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GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

STABILITY ANALYSIS OF NONLINEAR FUZZY DYNAMIC SYSTEMS OF PERTURBED SYSTEMS WITH RESPECT TO UNPERTURBED SYSTEMS WITH INITIAL TIME DIFFERENCE

MUSTAFA BAYRAM GÜCEN

A THESIS SUBMITTED FOR THE DEGREE of
DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS

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ÜYE : Prof. Dr. Fatih TAŞÇI

ÜYE : Prof. Dr. Emil NOVRUZ

ÜYE : Doç. Dr. Yalçın YILMAZ

ÜYE : Doç. Dr. Mustafa AKKURT

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SUMMARY

In this thesis, fuzzy differential equations and qualitative properties are studied and the some results are obtained. This thesis have five chapters. In chapter one, we have given Fundamentals of fuzzy calculus and fuzzy differential equations. In the second chapter, we also given the basic concept, main stability definitions and some properties. In the third chapter, we have investigated the practical stability of fuzzy differential equations. In the fourth chapter, we have studied the stability of fuzzy differential equations with initial time difference. Last chapter, we have studied the strict stability of fuzzy differential equations with initial time difference. We have worked on the practical stability, stability of perturbed systems and initial time difference stability.

Key Words: Fuzzy Calculus, Fuzzy Differential Equations, Perturbed Differential Systems, Initial Time Difference Stability.

ÖZET

Bu tezde, bulanık diferansiyel denklemler ve nitel özellikleri ele alınmış ve bazı sonuçlar elde edilmiştir. Bu tez beş bölümden oluşmaktadır. İlk bölüm bulanık analizin temelleri ve bulanık diferansiyel denklemler ile ilgilidir. İkinci bölümde, temel kavramlar ve temel kararlılık tanımları ve kullandığımız bazı özellikler verilmiştir. Üçüncü bölümde, bulanık diferansiyel denklemlerin pratik kararlılığını araştırılmıştır. Dördüncü bölümde, bulanık diferansiyel denklemlerin kararlılığını başlangıç zaman farkı analizi ile birlikte çalıştık. Son bölümde kesin kararlılık çalışılmıştır. Pratik kararlılık, pertörb sistemlerin stabilitesi ve başlangıç zaman farklı kararlılık gibi farklı kararlılık üzerinde çalışılmıştır.

Anahtar Kelimeler: Bulanık Analiz, Bulanık Diferansiyel Denklemler, Pertörb Diferansiyel Sistemler, Başlangıç Zaman Farklı Kararlılık.

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LIST of ABBREVIATIONS and ACRONYMS

Abbreviations and Acronyms		**Explanations**
R^n	:	The *n*-dimensional real space.
$d_H(A,B)$	:	The Haussdorff metric is a distance between A and B.
$\mathbb{Q}$	:	The set of rational numbers.
$\mathbb{Z}$	:	The set of integers.
$\mathbb{N}$	:	The set of positive integers.
$\|.\|$	:	The usual Euclidean norm in R^n.
$P_k(R^n)$	:	The family of all non-empty compact and convex subsets of R^n.
ITD	:	Initial Time Difference

1. INTRODUCTION

Fuzzy sets first appeared in 60s by Lotfi Aliaskerzadeh who was the mathematician, computer scientist, electrical engineer, artificial intelligence researcher and professor. In 1965, the scientific journal Information and Control, in his article Fuzzy Sets, says: “A fuzzy set is a class of objects with a continuum of grades of membership” [Zadeh, 1965]. This idea is the very important to understand the fuzzy sets and fuzzy logic.

The fuzzy logic went beyond the logic of Aristotle. A more advanced version of Aristotle logic is adapted to today's technology. Fuzzy logic and fuzzy sets are the first step in machines that can think and make decisions. Fuzzy logic provides a wider rating than 0 and 1. That is very useful to provide broader thinking and decision making opportunities.

In classical set theory, an element belongs to the cluster or does not belong. It is either inside or outside. In other words, if we express the membership of something to the cluster, membership is either one or zero. Is this true? Or how much true?

For example, we think a man. Let the man be long. Let's think of another man. Second man is short. So what is the longest? What's short? How do we define it? We can define a set for long and a set for short. If the length of the man belongs to the long set, the man is long. But can we define all of those that belong to a long set? Can it be a little long, too long, medium long? How can we express it? How can we define this? At this point the fuzzy logic helps us with the concept of degree of membership. Degree of membership express belonging degree of a set.

When we think about 0 and 1, in a dishwasher, the dishes are cleaned or not cleaned. In an oven, the bread was either cooked or not. In a washing machine, clothes are not washed or washed, there are no intermediate numbers and ratings, but if we use fuzzy logic, we can know washing degrees of clothes. Furthermore, machines and computer programs that use fuzzy logic and use commands according to intermediate numbers and ratings can be made. Nowadays, we already have machines and computer programs that use fuzzy logic. We teach computers how to decide using fuzzy logic. We use fuzzy logic in machine learning, artificial intelligence, deep learning and other software applications.

In daily life, when we see a friend, we ask him: how is the weather. He responds to us: hot. The question here comes to mind. How hot is the weather? Too extreme hot, extreme hot, medium hot, few hot, less hot. The answer can be one of these expressions. We can express the answer mathematically. Suppose that too extreme hot is equal to 1 and less hot is equal to 0. That is hard computing. Hard computing is deterministic, precise and strict. Hard computing based on strict numbers and rules, mathematical analysis, numerical approximations and binary system. What is the soft computing? Soft computing is a special approach and tolerance selection of vague, approximate reasoning and partial accuracy. Soft computing based on artificial intelligence algorithms, probability theory and fuzzy approach. Fuzzy logic is a soft computing. In this example, let's take a cluster of these expressions: Too extreme hot, extreme hot, medium hot, few hot, less hot. These expressions can match 1, 0.8, 0.5, 0.3 and 0. These numbers belong to the interval [0,1]. We have further examples and applications in real life and computer science. This is a simple talking example that have not necessarily functions.

Fuzzy sets, fuzzy calculus and fuzzy differential equations have been studied. Some new results obtained and theorems have been proved. Therefore, the efficient results and studies have been available and developed in fuzzy calculus and fuzzy differential equations. We have numerical approaches and modeling studies of fuzzy differential equations in recent years. We have to say that definition and concept of fuzzy derivative and integration is very important for fuzzy calculus and the theory of fuzzy differential equations. Some studies and developments have been available on definition and concept of fuzzy derivative and integration. For instance, Bede and Gal [Bede and Gal, 2005] have presented the generalizations of differentiability. Based on this work, some other studies was used the generalized differentiability and the second type of hukuhara derivative was applied in fuzzy differential equations. Recently, the stability studies, theorems and the comparison results for fuzzy dynamic systems with the second type of hukuhara derivative have investigated and found some new approaches, results and models.

Stability of differential equations and initial time difference stability analysis of differential equations are very important for the theory of differential equations and mathematics. We have the very effective and useful results, theorems in the qualitative theory of differential equations and stability with initial time difference analysis.

If we have a differential system, we say that this system is stable or unstable. The system is stable or asymptotically stable in theory, but it is actually unstable in practice because the stable domain or the domain of attraction is not large enough to allow the desired deviation to cancel out. On the other hand, sometimes the desired state of a system may be mathematically unstable and yet the system may oscillate sufficiently near this state in which its performance is acceptable, that is, it is stable in practice. In many problems of practical importance, one is not only interested in the qualitative information provided by Lyapunov stability results, but also interested in quantitative information concerning the system behavior such as estimation of trajectory bounds. For example, a system could be asymptotically stable in the Lyapunov sense, yet be completely useless because of undesirable transient characteristics. However, the desired state of a system may be mathematically unstable, but the system may oscillate sufficiently near this state so that the performance is considered acceptable. To deal with such situations, the notions of practical stability [Lakshmikantham et al., 1990], are very effective.

We have worked on the practical stability of fuzzy differential equations by Lyapunov's direct method. Because fuzzy differential systems are powerful tool for modelling uncertainty and for processing vague or subjective information in mathematical models, which have been applied to a wide variety of real world problems such as computer engineering, medicine, some engineering problems [Lakshmikantham and Mohapatra, 2003], [Bede and Gal, 2005], [Bede et al., 2007], [Bede and Stefanini, 2013], [Zhang and Sun, 2015], [Gomes et al., 2015], [Ahmadian et al., 2017], [Ahmadian et al., 2018]. The concept of the fuzzy derivative was first introduced by Chang and Zadeh [Chang and Zadeh, 1972], followed by Dubois and Prade [Dubois and Prade, 1982], who used the extension principle. The use of fuzzy differential equations is a natural way to model dynamic systems under possibilistic uncertainty. The concept of differential equations in a fuzzy environment was formulated by Kaleva [Kaleva, 1987], [Kaleva, 1990] and several authors have produced a wide range of results in the theory and applied fields [Lakshmikantham and Mohapatra, 2003], [Bede et al., 2007], [Bede and Stefanini, 2013], [Zhang and Sun, 2015].

A variety of exact, approximate, and purely numerical methods are available to find the solution of a fuzzy initial value problem. If we use the scalar equations and the comparison principle for Lyapunov-like functions, we give some sufficient conditions for the stability, practical stability of solutions of fuzzy differential equations. To study some dynamic systems, it is necessary to take into account uncertainty. The lack of certainty can be modeled by incorporating it into the fuzzy differential equations. It was suggested that the fuzzy differential equations, as originally formulated by Kaleva [Kaleva, 1987] suffer from certain disadvantages because the solution possesses the property that the diameter is nondecreasing as time increases. It was supposed that this aspect of the solutions is due to the fuzzification of the derivative utilized in the formulation of the fuzzy differential equations. As a result, alternative formulations have been suggested. One of the well-received alternatives is to replace the fuzzy differential equations with a sequence of multivalued differential inclusions generated from the function involved in fuzzy differential equations.

We will extend the notion of fuzzy differential system and then develop the practical stability, boundedness and Lagrange stability theory by proving sufficient conditions in terms of Lyapunov-like functions in the framework of fuzziness. For this purpose, we develop suitable comparison results to deal with fuzzy differential systems in terms of Lyapunov-like functions and then employ the comparison results offer sufficient conditions. This new approach helps us to understand and resolve the complexities incorporated into the fuzziness in the theory of differential equations.

Lyapunov' s second method or direct method is a method to predict the behavior of a differential system. According to this technique, if we know the stability properties of a comparison system, then we can predict the behavior of a differential system [Lakshmikantham et al., 1990], [Lakshmikantham and Mohapatra, 2003]. There is a problem with this way when trying to apply it to unperturbed differential systems with respect to perturbed differential systems and this problem is in initial time difference stability analyses [Yakar and Shaw, 2008]. We solve this problem with initial time difference analysis for perturbed fuzzy differential systems with respect to unperturbed fuzzy differential systems. The problem appears because there is a distinction between initial time difference stability approach and the classical stability approach. The classical stability approach of fuzzy differential equations are in terms of the null solution. However, in the initial time difference stability analysis of fuzzy differential equations, initial time difference stability of fuzzy differential equations is in terms of the unperturbed fuzzy differential system where the perturbed fuzzy differential system and the unperturbed fuzzy differential system have different initial positions and initial times.

In this thesis, we have worked out these problems and we have some fuzzy comparison results. We have some new results in practical stability of fuzzy differential equations and initial time difference stability approach of fuzzy differential equations. This approaches creates new initial time difference stability applications and results of the theory of fuzzy differential equations.

2. FUNDAMENTALS OF FUZZY CALCULUS

In this section, we have given the notations of framework of the fuzzy calculus and fuzzy differential equations we will use. First of all, we define the sets and frameworks. In order to define the differential equations we need a derivative definition, therefore we should give a framework for fuzzy calculus and fuzzy differential equations. $P_k(R^n)$ is the family of all non-empty compact and convex subsets of R^n. If the scalars $\alpha, \beta \in R$ and sets $A, B \in P_k(R^n)$, then $\alpha(A+B) = \alpha A + \alpha B, \alpha(\beta A) = (\alpha\beta)A, 1A = A$. If $\alpha, \beta \geq 0,$ then $(\alpha+\beta)A = \alpha A + \beta A$.

Haussdorff metric is a distance between A and B. It is defined by as follows:

$$d_H(A,B) = max[sup_{a\in A} inf_{b\in B} \|a-b\|, sup_{b\in B} inf_{a\in A} \|b-a\|] \tag{2.1}$$

where $A, B \in P_k(R^n)$ and $\|.\|$ is the usual Euclidean norm in R^n and it is clear that $(P_k(R^n), d_H)$ is a metric space. $(P_k(R^n), d_H)$ is complete and separable. For details please see [Lakshmikantham and Mohapatra, 2003].

We let $\Omega = [t_0, \tau_0 + a]$ where $\tau_0 \geq t_0 \geq 0$ and $a > 0$ be a interval in R and denote

$$E^{\mathrm{n}} = [u: R^{\mathrm{n}} \rightarrow [0;1] | u \ satisfies \ (i) \ to \ (iv)]. \tag{2.2}$$

Below

i) u is normal if there exists an $x_0 \in R^{\mathrm{n}}$ such that $u(x_0) = 1$;

ii) u is fuzzy convex;

iii) u is upper semi-continuous;

iv) $[u]^0 = [x \in R^{\mathrm{n}}: u(x) > 0]$ is compact subset of in R^{n}.

If for $x, y \in R^{\mathrm{n}}$, $0 \leq \lambda \leq 1$ and $u(\lambda x + (1-\lambda)y) \geq min[u(x), u(y)]$, then u is fuzzy convex. For $0 < \alpha \leq 1$, we denote the α −level sets as $[u]^{\alpha} = [x \in R^{\mathrm{n}}: u(x) \geq \alpha]$. Then from (i) to (iv), it follows that $[u]^{\alpha} \in P_k(R^n)$ for $0 \leq \alpha \leq 1$.

Let $d_H(A, B)$ be the Hausdorff distance between the sets $A, B \in P_k(R^n)$. Then we define

$$d[u, v] = \sup_{0 \leq \alpha \leq 1} d_H[[u]^{\alpha}, [v]^{\alpha}] \tag{2.3}$$

which defines a metric in E^{n} and (E^{n}, d) is a complete metric space. For details please see [Lakshmikantham and Mohapatra, 2003]. We have the following properties of $d[u, v]$:

i) $d[u + w, v + w] = d[u, v]$

ii) $d[u, v] = d[v, u]$

iii) $d[\lambda u, \lambda v] = |\lambda| d[u, v]$

iv) $d[u, v] \leq d[u, w] + d[w, v]$

for all $u, v, w \in E^{\mathrm{n}}$ and $\lambda \in R$.

