

Design and Comparison of H_{∞} Loop-Shaping and LQR/LQG control systems for a Quadrotor

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CONTENTS

CONTENTS	2
LIST OF FIGURES	5
LIST OF TABLES	6
ABSTRACT	7
DECLARATION	8
COPYRIGHT	8
ACKNOWLEDGEMENTS	9
1. INTRODUCTION	10
1.1. Background	10
1.2. Aim and objectives	11
1.2.1. Aim.....	11
1.2.2. Objectives.....	11
2. LITERATURE REVIEW	12
2.1. PD/ PID Controllers	12
2.2. Backstepping and sliding model controllers.....	12
2.3. Linear Quadratic Regulator and Linear Quadratic Gaussian Controllers	14
2.4 H-Infinity Controller	14
2.5. Alternative Control Strategies	15
3. QUADROTOR DYNAMICS MODELLING	16
3.1. Quadrotor Model	16
3.2. Motion Equations	17
3.3. Presentation of Motion Equations in State-Space Form	21
3.4. Jacobian Linearization around Equilibrium Points	22
4. H^∞ LOOP-SHAPING CONTROL SYSTEM DESIGN.....	26
4.1. Background Theory	26
4.1.1. H-infinity Loop-shaping Design Procedure	26

4.1.2. Coprime Factor Uncertainty.....	27
4.1.3. Weights selection	28
4.2. Control Systems Design	30
4.2. 1.Attitude Controller Design.....	31
4.2. 2. Height Controller Design	34
4.2. 3. Lateral and Longitudinal Position Controller Design	36
5. LINEAR QUADRATIC REGULATOR / GAUSSIAN CONTROL SYSTEMS DESIGN	40
5.1. Background Theory	40
5.1.1. Linear Quadratic Regulator (LQR)	40
5.1.2. Q and R weight matrices selection.....	41
5.1.3. Linear Quadratic Gaussian (LQG)	42
5.2. Linear Quadratic Regulator (LQR) Control System Design	43
5.2.1. Attitude Controller Design.....	43
5.2.2 .Height Controller Design	45
5.2.3. Lateral and Longitudinal Position Controller Design	46
5.3. Linear Quadratic Gaussian (LQG) Control System Design	47
5.3.1. Attitude Controller Design.....	47
5.3.2 .Height Controller Design	49
5.3.3. Lateral and Longitudinal Position Controller Design	50
6. COMPARISON of DESIGNED CONTROL SYSTEMS	52
6.1 Experiment 1: Effect of Sensor Noise	52
6. 2. Experiment 2: Effect of Aggressive Disturbance	53
6. 3. Experiment 3: Effect of Initial Conditions	54
6. 4. Experiment 4: Effect of parameter uncertainty	55
6. 5. Experiment 5: Effect of Rotor saturations.....	56
6. 6. Experiment 6: Effect of a multiplicative input uncertainty	57

7. CONCLUSION and FUTUREWORK	58
7.1. Conclusion.....	58
7.2. Future work	58
BIBLIOGRAPHY	59



LIST OF FIGURES

Figure 3.1: A quadrotor scheme.....	16
Figure 4.1: Typical loop shaping procedure	26
Figure 4.2: Coprime factor robust stabilization problem	27
Figure 4.3: A typical desired loop-shape	29
Figure 4.4: Hierarchy of the proposed control strategy	30
Figure 4.5: Comparison singular values of actual and shaped open loop plants	32
Figure 4.6: Design specifications of attitude control	32
Figure 4.7: Step responses of the attitude controller & control input	33
Figure 4.8: Comparison singular values of actual and shaped open loop plants for height control	34
Figure 4.9: Design specifications of height control	35
Figure 4.10: Step response of height control& control input.....	36
Figure 4.11: Comparison singular values of actual and shaped open loop plants for position control x and y axes control	37
Figure 4.12: Design specifications of position control on x and y axes	38
Figure 4.13: Step response of position control on x and y axes.....	38
Figure 4.14: Trajectory tracking of the quadrotor.....	39
Figure 5.1: Output feedback structure.....	41
Figure 5.2: Structure for LQG control	43
Figure 5.3: Step response of LQR attitude control & control inputs	44
Figure 5.4: Step response of LQR height control & control input.....	45
Figure 5.5: Step response of LQR position control on x and y axes.....	46
Figure 5.6: Step response of LQG attitude control & control inputs	48
Figure 5.7: Step response of LQG height control & control input	49
Figure 5.8: Step response of LQG position control on x and y axes	50
Figure 5.9: Trajectory tracking of the Quadrotor with LQR control.....	51
Figure 5.10: Trajectory tracking of the Quadrotor with LQG control	51
Figure 6.1: sensor noise	52
Figure 6.2: Step response of positions on x, y, z axes and yaw angel with sensor noise....	52
Figure 6.3: Step response of positions on x, y, z axes and yaw angel with aggressive disturbance	53
Figure 6.4: Step response of positions on x, y, z axes and yaw angel with initial condition	54
Figure 6.5: Step response of position on x, y, z axes and yaw angel with parameter uncertainty	55
Figure 6.6: Step response of position on x, y, z axes and yaw angel with rotors saturation	56
Figure 6.7: Step response of position on x, y, z axes and yaw angel with multiplicative input uncertainty.....	57

LIST OF TABLES

Table 4.1: Quadrotor parameter	30
Table 4.2: The performance analysis of H_{∞} loop-shaping attitude control	33
Table 4.3: The performance analysis of H_{∞} loop-shaping height control	35
Table 4.5: The performance analysis of H_{∞} loop-shaping position controllers on x and y axes.....	39
Table 5.1: Performance analysis of LQR attitude control.....	45
Table 5.2: Performance analysis of LQR height control.....	45
Table 5.3: Performance analysis of LQR position control on x and y axes.....	46
Table 5.4: Performance analysis of LQG attitude control	48
Table 5.5: Performance analysis of LQG height control	49
Table 5.6: Performance analysis of LQG position control on x and y axes.....	50

ABSTRACT

Design and implementation of H_∞ loop-shaping, linear quadratic regulator and linear quadratic gaussian control systems for a Quadrotor are addressed in this thesis. The overall control systems are implemented in a Simulink/MATLAB and compared their robustness. The controllers are designed to operate in bad weather conditions and handle parameter uncertainties and disturbances. Different experiments such as sensor noise, initial conditions, parametric uncertainty, aggressive disturbance, multiplicative input uncertainty are conducted out to measure the robustness of each controller. The weight of the Quadrotor is chosen as parameter uncertainty. During the development of the controllers, the main purpose is to obtain good robustness with performance. Finally controllers are developed and their robustness is compared using nonlinear model in Simulink. Three controllers are robust against disturbance and parameter uncertainty. Nevertheless; H_∞ Loop-shaping has quite small state steady error with parameter uncertainty and multiplicative input uncertainty. The conducted tests showed that H_∞ loop-shaping is more promising to implement on a real Quadrotor. The controller gives the best robustness with good performance and hence is recommended for further studies.

Key words: Quadrotor, H_∞ loop shaping, robustness, LQR, LQG, control systems design.

DECLARATION

I, Basak Hasan, hereby declare that no portion of the work referred to in the dissertation has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Signed: BASAK H.

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1. INTRODUCTION

1.1. Background

Unmanned Aerial Vehicles (UAVs) have been developed to service in many fields such as military, search and rescue. The biggest part of the UAVs has been designed for military applications and this industrial sector has been consisting of an important part of defence commerce. They have been used for reconnaissance, surveillance, and target acquisition requirements.

In the past decade, a vertical take-off and landing Quadrotor aircrafts have been very popular to be used as UAVs in academic platform and commercial sectors because of relatively simple mechanical design and hovering capability, high maneuverability and low maintenance cost. They are ideal solutions to operate in dangerous environments such as reconnaissance for forest fires, surveillance and rescue by using geographic information systems mapping, global positioning system technology and infrared cameras.

A Quadrotor dynamics is highly sensitivity to immeasurable disturbances and noise. These facts enhance difficulty to control the Quadrotor. Even though the purpose of the most dynamic modelling is to simulate the dynamic as possible to reality, the dynamic of the quadrotor cannot be precisely modelled. Consequently; the quadrotor would have an imperfect modelling and uncertain parameters.

Robust and optimal control methodologies are proposed to solve this problem in this thesis. Firstly, H_∞ loop-shaping control strategy which minimises the H_∞ norm to obtain robust stability in the face of immeasurable disturbance. This method uses robust stability margin value as a performance measure. Choosing weights function such that shaped plant has desirable properties. Loop shaping design procedure of McFarlane and Glover gives sharp bounds on the attained sensitivity and complementary sensitivity functions in terms of robust stability margin and the frequency response of the plant [38]. Secondly, Linear Quadratic Regulator (LQR) is known as a very robust controller by guaranteeing infinity gain margin and at least 60 degree phase margin. This control algorithm can deal with time-varying systems by solving an optimization problem. In order to solve the optimization problem which will results in minimum value of cost function, symmetric semi-definite positive state weighting and symmetric positive definite input weighs matrices are selected to minimize cost function for a Quadrotor aircraft. LQR provides

optimal state trajectory with close loop robustness because any deviation from optimal state trajectory that is automatically accounted for. In order to find optimal gain, the Algebraic Riccati equation needs to be solved for positive definite matrix by using suitable weighting matrices Q and R. Another proposed optimal control methodology is Linear Quadratic Gaussian (LQG) control that is a combination Kalman filter and LQR feedback gain. Kalman gain is given in term of the Riccati equation solution and the system and weighting matrices. Kalman filter estimates all the state variables. This property is good for roll, pitch and yaw angle rates that cannot be measured. Kalman filter attains suitable estimator behaviour results by using trail error method to obtain Q and R weights. Furthermore, Kalman filter has some robustness to modelling inaccuracies, disturbance or noise. [36].

1.2. Aim and objectives

1.2.1. Aim

The aim of this project is to solve the stabilization problem of a Quadrotor in presence of parametric uncertainty, disturbance and robust trajectory tracking and stable hovering by using optimal and robust controllers and also to select a suitable control methodology which deals with these problems. It is desired that each controller will be designed for multivariable control of the Quadrotor system with guaranteed stability.

1.2.2. Objectives

The scope of the thesis presented here includes the following;

- Understanding the presence of a Quadrotor mathematical model in literature for simulation and designing control systems.
- Designing H_∞ loop-shaping controller for attitude, height, lateral and longitudinal position of a quadrotor.
- Designing LQR and LQG controllers for attitude, height lateral and longitudinal position of a quadrotor.
- Comparing their robust performance and suggesting a control system for real applications.

2. LITERATURE REVIEW

In recent years, there has been an increasing amount of literature on Quadrotor control methodology. While some researchers or designers prefer to use linear controllers, some of them prefer to use nonlinear controllers. One has advantages over others. This chapter gives information about existing controllers which have applied to quadrotor in the literature.

2.1. PD/ PID Controllers

The authors in [13] design PD (proportional-derivative) controller for position control on x and y axes and also proposes a vision based controller by using a camera that can see the pattern on the ground and hover over on that pattern at a desired altitude. Furthermore, [14] suggests a PID (proportional-integral-derivative) control method which can stabilise successfully a Quadrotor. In the reference [15] PID controller is sufficient for good tracking of commanded attitude and PD controllers provides good tracking in indoor flight. In [16] authors carry out tests of PD attitude control algorithm for a quadrotor mini UAV by using hardware-in-the-loop setup.[17] designs two linear proportional derivative (PD) passive controllers by combining with a nonlinear saturation element. PD control strategy provides very good manoeuvrability for a Quadrotor. [18] Compares the advantages and disadvantages of a PD controller with sensor drift and noise, acceptable payload and inertial values. The absence of the integral term in the PD controller makes it more sensitive to noise, dynamics uncertainty and disturbance. As a result, the Quadrotor has an inability of reaching the desired target. In reference [19] the study proposes PID controller method which makes the transient response of the system fast for the Quadrotor. [20] presents a PID controller for posture Control of Micro Quadrotor and tests controller with fan disturbance. Experimental results show that the accelerations and angular velocities balance fast.

2.2. Backstepping and sliding model controllers

In reference [2] , the authors develop a new control strategy based on robust integral Backstepping using sliding mode by taking into account the actuator and sensor faults which result in undesired behaviour of quadrotor, or instability .An integral compensation

term and "sign" function involve this control strategy. Simulation results show that this control strategy has high efficiency in stability and performance of a Quadrotor during the actuator and sensor faults.

In reference [3,4] the authors compare stability and performance of backstepping and sliding mode controllers and experiments on a Quadrotor prototype. Simulation results of both controllers are promising to implement on the real system. The backstepping control methodology can stabilise the orientation angles in the presence of relatively high disturbance. In reference [5] authors use Backstepping approach for trajectory tracking errors by using Lyapunov function and also develop a sliding mode controller to guarantee Lyapunov stability and deal with system nonlinearities and desired tracking trajectories. A Quadrotor dynamics is developed by taking into account the high order nonholonomic constraints imposed to the system. In [6], a Backstepping controller is applied based on the Lagrangian form of the dynamics. This control approach deals with unmodelled disturbances and forces. In [7] sliding mode controller is presented in order to perform best tracking results and the nonlinear observer is introduced to estimate unmeasured states and the effects of additive uncertainties. Authors in [8] present a new approach for the attitude control of a vertical take-off and landing (VTOL) quadrotor aircraft based on the Euler angles parameterization and the combination of the Backstepping technique and a nonlinear robust PI controller.

The proposed control strategy is robust against parameter uncertainties. [9] proposes the sliding mode controller which is insensitivity to the quadrotor model errors, parametric uncertainties and controller is robust to stabilize a quadrotor. [10] presents sliding mode control and PD control strategies which are used to control altitude and yaw angle. In [11] the authors design a robust flight controller based on Sliding Mode Disturbance Observer approach for a small quadrotor. This control strategy is robust to external disturbance and model uncertainties. [12] Presents Backstepping and sliding mode techniques for miniature quadrotor helicopter based on the Lyapunov stability. The whole system was divided into three subsystems: an under-actuated subsystem, a fully-actuated subsystem and a rotors subsystem.

In reference [1], Classically PD controller, Inverse dynamic control, feedback linearization control and sliding mode control methods are derived and applied for altitude stabilization and hovering control of quadrotor. Derived linear and nonlinear control techniques are performed by using computer simulations and compared the results. Sliding mode

technique has better performance and inverse dynamic control has the worst performance on high initial conditions.

2.3. Linear Quadratic Regulator and Linear Quadratic Gaussian Controllers

In reference [21] presents Linear Quadratic tracking controller with integral action, and Optimal Linear Quadratic Gaussian (LQG) control with integral action in order to stabilize the attitude of the quadrotor MAV. Proposed controllers have the ability to perform output tracking control objective in hover.

In [22] authors propose LQR controller for all possible yaw angles which is quite effective and a sliding mode controller (SMC) with recursive nature which stabilizes the vehicle's attitude and altitude. The proposed controllers have satisfactory performance in tracking the reference inputs. The sliding mode control of altitude has almost zero overshoot in comparison to LQR. Although the sliding mode control of altitude response is almost as fast as LQR controller, it has almost zero overshoot compared to LQR. In reference [23] Integral-LQR Controller has the slower settling time and the smoother response compared to the PID controller.

2.4 H-Infinity Controller

In reference [24] presents a nonlinear H_∞ control law for quadrotor with input coupling. Cascade control strategies are used to avoid input coupling and this controller is effective with respect to the presence of aerodynamic disturbances, structural and parametric uncertainties. [25] proposes h-infinity and loop-shaping methods to control an attitude of an unmanned helicopter. The simulation results show this control strategy has a good robust performance. [26] The presents hierarchical control structure which is consisting of a model predictive controller (MPC) and nonlinear H_∞ controller to track the reference trajectory and stabilize the rotational movements. The integral of the position error achieve a null steady-state error when disturbances are introduced into the system consistently.

2.5. Alternative Control Strategies

In reference [27] authors propose feedback linearization approach to design a controller and double control loop architecture which allow quadrotor to perform the trajectory tracking and roll/pitch control during a rotor failure. [28] develops a new adaptive neural network control to deal with large uncertainties, modelling error and significant wind disturbance. This method is compared with deadzone and e -modification adaptive techniques. Even though this adaptive neural network controller provides less error in terms of obtaining a desired attitude, the response of attitude has large oscillation. [29] proposes a combined or composite model reference adaptive controller (MRAC) approach which increases robustness to parametric uncertainties. When a sudden change in mass, or alternatively a battery failure or electrical fault occurs, CMRAC get used to the failure and tracks the desired altitude.[30] presents mixed robust feedback linearization with linear GH_∞ controller against uncertainties on mass and inertia. This control strategy has good robustness properties because of not only the robustness of the H_∞ controller but also robust feedback linearization. Simulation result shows that this control strategy overcomes easily disturbances.

3. QUADROTOR DYNAMICS MODELLING

3.1. Quadrotor Model

In this chapter, the mathematical modelling of quadrotor presented in [27] is reviewed by considering four rotors without any faults. Opposing rotors M_1 and M_3 rotate in the clockwise direction and the other pair rotors M_2 and M_4 rotate in the counter-clockwise direction.

