REPUBLIC OF TURKEY
YILDIZ TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON THE ANALYTICAL SOLUTION METHODS OF MULTIDIMENSIONAL HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

Burcu GÖNÜL

MASTER OF SCIENCE THESIS
Department of Mathematics
Mathematics Program

Advisor
Assoc. Prof. Dr. Özgür YILDIRIM

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A thesis submitted by Burcu GÖNÜL in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE** is approved by the committee on 06.07.2020 in Department of Mathematics, Mathematics Program.

Assoc. Prof. Dr. Özgür YILDIRIM
Yıldız Technical University
Advisor

Approved By the Examining Committee

Assoc. Prof. Dr. Özgür YILDIRIM, Advisor
Yıldız Technical University

Prof. Dr. Ömer GÖK, Member
Yıldız Technical University

Prof. Dr. Allaberen ASHYRALYEV, Member
Near East University

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Burcu GÖNÜL

Signature

Dedicated to my family

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Burcu GÖNÜL

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LIST OF SYMBOLS

Δ	Discriminant
H	Fluid Velocity Vector
∇	Gradient
c	Heat Transmission Coefficient
ρ	Intensity of the Solid Substance
C	Specific Heat
P	Static Pressure
T	Temperature

LIST OF ABBREVIATIONS

BVP	Boundary Value Problem
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation

ABSTRACT

ON THE ANALYTICAL SOLUTION METHODS OF MULTIDIMENSIONAL HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

Burcu GÖNÜL

Department of Mathematics
Master of Science Thesis

Advisor: Assoc. Prof. Dr. Özgür YILDIRIM

In this thesis, the analytical solution methods for solving multidimensional partial differential equations (PDEs) in particular two dimensional hyperbolic partial differential equations are investigated. In modeling of the partial differential equations, the transport of some physical quantities, e.g. liquids or waves are defined by hyperbolic equations. In this work, wave equations are solved by using Fourier series and Fourier transform methods. Fourier series help us to model any periodic function as a combine of sine and cosine trigonometric functions. Fourier series make use of the orthogonality relationships of the sine and cosine functions. On the other hand, the Fourier transform is an extending of the Fourier series adapted to non periodic functions. The Fourier Transform shows that any waveform can be rewritten as the sum of sinusoidal functions. The first chapter of the thesis consists of the introduction part, literature review, objective and hypothesis of the thesis. In the second chapter Fourier series are considered and solutions are obtained for one and two dimensional hyperbolic partial differential equations by Fourier series method. In the third chapter Fourier transform method is introduced and one and two dimensional wave equations are solved by using the Fourier transform method. Finally, the last chapter includes the results and discussion parts.

Keywords: Hyperbolic partial differential equations, Fourier series, Fourier transform

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ÖZET

ÇOK BOYUTLU HİPERBOLİK KISMİ TÜREVLİ DİFERANSİYEL DENKLEMLERİN ANALİTİK ÇÖZÜM METOTLARI

Burcu GÖNÜL

Matematik Anabilim Dalı
Yüksek Lisans Tezi

Danışman: Doç. Dr. Özgür YILDIRIM

Bu tezde çok boyutlu kısmi diferansiyel denklemlerin, özellikle iki boyutlu hiperbolik denklemlerin, analitik çözümleri araştırılmıştır. Kısmi diferansiyel denklemlerin modellenmesinde, bazı fiziksel miktarların, örneğin sıvılar veya dalgaların taşınması hiperbolik denklemlerle tanımlanır. Bu çalışmada dalga denklemleri Fourier serileri ve Fourier dönüşüm yöntemleri kullanılarak çözülmüştür. Fourier serisi herhangi bir periyodik fonksiyonu sinüs ve kosinüs trigonometrik fonksiyonların bir kombinasyonu olarak modellememizi sağlar. Fourier serileri sinüs ve kosinüs fonksiyonlarının diklik ilişkilerinden yararlanır. Öte yandan, Fourier dönüşümü periyodik olmayan fonksiyonlara uyarlanan Fourier serisinin bir uzantısıdır. Fourier Dönüşümü, herhangi bir dalga formunun sinüzoidal fonksiyonların toplamı olarak yeniden yazılabileceğini gösterir. Tezin ilk bölümü; giriş bölümü, literatür taraması, tezin amacı ve hipotezinden oluşmaktadır. İkinci bölümde Fourier serileri ele alınmış ve Fourier serileri yöntemi ile bir ve iki boyutlu hiperbolik kısmi diferansiyel denklemler için çözümler elde edilmiştir. Üçüncü bölümde Fourier dönüşüm yöntemi tanıtılmış ve bir ve iki boyutlu dalga denklemleri Fourier transform metodunu kullanarak çözülmüştür. Son olarak, son bölüm sonuçları ve tartışma bölümlerini içerir.

Anahtar Kelimeler: Hiperbolik kısmi diferansiyel denklemler, Fourier serileri, Fourier transform

YILDIZ TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

1
INTRODUCTION

1.1 Literature Review

Studies on partial differential equations (PDE) began by 1700s by Euler, Lagrange and d'Alembert for the science of physics. From the mid-19th century to the present, PDE's have become important in other branches of mathematics, especially with Riemann's work on partial differential equations. This dilemma led to the work of PDE's throughout the nineteenth and twentieth centuries. Some used PDE's for modeling in applied branches such as physics and engineering, while others used them for the progress of fields of mathematics. This binary viewpoint was first explicitly remarked in an article by H. Poincare in the 19th century [1]. Poincare pointed out that there are various physically important problems that arise in many different areas. For example; electricity, hydrodynamics, heat, magnetism, optics, etc. In the same paper, it is emphasized that the equations used in mathematical physics are also important in mathematics. This is actually a bridge throughout the 20th century between the problems of mathematics and physics and the progress of mathematical opinions in the fields of pristine mathematics.

First, Dirichlet principle and its uses, which Riemann's technics is to develop an analytical functions theory of a complex variable, can be given as an sample. The next important example is differential geometry, which promotes the assay of PDE's. Otherwise, the theory of first order PDE systems interacts with the theory of Lie in the study of S. Lie [2].

Since the beginning of the 20th century, the desire to develop base means in real and functional analysis has required addressing solutions of boundary value problems (BVP's)[3]. The development of the Fourier analysis took place widely in the 20th century with the studies of linear PDE's and its BVP's. However, the linear PDE theory has been greatly expanded by the application of Fourier analysis to PDE's with Fourier integral operators.

A partial differential equation which are consider as continually in all fields of physics and mathematics determines a connection among an unknowable function and its derivatives [4]. In addition to being frequently sighted in the field of physics and engineering, we have also seen a major raise in the utilization of PDE's in fields like biological science, chemical science, computer science and economics of late years.

Actually, we try to describe the functions in these variables in the field where there are independent variables. And then we model the various processes by establishing equations [5]. In the most general description of a PDE's for $w(x_1, x_2, ..., x_n)$ is

$$F\left(x_1, ..., x_{n-1}, x_n, w, w_{x_1}, w_{x_2}, ..., w_{x_{11}}, ...\right) = 0. \tag{1.1}$$

In this equation, the independent variables are indicated by $x_1, x_2, ..., x_n$, the unknown function is indicated by w, and the partial derivative $\partial w/\partial x_i$ is indicated by w_{x_i}. By giving the initial conditions and boundary conditions the equation is supported.

There are many methods to analyze PDE's. The aim of the classic approach in the 19th centenary was to improve technics to find explicit solutions. We have said that PDE's are used in various fields of mathematics. It is also of great importance in different branches of physics. For example, the Fourier method provided the solution of heat convection. At the same time, this method provided a solution for wave transmission. Method of Green's contributed to the improvement of electromagnetic theory and was effective. But the biggest improvement in the expansion of PDE's came when it entered computer science. The use of numerical methods, which allows computers to be used to solve almost any type of PDE in general geometries and arbitrary external conditions, proves this.

Now, let us examine classification of PDE.

1.2 Classification of PDE

The well known form of the second order linear PDE is

$$A\frac{\partial^2 w}{\partial x^2} + B\frac{\partial^2 w}{\partial x \partial y} + C\frac{\partial^2 w}{\partial y^2} + D\frac{\partial w}{\partial x} + E\frac{\partial w}{\partial y} + Fw = G \tag{1.2}$$

such that A, B, C constants are functions of x, y. Also, the value of D, E, F are supposed to be functions of x and y.

Second order PDE's classification depend on the manner of the equation comprising of second order terms. Therefore, for convenience, we reunite lower order terms. Then we rewrite the equation (1.2) as

$$A(x,y)\frac{\partial^2 w}{\partial x^2}+B(x,y)\frac{\partial^2 w}{\partial x\partial y}+C(x,y)\frac{\partial^2 w}{\partial y^2}=\varphi\left(x,y,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\right). \tag{1.3}$$

We can also write briefly as follows:

$$A(x,y)w_{xx}+B(x,y)w_{xy}+C(x,y)w_{yy}=\varphi\left(x,y,w,w_x,w_y\right). \tag{1.4}$$

As we have seen, there are three basic types of PDE: elliptic, parabolic and hyperbolic. Physically, these equations represent steady state or equilibrium operations, diffusion processes, and wave propagation, respectively. Therefore, elliptic equations relate to the aspect of a system corresponding to the minimum power. Parabolic equations identify evolutive phenomena that cause to described a stable state with an elliptic equation. And hyperbolic equations represent the transportation of some physical quantities like liquids or waves.

The classification of second order PDE's is predicated on reducing the equation (1.4) and transforming it into canonical or standard type at a point in mathematics. At this place, we can point out that it is not essential to limit the linear equations for classification purposes. It is also applicable to quasilinear second order PDE. These PDE's are linear merely in the second derivatives.

From equation (1.4), the type of the second order PDE at (x_0, y_0) is determined by the symbol of the discriminant and

$$\Delta(x_0,y_0)=\begin{vmatrix} B & 2A \\ 2C & B \end{vmatrix}=B^2-4AC. \tag{1.5}$$

The second order linear PDE's classification is depend on Δ. Thus, the equation is

$$\begin{aligned} \text{hyperbolic, if } \Delta &> 0, \\ \text{parabolic, if } \Delta &= 0, \\ \text{elliptic, if } \Delta &< 0. \end{aligned}$$

Now, we will examine what the elliptic, parabolic and hyperbolic equations mean physically.

1.2.1 Elliptic PDE

The Laplace equation is a usual example for elliptic PDE:

$$\nabla^2 f = 0. \tag{1.6}$$

The Laplace equation concern about matters in fluid flow, mass flux, heat propagation, etc. The general properties of the Laplace equation for the stable heat diffusion problem in a solid are shown below.

