# COMPARISON OF ROBUST OPTIMIZATION MODELS FOR PORTFOLIO OPTIMIZATION

by

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## ABSTRACT

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Using optimization techniques in portfolio selection has attracted significant attention in financial decisions. However, one of the main challenging aspects faced in optimal portfolio selection is that the models are sensitive to the estimations of the uncertain parameters. In this thesis, we focus on the robust optimization problems to incorporate uncertain parameters into the standard portfolio problems. First, we provide an overview of well-known optimization models when risk measures considered are variance, Value-at-Risk, and Conditional Value-at-Risk. Then, we provide reformulations of the robust versions of these portfolio optimization problems as conic programs when the uncertainty sets involve polytopic, ellipsoidal, or budgeted uncertainty for either mean return vector or covariance matrix or both. Finally, we conduct a computational study on two real data sets to evaluate and compare the effectiveness of the robust optimization approaches.

# ÖZET

PORTFÖY ENIYILEMESI IÇIN GÜRBÜZ ENIYILEME MODELLERININ KARŞILAŞTIRMASI

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ENDÜSTRİ MÜHENDİSLİĞİ YÜKSEK LİSANS TEZİ, AĞUSTOS 2020

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Anahtar Kelimeler: portföy eniyilemesi, gürbüz eniyileme, konik programlama

Portföy seçiminde eniyileme tekniklerinin kullanılması finansal kararlarda büyük ilgi görmüştür. Bununla birlikte, optimal portföy seçiminde karşılaşılan temel zorluklardan biri, modellerin belirsiz parametrelerin tahminlerine duyarlı olmasıdır. Bu tezde, belirsiz parametreleri standart portföy problemine dahil etmek için gürbüz eniyileme problemlerine odaklanıyoruz. İlk olarak, dikkate alınan risk önlemleri varyans, Riske Maruz Değer ve Koşullu Riske Maruz Değer olduğunda, bilinen eniyileme modellerine genel bir bakış sunuyoruz. Ardından, belirsizlik kümeleri, ortalama getiri vektörü veya kovaryans matrisi veya her ikisi için politopik, elipsoidal veya bütçelenmiş belirsizlik içerdiğinde, bu portföy eniyileme problemlerinin gürbüz versiyonlarının konik program olarak yeniden gösterilmesini sağlıyoruz. Son olarak, gürbüz eniyileme yaklaşımlarının etkinliğini değerlendirmek ve karşılaştırmak için iki gerçek veri seti üzerinde sayısal bir çalışma yürütüyoruz.

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# 1. INTRODUCTION

Portfolio selection problem seeks to determine the best investment to be made in a number of risky assets given a certain amount of fund. Due to the uncertain nature of asset returns, future performance of the selected portfolio may have poor outcomes, hence, investors also need to consider the risk associated with their decisions. Therefore, portfolio optimization has become one of the most popular methods used in financial portfolio decisions. In the early years of 1950's, the theory of optimal portfolio selection was developed by Markowitz (1952). According to the theory, the optimal portfolio problem aims to construct a portfolio which achieves maximum expected return with a minimum risk. However, the existence of these two conflicting objectives has become one of the most challenging aspects of the optimal portfolio problem. Thus, risk adjusted models are considered to combine risk and return to present a trade-off.

Although the Markowitz model has been used as a framework to find the optimal portfolios for decades, it suffers from a number of shortcomings. As one of the shortcomings, variance is considered as not an adequate risk measure. Therefore, models with different measures of risks such as Value-at-Risk and Conditional Value-at-Risk are considered in literature. Moreover, despite the importance of the Markowitz model in theory, portfolios determined by this model are sensitive to the estimations of the parameters. In this thesis, we illustrate this sensitivity issue related to the portfolio optimization in Figure 5.1. This example demonstrates that "optimal" decisions obtained from the Markowitz model might have poor performance in an out-of-sample test due to the existence of estimation errors. In order to incorporate estimation errors or data perturbations into the portfolio optimization process, we consider robust optimization. More specifically, we present an analysis of several robust portfolio optimization problems with different uncertainty sets and reformulated the associated two-stage problems into their single-stage conic program equivalents. We also compare the performance of these models using two real datasets from the literature.

The remainder of the thesis is organized as follows: In Chapter 2, we review the related literature on different approaches of portfolio optimization. In Chapter 3, we review some of the well-known optimization problems to compromise the conflicting objectives of the standard optimization problems. In addition to these acknowledged optimization models, we present robust optimization models involving uncertain parameters for a variety of financial risks in Chapter 4. Here, we cast the robust optimization problems as conic program when the uncertainty sets involve polytopic, ellipsoidal or budgeted uncertainty for the parameters. In Chapter 5, we provide our computational experiments on S&P 500 and MIBTEL data sets and compare the resulting optimal portfolios. Finally, we present the concluding remarks of this thesis in Chapter 6.

## 2. LITERATURE REVIEW

In the 1950's, Markowitz (1952) developed the theory of optimal selection of portfolios. Markowitz porfolio optimization problem, also called the mean-variance problem, adopts variance as the risk measure. The paper acknowledges that the expected return is desirable while the variance is undesirable, so there is a trade off between risk and return. Although Markowitz portfolio problem is used as the primary framework, there are many studies that point out its shortcomings. Black & Litterman (1992) show that the decision of optimal portfolio is sensitive to the mean vector and covariance matrix estimations in the classical mean-variance model. The authors point out that even a small change in the mean estimation can result in a large change in the optimal portfolio selection. Best & Grauer (1991) focus on sensitivity of optimal portfolios. Their analysis shows that optimal portfolio weights obtained from the mean-variance model are highly sensitive to changes in asset means. Chopra & Ziemba (2013) also focus on the effect of errors in inputs on optimal portfolio. They show that although errors in mean have extreme effects on optimal portfolio, errors in variance and covariance matrix also affect the optimal portfolio choice. Broadie (1993) investigates the effects of estimation errors of parameters on the results of the mean-variance model using simulation. According to the paper, using estimated parameters can cause significant errors in efficient frontiers due to the error-maximization property and the estimated frontier results with larger errors than the actual frontier.

In the literature, there are many different approaches on portfolio optimization to resolve the sensitivity issue experiences in mean-variance optimal portfolio problems due to slight changes in inputs. Frost & Savarino (1988) suggest to conduct Bayesian estimation of mean and covariance to reduce the errors in estimation and improve the portfolio performance. Even though their approach reduces the sensitivity of the parameter estimates, it does not provide any optimality guarantee on the portfolio. Black & Litterman (1990) propose an approach to overcome the sensitivity issue by combining the classical Markowitz approach with a prior information on the market. Thus, investors can combine and compare their view for currencies and fixed-income securities' expected returns, which are generated by using International Capital Asset Pricing Model (ICAPM) equilibrium. Idzorek (2007) claims that the shortcomings of Markowitz portfolio optimization can be overcome by the Black–Litterman model. The author gives information about relatively few works related to the Black-Litterman model and combine them step by step. This paper introduces a new method to control tilts and the final portfolio weights caused by views.

All the studies mentioned above investigate the problem with no consideration on time varying effect on data. Different from these studies, some construct weighted portfolio problems to focus on time varying effect on data. According to Perry (2010), the Markowitz model generally uses the historical data as equally weighted but this model neglects the current market conditions. In order to take current market conditions into consideration more accurately, this paper uses exponentially weighted moving average (EWMA). In their approach, EWMA assigns weight factors to data points that decrease exponentially for the older observations. Lee & Stevenson (2003) also use time weighted returns in order to take current market conditions into consideration more accurately. In order to apply weights to the data, their method uses the length of historical estimation period and forms a Fisher distributed lag model. For the optimal portfolio selection, Horasanlı & Fidan (2007) consider exponentially weighted moving averages and generalize autoregressive conditional heteroscedasticity techniques.

## 2.1 Portfolio Optimization with Robust Approaches

Robust optimization, which considers uncertainty in parameters, is a suitable approach to construct optimal portfolios. Ben-Tal & Nemirovski (1998) has a significant effect on the robust optimizations' progress. Throughout the years, different researchers show that the robust optimization can incorporate the perturbations in the parameters into the optimization process and avoid infeasible solutions. For example, Bertsimas, Brown & Caramanis (2011) study the robust optimization theory by focusing on the computational attractiveness and applicability of approaches. As one of the applications of robust optimization, they develop uncertainty models for mean return and covariance in portfolio selection. Goldfarb & Iyengar (2003) also consider robust portfolio selection problems and develop a second-order cone program that captures the uncertainty structures for market parameters. Lobo &

Boyd (2000) focus on the worst-case analysis and robustness. The authors construct the problem of minimizing the worst-case variance with box and ellipsoidal uncertainty sets for mean and covariance. They show that computing the worst-case variance can be formulated as semidefinite programming problem and it can be efficiently computed by using interior point methods. Ceria & Stubbs (2006) explore the negative effect of estimation error on mean-variance optimal portfolios. The authors show that estimation errors in expected returns can cause optimal portfolios to significantly overestimate the *true* optimal portfolio. They propose to use robust optimization in order to decrease the sensitivity of asset weights in mean-variance optimal portfolios against slight changes in input parameters and constraints. Their analysis shows that the efficient frontier of the portfolios obtained by using robust optimization can be closer to the true efficient frontier and the realized returns can be better. Instead of focusing on mean vector, DeMiguel & Nogales (2009) propose a robust approach for portfolio selection problem by using robust estimators. Their method is performed by solving a single nonlinear program and show that their solution to the portfolio construction problem has better stability. Tütüncü & Koenig (2004) focus on finding optimal asset allocation under the worst possible realizations of the uncertain inputs. They construct componentwise uncertainty sets for the mean return vector and the covariance matrix. Their paper formulates resulting problem as a saddle-point problem with semidefinite constraints.

### 2.2 Portfolio Optimization with Value-at-Risk

Despite the fact that the mean-variance portfolio problem is accepted as the primary framework, it is argued that the variance is not an adequate measure of risk. Instead of variance, Value at Risk (VaR) is embraced as a better risk measure for downside risk in a portfolio. Goldfarb & Iyengar (2003) consider the robust VaR portfolio selection problem and recast it as a second-order cone program under normal distribution. This paper proposes models that are robust to parameter uncertainty and estimation errors. The authors conducted two types of tests; performance on simulated data and sample path performance on real market data. In order to avoid user-defined parameters, they select both the classical and robust portfolios by maximizing the Sharpe ratio. Ghaoui, Oks & Oustry (2003) investigate the robust portfolio optimization using worst-case VaR. They assume that the distribution is only partially known with information on the mean vector and covariance matrix are available through box and ellipsoidal sets. Their approach computes and optimizes the worst-case VaR, the largest VaR accessible. Models are cast as semidefinite programs along with the uncertainty in moments, factor models, support constraints and relative entropy information.

### 2.3 Portfolio Optimization with Conditional Value-at-Risk

Although VaR is a popular measure of risk, it suffers from its shortcoming as instability and difficult to work with different distributions than normal distribution. Therefore, many researchers have taken Conditional Value-at-Risk (CVaR), which is defined as the mean of the tail distribution exceeding VaR, into consideration due to its desirable properties. Moreover, VaR is neither a coherent risk measure (since it is not subadditive) nor a convex function (Artzner, Delbaen, Eber & Heath, 1999; Pflug, 2000) Rockafellar & Uryasev (2002); Rockafellar, Uryasev & others (2000) introduce an approach on optimizing or hedging a portfolio by minimizing CVaR. Although they focus on minimizing CVaR instead of VaR, they indicate that portfolios with low CVaR have also low VaR. They show that using linear programming and nonsmooth optimization, CVaR can be minimized efficiently.

Krokhmal, Palmquist & Uryasev (2002) focus on extending the approach for optimization of CVaR to solve optimization problems with CVaR constraints. Instead of minimizing CVaR, they suggest to maximize expected returns with a set of constraints on CVaR. The authors show that by using multiple CVaR constraints with different confidence levels, loss distribution can be changed. Their approach provides a Monte Carlo simulation to avoid making assumptions on distribution. Therefore, the approach can be used for large number of instruments and scenarios. Furthermore, the comparison with the standard mean-variance approach shows that using CVaR in constraints for given expected returns results with smaller risk than the mean-variance approach. Zhu & Fukushima (2009) propose a robust portfolio problem where they consider the worst-case CVaR with partial information on the underlying probability distribution. The authors formulate the portfolio problem either as linear or second-order cone program depending on which type of uncertainty (box or ellipsoidal) set is considered. This paper shows that the larger risk is usually rewarded by a higher return. As a result of their market data simulation and Monte Carlo simulation, the author argue that the risk increases as the value of the uncertainty parameter increases. Although a robust portfolio policy usually depends on the structure of the uncertainty set, they claim that their approach has more flexibility in portfolio selection. Different from the studies above, Pang & Karan (2018) prefer to use multi-variate elliptical distributions rather than the multi-variate normal distribution on returns to analyze the non-normal behavior of data. The portfolio problem is constructed as the Black-Litterman model with upper bound on the risk measure CVaR. Kocuk & Cornuéjols (2020) consider the portfolio optimization that minimizes the Conditional Value-at-Risk under a mixture of normal distribution. In order to incorporate market information into a portfolio, they propose a Black-Litterman approach using an inverse optimization framework. Their approach show that the portfolio risk can be reduced while achieving similar returns with the classical market-based approaches.

# 3. STANDARD OPTIMIZATION MODELS

In this chapter, we review the standard optimization problems for portfolio construction. We assume throughout this chapter that the following pieces of information are known and given:

- $n$ is the total number of assets.
- $\mu$ is mean vector of returns of $n$ assets.
- $\Sigma$ is positive semidefinite covariance matrix of returns of $n$ assets.

In this chapter, decision variable $x$ denotes a portfolio vector.

In an ideal situation, the aim of an investor is to achieve minimum risk and maximum expected return. However, since these two objectives might be conflicting, a compromise has to be made. We will now review some of the well-known optimization problems proposed for this purpose.

## 3.1 Markowitz Model

The theory of optimal selection of portfolios is developed by Markowitz (1952). Markowitz porfolio optimization problem, also called mean-variance problem, adopts variance as the risk measure. The theory presents a trade-off between risk and return as follows.

$$\min_{x} \; x^T \Sigma x - \tau \mu^T x \tag{3.1a}$$

$$\text{s.t.} \; e^T x = 1 \tag{3.1b}$$

$$x \geq 0. \tag{3.1c}$$

The first and the second parts of the objection function (3.1a) refer to risk, which is measured by the variance of the return, and the expected return of the portfolio, respectively. Since minimizing risk and maximizing expected return at the same time might be conflicting, the expected return is multiplied with a constant factor $\tau > 0$ to combine risk and return into a single objective function. Here, $\frac{1}{\tau}$ is a risk-aversion constant used to quantify the trade-off between the expected return and risk. The first constraint (3.1b) corresponds to the summation of the proportions of the total funds invested in portfolio vector $x_i$ equals to 1. In order to prevent short sales, we also introduce the non-negativity constraint (3.1c).

## 3.2 Worst-Case Value-at-Risk Model

Value-at-risk (VaR) is a statistical measure which quantifies the level of risk within a portfolio. Cornuejols & Tütüncü (2005) exhibits the general definition of $\alpha$-level VaR as where $X$ is a random variable that stands for loss from a portfolio for a certain period of time. Moreover, it is stated that the negative value of X represents return of a portfolio. The following equation gives the formulation of the $\alpha$-level VaR where $\alpha \in (0, 1)$ :

$$\text{VaR}_\alpha(X) := \min\{\gamma : P(X \geq \gamma) \leq 1 - \alpha\}. \tag{3.2}$$

Throughout the years, many studies have focused on optimizing the portfolio by using VaR as the risk measure. As discussed in the literature review, VaR is seen as a 'better' risk measure than variance since it is directly related to the quantification of the loss of a portfolio. We use the formulation of the worst-case VaR as the largest VaR attainable problem from Ghaoui et al. (2003), which does not require any distributional assumption.

$$\text{(3.3a)} \qquad \min \; K(\alpha)\sqrt{x^T\Sigma x} - \mu^T x$$
$$\text{s.t.} \; (3.1b) - (3.1c).$$

Independent from any assumptions on distribution to the random returns, the objective function (3.3a) refers to largest value that can be assigned to $VaR_\alpha(X)$. Thus, this defines an upper bound on VaR. The risk factor $K(\alpha)$ is defined as $K(\alpha) = \sqrt{\left(\frac{1-\alpha}{\alpha}\right)}$, proposed in the papers Ghaoui et al. (2003) and Bertsimas & Popescu (2002) for finding the upper bound on VaR.

### 3.3 Conditional Value-at-Risk Model

Although VaR is a popular risk measure, it is neither coherent nor convex in general. Instead, many practitioners prefer to use Conditional Value-at-Risk (CVaR), which has these two desirable features.

One can obtain CVaR, also called the expected loss given that the loss exceeds VaR, using the following formula:

$$\text{(3.4)} \qquad \mathrm{CVaR}_\alpha(X) = -\mathrm{E}\left[X \mid X \leq -\mathrm{VaR}_\alpha(X)\right].$$

Here, the random variable $X$ represents the return of a portfolio investment.

As an example of a portfolio optimization problem involving CVaR, let us assume that the return vector is distributed as a multivariate normal with parameters $\mu$ and $\Sigma$, and replace the variance term in the Markowitz model with CVaR. Then, we obtain the following convex program:

$$\text{(3.5a)} \qquad \min_x \; \left(-\mu^T x + \frac{\phi(\Phi^{-1}(\alpha))}{\alpha}\sqrt{x^T\Sigma x}\right) - \tau\mu^T x$$
$$\text{s.t.} \; (3.1b) - (3.1c).$$

Here, $\phi$ and $\Phi$ are the probability density function (pdf) and cumulative distribution function (cdf) of the standard normal distribution, and $\alpha$ is a predetermined constant. The objective function (3.5a) refers to minimizing the CVaR as a risk measure and combining the risk with the expected return for some $\tau > 0$.

We note that there may not exist a closed form expression for CVaR under an arbitrary distribution.

# 4. ROBUST OPTIMIZATION MODELS

Robust optimization considers uncertainty in problem parameters. One can describe the uncertainty of the parameters by defining uncertainty sets. Our main motivation to use robust optimization approaches in portfolio optimization is to overcome the sensitivity problems caused by the uncertainty in data. To simply put, we have no complete knowledge on parameters of portfolio problem in real life. Therefore, estimating these unknown parameters can result in errors which have negative effects on the *optimal* portfolios obtained through optimization.

The purpose of this chapter is to present an analysis of robust portfolio optimization problems involving uncertain parameters. We show how to build robust portfolio problems where objective function has robustness and minimizes the risk with a trade-off between risk and return. Even tough we cannot know the exact value of true parameters in reality, we also cannot expect to solve portfolio optimization problems with high accuracy with fully unknown parameters (Lobo & Boyd (2000)). Therefore, we build the two-stage problems where the parameters are partially known with different uncertainty sets and then obtain their single-stage equivalents as conic programs.

Throughout this chapter, we will denote the sample mean and sample covariance as $\hat{\mu}$ and $\hat{\Sigma}$, respectively.

Table 4.1 Portfolio Optimization Models with Uncertainty Sets.

| Uncertainty Set \ Optimization Model | Markowitz (1952) | Worst Case VaR (Ghaoui et al. (2003)) | CVaR Mixture (Kocuk & Cornuéjols (2020)) |
|---|---|---|---|
| Polyhedral for mean | Lobo & Boyd (2000) | Ghaoui et al. (2003) | |
| Budgeted for mean (Ben-Tal & Nemirovski (2001), Bertsimas & Sim (2004)) | | | |
| Ellipsoidal for mean | Ceria & Stubbs (2006), Lobo & Boyd (2000) | Ghaoui et al. (2003) | |
| Covariance | Lobo & Boyd (2000) | | |
| Mean-Covariance | Lobo & Boyd (2000) | Ghaoui et al. (2003) | |

We note that Table 4.1 shows some well-known portfolio optimization models and uncertainty sets for the parameters considered in this thesis.