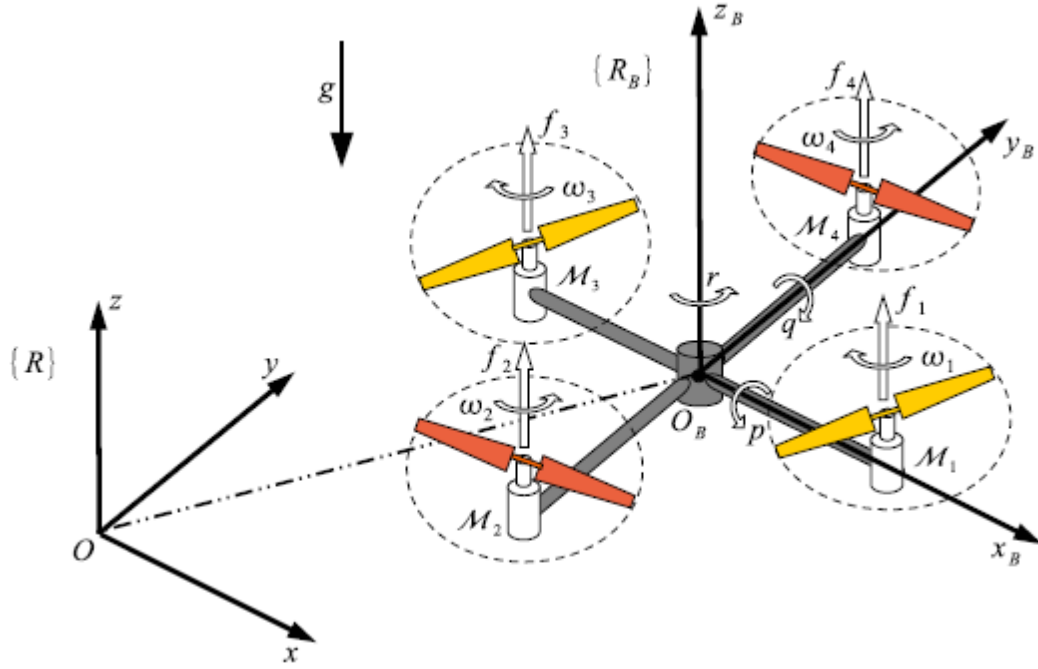


Figure 3.1: A quadrotor scheme [27].

When angular velocities of M_1 and M_3 remain the same increasing or decreasing the angular velocity of the rotors M_2 and M_4 creates roll rotation. Pitch rotation is achieved by increasing or decreasing angular velocities of M_1 and M_3 while angular velocity of the rotors M_2 and M_4 maintain the same. If the forces produced by the rotors mismatch This creates yaw rotation. That is, increasing or decreasing the angular velocities of rotors M_1 and M_3 while decreasing or increasing angular velocities of rotors M_2 and M_4 . This leads to rotation around the z-body axis. In order to model a quadrotor, following assumptions are taken into account [27]:

- Quadrotor is designed symmetrically.
- Quadrotor body and propellers are rigid.
- Free stream air velocity is zero.

- The motors dynamics can be neglected as it is very fast.
- The blade is not very flexible thus can be neglected.
- Drag is considered to be linear, hence obeying Stoke's law.

3.2. Motion Equations

Earth frame $\{R\}$ (O, x, y, z) and body-fixed frame $\{R_B\}$ (O_B, x_B, y_B, z_B), are used to model the system. Where O_B is the centre of the quadrotor [27].

A position vector of body-fixed frame related to inertial frame is

$$\xi = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (3.1)$$

Orientation of the body fixed frame $\{R_B\}$ (O_B, x_B, y_B, z_B), with respect to the earth frame $\{R\}$ (O, x, y, z) is expressed by a vector of three independent angles

$$\eta = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \quad (3.2)$$

Euler's rotation theorem is used to obtain the translational and the rotational movement of the quadrotor with respect to the earth frame given $\xi = [x \ y \ z]^T$ and $\eta = [\phi \ \theta \ \psi]^T$ vectors and three rotation matrices $R_x(\psi)$, $R_y(\theta)$ and $R_z(\phi)$.

Transforming body-fixed frame coordinates to earth frame coordinates can be obtained as follows [31].

$$\begin{aligned} R_{B \rightarrow E} &= R_z(\psi) \cdot R_y(\theta) \cdot R_x(\phi) \\ &= \begin{bmatrix} C_\psi & -S_\psi & 0 \\ S_\psi & C_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_\theta & 0 & S_\theta \\ 0 & 1 & 0 \\ -S_\theta & 0 & C_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\phi & -S_\phi \\ 0 & S_\phi & C_\phi \end{bmatrix} \end{aligned} \quad (3.3)$$

where $S(\cdot)$ and $C(\cdot)$ represent $\sin(\cdot)$ and $\cos(\cdot)$ respectively

$$R_{B \rightarrow E} \doteq \begin{bmatrix} C_\theta C_\psi & C_\psi S_\theta S_\phi - C_\phi S_\psi & C_\phi C_\psi S_\theta + S_\phi S_\psi \\ C_\theta S_\psi & S_\theta S_\phi S_\psi + C_\phi C_\psi & C_\phi S_\theta S_\psi - C_\psi S_\phi \\ -S_\theta & C_\theta S_\phi & C_\theta C_\phi \end{bmatrix} \quad (3.4)$$

in which $R_{B \rightarrow E}$ is the rotation transformation from vectors read in the body reference frame to vectors read in the earth reference frame.

In a similar way, given the angular velocity vector $[p \ q \ r]^T$, where p , q and r represent the direct angular velocities around the x_B -axis, y_B -axis and z_B -axis respectively, it is related to the rate of change of the yaw, pitch and roll angles.

In this sense, given an attitude vector $\eta = [\phi \ \theta \ \psi]^T$ and angular velocities around the x_B -axis, y_B -axis and z_B -axis $[p \ q \ r]^T$ where p , q and r is related to the rate of change of the

yaw, pitch and roll angles. By using three rotation matrices R_x , R_y , R_z defining rotations by ϕ , θ , ψ around x , y , z respectively, and W_η denoting an unknown Jacobian matrix, a rotation matrix for transforming body fixed angular velocities to earth fixed angular velocities can be expressed in the following form [31]:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + R_x(\phi)^T \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + R_x(\phi)^T \cdot R_y(\theta)^T \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \quad (3.5)$$

The Jacobian matrix can be obtained by solving following the equation.

$$\begin{aligned} &= \begin{bmatrix} \dot{\phi} - \dot{\psi} S\theta \\ \dot{\theta} C\phi + \dot{\psi} S\phi C\theta \\ -\dot{\theta} S\phi + \dot{\psi} C\phi C\theta \end{bmatrix} \equiv W_\eta \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \\ W_\eta &= \begin{bmatrix} 1 & 0 & -S\theta \\ 0 & C\phi & S\phi C\theta \\ 0 & -S\phi & C\phi C\theta \end{bmatrix} \end{aligned} \quad (3.6)$$

The desired rotation matrix is the inverse Jacobian matrix and can be given by:

$$W_\eta^{-1} = \begin{bmatrix} 1 & S\phi T_\theta & C\phi T_\theta \\ 0 & C\phi & -S\phi \\ 0 & \frac{S\phi}{C_\theta} & \frac{C_\phi}{C_\theta} \end{bmatrix} \quad (3.7)$$

where $S(\cdot)$, $C(\cdot)$ and $T(\cdot)$ present $\sin(\cdot)$, $\cos(\cdot)$ and $\tan(\cdot)$ respectively and angular velocities can be calculated as followings [27].

$$\begin{aligned} &\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = W_\eta^{-1} \cdot \begin{bmatrix} p \\ q \\ r \end{bmatrix} \\ &\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & S\phi T_\theta & C\phi T_\theta \\ 0 & C\phi & -S\phi \\ 0 & \frac{S\phi}{C_\theta} & \frac{C_\phi}{C_\theta} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \end{aligned} \quad (3.8)$$

Each Angular velocity can be expressed in following form:

$$\begin{aligned} \dot{\phi} &= p + q S\phi T_\theta + r C\phi T_\theta \\ \dot{\theta} &= q C\phi - r S\phi \\ \dot{\psi} &= q \frac{S\phi}{C_\theta} + r \frac{C_\phi}{T_\theta} \end{aligned} \quad (3.9)$$

The dynamics of the quadrotor can be described by taking into consideration the weight force, the thrust forces and the drag terms acting on it. The weight force is applied to the centre of gravity and directed along the negative z -axis in the earth frame.

Each motor produces thrust force which is directed along the positive z_B -axis (f_1, f_2, f_3 and f_4) and they are applied from distant l which is from the centre of each motor to the centre of the quadrotor. The drag terms Stoke's law is used for the drag terms. The translational drag is proportional to the linear velocity and the rotational drag term is proportional to the angular velocity.

An Euler-Lagrange approach is adopted in order to write the equations which describe the translational motion of the quadrotor. Translational and rotational components consist of the kinetic energy of the Quadrotor and these components can be expressed as followings [27].

The translational component is

$$T_{trans} = \frac{1}{2} m \dot{\xi}^T \dot{\xi} \quad (3.10)$$

where m is the total mass of the quadrotor.

The rotational component is

$$T_{rot} \triangleq \frac{1}{2} \dot{\eta}^T \mathbb{J} \dot{\eta} \quad (3.11)$$

Where $\mathbb{J}$ donates the inertia matrix in terms of the generalized coordinates η . The inertia matrix in the body frame can be defined as

$$\mathbb{J} \triangleq \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \quad (3.12)$$

The inertia matrix is diagonal because of the symmetrical design of the quadrotor; the inertia matrix in terms of the generalized coordinates can be expressed as followings;

$$\mathbb{J} = W_{\eta}^T \mathbb{J} W_{\eta} \quad (3.13)$$

The potential energy has to be taken into consideration due to the gravitational field.

Hence, this energy is defined as

$$U = -m g z \quad (3.14)$$

$[x, y, z, \phi, \theta, \psi]^T \in R^6$ be the generalized coordinates vector for the quadrotor ,

Lagrangian is expressed by

$$\mathcal{L}(q, \dot{q}) = T_{trans} + T_{rot} + U = \frac{1}{2} m \dot{\xi}^T \dot{\xi} + \frac{1}{2} \dot{\eta}^T \mathbb{J} \dot{\eta} - m g z \quad (3.15)$$

The Lagrangian contains no cross-terms in the kinetic energy combination of $\dot{\xi}$ and $\dot{\eta}$. Therefore, translational motion can be expressed as following by using the Euler-Lagrange equation [27]:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\xi}} - \frac{\partial L}{\partial \xi} = F_{\xi} \quad (3.16)$$

In which F_{ξ} donates the translational generalized force acting on the Quadrotor and relative to the earth frame $\{R\}$. The upward lifting forces generated by the propellers are f_1, f_2, f_3 and f_4 , the force in the z_B direction as the control inputs can be written as

$$u_f = f_1 + f_2 + f_3 + f_4 \quad (3.17)$$

Substituting (4.15) in (4.16) and expanding F_{ξ} results in

$$m\ddot{\xi} + \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = R_{B \rightarrow E} \begin{bmatrix} 0 \\ 0 \\ u_f \end{bmatrix} - k_t \dot{\xi} \quad (3.18)$$

Substituting required the matrices; eq. (3.18) can be expressed as

$$m \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = \begin{bmatrix} C_{\theta}C_{\psi} & C_{\psi}S_{\theta}S_{\phi} - C_{\phi}S_{\psi} & C_{\phi}C_{\psi}S_{\theta} + S_{\phi}S_{\psi} \\ C_{\theta}S_{\psi} & S_{\theta}S_{\phi}S_{\psi} + C_{\phi}C_{\psi} & C_{\phi}S_{\theta}S_{\psi} - C_{\psi}S_{\phi} \\ -S_{\theta} & C_{\theta}S_{\phi} & C_{\theta}C_{\phi} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ u_f \end{bmatrix} - k_t \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \quad (3.19)$$

Simplifying the eq. (3.19)

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \frac{u_f}{m} \begin{bmatrix} C_{\phi}C_{\psi}S_{\theta} + S_{\phi}S_{\psi} \\ C_{\phi}S_{\theta}S_{\psi} - C_{\psi}S_{\phi} \\ C_{\theta}C_{\phi} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} - \frac{k_t}{m} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \quad (3.20)$$

Accelerations for Cartesian coordinate can be expressed as followings;

$$\begin{aligned} \ddot{x} &= \frac{u_f(C_{\phi}C_{\psi}S_{\theta} + S_{\phi}S_{\psi})}{m} - \frac{k_t}{m} \dot{x} \\ \ddot{y} &= \frac{u_f(C_{\phi}S_{\theta}S_{\psi} - C_{\psi}S_{\phi})}{m} - \frac{k_t}{m} \dot{y} \\ \ddot{z} &= \frac{u_f(C_{\theta}C_{\phi})}{m} - \frac{k_t}{m} \dot{z} - g \end{aligned} \quad (3.21)$$

In which k_t donates translational drag coefficient, which is assumed to be equal in all directions for the sake of simplicity. The rotational motion of the quadrotor referring to the body frame coordinates can be expressed by

$$\tau_B = \begin{bmatrix} \tau_p \\ \tau_q \\ \tau_r \end{bmatrix} = \mathbb{I}\dot{\omega} + \omega \times (\mathbb{I}\omega) + k_r \omega \quad (3.22)$$

Where k_r is the rotational drag coefficient, and supposed to be constant in all directions for the sake of simplicity and “ $\times$ ” presents the vector cross product. Substituting required the matrices equation () can be expressed as

$$\begin{bmatrix} \tau_p \\ \tau_q \\ \tau_r \end{bmatrix} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} + k_r \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad (3.23)$$

The torques can be presented as

$$\begin{aligned} \tau_p &= I_{xx} \dot{p} + qr(I_{zz} - I_{yy}) + k_r p \\ \tau_q &= I_{yy} \dot{q} + pr(I_{xx} - I_{zz}) + k_r q \\ \tau_r &= I_{zz} \dot{r} + pq(I_{yy} - I_{xx}) + k_r r \end{aligned} \quad (3.24)$$

Angular accelerations can be calculated as;

$$\begin{aligned} \dot{p} &= \frac{1}{I_{xx}} [\tau_p - qr(I_{zz} - I_{yy}) - k_r p] \\ \dot{q} &= \frac{1}{I_{yy}} [\tau_q - pr(I_{xx} - I_{zz}) - k_r q] \\ \dot{r} &= \frac{1}{I_{zz}} [\tau_r - pq(I_{yy} - I_{xx}) - k_r r] \end{aligned} \quad (3.25)$$

Where

$$\begin{bmatrix} \tau_p \\ \tau_q \\ \tau_r \end{bmatrix} = \begin{bmatrix} l(f_4 - f_1) \\ l(f_3 - f_1) \\ d(f_1 - f_2 + f_3 - f_4) \end{bmatrix} \quad (3.26)$$

In which l is the arm length and d is the ratio between the drag and the thrust coefficients of the blade.

3.3. Presentation of Motion Equations in State-Space Form

Choosing the state vector as

$$\mathbf{x} = [\phi \ \theta \ \psi \ p \ q \ r \ z \ \dot{z} \ x \ \dot{x} \ y \ \dot{y}]^T$$

$$= [x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7 \ x_8 \ x_9 \ x_{10} \ x_{11} \ x_{12}]^T \quad (3.27)$$

and the input vector as

$$\mathbf{u} = [\tau_p \ \tau_q \ \tau_r \ u_f]^T \quad (3.28)$$

Then the dynamics equations (3.9), (3.21) and (3.25) can be written in state-space form;

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x, t)\end{aligned}\tag{3.29}$$

with $y(t)$ is measured output vector.

$$\dot{x}_1 = x_4 + x_5 \sin(x_1) \tan(x_2) + x_6 \cos(x_1) \tan(x_2) \tag{3.30}$$

$$\dot{x}_2 = x_5 \cos(x_1) - x_6 \sin(x_1) \tag{3.31}$$

$$\dot{x}_3 = x_5 \frac{\sin(x_1)}{\cos(x_2)} + x_6 \frac{\cos(x_1)}{\cos(x_2)} \tag{3.32}$$

$$\dot{x}_4 = \frac{1}{I_{xx}} [\tau_p - x_5 x_6 (I_{zz} - I_{yy}) - k_r x_4] \tag{3.33}$$

$$\dot{x}_5 = \frac{1}{I_{yy}} [\tau_q - x_4 x_6 (I_{xx} - I_{zz}) - k_r x_5] \tag{3.34}$$

$$\dot{x}_6 = \frac{1}{I_{zz}} [\tau_r - x_4 x_5 (I_{yy} - I_{xx}) - k_r x_6] \tag{3.35}$$

$$\dot{x}_7 = x_8 \tag{3.36}$$

$$\dot{x}_8 = \frac{u_f(\cos(x_1)\cos(x_2))}{m} - \frac{k_t}{m} x_8 - g \tag{3.37}$$

$$\dot{x}_9 = x_{10} \tag{3.38}$$

$$\dot{x}_{10} = \frac{u_f(\cos(x_1)(\cos(x_3)\sin(x_2) + \sin(x_1)\sin(x_3)))}{m} - \frac{k_t}{m} x_{10} \tag{3.39}$$

$$\dot{x}_{11} = x_{12} \tag{3.40}$$

$$\dot{x}_{12} = \frac{u_f(\cos(x_1)\sin(x_2)\sin(x_3) - \cos(x_3)\sin(x_1))}{m} - \frac{k_t}{m} x_{12} \tag{3.41}$$

3.4. Jacobian Linearization around Equilibrium Points

The quadrotor dynamics is highly nonlinear, in order to design linear time invariant (LTI) controller, jacobian linearization method can be used to obtain linear time invariant state space form for the system.

