SALES AND RETURNS FORECASTING FOR INVENTORY CONTROL

OKTAY KARABAĞ

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SALES AND RETURNS FORECASTING FOR INVENTORY CONTROL

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(Prof. Dr. Murat Aşkar)
Director

I certify that this thesis satisfies all the requirements for the degree of **Master of Science** in **Intelligent Production Systems** option of **Intelligent Engineering Systems**.

(Assoc. Prof. Dr. Arslan Örnek)
Head of Department

We have read the thesis entitled **Sales and Returns Forecasting for Inventory Control** prepared by **Oktay KARABAĞ** under supervision of **Assoc. Prof. Dr. Deniz TÜRSEL ELİİYİ** and **Assoc. Prof. Dr. Murat FADILOĞLU**, and we hereby agree that it is fully adequate, in scope and in quality, as a thesis for the degree of **Master of Science** in **Intelligent Production Systems** option of **Intelligent Engineering Systems**.

(Assoc. Prof. Dr. Deniz Türsel Eliiyi)
Advisor

(Assoc. Prof. Dr. Murat Fadıloğlu)
Co-Advisor

Examining Committee Members:
(Chairman, Supervisor and Members)

Asst. Prof. Dr. Zeynep SARGUT
Industrial Engineering Dept., IUE

Assoc. Prof. Dr. Deniz TÜRSEL ELİİYİ
Industrial Engineering Dept., IUE

Assoc. Prof. Dr. Murat FADILOĞLU
Industrial Engineering Dept., Yaşar Uni.

Assoc. Prof. Dr. Arslan ÖRNEK
Industrial Engineering Dept., IUE

Asst. Prof. Dr. Erdinç ÖNER
Industrial Engineering Dept., IUE

ABSTRACT

SALES AND RETURNS FORECASTING FOR INVENTORY CONTROL

Karabağ, Oktay

M.Sc. in Intelligent Engineering Systems
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Advisor: Assoc. Prof. Dr. Deniz Türsel Eliiyi
Co-Advisor: Assoc. Prof. Dr. Murat Fadıloğlu

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Nowadays, environmental concerns are forcing the manufacturers, which use conventional manufacturing systems, to adopt remanufacturing practices. When a company collects their own used products from their customers and reprocesses them, it is called remanufacturing. In the remanufacturing systems, the demand and returns forecasting is indispensable for the management activities such as procurement decisions, production planning and inventory management, etc. This study aims to review the current literature and to develop new forecasting methods for demand and returns. Firstly, the Holt and Winters method, which is well-known and easy to apply for the demand forecasting, is investigated. However, this method is not sufficient to capture demand pattern when the simultaneous effects of two different asynchronous calendars, such as Gregorian and Islamic (Hijri), manifest themselves on a specific market. For this reason, the Holt–Winters method is extended by considering the seasonalities that are due to two asynchronous calendars; and the extended new method is called as the Augmented Holt–Winters method. Secondly, the methods, which are developed by Kelle and Silver for the return forecasting, are examined and rewritten more clearly. The methods of Kelle and Silver are not appropriate for the non-stationary demand environment, since they proposed for a stationary demand. In order to address this issue, their return forecasting methods are revised to allow the non-stationary demand. Finally, we compare the developed new forecasting methods for demand and returns with previous ones by using the real life time series data. The obtained results indicate that better forecasts can be achieved using the proposed new methods for the demand and returns forecasting.

Keywords: Holt–Winters Method, Kelle and Silver Methods, Returns forecasting, Demand forecasting, Return-time distribution

ÖZ

ENVANTER KONTROLÜNDE SATIŞ VE GERİ DÖNÜŞ TAHMİNİ

Karabağ, Oktay

Akıllı Mühendislik Sistemleri Yüksek Lisans Programı
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Tez Danışmanı: Doç. Dr. Deniz Türsel Eliiyi
Ortak Tez Danışmanı: Doç. Dr. Murat Fadıloğlu

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Gelişen çevre duyarlılığı, geleneksel üretim sistemlerini kullanan üreticileri yeni stratejiler benimsemeye zorlamaktadır. Piyasaya verilen ürünlerin kullanımlarından sonra toplanıp, işlenerek tekrar tüketiciye ulaştırılmasına, yeniden üretim (remanufacturing) ismi verilmektedir. Yeniden üretim sistemlerinde talep ve geri dönüş tahmini, satın alma kararları, üretim planlama ve envanter yönetimi vb. gibi yönetim konuları için vazgeçilemez öğelerdir. Bu çalışma, talep ve geri dönüş tahmini için yeni metotlar geliştirmeyi ve mevcut literatürü incelemeyi amaçlamaktadır. İlk olarak, talep tahmini için en bilindik ve uygulaması kolay olan "Holt–Winters" metodu incelenmiştir. Ancak, Gregoryen ve Hicri takvim gibi iki asenkron takvimin ortak etkisi belirli bir pazarda kendini gösterdiğinde, bu metodun talep eğilimini yakalamakta yeterli olmadığı gerçeği fark edilmiştir. Bu nedenle, "Holt–Winters" metodu iki asenkron takvim nedeniyle oluşan sezonluk etkiler dikkate alınarak geliştirilmiştir ve oluşturulan bu yeni yöntem "Augmented Holt–Winters" metodu olarak adlandırılmıştır. İkinci olarak, geri dönüş tahmini için Kelle ve Silver tarafından geliştirilen metotlar incelenmiş ve farklı bir bakış açısıyla yeniden sunulmuşlardır. Kelle ve Silver'a ait bu yöntemler durağan talep şartıyla sınırlandırıldıklarından, talebin durağan olmadığı durumlarda etkili olmayacaktır. Bu sorunu gidermek için, ilgili yöntemler durağan olmayan talebe izin veren daha gerçekçi bir bakış açısı ile revize edilmiştir. Son olarak, talep ve geri dönüşler için geliştirilen yeni tahmin yöntemleri önceki halleri ile gerçek zaman serisi kullanılarak karşılaştırılmıştır. Elde edilen sonuçlar, talep ve geri dönüş tahmini için önerilen yeni metotlar kullanıldığında önemli iyileştirmeler sağlanabileceğini göstermiştir.

Anahtar Kelimeler: Holt–Winters Metodu, Kelle - Silver Metodu, Geri dönüş tahmini, Talep tahmini, Geri dönüş süresi dağılımı

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CHAPTER - 1

INTRODUCTION

In today's business world, firms are becoming ever more concerned to environmental issues due to pressures from society and government regulations. They also know that more environmentally conscious strategies and practices can bring additional benefits in terms of cost savings and the market opportunities. Companies can save money by reducing the amount of raw material and energy consumption. Also, offering environmental friendly products may result in increased market share as well as an improved firm image. To achieve these tasks in a competitive and global setting, firms need to devise strategies that incorporate environmental concerns. This environmental awareness in the management perspective has a direct impact on their production processes in various contexts such as waste management, recycling and remanufacturing options, selection of environment friendly and reusable materials etc.

All these issues are an integral part of product recovery management (PRM) literature. Product recovery management is the collective noun for a broad area of managerial problems such as product design, production planning, inventory control, logistics, information systems and marketing. In the work of Thierry et al. (1995), PRM is defined as the management of all used and discarded products that are produced by a manufacturing company. The main purpose is to minimize the amount of disposal and maximize the amount of the recycled or reused products (Gupta and Gungor, 1998).

Recovered products can be utilized as a valuable source of raw or semi-finished material. This provides an additional channel to replenish raw/semi-finished material inventories. Hence, such systems have two channels for inventory replenishment: procurement and recovery. Controlling and operating the inventory levels in such systems is more complex since return flows that has directly impact on the inventory level are inherently uncertain. So, an efficient inventory management of stock keeping units (SKUs) under the recovering policies depends on the accurate forecasts of both demand and return quantities. This work contributes to these research areas by proposing two new methodologies for the demand and return forecasting.

Demand forecasting is one of the key drivers of every planning decision made under the both traditional and PRM circumstances. A various forecasting methods for the demand patterns are developed by some authors. Holt (1957) and Winters (1960) methods are most well-known methodologies in this literature. However, these forecasting methods are not sufficient to capture demand patterns when the two different seasonality effects, such as climate and cultural, manifest themselves on the specific market. In this thesis, the Holt and Winters works are re-examined and are extended through the addition of seasonalities that composed due to asynchronous calendars effects.

Developing an accurate return forecasting method is another imperative task for the inventory management under the recovering policies. The key to forecasting returns is the observation that returns in any one period are generated by demand in the preceding periods (Clottey et. al, 2012). In the Kelle and Silver (1989) works', four prominent forecasting methods are proposed by taking into account this point of view. However, their methods are limited to settings with stationary demand. In this work, we assume a more realistic perspective by allowing the non-stationary demand.

The rest of this work is organized as follows. Chapter 2 provides a review of the literature on the theory and methodology for inventory control with return flows and positions our contribution. Chapter 3 presents Augmented Holt-Winters' methods we propose for the demand forecasting and provides a guideline for the choice of smoothing parameters, starting values and estimation procedures. Chapter 4 introduces the four different forecasting methods proposed by Kelle and Silver for return forecasting and indicates how these methods can be extended to use in dynamic environment. In Chapter 5, we provide a case study of an inventory control system with return flows as it is used in a major brewery in Turkey. The next chapter, Chapter 6, summarizes the results and findings of this thesis.

CHAPTER - 2

LITERATURE REVIEW

In this chapter, we provide a literature review that covers the theory and methodology on inventory control with return flows. This literature review can be divided into three main categories: (1) Forecasting Methods (2) Return Forecasting (3) Inventory Controls under Returns.

2.1. Forecasting Methods

With the increasing uncertainty and rapid change in today's business environment, predicting future is the key to the survival of companies. This area has been widely researched, resulting in varied techniques and methods for predict on of future. Both time-series (Holt, Winters, Box&Jenkins etc.) and casual methods (Regression, Econometric methods etc.), as well as newer methods such as neural networks and generic algorithms are suggested by the literature. These techniques and models are utilized in different settings with different prerequisites. By taking into consideration all of the above mentioned aspects; the most relevant articles for our case are examined in the following paragraphs.

Holt (1957) extends simple exponential smoothing to double exponential smoothing by introducing a new component, linear trend factor. This method is similar to simple smoothing method except for the presence of the basic and trend components, which must be updated in each period. If there is a trend, Holt's Method is more suitable since it combines the concept of exponential smoothing with the ability to track linear trends in the data. The forecast in Holt's Method is found using two smoothing constants and equations. One of these equations provides exponential smoothing for the current level. The other one estimates the rate of growth at the end of each period.

Winter (1960) defines and discusses three exponentially weighted forecasting methods frame theoretical and practical perspectives. All three forecasting methods are used in his paper to predict three different sales time series. One of these methods is the triple (complete) exponential method; the other two are a naive method and a simple forecasting method previously used. In the triple exponential smoothing method, a third equation is added to the equation system of the Holt's Method and that captured the seasonal dynamics of the time-series data. The method that incorporates these three equations is called Holt & Winters Method. This method is applicable to data with linear trend and seasonal components. The paper also provides comparison of the performance of three methodologies on the three time series. These comparisons show that exponential method provides more accurate forecasts, require less information storage while it requires slightly more computation time than the two conventional methods.

Newbold and Granger (1974) compare the forecasting performance of the Box-Jenkins, Holt-Winters and Stepwise Regression methods using seven time-series data. Their results show that each method possesses some advantages over the others. The authors conclude that the Box-Jenkins method gives the better forecasts in the short run while it requires more computation time and skill to compute. Their results indicate that for the time series data with less than 30 observations, stepwise regression is better than the others. For data 40-50 observations, a combination of Holt-Winters and stepwise regression are applicable. For series of 50 observations the Box-Jenkins model performs well. Authors note that for the data with strong seasonal fluctuation and long fluctuations the Holt Winters method should not be used.

Chatfield (1978) reanalyzes seven series from the Newbold-Granger study for which Box-Jenkins forecasts were reported to be much superior to the (automatic) Holt-Winters forecasts. Chatfield indicates that the automatic Holt-Winters forecasts can often be improved by subjective adjustments. The author argues that a fairer comparison would be that between Box-Jenkins and a non-automatic version of Holt-Winters. Some general recommendations are made concerning the choice of a univariate forecasting procedure. The paper also makes suggestions regarding the implementation of the Holt-Winters procedure, including a choice of starting values.

Chatfield and Yar (1988) discuss the practical problems in implementing the Holts & Winters Method; including the normalization of seasonal indices, the choice of starting values and the choice of smoothing parameters. The authors point out that there is a substantial distinction between an automatic and a non-automatic approach to forecasting and detailed suggestions are made for implementing Holt-Winters in both ways.

2.2. Return Forecasting

The key issue in the return forecasting literature is the development of techniques capable of handling both quantity and timing uncertainty. Forecasting models designed to predict the availability of returns are needed to reduce some of the uncertainty (V.D.R. Guide Jr., 1999). The following papers have helped to fill this research gap.

Goh and Varaprasad (1986) develop a transfer function model for forecasting the timing and quantity of future returns and apply it to data on soft drinks as Coca-Cola and Fanta from Southeast Asia market. Using only the series of aggregated demand and aggregated return data, they obtain important parameters of container life cycle such as trip duration, trippage, loss rate etc. The precision of this method depends on the quality of the estimation of the return distribution. This estimation is based on Box and Jenkins time series techniques. Consequently, a series with less than 50 points is too short to employ this methodology. However if a series covers too long a period, patterns of usage and returns may undergo changes, resulting in estimation problems for method parameters. Hence, a data set with 50 points, or a period of four years from a stable marketing region is recommended.

Kelle and Silver (1989) describe four forecasting procedures to estimate the returns and net demand during a given lead time in the case of reusable SKUs. The paper assumes stationary demand and return distributions. Each of these methods requires a different level of information. The method (Method-1) with the least information requirement utilizes the expectation and variance of the net demand together with the return probability. The method (Method-3) with the highest information requirement necessitates individual tracing and tracking of SKUs. To evaluate the forecasting procedures, Method-3 is used as a benchmark.

Overall, one can easily conclude that the proposed procedures are all necessitate reliable data. When reliable data is available, the most sophisticated method leads to the superior performance. In the following papers, authors take a more realistic perspective of misinformation and analyze its impact.

De Brito and van der Laan (2007) investigate the performance of the forecasting methods discussed in Kelle and Silver (1989) by considering the impact of imperfect information on inventory costs. They indicate that in the presence of imperfect information, the most sophisticated method does not necessarily lead to best forecasting performance.

Ketzenberg et al. (2006) explore the impact of information regarding demand, return flows and yield on the operating cost of a reverse supply chain by comparing a number of alternative models, which differ in the type of information (full information and partial information) available before the purchasing decisions are made. The authors report that there is no dominance in value amongst the different types of information, when economic aspects of the required investment are considered.

Karaer and Lee (2007) analyze the inventory decisions of a manufacturer who has ample production capacity and who also uses returned products to satisfy customer demand. The authors discuss the value of information on the reverse channel for the manufacturer by making comparisons among three approaches: No information/naive; no visibility/enlightened; and full visibility. They find the value of visibility increases with the comparative length of the reverse channel and volume, volatility, and usability of returns. Via this analysis, the authors also quantify the visibility savings of RFID (Radio Frequency Identification) in the reverse channel as a candidate enabler technology. And, in their paper, numerical examples demonstrate that full visibility does not bring lots of savings.

2.3. Inventory Control Under Returns

Inventory Control in recycling and remanufacturing environments cannot be handled with techniques devised for conventional manufacturing systems. These environments are inherently more difficult to control. The complexity in a remanufacturing system is explained in the following list given by Gupta and Gungor (1998):

– a high degree of uncertainty in material planning due to the probabilistic recovery rates of parts,

– stochastic routings and lead times due to unknown conditions of the recovered parts until inspected,

– the part matching problem,

– the complexity of a remanufacturing shop structure,

– the problem of imperfect correlation between supply of cores and demand for remanufactured units,

– and uncertainties in the quantity and timing of returned products.

Inventory control models for recycling and remanufacturing environments have to keep track of such things as returned products, the quantity of remanufactured products as well as new products, arriving time etc. Due to a high degree of uncertainty in the quantity and quality of retired products, arrival times and demand occurrences for the remanufactured parts and products these models, are further sophisticated (Gupta and Gungor, 1998). The rest of this section discusses techniques devised to handle these complexities.

Buchanan and Abad (1998) formulate a periodic review model of a returnable system. The authors create a model similar to that of Kelle and Silver (1989). The returns are assumed to be a random fraction of the number of units in the field. Also the time of product presence on the market is described by the exponential distribution. They derive the structure of the optimal policy and enhance a procedure for determining the optimal policy. Also the authors noted that their model can be used to determine the minimum cost plan for the finite horizon.

