Portfolio Selection with Prospect Models

by

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This is to certify that I have examined this copy of a doctoral dissertation by

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and have found that it is complete and satisfactory in all respects,
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ABSTRACT

Decision making under uncertainty has always been important among the research community in industrial engineering. Many models which try to explain the attitude of decision maker under uncertainty have been proposed until today. The expected utility model is one of the most widely used models in this research area. However, some surveys show that people do not behave as stated in utility theory. So, non-expected utility models have recently become popular and the prospect model is one of them.

The prospect model contradicts the expected utility model in a number of ways. Main theme of our research is to investigate the choices of prospect investors in a market that contains one risky and one risk-free asset. Exhibiting the differences between structures of the value functions, we get some indications about the portfolio choices of prospect investors.

Firstly, we constructed a general model for the portfolio optimization problem within the frame of the prospect theory. Then, we analyzed it with the different types of value functions: piecewise linear, exponential and piecewise exponential, sequentially. Using different return distributions, each value function is investigated in more details. We derived the solution of the portfolio optimization problem and we obtained some interesting properties of optimal prospect portfolios. Looking at the relationship between the optimal portfolios and asset means, we show that there is a mean interval for a portfolio where it is optimal not to buy or shortsell the risky asset.

Finally, we presented numerical examples to illustrate the shapes of the objective function. Moreover, comparing piecewise exponential and exponential optimal solutions, we analyzed the effects of the prospect value functions.

ÖZETÇE

Belirsizlik altında karar verme uzun zamandır endüstri mühendisliği araştırma dünyasında önemli bir yer edinmiştir. Günümüze kadar, karar vericinin belirsizlik altındaki davranışını açıklayan birçok model ortaya atılmıştır. Beklenen fayda modeli yazında en yaygın kullanıma sahip olan modellerden bir tanesidir. Fakat bazı araştırmalar göstermiştir ki insanlar beklenen fayda modelinde bahsedildiği gibi davranmamaktadırlar. Bu yüzden son zamanlarda beklenen fayda harici modeller yaygınlaşmaya başlamış ve ümit teorisi bunlardan bir tanesi olmuştur.

Ümit modeli bazı açılardan beklenen fayda modeli ile ters düşmektedir. Araştırmamızın ana temasını ise bir riskli ve bir risksiz yatırım araçlarını içeren bir markette ümit modeli yatırımcısının seçimlerini incelemek oluşturmaktadır. Değer fonksiyonlarının yapıları arasındaki farklar incelenerek ümit modeli yatırımcısı hakkında birtakım bulgular elde edilmiştir.

İlk olarak, portföy eniyilemesi problemi için ümit modeli çerçevesinde bir genel model oluşturulmuştur. Sonrasında, bu model farklı tiplerdeki değer fonksiyonları olan parçalı doğrusal, üssel ve parçalı üssel fonksiyonları ile analiz edilmiştir. Farklı dağılımlar ile her bir değer fonksiyonu daha detaylı bir biçimde incelenmiştir. İlgili portföy eniyilemesi problemi için çözümler elde edilmiş ve en iyi portföylerin bazı enteresan yapıları bulunmuştur. Eniyi portföyler ve beklenen değerler arasındaki ilişkiye bakıldığında, bir portföy için en iyi çözümün riskli yatırım aracından alıp veya satmamanın olduğu bir beklenen değer aralığının var olduğu gösterilmiştir.

Son olarak, birtakım örnekler yardımıyla amaç fonksiyonunun yapısı gösterilmiştir. Dahası parçalı üssel ve üssel fonksiyonların eniyi çözümleri karşılaştırılarak değer fonksiyonunun parçalı oluşunun etkileri araştırılmıştır.

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W	:	The wealth level after one period
w_0	:	Initial wealth before investment
w^{ref}	:	Reference wealth level
u	:	The investment policy
u^*	:	The optimal investment policy
$V(u)$	:	Value function for the investment policy u
v^+	:	Gain part of the value function V
v^-	:	Loss part of the value function V
λ^+	:	Multiplier of the gain part
λ^-	:	Multiplier of the loss part
α	:	Risk aversion parameter
$T^+(p)$	:	Probability distortation function for the gain part
$T^-(p)$	:	Probability distortation function for the loss part
r_f	:	The return of the riskless asset
R	:	The return vector of the risky assets
R^e	:	The excess return vector of the risky assets
$E[.]$	:	Expectation operator
μ	:	Mean of the risky asset return
σ^2	:	Variance of the risky asset return

Chapter 1

INTRODUCTION

In today's financial markets, there are many types of financial instruments that investors buy in order to make some profit. In general, stocks, bonds, commodities are the primary instruments. When we look at these instruments, we can see that they show varieties even in themselves. For instance, there are more than 400 stocks traded in Istanbul Stock Exchange (BIST). Imagine an investor who tries to make any investment in BIST. Which stocks or stock must he choose for this aim? This is a question that we are frequently faced with. Portfolio optimization deals with this type of problems. In this respect, we can describe portfolio optimization as the determination of the best proportions of assets that optimize a given objective under some restrictions.

Markowitz (1952) provided the fundamental model in portfolio optimization. Since portfolios have a stochastic nature, he uses both expected return and risk of a portfolio as evaluation criteria. In this model, risk of a portfolio is measured by the variance of the portfolio's rate of return. As stated by Markowitz, an investor wants to maximize expected return of a portfolio for a given amount of portfolio risk, or minimize risk for a given expected return level. Deriving all efficient expected return-risk pairs, one obtains the efficient frontier. The main assumption is that investors are risk-averse and they prefer a certain portfolio to a risky portfolio whose expected return equals the certain return.

Another common approach in portfolio optimization is expected utility theory. Every investor has a utility function which is defined over the final portfolio wealth. Here, utility is a measure of satisfaction and the aim is to maximize the investor's satisfaction which is the expected value of utility. This utility function is concave increasing in wealth. The structure of utility function reflects a preference that attempts higher rather than lower returns by avoiding risky alternatives.

In the literature of portfolio optimization, Markowitz's mean-variance portfolio theory and utility theory are dominant models. However, non-utility models has become more popular recently. Among these non-utility models, the prospect model of Kahneman and Twerksky (1979) is the most familiar. They make a survey and investigate how people react when they are faced with uncertainty. Their works led the field of behavioral finance.

However, their findings contradict modern utility theory.

Firstly, they show that investors do not evaluate their portfolios according to the their final wealth. Instead, investors have a reference wealth point and they evaluate their satisfaction by the difference between these two points. Furthermore , the behavior of investors indicates that they do not simply have risk-averse attitude and their risk sensitivity changes with respect to the reference point. If the level of wealth is less than the reference point so that there is a loss, then an investor takes more risks and tries to rise up to the reference level; or equivalently he exhibits risk-seeking behavior. If the level of wealth is greater than the reference point so that there is a gain, then the investor's attitude turns into risk-averse behavior. Secondly, investors are more sensitive to losses than gains. Empirical results shows that dissatisfaction of losses is as double as likeness of gains for the same amount.

To represent the behavior of an investor, Kahneman and Tversky (1979) suggested a piecewise power value function kinked at zero. This value function is convex increasing up to zero. This part of function is the loss part and the investor is risk-seeking. After zero, the value function is concave increasing since this part of the function is the gain part and the investor is risk-averse.

Rather than the piecewise power function as in the original prospect model, the piecewise exponential function is also used as a value function in the literature. De Giorgi and Hens (2006) examine these two types of value functions and find that there are some benefits in using piecewise exponential value functions. They state that there is a CAPM equilibria for the piecewise exponential value function while there is no equilibria for the piecewise power value function. CAPM equilibrium specifies a relation for the expected rates of return of financial assets.

The motivation of this thesis is to follow the prospect theory approach for the portfolio optimization problem in a market which consists of a single risky asset and one risk-free asset. This study consists of three main parts. In the first part, we take the piecewise linear function as the value function. After analyzing this model with different distributions, we change the value function to the piecewise exponential value function in the second part. Similarly, we analyze this value function with the same distributions. In the final part, we compare expected utility models and prospect models. We try to answer the question of how the prospect model differs from the expected utility model.

This thesis is organized as follows. In Chapter 2, we review the literature on relevant portfolio optimization problems. In Chapter 3, we introduce the prospect model and formulate the portfolio optimization problem using the prospect perspective. In Chapter 4, we characterize optimal prospect portfolios for the single asset model with various distributions. In Chapter 5, we illustrate the models discussed in Chapter 3 and Chapter 4 by some numerical examples. Finally, we make concluding remarks in Chapter 6.

Chapter 2

LITERATURE REVIEW

For decision-making under uncertainty, expected utility theory is perhaps the most widely used approach. In the expected utility model, investors have uniformly risk-averse behavior and they evaluate their wealth according to final absolute outcomes. However, some experimental results show that peoples behaviors violate the expected utility hypothesis in a number of ways. Kahneman and Tversky (1979) introduced a new theory (prospect theory) for dealing with these violations. Prospect theory differs from expected utility theory with some significant differences as explained below:

- Investors evaluate outcomes according to deviations (losses and gains) from certain benchmarks rather than final wealth positions.

- Investors have asymmetric risk averse behavior. The value function is concave for gains and convex for losses.

- Investors are more sensitive for losses and it is empirically shown that investors dislike losses by a factor of 2.25 when compared to likeness of gains.

- Probability assessments of investors are not objective. They tend to overweight small probabilities and underweight large probabilities.

Prospect theory is later extended by Tversky and Kahneman (1992) who employ cumulative decision weights. The difference from prospect theory is that the new version, cumulative prospect theory, basically transforms cumulative probabilities using weighting functions rather than individual probabilities. Moreover, different weighting functions are used for losses and gains.

There is limited literature about portfolio choice under prospect theory, though it is well-known model of decision-making. Benartzi and Thaler(1995) make an explanation for the "equity premium puzzle" with loss-aversion. Actually, the term is brought into the literature by Mehra and Prescott (1985). They observed that real returns of U.S. government bonds have been estimated at one percent per year, while real returns of stock in U.S. have been estimated at seven percent per year. The difference is too large to explain as a result of investor risk aversion. According to the standard economics models, the premium should be much lower. While Benartzi and Thaler(1995) settle to explain “equity premium puzzle” with only loss-aversion, Barberis et al. (2001) combine loss-aversion with the effect of prior outcomes of investment. As another difference from Benartzi and Thaler (1995), they consider models in a multiperiod context. However, in both articles probability weights are not used.

Gomes (2005) represented a new value function for loss-aversion. With piecewise power functions, value function is concave for gains, convex for small losses and concave again for large losses. Gomes (2005) claimed that decreasing marginal utility dominates psychological effect of the loss. In a market which consists of two assets, one risk free bond and one risky asset, Gomes (2005) provides an exact solution with the specified value function. To evaluate this result, he uses the result of power utility function as a benchmark. However, risky asset return is binomial and the probability distortation or weighting function is dismissed for simplicity.

De Giorgi and Hens (2006) modeled the portfolio selection problem with piecewise exponential value function rather than piecewise power function as in Tversky and Kahneman (1992). But, they choose appropriate parameters to satisfy the main features of prospect theory. Their value function is also convex for loses and concave for gains, but more risk-averse than the piecewise power function. They claim that, the marginal utility of wealth does not decrease sufficiently fast with piecewise power functions and, for almost all asset prices, optimal portfolios are unbounded. Since the marginal utility of wealth decreases sufficiently fast with piecewise exponential functions, there is a unique bounded solution. For normally distributed risky asset returns and the risk-free return as the reference point, De Giorgi and Hens (2006) show that CAPM equilibria exists for the piecewise exponential value function. However, this equilibria does not exist for prospect theory models with piecewise power value function.

Berkelaar et al. (2004) derived closed-form solutions to loss-averse preferences and investigated the impact of loss-averse behaviors on portfolio choices. They deal with the kinked concave function and the piecewise power function. Similar to Barberis et al. (2001) and Gomes (2005), they prefer to use dynamic reference point in their model. Since Berkelaar et al. (2004) is interested in the impact of loss-aversion, the probability weights are ignored.

Jin and Zhou (2008) is based on continuous-time portfolio selection under cumulative prospect theory. As in original prospect theory, they use piecewise power value function and probability weights. Because of the shape of the value function and probability weights, they claimed that the model could have an unbounded solution. When the model has a finite unique solution, they solved the portfolio selection problem in a market consisting of log-normal asset returns. However, they assumed that the reference point at terminal time is 0. As the crucial point in their research, they decompose the original portfolio selection problem into 2 parts; gain part and loss part. Then, the two problems are solved separately. At the last step, the solution to the original portfolio selection problem is derived.

Bernard and Ghossoub (2010) and He and Zhou (2011) consider single-period portfolio selection under cumulative prospect theory. They find the optimal portfolios for the one risky asset case. However, Bernard and Ghossoub (2010) does not use probability weights. While He and Zhou (2011) consider the cases when the reference point is the risk-free return and when it is not, Bernard and Ghossoub (2010) is based on only the risk-free return reference point. Also, He and Zhou (2011) introduces a new term, “large-loss aversion degree”, which measures the ratio between the pain of a substantial loss and the pleasure of a gain of the same magnitude. They uses this term for determining whether the model has a bounded or unbounded solution. Furthermore, He and Zhou (2011) makes some evaluations for different reference point cases using linear (not power) value functions.

Pirvu and Schulze (2012) also consider the single-period portfolio selection problem. They represent optimal portfolios for multiple risky assets which have multivariate elliptical distributions. They use the term “large-loss aversion degree” based on He and Zhou (2011) to determine whether the model has a finite unique solution or not. For different value functions, if there is a unique finite solution, they create semi-closed solutions directly. When the model has an infinite solution, they have to add a regularity constraint for limiting investors’ risk-lover behavior. Then, they find a semi-closed solution. Some different reference points are discussed in their study and they examine models with multivariate normal distributed and t-distributed asset returns for the risk-free return reference case.

Chapter 3

PROSPECT MODEL

Consider the single-period portfolio optimization problem under cumulative prospect theory where the market consists of one risk-free asset and n risky assets. Risk-free rate of return is r_f and risky assets have a stochastic rate of return vector $R = (R_1, R_2, \cdots, R_n)$ over the period. An investor allocates vector $u = (u_1, u_2, \cdots, u_n)$ from his initial wealth w_0 to the risky assets, and remaining amount $w_0 - u^T 1$ to the risk free asset. At the end of the period, the investor's wealth is given by

$$W = (w_0 - u^T 1)(1 + r_f) + u^T(1 + R)$$

or

$$W = w_0(1 + r_f) + u^T R^e \tag{3.1}$$

where $R^e = R - r_f$ is the vector of excess returns over the risk-free asset. Our notation is such that any vector x is a column vector unless stated otherwise and we will use its transpose x^T to denote a row vector.

The investor has a reference point in evaluating the final wealth denoted by w^{ref}. He makes assessments with respect to the reference point and distinguishes losses from gains. We suppose that the reference point has the form

$$w^{ref} = \theta w_0 + (1 - \theta)(1 + r_f)w_0 \tag{3.2}$$

for some $0 \leq \theta \leq 1$. Actually, we can say θ is relevance factor of initial wealth for the reference point. When $\theta = 1$ or $\theta = 0$, these are special cases and we call them as "initial wealth reference" and "risk-free" reference, respectively.

It follows from (3.2) that gains or losses with respect to the reference point is

$$W - w^{ref} = u^T R^e + \theta w_0 r_f. \tag{3.3}$$

When $\theta = 0$, (3.3) takes the simpler form $u^T R^e$ and gains/losses become independent of the initial wealth. In other words, gains/losses only depend on the portfolio. When $\theta > 0$, gains/losses depend on the initial wealth since the reference wealth depends on w_0.

In cumulative prospect theory, gains and losses are evaluated in a different way using different valuation functions and event probabilities. Let v^+ denote the value function for gains and v^- denote the value function for losses. Two additional functions: $T^+ : [0,1] \to [0,1]$ and $T^- : [0,1] \to [0,1]$ are used to describe the probability distortion for gains and losses, respectively. Tversky and Kahnemann (1992) suggest the functional forms

$$T^+(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}} \tag{3.4}$$

and

$$T^-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{1/\delta}} \tag{3.5}$$

for $0 \leq p \leq 1$. They propose the median values of γ and δ as 0.61, 0.69 respectively. With these parameters, shapes of $T^+(p)$ and $T^-(p)$ are shown in Figure 3.1.

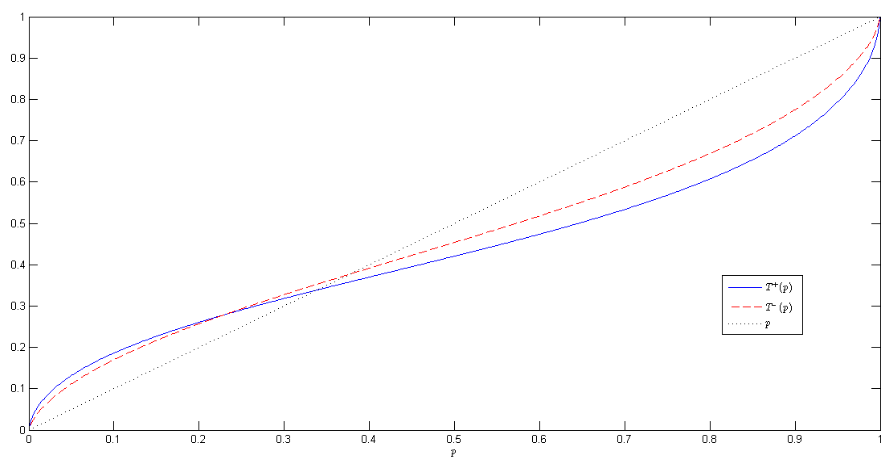

Figure 3.1: Probability distortion functions for gains and losses

In this thesis, we will use various value functions. Tversky and Kahneman (1992) model the decision-making problem with piecewise power value functions. However, we will analyze piecewise linear and piecewise exponential forms. Also, standard linear and exponential utility functions are used for comparisons as a benchmark.

Since behavior of investors have different pattern for gains and losses, value functions have different forms for both parts. So, they are stated as a partial function with respect to the reference point. Firstly, piecewise linear function has the form

$$v(x) = \begin{cases} v^+(x) = \lambda^+ x & x \geq 0 \\ v^-(x) = \lambda^- x & x < 0 \end{cases} \tag{3.6}$$

for any gain $x \geq 0$ or loss $x < 0$. Similarly, the piecewise exponential function has the form

$$v(x) = \begin{cases} v^+(x) = \lambda^+(1 - \exp(-\alpha x)) & x \geq 0 \\ v^-(x) = -\lambda^-(1 - \exp(\alpha x)) & x < 0 \end{cases} . \tag{3.7}$$

The shape of the piecewise linear and exponential value functions are also shown graphically as below,

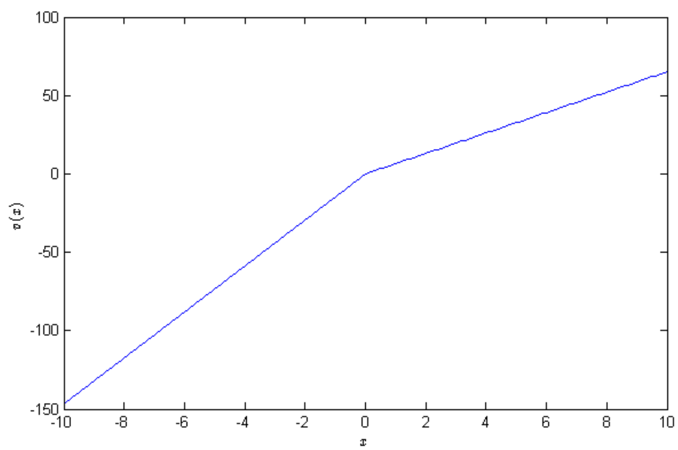

Figure 3.2: Piecewise linear value function

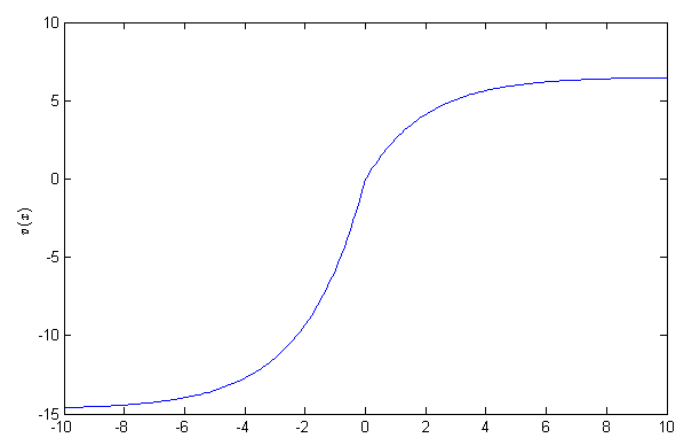

Figure 3.3: Piecewise exponential value function

For stated value functions in (3.6) and (3.7), the expected prospect value of the portfolio is

$$V(u) = \int_{-\infty}^{0} v^{-}(x) d[T^{-}(F_u(x))] + \int_{0}^{+\infty} v^{+}(x) d[-T^{+}(1 - F_u(x))]$$

where $F_u(x) = P\{u^T R^e + \theta w_0 r_f \leq x\}$.