Supposing an equilibrium point and input $(\bar{x}, \bar{u})$. If the system is around an equilibrium state, then state and input can be expressed as

$$\begin{aligned}\Delta_x &= x - \bar{x} \\ \Delta_u &= u - \bar{u} \\ \Delta_y &= y - \bar{y}\end{aligned}\tag{3.42}$$

in which Δ_u , Δ_x and Δ_y are supposed to be sufficiently small

$$\dot{\Delta}_x = f(\Delta_x + \bar{x}, (\Delta_u + \bar{u})) \tag{3.43}$$

Then, expanding Taylor expansion of the right-hand side of eq. (3.43) and neglecting all higher order terms

$$\dot{\Delta}_x \approx f(\bar{x}, \bar{u}) + \frac{\partial f}{\partial x} \bigg|_{\substack{x = \bar{x} \\ u = \bar{u}}} \Delta_x(t) + \frac{\partial f}{\partial u} \bigg|_{\substack{x = \bar{x} \\ u = \bar{u}}} \Delta_u(t) \quad (3.44)$$

At equilibrium point leads to $f(\bar{x}, \bar{u}) = 0$

$$\dot{\Delta}_x \approx \frac{\partial f}{\partial x} \bigg|_{\substack{x = \bar{x} \\ u = \bar{u}}} \Delta_x(t) + \frac{\partial f}{\partial u} \bigg|_{\substack{x = \bar{x} \\ u = \bar{u}}} \Delta_u(t) \quad (3.45)$$

This equation can be expressed as following;

$$\dot{\Delta}_x = A \Delta_x + B \Delta_u \quad (3.46)$$

where $A \in R^{n \times n}$, $B \in R^{n \times m}$ and $C \in R^{n \times n}$ are the *Jacobian* matrices, given by

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \quad B = \frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \dots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \dots & \frac{\partial f_n}{\partial u_m} \end{bmatrix} \quad (3.47)$$

Similarly, output deviation variable is

$$\Delta_y = C \Delta_x \quad (3.48)$$

In which C is the Jacobian matrix of h given by

$$C = \frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial x_1} & \dots & \frac{\partial h_n}{\partial x_n} \end{bmatrix} \quad (3.49)$$

Jacobian matrices are calculated at equilibrium points $(\bar{x}, \bar{u})$

In order to design an attitude controller, equations from (4.1) to 4.6 needs to be linearized

The input vector can be chosen $u = [\tau_p \ \tau_q \ \tau_r]^T$, since attitude equations are not related to u_f . Then, choosing input vector at equilibrium points as

$$u = [\tau_p \ \tau_q \ \tau_r]^T = [0 \ 0 \ 0]^T \quad (3.50)$$

and equilibrium states can be chosen as followings.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.51)$$

That yields following Jacobian matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{-k_r}{I_{xx}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-k_r}{I_{yy}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-k_r}{I_{zz}} \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{I_{xx}} & 0 & 0 \\ 0 & \frac{1}{I_{yy}} & 0 \\ 0 & 0 & \frac{1}{I_{zz}} \end{bmatrix} \quad (3.52)$$

Measureable output matrix is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (3.53)$$

In order to design altitude and position control equations from (3.35) to (3.40) needs to be linearized. Since the quadrotor is an under-actuated system, additional to inputs, two virtual inputs are given by

$$u_x = \cos(x_1)(\cos(x_3)\sin(x_2) + \sin(x_1)\sin(x_3)) \quad (3.54)$$

$$u_y = \cos(x_1)\sin(x_2)\sin(x_3) - \cos(x_3)\sin(x_1) \quad (3.55)$$

Substituting the eq. (3.53) and (3.54) into (3.38) and (3.40) respectively ,then eq. (3.38) and (3.40) can be written as follow

$$\dot{x}_{10} = \frac{u_f u_x}{m} - \frac{k_t}{m} x_{10} \quad (3.56)$$

$$\dot{x}_{12} = \frac{u_f u_y}{m} - \frac{k_t}{m} x_{12} \quad (3.57)$$

The movement through the x and y axes can be obtained by orientations pitch and roll angels so two new inputs can be considered as the directions of u_f that results in the movement through the x and y axes, respectively.

$\hat{u}_f$, $\hat{u}_x$ and $\hat{u}_y$ input can be given by

$$\hat{u}_f = \frac{u_f \cos(x_1) \cos(x_2)}{m} - g \quad (3.58)$$

$$\hat{u}_x = \frac{u_f u_x}{m} \quad (3.59)$$

$$\hat{u}_y = \frac{u_f u_y}{m} \quad (3.60)$$

Eq.(3.36) , (3.55) and (3.56) can be expressed as follows

$$\dot{x}_8 = \hat{u}_f - \frac{k_t}{m} x_8 \quad (3.61)$$

$$\dot{x}_{10} = \hat{u}_x - \frac{k_t}{m} x_{10} \quad (3.62)$$

$$\dot{x}_{12} = \hat{u}_y - \frac{k_t}{m} x_{12} \quad (3.63)$$

Linearized height model can be obtained by using eq. (3.60) and (3.35) that lead to following state space form;

$$\dot{x} = Ah \begin{bmatrix} x_7 \\ x_8 \end{bmatrix} + Bh \hat{u}_f \quad (3.64)$$

$$y = Ch \begin{bmatrix} x_7 \\ x_8 \end{bmatrix} \quad (3.65)$$

where Ah ,Bh and Ch are jacobian matrices given by

$$Ah = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{k_t}{m} \end{bmatrix}, Bh = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } Ch = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (3.66)$$

Control input can be calculated by

$$u_f = \frac{(\hat{u}_f + g)m}{\cos(x_1) \cos(x_2)} \quad (3.67)$$

Linearized position model can be obtained by using eq.(3.61),(3.62), (3.37) and 3.39 .state space form of position controller is

$$\dot{x} = Ap \begin{bmatrix} x_9 \\ x_{10} \\ x_{11} \\ x_{11} \end{bmatrix} + Bp \begin{bmatrix} \hat{u}_x \\ \hat{u}_y \end{bmatrix} \quad (3.68)$$

$$y = Cp \begin{bmatrix} x_9 \\ x_{10} \\ x_{11} \\ x_{11} \end{bmatrix} \quad (3.69)$$

where Ap , Bp and Cp jacobian matrices are given by

$$Ap = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{k_t}{m} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{k_t}{m} \end{bmatrix}, Bp = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad Cp = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.70)$$

4. H_∞ LOOP-SHAPING CONTROL SYSTEM DESIGN

4.1. Background Theory

4.1.1. H-infinity Loop-shaping Design Procedure

H-infinity robust control is a powerful technique for system under conditions of uncertainty modelling and parameter uncertainty, disturbance and noise. In order to obtain robust performance, H_2/H_∞ optimal control, the mixed sensitivity function, H_∞ loop shaping mu-synthesis technique are presented in literature.

H_∞ loop-shaping control method was originally proposed by McFarlane and Glover [32] which is combine loop shaping and stabilization to design robust controllers. In This method, in order to shape the singular values of the nominal plant, a pre-compensator W_1 and a post-compensator W_2 need to be selected in the frequency domain. The nominal plant P and weights W_1, W_2 form the shaped plant, P_s where $P_s = W_2 P W_1$.

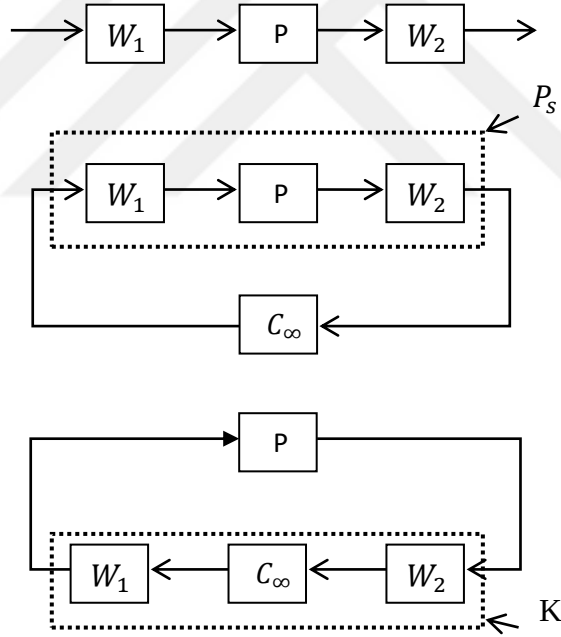


Figure 4.1: Typical loop shaping procedure [32].

Next, the robust stability margin $b(P_s, C_\infty)$ and the optimal robust stability margin $b_s(P_s)$

are defined in [34]. The robust stability margin is

$$b(P_s, C_\infty) = \begin{cases} \left\| \begin{bmatrix} P_s \\ I \end{bmatrix} (I - C_\infty P_s)^{-1} [-C_\infty \ I] \right\|_\infty^{-1} & \text{if } b(P_s, C_\infty) \text{ is internally stable} \\ 0 & \text{otherwise} \end{cases} \quad (4.1)$$

The largest value of the robust stability margin is

$$b_s(P_s) = \sup_{C_\infty} b(P_s, C_\infty) \quad (4.2)$$

The value of the robust stability margin indicates the achievement of the loop-shaping method. If $b_s(P_s)$ is small, then the shaped-loop is incompatible with robust stability requirements, and should be adjusted and C_∞ is calculated again. Furthermore, in [34] the robust stability margin is defined as some number always between zero and one . if $b(P_s, C_\infty)$ is greater than 0.3 , that guarantees at least a phase margin of 60° and a gain margin of 6 dB.

4.1.2. Coprime Factor Uncertainty

In order to show that normalized coprime factor robust stability problem guarantees a robust stability for the closed-loop system, normalized left coprime factorization (LCF) robust stabilization H_∞ problem is defined [32].

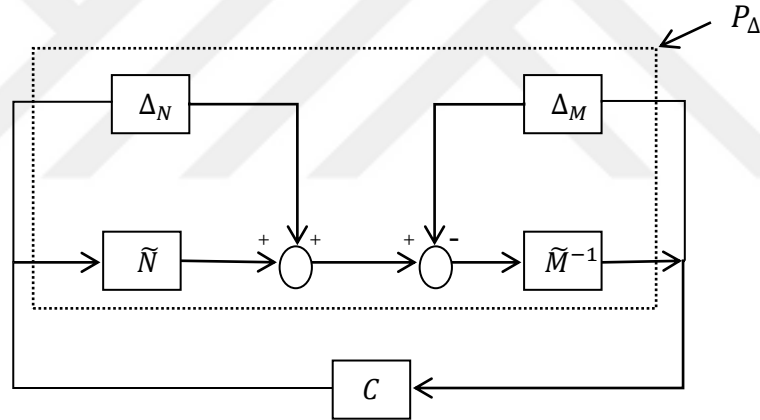


Figure 4.2: Coprime factor robust stabilization problem [32].

In order to represent unstructured uncertainty in a plant by using coprime factor perturbations, the nominal plant is

$$P = \tilde{M}^{-1} \tilde{N} \quad (4.3)$$

Perturbed plant can be expressed as

$$P_\Delta = (\tilde{M} + \Delta_M)^{-1}(\tilde{N} + \Delta_N) \quad (4.4)$$

In which $\tilde{M}, \tilde{N}$ is a left coprime factorization of the plant and Δ_M, Δ_N are unknown and stable uncertainties which satisfy that

$$\|[\Delta_M, \Delta_N]\|_\infty < \epsilon \quad (4.5)$$

Where $0 < \epsilon < 1$

The purpose of design is to find a feedback controller K so that it stabilizes P_Δ for a given ϵ . This can be expressed for an H_∞ optimization problem. A stabilizing controller C needs to be found such that

$$\left\| \begin{bmatrix} (I - PC)^{-1} \tilde{M}^{-1} \\ C(I - PC) \tilde{M}^{-1} \end{bmatrix} \right\|_\infty < b(P_s, C_\infty) \quad (4.6)$$

Where C is a controller which is chosen so that it stabilizes the nominal plant and satisfies the requirement on the, H_∞ norm of the combination of closed-loop transfer functions in eq.(4.6). Finding $b_s(P_s)$ is generally iterative .However, if the left coprime factorization (LCF) of P is normalized .That is

$$\tilde{M}(j\omega)\tilde{M}(j\omega)^* + \tilde{N}(j\omega)\tilde{N}(j\omega)^* = I \text{ for all } \omega \quad (4.7)$$

Then,the maximum value of stability margin can be obtained without iterative method, and is given as

$$b_s(P_s) = \sqrt{(1 - \|\tilde{M}, \tilde{N}\|_H^2)} \quad (4.8)$$

where $\|\cdot\|_H$ shows the Hankel norm. A stabilizing controller achieving $b_s(P_s)$ is called an optimal controller for this problem. Further details can be found in [32].

4.1.3. Weights selection

Weights Selection is essential for a good performance in both frequency domain and time domain. Weights W_1 and W_2 are selected so that the closed loop shaped plant achieves the performance tracking, good disturbance rejection, robust stability and sensor noise rejection.

Recalling some common terminology

$$L(j\omega) = P(j\omega)C(j\omega) \quad (4.9)$$

$$S(j\omega) = [I + P(j\omega)C(j\omega)]^{-1} \quad (4.10)$$

$$T(j\omega) = [I + P(j\omega)C(j\omega)]^{-1} P(j\omega)C(j\omega) \quad (4.11)$$

where $L(j\omega)$, $S(j\omega)$ and $T(j\omega)$ are open loop, sensitivity and complementary transfer function respectively. $S(j\omega) + T(j\omega) = I$ which is an important limit on achievable performance. In order to achieve the good tracking and disturbance rejection, large open loop $L(j\omega)$ gain requires at a low frequency range. For insensitivity against sensor noise and high-frequency unmodelled dynamics of the plant, it is desired that the sensor noise and unmodelled dynamics have as small as possible effect on the system outputs. Thus, $T(j\omega)$ must be small in the frequency range that the noise occurs. This needs for a large roll-off rate at high the frequency range. That is $L(j\omega) \ll 1$. The open loop of the shaped plant intersects with the 0dB line which is crossover frequency determines the speed of the response. Roll-off rate around crossover frequency affects overshoot and stability.

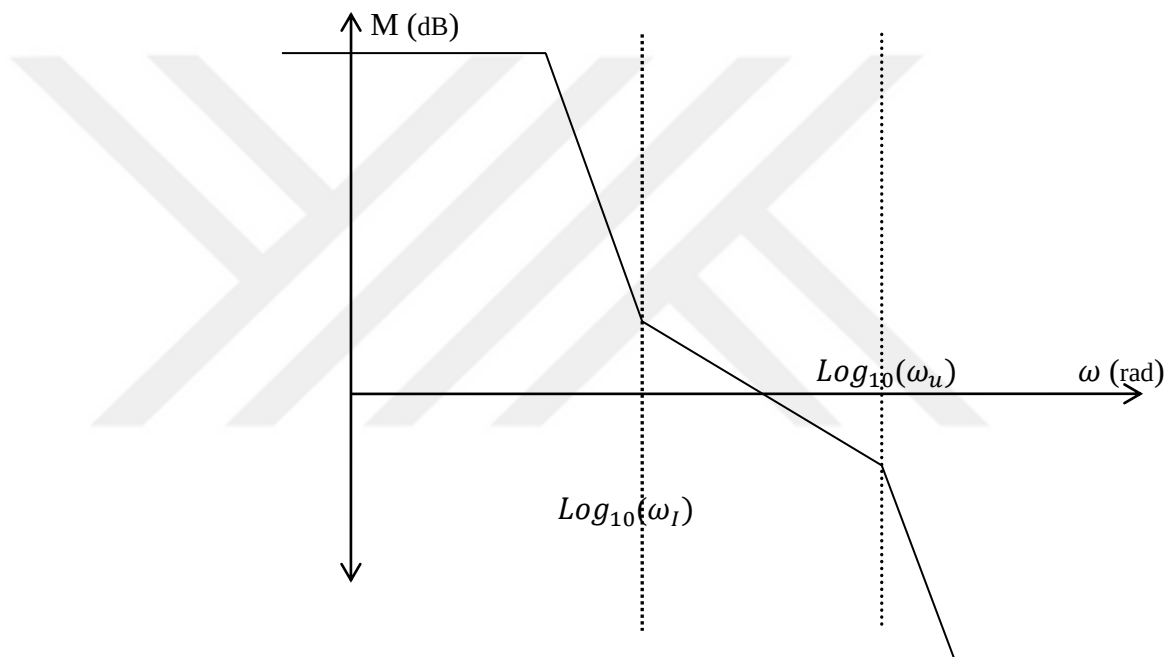


Figure 4.3: A typical desired loop-shape [42].

4.2. Control Systems Design

It is clearly seen from eq. (3.29 -3.40) that roll, pitch and yaw angles and their time derivatives are not based on translation components. However, position equations depend on the angles. In order to design for attitude, altitude and lateral and longitudinal controllers, eq. (3.29 -3.40) can be divided four subsystems.

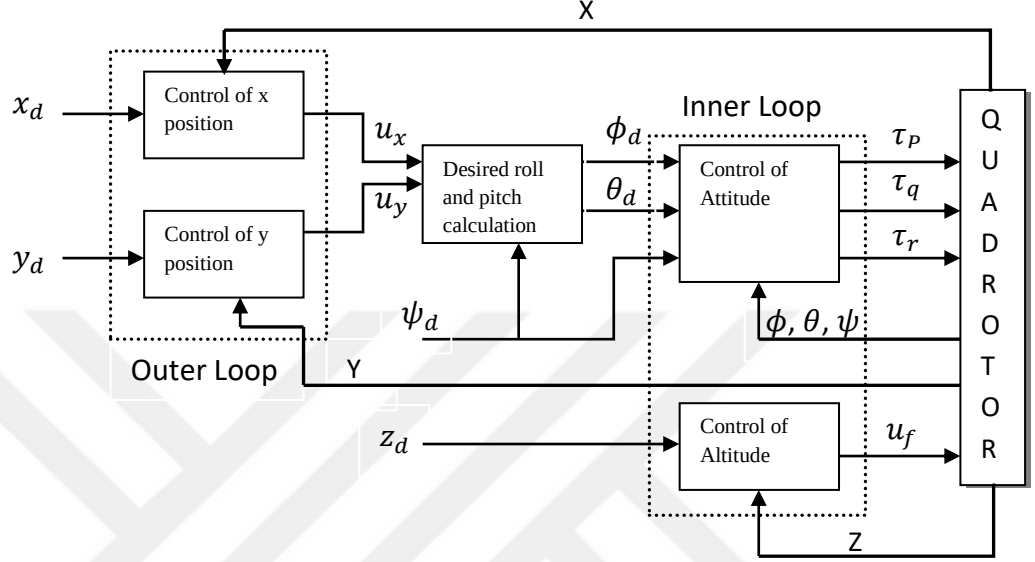


Figure 4.4: Hierarchy of the proposed control strategy

Parameters	values
m	0.486 kg
g	9.806 m/s ²
l	0.25 m
b	$2.9842 \cdot 10^{-5}$ N/rad/s
d	$3.2320 \cdot 10^{-5}$ N.m/rad/s
diag[I_{xx} , I_{yy} , I_{zz}]	diag [3.8278, 3.8278, 7.1345]. 10^{-3} kg.m ²
k_t	$5.5670 \cdot 10^{-3}$ N/rad/s
k_r	0.0320 N/m/s

Table 4.1: Quadrotor parameter

For design of control systems, jacobian matrices need to be calculated by substituting parameter values. These values can be found in reference [2].