Toktay et al. (2000) also examine the value of information relating return flows through two informational structures and the five different inventory control policies. They use a closed queuing network model to decide on periodic ordering decisions. Their aim is to find an ordering policy to minimize the total expected procurement inventory holding and lost sales costs in remanufacturing system. They explore the data on returns of Kodak single-use cameras and model the time to return with a discrete time distributed-lag model using geometric and pascal distributions. A Bayesian approach and the Expectation Maximization (EM) algorithm are used to estimate the probability that a product will be returned and the probability that a sold item is returned in the next period given that it will be returned.

Kiesmüller and van der Laan (2001) develop and investigate an inventory model for a single, reusable product in which the random returns correlate explicitly on the demand stream. In their paper, under the assumption that demands and returns are assumed to be poisson processes, the influence of return probability and the length of the planning horizon on policy performance are examined. Due to these assumptions, the model distinguishes itself from most other publications in this field. By benchmarking the performance on the total average costs with the models neglecting the dependency, the authors report that it is significant to use the dependency information between demand and returns.

M. Fleischmann et al. (2001) present a model extending a traditional single item poisson-demand inventory model with a poisson return-flow of items. For this model, the authors derive an optimal control policy and point out how to determine optimal values of the control parameters. In particular, comparison with traditional *(s,Q)* inventory model is central throughout the numerical analysis. Also, the impact of the return ratio on system costs and on relative performance of alternative order policies quantitatively is discussed in their study.

De Brito and Dekker (2003) discuss that the validity of the simplified assumptions in the literature on inventory modeling with product returns. These simplified assumptions are listed as follows: (1) The demand and the return flows are a homogeneous Poisson Process, (2) demand and returns are independent from each other. In their work, the empirical validity of these two assumptions is examined. They used the framework with three sets of real data and conclude that there is the need to adjust these assumptions to develop models suitable for non-stationary real situations, such as the seasonality of particular products.

Mostard and Teunter (2004) analyze the newsvendor problem where undamaged returned products from customers during the selling season are still resalable before the season ends. They derive a simple closed form news-vendor equation that determines the optimal order quantity given the demand distribution, the probability that a sold product is returned, and all of the relevant revenues and costs. And the use of their model is illustrated by real data from a Dutch mail order company.

Galbreth and Blackburn (2009) enhance models that capture the trade-off between acquisition costs and remanufacturing costs when the used product condition is uncertain. They determine acquisition quantities that minimize the total expected costs for the different conditions of returned product (continuous and discrete) and remanufacturing cost functions (linear and non-linear). The authors use real data from a phone remanufacturer. For each case, they point out that a unique optimal acquisition quantity exists and can be found via a simple computational procedure.

Clottey et al. (2012) consider an electronics original equipment manufacturer which also acts as a remanufacturer. The authors develop a generalized forecasting approach to determine the distribution of the returns of used products, as well as integrate it with an inventory model to enable production planning and control. Also they compare their approach to previous models and indicate that their approach is more consistent with continuous time, provides accurate estimates when the return lags are exponential in nature, and results in fewer units being held in inventory on average. The analysis reveals that these gains in accuracy resulted in the most cost savings when demand volumes for remanufactured products are high compared to the volume of returned products.

This thesis contributes to the literature in two dimensions. First, it extends the work of Winter (1960) on demand forecasting by allowing joint incorporation of seasonalities due to asynchronous calendars. We believe this contribution could be of use in many forecasting situations where demand is shaped by the climate seasonality as well as cultural seasonality for a specific market. Second, the four methods proposed by Kelle and Silver (1989) for forecasting returns are adapted to a dynamic environment where average demand is not stationary. We also complement these methods with a new linear regression approach to obtain the return lag distribution needed for return forecasting.

CHAPTER - 3

FORECASTING METHODS

Silver et al (2004) reports that statistically forecasting process consists of four basic steps:

1. An appropriate underlying model of the data pattern through time is selected.
2. The values for the parameters inherent in the model are selected.
3. The model having been identified and the parameters estimated, diagnostic checks are then applied to the fitted model.
4. The model selected in Step-1 and the parameters values chosen in Step-2 are used to forecast the future data.

To select an appropriate underlying model for time-series, various statistical and graphic techniques can be used. The best place to start with any time-series forecasting analysis is to graph the line-graph of the time-series to be forecasted. The purpose of this plot is to extract the information embedded in the time-series data. Based on the plot the analyst should decide whether there are certain behavioral components present within the observed data. The presence or absence of such components can help the analyst in selecting the model with the potential to produce the good forecasts.

Silver et al (2004) argues that any time series is composed of five components:

level, trend, seasonal variations, cyclical movements and random fluctuations. The scale of a time series is captured by the level. The rate of growth or decline of a series over time is identified by the trend. There are two different types of seasonal variations due to the natural forces and due to human decisions or customs. Cyclical variations between expansion and contraction of economic activity are the result of business cycles. The residual that remain after the effects of the other four components are identified and removed from the time series is called as random fluctuations.

After observing the behavior of the series, the next step is the selection of an appropriate model. The process of specifying a forecasting model involves selection of the variables to be included, selection of the functional form and estimation of the parameters. In the following sections, we describe and list the three smoothing methods that are effective under different circumstances. One of these methods, Holts' method, is used when the time-series data has significant linear-trend and level components. The other one, Holts-Winters' method, is effective when linear-trend, level and seasonal effects manifest themselves in the time-series data. Finally, we propose a new addition to the available two methods that we name as Augmented Holt-Winters' method. This new method extends the Holts-Winters' method by allowing joint incorporation of seasonalities due to asynchronous calendars.

3.1. Holt's Method

When the time-series data exhibits either an increasing or decreasing the linear-trend overtime, the application of Holt's method is indicated. This forecasting method has two updating equations, one for the level and one for the linear-trend. And these two equations make use of individual smoothing factors, α and β.

The forecasting procedure of Holt's method can be outlined as follows:

The level estimate:

$$S_t = \alpha\left(D_t\right) + (1-\alpha)(S_{t-1} - G_{t-1}) \quad (3.1)$$

The trend estimate:

$$G_t = \beta\left(S_t - S_{t-1}\right) + (1-\beta)G_{t-1} \tag{3.2}$$

The forecast for τ periods ahead:

$$F_{t,t+\tau} = (S_t + G_t\tau) \tag{3.3}$$

where,

S_t : *Level estimate*
α : *Smoothing factor for the level* $(0 \le \alpha \le 1)$
D_t : *Actual volume in period t*
β : *Smoothing factor for the trend* $(0 \le \beta \le 1)$
G_t : *Trend estimate*
τ : *Number of periods to be forecasted ahead*
$F_{t,t+\tau}$: *Forecasted volume for* τ *periods ahead*

In general, smoothing constants of this method are selected by either subjectively or the trial-and-error approach with one of the measure of forecast error such as mean square error (MSE). To initialize Holt's method, an initial estimation of level and slope are also required. Intuitively, the first values of the time-series data can be used in place of the initial values. But recommended approach is to establish some set of initial periods as a baseline and use the linear regression analysis to determine estimates of trend and level using the baseline data. (Steven Nahmias, 2004).

3.2. Holt and Winters' Method

When data exhibits seasonal fluctuations, typical forecasting methods such as exponential smoothing, Holt's method will not be able to track the series. To deal with seasonality, Winters developed an elaboration on the Holt's method. More details can be found in (Winters, 1960) and (Holt, 1957). This forecasting method accommodates level, trend and seasonality. And these three components are updated via the use of their individual smoothing factors. Holt-Winters' method is a type of triple exponential smoothing, and has the important advantage of being easy to update as new data become available (Steven Nahmias, 2004).

The four equations that define Holt-Winters' method are:

The level estimate:

$$S_t = \alpha\left(D_t / c_{t-N}\right) + (1-\alpha)(S_{t-1} - G_{t-1}) \tag{3.4}$$

The trend estimate:

$$G_t = \beta\left(S_t - S_{t-1}\right) + (1-\beta)G_{t-1} \tag{3.5}$$

The seasonal estimate:

$$c_t = \gamma(D_t / S_t) + (1-\gamma)c_{t-N} \tag{3.6}$$

The forecast for τ periods ahead:

$$F_{t,t+\tau} = (S_t + G_t\tau)c_{t+\tau-N} \tag{3.7}$$

where,

S_t : *Level estimate*
α : *Smoothing factor for the level* $(0 \le \alpha \le 1)$
D_t : *Actual volume in period t*
β : *Smoothing factor for the trend* $(0 \le \beta \le 1)$
G_t : *Trend estimate*
γ : *Smoothing factor for the seasonality estimate* $(0 \le \gamma \le 1)$
c_t : *Seasonality estimate*
τ : *Number of periods to be forecasted ahead*
N : *Seasonality length*
$F_{t,t+\tau}$: *Forecasted volume for* τ *periods ahead*

The smoothing factors which are denoted by *α, β* and *γ* is usually determined by minimizing one of the error measures mean square error (MSE), mean absolute deviation (MAD) or mean absolute percentage error (MAPE). To initialize the forecasts, we need to set the initial values for level, trend and seasonal factors. To this end, several different initialization procedures have been suggested in literature, but we restrict our attention to the original suggestion of Winters. The procedure requires a minimum of two seasons of observations for initialization. The procedure given in Nahmias (2004) is summarized below.

1. Calculate the sample means for the two separate seasons of data with

$$V_1 = \frac{1}{N} \sum_{j=-2N+1}^{-N} D_j \tag{3.8}$$

$$V_2 = \frac{1}{N} \sum_{j=-N+1}^{0} D_j \tag{3.9}$$

where the present period is $t=0$, and the past observations are identified by negative indices.

2. Estimate the initial slope (G_o) and the level of the series (S_o) with

$$G_0 = (V_2 - V_1) / N \tag{3.10}$$

$$S_0 = V_2 + G_0 \left[\left(N-1 \right) / 2 \right] \tag{3.11}$$

3. Compute the seasonal factors. The initial seasonal indices are estimated for each initialization period and then averaged to obtain better estimators for seasonal factors. These estimates are

$$c_t = \frac{D_t}{V_i - \left[\left(N+1 \right) / 2 - j \right] G_0} \quad , \quad for \;\; -2N+1 \le t \le 0 \;,\; i = 1,2 \tag{3.12}$$

$$c_{-N+1} = \frac{c_{-2N+1} + c_{-N+1}}{2}, \ldots, c_0 = \frac{c_{-N} + c_0}{2} \tag{3.13}$$

where i is the order of season and j is the period of the season.

4. Finally, the initial seasonal factors are normalized so that their average is 1.

$$\textit{Normalizing procedure}\ :\ c_j = \left[\frac{c_j}{\sum_{i=0}^{-N+1} c_i} \right] N\ ,\ \textit{for}\ \ -N+1 \le j \le 0 \qquad (3.14)$$

3.3. Augmented Holt and Winters' Method

The Holt and Winter's method allows the incorporation of seasonalities. The assumption for this method is that there is a number of periods *N*, such that every period *i+kN* where *i=1,2,…,N* and *k* is an integer shows the same seasonal behavior. Hence, the same seasonality factor comes into play every *N* periods. Typically, demand seasonalities are in synch with the calendar in use. Temperature variations within the year specifies the times for change of collections in the fashion industry due to their inherent effect in the demand pattern. Similarly, there is additional demand for many types of good in well-known periods of the year due to festivities and accompanying gift-exchange traditions. All these phenomena with pronounced effect on demand patterns happen at the same period, each year. Hence, they can all be captured within the framework of Holt and Winter's.

However, there is a problem when two different calendars that are not synchronized come into effect. The best example of this is the asynchronicity between the western (Gregorian) and the Islamic (Hijri) calendars. With the influence of globalization as well as nature, these two calendars exert their influences simultaneously in many countries. The Gregorian calendar captures the rhythm of seasons in nature as well as the rhythm of – now global- festivities such as the new year. However, in Muslim countries (as in the example of Turkey), Islamic traditions also have profound effect in demand patterns. During the month of Ramadan, Muslims fast and their eating patterns are different to those during the rest of the year. Of particular significance is that alcohol consumption significantly drops during this month and soft beverage consumption shows an opposite characters.

To capture simultaneous effects of two asynchronous calendars, we propose a new approach which we name as the Augmented Holt and Winter's method. In this method, we use two sets of seasonal factors to forecast the demand in a given period. The forecasting periods are according to the first calendar (Gregorian). The main difficulty is that the months of the two calendars do not usually coincide. For example, the month of Ramadan of Islamic calendar usually occurs at two consecutive months of the Gregorian calendar. Moreover, the number of days of Ramadan in each of these Gregorian months varies from year to year. Therefore, the methodology has to partition the effect of Ramadan to the two forecasting periods.

In the rest of the section, we are going to explain Augmented Holt and Winter's method in detail. In this exposition, we are not going to treat the months' of the second (Islamic) calendar separately, but we are going to classify them as Ramadan month and the other months. Hence, we are going to have only two seasonal factors for the second calendar. These factors are represented with two coefficients. We call these coefficients *kr* and *ko*, where *kr* is the coefficient for Ramadan months and *ko* is the coefficient for the Non-Ramadan months.

In this methodology, we treat the demand as a time-series with multiple layers of seasonality in play. There is underlying linear model in the spirit of Holt. Then, the layer of first calendar seasonality is added just like in Holt-Winter. Finally, the new layer of second calendar seasonality is placed again via a multiplicative factor. In order to dynamically estimate the factors involved, we need to peel the data layer by layer.

In a given forecasting period, the number of Ramadan day varies. Some years the Ramadan month can fully be on top of a Gregorian month. In this case the effect of Ramadan will be fully felt in this Gregorian month. However, in many years, the month Ramadan is divided between two Gregorian months. In this case, we need a method to divide its effect between those months as well. We postulate that the effect of Ramadan is distributed between the Gregorian months in proportion with the number of Ramadan days within each month.

Let r_t be the ratio of Ramadan days in the Gregorian month t and o_t be the ratio of Non-Ramadan days in the Gregorian month t. Note that r_t+o_t=*1* for any given month *t*. The higher r_t is the more Ramadan days there is in month *t* and the Ramadan effect is more significant.

Now, we can proceed to the process of peeling the second calendar effects out of the demand data using the equation (3.15).

$$R_t = \left(\frac{D_t}{kr_{last}}\right) r_t + \left(\frac{D_t}{ko_{last}}\right) o_t \tag{3.15}$$

where,

D_t : *Actual volume in period t*
R_t : *De – Ramadanized series in period t*
kr_{last} : *Coefficient for the Ramadan month. (Ramadan factor)*
ko_{last} : *Coefficient for the other months of Islamic calendar. (Non – Ramadan factor)*
r_t : *Ratio of Ramadan days in the current forecast period*, $0 \le r_t \le 1$
o_t : *Ratio of Non – Ramadan days in the current forecast period*, $0 \le o_t \le 1$

In the equation, $\frac{D_t}{kr_{last}}$ represents what the de-Ramadanized demand would be if the whole Gregorian month coincided with Ramadan. The second part, $\frac{D_t}{ko_{last}}$ represents de-Ramadanized data if the whole month consists of Non-Ramadan days. If the month has both Ramadan and Non-Ramadan days, the de-Ramadanized demand would is somewhere between these two numbers. Hence, we compute the de-Ramadanized demand by using a convex combination of the two numbers, in accordance with our postulate about how the effect of Ramadan is distributed.