## 4.1 Markowitz Model

Let us recall problem (3.1) we stated as in the Markowitz framework that combines the expected return and variance in the objective function. In order to incorporate robustness into the objective function, we consider the following general form of a two-stage problem:

$$\min_{x} \quad \max_{(\Sigma,\mu)\in\mathcal{S}} \quad x^T\Sigma x - \tau\mu^T x \tag{4.1a}$$

$$\text{s.t.} \quad (3.1\text{b}) - (3.1\text{c})$$

Here, $\mathcal{S}$ denotes the uncertainty set and $\tau$ is a given positive number. In the sequel, we reformulate problem (4.1) as a single-stage conic program when the uncertainty set $\mathcal{S}$ involves polytopic, budgeted or ellipsoidal uncertainty for either mean $\mu$ or covariance $\Sigma$ or both.

### 4.1.1 Polyhedral Uncertainty for Mean

In this section, we first look at a generic polyhedral uncertainty for mean $\mu$ while assuming that $\Sigma$ is known (or estimated from data) as $\hat{\Sigma}$ and $\bar{\mu} = \hat{\mu}$ (although other choices are allowed). In particular, let us consider the following uncertainty set:

$$\mathcal{S} := \{(\mu, \Sigma) : A\mu \leq b,\ \Sigma = \hat{\Sigma}\},$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^n$ are given.

The inner problem in (4.1) can be written as below:

$$\max_{\mu} -\mu^T x \tag{4.2a}$$

$$\text{s.t. } A\mu \leq b. \qquad : \lambda \tag{4.2b}$$

Let us associate a dual variable $\lambda$ with primal constraint (4.2b) and obtain the dual problem as follows:

$$\min_{\lambda}\ \lambda^T b \tag{4.3a}$$

$$\text{s.t. } A^T \lambda = -x \tag{4.3b}$$

$$\lambda \geq 0. \tag{4.3c}$$

Assuming that the strong duality holds between problems (4.2) and (4.3), we obtain a single-stage reformulation of problem (4.1) as the following convex quadratic problem:

$$\min_{x,\lambda}\ \{x^T \Sigma x + \tau \lambda^T b : A^T \lambda = -x, e^T x = 1, \lambda, x \geq 0\}. \tag{4.4}$$

Next, we focus on a special case of polyhedral uncertainty, which we will refer to as *box uncertainty.* In particular, let us consider the following uncertainty set:

$$S := \{(\mu, \Sigma) : \|\mu - \bar{\mu}\|_\infty \leq \Upsilon,\ \Sigma = \hat{\Sigma}\},$$

where $\Upsilon$ is a positive scalar controlling the robustness level.

Firstly, the inner problem of (4.1) can be written as follows:

$$\max_{\mu} -\mu^T x \tag{4.5a}$$

$$\text{s.t. } \|\mu - \bar{\mu}\|_\infty \leq \Upsilon. \tag{4.5b}$$

Since constraint (4.5b) involves the infinity-norm, we linearize the inner problem as below:

$$\max_{\mu} -\mu^T x \tag{4.6a}$$

$$\text{s.t. } \mu - \bar{\mu} \leq \Upsilon e \qquad : \lambda^+ \tag{4.6b}$$

$$-\mu + \bar{\mu} \leq \Upsilon e. \qquad : \lambda^- \tag{4.6c}$$

Here, $e$ is defined as the vector of ones. By introducing the dual variables $\lambda^+$ and $\lambda^-$ for constraints (4.6b) and (4.6c) respectively, we obtain the dual problem as follows:

$$\min_{\lambda^+, \lambda^-} \ \lambda^{+T}(\Upsilon + \bar{\mu}) + \lambda^{-T}(\Upsilon - \bar{\mu}) \tag{4.7a}$$

$$\text{s.t. } \lambda^+ - \lambda^- = -x \tag{4.7b}$$

$$\lambda^+, \lambda^- \geq 0. \tag{4.7c}$$

Since strong duality always holds between problems (4.6) and (4.7) (due to the fact the feasible region of the primal problem is a nonempty polytope), we obtain the single-stage reformulation of problem (4.1) as follows:

$$\min_{x,\lambda^+,\lambda^-} \quad \{x^T\Sigma x + \tau(\lambda^{+T}(\Upsilon + \bar{\mu}) + \lambda^{-T}(\Upsilon - \bar{\mu})) :$$

$$\lambda^+ - \lambda^- = -x, e^T x = 1, x, \lambda^+, \lambda^- \geq 0\}. \tag{4.8}$$

Note that the problem (4.8) is again a convex quadratic optimization problem.

### 4.1.2 Budgeted Uncertainty for Mean

Here, we consider a *budgeted uncertainty* (Ben-Tal & Nemirovski (2001)), which can be treated as the intersection of the infinity-norm and the 1-norm for mean $\mu$, while assuming that $\Sigma$ is known (or estimated from data) as $\hat{\Sigma}$ and $\bar{\mu} = \hat{\mu}$ (although other choices are allowed). In particular, let us consider the following uncertainty set:

$$\mathcal{S} := \left\{ (\mu, \Sigma) : \sum_{j=1}^{n} \frac{|\mu_j - \bar{\mu_j}|}{\bar{\mu_j}} \leq \Upsilon,\ \Sigma = \hat{\Sigma} \right\},$$

where $\Upsilon$ is a positive scalar controlling the robustness level.

Assuming that $\bar{\mu}$ is a positive vector, we first write the inner problem of (4.1) as follows:

$$\max_{\mu} \quad -\mu^T x \tag{4.9a}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} \frac{|\mu_j - \bar{\mu_j}|}{\bar{\mu_j}} \leq \Upsilon. \tag{4.9b}$$

Since constraint (4.9b) involves absolute values, we linearize the inner problem as follows:

$$\max_{\mu} \quad -\mu^T x \tag{4.10a}$$

$$\text{s.t.} \quad \frac{\mu_j - \bar{\mu_j}}{\bar{\mu_j}} \leq u_j \qquad j = 1, ..., n \quad : u_j^+ \tag{4.10b}$$

$$\frac{\mu_j - \bar{\mu_j}}{\bar{\mu_j}} \leq -u_j \qquad j = 1, ..., n \quad : u_j^- \tag{4.10c}$$

$$\text{(4.10d)} \qquad \sum_{j=1}^{n} u_j \leq \Upsilon. \qquad : \gamma$$

By introducing the dual variables $u_j^+, u_j^-$, and $\gamma$ for constraints (4.10b), (4.10c), and (4.10d) respectively, we obtain the dual problem as the following form:

$$\text{(4.11a)} \qquad \min \; e^T(u^+ - u^-) + \gamma\Upsilon$$

$$\text{(4.11b)} \qquad \text{s.t.} \; \frac{1}{\bar{\mu_j}} u^+ - \frac{1}{\bar{\mu_j}} u^- = -x$$

$$\text{(4.11c)} \qquad \gamma e - u^+ - u^- = 0$$

$$\text{(4.11d)} \qquad \gamma, u^+, u^- \geq 0$$

Finally, we reformulate the single-stage equivalent of problem (4.1) as the following convex quadratic problem:

$$\min_{x,\gamma,u^+,u^-} \Big\{ (x^T \Sigma x + \tau(e^T(u^+ - u^-) + \gamma\Upsilon)) :$$

$$\text{(4.12)} \qquad e^T x = 1, \frac{1}{\bar{\mu_j}} u_j^+ - \frac{1}{\bar{\mu_j}} u_j^- = -x_j, \gamma e^T - u^+ - u^- = 0, \gamma, u^+, u^- x \geq 0 \Big\}.$$

### 4.1.3 Ellipsoidal Uncertainty for Mean

In this section, an *ellipsoidal uncertainty* is considered for mean $\mu$ while assuming that $\Sigma$ is known (or estimated from data) as $\hat{\Sigma}$. Let us consider the following uncertainty set specifically:

$$\mathcal{S} := \{(\mu, \Sigma) : (\mu - \bar{\mu})^T \Omega^{-1} (\mu - \bar{\mu}) \leq \Upsilon^2, \; \Sigma = \hat{\Sigma}\},$$

where $\Upsilon$ is a positive scalar controlling the robustness level and $\Omega \succ 0$ is given (in the computational experiments, we will take $\Omega = \hat{\Sigma}$ and $\bar{\mu} = \hat{\mu}$ although other choices are also allowed).

Firstly, we write the inner problem of (4.1) as below:

$$\max_{\mu} \; -\mu^T x \tag{4.13a}$$

$$\text{s.t.} \; \|\Omega^{-1/2}(\mu - \bar{\mu})\|_2 \leq \Upsilon. \tag{4.13b}$$

Let us introduce $\mu' := \Omega^{1/2}(\mu - \bar{\mu})$ and rewrite the problem (4.13) as follows:

$$-x^T\bar{\mu} + \max_{\mu'} \; -(x^T\Omega^{1/2})\mu' \tag{4.14a}$$

$$\text{s.t.} \; \|\mu'\|_2 \leq \Upsilon. \tag{4.14b}$$

By using Karush-Kuhn-Tucker (KKT) optimality conditions, we obtain the optimal value of problem (4.14) as below:

$$-x^T\bar{\mu} + \|\Upsilon\Omega^{1/2}x\|_2.$$

Finally, we obtain a single-stage reformulation of problem (4.1) as the following:

$$\min_x \{x^T\Sigma x - \tau x^T\bar{\mu} + \tau\|\Upsilon\Omega^{1/2}x\|_2 : e^T x = 1, x \geq 0\}. \tag{4.15}$$

Note that problem (4.15) is a convex program. In particular, it can be recast as a second-order cone program.

### 4.1.4 Uncertainty for Covariance Matrix

Now, we look at the case in which the covariance matrix $\Sigma$ is uncertain while assuming that $\mu$ is known (or estimated from data) as $\hat{\mu}$. In particular, let us consider the following uncertainty set:

$$\mathcal{S} := \{(\mu, \Sigma) : (1-\beta)\hat{\Sigma} \preceq \Sigma \preceq (1+\beta)\hat{\Sigma}, \; \mu = \hat{\mu}\},$$

where the matrix $\hat{\Sigma}$ is the sample covariance estimated from data and the scalar $\beta \in [0,1]$ controls the robustness level.

The inner problem of (4.1) can be written as follows:

$$\max_{\Sigma \succeq 0} \ \mathrm{Tr}(xx^T\Sigma) \tag{4.16a}$$

$$\text{s.t. } \Sigma \preceq (1+\beta)\hat{\Sigma} \qquad : \Lambda^+ \tag{4.16b}$$

$$-\Sigma \preceq -(1-\beta)\hat{\Sigma} \qquad : \Lambda^- \tag{4.16c}$$

By associating dual variables $\Lambda^+$ and $\Lambda^-$ with constraints (4.16b) and (4.16c) respectively, we obtain the dual problem as below:

$$\min_{\Lambda^+,\Lambda^-} \ (1+\beta)\mathrm{Tr}(\hat{\Sigma}\Lambda^+) - (1-\beta)\mathrm{Tr}(\hat{\Sigma}\Lambda^-) \tag{4.17a}$$

$$\text{s.t. } \begin{bmatrix} \Lambda^+ - \Lambda^- & x \\ x^T & 1 \end{bmatrix} \succeq 0 \tag{4.17b}$$

$$\Lambda^+, \Lambda^- \succeq 0. \tag{4.17c}$$

We note that we use the Schur's Complement Lemma to rewrite the constraint $\Lambda^+ - \Lambda^- \succeq xx^T$ as in (4.17b).

Finally, the single-stage problem is written as follows:

$$\min_{x,\Lambda^+,\Lambda^-} \Bigg\{ (1+\beta)\mathrm{Tr}(\hat{\Sigma}\Lambda^+) - (1-\beta)\mathrm{Tr}(\hat{\Sigma}\Lambda^-) - \tau x^T\bar{\mu}) :$$

$$e^Tx = 1, \begin{bmatrix} \Lambda^+ - \Lambda^- & x \\ x^T & 1 \end{bmatrix} \succeq 0, \Lambda^+, \Lambda^- \succeq 0, x \geq 0 \Bigg\}. \tag{4.18}$$

Note that problem (4.18) is a semidefinite program.

### 4.1.5 Uncertainty for Mean and Covariance Matrix

In this section, we consider uncertainty for both covariance matrix $\Sigma$ and mean vector $\mu$. Our uncertainty set involves ellipsoidal uncertainty for $\mu$ and the upper and lower bounds for $\Sigma$ (in matrix sense). We particularly consider the following uncertainty set:

$$\mathcal{S} := \{(\mu, \Sigma) : (\mu - \bar{\mu})^T \Sigma^{-1} (\mu - \bar{\mu}) \leq \Upsilon^2,\ (1-\beta)\hat{\Sigma} \preceq \Sigma \preceq (1+\beta)\hat{\Sigma}\},$$

where the matrix $\hat{\Sigma}$ and $\hat{\mu}$ are the sample covariance and mean estimated from data, $\beta \in [0,1]$ and $\Upsilon$ are positive scalars, controlling the robustness level.

Firstly, the inner problem of (4.1) can be written as follows:

$$\max_{\Sigma,\mu} \ \mathrm{Tr}(xx^T\Sigma) - \tau\mu^T x \tag{4.19a}$$

$$\text{s.t.} \ \Sigma \preceq (1+\beta)\hat{\Sigma} \tag{4.19b}$$

$$-\Sigma \preceq -(1-\beta)\hat{\Sigma} \tag{4.19c}$$

$$\begin{bmatrix} \Sigma & \mu - \bar{\mu} \\ \mu^T - \bar{\mu}^T & \Upsilon^2 \end{bmatrix} \succeq 0. \tag{4.19d}$$

Note that (4.19d) is constructed by using Schur's Complement. We rewrite the above problem in the canonical conic program form as follows:

$$\max_{\Sigma,\mu} \ \mathrm{Tr}(xx^T\Sigma) - \tau\mu^T x \tag{4.20a}$$

$$\text{s.t.} \ \Sigma \preceq (1+\beta)\hat{\Sigma} \qquad : \Lambda^+ \tag{4.20b}$$

$$-\Sigma \preceq -(1-\beta)\hat{\Sigma} \qquad : \Lambda^- \tag{4.20c}$$

$$\begin{bmatrix} -\Sigma & -\mu \\ -\mu^T & 0 \end{bmatrix} \preceq \begin{bmatrix} 0 & -\bar{\mu} \\ -\bar{\mu}^T & \Upsilon^2 \end{bmatrix}. \qquad : \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \tag{4.20d}$$

After associating the dual variables to the primal constraints in problem (4.20), we obtain the dual problem as follows:

$$\min_{\Lambda^+,\Lambda^-,\gamma} \ (1+\beta)\mathrm{Tr}(\hat{\Sigma}\Lambda^+) - (1-\beta)\mathrm{Tr}(\hat{\Sigma}\Lambda^-) + \Upsilon^2\gamma_{22} - 2\bar{\mu}\gamma_{12} \tag{4.21a}$$

$$\text{s.t.} \ \begin{bmatrix} \Lambda^+ - \Lambda^- - \gamma_{11} & x \\ x^T & 1 \end{bmatrix} \succeq 0 \tag{4.21b}$$

(4.21c) $$-2\gamma_{12} = -\tau x$$

(4.21d) $$\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \succeq 0$$

(4.21e) $$\Lambda^+, \Lambda^-, \gamma \succeq 0.$$

Finally, we reformulate the single-stage equivalent of problem (4.1) as the the following semidefinite program:

(4.22a) $$\min_{x,\Lambda^+,\Lambda^-,\gamma} \; (1+\beta)\mathrm{Tr}(\hat{\Sigma}\Lambda^+) - (1-\beta)\mathrm{Tr}(\hat{\Sigma}\Lambda^-) + \Upsilon^2\gamma_{22} - 2\bar{\mu}\gamma_{12}$$

$$\text{s.t.} \; (4.21b) - (4.21e)$$

$$(3.1b) - (3.1c).$$

## 4.2 Worst Case Value-at-Risk Model

In this section, we focus on the robust optimization version of the worst-case VaR model presented in Section 3.2. We will assume that the parameters of problem (3.3), that is $\mu$ and $\Sigma$, are uncertain. In order to incorporate robustness into the objective function, we formulate the following general form of a two-stage problem:

(4.23a) $$\min_{x} \; \max_{(\mu,\Sigma)\in\mathcal{S}} \; K(\alpha)\sqrt{x^T\Sigma x} - \mu^T x$$

$$\text{s.t.} \; (3.1b) - (3.1c).$$

In the sequel, we reformulate problem (4.23) as a single-stage conic program when the uncertainty set $\mathcal{S}$ involves polytopic, budgeted or ellipsoidal uncertainty for either mean $\mu$ or covariance $\Sigma$ or both.

### 4.2.1 Polyhedral Uncertainty for Mean

In this section, we consider box uncertainty as a special case of polyhedral uncertainty for mean $\mu$ while covariance $\Sigma$ is assumed to be known (or estimated from data) as $\hat{\Sigma}$ and $\bar{\mu} = \hat{\mu}$ (although other choices are allowed). Now, let us consider the following uncertainty set:

$$\mathcal{S} := \{(\mu, \Sigma) : \|\mu - \bar{\mu}\|_\infty \leq \Upsilon,\ \Sigma = \hat{\Sigma}\},$$

where $\Upsilon$ is a positive scalar controlling the robustness level.

Firstly, we write the inner problem of (4.23) as the following:

$$\max_{\mu} - \mu^T x \tag{4.24a}$$

$$\text{s.t. } \|\mu - \bar{\mu}\|_\infty \leq \Upsilon. \tag{4.24b}$$

Since problem (4.24) is identical to the inner problem in Section 4.1.1, the remaining derivations regarding the reformulation are also the same. Therefore, we directly give the single-stage reformulation of problem (4.23) as follows:

$$\min_{x,\lambda^+,\lambda^-} \{K(\alpha)\sqrt{x^T \Sigma x} + (\lambda^{+T}(\Upsilon + \bar{\mu}) + \lambda^{-T}(\Upsilon - \bar{\mu})) :$$
$$\lambda^+ - \lambda^- = -x, e^T x = 1, x, \lambda^+, \lambda^- \geq 0\}. \tag{4.25}$$

Note that the problem (4.25) is a convex optimization problem.

### 4.2.2 Budgeted Uncertainty for Mean

In this section, we consider the budgeted uncertainty for mean vector $\mu$, assuming that covariance matrix $\Sigma$ is known (or estimated from data) as $\hat{\Sigma}$ and $\bar{\mu} = \hat{\mu}$. Here, let us consider the following uncertainty set particularly:

$$\mathcal{S} := \left\{ (\mu, \Sigma) : \sum_{j=1}^{n} \frac{|\mu_j - \bar{\mu_j}|}{\bar{\mu_j}} \leq \Upsilon,\ \Sigma = \hat{\Sigma} \right\},$$

where $\Upsilon$ is a positive scalar controlling the robustness level.