Heat flow in a solid is act through Fourier's statute of transmission [6].

$$\hat{Q} = -cA\frac{dT}{dn}. \tag{1.7}$$

In this equation, the energy transfer in unit of time is indicated by $\hat{Q}$, the temperature is indicated by T, the across which the energy flows are indicated by A, the temperature gradient normal to the area A is indicated by dT/dn, and the heat transmission coefficient of the solid is indicated by c. The total ratio of flow energy into the solid in the x axis way is

$$\hat{Q}_{net,x} = \hat{Q}(x) - \hat{Q}(x+dx) = \hat{Q}(x) - \left[\hat{Q}(x) + \frac{\partial \hat{Q}(x)}{\partial x}dx\right] = -\frac{\partial \hat{Q}(x)}{\partial x}. \tag{1.8}$$

Consider the differential cube of solid material where heat diffusion takes place. If $\hat{Q}$ value in the equation (1.7) is written in the equation (1.8),we get

$$\hat{Q}_{net,x} = -\frac{\partial}{\partial x}\left(-cA\frac{\partial T}{\partial x}\right)dx = \frac{\partial}{\partial x}\left(c\frac{\partial T}{\partial x}\right)dV. \tag{1.9}$$

where $dV = Adx$ is the differential cube volume. Likewise,

$$\hat{Q}_{net,y} = \frac{\partial}{\partial y}\left(c\frac{\partial T}{\partial y}\right)dV, \tag{1.10}$$

$$\hat{Q}_{net,z} = \frac{\partial}{\partial z}\left(c\frac{\partial T}{\partial z}\right)dV. \tag{1.11}$$

There is no total alteration in the quantity of energy reserved in the solid for stable heat flow. For this reason, the sum of the net proportion of flow of energy in the three way is zero. Therefore,

$$\frac{\partial}{\partial x}\left(c\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(c\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(c\frac{\partial T}{\partial z}\right) = 0. \tag{1.12}$$

The stable diffusion of heat in a solid is checked by equation (1.12). If heat transmission coefficient c is eliminated in the equation (1.12) , we get the equation

$$T_{xx} + T_{yy} + T_{zz} = \nabla^2 T = 0 \tag{1.13}$$

which is the Laplace equation.

The two dimensional heat distribution equation is

$$T_{xx} + T_{yy} = 0. \tag{1.14}$$

The terms of the general second order PDE's described in the equation (1.4) appear as $A = 1$, $B = 0$, $C = 1$. Hence, we find that $\Delta < 0$. Eventually, (1.14) is an elliptic PDE.

In brief, the partial differential equation governing stable heat transmission is a classic example for an elliptic PDE.

1.2.2 Parabolic PDE

Usual example for parabolic PDE is the diffusion equation

$$f_t = \alpha \nabla^2 f. \tag{1.15}$$

The diffusion equation usually concern about problems in mass, momentum, heat diffusion.

The equations which (1.9) to (1.11) show the net afflux of heat in the x, y, z directions. About stable heat flow, there is no clear alteration in the quantity of energy reserved in the solid. Therefore, the total of net heat flow constituents is zero. Contrary to this, there may be a clear alteration over time in the quantity of energy reserved in the solid in an inconsistent state [6]. The energy E reserved in the solid mass dm is shown as follows:

$$E_{stored} = dmCT = (\rho dV)CT = (\rho CT)dV. \tag{1.16}$$

In this equation, the intensity of the solid substance is indicated by ρ, the differential volume is indicated by dV, the temperature is indicated by T, and the specific heat is indicated by C. Specific heat is a physical speciality the solid substance. The total of the net heat afflux should equal the time rate of alteration of the reserved energy.

Thereby,

$$\frac{\partial(\rho CT)}{\partial t}=\frac{\partial}{\partial x}\left(c\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(c\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(c\frac{\partial T}{\partial z}\right). \tag{1.17}$$

The inconsistent diffusion of heat in a solid is checked by the equation (1.17). When the heat transmission coefficient c, intensity ρ and specific heat C in the equation (1.17) are considered constant and these values are eliminated, we get the equation

$$T_t=\alpha(T_{xx}+T_{yy}+T_{zz})=\alpha\nabla^2 T \tag{1.18}$$

where α is the thermal diffusivity coefficient. The equation (1.18) is named the diffusion equation.

The one dimensional inconsistent heat diffusion equation is

$$T_t=\alpha T_{xx}. \tag{1.19}$$

The terms of the general second order PDE's described in the equation (1.4) appear as $A=\alpha$, $B=0$, and $C=0$. So, discriminant is $\Delta=0$. Eventually, the equation (1.19) is an parabolic PDE.

In brief, the heat equation managing unstable heat diffusion is an usual example for parabolic PDE.

1.2.3 Hyperbolic PDE

Usual example for hyperbolic PDE is the wave equations such that

$$f_{tt}=c^2\nabla^2 f. \tag{1.20}$$

Electrostatic, vibration, gas dynamics, acoustics, etc. problems are modeled with the wave equation. Now, let us examine some uses of the wave equation in physics.

Fluid afflux is ruled by the law of preservation of mass, second law of Newton which the momentum equation, and the first law which the energy equation [6]. These basic physical laws are expressed by a quasilinear first order PDE's as follows:

$$\rho_t+\nabla.(\rho\mathbf{H})=0, \tag{1.21}$$

$$\rho\mathbf{H}_t+\rho\left(\mathbf{H}.\nabla\right)\mathbf{H}+\nabla P=0, \tag{1.22}$$

$$P_t + \mathbf{H}.\nabla P - \alpha^2 (\rho_t + \mathbf{H}.\nabla P) = 0. \tag{1.23}$$

In these equations, fluid density is indicated by ρ, fluid velocity vector is indicated by **H**, static pressure is indicated by P and propagation velocity of small disturbances is indicated by α. Equations (1.21) to (1.23) are limited by the flow of a pure substance without mass, momentum, or energy diffusion. The following equations are unstable one dimensional flow equations.

$$\rho_t + \rho u_x + u\rho_x = 0, \tag{1.24}$$

$$\rho u_t + \rho u_x + P_x = 0, \tag{1.25}$$

$$P_t + uP_x - \alpha^2 (\rho_t + u\rho_x) = 0. \tag{1.26}$$

The equations (1.24) to (1.26) are more general examples of the following equation, which is a simple one dimensional convection equation

$$f_t + uf_x = 0 \tag{1.27}$$

where the function f is being convected by the speed u over the solution domain $D(x,t)$. Equation (1.27) in three independent variables is

$$f_t + uf_x + vf_y + wf_z = f_t + \mathbf{H}.\nabla f = \frac{Df}{Dt} = 0 \tag{1.28}$$

such that u, v, w are speed constituents in the x, y, z ways, in turn. Operator of the vector is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \mathbf{H}.\nabla. \tag{1.29}$$

Generally, the equations (1.24) and (1.26) are combined and derivatives of density are eliminated. Therefore,

$$P_t + uP_x + \rho\alpha^2 u_x = 0. \tag{1.30}$$

Usual samples for nonlinear first order PDE's can show the equations (1.24) to (1.26) or the equations (1.25) and (1.30).

Acoustic is a science that examines mechanical waves such as vibration, sound, ultrasound in a liquid medium. Think of the classical situation of infinitesimally little confusions in velocity, pressure, and intensity in a stagnant afflux. Then,

$$u = u_0 + u' = u', \quad P = P_0 + P', \quad \rho = \rho_0 + \rho', \quad \alpha = \alpha_0 + \alpha' \tag{1.31}$$

where u_0, ρ_0, P_0, and α_0 are the unimpaired specifications of the fluid, and u', ρ', P', α' are infinitesimal confusion. For a still liquid, $u_0 = 0$. Putting the equation (1.31) in equations (1.25) and (1.30) and ignoring all product of confusion quantities, we get the following system of linear PDEs:

$$\rho_0 u'_t + P'_x = 0, \tag{1.32}$$

$$P'_t + \rho_0 \alpha_0^2 u'_x = 0. \tag{1.33}$$

The equations (1.32) and (1.33) can be united to unfasten the pressure confusion P' or the velocity confusion u'. If the equation (1.32) is taken derivative in regard to x and the equation (1.33) in regard to t and eliminates u'_{xt}, it manages the wave equation P' for pressure confusion [7]

$$P'_{tt} = \alpha_0^2 P'_{xx}. \tag{1.34}$$

If the equation (1.32) is differentiated with respect to t and the equation (1.33) with respect to x and eliminate P'_{xt}, it leads to the wave equation u' for velocity confusion

$$u'_{tt} = \alpha_0^2 u'_{xx}. \tag{1.35}$$

The equations (1.34) and (1.35) indicate that the specifications of a linearised acoustic area are ruled by the wave equation. The terms of the general second order PDE's described in the equation (1.4) appear as $A = 1$, $B = 0$, and $C = -\alpha_0^2$. So, we find that $\Delta > 0$. Eventually, the equation (1.34) and (1.35) are hyperbolic PDEs.

In brief, the wave equation managing unstable wave movement is a usual sample for a hyperbolic PDE.

1.2.3.1 One Dimensional Wave Equations

One dimensional hyperbolic PDE indicates wave equation. The equation expresses that the second derivative of the height of a rope $u(t,x)$ in regard to time (t) is equivalent to the velocity of the dissemination of the wave (c) in the medium it's in multiplied by the second derivative of the height of the rope with respect to position

[8]. $u(t,x)$ satisfies the one dimensional wave equation

$$\frac{\partial^2}{\partial t^2}u(t,x) = c^2\frac{\partial^2}{\partial x^2}u(t,x) \tag{1.36}$$

such that $0 < x < \ell$, $t > 0$. The PDE fundamentally says that the acceleration of the height of the string is equal to the speed of the string squared multiplied by the concavity of the graph. To start solving partial differential equation, we need some conditions.

$$u(t,0) = u(t,\ell) = 0 \tag{1.37}$$

for every $t > 0$. Conditions (1.37) are named as the boundary conditions. Also, the initial conditions are

$$\begin{aligned} &u(0,x) = h(x), \\ &\frac{\partial u}{\partial t}(0,x) = g(x) \end{aligned} \tag{1.38}$$

for $0 < x < \ell$ at $t = 0$. These conditions combine the initial value of the height of the rope and slope of the rope. If we do not know these values, it is not possible to get the $u(t,x)$ solution. Separating of variables method is one of the extensive methods used to solve the equation (1.36). This method represent $u(t,x)$ as two independent functions that

$$u(t,x) = T(t)X(x). \tag{1.39}$$

When the equation (1.39) is substituted in the PDE (1.36), the following ordinary differential equation (ODE) is obtained [9]:

$$X(x)\frac{d^2}{dt^2}T(t) = c^2\frac{d^2}{dx^2}X(x)T(t) \tag{1.40}$$

or

$$X(x)T^{''}(t) = c^2X^{''}(x)T(t). \tag{1.41}$$

When we divide either parts by c^2XT, we get

$$\frac{T^{''}(t)}{c^2T(t)} = \frac{X^{''}(x)}{X(x)}.$$

We can equate these differential equations to a constant. Then,

$$\frac{T^{''}(t)}{c^2T(t)} = \frac{X^{''}(x)}{X(x)} = r \tag{1.42}$$

such that r is a constant. Also, we write

$$X^{''}(x) - rX(x) = 0 \tag{1.43}$$

and

$$T^{''}(t) - rc^2T(t) = 0. \tag{1.44}$$

Now, substituting the equation (1.37) into the equation (1.39), we get

$$\begin{aligned} X(0)T(t) &= 0, \\ X(\ell)T(t) &= 0 \end{aligned} \tag{1.45}$$

for $t > 0$. If $X(0)$ or $X(\ell)$ do not equal zero, we have trivial solution. So, $X(0)$ and $X(\ell)$ must be zero. Thus, we reach the BVP

$$X^{''}(x) - rX(x) = 0, \quad X(0) = 0 \text{ and } X(\ell) = 0. \tag{1.46}$$

Now, we will examine whether r is positive, negative or zero.

