})^\dagger e^{-iq.x}], \quad (2.121)$$

where $q^0 = w_q = |\vec{q}|$, $\alpha = \pm$ are the two helicities. Now to parameterize the graviton four-momentum, one uses $(w_q, w, \overline{w})$

$$q^\mu = \frac{w_q}{1 + w\overline{w}}(1 + w\overline{w}, w + \overline{w}, -i(w - \overline{w}), 1 - w\overline{w}), \quad (2.122)$$

where $\epsilon^{\pm\mu\nu} = \epsilon^{\pm\mu} + \epsilon^{\pm\nu}$, this gives the polarization tensors as

$$\epsilon^{+\mu}(\vec{q}) = \frac{1}{\sqrt{2}}(\overline{w}, 1, -i, -\overline{w}), \qquad \epsilon^{-\mu}(\vec{q}) = \frac{1}{\sqrt{2}}(w, 1, i, -w). \quad (2.123)$$

Obeying the conditions $\epsilon^{\pm\mu\nu} q_\nu = \epsilon_\mu^{\pm\mu} = 0$. In retarded Bondi coordinates, on $\mathcal{I}^+$,

$$C_{zz}(u, z, \overline{z}) = \kappa \lim_{r\to\infty} \frac{1}{r} h_{zz}^{out}(r, u, z, \overline{z}). \quad (2.124)$$

Using $h_{zz} = \partial_z x^\mu \partial_z x^\nu h_{\mu\nu}$ and (2.121)

$$C_{zz} = \kappa \lim_{r\to\infty} \frac{1}{r} \partial_z x^\mu \partial_z x^\nu \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2w_q} [\epsilon_{\mu\nu}^{\alpha *}(\vec{q}) a_\alpha^{out}(\vec{q}) e^{-iw_q u - iw_q r(1-\cos\theta)} + h.c.], \quad (2.125)$$

where θ is the angle between $\vec{x}$ and $\vec{q}$. The integrand has stationary points at $\theta = 0, \pi$. Over the momentum space S^2, the stationary phase approximation to the integral results in

$$C_{zz} = -\frac{i\kappa}{4\pi^2(1 + z\overline{z})^2} \int_0^\infty dw_q [a_+^{out}(w_q \hat{x}) e^{-iw_q u} - a_-^{out}(w_q \hat{x})^\dagger e^{iw_q u}]. \quad (2.126)$$

In the large r limit, the additional part coming from stationary part $\theta = \pi$ vanishes (T. He et al., 2015).

Defining

$$N^w_{zz}(z,\overline{z}) = \int_{-\infty}^{+\infty} due^{iwu}\partial_u C_{zz}. \tag{2.127}$$

One gets

$$N^w_{zz}(z,\overline{z}) = -\frac{\kappa}{2\pi(1+z\overline{z})^2}\int_0^{\infty} dw_q w_q[a^{out}_{+}(w_q\hat{x})\delta(w_q-w)+a^{out}_{-}(w_q\hat{x})^{\dagger}\delta(w_q+w)]. \tag{2.128}$$

When w is positive one gets a contribution from the first term only and similarly when w is negative one gets a contribution from the second term only. So for $w > 0$

$$N^w_{zz}(z,\overline{z}) = -\frac{\kappa w a^{out}_{+}(w\hat{x})}{2\pi(1+z\overline{z})^2}, \qquad N^{-w}_{zz}(z,\overline{z}) = -\frac{\kappa w a^{out}_{+}(w\hat{x})^{\dagger}}{2\pi(1+z\overline{z})^2}. \tag{2.129}$$

One needs a definition in a hermitian way when dealing with the zero mode case

$$N^0_{zz} \equiv \lim_{w\to 0^+}\frac{1}{2}(N^w_{zz}+N^{-w}_{zz}). \tag{2.130}$$

Continuing with this definition, one gets

$$N^0_{zz}(z,\overline{z}) = \frac{\kappa}{4\pi(1+z\overline{z})^2}\lim_{w\to 0^+}[wa^{out}_{+}(w\hat{x}) + wa^{out}_{-}(w\hat{x}^{\dagger})]. \tag{2.131}$$

On $\mathcal{I}^-$, one has an analogous term

$$M^w_{zz}(z,\overline{z}) \equiv \int_{-\infty}^{\infty} d\nu e^{iw\nu}\partial_\nu D_{zz}. \tag{2.132}$$

Again for $w > 0$ it gives

$$M^w_{zz}(z,\overline{z}) = -\frac{\kappa w a^{in}_{+}(w\hat{x})}{2\pi(1+z\overline{z})^2}, \qquad M^{-w}_{zz}(z,\overline{z}) = -\frac{\kappa w a^{in}_{+}(w\hat{x})^{\dagger}}{2\pi(1+z\overline{z})^2}, \tag{2.133}$$

where $a_\pm^{in}$ and $a_\pm^{in\dagger}$ annihilate and create incoming gravitons on $\mathcal{I}^-$ respectively. At $w = 0$

$$M^0_{zz}(z,\bar{z}) = -\frac{\kappa}{4\pi(1+z\bar{z})^2} \lim_{w\to 0^+}[wa_+^{in}(w\hat{x}) + wa_-^{in}(w\hat{x})^\dagger]. \tag{2.134}$$

From (2.127) and (2.132)

$$N^0_{zz}(z,\bar{z}) = D_z^2 N, \qquad M^0_{zz}(z,\bar{z}) = D_z^2 M. \tag{2.135}$$

Defining $\mathcal{O}_{zz}$ as

$$\mathcal{O}_{zz} \equiv N^0_{zz}(z,\bar{z}) + M^0_{zz}(z,\bar{z}) = D_z^2 N + D_z^2 M. \tag{2.136}$$

Remembering the future supertranslation charge (2.114), it can be rewritten as

$$\mathcal{Q}_f^+ = \frac{1}{4\pi G}\int dud^2z f\left[\gamma_{z\bar{z}}T_{uu} + \frac{1}{2}\partial_u(\partial_z U_{\bar{z}} + \partial_{\bar{z}}U_z)\right], \tag{2.137}$$

where

$$U_z = \frac{1}{2}D^2 C_{zz}, \qquad V_z = \frac{1}{2}D^z D_{zz}. \tag{2.138}$$

Using the matching conditions and energy conservation one gets

$$V_z|_{\mathcal{I}^-_+} = U_z|_{\mathcal{I}^+_-}. \tag{2.139}$$

Now with the choice of $f = \delta^2(z-w)$ one can show that

$$\int_{\mathcal{I}^+} du\left[\gamma_{z\bar{z}}T_{uu} + \frac{1}{2}\partial_u\big(\partial_z U_{\bar{z}} + \partial_{\bar{z}}U_z\big)\right] = \int_{\mathcal{I}^-} d\nu\left[\gamma_{z\bar{z}}T_{\nu\nu} + \frac{1}{2}\partial_\nu\big(\partial_z V_{\bar{z}} + \partial_{\bar{z}}V_z\big)\right]. \tag{2.140}$$

With few manipulations this can be rearranged as

$$\gamma_{z\bar{z}}\left(\int_{\mathcal{I}^+} duT_{uu} - \int_{\mathcal{I}^-} d\nu T_{\nu\nu}\right) = \partial_{\bar{z}}V_z|^{\mathcal{I}^-_+}_{\mathcal{I}^-_-} - \partial_{\bar{z}}U_z|^{\mathcal{I}^+_+}_{\mathcal{I}^+_-}. \tag{2.141}$$

Then the soft graviton current which can be defined as the difference of the incoming and outgoing energy fluxes at a point $(z, \bar{z})$ on CS^2 (Strominger, 2014). This current takes the form

$$P_z = \frac{1}{2G}\left(V_z\,|^{\mathcal{I}^-_+}_{\mathcal{I}^-_-} - U_z\,|^{\mathcal{I}^+_+}_{\mathcal{I}^+_-}\right) = \frac{1}{4G}\gamma^{z\bar{z}}\partial_{\bar{z}}\mathcal{O}_{zz}. \tag{2.142}$$

2.6.1 Soft Graviton Theorem as Ward Identity

In (2.142), one can see that the soft graviton current P_z is expressed in terms of standard creation and annihilation operators of the momentum space. To form a Ward Identity for soft graviton, one needs to consider an S-matrix. An S-matrix element is given as

$$\langle z_1^{out}, ...|\, S\, |z_1^{in}, ...\rangle\,, \tag{2.143}$$

where the in (out) momenta are parameterized by $z^{in}(z^{out})$. Consider the element $\langle z_1^{out}, ...| : \mathcal{O}S : |z_1^{in}, ...\rangle$ with a time ordered insertion. Now equipped with (2.136) and the knowledge that $a_-^{out}(w\hat{x})^\dagger$ $(a_+^{in}(w\hat{x}))$ annihilates the out (in) state for $w \to 0$, the S-matrix element takes the form

$$\langle z_1^{out}, ...| : \mathcal{O}_{zz}S : |z_1^{in}, ...\rangle = -\frac{\kappa}{4\pi(1+z\bar{z})^2}\lim_{w\to 0^+}[w\,\langle z_1^{out}, ...|\, a_+^{out}(w\hat{x})S\,|z_1^{in}, ...\rangle$$
$$+\, w\,\langle z_1^{out}, ...|\, Sa_-^{in}(w\hat{x})^\dagger\,|z_1^{in}, ...\rangle]. \tag{2.144}$$

The first term has a single outgoing positive helicity soft graviton with spatial momentum $w\hat{x}$ and the second term has a single incoming negative helicity soft graviton with spatial momentum (T. He et al., 2015). These two amplitudes are the same. So simplifying the relationship gives

$$\langle z_1^{out},...| : \mathcal{O}_{zz} S : |z_1^{in},...\rangle = -\frac{\kappa}{2\pi(1+z\overline{z})^2} \lim_{w\to 0^+} [w \langle z_1^{out},...| \, a_+^{out}(w\hat{x}) S \, |z_1^{in},...\rangle]. \tag{2.145}$$

With a positive helicity outgoing graviton, the soft graviton theorem follows as

$$\begin{aligned}
&\lim_{w\to 0}[w \langle z_1^{out},...| \, a_+^{out}(\vec{q}) S \, |z_1^{in},...\rangle] \\
&= \frac{\kappa}{2} \lim_{w\to 0^+} \left[\sum_{k=1}^{m} \frac{w[p_k^{'}.\epsilon^+(q)]^2}{p_k^{'}.q} - \sum_{k=1}^{n} \frac{w[p_k^{'}.\epsilon^+(q)]^2}{p_k.q} \right] \langle z_1^{out},...| \, S \, |z_1^{in},...\rangle \, .
\end{aligned} \tag{2.146}$$