Firstly, we write the inner problem of (4.23) as follows:

$$\max_{\mu} \ -\mu^T x \tag{4.26a}$$

$$\text{s.t.} \ \sum_{j=1}^{n} \frac{|\mu_j - \bar{\mu_j}|}{\bar{\mu_j}} \leq \Upsilon. \tag{4.26b}$$

Note that the inner problem in Section 4.1.2 is identical to problem (4.26) due to the consideration of the same uncertainty set for the mean vector. Hence, we omit the rest of the derivations and provide the single-stage reformulation of problem (4.23) as the following convex optimization problem:

(4.27)

$$\min_{x,\gamma,u^+,u^-} \Big\{ K(\alpha)\sqrt{x^T \Sigma x} + (e^T(u^+ - u^-) + \gamma \Upsilon) :$$

$$e^T x = 1, \frac{1}{\bar{\mu_j}} u_j^+ - \frac{1}{\bar{\mu_j}} u_j^- = -x_j, \gamma e^T - u^+ - u^- = 0, \gamma, u^+, u^- x \geq 0 \Big\}.$$

### 4.2.3 Ellipsoidal Uncertainty for Mean

Here, we consider an ellipsoidal uncertainty for mean $\mu$ while assuming that covariance $\Sigma$ is known (or estimated from data) as $\hat{\Sigma}$. Let us consider the following uncertainty set specifically:

$$\mathcal{S} := \{(\mu, \Sigma) : (\mu - \bar{\mu})^T \Omega^{-1} (\mu - \bar{\mu}) \leq \Upsilon^2,\ \Sigma = \hat{\Sigma}\},$$

where $\Upsilon$ is a positive scalar controlling the robustness level and $\Omega \succ 0$ is given (in the computational experiments, we will take $\Omega = \hat{\Sigma}$ and $\bar{\mu} = \hat{\mu}$).

The inner problem of (4.23) is written as the following formulation:

$$\max_{\mu} \; -\mu^T x \tag{4.28a}$$

$$\text{s.t. } \|\Omega^{-1/2}(\mu - \bar{\mu})\|_2 \le \Upsilon. \tag{4.28b}$$

Here, the inner problem in Section 4.1.3 and problem (4.28) are identical since the uncertainty sets for mean vector are the same for both problems. This leads the derivations to be equivalent. Therefore, we can obtain a single-stage reformulation of problem (4.23) as the follows:

$$\min_{x} \left\{ K(\alpha)\sqrt{x^T \Sigma x} - x^T \bar{\mu} + \|\Upsilon \Omega^{1/2} x\|_2 : e^T x = 1, x \ge 0 \right\}. \tag{4.29}$$

Note that problem (4.29) is a convex program. In particular, it can be recast as a second-order cone program.

### 4.2.4 Uncertainty for Covariance Matrix

In this section, we look at the case in which the covariance matrix $\Sigma$ is uncertain while assuming that $\mu$ is known (or estimated from data) as $\bar{\mu}$. In particular, let us consider the following uncertainty set:

$$\mathcal{S} := \{(1-\beta)\hat{\Sigma} \preceq \Sigma \preceq (1+\beta)\hat{\Sigma}, \quad \mu = \hat{\mu}\},$$

where the matrix $\hat{\Sigma}$ and $\hat{\mu}$ are the sample covariance and sample mean estimated from data and the scalar $\beta \in [0,1]$ controls the robustness level.

Firstly, the inner problem of (4.23) can be written as the following:

$$\max_{\Sigma \succeq 0} \; K(\alpha)\sqrt{x^T \Sigma x} \tag{4.30a}$$

$$\text{s.t. } \Sigma \preceq (1+\beta)\hat{\Sigma} \tag{4.30b}$$

$$-\Sigma \preceq -(1-\beta)\hat{\Sigma}. \tag{4.30c}$$

Since the objective function (4.30a) involves square root, we introduce a variable $y$ as below:

$$\max_{\Sigma,y} \; K(\alpha)y \tag{4.31a}$$

$$\text{s.t.} \; \Sigma \preceq (1+\beta)\hat{\Sigma} \tag{4.31b}$$

$$\Sigma \preceq -(1-\beta)\hat{\Sigma} \tag{4.31c}$$

$$x^T\Sigma x \geq y^2. \tag{4.31d}$$

We note that constraint (4.31d) is second-order cone representable, it can be written as in constraint (4.32d):

$$\max_{\Sigma,y} \; K(\alpha)y \tag{4.32a}$$

$$\text{s.t.} \; \Sigma \preceq (1+\beta)\hat{\Sigma} \qquad : \Lambda^+ \tag{4.32b}$$

$$-\Sigma \preceq -(1-\beta)\hat{\Sigma} \qquad : \Lambda^- \tag{4.32c}$$

$$\begin{pmatrix} -2 & 0 \\ 0 & (\text{vec}(xx^T))^T \\ 0 & -(\text{vec}(xx^T))^T \end{pmatrix} \begin{pmatrix} y \\ \text{vec}(\Sigma) \end{pmatrix} \leq_{L^3} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \qquad : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{4.32d}$$

Here, $\text{vec}(U)$ is the vectorized version of a matrix $U$.

After introducing the dual variables as in problem (4.32), we obtain the dual problem as follows:

$$\min_{\Lambda^+,\Lambda^-,a,b,c} \; (1+\alpha)\text{Tr}(\hat{\Sigma}\Lambda^+) - (1-\alpha)\text{Tr}(\hat{\Sigma}\Lambda^-) + b + c \tag{4.33a}$$

$$\text{s.t.} \; \begin{bmatrix} \Lambda^+ - \Lambda^- & x \\ x^T & (c-b)^{-1} \end{bmatrix} \succeq 0 \tag{4.33b}$$

$$a \leq -\frac{K(\alpha)}{2} \tag{4.33c}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in L^3 \tag{4.33d}$$

$$\Lambda^+, \Lambda^- \succeq 0. \tag{4.33e}$$

Note that we use the Schur's Complement Lemma together with Claims 1-3 to rewrite the constraint $\Lambda^+ - \Lambda^- - \gamma_{11} \succeq x(c-b)x^T$ as in (4.33b).

**Claim 1** *There exists an optimal solution to* (4.33) *such that*

$$a^* = -\frac{K(\alpha)}{2}.$$

**Proof 1** *We know that a only appears in* $a \leq -\frac{K(\alpha)}{2}$ *and* $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in L^3$. *Suppose* $a^* < -\frac{K(\alpha)}{2}$ *and* $a^{*2} + b^2 \leq c^2$. *In this case, we can choose* $a^{**} = -\frac{K(\alpha)}{2}$ *and obtain* $a^{**2} + b^2 < c^2$ *(note that* $a^* < a^{**} \implies \|a^*\| > \|a^{**}\|$*).*

*Hence, we can always set* $a^* = \frac{-K(\alpha)}{2}$.

**Claim 2** *In an optimal solution to* (4.33), *we have* $a^2 + b^2 = c^2$.

**Proof 2** *Suppose* $a^2 + b^2 \leq c^2$. *Instead we consider* $\tilde{c} = \sqrt{a^2 + b^2} < c$.

*Observe that* $\begin{pmatrix} a \\ b \\ \tilde{c} \end{pmatrix} \in L^3$ *and we have*

$$\begin{bmatrix} \Lambda^+ - \Lambda^- - \gamma_{11} & x \\ x^T & (c-b)^{-1} \end{bmatrix} \succeq 0 \implies \begin{bmatrix} \Lambda^+ - \Lambda^- - \gamma_{11} & x \\ x^T & (\tilde{c}-b)^{-1} \end{bmatrix} \succeq 0,$$

*since* $(c-b)^{-1} < (\tilde{c}-b)^{-1}$ *due to* $\tilde{c} < c$.

*However, this contradicts to optimality as objective contribution of* $(b+c)$ *is larger than objective contribution of* $(b+\tilde{c})$.

**Claim 3** *We can rewrite the positive semidefinite constraint*

$$\begin{bmatrix} \Lambda^+ - \Lambda^- - \gamma_{11} & x \\ x^T & (c-b)^{-1} \end{bmatrix} \succeq 0 \quad as \quad \begin{bmatrix} \Lambda^+ - \Lambda^- - \gamma_{11} & x \\ x^T & \frac{4}{K^2(\epsilon)}(b+c) \end{bmatrix} \succeq 0.$$

**Proof 3** *Due to Claims 1 and 2, we have*

$$c^2 = a^2 + b^2 = \frac{K^2(\epsilon)}{4} + b^2$$

$$\implies (c-b)(c+b) = \frac{K^2(\epsilon)}{4}$$

$$\implies (c-b)^{-1} = \frac{4}{K^2(\epsilon)}(b+c)$$

*In short, we can add* $\begin{bmatrix} \Lambda^+ - \Lambda^- - \gamma_{11} & x \\ x^T & \frac{4}{K^2(\epsilon)}(b+c) \end{bmatrix} \succeq 0$ *and* $a^* = -\frac{K(\alpha)}{2}$ *as constraints* (4.34b) *and* (4.34c)*, respectively.*

We rewrite problem (4.33) due to Claims 1-3 as the following semidefinite program:

$$\min_{\Lambda^+,\Lambda^-,a,b,c} \quad (1+\alpha)\mathrm{Tr}(\hat{\Sigma}\Lambda^+) - (1-\alpha)\mathrm{Tr}(\hat{\Sigma}\Lambda^-) + b + c \tag{4.34a}$$

$$\text{s.t.} \quad \begin{bmatrix} \Lambda^+ - \Lambda^- & x \\ x^T & \frac{4}{K^2(\epsilon)}(b+c) \end{bmatrix} \succeq 0 \tag{4.34b}$$

$$a = -\frac{K(\alpha)}{2} \tag{4.34c}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in L^3 \tag{4.34d}$$

$$\Lambda^+, \Lambda^- \succeq 0. \tag{4.34e}$$

Finally, the single-stage reformulation is obtained as follows:

$$\min_{x,\Lambda^+,\Lambda^-,a,b,c} \quad (1+\alpha)\mathrm{Tr}(\hat{\Sigma}\Lambda^+) - (1-\alpha)\mathrm{Tr}(\hat{\Sigma}\Lambda^-) + b + c - x^T\mu \tag{4.35a}$$

$$\text{s.t.} \quad (4.34\text{b}) - (4.34\text{e})$$

$$(3.1\text{b}) - (3.1\text{c}).$$

Note that problem (4.35) is a semidefinite program.

### 4.2.5 Uncertainty for Mean and Covariance Matrix

Now, we consider uncertainty for both covariance matrix $\Sigma$ and mean vector $\mu$ in this section. Our uncertainty set involves ellipsoidal uncertainty for $\mu$ and the upper and lower bounds for $\Sigma$ (in matrix sense). We consider the following uncertainty set particularly:

$$\mathcal{S} := \{(\mu,\Sigma) : (\mu-\bar{\mu})^T\Sigma^{-1}(\mu-\bar{\mu}) \leq \Upsilon^2,\ (1-\beta)\hat{\Sigma} \preceq \Sigma \preceq (1+\beta)\hat{\Sigma}\},$$

where the matrix $\hat{\Sigma}$ and $\hat{\mu}$ are the sample covariance and sample mean estimated from data, $\beta \in [0,1]$ and $\Upsilon$ are positive scalars, controlling the robustness level.

The inner problem of (4.23) can be written as follows:

$$\max_{\mu,\Sigma}\ K(\alpha)\sqrt{x^T\Sigma x} - \mu^T x \tag{4.36a}$$

$$\text{s.t. } \Sigma \preceq (1+\beta)\hat{\Sigma} \tag{4.36b}$$

$$-\Sigma \preceq -(1-\beta)\hat{\Sigma} \tag{4.36c}$$

$$\begin{bmatrix} \Sigma & \mu-\bar{\mu} \\ \mu^T-\bar{\mu}^T & \Upsilon^2 \end{bmatrix} \succeq 0. \tag{4.36d}$$

Note that (4.36d) is constructed by using Schur's Complement. Here, we introduce a variable $y$ in objective function (4.36a) such that $x^T\Sigma x \geq y^2$. This leads the derivations of problem (4.36) to be the same with problem (4.31). Therefore, we can write the canonical conic problem as the following:

$$\max_{\mu,\Sigma,y}\ K(\alpha)y - \mu^T x \tag{4.37a}$$

$$\text{s.t. } \Sigma \preceq (1+\beta)\hat{\Sigma} \qquad : \Lambda^+ \tag{4.37b}$$

$$-\Sigma \preceq -(1-\beta)\hat{\Sigma} \qquad : \Lambda^- \tag{4.37c}$$

$$\begin{pmatrix} -2 & 0 \\ 0 & (\text{vec}(xx^T))^T \\ 0 & -(\text{vec}(xx^T))^T \end{pmatrix} \begin{pmatrix} y \\ \text{vec}(\Sigma) \end{pmatrix} \leq_{L^3} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{4.37d}$$

$$\begin{bmatrix} -\Sigma & -\mu \\ -\mu^T & 0 \end{bmatrix} \preceq \begin{bmatrix} 0 & -\bar{\mu} \\ -\bar{\mu}^T & \Upsilon^2 \end{bmatrix} \quad : \begin{bmatrix} \cdot\gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \tag{4.37e}$$

After associating the dual variables in problem (4.37), the dual problem is obtained as follows:

$$\min_{\Lambda^+,\Lambda^-,a,b,c,\gamma} \quad (1+\beta)\mathrm{Tr}(\hat{\Sigma}\Lambda^+) - (1-\beta)\mathrm{Tr}(\hat{\Sigma}\Lambda^-) + b + c + \Upsilon^2\gamma_{22} - 2\bar{\mu}^T\gamma_{12} \tag{4.38a}$$

$$\text{s.t.} \quad \begin{bmatrix} \Lambda^+ - \Lambda^- - \gamma_{11} & x \\ x^T & \frac{4}{K^2(\epsilon)}(b+c) \end{bmatrix} \succeq 0 \tag{4.38b}$$

$$a = -\frac{K(\alpha)}{2} \tag{4.38c}$$

$$-2\gamma_{12} = -x \tag{4.38d}$$

$$\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \succeq 0 \tag{4.38e}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in L^3 \tag{4.38f}$$

$$\Lambda^+, \Lambda^-, \gamma \succeq 0. \tag{4.38g}$$

We note that we reformulate constraints (4.38b) and (4.38c), due to Claims 1-3. Finally, the single-stage reformulation of the problem (4.23) is written as the following semidefinite program:

$$\min_{x,\Lambda^+,\Lambda^-,a,b,c,\gamma} \quad (1+\beta)\mathrm{Tr}(\hat{\Sigma}\Lambda^+) - (1-\beta)\mathrm{Tr}(\hat{\Sigma}\Lambda^-) + b + c + \Upsilon^2\gamma_{22} - 2\bar{\mu}^T\gamma_{12} \tag{4.39a}$$

$$\text{s.t.} \quad (4.38b) - (4.38g)$$

$$(3.1b) - (3.1c).$$

## 4.3 Conditional Value-at-Risk Model under Mixture Distribution

In this section, we focus on the robust optimization version of the CVaR model presented in Section 3.3 under the assumption that the return vector is distributed according to a mixture of two multivariate normals. The reasons we consider a mixture distribution with the CVaR optimization model are two-fold: i) The Markowitz and worst-case VaR models are distribution independent, ii) the merits of mixture distribution with the CVaR optimization model are discussed in a recent paper by Kocuk & Cornuéjols (2020). In this probabilistic model, random returns come from the normal distributions $\mathrm{N}\left(\mu^1, \Sigma^1\right)$ with probability $\rho_1$ and $\mathrm{N}\left(\mu^2, \Sigma^2\right)$ with probability $\rho_2$. The motivation for such a model is that although most of the time (say with probability $\rho_1$) stock returns behave as normally distributed as $\mathrm{N}\left(\mu^1, \Sigma^1\right)$, every once in a while (say with probability $\rho_2$) a shock happens and shifts the mean of the normal distribution to the left with a higher variance as $\mathrm{N}\left(\mu^2, \Sigma^2\right)$ (see the discussions in Kocuk & Cornuéjols (2020) for details). We note that if a data set is given, we can compute the parameters of a mixture distribution by using the Expectation-Maximization (EM) Algorithm (Dempster, Laird & Rubin (1977)).

Since the CVaR function does not have a closed form expression in this case, we utilize a second-order cone representable approximation proposed in Kocuk & Cornuéjols (2020). Throughout this section, we will assume that the parameters of problem (3.5), that is $\mu^1$, $\Sigma^1$, $\mu^2$ and $\Sigma^2$, are uncertain. In order to incorporate robustness into the objective function, we formulate the following general form of a two-stage problem:

$$\text{(4.40a)} \qquad \min_{x} \quad \sum_{i=1}^{2} \max_{(\mu^i, \Sigma^i) \in \mathcal{S}^i} \left( z_i(\rho_i) \sqrt{x^T \Sigma^i x} - {\mu^i}^T x - \tau \rho_i {\mu^i}^T x \right)$$

$$\text{s.t.} \quad (3.1\text{b}) - (3.1\text{c}).$$

Here, the parameter $\rho_i$ represents the probability of $i$-th normal random variable and is assumed to be known, and the function $z_i(\rho_i)$ is defined as

$$z_i\left(\rho_i\right) := \frac{\phi\left(\Phi^{-1}\left(\alpha/\rho_i\right)\right)}{\alpha/\rho_i},$$

for $i = 1, 2$. We will assume that $\alpha < 0.5$ in model (4.40), which is not a restrictive assumption.

In the sequel, we reformulate problem (4.40) as a single-stage conic program when the uncertainty set $\mathcal{S}^i$ involves polytopic, budgeted or ellipsoidal uncertainty for either mean $\mu^i$ or covariance $\Sigma^i$ or both.

### 4.3.1 Polyhedral Uncertainty for Mean

In this section, we consider box uncertainty as a special case of polyhedral uncertainty for mean vectors $\mu^i$ while covariance matrices $\Sigma^i$ are assumed to be known (or estimated from data) as $\hat{\Sigma}^i$. Now, let us consider the following uncertainty sets:

$$\mathcal{S}^i := \{(\mu^i, \Sigma^i) : \|\mu^i - \bar{\mu}^i\|_\infty \leq \Upsilon_i,\ \Sigma^i = \hat{\Sigma}^i\},$$

where $\Upsilon_i$ is a positive scalar controlling the robustness level for $i = 1, 2$. (In the computational experiments we will take $\bar{\mu}^i = \hat{\mu}^i$).

The inner problem of (4.40) is written as follows:

$$\text{(4.41a)} \qquad \max_{\mu^1, \mu^2} \sum_{i=1}^{2} -{\mu^i}^T x$$

$$\text{(4.41b)} \qquad \text{s.t.} \ \|\mu^i - \bar{\mu}^i\|_\infty \leq \Upsilon_i \qquad i = 1, 2.$$

Here, we note that the remaining derivations of the inner problem in Section 4.1.1 are the same, we obtain the single-stage reformulation of the problem (4.40) as follows:

$$\min_{x, {\lambda_1}^+, {\lambda_1}^-, {\lambda_2}^+, {\lambda_2}^-} \sum_{i=1}^{2} [z_i(\rho_i)\sqrt{x^T \Sigma^i x} + (\tau\rho_i + 1){\lambda_i}^{+T}(\Upsilon_i + \bar{\mu}^i)$$

$$\text{(4.42a)} \qquad + {\lambda_i}^{-T}(\Upsilon_i - \bar{\mu}^i)]$$

$$\text{(4.42b)} \qquad \text{s.t.} \ {\lambda_i}^+ - {\lambda_i}^- = -x \qquad i = 1, 2$$

$$\text{(4.42c)} \qquad {\lambda_i}^+, {\lambda_i}^- \geq 0 \qquad i = 1, 2$$

$$(3.1b) - (3.1c).$$

Note that problem (4.42) is a convex optimization problem.

### 4.3.2 Budgeted Uncertainty for Mean

Now, we consider a budgeted uncertainty for mean vectors $\mu^i$, while assuming that covariance matrices $\Sigma^i$ are known (or estimated from data) as $\hat{\Sigma}^i$. In particular, let us consider the following uncertainty sets:

$$\mathcal{S}^i := \left\{ (\mu^i, \Sigma^i) : \sum_{j=1}^{n} \frac{|\mu^i_j - \bar{\mu}^i_j|}{\bar{\mu}^i_j} \leq \Upsilon_i, \ \Sigma^i = \hat{\Sigma}^i \right\},$$

where $\Upsilon_i$ is a positive scalar controlling the robustness level for $i = 1,2$. In the computational experiments we will take $\bar{\mu}^i = \hat{\mu}^i$.

