DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

SOME EIGENVALUES AND TRACE INEQUALITIES FOR MATRICES

by
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İZMİR

SOME EIGENVALUES AND TRACE INEQUALITIES FOR MATRICES

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M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"SOME EIGENVALUES AND TRACE INEQUALITIES FOR MATRICES"** completed by **DİLEK VAROL** under supervision of **ASSIST. PROF. DR. MUSTAFA ÖZEL** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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SOME EIGENVALUES AND TRACE INEQUALITIES FOR MATRICES

ABSTRACT

In matrix theory, the more information about the spectrum gives the power of understanding matrices better. The spectrum can be identified by the knowledge of the spectral radius and the minimum eigenvalue of a matrix. In this thesis, the studies about the lower and upper bounds for the spectral radius and the lower bounds for the minimum eigenvalue of a matrix are investigated. In these studies, the Hadamard, the Kronecker and the Fan product of matrices are widely used to establish new types of matrices. Then several existing results are improved for these products and their algebraic characterizations. Furthermore, the lower bounds for the minimum eigenvalue of the Hadamard product of an *M*-matrix and its inverse are examined and some new lower bounds are computed.

Keywords: Spectral radius, minimum eigenvalue, inverse of a matrix, Hadamard product, *M*-matrix.

MATRİSLER İÇİN ÖZDEĞER VE İZ EŞİTSİZLİKLERİ

ÖZ

Matris Teori'de spectrum hakkında sahip olunulan her bilgi bize matrisleri daha iyi anlama gücü verir. Spektrumu, spectral yarıçap ve en küçük özdeğer hakkında edinilen bilgilerle tanımlayabiliriz. Bu tezde, bir matrisin spectral yarıçapı için alt ve üst sınırlar belirleme ile en küçük yarıçapı için alt sınır belirleme konularındaki çalışmalar araştırılmıştır. Bu çalışmalarda Hadamard, Kronecker ve Fan çarpımlarının yeni tip matrisler oluşturmak için sıkça kullanıldığı görülmüştür. Ayrıca, mevcut birçok sonuçlar da bu çarpımlar ve onların cebirsel özellikleri için geliştirilmiştir. Bunlara ek olarak, bir M-matris ve tersinin Hadamard çarpımlarının en küçük özdeğeri için alt sınırlar incelenmiş ve yeni alt sınırlar bulunmuştur.

Anahtar sözcükler: Spektral yarıçap, en küçük özdeğer, matris tersi, Hadamard Çarpım,M-matris.

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CHAPTER ONE

INTRODUCTION

Nowadays, matrices are frequently used not only in mathematics but also in the most of areas of sciences since they are quite helpful to understand, to classify and to solve the problems in these areas. However, in each area people can need different types of matrices and different properties of these matrices. Therefore, the main goal in Matrix Theory is to classify matrices, to determine the elements of matrices that will be helpful and to be able to generate new types of matrices in the sense of necessity. The widest information about matrices can be found in the books which are written by Horn and Johnson in 1985 and 1991. There is a great deal of types of matrices and there are plenty of significant elements of these matrices. In this thesis, we especially focus on the eigenvalues, the norms and the trace of a matrix. Moreover, we concentrate on the positive definite, positive semidefinite, irreducible matrices and M-matrices. Furthermore, we investigate the matrices obtained by the special products of matrices which are Hadamard, Fan, Kronecker, Khatri-Rao and Tracy-Singh products.

There are studies for bounding the trace (which is the sum of the diagonal entries of a matrix) of product of two matrices. Yang (1988) has proved a bound for product of two positive definite matrices and Neudecker (1990), Coope (1993) and Yang (2000) have stated and proved bounds for the product of two positive semidefinite matrices. Moreover, Horn and Johnson (1991) proposed an upper bound for the spectral radius of the Hadamard product of two nonnegative matrices. Almost after one decade, Audenaert (2010) has stated a theorem giving a better upper bound for the spectral radius of the Hadamard product of two non-negative matrices. For proof of this theorem, Audenaert has proposed a new definition of the spectal radius by use of the trace of a matrix. Contemporaneously, Horn and Zhang (2010) have also demonstrated a new inequality involving the spectral radius of the Hadamard product of two nonnegative matrices. Furthermore, not only he has given the relation between the spectral radius of the Hadamard product and the usual product of a positive definite matrix and a positive semidefinite matrix, but also he has established nice results for the principal submatrices of nonnegative and positive semidefinite matrices. Furthermore, Du (2010) has come up with inequalities involving the Hadamard product and the norms of matrices such as the spectral norm, the trace norm and the Frobenius norm. These studies have inspired us to have results on both the spectral radius and the norms of the Fan product of two nonnegative matrices.

In recent years, one of the most popular topics in Matrix Theory is to have a lower bound for the minimum eigenvalue of the Hadamard product of an *M*-matrix and its inverse. The first opinion for this topic has come from Fiedler and Markham (1988) only by use of the dimension of the matrix. In the process of time, there have been significant improvements for this lower bound. Most recently, Li et. al. (2007) and Li et. al. (2009) have developed new lower bounds for the minimum eigenvalue of the Hadamard product of an *M*-matrix and its inverse including the condition for the matrix to have a doubly stochastic inverse. Furthermore, Li et. al. (2011) have had an important result in finding the lower bound by letting the *M*-matrix with a doubly stochastic inverse to be also irreducible. In this thesis, we have achieved to establish and to prove the results for the new lower bounds for the minimum e eigenvalue of the Hadamard product of an *M*-matrix and its inverse.

Now, we will look at brief information about how this thesis has been organized:

In Chapter 2, the definitions of the basic elements of a matrix which will be commonly used in each chapter of this thesis such as the spectral radius, the minimum eigenvalue, the singular values and the norms will be given. In addition, the special products such as Hadamard, Fan, Kronecker, Khatri-Rao and Tracy-Singh products will be introduced to be able to understand the lemmas, theorems and corollaries in this thesis more efficiently. The second chapter will come to the end with the lemmas and theorems in inequalities involving the trace of a matrix.

In Chapter 3, our focus mostly will be on the well-known products of matrices such as the Hadamard product, the Fan product and the Kronecker product which are briefly introduced in Chapter 2. Thereafter, we will analyze the inequalities for the spectral radius and the norm of the Hadamard product of two matrices. After examining these inequalities, we will try to construct bounds for the spectral radius and the norm of the Fan product of two matrices in consideration of the results for the Hadamard product. Furthermore, we will present the definitions of Khatri-Rao and Tracy-Singh products and we will try to figure out the relations between these products and the Kronecker product.

In Chapter 4, we will be specifically focused on the inequalities involving *M*-matrices. In the former section, we will define *M*-matrices and we will mention the basic properties of *M*-matrices. Furthermore, we will mention the inequalities involving Jacobi iterative matrix and we will try to construct new bounds for the spectral radius of the Jacobi iterative matrix of a nonsingular *M*-matrix. In the last section, we will more particularly pay attention to the lower bounds for the minimum eigenvalue of the Hadamard product of an *M*-matrix and its inverse. For the beginning, we will present the results which have been studied so far, after that we will put emphasis on the new lower bounds that we tried to build for the minimum eigenvalue of the Hadamard product of an *M*-matrix and its inverse.

CHAPTER TWO

PRELIMINARIES

This chapter provides fundamental facts for us in matrix theory to introduce our definitions and main theorems. For the proofs, we refer to the books and articles being in references.

Let $M_{m,n}$ denote an $m \times n$ matrix and let M_n denote an $n \times n$ matrix. Also, let an $m \times n$ complex matrix be denoted by $\mathbb{C}^{m\times n}$ and let an $n \times n$ complex matrix be denoted by $\mathbb{C}^{n\times n}$. Furthermore, let an $m \times n$ real matrix be denoted by $\mathbb{R}^{mxn}$ and let an $n \times n$ complex matrix be denoted by $\mathbb{R}^{nxn}$.

2.1 Definitions for Spectrum and Types of Matrices

Since the thesis is based on inequalities involving eigenvalues of a matrix, it will be very useful to give basic definitions which are spectral radius and minimum eigenvalue and singular values of a matrix:

Definition 2.1.1 (Horn & Johnson, 1985) The *spectral radius* $\rho(A)$ of a matrix $A \in M_n$ is

$$\rho(A) = max\,\{\,|\lambda|: \lambda \; is\ an\ eigenvalue\ of\ A\,\}.$$

Definition 2.1.2 (Li, Liu, Yang & Li, 2011) For $A = [a_{ij}] \in \mathbb{R}^{nxn}$, the minimum eigenvalue of the matrix A is defined by

$$\tau(A) = \min\{|\lambda|: \lambda \in \sigma(A)\}$$

where $\sigma(A)$ denotes the spectrum of A.

Definition 2.1.3 (Meyer, 2004) The nonzero singular values $s_i(A)$ of an $m \times n$ matrix A are the positive square roots of the nonzero eigenvalues of A^*A and AA^*, where A^* is the conjugate transpose of the matrix A .