With the aid of momenta parametrization

$$\begin{aligned}
p_k^{\mu} &= E_k^{in}\left(1, \frac{z_k^{in}+\overline{z}_k^{in}}{1+z_k^{in}\overline{z}_k^{in}}, \frac{-i(z_k^{in}-\overline{z}_k^{in})}{1+z_k^{in}\overline{z}_k^{in}}, \frac{1-z_k^{in}\overline{z}_k^{in}}{1+z_k^{in}\overline{z}_k^{in}}\right), \\
p_k^{'\mu} &= E_k^{out}\left(1, \frac{z_k^{out}+\overline{z}_k^{out}}{1+z_k^{out}\overline{z}_k^{out}}, \frac{-i(z_k^{out}-\overline{z}_k^{out})}{1+z_k^{out}\overline{z}_k^{out}}, \frac{1-z_k^{out}\overline{z}_k^{out}}{1+z_k^{out}\overline{z}_k^{out}}\right), \\
q^{\mu} &= w\left(1, \frac{z+\overline{z}}{1+z\overline{z}}, \frac{-i(z-\overline{z})}{1+z\overline{z}}, \frac{1-z\overline{z}}{1+z\overline{z}}\right), \\
\epsilon^{+\mu}(q) &= \frac{1}{\sqrt{2}}\big(\overline{z}, 1, -i, -\overline{z}\big).
\end{aligned} \tag{2.147}$$

Using these relations, and inserting them into the S-matrix element (2.145)

$$\begin{aligned}
\langle z_1^{out},...| : \mathcal{O}_{zz} S : |z_1^{in},...\rangle = &-\frac{8G}{(1+z\overline{z})} \langle z_1^{out},...| \, S \, |z_1^{in},...\rangle \\
&\times \left[\sum_{k=1}^{m} \frac{E_k^{out}(\overline{z}-\overline{z}_k^{out})}{(z-z_k^{out})(1+z_k^{out}\overline{z}_k^{out})} - \sum_{k=1}^{n} \frac{E_k^{in}(\overline{z}-\overline{z}_k^{in})}{(z-z_k^{in})(1+z_k^{in}\overline{z}_k^{in})} \right].
\end{aligned} \tag{2.148}$$

Utilizing the connection between P_z and $\mathcal{O}_{zz}$ (2.142) leads to

$$\begin{aligned}
\langle z_1^{out},...| : P_z S : |z_1^{in},...\rangle &= \frac{1}{4G}\gamma^{z\overline{z}}\partial_{\overline{z}} \langle z_1^{out},...| : \mathcal{O}_{zz} S : |z_1^{in},...\rangle \\
&= \langle z_1^{out},...| \, S \, |z_1^{in},...\rangle \left[\sum_{k=1}^{m} \frac{E_k^{out}}{(z-z_k^{out})} - \sum_{k=1}^{n} \frac{E_k^{in}}{(z-z_k^{in})} \right] \\
&+ \langle z_1^{out},...| \, S \, |z_1^{in},...\rangle \left[\sum_{k=1}^{m} \frac{E_k^{out}\overline{z}_k^{out}}{(1+z_k^{out}\overline{z}_k^{out})} - \sum_{k=1}^{n} \frac{E_k^{in}\overline{z}_k^{in}}{(1+z_k^{in}\overline{z}_k^{in})} \right].
\end{aligned} \tag{2.149}$$

Due to total momentum conservation, the last square bracket vanishes. So one ends up with

$$\langle z_1^{out}, ...| : P_z S : |z_1^{in}, ...\rangle = \langle z_1^{out}, ...| \, S \, |z_1^{in}, ...\rangle \left[\sum_{k=1}^{m} \frac{E_k^{out}}{(z - z_k^{out})} - \sum_{k=1}^{n} \frac{E_k^{in}}{(z - z_k^{in})} \right], \tag{2.150}$$

which completely regenerates the supertranslation Ward identity.

2.7 Superrotations

So far the investigation was built on the m_B and C_{zz}, the first nontrivial corrections to the metric near $\mathcal{I}$. And it has been observed that m_B has a relation with the total mass and N_z has a similar relationship with the angular momentum. As a result, from the matching conditions of m_B, supertranslation charges emerged. Now it is time to investigate matching conditions for N_z and discover the superrotation charges.

The angular momentum characteristic N_z has the constraint equation $G_{uz} = 8\pi G T^M_{uz}$. The leading uz component of this equation is

$$\partial_u N_z = \frac{1}{4}\partial_z (D_z^2 C^{zz} - D_{\overline{z}}^2 C^{\overline{zz}}) - u\partial_u \partial_z m_B - T_{uz}, \tag{2.151}$$

where the T_{uz} is

$$T_{uz} \equiv 8\pi G \lim_{r\to\infty} \left[r^2 T^M_{uz} \right] - \frac{1}{4}\partial_z (C_{zz} N^{zz}) - \frac{1}{2} C_{zz} D_z N^{zz}. \tag{2.152}$$

N_z is constrained in relation to a momentum density T_{uz}, in contrast to m_B, which was constrained in relation to the energy density T_{uu}. This fixes N_z up to an integration function. One can fix this function by the matching condition

$$N_z(z, \overline{z}) \mid_{\mathcal{I}^+_-} = N_z(z, \overline{z}) \mid_{\mathcal{I}^-_+}, \tag{2.153}$$

resembling the matching condition for m_B, which is not a surprise. And again, this indicates the existence of an infinity of conserved charges. They can be produced from a random vector field Y^z on S^2. By the matching condition

$$\mathcal{Q}_Y^+ = \frac{1}{8\pi G}\int_{\mathcal{I}_-^+} d^2z(Y_{\bar{z}}N_z + Y_zN_{\bar{z}}) = \frac{1}{8\pi G}\int_{\mathcal{I}_+^-} d^2z(Y_{\bar{z}}N_z + Y_zN_{\bar{z}}) = \mathcal{Q}_Y^-, \quad (2.154)$$

one gets the sign of the conservation of the superrotation charge. If the vector field is taken to be a delta function, for every angle the new conservation laws associate net in and out angular momentum flux.

2.7.1 Symmetries

Lorentz Killing vectors are of the general form

$$\zeta_Y = \big(1+\frac{u}{2r}\big)Y^z\partial_z - \frac{u}{2r}D^{\bar{z}}D_zY^z\partial_{\bar{z}} - \frac{1}{2}(u+r)D_zY^z\partial_r + \frac{u}{2}D_zY^z + c.c., \quad (2.155)$$

where $(Y^z, Y^{\bar{z}})$ is a two dimensional vector field on CS^2. At null infinity, ζ_Y simplifies to

$$\zeta_Y \mid_{\mathcal{I}^+} = Y^z\partial_z + \frac{u}{2}D_zY^z\partial_u + c.c.. \quad (2.156)$$

So now one needs to check that in Minkowski space, after choosing the vector field as

$$Y^z = 1, z, z^2, i, iz, iz^2, \quad (2.157)$$

do they still generate the Lorentz transformations or not. Without restrictions, computing for a general Y^z, the Lie derivative with respect to ζ_Y of distinct metric components gives

$$
\begin{aligned}
\mathcal{L}_Y g_{ur} &= \mathcal{O}\left(\frac{1}{r^2}\right),\\
\mathcal{L}_Y g_{zr} &= \mathcal{O}\left(\frac{1}{r}\right),\\
\mathcal{L}_Y g_{z\bar{z}} &= \mathcal{O}\big(r\big),\\
\mathcal{L}_Y g_{uu} &= \mathcal{O}\left(\frac{1}{r}\right),\\
\mathcal{L}_Y g_{\bar{z}\bar{z}} &= 2r^2\gamma_{z\bar{z}}\partial_{\bar{z}}Y^z\mathcal{O}\big(r\big).
\end{aligned}
\tag{2.158}
$$

If the first $\mathcal{O}(r^2)$ term vanishes, the large r fall-off conditions are obeyed. For this to be satisfied, Y^z has to be a holomorphic vector field (Strominger, 2018). This is locally resolved if $Y^z = z^n$ for any integer n. On the other hand, to obtain the globally defined vector fields, one has to go with the restrictive choice $Y^z \sim 1, z, z^2$.

2.7.2 Canonical Formalism

In a canonical way, superrotation symmetries are produce by the superrotation charges, but only at linearized order. To see this one needs to start by checking the boundary data representing the geometry, which will be changed under superrotations. The Lie derivative with respect to Y (represented as δ_Y) of the C_{zz} term of the metric is

$$
\delta_Y C_{zz} = \frac{u}{2}D\cdot Y N_{zz} + Y\cdot DC_{zz} - \frac{1}{2}D\cdot YC_{zz} + 2D_zY^zC_{zz} - uD_z^3Y^z. \tag{2.159}
$$

Now taking the u derivative gives

$$
\delta_Y N_{zz} = \frac{u}{2}D\cdot Y\partial_u N_{zz} + Y\cdot DN_{zz} + 2D_zY^zN_{zz} - D_z^3Y^z. \tag{2.160}
$$

From these, the conserved superrotation charge follows as

$$
\mathcal{Q}_Y^+ = \frac{1}{8\pi G}\int_{\mathcal{I}_-^+} d^2z\big[Y_{\bar{z}}N_z + Y_zN_{\bar{z}}\big]. \tag{2.161}
$$

Integrating it by parts and remembering (2.151) (the constraint equation for the angular momentum aspect N_z)

$$\mathcal{Q}_Y^+ = \mathcal{Q}_H^+ + \mathcal{Q}_S^+, \tag{2.162}$$

$$\mathcal{Q}_S^+ = -\frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z[D_z^3Y^z uN_{\overline{z}}^z + D_{\overline{z}}^3Y^{\overline{z}}uN_z^{\overline{z}}], \tag{2.163}$$

$$\mathcal{Q}_H^+ = \frac{1}{8\pi G}\int_{\mathcal{I}^+} dud^2z\big(Y_{\overline{z}}T_{uz} + Y_zT_{u\overline{z}} + u\partial_zY_{\overline{z}}T_{uu} + u\partial_{\overline{z}}Y_zT_{uu}\big). \tag{2.164}$$

Looking at the last two equations, one can see that while the soft charges are linear in the C_{zz}, the hard charge is quadratic. To check if the symmetries are generated or not one needs commutator relations, which require this commutator

$$[N_{\overline{z}\overline{z}}(u,z,\overline{z}), C_{ww}(u^{'},w,\overline{w})] = 16\pi Gi\gamma_{z\overline{z}}\delta^2(z-w)\delta(u-u^{'}). \tag{2.165}$$

Equipped with this, the desired commutators are

$$\begin{aligned}
&[\mathcal{Q}_S^+, C_{zz}] = -iuD_z^3Y^z, \\
&[\mathcal{Q}_H^+, C_{zz}] = \frac{iu}{2}D\cdot YN_{zz} + iY\cdot DC_{zz} - \frac{i}{2}D\cdot YC_{zz} + 2iD_zY^zC_{zz}.
\end{aligned} \tag{2.166}$$

These prove that

$$[\mathcal{Q}_Y^+, ...] = i\delta_Y. \tag{2.167}$$

So the symmetry is generated by the conserved charge. But there is a very important outcome rising from these commutators. Commutators with $\mathcal{Q}_Y^+$ shift the news by a function that approaches to a constant at $\mathcal{I}_\pm^+$, in the mean time C_{zz} diverges linearly. Looking back at the boundary conditions that were used thorough out this thesis, one sees that they are violated with such behaviour. It basically maps the point in the phase space to points outside that phase space. This is physically an unacceptable result. To make sense of it, it is reasonable to demand a larger phase space. Having a larger phase space which permits so called defects, one can understand the superrotations in a more explicit manner.