Firstly, the inner problem of (4.40) can be written as follows:

$$\text{(4.43a)} \qquad \max_{\mu^1,\mu^2} \sum_{i=1}^{2} -{\mu^i}^T x$$

$$\text{(4.43b)} \qquad \text{s.t.} \sum_{j=1}^{n} \frac{|\mu^i_j - \bar{\mu}^i_j|}{\bar{\mu}^i_j} \leq \Upsilon_i \qquad i = 1,2.$$

Let us recall the inner problem in Section 4.1.2. Since the rest of the derivations in that section are the same (due to the fact that the uncertainty sets considered are identical), we obtain the single-stage reformulation of the problem (4.40) as the following convex optimization problem:

$$\text{(4.44a)} \qquad \min_{x,u^+,u^-,\gamma} \sum_{i=1}^{2} [z_i(\rho_i)\sqrt{x^T \Sigma^i x} + (\tau\rho_i + 1)(e^T(u^{i+} - u^{i-}) + \gamma^i \Upsilon_i)]$$

$$\text{(4.44b)} \qquad \text{s.t.} \ \frac{1}{\bar{\mu}^i_j} u^{i+} - \frac{1}{\bar{\mu}^i_j} u^{i-} = -x \qquad i = 1,2$$

$$\text{(4.44c)} \qquad \gamma^i e - u^{i+} - u^{i-} = 0 \qquad i = 1,2$$

$$\text{(4.44d)} \qquad \gamma^i, u^{i+}, u^{i-} \geq 0 \qquad i = 1,2$$

$$(3.1b) - (3.1c).$$

### 4.3.3 Ellipsoidal Uncertainty for Mean

In this section, an ellipsoidal uncertainty is considered for mean vectors $\mu^i$ while assuming that covariance matrices $\Sigma^i$ are known (or estimated from data) as $\hat{\Sigma}^i$. Let us consider the following uncertainty sets specifically:

$$\mathcal{S}^i := \{(\mu^i, \Sigma^i) : (\mu^i - \bar{\mu}^i)^T {\Omega^i}^{-1} (\mu^i - \bar{\mu}^i) \leq {\Upsilon_i}^2,\ \Sigma^i = \hat{\Sigma}^i\},$$

where $\Upsilon_i$ is a positive scalar controlling the robustness level and $\Omega^i \succ 0$ is given (in the computational experiments, we will take $\Omega^i = \hat{\Sigma}^i$ and $\bar{\mu}^i = \hat{\mu}^i$ although other choices are also allowed).

Firstly, we write the inner problem of (4.40) as below:

$$\max_{\mu^1, \mu^2} \sum_{i=1}^{2} -{\mu^i}^T x \tag{4.45a}$$

$$\text{s.t. } \|{\Omega^i}^{-1/2} (\mu^i - \bar{\mu}^i)\|_2 \leq \Upsilon_i \qquad i = 1, 2. \tag{4.45b}$$

Let us recall the inner problem in Section 4.1.3. the rest of the derivations in that section are the same (due to the fact that the uncertainty sets considered are identical), we obtain the single-stage reformulation of the problem (4.40) as the following:

(4.46)

$$\min_x \left\{ \sum_{i=1}^{2} [z_i(\rho_i) \sqrt{x^T \Sigma^i x} - (\tau\rho_i + 1) x^T \bar{\mu}^i + (\tau\rho_i + 1) \| \Upsilon_i {\Omega^i}^{1/2} x \|_2] : e^T x = 1, x \geq 0 \right\}.$$

Note that problem (4.46) is a convex program. In particular, it can be recast as second-order cone program.

### 4.3.4 Uncertainty for Covariance Matrix

Now, we look at the case in which the covariance matrix $\Sigma^i$ is uncertain while assuming that $\mu^i$ is known (or estimated from data) as $\hat{\mu}^i$. In particular, let us consider the following uncertainty sets:

$$\mathcal{S}^i := \{(\mu^i, \Sigma^i) : (1-\beta_i)\hat{\Sigma}^i \preceq \Sigma^i \preceq (1+\beta_i)\hat{\Sigma}^i,\ \mu^i = \hat{\mu}^i\},$$

where the matrix $\hat{\Sigma}^i$ is the sample covariance estimated from data and the scalar $\beta_i \in [0,1]$ controls the robustness level for $i = 1,2$.

The inner problem of (4.40) can be written as follows:

$$\text{(4.47a)} \qquad \max_{\Sigma^1,\Sigma^2 \succeq 0} \sum_{i=1}^{2} (z_i(\rho_i)\sqrt{x^T\Sigma^i x})$$

$$\text{(4.47b)} \qquad \text{s.t. } \Sigma^i \preceq (1+\beta_i)\hat{\Sigma}^i \qquad i = 1,2$$

$$\text{(4.47c)} \qquad -\Sigma^i \preceq -(1-\beta_i)\hat{\Sigma}^i \qquad i = 1,2.$$

Since the objective function (4.47a) involves square root, we introduce a variable $y_i$ such that $x^T\Sigma^i x \geq {y_i}^2$ as in the following canonical conic problem:

$$\text{(4.48a)} \quad \max_{\Sigma,y} \sum_{i=1}^{2} [z_i(\rho_i)y_i]$$

$$\text{(4.48b)} \quad \text{s.t. } \Sigma^i \preceq (1+\beta_i)\hat{\Sigma}^i \qquad i = 1,2 : \Lambda^{i^+}$$

$$\text{(4.48c)} \quad -\Sigma^i \preceq -(1-\beta_i)\hat{\Sigma}^i \qquad i = 1,2 : \Lambda^{i^-}$$

$$\text{(4.48d)} \quad \begin{pmatrix} -2 & 0 \\ 0 & (\text{vec}(xx^T))^T \\ 0 & -(\text{vec}(xx^T))^T \end{pmatrix} \begin{pmatrix} y_i \\ \text{vec}(\Sigma^i) \end{pmatrix} \leq_{L^3} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad i = 1,2. : \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix}$$

After introducing the dual variables as in problem (4.48), we obtain the dual problem after introducing the dual variables $\Lambda^{i^+}$ and $\Lambda^{i^-}$ as below:

$$\text{(4.49a)} \quad \min_{\Lambda^+,\Lambda^-,a,b,c} \sum_{i=1}^{2}[(1+\beta_i)\text{Tr}(\hat{\Sigma}^i\Lambda^{i^+}) - (1-\beta_i)\text{Tr}(\hat{\Sigma}^i\Lambda^{i^-}) + b_i + c_i]$$

$$\text{(4.49b)} \qquad \text{s.t. } \begin{bmatrix} \Lambda^{i^+} - \Lambda^{i^-} - {\gamma^i}_{11} & x \\ x^T & \frac{4}{z_i^2(\rho_i)}(b_i + c_i) \end{bmatrix} \succeq 0 \qquad i = 1,2$$

$$a_i = -\frac{z_i(\rho_i)}{2} \qquad i = 1,2 \tag{4.49c}$$

$$\begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} \in L^3 \qquad i = 1,2 \tag{4.49d}$$

$$\Lambda^{i+}, \Lambda^{i-} \succeq 0 \qquad i = 1,2. \tag{4.49e}$$

We note that constraints (4.49b) and (4.49c) are reformulated according to Claims 1-3. Finally, we reformulate the single-stage equivalent of problem (4.40) as the following semidefinite program:

$$\begin{aligned} \min_{x,\Lambda^+,\Lambda^-,a,b,c} \quad & \sum_{i=1}^{2} [(1+\beta_i)\mathrm{Tr}(\hat{\Sigma}_i {\Lambda_i}^+) - (1-\beta_i)\mathrm{Tr}(\hat{\Sigma}_i {\Lambda_i}^-) + b_i + c_i \\ & - {\mu^i}^T x - \tau \rho_i {\mu^i}^T x] \\ \text{s.t.} \quad & (4.49b) - (4.49e) \\ & (3.1b) - (3.1c). \end{aligned} \tag{4.50a}$$

### 4.3.5 Uncertainty for Mean and Covariance Matrix

In this section, we consider uncertainty for both covariance matrices $\Sigma^i$ and mean vectors $\mu^i$. Our uncertainty set involves ellipsoidal uncertainty for $\mu^i$ and the upper and lower bounds for $\Sigma^i$ (in matrix sense). In particular, let us consider the following uncertainty sets:

$$\mathcal{S}^i := \{(\mu^i, \Sigma^i) : (\mu^i - \bar{\mu}^i)^T {\Sigma^i}^{-1} (\mu^i - \bar{\mu}^i) \le {\Upsilon_i}^2, \ (1-\beta_i)\hat{\Sigma}^i \preceq \Sigma^i \preceq (1+\beta_i)\hat{\Sigma}^i\},$$

where the matrix $\hat{\Sigma}^i$ is the sample covariance estimated from data, $\beta_i \in [0,1]$ and $\Upsilon_i$ are positive scalars, controlling the robustness level for $i = 1,2$ (in the computational experiments, we will take $\bar{\mu}^i = \hat{\mu}^i$).

The inner problem is written as the following:

$$\max_{(\mu,\Sigma)} \sum_{i=1}^{2} \Big( z_i(\rho_i)\sqrt{x^T\Sigma^i x} - {\mu^i}^T x - \tau\rho_i {\mu^i}^T x \Big) \tag{4.51a}$$

$$\text{s.t. } \Sigma^i \preceq (1+\beta_i)\hat{\Sigma}^i \qquad i = 1,2 \tag{4.51b}$$

$$-\Sigma^i \preceq -(1-\beta_i)\hat{\Sigma}^i \qquad i = 1,2 \tag{4.51c}$$

$$\begin{bmatrix} \Sigma^i & \mu^i - \bar{\mu}^i \\ {\mu^i}^T - {\bar{\mu}^i}^T & {\Upsilon_i}^2 \end{bmatrix} \succeq 0 \qquad i = 1,2. \tag{4.51d}$$

Note that (4.51) is constructed by using Schur's Complement. Here, the following derivations will be the equivalent versions of the problems (4.48)-(4.49) respectively. Therefore, we introduce the dual variables in the canonical conic problem (4.52) as the following:

(4.52a)

$$\max_{\mu,\Sigma,y} \sum_{i=1}^{2} \Big( z_i(\rho_i) y_i - {\mu^i}^T x - \tau\rho_i {\mu^i}^T x \Big)$$

(4.52b)

$$\text{s.t. } \Sigma^i \preceq (1+\beta_i)\hat{\Sigma}^i \qquad i = 1,2 \quad : {\Lambda^i}^+$$

(4.52c)

$$-\Sigma^i \preceq -(1-\beta_i)\hat{\Sigma}^i \qquad i = 1,2 \quad : {\Lambda^i}^-$$

(4.52d)

$$\begin{pmatrix} -2 & 0 \\ 0 & (\mathrm{vec}(xx^T))^T \\ 0 & -(\mathrm{vec}(xx^T))^T \end{pmatrix} \begin{pmatrix} y_i \\ \mathrm{vec}(\Sigma^i) \end{pmatrix} \leq_{L^3} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad i = 1,2 \quad : \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix}$$

(4.52e)

$$\begin{bmatrix} -\Sigma^i & -\mu^i \\ -{\mu^i}^T & 0 \end{bmatrix} \preceq \begin{bmatrix} 0 & {\bar{\mu}^i}^T \\ -{\bar{\mu}^i}^T & {\Upsilon_i}^2 \end{bmatrix} \qquad i = 1,2. \quad : \begin{bmatrix} {\gamma^i}_{11} & {\gamma^i}_{12} \\ {\gamma^i}_{21} & {\gamma^i}_{22} \end{bmatrix}$$

After introducing the dual variables, the dual problem can be obtained as follows:

$$\text{(4.53a)} \quad \min_{x,\Lambda^+,\Lambda^-,a,b,c} \sum_{i=1}^{2}[(1+\beta_i)\mathrm{Tr}(\hat{\Sigma}_i{\Lambda_i}^+)-(1-\beta_i)\mathrm{Tr}(\hat{\Sigma}_i{\Lambda_i}^-)+b_i+c_i$$

$$+\Upsilon^2{\gamma^i}_{22}-2{\bar{\mu}_i}^T{\gamma^i}_{12}]$$

$$\text{(4.53b)} \quad \text{s.t.} \begin{bmatrix} {\Lambda_i}^+-{\Lambda_i}^--{\gamma^i}_{11} & x \\ x^T & \frac{4}{z_i^2(\rho_i)}(b+c) \end{bmatrix} \succeq 0 \qquad i=1,2$$

$$\text{(4.53c)} \quad a_i=-\frac{z_i(\rho_i)}{2}$$

$$\text{(4.53d)} \quad -2{\gamma^1}_{12}=-(\tau\rho_i+1)x \qquad i=1,2$$

$$\text{(4.53e)} \quad \gamma^i=\begin{bmatrix} {\gamma^i}_{11} & {\gamma^i}_{12} \\ {\gamma^i}_{21} & {\gamma^i}_{22} \end{bmatrix} \succeq 0 \qquad i=1,2$$

$$\text{(4.53f)} \quad \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} \in L^3 \qquad i=1,2$$

$$\text{(4.53g)} \quad {\Lambda^i}^+,{\Lambda^i}^-,\gamma^i \succeq 0 \qquad i=1,2.$$

Finally, the single-stage reformulation of the problem (4.40) is obtained as the following semidefinite problem:

$$\text{(4.54a)} \quad \min_{x,\Lambda^+,\Lambda^-,a,b,c} \sum_{i=1}^{2}[(1+\beta_i)\mathrm{Tr}(\hat{\Sigma}^i{\Lambda^i}^+)-(1-\beta_i)\mathrm{Tr}(\hat{\Sigma}^i{\Lambda^i}^-)+b_i+c_i]$$

$$+{\Upsilon_i}^2{\gamma^i}_{22}-2{\bar{\mu}^i}^T{\gamma^i}_{12}$$

$$\text{s.t.} \quad (4.53\text{b})-(4.53\text{g})$$

$$(3.1\text{b})-(3.1\text{c}).$$

# 5. COMPUTATIONAL EXPERIMENTS

In this chapter, we present the results of our computational experiments, which have been conducted to investigate the effectiveness of robust optimization approaches and assess their impact on optimal portfolios. We first explain the two data sets used in our computational experiments in Section 5.1. Then, we utilize two approaches to evaluate the performance of the robust optimization models: i) A rolling horizon based evaluation in Section 5.2, and ii) an efficient frontier based evaluation in Section 5.3.

## 5.1 Data Sets

In our computational experiments, we use two data sets listed below:

- Standard & Poors 500: We use a real data set provided by Kocuk & Cornuéjols (2020) from Standard & Poors (S&P) 500 index spanning 30 years between January 1987–December 2016 with monthly resolution. This gives us 360 asset return realizations for 11 sectors considered in this paper.
- MIBTEL: We use data set from MIBTEL index of Borsa Italiana provided by (Ces) between March 2003–March 2008 with weekly resolution. We categorize 146 assets into sectors based on (YF). This gives us 264 asset return realizations for 11 sectors considered in this paper.

In Tables 5.1 and 5.2, we provide the estimates for normal fits of S&P 500 and MIBTEL data sets. We note that the tables are in percentages and covariance matrices are provided in Tables A.1 and A.2.

Table 5.1 S&P 500 Data Set.

| Sector | Normal | |
|---|---|---|
| | $\mu$ | $\sigma$ |
| Energy | 1.177 | 6.274 |
| Consumer discretionary | 1.511 | 5.312 |
| Consumer staples | 1.391 | 4.141 |
| Real estate | 1.151 | 7.235 |
| Industrials | 1.289 | 5.143 |
| Financials | 1.332 | 6.272 |
| Telecomunication services | 1.032 | 5.470 |
| Information technology | 1.726 | 7.093 |
| Materials | 1.390 | 5.688 |
| Health care | 1.416 | 4.637 |
| Utilities | 1.014 | 4.274 |

Table 5.2 MIBTEL Data Set.

| Sector | Normal | |
|---|---|---|
| | $\mu$ | $\sigma$ |
| Energy | 0.569 | 3.075 |
| Consumer Cyclical | 0.181 | 1.904 |
| Capital Goods | 0.297 | 6.706 |
| Real Estate | 0.116 | 2.772 |
| Industrials | 0.360 | 2.076 |
| Financials | 0.189 | 1.678 |
| Communication | 0.016 | 2.409 |
| Technology | 0.231 | 2.295 |
| Basic Materials | 0.263 | 2.385 |
| Health care | 0.171 | 2.362 |
| Utilities | 0.298 | 2.222 |

We implement all the experiments using CVXPY which is a Python-embedded modeling language. The solver MOSEK is used in the code to solve second-order cone and semidefinite programs. All computational experiments are carried out on a 64-bit machine with Intel Xeon E5-2640 v3 processor at 2.60 GHz using 12.4 GB of RAM.

## 5.2 Rolling Horizon Based Evaluation

The first evaluation scheme we use in this thesis is based on an out-of-sample analysis using the real data sets. Let $T$ be the number of data points available, and $H$ and $m$ be positive integers. The main idea of the rolling horizon based evaluation scheme is to use the return vectors of the last $H$ periods, namely, $r^{t-H+1},\dots,r^t$ to estimate the parameters of a portfolio optimization problem, which we use to determine the portfolio decision $x^{*,t}$ for time period $t+1$. Then, we evaluate the performance of this decision using the return vector $r^{t+1}$ as $\gamma^{t+1} = {r^{t+1}}^T x^{*,t}$. Finally, we repeat this procedure for $t = T/2,\dots,T$ and for different values of $\tau$ and evaluate the overall performance.

Algorithm 1 outlines the main steps of the procedure summarized above. We fix $H = T/2$ and $\alpha = 0.01$ throughout this chapter.

**Algorithm 1**

**Input:** *Distribution*, Optimization Model, *Realizations* $=\{r^1,...,r^T\},\ T,\ H,\ \alpha$.

**Output** $x^*$.

**for** $\tau=\tau_1$ to $\tau_m$ **do**
    **for** $t=H$ to $T-1$ **do**
        Compute parameters $(\hat{\Theta}^t)$ of *Distribution* using $r^{t-H+1},\dots,r^t$
        Solve Optimization Model $(\hat{\Theta}^t,\tau,\alpha)$ to obtain $x^{*,t}$
        Compute $\gamma^{t+1}={r^{t+1}}^T x^{*,t}$
    **end for**
    Report statistical measures using $\gamma^{\frac{T}{2}+1}(\tau),...,\gamma^T(\tau)$ (such as $\text{avg}^\tau,\text{stdev}^\tau,\text{VaR}^\tau,\text{CVaR}^\tau$)
**end for**

In the remainder of this section, we provide a comparison of the various robust optimization models constructed in Chapter 4 with respect to different uncertainty sets and robustness levels.

### 5.2.1 Markowitz Model

In this subsection, we will present the performance of the standard Markowitz model (3.1) and the robust Markowitz models in Section 4.1 on the data sets we mentioned earlier. In the following figures, we will provide the average expected return of the optimal portfolios with respect to risk measures; standard deviation, $\text{CVaR}_{0.01}$, and $\text{VaR}_{0.01}$ for different values of risk aversion constant $\tau$.

#### 5.2.1.1 Performance of the Standard Markowitz Model

As a benchmark, we first solve the standard Markowitz model (3.1) The following Figures 5.1 and 5.2 will illustrate the results of this problem for two data sets.