For $x, y \in E^{\mathrm{n}}$ if there exists a $z \in E^{\mathrm{n}}$ such that $x = y + z$, then z is called the H −difference of x and y and is denoted by $x - y$. For details please see [Lakshmikantham and Mohapatra, 2003].

2.1. Fuzzy Derivative Definitions

In this section, the certain definitions of fuzzy derivatives are introduced. First of all, for fuzzy differential equations we need a derivative definition and we have a derivative definition in the literature [Bede and Gal, 2005]. These definitions in fuzzy calculus and fuzzy differential equations will be used. Let us give some examples.

Definition 2.1: Let $F:(a,b) \rightarrow E^{\mathrm{n}}, t_0 \in (a,b)$ *and* F *is differentiable at* t_0. *If there exists* $F\,'(t_0) \in E^{\mathrm{n}}$ *such that*

i) For all $h > 0$ *sufficiently small there exist* $F(t_0 + h) - F(t_0), F(t_0) - F(t_0 - h)$ *and the limits in* (E^{n}, d), $\lim_{h\to 0^+}((F(t_0 + h) - F(t_0))/h) = \lim_{h\to 0^+}((F(t_0) - F(t_0 - h))/h) = F\,'(t_0)$ *under the similar conditions, and other definion is,*

ii) For all $h < 0$ *sufficiently small there exist* $F(t_0 + h) - F(t_0), F(t_0) - F(t_0 - h)$ *and the limits in* (E^{n}, d), $\lim_{h\to 0^-}((F(t_0 + h) - F(t_0))/h) = \lim_{h\to 0^-}((F(t_0) - F(t_0 - h))/h) = F\,'(t_0)$. For details please see [Bede and Gal, 2005].

3. PRACTICAL STABILITY OF FUZZY DIFFERENTIAL EQUATIONS

Consider the initial value problem for the fuzzy differential equations

$$u' = f(t,u), u(t_0) = u_0 \; for \; t \geq t_0 \geq 0 \tag{3.1}$$

where $f \in C[R_+ \times S_\rho, E^{\mathrm{n}}]$, and where the set $S_\rho = [u \in E^{\mathrm{n}}: d[u,0] < \rho < \infty]$, $f(t,0) \equiv 0$ for $t \geq 0$. The above assumptions imply the existence of trivial solution of (3.1) through (t_0, u_0). Before, we can establish our comparison theorem and Lyapunov stability criteria for initial time difference. We need to introduce the following definitions.

Definition 3.1: The trivial solution $u = 0$ *of (3.1) is said to be practical-stable, if and only if given* (λ, A) *with* $0 < \lambda < A$ *such that* $d[u(t), 0] < A$ *whenever* $d[u_0, 0] < \lambda$ *for* $t \geq t_0$ *and uniformly practically stable, if (PS1) holds for all* $t_0 \in R_+$*. Practically quasi-stable, if and only if given* (λ, B, T) *with* $0 < \lambda < B$ *such that* $d[u(t), 0] < B$ *whenever* $d[u_0, 0] < \lambda$ *for* $t \geq t_0 + T$ *for some* $t_0 \in R_+$*. The trivial solution* $u = 0$ *of (3.1) is said to be Uniformly practically quasi stable, if definition of practically quasi-stable holds for each* $t_0 \in R_+$*.*

Definition 3.2: The trivial solution $u = 0$ *of (3.1) is said to be equi-stable if, for each* $\varepsilon > 0, t_0 \in R_+$*, there exists a positive function* $\delta = \delta(t_0, \varepsilon)$ *that is continuous in* t_0 *for each ε such that the inequality* $d[u_0, 0] \leq \delta$ *implies* $d[u(t), 0] < \varepsilon$ *for* $t \geq t_0$ *where* $u(t) = u(t, t_0, u_0)$ *is the solution of (3.1). Uniformly stable, if the* δ *is independent of* t_0*. Quasi-equi-asymptotically stable, if for each* $\varepsilon > 0$ *and,* $t_0 \in R_+$ *there exist positive functions* $\delta_0 = \delta_0(t_0)$ *and* $T = T(t_0, \varepsilon)$ *such that* $d[u_0, 0] \leq \delta_0$ *implies* $d[u(t), 0] < \varepsilon$ *for* $t \geq t_0 + T$*. Quasi-uniformly asymptotically stable if* δ_0 *and* T *in definition of Quasi-equi-asymptotically stable is independent of* t_0*.*

Corresponding to the definitions of various stability notions, we have practical stability definitions and the classical stability concepts for fuzzy differential equations. The solutions of (3.1) are said to be stable with respect to Definition 3.2.

Now, we define and explain the fundamental concepts of stability and comparison theorems. We express basic formulation and notation which we use.

Definition 3.3: For any Lyapunov-like function $V(t,u) \in C[R_+ \times E^{\mathrm{n}}, R_+]$ *we define the fuzzy Dini derivative* $D^{+}V(t,u)$ *as follows* $D^{+}V(t,u) = \lim_{h\to 0^{+}} sup(1/h)[V(t+h, u+hf(t,u)) - V(t,u))]$, $(t,u) \in R_+ \times E^{\mathrm{n}}$.

Definition 3.4: A function a *is said to belong to the class* K *such that* $K = [a \in C[[0,\rho), \mathbb{R}_+]: a(0) = 0 \; and \; a \; is \; strictly \; increasing]$.

Definition 3.5: A real-valued function $V(t,u)$ *defined on* $R_+ \times S_\rho$ *with* $V(t,0) = 0$ *for* $t > 0$ *is said to be positive definite if there exists a function* $b(r) \in K$ *such that the relation* $b(d[u,0]) \leq V(t,u)$ *for* $(t,u) \in R_+ \times S_\rho$*. A function* $V(t,u)$ *is called decrescent if there exists a function* $a \in K$ *such that* $V(t,u) \leq \psi(d[u,0])$ *for* $(t,u) \in R_+ \times S_\rho$*. To use the second method of Lyapunov, which attempts to make statements about the stability properties directly by using suitable functions, we need to study the scalar differential equation*

$$w' = g(t,w), w(t_0) = w_0 \geq 0 \tag{3.2}$$

where $g(t,w) \in C[{R_+}^2, R]$*. Corresponding to the stability definitions of Definition 3.1 and we designate by Definition 3.1 and Definition 3.2 the concepts concerning the stability.*

Definition 3.6: The trivial solution $w = 0$ *of (3.2) is said to be equi-stable if, for each* $\varepsilon > 0, t_0 \in \mathbb{R}_+$*, there exists a positive function* $\delta = \delta(t_0, \varepsilon)$ *that is continuous in* t_0 *for each ε such that the inequality* $w(t, t_0, w_0) < \varepsilon, t \geq t_0$*, provided* $w_0 \leq \delta$*.*

We use Lyapunov-like functions for finding behavior of systems' solutions. We study practical stability of fuzzy differential equations. We have a comparison result that is in terms of a Lyapunov function. Lyapunov function is very important and Lyapunov function aproaches are proved by the theory of differential equations and differential inequalities. Finally, we define the Lyapunov function for transform the fuzzy differential equation into a scalar comparison differential equation. Therefore, it is enough to consider the stability properties of the more simple comparison equation for detection of stability. Scalar differential equation is very usefull for this reason. This is known as the comparison principle, in general. We can now formulate the basic comparison results via Lyapunov-like function.

Theorem 3.1: Assume that, $V \in C[R_+ \times S_\rho, R_+]$ *and* $|V(t,u_1) - V(t,u_2)| \leq Ld[u_1, u_2]$ *where* $L > 0$ *and where the set* S_ρ; $S_\rho = [u \in E^n : d[u, 0] < \rho]$. *Dini derivative is* $D^+V(t,u) = \lim\limits_{h \to 0^+} sup \left(\frac{1}{h}\right) \left[V\big(t+h, u+hf(t,u)\big) - V(t,u)\right] \leq g\big(t, V(t,u)\big)$, *where* $g \in C[{R_+}^2, R]$. *Then, if* $u(t)$ *is any solution of (3.1) existing on* $[t_0, \infty)$ *such that* $V(t_0, u_0) \leq w_0$, *we have* $V(t, u(t)) \leq r(t, t_0, w_0), t \geq t_0$ *where* $r(t, t_0, w_0)$ *is maximal solution of the scalar differential equation* $w' = g(t, w), w(t_0) = w_0 \geq 0$, *existing on* $[t_0, \infty)$. *For details please see* [Lakshmikantham and Mohapatra, 2003].

Corollary 3.1: [Lakshmikantham and Mohapatra, 2003]. *If, in Theorem 3.1, we suppose that* $g(t,u) \equiv 0$, *then we arrive at*

$$V(t, u(t)) \leq V(t_0, w_0), t \geq t_0. \tag{3.3}$$

For details please see [Lakshmikantham and Mohapatra, 2003].

Corollary 3.2: If in Theorem 3.1, we strengthen the assumption on $D^+V(t,u)$ *to* $D^+V(t,u) = \lim\limits_{h \to 0^+} sup \left(\frac{1}{h}\right) [V(t+h, u+hf(t,u)) - V(t,u))] \leq -c(w(t,u)) + g(t, V(t,u)), (t,u) \in R_+ \times S_\rho$ *where* $w \in C\big[R_+ \times S_\rho, R_+\big]$ *and* $c \in K$ *and* $K = [a \in C\big[[0,\rho), R^+\big] : a(0) = 0$ *and* $a(w)$ *is increasing in* $w]$ *and* $g(t,w)$ *is nondecreasing in* w *for each* $t \in R_+$, *then we get the estimate* $V(t, u(t)) + \int_{t_0}^{t} c(w(s, u(s)))ds \leq r(t, t_0, w_0)$, $t \geq t_0$ *whenever* $V(t_0, u(t_0)) \leq w_0$.

For details please see [Lakshmikantham and Mohapatra, 2003].

3.1. Practical Stability of the Fuzzy Differential Systems

Theorem 3.2: Assume that the following hold

(i) Let $V \in C[R_+ \times S(\rho), R_+], |V(t,u_1) - V(t,u_2)| \leq Ld[u_1,u_2], L > 0$ *and for* $(t,u) \in R_+ \times S_\rho$ *such that*

$$D^+V(t,u) = \lim_{h \to 0^+} \sup \left(\frac{1}{h}\right)[V(t+h, u+hf(t,u)) - V(t,u))] \leq 0 \tag{3.4}$$

(ii) and suppose that

$$b(d[u(t),0]) \leq V(t,u(t)) \leq a(t, d[u(t),0]) \tag{3.5}$$

where $(t,u) \in R_+ \times S_\rho$ *and* $a, b \in K$.

Then the trivial solution $u = 0$ *of fuzzy differential equation of (3.1) is practical stable.*

Proof of Theorem 3.2: Let us assume that $0 < \lambda < A$ *and* $t_0 \in R_+$ *is given. Then it is possible to find* $a(t_0, \lambda) > 0$ *and* $b = b(A) > 0$ *such that*

$$a(t_0, \lambda) < b(A). \tag{3.6}$$

We claim that with this λ*, practical stability holds such that*

$$d[u_0, 0] < \lambda \text{ whenever } d[u(t),0] < A \text{ for } t \geq t_0. \tag{3.7}$$

If the solution of fuzzy differential equation (3.1) is not practically stable and then there would exist a solution of fuzzy differential equation (3.1) $u(t) = u(t, t_0, u_0)$ *exist a* $t_1 > t_0$ *and with* $d[u_0, 0] < \lambda$ *satisfying*

$$d[u(t_1),0] = A \text{ and } d[u(t),0] \leq A < \rho \text{ for } t \in [t_0, t_1]. \tag{3.8}$$

Then we have, because of (3.4) and (3.5)

$$b(A) \leq V(t_1, u(t_1)), t_1 > t_0. \tag{3.9}$$

This means that $d[u(t),0] < \rho$ *for* $t \in [t_0, t_1]$ *and hence we get from the assumption (3.4) and Corollary 3.1, the estimate*

$$V(t_1, u(t_1)) \leq V(t_0, u(t_0)), t \geq t_0. \tag{3.10}$$

By using (3.6), (3.8), (3.9) and (3.10), we get $b(A) = b(d[u(t_1),0]) \leq V(t_1, u(t_1)) \leq V(t_0, u(t_0)) \leq a(t_0, d[u_0, 0] \leq a(t_0, \lambda) < b(A)$. *This is a contradiction. Hence, (3.7) is valid and we have practical stability of the trivial solution* $u = 0$ *of fuzzy differential equation (3.1).*

Corollary 3.3: If V(t,u) is only decrescent, we get by $V(t,u) \leq a(d[u(t),0]), V(t_0, u_0) \leq a(t_0, d[u_0, 0])$ *and choose* $\lambda = \lambda(A) > 0$ *such that* $a(\lambda) < b(A)$. *Since* λ *is now independent of* t_0*, we have uniformly practical stability of the trivial solution* $u = 0$ *of fuzzy differential equation (3.1).*

Theorem 3.3: Assume that the following holds

(i) Let $V \in C[R_+ \times S_\rho, R_+], |V(t,u_1) - V(t,u_2)| \leq Ld[u_1, u_2], L > 0$ *and for* $(t,u) \in R_+ \times S(\rho)$, *where* $S_\rho = [u \in E^n: d[u(t),0] < \rho]$ *such that*

$$D^+V(t,u) \leq -\mu V(t,u). \tag{3.11}$$

(ii) Let $V(t,u) \in C[R_+ \times S_\rho, R_+]$,

$$b(t, d[u(t),0]) \leq V(t,u) \leq a(t, d[u(t),0] \tag{3.12}$$

where $a, b \in K$ *and* $(t,u) \in R_+ \times S_\rho$. *Then the trivial solution* $u(t) = 0$ *of fuzzy differential equation (3.1) is practically quasi stable.*

Proof of Theorem 3.3: It is clear that we have practical stability of the trivial solution $u = 0$ *of fuzzy differential equation* (3.1) *from* (3.11). *Taking B=ρ and there exist* $\lambda_0 =$

$\lambda_0(t_0, \rho) > 0$. *Then we have quasi-practical stability of the trivial solution* $u = 0$ *of fuzzy differential equation* (3.1) *with this* (λ_0, B)*; practical stability holds such that*

$$d[u(t),0] < \rho \text{ whenever } d[u_0, 0] < \lambda_0 \text{ for } t \geq t_0 + T. \quad (3.13)$$

If the solution of fuzzy differential equation (3.1) *is not practically quasi-stable and then there would exist a solution of fuzzy differential equation* (3.1) $u(t) = u(t, t_0, u_0)$ *exist a* $t_1 > t_0 + T$ *and with* $d[u_0, 0] < \lambda_0$ *satisfying*

$$d[u(t_1),0] = B \text{ and } d[u(t),0] \leq B < \rho \text{ for } t \in [t_0 + T, t_1]. \quad (3.14)$$

Now, (3.11) *and Corollary 3.2 yields the estimate*

$$V(t, u(t)) \leq V(t_0, u(t_0))exp[-\mu(t - t_0)], t \geq t_0 + T. \quad (3.15)$$

For B>0, we choose $T = T(t_0, B) = (1/\mu)ln[((a(t_0, \lambda_0))/(b(B)))] + 1$. *Then we have from* (3.11), (3.12), (3.14) *and* (3.15), *we get* $b(B) = b(d[u(t_1), 0]) \leq V(t_1, u(t_1)) \leq V(t_0, u(t_0))exp[-\mu(t_1 - t_0)] \leq a(t_0, \lambda_0)\, exp[-\mu(t_1 - t_0)] < b(B)$. *We have the strict inequality* $b(B) < b(B)$. *This contradiction which gives us practical quasi-stability of the solution* $u(t) = u(t, t_0, u_0)$ *of the fuzzy differential equation (3.1).*

Corollary 3.4: If $V(t, u)$ *is only decrescent, we get by* $V(t, u) \leq a(d[u(t),0])$, $V(t_0, u_0) \leq a(d[u_0, 0])$ *and choose* $\lambda = \lambda(B) > 0$. *Since* λ *is now independent of* t_0, *we have uniformly practical stability of the trivial solution* $u = 0$ *of the fuzzy differential equation* (3.1).