We try to solve the optimization problem of maximizing the prospect value

$$\max_{u} \; V(u) = E[v(u^T R^e + \theta w_o r_f)]$$

where the value function v is given by (3.6), (3.7).

Tversky and Kahnemann (1992) provide evidence that $\lambda^{-} > \lambda^{+}$ and extended their research based on this assumption. Like them, we assume $\lambda^{-} > \lambda^{+}$ in our main research. But, we also take $\lambda^{-} \leq \lambda^{+}$ for some models and analyzed the effects of this assumption. Tversky and Kahnemann (1992) used the piecewise power function in their behavioral model. However, De Giorgi and Hens (2006) stated that the piecewise exponential value function is more suitable than the piecewise power value function for portfolio optimization problem. So, we concentrated on the models with the piecewise exponential value function. Moreover, as in Berkelaar et al. (2004), we analyzed the effect of the value function rather than the probability weights. Therefore, we assumed $T^{+}(p) = T^{-}(p) = p$ throughout this thesis.

Chapter 4

SINGLE ASSET MODEL

In this chapter, we analyze the portfolio selection problem where the market consists of the risk-free asset and only one risky asset. For the single period setting, we generate closed-form solutions for a number of value functions.

4.1 Piecewise Linear Value Function

Using (3.6), optimal choices of prospect investors is analyzed. The structure of value function is specified by λ^+ and λ^-. More precisely, when $\lambda^+ \leq \lambda^-$ the value function is concave and when $\lambda^+ \geq \lambda^-$ the value function is convex. According to the structure of the value function, characterizations of the optimal portfolio differ.

4.1.1 Risk-free Reference Point

Taking $\theta = 0$, (3.3) has a simpler form and the term reduces to $W - w^{ref} = uR^e$. So, optimal portfolios are independent from the initial wealth. We now consider a number of cases with different asset returns.

General Return Model

Portfolio optimization problem with the piecewise linear value function and risk-free reference point can be stated as

$$\max_u \; V(u) = E[v(uR^e)]$$

where v has the form denoted by (3.6).

The objective function is $V(u) = E[v(uR^e)]$ which we want to maximize by choosing u. Note that we can also write

$$\begin{aligned} V(u) &= E[v^-(uR^e)1_{\{uR^e<0\}}] + E[v^+(uR^e)1_{\{uR^e\geq 0\}}] \\ &= \lambda^- E[uR^e 1_{\{uR^e<0\}}] + \lambda^+ E[uR^e 1_{\{uR^e\geq 0\}}] \\ &= \begin{cases} \lambda^- uE[R^e 1_{\{R^e<0\}}] + \lambda^+ uE[R^e 1_{\{R^e\geq 0\}}] & u \geq 0 \\ \lambda^- uE[R^e 1_{\{R^e>0\}}] + \lambda^+ uE[R^e 1_{\{R^e\leq 0\}}] & u < 0 \end{cases} . \end{aligned} \tag{4.1}$$

It follows trivially that $V(0) = 0$ and $V(u)$ is continuos on $(-\infty, +\infty)$. Moreover, it is piecewise linear with a possible kink at $u = 0$.

Taking the derivative of V with respect to u, we obtain

$$\begin{aligned}\frac{dV(u)}{du} &= \begin{cases} \lambda^- E[R^e 1_{\{R^e<0\}}] + \lambda^+ E[R^e 1_{\{R^e\geq 0\}}] & u \geq 0 \\ \lambda^- E[R^e 1_{\{R^e>0\}}] + \lambda^+ E[R^e 1_{\{R^e\leq 0\}}] & u < 0 \end{cases} \\ &= \begin{cases} \lambda^- E[R^e] - (\lambda^- - \lambda^+) E[R^e 1_{\{R^e\geq 0\}}] & u \geq 0 \\ \lambda^- E[R^e] - (\lambda^- - \lambda^+) E[R^e 1_{\{R^e\leq 0\}}] & u < 0 \end{cases} .\end{aligned} \tag{4.2}$$

If we take the second derivative of V, then it is obvious that it is equal to 0 for $u \neq 0$. Note also that $E[R^e 1_{\{R^e>0\}}\}] \geq 0$ and $E[R^e 1_{\{R^e\leq 0\}}\}] \leq 0$. Moreover, the derivative of V is constant on $u > 0$ and $u < 0$, and it depends on $E[R^e]$, $E[R^e 1_{\{R^e\geq 0\}}\}]$, and $E[R^e 1_{\{R^e\leq 0\}}\}]$ as well as λ^+ and λ^-. According to the superiority relation between $\lambda^- E[R^e]$, $(\lambda^- - \lambda^+) E[R^e 1_{\{R^e\geq 0\}}\}]$, and $(\lambda^- - \lambda^+) E[R^e 1_{\{R^e\leq 0\}}\}]$; optimal portfolio is equal to $-\infty$, 0 or $+\infty$. This follows from the fact that the objective function $V(u)$ is piecewise linear with a possible kink at $u = 0$ and the optimal portfolio u^* depend on the following cases. We let $\lambda = \lambda^+/\lambda^-$ to simplify the notation.

Case I : $\lambda < 1$.

1. If $E[R^e] \geq 0$, then

$$u^* = \begin{cases} 0 & E[R^e] < (1-\lambda) E[R^e 1_{\{R^e\geq 0\}}] \\ [0, +\infty) & E[R^e] = (1-\lambda) E[R^e 1_{\{R^e\geq 0\}}] \\ +\infty & E[R^e] > (1-\lambda) E[R^e 1_{\{R^e\geq 0\}}] \end{cases} . \tag{4.3}$$

2. If $E[R^e] < 0$, then

$$u^* = \begin{cases} 0 & E[R^e] > (1-\lambda) E[R^e 1_{\{R^e<0\}}] \\ (-\infty, 0] & E[R^e] = (1-\lambda) E[R^e 1_{\{R^e<0\}}] \\ -\infty & E[R^e] < (1-\lambda) E[R^e 1_{\{R^e<0\}}\}] \end{cases} . \tag{4.4}$$

Case II : $\lambda = 1$.

1. If $E[R^e] > 0$, then

$$u^* = +\infty. \tag{4.5}$$

2. If $E[R^e] = 0$, then

$$u^* = (-\infty, +\infty). \tag{4.6}$$

3. If $E[R^e] < 0$, then
$$u^* = -\infty. \tag{4.7}$$

Case III : $\lambda > 1$.

1. If $E[R^e] > 0$, then
$$u^* = \begin{cases} +\infty, -\infty & E[R^e] \leq (1-\lambda)E[R^e 1_{\{R^e \leq 0\}}] \\ +\infty & E[R^e] > (1-\lambda)E[R^e 1_{\{R^e \leq 0\}}] \end{cases}. \tag{4.8}$$

2. If $E[R^e] = 0$, then
$$u^* = -\infty, +\infty. \tag{4.9}$$

3. If $E[R^e] < 0$, then
$$u^* = \begin{cases} -\infty & E[R^e] < (1-\lambda)E[R^e 1_{\{R^e \geq 0\}}] \\ -\infty, +\infty & E[R^e] \geq (1-\lambda)E[R^e 1_{\{R^e \geq 0\}}] \end{cases}. \tag{4.10}$$

We demonstrate all possible optimal portfolios under the piecewise linear value function and the risk-free reference point. However, for limited cases portfolio selection problem is bounded and the optimal portfolio is $u^* = 0$ and the portfolio does not include the risky asset. Since the objective function is piecewise linear with a kink at $u = 0$, the optimal solution will be unbounded with $u^* = +\infty$ or $u^* = -\infty$ depending on the slopes for $u > 0$ and $u < 0$. Clearly, $u^* = +\infty$ or $u^* = -\infty$ is not realistic since it is not possible to invest or short-sell on infinite amount because of resource limitations. Therefore, one should interpret such solutions so that $u^* = +\infty$ implies that one should invest “all he could” in the risky asset, and $u^* = -\infty$ implies that one should short-sell “all he could” the risky asset in favor of the risky asset. When $\lambda < 1$, there are some levels of the expected excess return beyond which investors start buying or selling. It follows that for $E[R^e] \geq 0$, $(1-\lambda)E[R^e 1_{\{R^e > 0\}}]$ is a level to buy and if $E[R^e]$ exceeds or is equal to this level then the investor buys the risky asset. Similarly, $(1-\lambda)E[R^e 1_{\{R^e \leq 0\}}]$ is a level to sell, if $E[R^e]$ drops under this level investor start to sell. If $E[R^e]$ is between the two levels, then the investor sells or buys nothing and optimal portfolio becomes zero.

Note that the case when $\lambda = 1$ corresponds to the risk-neutral model. In this case, $u^* = +\infty$ if $E[R^e] > 0$, $u^* = -\infty$ if $E[R^e] < 0$ and any $-\infty \leq u < +\infty$ is optimal when $E[R^e] = 0$.

Finally, take $\lambda > 1$. We check the possible cases for $E[R^e] > 0$ using (4.2). Since $\lambda^- E[R^e] - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e \geq 0\}}] > 0$, $V(u)$ is a linear increasing function for $u \geq 0$.

However, $\lambda^- E[R^e] - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e \leq 0\}}]$ can take positive or negative values. If it takes a positive value, then $V(u)$ is a linear increasing function for $u < 0$. Clearly, $V(u)$ is an increasing function for all u and $V(u)$ has the different slopes for $u \geq 0$ and $u < 0$. In that case, the optimal portfolio is $u^* = +\infty$. If $\lambda^- E[R^e] - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e \leq 0\}}]$ takes a negative value, then $V(u)$ is a linear decreasing function for $u < 0$. In other words, $V(u)$ is a V-shaped function and optimal portfolios are $u^* = -\infty, +\infty$. However, $V(u)$ has the different absolute slopes for $u \geq 0$ and $u < 0$. Since $|\lambda^- E[R^e] - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e \leq 0\}}]| < |\lambda^- E[R^e] - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e \geq 0\}}]|$, $V(u)$ goes to infinity faster in the positive region. Due to the limitations of investor's resource, only $+\infty$ can be applied as an optimal portfolio practically. For $E[R^e] < 0$, it is obvious that $\lambda^- E[R^e] - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e \leq 0\}}] < 0$. Therefore, $V(u)$ is an increasing function for $u < 0$. For $u \geq 0$, $dV(u)/du$ can be negative or positive. If $dV(u)/du$ is negative then $V(u)$ is decreasing for all u and $u^* = -\infty$. If $dV(u)/du$ is positive then $V(u)$ is increasing for $u \geq 0$ and $V(u)$ is a V-shaped function where the relation between slopes is $|\lambda^- E[R^e] - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e \leq 0\}}]| > |\lambda^- E[R^e] - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e > 0\}}]|$. In this case, there are two optimal portfolios $u^* = -\infty, +\infty$. But, $V(u)$ goes to infinity faster for negative region and the optimal portfolio u^* becomes $-\infty$ practically. When $E[R^e] = 0$, $dV(u)/du = -(\lambda^- - \lambda^+)E[R^e 1_{\{R^e \geq 0\}}]$ for $u \geq 0$ and $dV(u)/du = -(\lambda^- - \lambda^+)E[R^e 1_{\{R^e < 0\}}]$ for $u < 0$. Also, we know that $|E[R^e 1_{\{R^e \geq 0\}}]| = |E[R^e 1_{\{R^e < 0\}}]|$ for $E[R^e] = 0$. Then, $V(u)$ is again a V-shaped function but it has same absolute slope and there are two optimal portfolios $u^* = -\infty, +\infty$.

In more detail, we now make some observations using different distributions for the excess return in the following subsections. Since, when $\lambda \geq 1$ the optimal portfolios are trivial, we concentrate on the case $\lambda < 1$.

Binomial Return Model

Suppose that excess return R^e comes from the binomial distribution. With probability p_1, R^e equals $\delta_1 > 0$ and with probability p_2, R^e equals $-\delta_2 < 0$ where $\lambda^-, \lambda^+ > 0$. Expected excess return is

$$E[R^e] = p_1\delta_1 - p_2\delta_2$$

and the value function becomes

$$V(u) = \begin{cases} (\lambda^+ p_1\delta_1 - \lambda^- p_2\delta_2)u & u \geq 0 \\ (\lambda^- p_1\delta_1 - \lambda^+ p_2\delta_2)u & u < 0 \end{cases} .$$

Clearly, the value function is linear and the optimal solution can be $-\infty, 0, +\infty$ depending on the sign of $(\lambda^+ p_1\delta_1 - \lambda^- p_2\delta_2)$ and $(\lambda^- p_1\delta_1 - \lambda^+ p_2\delta_2)$, using the results in (4.3)-(4.10).

To simplify the notation, let p and δ, denote p_1/p_2 and δ_1/δ_2 respectively. Optimal portfolios are classified below.

1. If $E[R^e] \geq 0$, then

$$u^* = \begin{cases} 0 & \lambda p\delta < 1 \\ [0, +\infty) & \lambda p\delta = 1 \\ +\infty & \lambda p\delta > 1 \end{cases} .$$

2. If $E[R^e] < 0$, then

$$u^* = \begin{cases} 0 & \lambda < p\delta \\ (-\infty, 0] & \lambda = p\delta \\ -\infty & \lambda > p\delta \end{cases} .$$

Exponential Return Model

In this chapter, we assume that the risky asset returns R is exponentially distributed with rate μ so that $E[R] = 1/\mu$. Let $q = 1 - \lambda$ $(0 \leq q \leq 1)$ and $R_f = 1 + r_f$. If $E[R] \geq R_f$ or $\mu R_f \leq 1$, by (4.3), the optimal portfolio is $u^* = 0$ provided that

$$\begin{aligned} E[R^e] &< qE[R^e 1_{\{R^e > 0\}}] \\ &< qE[R^e 1_{\{R - R_f > 0\}}] \\ &< q\int_{R_f}^{+\infty} (x - R_f)\mu e^{-\mu x} dx \\ &< q\int_{0}^{+\infty} y\mu e^{-\mu(y+Rf)} dy \\ &< qe^{-\mu R_f}\int_{0}^{+\infty} y\mu e^{-\mu y} dy \\ &< q(\frac{e^{-\mu R_f}}{\mu}). \end{aligned} \tag{4.11}$$

Noting that $E[R^e] = E[R - R_f] = 1/\mu - R_f$, we can rewrite (4.11) as

$$R_f > \frac{1}{\mu}(1 - qe^{-\mu R_f})$$

or

$$\mu R_f > 1 - qe^{-\mu R_f}$$

or

$$\hat{\mu} > 1 - qe^{-\hat{\mu}}$$

where $\hat{\mu} = \mu R_f$.

Let $h_1(x) = 1 - qe^{-x}$. Then, $dh_1(x)/dx = qe^{-x} \geq 0$ and $d^2h_1(x)/dx^2 = -qe^{-x} \leq 0$ so that h_1 is a concave increasing function with $h_1(0) = 1 - q$ and $h_1(1) = 1 - qe^{-1} < 1$. As a result, on $[0, 1]$, there is only one solution $\hat{\mu}_1$ which satisfies

$$\hat{\mu}_1 = 1 - qe^{-\hat{\mu}_1}. \tag{4.12}$$

Thus, we can say that there is a level $\hat{\mu}_1$ such that if $\hat{\mu}$ is greater than $\hat{\mu}_1$, optimal portfolio is 0, otherwise it is unbounded. In other words,

$$u^* = \begin{cases} 0 & \mu R_f > \hat{\mu}_1 \text{ or } E[R^e] < \frac{1-\hat{\mu}_1}{\mu} \\ [0, +\infty) & \mu R_f = \hat{\mu}_1 \text{ or } E[R^e] = \frac{1-\hat{\mu}_1}{\mu} \\ +\infty & \mu R_f < \hat{\mu}_1 \text{ or } E[R^e] > \frac{1-\hat{\mu}_1}{\mu} \end{cases} . \tag{4.13}$$

If $E[R] < R_f$ or $\mu R_f > 1$, by (4.4), the optimal portfolio is $u^* = 0$ provided that

$$\begin{aligned} E[R^e] &\geq qE[R^e 1_{\{R^e \leq 0\}}] \\ &\geq qE[R^e 1_{\{R - R_f \leq 0\}}] \\ &\geq q\int_0^{R_f} (x - R_f)\mu e^{-\mu x} dx \\ &\geq q(\frac{1 - e^{-\mu R_f}}{\mu} - R_f) \\ &\geq -\frac{qe^{-\mu R_f}}{(1-q)\mu}. \end{aligned} \tag{4.14}$$

Noting that $E[R - R_f] = 1/\mu - R_f$, we can rewrite (4.14) as

$$R_f \leq \frac{1}{\mu}(1 + \frac{qe^{-\mu R_f}}{(1-q)})$$

or

$$\mu R_f \leq 1 + \frac{q}{(1-q)} e^{-\mu R_f}$$

or

$$\hat{\mu} \leq 1 + \frac{q}{(1-q)} e^{-\hat{\mu}}.$$

Let $h_2(x) = 1 + qe^{-x}/(1-q)$. Then, $dh_2(x)/dx = -qe^{-x}/(1-q) \leq 0$ and $d^2h_2(x)/dx^2 = qe^{-x}/(1-q) \geq 0$ so that h_2 is convex decreasing function with $h_2(1) = 1 + qe^{-1}/(1-q)$ and $h_2(+\infty) = 1$. As a result, on $[1, +\infty)$, there is only one solution $\hat{\mu}_2$ which satisfies

$$\hat{\mu}_2 = 1 - \frac{q}{(1-q)} e^{-\hat{\mu}_2}. \tag{4.15}$$

Thus, we can say that there is a level such that if $\overset{\wedge}{\mu}$ is less than $\overset{\wedge}{\mu}_2$, optimal portfolio is 0, otherwise unbounded. In other words,

$$u^* = \begin{cases} 0 & \mu R_f < \overset{\wedge}{\mu}_2 \text{or } E[R^e] > \frac{1-\overset{\wedge}{\mu}_2}{\mu} \\ (-\infty, 0] & \mu R_f = \overset{\wedge}{\mu}_1 \text{or } E[R^e] = \frac{1-\overset{\wedge}{\mu}_2}{\mu} \\ -\infty & \mu R_f > \overset{\wedge}{\mu}_2 \text{or } E[R^e] < \frac{1-\overset{\wedge}{\mu}_2}{\mu} \end{cases} . \tag{4.16}$$

Normal Return Model

We now suppose that the excess return has the normal distribution. First, we will define some notations to simplify the expressions for the standard normal distribution. After introducing the new notations, we will mention about some properties of the standard normal distribution. Later on, we use these properties for our inferences.