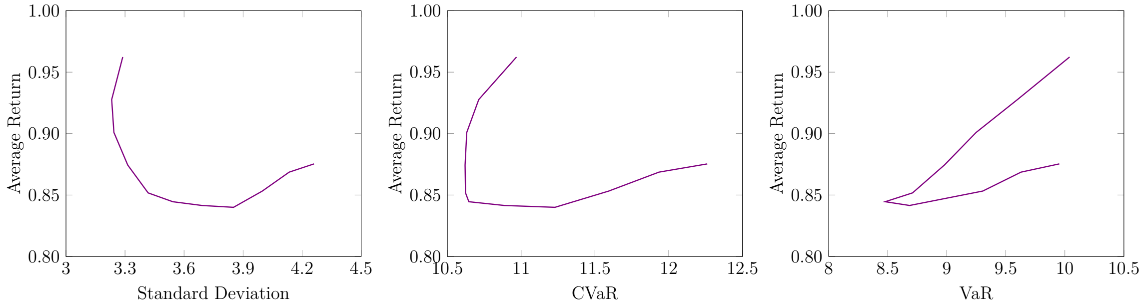


Figure 5.1 Performance comparison of the standard Markowitz model for the S&P 500 data set.

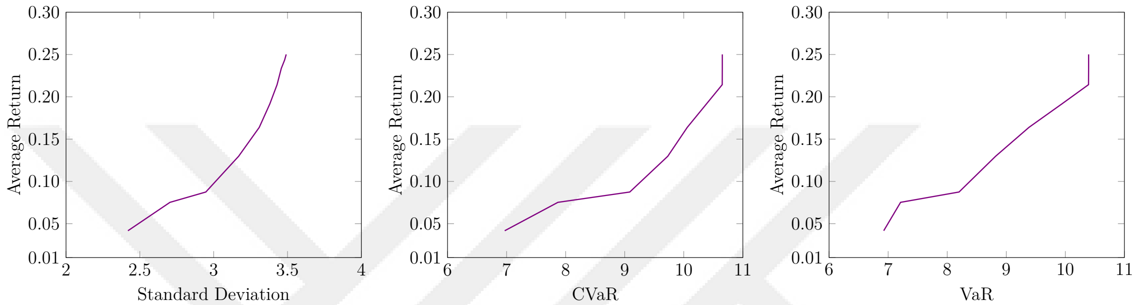


Figure 5.2 Performance comparison of the standard Markowitz model for the MIBTEL data set.

In Figure 5.1, one can observe that as the value of $\tau$ increases towards the right side on the horizontal axis, the average return starts to decrease while the standard deviation starts to increase. As a result, even the slightest difference in $\tau$ leads different portfolios. Counter-intuitively, Figure 5.1 shows that $\tau = 0$ (which puts no weight at all to mean return) yields the best mean return. We attribute this to the fact that mean returns are very hard to estimate. On the other hand, Figure 5.2 demonstrates a behavior in which return and risk increase at the same time. Thus, the MIBTEL data set shows more reliable results than the S&P 500 data set does. This might be due to the fact that the MIBTEL data set is provided spanning 5 years with weekly resolution whereas the S&P 500 is given spanning 30 years with monthly resolution.

In the remainder of this subsection, we solve several robust versions of the Markowitz optimization model and provide the corresponding results in comparison with the results of the standard Markowitz model.

#### 5.2.1.2 Performance of the Robust Markowitz Model with Polyhedral Uncertainty

Here, the following figures will show the performance of the robust Markowitz model with polyhedral uncertainty discussed in Section 4.1.1 on the data sets.

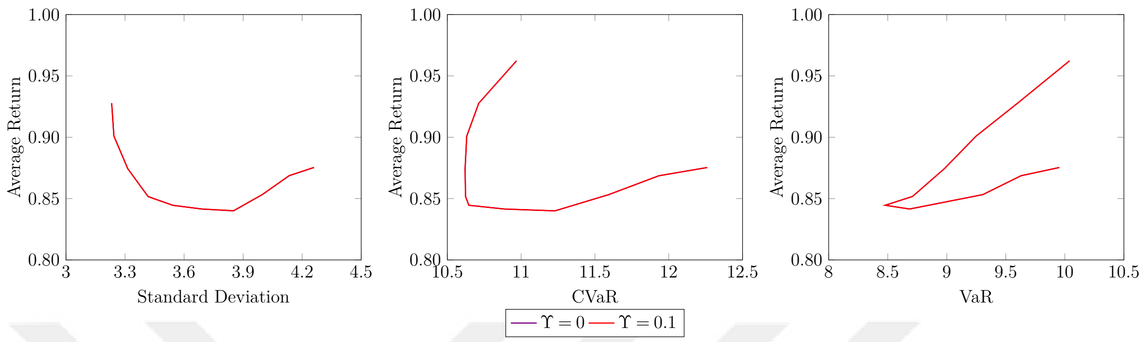


Figure 5.3 Performance comparison of the robust Markowitz model with polyhedral uncertainty for mean for the S&P 500 data set.

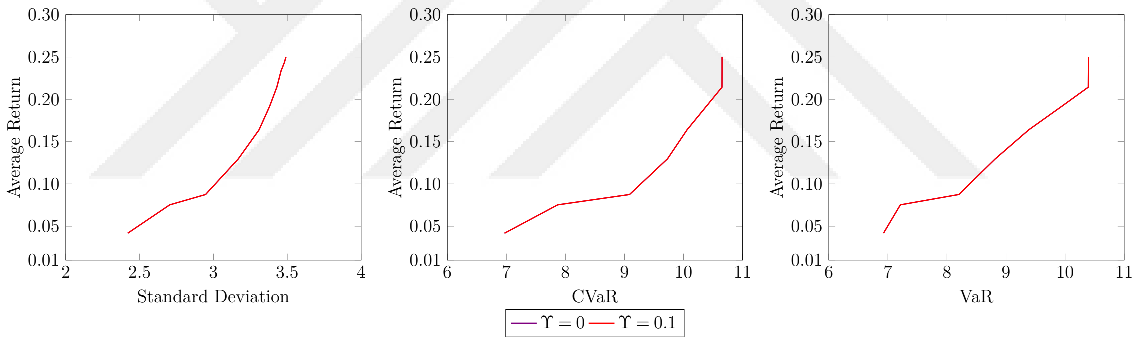


Figure 5.4 Performance comparison of the robust Markowitz model with polyhedral uncertainty for mean for the MIBTEL data set.

As can be seen from Figures 5.3 and 5.4, the frontiers of standard and robust models are overlapping, meaning that for the same level for risk, the return does not change.

#### 5.2.1.3 Performance of the Robust Markowitz Model with Budgeted Uncertainty

In this subsection, we will present the performance of the robust Markowitz model with budgeted uncertainty discussed in Section 4.1.2 on the data sets.

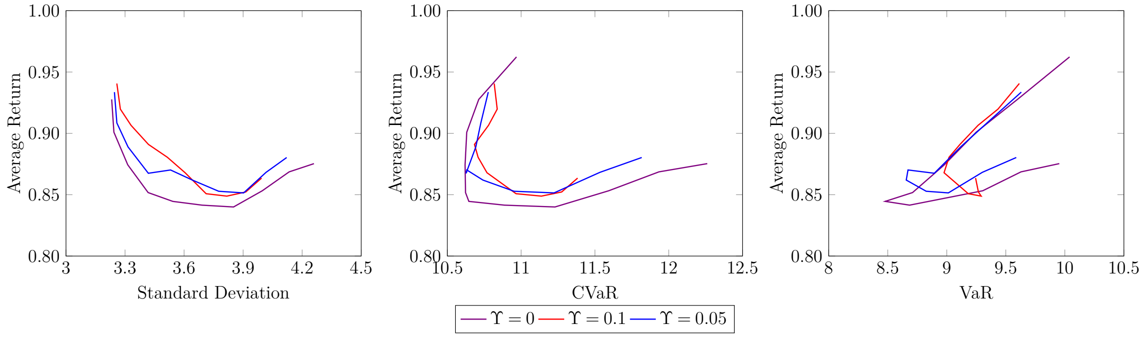


Figure 5.5 Performance comparison of the robust Markowitz model with budgeted uncertainty for mean for the S&P 500 data set.

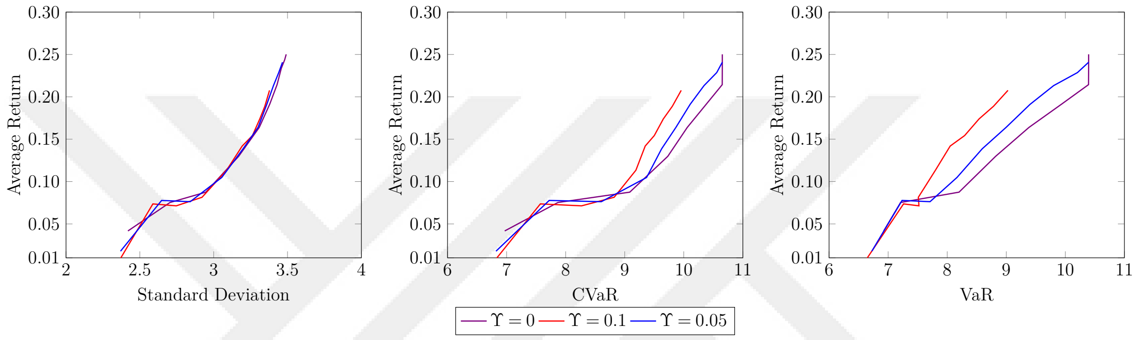


Figure 5.6 Performance comparison of the robust Markowitz model with budgeted uncertainty for mean for the MIBTEL data set.

In Figure 5.5, one can say that as $\Upsilon$ increases, for the same level of standard deviation, the average return of the robust models' portfolios dominates the standard model slightly. However, Figure 5.6 shows that for different $\Upsilon$ values, the results of optimal portfolios do not change significantly. According to this figure, if the investor is more inclined to take risks, then using robust models may yield higher returns for a same level standard deviation, CVaR or VaR.

#### 5.2.1.4 Performance of the Robust Markowitz Model with Ellipsoidal Uncertainty

In this subsection, we present the performance of the robust Markowitz model with ellipsoidal uncertainty discussed in Section 4.1.3.

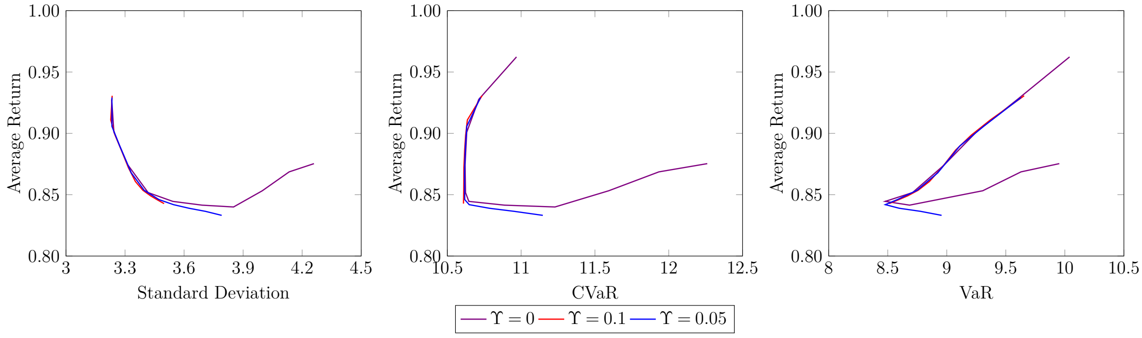


Figure 5.7 Performance comparison of the robust Markowitz model with ellipsoidal uncertainty for mean for the S&P 500 data set.

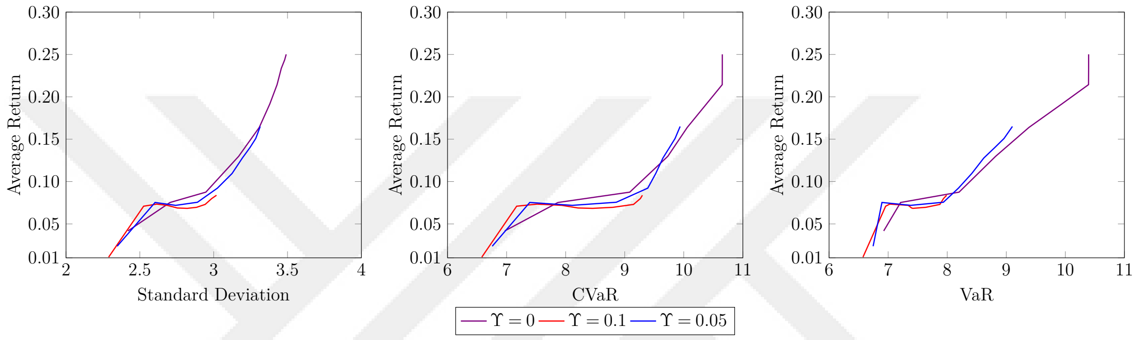


Figure 5.8 Performance comparison of the robust Markowitz model with ellipsoidal uncertainty for mean for the MIBTEL data set.

According to Figure 5.7, one can say that even the slightest increase in the robustness level $\Upsilon$ results in a lower average return and risk. However, in Figure 5.8, the performance of the robust model on the MIBTEL data set shows that the lower risk levels may yield higher returns.

#### 5.2.1.5 Performance of the Robust Markowitz Model with Uncertainty for Covariance

Here, the following figures will demonstrate the performance of the robust Markowitz model with uncertainty for covariance discussed in Section 4.1.4 for the data sets.

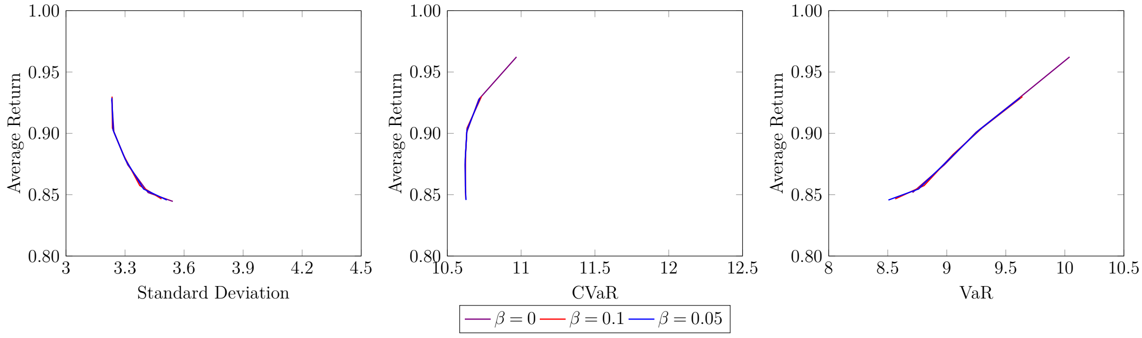


Figure 5.9 Performance comparison of the robust Markowitz model with uncertainty for covariance for the S&P 500 data set.

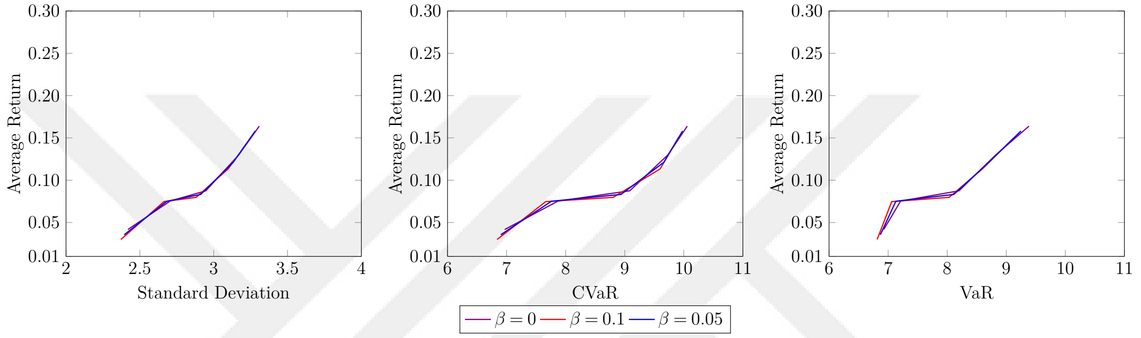


Figure 5.10 Performance comparison of the robust Markowitz model with uncertainty for covariance for the MIBTEL data set.

Figures 5.9 and 5.10 show that although $\beta$ increases, it does not lead to a significant change in the average return and risk. This is due to the fact that sensitivity in estimating the covariance matrix is comparably less than the sensitivity in estimating the mean return vector.

#### 5.2.1.6 Performance of the Robust Markowitz Model with Uncertainty for Mean and Covariance

In this subsection, the following figures will demonstrate the performance of the robust Markowitz model with uncertainty for mean and covariance discussed in Section 4.1.5 on the data sets.

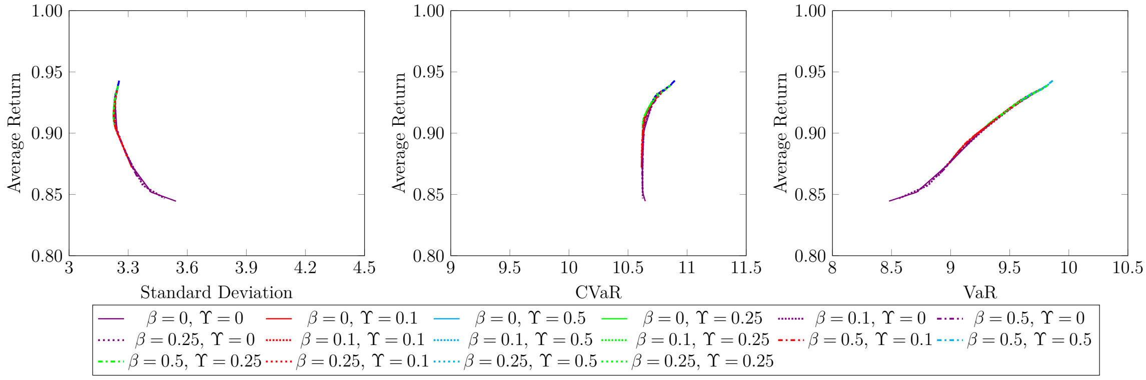


Figure 5.11 Performance comparison of the robust Markowitz model with uncertainty for mean and covariance for the S&P 500 data set.

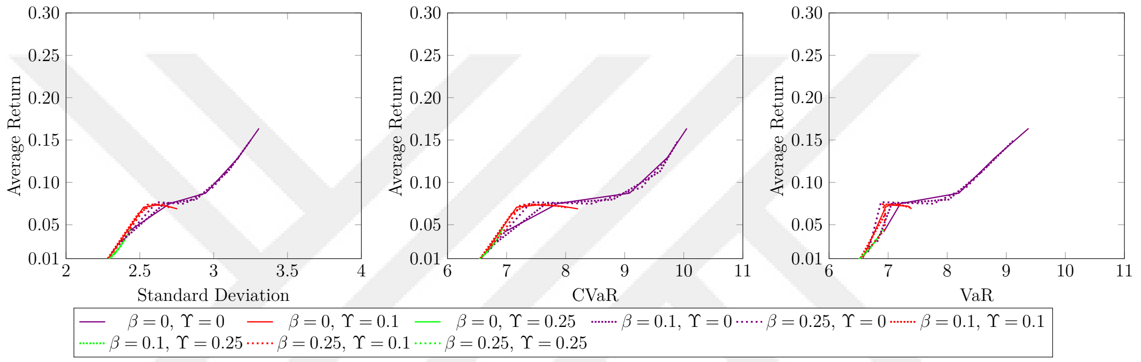


Figure 5.12 Performance comparison of the robust Markowitz model with uncertainty for mean and covariance for the MIBTEL data set.

Figure 5.11 shows that although the difference is not much, the average return increases as the robustness levels of both $\beta$ and $\gamma$ increase. On the other hand, Figure 5.12 suggests that using the robust model on the MIBTEL data for the lower risk levels may yield higher returns.

### 5.2.2 Robust Worst Case Value-at-Risk Models

In this subsection, we provide the results of the robust versions of the worst case Value-at-Risk model discussed in Section 4.2 on the data sets we mentioned earlier. In sequel, following figures will demonstrate the average expected return of the optimal portfolios with respect to risk measures; standard deviation, $\text{CVaR}_{0.01}$, and $\text{VaR}_{0.01}$ for different values of $K(\alpha)$.