If $r > 0$, let $r = \mu^2$ with $\mu > 0$, we get

$$X^{''}(x) - \mu^2 X(x) = 0 \tag{1.47}$$

and the general solution is

$$X(x) = c_1 \cosh \mu x + c_2 \sinh \mu x. \tag{1.48}$$

If the boundary condition $X(0) = 0$ is used, $c_1 = 0$ is found. So, $X(x) = c_2 \sinh \mu x$. Next utilizing the other condition $X(\ell) = 0$, $c_2 \sinh(\mu \ell) = 0$ is obtained. Here, $\sinh(\mu\ell) \neq 0$, and hence $c_2 = 0$. Since $c_1 = c_2 = 0$, we have trivial solution. But we do not want this.

For $r = 0$, we obtain $X^{''}(x) = 0$. Then the general solution is $X(x) = c_1 x + c_2$. If we use the conditions, we find again $c_1 = c_2 = 0$ which leads to trivial solution.

Finally, let us examine the situation when r is negative. Let say $r = -\mu^2 < 0$. The corresponding BVP in X is

$$X^{''}(x) + \mu^2 X(x) = 0, \quad X(0) = 0 \text{ and } X(\ell) = 0. \tag{1.49}$$

The characteristic equation is $m^2 + \mu^2 = 0$. Hence, we find the characteristic of the equation that $m = \pm i\mu$. Thereby, the general solution is

$$X(x) = d_1 \cos \mu x + d_2 \sin \mu x. \tag{1.50}$$

The condition that $X(0) = 0$ refers that $d_1 = 0$, and hence $X(x) = d_2 \sin \mu x$. Another condition $X(\ell) = 0$ refers

$$d_2 \sin \mu \ell = 0. \tag{1.51}$$

In order not to get trivial solution, we take $d_2 = 1$ and $\sin \mu \ell = 0$. Since the sine function eliminates at the integer multiples of π, we conclude that

$$\mu = \mu_n = \frac{n\pi}{\ell} \tag{1.52}$$

where n is an integer. And

$$X(x) = X_n(x) = \sin \frac{n\pi}{\ell} x \tag{1.53}$$

such that n is a positive integer.

Now, let us solve (1.44). Substituting $r = -\mu^2 = -\left(\frac{n\pi}{\ell}\right)^2$, we acquire

$$T^{''}(t) + \left(c\frac{n\pi}{\ell}\right)^2 T(t) = 0. \tag{1.54}$$

Let say $c\frac{n\pi}{\ell} = \eta_n$. The general solution is

$$T(t) = S_n \cos \eta_n t + S_n^* \sin \eta_n t. \tag{1.55}$$

Thus, substituting the solutions in (1.39), we get

$$u_n(t,x) = \left(S_n \cos \eta_n t + S_n^* \sin \eta_n t\right) \sin \frac{n\pi}{\ell} x. \tag{1.56}$$

At the same time, on the purpose of to reach a solution that provides the initial conditions, we sum all the product solutions and we acquire the following solution

$$u(t,x) = \sum_{n=1}^{\infty} \left(S_n \cos \eta_n t + S_n^* \sin \eta_n t\right) \sin \frac{n\pi}{\ell} x. \tag{1.57}$$

If the initial conditions are used, we get

$$u(0,x) = h(x) = \sum_{n=1}^{\infty} S_n \sin \frac{n\pi}{\ell} x \tag{1.58}$$

with

$$S_n = \frac{2}{\ell}\int_0^\ell h(x)\sin\frac{n\pi}{\ell}x dx, \quad n = 1,2,... \tag{1.59}$$

and

$$\frac{\partial}{\partial t}u(0,x) = g(x) = \sum_{n=1}^{\infty} S_n^* \eta_n \sin\frac{n\pi}{\ell}x \tag{1.60}$$

with

$$S_n^* = \frac{2}{cn\pi}\int_0^\ell g(x)\sin\frac{n\pi}{\ell}x dx, \tag{1.61}$$

for $n = 1,2,....$ Eventually, we find the solution when S_n and S_n^* are substituted in the equation (1.57). While S_n and S_n^* is found, it takes advantage of the Fourier series, which we will explain more broadly in the next section.

1.2.3.2 Two Dimensional Wave Equations

In this section we consider two dimensional wave equations. Assume that a slim elastic cover is strained over a rectangular frame. Suppose dimension of the frame are a and b. Also, let its sides are held steady. The two dimensional wave equation refers to the ripple that occurs when the cover vibrates and is released vertically [8] and the two dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1.62}$$

such that $0 < x < a$, $0 < y < b$ and $t > 0$. Also, $u(t,x,y)$ states the deviation at (x,y) at time t. The fixed edges are shown with $u(t,x,y) = 0$ for $t \geq 0$. In other words, the boundary conditions are

$$\begin{aligned} u(t,0,y) = u(t,a,y) = 0, &\quad \text{for } 0 \leq y \leq b \text{ and } t \geq 0, \\ u(t,x,0) = u(t,x,b) = 0, &\quad \text{for } 0 \leq x \leq a \text{ and } t \geq 0 \ . \end{aligned} \tag{1.63}$$

Also, initial conditions shown as

$$\begin{aligned} &u(0,x,y) = h(x,y), \\ &\frac{\partial u}{\partial t}(0,x,y) = g(x,y) \end{aligned} \tag{1.64}$$

that represent the shape and speed of the cover at $t = 0$. Now, let us find the function u that provides (1.62), (1.63) and (1.64). Using the separation of variables method, we will solve the BVP.

According to the method of separating variables, we denote

$$u(t,x,y)=T(t)X(x)Y(y). \tag{1.65}$$

Differentiating and substituting $u(t,x,y)$ into (1.62), we obtain

$$T^{''}(t)X(x)Y(y)=c^2(T(t)X^{''}(x)Y(y)+T(t)X(x)Y''(y)) \tag{1.66}$$

[10]. Dividing both sides by c^2XYT, we get

$$\frac{T^{''}(t)}{c^2T(t)}=\frac{X^{''}(x)}{X(x)}+\frac{Y^{''}(y)}{Y(y)}. \tag{1.67}$$

Here, one part of the equation just consists of t and the other part merely comprises of x and y functions. For this reason, we have to equalize the expressions on both sides to one constant

$$\frac{T^{''}(t)}{c^2T(t)}=-r^2 \text{ and } \frac{X^{''}(x)}{X(x)}+\frac{Y^{''}(y)}{Y(y)}=-r^2. \tag{1.68}$$

When finding a solution in terms of t, we only consider the negative separation constants. For $r=0$ and $r>0$ the BVP has solely trivial solution. We do not want this. Thus, we obtain

$$T^{''}(t)+r^2c^2T(t)=0 \tag{1.69}$$

and

$$\frac{X^{''}(x)}{X(x)}=-\frac{Y^{''}(y)}{Y(y)}-r^2. \tag{1.70}$$

Here, we must define a new parameter. So,

$$\frac{X^{''}(x)}{X(x)}=-\mu^2 \text{ and } -\frac{Y^{''}(y)}{Y(y)}-r^2=-\mu^2 \tag{1.71}$$

or

$$X^{''}(x)+\mu^2X(x)=0 \text{ and } Y^{''}(y)+\gamma^2Y(y)=0 \tag{1.72}$$

where $\gamma^2=r^2-\mu^2$. Here again we only consider the negative separation constants in order not to get trivial solutions. So, we get the following equations by separating variables in the boundary conditions

$$\begin{aligned} &X^{''}(x)+\mu^2X(x)=0,\ X(0)=0,\ X(a)=0,\\ &Y^{''}(y)+\gamma^2Y(y)=0,\ Y(0)=0,\ Y(b)=0,\\ &T^{''}(t)+r^2c^2T(t)=0\ ,\ r^2=\ \gamma^2+\mu^2. \end{aligned} \tag{1.73}$$

From hence , the general solutions are

$$\begin{aligned}
&X(x) = c_1 \cos \mu x + c_2 \sin \mu x, \\
&Y(y) = d_1 \cos \gamma y + d_2 \sin \gamma y, \\
&T(t) = e_1 \cos crt + e_2 \sin crt, \ (r^2 = \ \gamma^2 + \mu^2).
\end{aligned} \tag{1.74}$$

If we apply the boundary conditions for X and Y, we find $c_1 = 0$ and $c_2 \sin \mu a = 0$, $d_1 = 0$ and $d_2 \sin \gamma a = 0$. Thus,

$$\begin{aligned}
\mu = \mu_m = \frac{m\pi}{a}, \\
\gamma = \gamma_n = \frac{n\pi}{b}
\end{aligned} \tag{1.75}$$

such that $m, n = 1, 2, ...$, and

$$\begin{aligned}
X_m(x) &= \sin \frac{m\pi}{a} x, \\
Y_n(y) &= \sin \frac{n\pi}{b} y.
\end{aligned} \tag{1.76}$$

For $m, n = 1, 2, ...$, we have

$$r = r_{mn} = \sqrt{\mu_m^2 + \gamma_n^2} = \sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}} \tag{1.77}$$

and so

$$T(t) = T_{mn}(t) = S_{mn} \cos \eta_{mn} t + S_{mn}^* \sin \eta_{mn} t \tag{1.78}$$

where

$$\eta_{mn} = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}. \tag{1.79}$$

Here, η_{mn} is characteristic frequency of the membrane.

Thus, we obtained the product solutions providing (1.62) and (1.63):

$$u_{mn}(t, x, y) = \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \left(S_{mn} \cos \eta_{mn} t + S_{mn}^* \sin \eta_{mn} t \right). \tag{1.80}$$

At the same time, in order to reach a solution that provides the initial conditions, we sum all the product solutions and we acquire the following solution.

$$u_{mn}(t, x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (S_{mn} \cos \eta_{mn} t + S_{mn}^* \sin \eta_{mn} t) \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y. \tag{1.81}$$

If the initial conditions are used, we achieve

$$u(0,x,y)=h(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}S_{mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \tag{1.82}$$

with

$$S_{mn}=\frac{4}{ab}\int_0^b\int_0^a h(x,y)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,dx\,dy \tag{1.83}$$

and

$$\frac{\partial u}{\partial t}(0,x,y)=g(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}S^*_{mn}\eta_{mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \tag{1.84}$$

with

$$S^*_{mn}=\frac{4}{ab\eta_{mn}}\int_0^b\int_0^a g(x,y)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,dx\,dy \tag{1.85}$$

Thereby, we find $u(t,x,y)$ when S_{mn} and S^*_{mn} are substituted in the equation (1.81). While S_{mn} and S^*_{mn} is found, it takes advantage of the Fourier series, which we will explain more broadly in the next chapter.

1.3 Objective of the Thesis

The goal of the thesis is to investigate the analytical solutions of the multidimensional PDE's in particular two dimensional hyperbolic PDEs. Some of the well known methods namely Fourier series and Fourier transform methods will be considered.

1.4 Hypothesis

In this thesis, one and two dimensional problems of the hyperbolic type with boundary and initial conditions will be studied. In the first step problems with boundary and initial conditions are solved by the Fourier series method. Next one and two dimensional problems of the hyperbolic type with initial conditions will be considered and the Fourier transform method will be applied for solving the problems.