2.7.3 Subleading Soft Theorem

The soft theorem is already established for supertranslations. But with the existence of the infinities of conserved superrotation charges and symmetries, it is reasonable to assume there exists a correlated soft theorem which will be the second soft theorem in gravity.

Superrotation charge conservation at the level of the quantum S-matrix indicates that

$$\langle out| \left(\mathcal{Q}_Y^+ S - S\mathcal{Q}_Y^-\right) |in\rangle = 0. \tag{2.168}$$

This is conventionally written in terms of the coordinates $(z_k, \bar{z}_k)$. Now by the decomposition of $\mathcal{Q}_Y^+$, $\mathcal{Q}_S^+$ and $\mathcal{Q}_H^+$, it becomes obvious that above expression indicates a soft graviton inclusion to a hard term.

Using the previous conclusions and setting $Y^z = \frac{1}{z-(q^1+iq^2/q^0+q^3)}$ (Kapec et al., 2014), one finds this equation can be re-expressed in momentum space as

$$\lim_{w\to 0}(1 + w\partial_w) \langle p_{n+1}, p_{n+2}, ...| \, a_-(q)S \, |p_1, p_2, ...\rangle = \sqrt{8\pi G} S^{(1)} \langle p_{n+1}, p_{n+2}, ...| \, S \, |p_1, p_2, ...\rangle \,, \tag{2.169}$$

where $a_-(q)$ is the annihilation operator for a negative helicity graviton of four momentum $q = w(1, \hat{q})$, and the subleading soft factor is

$$S^{(1)} = -i \sum_k \frac{p_{k\mu}\epsilon^{-\mu\nu} q^\lambda J_{k\lambda\nu}}{p_k \cdot q}, \qquad J_{k\mu\nu} \equiv L_{k\mu\nu} + S_{k\mu\nu}. \tag{2.170}$$

$L_{k\mu\nu}$ is the orbital angular momentum and $S_{k\mu\nu}$ the helicity of the internal spin of the kth particle (Strominger, 2018). By replacing p_ν with $q^\mu J_{k\mu\nu}$, the subleading soft factor $S^{(1)}$ can be obtained from the leading soft factor, so basically about q, changing the position of translations with rotations does the trick.

This is logical since the first term in Q_H^+ is $Y^z T_{uz}$ (2.164), which is responsible for generating a rotation, whereas the charge which generates supertranslation has a hard term involving fT_{uu} (2.116).

2.8 Goldstone Bosons

No matter how complex the dynamics are, the conserved quantities coming from the symmetry are exactly conserved. So the symmetries offer non-perturbative outcomes and they are not valid for special physical cases only. Spontaneously broken symmetries do not make physics duller, since many physical scenarios occur on a background which spontaneously breaks the fundamental symmetries. Also by use of Goldstone's theorem (Goldstone, 1961), it has been proven that one does not lose the exactness of the results related to symmetries. Being non-linearly realized, it is harder to see the invariance of the Lagrangian, but spontaneously broken symmetries still constrain the Lagrangian.

In classical field theory, one can distinguish the state of the system by one of the solutions of the equation of motion. This solution can be referred to as the background (or vacuum). It is stable if it stays finite and has small perturbations around it during its evolution. Conventionally, the solutions that are chosen to be background are the ones that minimise the energy.

At the quantum level, consider a field theory, with a global continuous symmetry group C so that it is spontaneously broken to a subgroup L other than $C(L \nsubseteq C)$ and let there exist an explicit notion of the gap. This indicates that the span of the theory involves minimum one gapless mode (Goldstone et al., 1962).

To prove this, one can use the conventional spectral decomposition method. Assume that group C is uniform and globally continuous. C being continuous lets one construct the currents and C being global lets one avoid gauge symmetries which makes the conserved currents trivial. C has been chosen as global to get at least one physical massless mode. And by assuming C is uniform one rules out the spacetime symmetries, implying that the vacuum is an eigenstate of unbroken translation generators P_μ (it is homogeneous).

Now letting Q to be a generator of C so that Q is spontaneously broken, one automatically obtains a field ψ leading to the relation

$$\langle 0| \left[Q, \psi(x)\right] |0\rangle \neq 0. \tag{2.171}$$

By using the basis vectors $|n_{\vec{k}}\rangle$ (eigenvectors of P_μ), one injects a closure relation and spectrally decomposes the above relation (Naegels, 2021).

$$\begin{aligned}\langle 0| \left[Q, \psi(x)\right] |0\rangle &= \int d^{d-1}x' \, \langle 0| \left[j^0(x'), \psi(x)\right] |0\rangle \\ &= \int d^{d-1}x' \sum_n \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \Big(\langle 0| j^0(x') |n_{\vec{k}}\rangle \langle n_{\vec{k}}| \psi(x) |0\rangle \\ &- \langle 0| \psi(x) |n_{-\vec{k}}\rangle \langle n_{-\vec{k}}| j^0(x') |0\rangle \Big).\end{aligned} \tag{2.172}$$

Since Q generates a uniform symmetry, one can translate the conserved current to the origin. Keeping in mind that with having a uniform symmetry one has $j^0(x) = e^{ix^\mu P_\mu} j^0(0) e^{-ix^\mu P_\mu}$. Equipped with this relation and (2.172) one gets the conserved current as

$$\begin{aligned}\langle 0| \left[Q, \psi(x)\right] |0\rangle &= \int d^{d-1}x' \sum_n \int \frac{d^{d-1}k}{(2\pi)^{d-1}} e^{-ik_\mu x'^\mu} (\langle 0| j^0(0) |n_{\vec{k}}\rangle \langle n_{\vec{k}}| \psi(x) |0\rangle \\ &- \langle 0| \psi(x) |n_{-\vec{k}}\rangle \langle n_{-\vec{k}}| j^0(0) |0\rangle) \\ &= \sum_n \int d^{d-1}k e^{-iE_n(\vec{k})t} \phi(\vec{k}) \big[\langle 0| j^0(0) |n_{\vec{k}}\rangle \langle n_{\vec{k}}| \psi(x) |0\rangle \\ &- \langle 0| \psi(x) |n_{-\vec{k}}\rangle \langle n_{-\vec{k}}| j^0(0) |0\rangle \big],\end{aligned} \tag{2.173}$$

where

$$\int \frac{d^{d-1}x'}{(2\pi)^{d-1}} e^{i\vec{k}\vec{x}} = \phi(\vec{k}) \xrightarrow[V \to \infty]{} \delta^{d-1}(\vec{k}). \tag{2.174}$$

So one has only the modes in the zero momentum limit interfere in the spectral decomposition equation. It is also evident that $\frac{dQ}{dt} = 0$, meaning that $\psi(x)$ is the only time dependent variable in the equation. Thus only the modes with

$$E_n(\vec{k}) \xrightarrow[\vec{k}\to\vec{0}]{} 0, \tag{2.175}$$

which means, the contribution is coming from the massless modes summed over n and this sum should be non zero. So one guarantees that there exists at least one massless particle satisfying this relation. These are so called Nambu-Goldstone (NG) modes, products of acting on the vacuum with a broken symmetry.

By enforcing the spontaneously broken symmetries successionally on the broken states, one can get a set of broken states. This set is called the coset space and it is continuous (Naegels, 2021). All the states in this set have the same energy. Since the spacetime symmetries are neglected, the states with the same potential energy means there is no potential barrier separating the states. At quadratic order, considering the perturbed Lagrangian, Lagrangian of the fluctuations in the vicinity of a broken state, in the directions of the broken symmetries expected to do not contain potential terms. So these fluctuations are massless. Now having these fluctuations independent of each other, once can classify them as NG modes. Intuitively they can be understood as spacetime modulated action of the spontaneously broken symmetries on the background. One can see that in this discussion the continuity of C played a very vital role to obtain masslessness.

In many cases, only the internal spontaneously broken symmetries are involved, which indicates that the algebra of C is spin 0. So by acting on the vacuum, these fluctuations generate scalars, which are historically called Goldstone bosons.

Relating this to the conserved charges discussion in (subsection 2.3.3), one has a charge $\mathcal{Q}_\epsilon^+$ that generates a symmetry of the Lagrangian of any Abelian gauge theory. And this charge creates a mode which transforms inhomogeneously under a broken symmetry. This mode is a soft photon mode (Strominger, 2018). This indicates that soft photons are Goldstone bosons corresponding to the spontaneous breaking of global symmetry. Since one can add any number of soft photons and still get the same zero energy, there is an infinite vacuum degeneracy.