#### 5.2.2.1 Performance of the Robust Worst Case Value-at-Risk Model with Polyhedral Uncertainty

The following figures present the performance of the robust worst case VaR model with polyhedral uncertainty discussed in Section 4.2.1 on the data sets.

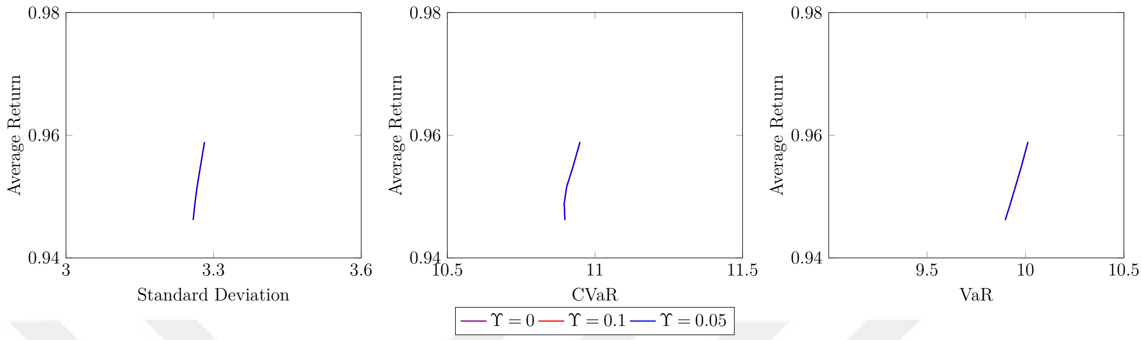


Figure 5.13 Performance comparison of the robust worst case VaR model with polyhedral uncertainty for mean for the S&P 500 data set.

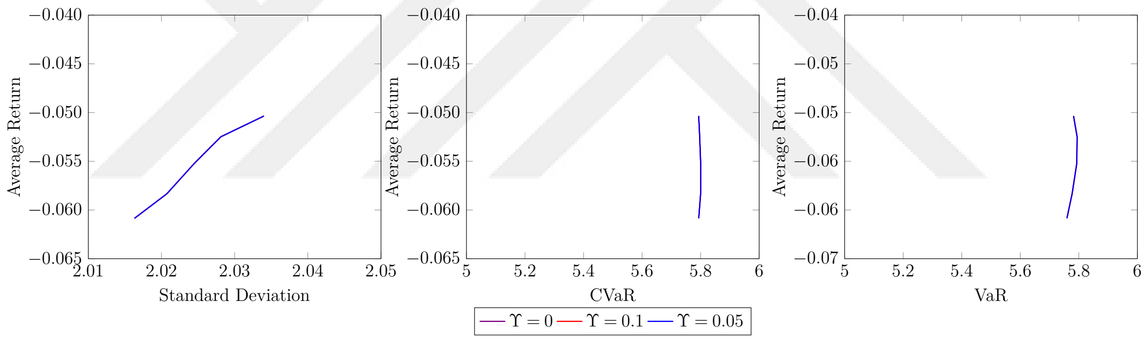


Figure 5.14 Performance comparison of the robust worst case VaR model with polyhedral uncertainty for mean for the MIBTEL data set.

As can be seen from Figures 5.13 and 5.14, this robust model is not very sensitive to changes in the $\Upsilon$ parameter. Although resulting portfolios change according to $K(\alpha)$, there is no significant difference.

#### 5.2.2.2 Performance of the Robust Worst Case Value-at-Risk Model with Budgeted Uncertainty

In this subsection, we present the performance of the robust model with budgeted uncertainty discussed in Section 4.2.2. The following Figures 5.15 and 5.16 show that although the resulting portfolios change slightly according to parameter $\alpha$, this robust model is not very sensitive to the changes in parameter $\Upsilon$.

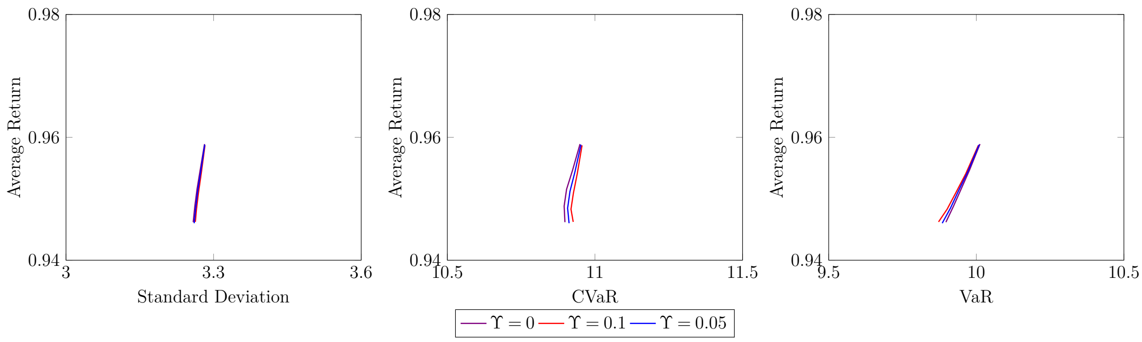


Figure 5.15 Performance comparison of the robust worst case VaR model with budgeted uncertainty for mean for the S&P 500 data set.

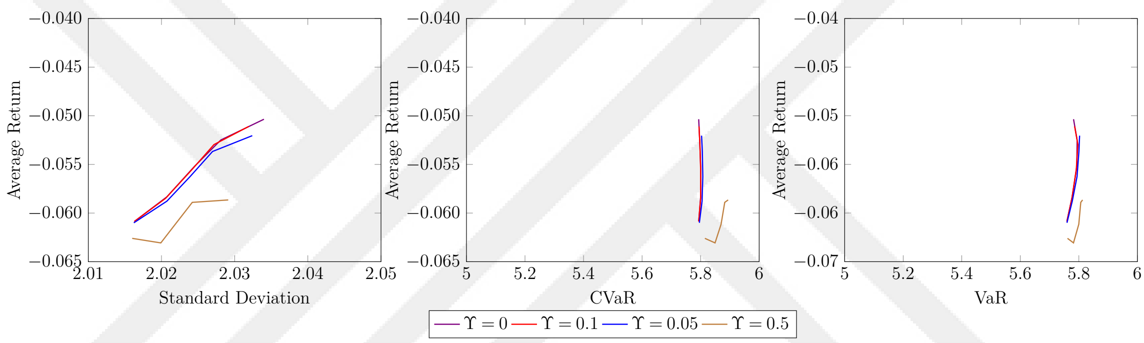


Figure 5.16 Performance comparison of the robust worst case VaR model with budgeted uncertainty for mean for the MIBTEL data set.

As can be seen from Figure 5.16, robust models result in poor portfolios with negative returns. We have the following explanation for this observation: Let us recall the worst-case value-at-risk model (4.23). Since this model does not consider return in the objective function, it gives more emphasis in reducing the risk, which eventually results in negative returns.

#### 5.2.2.3 Performance of the Robust Worst Case Value-at-Risk Model with Ellipsoidal Uncertainty

Here, the following figures will demonstrate the performance of the robust model with ellipsoidal uncertainty discussed in Section 4.2.3 on the data sets.

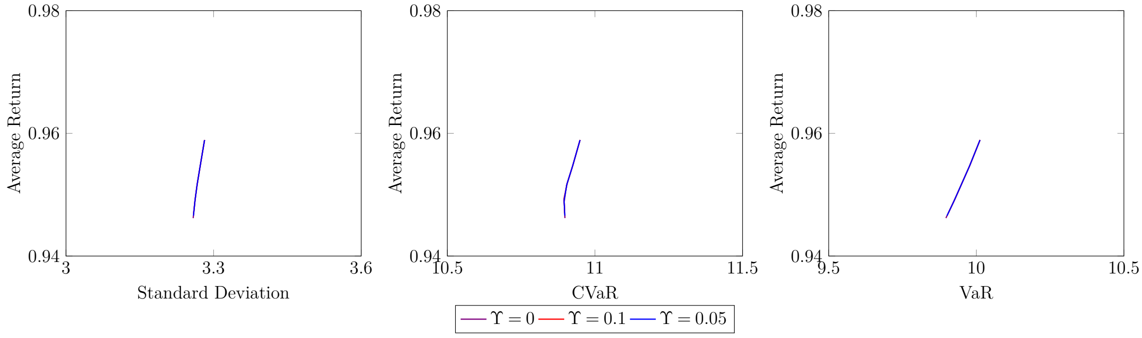


Figure 5.17 Performance comparison of the robust worst case VaR model with ellipsoidal uncertainty for mean for the S&P 500 data set.

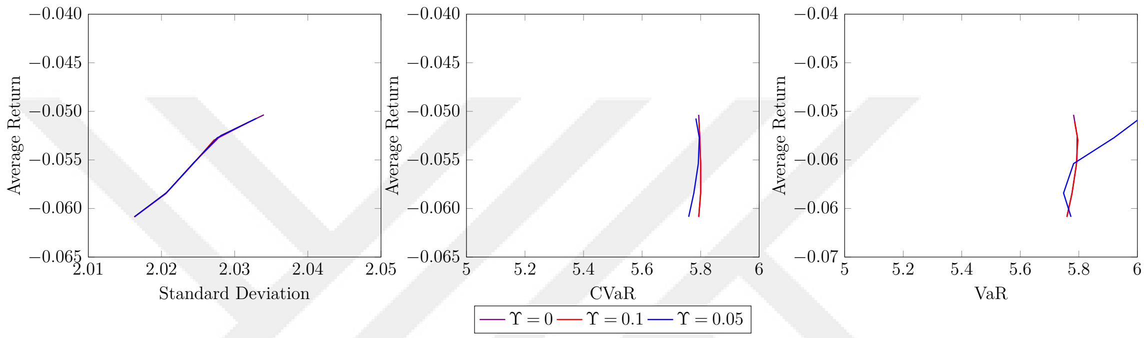


Figure 5.18 Performance comparison of the robust worst case VaR model with ellipsoidal uncertainty for mean for the MIBTEL data set.

Figures 5.17 and 5.18 depict that the robust model is not very sensitive to changes in parameter $\Upsilon$.

#### 5.2.2.4 Performance of the Robust Worst Case Value-at-Risk Model with Uncertainty for Covariance

In this subsection, we present the performance of the robust problem with uncertainty for covariance discussed in Section 4.2.4 on the data sets. The following Figures 5.19 and 5.20 demonstrate that changes in $\beta$ does not lead to a significant change in the resulting portfolios.

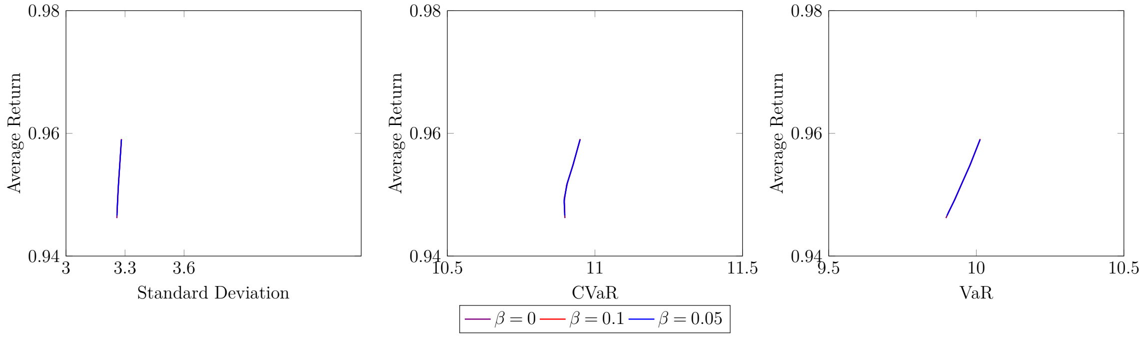


Figure 5.19 Performance comparison of the robust worst case VaR model with uncertainty for covariance matrix for the S&P 500 data set.

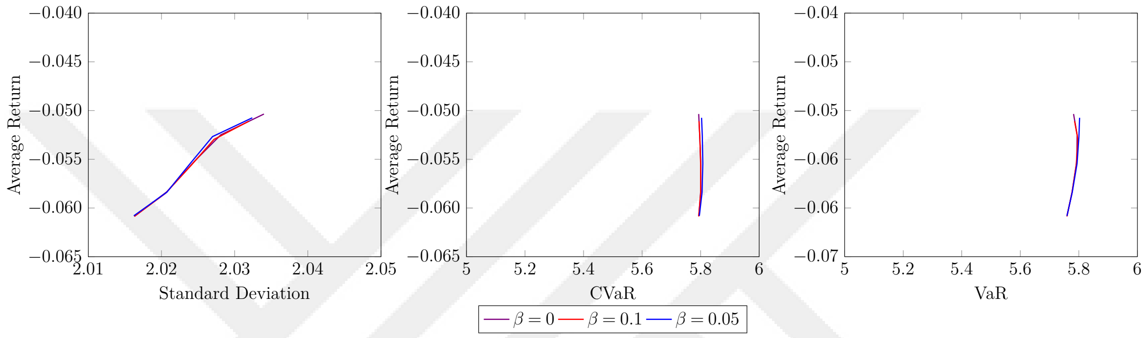


Figure 5.20 Performance comparison of the robust worst case VaR model with uncertainty for covariance for the MIBTEL data set.

#### 5.2.2.5 Performance of the Robust Worst Case Value-at-Risk Model with Uncertainty for Mean and Covariance

Here, the following figures will demonstrate the performance of the robust model with uncertainty for mean and covariance discussed in Section 4.2.5 on the data sets.

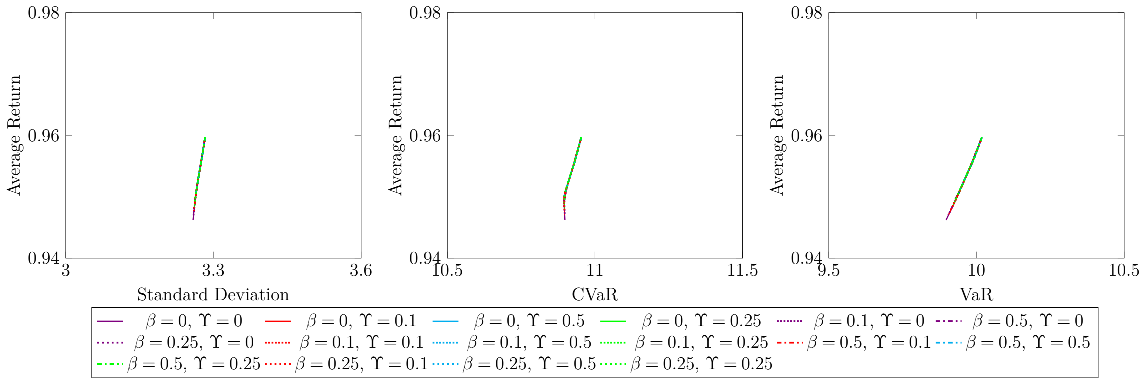


Figure 5.21 Performance comparison of the robust worst case VaR with uncertainty for mean and covariance for the S&P 500 data set.

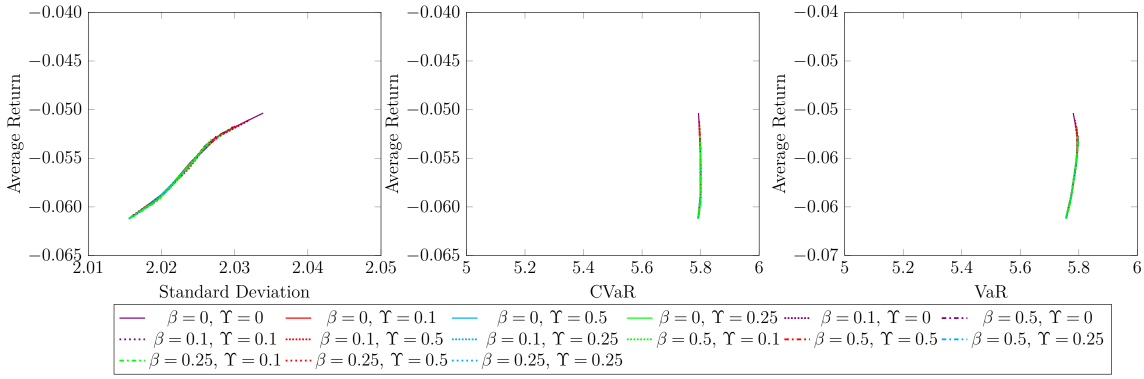


Figure 5.22 Performance comparison of the robust worst case VaR model with uncertainty for mean and covariance for the MIBTEL data set.

As can be seen form the Figures 5.21 and 5.22, robust model is not very sensitive to changes in the robustness levels $\Upsilon$ and $\beta$.

### 5.2.3 Robust Conditional Value-at-Risk Model under Mixture Distribution

In this subsection, we solve the robust versions of the CVaR model assuming the random return vector is distributed as a mixture of normals discussed in Section 5.2.3. In order to estimate the parameters of the two mixtures, we employ the EM Algorithm on S&P 500 and MIBTEL data sets. However, we could obtain the following relations between the parameters only for the S&P 500 data set:

$$\hat{\rho}_1 > \hat{\rho}_2,\ \hat{\mu}^1 > \hat{\mu}^2,\ \hat{\Sigma}^2 > \hat{\Sigma}^1.$$

Therefore, in our experiments, we only use this data set as the above relations do not hold for the MIBTEL dat set (see Section 4.3 and Kocuk & Cornuéjols (2020) for the details about the significance of these relations). In sequel, the following figures will demonstrate the average expected return of the optimal portfolios with risk measures; standard deviation, $\text{CVaR}_{0.01}$, and $\text{VaR}_{0.01}$ for different values of risk aversion constant $\tau$.

#### 5.2.3.1 Performance of the Robust CVaR Model under Mixture Distribution with Polyhedral Uncertainty

Here, we will present the performance of the robust model with polyhedral uncertainty discussed in Section 4.3.1.

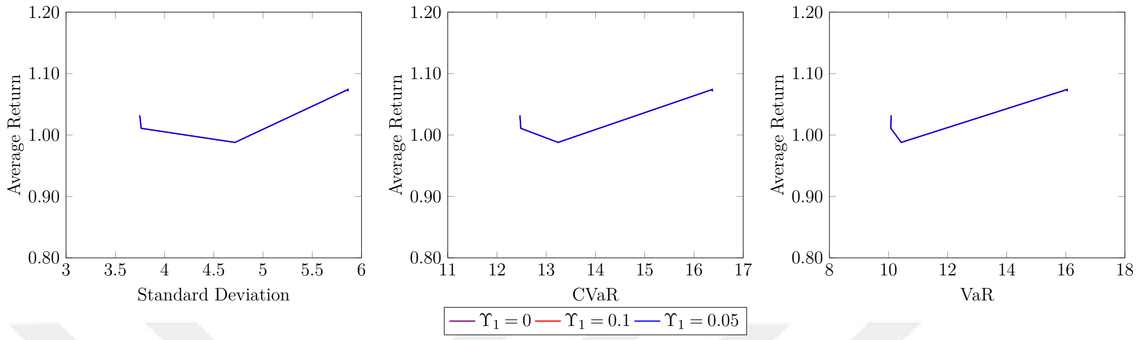


Figure 5.23 Performance comparison of the robust CVaR model with polyhedral uncertainty for mean for the normal with $\rho_1$ of the mixture for the S&P 500 data set.