As is with seasonal factors, the Ramadan and Non-Ramadan coefficients need to be updated once new data is available. This is done each year at the end of Ramadan using the equations (3.16) and (3.17):

$$kr_{new} = \rho \left(\frac{\sum_{i \in S_R} r_i D_i}{\sum_{i \in S_R} r_i R_i} \right) + (1-\rho) kr_{last} \;\; where \;\; 0 \le \rho \le 1 \;\; and \;\; S_R = \{ i \mid r_i > 0 \} \tag{3.16}$$

$$ko_{new} = \rho\left(\frac{\sum_{i\in S_R} o_i D_i}{\sum_{i\in S_R} o_i R_i}\right) + (1-\rho)kr_{last} \;\; where \;\; 0 \le \rho \le 1 \; and \; S_R = \{i \mid r_i > 0\} \qquad (3.17)$$

Also, Ramadan and Non-Ramadan factors are normalized so that their weighted averages are 1. The normalizing equation is explicitly expressed as follows

$$\left(\frac{11ko + kr}{12}\right) = 1. \qquad (3.18)$$

Once the effect of the second calendar is taken out, the rest of the forecasting follows Holt and Winter's method. The first calendar seasonalities are updated with equation (3.19).

$$c_t = \gamma\left(\frac{R_t}{S_t}\right) + (1-\gamma)c_{t-N} \qquad (3.19)$$

where,

S_t : *Level estimate*

γ : *Smoothing factor for the seasonality estimate* $(0 \le \gamma \le 1)$

c_t : *Seasonality estimate for period* t

N : *Seasonality length*

Moreover, the sum of any *N* successive seasonal factors should always be *N*. In another words, after estimating each seasonal factor, it is good implementation to normalize the most recent seasonal factors such that

$$c_j = \left[\frac{c_j}{\sum_{i=t-N+1}^{t} c_i}\right] N \; , \; for \;\; t - N + 1 \le j \le t \qquad (3.20)$$

The fully deseasonalized series is now used to update level and trend estimates;

$$S_t = \alpha\left(\frac{R_t}{c_{t-N}}\right) + (1-\alpha)(S_{t-1} + G_{t-1}) \tag{3.21}$$

where

S_t : *Level estimate*
α : *Smoothing factor for the level* $(0 \le \alpha \le 1)$
R_t : *De – Ramadanized value in period t*
G_t : *Trend estimate*
c_t : *Seasonality estimate*
N : *Seasonality length*

The estimation of the trend component is simply the smoothed difference between two consecutive estimations of the deseasonalized level.

$$G_t = \beta(S_t - S_{t-1}) + (1-\beta)(G_{t-1}) \tag{3.22}$$

where $0 \le \beta \le 1$ is the smoothing constant, which determines the relative weight placed on the current estimation of the trend.

The forecast made in period t for any future period $t+\tau$ is given by the equation (3.23).

$$F_{t,t+\tau} = (S_t + \tau G_t)c_t(ko_{last}o_{t+\tau} + kr_{last}r_{t+\tau}) \tag{3.23}$$

Clearly, there is a need for a more sophisticated initialization procedure for the described method since there are additional coefficients to be estimated. For this new method, we propose the following initialization procedure. The first steps of the procedure follows the initialization suggested for Holt and Winter's method:

First of all, as with Holt-Winters' method, a minimum of two full seasons ($2N$ periods) of historical data is required to initialize a set of seasonal factors. Then,

1. Calculate the sample means for the two separate seasons of data with

$$V_1 = \frac{1}{N} \sum_{j=-2N+1}^{-N} D_j \tag{3.24}$$

$$V_2 = \frac{1}{N} \sum_{j=-N+1}^{0} D_j \tag{3.25}$$

2. Estimate the initial slope (G_o) and the level of the series (S_o) with

$$G_0 = (V_2 - V_1) / N \tag{3.26}$$

$$S_0 = V_2 + G_0 \left[\left(N - 1 \right) / 2 \right] \tag{3.27}$$

3. The initial seasonal indices are estimated for each initialization period.

$$\hat{c}_t = \frac{D_t}{V_i - \left[\left(N + 1 \right) / 2 - j \right] G_0} \quad , \quad for \;\; -2N + 1 \le t \le 0 \;,\; i = 1,2 \tag{3.28}$$

The initial estimates of trend and level components are calculated in the same manner with Holt-Winter. The raw seasonal indices are estimated via equation (3.28) again just like in Holt-Winter's initialization. Holt-Winter's initialization continues by averaging the raw seasonal indices belonging to the same season of the two years. At this point, our method deviates from Holt-Winter. Note that our raw estimates reflect the joint effects of the seasonalities of two asynchronous calendars. Hence, we need a more sophisticated approach to decompose the effect of each calendar's seasonality. Our suggestion for this problem is a least-square formulation. We would like to obtain two sets of seasonal factors minimizing

$$\min \left\{ \sum_{t=-2N+1}^{-N} \left(\hat{c}_t - c_{t+12} \left(o_t ko + r_t kr \right) \right)^2 + \sum_{t=-N+1}^{0} \left(\hat{c}_t - c_t \left(o_t ko + r_t kr \right) \right)^2 \right\}. \tag{3.29}$$

In the equation (3.29), $c_t(o_t ko + r_t kr)$ is the joint seasonal factor for month t of the second year of initialization, whereas $c_{t+12}(o_t ko + r_t kr)$ is the joint seasonal factor for month t of the first year of initialization.

When the normalizing equation (3.18) is substituted in (3.29), we obtain the equation (3.30).

$$\min\left\{\sum_{t=-2N+1}^{-N}\left(\hat{c}_t - c_{t+12}\left(o_t(12-kr)/11+r_t kr\right)\right)^2 + \sum_{t=-N+1}^{0}\left(\hat{c}_t - c_t\left(o_t(12-kr)/11+r_t kr\right)\right)^2\right\} \quad (3.30)$$

We obtain the optimal set $(c_1, c_2, \dots, c_N)$ and *kr* from the equation (3.30). After that *ko* is calculated by substituting *kr* in the equation (3.18). Also, the obtained set $(c_1, c_2, \dots, c_N)$ is normalized so that their average is equal to 1. This normalization is performed by using the equation (3.31).

$$c_j = \left[\frac{c_j}{\sum_{i=t-N+1}^{t} c_i}\right] N \text{ , } for\ t - N + 1 \le j \le t \qquad (3.31)$$

The additional least-square formulation brings an extra burden to the initialization procedure. Note that the least-square formulation is of non-linear nature. However, the non-linear interaction between seasonal factors does not allow for a simpler initialization procedure. After attempting several different approaches, this method was clearly most viable in terms of estimation quality. We propose the use of an off-the-shelf optimization software such as MATLAB to minimize the least square error.

CHAPTER - 4

RETURN FORECASTING

4.1. Methods of Kelle and Silver

Kelle and Silver (1989) propose four forecasting procedures to estimate net demand during a given lead time in the case of reusable SKUs. Their work assumes stationary demand and return distribution. As well as the expectation and variance of the demand during lead time, each of these methods has a different level of information requirement. Method-1 is an approximation in the sense that all the returns during the lead time are considered to be correlated with the demand during that same lead time and independent of previous demand volumes (M.P. de Brtio, 2007). This naïve method only utilizes the expectation and variance of the net demand together with the return rate. On top of that, Method-2 makes use of the information on previous demand per period and the probability mass function of returns. To employ Method-3 one needs to invest in a system that allows to scan individual returns and track them back to the period in which they were originally issued (M.P. de Brtio, 2007). However, this sophisticated method is very expensive or even impossible in practice. Method-4 is developed to address this issue. In order to use this methodology we need to record of the aggregated returns of per period on top of the requirements of Method-2. In the rest of the section, we are going to examine and list the methods of Kelle & Silver in increasing order of information.

4.1.1. Method-1

In this naïve forecasting method, Kelle & Silver (1989) assume that each demand units has the same probability p_{return} of an accompanying return of reusable SKU's and the returns during the lead time are only dependent on the demand during the same lead time. To employ this methodology two pieces of information are utilized. These are listed as follows

1. Probability p_{return} that a demand unit is ever returned.
2. Expected value and the variance of the lead time demand (DL_t) are known and are respectively denoted by $E(DL_t)$, $Var(DL_t)$.

In deciding on the need to acquire new SKUs, expectation and variability of net demand during the replenishment lead time has to be obtained (Kelle and Silver, 1989). The authors define the lead time net demand (NDL_t) as the difference between lead time demand and returns. Also, thanks to the assumptions that are used in Method-1, the lead time net demand is an inherently random variable that follows the Binomial distribution. This special distribution has two parameters such that the number of trials and the success probability.

One can observe that if the lead time demand units does not come back in a considered period the amount of lead time net demand increases due to the its definition. Therefore, the success probability for the distribution of NDL_t is written as $(1-p_{return})$. Also, by taking consideration to the assumptions that are mentioned in the section introducing, it is seen that the number of trials for the distribution of lead time net demand exactly depends on the DL_t. The complicating factor is that DL_t, another parameter of the binomial distribution, is itself a random variable (Kelle and Silver, 1989). And this fact leads to the mixed binomial distribution that can be expressed in a mathematical form as follows

$$NDL_t \mid DL_t \sim Binomial\left(\left(1-p_{return}\right), DL_t\right), \qquad (4.1)$$

$$P\left\{NDL_t = m\right\} = \sum_{d=1}^{\infty} \binom{d}{m} \, p_{return}^{d-m} \left(1-p_{return}\right)^m P\left\{DL_t = d\right\}. \qquad (4.2)$$

Hence, expected lead time net demand under the Method-1 can be written as

$$E\left[NDL_t\right]=E\left[E\left(DL_t-RL_t\right)\mid DL_t\right]=\left(1-p_{return}\right)E\left[DL_t\right]. \tag{4.3}$$

In the calculation of the variance of lead time net demand, $Var(NDL_t)$, we have to accounts for the correlation between the lead time demand and random lead time return (Kelle & Silver, 1989). As a result of this consideration, the law of total variance must be used. So, the variance of the lead time net demand has the form as follows

$$\begin{aligned}Var\left(NDL_t\right)&=E\left[Var\left(NDL_t\mid DL_t\right)\right]+Var\left(E\left[NDL_t\mid DL_t\right]\right)\\&=E\left[Var\left(\left(DL_t-RL_t\right)\mid DL_t\right)\right]+Var\left(E\left[\left(DL_t-RL_t\right)\mid DL_t\right]\right)\\&=\left(1-p_{return}\right)p_{return}E\left[DL_t\right]+\left(1-p_{return}\right)^2 Var\left(DL_t\right).\end{aligned} \tag{4.4}$$

4.1.2. Method-2

The underlying assumption in this methodology each unit sold returns after a period of random length with a known distribution. This method therefore requires information on preceding demand volume per period and the knowledge of the return distribution. The requirements are particularly explained in the following list given by Kelle & Silver (1989):

1. The actual demand volume, D_u, during each preceding period $u\leq t$ where t represents the last period that has been observed.
2. The probability distribution of the number of periods from demands to return of any individual demand units. In this methodology, the amount of return at time period u, R_u, is assumed to be a function of preceding demand volumes; in fact, the returns in a given time period would comprise the proportion p_1 of the demand volumes from the previous period, the proportion p_2 of those two periods ago, and so on (Goh and Varaprasad, 1986). The return proportions for an individual demand unit are also assumed to be independent from the return of the other units. Consequently, these proportions thus constitute the probability mass function for the return times that is writ-

ten in a mathematical form as follows

$$\begin{cases} 0 \le p_j \le 1 & j = 1,2,3,\ldots,n \\ p_j = 0 & otherwise \end{cases} \quad . \tag{4.5}$$

In this probability mass function, p_j indicates the probability of return after exactly j periods. The largest j for which p_j is significant (non-zero) is denoted by *n*. Also, *n* typically attains the values in the range of 2 to 20 time periods. The probability mass function for the return times is an arbitrary discrete distribution that can be obtained from observations of returns of individually identified dispatched units. If the observations are not available, statistical methods such that regression analysis, time series analysis, Box-Jenkins procedure can be used. Also, in this methodology, the obtained probability distribution function for return times has to be satisfied to the following properties.

i. $\sum_{j=1}^{n} p_j = p_{return}$ is the total probability of a return, used in Method-1.

ii. $p_\infty = 1 - \sum_{j=1}^{n} p_j = \left(1 - p_{return}\right)$ is the probability of the demand units never being returned.

iii. The probability of returns from the same period is $p_0 = 0$.

Under the assumptions that the probability distribution function for return times satisfies the mentioned properties and the returns from different period are independent, the amount of products sold in period *u* and returned exactly at period *v*, $R_{u,v}$ *where* $u < v \le u+n$, is a sequence of multinomial trials with the respective distribution as follows

$$P\left\{R_{u,u+1} = k_1, R_{u,u+2} = k_2, \ldots, R_{u,\infty} = D_u - \sum_{j=1}^{n} k_j\right\} = \frac{D_u\,!}{k_1!\ldots k_n!\left(D_u - \sum_{j=1}^{n} k_j\right)} p_1^{k_1} \ldots p_n^{k_n} p_\infty^{\left(D_u - \sum_{j=1}^{n} k_j\right)} \tag{4.6}$$

where $R_{u,\infty}$ is the amount of products sold in period *u* which will never come back.

In this context, the total amount of returns at period *v*, R_v, can be expressed in terms of random $R_{u,v}$ as follows

$$R_v = \sum_{u=b_m}^{v-1} R_{u,v} \text{ where } b_m = \max\{(v-n),1\}. \tag{4.7}$$

In terms of these quantities, we can now express the net demand for a lead time starting from period t NDL_t, as following equation

$$NDL_t = \sum_{v=t+1}^{t+L} (D_v - R_v). \tag{4.8}$$

At any period *t*, we have the knowledge of the demand at all past periods which is represented in the information set.

$$I_t^D = (\{D_u,\ u = 1, 2, ..., t\}) \tag{4.9}$$

By using the information set I_t^D the expectation of lead time net demand is expressed as follows

$$E\left[NDL_t \mid I_t^D\right]. \tag{4.10}$$

The equation (4.10) is rewritten as (4.11) by using the law of iterated expectation.

$$E\left[NDL_t \mid I_t^D\right] = E\left[E\left[NDL_t \mid I_{t+L}^D\right] \mid I_t^D\right] \tag{4.11}$$

One can easily notice that the inner expectation conditions on a larger information set I_{t+L}^D where the demand volumes are available until the period (*t*+*L*). The demand volumes during the lead time are thus assumed to be known and it is explicitly written as (4.12).

$$E\left[NDL_t \mid I_{t+L}^D\right] = \sum_{v=t+1}^{t+L} D_v - \left(\sum_{u=i_m}^{t} D_u \sum_{k=LB}^{UB} p_k + \sum_{u=t+1}^{t+L-1} D_u \sum_{k=LB}^{UB} p_k \right). \tag{4.12}$$

The bounds in expression of (4.12), i_m, UB and LB, are defined as follows

$$i_m = \max\left\{\left(t-n+1\right),1\right\}, \tag{4.13}$$

$$UB = \min\left\{\left(t+L-u\right),n\right\}, \tag{4.14}$$

$$LB = \max\left\{\left(t+1-u\right),1\right\}. \tag{4.15}$$

However, when our information set is limited to I_t^D, the demand volume during the each lead time period is inherently random variable. Under such a circumstance, we assume that demand has a stationary distribution such that

$$E\left[D_v\right] = \mu_v = \mu,\ Var\left[D_v\right] = \sigma_v^2 = \sigma^2. \tag{4.16}$$

The expression (4.12) is substituted into the equation (4.11). Next, the outer expectation is distributed over the equation (4.11) by taking into account (4.16) and I_t^D, thereby the equation (4.17) is obtained.

$$E\left[NDL_t \mid I_t^D\right] = \sum_{v=t+1}^{t+L} \mu_v - \left(\sum_{u=i_m}^{t} D_u \sum_{k=LB}^{UB} p_k + \sum_{u=t+1}^{t+L-1} \mu_u \sum_{k=LB}^{UB} p_k \right) \tag{4.17}$$

The variance of lead time net demand is expressed as (4.18) by using the information set I_t^D.

$$Var\left(NDL_t \mid I_t^D\right) \tag{4.18}$$

Then, the equation (4.18) is rewritten as (4.19) by applying the law of total variance.

$$Var\left(NDL_t \mid I_t^D\right) = Var\left(E\left[NDL_t \mid I_{t+L}^D\right] \mid I_t^D\right) + E\left[Var\left(NDL_t \mid I_{t+L}^D\right) \mid I_t^D\right] \quad (4.19)$$

$Var\left(E[NDL_t \mid I_{t+L}^D] \mid I_t^D\right)$ which is the first component of the equation (4.19) can be explicitly written as (4.20).

$$Var\left(E[NDL_t \mid I_{t+L}^D] \mid I_t^D\right) = Var\left(\left\{\sum_{v=t+1}^{t+L} D_v - \left(\sum_{u=i_m}^{t} D_u \sum_{k=LB}^{UB} p_k + \sum_{u=t+1}^{t+L-1} D_u \sum_{k=LB}^{UB} p_k\right)\right\} \mid I_t^D\right) \quad (4.20)$$

The variance of $\left(\sum_{u=i_m}^{t} D_u \sum_{k=LB}^{UB} p_k\right)$ term is zero, since it is deterministic periods under the information set I_t^D. Also, there is no correlation between this term and other two terms, since demands are independent. By considering these two pieces of knowledge the expression (4.20) is expressed as follows

$$Var\left(\left\{\sum_{v=t+1}^{t+L} D_v - \sum_{u=t+1}^{t+L-1} D_u \sum_{k=LB}^{UB} p_k\right\} \mid I_t^D\right). \quad (4.21)$$

To remove the correlation between lead time returns and sales, we rearrange the terms as

$$Var\left(\left\{D_{t+L} - \sum_{u=t+1}^{t+L-1} D_u \left[1 - \sum_{k=LB}^{UB} p_k\right]\right\} \mid I_t^D\right). \quad (4.22)$$

Applying the outer conditional variance, we obtain following expression.

$$Var\left(E\left[NDL_t \mid I_{t+L}^D\right] \mid I_t^D\right) = \sigma_{t+L}^2 + \sum_{u=t+1}^{t+L-1} \sigma_u^2 \left(1 - \sum_{k=LB}^{UB} p_k\right)^2 \quad (4.23)$$

$E\left[Var\left(NDL_t \mid I_{t+L}^D\right) \mid I_t^D\right]$ which is the second component of the equation (4.19) can be expressed as (4.24).

$$E\left[Var\left(NDL_t \mid I_{t+L}^D\right) \mid I_t^D\right] = E\left[\left\{\sum_{u=i_m}^{t} D_u \sum_{k=LB}^{UB} p_k \left(1 - \sum_{k=LB}^{UB} p_k\right) + \sum_{u=t+1}^{t+L-1} D_u \sum_{k=LB}^{UB} p_k \left(1 - \sum_{k=LB}^{UB} p_k\right)\right\} \mid I_t^D\right] \quad (4.24)$$

Note that all the variance terms in the equation (4.24) is due to returns, since demands are all known under the information set I_{t+L}^D.