3.2. A Comparison Result in Practical Stability of Fuzzy Differential Equations

In this section, we have a useful comparison theorem in practical stability of fuzzy differential systems via scalar differential equation and proof of this theorem.

Theorem 3.4: Assume that

(i) $V \in C[R_+ \times S_\rho, R_+], |V(t,u_1) - V(t,u_2)| \leq Ld[u_1,u_2], L > 0$ *and for* $(t,u) \in R_+ \times S_\rho$ *and*

$$b(t, d[u,0]) \leq V(t,u) \leq a(t, d[u,0]) \qquad (3.16)$$

where $(t,u) \in R_+ \times S_\rho$ *and* $a, b \in K$*.(3.16) and the Dini derivatives of Lyapunov functions*

$$D^+V(t,u) = \lim_{h \to 0^+} sup(1/h)[V(t+h, u+hf(t,u)) - V(t,u))] \leq g(t, V(t,u) \quad (3.17)$$

where $(t,u) \in R_+ \times S_\rho$ *and* $g(t,u) \in C[{R_+}^2, R]$*;*

(ii) $r(t) = r(t, t_0, w_0)$ *be the maximal solution of the scalar differential equation*

$$w' = g(t,w), w(t_0) = w_0 \geq 0. \qquad (3.18)$$

Then the practical stability properties of the comparison differential equation (3.2) imply the corresponding practical stability properties of the trivial solution $u = 0$ *of (3.1) fuzzy differential equation.*

Proof of Theorem 3.4: Suppose that the comparison differential equation (3.18) is practically stable, then for given (λ, A) *with* $0 < \lambda < A$ *and there exists* $b(t_0, A) > 0$ *such that*

$$w(t, t_0, w_0) < b(t_0, A) \ \text{for}\ d[w_0, 0] < \lambda\,, t \geq t_0 \qquad (3.19)$$

for some $t_0 \in R_+$. *We claim that with this λ, practical stability* holds such that

$$d[u_0, 0] < \lambda \ \text{whenever}\ d[u(t), 0] < A \ \text{for}\ t \geq t_0 \qquad (3.20)$$

If the fuzzy differential equation (3.1) is not practically stable and then there would exist a solution of fuzzy differential equation (3.1) $u(t) = u(t, t_0, u_0)$ *exist a* $t_1 > t_0$ *and with* $d[u_0, 0] < \lambda$ *satisfying*

$$d[u(t_1),0] = A \text{ and } d[u(t),0] \leq A < \rho \text{ for } t \in [t_0, t_1]. \tag{3.21}$$

Choose $w_0 = a(d[u_0, 0])$*, we get the inequality, from Theorem 3.1,*

$$V(t, u(t)) \leq r(t, t_0, w_0), t_0 \leq t \leq t_1. \tag{3.22}$$

So that we have, because of (3.16) and (3.21)

$$b(t_0, A) \leq V(t_1, u(t_1)), t_1 > t_0 \tag{3.23}$$

This means that $d[u(t),0] < \rho$ *for* $t \in [t_0, t_1]$ *and hence we have the inequality*

$$V(t_1, u(t_1)) \leq r(t_1, t_0, w_0), t \geq t_0. \tag{3.24}$$

By using (3.19), (3.22), (3.23) and (3.24), we get $b(t_0, A) = b(t_0, d[u(t_1),0] \leq V(t_1, u(t_1)) \leq r(t_1, t_0, w_0) \leq r(t_1, t_0, a(d[u_0, 0])) \leq r\big(t, t_0, a(\lambda)\big) < b(t_0, A)$ *and we have the contradiction*

$$b(t_0, A) < b(t_0, A). \tag{3.25}$$

This is mean that we have practical stability of fuzzy differential equation (3.1). If $V(t, u)$ *is only decrescent, we get by* $V(t, u) \leq a(d[u(t),0]), V(t_0, u_0) \leq a(d[u_0, 0])$ *and choose* $\lambda = \lambda(A) > 0$ *such that* $a(\lambda) < b(A)$*. Since* λ *is now independent of* t_0*, we have uniformly practically stability of the trivial solution* $u = 0$ *of fuzzy differential equation (3.1).*

Assume that the comparison system is practical quasi-stable. Any given $b(B) > 0$*, there exists a* $T = T(t_0, B) > 0$ *such that* $0 < w_0 < \lambda$ *implies* $w(t, t_0, w_0) < b(B), t \geq t_0 + T$.

Setting $w_0 = a(d[u_0, 0]) < \lambda$ *and using (i),* $V(t_0, u_0) \leq a(d[u_0, 0]) < \lambda$. *By using Theorem 3.1 and using (i),* $b(d[u(t),0]) \leq V(t, u(t)) \leq r(t, t_0, w_0) < b(B)\ for\ t \geq t_0 + T$. *We obtain* $d[u(t),0] < B, t \geq t_0 + T$ *since* $b \in K$. *This proves the practically quasi stability.*

Remark 3.1: In this process, if $g(t, w) \equiv 0$ *has been choosed in (3.2), we have practical stability features of scalar equation (3.18) and this has been given the practical stability features of the solution* $u = 0$ *of (3.1) fuzzy differential equation as we have in Theorem 3.1 and Theorem 3.2.*

4. PERTURBED FUZZY DIFFERENTIAL EQUATIONS AND STABILITY DEFINITIONS OF FUZZY DIFFERENTIAL SYSTEMS WITH INITIAL TIME DIFFERENCE

4.1. Introduction to Initial Time Difference Stability Analysis of the Fuzzy Systems

Fuzzy sets and fuzzy differential equations have been studied recently. Moreover, efficient studies have been available and developed of fuzzy calculus, fuzzy differential equations, numerical approaches and modeling of fuzzy differential equations in recent years [Ahmadian et al. 2017], [Ahmadian et al. 2018], [Zhang and Sun, 2015], [Gomes et al., 2015]. We have to say that definition and concept of fuzzy derivative and integration is very important for fuzzy differential equations. Bede and Gal [Bede and Gal, 2005] presented the generalizations of differentiability. Based on this work, some studies [Bede et al., 2007], [Bede and Stefanini, 2013] used and improved the generalized differentiability and the second type of Hukuhara derivative was applied in fuzzy differential equations.

The stability theorems and the comparison results for fuzzy systems with the second type of Hukuhara derivative have investigated and found some new results [Zhang and Sun, 2015], [Gomes et al., 2015].

Lyapunov's second method is a effective method for the qualitative theory of differential equations. The method is useful to predict the behavior and qualitative properties of solutions and a solution of a differential system. In this technique, if we have information the stability properties of a comparison system, then we can predict the behavior of a differential system [Lakshmikantham et al., 1989].

Normally, stability approach can work with standart initial conditions that have initial times and initial positions. If we have different initial conditions that have initial times and initial positions, there is a problem with this way when trying to apply it to perturbed fuzzy differential systems. The problem appears because there is a distinction between initial time difference (ITD) stability and the classical stability approach [Yakar and Shaw, 2005], [Yakar et al., 2013]. The standart stability approach of fuzzy differential equations are in terms of the null solution, but initial time difference stability approach of fuzzy differential equations is in terms of the unperturbed fuzzy differential system where the perturbed fuzzy differential system and the unperturbed fuzzy differential system have different initial positions and initial times [Yakar and Shaw, 2005] (and for fuzzy differential systems please see [Yakar et al., 2012]). In this study, we have worked out this problem and we have a fuzzy comparison result. This approaches creates new initial time difference stability applications of fuzzy differential systems.

4.2. The Concept of Fuzzy Calculus and Initial Time Difference

In this section, we have some definitions for fuzzy calculus and similarly fuzzy derivative definitions. We use the similar concepts with little differences for initial time difference concept. However, initial time difference concept is different from our previous definitions and explanations. Before initial time difference concept explanations, we give an information about fuzzy logic, fuzzy calculus and fuzzy differential equations.

Let the base space R^{n} for fuzzy sets and we let a fuzzy set u and $u: R^{\mathrm{n}} \to [0,1]$ is a function. The function $u(x)$ for $x \in R^{\mathrm{n}}$ taking values in the interval $[0,1]$ called membership grades. Membership grades are very important for fuzzy sets. Now we state the following membership grade of the $u(x)$, $u(x) = 0$ is meaning to nonmembership, $0 < u(x) < 1$ is meaning to partial membership and $u(x) = 1$ is meaning to full membership.

E^{n} is the space of all fuzzy subsets of R^{n}. We give the description of the set E^{n}, $E^{\mathrm{n}} = [u: R^{\mathrm{n}} \to [0;1] | u \ \ satisfies\ (i)\ \ to\ (iv)\ ]$

We define the set $P_k(R^{\mathrm{n}})$ be the family of all nonempty compact, convex subsets of R^{n}.

Haussdorff metric as follows: $d_H(A,B) = max[\sup_{a\in A} \inf_{b\in B} \|a-b\|, \sup_{b\in B} \inf_{a\in A} \|b-a\|]$ where $A,B \in P_k(R^{\mathrm{n}})$ and $\|.\|$ is the usual Euclidean norm in R^{n}.

The metric space $(P_k(R^{\mathrm{n}}), d_H)$ is complete and separable (see [Lakshmikantham and Mohapatra, 2003]). Let $\Omega = [t_0, \tau_0 + a]$ where $\tau_0 \geq t_0 \geq 0$ and $a > 0$ be a interval in R and denote $E^{\mathrm{n}} = [u: R^{\mathrm{n}} \to [0;1] | u \ \ satisfies\ (i)\ \ to\ (iv)\ ]$ below

i) u is normal if there exists an $x_0 \in R^{\mathrm{n}}$ such that $u(x_0) = 1$;

ii) u is fuzzy convex;

iii) u is upper semi-continuous;

iv) $[u]^0 = [x \in R^n : u(x) > 0]$ is compact subset of in R^n.

Where u is fuzzy convex and we give the definition of convexity $x, y \in R^n, 0 \leq \lambda \leq 1, u(\lambda x + (1-\lambda)y) \geq min[u(x), u(y)]$. For $0 < \alpha \leq 1$, we denote the α_level sets as $[u]^\alpha = [x \in R^n : u(x) \geq \alpha]$. Then from (i) to (iv), it follows that $[u]^\alpha \in P_k(R^n)$ for $0 \leq \alpha \leq 1$.

Let $d_H(A, B)$ be the Hausdorff distance between the sets $A, B \in P_k(R^n)$. Then we define $d[u, v] = \sup_{0 \leq \alpha \leq 1} d_H\, [[u]^\alpha, [v]^\alpha]$ which defines a metric in E^n and (E^n, d) is a complete metric space. We have the following properties of $d[u, v]$:

i) $d[u + w, v + w] = d[u, v]$,

ii) $d[u, v] = d[v, u]$,

iii) $d[\lambda u, \lambda v] = |\lambda| d[u, v]$,

iv) $d[u, v] \leq d[u, w] + d[w, v]$

for all $u, v, w \in E^n$ and $\lambda \in R$.

For $x, y \in E^n$ if there exists a $z \in E^n$ such that $x = y + z$, then z is called the H-difference of x and y and is denoted by $x - y$. For further details, see [Lakshmikantham and Mohapatra, 2003].

Now we define the fuzzy derivative for our fuzzy differential equations and the initial time difference concept.

Definition 4.1: Let $F: (a, b) \to E^n, t_0 \in (a, b)$ and F is differentiable at t_0. If there exists $F'(t_0) \in E^n$ such that *(i) For all* $h > 0$ *sufficiently small there exist* $F(t_0 + h) - F(t_0), F(t_0) - F(t_0 - h)$ *and the limits in* (E^n, d) $\lim_{h \to 0^+}((F(t_0 + h) - F(t_0))/h) = \lim_{h \to 0^+}((F(t_0) - F(t_0 - h))/h) = F'(t_0)$. *Under the similar conditions, and other definion is,*

(ii) For all $h < 0$ *sufficiently small there exist* $F(t_0 + h) - F(t_0), F(t_0) - F(t_0 - h)$ *and the limits in* (E^n, d), $\lim_{h \to 0^-}((F(t_0 + h) - F(t_0))/h) = \lim_{h \to 0^-}((F(t_0) - F(t_0 - h))/h) = F'(t_0)$

For details please see [Bede and Gal 2005].

4.3. Concept of Fuzzy Differential Equations and Definitions of Initial Time Difference Stability

Consider the initial value problem for the fuzzy differential equations

$$u' = f(t, u), u(t_0) = u_0 \, for \; t \geq t_0 \geq 0 \tag{4.1}$$

$$u' = f(t, u), u(\tau_0) = v_0 \, for \; t \geq \tau_0 \geq t_0 \tag{4.2}$$

the perturbed system of (4.2)

$$v' = F(t, v), v(\tau_0) = v_0 \text{ for } t \geq \tau_0 \tag{4.3}$$

where the functions $f, F \in C[R_+ \times S\rho, E^n], S_\rho = [u \in E^n : d[u, 0] < \rho < \infty]$, $f(t, 0) \equiv 0$ for $t \geq 0$. We have a special case of (4.3) that is perturbation equation of (4.2) if $F(t, v) = f(t, v) + R(t, v)$ where $R(t, v)$ is the perturbation term. The above assumptions imply the existence of trivial solutions of (4.1) and (4.2) through (t_0, u_0) and (τ_0, v_0), respectively.