4.2. 1.Attitude Controller Design

By using Matlab Symbolic Math Toolbox in order to linearize the nonlinear attitude equations around [0 0 0 0 0 0] operating point. A, B and C matrices can be obtained as followings.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -.1454 & 0 & 0 \\ 0 & 0 & 0 & 0 & -.1454 & 0 \\ 0 & 0 & 0 & 0 & 0 & -.078 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 261.25 & 0 & 0 \\ 0 & 261.25 & 0 \\ 0 & 0 & 140.2 \end{bmatrix} \quad (4.12)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (4.13)$$

In reference [40,41] authors recommend scaling output and input plants such that all channels of the plant are posed by the same amount of magnitude of a control signal so an external scaling matrix needs to scale accurately inputs and outputs. The scaling matrix needs to ensure that singular values of linearized attitude model have roughly the same. If the scaling matrix is chosen as follows.

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (4.14)$$

For the sake of simplicity, if weight $W_2 = I^{3 \times 3}$ is selected. By using trial and error method, two zeros are chosen at 0.001 which helps low gain stay at 130dB. Two zeros are selected at 1 rad/s which reduce slope around crossover frequency and two poles at 0.01 rad/s reduces the low gain and finally, two poles at 80 rad/s whose aim to increases the slop at high frequency. Weight W_1 is given as follows

$$W_1 = \frac{550[s+0.001](s+1)^2}{(s+0.01)^2 (s+80)^2} \quad (4.15)$$

Fig.4.5 shows the comparison of nominal and shaped plant singular values, Roll-off rate around crossover frequency is decreased from 40 dB/dec to 20.5 dB /dec and shaped plant is maintained at 130 dB until 0.01 rad/sec. High roll-off rates is increased from

40dB/dec to and 58.5 dB/dec . The bandwidth is 20.6 rad/s .These values are sufficient to obtain a robust performance against parameter uncertainty, noise and disturbance rejection.

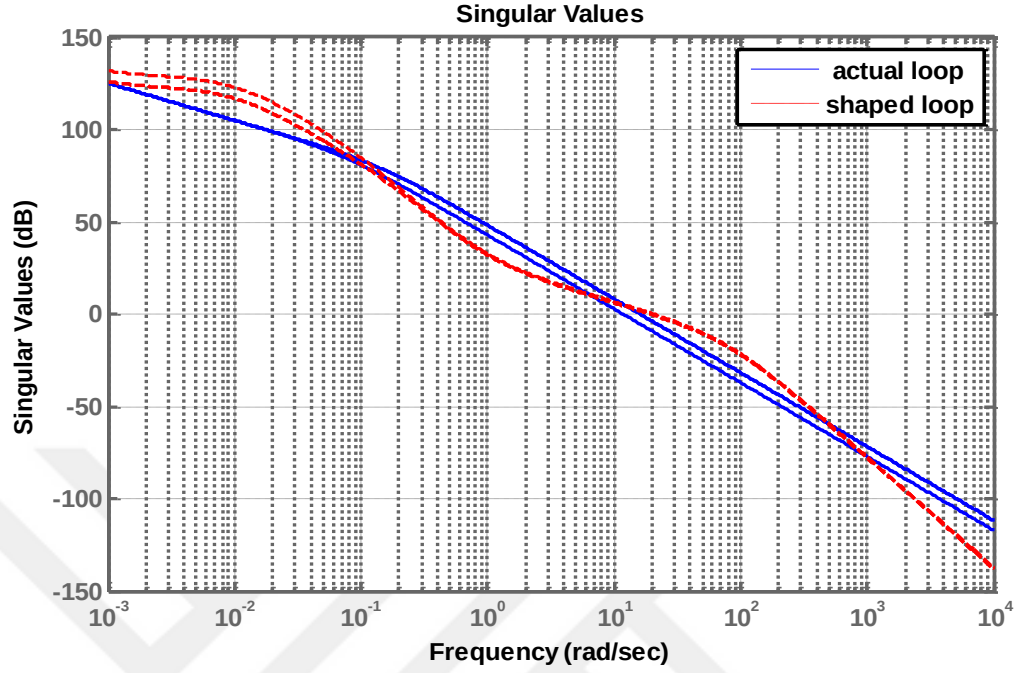


Figure 4.5: Comparison singular values of actual and shaped open loop plants
The Controller is synthesized by using *ncfsyn* command in Matlab robust toolbox. This also provides information about robust stability margin which is 0.5531 ($b(P_s, C_\infty) > 0.3$)
From robust stability margin, the attitude control is robustly stable at least 55% of change in parameters.

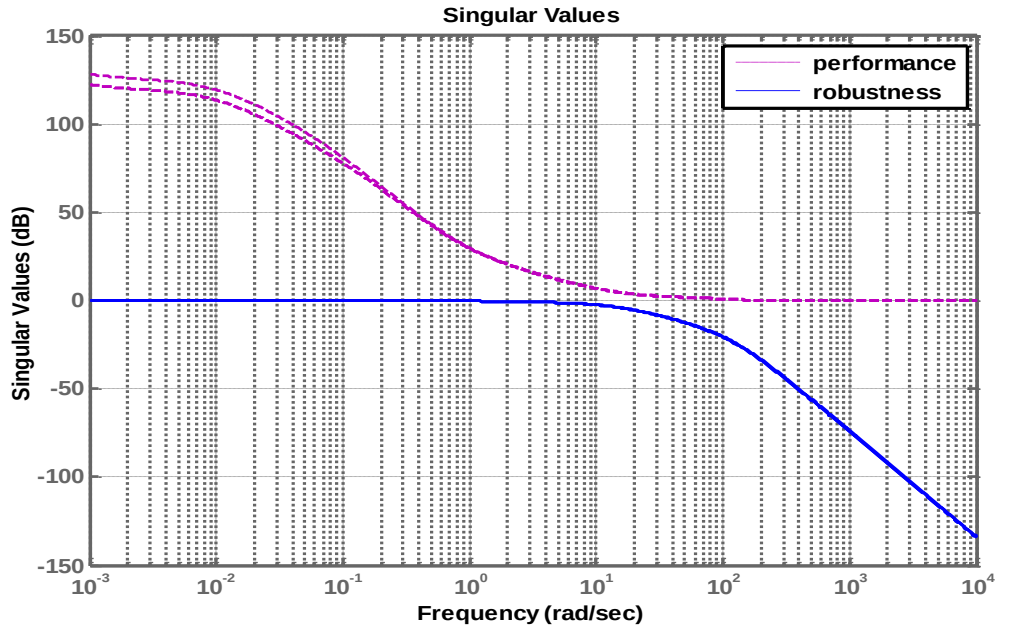


Figure 4.6: Design specifications of attitude control

Stability robustness and performance can be seen from Fig. 4.6 attitude control has low gain at high frequencies where typically unmodelled dynamics and noise occur. High gain at low frequency range guarantees a good tracking and good disturbance attenuation.

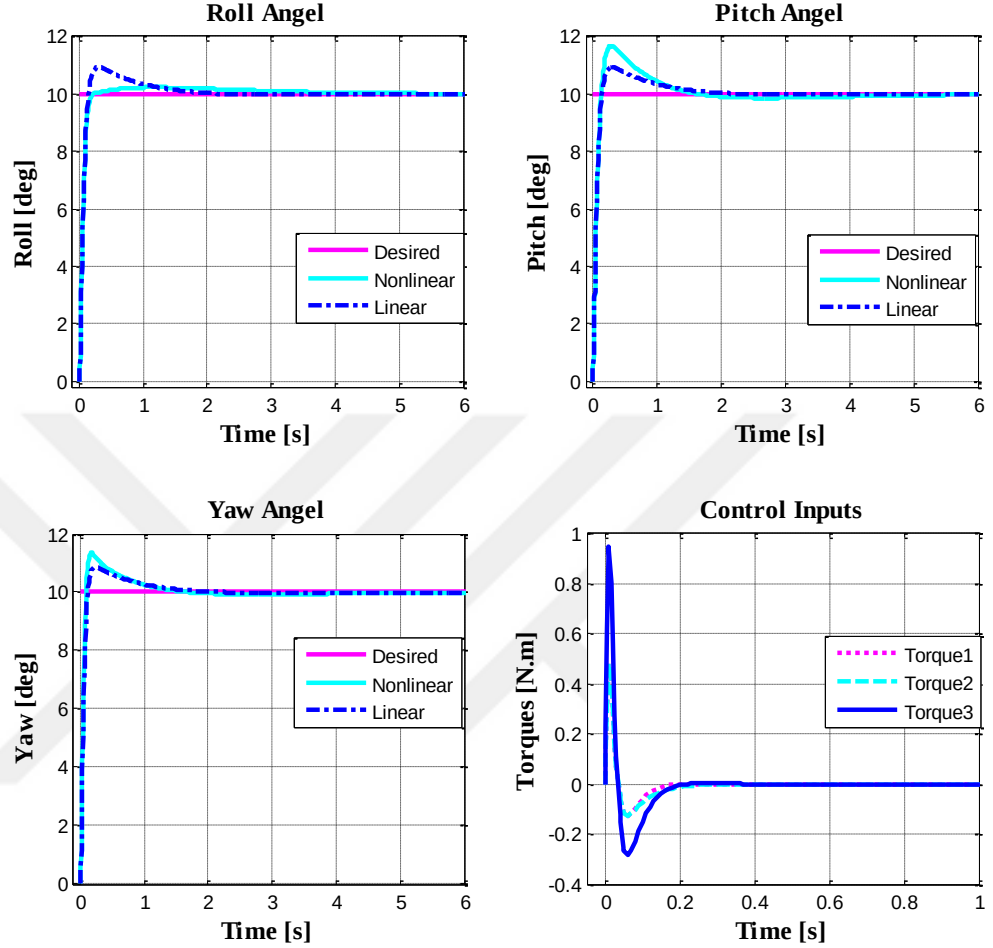


Figure 4.7: Step responses of the attitude controller & control input

It can be seen from fig. 4.7, linear and nonlinear systems are almost have the same response.

Angels	Rise-time Nolinear-linear	Settling time Nolinear-linear	Overshoot Nolinear/linear	Steady-state error Nolinear/linear
Roll [deg]	0.05 s	11.8 s	2 % - 8 %	0.1% -none
Pitch [deg]	0.05 s	13.6 -11.8s	15% - 8.4%	0.1% -none
Yaw [deg]	0.05 s	11.6 -11.7s	12% -8.2%	none

Table 4.2: The performance analysis of H_{∞} loop-shaping attitude control

Linear and nonlinear systems have the same rising time and settling time because they are overlapping during these time periods. Nonetheless, they have a little different overshoots.

4.2. 2. Height Controller Design

By using jacobian linearization method, the matrices can be obtained around $[x_7 \ x_8]^T = [0 \ 0]^T$ operating point and input $\hat{u}_f = 0$ as followings.

$$A_h = \begin{bmatrix} 0 & 1 \\ 0 & -0.0658 \end{bmatrix}, B_h = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } C_h = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (4.16)$$

When the plant is shaped to obtain desired properties, scaling matrix is not necessary as the plant has the single input. Similarly weight $W_2 = I^{2 \times 2}$ is selected and weight W_1 is considered to obtain desired performance. By using trial and error method, W_1 is selected by increasing the gain low frequency and two poles at 80 rad/s are used to increase the slope at high frequency for the disturbance rejection and avoiding unmodelled dynamics.

$$W_1 = \frac{130000}{(s+80)^2} \quad (4.17)$$

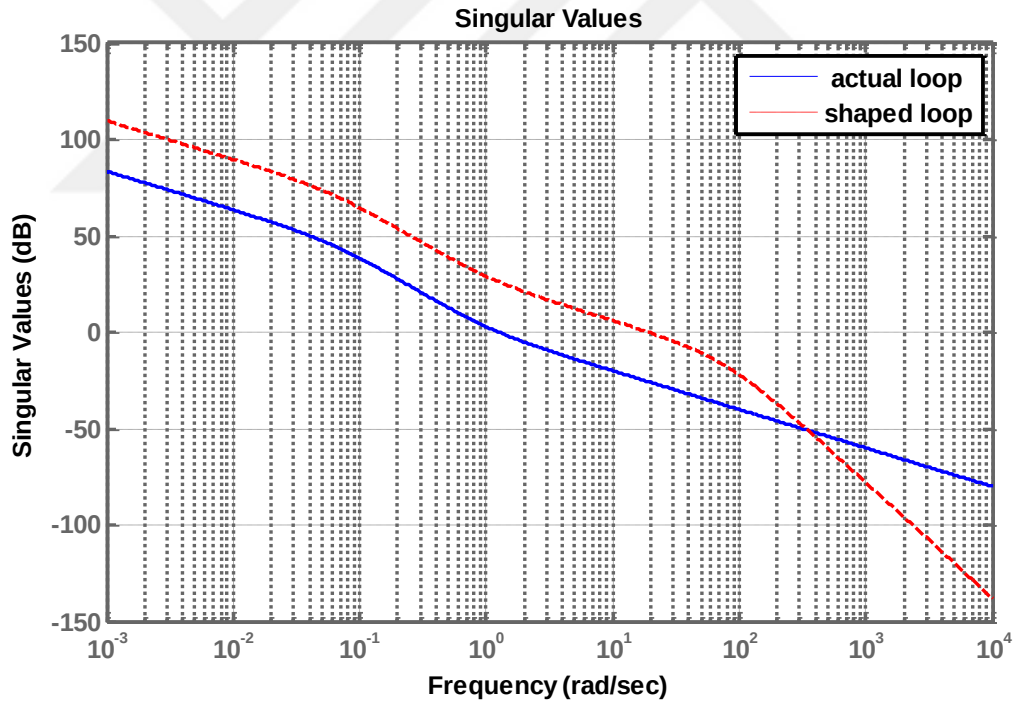


Figure 4.8: Comparison singular values of actual and shaped open loop plants for height control

The gain at low frequency was increased from 83 dB to 103 dB. Bandwidth was improved from 1.3 rad/sec to 20.2 rad/sec in order to obtain fast response. Two poles at 80 rad/sec helps high roll off rate increase from 22 dB/dec to 58 dB/dec. This high roll-off rate is quite

good for disturbance and noise attenuation. Roll-off rate around crossover frequency is decreased from 33dB/dec to 20.5 dB/dec .By using *ncfsyn* to compute an optimal positive feedback height controller in Matlab robust toolbox , which provides 0.5449 robust stability ($b(P_s, C_\infty) > 0.3$) . It can be said that the height controller is robust at least 54% of change in parameters.

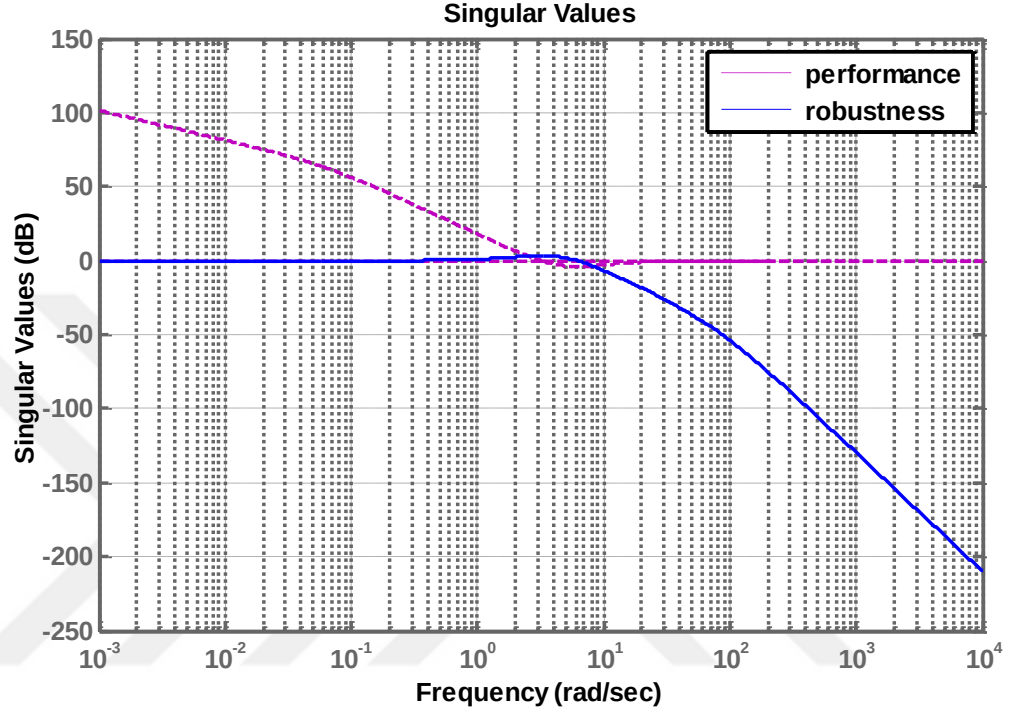


Figure 4.9: Design specifications of height control

It can be seen from Fig. 4.9 the sensitivity (singular value of performance) of height control decreases at high frequency and at high frequency roll-off rate is about 58dB/dec they are good for avoiding unmodelled dynamics and noise.

When quadrotor hovers, roll and pitch angels are supposed to be zero then if step response is applied to height controller .that yields step response in fig. 10. Linear and nonlinear model is overlapping.