After simple mathematical manipulation, the equation (4.24) is rewritten as follows when I_t^D is considered.

$$E\left[Var\left(NDL_t \mid I_{t+L}^D\right) \mid I_t^D\right] = \sum_{u=i_m}^{t} D_u \sum_{k=LB}^{UB} p_k \left(1 - \sum_{k=LB}^{UB} p_k\right) + \sum_{u=t+1}^{t+L-1} \mu_u \sum_{k=LB}^{UB} p_k \left(1 - \sum_{k=LB}^{UB} p_k\right) \quad (4.25)$$

We substitute the obtained both of two components, (4.23) and (4.25), into the expression (4.19) and thereby obtain

$$\begin{aligned} Var\left(NDL_t \mid I_t^D\right) = \sigma_{t+L}^2 + \sum_{u=t+1}^{t+L-1} \sigma_u^2 \left(1 - \sum_{k=LB}^{UB} p_k\right)^2 + \sum_{u=i_m}^{t} D_u \sum_{k=LB}^{UB} p_k \left(1 - \sum_{k=LB}^{UB} p_k\right) \\ + \sum_{u=t+1}^{t+L-1} \mu_u \sum_{k=LB}^{UB} p_k \left(1 - \sum_{k=LB}^{UB} p_k\right) \end{aligned} \quad (4.26)$$

4.1.3. Method-3

In this methodology, Kelle and Silver assume that each unit sold can be individually tracked. Hence, we know when each unit is dispatched and returned back. To employ this methodology, we need to know:

1. The actual demand volume, D_u, during each preceding period u≤t.
2. The probability distribution of the number of periods from demands to return of any individual demand units as defined in (4.5)
3. $CR_{u,t}$, cumulative returns up to period *t* of the units dispatched at period *u*.

$$CR_{u,t} = \sum_{v=u+1}^{h_m} R_{u,v} \text{ where } h_m = min\{(u+n),t\} \text{ and } u < t \quad (4.27)$$

The returns in different periods that belong to a specific demand volume D_u are correlated. This information on the realized returns from a specific dispatch allows us a more accurate forecast. For this method the information set on-hand is defined as

$$I_t^{CRD} = \left(\left\{ D_u,\ \text{u=1,2,...,t} \right\}, \left\{ CR_{u,t},\ \text{u=1,2,...,t-1} \right\} \right). \tag{4.28}$$

The expected lead time net demand for this case is defined as follows

$$E\left[NDL_t \mid I_t^{CRD} \right]. \tag{4.29}$$

Expected value for the lead time returns from demand during the lead time and period t are obtained as in Method-2. However, expected value for the lead time returns that are dispatched from previous sales is computed in different way using the extra information on returns. The expectation of the lead time returns from previous periods is therefore conditioned on information vector I_t^{CRD} as follows

$$E\left[R_{u,v} \mid CR_{u,t} = cr_{u,t}, D_u = d_u \right] = \left(d_u - cr_{u,t} \right) \frac{p_{v-u}}{1 - \sum_{j=1}^{t-u} p_j}\ . \tag{4.30}$$

By considering the previous paragraph, the expression (4.29) is explicitly written as follows

$$E\left[NDL_t \mid I_t^{CRD} \right] = \sum_{v=t+1}^{t+L} \mu_v - \left\{ \sum_{u=i_m}^{t-1} \left(D_u - CR_{u,t} \right) \left(\frac{\sum_{k=LB}^{UB} p_k}{1 - \sum_{j=1}^{t-u} p_j} \right) + \left(D_t \sum_{k=LB}^{UB} p_k \right) + \sum_{u=t+1}^{t+L-1} \mu_u \sum_{k=LB}^{UB} p_k \right\}. \tag{4.31}$$

Note that the expectation of the lead time net demand is similar to (4.17) if the expected lead time returns dispatched from previous sales is rewritten according to the expression (4.30).

The variance of the lead time net demand is expressed as (4.32) by using the information set I_t^{CRD}.

$$Var\left(NDL_t \mid I_t^{CRD} \right) \tag{4.32}$$

Similarly, the variance of the lead time returns from previous issues can be more exactly estimated under the information vector I_t^{CRD}. Thus, the variance of the lead time returns dispatched from previous periods is described as follows

$$Var\left(R_{u,v} \mid CR_{u,t} = cr_{u,t}, D_u = d_u\right) = \left(d_u - cr_{u,t}\right)\left(\frac{p_{v-u}}{1-\sum_{j=1}^{t-u} p_j}\right)\left(1-\frac{p_{v-u}}{1-\sum_{j=1}^{t-u} p_j}\right). \quad (4.33)$$

The variance of lead time net demand is also similar to (4.26) with the variance of returns dispatched from previous demand updated by the equation (4.33) since the random lead time demand and their returns are not influenced by the earlier observations (Kelle & Silver, 1989).

$$Var\left(NDL_t \mid I_t^{CRD}\right) = \sigma_{t+L}^2 + \sum_{u=t+1}^{t+L-1} \sigma_u^2 \left(1-\sum_{k=LB}^{UB} p_k\right)^2 + \sum_{u=i_m}^{t-1}\left[\left(D_u - CR_{u,t}\right)\left(\frac{\sum_{k=LB}^{UB} p_k}{1-\sum_{j=1}^{t-u} p_j}\right)\left(1-\frac{\sum_{k=LB}^{UB} p_k}{1-\sum_{j=1}^{t-u} p_j}\right)\right] \quad (4.34)$$
$$+\left[D_t \sum_{k=LB}^{UB} p_k \left(1-\sum_{k=LB}^{UB} p_k\right)\right] + \sum_{u=t+1}^{t+L-1}\left[\mu_u \sum_{k=LB}^{UB} p_k \left(1-\sum_{k=LB}^{UB} p_k\right)\right]$$

4.1.4. Method-4

Individually tracing of each unit sold is usually very expensive or even impossible in practice. In order to address this problem, Kelle & Silver (1989) propose Method-4. In this method, the total amount of returns and demand for each previous period is assumed to be observed. But, it is not known the dispatch periods of the returned items. Method-4 assumes that we have knowledge of the following:

1. The actual demand volume, D_u, during each preceding period $u \leq t$.
2. The probability distribution of the number of periods from demands to return of any individual demand units as defined in (4.5)
3. The total returns in period u, R_u where $u \leq t$.

$$R_u = \sum_{k=\max\{u-n,1\}}^{u-1} R_{k,u} \quad ,where\ u \le t \tag{4.35}$$

Method-4 aims at improving Method-2 by taking into account the correlations between the observed total returns in recent periods and the future lead time returns (M.P. de Brtio, 2007). For this methodology the available information set is described in the expression (4.36).

$$I_t^{ARD} = \left(\{D_u,\ u = 1, 2, ..., t\}, \{R_u,\ u = 1, 2, ..., t\}\right) \tag{4.36}$$

The expected value of the lead time net demand under the information vector I_t^{ARD} will be computed by applying the law of iterated expectation.

$$E\left[NDL_t \mid I_t^{ARD}\right] = E\left[E\left[NDL_t \mid I_{t+L}^{ARD}\right] \mid I_t^{ARD}\right] \tag{4.37}$$

One can easily notice that the inner expectation conditions on a larger information set I_{t+L}^{ARD} where the amounts of demand and returns for each period are available until the period ($t+L$). The demand and returns volumes during the lead time are thus assumed to be known and it is explicitly written as follows

$$E\left[NDL_t \mid I_{t+L}^{ARD}\right] = \sum_{v=t+1}^{t+L} \left(D_v - R_v\right). \tag{4.38}$$

The expression (4.38) is substituted into the expression (4.37). Then, the returns during the lead time can be partitioned as returns that dispatched within periods until t, and returns that dispatched within the lead time. Both of these two quantities can be explicitly written in terms of $R_{k,u}$ as follows

$$E\left[NDL_t \mid I_t^{ARD}\right] = E\left[\left\{\sum_{v=t+1}^{t+L} D_v - \left(\sum_{u=t+1}^{t+L}\left[\sum_{k=j_m}^{t} R_{k,u} + \sum_{k=h_m}^{u-1} R_{k,u}\right]\right)\right\} \mid I_t^{ARD}\right]. \tag{4.39}$$

where j_m=max{(u-n),1}, h_m=max{(u-n),(t+1)}.

When the outer conditional expectation is distributed over the summations, we obtain the equation (4.40).

$$E\left[NDL_t \mid I_t^{ARD}\right] = \left(\sum_{v=t+1}^{t+L} E\left(D_v \mid I_t^{ARD}\right) - \underbrace{\sum_{u=t+1}^{t+L} \sum_{k=j_m}^{t} E\left(R_{k,u} \mid I_t^{ARD}\right)}_{returns\ from\ periods\ until\ t} - \underbrace{\sum_{u=t+1}^{t+L} \sum_{k=h_m}^{u-1} E\left(R_{k,u} \mid I_t^{ARD}\right)}_{returns\ from\ during\ lead\ time} \right) \quad (4.40)$$

One can notice that returns during the lead time of the units dispatched at previous periods are directly affected by the total returns R_u where $u \leq t$. This is due to the fact that the units that arrive before the current time t would not arrive after t. However, we cannot pinpoint the dispatch period of the arriving units. Hence, we can only quantify this relation with a correlation. The conditional expectation can be expressed as follows

$$\sum_{u=t+1}^{t+L} \sum_{k=j_m}^{t} E\left(R_{k,u} \mid I_t^{ARD}\right) = \sum_{u=t+1}^{t+L} \sum_{k=j_m}^{t} E\left(R_{k,u} \mid R_t, R_{t-1}, \ldots R_{\max\{(t-n+2),2\}}, D_t, D_{t-1}, \ldots D_{\max\{(t-n+1),1\}}\right) \quad (4.41)$$

The first period of the total returns that has a bearing the current lead time is *(t-n+2)* since the lead time is never *0*.

The expression (4.41) cannot be expressed in an exact analytical form. However, a multi-dimensional normal vector gives a good approximation, if the demand is reasonably large and the return distribution has several significant positive terms (Kelle & Silver, 1989). Thus, the approximation allows us to express the conditional expectation (4.41) in an analytic form as follows

$$E\left[\sum_{u=t+1}^{t+L} \sum_{k=j_m}^{t} R_{k,u} \mid D_t, D_{t-1}, \ldots D_{\max\{(t-n+1),1\}}\right] + \underline{c} \times T^{-1} \times \left(\underline{R_u} - E\left[\underline{R_u}\right]\right). \quad (4.42)$$

where $\underline{c}$ is the covariance vector characterizing covariance between observed aggregate returns R_u and the lead time returns from the previous demand, $\underline{R_u}$ is the vector of recent return observations, and T^{-1} is the variance – covariance matrix of the vector $\underline{R_u}$.

The vector of recent return observations, $\underline{R_u}$ can be described by the following expression

$$\underline{R_u} = \left(R_t, R_{t-1}, \ldots, R_{\max\{t-n+2,2\}}\right)^T \tag{4.43}$$

The components of the variance – covariance matrix, T are defined according to the following function

$$\begin{cases} T_{uu} = Var\left(\sum_{k=\max\{1,(u-n)\}}^{u-1} R_{k,u}\right) = \sum_{k=\max\{1,(u-n)\}}^{u-1} D_k\, p_{(u-k)}\left(1-p_{(u-k)}\right) \\ for\ u = \left\{\max\left\{2,(t-n+2)\right\}, t\right\} \\ T_{uv} = Cov\left(\sum_{k=\max\{1,(v-n)\}}^{u-1} R_{k,u}, R_{k,v}\right) = \sum_{k=\max\{1,(v-n)\}}^{u-1} D_k\, p_{(u-k)}\, p_{(v-k)} \\ for\ u = \left\{\max\left\{2,(t-n+2)\right\}, t\right\}\ and\ u < v \le t \\ T_{uv} = T_{vu} \end{cases} \tag{4.44}$$

The elements of the covariance vector, $\underline{c}$ are also constructed by the expression (4.45).

$$\begin{aligned} c_u &= Cov\left(\sum_{u'=t+1}^{t+L}\sum_{k=i_m}^{u-1} R_{k,u'}, \sum_{k'=i_m}^{u-1} R_{k',u}\right) = -\left(\sum_{k=i_m}^{u-1} Cov\left(\sum_{u'=t+1}^{t+L} R_{k,u'}, \sum_{k'=i_m}^{u-1} R_{k',u}\right)\right) \\ &= -\left(\sum_{k=i_m}^{u-1} D_k\, p_{(u-k)} \sum_{i=t+1-k}^{min\{n,(t+L-k)\}} p_i\right) \end{aligned} \tag{4.45}$$

$$where\ u = \left\{\max\left\{(t-n+2),2\right\}, t\right\}\ and\ i_m = \max\left\{(t-n+1),1\right\}$$

Then, the analytical form of the conditional expectation which is obtained in the equation (4.42) is substituted into the expression (4.40). After simple algebraic manipulations the lead time net demand expectation is equal to

$$E\left[NDL_t \mid I_t^{ARD}\right] = \sum_{v=t+1}^{t+L} \mu_v - \left\{\sum_{u=i_m}^{t} D_u \sum_{k=LB}^{UB} p_k + \underline{c} \times T^{-1} \times \left(\underline{R_u} - E\left[\underline{R_u}\right]\right)\right\} - \left\{\sum_{u=t+1}^{t+L-1} \mu_u \sum_{k=LB}^{UB} p_k\right\} \tag{4.46}$$

The variance of lead time returns from previous sales for this information set is

$$\sum_{u=t+1}^{t+L}\sum_{k=j_m}^{t} Var\left(R_{k,u} \mid I_t^{ARD}\right) = \sum_{u=t+1}^{t+L}\sum_{k=j_m}^{t} Var\left(R_{k,u} \mid R_t, R_{t-1}, \ldots R_{\max\{(t-n+2),2\}}, D_t, D_{t-1}, \ldots D_{\max\{(t-n+1),1\}}\right) \quad (4.47)$$

The equation (4.47) cannot be expressed in an exact analytical form. However, a multi-dimensional normal vector can be used to address this issue as used in the expression (4.42). The normal vector allows us to rearrange the conditional variance (4.47) in an analytical form as follows

$$Var\left[\sum_{u=t+1}^{t+L}\sum_{k=j_m}^{t} R_{k,u} \mid D_t, D_{t-1}, \ldots D_{\max\{(t-n+1),1\}}\right] - \underline{c} \times T^{-1} \times \underline{c}^T . \quad (4.48)$$

where $\underline{c}^T$ is the transpose of covariance vector $\underline{c}$.

By applying the obtained result (4.48) and mathematical manipulations the variance of lead time net demand can be explicitly written as follows

$$\begin{aligned} Var\left(NDL_t \mid I_t^{ARD}\right) = \sigma_{t+L}^2 + \sum_{u=t+1}^{t+L-1} \sigma_u^2 \left(1 - \sum_{k=LB}^{UB} p_k\right)^2 + \sum_{u=t+1}^{t+L-1} \mu_u \sum_{k=LB}^{UB} p_k \left(1 - \sum_{k=LB}^{UB} p_k\right) \\ + \sum_{u=i_m}^{t} D_u \sum_{k=LB}^{UB} p_k \left(1 - \sum_{k=LB}^{UB} p_k\right) - \underline{c} \times T^{-1} \times \underline{c}^T . \end{aligned} \quad (4.49)$$

This is seen to be the variance of lead time net demand given by Method-2 adjusted by a term depending upon only the variance – covariance matrix of returns (Kelle & Silver, 1989).