Before, we can establish our comparison theorem and Lyapunov stability criteria for initial time difference we need to introduce the following definitions.

Definition 4.2: Stability definition of $v(t,\tau_0,v_0)$ *of the system (4.3) through* (τ_0,v_0) *for* $t \geq \tau_0, \tau_0 \in R_+$ *is called initial time difference*

(S₁) stable in terms of $u(t,\tau_0,u_0) = u(t-\eta,t_0,u_0)$*, here* $u(t,t_0,u_0)$ *is any solution of the system (4.1) for* $t \geq \tau_0$*,* $t_0 \in R_+$ *and* $\eta = \tau_0 - t_0$ *if and only if given* $\forall\varepsilon > 0$ *there exists* $\delta_1(\varepsilon,\tau_0) > 0$ *and* $\delta_2(\varepsilon,\tau_0) > 0$ *such that* $d[v(t,\tau_0,v_0) - u(t-\eta,t_0,u_0),0] < \varepsilon$ *provided that* $d[\,v_0 - u_0,0] < \delta_1$ *and* $d_s[\tau_0 - t_0,0] < \delta_2$ *for* $t \geq \tau_0$*; where* d_s *is a standart metric in R,*

(S₂) uniformly stable in terms of the solution $u(t-\eta,t_0,u_0)$ *If* δ_1 and δ_2 *in (S₁) are not dependent on* $\tau_0 \in R_+$*,*

(S₃) attractive in terms of the solution $u(t,\tau_0,u_0) = u(t-\eta,t_0,u_0)$ *for* $t \geq \tau_0$ *if and only if given* $\forall\varepsilon > 0$ *and* $\tau_0 \in R_+$ *there exists* $\delta_1(\tau_0) > 0$*,* $\delta_2(\tau_0) > 0$ *and a* $T = T(\varepsilon,\tau_0) > 0$ *such that* $d[v(t,\tau_0,v_0) - u(t-\eta,t_0,u_0),0] < \varepsilon$ *provided that* $d[v_0 - u_0,0] < \delta_1$ *and* $d_s[\tau_0 - t_0,0] < \delta_2$ *for* $t \geq \tau_0 + T$*,*

(S₄) uniformly attractive according to the solution $u(t,\tau_0,u_0)$ *if* $\delta_1 > 0$*,* $\delta_2 > 0$ *and a* $T > 0$ *in (S₃) are independent of* $\tau_0 \in R_+$*,*

(S₅) asymptotically stable according to the solution $u(t,\tau_0,u_0)$ *if (S₁) and (S₃) hold simultaneously. Or, if (S₁) holds and there exists* $\gamma_1(\tau_0)$ *and* $\gamma_2(\tau_0)$ *are positive functions such that* $\lim_{t\to\infty} d[v(t,\tau_0,v_0) - u(t-\eta,t_0,u_0),0] = 0$ *with* $d[v_0 - u_0,0] < \gamma_1, d_s[\tau_0 - t_0,0] < \gamma_2$ *where* $t \geq \tau_0$*,*

(S₆) uniformly asymptotically stable according to the solution $u(t,\tau_0,u_0)$ *if (S₂) and (S₄) hold simultaneously. Or, if (S₂) holds and given any ε>0 there exist* $\gamma_1 = \gamma_1(\varepsilon) > 0, \gamma_2 = \gamma_2(\varepsilon) > 0$ *and a* $T(\varepsilon) > 0$ *such that* $d[v(t,\tau_0,v_0) - u(t-\eta,t_0,u_0),0] < \varepsilon$ *provided that* $d[v_0 - u_0,0] < \gamma_1$ *and* $d_s[\tau_0 - t_0,0] < \gamma_2$ *for each* $t \geq \tau_0 + T(\varepsilon)$*.*

Definition 4.3: For a real-valued function $V(t,u) \in C[R_+ \times S_\rho, R_+]$*, we define the Dini derivatives as follows* $D^+V(t,u) \equiv \lim_{h\to 0^+} sup(1/h)[V(t+h,u+hf(t,u)) - V(t,u)]$ *for* $(t,x) \in R_+ \times S_\rho$*.*

Definition 4.4: For a real-valued function $V(t,u) \in C[R_+ \times E^{\mathrm{n}}, R_+]$, we define the generalized derivatives (Dini-like derivatives) as follows $^{*}D^{+}V(t, v-u) = \lim_{h\to 0^+} sup(1/h)[V(t+h, v-u+h(F(t,v)-f(t,u))) - V(t, v-u)]$ *for* $(t,u) \in R_+ \times S_\rho$ *where* $v(t,\tau_0,v_0)$ *is the solution of the system (4.3) and* $u(t,\tau_0,u_0) = u(t-\eta, t_0, u_0)$*, where* $u(t,t_0,u_0)$ *is any solution of the system (4.1) for* $t \geq \tau_0 \geq 0, t_0 \in R_+$ *and* $\eta = \tau_0 - t_0$.

4.4. Comparing Classical Stability with (ITD) Stability of Fuzzy Systems

4.4.1. Classical Notion of Stability of Fuzzy Systems

Let $u(t,t_0,u_0)$ and $u(t,t_0,v_0)$ be any solutions of the fuzzy differential system (4.1) with $u' = f(t,u), u(t_0) = v_0\, for\ t \geq t_0, t_0 \in R_+$ where $f \in C[R_+ \times S_\rho, E^{\mathrm{n}}]$, and S_ρ is the set $S_\rho = [u \in E^{\mathrm{n}}: d[u,0] < \rho < \infty]$. Assume that $f(t,0) = 0, t \in R_+$ so that $u = 0$ is a null solution of (4.1) through $(t_0, 0)$. We can define the stability properties of null solution.

Definition 4.5: The null solution $u = 0$ *of equation (4.1) is said to be stable if and only if for each* $\varepsilon > 0$ *and for all* $t_0 \in R_+$, *there exist two positive functions* $\delta_1 = \delta_1(\varepsilon, t_0)$ *and* $\delta_2 = \delta_2(\varepsilon, t_0)$ *that are continuous in* t_0 *for each* ε *such that* $d[u(t,t_0,u_0),0] < \varepsilon$ *provided that* $d[u_0, 0] < \delta_1$ *and* $d_s[t_0, 0] < \delta_2$ *for* $t \geq t_0$. *If* δ_1 *and* δ_2 *are independent of* t_0, *then the null solution* $u = 0$ *of (4.1) is said to be uniformly stable.*

Definition 4.6: The solution $u(t,t_0,v_0)$ *of (4.1) through* (t_0, v_0) *is said to be stable with respect to the solution* $u(t,t_0,u_0)$ *of (4.1) for* $t \geq t_0 \in R_+$ *if and only if given any* $\varepsilon > 0$, *there exist two positive functions* $\delta_1 = \delta_1(\varepsilon, t_0)$ *and* $\delta_2 = \delta_2(\varepsilon, t_0)$ *are continuous in* t_0 *for each* ε *such that* $d[u(t,t_0,u_0) - u(t,t_0,v_0),0] < \varepsilon$ *provided* $d[v_0 - u_0, 0] < \delta_1$ *and* $d_s[t_0, 0] < \delta_2$ *for* $t \geq t_0$. *If* δ_1 *and* δ_2 *are independent of* $\tau_0 \in$

R_+, *then the solution of the system (4.1) is uniformly stable with respect to the solution* $u(t, t_0, u_0)$ [Yakar and Shaw, 2005], [Yakar et al., 2012].

We remark that for the purpose of studying the classical stability of a given solution $u(t, t_0, v_0)$ of the system (4.1), it is convenient to make a change of variable. Let $u(t, t_0, v_0)$ and $u(t, t_0, u_0)$ be the unique solutions of (4.2) and set $w_1(t, t_0, v_0 - u_0) = u(t, t_0, v_0) - u(t, t_0, u_0)$ for $t \geq t_0$. Then $w_1{}'(t, t_0, v_0 - u_0) = u'(t, t_0, v_0) - u'(t, t_0, u_0)$, $w_1{}'(t, t_0, v_0 - u_0) = f(t, w_1(t, t_0, v_0 - u_0) + u(t, t_0, u_0)) - f(t, u(t, t_0, u_0))$, $w_1{}'(t, t_0, v_0 - u_0) = f(t, w_1(t, t_0, v_0 - u_0))$ and we say that $w_1(t, t_0, v_0 - u_0) \equiv 0$ is a solution of the transformed system if $v_0 - u_0 = 0$ which implies $f(t, 0) = 0$. Since $w_1{}'(t) = 0$ and $w_1(t, t_0, 0) = 0$ is the null solution, $u(t, t_0, u_0)$ the solution of (4.1) of corresponds to the identically null solution of $w_1{}' = f(t, w_1)$ where $f(t, w_1(t, t_0, v_0 - u_0)) = f(t, w_1(t, t_0, v_0 - u_0) + v(t, \tau_0, u_0)) - f(t, v(t, \tau_0, u_0))$.

Hence, we can always assume, without any loss of generality, that $u(t, t_0, u_0) \equiv 0$ is the null solution of the given the equation (4.1) and we can limit our study of stability to that of the null solution [Yakar and Shaw, 2005], [Yakar et al., 2013], [Yakar, 2010]. However; it is impossible to do the same transformation for initial time difference stability.

4.4.2. New Notion of (ITD) Stability of Fuzzy Systems

Let $u(t, \tau_0, v_0)$ be a solution of equation (4.2) and $u(t, \tau_0, u_0) = u(t - \eta, t_0, u_0)$ where $u(t, t_0, u_0)$ is any solution of the system equation (4.1) for $t \geq \tau_0 \geq 0$. Let us make a transformation similar to that in (4.2). Set $w_2(t, \tau_0, v_0 - u_0) = u(t, \tau_0, v_0) - u(t - \eta, t_0, u_0)$ for $t \geq \tau_0$. Then $w_2{}'(t, \tau_0, v_0 - u_0) = u'(t, \tau_0, v_0) - u'(t - \eta, t_0, u_0)$, $w_2{}'(t, \tau_0, v_0 - u_0) = f(\eta; t, w_2(t, \tau_0, v_0 - u_0))$.

One can observe that even if $v_0 = u_0$, $w_2(t, \tau_0, 0)$ is not zero and is not the null solution of the transformed system and the solution $u(t - \eta, t_0, u_0)$ does not correspond to the identically zero solution of $w_2{}' = f(\eta; t, w_2)$.

Therefore, we cannot use stability properties of the null solution in order to find (ITD) stability properties using this approach.

4.5. Initial Time Difference Comparison Theorems

We know that the differences between the classical notion of stability and initial time difference(ITD) stability did not allow the use of the behavior of the null solution in our ITD stability analysis. The main result presented in this section resolves those difficulties with a new approach that allows the use of the stability of the null solution of the comparison system to predict the stability properties of $v(t,\tau_0,v_0)$ the solution of equation (4.3) with respect to $u(t) = u(t-\eta,t_0,u_0)$ where $u(t,t_0,u_0)$ is any solution of the system equation (4.1).

Theorem 4.1: Assume that $f, F \in C[J \times S_\rho, E^{\mathrm{n}}]$ *and*

i) $d[F(t,v) - f(t,u), 0] \leq G(t, d[v(t,\tau_0,v_0) - u(t-\eta,t_0,u_0), v_0 - u_0])$ *for* $t \in J$ *and* $u, v \in E^{\mathrm{n}}$ *where* $G \in C[J \times R_+, R]$*;*

ii) $r(t) = r(t,\tau_0,w_{10})$ *is the maximal solution of the scalar differential equation exists on J*

$$w_1{}' = G(t,w_1), w_1(\tau_0) = w_{10} \geq 0 \;\; \textit{for}\; t \geq \tau_0 \;\textit{in}\; J. \tag{4.4}$$

Then, if $u(t)$ *and* $v(t)$ *are any solutions of equation (4.1) and equation (4.2) through* (t_0,u_0) *and* (τ_0,v_0)*; respectively on* J*, we have* $d[v(t,\tau_0,v_0) - u(t-\eta,t_0,u_0), 0] \leq r(t,\tau_0,w_{10})$, $t \in J$ *provided that* $d[v_0 - u_0, 0] \leq w_{10}$.

Proof of Theorem 4.1: For small enough $h > 0$ *the H-difference of* $u(t+h) - u(t), v(t+h) - v(t)$ *exist. Define* $m(t) = d[v(t) - u(t), 0]$, *and we have for* $t \in J$, $m(t+h) - m(t) = d[v(t+h) - u(t+h), 0] - d[v(t) - u(t), 0]$. *By using the triangular inequality for d, we obtain the inequality* $d[v(t+h) - u(t+h), 0] \leq d\big[v(t+h) - u(t+h), v - u + h\big(F(t,v) - f(t,u)\big)\big] + d\Big[v(t) - u(t) + h\Big(F\big(t,v(t)\big) - f\big(t,u(t)\big)\Big), 0\Big]$. *We obtain the inequality* $d\Big[v(t) - u(t) + h\Big(F\big(t,v(t)\big) - f\big(t,u(t)\big)\Big),\ 0\Big] \leq d\big[v - u + h\big(F(t,v) - f(t,u)\big),\ v - u + h\big(F(t,0) - f(t,0)\big)\big] + d\big[v(t) - u(t) + h\big(F(t,0) - f(t,0)\big),\ h\big(F(t,0) - f(t,0)\big)\big] = d[h(F(t,v(t)) - f(t,u(t))),\ h(F(t,0) - f(t,0))]$.

Hence, it follows that $((m(t+h)-m(t))/h) \leq (1/h)d[v(t+h)-u(t+h), v - u + h(F(t,v)-f(t,u))] + (1/h)d[h(F(t,0)-f(t,0)),0] + (1/h)d[h(F(t,v(t))-f(t,u(t))), h(F(t,0)-f(t,0))]$ *since the properties of d and the fact that u(t) and v(t) are the solutions of equation (4.1) and equation (4.3) respectively, we get the Dini derivative* $D^{+}m(t) = \lim_{h\to 0^{+}} sup(1/h)[m(t+h) - m(t)] \leq \lim_{h\to 0^{+}} sup\, d[(1/h)[v(t+h)-u(t+h)-(v-u),\ F(t,v)-f(t,u)] + d[(F(t,0)-f(t,0)),\ 0)] + d[(F(t,v(t))-f(t,u(t))),\ (F(t,0)-f(t,0))]$.