2
FOURIER SERIES

The problem of vibration motion of a strained rope is indicated by PDE's that have no solution in terms of simple functions as a result of the researches of Euler, d'Alembert, and Bernoulli. The objective of their work was to define the series of infinite sine and cosine functions that provide these equations. At the commencing of the 19th centenary, Joseph Fourier improved the theory of this series while working the heat flow problem, and this theory was named after him. Although the main goal is to solve the heat flow problem, it has been revealed that the similar methods can be adapted to a diverse range of problems of mathematics and physics [11]. Fourier series are used in many areas such as electrical engineering, acoustics, optics, quantum mechanics, econometrics etc. In this chapter, Fourier series that allow us to model any periodical function as combination of sine and cosine are studied. Then, after the solving of the one dimensional PDE has been reached, the solution of the two dimensional wave equation has also been reached by the Fourier series method. But firstly, some definitions are given to define the Fourier series.

Definition 2.1. If $h(x)$ provides $h(x+P)=h(x)$ for every x, it is told to have a period P such that $P>0$. The minimal value of P is described as the period of $h(x)$.

For example, the period of the sine is 2π, which infers that the value of the function is same for every 2π units. In fact, for $\sin(x+4\pi)$, $\sin(x+6\pi)$, ...is equal to $\sin x$, we can say that 4π, 6π,... are periods of $\sin x$. But, since the minimal value is taken as a period, 2π is the period of $\sin x$. The periods of $\sin nx$ or $\cos nx$ functions are $2\pi/n$ such that $n>0$. And the period of $\tan x$ is π.

Definition 2.2. A function $h(x)$ is named piecewise continuous in an interval if the function satisfies the following conditions,

(i) The interval must be splitted into a limited number of subinterval in every one of which $h(x)$ is continuous.

(ii) The interval has a finite limit at the endpoints for every subinterval.

2.1 Definition of Fourier Series

Suppose that $h(x)$ be piecewise continuous and periodical function which is described on $[-\ell,\ell]$ is such that the integrals

$$\int_{-\ell}^{\ell} h(x)\cos\left(\frac{n\pi x}{\ell}\right)dx \quad \text{and} \quad \int_{-\ell}^{\ell} h(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

exists. Then the series

$$h(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right) \tag{2.1}$$

is called the Fourier series of $h(x)$ on $[-\ell,\ell]$. Here, a_n and b_n are Fourier coefficients of $h(x)$. These coefficients defined as

$$\begin{aligned} a_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} h(x)\cos\left(\frac{n\pi x}{\ell}\right)dx \\ b_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} h(x)\sin\left(\frac{n\pi x}{\ell}\right)dx \end{aligned} \tag{2.2}$$

where $n = 0,1,2,\ldots$. In the series (2.1), the average of $h(x)$ through a period is indicated by a_0 and is equal to

$$a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} h(x)dx. \tag{2.3}$$

Note that the series corresponding to $h(x)$ is only the series (2.1).

2.2 Convergence of Fourier Series

In this section we consider the convergence of the Fourier series. We do not have information about whether this series is convergent and its convergence to $h(x)$. Dirichlet who has worked on the convergence of the Fourier series, has developed some provisions to this convergence problem. Let us give a theorem about convergence of Fourier series.

Theorem 2.1. *If $h(x)$ is piecewise continuous and periodical function on $[-\ell,\ell]$, then the Fourier series (2.1) of $h(x)$ converges*

(i) *To the periodical extension of $h(x)$ where periodical extension is continuous,*

(ii) *To the average of two limits usually* $[h(x+)+h(x-)]\backslash 2$ *where is a periodic extension has a sump discontinuity [12].*

Here, $h(x+)$ is right hand limit of $h(x)$ and represent by $\lim_{\epsilon \to 0+} h(x+\epsilon)$. Also, $h(x-)$ is left hand limit of $h(x)$ and represent by $\lim_{\epsilon \to 0-} h(x+\epsilon)$[12].

Now, assume that we have a series $\sum_{n=1}^{\infty} u_n(x)$. The $M-$th partial sum of the series is determined as the sum of the first M terms as follows,

$$\sum_{n=1}^{M} u_n(x). \tag{2.4}$$

According to the convergence theorem in infinite series if given any positive number ϵ, there exists for every x in the interval a positive number N such that

$$|S_M(x) - h(x)| < \epsilon \ \text{ whenever } \ M > N. \tag{2.5}$$

Here, N depends on both ϵ and x. Also, $h(x)$ is the sum of the series.

A special case take places when N is depend on ϵ but independent of x in the interval. In view of such a situation, the series is called uniformly convergent to $h(x)$.

In the two theorems below, two very significant features of the series uniformly convergent are indicated [12].

Theorem 2.2. *Suppose that every term of an infinite series be continuous on* (a, b) *and the series in this interval uniformly convergent to the sum of* $h(x)$*. Thereby,*

(i) $h(x)$ *is also continuous in the interval,*

(ii) *The series can integrate term by term. In other words,*

$$\int_a^b \left\{ \sum_{n=1}^{\infty} u_n(x) \right\} dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\, dx. \tag{2.6}$$

Theorem 2.3. *Suppose that every term of an infinite series is differentiable and the series of derivatives is uniformly convergent. In that case, derivative of the series can be take term by term*

$$\frac{d}{dx} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x). \tag{2.7}$$

2.3 Orthogonality of the Sine and Cosine Functions

The orthogonality conditions are

$$\begin{aligned}
\int_{-\ell}^{\ell} \cos\left(\frac{m\pi x}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right)dx &= \begin{cases} 0, & m \neq n \\ \ell, & m = n, \end{cases} \\
\int_{-\ell}^{\ell} \sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)dx &= \begin{cases} 0, & m \neq n \\ \ell, & m = n, \end{cases} \\
\int_{-\ell}^{\ell} \cos\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)dx &= \begin{cases} 0, & m \neq n \\ 0, & m = n. \end{cases}
\end{aligned} \tag{2.8}$$

It is clear from the formulas that the Fourier coefficients are defined in integrals. Using these properties, Fourier coefficients in the series (2.1) are obtained.

To procure the Fourier coefficient a_n, we must multiply either parts of series (2.1) by $\cos\left(\frac{m\pi x}{\ell}\right)$ and integrate term by term from $-\ell$ to ℓ.

$$\begin{aligned}
\int_{-\ell}^{\ell} h(x)\cos\left(\frac{m\pi x}{\ell}\right)dx &= \int_{-\ell}^{\ell}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)\right. \\
&\quad \left. + b_n \sin\left(\frac{n\pi x}{\ell}\right)\right]\cos\left(\frac{m\pi x}{\ell}\right)dx \\
&= a_0 \int_{-\ell}^{\ell} \cos\left(\frac{m\pi x}{\ell}\right)dx \\
&\quad + \sum_{n=1}^{\infty} a_n \int_{-\ell}^{\ell} \cos\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right)dx \\
&\quad + \sum_{n=1}^{\infty} b_n \int_{-\ell}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right)dx.
\end{aligned}$$

Here, we have seen that $a_0 \int_{-\ell}^{\ell} \cos\left(\frac{m\pi x}{\ell}\right)dx = 0$. With the help of orthogonality conditions (2.8), we get

$$\begin{aligned}
\int_{-\ell}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right)dx &= 0, \\
\int_{-\ell}^{\ell} \cos\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right)dx &= \ell.
\end{aligned} \tag{2.9}$$

Therefore,

$$\int_{-\ell}^{\ell} h(x)\cos\left(\frac{m\pi x}{\ell}\right)dx = a_m \ell. \tag{2.10}$$

Replacing m by n, we have

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} h(x)\cos\left(\frac{n\pi x}{\ell}\right)dx. \tag{2.11}$$

Similarly, to determine the Fourier coefficient b_n, we can multiply either parts of series (2.1) by $\sin\left(\frac{m\pi x}{\ell}\right)$ and integrate term by term from $-\ell$ to ℓ.

$$\begin{aligned}
\int_{-\ell}^{\ell} h(x)\sin\left(\frac{m\pi x}{\ell}\right)dx &= \int_{-\ell}^{\ell}\left[a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{\ell}\right)\right.\\
&\quad \left.+b_n\sin\left(\frac{n\pi x}{\ell}\right)\right]\sin\left(\frac{m\pi x}{\ell}\right)dx\\
&= a_0\int_{-\ell}^{\ell}\sin\left(\frac{m\pi x}{\ell}\right)dx\\
&\quad +\sum_{n=1}^{\infty}a_n\int_{-\ell}^{\ell}\cos\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{m\pi x}{\ell}\right)dx\\
&\quad +\sum_{n=1}^{\infty}b_n\int_{-\ell}^{L}\sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{m\pi x}{\ell}\right)dx.
\end{aligned}$$

Here, we have seen that $a_0\int_{-\ell}^{\ell}\sin\left(\frac{m\pi x}{\ell}\right)dx = 0$. With the help of orthogonality conditions (2.8), we get

$$\begin{aligned}
\int_{-\ell}^{\ell}\sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right)dx = 0,\\
\int_{-\ell}^{\ell}\sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{m\pi x}{\ell}\right)dx = \ell.
\end{aligned} \tag{2.12}$$

Therefore,

$$\int_{-\ell}^{\ell} h(x)\sin\left(\frac{m\pi x}{\ell}\right)dx = b_m\ell \tag{2.13}$$

Replacing m by n, we reach

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} h(x)\sin\left(\frac{n\pi x}{\ell}\right)dx. \tag{2.14}$$

2.4 Sine and Cosine Fourier Series

As mentioned earlier, the Fourier series is written as a combine of sine and cosine functions to symbolize any periodic function. It is also simple to obtain Fourier coefficients using the formulas (2.2) if the conditions required to obtain a Fourier series are also provided. However, this integration may force us a little. In this part, we will considerably simplify the calculations required to create a Fourier series representation using symmetries in the $h(x)$ function. When we examine the sine and cosine graphs, we observe that both functions have certain natural symmetries. It is seen that the cosine is a perfect symmetry in regard to the y axis and the sine is a antisymmetric in regard to the x axis. Therefore, we can say that there is a strong relation among the symmetries in some periodic functions $h(x)$ and the existence of sine or cosine terms in its Fourier series. The symmetry and antisymmetry defined above are the same as the several powers of x, depending on whether the base is even or odd. Thereby, instead of symmetry and antisymmetry, we can call it even and odd. To sum up briefly, we can write the following definitions.

Definition 2.3. If $h(x)$ satisfies $h(-x) = -h(x)$ for every x , the function is named odd function. If $h(x)$ satisfies $h(-x) = h(x)$ for every x , the function is named even function.

At present, for odd and even functions, let us examine the Fourier series.

Definition 2.4. If $h(x)$ is odd described on $[-\ell, \ell]$, then

$$\int_{-\ell}^{\ell} h(x)\cos\left(\frac{k\pi x}{\ell}\right)dx = 0, \quad k = 0,1,2,.... \tag{2.15}$$

and

$$\int_{-\ell}^{\ell} h(x)\sin\left(\frac{k\pi x}{\ell}\right)dx = 2\int_{0}^{\ell} h(x)\sin\left(\frac{k\pi x}{\ell}\right)dx, \quad k = 0,1,2,... \tag{2.16}$$

For this reason, for $h(x)$, we have the series in the form

$$\sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{\ell}\right) \tag{2.17}$$

such that $k = 0,1,2,....$ Here,

$$b_k = \frac{2}{\ell}\int_{0}^{\ell} h(x)\sin\left(\frac{k\pi x}{\ell}\right)dx. \tag{2.18}$$

In the Fourier series of an odd function, merely sine terms exist. For this reason, this series are named Fourier sine series.