4.2. Modified Kelle and Silver Methods

Four different return forecasting methods proposed by Kelle and Silver are examined in the previous section. The underlying assumption for these methods is that the demand is stationary. This is mainly because the assumption of stationarity greatly simplifies the characterization of the unobserved demand periods. A direct consequence of this assumption is that the expected demand is constant and does not change over time. In real life forecasting, demand usually exhibits high volatility and nonstationarity since it is subject to seasonality and changing trends. The methods developed under the stationary demand assumption, are not applicable to such circumstance. Furthermore, Kelle and Silver assume that products cannot return in the period they are dispatched. This means the return times are not allowed to be zero. However, if the periods are sufficiently long such as a month, this assumption is not usually valid. To cope with these inconsistencies that arise in practice, we modify the methods of Kelle and Silver in the following sections by considering nonstationarity in the demand series and altering the return time distribution.

4.2.1. Modified Method-1

As in the original Method-1, we assume that each demand units has the fixed probability that it is ever returned and the lead time returns are only correlated with the lead time demand. Also, demand is assumed to vary over time due to the different exogenous factors such as seasonality, trend and so on. This is in contrast to original Method-1 where the demand is assumed to be stationary. The requirements of this method therefore have slight changes. Below we list the information required to employ modified Method-1.

1. Probability p_{return} that a demand unit is ever returned.
2. Demand forecast for period v, $\hat{D}_v$ *where* $t < v \le t + L$.

In this context, the expected lead time net demand stated in (4.3) can be rephrased as follows

$$E\left[NDL_t\right] = E\left[E\left(DL_t - RL_t\right) | DL_t\right] = \left(1 - p_{return}\right)\sum_{v=t+1}^{t+L}\hat{D}_v \ . \qquad (4.50)$$

Note that the demand forecasting is an important input for the equation (4.50) since its accuracy determines to a large extent the quality of lead time net demand forecasting.

By considering these assumptions the variance of lead time net demand is re-written as follows

$$\begin{aligned} Var\left(NDL_t\right) &= E\left[Var\left(NDL_t \mid DL_t\right)\right] + Var\left(E\left[NDL_t \mid DL_t\right]\right) \\ &= E\left[Var\left(\left(DL_t - RL_t\right) \mid DL_t\right)\right] + Var\left(E\left[\left(DL_t - RL_t\right) \mid DL_t\right]\right) \\ &= \left(1 - p_{return}\right) p_{return} \sum_{v=t+1}^{t+L} \hat{D}_v + \left(1 - p_{return}\right)^2 \sum_{v=t+1}^{t+L} \hat{\sigma}^2 \end{aligned} \quad (4.51)$$

Note that the $\hat{\sigma}^2$ in the equation (4.51) is obtained from the forecasting method used for demand series.

4.2.2. Modified Method-2

As in the original Method-2, each dispatched unit is assumed to come back according to the return time distribution. However, we allow products to return in the period they are dispatched, whereas Kelle and Silver do not. Furthermore, in spite of its analytical convenience, the assumption of stationary demand is also dropped. In this context, the modified Method-2 utilizes the following information.

1. The actual demand volume, D_u, during each preceding period u≤t where t represents the last period that has been observed.
2. Demand forecast for period v, $\hat{D}_v$ *where* $t < v \leq t + L$.
3. The probability distribution of the number of periods from demands to return of any individual demand units. The distribution stated in the original Method-2, (4.5), is updated as in the following equation when the return times are allowed to be zero.

$$\begin{cases} 0 \leq p_j \leq 1 & j = 0, 1, 2, 3, \ldots, n \\ p_j = 0 & otherwise \end{cases} \quad (4.52)$$

In view of this change, the properties of the return time distribution is re-phrased as follows

i. $\sum_{j=0}^{n} p_j = p_{return}$ is the total probability of a return, used in modified Method-1.

ii. $p_\infty = 1 - \sum_{j=0}^{n} p_j = \left(1 - p_{return}\right)$ is the probability of the demand units never being returned.

So, the expected lead time net demand stated in the expression (4.17) is rewritten as follows

$$E\left[NDL_t \mid I_t^D\right] = \sum_{v=t+1}^{t+L} \hat{D}_v - \left(\sum_{u=i_m}^{t} D_u \sum_{k=LB}^{UB} p_k + \sum_{u=t+1}^{t+L} \hat{D}_u \sum_{k=LB}^{UB} p_k \right). \tag{4.53}$$

Bounds in the expression (4.53), i_m, UB and LB, are given in the following equations.

$$i_m = \max\left\{\left(t - n + 1\right), 1\right\} \tag{4.54}$$

$$UB = min\left\{\left(t + L - u\right), n\right\} \tag{4.55}$$

$$LB = \max\left\{\left(t + 1 - u\right), 0\right\} \tag{4.56}$$

In this case, the variance of lead time net demand is explicitly defined in the following equation.

$$\begin{aligned} Var\left(NDL_t \mid I_t^D\right) = &\sum_{u=t+1}^{t+L} \hat{\sigma}_u^2 \left(1 - \sum_{k=LB}^{UB} p_k\right)^2 + \sum_{u=i_m}^{t} D_u \sum_{k=LB}^{UB} p_k \left(1 - \sum_{k=LB}^{UB} p_k\right) \\ &+ \sum_{u=t+1}^{t+L} \hat{D}_u \sum_{k=LB}^{UB} p_k \left(1 - \sum_{k=LB}^{UB} p_k\right) \end{aligned} \tag{4.57}$$

Note that the demand forecast affects both the expected net demand during the lead time and its variance.

4.2.3. Modified Method-3

In this method, we assume to individually track each dispatched unit as in the original Method-3. The dispatch and return times for each unit are exactly known. We also allow the non-stationary demand and zero return times just like in the previous modified method. Thus, the modified Method-3 utilizes the following:

1. The actual demand volume, D_u, of past periods u≤t.
2. Demand forecast for period *v*, $\hat{D}_v$ *where* $t < v \leq t + L$.
3. The probability distribution of the number of periods from demands to return of any individual demand units as described in (4.52)
4. $CR_{u,t}$, cumulative returns up to period *t* of the units dispatched at period *u*. Note that since the return time is allowed to be zero, this quantity differs from the expression (4.27). The revised expression is given in the below.

$$CR_{u,t} = \sum_{v=u}^{h_m} R_{u,v} \text{ where } h_m = min\{(u+n),t\} \text{ and } u \leq t \tag{4.58}$$

When the changes in the expression (4.58) is taken into consideration, the expected lead time returns from previous periods must be revised as follows

$$E\left[R_{u,v} \mid CR_{u,t} = cr_{u,t}, D_u = d_u\right] = \left(d_u - cr_{u,t}\right)\frac{p_{v-u}}{1-\sum_{j=0}^{t-u} p_j} . \tag{4.59}$$

Consequently, the expected lead time net demand of (4.31) is adjusted as

$$E\left[NDL_t \mid I_t^{CRD}\right] = \sum_{v=t+1}^{t+L} \hat{D}_v - \left\{ \sum_{u=i_m}^{t} \left(D_u - CR_{u,t}\right) \left(\frac{\sum_{k=LB}^{UB} p_k}{1-\sum_{j=0}^{t-u} p_j} \right) + \sum_{u=t+1}^{t+L} \hat{D}_u \sum_{k=LB}^{UB} p_k \right\} \tag{4.60}$$

Similarly, in view of modification in the equation (4.58), the conditional variance of the lead time returns from previous issues is described as follows

$$Var\left(R_{u,v} \mid CR_{u,t} = cr_{u,t}, D_u = d_u\right) = \left(d_u - cr_{u,t}\right)\left(\frac{p_{v-u}}{1-\sum_{j=0}^{t-u} p_j}\right)\left(1-\frac{p_{v-u}}{1-\sum_{j=0}^{t-u} p_j}\right). \quad (4.61)$$

By means of the expression (4.61), the variance of the lead time net demand is reformed as follows

$$Var\left(NDL_t \mid I_t^{CRD}\right) = \sum_{u=t+1}^{t+L} \hat{\sigma}_u^2 \left(1-\sum_{k=LB}^{UB} p_k\right)^2 + \sum_{u=i_m}^{t}\left[\left(D_u - CR_{u,t}\right)\left(\frac{\sum_{k=LB}^{UB} p_k}{1-\sum_{j=0}^{t-u} p_j}\right)\left(1-\frac{\sum_{k=LB}^{UB} p_k}{1-\sum_{j=0}^{t-u} p_j}\right)\right] + \sum_{u=t+1}^{t+L} \hat{D}_u \sum_{k=LB}^{UB} p_k \left(1-\sum_{k=LB}^{UB} p_k\right). \quad (4.62)$$

Note that the demand forecast is a crucial input for this method.

4.2.4. Modified Method-4

As in the original Method-4, we assume that the total amount of returns and demand for previous periods are observed. But, it is not known the dispatch periods of the returned items. The non-stationarity of demand and zero return times are also considered in this method. We require the following information to employ the modified Method-4.

1. The actual demand volume, D_u, during each preceding period $u \le t$.
2. Demand forecast for period v, $\hat{D}_v$ *where* $t < v \le t + L$.
3. The probability distribution of the number of periods from demands to return of any individual demand units as defined in (4.52).
4. The total returns in period u, R_u where $u \le t$. Note that since the return time is allowed to be zero, this quantity is different than the expression (4.35). The revised expression is written as follows

$$R_u = \sum_{k=\max\{u-n,1\}}^{u} R_{k,u} \quad ,where\ u \le t. \quad (4.63)$$

In view of these changes, expectation of the lead time net demand expressed as follows

$$E\left[NDL_t \mid I_t^{ARD}\right] = \sum_{v=t+1}^{t+L} \hat{D}_v - \left\{ \sum_{u=i_m}^{t} D_u \sum_{k=LB}^{UB} p_k + \underline{c} \times T^{-1} \times \left(\underline{R_u} - E\left[\underline{R_u}\right] \right) \right\} - \left\{ \sum_{u=t+1}^{t+L} \hat{D}_u \sum_{k=LB}^{UB} p_k \right\}. \quad (4.64)$$

where $\underline{c}$ is the vector which defines covariance between observed aggregate returns R_u and the lead time returns from the previous demand, $\underline{R_u}$ is the vector of recent return observations, and T^{-1} is the variance – covariance matrix of the vector $\underline{R_u}$.

Since the units can return back in the period they are dispatched, the vector of recent return observations, $\underline{R_u}$ is revised as follows

$$\underline{R_u} = \left(R_t, R_{t-1}, \ldots, R_{\max\{t-n+1,1\}} \right)^T \quad (4.65)$$

The components of the variance – covariance matrix, T are

$$\begin{cases} T_{uu} = Var\left(\sum_{k=\max\{1,(u-n)\}}^{u} R_{k,u} \right) = \sum_{k=\max\{1,(u-n)\}}^{u} D_k \, p_{(u-k)} \left(1 - p_{(u-k)}\right) \\ for\ u = \left\{ \max\left\{1, (t-n+1)\right\}, t \right\} \\ T_{uv} = Cov\left(\sum_{k=\max\{1,(v-n)\}}^{u} R_{k,u}, R_{k,v} \right) = \sum_{k=\max\{1,(v-n)\}}^{u} D_k \, p_{(u-k)} \, p_{(v-k)} \\ for\ u = \left\{ \max\left\{1, (t-n+1)\right\}, t \right\}\ and\ u < v \le t \\ T_{uv} = T_{vu} \end{cases} \quad (4.66)$$

Reflecting the change in the return time distribution, the elements of the covariance vector, $\underline{c}$ are

$$\begin{aligned} c_u &= Cov\left(\sum_{u'=t+1}^{t+L} \sum_{k=i_m}^{u} R_{k,u'}, \sum_{k'=i_m}^{u} R_{k',u} \right) = -\left(\sum_{k=i_m}^{u} Cov\left(\sum_{u'=t+1}^{t+L} R_{k,u'}, \sum_{k'=i_m}^{u} R_{k',u} \right) \right) \\ &= -\left(\sum_{k=i_m}^{u} D_k \, p_{(u-k)} \sum_{i=t+1-k}^{min\{n,(t+L-k)\}} p_i \right) \end{aligned} \quad (4.67)$$

$$where\ u = \left\{ \max\left\{(t-n+1), 1\right\}, t \right\}\ and\ i_m = \max\left\{(t-n+1), 1\right\}$$

By considering these modifications, the variance of lead time net demand is also revised as follows

$$Var\left(NDL_t \mid I_t^{ARD}\right) = \sum_{u=t+1}^{t+L} \hat{\sigma}_u^2 \left(1 - \sum_{k=LB}^{UB} p_k\right)^2 + \sum_{u=t+1}^{t+L} \hat{D}_u \sum_{k=LB}^{UB} p_k \left(1 - \sum_{k=LB}^{UB} p_k\right) + \sum_{u=i_m}^{t} D_u \sum_{k=LB}^{UB} p_k \left(1 - \sum_{k=LB}^{UB} p_k\right) - \underline{c} \times T^{-1} \times \underline{c}^T \quad (4.68)$$

where $\underline{c}^T$ is the transpose of covariance vector $\underline{c}$.

CHAPTER - 5

A CASE STUDY IN A MAJOR BREWERY IN TURKEY

This research is motivated by a project at an important beer factory in Turkey. This factory uses three defining products attributes: brand, package type and package size. In the examined period, a total of twelve beer brands are being offered in Turkey market. Also, these are sold in three package types (returnable bottle, non-returnable bottle and can) and in two sizes (35cc, 50cc). But, in this study we restrict our consideration to only a specific returnable bottled beer. This product is dominant in the revenues of the factory.

5.1. Demand Forecasting

In the Turkish market, beer sales fluctuate within the year due to different cultural and seasonal factors. One of the major reasons behind the fluctuation is the decrease of alcohol consumption during the holy Ramadan month. Ramadan is a Hijra month (Islamic Calendar) and the fasting month for the all Muslims. Like all other months of Hijra, it moves cyclically by 10 or 11 days every Gregorian year. During this month, consumption habits are distinguishably affected and many consumers do not consume alcoholic beverages. Significant Gregorian seasonality effect is also present in beer sales, mainly due to ambient temperature and tourism activity. The average beer sales in summer months are much higher compared to the other months.

Monthly sales data for almost five-year period, encompassing 51 monthly data points, is used to reveal validity of the assertions. In this context, firstly, we try to capture mentioned factors from the date depicted in Figure-1. And three striking features of the sales data immediately come to light in this figure. First, there is an upward linear trend in the sales from January 2008 to March 2012. Second, over the entire 12-month periods, the numbers of sales show seasonal fluctuations with consistent peaks in summer months and slumps in Ramadan months which are denoted as yellow points in Figure-1. Third, a keen observer notices that trend of series increases notably after observation February 2010. If you look at the Figure-1 imagine a freehand line tracing the centre of data as you move across the graph, it clearly seems that while the mean of data for the first two periods (2008-2009) is below of the central line, the others (2010-2011-2012) are the above.