Although we are interested in all matrices, in most this thesis we will be dealing with positive, non-negative, positive definite and positive semidefinite matrices which can be defined as:

Definition 2.1.4 (Horn & Johnson, 1985) Let $A = [a_{ij}] \in M_{m,n}$ and $B = [b_{ij}] \in M_{m,n}$. We write

$$B \geq 0 \text{ if all } b_{ij} \geq 0$$

$$B > 0 \text{ if all } b_{ij} > 0$$

$$A \geq B \text{ if all } a_{ij} \geq b_{ij}$$

$$A > B \text{ if all } a_{ij} > b_{ij}$$

The reverse relations $\leq$ and $<$ are defined similarly. We define $|A| = [|a_{ij}|]$. If $A \geq 0$, we say A is a *nonnegative matrix*, and if $A > 0$, we say that A is a *positive matrix*.

Definition 2.1.5 (Horn & Johnson, 1991) An $n \times n$ Hermitian matrix A is said to be *positive definite* if

$$x^* A x > 0 \text{ for all nonzero } x \in \mathbb{C}^{n \times n}.$$

It is denoted by $A \succ 0$. Also $A \succ$ B means that $A - B \succ 0$.

Definition 2.1.6 (Horn & Johnson, 1991) An $n \times n$ Hermitian matrix A is said to be *positive semidefinite* if

$$x^* A x \geq 0 \text{ for all nonzero } x \in \mathbb{C}^{n \times n}.$$

It is denoted by $A \succeq 0$. Also, $A \succeq$ B means that $A - B \succeq 0$.

2.2 Definitions for Products of Matrices and Norms of Matrices

Now, we can give the definitions of some special products of matrices which are mentioned above:

Definition 2.2.1 (Horn & Johnson, 1985) If $A = [a_{ij}] \in M_{m,n}$ and $B = [b_{ij}] \in M_{m,n}$ are given, then the *Hadamard product* of A and B is the matrix

$$A \circ B = [\, a_{ij} b_{ij} \,] \in M_{m,n} .$$

Definition 2.2.2 (Horn & Johnson, 1991) The *Kronecker product* of an $m \times n$ matrix $A = [\, a_{ij} \,]$ and $p \times q$ matrix $B = [\, b_{kl} \,]$ is an $mp \times qn$ matrix denoted by $A \otimes B$ and defined to be the block matrix

$$A \otimes B = [a_{ij} B]$$

Definition 2.2.3 (Huang, 2008) Let $A = [a_{ij}] \in M_{m,n}$ and $B = [\, b_{ij} \,] \in M_{m,n}$. The *Fan product* of A and B is denoted by $A \star B \equiv C = [c_{ij}] \in M_{m,n}$ and is defined by

$$c_{ij} = \begin{cases} -a_{ij} b_{ij} & if\ i \neq j \,. \\ \ \ a_{ii} b_{ii} & if\ i = j \,. \end{cases}$$

Definition 2.2.4 (Liu & Trenkler, 2008) The *Khatri-Rao product* of an $m \times n$ matrix $A = [\, a_{ij} \,]$ and $p \times q$matrix $B = [\, b_{kl} \,]$ is an $(\sum m_i p_i) \times (\sum n_j q_j)$ matrix denoted by $A * B$ and defined to be the matrix

$$A * B = (A_{ij} \otimes B_{ij})_{ij}$$

where $A_{ij} \otimes B_{ij}$ is order of $m_i p_i \times n_j q_j$.

Definition 2.2.5 (Liu & Trenkler, 2008) The *Tracy-Singh product* of an $m \times n$ matrix $A = [\, a_{ij} \,]$ and an $p \times q$ matrix $B = [\, b_{kl} \,]$ is an $mp \times nq$ matrix denoted by $A \bowtie B$ and defined to be the matrix

$$A \bowtie B = (A_{ij} \bowtie B)_{ij} = ((A_{ij} \otimes B_{kl})_{kl})_{ij}$$

where $A_{ij} \otimes B_{kl}$ is of order $m_i p_k \times n_j q_l$, and $A_{ij} \bowtie B$ is of order $m_i p \times n_j q$.

In matrix theory, norm of a matrix is one of the most important and useful tools while trying to identify and to understand the behavior of the matrix. There are several ways and formulas to define a norm, but in this thesis we will look at some of them. For the following two definitions we will use the singular values of a matrix which are defined in Section 2.1 and denoted by

$$s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$$

in decreasing order.

Definition 2.2.6 (Du, 2010) The Fan k-norm of $A \in M_n$ is

$$\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A) .$$

where $k = 1,2, \dots, n$.

Thus, $\|A\|_{(1)}$ is the spectral norm and $\|A\|_{(n)}$ is the trace norm.

Definition 2.2.7 (Du, 2010; Huang, 2008) The Frobenius norm or the Euclidean norm of $A \in M_n$ is defined as and denoted by

$$\|A\|_2 = (\sum_{i=1}^{n} s_i^2(A))^{\frac{1}{2}} .$$

2.3 Basic Lemmas and Theorems in Trace Inequalities

In each main chapter, mostly we will be dealing with inequalities involving the spectral radius and the minimum eigenvalue of some special matrices. However, before introducing these main chapters, we will give some theorems and lemmas about the trace of a matrix:

Theorem 2.3.1 (Yang, 1988) If A and B are two $n \times n$ positive definite matrices, then

$$(1)\ tr(AB) > 0 \ \ and \tag{2.1}$$

$$(2)\ \sqrt{tr(AB)} < (tr\, A + tr\, B)/2 . \tag{2.2}$$

Theorem 2.3.2 (Neudecker, 1990) If A and B are positive semidefinite matrices of the same order, then

$$(1)\ tr(AB) \geq 0 \ \ and \tag{2.3}$$

$$(2)\ \sqrt{tr(AB)} \leq (tr\,A + tr\,B)/2\ . \tag{2.4}$$

Theorem 2.3.3 (Coope, 1993) If A and B are positive semidefinite then AB has positive eigenvalues.

Theorem 2.3.4 (Coope, 1993) The trace of a product of two Hermitian matrices of the same order is real.

Theorem 2.3.5 (Coope, 1993) For positive semidefinite matrices, A, B of the same order

$$0 \leq tr(AB) \leq tr(A)tr(B)\,. \tag{2.5}$$

Lemma 2.3.1 (Coope, 1993; Yang, 2000) If A and B are positive semidefinite matrices of the same order, then

$$0 \ \leq\ tr(\,AB\,) \leq tr\,A\,tr\,B \tag{2.6}$$

Theorem 2.3.6 (Yang, 2000) Let A and B be positive semidefinite matrices of the same order; then for $n\ =\ 1, 2, \ldots$

$$0\ \leq\ tr(AB)^{2n} \leq\ (tr\,A)^2(tr\,A^2)^{n-1}(tr\,B^2)^n\,, \tag{2.7}$$

$$0\ \leq\ tr(AB)^{2n+1} \leq (tr\,A)(tr\,B)(tr\,A^2)^n(tr\,B^2)^n. \tag{2.8}$$

Corollary 2.3.1 (Yang, 2000) If A and B are defined in Theorem 2.3.6, then

$$0\ \leq tr(AB)^n \leq\ (trA)^n(trB)^n\,. \tag{2.9}$$

Lemma 2.3.2 (Yang, 2000) If A and B are positive semidefinite matrices of the same order, then $(AB)^nB$ and $(BA)^nB$ are positive semidefinite matrices.

CHAPTER THREE

INEQUALITIES IN HADAMARD, FAN AND KRONECKER PRODUCTS

In this chapter of this thesis, we focused on the inequalities involving special products of matrices. Firstly, we investigated the relations between these products. After that we studied the inequalities of the spectral radius and the norm for the Hadamard product in detail and we tried to apply the analogous of them to Fan product of matrices.