2.9 Memory Effect

2.9.1 Gravitational Memory

Writing the metric near $\mathcal{I}^+$ using the Bondi coordinates and utilizing the corresponding constraint equations, one finds that the angular momentum characteristic is connected to the Weyl tensor component Ψ_1^0 on $\mathcal{I}^+$ by (Pasterski et al., 2016)

$$N_z = \lim_{r\to\infty} r^3 C_{zrru}, \tag{2.176}$$

where

$$\mathrm{Im}\Psi_2^0 = \mathrm{Im}\lim_{r\to\infty} r\gamma^{z\bar{z}} C_{z\bar{z}zr} = -\mathrm{Im}\left[\frac{1}{2}D_z^2 C^{zz} + \frac{1}{4}C_{zz}N^{zz}\right]. \tag{2.177}$$

The same analysis can be made near $\mathcal{I}^-$ using the respective Bondi coordinates and constraints. Now considering spacetimes that decay to vacuum at $\mathcal{I}^-_-$ and $\mathcal{I}^+_+$, one demands the condition

$$N_z \mid_{\mathcal{I}^+_+} = N_z \mid_{\mathcal{I}^-_-} = m_B \mid_{\mathcal{I}^+_+} = m_B \mid_{\mathcal{I}^-_-} = 0. \tag{2.178}$$

Additionally around $\mathcal{I}^\pm_\pm$, radiative modes should be unexcited

$$N_{zz} \mid_{\mathcal{I}^\pm_\pm} = \mathrm{Im}\Psi_2^0 \mid_{\mathcal{I}^\pm_\pm} = 0. \tag{2.179}$$

The general solution to this equation is

$$C_{zz} = -2D_z^2 C, \tag{2.180}$$

where C is an arbitrary function of $(z, \bar{z})$. Solutions of this kind are related to each other by supertranslations which indicates the degeneracy of the vacuum. Focusing on the continuity conditions for m_B and C_{zz} one gets

$$C_{zz} \mid_{\mathcal{I}^+_-} = C_{zz} \mid_{\mathcal{I}^-_+}, \qquad m_B \mid_{\mathcal{I}^+_-} = m_B \mid_{\mathcal{I}^-_+} . \tag{2.181}$$

Following above statement with the Bianchi identity for N_{zz}

$$\partial_{[z} N_{\overline{z}]} \mid_{\mathcal{I}^-_+} = \partial_{[z} N_{\overline{z}]} \mid_{\mathcal{I}^+_-} . \tag{2.182}$$

By integrating the constraint equation (2.180) and defining

$$\Delta^+ C_{zz} = C_{zz} \mid_{\mathcal{I}^+_+} - C_{zz} \mid_{\mathcal{I}^+_-}, \qquad \Delta^+ m_B = m_B \mid_{\mathcal{I}^+_+} - m_B \mid_{\mathcal{I}^+_-}, \tag{2.183}$$

one can find the dissimilarity amid the initial C function and the final C function as

$$D_z^2 \Delta^+ C^{zz} = 2 \int du \big(T_{uu} + 2\Delta^+ m_B \big). \tag{2.184}$$

Indicating that

$$\Delta^+ C(z, \overline{z}) = \int d^2 w \gamma_{w\overline{w}} G(z; w) \Big[\int du T_{uu}(w) + \Delta m_B \Big], \tag{2.185}$$

where the G is

$$\begin{aligned} G(z; w) &= -\frac{1}{\pi} \sin^2 \frac{\Theta}{2} \log \sin^2 \frac{\Theta}{2}, \\ \sin^2 \frac{\Theta(z, w)}{2} &\equiv \frac{\mid z - w \mid^2}{(1 + w\overline{w})(1 + z\overline{z})}. \end{aligned} \tag{2.186}$$

Using the same argument for the shift of C on $\mathcal{I}^-$ one can define

$$\Delta C = \Delta^+ C - \Delta^- C, \tag{2.187}$$

which gives the equality

$$\Delta C(z,\overline{z}) = \int d^2 w \gamma_{w\overline{w}} G(z;w) \Big[\int du T_{uu}(w) - \int d\nu T_{\nu\nu}(w) \Big]. \tag{2.188}$$

Now considering fixed angle detectors which travel along fixed r and $(z, \overline{z})$, they can be represented as

$$X^{\mu}_{BMS}(s) = (s, r_0, z_0, \overline{z}_0), \tag{2.189}$$

where r_0 is large (Strominger & Zhiboedov, 2016). Now considering another set of detectors which are inertial and moving along geodesics

$$\partial_s^2 X^{\mu}_{Geo}(s) + \Gamma^{\mu}_{\nu\lambda} \partial_s X^{\nu}_{Geo}(s) X^{\lambda}_{Geo}(s) = 0. \tag{2.190}$$

The relation between the X_{BMS} and X_{Geo} turns out to be

$$X^{u,r}_{BMS}(s) = X^{u,r}_{Geo}(s) + \mathcal{O}\Big(\frac{1}{r_0}\Big), \qquad X^{z}_{BMS}(s) = X^{z}_{Geo}(s) + \mathcal{O}\Big(\frac{1}{r_0^2}\Big). \tag{2.191}$$

For $u > r_0$ the radius can become small since truly inertial detectors do not remain at fixed r and z. Checking the detectors' worldlines considering that they encounter a pulse of radiation gives

$$L = \frac{2r_0 \mid \delta z \mid}{1 + z_1 \overline{z}_1}, \qquad\qquad \delta z \equiv z_1 - z_2, \tag{2.192}$$

where z_1 and z_2 are the respective initial positions of the detectors and δz is taken to be the order of $\frac{1}{r_0}$. Now one can see that the metric goes through a transition but z_1 and z_2 are fixed, indicating that the proper distance between the detectors (L) has to be changed. And this new distance is found to be

$$\Delta L = \frac{r_0}{2L} \Delta C_{zz}(z_1, \overline{z}_1) \delta z^2 + c.c. = \frac{(1 + z_1 \overline{z}_1)^2}{8} \frac{L}{r_0} \Big(\Delta C_{zz}(z_1, \overline{z}_1) \frac{\delta z}{\delta \overline{z}} + c.c. \Big). \tag{2.193}$$

In terms of the energy flux, it can be rewritten as

$$\Delta C_{zz}(z_1,\overline{z}_1) = \frac{4}{\pi}\int d^2z^{'}\gamma_{z^{'}\overline{z}^{'}}\frac{(1+z^{'}\overline{z})^2}{(1+\overline{z}^{'}z^{'})(1+z\overline{z})^3}\frac{\overline{z}-\overline{z}^{'}}{z-z^{'}}\bigg(\int_{u_i}^{u_f} duT_{uu}(z^{'},\overline{z}^{'})+\Delta m_B\bigg), \tag{2.194}$$

which is the standard gravitational memory formula. This gravitational memory is strongly connected to soft theorems which are discussed in this thesis so far. To see this one can analyze the transverse traceless part of the asymptotic metric at $\mathcal{I}^+$

$$\Delta h_{\mu\nu}^{TT}(\vec{k}) = \frac{1}{r_0}\sqrt{\frac{G}{2\pi}}\bigg(\sum_{j=1}^{n}\frac{p_{j\mu}^{'}p_{j\nu}^{'}}{wk\cdot p_j^{'}} - \sum_{j=1}^{m}\frac{p_{j\mu}p_{j\nu}}{wk\cdot p_j}\bigg)^{TT}, \tag{2.195}$$

where $p_{j\mu}$ is the asymptotic momenta of the n incoming particles and $p_{j\mu}^{'}$ is the asymptotic momenta of the m outgoing particles, k can be considered as a coordinate on S^2 (Wiseman & Will, 1991).

The Fourier transform of $h_{\mu\nu}^{TT}(w,\vec{k})$ on $\mathcal{I}^+$ can be expressed as

$$h_{\mu\nu}^{TT}(w,\vec{k}) = 4\pi i \lim_{r\to\infty} r\int du e^{iwu}h_{\mu\nu}^{TT}(u,r\vec{k}). \tag{2.196}$$

Adding the assumptions that $r = r_0$ is large and for $u \to \pm\infty$, $h_{\mu\nu}^{TT}(u,r\vec{k})$ gets different finite values

$$\Delta h_{\mu\nu}^{TT}(\vec{k}) = \frac{1}{4\pi i r_0}\lim_{w\to 0}\big(-iwh_{\mu\nu}^{TT}(w,\vec{k})\big). \tag{2.197}$$

In the process of $n\to m$ scattering, QFT scattering amplitudes can be written as

$$\begin{aligned}\lim_{w\to 0} & A_{m+n+1}(p_1,...,p_n;p_1^{'},...,p_m^{'},(wk,\epsilon_{\mu\nu})) \\ &= \sqrt{8\pi G}S_{\mu\nu}\epsilon^{\mu\nu}A_{m+n}(p_1,...,p_n;p_1^{'},...,p_m^{'}) + \mathcal{O}(w^0).\end{aligned} \tag{2.198}$$

Left hand side of the (2.198) represents the $n\to m+1$ scattering amplitude where the $+1$ contribution is coming from the soft graviton.

Then using (2.197) and (2.198), one sees that the metric fluctuation obeys (to linear order)

$$\begin{aligned}
\lim_{w\to 0} w h^{TT}_{\mu\nu}(w,k)\epsilon^{\mu\nu} &= \lim_{w\to 0} \frac{w A_{m+n+1}(p_1,...,p_n;p_1^{'},...,p_m^{'},(wk,\epsilon_{\mu\nu}))}{A_{m+n}((p_1,...,p_n;p_1^{'},...,p_m^{'})} \\
&= \sqrt{8\pi G}\epsilon^{\mu\nu} \lim_{w\to 0} w S_{\mu\nu}(wk) \\
&= \sqrt{8\pi G}\epsilon^{\mu\nu}\left(\sum_{j=1}^{m} \frac{p_{j\mu}p_{p\nu}}{k\cdot p_j} - \sum_{j=1}^{n} \frac{p_{j\mu}^{'}p_{j\nu}^{'}}{k\cdot p_j^{'}}\right),
\end{aligned} \tag{2.199}$$

where

$$S_{\mu\nu} = \left(\sum_{j=1}^{m} \frac{p_{j\mu}p_{j\nu}}{wk\cdot p_j} - \sum_{j=1}^{n} \frac{p_{j\mu}^{'}p_{j\nu}^{'}}{wk\cdot p_j^{'}}\right)^{TT}, \tag{2.200}$$

giving the direct relation between the memory effect and the soft theorem representation (Strominger & Zhiboedov, 2016).

2.9.2 Spin Memory

This memory effect is a specific type of gravitational memory effect that involves asymmetric changes in the angular momentum because of gravitational and massless fields (gravitational waves). To see this take a circle G of radius R near $\mathcal{I}^+$ centred around z_0 on a sphere of large $r = r_0$ where $R << r_0$. If G is taken to be the orbit of a light ray, one can define

$$Z(\phi) = z_0\left[1 + \frac{Re^{i\phi}}{2r_0}\frac{1+z_0\overline{z}_0}{\sqrt{z_o\overline{z}_0}}\right] + \mathcal{O}\left(\frac{R^2}{r_0^2}\right), \tag{2.201}$$

where $\phi \sim \phi + 2\pi$. A light ray on G has a path $\phi(u)$ obeying

$$\begin{aligned}
ds^2 &= 0 \\
&= 1 - 2r_0^2\gamma_{z\overline{z}}\partial_u Z\partial_u\overline{Z} - 2\frac{m_B}{r_0} - r_0 C_{zz}(\partial_u Z)^2 - r_0 C_{\overline{z}\overline{z}}(\partial_u\overline{Z})^2 \\
&- [D^z C_{zz}\partial_u Z + D^{\overline{z}} C_{\overline{z}\overline{z}}\partial_u\overline{Z}] + ...
\end{aligned} \tag{2.202}$$