The standard normal density function is

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

and the standard normal cumulative function is represented by the integral

$$\Phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

Noting that $\varphi(x) = \varphi(-x)$, φ is a symmetric function with respect to zero. Moreover, Φ is the cumulative function of φ and

$$\lim_{y \to +\infty} \Phi(y) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1.$$

Due to the symmetry of φ, $\Phi(-y) = 1 - \Phi(y)$. We, also define Ψ so that

$$\Psi(y) = \int_{y}^{+\infty} \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

or

$$\Psi(y) = \frac{1}{\sqrt{2\pi}} \int_{\frac{y^2}{2}}^{+\infty} e^{-t} dt$$

or

$$\Psi(y) = \left(-\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right) |_{y}^{+\infty}$$

or

$$\Psi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

or

$$\Psi(y) = \varphi(y).$$

Suppose that the excess return R^e is normally distributed with mean μ and standard deviation σ. To use our results in Chapter 4.1.1, we must determine $E[R^e 1_{\{R^e<0\}}]$, $E[R^e 1_{\{R^e\geq 0\}}]$ for the normal distribution. In that case,

$$\begin{aligned}
E[R^e 1_{\{R^e\geq 0\}}] &= \int_0^{+\infty} \frac{x}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
&= \int_{-\frac{\mu}{\sigma}}^{+\infty} \frac{(\mu+\sigma z)}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \mu \int_{-\frac{\mu}{\sigma}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \sigma \int_{-\frac{\mu}{\sigma}}^{+\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \mu \left(\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - \int_{-\infty}^{-\frac{\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \right) + \sigma \int_{-\frac{\mu}{\sigma}}^{+\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \mu \left(1 - \Phi\left(-\frac{\mu}{\sigma}\right)\right) + \sigma \Psi\left(-\frac{\mu}{\sigma}\right) \\
&= \mu \Phi\left(\frac{\mu}{\sigma}\right) + \sigma \varphi\left(\frac{\mu}{\sigma}\right) \qquad (4.17)
\end{aligned}$$

and

$$\begin{aligned}
E[R^e 1_{\{R^e<0\}}] &= \int_{-\infty}^{0} \frac{x}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
&= \int_{-\infty}^{-\frac{\mu}{\sigma}} \frac{\mu+\sigma z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \mu \int_{-\infty}^{-\frac{\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \sigma \int_{-\infty}^{-\frac{\mu}{\sigma}} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \mu \int_{-\infty}^{-\frac{\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \sigma \left(\int_{-\infty}^{+\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - \int_{-\frac{\mu}{\sigma}}^{+\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \right) \\
&= \mu \Phi\left(-\frac{\mu}{\sigma}\right) - \sigma \Psi\left(-\frac{\mu}{\sigma}\right) \\
&= \mu \Phi\left(-\frac{\mu}{\sigma}\right) - \sigma \varphi\left(\frac{\mu}{\sigma}\right). \qquad (4.18)
\end{aligned}$$

Then, for $\mu \geq 0$, it follows from (4.3) that $u^* = 0$ if

$$E[R^e] < (1-\lambda) E[R^e 1_{\{R^e\geq 0\}}]$$

or

$$\mu < (1-\lambda)(\mu\Phi(\frac{\mu}{\sigma}) + \sigma\varphi(\frac{\mu}{\sigma}))$$

or

$$\frac{1}{(1-\lambda)} < \Phi(\frac{\mu}{\sigma}) + \frac{\sigma}{\mu}\varphi(\frac{\mu}{\sigma}). \tag{4.19}$$

Let $x = \mu/\sigma$ and $h_1(x) = \Phi(x) + \varphi(x)/x$ for $x > 0$. Note that, h_1 is a decreasing function with respect to x since

$$\begin{aligned} \frac{dh_1(x)}{dx} &= \varphi(x) + \frac{\varphi'(x)x - \varphi(x)}{x^2} \\ &= \varphi(x) + \frac{-\varphi(x)x^2 - \varphi(x)}{x^2} \\ &= -\frac{\varphi(x)}{x^2} \leq 0. \end{aligned} \tag{4.20}$$

Moreover, $\lim_{x\downarrow 0+} h_1(x) = +\infty$ and $\lim_{x\longrightarrow+\infty} h_1(x) = 1$. Also, $1/(1-\lambda) > 1$ for $\lambda < 1$ and it is a constant level. So, there is a unique $\bar{x_1} \geq 0$ which satisfies $h_1(\bar{x_1}) = 1/(1-\lambda)$. Consequently, it is a critical level such that if $x < \bar{x_1}$ or $\mu < \sigma\bar{x_1}$ then $u^* = 0$, and if $x > \bar{x}$ or $\mu > \sigma\bar{x}$ then $u^* = +\infty$.

Now, suppose that $\mu < 0$, then it follows from (4.4) that $u^* = 0$ provided that

$$E[R^e] > (1-\lambda)E[R^e 1_{\{R^e<0\}}]$$

or

$$\mu > (1-\lambda)(\mu\Phi(-\frac{\mu}{\sigma}) - \sigma\varphi(\frac{\mu}{\sigma}))$$

or

$$\frac{1}{1-\lambda} > \Phi(-\frac{\mu}{\sigma}) - \frac{\sigma}{\mu}\varphi(\frac{\mu}{\sigma}). \tag{4.21}$$

Let $x = \mu/\sigma$ and $h_2(x) = \Phi(-x) - \varphi(x))/x$ for $x < 0$. Note that, h_2 is an increasing function with respect to x since

$$\begin{aligned} \frac{dh_2(x)}{dx} &= -\varphi(-x) - \frac{\varphi'(x)x - \varphi(x)}{x^2} \\ &= -\varphi(x) - \frac{-\varphi(x)x^2 - \varphi(x)}{x^2} \\ &= \frac{\varphi(x)}{x^2} \geq 0. \end{aligned} \tag{4.22}$$

Moreover, $\lim_{x\longrightarrow-\infty} h_2(x) = 1$ and $\lim_{x\uparrow 0-} h_2(x) = +\infty$. Since $1/(1-\lambda) > 1$ and h_2 is an increasing function, there is a unique $\bar{x_2} < 0$ which satisfies $h_2(\bar{x_2}) = 1/(1-\lambda)$. It is a critical level such that if $x > \bar{x_2}$ or $\mu > \sigma\bar{x_2}$ then $u^* = 0$, and if $x < \bar{x_2}$ or $\mu < \sigma\bar{x_2}$ then $u^* = -\infty$.

To sum up, optimal portfolios stated by (4.3)-(4.4) can be rewritten as follows:

Case $I : \lambda < 1$.

1. If $\mu \geq 0$, then

$$u^* = \begin{cases} 0 & \mu < \sigma \bar{x_1} \\ 0 \leq u^* \leq +\infty & \mu = \sigma \bar{x_1} \\ +\infty & \mu > \sigma \bar{x_1} \end{cases}.$$

2. If $\mu < 0$, then

$$u^* = \begin{cases} 0 & \mu > \sigma \bar{x_2} \\ -\infty \leq u^* \leq 0 & \mu = \sigma \bar{x_2} \\ -\infty & \mu < \sigma \bar{x_2} \end{cases}.$$

4.1.2 General Reference Point

In this part, we take $\theta > 0$ to analyze the effect of the initial wealth on the optimal portfolio. Also, we assume that the initial wealth of investor w_0 is greater than 0. Unlike the risk-free reference point model optimal portfolios now vary with the initial wealth.

General Return Model

If we take any point above the initial wealth as a reference point, the portfolio optimization problem gets more difficult. Using (3.3), our problem now becomes

$$\max_u \; V(u) = \max_u E[v(uR^e + \theta w_0 r_f)]$$

where $0 \leq \theta \leq 1$.

The objective function is $V(u) = E[v(uR^e + \theta w_0 r_f)]$ which we want to maximize by choosing u. Note that we can also write,

$$\begin{aligned} V(u) &= E[v^-(uR^e + \theta w_0 r_f)1_{\{uR^e < -\theta w_0 r_f\}}] + E[v^+((uR^e + \theta w_0 r_f))1_{\{uR^e \geq -\theta w_0 r_f\}}] \\ &= \lambda^- E[(uR^e + \theta w_0 r_f)1_{\{uR^e < -\theta w_0 r_f\}}] + \lambda^+ E[(uR^e + \theta w_0 r_f)1_{\{uR^e \geq -\theta w_0 r_f\}}] \\ &= \begin{cases} \lambda^- E[(uR^e + \theta w_0 r_f)1_{\{R^e < -\frac{\theta w_0 r_f}{u}\}}] + \lambda^+ E[(uR^e + \theta w_0 r_f)1_{\{R^e \geq -\frac{\theta w_0 r_f}{u}\}}] & u \geq 0 \\ \lambda^- E[(uR^e + \theta w_0 r_f)1_{\{R^e > -\frac{\theta w_0 r_f}{u}\}}] + \lambda^+ E[(uR^e + \theta w_0 r_f)1_{\{R^e \leq -\frac{\theta w_0 r_f}{u}\}}] & u < 0 \end{cases} \\ &= \begin{cases} \lambda^- E[uR^e + \theta w_0 r_f] - (\lambda^- - \lambda^+)E[(uR^e + \theta w_0 r_f)1_{\{R^e \geq -\frac{\theta w_0 r_f}{u}\}}] & u \geq 0 \\ \lambda^- E[uR^e + \theta w_0 r_f] - (\lambda^- - \lambda^+)E[(uR^e + \theta w_0 r_f)1_{\{R^e \leq -\frac{\theta w_0 r_f}{u}\}}] & u < 0 \end{cases}. \end{aligned} \quad (4.23)$$

In short form, derivatives of V with respect to u can be written as

$$\frac{dV(u)}{du} = E[R^e v^{'}(uRe + \theta w_0 r_f)] \tag{4.24}$$

and

$$\frac{d^2V(u)}{du^2} = E[(R^e)^2 v^{''}(uRe + \theta w_0 r_f)]$$

whenever the derivatives exists. Also, $V'' = 0$ since the value function is piecewise linear.

Depending on the structure of the function v, the second derivative of V will be nonpositive or nonnegative and the structure of V will change. If $\lambda < 1$, the second derivative of V is nonpositive and V is concave. If $\lambda > 1$, the second derivative of V is nonnegative and V is convex. Apparently, V is linear when $\lambda = 1$.

When V is concave, optimal portfolio u^* must satisfy the optimality condition

$$\frac{dV(u)}{du} = E[R^e v^{'}(uRe + \theta w_0 r_f)] = 0. \tag{4.25}$$

When V is convex, we must investigate the optimal portfolio at a boundary since our problem involves maximization.

The derivative of V helps us to characterize the optimal portfolio. Suppose f is a probability density function of excess return R^e. Then, we can rewrite (4.23) as

$$V(u) = \begin{cases} \lambda^-(uE[R^e] + \theta w_0 r_f) - (\lambda^- - \lambda^+) \displaystyle\int_{-\frac{\theta w_0 r_f}{u}}^{+\infty} (ux + \theta w_0 r_f) f(x) dx & u \geq 0 \\ \lambda^-(uE[R^e] + \theta w_0 r_f) - (\lambda^- - \lambda^+) \displaystyle\int_{-\infty}^{-\frac{\theta w_0 r_f}{u}} (ux + \theta w_0 r_f) f(x) dx & u < 0 \end{cases}. \tag{4.26}$$

Using (4.26), we obtain the derivative

$$\begin{aligned} \frac{dV(u)}{du} &= \begin{cases} \lambda^- E[R^e] - (\lambda^- - \lambda^+) \displaystyle\int_{-\frac{\theta w_0 r_f}{u}}^{+\infty} x f(x) dx & u \geq 0 \\ \lambda^- E[R^e] - (\lambda^- - \lambda^+) \displaystyle\int_{-\infty}^{-\frac{\theta w_0 r_f}{u}} x f(x) dx & u < 0 \end{cases} \\ &= \begin{cases} \lambda^- E[R^e] - (\lambda^- - \lambda^+) E[R^e 1_{\{R^e \geq -\frac{\theta w_0 r_f}{u}\}}] & u \geq 0 \\ \lambda^- E[R^e] - (\lambda^- - \lambda^+) E[R^e 1_{\{R^e < -\frac{\theta w_0 r_f}{u}\}}] & u < 0 \end{cases}. \end{aligned} \tag{4.27}$$

Therefore, the derivative of V is the difference of two parts. The first component $\lambda^- E[R^e]$ is constant and the second component is a function of u. The second derivative of V with respect to u is

$$\frac{d^2V(u)}{du^2} = \begin{cases} -(\lambda^- - \lambda^+)\frac{(\theta w_0 r_f)^2}{u^3} f(-\frac{\theta w_0 r_f}{u}) & u \geq 0 \\ (\lambda^- - \lambda^+)\frac{(\theta w_0 r_f)^2}{u^3} f(-\frac{\theta w_0 r_f}{u}) & u < 0 \end{cases}. \tag{4.28}$$

Firstly, consider the case where $\lambda < 1$ or $V(u)$ is concave. In (4.28), f is a probability density function and it only takes positive values. Then, it is clear that $d^2V(u)/du^2 < 0$. So, $dV(u)/du$ is a decreasing function with respect to u. It has maximum value at

$$\lim_{u \to -\infty} dV(u)/du = \lambda^- E[R^e] - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e < 0\}}]$$

and minimum value at

$$\lim_{u \to +\infty} dV(u)/du = \lambda^- E[R^e] - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e \geq 0\}}].$$

Also, we know that $dV(u)/du = \lambda^+ E[R^e]$ when $u = 0$. This information gives us a clue about optimal portfolios.

If there is a finite optimal portfolio, it must satisfy the optimality condition (4.25). Using (4.27), the optimality condition can be updated as

$$\left.\begin{array}{ll} \lambda^- E[R^e] - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e \geq -\frac{\theta w_0 r_f}{u}\}}] & u \geq 0 \\ \lambda^- E[R^e] - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e < -\frac{\theta w_0 r_f}{u}\}}] & u < 0 \end{array}\right\} = 0$$

or

$$\left.\begin{array}{ll} E[R^e 1_{\{R^e \geq -\frac{\theta w_0 r_f}{u}\}}]/E[R^e] & u \geq 0 \\ E[R^e 1_{\{R^e < -\frac{\theta w_0 r_f}{u}\}}]/E[R^e] & u < 0 \end{array}\right\} = \frac{1}{1-\lambda}. \tag{4.29}$$

Surely, if $\lambda^- E[R^e] - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e < 0\}}] \leq 0$ then $dV(u)/du < 0$ for all u. This implies that $V(u)$ is a decreasing function and the optimal portfolio is $u^* = -\infty$. When $\lambda^- E[R^e] - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e \geq 0\}}] \geq 0$, $V(u)$ is a increasing function and the optimal portfolio is $u^* = +\infty$. As mentioned previously, $u^* = -\infty$ and $u^* = +\infty$ are not realistic solutions. So, we use them to state that the investor should short-sell "all he could" the risky asset in favor of the risky asset and he should invest "all he could" in the risky asset, respectively.

However, if $\lambda^- E[R^e] - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e < 0\}}] > 0$ and $\lambda^- E[R^e] - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e \geq 0\}}] < 0$ then there is a finite solution u^* that satisfies the optimality condition (4.29). Clearly, when $E[R^e] = 0$ since $dV(u)/du = 0$ at $u = 0$, the optimal solution is $u^* = 0$. When $E[R^e] < 0$, there is a solution such that the optimality condition holds in the region $(-\infty, 0)$. That also make sense intuitively. It is obvious that if there is negative expected return in the risky asset, investor wants to short-sell the risky asset. When $E[R^e] > 0$, optimal portfolio u^* lays in the region $(0, +\infty)$ and investor wants to invest his money in the risky asset.

To summarize, in some cases there is an infinite optimal solution and in some cases there is a finite optimal solution. Moreover, that finite optimal solution u^* satisfies the optimality condition (4.29). All solutions are gathered in the following summary.

Case I. $\lambda < 1$.

1. If $E[R^e] \geq 0$, then

$$u^* = \begin{cases} 0 \leq u^* < +\infty & E[R^e] < (1-\lambda)E[R^e 1_{\{R^e>0\}}] \\ +\infty & E[R^e] \geq (1-\lambda)E[R^e 1_{\{R^e>0\}}] \end{cases} \tag{4.30}$$

   where u^* satisfies $E[R^e 1_{\{R^e \geq -\theta w_0 r_f/u^*\}}]/E[R^e] = 1/(1-\lambda)$. Here, note that $E[R^e 1_{\{R^e \geq -\theta w_0 r_f/u\}}]/E[R^e]$ is increasing in u for $u \geq 0$. It takes its minimum value at 0 which is 1 and maximum value at $+\infty$ which is $E[R^e 1_{\{R^e \geq 0\}}]/E[R^e] \geq 0$. If $E[R^e 1_{\{R^e \geq 0\}}]/E[R^e] < 1/(1-\lambda)$, then the objective function is continuously increasing in u and there is an unbounded solution. However, if $E[R^e 1_{\{R^e \geq 0\}}]/E[R^e] > 1/(1-\lambda)$, then there is a unique bounded portfolio that maximizes the objective function.

2. If $E[R^e] < 0$, then

$$u^* = \begin{cases} -\infty < u^* < 0 & E[R^e] > (1-\lambda)E[R^e 1_{\{R^e \leq 0\}}] \\ -\infty & E[R^e] \leq (1-\lambda)E[R^e 1_{\{R^e \leq 0\}}] \end{cases} . \tag{4.31}$$

   for u^* satisfies $E[R^e 1_{\{R^e < -\theta w_0 r_f/u^*\}}]/E[R^e] = 1/(1-\lambda)$. Now, one can observe that $E[R^e 1_{\{R^e < -\theta w_0 r_f/u\}}]/E[R^e]$ is decreasing in u for $u \leq 0$. It takes its minimum value at 0 and maximum value at $-\infty$ which is $E[R^e 1_{\{R^e < 0\}}]/E[R^e] \geq 0$. If $E[R^e 1_{\{R^e < 0\}}]/E[R^e] < 1/(1-\lambda)$, then the objective function is continuously decreasing in u and there is an unbounded solution. However, if $E[R^e 1_{\{R^e < 0\}}]/E[R^e] > 1/(1-\lambda)$, then there is a unique bounded portfolio that maximizes the objective function.

Consider $\lambda = 1$. Due to $dV(u)/du = \lambda^- E[R^e]$, $dV(u)/du$ is a constant function. By the sign of $dV(u)/du$, we can obtain the optimal portfolios. If $E[R^e] < 0$, then V is an decreasing function and optimal portfolio $u^* = -\infty$. However, if $E[R^e] > 0$, then V is a increasing function and optimal portfolio $u^* = +\infty$. When $E[R^e] = 0$, it is clear that $V(u) = 0$ for all u and all u's are optimal. Consequently,

Case II : $\lambda = 1$.

1. If $E[R^e] > 0$, then $u^* = +\infty$.

2. If $E[R^e] = 0$, then $u^* = (-\infty, +\infty)$.

3. If $E[R^e] < 0$, then $u^* = -\infty$.

Now, consider the case where $\lambda > 1$ or $V(u)$ is convex. Since our problem is maximizing the value function in (4.26), we can collect the optimal solutions by investigating the value of the objective function at the boundaries. From (4.28), we obtain that $dV(u)/du$ is an increasing function since $d^2V(u)/du^2 \geq 0$. Also, we know that $dV(u)/du = \lambda^+ E[R^e]$ at $u = 0$.

When $E[R^e] > 0$, it is clear that $dV(u)/du > 0$ for $u \geq 0$ and $V(u)$ is an increasing function in the positive region. However, for $u < 0$, $dV(u)/du$ takes its minimum value at $-\infty$ and it can be negative. If $dV(u)/du$ is positive for $u = -\infty$, then V is an increasing function for all u and the optimal portfolio u^* is $+\infty$. If $dV(u)/du$ is negative for $u = -\infty$, then V decreases up to the point $\bar{u}$ that satisfies $dV(u)/du = 0$. Additionally, V is increasing function for $u > \bar{u}$. Therefore, the optimal portfolios u^* are $-\infty$, $+\infty$.

When $E[R^e] < 0$, $dV(u)/du < 0$ for $u < 0$ and $V(u)$ is an increasing function in the positive region. $dV(u)/du$ takes its maximum value at $+\infty$ and it can be positive. If $dV(u)/du$ is negative for $u = +\infty$, then V is a decreasing function for all u and the optimal portfolio u^* is $-\infty$. If $dV(u)/du$ is positive for $u = +\infty$, then V decreases up to the point $\bar{u}$ that satisfies $dV(u)/du = 0$. Similarly, V is increasing function for $u > \bar{u}$ and the optimal portfolios u^* are $-\infty$, $+\infty$.

When $E[R^e] = 0$, since $dV(u)/du = \lambda^+ E[R^e] = 0$ at $u = 0$, V is decreasing function for $u < 0$ and increasing for $u > 0$. The optimal portfolios u^* are $-\infty$, $+\infty$. In brief,

$Case\ III : \lambda > 1.$

1. If $E[R^e] > 0$, then

$$u^* = \begin{cases} -\infty,\ +\infty & E[R^e] \leq (1-\lambda)E[R^e 1_{\{R^e<0\}}] \\ +\infty & E[R^e] > (1-\lambda)E[R^e 1_{\{R^e<0\}}] \end{cases} .$$

2. If $E[R^e] = 0$, then $u^* = -\infty$ and $+\infty$.

3. If $E[R^e] < 0$, then

$$u^* = \begin{cases} -\infty & E[R^e] < (1-\lambda)E[R^e 1_{\{R^e\geq 0\}}] \\ -\infty,\ +\infty & E[R^e] \geq (1-\lambda)E[R^e 1_{\{R^e\geq 0\}}] \end{cases} .$$

In the following subsections, since the optimal portfolios are trivial when $\lambda \geq 1$, we make some observations only for the case $\lambda < 1$.