Height	Rise-time	Settling time	Overshoot	Steady-state error
	Nolinear-linear	Nolinear-linear	Nolinear/linear	Nolinear/linear
Z [m]	2.5 s	9.8 s	none	0.18 %

Table 4.3: The performance analysis of H_∞ loop-shaping height control

H_∞ Loop-shaping height control has quite fast step response without overshoot

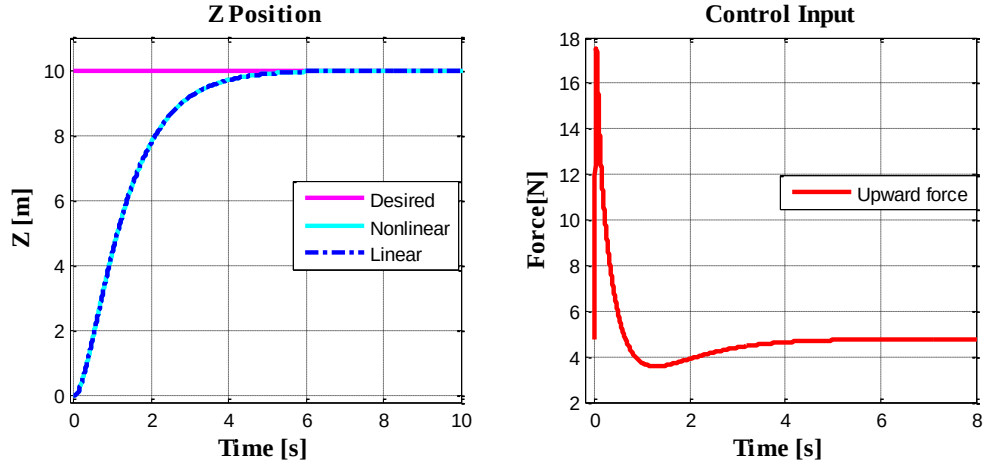


Figure 4.10: Step response of height control& control input

4.2. 3. Lateral and Longitudinal Position Controller Design

The movement through the x and y axes can be obtained by orientations pitch and roll angles so two new inputs can be considered as the directions of u_f that results in the movement through the lateral and longitudinal axes. Desired roll and pitch orientations are produced based on control virtual inputs u_x and u_y . Desired roll and pitch orientations are given by

$$x_{1d} = \arcsin(u_x \sin(x_3) - u_y \cos(x_3)) \quad (4.18)$$

$$x_{2d} = \arcsin\left(\frac{u_x \cos(x_3) + u_y \sin(x_3)}{\cos(x_{1d})}\right) \quad (4.19)$$

A controller will be designed for position controller on x and y axes but they will be implemented separately. Jacobian matrices of position control are

$$A_p = \begin{bmatrix} 0 & 1 \\ 0 & -0.0658 \end{bmatrix}, B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } C_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (4.20)$$

Similarly weight $W_2 = I^{2 \times 2}$ was selected and weight W_1 was considered to obtain desired performance. By using trial and error method, one zero at 0.07 rad/sec and one pole 0.7 rad/sec were selected to decrease the slope around crossover frequency. Two poles at

100rad/sec are chosen to increased roll off rate at high frequency. Weight W_1 can be given as follows

$$W_1 = \frac{100000(s+0.07)}{(s+0.7)(s+100)^2} \quad (4.21)$$

The Glover-McFarlane H_∞ normalized coprime factor loop-shaping controller, with weights W_1 and W_2 was calculated by using ncfsyn function in Matlab. 0.5728 robust stability margin obtained which is greater than 0.3.

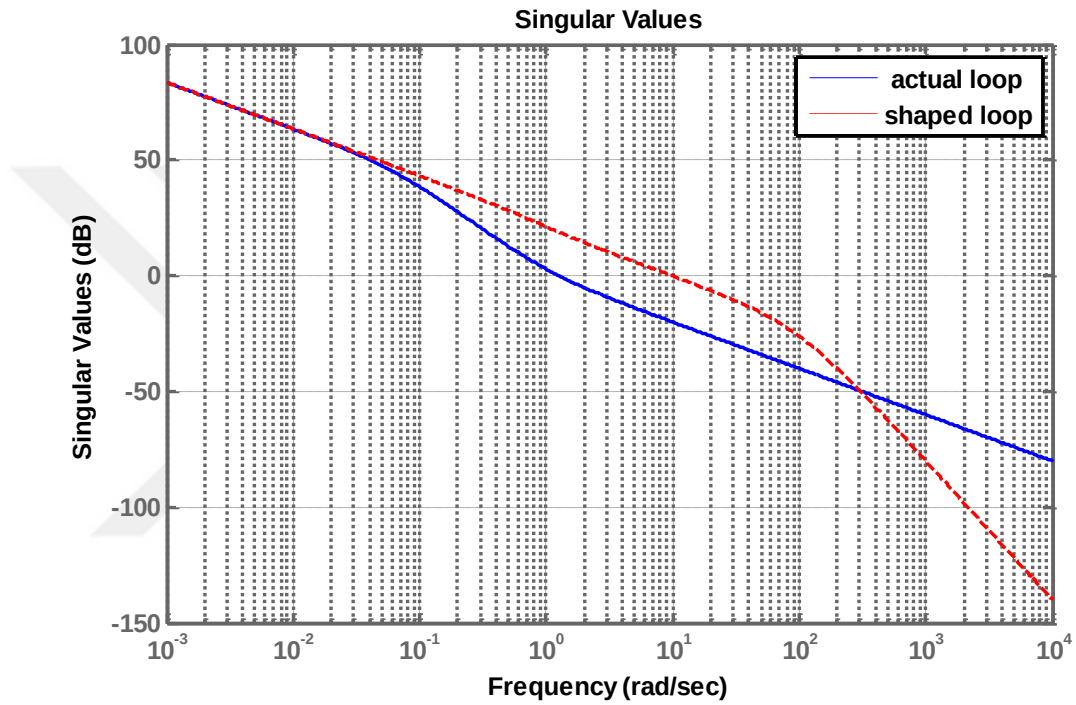


Figure 4.11: Comparison singular values of actual and shaped open loop plants for position control x and y axes control

The gain at low frequency stays stable at 83dB. Bandwidth is grown from 1.3 to 10.4 rad/sec. Outer loop have to be slower than inner loop so this bandwidth is sufficient to obtain fast response. Especially, the speed of outer loop has to be slower than attitude controller. On the other hand overall control will be instable. roll-off rate around crossover frequency is reduced from 33dB to 20.2 dB/dec and high roll off rate is improved from 20dB/dec to 60 dB/dec

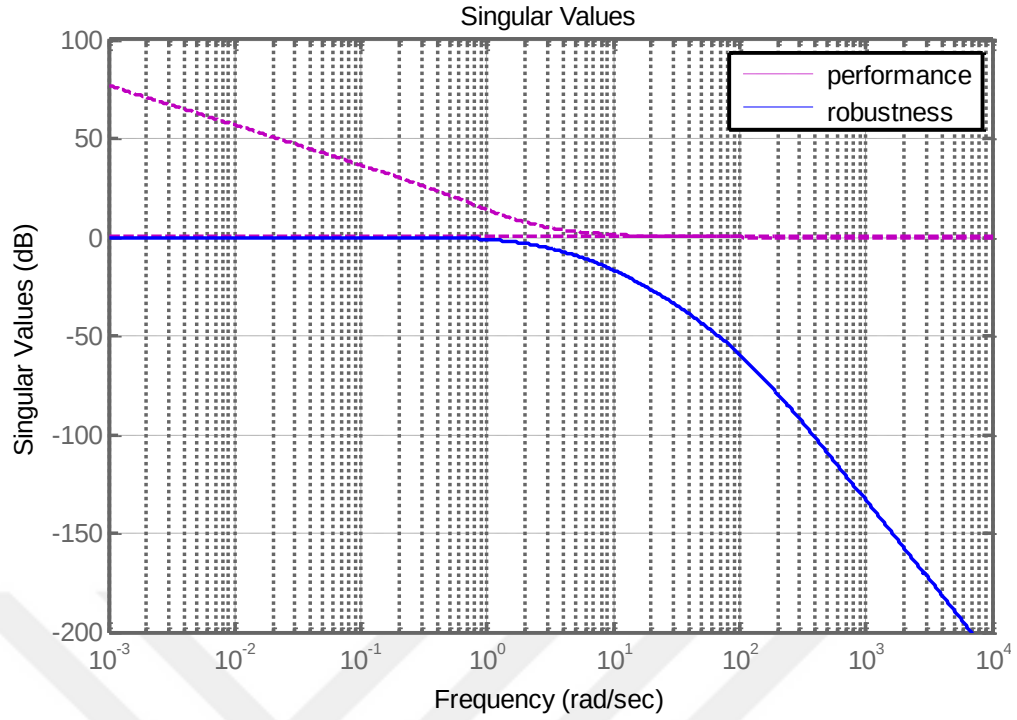


Figure 4.12: Design specifications of position control on x and y axes

Sensitivity decreases at high frequency and high roll of rate also can determine from fig.4.12. Disturbance attenuation and unmodelled dynamics rejection is provided by obtaining 60dB/dec roll-off rate at high frequency.

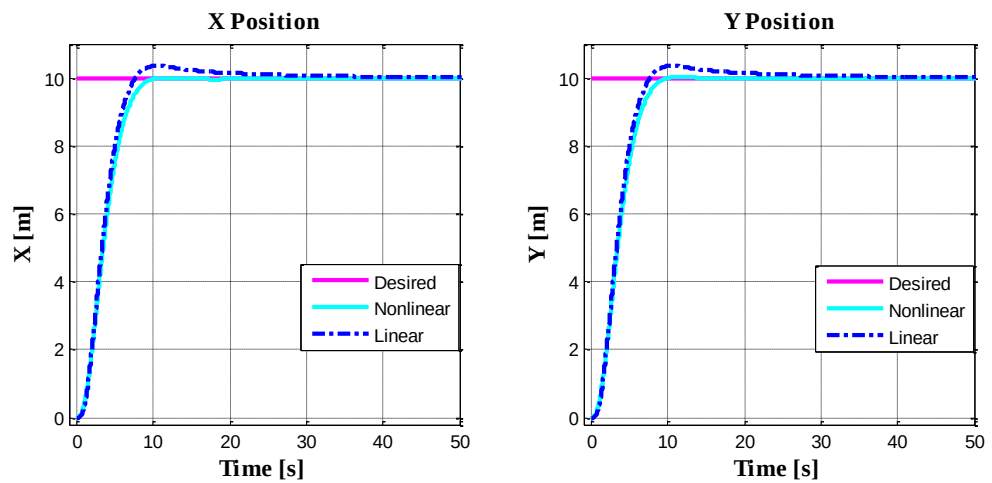


Figure 4.13: Step response of position control on x and y axes

positions	Rise-time Nolinear-linear	Settling time Nolinear-linear	Overshoot Nolinear/linear	Steady-state error Nolinear/linear
x [m]	5- 4.4 s	28-48 s	0.1 % - 3 %	0.05% -0.3%
y [m]	5- 4.4 s	26-48 s	0.2 % - 3.5 %	0.05% -0.2%

Table 4.5: The performance analysis of position controllers on x and y axes
As expected, lateral and longitudinal controllers have the same performance since the quadrotor is designed symmetrically. H_∞ Loop-shaping lateral and longitudinal control has a sufficient performance with 0.5728 robust stability margin.

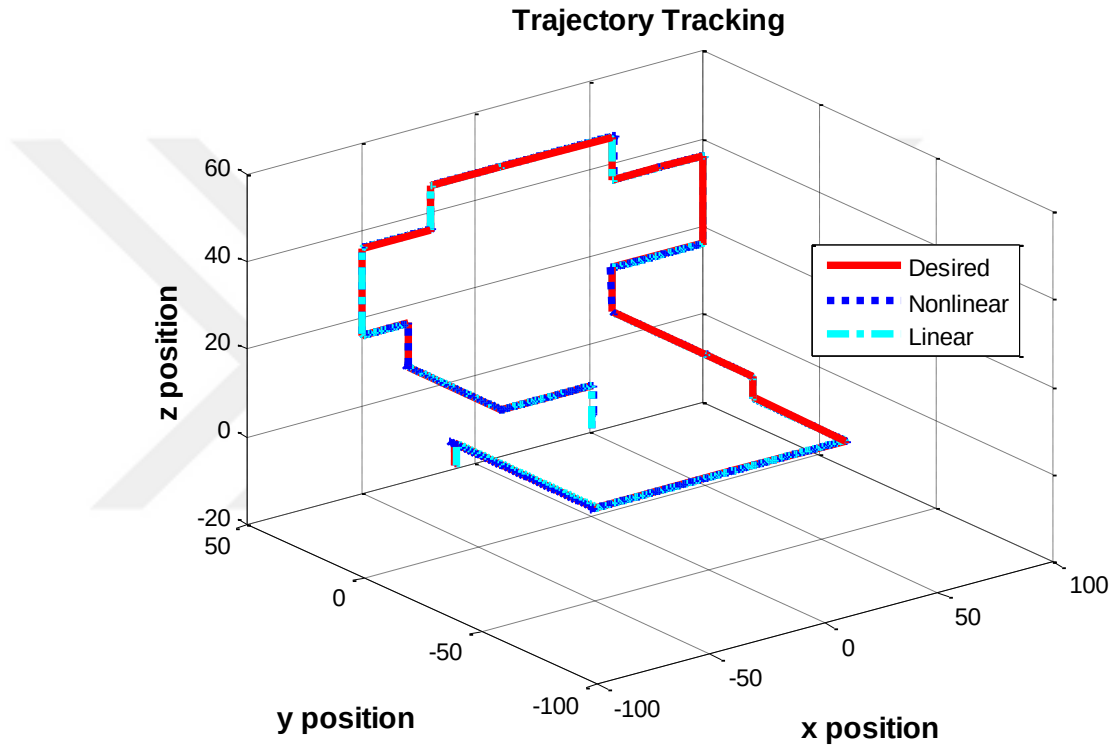


Figure 4.14: Trajectory tracking of the quadrotor.

5. LINEAR QUADRATIC REGULATOR / GAUSSIAN CONTROL SYSTEMS DESIGN

5.1. Background Theory

5.1.1. Linear Quadratic Regulator (LQR)

Considering a linear system given by

$$\dot{x} = Ax + Bu \quad (5.1)$$

In which the pair (A, B) is stabilizable

The cost function is

$$J = \int_0^{\infty} (x^T Q x + u^T R u) dt \quad (5.2)$$

with $Q \geq 0$ and $R > 0$

The linear quadratic regulation problem finds a control law $u = -Kx$ such that closed-loop is stable and keeps the cost function as small as possible.

Riccati equation is given by

$$Q + PA + A^T P - PBR^{-1}B^T P = 0 \quad (5.3)$$

Lemma 5.1: For any positive definite Q matrix, the close loop system $\dot{x} = Ax$ quadratic stable if and only if there exists a positive definite matrix P which satisfies the so called Lyapunov matrix equation:

$$A^T P + PA + Q = 0 \text{ or } A^T P + PA < 0 \quad (5.4)$$

Furthermore, this makes $V = x^T P x$ a valid quadratic Lyapunov function for the system as $P > 0 \rightarrow V > 0$ and $\dot{V} = x^T (A^T P + PA)x < 0$. [37]

Derivative of Lyapunov function is

$$\dot{V} = x^T A^T P x + x^T P A x + u^T B^T P x + x^T P B u \quad (5.5)$$

$$= x^T P (A - BR^{-1}B^T P)x + x^T (A - BR^{-1}B^T P)^T P x \quad (5.6)$$

Riccati equation can be rewritten by taking into consideration that $u = -R^{-1}B^T P x$

$$P(A - BR^{-1}B^T P)^T + (A - BR^{-1}B^T P)^T P = -Q - PBR^{-1}B^T P \quad (5.7)$$

$$\dot{V} < 0 \quad (5.8)$$

This guarantees Lyapunov stability of the system. State feedback gain can be defined as

$$K = R^{-1}B^T P \quad (5.9)$$

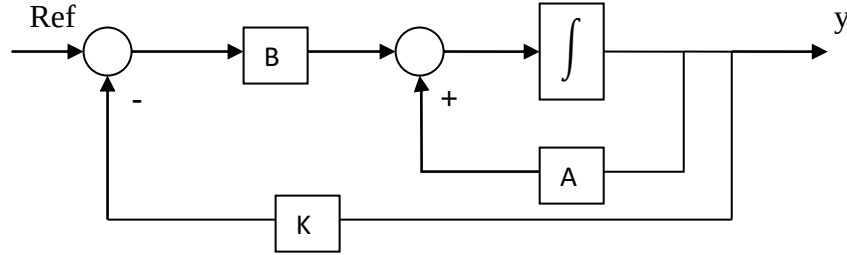


Figure 5.1: Output feedback structure

5.1.2. Q and R weight matrices selection

In order to minimize cost function J , state and control weights are chosen. When J is kept as small as possible then that yields

$$\lim_{t \rightarrow \infty} x = 0 \quad (5.10)$$

This guarantees stability of the closed loop systems. Positive semi-definite $Q \in R^{n \times n}$ and positive definite $R \in R^{m \times m}$ are selected by analysing location of the closed-loop poles in the s -plane and R matrices affect closed loop system behaviour differently. For instance, selecting Q determines the limitations of state variables. On the other hand selecting R limits control inputs. That is, larger values of Q lead to the poles of the closed-loop systems further left in the s -plane so the response will be faster. However, Larger R results in less control effort and poles get closer to the imaginary axis. Both matrices can be chosen diagonal but each element of input weight matrix must be positive and invertible but those of state weight matrix should be positive and might have some zeros on its diagonal.

5.1.3. Linear Quadratic Gaussian (LQG)

Considering the plant given by

$$\dot{x} = A x + B u + n_1 \quad (5.11)$$

$$y = C x + n_2 \quad (5.12)$$

In which n_1 is the vector of Gaussian noise processes and n_2 is the vector of Gaussian measurement noise processes. It is assumed that there is no correlation between n_1 and n_2 $E[n_1 n_2^T] = 0$. If the pair (A, B) is controllable and the pair (A, C) is observable, then the purpose of the control law is to minimize the cost function by finding the control input u.

$$(5.13)$$

Where E donates expectation and $Q \geq 0$ and $R > 0$ are weight matrices.