3.1 Spectral Radius and Norm Inequalities Involving Hadamard Product

While we were searching the articles written about the spectral radius of the Hadamard product of two matrices to investigate a bound for the spectral radius, first theorems we encountered gave us the inequalities for nonnegative matrices. These theorems can be introduced as in below:

Theorem 3.1.1 (Horn & Johnson, 1985) If $A, B \in M_n$, $A \geq 0$, and $B \geq 0$, then

$$\rho\,(A \circ B) \leq \rho\,(A)\rho(B). \tag{3.1}$$

Theorem 3.1.2 (Audenaert, 2010) For $n \times n$ non-negative matrices A and B,

$$\rho\,(A \circ B) \leq \rho\big((A \circ A)(B \circ B)\big)^{1/2} \leq \rho\,(AB). \tag{3.2}$$

Theorem 3.1.3 (Horn & Zhang, 2010) Let $A, B \in M_n$. Suppose that $A \geq 0$ and $B \geq 0$. Then

$$\rho\,(A \circ B) \leq \rho^{1/2}(AB \circ BA) \leq \rho\,(AB). \tag{3.3}$$

However, if we want to focus on positive matrices for the same bound searched in the first three theorems above, Theorem 3.1.3 can be changed slightly as follows:

Corollary 3.1.1 (Horn & Zhang, 2010) Let $A, B \in M_n$. Suppose that $A > 0$ and $B > 0$. Then

$$\rho\,(A \circ B) < \rho\,(AB). \tag{3.4}$$

By the next theorem, it can be seen that the upper bound for the spectral radius of the Hadamard product of a positive definite matrix and a positive semidefinite matrix will have a more concrete result involving the spectral radius of the Hadamard product of the positive definite matrix and its inverse:

Theorem 3.1.4 (Horn & Zhang, 2010) Let $A, B \in M_n$. Suppose that $A \succeq 0$ and $B \succ 0$. Then

$$\rho(A \circ B) \leq \beta\rho(AB),\ \beta = \rho(B^{-1} \circ B) \geq 1. \tag{3.5}$$

To understand the relation between the Hadamard product and the Kronecker product, we should acknowledge the following definition:

Definition 3.1.1 (Horn & Johnson, 1985) Let A be an $m \times n$ matrix. For index sets $\alpha \subseteq \{1, \dots, m\}$ and $\beta \subseteq \{1, \dots, n\}$, we denote the submatrix that lies in the rows of A indexed by α and in the columns indexed by β as $A[\alpha, \beta]$. If $m = n$ and $\alpha = \beta$, the submatrix $A[\alpha, \alpha]$ is called the *principal submatrix of* A and is abbreviated $A[\alpha]$.

Due to the fact that we are now familiar with the definition of principal submatrix, we can present the following proposition which is the key of the relation between the Hadamard product and the Kronecker product:

Proposition 3.1.1 (Horn & Zhang, 2010) The Hadamard product is a submatrix of the Kronecker product: if $A, B \in M_{m,n}$, then

$$A \circ B = (A \otimes B)[\alpha, \beta] \tag{3.6}$$

in which $\alpha = \{1, \mathrm{m}+2, 2\mathrm{m}+3, \dots, m^2\}$ and $\beta = \{1, \mathrm{n}+2, 2\mathrm{n}+3, \dots, n^2\}$. In particular, if $m = n$, $A \circ B$ is principal submatrix of $(A \otimes B)$.

From now on, we will be interested in some theorems and lemmas issuing inequalities related to the spectral radius of principal submatrices of some special matrices such as nonnegative matrices, irreducible matrices and positive semidefinite matrices:

Lemma 3.1.1 (Horn & Zhang, 2010) Let $A, B \in M_N$. be nonnegative and let $\alpha \in \{1, \ldots, N\}$ be given and nonempty.

$$(1)\ If\ A \geq B, then\ \rho(A) \geq \rho(B)\,. \tag{3.7}$$

$$(2)\ \rho(\,A[\alpha]\,) \leq \rho(A). \tag{3.8}$$

$$(3)\ A[\alpha]B[\alpha] \leq (AB)[\alpha]. \tag{3.9}$$

$$(4)\ \rho\,(A[\alpha]B[\alpha]\,) \leq \rho((AB\,)[\alpha]) \leq \rho(AB\,). \tag{3.10}$$

Lemma 3.1.2 (Horn & Zhang, 2010) Let $A = [a_{ij}] \in M_n$ with $n \geq 2$, and suppose that A is nonnegative and irreducible. Let $\alpha \subsetneq \{1, \ldots, n\}$ be nonempty. Then

$$\rho\,(A[\alpha]) < \rho\,(A). \tag{3.11}$$

Lemma 3.1.3 (Horn & Zhang, 2010) Let $A, B, C \in M_N$ be positive semidefinite and let $\alpha \subsetneq \{1, \ldots, N\}$ be given and nonempty.

$$(1)\ If\ A \geq B, then\ \rho(A) \geq \rho(B). \tag{3.12}$$

$$(2)\ \rho(\,A[\alpha]\,) \leq \rho(A). \tag{3.13}$$

$$(3)\ If\ B \succ 0, then\ \rho\,(\,A[\alpha]B[\alpha]^{-1}\,) \leq \rho\,(AB^{-1}). \tag{3.14}$$

$$(4)\ If\ A \geq B, then\ \rho\,(BC) \leq \rho\,(AC). \tag{3.15}$$

Lemma 3.1.4 (Horn & Johnson, 1985; Du, 2010) Let $A, B \in M_n$. If $|A| \leq B$, then

$$\rho(A) \leq \rho(|A|) \leq \rho(B). \tag{3.16}$$

Lemma 3.1.5 (Du, 2010) For any $A \in M_n$,

$$tr(A) \leq \sum_{i=1}^{n} s_i(A). \tag{3.17}$$

Theorem 3.1.5 (Du, 2010) The inequality

$$\|(A \circ B)(A \circ \mathrm{B})^*\| \leq \left\|\left(A \circ \overline{A}\right)\left(B \circ \overline{B}\right)^T\right\|\ for\ A, B \in M_n \tag{3.18}$$

is true when $\|\,.\,\|$ is the spectral norm, the trace norm or the Frobenius norm where $\overline{A}$ is the conjugate of the matrix A.

Theorem 3.1.6 (Horn & Johnson, 1991; Du, 2010) For any $A \in M_n$ and $B \in M_n$, we have

$$\|A \circ B\|_{(1)}^2 \leq \left\|A \circ \overline{A}\right\|_{(1)} \left\|B \circ \overline{B}\right\|_{(1)} \tag{3.19}$$

Theorem 3.1.7 (Horn & Johnson, 1991) Let $A \in M_n$ and $B \in M_m$. If $\lambda \in \sigma(A)$ and $x \in \mathbb{C}^n$ is a corresponding eigenvector of A, and if $\mu \in \sigma(B)$ and $y \in \mathbb{C}^m$ is a corresponding eigenvector of B, then $\lambda\mu \in \sigma(A \otimes B)$ and $x \otimes y \in \mathbb{C}^{nm}$ is a corresponding eigenvector of $A \otimes B$. Every eigenvalue of $A \otimes B$ arises as such a product of eigenvalue of A and B. If $\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and $\sigma(B) = \{\mu_1, \mu_2, \dots, \mu_n\}$, then $\sigma(A \otimes B) = \{\lambda_i \mu_j : i = 1, \dots, n\,; j = 1, \dots, m\}$ (including algebraic multiplicities in all three cases). In particular, $\sigma(A \otimes B) = \sigma(B \otimes A)$.

Lemma 3.1.6 (Audenaert, 2010) For $A \in M_n$ such that $A > 0$

$$\rho(A) = \lim_{m \to \infty} (TrA^m)^{1/m}. \tag{3.20}$$

3.2 Results on the Spectral Radius and the Norm of the Fan product

In this section, we will construct and prove three theorems for some upper bounds for the Fan product of two nonnegative matrices and an upper bound for the square of the norm of the Fan product of two matrices. Both in writing and proving these theorems, the lemmas and the theorems in section 3.1 with their proofs will be very useful.

Theorem 3.2.1 If $A, B \in M_n$; $A \geq 0$ and $B \geq 0$, then

$$\rho(A \star B) \leq \rho(A)\rho(B). \tag{3.21}$$

Proof. Since $A \geq 0$ and $B \geq 0$; $A \circ B \geq 0$ and therefore $|A \star B| = A \circ B$.

Using Lemma 3.1.4, $\rho(A \star B) \leq \rho(|A \star B|) = \rho(A \circ B)$. On the other hand, by theorem 3.1.1, $\rho(A \circ B) \leq \rho(A)\rho(B)$, since $A \circ B$ is principal submatrix of $A \otimes B$ and $\rho(A \circ B) \leq \rho(A \otimes B)$ by Lemma 3.1.2 and $\rho(A \otimes B) = \rho(A)\rho(B)$. Thus, $\rho(A \star B) \leq \rho(A)\rho(B)$.

Theorem 3.2.2 Let A, $B \in M_n$. Suppose that $A \geq 0$ and B is an M-matrix. Then

$$\rho\,(A \star B) \leq \rho(|(A \star A)(B \star B)|)^{1/2} \tag{3.22}$$

While trying to prove this theorem, firstly, we tried to understand the proof of Theorem 3.1.2 (Theorem 1 of Audenaert) and we recognized that it will be very easy to prove the above theorem by using the facts that for the nonnegative matrix A and M-matrix B , $Tr((A \circ B))^{2k} = Tr((A \star B))^{2k}$ and $Tr\left(\left((A \circ A)(B \circ B)\right)^k\right) = Tr((|(A \star A)(B \star B)|)^k)$.