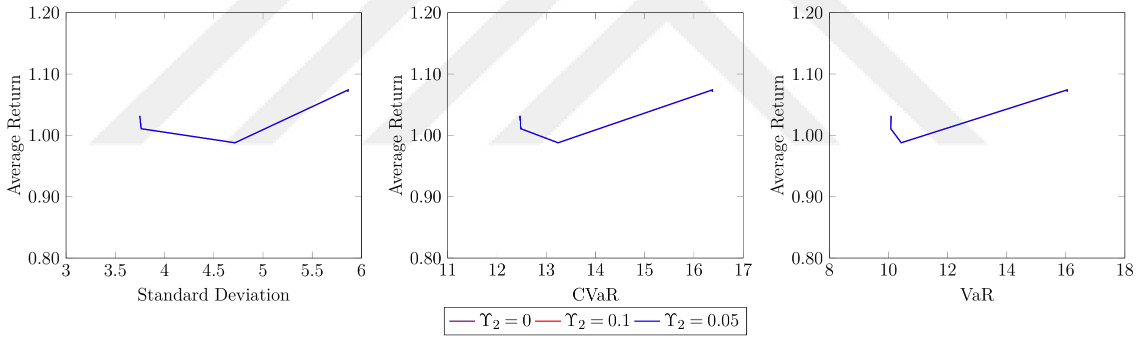


Figure 5.24 Performance comparison of the robust CVaR model with polyhedral uncertainty for mean for the normal with $\rho_2$ of the mixture for the S&P 500 data set.

According to Figures 5.23 and 5.24 one can say that, independent from the mixture, the robust problem is not very sensitive to any change in the robustness level.

#### 5.2.3.2 Performance of the Robust CVaR Model under Mixture Distribution with Budgeted Uncertainty

In this subsection, we will present the performance of the robust robust model with budgeted uncertainty discussed in Section 4.3.2.

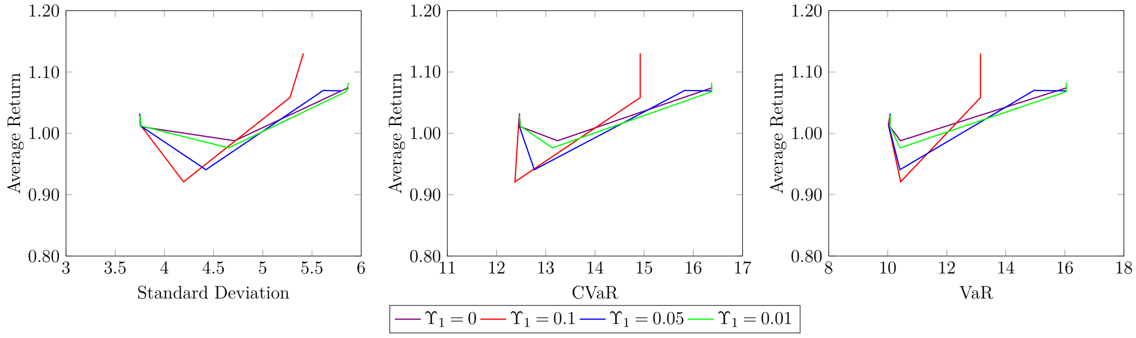


Figure 5.25 Performance comparison of the robust CVaR model with budgeted uncertainty for mean for the normal with $\rho_1$ of the mixture for the S&P 500 data set.

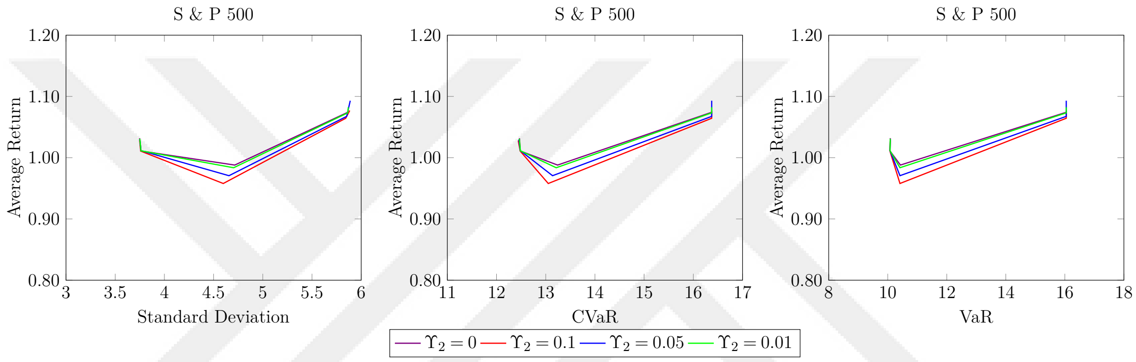


Figure 5.26 Performance comparison of the robust CVaR model with budgeted uncertainty for mean for the normal with $\rho_2$ of the mixture for the S&P 500 data set.

As can be seen from Figure 5.25, as $\Upsilon$ increases, the average return of the robust model starts to dominate the standard model while risk stays at similar levels for greater values of $\tau$. By contrast, Figure 5.26 shows that robust models result in poor portfolios as the robustness level increases.

#### 5.2.3.3 Performance of the Robust CVaR Model under Mixture Distribution with Ellipsoidal Uncertainty

The following figures will show the performance of the robust model with ellipsoidal uncertainty discussed in Section 4.3.2.

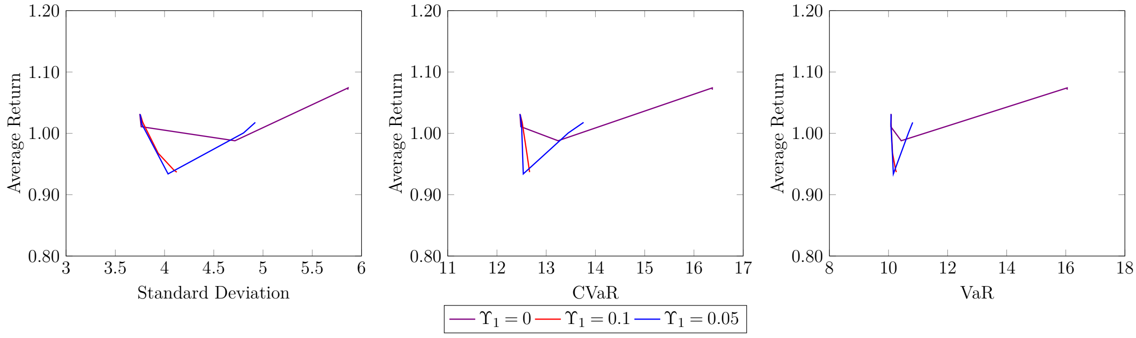


Figure 5.27 Performance comparison of the robust CVaR model with ellipsoidal uncertainty for mean for the normal with $\rho_1$ of the mixture for the S&P 500 data set.

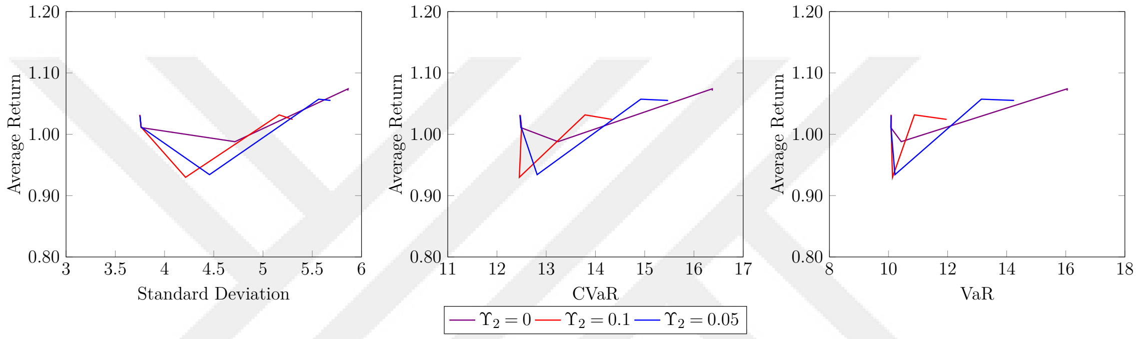


Figure 5.28 Performance comparison of the robust CVaR model with ellipsoidal uncertainty for mean for the normal with $\rho_2$ of the mixture for the S&P 500 data set.

In Figure 5.27, one can say that the resulting robust portfolios are inversely correlated with the increment in the robustness level in general. Moreover, for the same level of risk, the robust problem may result with a lower return. On the other hand, Figure 5.28 shows that for some same risk levels, the robust model results with higher returns. This may be due to the fact that the random returns come from the different normals of the mixture.

#### 5.2.3.4 Performance of the Robust CVaR Model under Mixture Distribution with Uncertainty for Covariance

In this subsection, we present the performance of the robust model with uncertainty for covariance discussed in Section 4.3.4.

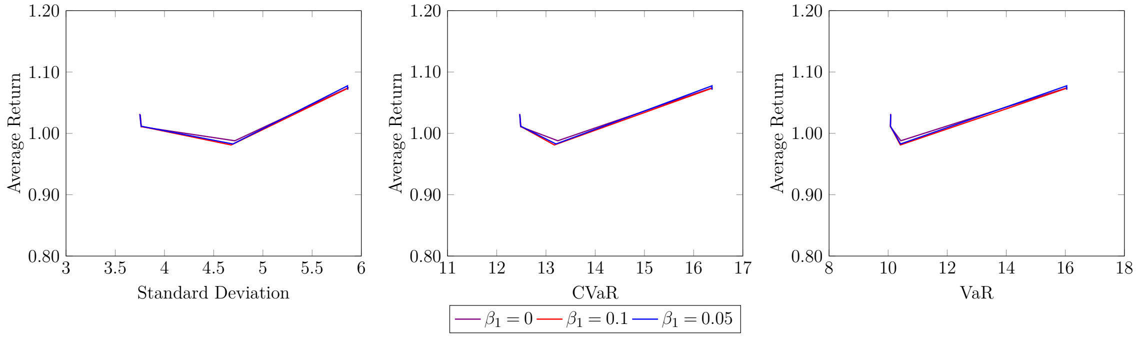


Figure 5.29 Performance comparison of the robust CVaR model with uncertainty for covariance for the normal with $\rho_1$ of the mixture for the S&P 500 data set.

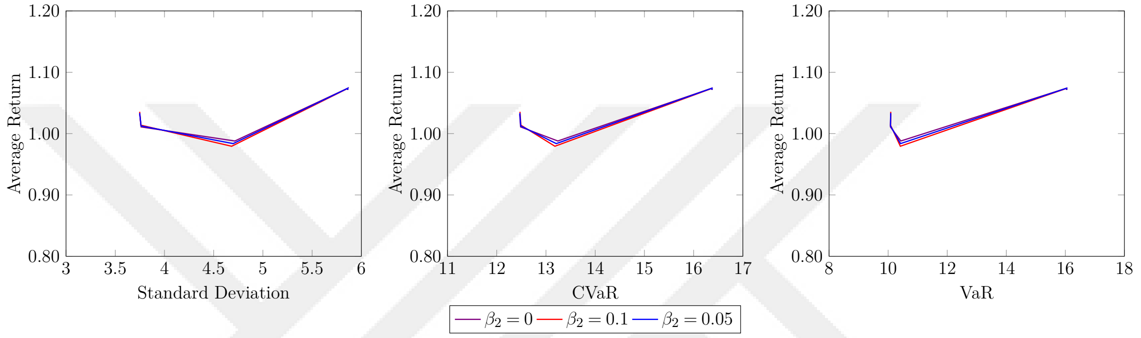


Figure 5.30 Performance comparison of the robust CVaR model with uncertainty for covariance for the normal with $\rho_2$ of the mixture for the S&P 500 data set.

According to Figures 5.29 and 5.30, one can say that the robust model remains nearly unchanged for the different values of $\beta$.

Unfortunately, due to some numerical problems, we cannot solve the robust CVaR model under mixture distribution with uncertainty for mean and covariance discussed in Section 4.3.5

### 5.3 Efficient Frontier Based Evaluation

Among all feasible portfolios at the same risk level, the one with the maximum expected return is called an efficient portfolio. An efficient frontier of portfolios can be obtained with a collection of efficient portfolios (Broadie (1993)). As shown by Ceria & Stubbs (2006), slightly different expected returns can cause portfolios to be very different. Therefore, to consider the error-maximization effect, we use an efficient frontier based analysis as another evaluation scheme.

Let again $T$ be the number of data points available and $m$ be a positive integer. As a first step in our approach, we compute the sample mean and sample covariance of the return vectors $r^1,...,r^T$ coming from real data. In the rest of the algorithm, these values are treated as *true* parameters. Then, by using these parameters, a synthetic data set is sampled, which is denoted $\hat{\xi}^1,...,\hat{\xi}^T$. Once we have the synthetic data set, we again estimate the parameters of the unknown true distribution. Then, we solve the optimization problem for each $\tau$ in order to obtain optimal portfolio vectors $x^*$ and $x^{**}$ of synthetic and true data respectively. In order to compute the estimated $({\gamma^t}_{est})$ and actual $({\gamma^t}_{act})$ expected returns, we evaluate the optimal portfolio vector $x^*$ with $\hat{\xi}^t$ and $r^t$ respectively. Then, true expected returns ${\gamma^t}_{true}$ are computed by evaluating the portfolio vector $x^{**}$ with $r^t$. Finally, we repeat this procedure for $t = 1,\ldots,T$ and for different values of $\tau$ and we evaluate the overall performance.

Algorithm 2 outlines the steps of the procedure summarized above.

---

**Algorithm 2**

**Input:** *Distribution*, Optimization Model, *Realizations* $= \{r^1,...,r^T\}$, $T$,$\alpha$.
**Output** $x^*$, $x^{**}$.

Estimate parameters $(\Theta^{true})$ of *Distribution* using $\{r^1,...,r^T\}$
Sample $T$ random return vectors from *Distribution* $(\Theta^{true})$ : $\hat{\xi}^1,...,\hat{\xi}^T$
Estimate $\Theta^{true}$ using $\hat{\xi}^1,...,\hat{\xi}^T$ as $\hat{\Theta}$
**for** $\tau = \tau_1$ to $\tau_m$ **do**
  Solve Optimization Model $(\Theta^{true},\tau,\alpha)$ to obtain $x^{**}$
  Solve Optimization Model $(\hat{\Theta},\tau,\alpha)$ to obtain $x^*$
  **for** $t = 1$ to $T$ **do**
    Compute ${\gamma^t}_{true} = x^{**T} r^t$
    Compute ${\gamma^t}_{est} = x^{*T} \hat{\xi}^t$
    Compute ${\gamma^t}_{act} = x^{*T} r^t$
  **end for**
  Report statistical measures using ${\gamma_{est}}^\tau, {\gamma_{act}}^\tau$, and${\gamma_{true}}^\tau$ (such as avg$^\tau$, stdev$^\tau$, VaR$^\tau$, CVaR$^\tau$)
**end for**
Draw frontiers

---

In sequel, figures will demonstrate the effects of changes in robustness levels for uncertain parameters on optimal portfolios.

### 5.3.1 Results of the Markowitz Model

In this subsection, we solve the standard and robust versions of the Markowitz model (3.1). The following figures will demonstrate the average expected return of the optimal portfolios with respect to risk measures; standard deviation, $\mathrm{CVaR}_{0.01}$, and $\mathrm{VaR}_{0.01}$ for different values of risk aversion constant $\tau$.

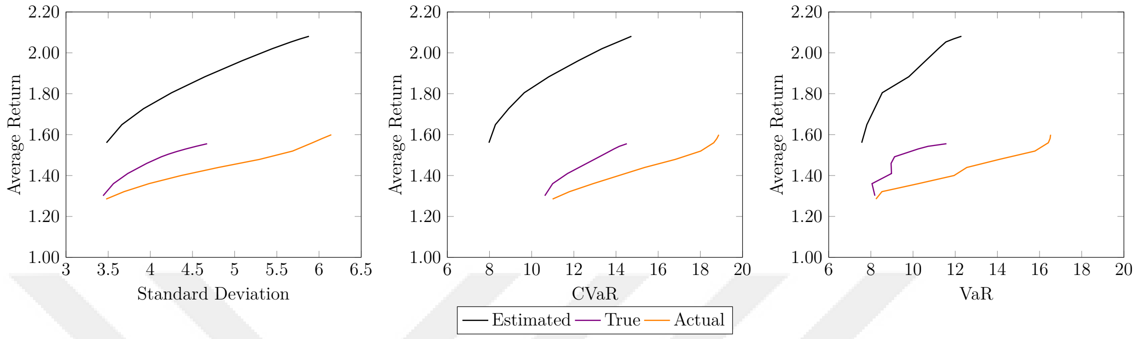


Figure 5.31 Efficient frontiers of the standard Markowitz model for the S&P 500 data set.

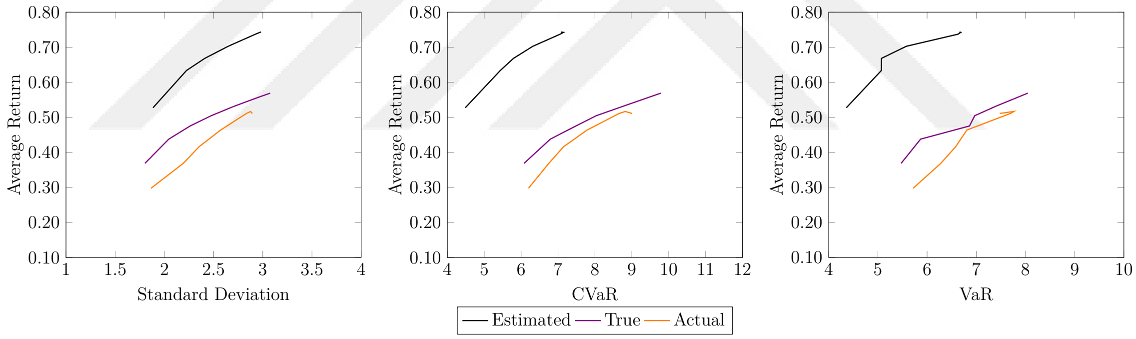


Figure 5.32 Efficient frontiers of the standard Markowitz model for the MIBTEL data set.

The actual frontier always lies below the true frontier by definition. In Figure 5.31 and 5.32, illustrated frontiers support this fact. Since expected returns are overestimated, the estimated frontier gives a result that is very different with higher expected returns from the true frontier. Therefore, estimation errors in mean return misleads the investor and results in dominated portfolios, especially when $\tau$ is large.

#### 5.3.1.1 Results of the Robust Markowitz Model with Polyhedral Uncertainty

Here, the following figures will present the results of the robust Markowitz model with polyhedral uncertainty discussed in Section 4.1.1 on the data sets.

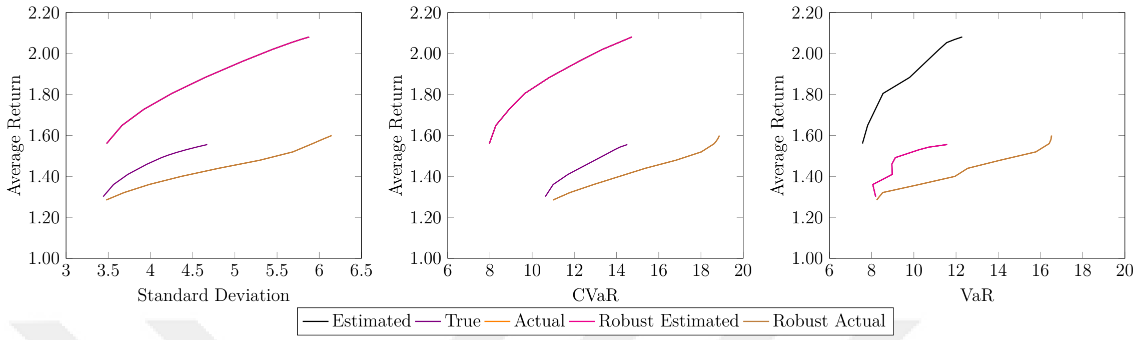


Figure 5.33 Efficient frontiers of the robust Markowitz model with polyhedral uncertainty for mean for the S&P 500 data set when $\Upsilon = 0.5$.