Here, we have used the fact that $d[v(t+h)-u(t+h),\ v-u+h(F(t,v)-f(t,u))] = d[v-u+w(t),\ v-u+h(F(t,v)-f(t,u))] = d[w(t)+v-u, v-u+h(F(t,v)-f(t,u))] = d[w(t),\ h(F(t,v)-f(t,u))] = d[v(t+h)-u(t+h)-(v-u),\ h(F(t,v)-f(t,u))]$ *where w(t) is the* H *−difference of* $v(t+h)$ *and* $u(t+h)$ *and* $v(t)-u(t)$. *This implies that* $D^{+}m(t) \leq G(t, d[v(t)-u(t),0]) = G(t,m(t)), t \in J$ *and by using the comparison result in* [Lakshmikantham and Leela, 1999].

Therefore, we have $m(t) = d[v(t,\tau_0,v_0) - u(t-\eta,t_0,u_0),0] \leq r(t,\tau_0,w_{10})$, *where* $t \in J$ *provided that* $d[v_0 - u_0, 0] \leq w_{10}$.

Theorem 4.2: Assume that $f, F \in C[J \times S_\rho, E^n]$ *and*

i) $\lim_{h\to 0^{+}} sup(1/h)[d[v-u+h(F(t,v)-f(t,u)),0] - d[v-u,0]] \leq G(t, d[v-u,0])$ *for* $t \in J$ *and* $u, v \in E^n$ *where* $G \in C[J \times R_+, R]$*;*

ii) $r(t) = r(t,\tau_0,w_{10})$ *is the maximal solution of the Fuzzy scalar differential equation exists on J,* $w_1' = G(t,w_1), w_1(\tau_0) = w_{10} \geq 0$ *for* $t \geq \tau_0$ *in* J.

Then, if $u(t)$ *and* $v(t)$ *are any solutions of (4.1) and (4.2) through* (t_0,u_0) *and* (τ_0,v_0) *respectively on* J, *we have* $d[v(t,\tau_0,v_0) - u(t-\eta,t_0,u_0),0] \leq r(t,\tau_0,w_{10})$, $t \in J$ *provided that* $d[v_0-u_0,0] \leq w_{10}$.

Proof of Theorem 4.2: As we proceed in the proof of Theorem 5.1, we get $m(t+h) - m(t) = d[v(t+h)-u(t+h),0] - d[v-u,0] \leq d\big[v(t+h)-u(t+h), v-u+h\big(F(t,v)-f(t,u)\big)\big] + d\big[h\big(F(t,v)-f(t,u)\big), h\big(F(t,0)-f(t,0)\big)\big] - d[v-u,0]$

and $D^{+}m(t) = \lim_{h\to 0^{+}} sup\left(\frac{1}{h}\right)[m(t+h)-m(t)] \leq lim \lim_{h\to 0^{+}} \sup(1/h)d[v-u+h(F(t,v)-f(t,u),0]-d[v-u,0]+\lim_{h\to 0^{+}} sup\left(\frac{1}{h}\right) d\left[\left(\frac{v(t+h)-u(t+h)-(v-u)}{h}\right),\right.$

$\left. F(t,v)\ -\ f(t,u)\right] \leq G(t, d[v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0)\ ),0]) = G(t,m(t)), t\in I.$

Similarly, we have $m(t) = d[v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0),0] \leq r(t,\tau_0,w_{10}),\ t\in J$ *provided that* $d[v_0-u_0,0]\leq w_{10}$. *This completes the proof.*

Corollary 4.1: Assume that $f,F\in C[J\times S_\rho, E^{\mathrm{n}}]$ *and either*

i) $d[F(t,v)-f(t,u),0]\leq G(t,d[v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0),v_0-u_0])$

or

ii) $\lim_{h\to 0^{+}} sup\left(\frac{1}{h}\right)\left[d\left[v-u+h\big(F(t,v)-f(t,u)\big),0\right]-d[v-u,0]\right]\leq G(t,d[v-u,0])$ *where* $G\in C[J\times R_+,R]$.

Then, if $d[v_0-u_0,0]\leq w_{10}$, *we have* $d[v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0),0]\leq r(t,\tau_0,w_{10})$ *for* $t\in J$ *where* $r(t)=r(t,\tau_0,w_{10})$ *is the maximal solution of the Fuzzy scalar differential equation* $w_1{}'=G(t,w_1), w_1(\tau_0)=w_{10}\geq 0$ *for* $t\geq\tau_0$ *in* J.

4.6. Stability Criteria of the Fuzzy Differential Systems

Now, we prove the main result.

Theorem 4.3: Assume that if $f,F\in C[R_+\times S_\rho, E^{\mathrm{n}}]$, $f(t,0)\equiv 0$ *for* $t\geq 0$ *and for* $h>0, t\in R_+, u,v\in S(\rho)$,

i) $\lim_{h\to 0^{+}} sup(1/h)[d[v-u+h(F(t,v)-f(t,u)),0]-d[v-u,0]]\leq G(t,d[v-u,0]);$

ii) $G\in C[J\times R_+,R], G(t,0)\equiv 0.$

Then the stability properties of the null solution of the Fuzzy differential equation (4.4) with $G(t,0)=0$ *imply the corresponding stability properties of* $v(t,\tau_0,v_0)$ *any solution of the fuzzy system equation (4.3) with respect to* $u(t,\tau_0,u_0)=u(t-\eta,t_0,u_0)$ *where* $u(t,t_0,u_0)$ *is any solution of the Fuzzy system (4.1).*

Proof of Theorem 4.3: Let $v(t,\tau_0,v_0)$ *any solution of the Fuzzy system of equation (4.3) be stable with respect to* $u(t,\tau_0,u_0)=u(t-\eta,t_0,u_0)$ *where* $u(t,t_0,u_0)$ *is any solution of the Fuzzy system (4.1).*

Then, we need to show that the trivial solution of equation (4.4) is stable. Then, given any $\varepsilon>0$ *and* $\tau_0\in R_+$, *there exists a* $\delta=\delta(\tau_0,\varepsilon)>0$ *such that* $w_1(t,\tau_0,w_{10})<\varepsilon$ *provided that* $0\le w_{10}<\delta$ *for* $t\ge\tau_0$ *where* $w_1(t,\tau_0,w_{10})$ *is any solution of equation (4.4). We claim that with these* $\varepsilon,\delta,v(t,\tau_0,v_0)$ *any solution of the Fuzzy system of (4.3) is stable with respect to* $u(t,\tau_0,u_0)$ *for* $t\ge\tau_0$.

If this is false, there would exist solutions $u(t,\tau_0,u_0)$ *and* $v(t,\tau_0,v_0)$ *for* $t\ge\tau_0$ *of the fuzzy systems (4.1) and (4.3) with given any* $\varepsilon>0$ *there exist* $\delta_1=\delta_1(\varepsilon,\tau_0)>0,\delta_2=\delta_2(\varepsilon,\tau_0)>0$ *and* $t_1>\tau_0$ *such that* $d[v(t_1,\tau_0,v_0)-u(t_1-\eta,t_0,u_0),0]=\varepsilon$ *and* $d[v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0),0]\le\varepsilon<\rho$ *for* $\tau_0\le t\le t_1$ *whenever* $d[v_0-u_0,0]<\delta_1$ *and* $d_s[\tau_0-t_0,0]<\delta_2$ *where* $\delta=min\{\delta_1,\delta_2\}$ *for* $t\ge\tau_0$. *By using condition of stability for* $\tau_0\le t\le t_1$ *and Corollary 4.1, we get* $d[v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0),0]\le r(t,\tau_0,d[w_{10},0])<\varepsilon\, for\ \tau_0\le t\le t_1$.

This completes the proof.

Lemma 4.1: First of all, we let $f,F\in C[J\times E^n,E^n]$ *and we define* $G(t,r)=\max\limits_{d[v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0),v_0-u_0]\le r} d[F(t,v)-f(t,u),0]$ *where* $G(t,r)\in C[J\times R_+,R_+]$. *Assume that* $r^*(t,\tau_0,d[v_0-u_0,0])$ *is the maximal solution of* $w_1{}'=G(t,w_1),w_1(\tau_0)=d[v_0-u_0,0]$ *through* (τ_0,v_0-u_0) $u(t,\tau_0,u_0)=u(t-\eta,t_0,u_0)$ *where* $u(t,t_0,u_0)$ *is any solution of the Fuzzy system (4.1) for* $t\ge\tau_0\ge0,t_0\in R_+$ *and* $\eta=\tau_0-t_0$ *and* $v(t,\tau_0,v_0)$ *is the solution of equation (4.3). Then, for* $t\ge\tau_0$ *in* J, $d[v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0)]\le r^*(t,\tau_0,d[v_0-u_0,0])$.

Proof of Lemma 4.1: Let $m(t)=d[v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0),v_0-u_0]$ *for* $t\ge\tau_0$ *in* J. *Then by using corollary 4.1, we have* $D^+m(t)=D^+d[v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0),v_0-u_0]\qquad\le d\big[D^+[v(t,\tau_0,v_0)-x(t-\eta,t_0,u_0),\ v_0-u_0]\big]=$

$d[v'(t,\tau_0,v_0)-u'(t-\eta,t_0,u_0),\ 0]=d\big[F\big(t,v(t,\tau_0,v_0)\big)\ -\ f\big(t,u(t,\tau_0,u_0)\big),0\big]$,

$\max_{d[v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0),v_0-u_0]\leq m(t)} d[F(t,v)-f(t,u),0]=G(t,m(t))$.

Therefore the inequality $D^+m(t)\leq G(t,m(t)), m(\tau_0)=d[v_0-u_0,0]$, *we obtain and the theory of differential inequalities* [Lakshmikantham and Mohapatra, 2003] *implies that* $m(t)=d[v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0),v_0-u_0]\leq r^*(t,\tau_0,d[v_0-u_0,0])$ *for* $t\in J$.

This completes the proof.

Lemma 4.2: Let $V(t,u)\in C[R_+\times S_\rho,R_+]$ *and* $V(t,u)$ *be locally Lipschitzian in* u. *Assume that the function* ${}^*D^+V(t,v-u)=\lim_{h\to 0^+} sup(1/h)[V(t+h,v-u+h(F(t,v)-f(t,u)))-V(t,v-u)]$ *satisfies* ${}^*D^+V(t,v-u)\leq G(t,V(t,v-u))$ *with* $(t,u),(t,v)\in R_+\times S_\rho$, *where* $G(t,u)\in C[J\times R_+,R]$. *Let* $r(t)=r(t,\tau_0,w_{10})$ *be the maximal solution of the scalar differential equation* $w_1{}'=G(t,w_1), w_1(\tau_0)=w_{10}\geq 0$, *for* $t\geq t_0$. *If* $u(t)=u(t-\eta,t_0,u_0)$ *where* $u(t,t_0,u_0)$ *is any solution of the system (4.1) for* $t\geq\tau_0\geq 0,\ t_0\in R_+$ *and* $\eta=\tau_0-t_0$ *and* $v(t)=v(t,\tau_0,v_0)$ *is any solution of (4.3) for* $t\geq\tau_0$ *such that* $V(\tau_0,v_0-u_0)\leq w_{10}$ *then* $V(t,v(t)-u(t))\leq r(t)$ *for* $t\geq\tau_0$.

Proof of Lemma 4.2: Let $u(t)=u(t-\eta,t_0,x_0)$ *where* $u(t,t_0,u_0)$ *is any solution of the system (4.1) for* $t\geq\tau_0\geq 0,\ t_0\in R_+$ *and* $\eta=\tau_0-t_0$ *and* $v(t)=v(t,\tau_0,v_0)$ *is any solution of (4.3) for* $t\geq\tau_0$ *such that* $V(\tau_0,v_0-u_0)\leq w_{10}$ *holds. Define* $m(t)=V(t,v(t)-u(t))$ *for* $t\geq\tau_0$ *so that* $m(t_0)\leq w_{10}$. *For* $h>0$ *we get* $m(t+h)-m(t)=V(t+h,v(t+h)-u(t+h))-V(t,v(t)-u(t))$.

Since V *is locally Lipschitzian in* x $m(t+h)-m(t)\ \leq L(\varepsilon_1(h)-\varepsilon_2(h))+V(t+h,y(t)-x(t)+h(F(t,y)-f(t,x)))-V(t,y(t)-x(t))$ *where* $L>0$ *is the Lipschitzian constant and* ε_1 *and* ε_2 *are error terms. Dini derivatives and* $D^+m(t)\leq \lim_{h\to 0^+} sup(1/h)[L(\varepsilon_1(h)-\varepsilon_2(h))]+\lim_{h\to 0^+} sup(1/h)[V(t+h,v(t)\,u(t))+h(F(t,v)-f(t,u)))\ -V(t,v(t)-u(t))]$ *where* $((\varepsilon_1(h)-\varepsilon_2(h)))/h)\to 0$ *as* $h\to 0$. $D^+m(t)\leq\ {}^*D^+V(t,v(t)-u(t))\ \leq G(t,V(t,v(t)-u(t)))=G(t,V(t,m(t)))$

We have $D^+m(t)\leq G(t,V(t,m(t)))$ *and* $m(\tau_0)=V(\tau_0,v(\tau_0)-u(\tau_0))\leq w_{10}$. *By using Lemma, this implies* $m(t)=V(t,v(t)-u(t))\leq r(t,\tau_0,d[v_0-$

$u_0, 0]$) *where* r *is the maximal solution of* $w_1{}' = G(t, w_1), w_1(\tau_0) = w_{10} \geq 0$ *for* $t \geq \tau_0$.

Now, we present the main comparison result that yields knowledge of ITD stability properties if we know the stability properties of the null solution of the comparison system.

Theorem 4.4: Assume that there exist a function $V(t, x)$ *satisfies:*

i) $V(t, u) \in C[R_+ \times R^{\mathrm{n}}, R_+]$ *be* $|V(t, u) - V(t, v)| \leq Ld[u, v]$ *for all* $(t, u), (t, v) \in R_+ \times S_\rho$ *and* $L > 0$ *and positive definite and decrescent where* $D^+m(t) \leq \lim_{h\to 0^+} sup(1/h)[V(t + h, v - u) + h(F(t, v) - f(t, u))) - V(t, v - u)]$ *satisfies* ${}^*D^+V(t, v - u) \leq G(t, V(t, v - u))$ *for all* (t, u) *and* $(t, v) \in R_+ \times S_\rho$, *where* $G(t, w_1) \in C[J \times R_+, R]$ *and* $\varphi(d[u, u_0]) \leq V(t, u) \leq \psi(d[u, u_0]), (t, u) \in R_+ \times S_\rho$ *and* $\varphi, \psi \in K$, $K = [\varphi \in C[(0, \rho), R_+], \varphi(0) = 0$, *and* $\varphi(r)$ *is strictly monotone increasing in* r*].*

ii) $r(t) = r(t, \tau_0, w_{10})$ *be the maximal solution of the scalar differential equation* $w_1{}' = G(t, w_1), w_1(\tau_0) = w_{10} \geq 0$ *for* $t \geq \tau_0$.