Proof. Let $A \geq 0$ and B be an M-matrix. Firstly, it will be very useful to prove the following inequalities for any positive integer k:

$$Tr((A \star B))^{2k} \leq Tr((|(A \star A)(B \star B)|)^k)$$

$$Tr((A \star B))^{2k} = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n} \dots \sum_{i_{2k}=1}^{n} \left(a_{i_1 i_2} b_{i_1 i_2}\right)\left(a_{i_2 i_3} b_{i_2 i_3}\right) \dots \left(a_{i_{2k} i_1} b_{i_{2k} i_1}\right)$$

$$= \sum_{i_1, i_2, \dots, i_{2k}} \left(a_{i_1 i_2} b_{i_1 i_2}\right)\left(a_{i_2 i_3} b_{i_2 i_3}\right) \dots \left(a_{i_{2k} i_1} b_{i_{2k} i_1}\right)$$

$$= \sum_{i_1, i_2, \dots, i_{2k}} (a_{i_1 i_2} b_{i_2 i_3} \dots a_{i_{2k-1} i_{2k}} b_{i_{2k} i_1}) \times (b_{i_1 i_2} a_{i_2 i_3} \dots b_{i_{2k-1} i_{2k}} a_{i_{2k} i_1})$$

It can be easily seen that this expression is an inner product between two vectors one with entries $a_{i_1 i_2} b_{i_2 i_3} \dots a_{i_{2k-1} i_{2k}} b_{i_{2k} i_1}$ and the other with entries $b_{i_1 i_2} a_{i_2 i_3} \dots b_{i_{2k-1} i_{2k}} a_{i_{2k} i_1}$. If we perform a cyclic permutation on the indices it is obvious that these vectors have the same sets of entries. Therefore, both vectors have the same Euclidean norm, so we can apply Cauchy-Schwarz Inequality and then we get:

$$\leq \sum_{i_1, i_2, \dots, i_{2k}} \left(a_{i_1 i_2} b_{i_2 i_3} \dots a_{i_{2k-1} i_{2k}} b_{i_{2k} i_1}\right)^2$$

$$= \sum_{i_1, i_2, \dots, i_{2k}} {a_{i_1 i_2}}^2 {b_{i_2 i_3}}^2 \dots {a_{i_{2k-1} i_{2k}}}^2 {b_{i_{2k} i_1}}^2$$

$$= Tr((|(A \star A)(B \star B)|)^k) \ .$$

Finally, taking the $(2k)^{th}$ root, taking the limit $k \to \infty$ and applying the lemma 3.1.6 the theorem follows from the inequality

$$Tr((A \star B))^{2k} \leq Tr((|(A \star A)(B \star B)|)^k).$$

Theorem 3.2.3 For any $A \in M_n$ and $B \in M_n$, we have

$$\|A \star B\|_{(1)}^2 \leq \left\|A \circ \overline{A}\right\|_{(1)} \left\|B \circ \overline{B}\right\|_{(1)} . \tag{3.23}$$

We tried to prove the above result using the proof of the Theorem 3.1.6:

Proof. The spectral norm $\|.\|_{(1)}$ is induced by the Euclidean norm $\|.\|_2$ on $\mathbb{C}^n$. Let $x = [x_j] \in \mathbb{C}^n$ be such that $\|x\|_2 = 1$ and $\|A \star B\|_{(1)} = \|(A \star B)x\|_2$. Also, let $A = [a_{ij}]$ and $B = [b_{ij}]$. Then,

$$\|(A \star B)x\|_2^2 = \sum_i \left|\sum_j a_{ij} b_{ij} x_j\right|^2$$

$$= \sum_i |a_{i1}b_{i1}x_1 - a_{i2}b_{i2}x_2 - \cdots - a_{in}b_{in}x_n|^2$$

$$\leq \sum_i [|a_{i1}||b_{i1}||x_1| + |a_{i2}||b_{i2}||x_2| + \cdots + |a_{in}||b_{in}||x_n|]^2$$

$$= \sum_i \left[\left(|a_{i1}||x_1|^{\frac{1}{2}}, |a_{i2}||x_2|^{\frac{1}{2}}, \ldots, |a_{in}||x_n|^{\frac{1}{2}}\right).\left(|b_{i1}||x_1|^{\frac{1}{2}}, |b_{i2}||x_2|^{\frac{1}{2}}, \ldots, |b_{in}||x_n|^{\frac{1}{2}}\right)\right]^2$$

$$\leq \sum_i [\sqrt{|a_{i1}|^2|x_1| + \cdots + |a_{in}|^2|x_n|} \ . \sqrt{|b_{i1}|^2|x_1| + \cdots + |b_{in}|^2|x_n|} \ ]^2$$

$$= \sum_i \left[\left[\sum_j \left|a_{ij}\right|^2 |x_j|\right]\left[\sum_k |b_{ik}|^2 |x_k|\right]\right]$$

$$\leq \left[\sum_i \left[\sum_j \left|a_{ij}\right|^2 |x_j|\right]^2\right]^{\frac{1}{2}} . \left[\sum_i \left[\sum_j \left|b_{ij}\right|^2 |x_j|\right]^2\right]^{\frac{1}{2}}$$

$$= \left\|\left(A \circ \overline{A}\right)|x|\right\|_{\mathbf{2}} . \left\|\left(B \circ \overline{B}\right)|x|\right\|_{\mathbf{2}}$$

$$\leq \left\|\left(A \circ \overline{A}\right)\right\|_{(1)} \|x\|_{(1)} . \left\|\left(B \circ \overline{B}\right)\right\|_{(1)} \|x\|_{(1)}$$

$$= \left\|\left(A \circ \overline{A}\right)\right\|_{(1)} \left\|(B \circ \overline{B})\right\|_{(1)} \quad .$$

3.3 Inequalities involving Kronecker, Khatri-Rao and Tracy-Singh Products

Observation 3.3.1 The Khatri-Rao product is a submatrix of the Tracy-Singh product: if $A, B \in M_{m,n}$, then

$$A * B = (A \bowtie B)\,[\alpha,\beta] \tag{3.24}$$

In particular, if $m = n$, $A * B$ is principal submatrix of $(A \bowtie B)$.

Observation 3.3.2 The Khatri-Rao product is a submatrix of the Kronecker product: if $A, B \in M_{m,n}$, then

$$A * B = (A \otimes B)\,[\alpha,\beta] \tag{3.25}$$

In particular, if $m = n$, $A * B$ is principal submatrix of $(A \otimes B)$.

For instance, let an $m \times n$ matrix A is partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

and let an $m \times n$ matrix B is partitioned as

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

Now, the Tracy-Singh product of these two matrices A and B is defined to be:

$$A \bowtie B = \begin{bmatrix} A_{11} \bowtie B & A_{12} \bowtie B \\ A_{21} \bowtie B & A_{22} \bowtie B \end{bmatrix} = \begin{bmatrix} A_{11}\otimes B_{11} & A_{11}\otimes B_{12} & A_{12}\otimes B_{11} & A_{12}\otimes B_{12} \\ A_{11}\otimes B_{21} & A_{11}\otimes B_{22} & A_{12}\otimes B_{21} & A_{12}\otimes B_{22} \\ A_{21}\otimes B_{11} & A_{21}\otimes B_{12} & A_{22}\otimes B_{11} & A_{22}\otimes B_{12} \\ A_{21}\otimes B_{21} & A_{21}\otimes B_{22} & A_{22}\otimes B_{21} & A_{22}\otimes B_{22} \end{bmatrix}.$$

and the Khatri-Rao product of these matrices can be defined as:

$$A * B = \begin{bmatrix} A_{11}\otimes B_{11} & A_{12}\otimes B_{12} \\ A_{21}\otimes B_{21} & A_{22}\otimes B_{22} \end{bmatrix}.$$

It can be easily seen that the Khatri-Rao product is a submatrix of the Tracy-Singh product.

In the light of observations 3.3.1 and 3.3.2 we can also observe some relationships between the spectral radius of Tracy-Singh product of two matrices and the spectral radius of the Khatri-Rao product of these given matrices:

Theorem 3.3.3 If $A, B \in M_n$, $A \geq 0$, and $B \geq 0$, then

$$\rho\ (A * B) \leq \rho\ (A \bowtie B). \tag{3.26}$$

Proof. By Observation 3.3.1 it is obvious that $A * B$ is a principal submatrix of $A \bowtie B$. Therefore, by using Lemma 3.1.1 (2) the inequality above concludes.

Theorem 3.3.4 If $A, B \in M_n$, $A \geq 0$, and $B \geq 0$, then

$$\rho\ (A * B) \leq \rho\ (A \otimes B). \tag{3.27}$$

Proof. By Observation 3.3.2 it is obvious that $A * B$ is a principal submatrix of $A \otimes B$. Therefore, by using Lemma 3.1.1 (2) the inequality above concludes.

CHAPTER FOUR

INEQUALITIES FOR *M*-MATRICES

In this chapter, we will be interested in inequalities involving not only the spectral radius but also the minimum eigenvalue of a matrix. Furthermore, the focus will be on M-matrices. To be able to follow the lemmas and theorems given and comprehend the results, we will study the definitions and basic properties of Z-matrices and M-matrices.

4.1 General Information of *M*-matrices and Inequalities involving Jacobi Iterative Matrix

Definition 4.1.1 (Horn & Johnson, 1991) The set $Z_n \subset M_n(\mathbb{R})$ is defined by

$$Z_n = \{A = [a_{ij}] \in M_n(\mathbb{R}): a_{ij} \leq 0 \; if \; i \neq j; i, j = 1, \dots, n\}$$

Definition 4.1.2 (Horn &Johnson, 1991) A matrix A is called an M-matrix if $A \in Z_n$ and A positive stable.