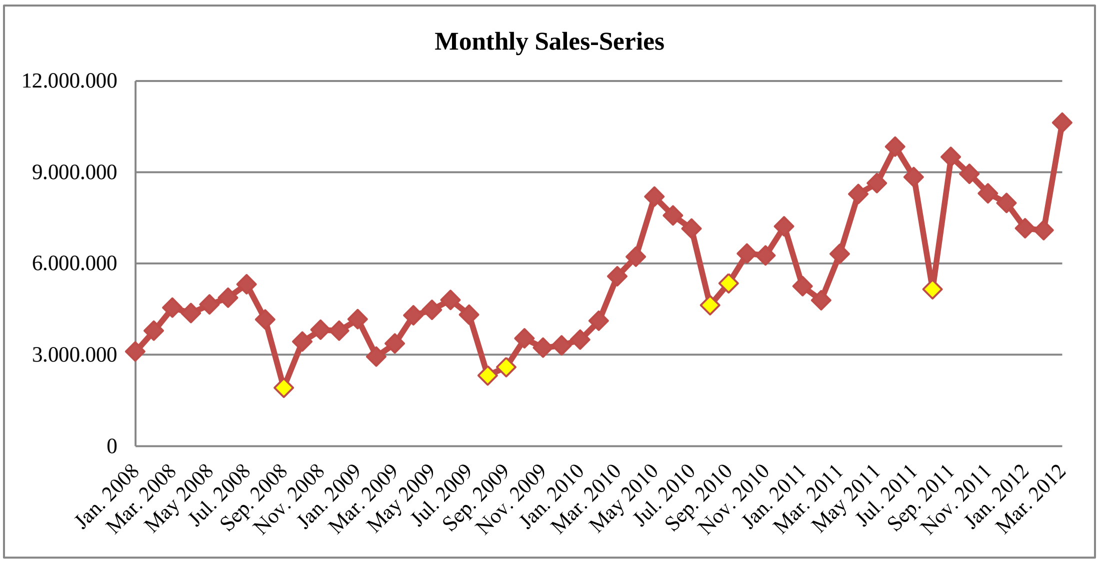

Figure 1 Monthly Sales-Series

These results can also be reconfirmed in detail from Figure-2. The sales volumes are clearly less than what is to be expected without the presence of the Ramadan factor. Moreover, it is clearly seen that the effect of Ramadan does not manifest itself in a single month of the Gregorian calendar. Since the month of Ramadan is usually divided between two consecutive Gregorian months, this effect is also distributed to these months. (Yellows are again Ramadan months). Furthermore, it is observed that the sales of each year are higher than the previous one. This can be explained by the fact that the lines connecting the months in Figure-2 are almost parallel. The trend may have increasing slope over time, possibly being linear. The beer sales volumes also exhibit a seasonal pattern. Most importantly, for forecasting issues, the pattern seems to be consistent from year to year.

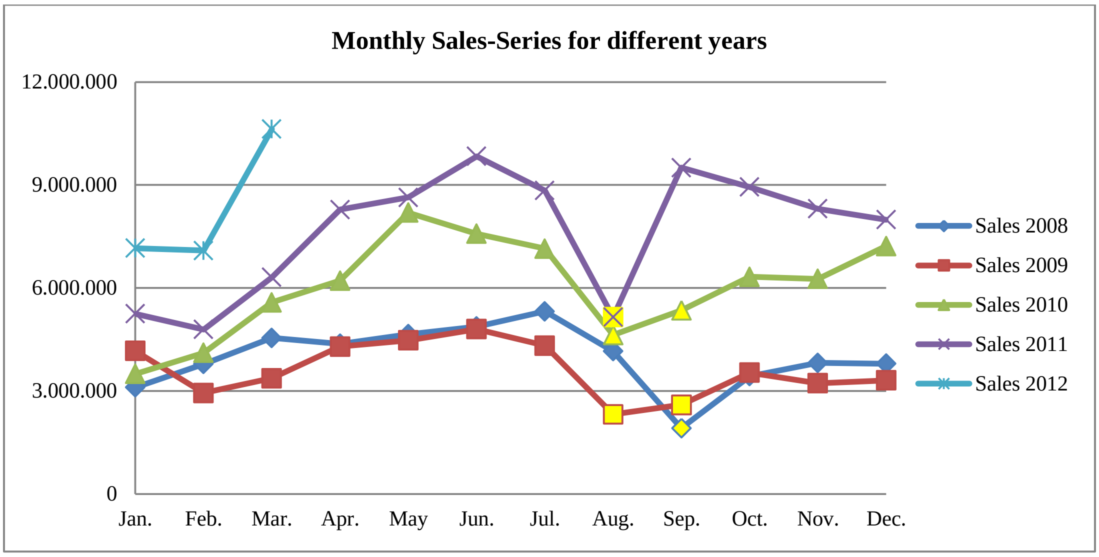

Figure 2 Monthly Sales-Series for different years

The aforementioned demand forecasting methods and their performance are examined here by applying them to the monthly beer sales data. We use the data by dividing into two parts; the first one which is from January 2008 to December 2009 is used for initialization procedures and the second one, from January 2010 to March 2012, is used to perform the proposed forecasting procedures so as to test their effectiveness. The smoothing parameters (α, β, γ, ρ) used in these methods are chosen using the common procedure of minimizing the mean square error. Also, to obtain the initial seasonal indices c_i and Ramadan and Non-Ramadan months factors (*ko*, *kr*), nonlinear least squares optimization routine is employed on the MATLAB as explain in section 3.3. Their values are illustrated in the following Table-1 and Table-2.

Forecasting	*Smoothing Parameters*				*Ramadan Factors*	
Methods	**α**	**β**	**γ**	**ρ**	*ko*	*kr*
HM	*0.3*	*0.1*	-	-	-	-
HWM	*0.3*	*0.1*	*0.4*	-	-	-
AHWM	*0.3*	*0.1*	*0.4*	*0.5*	*1.06*	*0.32*

Table 1 Initial Ramadan Factors and Smoothing Parameters

Forecasting Methods	Initial Seasonal Factors											
	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}
HM	-	-	-	-	-	-	-	-	-	-	-	-
HWM	*0.92*	*0.85*	*1.00*	*1.21*	*1.18*	*1.27*	*1.27*	*0.85*	*0.61*	*0.94*	*0.96*	*0.97*
AHWM	*0.87*	*0.79*	*0.94*	*1.05*	*1.10*	*1.19*	*1.19*	*0.91*	*1.23*	*0.88*	*0.89*	*0.89*

Table 2 Initial Seasonal Factors

In order to quantitatively determine the most appropriate forecasting method and to compare their effectiveness, three accuracy measures are employed such that Mean Square Error (MSE), Mean Absolute Deviation (MAD) and Mean Absolute Percentage Error (MAPE). These statistical error measures are defined in the following paragraph.

Mean Squared Error which averages the square of the error over n time periods is given as

$$MSE = \frac{1}{n}\sum_{t=1}^{n}(D_t - \hat{D}_t)^2 . \tag{5.1}$$

Mean Absolute Deviation which averages absolute deviations over the n time periods and provides the absolute measure of forecasting error is given as

$$MAD = \frac{1}{n}\sum_{t=1}^{n}\left|D_t - \hat{D}_t\right| . \tag{5.2}$$

Mean Absolute Percentage Error (MAPE) is defined as

$$MAPE = \frac{1}{n}\sum_{t=1}^{n}\left|\left(\frac{D_t - \hat{D}_t}{D_t}\right)100\right| . \tag{5.3}$$

In order to examine the improvement performance of the forecasting methods, Holt's Method (HM) is used as a benchmark method. "Percentage Change" columns indicated in Table-3 are computed by following equation.

$$\begin{matrix} Percentage \\ Change \end{matrix} = \frac{Accuracy\ measure\ of\ forecasting\ method - Accuracy\ measure\ of\ HM}{Accuracy\ measure\ of\ HM} \times 100 \qquad (5.4)$$

Thus, a positive value in "Percentage Change" columns implies that the benchmark method (HM) performs better than the compared method. A negative value signifies an improvement.

For one-step-ahead forecasts, the accuracy measures of three demand forecasting methods are given in the Table-3.

	HM	*HWM*	*Percentage Change*	*AHWM*	*Percentage Change*
MSE:	*2.8814E+12*	*2.0010E+12*	*-31%*	*1.7998E+12*	*-38%*
SqrtMSE:	*1,697,473*	*1,414,571*	*-17%*	*1,341,580*	*-21%*
MAD:	*1,398,948*	*1,053,097*	*-25%*	*1,098,146*	*-22%*
MAPE:	*21.34%*	*15.89%*	*-26%*	*18.01%*	*-16%*
MaxError:	*3,729,326*	*4,746,513*	*27%*	*3,181,837*	*-15%*

Table 3 Accuracy measures results

Also, to obtain a visual impression about the performance of the forecasting methods Figure-3, Figure-4 and Figure-5 is given in the following paragraphs.

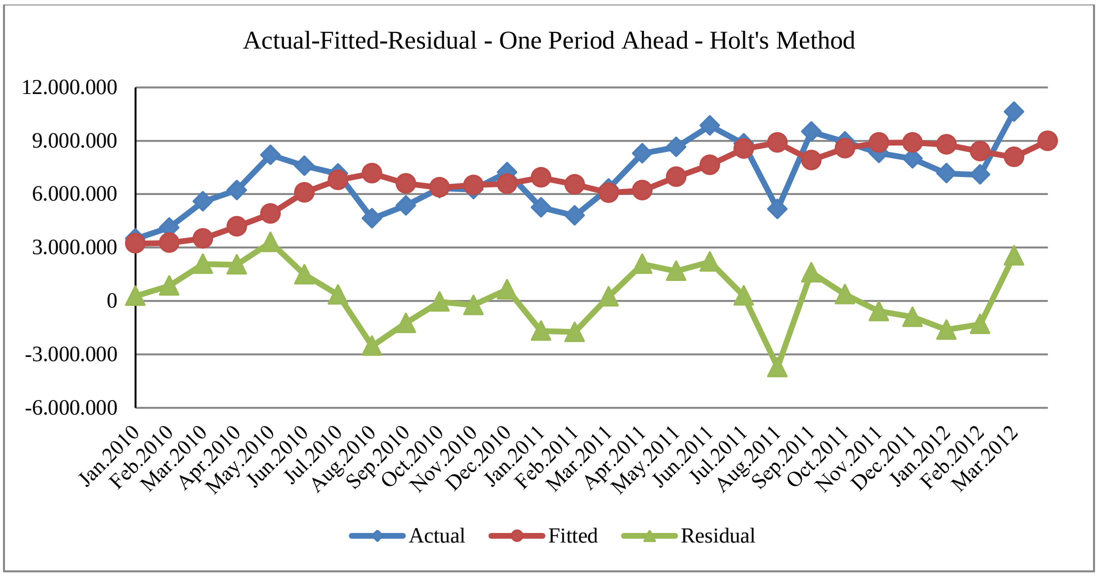

Figure 3 Actual-Fitted-Residual Graph - One period ahead by using HM

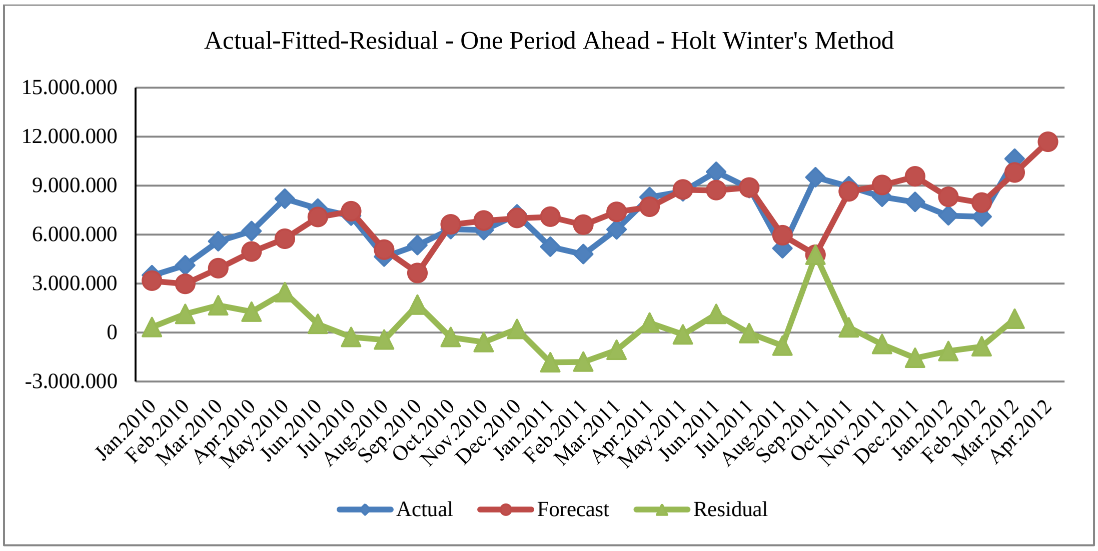

Figure 4 Actual-Fitted-Residual Graph - One period ahead by using HWM

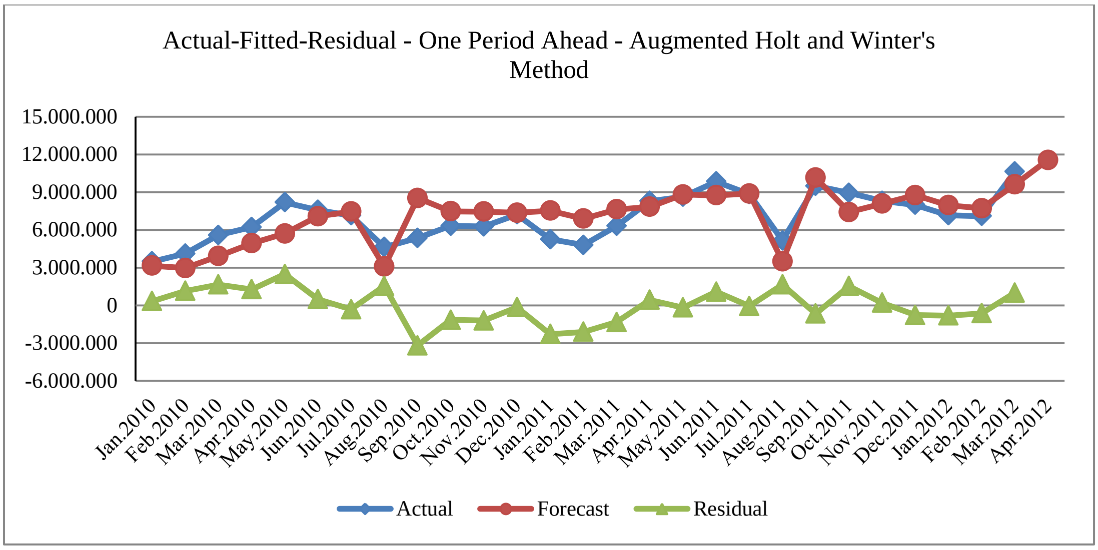

Figure 5 Actual-Fitted-Residual Graph - One period ahead by using AHWM

Unsurprisingly, HM seems to be inferior, to the other methods. This result can be attributed to fact that Holt's Method does not respond to seasonal and cyclic changes. If the Table-3 is examined in detail, it appears that HWM and AHWM yield a large decrease in MSE, MAD and MAPE compared to HM. By using both of these two methods, reductions of 31-38% in MSE are achieved compared with the classical HM. However, in terms of MAPE the table indicates that HWM gives an improvement of 26%, only slightly better than the AHWM at 16%. Although substantial improvement is achieved by AHWM, it shows almost similar results with HWM. The reason that lay behind of this result could be explained as follows. In the considered period, the Ramadan effects typically manifest itself on the similar months. Thus, recognizing such a cyclic pattern becomes relatively easy for HWM. Otherwise, it cannot produce estimation as well as AHWM. Also, in our case, the warm-up period for AHWM isn't enough to capture the natural behavior of the series. So, indicators show that AHWM and HWM nearly give similar results.

5.2. Return Forecasting

The main purpose of this study was to come up with reliable return forecasts. However, this necessitated good forward forecasts, since forward forecasts are used as input for the return forecasting models. In the previous section, we explained our work for the forward forecasts. In this section, we present the work we did for return forecasting using the methods given in Chapter 4.

The returned items are dispatched items from the past periods that are sent back after a stochastic sojourn time outside the inventory system. This sojourn time is modeled by the return time distribution which is crucial in estimating the returns. As mentioned in the section 4.1.2, the return times follow an arbitrary discrete distribution that can be easily estimated by observing returns of individually identified units. Since it is difficult to track every individual bottle in the factory, the sampling issue is of importance in our research. To this end, the company used the simple random sampling technique where is each member of a population has an equal chance of being selected. Between 29/09/2010 and 03/04/2012, samples were randomly drawn from each vehicle which returned to factory after carrying out its operation. The sample size selected by the company is five bottles per batch. The company had collected 20,541 data points via simple random sampling. For each selected item, dispatch and return date were recorded in a spread-sheet. Hence, the sojourn times of the sampled bottles can be computed in terms of days. However, since the factory's forecasting activities is on a monthly basis, monthly intervals for return times are required. To address this issue, we discretize the each data points by using equation (5.5).

$$Sojourn\ Time = Round\left(\frac{Return\ Time\ -\ Dispatch\ Time}{30}\right) \tag{5.5}$$

Note that the data set also has some shortcomings. For instance, there is no information on what happened at the retailer, no information about the lost bottles, etc. Especially, the lack of information on the lost bottles is problematic in the estimation of the return time distribution. Remember that the sum of the probability masses for the distribution used in return forecasting methods is assumed to be less than 1, since items may never return back. However, we cannot obtain this distribution since the data at hand has only information on returned items. Therefore, the probability distributions obtained using discretized data points have to be defined differently than the ones proposed in Chapter 4, (4.5) and (4.52). We estimate the censored distribution for the sojourn times which is

$$\begin{cases} 0 \le p'_j \le 1 & if\ j = 0,1,2,...,n \\ p'_j = 0 & if\ j > n\ or\ j < 0 \end{cases} \tag{5.6}$$

Note that $\sum_{j=0}^{n} p_j = 1$ for this distribution. Hence, to use the probability distribution (5.6) in return forecasting methods, p_{return} have to be estimated separately, and then the probability distribution obtained from the sampled data must be adjusted multiplying by the p_{return}.