LQG control requires only measurable information. Structure of the LQG control consists of two sub-problems: first of those is Kalman state estimation and other is state feedback problem. Kalman state estimator $\hat{x}$ is formularized as follows.

$$\dot{\hat{x}} = A x + B u + L(C x - C \hat{x}) \quad (5.14)$$

for state feedback problem Riccati equation is

$$Q + P A + A^T P - P B R^{-1} B^T P = 0 \quad (5.15)$$

Then optimal gain matrix is

$$K = R^{-1} B^T P \quad (5.16)$$

Substituting $u = -R^{-1} B^T P x$ yields

$$\dot{x} = A x + B u = (A - B K) x \quad (5.17)$$

For State estimation problem Riccati equation is written as follows

$$Q_f + P_f A + A^T P_f - P_f C^T R_f^{-1} C P_f = 0 \quad (5.18)$$

Kalman estimation gain matrix is

$$L = P_f C^T R_f^{-1} \quad (5.19)$$

Then forming control law is $u = -K \hat{x}$ and Linear quadratic Gaussian is given by

$$\dot{\hat{x}} = (A - L C) \hat{x} + L C x - B K \quad (5.20)$$

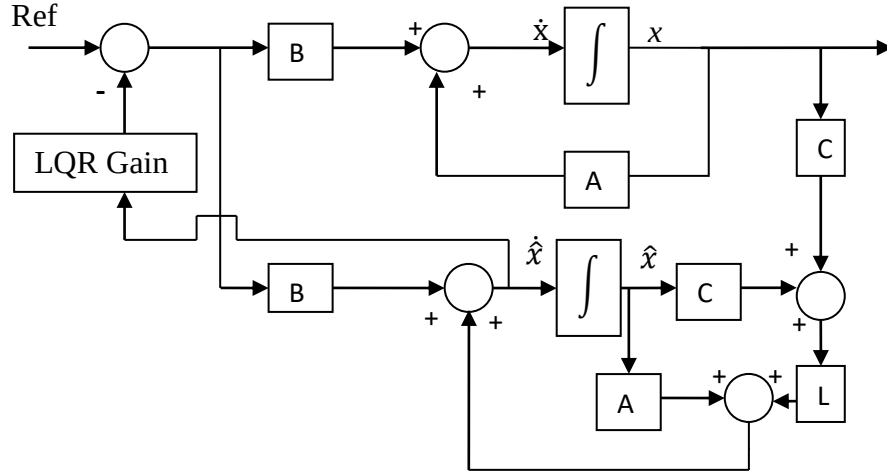


Figure 5.2: Structure for LQG control

Eigenvalues of the closed-loop poles ($A - LC$) is left hand side in s plane and Kalman state estimator is stable and optimal state feedback and state Kalman filter can be combined as

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \quad (5.21)$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \quad (5.22)$$

with $e = x - \hat{x}$ state estimation error, the closed-loop eigenvalues of state feedback is ($A - BK$) and those of Kalman state estimator is $A - LC$. When they are in the left half side in s plane, then eq.5.11 is stable. Further details can be found in [37].

5.2. Linear Quadratic Regulator (LQR) Control System Design

5.2.1. Attitude Controller Design

Assuming that, all outputs can be measurable. C jacobian output matrix constructed as followings additional to eq. (4.12).

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (5.23)$$

Typically choosing Q is $C^T C$ and by using trial and error method, Q and R matrices can be obtained as follows

$$Q = \text{diag} [1 \ 1 \ 1 \ .01 \ .01 \ .01] \text{ and } R = \text{diag} [1 \ 1 \ 1] \quad (5.24)$$

By using lqr function in MATLAB, feedback gain matrix $K = \text{lqr}(A, B, Q, R)$ can be computed. These weights lead to feedback gain and closed loop eigenvalues as

$$K = \begin{bmatrix} 1 & 0 & 0 & .1323 & 0 & 0 \\ 0 & 1 & 0 & 0 & .1323 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0.1552 \end{bmatrix} \text{ and } E = \begin{bmatrix} -11.0316 \\ -23.6817 \\ -10.9178 + 4.5788i \\ -10.9178 - 4.5788i \\ -11.0316 \\ -23.6817 \end{bmatrix} \quad (5.25)$$

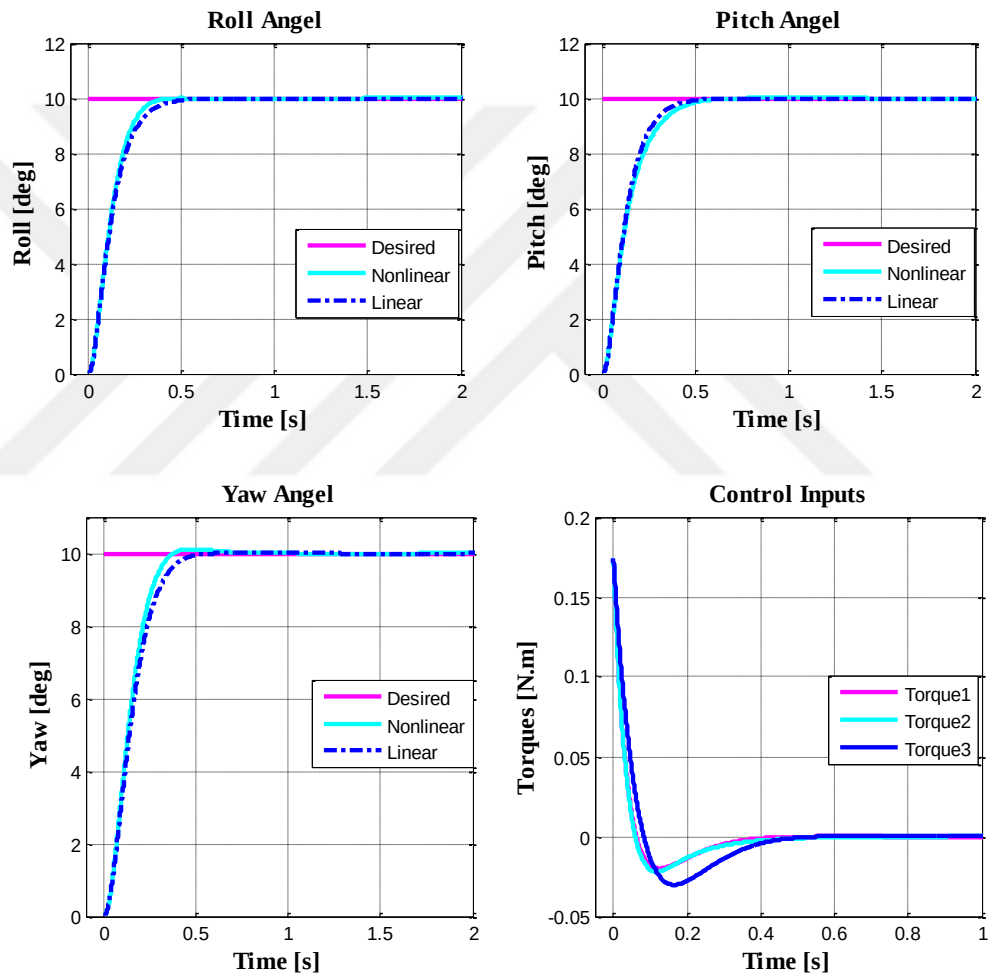


Figure 5.3: Step response of LQR attitude control & control inputs

Angels	Rise-time	Settling time	Overshoot	Steady-state error
	Nonlinear-linear	Nonlinear-linear	Nonlinear/linear	Nonlinear/linear

Roll [deg]	0.2 s	1 s	none	none
Pitch [deg]	0.2 s	1.1 s	none	none
Yaw [deg]	0.2 s	0.8 s	1%-none	none

Table 5.1: Performance analysis of LQR attitude control

LQR attitude control was designed and simulated .Fig 5.3 and 5.4 graphs were obtained. And real time performance analysis presents in Tab.5.1. Step responses of roll, pitch and yaw angels are quite fast without steady state error or overshoot.

5.2.2 .Height Controller Design

By using Eq. (4.16) and weight Q and R matrices are selected by

$$Q = \text{diag} [1 \ 5] \text{ and } R = 1 \quad (5.26)$$

Feedback gain and closed-loop eigenvalues can be computed as follows

$$K=[1.0000 \ 2.5807] \text{ and } E=[-0.4566 \ -2.1899]^T \quad (5.27)$$

positions	Rise-time	Settling time	Overshoot	Steady-state error
	Nolinear-linear	Nolinear-linear	Nolinear/linear	Nolinear/linear
z [m]	5 s	23 s	none	none

Table 5.2: Performance analysis of LQR height control

LQR height controller provides Tab.5.2 performances. Height controller was simulated with lateral and longitudinal controller and it can be seen in Fig 5.4 that control input goesto 10 N which recovers gravitational force.

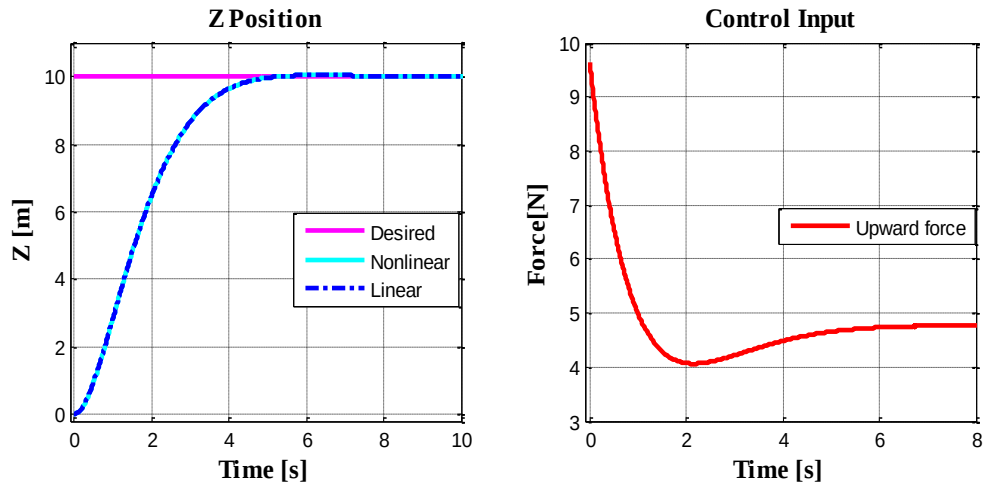


Figure 5.4: Step response of LQR height control & control input

5.2.3. Lateral and Longitudinal Position Controller Design

Based on Eq.(4.20) LQR lateral and longitudinal control can be developed. Weight Q and R matrices is selected by

$$Q = \text{diag} [1 \ 80] \text{ and } R = 1 \quad (5.28)$$

These selections yield feedback gain and closed –loop eigenvalues as

$$K = [1.0000 \ 8.9898] \text{ and } E = [-0.1118 \ -8.9438]^T \quad (5.29)$$

positions	Rise-time Nolinear-linear	Settling time Nolinear-linear	Overshoot Nolinear/linear	Steady-state error Nolinear/linear
x [m]	23- 22 s	75-62 s	none	0.03% -0.04%
y [m]	24- 24.5 s	77-77 s	none	0.05% -0.08%

Table 5.3: Performance analysis of LQR position control on x and y axes

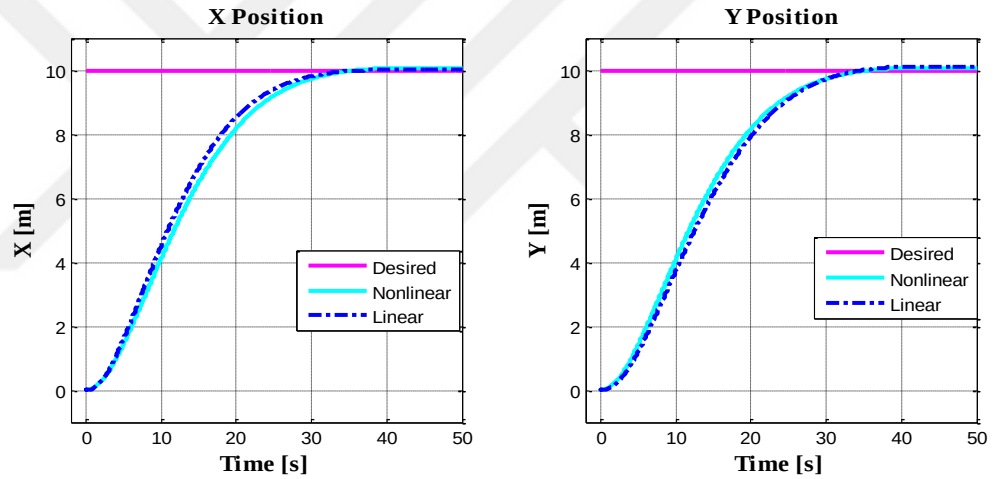


Figure 5.5: Step response of LQR position control on x and y axes

LQR lateral and longitudinal control has slower step response with compared height and attitude control with a bit steady –state error. Linear and nonlinear model has almost the same response with three controllers. Robustness analysis will be done in section 6 by comparing LQR robustness performance with H-infinity and LQG controllers.

5.3. Linear Quadratic Gaussian (LQG) Control System Design

5.3.1. Attitude Controller Design

Eq.(4.12) and (4.13) can be used to design LQG attitude control since measureable output are roll, pitch and yaw angels. LQR feedback gain and L Kalman gain matrix are computed by selecting input and state weight matrices. If Q and R weights are is selected as followings

$$Q = \text{diag} [1 \ 1 \ 1 \ 0.1 \ 0.1 \ 0.1] \text{ and } R = \text{diag} [1 \ 1 \ 1] \quad (5.30)$$

Optimal feedback gain and closed-loop eigenvalues a can be found by

$$K = \begin{bmatrix} 1 & 0 & 0 & 0.3276 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0.3276 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0.3375 \end{bmatrix} \quad E = \begin{bmatrix} -3.1646 \\ -82.5529 \\ -44.2103 \\ -3.1704 \\ -82.5529 \\ -3.1646 \end{bmatrix} \quad (5.31)$$

Typically choosing Q is BB^T in order to calculate Kalman gain matrix and by using trial and error method, Q and R matrices can be selected as follows

$$Q = \text{diag} [0 \ 0 \ 0 \ 6825000 \ 6825000 \ 6825000] \text{ and } R = \text{diag} [0.3 \ 0.3 \ 0.3] \quad (5.32)$$

By using lqr function in MATLAB, Kalman gain matrix $L = \text{lqr} (A', C', Q, R)'$ can be calculated. In eq(5.32) weights selection lead to Kalman gain and closed loop eigenvalues as

$$L = 10^3 \begin{bmatrix} 0.1284 & 0 & 0 \\ 0 & 0.1284 & 0 \\ 0 & 0 & 0.1285 \\ 8.2427 & 0 & 0 \\ 0 & 8.2427 & 0 \\ 0 & 0 & 8.2513 \end{bmatrix} \quad E = \begin{bmatrix} -64.2704 + 64.2703i \\ -64.2704 - 64.2703i \\ -64.2704 + 64.2703i \\ -64.2704 - 64.2703i \\ -64.2704 + 64.2703i \\ -64.2704 - 64.2703i \end{bmatrix} \quad (5.33)$$

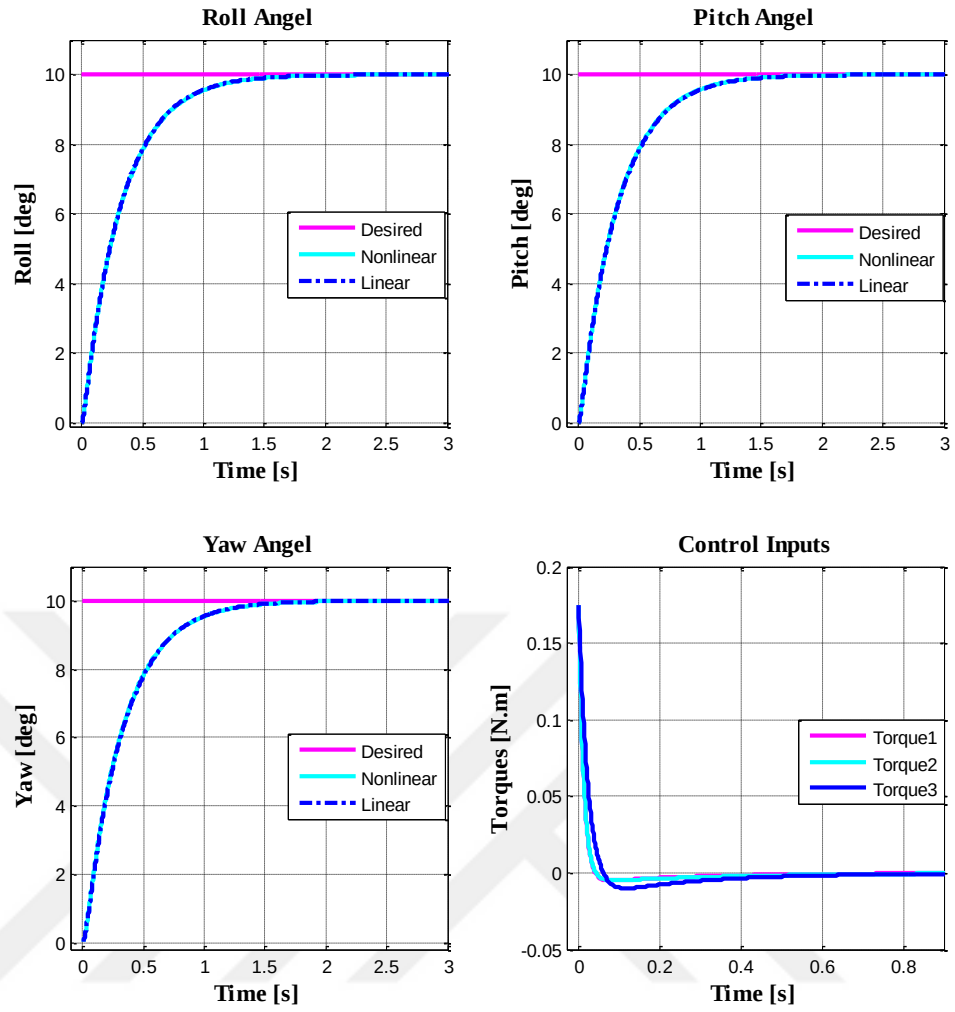


Figure 5.6: Step response of LQG attitude control & control inputs

Angels	Rise-time	Settling time	Overshoot	Steady-state error
	Nolinear-linear	Nolinear-linear	Nolinear/linear	Nolinear/linear
Roll [deg]	0.75 s	4 s	none	none
Pitch [deg]	0.7 s	4 s	none	none
Yaw [deg]	0.68 s	2.8 s	none	none

Table 5.4: Performance analysis of LQG attitude control

LQG attitude controller is slower than LQR attitude controller. Apart from LQR has 1% overshoot of nonlinear yaw response. LQG and LQR have the same overshoot or steady state error.