Binomial Return Model

Up to this point, we consider the case where excess return has any distribution. Now, we assume that excess return has the binomial distribution specifically. All parameters stated in Chapter (4.1.1) are again valid. But, we analyze the effect of the initial wealth on the optimal portfolio in this chapter. To determine the optimal portfolios, we must update $dV(u)/du$. It follows from (4.27) that

$$\begin{aligned}
\frac{dV(u)}{du} &= \begin{cases} \lambda^-(p_1\delta_1 - p_2\delta_2) - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e \geq -\frac{\theta w_0 r_f}{u}\}}] & u \geq 0 \\ \lambda^-(p_1\delta_1 - p_2\delta_2) - (\lambda^- - \lambda^+)E[R^e 1_{\{R^e < -\frac{\theta w_0 r_f}{u}\}}] & u < 0 \end{cases} \\
&= \begin{cases} \lambda^-(p_1\delta_1 - p_2\delta_2) - (\lambda^- - \lambda^+)p_1\delta_1 & u > \frac{\theta w_0 r_f}{\delta_2} \\ \lambda^-(p_1\delta_1 - p_2\delta_2) - (\lambda^- - \lambda^+)(p_1\delta_1 - p_2\delta_2) & -\frac{\theta w_0 r_f}{\delta_1} < u \leq \frac{\theta w_0 r_f}{\delta_2} \\ \lambda^-(p_1\delta_1 - p_2\delta_2) - (\lambda^- - \lambda^+)(-p_2\delta_2) & u \leq -\frac{\theta w_0 r_f}{\delta_1} \end{cases} \\
&= \begin{cases} \lambda^+ p_1\delta_1 - \lambda^- p_2\delta_2 & u > \frac{\theta w_0 r_f}{\delta_2} \\ \lambda^+(p_1\delta_1 - p_2\delta_2) & -\frac{\theta w_0 r_f}{\delta_1} < u \leq \frac{\theta w_0 r_f}{\delta_2} \\ \lambda^- p_1\delta_1 - \lambda^+ p_2\delta_2 & u \leq -\frac{\theta w_0 r_f}{\delta_1}. \end{cases} \quad . \qquad (4.32)
\end{aligned}$$

Note that $dV(u)/du$ is a piecewise constant function and it takes different values in three regions. From (4.32), there are two critical points that $dV(u)/du$ changes and these points are candidates to be optimal. As mentioned, optimal portfolios are trivial when $\lambda \geq 1$. So, we just remark on the case when $\lambda < 1$ or $\lambda^- > \lambda^+$.

Suppose $E[R^e] \geq 0$ or $p_1\delta_1 \geq p_2\delta_2$. Since $\lambda^- p_1\delta_1 - \lambda^+ p_2\delta_2 > 0$ and $\lambda^+(p_1\delta_1 - p_2\delta_2) \geq 0$, $V(u)$ is an increasing function on the interval $(-\infty, \frac{\theta w_0 r_f}{\delta_2}]$. However, we cannot make a certain observation for $u > \theta w_0 r_f/\delta_2$. If $\lambda^+ p_1\delta_1 - \lambda^- p_2\delta_2 > 0$, then $V(u)$ is an increasing function for $u > \theta w_0 r_f/\delta_2$. If $\lambda^+ p_1\delta_1 - \lambda^- p_2\delta_2 < 0$, then $V(u)$ is an decreasing function for $u > \theta w_0 r_f/\delta_2$. When $\lambda^+ p_1\delta_1 - \lambda^- p_2\delta_2 = 0$, $V(u)$ is a constant function for the same interval. Moreover, optimal portfolio u^* depends on the structure of $V(u)$ for $u > \theta w_0 r_f/\delta_2$. Increasing in this interval implies that optimal portfolio is $u^* = +\infty$ and decreasing implies that optimal portfolio $u^* = \theta w_0 r_f/\delta_2$. If $V(u)$ is a constant function for the same interval, there are alternative optimal portfolios and $u^* \epsilon [\theta w_0 r_f/\delta_2, +\infty)$.

Similar reviews can be obtained for $E[R^e] < 0$ or $p_1\delta_1 < p_2\delta_2$. Since $\lambda^+ p_1\delta_1 - \lambda^- p_2\delta_2 < 0$ and $\lambda^+(p_1\delta_1 - p_2\delta_2) < 0, V(u)$ is a decreasing function on the interval $[-\theta w_0 r_f/\delta_1, +\infty)$. If $\lambda^- p_1\delta_1 - \lambda^+ p_2\delta_2 > 0$, then $V(u)$ is an increasing function for $u < -\theta w_0 r_f/\delta_1$ and optimal portfolio is $u^* = -\theta w_0 r_f/\delta_1$. If $\lambda^- p_1\delta_1 - \lambda^+ p_2\delta_2 < 0$, then $V(u)$ is an decreasing function for $u < -\theta w_0 r_f/\delta_1$ and optimal portfolio u^* is $-\infty$. We can summarize the conclusion as follows.

1. If $E[R^e] \geq 0$, then

$$
\begin{aligned}
u^* &= \begin{cases} \theta w_0 r_f/\delta_2 & p_1\delta_1 - p_2\delta_2 < (1-\lambda)p_1\delta_1 \\ [\theta w_0 r_f/\delta_2, +\infty) & p_1\delta_1 - p_2\delta_2 = (1-\lambda)p_1\delta_1 \\ +\infty & p_1\delta_1 - p_2\delta_2 > (1-\lambda)p_1\delta_1 \end{cases} \\
&= \begin{cases} \theta w_0 r_f/\delta_2 & \lambda p_1\delta_1 < p_2\delta_2 \\ [\theta w_0 r_f/\delta_2, +\infty) & \lambda p_1\delta_1 = p_2\delta_2 \\ +\infty & \lambda p_1\delta_1 > p_2\delta_2 \end{cases} \\
&= \begin{cases} \theta w_0 r_f/\delta_2 & \lambda p\delta < 1 \\ [\theta w_0 r_f/\delta_2, +\infty) & \lambda p\delta = 1 \\ +\infty & \lambda p\delta > 1 \end{cases} .
\end{aligned}
$$

2. If $E[R^e] < 0$, then

$$
\begin{aligned}
u^* &= \begin{cases} -\theta w_0 r_f/\delta_1 & p_1\delta_1 - p_2\delta_2 > (1-\lambda)(-p_2\delta_2) \\ (-\infty, -\theta w_0 r_f/\delta_1] & p_1\delta_1 - p_2\delta_2 = (1-\lambda)(-p_2\delta_2) \\ -\infty & p_1\delta_1 - p_2\delta_2 < (1-\lambda)(-p_2\delta_2) \end{cases} \\
&= \begin{cases} -\theta w_0 r_f/\delta_1 & \lambda p_2\delta_2 < p_1\delta_1 \\ (-\infty, -\theta w_0 r_f/\delta_1] & \lambda p_2\delta_2 = p_1\delta_1 \\ -\infty & \lambda p_2\delta_2 > p_1\delta_1 \end{cases} \\
&= \begin{cases} -\theta w_0 r_f/\delta_1 & \lambda < p\delta \\ (-\infty, -\theta w_0 r_f/\delta_1] & \lambda = p\delta \\ -\infty & \lambda > p\delta \end{cases} .
\end{aligned}
$$

Exponential Return Model

In this chapter, we assume that risky asset returns are exponentially distributed with rate μ so that $E[R] = 1/\mu$ as in Chapter(4.1.1).

We concentrate on the case when $\lambda < 1$ as usual. To update (4.30)-(4.31), we must determine $E[R^e 1_{\{R^e \geq -\theta w_0 r_f/u\}}]$ and $E[R^e 1_{\{R^e < -\theta w_0 r_f/u\}}]$ for exponentially distributed risky asset returns. Then,

$$\begin{aligned}
E[R^e 1_{\{R^e \geq -\theta w_0 r_f/u\}}] &= E[(R-R_f)1_{\{R \geq R_f - \theta w_0 r_f/u\}}] \\
&= \begin{cases} \displaystyle\int_{R_f-\theta w_0 r_f/u}^{+\infty} (x-R_f)\mu e^{-\mu x}dx & R_f \geq \theta w_0 r_f/u \\ \displaystyle\int_0^{+\infty} (x-R_f)\mu e^{-\mu x}dx & R_f < \theta w_0 r_f/u \end{cases} \\
&= \begin{cases} \displaystyle\int_0^{+\infty} (y-\theta w_0 r_f/u)\mu e^{-\mu(y+R_f-\theta w_0 r_f/u)}dy & R_f \geq \theta w_0 r_f/u \\ 1/\mu - R_f & R_f < \theta w_0 r_f/u \end{cases} \\
&= \begin{cases} e^{-\mu(R_f-\theta w_0 r_f/u)}(1/\mu - \theta w_0 r_f/u) & u \geq \theta w_0 r_f/R_f \\ 1/\mu - R_f & u < \theta w_0 r_f/R_f \end{cases} \qquad (4.33)
\end{aligned}$$

and

$$\begin{aligned}
E[R^e 1_{\{R^e < -\theta w_0 r_f/u\}}] &= E[(R-R_f)] - E[(R-R_f)1_{\{R \geq R_f - \frac{\theta w_0 r_f}{u}\}}] \\
&= \begin{cases} \begin{array}{c} 1/\mu - R_f - \\ e^{-\mu(R_f-\theta w_0 r_f/u)}(1/\mu - \theta w_0 r_f/u) \end{array} & u \geq \theta w_0 r_f/R_f \\ 0 & u < \theta w_0 r_f/R_f \end{cases} \\
&= \begin{cases} \begin{array}{c} 1/\mu - R_f - \\ e^{-\mu(R_f-\theta w_0 r_f/u)}(1/\mu - \theta w_0 r_f/u) \end{array} & u \geq \theta w_0 r_f/R_f \\ 0 & u < \theta w_0 r_f/R_f \end{cases} \qquad (4.34)
\end{aligned}$$

Using (4.33) for $u \geq 0$, and (4.34) for $u < 0$, we can determine the optimality condition (4.29) for the exponentially distributed returns as

$$\left.\begin{array}{cc} \frac{e^{-\mu(R_f-\theta w_0 r_f/u)}(1/\mu-\theta w_0 r_f/u)}{1/\mu - R_f} & u \geq \theta w_0 r_f/R_f \\ 1 & 0 \leq u < \theta w_0 r_f/R_f \\ 1 - \frac{e^{-\mu(R_f-\theta w_0 r_f/u)}(1/\mu-\theta w_0 r_f/u)}{1/\mu - R_f} & u < 0 \end{array}\right\} = \frac{1}{1-\lambda}.$$

Note that $E[R^e 1_{\{R^e \geq -\theta w_0 r_f/u\}}]/E[R^e] = 1$ for $0 \leq u < \theta w_0 r_f/R_f$. So, there is not any u such that $E[R^e 1_{\{R^e \geq -\theta w_0 r_f/u\}}]/E[R^e] = 1/(1-\lambda)$ in the interval $[0, \theta w_0 r_f/R_f)$.

In (4.30), for $E[R^e] \geq 0$, we stated that if $E[R^e] < (1-\lambda)E[R^e 1_{\{R^e > 0\}}]$, then there is a finite optimal portfolio. We can rewrite this condition for exponentially distributed asset returns. Actually, we made the same adaptation in Chapter (4.1.1). We can get similar results as in Chapter (4.1.1). Differently, there is a level $\overset{\wedge}{\mu}_1$ satisfying (4.12) that if μR_f

is greater than $\overset{\wedge}{\mu}_1$, the optimal portfolio is u^* is a finite positive number, otherwise it is unbounded. In other words,

$$u^* = \begin{cases} 0 \leq u^* < +\infty & \mu R_f > \overset{\wedge}{\mu}_1 \text{ or } E[R^e] < \frac{1-\overset{\wedge}{\mu}_1}{\mu} \\ +\infty & \mu R_f \leq \overset{\wedge}{\mu}_1 \text{ or } E[R^e] \geq \frac{1-\overset{\wedge}{\mu}_1}{\mu} \end{cases}$$

where u^* satisfies $e^{-\mu(R_f - \theta w_0 r_f/u)}((1/\mu) - \theta w_0 r_f/u)/\left((1/\mu) - R_f\right) = 1/(1-\lambda)$.

In (4.31), for $E[R^e] < 0$, we stated that if $E[R^e] > (1-\lambda)E[R^e 1_{\{R^e \leq 0\}}]$, then there is a finite optimal portfolio. We again get similar results with Chapter (4.1.1). Differently, there is a level $\overset{\wedge}{\mu}_2$ satisfying (4.15) that if μR_f is less than $\overset{\wedge}{\mu}_2$, the optimal portfolio is u^* is a finite negative number, otherwise it is unbounded. In other words,

$$u^* = \begin{cases} -\infty < u^* < 0 & \mu R_f < \overset{\wedge}{\mu}_2 \text{or } E[R^e] > \frac{1-\overset{\wedge}{\mu}_2}{\mu} \\ -\infty & \mu R_f \geq \overset{\wedge}{\mu}_2 \text{or } E[R^e] \leq \frac{1-\overset{\wedge}{\mu}_2}{\mu} \end{cases}$$

where u^* satisfies $1 - e^{-\mu(R_f - \theta w_0 r_f/u)}((1/\mu) - \theta w_0 r_f/u)/((1/\mu) - R_f) = 1/(1-\lambda)$.

Normal Return Model

Like in Chapter (4.1.1), excess return R^e is normally distributed with μ and standard deviation σ. However, the effect of the reference point on the optimal portfolio is analyzed here. To use general inferences that we obtained at chapter (4.1.2), we must update both

$E[R^e 1_{\{R^e \geq -\frac{\theta w_0 r_f}{u}\}}]$ and $E[R^e 1_{\{R^e < -\frac{\theta w_0 r_f}{u}\}}]$. So,

$$\begin{aligned}
E[R^e 1_{\{R^e \geq -\frac{\theta w_0 r_f}{u}\}}] &= \int_{-\frac{\theta w_0 r_f}{u}}^{+\infty} \frac{x}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
&= \int_{-\frac{\theta w_0 r_f}{u\sigma}-\frac{\mu}{\sigma}}^{+\infty} \frac{(\mu+\sigma z)}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \mu \int_{-\frac{\theta w_0 r_f}{u\sigma}-\frac{\mu}{\sigma}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \sigma \int_{-\frac{\theta w_0 r_f}{u\sigma}-\frac{\mu}{\sigma}}^{+\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \mu \left(\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - \int_{-\infty}^{-\frac{\theta w_0 r_f}{u\sigma}-\frac{\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \right) + \\
&\qquad \sigma \int_{-\frac{\theta w_0 r_f}{u\sigma}-\frac{\mu}{\sigma}}^{+\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \mu \left(1 - \Phi\left(-\frac{\theta w_0 r_f}{u\sigma} - \frac{\mu}{\sigma}\right)\right) + \sigma\Psi\left(-\frac{\theta w_0 r_f}{u\sigma} - \frac{\mu}{\sigma}\right) \\
&= \mu\Phi\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right) + \sigma\Psi\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right) \\
&= \mu\Phi\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right) + \sigma\varphi\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right)
\end{aligned}$$

and

$$
\begin{aligned}
E[R^e 1_{\{R^e < -\frac{\theta w_0 r_f}{u}\}}] &= \int_{-\infty}^{-\frac{\theta w_0 r_f}{u}} \frac{x}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
&= \int_{-\infty}^{-\frac{\theta w_0 r_f}{u\sigma}-\frac{\mu}{\sigma}} \frac{(\mu+\sigma z)}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \mu \int_{-\infty}^{-\frac{\theta w_0 r_f}{u\sigma}-\frac{\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \sigma \int_{-\infty}^{-\frac{\theta w_0 r_f}{u\sigma}-\frac{\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \mu \int_{-\infty}^{-\frac{\theta w_0 r_f}{u\sigma}-\frac{\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \\
&\quad \sigma \left(\int_{-\infty}^{+\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - \int_{-\frac{\theta w_0 r_f}{u\sigma}-\frac{\mu}{\sigma}}^{+\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \right) \\
&= \mu\Phi\left(-\frac{\theta w_0 r_f}{u\sigma} - \frac{\mu}{\sigma}\right) - \sigma\Psi\left(-\frac{\theta w_0 r_f}{u\sigma} - \frac{\mu}{\sigma}\right) \\
&= \mu\Phi\left(-\frac{\theta w_0 r_f}{u\sigma} - \frac{\mu}{\sigma}\right) - \sigma\varphi\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right).
\end{aligned}
$$

Using (4.29), the optimality condition can be stated as

$$
\left.\begin{array}{ll}
\Phi\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right) + \frac{\sigma}{\mu}\varphi\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right) & u \geq 0 \\
\Phi\left(-\frac{\theta w_0 r_f}{u\sigma} - \frac{\mu}{\sigma}\right) - \frac{\sigma}{\mu}\varphi\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right) & u < 0
\end{array}\right\} = \frac{1}{1-\lambda}. \tag{4.35}
$$

From Chapter (4.1.2), we know that if $E[R^e] = \mu \geq 0$, the optimal portfolio u^* must be greater than zero. Furthermore, the optimal portfolio u^* can be finite or infinite. If it is finite, it must satisfy the first optimality condition (4.35) that is available for $u \geq 0$. To derive optimal portfolios, we must check the optimality condition (4.35) in more detail. Lets define $h_1(u) = \Phi\left(\theta w_0 r_f/(u\sigma) + (\mu/\sigma)\right) + \sigma\varphi\left(\theta w_0 r_f/(u\sigma) + (\mu/\sigma)\right)/\mu$ on the set $u \geq 0$. It follows that h_1 is an increasing function since

$$
\begin{aligned}
\frac{dh_1(u)}{du} &= \Phi'\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right) + \frac{\sigma}{\mu}\varphi'\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right) \\
&= -\varphi\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right)\frac{\theta w_0 r_f}{u^2\sigma} + \frac{\sigma}{\mu}\left[-\varphi\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right)\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right)\right]\left(-\frac{\theta w_0 r_f}{u^2\sigma}\right) \\
&= \frac{\sigma}{\mu}\varphi\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right)\frac{\theta^2 w_0^2 r_f^2}{u^3\sigma^2} \geq 0.
\end{aligned}
$$

Moreover, $h_1(0) = 1$ and $\lim_{u\to+\infty} h_1(u) = \Phi(\mu/\sigma) + \sigma\varphi(\mu/\sigma)/\mu$. Obviously, if $\Phi(\mu/\sigma) + \sigma\varphi(\mu/\sigma)/\mu \le 1/(1-\lambda)$, there is not any finite $u \ge 0$ that satisfies the optimality condition (4.35). In other words, $dV(u)/d(u) \ge 0$ for $u \ge 0$ and the optimal portfolio is $u^* = +\infty$. However, if $\Phi(\mu/\sigma) + \sigma\varphi(\mu/\sigma)/\mu > 1/(1-\lambda)$, then there is a unique $u^* \ge 0$ such that it satisfies the optimality condition. In that case, the optimal portfolio is finite.

Also, we know that if $E[R^e] = \mu < 0$, the optimal portfolio u^* must be less than zero. Similarly, the optimal portfolio u^* can be finite or infinite. If it is finite, it must satisfy the second optimality condition (4.35). Let $h_2(u) = \Phi\left(-\theta w_0 r_f/(u\sigma) - (\mu/\sigma)\right) - \sigma\varphi\left(\theta w_0 r_f/(u\sigma) + (\mu/\sigma)\right)/\mu$ on the set $u < 0$. Then, h_2 is an decreasing function since

$$\begin{aligned}
\frac{dh_2(u)}{du} &= \Phi'\left(-\frac{\theta w_0 r_f}{u\sigma} - \frac{\mu}{\sigma}\right) - \frac{\sigma}{\mu}\varphi'\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right) \\
&= \varphi\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right)\frac{\theta w_0 r_f}{u^2\sigma} - \frac{\sigma}{\mu}\left[-\varphi\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right)\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right)\right]\left(-\frac{\theta w_0 r_f}{u^2\sigma}\right) \\
&= -\frac{\sigma}{\mu}\varphi\left(\frac{\theta w_0 r_f}{u\sigma} + \frac{\mu}{\sigma}\right)\frac{\theta^2 w_0^2 r_f^2}{u^3\sigma^2} \le 0.
\end{aligned}$$

Moreover, $\lim_{u\to-\infty} h_2(u) = \Phi\left(-\mu/\sigma\right) - \sigma\varphi\left(\mu/\sigma\right)/\mu$ and $\lim_{u\to0^-} h_2(u) = 1$. Obviously, if $\Phi(-\mu/\sigma) - \sigma\varphi(\mu/\sigma)/\mu < 1/(1-\lambda)$ there is not any finite $u < 0$ that satisfies the optimality condition (4.35). In other words, $dV(u)/d(u) \le 0$ for $u < 0$ and the optimal portfolio $u^* = -\infty$. If $\Phi(-\mu/\sigma) - \sigma\varphi(\mu/\sigma)/\mu > 1/(1-\lambda)$, then there is a unique $u^* < 0$ such that it satisfies the optimality condition. In that case, the optimal portfolio is finite.

To sum up, the optimal portfolio is characterized below.

1. If $\mu \ge 0$, then

$$u^* = \begin{cases} 0 \le u^* < +\infty & \frac{1}{(1-\lambda)} < \Phi(\frac{\mu}{\sigma}) + \frac{\sigma}{\mu}\Psi(\frac{\mu}{\sigma}) \\ +\infty & \frac{1}{(1-\lambda)} \ge \Phi(\frac{\mu}{\sigma}) + \frac{\sigma}{\mu}\Psi(\frac{\mu}{\sigma}) \end{cases}$$

   where u^* satisfies $\Phi\left(\theta w_0 r_f/(u\sigma) + \mu/\sigma\right) + \sigma\varphi\left(\theta w_0 r_f/(u\sigma) + \mu/\sigma\right)/\mu = 1/(1-\lambda)$.