Theorem 4.1.1 (Horn & Johnson, 1991) If $A \in Z_n$, the following statements are equivalent:

(1) A is positive stable, that is, A is an M-matrix.
(2) $A = \alpha I - P, P \geq 0, \alpha > \rho(P)$.
(3) Every real eigenvalue of A is positive.
(4) $A + tI$ is nonsingular for all $t \geq 0$.
(5) $A + D$ is nonsingular for every nonnegative diagonal matrix D.
(6) All principal minors of A are positive.
(7) The sum of all k-by-k principal minors of A is positive for $k = 1, \dots, n$.
(8) The leading principal minors of A are positive.
(9) The diagonal entries of A are positive and AD is strictly row diagonally dominant for some positive diagonal matrix D.

(10) A is nonsingular and $A^{-1} \geq 0$.

(11) $Ax \geq 0$ implies $x \geq 0$.

Through the following corollary, we will acknowledge a very useful definition of the spectral radius of a matrix in two different ways.

Corollary 4.1.1 (Horn & Johnson, 1985; Huang, 2008) If $A \in M_n$ and $A \geq 0$, then

$$\rho(A) = \max_{x \geq 0, x \neq 0} \min_{x_i \neq 0} \frac{(Ax)_i}{x_i}.$$

or, equivalently

$$\rho(A) = \max_{x \geq 0, x \neq 0} \min_{x_i \neq 0} \frac{1}{x_i} \sum_{j=1}^{n} a_{ij} x_j \,.$$

Rather than the above definitions of the spectral radius of a matrix, there is another way to define it if the matrix is irreducible. Firstly, we need to define irreducible matrices. Hence, to be able to understand the notion of irreducibility of matrices we can look at the definition of reducible matrices:

Definition 4.1.3 (Horn & Johnson, 1985) A matrix $A \in M_n$ is said to be *reducible* if either

(a) $n = 1$ and $A = 0$; or

(b) $n \geq 2$, there is a permutation matrix $P \in M_n$, and there is some integer r with $1 \leq r \leq n - 1$ such that

$$P^T A P = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

where $B \in M_r, D \in M_{n-r}, C \in M_{r,n-r}$, and $0 \in M_{n-r,r}$ is a zero matrix.

Definition 4.1.4 (Horn &Johnson, 1985) A matrix $A \in M_n$ is said to be *irreducible* if it is not reducible.

Corollary 4.1.2 (Horn &Johnson, 1985; Huang, 2008) If A is an irreducible nonnegative matrix; then

$$\rho(A) = \min_{x \geq 0, x \neq 0} \max_{x_i \neq 0} \frac{(Ax)_i}{x_i}.$$

In 2008, Huang stated a significant theorem which gives an upper bound and a lower bound for the spectral radius of Jacobi iterative matrix of a given matrix in regard of the minimum eigenvalue and the entries of the given matrix. Before stating that theorem, we should look at the way that the matrix J_A defined. Hence if $A = [a_{ij}]$ is an $n \times n$ M-matrix, we write $N = D - A$, where $D = \text{diag}(a_{ii})$. Note that $a_{ii} > 0$ for all i, if A is an $n \times n$ M-matrix. Thus we define $J_A = D^{-1}N$. Obviously, J_A is nonnegative. (Huang, 2008)

Theorem 4.1.2 (Huang, 2008) Let $A = [a_{ij}]$ be an $n \times n$ nonsingular M-matrix. Then

$$1 - \frac{\tau(A)}{\min\limits_{1 \leq i \leq n} a_{ii}} \leq \rho(J_A) \leq 1 - \frac{\tau(A)}{\max\limits_{1 \leq i \leq n} a_{ii}} \tag{4.1}$$

In particular, $\rho(J_A) < 1$.

Lemma 4.1.1 (Huang, 2008) If A is an irreducible M-matrix, and $Az \geq kz$ for a nonnegative nonzero vector z, then $\tau(A) \geq k$.

Lemma 4.1.2 (Horn &Johnson, 1985; Huang, 2008) If A is an irreducible nonnegative matrix, and $Az \leq kz$ for a nonnegative nonzero vector z, then

$$\rho(A) \leq k.$$

Lemma 4.1.3 (Huang, 2008; Berman & Plemmons, 1994) Let A be a nonnegative matrix, and $|B| \leq A$. Then $\rho(B) \leq \rho(A)$.

Theorem 4.1.3 (Huang, 2008) Let A and B be $n \times n$ nonsingular M-matrices. Then

$$\tau(A \star B) \geq \big(1 - \rho(J_A)\rho(J_B)\big) \min_{1 \leq i \leq n} (a_{ii}\, b_{ii}). \tag{4.2}$$

Theorem 4.1.4 (Huang, 2008) Let $A, B \in M_n$ be two real nonnegative matrices. Then

(1) If $a_{ii}b_{ii} \neq 0$ for all i, then

$$\rho(A \circ B) \leq \left(1 + \rho(J'_A)\rho(J'_B)\right) \max_{1 \leq i \leq n} a_{ii}b_{ii}. \tag{4.3}$$

(2) If $a_{i_0 i_0} \neq 0 \; or \; b_{i_0 i_0} \neq 0$ for some i_0, but $a_{ii}b_{ii} = 0$ for all i, then

$$\rho(A \circ B) \leq \left(\rho(J'_A)\rho(J'_B)\right) \max_{1 \leq i \leq n} \{a_{ii}, b_{ii}\}. \tag{4.4}$$

(3) If $a_{ii} = 0 \; and \; b_{ii} = 0$ for all i, then

$$\rho(A \circ B) \leq \left(\rho(J'_A)\rho(J'_B)\right). \tag{4.5}$$

(4) If $a_{i_0 i_0}b_{i_0 i_0} \neq 0 \; and \; a_{j_0 j_0}b_{j_0 j_0} = 0$ for some $i_0 and \; j_0$, then the upper bound on $\rho(A \circ B)$ is the maximum value of the upper bounds of (1)-(3).

Before stating the following theorem, it will be very useful to give the notations below: Let $N = \{1,2, \dots n\}$ and let $A = [a_{ij}]$ be nonsingular with $a_{ii} \neq 0$ for all i, and $A^{-1} = [b_{ij}]$;

$$R(A) = \max_{i \in N} \sum_{j=1}^{n} a_{ij} \;, \qquad r(A) = \min_{i \in N} \sum_{j=1}^{n} a_{ij}$$

and

$$M = \max_{i \in N} \sum_{j=1}^{n} b_{ij} \;, \qquad m = \min_{i \in N} \sum_{j=1}^{n} b_{ij}$$

Theorem 4.1.5 (Tian & Huang, 2010) Let $A = [a_{ij}]$ be a nonsingular *M*-matrix and $A^{-1} = [b_{ij}]$. Then

$$r(A) \leq \frac{1}{M} \leq \tau(A) \leq \frac{1}{m} \leq R(A). \tag{4.6}$$

So far, in this section we have investigated the lemmas and theorems about the bounds for the spectral radius of the Jacobi iterative matrix for an *M*-matrix, for the minimum eigenvalue of the Fan product of two *M*- matrices and for the spectral radius of the Hadamard product of two nonnegative matrices. After we analyzed all of them, we have concentrated on the Theorem 4.1.2 and Theorem 4.1.5; in the light of these bounds we have tried to find new lower and upper bounds for the spectral radius of the Jacobi iterative matrix of an M-matrix using the entries of the given M-matrix and the entries of its inverse. These bounds can be formalized as in the following theorems:

Theorem 4.1.6 Let $A = [a_{ij}]$ be an $n \times n$ nonsingular M-matrix and $A^{-1} = [b_{ij}]$. Then

$$1 - \frac{1}{\min\limits_{1 \le i \le n} \sum_{j=1}^{n} a_{ii} b_{ij}} \le \rho(J_A) \le 1 - \frac{1}{\max\limits_{1 \le i \le n} \sum_{j=1}^{n} a_{ii} b_{ij}} \quad . \tag{4.7}$$

Proof Let $A = [a_{ij}]$ be an $n \times n$ nonsingular M-matrix and $A^{-1} = [b_{ij}]$. Firstly, we will try to prove the left hand side of the inequality:

Since A is a nonsingular M-matrix, it follows from Theorem 4.1.2 that

$$1 - \frac{\tau(A)}{\min\limits_{1 \le i \le n} a_{ii}} \le \rho(J_A)$$

and it can be easily seen that $\tau(A) \ge \min_{1 \le i \le n} a_{ii} \ (1 - \rho(J_A))$.