5.3.2 .Height Controller Design

By using eq.(4.16) and Q and R matrices can be selected as follows in order to design height control system

$$Q = \text{diag} [1 \ 1] \text{ and } R = 1 \quad (5.34)$$

Those yield feedback gain and closed-loop eigenvalues as

$$K=[1.0000 \quad 1.6675] \text{ and } E=\begin{bmatrix} -0.8667 + 0.4989i \\ -0.8667 - 0.4989i \end{bmatrix} \quad (5.35)$$

$$Q = \text{diag} [0 \ 388000] \text{ and } R = \text{diag} [1 \ 1] \quad (5.36)$$

These weights yield Kalman gain matrix and closed-loop eigenvalues

$$L=\begin{bmatrix} 1 & 0.9983 \\ 0.9983 & 622.8298 \end{bmatrix}, E=\begin{bmatrix} -1 \\ -622.8957 \end{bmatrix} \quad (5.37)$$

$$(5.38)$$

positions	Rise-time	Settling time	Overshoot	Steady-state error
	Nolinear-linear	Nolinear-linear	Nolinear/linear	Nolinear/linear
z [m]	3.5 s	10.5 s	none	none

Table 5.5: Performance analysis of LQG height control

Even though LQG height control has faster step response than LQR control, it has 0.01% overshoot.

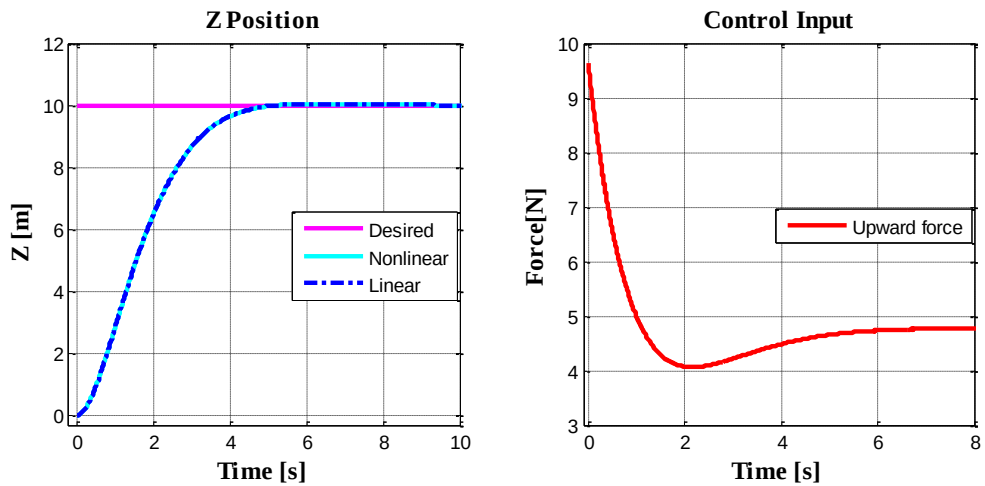


Figure 5.7: Step response of LQG height control & control input

It can be seen from Fig. 5.7 the control input recovers gravitational force which is a minimum requested force for the quadrotor.

5.3.3. Lateral and Longitudinal Position Controller Design

Eq.(4.20) can be used to design the controller .Q and R matrices can be selected as follows

$$Q = \text{diag} [1 \ 90] \text{ and } R = 1 \quad (5.39)$$

That result in

$$K = [1.0000 \ 9.5260] \text{ and } E = \begin{bmatrix} -0.1054 \\ -9.4865 \end{bmatrix} \quad (5.40)$$

In order to calculate Kalman gain matrix, state and input weights matrices can be selected by

$$Q = \text{diag} [0 \ 320] \text{ and } R = 0.2 \quad (5.41)$$

Kalman gain and closed-loop eigenvalues are

$$L = \begin{bmatrix} 0.9998 & 0.9815 \\ 0.9815 & 56.4942 \end{bmatrix}, E = \begin{bmatrix} -1.0002 \\ -56.5597 \end{bmatrix} \quad (5.42)$$

positions	Rise-time Nolinear-linear	Settling time Nolinear-linear	Overshoot Nolinear/linear	Steady-state error Nolinear/linear
x [m]	22- 20 s	77-64 s	none	0.019% - 0.002%
y [m]	22.5- 23 s	72-72s	0.5 % -1%	0.019% - 0.021%

Table 5.6: Performance analysis of LQG position control on x and y axes

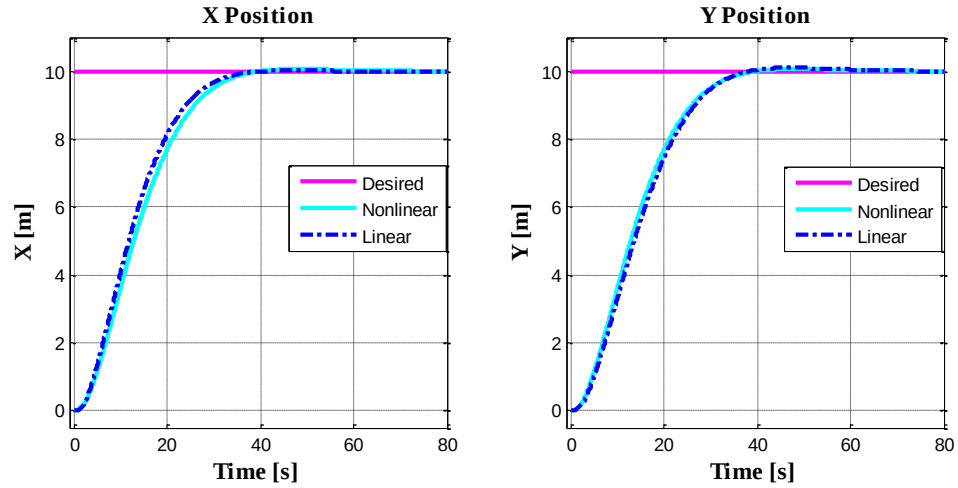


Figure 5.8: Step response of LQG position control on x and y axes

It can be said that LQG and LQR x and y position controllers has the same performance with slightly differences.

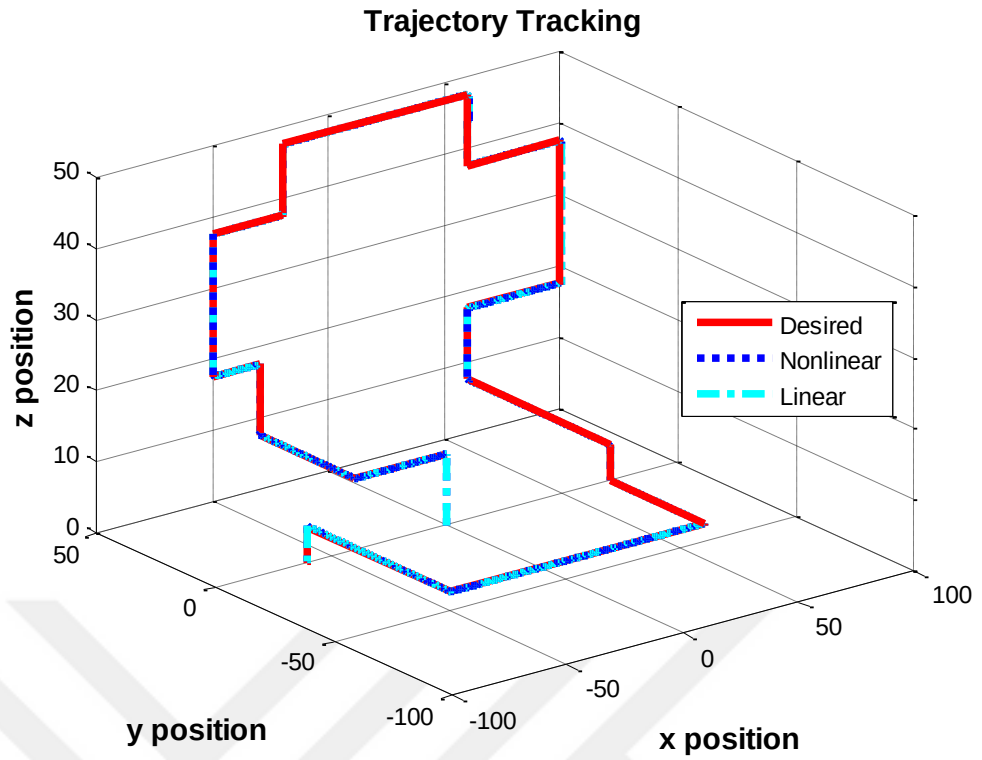


Figure 5.9: Trajectory tacking of the Quadrotor with LQR control

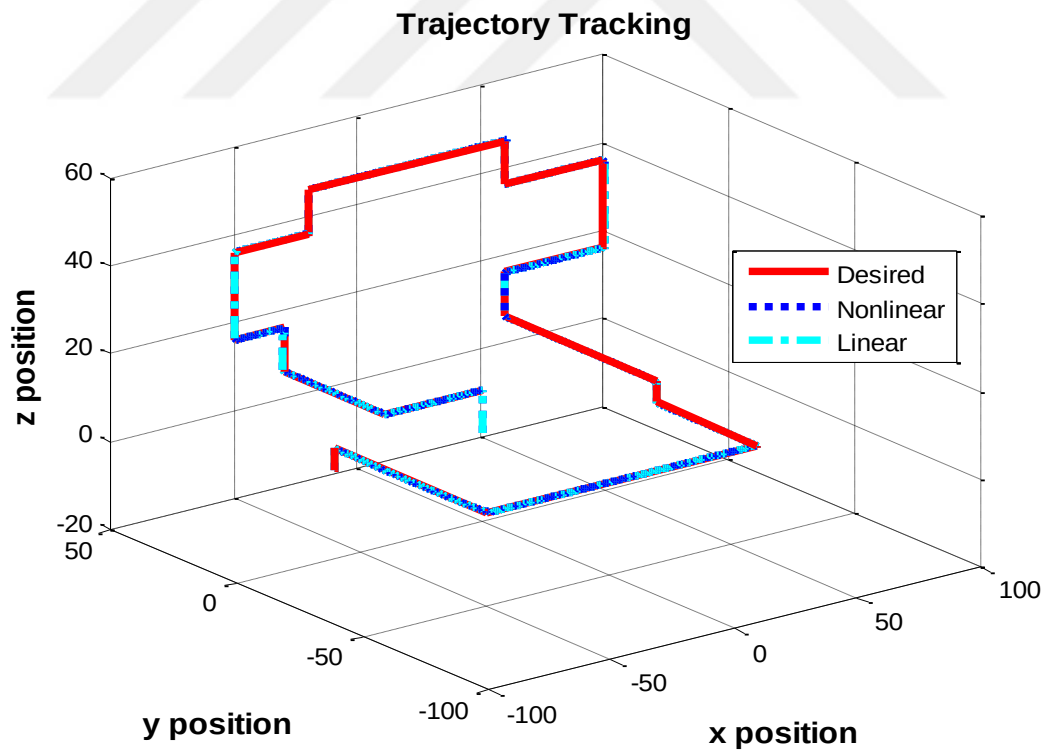


Figure 5.10: Trajectory tacking of the Quadrotor with LQG control

6. COMPARISON of DESIGNED CONTROL SYSTEMS

6.1 Experiment 1: Effect of Sensor Noise

A Gaussian white noise with variance =0.0001, mean =0 and sampling time=0.001 seconds is used to simulate sensor noise.

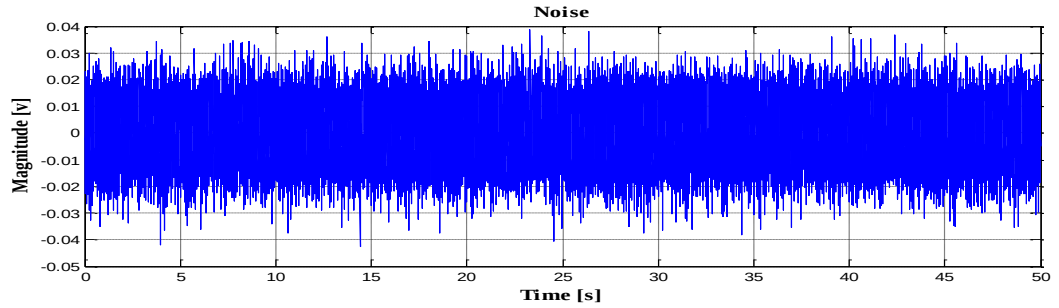


Figure 6.1: sensor noise

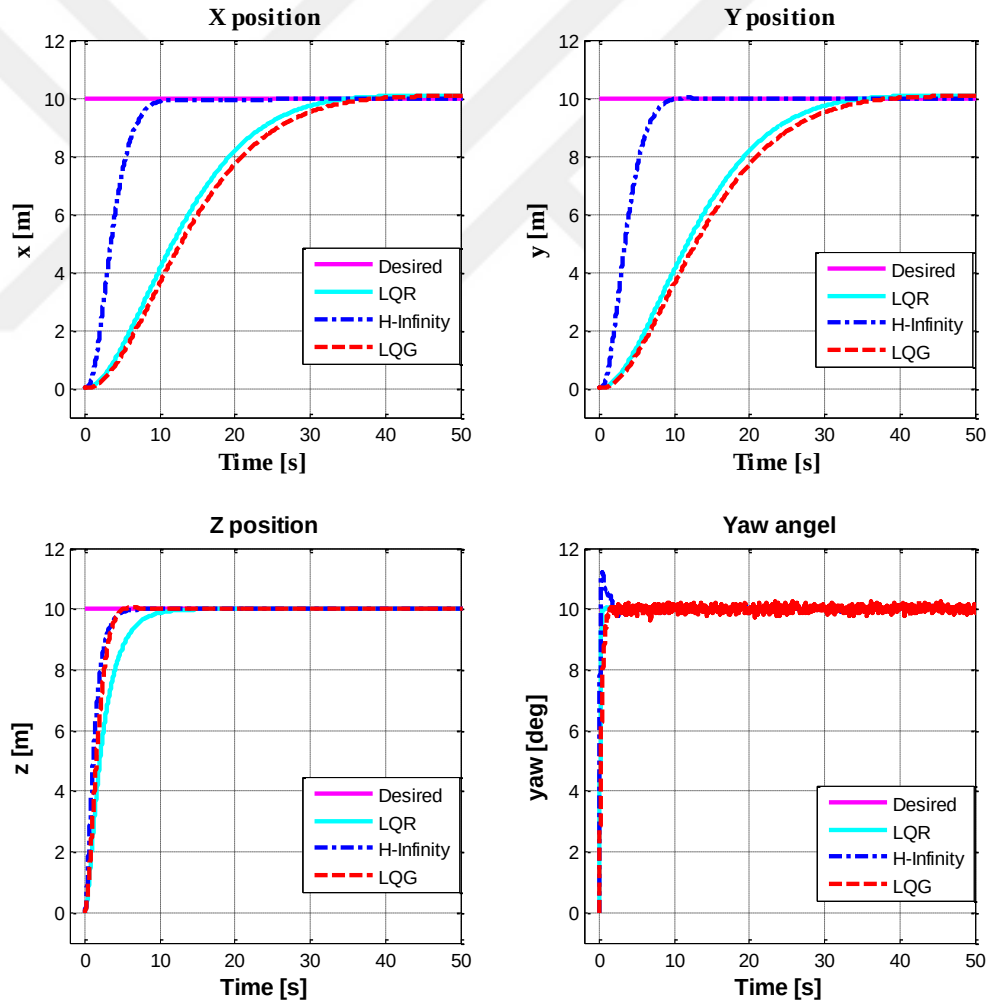


Figure 6.2: Step response of positions on x, y, z axes and yaw angel with sensor noise

Three controllers are robust relative with sensor noise. However, insensitivity of LQG is less than others .When noise is introduced , attitude controller of LQG has more sensitivity towards sensor noise. In this experiment, all three controllers achieve robustness and noise rejection.

6. 2. Experiment 2: Effect of Aggressive Disturbance

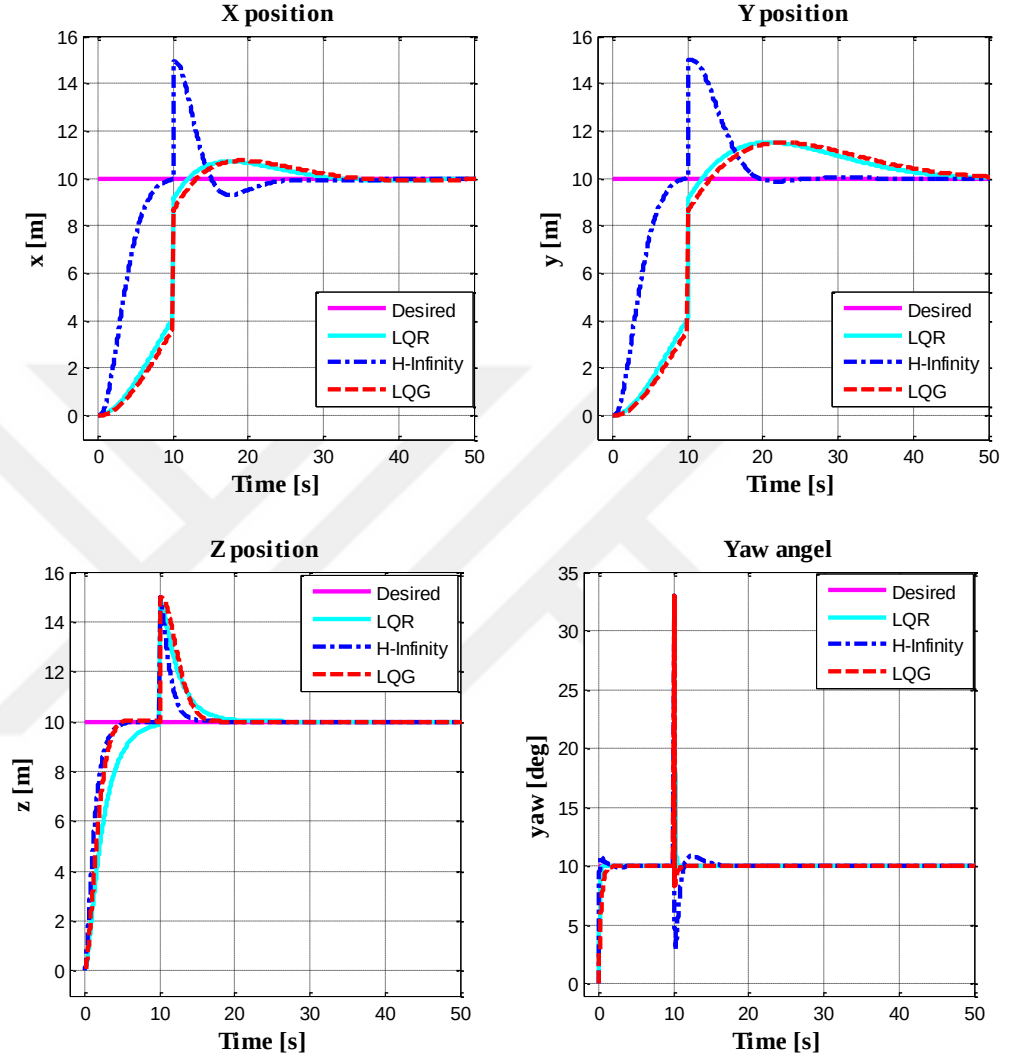


Figure 6.3: Step response of positions on x, y , z axes and yaw angle with aggressive disturbance

5 m/s step disturbance which can be considered as a wind disturbance with related to axes. This aggressive disturbance was applied to outputs of x, y and z positions on nonlinear model and 23 degrees/s disturbance was implemented to yaw angle at 10 seconds. H_{∞} loop –shaping controller with disturbance on x and y axes, has faster response .It can be seen from simulation results, all three controller are robust against disturbance .H-infinity controller had nearly 50 % overshoot at response of yaw angle to recovery the disturbance. LQG has 20% overshoot to recovery the disturbance at yaw

angel. It can be concluded that LQR has the best robustness against aggressive disturbance.