Definition 2.5. If $h(x)$ is even function described on $[-\ell, \ell]$, then

$$\int_{-\ell}^{\ell} h(x)\sin\left(\frac{k\pi x}{\ell}\right)dx = 0, \quad k = 0,1,... \tag{2.19}$$

and

$$\int_{-\ell}^{\ell} h(x)\cos\left(\frac{k\pi x}{\ell}\right)dx = 2\int_{0}^{\ell} h(x)\cos\left(\frac{k\pi x}{\ell}\right)dx, \quad k = 0,1,2,.... \tag{2.20}$$

For this reason, for $h(x)$, we have the series as

$$a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{\ell}\right) \tag{2.21}$$

such that $k = 0,1,....$ Here,

$$a_k = \frac{2}{\ell}\int_{0}^{\ell} h(x)\cos\left(\frac{k\pi x}{\ell}\right)dx. \tag{2.22}$$

In the Fourier series of an even function, merely cosine terms exist. For this reason, this series are named Fourier cosine series.

Based on these results, we can reach the following corollary.

Corollary 2.1. *The Fourier series of $h(x)$ equals to the sum of Fourier sine series and Fourier cosine series.*

2.5 Differentiation and Integration of Fourier Series

Theorem 2.4. *[8] Assume $h(x)$ be a piecewise continuous, periodic function described on $[-\ell,\ell]$. The Fourier series of $h(x)$ is*

$$h(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right).$$

If $h(x)$ is piecewise continuous and first derivative of $h(x)$ procures the periodicity conditions, the Fourier series representation of $h'(x)$ is

$$h'(x) = \sum_{n=1}^{\infty}\left\{-\frac{n\pi}{\ell}a_n \sin\left(\frac{n\pi x}{\ell}\right) + \frac{n\pi}{\ell}b_n \cos\left(\frac{n\pi x}{\ell}\right)\right\}. \tag{2.23}$$

Theorem 2.5. *[8] Suppose $h(x)$ be a periodical piecewise continuous function described on $[-\ell,\ell]$. Then this function can be integrated term by term.*

2.6 Complex Form of Fourier Series

By the Euler's formula, we can write

$$\cos\left(\frac{n\pi x}{\ell}\right)=\frac{e^{\frac{in\pi x}{\ell}}+e^{-\frac{in\pi x}{\ell}}}{2},\quad \sin\left(\frac{n\pi x}{\ell}\right)=\frac{e^{\frac{in\pi x}{\ell}}-e^{-\frac{in\pi x}{\ell}}}{2i}.$$

Substituting in the Fourier series (2.1), we get

$$\begin{aligned} & a_0+\sum_{n=1}^{\infty}\left\{a_n\left(\frac{e^{\frac{in\pi x}{\ell}}+e^{-\frac{in\pi x}{\ell}}}{2}\right)+b_n\left(\frac{e^{\frac{in\pi x}{\ell}}-e^{-\frac{in\pi x}{\ell}}}{2i}\right)\right\} \\ = \;& a_0+\sum_{n=1}^{\infty}\left(\frac{a_n-ib_n}{2}\right)e^{\frac{in\pi x}{\ell}}+\sum_{n=1}^{\infty}\left(\frac{a_n+ib_n}{2}\right)e^{-\frac{in\pi x}{\ell}} \\ = \;& C_0+\sum_{n=1}^{\infty}C_n e^{\frac{in\pi x}{\ell}}+\sum_{n=1}^{\infty}C_{-n}e^{-\frac{in\pi x}{\ell}} \\ = \;& \sum_{n=-\infty}^{\infty}C_n e^{\frac{in\pi x}{\ell}}. \end{aligned}$$

Here,

$$\begin{aligned} C_0 &= a_0=\frac{1}{2\ell}\int_{-\ell}^{\ell}h(x)dx \\ C_n &= \left(\frac{a_n-ib_n}{2}\right)=\frac{1}{2\ell}\int_{-\ell}^{\ell}h(x)e^{-\frac{in\pi x}{\ell}}dx \qquad (2.24) \\ C_{-n} &= \left(\frac{a_n+ib_n}{2}\right)=\frac{1}{2\ell}\int_{-\ell}^{\ell}h(x)e^{\frac{in\pi x}{\ell}}dx. \end{aligned}$$

So,

$$\sum_{n=-\infty}^{\infty}C_n e^{\frac{in\pi x}{\ell}}=a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{\ell}\right)+b_n\sin\left(\frac{n\pi x}{\ell}\right) \qquad (2.25)$$

is named complex form of Fourier series.

2.7 Double Fourier Series

In this section we will consider the extension of Fourier series to two variables. In addition to being a function of a single variable, the Fourier series can be expanded to the state of two variables [12]. In this case, this series is named a double Fourier series. For example,

$$h(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\sin\left(\frac{m\pi x}{\ell_1}\right)\sin\left(\frac{n\pi x}{\ell_2}\right) \qquad (2.26)$$

where

$$B_{mn}=\frac{4}{\ell_1\ell_2}\int_0^{\ell_1}\int_0^{\ell_2}h(x,y)\sin\left(\frac{m\pi x}{\ell_1}\right)\sin\left(\frac{n\pi x}{\ell_2}\right)dxdy \tag{2.27}$$

is a double Fourier sine series. Similarly, we can write double Fourier cosine series,

$$h(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\cos\left(\frac{m\pi x}{\ell_1}\right)\cos\left(\frac{n\pi x}{\ell_2}\right) \tag{2.28}$$

where

$$B_{mn}=\frac{4}{\ell_1\ell_2}\int_0^{\ell_1}\int_0^{\ell_2}h(x,y)\cos\left(\frac{m\pi x}{\ell_1}\right)\cos\left(\frac{n\pi x}{\ell_2}\right)dxdy. \tag{2.29}$$

In the remainder of this section, the period is accepted as π.

2.8 Fourier Series Solutions of One Dimensional PDEs

In the present section we consider a mixed type BVP for one dimensional hyperbolic PDE

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=\left(t^2+2t+3\right)\sin x,\ 0<t<1\ ,0\leq x\leq\pi, \\ u(0,x)=\sin x,\ \ 0\leq x\leq\pi, \\ u_t(0,x)=2\sin x,\ 0\leq x\leq\pi, \\ u(t,0)=u(t,\pi)=0,\ 0\leq t\leq 1. \end{cases} \tag{2.30}$$

We will obtain the solution of problem (2.30) by the Fourier series method. Using the Fourier series formula

$$u(t,x)=a_0(t)+\sum_{n=1}^{\infty}a_n(t)\cos(nx)+b_n(t)\sin(nx). \tag{2.31}$$

Here, $a_0(t)$, $a_n(t)$, and $b_n(t)$ are unknown functions and we must find this. Firstly, applying the initial condition $u(0,x)=\sin x$, we can write

$$a_0(0)+\sum_{n=1}^{\infty}a_n(0)\cos(nx)+b_n(0)\sin(nx)=\sin x \tag{2.32}$$

or

$$a_0(0)+\sum_{n=1}^{\infty}[a_n(0)\cos(nx)]+\sum_{n=2}^{\infty}[b_n(0)\sin(nx)]+b_1(0)\sin x=\sin x. \tag{2.33}$$

From that, we have

$$\begin{aligned}
&a_0(0) = 0.\\
&a_n(0) = 0, \quad n = 1, 2, ...,\\
&b_n(0) = 0, \quad n = 2, 3, ...,\\
&b_1(0) = 1.
\end{aligned} \tag{2.34}$$

And applying the other initial condition $u_t(0, x) = 2\sin x$, we get

$$a_0^{'}(0) + \sum_{n=1}^{\infty} a_n^{'}(0)\cos(nx) + b_n^{'}(0)\sin(nx) = 2\sin x. \tag{2.35}$$

From that, we have

$$\begin{aligned}
&a_0^{'}(0) = 0,\\
&a_n^{'}(0) = 0, \quad n = 1, 2, ...,\\
&b_n^{'}(0) = 0, \quad n = 2, 3, ...,\\
&b_1^{'}(0) = 2.
\end{aligned} \tag{2.36}$$

Now, substituting (2.31) into (2.30),

$$\begin{aligned}
&a_0^{''}(t) + \sum_{n=1}^{\infty}\left[a_n^{''}(t)\cos(nx) + b_n^{''}(t)\sin(nx)\right]\\
&+ \sum_{n=1}^{\infty}\left[n^2 a_n(t)\cos(nx) + n^2 b_n(t)\sin(nx)\right]\\
&= \left(t^2 + 2t + 3\right)\sin x.
\end{aligned} \tag{2.37}$$

We can rewrite this equation as

$$\begin{aligned}
&a_0^{''}(t) + \sum_{n=1}^{\infty}\left[a_n^{''}(t) + n^2 a_n(t)\right]\cos(nx)\\
&+ \sum_{n=1}^{\infty}\left[b_n^{''}(t) + n^2 b_n(t)\right]\sin(nx).\\
&= \left(t^2 + 2t + 3\right)\sin x.
\end{aligned} \tag{2.38}$$

So,

$$\begin{aligned}
&a_0^{''}(t) = 0,\\
&a_n^{''}(t) + n^2 a_n(t) = 0, \quad n = 1, 2, ...,\\
&b_n^{''}(t) + n^2 b_n(t) = 0, \quad n = 2, 3, ...
\end{aligned} \tag{2.39}$$

and

$$b_1^{''}(t) + b_1(t) = t^2 + 2t + 3. \tag{2.40}$$

We will solve the second order ODE (2.40). We consider our solution in two parts that are complementary and particular solution.