Then the stability properties of the null solution of the system (4.1) with $G(t, 0) = 0$ *imply the corresponding stability properties of* $v(t, \tau_0, y_0)$ *any solution of (4.3) with respect to* $u(t, \tau_0, u_0) = u(t - \eta, t_0, u_0)$ *where* $u(t, t_0, u_0)$ *is any solution of the system (4.1).*

Proof of Theorem 4.4: Let $u(t) = u(t - \eta, t_0, u_0)$ *where* $u(t, t_0, u_0)$ *is any solution of the system (4.1) for* $t \geq t_0 \geq 0, t_0 \in R_+$ *and* $\eta = \tau_0 - t_0$ *and* $v(t) = v(t, \tau_0, v_0)$ *is any solution of (4.3) for* $t \geq t_0$ *such that* $V(\tau_0, v_0 - u_0) \leq w_{10}$ *holds. If we define* $m(t) = V(t, v(t) - u(t))$ *for* $t \geq \tau_0$ *so that* $m(\tau_0) \leq w_{10}$, *then Lemma 4.1 and Lemma 4.2 imply that* $D^+m(t) \leq G(t, V(t, m(t))), m(\tau_0) = V(\tau_0, v(\tau_0) - u(\tau_0)) \leq w_{10}$.

Thus $m(t) = V(t, v(t) - u(t)) \leq r(t, \tau_0, d[v_0 - u_0, 0])$ *where* r *is the maximal solution of the comparison system (4.4). Let the null solution of the comparison system (4.4) be stable. Given any* $\varepsilon > 0$, *since* V *is positive definite and* $\varphi \in K$ *by* (i), *we have*

$\varphi(d[v-u,0]) \leq V(t,v-u)$ *for* $(t,u),(t,v) \in R_{+} \times R^{n}$ *and* $V(t,v(t)-u(t)) \leq r(t,\tau_0,d[v_0-u_0,0]) < \varphi(\varepsilon)$ *for* $t \geq \tau_0$ *provided that* $d[v_0-u_0,0] < \delta(\varepsilon,\tau_0)$.

Thus $V(t,v(t)-u(t)) \leq r(t,\tau_0,d[v_0-u_0,0] < \varphi(\varepsilon)$. *Since* φ^{-1} *exists, we have* $d[v(t)-u(t),v_0-u_0] < \varepsilon$ *provided that* $d[v_0-u_0,0] < \delta_1(\varepsilon,\tau_0)$ *and* $d_s[\tau_0-t_0,0] < \delta_2(\varepsilon,\tau_0)$ *for* $t \geq \tau_0$.

So, $v(t,\tau_0,v_0)$ *the solution of (4.3) is stable with respect to* $u(t) = u(t-\eta,t_0,u_0)$. *If the null solution of the comparison system is asymptotically stable, then* $\varphi(d[v(t)-u(t),v_0-u_0]) \leq r(t,\tau_0,\| v_0-u_0 \|)$ *holds for* $t \geq \tau_0$ *and* $d[v_0-u_0,0]$ *and* $d_s[\tau_0-t_0,0]$ *small enough. Which implies for* $T = T(\varepsilon,\tau_0) > 0$ *such that* $d[v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0),v_0-u_0] < \varepsilon$ *for* $t \geq \tau_0 + T$ *since* $\varphi \in K$.

Thus, $v(t,\tau_0,v_0)$ *the solution of equation (4.3) is asymptotically stable with respect to* $u(t) = u(t-\eta,t_0,u_0)$. *Since* V *is decrescent and* $\psi \in K$ *by (i) we have* $V(t,v-u) \leq \psi(d[v-u,v_0-u_0])$ *for* $(t,u),(t,v) \in R_{+} \times R^{n}$ *and the choice of* $\delta = \delta(\varepsilon)$ *is independent of* τ_0.

Thus, uniform stability and uniform asymptotic stability of the comparison system (4.4) imply the corresponding uniform stability and uniform asymptotic stability of the solution of (4.3) with respect to $u(t) = u(t-\eta,t_0,u_0)$.

5. STRICT STABILITY CRITERIA OF FUZZY DIFFERENTIAL SYSTEMS WITH INITIAL TIME DIFFERENCE

5.1. Fundamental Definitions of Strict Stability Criteria of Fuzzy Differential Systems

Consider the initial value problem for the fuzzy differential equations

$$u' = f(t,u), u(t_0) = u_0 \, for \;\; t \geq t_0 \geq 0 \tag{5.1}$$

where $f \in C\left[R^+ \times S_\rho, E^{\mathrm{n}}\right], S\rho = [u \in E^{\mathrm{n}} \colon d[u,0] < \rho < \infty], f(t,0) \equiv 0$ for $t \geq 0$.

Before we can establish our comparison theorem and Lyapunov stability criteria we need to introduce the following definitions.

Definition 5.1: The trivial solution of the system (5.1) is said to be strictly stable, if given $\varepsilon_1 > 0$ and $t_0 \in R_+$, there exist a $\delta_1 = \delta_1(\varepsilon_1, t_0) > 0$ such that $d[\, u_0, 0] < \delta_1$ implies $d[\, u(t),0] < \varepsilon_1, t \geq t_0$, and for every $0 < \delta_2 \leq \delta_1$, there exists an $0 < \varepsilon_2 < \delta_2$ such that $\delta_2 < d[\, u_0, 0]$ implies $\varepsilon_2 < d[\, u(t),0]$ for $t \geq t_0$. If δ_1, δ_2 and ε_2 in Definition 5.1 are independent of t_0, then the trivial solution of the system (5.1) is said to be strictly uniformly stable for $t \geq t_0$.

Definition 5.2: The trivial solution of the system (5.1) is said to be strictly attractive, if given $\alpha_1 > 0, \varepsilon_1 > 0$ and $t_0 \in R_+$, for every $\alpha_2 \leq \alpha_1$, there exists $\varepsilon_2 < \varepsilon_1$ and $T_1 = T_1(\varepsilon_1, t_0)$ and $T_2 = T_2(\varepsilon_2, t_0)$ such that $\alpha_2 < d[\, u_0, 0] < \alpha_1$ implies $\varepsilon_2 < d[\, u(t),0] < \varepsilon_1$ for $T_1 + t_0 \leq t \leq T_2 + t_0$.

If T_1, T_2 in Definition 5.2 are independent of t_0, then the trivial solution of the system (5.1) is said to be uniformly strictly attractive.

Definition 5.3: The trivial solution of the system (5.1) is said to be strictly asymptotically stable if Definition 5.1 holds and the trivial solution is stable. The *trivial solution of the system (5.1) is said to be strictly uniformly asymptotically stable if the system (5.1) is uniformly strictly attractive and the trivial solution is strictly uniformly stable.*

Definition 5.4: A function $\varphi(r)$ *is said to belong to the class* K *if* $\varphi \in C[(0,\rho), \mathbb{R}_+], \varphi(0) = 0$, *and* $\varphi(r)$ *is strictly monotone increasing in* r. *It is said to belong to class* K_∞ *if* $\rho = \infty$ *and* $\varphi(r) \to \infty$ *as* $r \to \infty$.

Definition 5.5: For any Lyapunov-like function $V(t,u) \in C[R_+ \times E^{\mathrm{n}}, R_+]$ *we define the fuzzy Dini derivative* $D^+V(t,u)$ *as follows* $D^+V(t,u) = \lim_{h\to 0^+} sup(1/h)[V(t+h, u+hf(t,u)) - V(t,u))], (t,u) \in R_+ \times E^{\mathrm{n}}$ *and* $D\_V(t,u) = \lim_{h\to 0^-} inf(1/h)[V(t+h, u+hf(t,u)) - V(t,u))], (t,u) \in R_+ \times E^{\mathrm{n}}$ *for* $(t,x) \in R_+ \times E^{\mathrm{n}}$.

5.2. Fundamental Theorems of Stability for Strict Stability Criteria of Fuzzy Differential Systems with Initial Time Difference

In this section, we have some definitions for fuzzy calculus and similarly fuzzy derivative definitions. We use the similar concepts with little differences for initial time difference concept. However, initial time difference concept is different of our previous definitions and explanations.

Before initial time difference concept explanations, we give an infomation about fuzzy logic, fuzzy calculus and fuzzy differential equations.

Consider the initial value problem for the fuzzy differential equations

$$u' = f(t,u), u(t_0) = u_0 \, for \; t \geq t_0 \geq 0 \tag{5.1}$$

$$u' = f(t,u), u(\tau_0) = v_0 \, for \; t \geq \tau_0 \geq t_0 \tag{5.2}$$

the perturbed system of (5.2)

$$v' = F(t,v), v(\tau_0) = v_0 \text{ for } t \geq \tau_0 \tag{5.3}$$

where the functions $f, F \in C[R_+ \times S\rho, E^n], S\rho = [u \in E^n: d[u, 0] < \rho < \infty]$, $f(t, 0) \equiv 0$ for $t \geq 0$. We have a special case of (5.3) that is perturbation equation of (5.2) if $F(t, v) = f(t, v) + R(t, v)$ where $R(t, v)$ is the perturbation term. The above assumptions imply the existence of trivial solutions of (5.1) and (5.2) through (t_0, u_0) and (τ_0, v_0) respectively.

If the perturbation term is equal to zero, then we have another unperturbed fuzzy differential equation. We can investigate stability properties of this equation similar way.

In this work, the differences between the classical notion of stability and ITD stability did not allow the use of the behavior of the null solution in our ITD stability analysis.

The main result presented in this section resolves those difficulties with a new approach that allows the use of the stability of the null solution of the comparison system to predict the stability properties of $v(t, \tau_0, v_0)$ the solution of (5.3) with respect to $u(t) = u(t - \eta, t_0, u_0)$ where $u(t, t_0, u_0)$ is any solution of the system (5.1).

Theorem 5.1:

i) $d[F(t, v) - f(t, u), 0] \leq G(t, d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), v_0 - u_0])$ *for* $t \in J$ *and* $u, v \in E^n$ *where* $G \in C[J \times R_+, R]$;

ii) $r(t) = r(t, \tau_0, w_{10})$ *is the maximal solution of the scalar differential equation exists on* J

$$w_1' = G(t, w_1), w_1(\tau_0) = w_{10} \geq 0 \; t \geq \tau_0 \; in \, J. \tag{5.4}$$

Then, if $u(t)$ *and* $v(t)$ *are any solutions of equation (5.1) and equation (5.2) through* (t_0, u_0) *and* (τ_0, v_0)*; respectively on* J, *we have* $d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), 0] \leq r(t, \tau_0, w_{10})$, $t \in J$ *provided that* $d[v_0 - u_0, 0] \leq w_{10}$.

Proof of Theorem 5.1: For small enough $h > 0$ *the H-difference of* $u(t+h) - u(t)$, $v(t+h) - v(t)$ *exist. Define* $m(t) = d[v(t) - u(t), 0]$, *and we have for* $t \in J$, $m(t+h) - m(t) = d[v(t+h) - u(t+h), 0] - d[v(t) - u(t), 0]$.

By using the triangular inequality for d, we obtain $d[v(t+h) - u(t+h), 0] \leq d[v(t+h) - u(t+h),\ v - u + h(F(t,v) - f(t,u))] + d[v(t) - u(t) + h(F(t,v(t)) - f(t,u(t))), 0]$ *and* $d[v(t) - u(t) + h(F(t,v(t)) - f(t,u(t))), 0] \leq d[v\ - u + h(F(t,v) - f(t,u)), v\ - u + h(F(t,0) - f(t,0))] + d[v(t) - u(t) + h(F(t,0) - f(t,0)),\ h(F(t,0) - f(t,0))] = d[h(F\big(t,v(t)\big) - f(t,u(t))),$ $h(F(t,0) - f(t,0))]$.

Hence, it follows that $\left(\frac{m(t+h)-m(t)}{h}\right) \leq \left(\frac{1}{h}\right) d\big[v(t+h) - u(t+h),\ v - u + h\big(F(t,v) - f(t,u)\big)\big] + \left(\frac{1}{h}\right) d\big[h\big(F(t,0) - f(t,0)\big),\ 0\big] + \left(\frac{1}{h}\right) d\left[h\left(F\big(t,v(t)\big) - f\big(t,u(t)\big)\right),\ h\big(F(t,0) - f(t,0)\big)\right]$ *since the properties of d and the fact that u(t) and v(t) are the solutions of equation (5.1) and equation (5.3) respectively, we get* $D^{+}m(t)$ $= \lim_{h\to 0^{+}} sup(1/h)[m(t+h) - m(t)] \leq \lim_{h\to 0^{+}} \sup d[(1/h)[v(t+h) - u(t+h) - (\ v\ - u),\ F(t,v) - f(t,u)] + d[(F(t,0) - f(t,0)), 0)\ ] + d[(F(t,v(t)) - f(t,u(t))), (F(t,0) - f(t,0))]$.

Here, we have used the fact that $d[v(t+h) - u(t+h),\ v\ - u + h(F(t,v\ ) - f(t,u))] = d[v\ - u + w(t),\ v\ - u + h(F(t,v) - f(t,u))] = d[w(t) + v\ - u, v - u + h(F(t,v) - f(t,u))] = d[w(t),\ h(F(t,v) - f(t,u))] = d[v(t+h) - u(t+h) - (v\ - u),\ h(F(t,v\ ) - f(t,u))]$ *where w(t) is the* H *−difference of* $v(t+h)$ *and* $u(t+h)$ *and* $v(t) - u(t)$.

This implies that $D^{+}m(t) \leq G(t, d[v(t) - u(t), 0]) = G(t, m(t)), t \in J$ *and by using the comparison result in [Lakshmikantham and Leela, 1999]. Therefore, we have* $m(t) = d[v(t,\tau_0,v_0) - u(t-\eta,t_0,u_0), 0] \leq r(t,\tau_0,w_{10}), t \in J$ *provided that* $d[v_0 - u_0, 0] \leq w_{10}$.