6. 3. Experiment 3: Effect of Initial Conditions

Assuming that quadrotor has initial values $x_0 = [0 \ 0 \ 0.4 \ 0 \ 0 \ 0 \ 15 \ 0 \ 15 \ 0 \ 15 \ 0]$ yaw angel is 23 degrees(0.4 rad) and x, y positions and height is 15 meters .

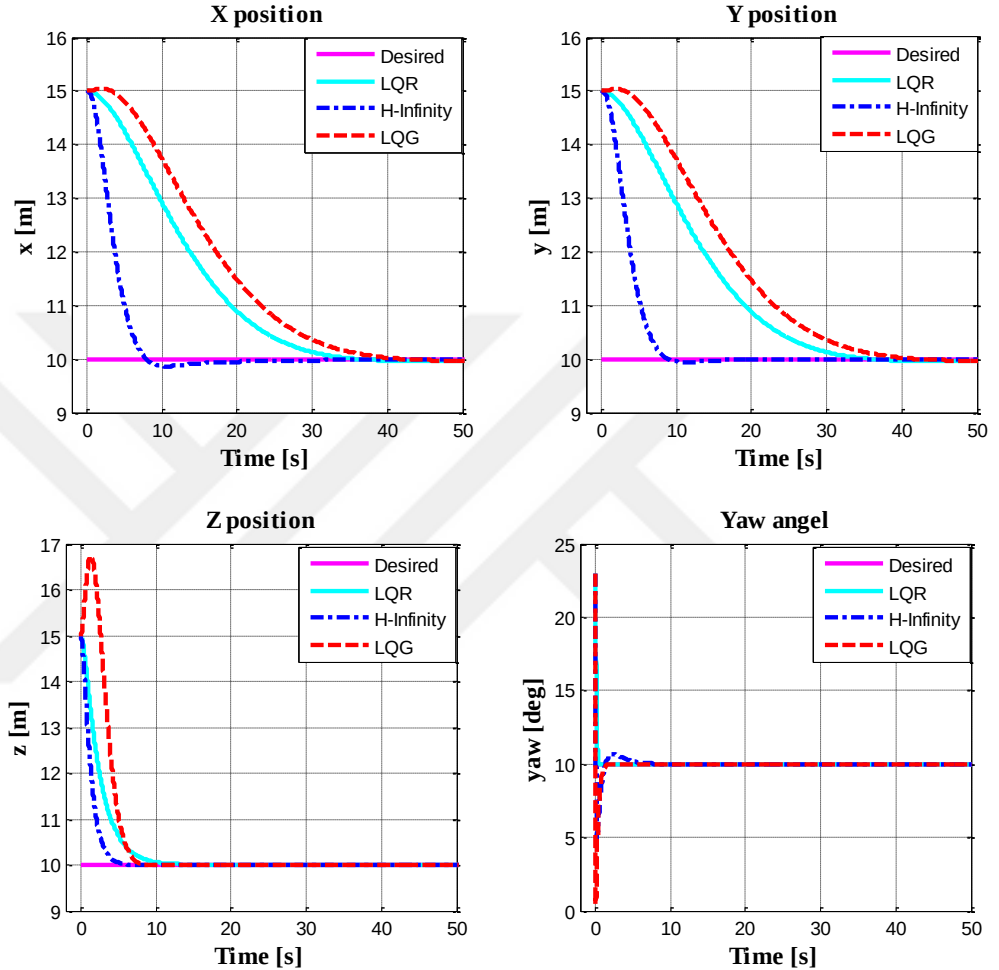


Figure 6.4: Step response of positions on x, y, z axes and yaw angle with initial condition

H_∞ loop-shaping controller has about 1 % overshoot to track x and y positions which is worth obtaining fast response. For initial condition, three controllers can stabilize the quadrotor. However, yaw angle with H_∞ loop-shaping control has about 38 % overshoot to reach the desired target from initial value. Yaw angle with LQG controller has worse (80%) overshoot. At Control of height and heading angel LQG has less robustness than others. LQR has the greatest robustness amongst them.

6. 4. Experiment 4: Effect of parameter uncertainty

Assuming that quadrotor has an increase in mass by 30 percentage of the weight. Quadrotor might carry a camera or an additional payload is attached to the Quadrotor. During H_∞ loop shaping control, robust stability margin showed that H_∞ loop shaping has robustness against at least 50% of parameter uncertainty.

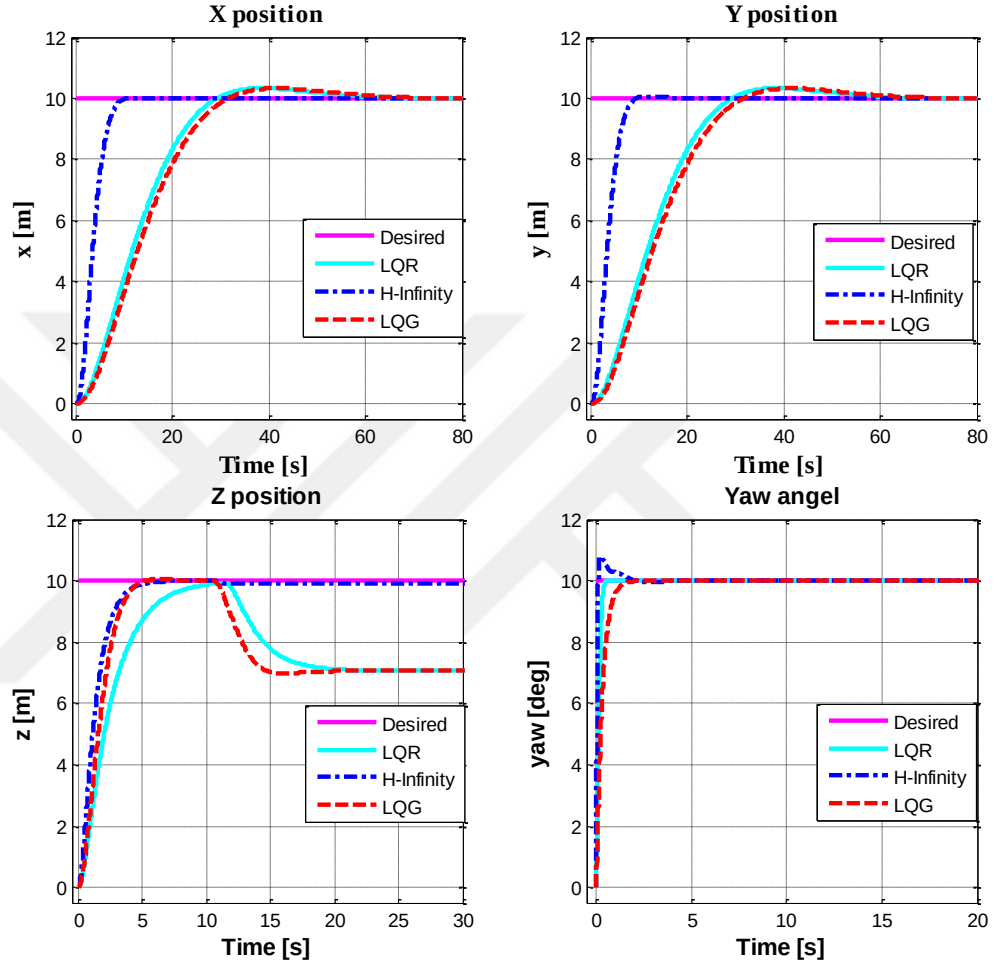


Figure 6.5: Step response of position on x, y, z axes and yaw angle with parameter uncertainty

LQR and LQG controller has similar performance and robustness when the mass parameter is increased manually at about 10 seconds. LQR/G position control on x and y axes has 3% overshoot but H_∞ loop shaping control is not affected by parameter uncertainty. When LQR/G height controllers fail by obtaining 30% steady state errors to stabilise height position of quadrotor, H_∞ loop shaping height control has acceptable state error (0.03%).

6. 5. Experiment 5: Effect of Rotor saturations

Three Controllers without the using of limiters are simulated in order to identify the range input torques and upward lift force for limitation in percentages of maximum values. Control inputs are limited by 50 % producing cut off.

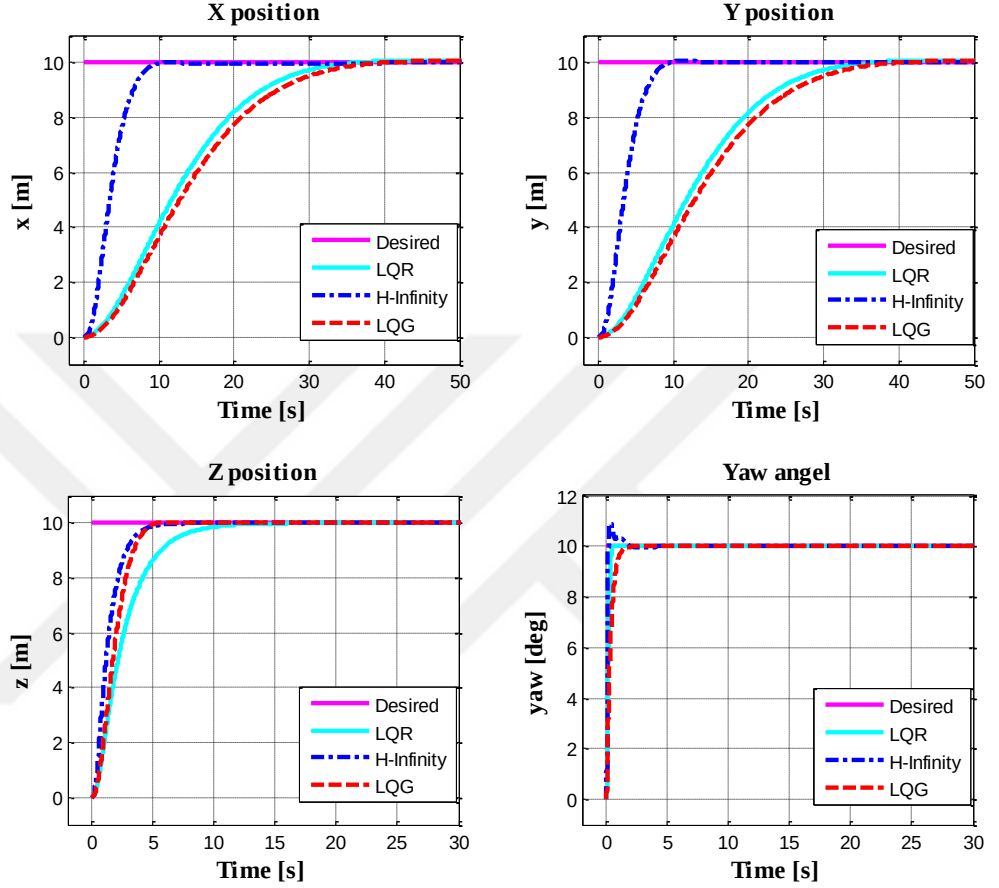


Figure 6.6: Step response of position on x, y, z axes and yaw angel with rotors saturation

Results in Fig. 6.6 shows that overall LQR, LQG and H_{∞} loop shaping control can stabile the Quadrotor even if there are rotor saturations. In spite of the fact that H_{∞} loop-shaping control needs more upward force than other both controllers. This controller does not have any problem to track and stabilise the quadrotor.

6. 6. Experiment 6: Effect of a multiplicative input uncertainty

Aim of this experiment to measure robustness against high frequency disturbance. A sinusoidal signal with 5 magnitude and 0.3 kHz frequency was used to obtain the multiplicative input uncertainty.

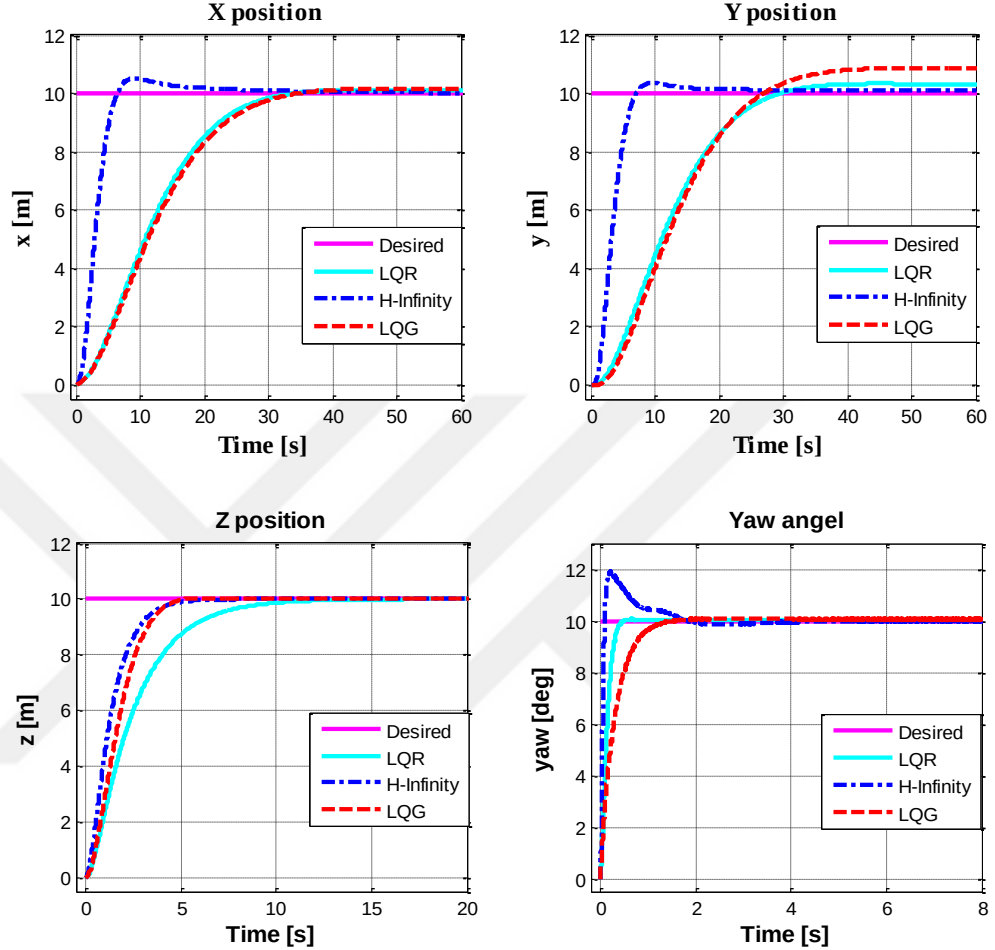


Figure 6.7: Step response of position on x, y, z axes and yaw angle with multiplicative input uncertainty

Although LQR and LQG give steady state error with 3% and 8.3 % respectively to track y position, the steady state error of H_{∞} loop-shaping control is 0.8%. On the other hand, H_{∞} loop-shaping control has biggest overshoot to stabilise yaw angel. LQR and LQG height controllers give steady state error 0.001% and 0.002% respectively. However, H_{∞} loop-shaping control does not have any steady state error to control height of Quadrotor. It can be said that in this experiment H_{∞} loop-shaping control is more successful to control of the Quadrotor in such a uncertainty with high frequency.

7. CONCLUSION and FUTUREWORK

7.1. Conclusion

In this thesis, H_{∞} loop-shaping and optimal LQR /LQG control systems have been designed and compared for robust stabilization and trajectory tracking based on Quadrotor nonlinear model. Highly nonlinear quadrotor model was reviewed without considering any rotor faults. 12 state variables were divided 4 sub-systems and linearized by using Jacobian linearization method. Double control structure was used to control the Quadrotor. Inner loop stabilise attitude and height position of the autonomous quadrotor vehicle. Speed of inner loop was designed slower than outer loop. During design of, H_{∞} loop-shaping and optimal LQR /LQG control systems their performances were analyzed on Quadrotor linear and nonlinear model. It was observed that each controller has similar performances on linear and nonlinear model. Robust stabilization of controllers was tested and compared with each other. LQR technique is known as a very robust control strategy and LQR guarantees infinity gain margin and at least 60° phase margin. However experiments simulations show that H_{∞} loop-shaping control has more robustness against parameter uncertainty and multiplicative input disturbance. H_{∞} Loop-shaping control was quite capable of handling with disturbances and it is expected to be capable of handling with even larger disturbance. Hence, H_{∞} loop-shaping is very good candidate to implement on real Quadrotor.

7.2. Future work

Successful designed H_{∞} loop shaping controller and LQG controller can be tested on the real quadrotor. LQR and LQG showed similar performance so it can be suggested that LQG can be used instead of LQR by optimizing R and Q weights and using integral action because LQR control requests measurable all states so LQR is lack of implementation in real life on such a Quadrotor systems. H_{∞} Loop-shaping control can be designed by considering sensor failure and actuator. Furthermore, Fault detection filters can be designed to detect sensor failure or actuator faults.

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