Looking at this one can see that only the square bracket term is odd under $\partial_u Z \to -\partial_u Z$. Simultaneously setting two light rays to opposite orbits, the time they return will digress by the integral of the odd term discussed above as

$$\Delta P = \oint_G (D^z C_{zz} dz + D^{\bar{z}} C_{\bar{z}\bar{z}} d\bar{z}), \tag{2.203}$$

for any contour G. For $C_{zz} = -2D_z^2 C$ i.e. vacuum

$$\Delta P_{\text{vacuum}} = -2 \oint_G d(D^z D_z C + C) = 0, \tag{2.204}$$

indicating that desynchronization appears only when the radiation passes through $\mathcal{I}^+$. Then the total time delay over all orbits is

$$\Delta^+ u = \frac{1}{2\pi R} \int du \oint_G (D^z C_{zz} dz + D^{\bar{z}} C_{\bar{z}\bar{z}} d\bar{z}), \tag{2.205}$$

hinting that between oppositely orbiting pulses, there is a shift in the interference pattern, which is called the spin memory effect. The displacement memory effect was established as a Green's function convoluted with an integral of the net local asymptotic energy flux, similar construction can be made for spin memory by obtaining an integral containing net local asymptotic angular momentum flux. Using the constraints and the properties of the selected Green's function one has

$$\text{Im}[\partial_{\bar{z}} D_z^3 C^{zz}] = 2\text{Im}[\partial_u \partial_{\bar{z}} N_z + \partial_{\bar{z}} T_{uz}], \tag{2.206}$$

$$\pi \text{Im}[D_w^2 C^{ww}] = -\text{Im} \int d^2 z \partial_{\bar{z}} G(z; w)[\partial_u N_z + T_{uz}]. \tag{2.207}$$

Using the Stokes theorem and integrating over D_G with boundary G gives

$$\pi \oint_G (D^w C_{ww} dw + D^{\bar{w}} C_{\bar{w}\bar{w}} d\bar{w}) = -2\text{Im} \int_{D_G} d^2 w \gamma_{w\bar{w}} \int d^2 z \partial_{\bar{z}} G(z; w)[\partial_u N_z + T_{uz}]. \tag{2.208}$$

After integration over u, it becomes

$$\Delta^+ u = -\frac{1}{\pi^2 R}\text{Im}\int_{D_G} d^2w\gamma_{w\overline{w}}\int d^2z\partial_{\overline{z}}G(z;w)\left[\Delta^+ N_z + \int duT_{uz}\right], \qquad (2.209)$$

where the shift in the angular momentum aspect is $\Delta^+ N_z \equiv N_z \mid_{\mathcal{I}^+_+} - N_z \mid_{\mathcal{I}^+_-}$.

After applying the same procedure near $\mathcal{I}^-$, one gets a similar formula for $\Delta^- \nu$ and by combining this result with $\Delta^+ u$, a formulation for the time delay ($\Delta\tau$) can be obtained as (Pasterski et al., 2016)

$$\begin{aligned}\Delta\tau &\equiv \Delta^+ u - \Delta^- \nu \\ &= -\frac{1}{\pi^2 R}\text{Im}\int_{D_G} d^2w\gamma_{w\overline{w}}\int d^2z\partial_{\overline{z}}G(z;w)\left[\int duT_{uz} - \int d\nu T_{\nu z}\right].\end{aligned} \qquad (2.210)$$

For localized wave packets penetrating $\mathcal{I}^+$ or massless particles, this relation simplifies to

$$\Delta\tau = -\frac{8G}{\pi R}\sum_k \left(\gamma^{z_k\overline{z}_k}\text{Im}\int_{D_G} d^2w\gamma_{w\overline{w}}L_{uz}(z_k)\partial_{\overline{z}_k}G(z_k;w) + \pi h_{k\in G}\right). \qquad (2.211)$$

This indicates a time-dragging effect since the second term on the right hand side hinting a time delay of order $\frac{h_k}{L}$ where h_k is the spin of the object passing through the boundary.

2.10 Soft Hair

The study of black hole physics has revealed the existence of a unique and intriguing phenomenon called soft hair. This term refers to low-energy excitations that can be associated with a black hole and act as a record of its previous interactions. The construction of soft hair is highly related to the supertranslations, superrotations and memory effect, as its name suggests.

The investigation starts with constructing supertranslations in Schwarzschild geometry. The Schwarzschild line element in advanced coordinates is

$$ds^2 = -Vd\nu^2 + 2d\nu dr + r^2\gamma_{AB}d\Theta^A d\Theta^B, \qquad (2.212)$$

where $V = 1 - \frac{2m_B}{r}$. Now the supertranslations ζ must satisfy the fall-off conditions and the Bondi gauge, so for Schwarzschild metric

$$\begin{aligned} \mathcal{L}_\zeta g_{rA} &= \partial_A \zeta^\nu + g_{AB}\partial_r \zeta^B = 0, \\ \mathcal{L}_\zeta g_{rr} &= 2\partial_r \zeta^\nu = 0, \\ \frac{r}{2} g^{AB}\mathcal{L}_\zeta g_{AB} &= rD_A\zeta^A + 2\zeta^r = 0. \end{aligned} \tag{2.213}$$

Giving the general supertranslation solution as

$$\zeta_f = f\partial_\nu + \frac{1}{r}D^A f \partial_A - \frac{1}{2}D^2 f \partial_r, \tag{2.214}$$

where $f = f(z, \bar{z})$. After taking the Lie derivative

$$\begin{aligned} ds^2 = &-\left(V - \frac{m_B D^2 f}{r^2}\right)d\nu^2 + d\nu dr - d\nu d\Theta^A D_A(2Vf + D^2 f) \\ &+ (r^2\gamma_{AB} + 2rD_A D_B f - r\gamma_{AB}D^2 f)d\Theta^A d\Theta^B, \end{aligned} \tag{2.215}$$

giving the horizon at $r = m_B + \frac{1}{2}D^2 f$ (Hawking et al., 2017). Another important detail is the supertranslated black hole metric is exact in r but only linear in f. The supertranslated black hole is a non-identical physical structure than the black hole that was developed at the start of this section since it involves a non-zero superrotation charge

$$\mathcal{Q}_Y^- = \frac{1}{8\pi}\int_{\mathcal{I}_+^-} d^2\Theta\sqrt{\gamma}Y^A N_A = -\frac{3m_B}{8\pi}\int_{\mathcal{I}_+^-} d^2\Theta\sqrt{\gamma}Y^A\partial_A f, \tag{2.216}$$

where Y^A is any smooth vector field on S^2 and $\delta_f N_A = -3m_B\partial_A f$ (Donnay et al., 2018).

2.10.1 Horizon Charges

In the case of the Schwarzschild metric, $\mathcal{I}^+$ is no longer a Cauchy surface. The new Cauchy surface can be denoted as $\mathcal{I}^+ \cup \mathcal{H}^+$ which indicates the existence of the charge conservation form (Hawking et al., 2016)

$$\hat{Q}_f^+ = \hat{Q}_f^{\mathcal{I}^+} + \hat{Q}_f^{\mathcal{H}^+}. \tag{2.217}$$

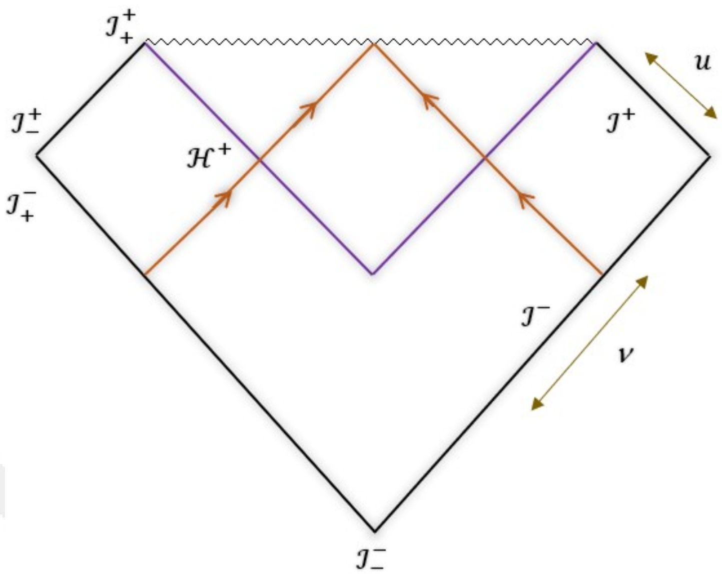

Figure 2.5: Diagram of a black hole formation under gravitational collapse. The orange lines indicate the shock wave and the purple line indicates the horizon. $\mathcal{I}^+ \cup \mathcal{H}^+$ is the Cauchy surface for the massless fields.

One has to be careful with the contribution coming from $\hat{Q}_f^{\mathcal{H}^+}$ since it depends on the choice of coordinates. In the rest of the analysis, the Bondi coordinates will be used.

Continuing with the Schwarzschild metric (perturbed), under the Bondi gauge

$$\hat{Q}_\zeta^{\partial\Sigma} = -\frac{1}{16\pi}\int_{\partial\Sigma} d^2\Theta\sqrt{\gamma}r^2 F_{r\nu}, \tag{2.218}$$

where $\partial\Sigma$ is an S^2 with constant ν and r. The $F_{r\nu}$ term turns out to be (Hawking et al., 2017)

$$\begin{aligned} F_{r\nu} = {} & \zeta^A\big(\partial_r h_{A\nu} - \frac{2}{r}h_{A\nu}\big) + \zeta^\nu\big(-\frac{1}{r^2}D^A h_{A\nu} - \frac{2}{r}h_{\nu\nu}\big) + \partial_r\zeta^\nu h_{\nu\nu} \\ & + \frac{1}{r^2}D^A\zeta^\nu h_{\nu A} + \partial_r\zeta^\nu V h_{\nu r} + \zeta^r\frac{2}{r}h_{\nu r}. \end{aligned} \tag{2.219}$$

For $\zeta = \zeta_f$ one obtains

$$F_{r\nu} = \frac{1}{r} D^A f \partial_r h_{A\nu} - f\left(\frac{2}{r} h_{\nu\nu} + \frac{4V}{r} h_{\nu r}\right) - D^2 f \frac{1}{r} h_{\nu r}. \tag{2.220}$$

As r goes to infinity one ends up with the incoming linearized supertranslation charge

$$\hat{\mathcal{Q}}_{\zeta_f}^{\mathcal{I}_+^-} = \frac{1}{4\pi} \int_{\mathcal{I}_+^-} d^2\Theta \sqrt{\gamma} f \hat{m}, \tag{2.221}$$

where $\hat{m}$ is the deviation of the Bondi mass aspect. Using the charge conservation, one has

$$\hat{\mathcal{Q}}_{\zeta_f}^{\mathcal{I}_+^-} = \hat{Q}_{\zeta_f}^{\mathcal{I}_-^+} = \frac{1}{4\pi} \int_{\mathcal{I}_-^+} d^2\Theta \sqrt{\gamma} f \hat{m}. \tag{2.222}$$