2. If $\mu < 0$, then

$$u^* = \begin{cases} -\infty < u^* < 0 & \frac{1}{(1-\lambda)} < \Phi(-\frac{\mu}{\sigma}) - \frac{\sigma}{\mu}\Psi(\frac{\mu}{\sigma}) \\ -\infty & \frac{1}{(1-\lambda)} \ge \Phi(-\frac{\mu}{\sigma}) - \frac{\sigma}{\mu}\Psi(\frac{\mu}{\sigma}) \end{cases}$$

   where u^* satisfies $\Phi\left(-\theta w_0 r_f/(u\sigma) - \mu/\sigma\right) - \sigma\varphi\left(\theta w_0 r_f/(u\sigma) + \mu/\sigma\right)/\mu = 1/(1-\lambda)$.

The ratio μ/σ is the Sharpe ratio of the risky asset. When $\mu \ge 0$, $\Phi(\mu/\sigma) + \sigma\Psi(\mu/\sigma)/\mu$ is a critical level that determines the structure of the optimal portfolio. As mentioned in Chapter (4.1.1), $\Phi(\mu/\sigma) + \sigma\Psi(\mu/\sigma)/\mu$ is a decreasing function with respect to μ/σ.

Increases in the Sharpe ratio causes decreases in $\Phi(\mu/\sigma)+\sigma\Psi(\mu/\sigma)/\mu$. So, $\Phi(\theta w_0 r_f/(u\sigma)+\mu/\sigma)+\sigma\Psi(\theta w_0 r_f/(u\sigma)+\mu/\sigma)/\mu$ intersects with the line $1/(1-\lambda)$ at a higher u and u^* gets bigger. If $\mu/\sigma > x_1$ where x_1 holds the equality that $1/(1-\lambda)=\Phi(x_1)+\sigma\Psi(x_1)/x_1$, then optimal portfolio is $u^*=+\infty$. When $\mu<0$, $\Phi(-\mu/\sigma)-\sigma\Psi(\mu/\sigma)/\mu$ is a critical level that it determines the structure of the optimal portfolio. Now, $\Phi(-\mu/\sigma)-\sigma\Psi(\mu/\sigma)/\mu$ is an increasing function with respect to μ/σ. So, increases in the Sharpe ratio causes increases in $\Phi(-\mu/\sigma)-\sigma\Psi(\mu/\sigma)/\mu$. In that case, $\Phi(-\theta w_0 r_f/(u\sigma)-\mu/\sigma)-\sigma\Psi(\theta w_0 r_f/(u\sigma)+\mu/\sigma)/\mu$ intersects with the line $1/(1-\lambda)$ at a smaller u where $u<0$. Consequently, the optimal portfolio u^* gets smaller. If $\mu/\sigma < x_2$ where x_2 holds the equality that $1/(1-\lambda)=\Phi(-x_2)-\sigma\Psi(x_2)/x_2$, then optimal portfolio is $u^*=-\infty$.

4.2 Exponential Value Function

4.2.1 General Reference Point

In this chapter, we analyzed the effect of the exponential value function and the initial wealth on optimal portfolio u^*. The objective function $V(u)$ which we want to maximize by choosing u is

$$V(u) = E[v(uR^e+\theta w_0 r_f] \tag{4.36}$$

where $v(x)=\lambda^+(1-\exp(-\alpha x))$ with λ^+, $\alpha>0$.

General Return Model

Suppose that the excess return R^e comes from any distribution. Then, taking derivative of (4.36),

$$\begin{aligned}\frac{dV(u)}{du} &= \alpha\lambda^+E[R^e e^{-\alpha(uR^e+\theta w_0 r_f)}]\\ &= \alpha\lambda^+e^{\theta w_0 r_f}E[R^e e^{-\alpha u R^e}]\end{aligned}$$

and the second derivative is

$$\frac{d^2V(u)}{du^2} = -\alpha^2\lambda^+e^{\theta w_0 r_f}E[(R^e)^2e^{-\alpha u R^e}].$$

Since $d^2V(u)/du^2\le 0$, V is a concave function. So, setting $dV(u)/du$ to zero gives the optimality condition. The optimality condition is

$$\alpha\lambda^+e^{\theta w_0 r_f}E[R^e e^{-\alpha u R^e}]=0$$

or

$$E[R^e e^{-\alpha u R^e}]=0. \tag{4.37}$$

Lemma 4.2.1 *a) If $E[R^e] \geq 0$, then $u^* \geq 0$,*

b) If $E[R^e] < 0$, then $u^* < 0$.

Proof. *Firstly, $E[R^e e^{-\alpha u R^e}]$ is decreasing in u, since the derivative is $-\alpha E[(R^e)^2 e^{-\alpha u R^e}] \leq 0$.*

Also, it is obvious that $E[R^e e^{-\alpha u R^e}] = E[R^e]$ for $u = 0$.

a) If $E[R^e] \geq 0$, then the optimality condition (4.37) holds only for $u \geq 0$.

b) If $E[R^e] < 0$, then the optimality condition (4.37) holds only for $u < 0$. ■

In the following cases, we derive the optimal portfolios using different distributions.

Binomial Return Model

Suppose that the excess return R^e has the binomial distribution. With probability p_1, R^e equals to δ_1 and with probability p_2, R^e equals to $-\delta_2$ where $p_1 + p_2 = 1$ and $\delta_1, \delta_2 \geq 0$. The optimality condition (4.37) can be rewritten as

$$p_1 \delta_1 e^{-\alpha u \delta_1} - p_2 \delta_2 e^{\alpha u \delta_2} = 0.$$

Therefore, the optimal solution is

$$\begin{aligned} u^* &= \frac{1}{\alpha(\delta_1 + \delta_2)} \ln(\frac{p_1 \delta_1}{p_2 \delta_2}) \\ &= \frac{1}{\alpha(\delta_1 + \delta_2)} \ln(p\delta). \end{aligned}$$

Exponential Return Model

In this chapter, we assume that risky asset returns are exponentially distributed with rate μ as before. The optimality condition is

$$E[(R - R_f) e^{-\alpha u (R - R_f)}] = 0$$

or

$$\int_0^{+\infty} (x - R_f) e^{-\alpha u (x - R_f)} \mu e^{-\mu x} dx = 0$$

or

$$e^{\alpha u R_f} \int_0^{+\infty} (x - R_f) \mu e^{-(\alpha u + \mu) x} dx = 0$$

or

$$e^{\alpha u R_f} \frac{\mu}{\alpha u + \mu} \int_0^{+\infty} (x - R_f)(\alpha u + \mu) e^{-(\alpha u + \mu) x} dx = 0$$

or

$$e^{\alpha u R_f} \frac{\mu}{\alpha u + \mu} \left(\frac{1}{\alpha u + \mu} - R_f \right) = 0$$

or

$$\frac{1}{\alpha u + \mu} - R_f = 0.$$

Then, the optimal portfolio is

$$u^* = \frac{1 - \mu R_f}{\alpha R_f}.$$

Normal Return Model

As before, the excess return R^e is normally distributed with μ and standard deviation σ. The optimality condition (4.37) can be updated as

$$\int_{-\infty}^{+\infty} \frac{x}{\sqrt{2\pi}\sigma} e^{-\alpha u x} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx = 0$$

or

$$\int_{-\infty}^{+\infty} \frac{(\sigma z + \mu)}{\sqrt{2\pi}} e^{-\alpha u(\sigma z + \mu)} e^{-\frac{1}{2}z^2} dz = 0$$

or

$$e^{-\alpha u \mu} \int_{-\infty}^{+\infty} \frac{(\sigma z + \mu)}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2 + 2\alpha u \sigma z)} dz = 0$$

or

$$e^{-\alpha u \mu + \frac{\alpha^2 u^2 \sigma^2}{2}} \int_{-\infty}^{+\infty} \frac{(\sigma z + \mu)}{\sqrt{2\pi}} e^{-\frac{1}{2}(z + \alpha u \sigma)^2} dz = 0$$

or

$$e^{-\alpha u \mu + \frac{\alpha^2 u^2 \sigma^2}{2}} \int_{-\infty}^{+\infty} \frac{(\sigma(y - \alpha u \sigma) + \mu)}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy = 0$$

or

$$e^{-\alpha u \mu + \frac{\alpha^2 u^2 \sigma^2}{2}} (\mu - \alpha u \sigma^2) = 0$$

or

$$\mu - \alpha u \sigma^2 = 0.$$

Then, the optimal portfolio is

$$u^* = \frac{\mu}{\alpha \sigma^2}.$$

4.3 Piecewise Exponential Value Function

In this chapter, the optimal choices of prospect investors are analyzed for the piecewise exponential model in (3.7). We assumed $\lambda < 1$ as in Tversky and Kahnemann (1992). Since the value function is neither convex nor concave, some difficulties arise. So, we examine the objective function in two parts for $u \geq 0$ and $u < 0$. Then, we derive the optimal solution and make our analysis.

4.3.1 Risk-free Reference Point

Portfolio optimization problem with piecewise exponential value function and risk-free reference point can be stated as

$$\max_u V(u) = \max_u E[v(uR^e)]$$

where $v(x)$ has the form shown as (3.7) with $\lambda^+, \lambda^-, \alpha > 0$.

Binomial Return Model

Suppose that excess return R^e comes from the binomial distribution. Then,

$$E[R^e] = p_1\delta_1 - p_2\delta_2$$

and

$$V(u) = \begin{cases} \lambda^+ p_1(1 - e^{-\alpha u\delta_1}) - \lambda^- p_2(1 - e^{-\alpha u\delta_2}) & u \geq 0 \\ \lambda^+ p_2(1 - e^{\alpha u\delta_2}) - \lambda^- p_1(1 - e^{\alpha u\delta_1}) & u < 0 \end{cases}.$$

Note that V is continuos at zero with $V(0) = 0$.

Taking the derivative of v with respect to u, we obtain

$$\frac{dV(u)}{du} = \begin{cases} \alpha\lambda^+\delta_1 p_1 e^{-\alpha u\delta_1} - \alpha\lambda^-\delta_2 p_2 e^{-\alpha u\delta_2} & u \geq 0 \\ \alpha\lambda^+\delta_2 p_2 e^{\alpha u\delta_2} - \alpha\lambda^-\delta_1 p_1 e^{\alpha u\delta_1} & u < 0 \end{cases}.$$

It is clearly seen that the derivative of V changes its sign at most once for each positive and negative part of u, separately. If there is an extreme point at the relevant region of u, setting the derivative of V to zero gives us that point. This point can be optimal, if some conditions are satisfied. For the positive region of u, the extreme point $\bar{u}$ satisfies

$$\alpha\lambda^+\delta_1 p_1 e^{-\alpha\bar{u}\delta_1} - \alpha\lambda^-\delta_2 p_2 e^{-\alpha\bar{u}\delta_2} = 0$$

$$\lambda^+\delta_1 p_1 e^{-\alpha\bar{u}\delta_1} = \lambda^-\delta_2 p_2 e^{-\alpha\bar{u}\delta_2}$$

$$e^{-\alpha\bar{u}(\delta_1-\delta_2)} = \frac{\lambda^-\delta_2 p_2}{\lambda^+\delta_1 p_1}$$

$$\alpha\bar{u}(\delta_1-\delta_2) = -\ln(\frac{\lambda^-\delta_2 p_2}{\lambda^+\delta_1 p_1})$$

which yields the explicit solution

$$\bar{u} = \frac{1}{\alpha(\delta_1-\delta_2)}\ln(\frac{\lambda^+\delta_1 p_1}{\lambda^-\delta_2 p_2})$$

or

$$\bar{u} = \frac{1}{\alpha(\delta_1-\delta_2)}\ln(\lambda\delta p).$$

For the negative region of u, the extreme point $\bar{u}$ satisfies

$$\alpha\lambda^+\delta_2 p_2 e^{\alpha\bar{u}\delta_2} - \alpha\lambda^-\delta_1 p_1 e^{\alpha\bar{u}\delta_1} = 0$$

$$\lambda^+\delta_2 p_2 e^{\alpha\bar{u}\delta_2} = \lambda^-\delta_1 p_1 e^{\alpha\bar{u}\delta_1}$$

$$e^{\alpha\bar{u}(\delta_2-\delta_1)} = \frac{\lambda^-\delta_1 p_1}{\lambda^+\delta_2 p_2}$$

$$\alpha\bar{u}(\delta_2-\delta_1) = \ln(\frac{\lambda^-\delta_1 p_1}{\lambda^+\delta_2 p_2})$$

which yields the explicit solution

$$\bar{u} = \ln(\frac{\lambda^+\delta_2 p_2}{\lambda^-\delta_1 p_1})\frac{1}{\alpha(\delta_1-\delta_2)}$$

or

$$\bar{u} = \ln(\frac{\lambda}{\delta p})\frac{1}{\alpha(\delta_1-\delta_2)}$$

In summary,

$$\bar{u} = \begin{cases} \frac{1}{\alpha(\delta_1-\delta_2)}\ln(p\delta\lambda) & p\delta > \frac{1}{\lambda},\ \delta_1 > \delta_2 \text{ or } p\delta < \frac{1}{\lambda},\ \delta_1 < \delta_2 \\ \frac{1}{\alpha(\delta_1-\delta_2)}\ln(\frac{\lambda}{p\delta}) & p\delta > \lambda,\ \delta_1 > \delta_2 \text{ or } p\delta < \lambda,\ \delta_1 < \delta_2 \end{cases}. \tag{4.38}$$

Using the extreme points $\bar{u}$ given by (4.38), we can identify the optimal portfolio u^* as follows for $\lambda < 1$:

Case I. $E[R^e] > 0$.

1. If $p > \lambda$, then

$$u^* = \begin{cases} 0 & p < \frac{1}{\lambda}\min\{1, \frac{1}{\delta}\} \\ \frac{1}{\alpha(\delta_1-\delta_2)}\ln(p\delta\lambda) & p > \frac{1}{\lambda},\ \delta > 1 \\ +\infty & p > \frac{1}{\lambda\delta},\ \delta < 1 \end{cases}.$$

2. If $p < \lambda$, then

$$u^* = \begin{cases} \frac{1}{\alpha(\delta_1-\delta_2)}\ln(p\delta\lambda) & p\lambda^+[1-(\frac{1}{p\delta\lambda})^{\frac{\delta_1}{\delta_1-\delta_2}}] - \lambda^-[1-(\frac{1}{p\delta\lambda})^{\frac{\delta_2}{\delta_1-\delta_2}}] > \lambda^+ - p\lambda^- \\ -\infty & p\lambda^+[1-(\frac{1}{p\delta\lambda})^{\frac{\delta_1}{\delta_1-\delta_2}}] - \lambda^-[1-(\frac{1}{p\delta\lambda})^{\frac{\delta_2}{\delta_1-\delta_2}}] < \lambda^+ - p\lambda^- \end{cases}.$$

Case II. $E[R^e] < 0$.

1. If $\lambda p < 1$, then

$$u^* = \begin{cases} 0 & \frac{1}{p} < \frac{1}{\lambda}\min\{1, \delta\} \\ \frac{1}{\alpha(\delta_1-\delta_2)}\ln(\frac{\lambda}{p\delta}) & \frac{1}{p} > \frac{\delta}{\lambda},\ \delta > 1 \\ -\infty & \frac{1}{p} > \frac{1}{\lambda},\ \delta < 1 \end{cases}.$$

2. If $\lambda p > 1$, then

$$u^* = \begin{cases} \frac{1}{\alpha(\delta_1-\delta_2)}\ln(\frac{\lambda}{p\delta}) & \lambda^+[1-(\frac{\lambda}{p\delta})^{\frac{\delta_2}{\delta_2-\delta_1}}] - p\lambda^-[1-(\frac{\lambda}{p\delta})^{\frac{\delta_1}{\delta_2-\delta_1}}] > p\lambda^+ - \lambda^- \\ +\infty & \lambda^+[1-(\frac{\lambda}{p\delta})^{\frac{\delta_2}{\delta_2-\delta_1}}] - p\lambda^-[1-(\frac{\lambda}{p\delta})^{\frac{\delta_1}{\delta_2-\delta_1}}] < p\lambda^+ - \lambda^- \end{cases}.$$

Case III. $E[R^e] = 0$.

Then,

$$u^* = \begin{cases} -\infty & p < \lambda \\ 0 & \lambda \le p \le \frac{1}{\lambda} \\ +\infty & p > \frac{1}{\lambda} \end{cases}.$$

Normal Return Model

Suppose that excess return R^e is normally distributed with mean μ and standard deviation σ. Then,

$$
\begin{aligned}
V(u) &= E[v^-(uR^e)1_{\{uR^e<0\}}] + E[v^+(uR^e)1_{\{uR^e\geq 0\}}] \\
&= -\lambda^- E[(1-e^{\alpha uR^e})1_{\{uR^e<0\}}] + \lambda^+ E[(1-e^{-\alpha uR^e})1_{\{uR^e\geq 0\}}] \\
&= \begin{cases} -\lambda^- E[(1-e^{\alpha uR^e})1_{\{R^e<0\}}] + \lambda^+ E[(1-e^{-\alpha uR^e})1_{\{R^e\geq 0\}}] & u \geq 0 \\ -\lambda^- E[(1-e^{\alpha uR^e})1_{\{R^e>0\}}] + \lambda^+ E[(1-e^{-\alpha uR^e})1_{\{R^e\leq 0\}}] & u < 0 \end{cases} \\
&= \begin{cases} -\lambda^- \int\limits_{-\infty}^{0} (1-e^{\alpha ux}))f(x)dx + \lambda^+ \int\limits_{0}^{+\infty} (1-e^{-\alpha ux})f(x)dx & u \geq 0 \\ -\lambda^- \int\limits_{0}^{+\infty} (1-e^{\alpha ux})f(x)dx + \lambda^+ \int\limits_{-\infty}^{0} (1-e^{-\alpha ux})f(x)dx & u < 0 \end{cases}
\end{aligned} \tag{4.39}
$$

where $f(x)$ is the probability density function of normal distribution with mean μ and variance σ^2. We consider the value function stated in (4.39) under the assumption $\lambda < 1$.

Lemma 4.3.1 *a) If $\mu \geq 0$, then $u^* \geq 0$.*
b) If $\mu < 0$, then $u^* < 0$.

Proof. *a) When $\mu \geq 0$, $f(x) \geq f(-x)$ for $x > 0$. Owing to the symmetry between* $1 - \exp(\alpha ux)$ *for $x < 0$ and* $1 - \exp(-\alpha ux)$ *for $x > 0$,*

$$\int\limits_{0}^{+\infty} (1-e^{\alpha ux})f(x)dx > \int\limits_{-\infty}^{0} (1-e^{-\alpha ux})f(x)dx$$

and

$$-\lambda^- \int\limits_{0}^{+\infty} (1-e^{\alpha ux})f(x)dx + \lambda^+ \int\limits_{-\infty}^{0} (1-e^{-\alpha ux})f(x)dx < 0$$

or

$$V(u) < 0$$

for $u < 0$. Consequently, $V(0) \geq V(u)$ for all $u \leq 0$ and $u^* \geq 0$.

b) When $\mu < 0$, $f(-x) > f(x)$ for $x > 0$. Owing to the symmetry between $1 - \exp(\alpha ux)$ *for $x < 0$ and* $1 - \exp(-\alpha ux)$ *for $x > 0$,*

$$\int\limits_{0}^{+\infty} (1-e^{\alpha ux})f(x)dx < \int\limits_{-\infty}^{0} (1-e^{-\alpha ux})f(x)dx$$

and

$$-\lambda^- \int\limits_{-\infty}^{0} (1-e^{-\alpha ux})f(x)dx + \lambda^+ \int\limits_{0}^{+\infty} (1-e^{\alpha ux})f(x)dx < 0$$

or

$$V(u) < 0$$

for $u > 0$*. Consequently,* $V(0) \geq V(u)$ *for all* $u \geq 0$ *and* $u^* < 0$. ■

Lemma 4.3.1 also implies that $u^* = 0$ when $\mu = 0$ and it is optimal not to invest in the risky asset when its expected excess return is zero.