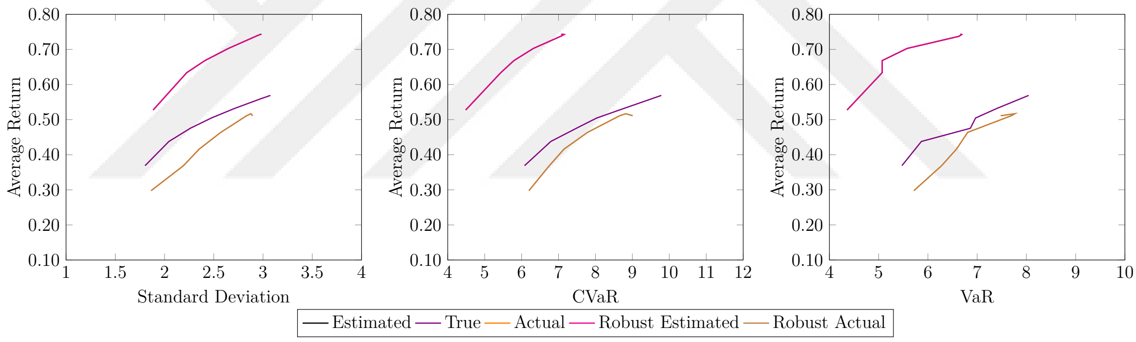


Figure 5.34 Efficient frontiers of the robust Markowitz model with polyhedral uncertainty for mean for the MIBTEL data set when $\Upsilon = 0.5$.

As can be seen from Figures 5.33 and 5.34, robust portfolio models are not very sensitive to changes in the $\Upsilon$.

#### 5.3.1.2 Results of the Robust Markowitz Model with Budgeted Uncertainty

In this subsection, the following figures will present the results of the robust Markowitz model with budgeted uncertainty discussed in Section 4.1.2 on the data sets. Although the aim is to get true and actual frontiers closer as in Ceria & Stubbs

(2006), experimental computations show that actual frontiers outperform the robust actual frontiers.

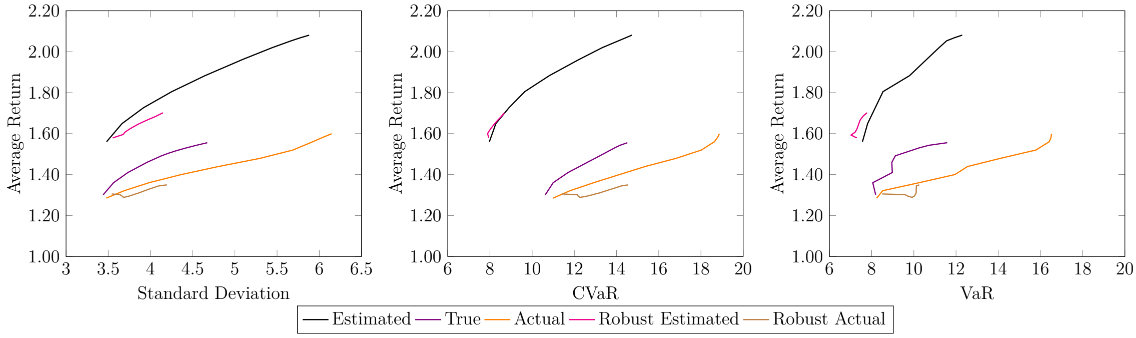


Figure 5.35 Efficient frontiers of the robust Markowitz model with budgeted uncertainty for mean for the S&P 500 data set when $\Upsilon = 0.5$.

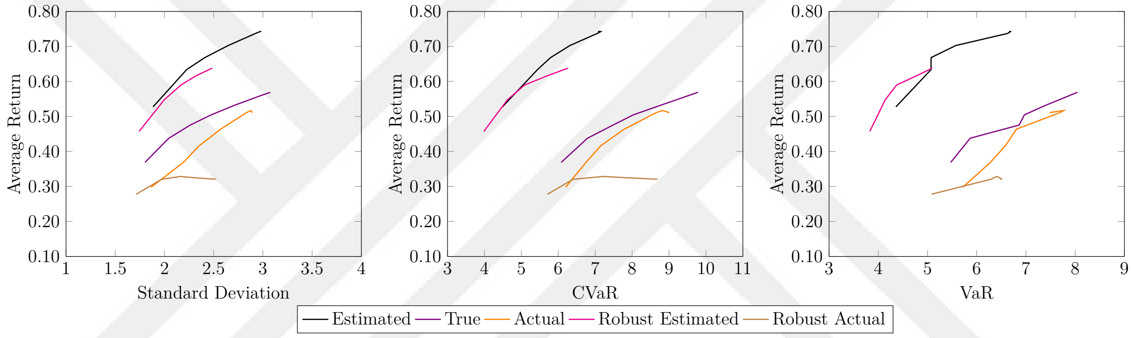


Figure 5.36 Efficient frontiers of the robust Markowitz model with budgeted uncertainty for mean for the MIBTEL data set when $\Upsilon = 0.5$.

Figures 5.35 and 5.36 depict that, as the result of using robust optimization, the estimated frontier lie closer to the true frontier.

#### 5.3.1.3 Results of the Robust Markowitz Model with Ellipsoidal Uncertainty

Here, we present the results of the robust Markowitz model with ellipsoidal uncertainty discussed in Section 4.3.2 on the S&P 500 data set.

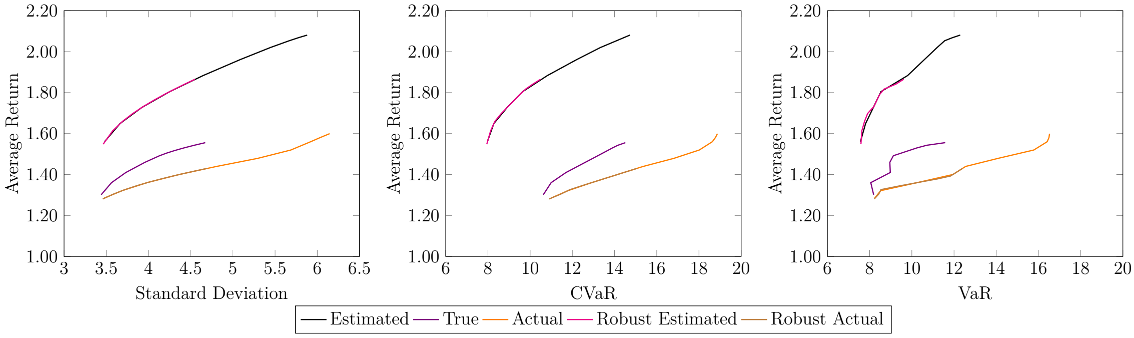


Figure 5.37 Efficient frontiers of the robust Markowitz model with ellipsoidal uncertainty for mean for the S&P 500 data set when $\Upsilon = 0.1$.

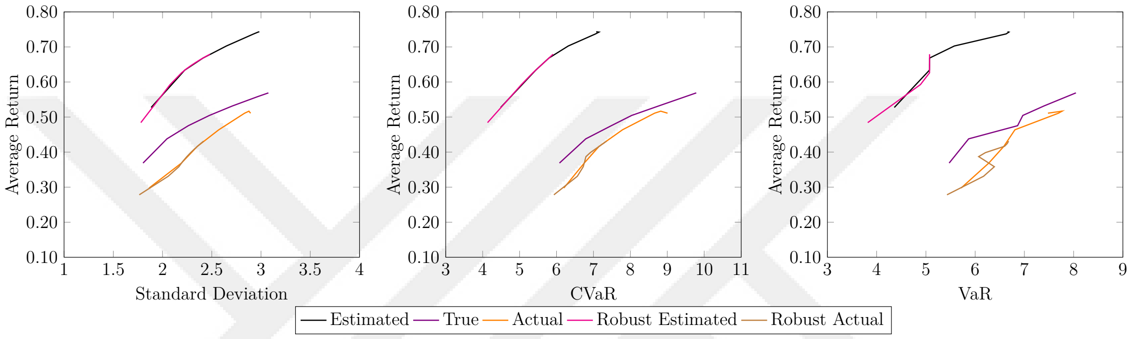


Figure 5.38 Efficient frontiers of the robust Markowitz model with ellipsoidal uncertainty for mean for the MIBTEL data set when $\Upsilon = 0.1$.

Figures 5.37 and 5.38 shows that the robust frontiers result in same average return with the standard frontiers for the same risk levels; standard deviation, CVaR, and VaR.

In this chapter, one of the general conclusions regarding the performances of the experiments is that if the investor is inclined to take more risk (or is a risk-taker), then using robust optimization models cause the optimal portfolio to yield with higher returns. Second, using robust Markowitz model with budgeted uncertainty for the same levels of risk may result with higher returns. Third, the parameter $\beta$ which controls robustness level for covariance, does not have a significant effect on portfolios. Lastly, optimal portfolios of the MIBTEL data set shows more reliable results than the S&P 500 data set does for the standard Markowitz model. This might be due to the fact that the difference in the time spanning of the data sets.

# 6. CONCLUSION

In this thesis, we address a number of challenging aspects of portfolio optimization problem including conflicting objectives, inadequate risk measures, and sensitivity issue in parameter estimation. We provide an overview of risk adjusted optimization models with different risk measures including variance, Value-at-Risk, and Conditional Value-at-Risk. We then adapt robust optimization to incorporate estimation errors or perturbations into the standard portfolio optimization problem. We present an analysis on robust portfolio optimization problems with uncertainty sets involving polytopic, ellipsoidal, or budgeted uncertainty for either mean or covariance or both, and cast these problems as conic programs. Moreover, we provide computational experiments on two different data sets and compare the performances of the resulting portfolios. Our computational experiments show that optimal portfolios constructed with the robust optimization models yield higher returns and higher risk than the standard portfolio models. Furthermore, employing robust models with budgeted uncertainty for the same levels of risk results in portfolios with higher returns. Finally, we conclude that changes in the robustness levels for covariance matrix have relatively limited effect on resulting portfolios than the mean return vector.

# BIBLIOGRAPHY


https://host.uniroma3.it/docenti/cesarone/DataSets.htm.

Industries. https://finance.yahoo.com/.

Artzner, P., Delbaen, F., Eber, J.-M., & Heath, D. (1999). Coherent measures of risk. *Mathematical finance*, *9*(3), 203–228.

Ben-Tal, A. & Nemirovski, A. (1998). Robust convex optimization. *Mathematics of operations research*, *23*(4), 769–805.

Ben-Tal, A. & Nemirovski, A. (2001). *Lectures on modern convex optimization: analysis, algorithms, and engineering applications*. SIAM.

Bertsimas, D., Brown, D. B., & Caramanis, C. (2011). Theory and applications of robust optimization. *SIAM review*, *53*(3), 464–501.

Bertsimas, D. & Popescu, I. (2002). *Optimal inequalities in probability theory: a convex optimization approach*. INSEAD.

Bertsimas, D. & Sim, M. (2004). The price of robustness. *Operations research*, *52*(1), 35–53.

Best, M. J. & Grauer, R. R. (1991). On the sensitivity of mean-variance-efficient portfolios to changes in asset means: some analytical and computational results. *The review of financial studies*, *4*(2), 315–342.

Black, F. & Litterman, R. (1990). Asset allocation: combining investor views with market equilibrium. Technical report, Discussion paper, Goldman, Sachs & Co.

Black, F. & Litterman, R. (1992). Global portfolio optimization. *Financial analysts journal*, *48*(5), 28–43.

Broadie, M. (1993). Computing efficient frontiers using estimated parameters. *Annals of Operations Research*, *45*(1), 21–58.

Ceria, S. & Stubbs, R. (2006). Incorporating estimation errors into portfolio selection: Robust portfolio construction. In *Asset Management* (pp. 270–294). Springer.

Chopra, V. K. & Ziemba, W. T. (2013). The effect of errors in means, variances, and covariances on optimal portfolio choice. In *Handbook of the Fundamentals of Financial Decision Making: Part I* (pp. 365–373). World Scientific.

Cornuejols, G. & Tütüncü, R. (2005). Optimization methods in finance.

DeMiguel, V. & Nogales, F. J. (2009). Portfolio selection with robust estimation. *Operations Research*, *57*(3), 560–577.

Dempster, A. P., Laird, N. M., & Rubin, D. B. (1977). Maximum likelihood from incomplete data via the em algorithm. *Journal of the Royal Statistical Society: Series B (Methodological)*, *39*(1), 1–22.

Frost, P. A. & Savarino, J. E. (1988). For better performance: Constrain portfolio weights. *Journal of Portfolio Management*, *15*(1), 29.

Ghaoui, L. E., Oks, M., & Oustry, F. (2003). Worst-case value-at-risk and robust portfolio optimization: A conic programming approach. *Operations research*, *51*(4), 543–556.

Goldfarb, D. & Iyengar, G. (2003). Robust portfolio selection problems. *Mathematics of operations research*, *28*(1), 1–38.

Horasanlı, M. & Fidan, N. (2007). Portfolio selection by using time varying covariance matrices. *Journal of Economic and Social Research*, *9*(2), 1–22.

Idzorek, T. (2007). A step-by-step guide to the black-litterman model: Incorporating user-specified confidence levels. In *Forecasting expected returns in the financial markets* (pp. 17–38). Elsevier.

Kocuk, B. & Cornuéjols, G. (2020). Incorporating black-litterman views in portfolio construction when stock returns are a mixture of normals. *Omega*, *91*, 102008.

Krokhmal, P., Palmquist, J., & Uryasev, S. (2002). Portfolio optimization with conditional value-at-risk objective and constraints. *Journal of risk*, *4*, 43–68.

Lee, S. & Stevenson, S. (2003). Time weighted portfolio optimisation. *Journal of Property Investment & Finance*, *21*(3), 233–249.

Lobo, M. S. & Boyd, S. (2000). The worst-case risk of a portfolio. *Unpublished manuscript. Available from http://faculty. fuqua. duke. edu/%7Emlobo/bio/researchfiles/rsk-bnd. pdf.*

Markowitz, H. (1952). Portfolio selection. *The journal of finance*, *7*(1), 77–91.

Pang, T. & Karan, C. (2018). A closed-form solution of the black–litterman model with conditional value at risk. *Operations Research Letters*, *46*(1), 103–108.

Perry, M. B. (2010). The exponentially weighted moving average. *Wiley Encyclopedia of Operations Research and Management Science.*

Pflug, G. C. (2000). Some remarks on the value-at-risk and the conditional value-at-risk. In *Probabilistic constrained optimization* (pp. 272–281). Springer.

Rockafellar, R. T. & Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. *Journal of banking & finance*, *26*(7), 1443–1471.

Rockafellar, R. T., Uryasev, S., et al. (2000). Optimization of conditional value-at-risk. *Journal of risk*, *2*, 21–42.

Tütüncü, R. H. & Koenig, M. (2004). Robust asset allocation. *Annals of Operations Research*, *132*(1-4), 157–187.

Zhu, S. & Fukushima, M. (2009). Worst-case conditional value-at-risk with application to robust portfolio management. *Operations research*, *57*(5), 1155–1168.

# APPENDIX A

Table A.1 Covariance Martix of the S&P 500 Data.

| Sector | Energy | Consumer discretionary | Consumer staples | Real estate | Industrials | Financials | Telecomunication services | Information technolegy | Materials | Health care | Utilities |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Energy | 0.39467 | 0.15328 | 0.09222 | 0.17256 | 0.17998 | 0.18296 | 0.10578 | 0.16700 | 0.21803 | 0.11611 | 0.08835 |
| Consumer discretionary | 0.15328 | 0.28297 | 0.13578 | 0.26924 | 0.22956 | 0.24931 | 0.14119 | 0.26058 | 0.22964 | 0.13237 | 0.05476 |
| Consumer staples | 0.09222 | 0.13578 | 0.17198 | 0.15055 | 0.13905 | 0.15840 | 0.10254 | 0.11175 | 0.13007 | 0.13338 | 0.07370 |
| Real estate | 0.17256 | 0.26924 | 0.15055 | 0.52492 | 0.26411 | 0.27134 | 0.14250 | 0.26131 | 0.29716 | 0.13643 | 0.08078 |
| Industrials | 0.17998 | 0.22956 | 0.13905 | 0.26411 | 0.26528 | 0.25398 | 0.13513 | 0.23726 | 0.24327 | 0.14576 | 0.07437 |
| Financials | 0.18296 | 0.24931 | 0.15840 | 0.27134 | 0.25398 | 0.39451 | 0.14458 | 0.22606 | 0.24142 | 0.17601 | 0.08495 |
| Telecomunication services | 0.10578 | 0.14119 | 0.10254 | 0.14250 | 0.13513 | 0.14458 | 0.30009 | 0.16993 | 0.12452 | 0.10015 | 0.07304 |
| Information technolegy | 0.16700 | 0.26058 | 0.11175 | 0.26131 | 0.23726 | 0.22606 | 0.16993 | 0.50455 | 0.22041 | 0.13847 | 0.02584 |
| Materials | 0.21803 | 0.22964 | 0.13007 | 0.29716 | 0.24327 | 0.24142 | 0.12452 | 0.22041 | 0.32438 | 0.13312 | 0.05912 |
| Health care | 0.11611 | 0.13237 | 0.13338 | 0.13643 | 0.14576 | 0.17601 | 0.10015 | 0.13847 | 0.13312 | 0.21559 | 0.07575 |
| Utilities | 0.08835 | 0.05476 | 0.07370 | 0.08078 | 0.07437 | 0.08495 | 0.07304 | 0.02584 | 0.05912 | 0.07575 | 0.18319 |

Table A.2 Covariance Martix of the MIBTEL Data.

| Sector | Energy | Capital Goods | Consumer Cyclical | Real Estate | Industrials | Financials | Communication | Basic Materials | Technology | Healthcare | Utilities |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Energy | 0.094908 | 0.027610 | 0.014517 | 0.033282 | 0.027966 | 0.022030 | 0.027298 | 0.029921 | 0.028578 | 0.023419 | 0.022406 |
| Capital Goods | 0.027610 | 0.036395 | 0.015373 | 0.033163 | 0.031305 | 0.024230 | 0.032785 | 0.032171 | 0.032232 | 0.025560 | 0.026245 |
| Consumer Cyclical | 0.014517 | 0.015373 | 0.451436 | 0.029711 | 0.022663 | 0.017910 | 0.027925 | 0.020968 | 0.026278 | 0.013837 | 0.018758 |
| Real Estate | 0.033282 | 0.033163 | 0.029711 | 0.077146 | 0.036037 | 0.027016 | 0.037380 | 0.033850 | 0.036737 | 0.027750 | 0.030204 |
| Industrials | 0.027966 | 0.031305 | 0.022663 | 0.036037 | 0.043266 | 0.023052 | 0.033021 | 0.032831 | 0.034763 | 0.027804 | 0.026924 |
| Financials | 0.022030 | 0.024230 | 0.017910 | 0.027016 | 0.023052 | 0.028269 | 0.028018 | 0.026276 | 0.025289 | 0.022582 | 0.022985 |
| Communication | 0.027298 | 0.032785 | 0.027925 | 0.037380 | 0.033021 | 0.028018 | 0.058229 | 0.038335 | 0.034061 | 0.028857 | 0.030853 |
| Basic Materials | 0.029921 | 0.032171 | 0.020968 | 0.033850 | 0.032831 | 0.026276 | 0.038335 | 0.052891 | 0.032656 | 0.030330 | 0.026788 |
| Technology | 0.028578 | 0.032232 | 0.026278 | 0.036737 | 0.034763 | 0.025289 | 0.034061 | 0.032656 | 0.057101 | 0.028595 | 0.033516 |
| Healthcare | 0.023419 | 0.025560 | 0.013837 | 0.027750 | 0.027804 | 0.022582 | 0.028857 | 0.030330 | 0.028595 | 0.055992 | 0.023332 |
| Utilities | 0.022406 | 0.026245 | 0.018758 | 0.030204 | 0.026924 | 0.022985 | 0.030853 | 0.026788 | 0.033516 | 0.023332 | 0.049563 |