Also, from Theorem 4.1.5 it can be seen that $\tau(A) \le \frac{1}{m}$ which is equal to the inequality

$$\tau(A) \le \frac{1}{\min\limits_{1 \le i \le n} \sum_{j=1}^{n} b_{ij}} .$$

By combining these two inequality and by removing $\tau(A)$, it is obvious that

$$\min_{1 \le i \le n} a_{ii} \ \left(1 - \rho(J_A)\right) \le \frac{1}{\min\limits_{1 \le i \le n} \sum_{j=1}^{n} b_{ij}} \quad .$$

Since both $a_{ii} > 0$ and $\sum_{j=1}^{n} b_{ij} > 0$, the left hand side of the inequality in (4.7) follows by simple calculations:

$$1 - \frac{1}{\min_{1 \le i \le n} \sum_{j=1}^{n} a_{ii} b_{ij}} \le \rho(J_A) . \qquad .$$

The right hand side of the inequality in (4.7) can be proved in the same way. By Theorem 4.1.2, we know that

$$\tau(A) \leq \max_{1 \leq i \leq n} a_{ii} \ (1 - \rho(J_A))$$

and by Theorem 4.1.5, we have that

$$\tau(A) \geq \frac{1}{\max_{1 \leq i \leq n} \sum_{j=1}^{n} b_{ij}} \ .$$

Therefore, by combining these two inequalities and applying simple calculations, it is obviously seen that

$$\rho(J_A) \leq 1 - \frac{1}{\max_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ii} b_{ij}} \ .$$

Thus, since we have proved both sides of the inequality, it is clear that

$$1 - \frac{1}{\min_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ii} b_{ij}} \leq \rho(J_A) \leq 1 - \frac{1}{\max_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ii} b_{ij}} \ .$$

Theorem 4.1.7 Let $A = [a_{ij}]$ be an $n \times n$ nonsingular *M*-matrix and $A^{-1} = [b_{ij}]$. Then

$$1 - \frac{\max_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ij}}{\min_{1 \leq i \leq n} a_{ii}} \leq \rho(J_A) \leq 1 - \frac{\min_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ij}}{\max_{1 \leq i \leq n} a_{ii}} \ . \qquad (4.8)$$

Proof Let $A = [a_{ij}]$ be an $n \times n$ nonsingular *M*-matrix and $A^{-1} = [b_{ij}]$. It follows from Theorem 4.1.2 that

$$\tau(A) \geq \min_{1 \leq i \leq n} a_{ii} \ (1 - \rho(J_A))$$

and from Theorem 4.1.5

$$\tau(A) \leq R(A) = \max_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ij} \ .$$

By combining these two inequalities and using simple calculations, it can be written that

$$1-\frac{\max\limits_{1\le i\le n}\sum_{j=1}^{n}a_{ij}}{\min\limits_{1\le i\le n}a_{ii}}\le\rho(J_A)\ .$$

Moreover, the right hand side of the inequality in Theorem 4.1.7 can be easily proved in the same way. The inequalities below can be written by using Theorem 4.1.2 and Theorem 4.1.5 respectively:

$$\tau(A)\le\max_{1\le i\le n}a_{ii}\ \left(1-\rho(J_A)\right)$$

and

$$\min_{i\in N}\sum_{j=1}^{n}a_{ij}=r(A)\le\tau(A).$$

Therefore, it is clear that

$$\rho(J_A)\le 1-\frac{\min\limits_{1\le i\le n}\sum_{j=1}^{n}a_{ij}}{\max\limits_{1\le i\le n}a_{ii}}$$

Thus the result of the theorem follows:

$$1-\frac{\max\limits_{1\le i\le n}\sum_{j=1}^{n}a_{ij}}{\min\limits_{1\le i\le n}a_{ii}}\le\rho(J_A)\le 1-\frac{\min\limits_{1\le i\le n}\sum_{j=1}^{n}a_{ij}}{\max\limits_{1\le i\le n}a_{ii}}\ .$$

4.2 On the Minimum Eigenvalue of the Hadamard Product of an *M*-Matrix and Inverse *M*-matrix

In this section, we will be dealing with the lower bounds for the minimum eigenvalue of the Hadamard product of an *M*- matrix and its inverse. For that purpose, different from the first part of this chapter, our bounds will be based on only the entries of the given matrix. Firstly, we will try to figure out some important relations between the entries of the given matrix and the entries of the inverse of that matrix.

For convenience, we will give some notations that will be used in lemmas and theorems of this part. These notations are composed of some different sums of the entries of a matrix.

For $i, j, k, l \in N$;

$$R_i = \sum_{k \neq i} |a_{ik}| \ , \qquad d_i = \frac{R_i}{|a_{ii}|} \ ;$$

$$r_{li} = \frac{|a_{li}|}{|a_{ll}| - \sum_{k \neq l,i} |a_{lk}|} \ , \qquad l \neq i \ ; \quad r_i = \max_{l \neq i} \{r_{li}\} \ , \quad i \in N;$$

$$m_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_i}{|a_{jj}|} \ , j \neq i \ ; \quad m_i = \max_{j \neq i} \{m_{ij}\} \ , \ i \in N;$$

$$n_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_k}{|a_{jj}|} \ , j \neq i \ ; \quad n_i = \max_{j \neq i} \{n_{ij}\} \ , \ i \in N;$$

$$s_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| d_k}{|a_{jj}|} \ , j \neq i \ ; \quad s_i = \max_{j \neq i} \{s_{ij}\} \ , \ i \in N;$$

$$T_{ji} = \min\{m_{ji}, n_{ji}\} \ , \ j \neq i \ ; \qquad T_i = \max_{i \neq j} \{T_{ij}\} \ , \ i \in N.$$

From now on, we will continue this section by giving the useful definitions and the convenient lemmas and theorems proved so far in various references that will lead us to the main theorems of this section:

Definition 4.2.1 (Horn & Johnson, 1991) For $A = [a_{ij}]$ be an $n \times n$ matrix, we say that A is strictly row diagonally dominant if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \qquad for \;\; i = 1, \dots, n$$

and we say that A is strictly column diagonally dominant if A^T is strictly row diagonally dominant.

In following lemmas, we will be focused on the inequalities for the off-diagonal entries of the inverse of the given matrix using the entries of the given matrix and the diagonal entries of the given matrix. These lemmas are strategically important for the proofs of the theorems in section 4.2.2.

Lemma 4.2.1 (Li, Liu Yang & Li, 2011) Let $A = [a_{ij}] \in \mathbb{R}^{nxn}$.

a) If $A = [a_{ij}]$ is a strictly row diagonally dominant M-matrix, then $A^{-1} = [b_{ij}]$ satisfies

$$b_{ji} \leq \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_i}{a_{jj}} b_{ii}, \qquad i, j \in \mathbb{N}, \;\; i \neq j. \tag{4.9}$$

b) If $A = [a_{ij}]$ is a strictly column diagonally dominant M-matrix, then $A^{-1} = [b_{ij}]$ satisfies

$$b_{ij} \leq \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{kj}| c_i}{a_{jj}} b_{ii}, \qquad i, j \in \mathbb{N}, \;\; i \neq j. \tag{4.10}$$

Lemma 4.2.2 (Li, Liu Yang & Li, 2011) Let A= $[a_{ij}] \in \mathbb{R}^{nxn}$.

a) If $A = [a_{ij}]$ is a strictly row diagonally dominant M-matrix, then $A^{-1} = [b_{ij}]$ satisfies

$$b_{ji} \leq \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|r_k}{a_{jj}} b_{ii}\,, \qquad i,j \in \mathbb{N},\ \ i \neq j. \tag{4.11}$$

b) If $A = [a_{ij}]$ is a strictly column diagonally dominant M-matrix, then $A^{-1} = [b_{ij}]$ satisfies

$$b_{ij} \leq \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{kj}|c_k}{a_{jj}} b_{ii}\,, \qquad i,j \in \mathbb{N},\ \ i \neq j. \tag{4.12}$$

Although we stated both part a) and part b) of the previous lemmas, since in this section we will be dealing with not the column diagonally dominant matrices but the row diagonally dominant matrices, part a) of Lemma 4.2.1 and part a) of Lemma 4.2.2 will be the focus in proofs of the theorems in this section. Therefore, the critical point is to realize the fact that the inequality in Lemma 4.2.1a) is basically can be rewritten by using the equalities in the notations section as $b_{ji} \leq m_{ji}b_{ii}$ and the inequality in Lemma 4.2.1.2 a) is basically can be rewritten as $b_{ji} \leq n_{ji}b_{ii}$ where $i,j \in \mathbb{N},\ \ i \neq j$.