Let $V_-(u)$ denote the loss part and $V_+(u)$ denote the gain part of the objective function. Firstly we analyze the loss part. We can now write

$$\begin{aligned}
V_-(u) &= E[v^-(uR^e)1_{\{uR^e<0\}}] \\
&= -\lambda^- E[(1-e^{uR^e})1_{\{uR^e<0\}}] \\
&= \begin{cases} -\lambda^- E[(1-e^{uR^e})1_{\{R^e<0\}}] & u \geq 0 \\ -\lambda^- E[(1-e^{uR^e})1_{\{R^e>0\}}] & u < 0 \end{cases} \\
&= \begin{cases} -\lambda^- E[1_{\{R^e<0\}}] + \lambda^- E[e^{uR^e}1_{\{R^e<0\}}] & u \geq 0 \\ -\lambda^- E[1_{\{R^e>0\}}] + \lambda^- E[e^{uR^e}1_{\{R^e>0\}}] & u < 0 \end{cases}.
\end{aligned}$$

Note that both $E[1_{\{R^e<0\}}]$ and $E[1_{\{R^e>0\}}]$ are constant terms and these terms do not effect the optimal solution. So, we will continue to analyze $E[\exp(uR^e)1_{\{R^e<0\}}]$ for $u \geq 0$ and $E[\exp(uR^e)1_{\{R^e>0\}}]$ for $u < 0$. Taking $R^e = \mu + \sigma Z$ where $Z \sim N(0,1)$, we obtain

$$\begin{aligned}
E[e^{uR^e}1_{\{R^e<0\}}] &= E[e^{u\mu+u\sigma Z}1_{\{\mu+\sigma Z<0\}}] \\
&= e^{\mu u}E[e^{u\sigma Z}1_{\{Z<-\frac{\mu}{\sigma}\}}] \\
&= e^{\mu u}\int_{-\infty}^{-\frac{\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{u\sigma z} e^{-\frac{1}{2}z^2} dz \\
&= e^{\mu u+\frac{1}{2}\sigma^2u^2}\int_{-\infty}^{-\frac{\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-u\sigma)^2} dz \\
&= e^{\mu u+\frac{1}{2}\sigma^2u^2}\int_{-\infty}^{-\frac{\mu+u\sigma^2}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\
&= e^{\mu u+\frac{1}{2}\sigma^2u^2}\Phi(-(\frac{\mu+\sigma^2u}{\sigma})) \qquad (4.40)
\end{aligned}$$

for $u \geq 0$ where Φ is the standard normal cumulative distribution function.

Similarly,

$$\begin{aligned}
E[e^{uR^e}1_{\{R^e>0\}}] &= E[e^{u\mu+u\sigma Z}1_{\{\mu+\sigma Z>0\}}] \\
&= e^{\mu u}E[e^{u\sigma Z}1_{\{Z>-\frac{\mu}{\sigma}\}}] \\
&= e^{\mu u}\int_{-\frac{\mu}{\sigma}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{u\sigma z}e^{-\frac{1}{2}z^2}dz \\
&= e^{\mu u+\frac{1}{2}\sigma^2u^2}\int_{-\frac{\mu}{\sigma}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-u\sigma)^2}dz \\
&= e^{\mu u+\frac{1}{2}\sigma^2u^2}\int_{-\frac{\mu+u\sigma^2}{\sigma}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx \\
&= e^{\mu u+\frac{1}{2}\sigma^2u^2}[1-\Phi(-(\frac{\mu+\sigma^2u}{\sigma}))] \\
&= e^{\mu u+\frac{1}{2}\sigma^2u^2}\Phi(\frac{\mu+\sigma^2u}{\sigma}). \qquad (4.41)
\end{aligned}$$

Taking the derivative of (4.40), we obtain

$$\begin{aligned}
\frac{d(e^{\mu u+\frac{1}{2}\sigma^2u^2}\Phi(-\frac{\mu+\sigma^2u}{\sigma}))}{du} &= (\mu+\sigma^2u)e^{\mu u+\frac{1}{2}\sigma^2u^2}\Phi(-\frac{\mu+\sigma^2u}{\sigma})-\sigma e^{\mu u+\frac{1}{2}\sigma^2u^2}\varphi(-\frac{\mu+\sigma^2u}{\sigma}) \\
&= e^{\mu u+\frac{1}{2}\sigma^2u^2}[(\mu+\sigma^2u)\Phi(-\frac{\mu+\sigma^2u}{\sigma})-\sigma\varphi(-\frac{\mu+\sigma^2u}{\sigma})] \\
&= e^{\mu u+\frac{1}{2}\sigma^2u^2}\sigma[(\frac{\mu+\sigma^2u}{\sigma})\Phi(-\frac{\mu+\sigma^2u}{\sigma})-\varphi(-\frac{\mu+\sigma^2u}{\sigma})]. \qquad (4.42)
\end{aligned}$$

for $u \geq 0$ where φ is the standard normal density function.

Similarly, the derivative of (4.41) is

$$\begin{aligned}
\frac{d(e^{\mu u+\frac{1}{2}\sigma^2u^2}\Phi(\frac{\mu+\sigma^2u}{\sigma}))}{du} &= (\mu+\sigma^2u)e^{\mu u+\frac{1}{2}\sigma^2u^2}\Phi(\frac{\mu+\sigma^2u}{\sigma})+\sigma e^{\mu u+\frac{1}{2}\sigma^2u^2}\varphi(\frac{\mu+\sigma^2u}{\sigma}) \\
&= e^{\mu u+\frac{1}{2}\sigma^2u^2}[(\mu+\sigma^2u)\Phi(\frac{\mu+\sigma^2u}{\sigma})+\sigma\varphi(\frac{\mu+\sigma^2u}{\sigma})] \\
&= e^{\mu u+\frac{1}{2}\sigma^2u^2}\sigma[(\frac{\mu+\sigma^2u}{\sigma})\Phi(\frac{\mu+\sigma^2u}{\sigma})+\varphi(\frac{\mu+\sigma^2u}{\sigma})]. \qquad (4.43)
\end{aligned}$$

To simplify our notation, we define

$$f(x) = x\Phi(-x) - \varphi(-x)$$

and

$$g(x) = x\Phi(x) + \varphi(x).$$

Then (4.42) and (4.43) can be written as

$$\frac{d(e^{\mu u+\frac{1}{2}\sigma^2u^2}\Phi(-(\frac{\mu+\sigma^2 u}{\sigma})))}{du} = e^{\mu u+\frac{1}{2}\sigma^2u^2}f(\frac{\mu+\sigma^2u}{\sigma}) \tag{4.44}$$

and

$$\frac{d(e^{\mu u+\frac{1}{2}\sigma^2u^2}\Phi(\frac{\mu+\sigma^2 u}{\sigma}))}{du} = e^{\mu u+\frac{1}{2}\sigma^2u^2}g(\frac{\mu+\sigma^2u}{\sigma}). \tag{4.45}$$

For the gain part, we can write

$$\begin{aligned}
V_+(u) &= E[v^+(uR^e)1_{\{uR^e\geq 0\}}] \\
&= \lambda^+E[(1-e^{-uR^e})1_{\{uR^e\geq 0\}}] \\
&= \begin{cases} \lambda^+E[(1-e^{-uR^e})1_{\{R^e\geq 0\}}] & u\geq 0 \\ \lambda^+E[(1-e^{-uR^e})1_{\{R^e\leq 0\}}] & u<0 \end{cases} \\
&= \begin{cases} \lambda^+E[1_{\{R^e>0\}}]-\lambda^+E[e^{-uR^e}1_{\{R^e\geq 0\}}] & u\geq 0 \\ \lambda^+E[1_{\{R^e<0\}}]-\lambda^+E[e^{-uR^e}1_{\{R^e\leq 0\}}] & u<0 \end{cases}.
\end{aligned}$$

Both $E[1_{\{R^e\geq 0\}}]$ and $E[1_{\{R^e\leq 0\}}]$ are constant terms and these terms do not effect the optimal solution. Now, we analyze $E[\exp(-uR^e)1_{\{R^e\geq 0\}}]$ for $u\geq 0$ and $E[\exp(-uR^e)1_{\{R^e\leq 0\}}]$ for $u<0$. Taking $R^e=\mu+\sigma Z$,

$$\begin{aligned}
E[e^{-uR^e}1_{\{R^e\geq 0\}}] &= E[e^{-u\mu-u\sigma Z}1_{\{\mu+\sigma Z\geq 0\}}] \\
&= e^{-\mu u}E[e^{-u\sigma Z}1_{\{Z\geq-\frac{\mu}{\sigma}\}}] \\
&= e^{-\mu u}\int_{-\frac{\mu}{\sigma}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-u\sigma z}e^{-\frac{1}{2}z^2}dz \\
&= e^{-\mu u+\frac{1}{2}\sigma^2u^2}\int_{-\frac{\mu}{\sigma}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z+u\sigma)^2}dz \\
&= e^{-\mu u+\frac{1}{2}\sigma^2u^2}\int_{\frac{u\sigma^2-\mu}{\sigma}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx \\
&= e^{-\mu u+\frac{1}{2}\sigma^2u^2}(1-\Phi(\frac{u\sigma^2-\mu}{\sigma})) \\
&= e^{-\mu u+\frac{1}{2}\sigma^2u^2}\Phi(-\frac{u\sigma^2-\mu}{\sigma})
\end{aligned} \tag{4.46}$$

for $u \geq 0$. Similarly,

$$\begin{aligned} E[e^{-uR^e}1_{\{R^e<0\}}] &= E[e^{-u\mu-u\sigma Z}1_{\{\mu+\sigma Z\leq 0\}}] \\ &= e^{-\mu u}E[e^{-u\sigma Z}1_{\{Z\leq -\frac{\mu}{\sigma}\}}] \\ &= e^{-\mu u}\int_{-\infty}^{-\frac{\mu}{\sigma}}\frac{1}{\sqrt{2\pi}}e^{-u\sigma z}e^{-\frac{1}{2}z^2}dz \\ &= e^{-\mu u+\frac{1}{2}\sigma^2u^2}\int_{-\infty}^{-\frac{\mu}{\sigma}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z+u\sigma)^2}dz \\ &= e^{-\mu u+\frac{1}{2}\sigma^2u^2}\int_{-\infty}^{\frac{u\sigma^2-\mu}{\sigma}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx \\ &= e^{-\mu u+\frac{1}{2}\sigma^2u^2}\Phi(\frac{u\sigma^2-\mu}{\sigma}) \end{aligned} \quad (4.47)$$

for $u < 0$. Taking derivative of (4.46) and (4.47), we obtain

$$\begin{aligned} \frac{d(e^{-\mu u+\frac{1}{2}\sigma^2u^2}\Phi(-\frac{u\sigma^2-\mu}{\sigma}))}{du} &= (-\mu+\sigma^2u)e^{-\mu u+\frac{1}{2}\sigma^2u^2}\Phi(-\frac{u\sigma^2-\mu}{\sigma}) - \\ &\quad \sigma e^{-\mu u+\frac{1}{2}\sigma^2u^2}\varphi(-\frac{u\sigma^2-\mu}{\sigma})) \\ &= e^{-\mu u+\frac{1}{2}\sigma^2u^2}[(-\mu+\sigma^2u)\Phi(-\frac{u\sigma^2-\mu}{\sigma}) - \\ &\quad \sigma\varphi(-\frac{u\sigma^2-\mu}{\sigma})] \\ &= e^{-\mu u+\frac{1}{2}\sigma^2u^2}\sigma[(\frac{-\mu+\sigma^2u}{\sigma})\Phi(-\frac{u\sigma^2-\mu}{\sigma}) - \\ &\quad \varphi(-\frac{u\sigma^2-\mu}{\sigma})] \end{aligned} \quad (4.48)$$

and

$$\begin{aligned} \frac{d(e^{-\mu u+\frac{1}{2}\sigma^2u^2}\Phi(\frac{u\sigma^2-\mu}{\sigma}))}{du} &= (-\mu+\sigma^2u)e^{-\mu u+\frac{1}{2}\sigma^2u^2}\Phi(\frac{u\sigma^2-\mu}{\sigma}) + \\ &\quad \sigma e^{-\mu u+\frac{1}{2}\sigma^2u^2}\varphi(\frac{u\sigma^2-\mu}{\sigma})) \\ &= e^{-\mu u+\frac{1}{2}\sigma^2u^2}[(-\mu+\sigma^2u)\Phi(\frac{u\sigma^2-\mu}{\sigma}) + \\ &\quad \sigma\varphi(\frac{u\sigma^2-\mu}{\sigma})] \\ &= e^{-\mu u+\frac{1}{2}\sigma^2u^2}\sigma[(\frac{-\mu+\sigma^2u}{\sigma})\Phi(\frac{u\sigma^2-\mu}{\sigma}) + \\ &\quad \varphi(\frac{u\sigma^2-\mu}{\sigma})]. \end{aligned} \quad (4.49)$$

Using f and g, we can rewrite (4.48) and (4.49) as

$$\frac{d(e^{-\mu u+\frac{1}{2}\sigma^2u^2}\Phi(-\frac{u\sigma^2-\mu}{\sigma}))}{du} = e^{-\mu u+\frac{1}{2}\sigma^2u^2}f(\frac{-\mu+\sigma^2 u}{\sigma}) \tag{4.50}$$

and

$$\frac{d(e^{-\mu u+\frac{1}{2}\sigma^2u^2}\Phi(\frac{u\sigma^2-\mu}{\sigma}))}{du} = e^{-\mu u+\frac{1}{2}\sigma^2u^2}g(\frac{-\mu+\sigma^2 u}{\sigma}). \tag{4.51}$$

From (4.44), (4.45), (4.50) and (4.51), $dV(u)/du$ can be rewritten as

$$\frac{dV(u)}{du} = \begin{cases} e^{\mu u+\frac{1}{2}\sigma^2u^2}\sigma\left[\lambda^- f\left(\frac{\mu+\sigma^2u}{\sigma}\right) - \lambda^+ e^{-2\mu u} f\left(\frac{-\mu+\sigma^2u}{\sigma}\right)\right] & u \geq 0 \\ e^{\mu u+\frac{1}{2}\sigma^2u^2}\sigma\left[\lambda^- g\left(\frac{\mu+\sigma^2u}{\sigma}\right) - \lambda^+ e^{-2\mu u} g\left(\frac{-\mu+\sigma^2u}{\sigma}\right)\right] & u < 0 \end{cases}. \tag{4.52}$$

In (4.52), since $e^{\mu u+\frac{1}{2}\sigma^2u^2}$ is positive for all u, it has no effect on the sign of $dV(u)/du$. Firstly, to investigate the sign of $dV(u)/du$ we must examine the structure of $f(x)$ and $g(x)$. Taking derivatives of both function,

$$\begin{aligned} f'(x) &= \Phi(-x) - x\varphi(-x) + x\varphi(-x) \\ &= \Phi(-x) \end{aligned}$$

and

$$\begin{aligned} g'(x) &= \Phi(x) + x\varphi(x) - x\varphi(-x) \\ &= \Phi(x). \end{aligned}$$

Since $f'(x) = \Phi(-x) > 0$ and $f''(x) = -\varphi(x) < 0$, $f < 0$ is strictly increasing concave function where $\lim_{x\longrightarrow+\infty} f(x) = 0$. Also, since $g'(x) = \phi(x) > 0$ and $g''(x) = \varphi(x) > 0$, $g > 0$ is strictly increasing convex function where $\lim_{x\longrightarrow-\infty} g(x) = 0$.

Let $\hat{f}(x) = -f(x)$, then $\hat{f} > 0$ is a strictly decreasing convex function. Then, the sign of $dV(u)/du$ becomes

$$\begin{aligned} sign\left(\frac{dV(u)}{du}\right) &= \begin{cases} sign\left(e^{\mu u+\frac{1}{2}\sigma^2u^2}\sigma\left[\lambda^- f\left(\frac{\mu+\sigma^2u}{\sigma}\right) - \lambda^+ e^{-2\mu u} f\left(\frac{-\mu+\sigma^2u}{\sigma}\right)\right]\right) & u \geq 0 \\ sign\left(e^{\mu u+\frac{1}{2}\sigma^2u^2}\sigma\left[\lambda^- g\left(\frac{\mu+\sigma^2u}{\sigma}\right) - \lambda^+ e^{-2\mu u} g\left(\frac{-\mu+\sigma^2u}{\sigma}\right)\right]\right) & u < 0 \end{cases} \\ &= \begin{cases} sign\left(\lambda^- f\left(\frac{\mu+\sigma^2u}{\sigma}\right) - \lambda^+ e^{-2\mu u} f\left(\frac{-\mu+\sigma^2u}{\sigma}\right)\right) & u \geq 0 \\ sign\left(\lambda^- g\left(\frac{\mu+\sigma^2u}{\sigma}\right) - \lambda^+ e^{-2\mu u} g\left(\frac{-\mu+\sigma^2u}{\sigma}\right)\right) & u < 0 \end{cases} \\ &= \begin{cases} -sign\left(\lambda^- \hat{f}\left(\frac{\mu+\sigma^2u}{\sigma}\right) - \lambda^+ e^{-2\mu u} \hat{f}\left(\frac{-\mu+\sigma^2u}{\sigma}\right)\right) & u \geq 0 \\ sign\left(\lambda^- g\left(\frac{\mu+\sigma^2u}{\sigma}\right) - \lambda^+ e^{-2\mu u} g\left(\frac{-\mu+\sigma^2u}{\sigma}\right)\right) & u < 0 \end{cases} \\ &= \begin{cases} -sign\left(\frac{\hat{f}\left(\frac{\mu+\sigma^2u}{\sigma}\right)e^{2\mu u}}{\hat{f}\left(\frac{-\mu+\sigma^2u}{\sigma}\right)} - \frac{\lambda^+}{\lambda^-}\right) & u \geq 0 \\ sign\left(\frac{g\left(\frac{\mu+\sigma^2u}{\sigma}\right)e^{2\mu u}}{g\left(\frac{-\mu+\sigma^2u}{\sigma}\right)} - \frac{\lambda^+}{\lambda^-}\right) & u < 0 \end{cases}. \end{aligned} \tag{4.53}$$

Theorem 4.3.2 *If $\mu \geq 0$, $V(u)$ is quasi-concave for $u \geq 0$. Moreover, the optimal solution is*

$$u^* = \begin{cases} 0 & \frac{\hat{f}\left(\frac{\mu}{\sigma}\right)}{\hat{f}\left(\frac{-\mu}{\sigma}\right)} \geq \frac{\lambda^+}{\lambda^-} \\ \bar{u} & \frac{\hat{f}\left(\frac{\mu}{\sigma}\right)}{\hat{f}\left(\frac{-\mu}{\sigma}\right)} < \frac{\lambda^+}{\lambda^-} \end{cases} \tag{4.54}$$

where $\bar{u}$ is the unique positive value that satisfies

$$\frac{\hat{f}\left(\frac{\mu+\sigma^2\bar{u}}{\sigma}\right)e^{2\mu\bar{u}}}{\hat{f}\left(\frac{-\mu+\sigma^2\bar{u}}{\sigma}\right)} = \frac{\lambda^+}{\lambda^-}. \tag{4.55}$$

Proof. *Consider the case for $u \geq 0$. We will first show that V is quasi-concave for any $\mu \geq 0$. It suffices to show that $\hat{f}\left((\mu+\sigma^2 u)/\sigma\right)e^{2\mu u}/\hat{f}\left((-\mu+\sigma^2 u)/\sigma\right)$ is an increasing function in u when $\mu \geq 0$.*

For simplicity, let $z = \sigma u$ and $\hat{\mu} = \mu/\sigma$. Firstly, we will show that

$$h_+(z,\hat{\mu}) = \frac{\hat{f}(z+\hat{\mu})e^{2\hat{\mu}z}}{\hat{f}(z-\hat{\mu})}$$

is an increasing function in z. Taking the partial derivative of h_+, we will show

$$\frac{\partial h_+(z,\hat{\mu})}{\partial z} = \left(\frac{\hat{f}(z+\hat{\mu})}{\hat{f}(z-\hat{\mu})}\right)' e^{2\hat{\mu}z} + \frac{\hat{f}(z+\hat{\mu})}{\hat{f}(z-\hat{\mu})}\left(e^{2\hat{\mu}z}\right)' \geq 0$$

or

$$\frac{\left(\frac{\hat{f}(z+\hat{\mu})}{\hat{f}(z-\hat{\mu})}\right)'}{\frac{\hat{f}(z+\hat{\mu})}{\hat{f}(z-\hat{\mu})}} \geq -\frac{\left(e^{2\hat{\mu}z}\right)'}{e^{2\hat{\mu}z}}$$

or

$$\frac{\frac{\left(\hat{f}'(z+\hat{\mu})\hat{f}(z-\hat{\mu}) - \hat{f}(z+\hat{\mu})\hat{f}'(z-\hat{\mu})\right)}{\left(\hat{f}(z-\hat{\mu})\right)^2}}{\frac{\hat{f}(z+\hat{\mu})}{\hat{f}(z-\hat{\mu})}} \geq -\frac{2\hat{\mu}e^{2\hat{\mu}z}}{e^{2\hat{\mu}z}}$$

or

$$\frac{\hat{f}'(z+\hat{\mu})}{\hat{f}(z+\hat{\mu})} - \frac{\hat{f}'(z-\hat{\mu})}{\hat{f}(z-\hat{\mu})} \geq -2\hat{\mu}. \tag{4.56}$$

Define

$$\begin{aligned} k(z) &= \frac{\hat{f}'(z)}{\hat{f}(z)} \qquad (4.57) \\ &= \frac{-\Phi(-z)}{\varphi(z) - z\left[1-\Phi(z)\right]} \\ &= \frac{1}{\frac{\varphi(z)}{\Phi(-z)} - z} \\ &= \frac{1}{z - r(z)} \end{aligned}$$

where $r(z) = \varphi(z)/(1-\Phi(z))$ *is the failure rate function of the standard normal distribution.*

Using k*, we rewrite (4.56) as*

$$k(z + \hat{\mu}) - k(z - \hat{\mu}) \geq -2\hat{\mu}. \tag{4.58}$$

For (4.58) to hold, it suffices to show that

$$k'(z) \geq -1.$$

Now, we will use some properties of failure rate function summarized, for example, in Bar-Isaac et al. (2014) to complete the analysis. The failure rate function is positive and convex increasing so that

a)

$$r(z) > 0$$

b)

$$r'(z) > 0$$

c)

$$r''(z) > 0$$

for all z*.*

Furthermore, we use the boundary condition

d)

$$z^+ < r(z) < \left| z + \frac{1}{z} \right| \tag{4.59}$$

given in Gordon (1941) where $z^+ = \max\{0, z\}$*.*

We continue with some analysis on r *and* k*, it is clear that*

$$r(x) = \frac{\varphi(x)}{(1-\Phi(x))} > 0.$$

Taking the derivative of r*, we get*

$$\begin{aligned} r'(x) &= \frac{\varphi'(x)(1-\Phi(x)) + \varphi^2(x)}{(1-\Phi(x))^2} \\ &= \frac{-x\varphi(x)(1-\Phi(x)) + \varphi^2(x)}{(1-\Phi(x))^2} \\ &= \frac{\varphi(x)\hat{f}(x)}{(1-\Phi(x))^2} > 0 \end{aligned}$$

since $\varphi(x) > 0$ *and* $\hat{f}(x) > 0$. *Also,* $r'(x)$ *can be rewritten as*

$$\begin{aligned} r'(x) &= -xr(x) + r(x)^2 \\ &= r(x)(r(x) - x) \end{aligned} \tag{4.60}$$

after some simple manipulations. Then, (4.60) implies that $r(x) > x$ *or* $k(x) < 0$, *since* $r'(x) > 0$.