Corollary 5.1: Assume that $f, F \in C[J \times E^{n}, E^{n}]$ *and either*

i) $d[F(t,v) - f(t,u), 0] \leq G(t, d[v(t,\tau_0,v_0) - u(t-\eta,t_0,u_0), v_0 - u_0])$

or

ii) $\lim\limits_{h\to 0^+} \sup \left(\frac{1}{h}\right)\Big[d\big[v-u+h\big(F(t,v)-f(t,u)\big),0\big]-d[v-u,0]\Big] \leq G(t,d[v-u,0])$ *where* $G \in C[J \times R_+, R]$.

Then, if $d[v_0-u_0,0] \leq w_{10}$, *we have* $d[v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0),0] \leq r(t,\tau_0,w_{10})$ *for* $t \in J$ *where* $r(t)=r(t,\tau_0,w_{10})$ *is the maximal solution of the Fuzzy scalar differential equation* $w_1{}'=G(t,w_1), w_1(\tau_0)=w_{10} \geq 0$ *for* $t \geq \tau_0$ *in* J.

5.3. Theorems of Strict Stability of Fuzzy Differential Systems with Initial Time Difference

In this section, we have given some theorems of strict stability of fuzzy differential systems with Initial Time Difference and we have proved these theorems.

Theorem 5.2: Assume that the following hold

(A1) Let $V_\mu \in C[R_+ \times S_\rho, R_+]$, $|V_\mu(t,u)-V_\mu(t,v)| \leq Ld[u,v], L>0$ *and for* $(t,u) \in R_+ \times S_\rho$, *where* $S_\rho = [u \in E^n : d[u(t),0] < \rho]$, *for each* $\mu, 0<\mu<\rho$, *and* V_μ *is locally Lipschitzian in* z *and for* $(t,z) \in R_+ \times S_\rho$ *and* $d[z,0] \geq \mu$, *such that* $b_1(d[z,0]) \leq V_\mu(t,z) \leq a_1(d[z,0])$ *where* $a_1, b_1 \in K$ *and* $D^+V_\mu(t,v-u) = \lim\limits_{h\to 0^+} \sup(1/h)[V_\mu(t+h,v-u+h(F(t,v)-f(t,u)))-V_\mu(t,v-u))]=0$ *where* $(t,z) \in R_+ \times S_\rho$.

(A2) Let $V_\theta \in C[R_+ \times S_\rho, R_+]$, *for each* $\theta, 0<\theta<\rho, V_\theta \in C\big[R_+ \times S_\rho, R_+\big]$ *and* V_θ *is locally Lipschitzian in* z *and for* $(t,z) \in R_+ \times S_\rho$ *and* $d[z,0] \leq \theta$, $b_2(d[z,0]) \leq V_\theta,(t,z) \leq a_2(d[z,0]), a_2, b_2 \in K$, $D^+V_\theta(t,v-u) = \lim\limits_{h\to 0^+} \sup(1/h)[V_\theta(t+h,v-u+h(F(t,v)-f(t,u)))-V_\theta(t,v-u))] \geq 0, (t,z) \in R_+ \times S_\rho$ *and where the function* $z(t)=v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0)$ *for* $t \geq \tau_0, v(t,\tau_0,v_0)$ *is the solution of the system (5.3) through* (τ_0,v_0) *and* $u(t-\eta,t_0,u_0)$, *where* $u(t,t_0,u_0)$ *is any solution of the system (5.1) for* $t \geq \tau_0 \geq 0, t_0 \in R_+$ *and* $\eta=\tau_0-t_0$.

Then the solution $v(t, \tau_0, v_0)$ of the system (5.3) is the initial time difference strictly stable with respect to $u(t-\eta, t_0, u_0)$ of the system (5.1) for $t \geq \tau_0 \geq 0, t_0 \in R_+$ and $\eta = \tau_0 - t_0$.

Proof of Theorem 5.2: Let us assume that $0 < \varepsilon_1 < \rho$ and $\tau_0 \in R_+$. Let us choose that $\delta_1 = \delta_1(\varepsilon_1, \tau_0) > 0$ such that

$$a_1(\delta_1) < b_1(\varepsilon_1). \tag{5.5}$$

Since we have $b_1(\varepsilon_1) \leq a_1(\delta_1)$ in (A1).

Then we claim that $d[z, 0] < \varepsilon_1$ for $t \geq \tau_0$ whenever $d[v_0 - u_0, 0] < \delta_1$ and $d[\,\tau_0 - t_0, 0] < \delta_2$.

If (5.5) is not true, then there exist $t_1 > t_2 > \tau_0$ and the solution of (5.1) and by using dini derivatives with $d[v_0 - u_0, 0] < \delta_1, d[\,\tau_0 - t_0, 0] < \delta_2$ satisfying $d[v(t_1) - u(t_1),0] = \varepsilon_1, d[v(t_2) - u(t_2),0] = \delta_1$ and $\delta_1 \leq d[v(t) - u(t),0] \leq \varepsilon_1$ for $t \in [t_2, t_1]$ where $u(t) = u(t-\eta, t_0, u_0)$.

Let us set $\mu = \delta_1$, we can obtain that $b_1(\varepsilon_1) = b_1(d[v(t_1) - u(t_1),0]) \leq V_\mu(t_1, v(t_1) - u(t_1)) \leq V_\mu(t_2, v(t_2) - u(t_2)) \leq a_1(d[v(t_2) - u(t_2),0]) = a_1(\delta_1)$, $b_1(\varepsilon_1) \leq a_1(\delta_1)$ which contradicts with (5.5).

Hence, strict stability is valid.

Now let $0 < \delta_1^* < \delta_1, 0 < \delta_2^* < \delta_2$ and $\varepsilon_2 < \delta = min\{\delta_1^*, \delta_2^*\}$ such that $a_2(\varepsilon_2) < b_2(\delta)$. Since we have $a_2(\varepsilon_2) \geq b_2(\delta)$ in (A_2).

Then we can prove that

$$\varepsilon_2 < d[v(t, \tau_0, v_0) - u(t-\eta, t_0, u_0),0] < \varepsilon_1 \text{ for } t \geq \tau_0 \tag{5.6}$$

whenever $\delta_1^* < d[v_0 - u_0, 0] < \delta_1$ and $\delta_2^* < d[\,\tau_0 - t_0, 0] < \delta_2$.

In fact, if (5.6) is not true, then there would exist $t_1 > t_2 > \tau_0$ and the solution of (5.1) and by using (5.2) with $\delta_1^* < d[v_0 - u_0, 0] < \delta_1$, $\delta_2^* < d[\,\tau_0 - t_0, 0] < \delta_2$ satisfying

$$d[v(t_1) - u(t_1),0] = \varepsilon_2, d[v(t_2) - u(t_2),0] = \delta. \tag{5.7}$$

We have $d[v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0),0]\leq\delta$ *for* $t\in[t_2,t_1]$.

Let us set $\theta=\delta$ *and by using (A₂), we get* $a_2(\varepsilon_2)\ =\ a_2(d[v(t_1)-u(t_1),0])\geq V_\theta(t_1,v(t_1)-u(t_1))\geq V_\theta(t_2,v(t_2)-u(t_2))\geq b_2(d[v(t_2)-u(t_2),0])=b_2(\delta)$, $a_2(\varepsilon_2)\geq b_2(\delta)$ *which contradicts with (5.5). Thus (5.6) is valid.*

Then the solution $v(t,\tau_0,v_0)$ *of the system (5.3) through* (τ_0,v_0) *is initial time difference strictly stable with respect to the solution* $u(t-\eta,t_0,u_0)$ *for* $t\geq\tau_0$. *This completes the proof.*

If δ_1,δ_2 and ε_2 is independent of τ_0, then the solution $v(t,\tau_0,v_0)$ of the system (5.3) is initial time difference uniformly strict stable with respect to the solution $u(t-\eta,t_0,u_0)$ for $t\geq\tau_0$.

Theorem 5.3: Assume that

(A1) Let $V_\mu\in C[R_+\times S_\rho,R_+]$, $|V_\mu(t,u)-V_\mu(t,v)|\leq Ld[u,v], L>0$ *and for* $(t,u)\in R_+\times S_\rho$, *where* $S_\rho=[u\in E^n: d[u(t),0]<\ \rho]$, *for each* μ ,$0<\mu\ <\rho$, *and* V_μ *is locally Lipschitzian in z and for* $(t,z)\in R_+\times S_\rho$ *and* $d[z,0]\geq\mu$, *such that* $b_1(d[z,0])\leq V_\mu(t,z)\leq a_1(d[z,0]), a_1,b_1\in K$

$$D^+V_\mu(t,v-u)\leq -c_1(d[z,0]), c_1\in K; \tag{5.8}$$

(A2) Let $V_\theta\in C[R_+\times S_\rho,R_+]$, *for each* $\theta, 0<\theta<\rho, V_\theta\in C\big[R_+\times S_\rho,R_+\big]$ *and* V_θ *is locally Lipschitzian in z and for* $(t,z)\in R_+\times S_\rho$ *and* $d[z,0]\leq\theta$, $b_2(d[z,0])\leq V_\theta,(t,z)\leq a_2(d[z,0]), a_2,b_2\in K$

$$D^+V_\theta(t,v-u)\geq -c_2(\|z\|)\ c_2\in K; \tag{5.9}$$

where $z(t)=v(t,\tau_0,v_0)-u(t-\eta,t_0,u_0)$ *for* $t\geq\tau_0, v(t,\tau_0,v_0)$ *of the system (5.3) through* (τ_0,v_0) *and* $u(t-\eta,t_0,u_0)$, *where* $u(t,t_0,u_0)$ *is any solution of the system (5.1) for* $t\geq\tau_0\geq0, t_0\in R_+$ *and* $\eta=\tau_0-t_0$. *Then the solution* $v(t,\tau_0,v_0)$ *of the system (5.3) through* (τ_0,v_0) *is the initial time difference uniformly strictly asymptotically stable with respect to* $u(t-\eta,t_0,u_0)$ *of the solution of the system (5.1) for* $t\geq\tau_0\geq0, t_0\in R_+$ *and* $\eta=\tau_0-t_0$.

Proof of Theorem 5.3: We note that (5.8) implies (A1). However, (5.9) does not yield (A2). As a result of these, we obtain because of (5.8) only uniformly stability with initial time difference with respect to $u(t-\eta, t_0, u_0)$ *that is for given any* $\varepsilon_1 \leq \rho$ *and* $\tau_0 \in R_+$ *there exist* $\delta_{10} = \delta_{10}(\varepsilon_1) > 0$ *and* $\delta_{20} = \delta_{20}(\varepsilon_1) > 0$ *such that* $d[v(t, \tau_0, v_0) - u(t-\eta, t_0, u_0), 0] < \varepsilon_1$ *for* $t \geq \tau_0$ *whenever*

$$d[v_0 - u_0, 0] < \delta_{10} \text{ and } d[\,\tau_0 - t_0, 0] < \delta_{20}. \tag{5.10}$$

To prove the conclusion of Theorem 5.3 we need to show that the solution $v(t, \tau_0, v_0)$ *of the system (5.3) through* (τ_0, v_0) *for* $t \geq \tau_0$ *is strictly uniformly attractive with respect to* $u(t-\eta, t_0, u_0)$ *for this purpose and for* $t \geq \tau_0$, *let* $\varepsilon_1 = \rho$ *and set* $\delta_{10} = \delta_1(\rho)$ *and* $\delta_{20} = \delta_2(\rho)$ *so that (5.10) yields* $d[v(t, \tau_0, v_0) - u(t-\eta, t_0, u_0), 0] < \rho$ *for* $t \geq \tau_0$ *whenever* $d[v_0 - u_0, 0] < \delta_1$ *and* $d[\,\tau_0 - t_0, 0] < \delta_2$.

Let $d[v_0 - u_0, 0] < \delta_{10}$ *and* $d[\,\tau_0 - t_0, 0] < \delta_{20}$. *We show, using standard argument, that there exists a* $t^* \in [\tau_0, \tau_0 + T]$, *we choose* $T = T(\varepsilon, \tau_0) \geq a_1(max\{\delta_{10}, \delta_{20}\})/c_1(min\{\delta_1, \delta_2\})$ *where* δ_{10} *and* δ_{20} *are the numbers corresponding to* ε_1 *in (5.10) that is in stability of unperturbed systems with initial time difference with respect to* $x(t-\eta, t_0, x_0)$ *such that* $d[v(t^*, \tau_0, v_0) - u(t^* - \eta, t_0, u_0), 0] < \delta_1, t^* \geq \tau_0$ *for any solutions of the systems (5.1) and (5.3) with* $d[v_0 - u_0, 0] < \delta_{10}$ *and* $d[\,\tau_0 - t_0, 0] < \delta_{20}$.

If this is not true, we will have $d[v(t^*, \tau_0, v_0) - u(t^* - \eta, t_0, u_0), 0] \geq \delta_1$ *for* $t^* \in [\tau_0, \tau_0 + T]$.

Then, $\mu = \delta_1$ *and using (A₁) with equation, we have in view of the choice of* T, $0 < b_1(\delta_1) \leq b_1(d[v(\tau_0 + T) - u(\tau_0 + T), u_0), 0]) \leq V_\mu(\tau_0 + T, y(\tau_0 + T) - x(\tau_0 + T)) \leq V_\mu(\tau_0, y_0 - x_0) - \int_{\tau_0}^{\tau_0 + T} c_1(d\,[v(s) - u(s), 0])ds \leq a_1(max\{\delta_{10}, \delta_{20}\}) - c_1(min\{\delta_1, \delta_2\})T \leq 0.$

This contradiction implies that there exist a $t^* \in [\tau_0, \tau_0 + T]$ *satisfying* $d[v(t^*, \tau_0, v_0) - u(t^* - \eta, t_0, u_0), 0] < \delta_1$ *for* $t^* \geq \tau_0$.

Because of the uniform stability $v(t, \tau_0, v_0)$ *of (5.3) with initial time difference with respect to* $u(t-\eta, t_0, u_0)$ *related to the solution of (5.1), this yields that* $d[v(t, \tau_0, v_0) - u(t-\eta, t_0, u_0), 0] < \varepsilon_1$ *for* $t \geq \tau_0 + T \geq t^*$ *which implies that there exists a* $\tau_0 < T_1 < T$ *such that* $d[v(\tau_0 + T, \tau_0, v_0) - u(\tau_0 + T - \eta, t_0, u_0), 0] = \varepsilon_1$.

Now, for any $\delta_{12}, 0 < \delta_{12} < \delta_{10}$ and $0 < \delta_{12} < \delta_{20}$ we can choose ε_2 such that $b_2(\varepsilon_1) > a_2(\varepsilon_2)$ and $0 < \varepsilon_2 < \varepsilon_1 < \delta_{12}$.