Lemma 4.2.3 (Li, Liu Yang & Li, 2011) If $A = [a_{ij}] \in \mathbb{R}^{nxn}$ is a strictly row diagonally dominant M-matrix, then $A^{-1} = [b_{ij}]$ satisfies

$$b_{ji} \leq T_{ji}b_{ii}\,, \qquad i,j \in \mathbb{N},\ \ i \neq j. \tag{4.13}$$

Lemma 4.2.4 (Li, Liu Yang & Li, 2011) If A= $[a_{ij}] \in \mathbb{R}^{nxn}$ is an M-matrix and $A^{-1} = [b_{ij}]$ is a doubly stochastic matrix, then

$$b_{ii} \geq \frac{1}{1 + \sum_{j\neq i} m_{ji}}\,, \ \ i \in \mathbb{N}. \tag{4.14}$$

Theorem 4.2.1 (Li, Liu Yang & Li, 2011) If $A = [a_{ij}] \in \mathbb{R}^{nxn}$ is an M-matrix, and $A^{-1} = [b_{ij}]$ is a doubly stochastic matrix, then

$$b_{ii} \geq \frac{1}{1 + \sum_{j \neq i} n_{ji}}\ ,\ \ i \in N;\ \ and\ \ b_{ii} \geq \frac{1}{1 + \sum_{j \neq i} T_{ji}}\ ,\ \ i \in N. \qquad (4.15)$$

Proof. We first prove $b_{ii} \geq \frac{1}{1+\sum_{j \neq i} n_{ji}}$, $i \in N$. Since $A^{-1} = [b_{ij}]$ is doubly stochastic, we know that $Ae = e,$ so A is strictly diagonally dominant matrix by row. By Lemma 4.2.2a, for $i \in N$;

$$1 = b_{ii} + \sum_{j \neq i} |b_{ji}|$$

$$\leq b_{ii} + \sum_{j \neq i} \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_k}{|a_{jj}|} b_{ii}$$

$$= \left(1 + \sum_{j \neq i} \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_k}{|a_{jj}|} \right) b_{ii}$$

$$= \left(1 + \sum_{j \neq i} n_{ji} \right) b_{ii}\ ,$$

i.e. ,

$$b_{ii} \geq \frac{1}{1 + \sum_{j \neq i} n_{ji}}\ .$$

Similarly, it can be proved that $b_{ii} \geq \frac{1}{1+\sum_{j \neq i} T_{ji}}$.

Lemma 4.2.5 (Li, Liu Yang & Li, 2011) Let A= $[a_{ij}] \in \mathbb{R}^{nxn}$.

a) If $A = [a_{ij}]$ is a strictly row diagonally dominant M-matrix, then $A^{-1} = [b_{ij}]$ satisfies

$$b_{ji} \leq \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| d_k}{a_{jj}} b_{ii}\ , \qquad i,j \in \mathbb{N},\ \ i \neq j. \qquad (4.16)$$

b) If $A = [a_{ij}]$ is a strictly column diagonally dominant M-matrix, then $A^{-1} = [b_{ij}]$ satisfies

$$b_{ij} \leq \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{kj}| \hat{c}_k}{a_{jj}} b_{ii}, \qquad i, j \in \mathbb{N}, \ i \neq j. \tag{4.17}$$

Lemma 4.2.6 (Li, Liu Yang & Li, 2011) If $A = [a_{ij}] \in \mathbb{R}^{nxn}$ is a strictly row diagonally dominant M-matrix, then $A^{-1} = [b_{ij}]$ satisfies

$$b_{ii} \leq \frac{1}{a_{ii}}, \ \ i \in \mathbb{N}. \tag{4.18}$$

Lemma 4.2.7 (Li, Liu Yang & Li, 2011) If A^{-1} is a doubly stochastic matrix, then $Ae = e, \ \ A^T e = e$ where $e = [1,1, \dots, 1]^T$.

Lemma 4.2.8 (Li, Liu Yang & Li, 2011) Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ and $x_1, x_2, \dots, x_n$ be positive real numbers. Then all the eigenvalues of A lie in the region

$$\bigcup_i \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \ x_i \sum_{j \neq i} \frac{1}{x_j} |a_{ji}|, \ \ i \in \mathbb{N} \right\}.$$

Lemma 4.2.9 (Li, Liu, Yang & Li, 2011) If P is an irreducible M-matrix, and $Pz \geq kz$ for a nonnegative vector z, then $\tau(P) \geq k$.

Lemma 4.2.10 (Li, Liu, Yang & Li, 2011) Let $A = [a_{ij}] \in \mathbb{R}^{nxn}$ is an M-matrix, then there exists a diagonal matrix D with positive diagonal entries such that $D^{-1}AD$ is a strictly row diagonally dominant M-matrix.

Lemma 4.2.11 (Li, Liu, Yang & Li, 2011) Let $A, B \in \mathbb{R}^{nxn}$, and suppose that $D \in \mathbb{R}^{nxn}$ and $E \in \mathbb{R}^{nxn}$ are diagonal matrices. Then

$$D(A \circ B)E = (DAE) \circ B = (AE) \circ (DB) = A \circ (\mathrm{DBE})$$

While we were investigating the studies for bounding the minimum eigenvalue of the Hadamard product of an *M*-matrix and its inverse carried out so far we have noticed that there were remarkable improvements in both developing the lower bounds and in the numerical solutions. Now, we will state these theorems to be able to see that improvement and to be close one more step to our main theorems. The first idea of inventing a lower bound for the minimum eigenvalue of the Hadamard product of an *M*-matrix and its inverse has been arisen from Fiedler and Markham in 1988 as in Theorem 4.2.2:

Theorem 4.2.2 (Fiedler & Markham, 1988) Let A be an $n \times n$ M-matrix. Then

$$\tau(A \circ A^{-1}) \geq \frac{1}{n}\,. \tag{4.19}$$

In years there have been various studies to prove the conjecture of Fiedler and Markham which is

$$\tau(A \circ A^{-1}) \geq \frac{2}{n} \tag{4.20}$$

and to improve these bounds to have a best approximation for $\tau(A \circ A^{-1})$. However, the most useful results have been achieved in the last few years. Now, we will state the theorems in order to be able to comprehend the improvements which we mentioned above:

Theorem 4.2.3 (Li, Huang, Shen & Li, 2007) If $A = [a_{ij}]$ is an M-matrix and $A^{-1} = [b_{ij}]$ is a doubly stochastic matrix, then

$$\tau(A \circ A^{-1}) \geq \min_i \left\{ \frac{a_{ii} - s_i R_i}{1 + \sum_{j \neq i} s_{ji}} \right\}. \tag{4.21}$$

Theorem 4.2.4 (Li, Chen & Wang, 2009) Let $A = [a_{ij}] \in \mathbb{R}^{nxn}$ be an M-matrix and let $A^{-1} = [b_{ij}]$ be doubly stochastic. Then

$$\tau(A \circ A^{-1}) \geq \min_i \left\{ \frac{a_{ii} - m_i R_i}{1 + \sum_{j \neq i} m_{ji}} \right\}. \tag{4.22}$$

Theorem 4.2.5 (Li, Liu, Yang & Li, 2011) Let $A = [a_{ij}] \in \mathbb{R}^{nxn}$ be an irreducible *M*-matrix and let $A^{-1} = [b_{ij}]$ be a doubly stochastic matrix. Then

$$\tau(A \circ A^{-1}) \geq \min_i \left\{ \frac{a_{ii} - T_i R_i}{1 + \sum_{j \neq i} T_{ji}} \right\}. \tag{4.23}$$

Theorem 4.2.6 (Li, Liu, Yang & Li, 2011) Let $A = [a_{ij}] \in \mathbb{R}^{nxn}$ be an irreducible *M*-matrix and let $A^{-1} = [b_{ij}]$ be a doubly stochastic matrix. Then

$$\tau(A \circ A^{-1}) \geq \min_i \left\{ \frac{a_{ii} - s_i \sum_{j \neq i} \frac{|a_{ji}| n_{ji}}{s_j}}{1 + \sum_{j \neq i} m_{ji}} \right\}. \tag{4.24}$$

In this section, so far we have dealed with the lower bounds for the minimum eigenvalue of the Hadamard product of an M-matrix and its inverse which are studied before to assist us with composing the main theorems of this section and the basic lemmas that will be very useful for us in proving these main theorems. We can refer to these theorems as Theorem 4.2.7 and Theorem 4.2.8, thereafter we can state and prove them as follows:

Theorem 4.2.7 Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be an irreducible *M*-matrix, and let $A^{-1} = [b_{ij}]$ be a doubly stochastic matrix. Then

$$\tau(A \circ A^{-1}) \geq \min_i \left\{ \frac{a_{ii} - T_i \sum_{j \neq i} \frac{|a_{ji}| n_{ji}}{T_j}}{1 + \sum_{j \neq i} T_{ji}} \right\}. \tag{4.25}$$

Proof. Since A^{-1} is a doubly stochastic, by Lemma 4.2.7, we have $Ae = e, \quad A^T e = e,$ so *A* is a strictly diagonally dominant *M*-matrix, and

$$a_{ii} = \sum_{k \neq i} |a_{ik}| + 1 = \sum_{k \neq i} |a_{ki}| + 1, \quad a_{ii} > 1 \qquad i \in N$$

and

$$r_{li} = \frac{|a_{li}|}{|a_{ll}| - \sum_{k \neq l,i} |a_{lk}|} < 1, \quad l \neq i\ ;$$

Therefore $r_i = \max_{l \neq i}\{r_{li}\} < 1, \quad i \in N$. Let

$$R_j^r = \sum_{k \neq j} |a_{jk}| r_i \ , \quad j \neq i \ , \quad i \in N.$$

Then, for any $j \in N$ with $j \neq i$, we have

$$R_j^r = \sum_{k \neq j} |a_{jk}| r_i \leq |a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_i \leq R_j = \sum_{k \neq j} |a_{jk}| \leq a_{jj}.$$

Therefore, there exists a real number α_{ji} $\left(0 \leq \alpha_{ji} \leq 1\right)$, such that

$$|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_i = \alpha_{ji} R_j + \left(1 - \alpha_{ji}\right) R_j^r \ .$$

Thus

$$m_{ji} = \frac{\alpha_{ji} R_j + \left(1 - \alpha_{ji}\right) R_j^r}{a_{jj}} \ .$$

It can be easily seen that $0 < \alpha_j \leq 1$ (if $\alpha_j = 0$, then A is reducible, which is a contradiction). Since A is irreducible, then $R_j > 0$, $R_j^r > 0$ and $0 < m_{ji} \leq 1$. In the same way, it can be proved that $0 < n_{ji} \leq 1$. So, from the definition of $T_{ji} = \min \ \{m_{ji}, n_{ji}\}$, $j \neq i$; we have also $0 < T_{ji} \leq 1$ .Therefore, $0 < T_j \leq 1$, where $T_j = \max_{i \neq j}\{T_{ji}\}$, $j \in N$.