Taking the second derivative of r, *we get*

$$\begin{aligned} r''(x) &= 2r(x)r'(x) - r(x) - xr'(x) \\ &= 2r(x)r(x)(r(x) - x) - r(x) - xr(x)(r(x) - x) \\ &= 2r(x)^3 - 2xr(x)^2 - r(x) - xr(x)^2 + x^2 r(x) \\ &= 2r(x)^3 - 3xr(x)^2 + x^2 r(x) - r(x) \\ &= r(x)(2r(x)^2 - 3xr(x) + x^2 - 1) > 0 \end{aligned} \tag{4.61}$$

by c). Then (4.61) implies that

$$2r(x)^2 - 3xr(x) + x^2 - 1 > 0. \tag{4.62}$$

Now, taking the derivative of k, *we obtain*

$$\begin{aligned} k'(x) &= \left(\frac{1}{x - r(x)}\right)' \\ &= \frac{-(1 - r'(x))}{(x - r(x))^2} \\ &= -\left(\frac{1 - r(x)(r(x) - x)}{(x - r(x))^2}\right). \end{aligned}$$

We are investigating whether $k'(x) \geq -1$. *It is indeed true if*

$$1 - r(x)(r(x) - x) \leq (x - r(x))^2$$

or

$$1 - r(x)^2 - r(x)x \leq x^2 - 2xr(x) + r(x)^2$$

or

$$2r(x)^2 - 3xr(x) + x^2 - 1 \geq 0$$

which is true by (4.62). Therefore, $k'(x) \geq -1$ *and* $V(u)$ *is quasi-concave for* $u \geq 0$ *Since the objective function* $V(u)$ *is quasi-concave for* $u \geq 0$, *the optimal solution* u^* *is obtained by setting (4.53) equal to zero. noting that* $h_+(z, \hat{\mu})$ *is increasing in* $z = \sigma u$, *this leads to the optimal solution given by (4.54) and (4.55).* ■

Now, we analyze how the optimal portfolio changes when μ increases for $\mu \geq 0$. The general expectation is that the optimal portfolio should be increasing in μ. From (4.53), it suffices to show that $h_+(z, \hat{\mu})$ is decreasing in $\hat{\mu}$ for $z = \sigma u \geq 0$. Note that this will be true if

$$\frac{\partial h_+(z, \hat{\mu})}{\partial \hat{\mu}} = \left(\frac{\hat{f}(z+\hat{\mu})}{\hat{f}(z-\hat{\mu})}\right)' e^{2\hat{\mu}z} + \frac{\hat{f}(z+\hat{\mu})}{\hat{f}(z-\hat{\mu})}\left(e^{2\hat{\mu}z}\right)' \leq 0$$

or

$$\frac{\left(\frac{\hat{f}(z+\hat{\mu})}{\hat{f}(z-\hat{\mu})}\right)'}{\frac{\hat{f}(z+\hat{\mu})}{\hat{f}(z-\hat{\mu})}} \leq -\frac{\left(e^{2\hat{\mu}z}\right)'}{e^{2\hat{\mu}z}}$$

or

$$\frac{\frac{\left(\hat{f}'(z+\hat{\mu})\hat{f}(z-\hat{\mu})+\hat{f}(z+\hat{\mu})\hat{f}'(z-\hat{\mu})\right)}{\left(\hat{f}(z-\hat{\mu})\right)^2}}{\frac{\hat{f}(z+\hat{\mu})}{\hat{f}(z-\hat{\mu})}} \leq -\frac{2ze^{2\hat{\mu}z}}{e^{2\hat{\mu}z}}$$

or

$$\frac{\hat{f}'(z+\hat{\mu})}{\hat{f}(z+\hat{\mu})} + \frac{\hat{f}'(z-\hat{\mu})}{\hat{f}(z-\hat{\mu})} \leq -2z. \tag{4.63}$$

Using the definition of k in (4.57), (4.63) can be rewritten as

$$k(z+\hat{\mu}) + k(z-\hat{\mu}) \leq -2z$$

or

$$\frac{1}{z+\hat{\mu}-r(z+\hat{\mu})} + \frac{1}{z-\hat{\mu}-r(z-\hat{\mu})} \leq -2z.$$

Now, it is sufficient to show that

$$\frac{1}{r(z+\hat{\mu})-(z+\hat{\mu})} + \frac{1}{r(z-\hat{\mu})-(z-\hat{\mu})} \geq 2z$$

for $\hat{\mu} \geq 0$.

This is true for $\hat{\mu} = 0$, since

$$\frac{1}{r(z)-z} > z$$

from (4.59). Therefore, it suffices to show that

$$-k(z+\hat{\mu}) - k(z-\hat{\mu})$$

is increasing in $\hat{\mu}$. In other words,

$$-k'(z+\hat{\mu}) + k'(z-\hat{\mu}) \geq 0$$

or

$$k'(z-\hat{\mu}) \geq k'(z+\hat{\mu}).$$

Although we are unable to prove that $k'' \leq 0$, we conjecture that it is true. Under this conjecture, u^* increases as μ increases.

Theorem 4.3.3 *If $\mu < 0$, $V(u)$ is quasi-concave for $u < 0$. Moreover, the optimal solution is*

$$u^* = \begin{cases} 0 & \frac{g\left(\frac{\mu}{\sigma}\right)}{g\left(\frac{-\mu}{\sigma}\right)} \geq \frac{\lambda^+}{\lambda^-} \\ \bar{u} & \frac{g\left(\frac{\mu}{\sigma}\right)}{g\left(\frac{-\mu}{\sigma}\right)} < \frac{\lambda^+}{\lambda^-} \end{cases} \tag{4.64}$$

where $\bar{u}$ is the unique negative value that satisfies

$$\frac{g\left(\frac{\mu+\sigma^2\bar{u}}{\sigma}\right)e^{2\mu\bar{u}}}{g\left(\frac{-\mu+\sigma^2\bar{u}}{\sigma}\right)} = \frac{\lambda^+}{\lambda^-}. \tag{4.65}$$

Proof. *Consider the case for $u < 0$. We will first show that V is quasi-concave for any $\mu < 0$. It suffices to show that $g\left((\mu+\sigma^2 u)/\sigma\right)e^{2\mu u}/g\left((-\mu+\sigma^2 u)/\sigma\right)$ is a decreasing function in u when $\mu < 0$. Now, we will show that*

$$h_-(z,\hat{\mu}) = \frac{g(z+\hat{\mu})e^{2\hat{\mu}z}}{g(z-\hat{\mu})}$$

is decreasing function in z. Taking the partial derivative of h_-, we will show

$$\frac{\partial h_-(z,\hat{\mu})}{\partial z} = \left(\frac{g(z+\hat{\mu})}{g(z-\hat{\mu})}\right)' e^{2\hat{\mu}z} + \frac{g(z+\hat{\mu})}{g(z-\hat{\mu})}\left(e^{2\hat{\mu}z}\right)' \leq 0$$

or

$$\frac{\left(\frac{g(z+\hat{\mu})}{g(z-\hat{\mu})}\right)'}{\frac{g(z+\hat{\mu})}{g(z-\hat{\mu})}} \leq -\frac{\left(e^{2\hat{\mu}z}\right)'}{e^{2\hat{\mu}z}}$$

or

$$\frac{\frac{(g'(z+\hat{\mu})g(z-\hat{\mu})-g(z+\hat{\mu})g'(z-\hat{\mu}))}{(g(z-\hat{\mu}))^2}}{\frac{g(z+\hat{\mu})}{g(z-\hat{\mu})}} \leq -\frac{2\hat{\mu}e^{2\hat{\mu}z}}{e^{2\hat{\mu}z}}$$

or

$$\frac{g'(z+\hat{\mu})}{g(z+\hat{\mu})} - \frac{g'(z-\hat{\mu})}{g(z-\hat{\mu})} \leq -2\hat{\mu}. \tag{4.66}$$

Define

$$
\begin{aligned}
l(z) &= \frac{g'(z)}{g(z)} \\
&= \frac{\Phi(z)}{\varphi(z)+z\Phi(z)} \\
&= \frac{1}{\frac{\varphi(z)}{\Phi(z)}+z} \\
&= \frac{1}{\frac{\varphi(-z)}{\Phi(z)}+z} \\
&= \frac{1}{r(-z)+z} \qquad (4.67)
\end{aligned}
$$

where $r(z)=\varphi(z)/[1-\Phi(z)]$. Actually, l is similar to k. The relation between k and l can be shown as

$$l(z) = -k(-z). \qquad (4.68)$$

Using l, we rewrite (4.66) as

$$l(z+\hat{\mu}) - l(z-\hat{\mu}) \leq -2\hat{\mu} \qquad (4.69)$$

where $\hat{\mu} < 0$. Then, it suffices to show that

$$l'(z) \geq -1.$$

But, (4.68) trivially implies that $l^{'}(z) = k^{'}(-z) \geq -1$ and $V(u)$ is quasi-concave for $u < 0$. The quasi-concavity of $V(u)$ implies that the optimal solution u^* is obtained by setting (4.53) equal to zero. Noting that $h_-(z,\hat{\mu})$ is decreasing in $z = \sigma u$, this leads to the optimal solution given by (4.64) and (4.65). ■

Now, we analyze how the optimal portfolio changes when μ increases for $\mu < 0$. The general expectation is that the optimal portfolio should be increasing in μ. From (4.53), it suffices to show that $h_-(z,\hat{\mu})$ is increasing in $\hat{\mu}$ for $z = \sigma u < 0$. Note that this will be true if

$$\frac{\partial h_-(z,\hat{\mu})}{\partial \hat{\mu}} = \left(\frac{g(z+\hat{\mu})}{g(z-\hat{\mu})}\right)' e^{2\hat{\mu}z} + \frac{g(z+\hat{\mu})}{g(z-\hat{\mu})}\left(e^{2\hat{\mu}z}\right)' \geq 0$$

or

$$\frac{\left(\frac{g(z+\hat{\mu})}{g(z-\hat{\mu})}\right)'}{\frac{g(z+\hat{\mu})}{g(z-\hat{\mu})}} \geq -\frac{\left(e^{2\hat{\mu}z}\right)'}{e^{2\hat{\mu}z}}$$

or

$$\frac{\frac{(g'(z+\hat{\mu})g(z-\hat{\mu})+g(z+\hat{\mu})g'(z-\hat{\mu}))}{(g(z-\hat{\mu}))^2}}{\frac{g(z+\hat{\mu})}{g(z-\hat{\mu})}} \geq -\frac{2ze^{2\hat{\mu}z}}{e^{2\hat{\mu}z}}$$

or

$$\frac{g'(z+\hat{\mu})}{g(z+\hat{\mu})} + \frac{g'(z-\hat{\mu})}{g(z-\hat{\mu})} \geq -2z. \tag{4.70}$$

Using l, (4.70) can be rewritten as

$$l(z+\hat{\mu}) + l(z-\hat{\mu}) \geq -2z$$

or

$$\frac{1}{z+\hat{\mu}+r(-z-\hat{\mu})} + \frac{1}{z-\hat{\mu}+r(-z+\hat{\mu})} \geq -2z$$

for $\hat{\mu} < 0$.

This is true for $\hat{\mu} = 0$, since

$$\frac{1}{z+r(-z)} > -z.$$

for $z < 0$ by (4.59). Therefore, it suffices to show that

$$l(z+\hat{\mu}) + l(z-\hat{\mu})$$

is decreasing in $\hat{\mu}$. In other words,

$$l'(z+\hat{\mu}) - l'(z-\hat{\mu}) \leq 0$$

or

$$l'(z+\hat{\mu}) \leq l'(z-\hat{\mu}).$$

where $\hat{\mu} < 0$. This is true since $l''(z) = -k''(-z) \geq 0$ by our conjecture.

From Theorem 4.3.2 and Theorem 4.3.3, there are some mean levels such that if the expected return μ is between these levels, the optimal solution equals to 0. To obtain both critical means, we can investigate the derivative of objective function around 0. From (4.52), we obtain

$$\begin{aligned} sign\left(\frac{dV(u)}{du}\right)\Big|_{u=0+} &= sign\left(\sigma\left[\lambda^- f\left(\frac{\mu}{\sigma}\right) - \lambda^+ f\left(\frac{-\mu}{\sigma}\right)\right]\right) \\ &= sign\left(\lambda^- f\left(\frac{\mu}{\sigma}\right) - \lambda^+ f\left(\frac{-\mu}{\sigma}\right)\right) \\ &= sign\left(\lambda^-\left(\frac{\mu}{\sigma}\Phi\left(-\frac{\mu}{\sigma}\right) - \varphi\left(-\frac{\mu}{\sigma}\right)\right) - \lambda^+\left(-\frac{\mu}{\sigma}\Phi\left(\frac{\mu}{\sigma}\right) - \varphi\left(\frac{\mu}{\sigma}\right)\right)\right) \\ &= sign\left(\lambda^-\left(\frac{\mu}{\sigma}\left(1-\Phi\left(\frac{\mu}{\sigma}\right)\right) - \varphi\left(-\frac{\mu}{\sigma}\right)\right) - \right. \\ &\qquad \left. \lambda^+\left(-\frac{\mu}{\sigma}\Phi\left(\frac{\mu}{\sigma}\right) - \varphi\left(\frac{\mu}{\sigma}\right)\right)\right) \end{aligned}$$

and

$$\begin{aligned}
sign\left(\frac{dV(u)}{du}\right)\Big|_{u=0-} &= sign\left(\sigma\left[\lambda^{-}g\left(\frac{\mu}{\sigma}\right)-\lambda^{+}g\left(\frac{-\mu}{\sigma}\right)\right]\right)\\
&= sign\left(\lambda^{-}g\left(\frac{\mu}{\sigma}\right)-\lambda^{+}g\left(\frac{-\mu}{\sigma}\right)\right)\\
&= sign\left(\lambda^{-}\left(\frac{\mu}{\sigma}\Phi\left(\frac{\mu}{\sigma}\right)+\varphi\left(\frac{\mu}{\sigma}\right)\right)-\lambda^{+}\left(-\frac{\mu}{\sigma}\Phi\left(-\frac{\mu}{\sigma}\right)+\varphi\left(-\frac{\mu}{\sigma}\right)\right)\right)\\
&= sign\left(\lambda^{-}\left(\frac{\mu}{\sigma}\left(1-\Phi\left(-\frac{\mu}{\sigma}\right)\right)+\varphi\left(\frac{\mu}{\sigma}\right)\right)-\right.\\
&\qquad \left.\lambda^{+}\left(-\frac{\mu}{\sigma}\Phi\left(-\frac{\mu}{\sigma}\right)+\varphi\left(-\frac{\mu}{\sigma}\right)\right)\right).
\end{aligned}$$

The first critic mean level must satisfy

$$sign\left(\frac{dV(u)}{du}\right)\Big|_{u=0+}=0$$

or

$$\lambda^{-}\left(\frac{\mu}{\sigma}\left(1-\Phi\left(\frac{\mu}{\sigma}\right)\right)-\varphi\left(-\frac{\mu}{\sigma}\right)\right)-\lambda^{+}\left(-\frac{\mu}{\sigma}\Phi\left(\frac{\mu}{\sigma}\right)-\varphi\left(\frac{\mu}{\sigma}\right)\right)=0$$

or

$$(\lambda^{-}-\lambda^{+})\frac{\mu}{\sigma}\Phi\left(\frac{\mu}{\sigma}\right)+(\lambda^{-}-\lambda^{+})\varphi\left(\frac{\mu}{\sigma}\right)=\lambda^{-}\frac{\mu}{\sigma}$$

or

$$(\lambda^{-}-\lambda^{+})\Phi\left(\frac{\mu}{\sigma}\right)+(\lambda^{-}-\lambda^{+})\frac{\sigma}{\mu}\varphi\left(\frac{\mu}{\sigma}\right)=\lambda^{-}$$

or

$$\Phi\left(\frac{\mu}{\sigma}\right)+\frac{\sigma}{\mu}\varphi\left(\frac{\mu}{\sigma}\right)=\frac{\lambda^{-}}{\lambda^{-}-\lambda^{+}}=\frac{1}{1-\lambda}. \tag{4.71}$$

The second critic mean level must satisfy

$$sign\left(\frac{dV(u)}{du}\right)\Big|_{u=0-}=0$$

or

$$\lambda^{-}\left(\frac{\mu}{\sigma}\left(1-\Phi\left(-\frac{\mu}{\sigma}\right)\right)+\varphi\left(\frac{\mu}{\sigma}\right)\right)-\lambda^{+}\left(-\frac{\mu}{\sigma}\Phi\left(-\frac{\mu}{\sigma}\right)+\varphi\left(-\frac{\mu}{\sigma}\right)\right)=0$$

or

$$(\lambda^{-}-\lambda^{+})\frac{\mu}{\sigma}\Phi\left(-\frac{\mu}{\sigma}\right)-(\lambda^{-}-\lambda^{+})\varphi\left(-\frac{\mu}{\sigma}\right)=\lambda^{-}\frac{\mu}{\sigma}$$

or

$$(\lambda^{-}-\lambda^{+})\Phi\left(-\frac{\mu}{\sigma}\right)-(\lambda^{-}-\lambda^{+})\frac{\sigma}{\mu}\varphi\left(\frac{\mu}{\sigma}\right)=\lambda^{-}$$

or

$$\Phi\left(-\frac{\mu}{\sigma}\right)-\frac{\sigma}{\mu}\varphi\left(\frac{\mu}{\sigma}\right)=\frac{\lambda^{-}}{\lambda^{-}-\lambda^{+}}=\frac{1}{1-\lambda}. \tag{4.72}$$

Chapter 5

NUMERICAL ILLUSTRATIONS

Up to this point, we discuss the portfolio optimization problem within the prospect framework. This chapter demonstrates the results of Chapter 4 by some illustrative numerical examples. We construct some examples to investigate the effects of parameters on the optimal portfolios. Then, we comment on the effect of the prospect approach. We take $\alpha = 1$, $\lambda^- = 14.7$, $\lambda^+ = 6.52$ in our analyses throughout this chapter unless stated otherwise.

Our illustrations concentrate on the normal distribution model. We will show the shape of some value functions that we discussed before. We will use normally distributed excess returns with different means. Moreover, we will fix the standard deviation $\sigma = 0.25$ for all illustrations. Then, we will show the graph of the optimal portfolios versus means with the same variance.