First, we obtain complementary solution. We have

$$b_1^{''}(t) + b_1(t) = 0. \tag{2.41}$$

The characteristic equation of (2.41) is $m^2 + 1 = 0$. Hence, we find the characteristics $m = \pm i$. Therefore,

$$b_1^c(t) = M \cos t + N \sin t. \tag{2.42}$$

Next, we will obtain the particular solution. We assume the solution as

$$b_1^p(t) = At^2 + Bt + C. \tag{2.43}$$

And substituting this solution into second order ODE (2.40), we get

$$2A + At^2 + Bt + C = t^2 + 2t + 3. \tag{2.44}$$

Thus, we find the coefficients of the solution (2.43) such that

$$\begin{aligned} A &= 1, \\ B &= 2, \\ C &= 1. \end{aligned} \tag{2.45}$$

Therefore, particular solution is

$$b_1^p(t) = t^2 + 2t + 1. \tag{2.46}$$

The general solution is

$$b_1(t) = b_1^c(t) + b_1^p(t). \tag{2.47}$$

So,

$$b_1(t) = M \cos t + N \sin t + t^2 + 2t + 1. \tag{2.48}$$

Using the conditions $b_1(0) = 1$ and $b_1'(0) = 2$ in (2.34) and (2.36), we have

$$\begin{aligned} M + 1 &= 1, \\ N + 2 &= 2. \end{aligned} \tag{2.49}$$

Therefore, we find $M = N = 0$. Putting M and N in equation (2.48), $b_1(t)$ can be written as

$$b_1(t) = t^2 + 2t + 1. \tag{2.50}$$

Let us find the solution of $a_n(t)$. In equations (2.39), we found

$$a_n^{''}(t) + n^2 a_n(t) = 0. \tag{2.51}$$

The characteristic equation of (2.51) is $m^2 + n^2 = 0$. Hence, we find the characteristic of the equation that $m = \pm in$. Therefore,

$$a_n(t) = F\cos nt + G\sin nt. \tag{2.52}$$

Using the conditions that $a_n(0) = 0$ and $a_n'(0) = 0$ in (2.34) and (2.36), we have $F = G = 0$. So, for $n = 1, 2, ...,$

$$a_n(t) = 0. \tag{2.53}$$

Finally, let us write the general solution

$$u(t,x) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos(nx) + b_n(t)\sin(nx) \tag{2.54}$$

or

$$u(t,x) = a_0(t) + b_1(t)\sin x + \sum_{n=1}^{\infty} a_n(t)\cos(nx) + \sum_{n=2}^{\infty} b_n(t)\sin(nx). \tag{2.55}$$

We find $a_0(t) = a_n(t) = 0$ and for $n > 1$, $b_n(t) = 0$. Then, substituting $b_1(t)$ in (2.55), we obtain

$$u(t,x) = \left(t^2 + 2t + 1\right)\sin x. \tag{2.56}$$

Thus, $u(t,x) = (1+t)^2 \sin x$ is the solution of the problem (2.30) obtained by the Fourier series method.

2.9 Fourier Series Solutions of Two Dimensional PDEs

In the present section we consider the mixed type BVP for two dimensional hyperbolic PDE

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \left(2t^2 + 4t + 4\right)\sin x \sin y, \\ 0 < t < 1,\ 0 \le x \le \pi,\ 0 \le y \le \pi, \\ u(0,x,y) = \sin x \sin y,\ 0 \le x \le \pi,\ 0 \le y \le \pi, \\ u_t(0,x,y) = 2\sin x \sin y,\ 0 \le x \le \pi,\ 0 \le y \le \pi, \\ u(t,0,y) = u(t,\pi,y) = 0,\ 0 \le t \le 1, \\ u(t,x,0) = u(t,x,\pi) = 0,\ 0 \le t \le 1. \end{cases} \tag{2.57}$$

Let us solve problem (2.57) with the Fourier series method. Assume that the solution is

$$u(t,x,y) = v(t,x,y) + w(t,x,y). \tag{2.58}$$

Here $v(t,x,y)$ is the solution of problem

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} - \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0, \\ 0 < t < 1,\ 0 < x < \pi,\ 0 \le y \le \pi, \\ v(0,x,y) = \sin x \sin y,\ 0 \le x \le \pi,\ 0 \le y \le \pi, \\ v_t(0,x,y) = 2\sin x \sin y,\ 0 \le x \le \pi,\ 0 \le y \le \pi, \\ v\,(t,0,y) = v\,(t,\pi,y) = 0,\ 0 \le t \le 1, \\ v\,(t,x,0) = v\,(t,x,\pi) = 0,\ 0 \le t \le 1 \end{cases} \tag{2.59}$$

and $w(t,x,y)$ is the solution of problem

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} - \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \left(2t^2 + 4t + 4\right)\sin x \sin y, \\ 0 < t < 1,\ 0 \le x \le \pi,\ 0 \le y \le \pi, \\ w(0,x,y) = 0,\ 0 \le x \le \pi,\ 0 \le y \le \pi, \\ w_t(0,x,y) = 0,\ 0 \le x \le \pi,\ 0 \le y \le \pi, \\ w(t,0,y) = w(t,\pi,y) = 0,\ 0 \le t \le 1, \\ w(t,x,0) = w(t,x,\pi) = 0,\ 0 \le t \le 1. \end{cases} \tag{2.60}$$

First, we solve problem (2.59). By the method of separation of variables we have

$$v(t,x,y) = T(t)X(x)Y(y) \neq 0. \tag{2.61}$$

Differentiating and substituting $v(t,x,y)$ into (2.59), we obtain

$$T^{''}(t)X(x)Y(y) - (T\,(t)X^{''}(x)Y(y) + T\,(t)X(x)Y^{''}(y)) = 0. \tag{2.62}$$

When we divide either parts by XYT, we get

$$\frac{T^{''}(t)}{T(t)}=\frac{X^{''}(x)}{X(x)}+\frac{Y^{''}(y)}{Y(y)}. \tag{2.63}$$

Here, the left hand consists of a function t only. Also, the right hand comprises of the x and y functions only. For this reason, we have to equalize the expressions on both sides to one constant.

$$\frac{T^{''}(t)}{T(t)}=-k^2 \text{ and } \frac{X^{''}(x)}{X(x)}+\frac{Y^{''}(y)}{Y(y)}=-k^2. \tag{2.64}$$

When finding a solution in terms of t, we only consider the negative separation constants. For $k=0$ and $k>0$ the BVP has only trivial solutions. We do not want this. Thereby, we get

$$T^{''}(t)+k^2T(t)=0 \tag{2.65}$$

and

$$\frac{X^{''}(x)}{X(x)}=-\frac{Y^{''}(y)}{Y(y)}-k^2. \tag{2.66}$$

Then

$$\frac{X^{''}(x)}{X(x)}=-\mu^2 \text{ and } -\frac{Y^{''}(y)}{Y(y)}-k^2=-\mu^2 \tag{2.67}$$

or

$$X^{''}(x)+\mu^2X(x)=0 \text{ and } Y^{''}(y)+\gamma^2Y(y)=0. \tag{2.68}$$

where $\gamma^2=k^2-\mu^2$. Here again we only consider the negative separation constants in order not to get trivial solutions. So, we get the following equations by separating variables in the boundary conditions

$$\begin{aligned} X^{''}(x)+\mu^2X(x) &=0,\ X(0)=X(\pi)=0,\\ Y^{''}(y)+\gamma^2Y(y) &=0,\ Y(0)=Y(\pi)=0,\\ T^{''}(t)+k^2T(t) &=0, k^2=\gamma^2+\mu^2. \end{aligned} \tag{2.69}$$

The general solutions of equations (2.69) are

$$\begin{aligned} &X(x)=c_1\cos\mu x+c_2\sin\mu x,\\ &Y(y)=d_1\cos\gamma y+d_2\sin\gamma y,\\ &T(t)=e_1\cos kt+e_2\sin kt,\ (k^2=\gamma^2+\mu^2). \end{aligned} \tag{2.70}$$

Using the boundary conditions for X and Y, we find $c_1=0$ and $c_2\sin\mu\pi=0$, $d_1=0$

and $d_2 \sin\gamma\pi = 0$. Thus

$$\mu = \mu_m = m \text{ and } \gamma = \gamma_n = n, \quad m,n = 1,2,... \tag{2.71}$$

and

$$X_m(x) = \sin mx \text{ and } Y_n(y) = \sin ny. \tag{2.72}$$

For $m,n = 1,2,...$, we have

$$k = k_{mn} = \sqrt{\mu_m^2 + \gamma_n^2} = \sqrt{m^2 + n^2} \tag{2.73}$$

and

$$T(t) = T_{mn}(t) = S_{mn}\cos\eta_{mn}t + S_{mn}^{*}\sin\eta_{mn}t \tag{2.74}$$

where

$$\eta_{mn} = \sqrt{m^2 + n^2}. \tag{2.75}$$

Here, η_{mn} is characteristic frequency. Thus, we obtained the following product solutions

$$v_{mn}(t,x,y) = \sin mx \sin ny \left(S_{mn}\cos\eta_{mn}t + S_{mn}^{*}\sin\eta_{mn}t\right). \tag{2.76}$$

This function is the normal mode of the two dimensional wave equation. At the same time, in order to reach a solution that also provides the initial conditions, all the product solutions collected and the following solution

$$v_{mn}(t,x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(S_{mn}\cos\eta_{mn}t + S_{mn}^{*}\sin\eta_{mn}t)\sin mx \sin ny \tag{2.77}$$

is obtained. If we use the initial condition

$$v(0,x,y) = \sin x \sin y,\ 0 \le x \le \pi,\ 0 \le y \le \pi, \tag{2.78}$$

we obtain the following equation

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} S_{mn}\sin mx \sin ny = \sin x \sin y. \tag{2.79}$$

By the equality we have

$$S_{11}\sin x \sin y = \sin x \sin y \tag{2.80}$$

for $m = n = 1$. Using the second initial condition

$$v_t(0,x,y) = 2\sin x \sin y,\ 0 \le x \le \pi, 0 \le y \le \pi, \tag{2.81}$$

we get

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} S^*_{mn}\eta_{mn} \sin mx \sin ny = 2\sin x \sin y. \tag{2.82}$$

By the last equality, we have

$$S^*_{11}\eta_{11} \sin x \sin y = 2\sin x \sin y, \tag{2.83}$$

for $m = n = 1$. Thus, the coefficients are

$$\begin{aligned} S_{11} &= 1, \\ S^*_{11} &= \sqrt{2}. \end{aligned} \tag{2.84}$$

Substituting S_{11} and S^*_{11} in (2.77), the following equation is written as

$$v_{11}(t,x,y) = (\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t)\sin x \sin y. \tag{2.85}$$

For $m, n \neq 1$, we do not have a solution. So, we can write

$$v(t,x,y) = (\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t)\sin x \sin y. \tag{2.86}$$

Now, we will achieve the solution for problem (2.60). For $w(t,x,y)$, we assume the solution as

$$w(t,x,y) = \sum_{k=1}^{\infty} A_k(t)\sin kx \sin ky. \tag{2.87}$$

Substituting this solution into (2.60), we get

$$\begin{aligned} &\sum_{k=1}^{\infty} A_k''(t)\sin kx \sin ky + \sum_{k=1}^{\infty} A_k(t)k^2 \sin kx \sin ky \\ &+ \sum_{k=1}^{\infty} A_k(t)k^2 \sin kx \sin ky \\ &= \left(2t^2 + 4t + 4\right)\sin x \sin y. \end{aligned} \tag{2.88}$$

From this point of view, we reach

$$A_k''(t) + 2k^2 A_k(t) = 0 \tag{2.89}$$

for $k \geq 2$. And we have

$$A_1''(t) + 2A_1(t) = 2t^2 + 4t + 4 \tag{2.90}$$

for $k = 1$. Now we will solve the last second order ODE. We consider our solution in two parts as complementary and particular solution.