We propose three different estimators for the total return probability, p_{return}. The first estimator given in (5.7) compares the total returns and the total dispatches using a delay of 2 months for the dispatches. This is due to the fact that returns arrive on the average about 2 months after their dispatch times.

$$\hat{p}_{return} = \frac{\sum_{i=3}^{T} R_i}{\sum_{i=1}^{T-2} D_i} \quad where\ T\ is\ the\ last\ period\ observed \tag{5.7}$$

We also propose another version of the estimation of (5.7) by using time windows of the length *n*, the maximum sojourn time. Here, we come up with different estimates for different windows to check if there is return probability change as a function of time.

$$\hat{p}^{(1)}_{return}(t) = \frac{\sum_{i=t+2}^{t+n+2} R_i}{\sum_{i=t}^{t+n} D_i} \quad where\ t = 1, 2, \ldots, (T - 2 - n) \tag{5.8}$$

Finally, we propose another set of estimators using initial censored distribution of returns. Here we assume that returns follow the time series

$$\begin{aligned} R_t &= p_0 D_t + p_1 D_{t-1} + \ldots + p_n D_{t-n} + \varepsilon \\ &= p_{return}\left(p'_0 D_t + p'_1 D_{t-1} + \ldots + p'_n D_{t-n} \right) + \varepsilon \end{aligned} \tag{5.9}$$

Based on this, the estimator for p_{return} of each period is given in (5.10).

$$\hat{p}^{(2)}_{return}(t) = \frac{R_t}{\sum_{i=1}^{n} p_i D_{t-i}} \quad where\ t = n, n+1, \ldots, T \tag{5.10}$$

Using the equation (5.7), the total return probability is estimated as 0.873. To use a single estimation value, one has to ensure that there is no change in the average of the return probabilities over time. To this end, the estimates obtained from equation (5.8) and (5.10) are examined by performing a test of hypothesis on the mean. Since the sample size is greater than 30, one sample z test is used. The results of the hypothesis tests for these two cases are given in the Table-4 and Table-5.

One sample Z – Test for $\mu_{p^{(1)}_{return}} = 0.873$ *vs* $\mu_{p^{(1)}_{return}} \neq 0.873$					
N	**Mean**	**SE Mean**	**95% CI**	**Z**	**P-Value**
43	*0.867*	*0.00807*	*(0.85078; 0.88242)*	*-0.81*	*0.415*

Table 4 One sample z test results for the average of $p^{(1)}_{return}$

One sample Z – Test for $\mu_{p^{(2)}_{return}} = 0.873$ *vs* $\mu_{p^{(2)}_{return}} \neq 0.873$					
N	**Mean**	**SE Mean**	**95% CI**	**Z**	**P-Value**
43	*0.869*	*0.0218*	*(0.8271; 0.9127)*	*-0.15*	*0.881*

Table 5 One sample z test results for the average of $p^{(2)}_{return}$

For two cases, one sample z test result yields that the null hypothesis is failed to reject with 5% significance level. This result can be reconfirmed by using the 95%

two-sided confidence interval on the mean. Consequently, the total return probability can be used as 0.873.

The censored distributions are obtained by analyzing the discretized data points in EXCEL, and then they are adjusted multiplying, p_{return}=0.873. Also, note that the zero sojourn times are ignored when the distribution for the original Kelle and Silver methods are estimated. The empirical distributions estimated for the original and modified return forecasting methods are given in the Table-6.

	p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
With Zero Return times	*0.047*	*0.327*	*0.222*	*0.122*	*0.064*	*0.039*	*0.023*	*0.015*	*0.009*	*0.006*
	p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
Without Zero Return times	-	*0.345*	*0.235*	*0.129*	*0.067*	*0.041*	*0.024*	*0.015*	*0.010*	*0.007*

Table 6 Empirical Distributions for the Modified and Original Kelle & Silver Methods

In the estimation, we use an upper bound for the return times distribution indices, namely n=9. Because, for the indices greater than the upper bound (n), the return probabilities are negligible.

Also, to identify a theoretical distribution which represents the sampled data, the discretized 20,541 data points are also analyzed using the Easy-Fit. Since return times are assumed to be discrete, goodness-of-fit test are performed using the theoretical discrete distributions such as poisson, binomial, negative binomial etc. To this end, two different goodness-of-fit tests such as Kolmogorov-Smirnov (K-S) and Anderson-Darling (A-D) are used in this study. The working concept behind the K-S test is based on the difference between the empirical cumulative distribution and the fitted cumulative distribution. If the maximum difference between these two quantities exceeds a critical K-S value, then the observed distribution cannot come from the theoretical distribution. Also, the principle behind the A-D test has similar spirit to the K-S test, but A-D is more sensitive to discrepancies in tails of distribution. (J. Banks et al., 2004)

In this context, the most appropriate three distributions for the return times are listed in the Table-7 and Table-8. The results of the K-S test indicate that Poisson distribution is the most suitable theoretical distribution for the collected sampling data. Also, one can easily notice that the A-D statistics for the Poisson distribution are noticeably lower than the other ones. These results are valid for both two cases.

		Kolmogorov-Smirnov		*Anderson-Darling*	
Distributions	**Parameters**	**Test Statistic**	**Rank**	**Test statistic**	**Rank**
D. Uniform	*a= 0 b= 5*	*0.291*	*2*	*7041.5*	*3*
Negative Binomial	*p= 0.764 n=7*	*0.351*	*3*	*2073.6*	*2*
Poisson	*λ= 2.2394*	*0.280*	*1*	*1515.5*	*1*

Table 7 Results of Goodness-of-fit Tests (with Zero Return times)

		Kolmogorov-Smirnov		*Anderson-Darling*	
Distributions	**Parameters**	**Test Statistic**	**Rank**	**Test statistic**	**Rank**
D. Uniform	*a= 0 b= 6*	*0.285*	*2*	*7431.2*	*3*
Negative Binomial	*p= 0.681 n=5*	*0.380*	*3*	*2683.1*	*2*
Poisson	*λ= 2.6185*	*0.263*	*1*	*1500.6*	*1*

Table 8 Results of Goodness-of-fit Tests (without Zero Return times)

Note that the zero sojourn times are ignored when the distribution for the original Kelle and Silver methods are fitted.

Consequently, Poisson distribution can be used for estimation of distributions that will be used in the original and modified return forecasting methods. In the estimation, the probability masses obtained from Poisson distribution are normalized so that their summation is 1, and then they are multiplied by p_{return} (0.873), in order to estimate uncensored distribution. In Table-9, results for both two cases are given.

With Zero Return Times	p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
	0.093	*0.208*	*0.233*	*0.174*	*0.097*	*0.044*	*0.016*	*0.005*	*0.001*	*0.0004*
Without Zero Return Times	p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
	-	*0.180*	*0.236*	*0.206*	*0.135*	*0.070*	*0.031*	*0.012*	*0.004*	*0.001*

Table 9 Fitted (Poisson) Distributions for the Modified and Original Kelle & Silver Methods

To estimate the return time distributions, we also propose another method that uses the monthly amounts of sales and returns. The original and modified Method-2 proposed in Chapter 4 model the returns as a function of past demands via a stationary return time distribution. By using a minimum of n+24 periods of historical data, this relationship between demand and returns can be explicitly expressed in the matrix form as follows

$$\textit{For the Original Kelle and Silver}: \begin{bmatrix} R_{n+1} \\ R_{n+2} \\ \vdots \\ R_{n+23} \\ R_{n+24} \end{bmatrix} = \begin{bmatrix} D_n & D_{n-1} & \cdots & D_1 \\ D_{n+1} & D_n & \cdots & D_2 \\ \vdots & \vdots & \ddots & \vdots \\ D_{n+22} & D_{n+21} & \cdots & D_{n+14} \\ D_{n+23} & D_{n+22} & \cdots & D_{n+15} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix} + \begin{bmatrix} \varepsilon_{n+1} \\ \varepsilon_{n+2} \\ \vdots \\ \varepsilon_{n+23} \\ \varepsilon_{n+24} \end{bmatrix} \quad (5.11)$$

$$\textit{For the Modified Kelle and Silver}: \begin{bmatrix} R_{n+1} \\ R_{n+2} \\ \vdots \\ R_{n+23} \\ R_{n+24} \end{bmatrix} = \begin{bmatrix} D_{n+1} & D_n & \cdots & D_1 \\ D_{n+2} & D_{n+1} & \cdots & D_2 \\ \vdots & \vdots & \ddots & \vdots \\ D_{n+23} & D_{n+22} & \cdots & D_{n+14} \\ D_{n+24} & D_{n+23} & \cdots & D_{n+15} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix} + \begin{bmatrix} \varepsilon_{n+1} \\ \varepsilon_{n+2} \\ \vdots \\ \varepsilon_{n+23} \\ \varepsilon_{n+24} \end{bmatrix} \quad (5.12)$$

If the previous monthly demand and returns data is available, it is possible to estimate the p_j's using linear least squares optimization routine. However, in such a case, a proper probability distribution function cannot be obtained due to fact that the solution of the least squares yields negative p_j values. In order to address this issue, following constraints must be considered while solving the linear equation systems (5.11) and (5.12).

$$\textit{For the Original Kelle and Silver}: \begin{cases} 0 \le p_j \le 1 \ , \ \forall_j \ \ j = 1, 2, \ldots, n \\ \sum_{j=1}^{n} p_j \le 1 \end{cases} \quad (5.13)$$

$$\textit{For the Modified Kelle and Silver}: \begin{cases} 0 \le p_j \le 1 \ , \ \forall_j \ \ j = 0, 1, \ldots, n \\ \sum_{j=0}^{n} p_j \le 1 \end{cases} \quad (5.14)$$

In this work, these constrained linear least squares problems are solved using the MATLAB, and the obtained return time distributions are given in Table-10.

	p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{return}
With Zero Return Times	*0.504*	*0.078*	*0.055*	*0*	*0.097*	*0*	*0*	*0.139*	*0*	*0.007*	*0.880*
	p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{return}
Without Zero Return Times	*-*	*0.513*	*0.049*	*0*	*0.052*	*0.085*	*0.056*	*0.037*	*0*	*0.112*	*0.904*

Table 10 Fitted (Linear Least Squares) Distributions for the Modified and Original Kelle & Silver Methods

At this point, we have three different return-time distributions to be used in return forecasting. However, the empirical and the fitted distributions estimated by using the sampled data, have an important drawback due to the lack of information on the lost bottles. Hence, the use of distribution obtained via linear least squares methods is more appropriate. Also, remember that methods proposed in Chapter 4 provide estimation for the lead time net demand. However, these methods can be converted to one-step-ahead return forecasting methods by assuming that the lead time parameter, L, is equal to 1 and taking out the lead time demand term. Here, for the modified methods, the demand estimates obtained via Augmented Holt-Winter's Method are still present since the lead-time returns depend on them. When these two assumptions are used, only the return terms remain in the expectation of the lead time net demand, and thereby one-step-ahead return forecasting methods are obtained. But, two of these methods, Method-3 and Method-1, are less interesting in our case due to their drawbacks. Method-1 which is a rather naïve forecasting method, is not expected to perform very well in general. Method-3 also cannot be employed in our case since it requires tracking of each returned item. Hence, only Method-2 and Method-4 are quantitatively analyzed. One-step-ahead versions of these methods are coded in C++ programming language, and the obtained results are analyzed in EXCEL.

Monthly sales and return data for almost five-year period, encompassing 51 monthly data points for each series, is used to examine the performance of these methods. In order to have a fair comparison, the periods used in the estimation of the return-time distribution are not considered. In the Figure-6 and Figure-7, we can see how well the modified and original Method-2 fit the monthly return data.

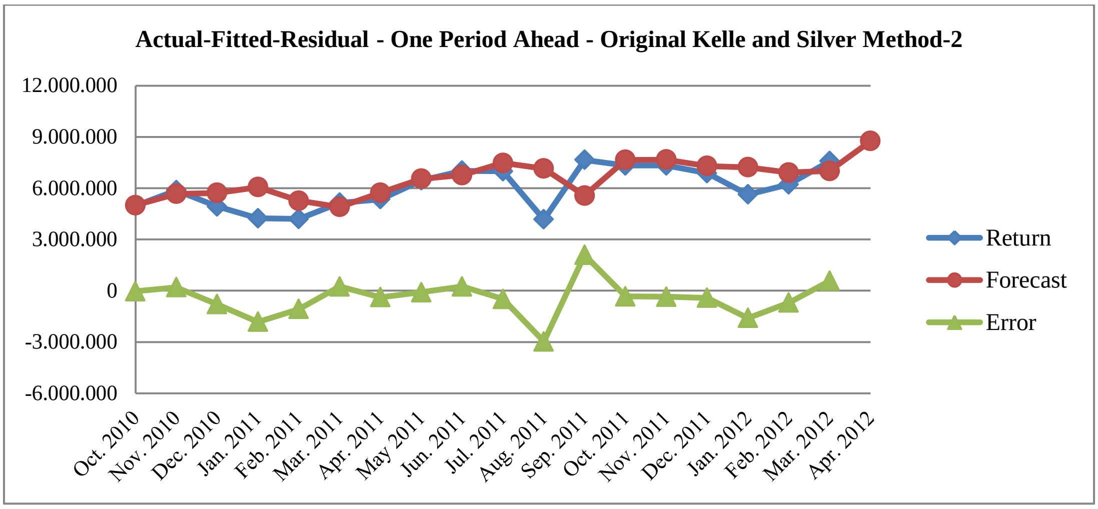

Figure 6 Actual-Fitted-Residual Graph for Original Kelle and Silver Method-2

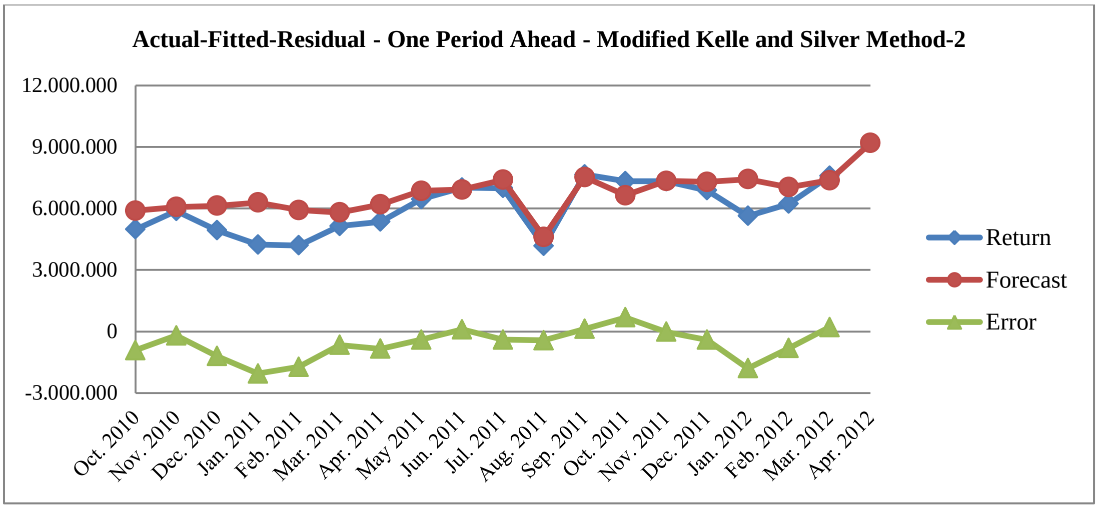

Figure 7 Actual-Fitted-Residual Graph for Modified Kelle and Silver Method-2

As it seen from the Figure-6 and Figure-7, modified method performs superior to the original one, since demand is non-stationary. To compare the methods quantitatively, five statistical accuracy measures which are defined in section 5.1 are presented in Table-11.