Following a similar construction, now in the vicinity of a black hole, let Υ^+ be a hypersurface extending from $\mathcal{I}_-^+$ to $\mathcal{H}_+^+$. Which makes the Cauchy surface under consideration to be the $\Upsilon^+ \cup \mathcal{H}^+$. Then

$$\hat{\mathcal{Q}}_f^- = \hat{\mathcal{Q}}_f^{\Upsilon^+} + \hat{\mathcal{Q}}_f^{\mathcal{H}^+}, \tag{2.223}$$

where the second term is the supertranslation charge contribution of the black hole. It can also be shown that

$$\hat{\mathcal{Q}}_f^{\mathcal{H}^+} = \frac{m_B}{8\pi} \int d^2\Theta \sqrt{\gamma} f \left[D^A \partial_r h_{A\nu} + 2h_{\nu\nu} + D^2 h_{\nu r}\right]_{\mathcal{H}_-^+}^{\mathcal{H}_+^+}. \tag{2.224}$$

Now to obtain the full picture of the horizon charges, one needs the horizon constraints

$$\partial_\nu (D^A h_{A\nu} + 2m_B h_{\nu\nu}) - \frac{1}{4m_B} D^A h_{A\nu} - \frac{1}{2} D^2 h_{\nu\nu} = 32\pi m_B^2 T_{\nu\nu}^M, \tag{2.225}$$

$$\partial_\nu\bigg(- D_A h_{\nu r} - \partial_r h_{A\nu} + \frac{1}{m_B} h_{\nu A} + \frac{1}{4m_B^2} D^B h_{BA}\bigg) + D_A \partial_r h_{\nu\nu} + \frac{1}{2m_B} D_A h_{\nu r}$$
$$+ \frac{1}{4m_B^2} D_A D^B h_{B\nu} - \frac{1}{4m_B^2} D^2 h_{A\nu} - \frac{1}{4m_B^2} h_{A\nu} = 16\pi T^{m_B}_{A\nu}. \tag{2.226}$$

Using the combination of these horizon constraints (2.225) & (2.226) (to the linearized order) one gets the horizon charges as (Hawking et al., 2017)

$$m_B \partial_\nu (2h_{\nu\nu} + D^2 h_{\nu r} + D^A \partial_r h_{A\nu}) - \frac{1}{2} D^2 h_{\nu\nu} - \frac{1}{4m_B} D^A D^B \partial_\nu h_{AB} - m_B D^2 \partial_r h_{\nu\nu}$$
$$- \frac{1}{2} D^2 h_{\nu r} + \frac{1}{4m_B} D^A h_{A\nu} = 32\pi m_B^2 T^{m_B}_{\nu\nu} - 16\pi m_B D^A T^{m_B}_{A\nu}, \tag{2.227}$$

after an integration one gets an equation for $\hat{\mathcal{Q}}_f^{\mathcal{H}^+}$. By setting the proper boundary conditions and gauge fixing, it becomes evident that this term generates horizon supertranslations.

2.11 BMS-like Structures in FLRW Spacetimes

So far, when the null infinity was under consideration, the background was either Minkowski or asymptotically flat spacetimes, whit the future null infinity ($\mathcal{I}^+$). Similarly, a flat FLRW spacetime with a decelerating expansion (radiation-dominated and matter-dominated universes) also has a future null infinity. Because of this, it might seem logical to use the Bondi coordinate system (which has been used in two previous chapters of this thesis so far) for this FLRW model as well. However one will see that this procedure leads to crucial errors since there are fundamental differences between the asymptotic behaviour of Minkowski and FLRW spacetimes. There are two points which make this claim obvious. The first one is, looking at the fall-off rates of the decompositions of the stress-energy tensor, one sees that they are too slow to let the stress-energy tensor to have a finite limit to $\mathcal{I}$ (it diverges). The second point is, when writing the diffeomorphisms of FLRW, one also obliged to transform the scale factor ($a(\eta)$) which will be evident later that meaning a different asymptotic symmetry algebra for FLRW spacetimes (this algebra will not be isomorphic to the $\mathfrak{bms}$). These differences call for an investigation on the null infinity once again, which will give rise to the concept of "cosmological null asymptote".

2.11.1 FLRW Spacetimes

The main contributor to the difference between the asymptotically flat spacetimes and FLRW spacetimes can be considered as the existence of homogeneous matter in the latter. To see this, one starts with the line element for the flat FLRW spacetimes

$$d\hat{s}^2 = a^2(\eta)(-d\eta^2 + dr^2 + r^2 S_{AB} dx^A dx^B). \tag{2.228}$$

Where S_{AB} is the round metric on S^2, η and r conformal time coordinate and radial coordinate, respectively. The spacetimes under consideration satisfy Einstein's equation, indicating that the related stress-energy tensor can be written as

$$\hat{T}_{ab} = a^2(p+\rho)\nabla_a\eta\nabla_b\eta + p\hat{g}_{ab}. \tag{2.229}$$

Following the conventional cosmological definitions, ρ represents the density and p represents pressure and the relation between them (i.e. the equation of state) is $p = w\rho$. One can use this relation to define the declaration parameter and the s parameter which will be important when defining the conformal completion of the FLRW spacetime. Starting with the former

$$q := \frac{1+3w}{2}, \tag{2.230}$$

and the s parameter is

$$s := \frac{2}{3(1+w)} = \frac{1}{1+q}. \tag{2.231}$$

Now focusing on constructing a conformal completion for FLRW spacetimes, one needs proper coordinates and rewrite conformal factor using them.

Defining two new pairs of coordinates (T, R) and (U, V) satisfying the relations

$$\begin{aligned} \eta &= \frac{\sin T}{\cos R + \cos T}, \quad r = \frac{\sin R}{\cos R + \cos T}, \\ U &:= T - R, \qquad\quad V := T + R. \end{aligned} \tag{2.232}$$

The range for these new coordinates are

$$\begin{aligned} 0 < T < \pi, \quad & 0 \le R < \pi - T, \\ -\pi < U < \pi, \quad & |U| < V < \pi, \end{aligned} \tag{2.233}$$

and the conformal factor turns out to be

$$\Omega = 2\bigg(\cos\frac{V}{2}\cos\frac{U}{2}\bigg)^{1/(1-s)}\bigg(\sin\frac{U+V}{2}\bigg)^{-s/(1-s)}. \tag{2.234}$$

From (2.234) one sees that at $V = -U$, Ω diverges and at $V = \pi$, Ω vanishes. $V = U = \pi$ corresponds to i^+ while $V = -U = \pi$ corresponds to i^0.

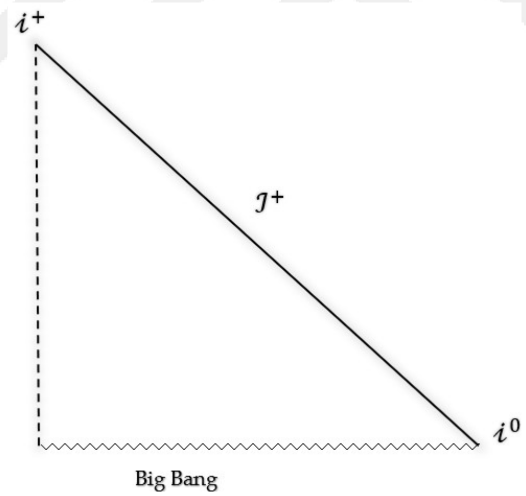

Figure 2.6: Conformal diagram for the decelerating FLRW spacetimes.

Comparing this to the conformal representation of the asymptotically flat spacetimes one can already start to see the difference. Now looking at how $a(\eta)$ behaves near $\mathcal{I}^+$

$$a(\eta) = \Omega^{-s} A^s, \qquad\qquad A \equiv 2\sin\frac{U+V}{2}, \tag{2.235}$$

one sees that it diverges, even though A is smooth at future null infinity. And similarly, near $\mathcal{I}^+$ the conformal factor Ω behaves like

$$\Omega \sim \cos\frac{U}{2}(\pi - V)^{\frac{1}{1-s}}. \tag{2.236}$$

Meaning that it is not smooth unless $s = 0$ (Minkowski) where $0 \leq s < 1$. Changing the Ω for another one which is smooth on the null infinity causes a serious technical problem which makes $\mathcal{I}^+$ to be no longer a null surface. So a logical decision is to allow Ω to be not smooth and use conformal completions where g_{ab} is smooth on the null infinity. The crucial way out from this hurdle is using Ω^{1-s} which is smooth on null infinity. Another important difference is rooted in the decaying of the stress-energy tensor as it reaches null infinity. Looking at the trace of a homogeneous stress-energy tensor of FLRW spacetime

$$\lim_{\to\mathcal{I}^+} 8\pi g^{ab}\hat{T}_{ab} = \frac{6s(1-2s)}{(1-s)^2}\left(\sec\frac{U}{2}\right)^2. \tag{2.237}$$

one sees that it is not vanishing on the null infinity. In addition to that, for some components of $\hat{T}_{ab}$

$$8\pi\hat{T}_{ab} = \underbrace{2s\Omega^{2(s-1)}n_a n_b}_{\text{divergent}} + 2s\Omega^{s-1}\tau_{(a}n_{b)} + \underbrace{\mathcal{O}(1)}_{\text{finite limit}}, \tag{2.238}$$

one sees that there is a divergent term $\big(n_a$ is normal to the null infinity and $\tau_a = \tan\frac{U}{2}\left[\nabla_a U + \nabla_a V\right]\big)$. From these deductions, it is evident that the null infinity needs a different treatment under these conformal completions, although that FLRW spacetimes are conformal to Minkowski spacetimes. This is where the concept 'cosmological null asymptote' presents itself.

2.11.2 Cosmological Null Asymptote

A spacetime $(\hat{M}, \hat{g}_{ab})$ satisfying $\hat{G}_{ab} = 8\pi\hat{T}_{ab}$ accepts a cosmological null asymptote at infinity if there is some other spacetime (M, g_{ab}) with the boundary $\mathcal{I} \cong \mathbb{R} \times \mathbb{S}^2$ and an embedding of $\hat{M}$ into $M - \mathcal{I}$ in such a way that

1. $\Omega > 0$, where Ω is smooth on M and continuously extendable to $\mathcal{I}$ such that

    - $\Omega|_{\mathcal{I}} = 0$ and $g_{ab} = \Omega^2 \hat{g}_{ab}$ is non-degenerate and smooth as well.
    - for $0 \leq s < 1$, Ω^{1-s} is smooth on M, where n_a is the non-vanishing normal on null infinity.

2. There exists $\hat{T}_{ab}$ such that

    - $\lim_{\to \mathcal{I}} g^{ab} \hat{T}_{ab}$ and,
    - $\lim_{\to \mathcal{I}} \Omega^{1-s} \left[8\pi \hat{T}_{ab} - 2s\Omega^{2(s-1)} n_a n_b \right]_{\mathcal{I}} = 2s\tau_{(a} n_{b)}$ for some smooth τ_a on the null infinity.