Suppose that $\delta_{12} < d[v_0 - u_0, 0] < min\{\delta_{10}, \delta_{20}\}$ and $\delta_{12} < d[\tau_0 - t_0, 0] < min\{\delta_{10}, \delta_{20}\}$. Let us define $\tau = [(b_2(\varepsilon_1) - a_2(\varepsilon_2))/c_2(\varepsilon_1)]$, and $T_2 = T_1 + \tau$. $a_2(d[v(t,\tau_0,v_0) - u(t-\eta,t_0,u_0),0]) \geq V_\theta(t, v(t) - u(t)) \geq V_\theta(\tau_0 + T_1, v(\tau_0 + T_1) - u(\tau_0 + T_1)) - \int_{\tau_0+T_1}^{t} c_2(d[v(s,\tau_0,v_0) - u(s-\eta,t_0,u_0),0])ds \geq b_2(\varepsilon_1) - \int_{\tau_0+T_1}^{t} c_2(d[v(s,\tau_0,v_0) - u(s-\eta,t_0,u_0),0])ds \geq b_2(\varepsilon_1) - c_2(\varepsilon_1)[t - (\tau_0 + T_1)]$.

Since, $t - (\tau_0 + T_1) > \tau$ and ${a_2}^{-1}$ exists, it follows that $a_2(d[v(t,\tau_0,v_0) - u(t-\eta,t_0,u_0),0]) > b_2(\varepsilon_1) - c_2(\varepsilon_1)[(b_2(\varepsilon_1) - a_2(\varepsilon_2)/c_2(\varepsilon_1)] = a_2(\varepsilon_2)$. This yields that $d[v(t,\tau_0,v_0) - u(t-\eta,t_0,u_0),0] > \varepsilon_2$ for $t \in [\tau_0 + T_1, \tau_0 + T_2]$ and therefore, $\varepsilon_2 < d[v(t,\tau_0,v_0) - u(t-\eta,t_0,u_0),0] < \varepsilon_1$ for $t \in [\tau_0 + T_1, \tau_0 + T_2]$.

This completes the proof. Then the solution $v(t,\tau_0,v_0)$ of the system (5.3) through (τ_0, v_0) is initial time difference uniformly strictly asymptotically stable in fractional case with respect to the solution $u(t-\eta,t_0,u_0)$, where $u(t,t_0,u_0)$ is any solution of the system (5.1) for $t \geq \tau_0 \geq 0, t_0 \in R_+$ and $\eta = \tau_0 - t_0$.

5.4. A Comparison Result in Strict Stability of Fuzzy Differential Systems

In this section, we have a comparison result of strict stability of fuzzy differential systems with Initial Time Difference and we use the scalar differential equations. We have proved the comparison result.

Before we prove the general result in terms of the comparison principle. Let us consider the uncoupled comparison differential systems:

$$w_1' = g_1(t, w_1), w_1(\tau_0) = w_{10} \geq 0, w_2' = g_2(t, w_2), w_2(\tau_0) = w_{20} \geq 0 \qquad (5.11)$$

where $g_1, g_2 \in C[R_+ \times R_+, R]$.

The comparison system (5.11) is said to be strictly stable: If given any $\varepsilon_1 > 0$ and $t \geq \tau_0, \tau_0 \in R_+$, there exist a $\delta_1 > 0$ such that $w_{10} \leq \delta_1$ implies $w_1(t) < \varepsilon_1$ for $t \geq \tau_0$ and for every $\delta_2 < \delta_1$ there exists an $\varepsilon_2 > 0, 0 < \varepsilon_2 < \delta_2$ such that $w_{20} \geq \delta_2$ implies $w_2(t) > \varepsilon_2$ for $t \geq \tau_0$. Here, $w_1(t)$ and $w_2(t)$ are any solutions of (5.11); respectively.

The comparison system (5.11) is said to be strictly attractive; if given any $\alpha_1 > 0, \gamma_1 > 0, \varepsilon_1 > 0$ and $\tau_0 \in R_+$, for every $\alpha_2 < \alpha_1$, there exist $\varepsilon_2 < \varepsilon_1, T_1 = T_1(\varepsilon_1, \tau_0) > 0$ and $T_2 = T_2(\varepsilon_1, \tau_0) > 0$ such that $w_1(t, \tau_0, w_{10}) < \varepsilon_1$ for $T_1 + \tau_0 \leq t \leq T_2 + \tau_0$ whenever $w_{10} \leq \alpha_1$ and $w_2(t, \tau_0, w_{20}) > \varepsilon_2 \, for \, T_2 + \tau_0 \geq t \geq T_1 + \tau_0$ whenever $w_{20} \geq \alpha_2$.

If T_1 and T_2 are independent of τ_0, then the comparison system (5.11) is initial time difference uniformly strictly attractive for $t \geq \tau_0$. Following main result based on this definition that result is formulated in terms of comparison principle.

In this section, we have a usefull comparison theorem in practical stability of fuzzy differential systems via scalar differential equation and proof of this theorem.

Theorem 5.4: Assume that

(A1) Let $V_\mu \in C[R_+ \times S_\rho, R_+], |V_\mu(t,u) - V_\mu(t,v)| \leq Ld[u,v], L > 0$ *and for* $(t,u) \in R_+ \times S_\rho$, *where* $S_\rho = [u \in E^{\mathrm{n}}: d[u(t),0] < \rho]$, *for each* μ ,$0 < \mu < \rho$, *and* V_μ *is locally Lipschitzian in z and for* $(t,z) \in R_+ \times S_\rho$ *and* $d[z,0] \geq \mu$, *such that* $b_1(d[z,0]) \leq V_\mu(t,z) \leq a_1(d[z,0]), a_1, b_1 \in K$

$$D^+ V_\mu(t, v-u) \leq g_1(t, V_\mu(t,z)); \tag{5.12}$$

(A2) Let $V_\theta \in C[R_+ \times S_\rho, R_+]$, *for each* $\theta, 0 < \theta < \rho, V_\theta \in C[R_+ \times S_\rho, R_+]$ *and* V_θ *is locally Lipschitzian in z and for* $(t,z) \in R_+ \times S_\rho$ *and* $d[z,0] \leq \theta$, $b_2(d[z,0]) \leq V_\theta, (t,z) \leq a_2(d[z,0]), a_2, b_2 \in K$

$$D^+ V_\theta(t, v-u) \geq g_2(t, V_\theta(t,z)); \tag{5.13}$$

where $g_2(t,w) \leq g_1(t,w), g_1, g_2 \in C[{R_+}^2, R], g_1(t,0) = g_2(t,0) = 0$ *and* $z(t) = v(t,\tau_0,v_0) - u(t-\eta, t_0, u_0)$ *for* $t \geq \tau_0, v(t,\tau_0,v_0)$ *of the system (5.3) through* (τ_0, v_0) *and* $u(t-\eta, t_0, u_0)$, *where* $u(t,t_0,u_0)$ *is any solution of the system (5.1) for* $t \geq \tau_0 \geq 0, t_0 \in R_+$ *and* $\eta = \tau_0 - t_0$.

Then any strict stability concept of the comparison system implies the corresponding strict stability concept of the solution $v(t,\tau_0,v_0)$ *of the system (5.3) through* (τ_0, v_0) *with respect to the solution* $u(t-\eta, t_0, u_0)$ *of the system (5.3) with* *initial time difference where* $u(t, t_0, u_0)$ *is any solution of the system (5.1) for* $t \geq \tau_0 \geq 0, t_0 \in R_+$.

Proof of Theorem 5.4: We will only prove the case of strict uniformly asymptotically stability. Suppose that the comparison differential systems in (5.11) is strictly uniformly asymptotically stable, then for any given $\varepsilon_1, 0 < \varepsilon_1 < \delta$, *there exist a* $\delta^* > 0$ *such that* $w_{10} \leq \delta^*$ *implies that* $w_1(t, \tau_0, w_{10}) < b_1(\varepsilon_1)$ *for* $t \geq \tau_0$

For this ε_1 *we choose* δ_1 *and* δ_{11}, *such that* $a_1({\delta_1}^*) \leq \delta^*$ *and* ${\delta_1}^* < \varepsilon_1$ *where* ${\delta_1}^* = max\{\delta_1, \delta_{11}\}$, *then we claim that* $d[v_0 - u_0, 0] < \delta_1, d[\tau_0 - t_0, 0] < \delta_{11}$ *imply that*

$$d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), 0] < \varepsilon_1 \text{ for } t \geq \tau_0. \tag{5.14}$$

If it is not true, then there exist t_1 *and* $t_2, t_2 > t_1 > \tau_0$ *and a solution* $z(t)$ *of* $z' = f(t, z), z(\tau_0) = v_0 - u_0$ *for* $t \geq \tau_0$ *with* $d[\tau_0 - t_0, 0] < \delta_{11}$ *and* $d[v_0 - u_0, 0] < \delta_1$, $d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), 0] < {\delta_1}^*, d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), 0] = \varepsilon_1$ *and* ${\delta_1}^* \leq d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), 0] < \varepsilon_1$ *for* $[t_1, t_2]$.

Choosing $\mu = {\delta_1}^*$ *and using the theory of differential inequalities we get* $b_1(\varepsilon_1) = b_1(d[v(t_2, \tau_0, v_0) - u(t_2 - \eta, t_0, u_0), 0]) \leq V_\mu(t_2, v(t_2, \tau_0, y_0) - u(t_2 - \eta, t_0, x_0)) \leq r(t_2, t_1, V_\mu(t_1, v(t_1, \tau_0, v_0) - u(t_1 - \eta, t_0, u_0))) \leq r(t_2, t_1, a_1({\delta_1}^*)) \leq r(t_2, t_1, \delta^*) < b_1(\varepsilon_1)$. *We have* $b_1(\varepsilon_1) < b_1(\varepsilon_1)$ *which is a contradiction. Here* $r(t, \tau_0, w_{10})$ *is the maximal solution of (5.11). Hence, (5.14) is true and we have uniformly stability with initial time difference.*

Now, we shall prove strictly uniformly attractive with initial time difference. For any given $\delta_2, \varepsilon_2 > 0, \delta_2 < \delta^*$, *we choose* δ_2 *and* ε_2 *such that* $a_1(\delta_2) < \delta_2$ *and* $b_1(\varepsilon_2) \geq \varepsilon_2$. *For these* δ_2 *and* ε_2, *since (5.11) is strictly uniformly attractive, for any* $\delta_3 < \delta_2$ *there exist* ε_3 *and* T_1 *and* T_2 *(we assume* $T_2 < T_1$*) such that* $\delta_3 < w_{10} = w_{20} < \delta_2$ *implies* $r(t, \tau_0, w_{10}) \leq r(t, \tau_0, \delta_2) < \varepsilon_2$, $\rho(t, \tau_0, w_{20}) \geq \rho(t, \tau_0, \delta_3) > \varepsilon_2$, *where* $r(t, \tau_0, w_{10})$ *and* $\rho(t, \tau_0, w_{20})$ *is the maximal solution and minimal solution of (5.11) (i) and (5.11) (ii); respectively.*

Now, for any δ_3, *let* $b_2(\delta_3) \geq \delta_3$. *We choose* ε_3 *such that* $a_2(\varepsilon_3) < \varepsilon_3$. *Then by using comparison principle in equations (5.11) and from (A1), we have* $b_1(d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), 0]) \leq V_\mu(t, v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0)) \leq$

$r(t, \tau_0, V_\mu(\tau_0, v_0 - u_0)) \leq r(t, \tau_0, a_1(d[v_0 - u_0, 0])) \leq r(t, \tau_0, \delta_2) < \varepsilon_2 \leq b_1(\varepsilon_2)$ *and* $b_1(d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), 0]) < b_1(\varepsilon_2)$ *since* ${b_1}^{-1}$ *exists which implies that* $d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), 0] < \varepsilon_2$ *for* $t \in [\tau_0 + T_2, \tau_0 + T_1]$.

Similarly, by using comparison principle in equations of (5.11) and from (A2) we get these inequalities $a_2(d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), 0]) \geq V_\theta(t, v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0)) \geq \rho(t, \tau_0, V_\theta(\tau_0, v_0 - u_0)) \geq \rho(t, \tau_0, b_2(\delta_3)) \geq \rho(t, \tau_0, \delta_3) > \varepsilon_3 \geq a_2(\varepsilon_3)$ *and* $a_2(d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), 0]) > a_2(\varepsilon_3)$ *since* ${a_2}^{-1}$ *exists which implies that for* $d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), 0] > \varepsilon_3$ *for* $t \in [\tau_0 + T_2, \tau_0 + T_1]$.

Hence, the solution $v(t, \tau_0, v_0)$ *of the system (5.3) through* (τ_0, v_0) *is strictly uniformly attractive with respect to the solution* $u(t - \eta, t_0, u_0)$ *is any solution of the system (5.1) for* $t \geq \tau_0 \geq 0, t_0 \in R_+$. *The proof is complete.*

6. CONCLUSION

In this thesis, we develope a new initial time difference comparison principle for nonlinear fuzzy differential equations, then we prove several stability criteria. It may provide a greater prospect of solving problems which exhibit initial time difference stability.

A new concept of stability, practical stability and comparison theorems of these stability types of fuzzy differential equations have been investigated in this work. Lyapunov stabilities of fuzzy differential equations have been obtained by using Lyapunov-like functions.

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BIOGRAPHY

Mustafa Bayram GÜCEN, was born in 05.08.1987 in BURSA. He completed primary and secondary education in BURSA. He graduated from Gebze Instıtute of Technology, Department of Mathematics, in 2009 and he has his Master Degree at Gebze Instıtute of Technology, Department of Mathematics, in 2011. He has been working as a research asistant at Department of Mathematics in the Yildiz Technical University.

APPENDICES

Appendix A: Publications Based on the Thesis

Yakar C., Çiçek M., Gücen M. B., (2012), “Practical Stability, Boundedness Criteria and Lagrange Stability of Fuzzy Differential Systems”, Journal of Computers and Mathematics with Applications, 64 (6), 2118-2127.

Yakar C., Gücen M. B., (2018), “Strict Stability of Fuzzy Differential Equations by Lyapunov Functions”, World Academy of Science, Engineering and Technology, International Journal of Computer and Information Engineering, 12 (5), 315-319.

Appendix B: Conference Presentations Based on the Thesis

Yakar C., Gücen M. B., (2018), “Strict Stability of Fuzzy Differential Equations by Lyapunov Functions”, ICFSNC 2018: International Conference on Fuzzy Systems and Neural Computing, 2327-2331, Berlin, Germany, 21-22 May.