	Original Kelle and Silver Method-2	*Modified Kelle and Silver Method-2*	*Percentage Change*
MSE:	*1.255E+12*	*8.768E+11*	*-30.1%*
SqrtMSE:	*1,120,298*	*936,374*	*-16.4%*
MAD:	*799,459*	*721,365*	*-9.7%*
MAPE:	*16.07%*	*14.87%*	*-7.4%*
MaxError:	*2,088,032*	*686,721*	*-67.1%*

Table 11 Accuracy measures results for Original and Modified Method-2

The most striking result of the Table-11 is that the modified method, which is specifically designed for situation with a non-stationary demand, yields the smallest errors in terms of the all accuracy measures. The remarkable success of modified Method-2 in terms of the MSE is noteworthy. Thanks to the modified return forecasting Method-2, a reduction of 30.1% in MSE is achieved compared with the original one. Also, the superiority of the modified method is much more pronounced in terms of Maximum Error. Maximum error of the modified Method-2 is 686,721 whereas original Method-2 yields a maximum error of 2,088,032. By using modified method, the maximum error thus decreases 67% compare to the original one.

For one-step-ahead forecasts, the accuracy measures of the original and modified Method-4 are presented in the Table-12.

	Original Kelle and Silver Method-4	*Modified Kelle and Silver Method-4*	*Percentage Change*
MSE:	*1.436E+12*	*1.380E+12*	*-3.9%*
SqrtMSE:	*1,198,229*	*1,174,593*	*-1.9%*
MAD:	*902,114*	*988,307*	*9.5%*
MAPE:	*18.29%*	*19.80%*	*8.2%*
MaxError:	*1,490,832*	*809,921*	*-45.6%*

Table 12 Accuracy measures results for Original and Modified Method-4

Table-12 indicates that the modified Method-4 yields a large decrease in maximum error compared to original one. By using this method, a reduction of 45.6% in maximum error is achieved. Also, in terms of MSE, the modified Method-4 gives an improvement that is approximately 4%. However, when MAPE and MAD are compared for the two forecasting methods, one can reveal that the modified method performs slightly worse than the original one. Furthermore, to get a visual impression of how each model performs over the forecasting period, Figure-8 and Figure-9 is depicted. Again, in order to have a fair comparison, the periods used in the estimation of distribution are not considered.

Moreover, it can be easily noticed that the two versions of the Method-2 outperform the modified and original Method-4. This result can be attributed the fact that in the presence of imperfect information, more sophisticated methods do not generally lead to best forecasting performance. This issue is extensively investigated in the work of D. Brito and van der Laan (2007). The authors reveal that given perfect information, forecasting performance increases as the level of information increases, resulting in the original Method-4 being the best model. However, they conclude that the original Method-2 presents a sufficient level of sophistication under perfect information and is exceptionally robust in the presence of misinformation in comparison with the others (D. Brito and van der Laan, 2007). Hence, our results are in agreement with their findings.

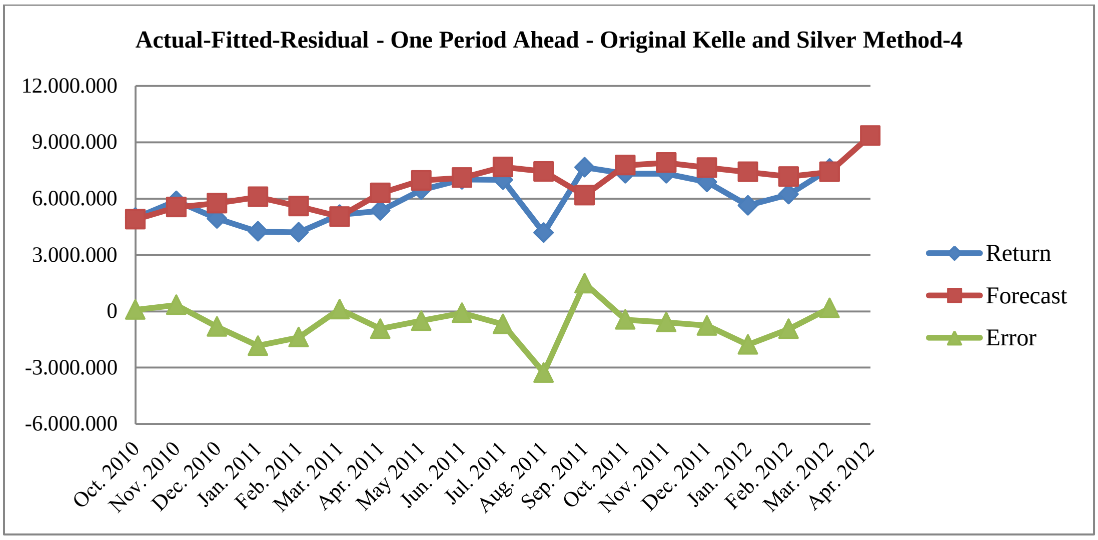

Figure 8 Actual-Fitted-Residual Graph for Original Kelle and Silver Method-4

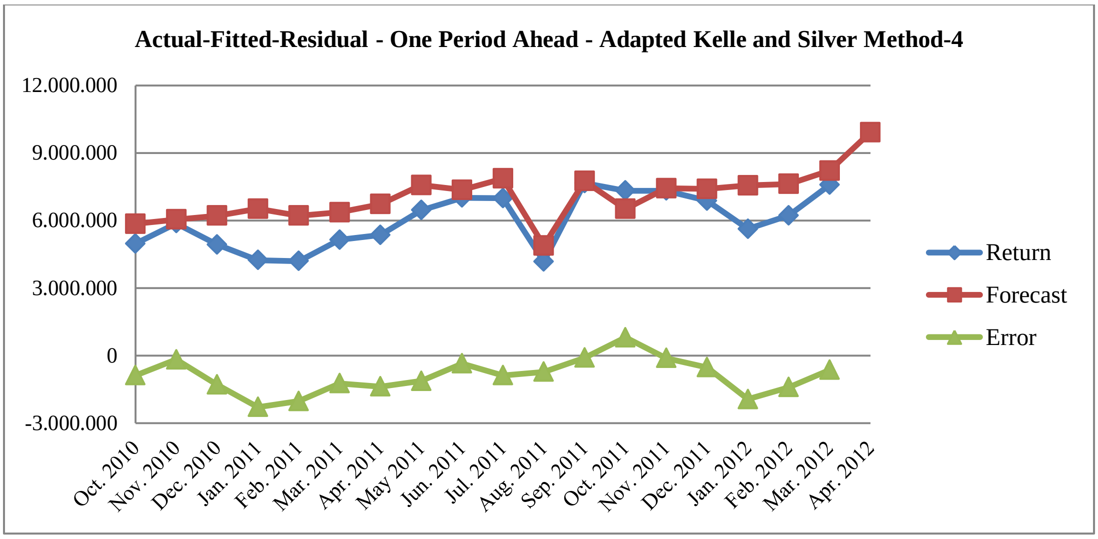

Figure 9 Actual-Fitted-Residual Graph for Modified Kelle and Silver Method-4

5.3. Net Demand Forecasting and Safety Stocks

Figure-10 demonstrates the details of the product recycling system for our case.

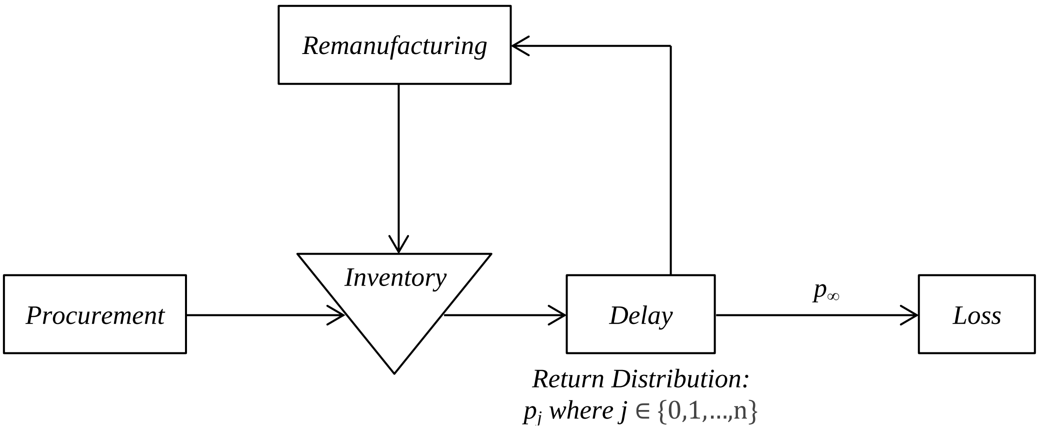

Figure 10 Remanufacturing Flow Diagram

The items are assumed to return back according to the return-time distribution. We also assume a constant probability, $p_\infty = 1 - \sum_{j=0}^{n} p_j$,that an item is lost since a customer does not return it to the manufacturer after use (Kiesmüller, 2001). The remanufacturing processes are assumed to be fit for reuse. After the remanufacturing process which consists of cleaning and labeling, the return bottle can be used just like a new bottle for filling. A common serviceable inventory for both new and remanufactured items is used, since we do not need to distinguish them.

In this work, we consider a periodic order-up-to -(R,S)- policy although it is not optimal. This policy is commonly used in practice as it does not require continuous monitoring of the inventory levels and the returns making it an easy to apply policy. Under the *(R,S)* policy, the inventory status is determined after prespecified periods of time *R* and a replenishment order is placed at the end of that period that is equal to difference between current inventory level and a prespecified order-up-to level *S*. For convenience, we assume that the lead time for a replenishment order is constant at *L*, and net demand is normally distributed.

The review interval (R) is determined as 1, since the factory's forecasting activities is on a monthly basis. The order-up-to level (S) is also determined by calculating the quantity needed between the time the order is placed and the time that the next period's order is received (A. Dolgui et. al, 2005). Since demand and returns are random variables during this time, the order-up-to level (S) is time-dependent and a function of net demand forecasts. Using the normality assumption, the order-up-to level (S) is expressed as follows

$$S = E\left[NDL_t\right] + k\sqrt{Var\left[NDL_t\right]}\,. \tag{5.15}$$

where k is the safety factor, based on the probability of not stocking out during a replenishment period.

The modified Method-2, which is superior to others in the return forecasting, is also used for the net demand estimation. We assume the lead time to be one period and we obtain the return-time distribution via linear least squares method. The lead time demand estimation, which is also required as an input for the modified Method-2, is obtained using Augmented Holt-Winter's Method. In this setting, the expectation and variance of the lead time net demand in the modified Method-2 can be calculated by using equations (4.53) and (4.57). However, the forecasting error of the net demand will be more appropriate to use as the standard deviation estimator for the setting safety stocks. The reason is that the process of forecasting introduces sampling error into the estimation process, and this sampling error is accounted for in the value of MSE. The forecasting error variance is higher than the net demand variance since the forecast is based on only a limited portion of the demand history (S. Nahmias 2004). The modified Method-2 is employed for the months between October 2010 and March 2011. This way, we obtain a sufficient historical data set to estimate the variance. By using the Mean Square Errors given in equation (5.1), standard deviation used for the setting safety stock is estimated as $\sqrt{MSE_{NDL_t}} = 836{,}259$.

CHAPTER - 6

CONCLUSION

Recently, environmental issues have become more important than ever for the manufacturers due to increasing demand for environmental friendly products and new government legislations. Many companies consider environmental issues as a threat to their business, since adapting the new regulations may require a fundamental change in doing business. However, there can be large opportunities for the manufacturers that succeed in embracing environmental policies in their business. In addition to enhanced environmental performance and a "green" image, environmentally conscious strategies and practices can prove beneficial to manufacturers in terms of cost savings and market opportunities. In this context, manufacturers, who realize the importance of environmental issues, start to reuse old products and to incorporate product recovery activities in their regular production systems. Reuse of old products raises a lot of challenges for the manufacturers, starting with collection and ending with sales of products with used components. In this work, we only focus on how inventory levels are controlled under the remanufacturing policies through improved demand and return forecasting accuracy.

In the traditional manufacturing and remanufacturing systems, demand forecasting is the main determinant of effectiveness and efficiency in inventory control. To this end, various forecasting methods are developed thus far. A review of the literature shows that methods of Holt (1957) and Winters (1960) are the most well-known methodologies, since they are easy to implement and can be quite effective in practice. Also, as it is known, these two methods are effective under the different circumstances. Whereas Holt's method (HM) does consider the trend within demand and adjust the forecasting based on the trend, the method of Winters (HWM) takes into account a seasonal factor as well as the trend. However, these two methods are not sufficient to capture demand pattern when the simultaneous effects of two different asynchronous calendars, such as Gregorian and Islamic (Hijri), manifest themselves on a specific market. We therefore adapt the Holt-Winters method so that it can accommodate such two different calendar effects. This adapted method, which is namely Augmented Holt-Winters (AHWM), involves the introduction of an additional seasonal index and an extra smoothing equation for this new seasonal index.

In order to examine the performance of these three forecasting methods, a real life time-series data, which is provided by a leading beer factory in Turkey, is utilized. Numerical studies show that Holt's method is inferior to the other methods, since it does not respond to seasonal and cyclic changes in data. HWM gives an improvement of 31% in terms of MSE, compared to classical Holt's Method. However, when the Holt-Winters Method is employed, Maximum Error is increased by 27%. Thanks to AHWM, an improvement of 38% in terms of MSE is observed whereas a reduction of maximum error amounting to 15% is achieved. Although important improvement is achieved by AHWM, it shows similar results with HWM. This can be attributed that the effects of Ramadan typically manifest itself on mostly the same months in the considered short period of time. Recognizing such a cyclic pattern thus becomes relatively easy for HWM. Otherwise, HWM cannot produce forecasts as well as AHWM. Also, the natural behavior of the demand pattern cannot be captured well enough by the AHWM, due to short warm-up period. Hence, accuracy measures show that AHWM is slightly better than HWM.

Under the recovering policies, another challenging issue in handling inventory efficiently is that returns are characterized by a considerable uncertainty mainly regarding time and quantity. To cope with this problematic issue, several return forecasting methods is proposed in the literature. The main idea in these methods is the observation that returns in any one period are generated by demand in the preceding periods. In this context, we investigate and explain the methods of Kelle and Silver in a way to clarify methodology. Kelle and Silver (1989) is one of the earliest and well-known work in this literature. Their methods are developed under the stationary demand assumption. This is mainly because the assumption of stationarity yields some statistical convenience characterizing the unobserved demand periods. Also, the authors assume that products cannot return in the period they are dispatched. This implies that the return times are not allowed to be zero. However, these two assumptions cause important inconsistencies in practice. Hence, we modify the four methods of Kelle and Silver by considering nonstationarity in the demand series and allowing zero return times.

Data provided by a leading beer factory, is utilized to compare the performance of original and modified methods. However, two of these methods, which are namely Method-3 and Method-1, are not considered in the numerical studies due to their drawbacks and shortcomings in data. Method-1 is not recommended to use for practical implementations, since this naïve method performs in general very poorly. Also, Method-3 cannot be employed in our case, since it needs to keep of the period in which each return is sold. Therefore, only Method-2 and Method-4 are analysed in this work. The results indicate that our modified Method-2 yields the smallest errors in terms of the all accuracy measures, compared to original one. The remarkable improvement of modified Method-2 in terms of the MSE is noteworthy. Thanks to the modified Method-2, a reduction of 30.1% in MSE is achieved compared with the original one. Furthermore, the superiority of the modified method is much more pronounced in terms of Maximum Error. Maximum error of the modified Method-2 is 686,721 whereas original Method-2 yields a maximum error of 2,088,032. By using modified one, the maximum error thus decreases 67% compare to the original Method-2. As in the modified Method-2, the modified Method-4 also yields a large decrease in maximum error compared to original one. Maximum error of the modified Method-4 is 809,921 whereas original Method-4 yields a maximum error of 1,490,832. By using the modified one, a reduction of 45.6% in maximum error is achieved. Also, in terms of MSE, the modified Method-4 gives an improvement that is approximately 4%. However, our modified method performs slightly worse than the original one in terms of MAPE and MAD. Consequently, we cannot say that the modified Method-4 establishes an overwhelming dominance in the forecasting performance, compared to original one. The numerical studies also show that the two versions of the Method-2 outperform to the modified and original Method-4, which are more information intensive methods. This result suggests that more information intensive methods do not generally lead to best forecasting performance in the presence of imperfect information as reported in D. Brito and Laan (2007). They reveal that since the forecasting performance increases as the level of information increases, Method-4 is the best forecasting technique in the perfect information setting, compared to others. However, the authors conclude that the original Method-2 presents a sufficient level of sophistication under perfect information and is exceptionally robust in the presence of misinformation in comparison with the others (D. Brito, 2007). Hence, our findings are in agreement with their results.

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