This defines a larger class of spacetimes than the asymptotically flat spacetimes. One way to see this is realizing that the above definition permits $\hat{T}_{ab}$ to have a limit to $\mathcal{I}$ while in asymptotically flat spacetimes one has a stronger condition which requires the existence of $\lim_{\to \mathcal{I}} \Omega^{-2} \hat{T}_{ab}$. Spacetimes having this cosmological null asymptote construction at null infinity are alike to decelerating FLRW spacetimes (Bonga & Prabhu, 2020).

For all the spacetimes equipped with the cosmological null asymptote with a given s value, one can define a common universal structure as

1. a smooth manifold $\mathcal{I} \cong \mathbb{R} \times \mathbb{S}^2$,
2. on $\mathcal{I}$ there is an equivalence class (q_{ab}, n^a) where former is a metric satisfying $q_{ab} n^b|_{\mathcal{I}} = 0$ and $\mathcal{L}_n q_{ab}|_{\mathcal{I}} = 0$ and latter is being a vector field.
3. there exists a map $(q_{ab}, n^a) \mapsto (w^2 q_{ab}, w^{-1-s} n^a)$ for some $w > 0$ with $\mathcal{L}_n w|_{\mathcal{I}} = 0$.

2.11.3 Asymptotic Symmetry Algebra

With the help of the definition of universal structure, now one can construct the asymptotic symmetry algebra, which will be the algebra of structure-preserving infinitesimal diffeomorphisms of $\mathcal{I}$. This algebra is expressed by all smooth vector fields ξ^a on $\mathcal{I}$ which maps one equivalence class to another one. Then ξ^a have to

$$\mathcal{L}_{\xi^a} q_{ab}|_{\mathcal{I}} = 2\alpha_{(\xi)} q_{ab}, \qquad \mathcal{L}_{\xi^a} n^a|_{\mathcal{I}} = -(1+s)\alpha_{(\xi)} n^a, \tag{2.239}$$

$\alpha_{(\xi)}$ being an arbitrary function with a ξ dependence on $\mathcal{I}$. The ξ^a obeying these relations form a Lie algebra $\mathfrak{b}_s$, this algebra has a strong resemblance to the $\mathfrak{bms}$ but it is not isomorphic to it.

Now let the vector fields take the form $\xi^a|_{\mathcal{I}} = fn^a$ (where fn^a is a supertranslation). This requires the conditions

$$\mathcal{L}_n f|_{\mathcal{I}} = 0, \qquad \alpha_{(fn)}|_{\mathcal{I}} = 0. \tag{2.240}$$

Then one can see that these vector fields form a subalgebra of supertranslations denoted as $\mathfrak{s}_s \subset \mathfrak{b}_s$. This subalgebra has a conformal weight $1+s$ for each parametrization function on S^2. Looking at the Lie bracket

$$[\xi, fn]^a = (\mathcal{L}_\xi f - (1+s)\alpha_{(\xi)} f) n^a, \tag{2.241}$$

one notices that the right hand side of the bracket is also a supertranslation, meaning that $\mathfrak{b}_s/\mathfrak{s}_s$ is a Lie algebra. Knowing that S^2 has a unique conformal structure, one can deduce that the algebra $\mathfrak{s}_s \subset \mathfrak{b}_s$ is the algebra of conformal isometries of S^2 which is isomorphic to $\mathfrak{so}(1,3)$. Which concluded that the asymptotic symmetry algebra is

$$\mathfrak{b}_s \cong \mathfrak{so}(1,3) \ltimes \mathfrak{s}_s. \tag{2.242}$$

As expected for $s = 0$ case, one gets $\mathfrak{b}_{s=0} = \mathfrak{bms}$ which is the BMS algebra. But for $s \neq 0$ cases, $\mathfrak{b}$ is not isomorphic to $\mathfrak{bms}$, the essential difference between these two algebras can be seen as the existence of the conformal weight $1+s$. One can also say that for the latter case, $\mathfrak{b}_s$ does not have any favored translation subalgebra. So indeed, BMS algebra and asymptotic symmetry algebra have deep affinities but one has to be careful with the technical details of the model at hand since the corresponding algebra may not be isomorphic to $\mathfrak{bms}$ which is mostly the case.

CHAPTER 3

DISCUSSION AND CONCLUSIONS

Starting from Weinberg's famous work on adiabatic modes in cosmology, this thesis encapsulated the theoretical and historical developments of the strongly correlated research areas such as asymptotically flat spacetimes, symmetries of the null infinity ($\mathcal{I}$), soft theorems, memory effects and the BMS group. Each one of these subjects are highly fruitful on their own. However, together they paint an undeniably astonishing picture of the cosmos, elegantly and simply. To be able to show this picture to the reader, this thesis is designed to be as clear as possible by building the theoretical background from the fundamentals, and by presenting complementary material such as chapters (2.4.1),(2.6.1),(2.8), (A.1), (A.2) and (A.3) when needed. With the aid of these chapters, the concepts which play a crucial role in this thesis such as soft particles, supertranslations, superrotations and gauge symmetries are believed to be put on solid ground. Before exposing the reader directly to gravity, in chapter (2.3), another model which is historically more developed (i.e. QED) has been chosen to bring to surface the necessary concepts including asymptotic expansion, charge conservation and Ward identities. Using the similarities between electrodynamics and gravity, it has been shown that these concepts also present themselves under certain spacetimes (2.4), and give rise to supertranslations (2.5) as well as superrotations (2.7), which are the focal points of this thesis. Equipped with the infrared triangle in mind, in chapter (2.9) the corresponding memory effects of the supertranslations and the superrotations were presented to the reader. These memory effects are believed to be vital observational candidates for testing the theory and they are expected to be discovered soon by the scientific community. After solidifying the theoretical background, in the last two chapters of the thesis, the focus is shifted to the black holes and their analysis using soft charges (2.10) and the analysis of FLRW spacetimes

(2.11), using the BMS group.

The null infinity boundaries ($\mathcal{I}^{\pm}$) are equipped with algebras and representations which give asymptotic states without dealing with the interior of the spacetime. This gives rise to the possibility of having a non-perturbative sum calculation over the line of spin foams. Another important detail involves the cosmological constant. For the careful reader, it should have been obvious that for each case in this thesis Λ was equal to 0. For the two other possible cases, $\Lambda < 0$ and $\Lambda > 0$, one needs to be careful. Both cases give rise to interesting results (known as Λ-BMS) and they are open research fields. Considering the observational results, it is known that Λ is very small, but it is positive (Aghanim et al., 2020). Hence, to analyse the real universe, eventually one needs to step into new territories where $\Lambda > 0$ and the expansion of the universe is accelerating. For such a case, near null infinity, the asymptotic symmetry corresponding to the time translation becomes spacelike and there is no term that matches with the Bondi news (Ashtekar et al., 2014), giving rise to many active research areas. Similarly, for one of the other corners of the infrared triangle, the memory effect, there are ongoing studies on perturbed FLRW spacetimes where the memory effect is defined locally, meaning that no $\mathcal{I}$ limit is required and no need for an explicit relation to the asymptotic symmetry algebra. These studies are based strictly on sources with stress-energy confined by delta functions and the corresponding memory effect is related to the derivative of this delta function in curvature, letting one define memory effect involving linear perturbations on any given background (Tolish & Wald, 2016).

Regarding the spacetimes that have black holes in the interior, they have their fair amount of ambiguities as well. Even though horizon charges and symmetries make sense mathematically, the physical interpretations of these concepts should be addressed carefully. Because of the Hawking radiation, it is well known that the black hole evaporates and its horizon vanishes, making it difficult to conceptualize the idea that the information about the bulk is stored on this horizon. Also, due to the existence of an infinity of conserved charges coming from the low energy symmetries, the outgoing Hawking radiation is greatly constrained. Meaning that the semi-classical black hole evaporation calculation needs a modification. Such a modification also raises questions over the black hole information paradox (Hawking et al., 2017).

The fact that the BMS group is indeed a Fréchet Lie group, it becomes evident that the mathematical structure of this group needs to be treated carefully. It is well-known that for such Lie groups, various results about finite-dimensional Lie groups are no longer satisfied and the association between the algebra and the group becomes unclear. These ambiguities are also present in the generalized BMS group and the extended BMS group (A.3). Resolution of these problems is expected to shed light on the nature of the BMS group and proposals such as celestial conformal field theory (Fotopoulos et al., 2020). This mathematical richness draws the attention of many scientists and it is an active research area.

As the final remark, just as in many other theories, one might be curious about whether the higher dimensional analysis of the BMS structure is fruitful or not. Looking at the research, it is evident that higher dimensional analogues of the BMS group are also promising candidates to answer some of the most fundamental questions about nature. The asymptotic structure of $\mathcal{I}^+$ and i^0 is believed to contribute in a coherent way to understanding the higher dimensional black hole physics. Which is still an open research area. In higher dimensions ($d > 4$), one must be careful with the geometric structure of the i^0 because its geometry depends on the number of dimensions. Unlike i^0, $\mathcal{I}^+$ is not that well understood in higher dimensions. To see that, one must look back at the conformal techniques that have been used in spacetime construction. While these techniques apply to i^0 in any dimension, they are only applicable to $\mathcal{I}^+$ if d is even. The problem with the odd dimensions arises from the existence of gravitational waves and their behaviour at $\mathcal{I}^+$ (for a conformal factor Ω, they behave like $\mathcal{O}(\Omega^{(d-2)/2})$) and another problem is the smoothness of Einstein's equations at $\mathcal{I}^+$ (Melas, 2011). In higher dimensions, the existence of supertranslations is also in jeopardy, but they can still be obtained by using a different asymptotic flatness definition. These difficulties also arise on one of the other corner of the infrared triangle, known as the memory effect. It is shown that for odd dimensions there is no memory effect. And another interesting result is the non-existence of gravitational or electromagnetic memory for spacetimes with $d > 6$ (Satishchandran & Wald, 2018). Some of these problems are still unresolved and remain as active research areas.

As the reader can see, the journey continues and it is far from ending. The collective understanding of cosmology evolves consistently and with the help of recent and upcoming groundbreaking observational results, one will be able to put the theory to test and keep moving forward. Unlike what has been told many times in the past, physics is far from complete and it is more exciting than ever.