First, we obtain complementary solution. Thus,

$$A_1^{''}(t)+2A_1(t)=0. \tag{2.91}$$

The characteristic equation of (2.91) is $m^2+2=0$. Hence, we find the characteristics as $m=\pm\sqrt{2}i$. Therefore,

$$A_1^c(t)=C_{11}\cos\sqrt{2}t+C_{11}^*\sin\sqrt{2}t. \tag{2.92}$$

Next we will obtain the particular solution. We assume the solution as

$$A_p(t)=at^2+bt+c. \tag{2.93}$$

And substituting this solution into second order ordinary differential equation (2.90), we get

$$2a+2at^2+2bt+2c=2t^2+4t+4. \tag{2.94}$$

Thus, we find the coefficients of the solution (2.93) such that

$$\begin{aligned} a&=1,\\ b&=2,\\ c&=1. \end{aligned} \tag{2.95}$$

Therefore, particular solution is

$$A_1(t)=t^2+2t+1. \tag{2.96}$$

Thus,

$$w(t,x,y)=(C_{11}\cos\sqrt{2}t+C_{11}^*\sin\sqrt{2}t+t^2+2t+1)\sin x\sin y. \tag{2.97}$$

Using the boundary conditions, we have

$$\begin{aligned} &(C_{11}+1)\sin x\sin y=0,\\ &(\sqrt{2}C_{11}^*\ +2)\sin x\sin y=0. \end{aligned} \tag{2.98}$$

From that we have

$$\begin{aligned} C_{11}&=-1\\ C_{11}^*&=-\sqrt{2}. \end{aligned} \tag{2.99}$$

Putting C_{11} and C_{11}^{*} in equation (2.97), $w(t,x,y)$ is written as

$$w(t,x,y)=(-\cos\sqrt{2}t-\sqrt{2}\sin\sqrt{2}t+t^2+2t+1)\sin x\sin y. \tag{2.100}$$

Then, substituting $v(t,x,y)$ and $w(t,x,y)$ in (2.58), we achieve

$$\begin{aligned} u(t,x,y)&=(\cos\sqrt{2}t+\sqrt{2}\sin\sqrt{2}t-\cos\sqrt{2}t-\sqrt{2}\sin\sqrt{2}t \\ &\quad+t^2+2t+1)\sin x\sin y \\ &=(1+t)^2\sin x\sin y. \end{aligned} \tag{2.101}$$

Thus, $u(t,x,y)=(1+t)^2\sin x\sin y$ is the solution of the problem (2.57) obtained by the Fourier series method.

3

FOURIER TRANSFORM

In this chapter we will study the Fourier transform method. Necessary information for the Fourier transformation is obtained with the Fourier series studies. In the Fourier series chapter, we have introduced that periodic functions are modeled as a combination of sine and cosine functions. The Fourier transform, which we will consider in this section, is an expansion of the Fourier series that occurs when the represented function is allowed to approach infinity [13]. To define the Fourier transform, the concept of the Fourier integral must be known. In this chapter, firstly, Fourier transform formulas are obtained from the Fourier integral. And then, using Fourier transform formulas, transform formulas of some important functions are obtained and properties of Fourier transform are mentioned. Finally, one and two dimensional hyperbolic problems are solved with the Fourier transform method.

3.1 Fourier Integral

In this section we obtain some important formulas. We regard the Fourier series for $h(x)$ on $[-\ell,\ell]$

$$h(x) \sim h_\ell(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right) \tag{3.1}$$

where

$$\begin{aligned} a_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} h(x)\cos\left(\frac{n\pi x}{\ell}\right)dx \\ b_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} h(x)\sin\left(\frac{n\pi x}{\ell}\right)dx \\ a_0 &= \frac{1}{2\ell}\int_{-\ell}^{\ell} h(x)dx. \end{aligned} \tag{3.2}$$

Denoting $w_n = \frac{n\pi}{\ell}$ and $\Delta w_n = w_{n+1} - w_n = \frac{(n+1)\pi}{\ell} - \frac{n\pi}{\ell} = \frac{\pi}{\ell}$. Then, we can write the follow equations:

$$\begin{aligned} a_n \cos\left(\frac{n\pi x}{\ell}\right) &= a_n \cos w_n x = \left(\frac{1}{\ell}\int_{-\ell}^{\ell} h_\ell(t)\cos(w_n t)\,dt\right)\cos w_n x, \\ b_n \sin\left(\frac{n\pi x}{\ell}\right) &= b_n \sin w_n x = \left(\frac{1}{\ell}\int_{-\ell}^{\ell} h_\ell(t)\sin(w_n t)\,dt\right)\sin w_n x. \end{aligned} \tag{3.3}$$

Substituting the equations (3.3) into the Fourier series (3.1), we obtain

$$\begin{aligned} h_\ell(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\Delta w_n \Bigg[& \cos(w_n x)\int_{-\ell}^{\ell} h_\ell(t)\cos(w_n t)\,dt \\ & + \sin(w_n x)\int_{-\ell}^{\ell} h_\ell(t)\sin(w_n t)\,dt \Bigg] \\ & + \frac{1}{2\pi}\Delta w_0 \int_{-\ell}^{\ell} h_\ell(t)\cos(w_0 t)\,dt. \end{aligned} \tag{3.4}$$

Passing to limit when $\ell \to \infty$, we can write

$$\begin{aligned} h(x) &= \lim_{\ell\to\infty} h_\ell(x) \\ &= \frac{1}{\pi}\lim_{\ell\to\infty}\Bigg\{\sum_{n=1}^{\infty}\Delta w_n \Bigg[\cos(w_n x)\int_{-\ell}^{\ell} h_\ell(t)\cos(w_n t)\,dt \\ &+ \sin(w_n x)\int_{-\ell}^{\ell} h_\ell(t)\sin(w_n t)\,dt\Bigg] \\ &+ \frac{1}{2\pi}\Delta w_0 \int_{-\ell}^{\ell} h_\ell(t)\cos(w_0 t)\,dt\Bigg\}. \end{aligned} \tag{3.5}$$

Since $\Delta w_0 = \frac{\pi}{\ell} \to 0$ when $\ell \to \infty$. Under the assumption $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$, we have

$$\lim_{\ell\to\infty} \frac{1}{2\pi}\Delta w_0 \int_{-\ell}^{\ell} h_\ell(t)\cos(w_0 t)\,dt = 0. \tag{3.6}$$

So,

$$\begin{aligned} h(x) = \frac{1}{\pi}\Bigg\{ & \int_0^{\infty} \cos(wx)\int_{-\infty}^{\infty} h(t)\cos(wt)\,dt\,dw \\ & + \int_0^{\infty} \sin(wx)\int_{-\infty}^{\infty} h(t)\sin(wt)\,dt\,dw\Bigg\} \end{aligned}$$

or

$$h(x) = \int_0^{\infty} [A(w)\cos(wx) + B(w)\sin(wx)]\,dw \tag{3.7}$$

where

$$\begin{aligned} A(w) = & \frac{1}{\pi}\int_{-\infty}^{\infty} h(t)\cos(wt)\,dt, \\ B(w) = & \frac{1}{\pi}\int_{-\infty}^{\infty} h(t)\sin(wt)\,dt \end{aligned} \tag{3.8}$$

is named the Fourier integral of $h(x)$ described on $(-\infty,\infty)$ for all $w \geq 0$,

Theorem 3.1. *[12] If*

(i) *$h(x)$ and first derivative of $h(x)$ are piecewise continuous function in every finite interval,*

(ii) *$\int_{-\infty}^{\infty} |h(t)|\,dt$ converges,*

then Fourier's integral theorem states the equations (3.7) and (3.8).

The validity of the result (3.7) depends on the fact that x is a point of continuity of f function. If it does not satisfy the continuity condition, we should replace by $[h(x+)+h(x-)]\setminus 2$ as in the Fourier series.

In addition, the Fourier's integral theorem can be written in different types. Substituting (3.8) into (3.7), we get

$$\begin{aligned} h(x) &= \frac{1}{\pi}\int_{0}^{\infty}\left[\cos(wx)\int_{-\infty}^{\infty} h(t)\cos(wt)\,dt\right. \\ &\quad \left. +\sin(wx)\int_{-\infty}^{\infty} h(t)\sin(wt)\,dt\right]dw \\ &= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} h(t)(\cos(wx)\cos(wt)+\sin(wx)\sin(wt))\,dt\,dw \\ &= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} h(t)\cos w(x-t)\,dt\,dw. \end{aligned}$$

Since cosine function is an even function, we can type

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t)\cos w(x-t)\,dt\,dw. \tag{3.9}$$

Also, sine function is an odd function so its integral is zero.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t)\sin w(x-t)\,dt\,dw = 0. \tag{3.10}$$

Applying (3.9) and (3.10), we can write

$$\begin{aligned} h(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t)\cos w(x-t)\,dt\,dw \\ &\quad + i\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t)\sin w(x-t)\,dt\,dw \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t)\left[\cos w(x-t) + i\sin w(x-t)\right]dt\,dw. \end{aligned}$$

In accordance with the Euler's identity, $\cos w(x-t) + i\sin w(x-t) = e^{iw(x-t)}$. Then,

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t)e^{iw(x-t)}dt\,dw \tag{3.11}$$

is named complex form of Fourier integral. Also, depending on whether $h(x)$ is even or odd, the following results can be written.

When $h(x)$ is odd,

$$h(x) = \frac{2}{\pi}\int_{0}^{\infty} \sin(wx)\,dw \int_{0}^{\infty} h(t)\sin(wt)\,dt. \tag{3.12}$$

When $h(x)$ is even,

$$h(x) = \frac{2}{\pi}\int_{0}^{\infty} \cos(wx)\,dw \int_{0}^{\infty} h(t)\cos(wt)\,dt. \tag{3.13}$$

3.2 The Fourier Transform

In this section we will introduce the Fourier transform formula. Using the formula (3.11) and identity $e^{iw(x-t)} = e^{iwx}e^{-iwt}$, we write

$$h(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iwx}\,\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(t)e^{-iwt}dt\,dw.}_{\mathbf{F}(w)} \tag{3.14}$$

We have

$$\mathbf{F}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x)e^{-iwx}dx, \quad (-\infty < w < \infty). \tag{3.15}$$

The function $\mathbf{F}(w)$ is called Fourier transform of $h(x)$. Here,

$$h(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iwx}\mathbf{F}(w)dw, \quad (-\infty < x < \infty) \tag{3.16}$$

is named the inverse Fourier transform of $\mathbf{F}(w)$.

The notation $F\{h(x)\}$ can be used instead of the Fourier transform $\mathbf{F}(w)$ and the notation $F^{-1}\{\mathbf{F}(w)\}$ can be used instead of the inverse Fourier transform $h(x)$. According to the Fourier's integral theorem, if h is not continuous at x, the left hand of (3.16) is to be replace by $[h(x+)+h(x-)]\setminus 2$.

Now, we will get some important and necessary formulas.

Let us obtain the Fourier transform of the function

$$h(x)=\begin{cases} k, & 0<x<a \\ 0, & x\notin(0,a) \end{cases} \tag{3.17}$$

where k is a constant.

Solution Applying the Fourier transform formula (3.15) for $h(x)=k$

$$\begin{aligned} F\{f(x)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x)e^{-iwx}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{0}^{a} ke^{-iwx}dx \\ &= \frac{k}{\sqrt{2\pi}}\left[\frac{e^{-iwx}}{-iw}\right]_0^a \\ &= \frac{k}{\sqrt{2\pi}}\frac{\left(e^{-iwa}-1\right)}{-iw}. \end{aligned}$$

So, Fourier transform of function (3.17) is

$$F\{h(x)\}=\mathbf{F}(w)=\frac{k}{\sqrt{2\pi}}\frac{\left(1-e^{-iwa}\right)}{iw}. \tag{3.18}$$

Now we obtain the Fourier transform of the function

$$h(x)=e^{-kx^2} \tag{3.19}$$

such that $k>0$.