Let $\tau(A \circ A^{-1}) = \lambda$. By Lemma 4.2.8, there exists $i_0 \in N$, such that

$$\left|\lambda - a_{i_0 i_0} b_{i_0 i_0}\right| \leq T_{i_0} \sum_{j \neq i_0} \frac{1}{T_j} \left|a_{j i_0} b_{j i_0}\right|$$

$$|\lambda| \geq a_{i_0 i_0} b_{i_0 i_0} - T_{i_0} \sum_{j \neq i_0} \frac{1}{T_j} \left|a_{j i_0} b_{j i_0}\right|$$

$$\geq a_{i_0 i_0} b_{i_0 i_0} - T_{i_0} \sum_{j \neq i_0} \frac{1}{T_j} \left|a_{j i_0}\right| \frac{\left|a_{j i_0}\right| + \sum_{k \neq j, i_0} |a_{jk}| r_k}{a_{jj}} b_{i_0 i_0}$$

$$= \left(a_{i_0 i_0} - T_{i_0} \sum_{j \neq i_0} \frac{1}{T_j} |a_{j i_0}| \, n_{j i_0} \right) b_{i_0 i_0}$$

$$\geq \frac{a_{i_0 i_0} - T_{i_0} \sum_{j \neq i_0} \frac{1}{T_j} |a_{j i_0}| \, n_{j i_0}}{1 + \sum_{j \neq i_0} T_{j i_0}} \; (by\, Theorem\, 4.2.1)$$

$$\geq \min_i \left\{ \frac{a_{ii} - T_i \sum_{j \neq i} \frac{|a_{ji}| n_{ji}}{T_j}}{1 + \sum_{j \neq i} T_{ji}} \right\}. \blacksquare$$

Theorem 4.2.8 Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be an irreducible *M*-matrix, and let $A^{-1} = [b_{ij}]$ be a doubly stochastic matrix. Then

$$\tau(A \circ A^{-1}) \geq \min_i \left\{ \frac{a_{ii} - s_i \sum_{j \neq i} \frac{|a_{ji}| n_{ji}}{s_j}}{1 + \sum_{j \neq i} T_{ji}} \right\}. \quad (4.26)$$

Proof. Since A^{-1} is a doubly stochastic, by Lemma 4.2.7, we have $Ae = e, \;\; A^T e = e,$ so *A* is a strictly diagonally dominant *M*-matrix, and

$$a_{ii} = \sum_{k \neq i} |a_{ik}| + 1 = \sum_{k \neq i} |a_{ki}| + 1, \;\; a_{ii} > 1$$

and

$$d_i = \frac{\sum_{k \neq i} |a_{ik}|}{|a_{ii}|} < 1, \qquad i \in N.$$

For convenience, we denote

$$\tilde{R}_j = \sum_{k \neq j} |a_{jk}| d_k \,, \qquad j \in N.$$

Then, for any $j \in N$ with $j \neq i$, we have

$$\tilde{R}_j \le |a_{ji}| + \sum_{k \ne j,i} |a_{jk}| d_k \le R_j = \sum_{k \ne j} |a_{jk}| \le a_{jj}.$$

Therefore, there exists a real number α_{ji} $(0 \le \alpha_{ji} \le 1)$, such that

$$|a_{ji}| + \sum_{k \ne j,i} |a_{jk}| d_k = \alpha_{ji} R_j + (1 - \alpha_{ji}) \tilde{R}_j .$$

Let $\alpha_j = \max_{i \ne j}\{\alpha_{ji}\}$. Then $0 < \alpha_j \le 1$, (if $\alpha_j = 0$, then A is reducible, which is a contradiction). So, from the definition of s_{ij}, we have

$$s_j = \max_{i \ne j}\{s_{ji}\} = \frac{\alpha_j R_j + (1 - \alpha_j) \tilde{R}_j}{a_{jj}} \; , \;\; j \in N.$$

Since $0 < \alpha_j \le 1$, we get $0 < s_j \le 1$.

Let $\tau(A \circ A^{-1}) = \lambda$.By Lemma 4.2.8, there exists $i_0 \in N$, such that

$$|\lambda - a_{i_0 i_0} b_{i_0 i_0}| \le s_{i_0} \sum_{j \ne i_0} \frac{1}{s_j} |a_{j i_0} b_{j i_0}|$$

$$|\lambda| \ge a_{i_0 i_0} b_{i_0 i_0} - s_{i_0} \sum_{j \ne i_0} \frac{1}{s_j} |a_{j i_0} b_{j i_0}|$$

$$\ge \; a_{i_0 i_0} b_{i_0 i_0} - s_{i_0} \sum_{j \ne i_0} \frac{1}{s_j} |a_{j i_0}| \frac{|a_{j i_0}| + \sum_{k \ne j, i_0} |a_{jk}| r_k}{a_{jj}} b_{i_0 i_0}$$

$$= \left(a_{i_0 i_0} - s_{i_0} \sum_{j \ne i_0} \frac{1}{s_j} |a_{j i_0}| \, n_{j i_0} \right) b_{i_0 i_0}$$

$$\ge \frac{a_{i_0 i_0} - s_{i_0} \sum_{j \ne i_0} \frac{1}{s_j} |a_{j i_0}| \, n_{j i_0}}{1 + \sum_{j \ne i_0} \mathrm{T}_{j i_0}} \; (by \;\; Theorem \; 4.2.1)$$

$$\geq \min_i \left\{ \frac{a_{ii} - s_i \sum_{j\neq i} \frac{|a_{ji}| n_{ji}}{s_j}}{1 + \sum_{j\neq i} T_{ji}} \right\}. \quad \blacksquare$$

Example 4.2.1

We will give the most known example of the irreducible *M*-matrix with a doubly stochastic inverse and we will try to see the numerical results by applying the theorems above in order:

$$A = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}$$

As a result of Theorem 4.2.2, we get

$$\tau(A \circ A^{-1}) \geq 0.25;$$

Applying Theorem 4.2.3, we have

$$\tau(A \circ A^{-1}) \geq 0.6624;$$

Applying Theorem 4.2.4, we have

$$\tau(A \circ A^{-1}) \geq 0.7999;$$

If we apply Theorem 4.2.5, we have

$$\tau(A \circ A^{-1}) \geq 0.85;$$

If we apply Theorem 4.2.6, we have

$$\tau(A \circ A^{-1}) \geq 0.8602.$$

But, as a result of Theorem 4.2.7, we have

$$\tau(A \circ A^{-1}) \geq 0.8622;$$

and, finally, as a result of Theorem 4.2.8, we have

$$\tau(A \circ A^{-1}) \geq 0.9098.$$

In fact, $\tau(A \circ A^{-1}) = 0.9755$.

In conclusion, it can be easily seen that the results of Theorem 4.2.6 and Theorem 4.2.7 are better lower bounds than the results of theorems 4.2.2 and also Theorem 3.2 is the best approximation to the exact value of the minimum eigenvalue of $A \circ A^{-1}$.

CHAPTER FIVE

CONCLUSION

In this thesis, we tried to characterize the eigenvalues of a given matrix. For this purpose, we have studied the articles on the bounds for the spectral radius and the minimum eigenvalue of some special matrices. Firstly, we have investigated the bounds for the spectral radius of the Hadamard product of two matrices and then we have tried to obtain bounds for the spectral radius and for the norm of the Fan product of two matrices. Secondly, we have tried to find both lower and upper bounds for the minimum eigenvalue of the Jacobi iterative matrix of an irreducible *M*-matrix in the light of the bounds for the spectral radius of the Jacobi iterative matrix. Finally, after examining the results for bounding the minimum eigenvalue of the Hadamard product of an *M*-matrix and its inverse, we have achieved to produce two new lower bounds for the minimum eigenvalue of the Hadamard product of an *M*-matrix and its inverse by improving the existing results.

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