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Appendix A

APPENDIX

A.1 Witt Algebra and Asymptotic Killing Vectors

A.1.1 Witt Algebra

Witt algebra can be considered as a space of derivations using the Laurent polynomials $\mathbb{C}[z, z^{-1}]$ satisfying some conditions such that

$$Witt = Der(\mathbb{C}[z, z^{-1}]) = \begin{cases} \text{all linear maps} & D : \mathbb{C}[z, z^{-1}] \to \mathbb{C}[z, z^{-1}] \\ \text{satisfying} & D(ab) = D(a)b + aD(b). \end{cases} \tag{A.1}$$

Focusing on the derivation, one sees that for all $n \geq 1$

$$D(z^n) = nz^{n-1}D(z). \tag{A.2}$$

Expanding this definition by use of Laurent polynomials $a(z)$ where $a(z) \in \mathbb{C}[z, z^{-1}]$ such that

$$D(a(z)) = \frac{da}{dz}D(z), \tag{A.3}$$

where $D(z)$ is also in $\mathbb{C}[z, z^{-1}]$. Now the Witt algebra takes the form

$$Witt = \left\{ p(z)\frac{d}{dz} \mid p(z) \in \mathbb{C}[z, z^{-1}] \right\}, \tag{A.4}$$

where $p(z)$ is any Laurent polynomial.

To understand the Witt algebra better, one conventional method is constructing a basis for it. It is evident that every element of the algebra is connected to the Laurent polynomial. So one can take the basis for Laurent polynomials and use it to get a basis for the Witt algebra. The relation can be considered as

$$
\begin{aligned}
&.\\
&.\\
z^{-1} &\to z^{-1}\frac{d}{dz}\\
1 &\to \frac{d}{dz}\\
z &\to z\frac{d}{dz}\\
z^{2} &\to z^{2}\frac{d}{dz}\\
&.\\
&.
\end{aligned}
\tag{A.5}
$$

Equipped with this, one can define $\{L_n \mid n \in \mathbb{Z}\}$ as a basis for Witt algebra where for all $n \in \mathbb{Z}$

$$
L_n = -z^{n+1}\frac{d}{dz}. \tag{A.6}
$$

Now as a final step, to see the algebra structure of this basis one can check the below derivation

$$
\begin{aligned}
[L_m, L_n]a(z) &= L_m L_n a(z) - L_n L_m a(z)\\
&= L_m(-z^{n+1}a^{'}(z)) - L_n(-z^{m+1}a^{'}(z))\\
&= -z^{m+1}\frac{d}{dz}(-z^{n+1}a^{'}(z)) - z^{n+1}\frac{d}{dz}(z^{m+1}a^{'}(z))\\
&= z^{m+1}\big((n+1)z^{n}a^{'}(z) + z^{n+1}a^{''}(z)\big) - z^{n+1}\big((m+1)z^{m}a^{'}(z) + z^{m+1}a^{''}(z)\big).
\end{aligned}
\tag{A.7}
$$

Where $a(z) \in \mathbb{C}[z, z^{-1}]$, $m, n \in \mathbb{Z}$ and the prime notation indicating derivative with respect to z. Looking at (A.7) one sees that the second derivatives cancels and the relation simplifies to

$$[L_m, L_n]a(z) = -(m-n)z^{n+m+1}a^{'}(z) = (m-n)L_{m+n}a(z). \tag{A.8}$$

Delivering the celebrated algebra

$$[L_m, L_n] = (m-n)L_{m+n}a(z). \tag{A.9}$$

This algebra is an example of an infinite dimensional Lie algebra and its central extension is the Virasoro algebra which has a very rich structure and it plays a crucial role in CFT and string theory.

A.1.2 Asymptotic Killing Vectors

One can consider the asymptotic symmetry algebra as the set of transformations that preserve the boundary conditions with trivial gauge transformations taken out. Working with the Bondi coordinates (in three spacetime dimensions) and required fall-off conditions, one sees that the boundary and gauge conditions are preserved by the asymptotic Killing vectors

$$\xi = (M(\phi) + uL^{'}(\phi))\partial_u + \big(L(\phi) - \frac{u}{r}L^{''}(\phi) - \frac{1}{r}M^{'}(\phi)\big)\partial_\phi - (rL^{'}(\phi) + \mathcal{O}(1/r))\partial_r, \tag{A.10}$$

where u is the retarded time, $\phi \sim \phi + 2\pi$ is the angular coordinate and $'$ notation is the derivative with respect to u. Focusing on the leading order in large r expansion and splitting the above Killing vector equation into L and M components one gets

$$\xi^L = uL^{'}(\phi)\partial_u + \big(L(\phi) - \frac{u}{r}L^{''}(\phi)\big)\partial_\phi - (rL^{'}(\phi) + \mathcal{O}(1/r))\partial_r, \tag{A.11}$$

$$\xi^M = M(\phi)\partial_u + \mathcal{O}(1/r). \tag{A.12}$$

Their Lie-bracket algebra consists of infinitely many generators and it is given as

$$\begin{aligned}
[\xi^L(L_1), \xi^L(L_2)] &= \xi^L(L_1 L_2^{'} - L_2 L_1^{'})\mathcal{O}(1/r), \\
[\xi^L(L_1), \xi^M(M_2)] &= \xi^M(L_1 M_2^{'} - M_2 L_1^{'})\mathcal{O}(1/r), \\
[\xi^M(M_1), \xi^M(M_2)] &= \mathcal{O}(1/r).
\end{aligned} \tag{A.13}$$

Looking at the first line , one can see that Witt algebra is recovered as subalgebra and from the last line it is obvious to the leading order, ξ^Ms commute with each other. The zero mode $\xi_0^M = \partial_u$ generates time translations, because of this, these asymptotic Killing vectors are known as supertranslations. Similarly, the zero mode $\xi_0^L = \partial_\phi$ generates rotations and because of this, they are called superrotations. They are also the generators of the Witt algebra. These two do not commute but they form something similar to Witt algebra.

A.2 BMS Algebra as Fréchet Lie Algebra

Fréchet Lie groups are infinite-dimensional Lie groups. While finite-dimensional Lie groups have Banach manifold structure, the Fréchet Lie groups have Fréchet manifold structure. There are significant differences between the finite-dimensional and infinite-dimensional cases. As already mentioned, the BMS group is an infinite-dimensional Lie group, making it a Fréchet Lie group. To understand the BMS group and diffeomorphisms better, an investigation of the Fréchet Lie group is required.

A Fréchet manifold can be defined as a Hausdorff topological space with an atlas of coordinate charts taking their value in Fréchet spaces. Because of its infinite-dimensional structure, one has to be careful with the differential geometric constructions on it. Coming back to the infinite-dimensional group structure, one can start by analysing the diffeomorphism group Diff(M) on a manifold M. This group can be defined as

$$\text{Diff}(M) = \{\phi \in C^\infty(M, M) : \phi \,\text{bijective},\ \phi^{-1} \in C^\infty(M, M)\}, \tag{A.14}$$

where M is a compact smooth manifold. This group is a Fréchet Lie group. The Banach manifold version of this group $\text{Diff}_{C^n}(M)$, which is called C^n-diffeomorphisms, is not a Lie group (more detail can be found in (Kriegl & Michor, 1998)). Now recalling that the BMS group can be written as

$$\text{BMS} = S \rtimes \text{SO}^+(3,1), \tag{A.15}$$

where $S := C^\infty(\mathbb{S}^2) := C^\infty((\mathbb{S}^2, \mathbb{R})$. S is called supertranslations and they form an abelian Fréchet Lie group. By the use of the right action

$$\sigma_k : C^k(\mathbb{S}^2, \mathbb{R}) \times \text{SO}^+(3,1) \to C^k(\mathbb{S}^2, \mathbb{R}), \tag{A.16}$$

keeping in mind that σ_∞ is smooth, for BMS, one gets

$$\sigma : S \times \text{SO}^+(3,1) \to S. \tag{A.17}$$

Meaning that

$$\text{BMS} \cong S \rtimes_\sigma \text{SO}^+(3,1). \tag{A.18}$$

Hence, it is shown that the BMS group is also an infinite-dimensional Fréchet Lie group (McCarthy, 1972; Prinz & Schmeding, 2022). However, this group is not real analytic and it does not accept local exponential coordinates.

A.3 Generalized BMS Group

Looking at the group definition (A.18), one can come up with a clever way to expand this group by replacing the $\text{SO}^+(3,1)$ with a larger symmetry group. Conventionally there are two ways to go. The first one is replacing the Lorentz group with $\text{Diff}(\mathbb{S}^1) \times \text{Diff}(\mathbb{S}^1)$, which after the semi-product, produces the so-called extended BMS group (BMS^{ext}). This extension emerged from the AdS/CFT correspondence (Barnich & Troessaert, 2010). The second way is replacing the Lorentz group with $\text{Diff}(\mathbb{S}^2)$ (recall that they are the superrotations), which after the semi-product, produces the so-called generalized BMS group (BMS^{gen}). This extension is assembled by use of gravitational scattering theorems. Focusing on the generalized BMS, one can find the asymptotic Killing vectors as

$$\xi^u = f(u, x^A), \tag{A.19}$$

$$\xi^A = Y^A(u, x^A) + I^A, \tag{A.20}$$

$$\xi^r = -\frac{1}{2} r(D_A Y^A + D_A I^A - U^B D_B f), \tag{A.21}$$

where $I^A = -D_B f \int_r^\infty dr' (e^{2\beta} g^{AB})$ and $\partial_r f = \partial_r Y^A = 0$. After the radial integration of these Killing vectors, they can be written as

$$\xi^u = f, \tag{A.22}$$

$$\xi^A = Y^A - \frac{1}{r} D^A f + \frac{1}{r^2}\left(\frac{1}{2} C^{AB} D_B f\right) + \frac{1}{r^3}\left(-\frac{1}{16} C_{BC} C^{BC} D^A f\right) + \mathcal{O}(r^{-4}), \tag{A.23}$$

$$\xi^r = -\frac{1}{2} r D_A Y^A + \frac{1}{2} D_A D^A f + \frac{1}{r}\left(-\frac{1}{2} D_A C^{AB} D_B f - \frac{1}{4} C^{AB} D_A D_B f\right) + \mathcal{O}(r^{-2}), \tag{A.24}$$

where C_{AB} and D_{AB} come from the metric equation $g_{AB} = r^2 q_{AB} + r C_{AB} + D_{AB} + \mathcal{O}(r^{-1})$ (q_{AB} is the unit S^2 metric). One can represent ξ as $\xi(T, Y)$ where T^A are the generators of supertranslations and Y^A are the generators of the superrotations (they are also referred as the pullback of super-Lorentz transformations) (Ruzziconi, 2020). Hence the generalized BMS group is

$$\text{BMS}^{gen} := S \rtimes_\alpha \text{Diff}(\mathbb{S}^2), \tag{A.25}$$

where the action α is

$$\alpha : S \times \text{Diff}(\mathbb{S}^2) \to S. \tag{A.26}$$

And its algebra is

$$\mathfrak{bms}^{gen} = \mathfrak{s} \rtimes \mathfrak{Diff}(\mathbb{S}^2), \tag{A.27}$$

where $\mathfrak{Diff}(\mathbb{S}^2)$ is the diffeomorphism algebra on the CS^2 (Campiglia & Laddha, 2014). Looking at (A.25), it is obvious that this group is also a Fréchet Lie group and it is not locally exponential.