5.1 Piecewise Linear Value Function

In this subsection, we present some examples for the piecewise linear value function. As done in Chapter 4.1.1, we obtain the solutions. Recall that there are some mean levels such that if the expected return is between these levels, there is a trivial solution and it is $u^* = 0$. If not, the solution is infinite. For these illustrations, the critical mean levels can be found easily. Using (4.19), the first critical point μ_1 must satisfy the equation

$$\Phi(\frac{\mu_1}{\sigma}) + \frac{\sigma\varphi(\frac{\mu_1}{\sigma})}{\mu_1} = \frac{1}{1-\lambda}$$

or

$$\Phi(4\mu_1) + \frac{\varphi(4\mu_1)}{4\mu_1} = 1.797 \tag{5.1}$$

where $\mu_1 \geq 0$. We know there is only one μ_1 that satisfies (5.1) and it is $\mu_1 = 0.081$. The second critical point μ_2 must satisfy

$$\Phi(-\frac{\mu_2}{\sigma}) - \frac{\sigma\varphi(\frac{\mu_2}{\sigma})}{\mu_2} = \frac{1}{1-\lambda}$$

or

$$\Phi(-4\mu_2) - \frac{\varphi(4\mu_2)}{4\mu_2} = 1.797 \tag{5.2}$$

where $\mu_2 < 0$. The two equations (5.1) and (5.2) have symmetric functions around 0 and $\mu_2 = -0.081$. Then, we can characterize the solution by using these two critical points so that

$$u^* = \begin{cases} -\infty & \mu < -0.081 \\ (-\infty, 0) & \mu = -0.081 \\ 0 & -0.081 < \mu < 0.081 \\ [0, +\infty) & \mu = 0.081 \\ +\infty & \mu > 0.081 \end{cases} . \tag{5.3}$$

Now, the shapes of the objective functions that we try to maximize for different expected rates of return are provided in Figure 5.1. Note that all objective functions are also piecewise linear and the optimal solution depends on the value of μ as given in (5.3).

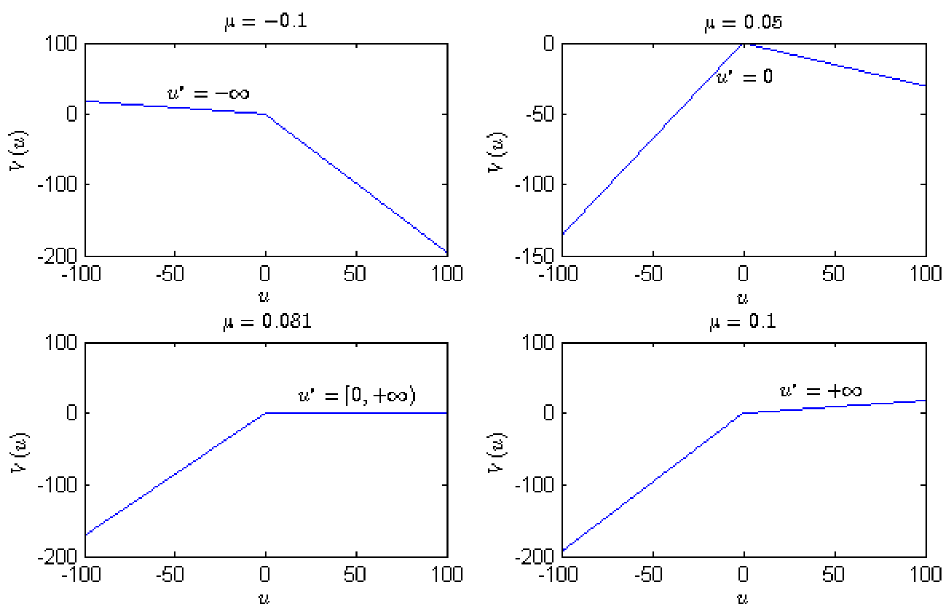

Figure 5.1: Objective functions for piecewise linear function

The optimal portfolios are depicted in Figure 5.2 as a function of the mean μ. As you see in Figure 5.2, there are three optimal solutions. Two of them are unbounded solutions ($u^* = -\infty$ or $+\infty$). They require selling or buying the risky asset as much as you can. The other optimal solution is do not buy any risky asset ($u^* = 0$) and invest all money in the risk-free asset. Here, the ratio λ specifies the interval where the optimal portfolio is $u^* = 0$. In (5.1) and (5.2), $1/(1-\lambda)$ is increasing in λ where $\lambda < 1$. Also, from (4.20) and (4.22), we know that $h_1(x) = \Phi(x) + \varphi(x)/x$ is a decreasing function for $x \geq 0$ and its symmetric

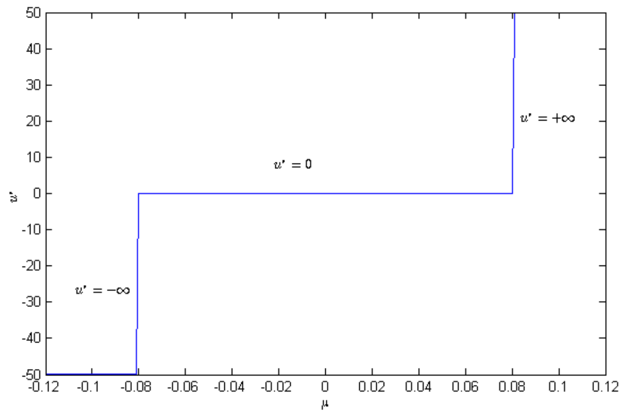

Figure 5.2: Optimal portfolios as a function of mean μ

function $h_2(x) = \Phi(-x) - \varphi(x)/x$ is an increasing function for $x < 0$. While λ increases, $1/(1-\lambda)$ increases and (5.1) holds for a smaller mean μ_1. By symmetry, (5.2) holds for a greater mean μ_2. In the other words, the interval that the optimal portfolio is not to buy any risky asset gets smaller. Also, the effect of the changing the interval with λ is shown in Figure 5.3. As you see, when λ increases to 1, the length of the interval goes to 0 and, if $\lambda = 1$ all optimal solutions are unbounded.

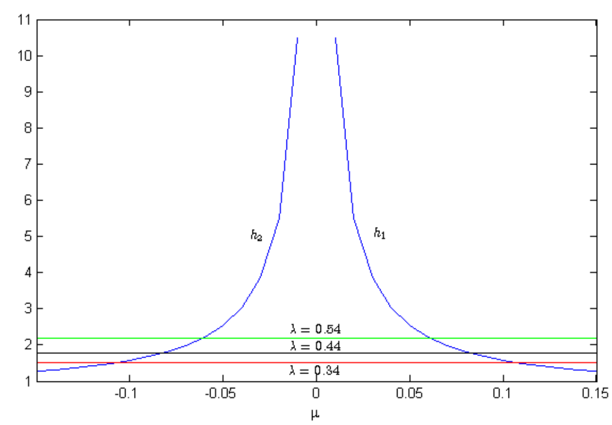

Figure 5.3: Critical points as a function of λ

5.2 Piecewise Exponential Value Function

Here, our aim is to show how the piecewise exponential value function effects the optimal portfolios. To make this analysis, we will examine the exponential value function first. The examination of the exponential value function helps us to understand the differences between the utility model and the prospect model.

In Chapter 4.2.1 we derived the optimal solutions as

$$u^* = \frac{\mu}{\alpha\sigma^2} = 16\mu$$

where the excess return is normally distributed with mean μ and variance $\sigma^2 = 0.25^2$. It is easily seen that optimal portfolios have a linear relation with the expected returns. In other words, the ratio between optimal portfolios and the standard normal distribution is constant. The linearity between optimal portfolios and the expected return μ is illustrated in Figure 5.4.

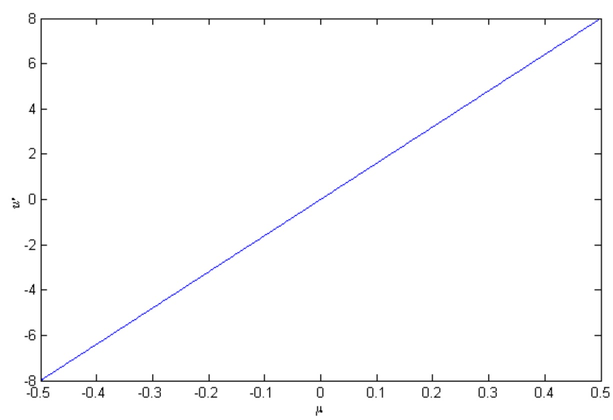

Figure 5.4: Optimal portfolios as a function of mean μ

We will now show some illustrations for the piecewise exponential function using the results in Chapter 4.3.1. As we proved before, the value function is quasi-concave for normally distributed excess return. Quasi-concavity of the objective function can be seen in Figures 5.5, 5.6, and 5.7 for different means.

When we gather the optimal portfolios for the different means in Figure 5.8, we see the interval that the optimal portfolio is not to buy any risky asset.

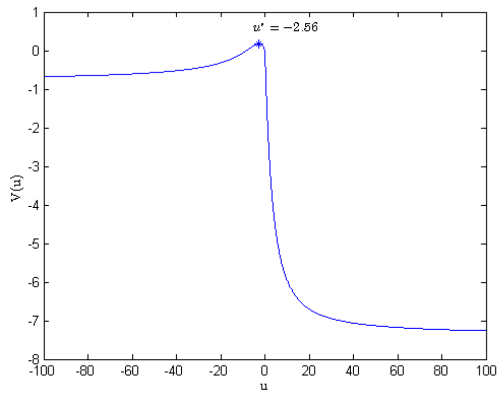

Figure 5.5: Objective function for piecewise exponential function ($\mu = -0.1$)

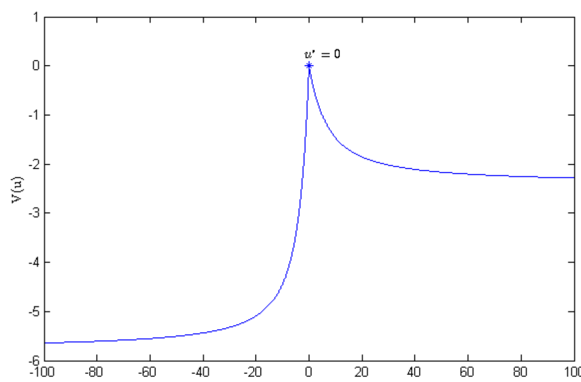

Figure 5.6: Objective function for piecewise exponential function ($\mu = 0.081$)

In Figure 5.8, there are two mean levels μ_1 and μ_2 such that $u^* = 0$ for $\mu_1 \leq \mu \leq \mu_2$. By (4.71) and (4.72), the endpoints of the interval $[\mu_1, \mu_2]$ can be obtained by solving.

$$\Phi\left(\frac{\mu}{\sigma}\right) + \frac{\sigma}{\mu}\varphi\left(\frac{\mu}{\sigma}\right) = \frac{1}{1-\lambda}$$

and

$$\Phi\left(-\frac{\mu}{\sigma}\right) - \frac{\sigma}{\mu}\varphi\left(\frac{\mu}{\sigma}\right) = \frac{1}{1-\lambda}.$$

Actually these conditions are the same as (5.1) and (5.2). So, $\mu_1 = 0.081$ and $\mu_2 = -0.081$.

In the exponential value function, the set of optimal solutions has a linear form given by $u^* = \mu/\alpha\sigma^2$. Using this closed form solution, we can determine the relation between

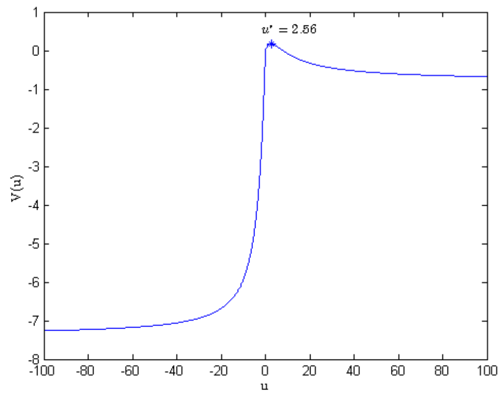

Figure 5.7: Objective function for piecewise exponential function ($\mu = 0.1$)

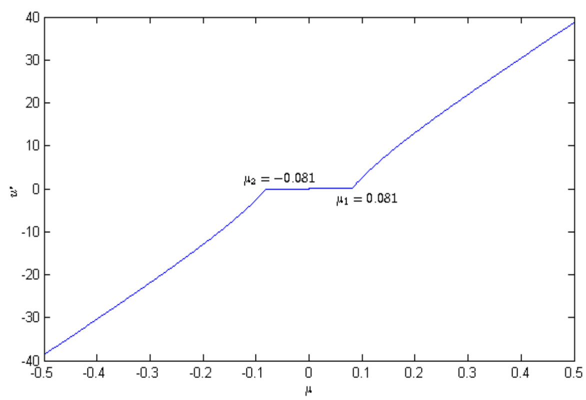

Figure 5.8: Optimal portfolios as a function of mean μ

the exponential and piecewise exponential solutions. If $\mu/\alpha\sigma^2$ is optimal for piecewise exponential value function, then the derivative of objective function must be equal to 0 at

$\mu/\alpha\sigma^2$. From (4.53), we get

$$sign\left(\frac{dV(u)}{du}\right)\Big|_{u=\frac{\mu}{\alpha\sigma^2}} = \begin{cases} -sign\left(\dfrac{\hat{f}\left(\frac{\mu+\sigma^2\left(\frac{\mu}{\sigma^2}\right)}{\sigma}\right)e^{2\mu\left(\frac{\mu}{\sigma^2}\right)}}{\hat{f}\left(\frac{-\mu+\sigma^2\left(\frac{\mu}{\sigma^2}\right)}{\sigma}\right)} - \frac{\lambda^+}{\lambda^-}\right) & \mu \geq 0 \\ sign\left(\dfrac{g\left(\frac{\mu+\sigma^2\left(\frac{\mu}{\sigma^2}\right)}{\sigma}\right)e^{2\mu\left(\frac{\mu}{\sigma^2}\right)}}{g\left(\frac{-\mu+\sigma^2\left(\frac{\mu}{\sigma^2}\right)}{\sigma}\right)} - \frac{\lambda^+}{\lambda^-}\right) & \mu < 0 \end{cases}.$$

$$= \begin{cases} -sign\left(\dfrac{\hat{f}\left(\frac{2\mu}{\sigma}\right)e^{\left(\frac{2\mu^2}{\sigma^2}\right)}}{\hat{f}(0)} - \frac{\lambda^+}{\lambda^-}\right) & \mu \geq 0 \\ sign\left(\dfrac{g\left(\frac{2\mu}{\sigma}\right)e^{\left(\frac{2\mu^2}{\sigma^2}\right)}}{g(0)} - \frac{\lambda^+}{\lambda^-}\right) & \mu < 0 \end{cases}$$

and letting $\hat{\mu} = \mu/\sigma$, we obtain

$$sign\left(\frac{dV(u)}{du}\right)\Big|_{u=\frac{\mu}{\alpha\sigma^2}} = \begin{cases} -sign\left(\dfrac{\hat{f}(2\hat{\mu})e^{2\hat{\mu}^2}}{\hat{f}(0)} - \frac{\lambda^+}{\lambda^-}\right) & \mu \geq 0 \\ sign\left(\dfrac{g(2\hat{\mu})e^{2\hat{\mu}^2}}{g(0)} - \frac{\lambda^+}{\lambda^-}\right) & \mu < 0 \end{cases}$$

$$= \begin{cases} -sign\left(\hat{f}\left(2\hat{\mu}\right)e^{2\hat{\mu}^2} - \frac{\lambda^+\hat{f}(0)}{\lambda^-}\right) & \mu \geq 0 \\ sign\left(g\left(2\hat{\mu}\right)e^{2\hat{\mu}^2} - \frac{\lambda^+ g(0)}{\lambda^-}\right) & \mu < 0 \end{cases}.$$

Here, $\hat{f}\left(0\right)\lambda^+/\lambda^-$ is a constant term and $\zeta_1(\hat{\mu}) = \hat{f}\left(2\hat{\mu}\right)e^{2\hat{\mu}^2}$ is a decreasing function in $\hat{\mu}$ or equally in μ. To show this, it is sufficient to show that

$$\zeta_1'(\hat{\mu}) = 2\hat{f}'\left(2\hat{\mu}\right)e^{2\hat{\mu}^2} + 4\hat{\mu}\hat{f}\left(2\hat{\mu}\right)e^{2\hat{\mu}^2} \leq 0$$

or

$$2e^{2\hat{\mu}^2}\left(\hat{f}'\left(2\hat{\mu}\right) + 2\hat{\mu}\hat{f}\left(2\hat{\mu}\right)\right) \leq 0$$

or

$$\hat{f}'\left(2\hat{\mu}\right) + 2\hat{\mu}\hat{f}\left(2\hat{\mu}\right) \leq 0$$

or

$$\frac{\hat{f}\left(2\hat{\mu}\right)}{-\hat{f}'\left(2\hat{\mu}\right)} \leq \frac{1}{2\hat{\mu}}$$

or

$$\frac{\varphi(2\hat{\mu}) - 2\hat{\mu}\Phi(-2\hat{\mu})}{\Phi(-2\hat{\mu})} \leq \frac{1}{2\hat{\mu}}$$

or

$$r(2\hat{\mu}) \leq \frac{1}{2\hat{\mu}} + 2\hat{\mu}$$

which is always true by (4.59) where $r(x)$ is hazard function of the standard normal distribution. It is also clear that, $\zeta_2(\hat{\mu}) = g\,(2\hat{\mu})\,e^{2\hat{\mu}^2}$ is an increasing function. To sum up there is only one positive and only one negative mean level such that solutions of exponential and piecewise exponential cases are equal. The comparison between the model is graphically demonstrated in Figure 5.9.

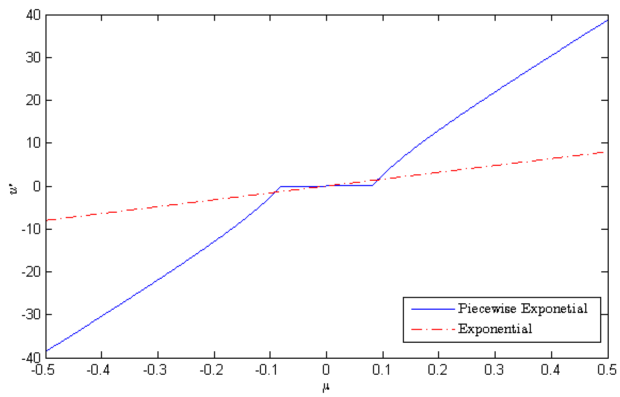

Figure 5.9: Optimal solutions for piecewise exponential and exponential functions

Chapter 6

CONCLUSIONS

Decision making under uncertainty has always been one of the important research areas in industrial engineering. In the research community, expected utility theory is perhaps the most widely used approach. However, non-utility theories have become more popular recently. The prospect theory of Daniel Kahneman and Amos Tversky is the most familiar one of non-utility theories. According to their theory, people are more sensitive to losses. This claim is quite different from the classical arguments. They supported their model with some empirical experiments.

Although utility based portfolio selection models make up the majority of the literature in portfolio optimization, non-utility based models are becoming more and more popular in this research area. We followed this new line of research in this thesis and used the prospect model approach in our analysis. However, we also referred to expected utility model for comparison. We tested preferences of prospect investors with the expected utility investors.

In the thesis, we focus on the single period, single risky asset portfolio selection problem. Although the prospect model is used, the distortation of the probability measure is omitted. Rather, studying the effect of the value functions is in the center of this thesis. Motivated with this idea, we deal with piecewise linear and piecewise exponential value functions. Also, we use exponential value function as a benchmark.

In the Chapter 3, we discussed the main features of prospect theory and formulated the portfolio optimization problem within the prospect theory framework. In Chapter 4, we derived the optimal solutions for piecewise linear value function and tested it with different distributions. We showed that there is a mean interval such that it is optimal not to make any investment on the risky asset for the prospect investor. This is a totally different attitude from the expected utility investors. In the next, we analyzed the case where the value function is exponential. In this case, there is a linear relation between the mean and the optimal portfolio. In other words, the more expected return gets the more optimal portfolio gets. For the piecewise exponential value function, we get similar results with the piecewise linear value function. There is a mean interval such that not to make any investment on the risky asset is optimal for the piecewise exponential value function.

In the numerical part, we provide some illustrative examples. The effects of the structure of the value function are examined using the normally distributed excess return. Also, we compared preferences of piecewise exponential and exponential investors. There is an interval for the mean such that exponential investors are more risk seeking than the prospect investors and out of this interval, the prospect investors are more risk seeking.

This research can be extended in several directions. Here, our analysis constructed on a single risky asset. Our analysis can be extended to multiple risky assets. Long-term behavior of a prospect investor is another issue. So, our setting of single period can be extended by considering multiple periods. Since the probability distortation of outcomes is a part of the prospect theory, another idea might be distorting the probabilities of the returns. Another extension may be the continuous time setting.

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