Solution Using the Fourier transform formula (3.15), we get

$$\mathbf{F}(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-kx^2}e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(kx^2+iwx\right)}dx.$$

We have $kx^2+iwx=\left(\sqrt{k}x\right)^2+2iw\sqrt{k}x\frac{1}{2\sqrt{k}}-\left(\frac{w}{2\sqrt{k}}\right)^2+\frac{w^2}{4k}=\left(\sqrt{k}x+\frac{iw}{2\sqrt{k}}\right)^2+\frac{w^2}{4k}$. Then,

$$\begin{aligned}
\mathbf{F}(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\sqrt{k}x+\frac{iw}{2\sqrt{k}}\right)^2-\frac{w^2}{4k}}dx \\
&= \frac{e^{-\frac{w^2}{4k}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\sqrt{k}x+\frac{iw}{2\sqrt{k}}\right)^2}dx \\
&= \frac{e^{-\frac{w^2}{4k}}}{\sqrt{2\pi k}}\int_{-\infty}^{\infty} e^{-y^2}dy.
\end{aligned}$$

Here , $\sqrt{k}x+\frac{iw}{2\sqrt{k}}=y$ and $dx=\frac{dy}{\sqrt{k}}$. Now, we have to compute the improper integral $I=\int_{-\infty}^{\infty}e^{-y^2}dy$. If we square the integral and use polar coordinates that $x^2+y^2=r^2$, $dxdy=rdrd\theta$, $0\le\theta\le 2\pi$, $0\le r<\infty$, we obtain

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-x^2}dx\int_{-\infty}^{\infty} e^{-y^2}dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)}dxdy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}rdrd\theta=\int_{0}^{2\pi}\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty}d\theta \\
&= \frac{1}{2}\int_{0}^{2\pi}d\theta=\pi.
\end{aligned}$$

Substituting this in the last Fourier transform formula gives

$$\mathbf{F}(w)=\frac{e^{-\frac{w^2}{4k}}}{\sqrt{2\pi k}}\sqrt{\pi}=\frac{e^{-\frac{w^2}{4k}}}{\sqrt{2k}},\quad k>0. \tag{3.20}$$

Next we obtain the Fourier transform of the function

$$h(x)=\begin{cases} e^{-x},\ x>0, \\ 0,\quad x\le 0. \end{cases} \tag{3.21}$$

Solution Using the Fourier transform formula

$$\begin{aligned}
\mathbf{F}(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} 0.e^{-iwx}dx+\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x}e^{-iwx}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x}e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x(1+iw)}dx \\
&= \left[\frac{-1}{\sqrt{2\pi}(1+iw)}e^{-iwx}e^{-x}\right]_0^{\infty}.
\end{aligned}$$

Since $\lim_{x\to\infty}\left|e^{-x(1+iw)}\right| = \lim_{x\to\infty} e^{-x} = 0$, we get the Fourier transform of the function (3.21)

$$\mathbf{F}(w) = \frac{1}{\sqrt{2\pi}\,(1+iw)} = \frac{1-iw}{\sqrt{2\pi}\,(1+w^2)}. \tag{3.22}$$

Let us obtain the Fourier transform of the function

$$h(x) = e^{-\delta|x|} \tag{3.23}$$

such that $\delta > 0$.

Solution Using the Fourier transform formula

$$\begin{aligned}
\mathbf{F}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\delta|x|} e^{-iwx} dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{-\delta(-x)} e^{-iwx} dx + \int_{0}^{\infty} e^{-\delta x} e^{-iwx} dx \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{-(iw-\delta)x} dx + \int_{0}^{\infty} e^{-(iw+\delta)x} dx \right] \\
&= \frac{1}{\sqrt{2\pi}} \left\{ \left. \frac{e^{-(iw-\delta)x}}{-(iw-\delta)} \right|_{-\infty}^{0} + \left. \frac{e^{-(iw+\delta)x}}{-(iw+\delta)} \right|_{0}^{\infty} \right\}.
\end{aligned}$$

Since $\lim_{x\to-\infty}\left|e^{-(iw-\delta)x}\right| = \lim_{x\to-\infty} e^{-x} = 0$ and $\lim_{x\to\infty}\left|e^{-(iw+\delta)x}\right| = \lim_{x\to\infty} e^{-x} = 0$, we get the Fourier transform of the function (3.23)

$$\mathbf{F}(w) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\delta - iw} - \frac{1}{\delta + iw} \right) = \frac{1}{\sqrt{2\pi}} \frac{2\delta}{\delta^2 + w^2}. \tag{3.24}$$

3.3 Some Algebraic Properties

3.3.1 Linearity of Fourier Transform

If given any integrable functions h and g and any reel numbers m and n, then

$$F\{mh(x) + ng(x)\} = mF\{h(x)\} + nF\{g(x)\}. \tag{3.25}$$

This equation indicates that Fourier transform operation is linear. Let us give a simple proof of this.

Applying the Fourier transform formula, we achieve

$$F\{mh(x) + ng(x)\} = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} [mh(x) + ng(x)]\, e^{-iwx} dx. \tag{3.26}$$

Based on the properties of integral, we type

$$\begin{aligned} F\{mh(x)+ng(x)\} &= \frac{1}{\sqrt{2\pi}}\int_0^{\infty}[mh(x)+ng(x)]e^{-iwx}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} mh(x)e^{-iwx}dx+\frac{1}{\sqrt{2\pi}}\int_0^{\infty} ng(x)e^{-iwx}dx \\ &= m\frac{1}{\sqrt{2\pi}}\int_0^{\infty} h(x)e^{-iwx}dx+n\frac{1}{\sqrt{2\pi}}\int_0^{\infty} g(x)e^{-iwx}dx \\ &= mF\{h(x)\}+nF\{g(x)\}. \end{aligned}$$

From hence,

$$F\{mh(x)+ng(x)\}=mF\{h(x)\}+nF\{g(x)\}.$$

3.3.2 Fourier Transforms of Derivatives of Functions

(i) Let $h(x)$ be a piecewise continuous function and $h(x)$ and $h'(x)$ are integrable. Then

$$F\left\{h'(x)\right\}=iwF\{h(x)\}$$

when $h(x)\to 0$ for $|x|\to\infty$.

(ii) In addition to (i), if $h'(x)$ is piecewise continuous and $h''(x)$ is integrable, then

$$F\left\{h''(x)\right\}=iwF\left\{h'(x)\right\}=-w^2F\{h(x)\}$$

when $h(x)\to 0$ for $|x|\to\infty$.

(iii) Generally, if $h(x)$ and $h^{(k)}(x)$ $(k=1,2,...m-1)$ are piecewise continuous and $h(x)$ and its derivatives of order up to m are integrable, then

$$F\left\{h^{(m)}(x)\right\}=(iw)^m F\{h(x)\}$$

when $h(x)\to 0$ for $|x|\to\infty$.

To prove (i), we use the formula of Fourier transform

$$F\left\{h'(x)\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}h'(x)e^{-iwx}dx \tag{3.27}$$

and integrate by parts

$$F\left\{h'(x)\right\}=\frac{1}{\sqrt{2\pi}}\left[h(x)e^{-iwx}\Big|_{-\infty}^{\infty}-(-iw)\int_{-\infty}^{\infty}h(x)e^{-iwx}dx\right]. \tag{3.28}$$

Since $h(x) \to 0$ as $|x| \to \infty$, and $\left|e^{\pm iwx}\right| = 1$, we acquire

$$F\left\{h'(x)\right\} = 0 + iwF\left\{h(x)\right\}. \tag{3.29}$$

So,

$$F\left\{h'(x)\right\} = iwF\left\{h(x)\right\}. \tag{3.30}$$

By induction, we can prove (ii) and (iii).

3.3.3 Fourier Transforms of Convolutions

The convolution of $h * g$ functions is defined by

$$(h*g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(p)g(x-p)dp = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x-p)g(p)dp. \tag{3.31}$$

The convolution of h and g is a binary operation that reunites multiplication, translation and integration. In addition, we see that the convolution is a commutative operation in formula (3.31). This means that $(h*g)(x) = (g*h)(x)$.

Theorem 3.2. *Assume that h and g are integrable, then*

$$F\left\{(h*g)(x)\right\} = F\left\{h(x)\right\} F\left\{g(x)\right\}. \tag{3.32}$$

Proof *Applying (3.31) and (3.15), we get*

$$\begin{aligned} F\left\{(h*g)(x)\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (h*g)(x)e^{-iwx}dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(p)g(x-p)dpdx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-p)e^{-iwx}dxh(p)dp. \end{aligned}$$

Applying integration by parts with $x - p = u$ and $dx = du$, we get

$$\begin{aligned} F\left\{(h*g)(x)\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(u)e^{-iw(u+p)}duh(p)dp \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(p)e^{-iwp}dp \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(u)e^{-iwu}du \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(p)e^{-iwp}dp \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)e^{-iwx}dx \\ &= F\left\{h(x)\right\} F\left\{g(x)\right\}. \end{aligned}$$

It follows that

$$F\left\{(h*g)(x)\right\} = F\left\{h(x)\right\}F\left\{g(x)\right\}.$$

3.4 The Fourier Transform Method for the Solution of PDEs

The PDE's that we encountered in the previous chapter are defined in finite regions, and their solutions depend on boundary conditions. However, we can mathematically model differential equations defined in infinite regions by the Fourier transform method. Now, let us look at how the Fourier transform method is applied for the one dimensional equation.

Suppose that $u(t,x)$ is a function of x and t, where in the interval $(-\infty,\infty)$ and $t > 0$. Since we have two variables, the variable to which the Fourier transform is calculated needs to be defined. For instance, if t is kept constant, x becomes the function of the spatial variable and we can apply the Fourier transform in regard to the x. This transform is shown as $u(t,w)$. If we formulate all this, we get

$$F\left\{u(t,x)\right\} = u(t,w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(t,x)e^{-iwx}dx. \tag{3.33}$$

Now, we will examine some properties that will help in applying this method:

$$F\left\{\frac{\partial}{\partial t}u(t,x)\right\} = \frac{d}{dt}u(t,w), \tag{3.34}$$

$$F\left\{\frac{\partial^m}{\partial t^m}u(t,x)\right\} = \frac{d^m}{dt^m}u(t,w), \quad m = 1,2,..., \tag{3.35}$$

$$F\left\{\frac{\partial}{\partial t}u(t,x)\right\} = iwu(t,w), \tag{3.36}$$

$$F\left\{\frac{\partial^m}{\partial t^m}u(t,x)\right\} = (iw)^m u(t,w), \quad m = 1,2,\dots. \tag{3.37}$$

Note that the PDE in the equation (3.34) and (3.35) turns into the ODE. And, the formulas of the Fourier transform of the PDE derivatives are given in equations (3.36) and (3.37).

3.4.1 Fourier Transform Solutions of One Dimensional PDEs

In this section firstly, we will consider the one dimensional hyperbolic PDE. Consider the following one dimensional hyperbolic PDE

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - (9x^4 - 6x)e^{-x^3-t} + e^{-x^3-t}, 0 < t < 1\ , -\infty < x < \infty, \\ u(0,x) = e^{-x^3}\ , -\infty < x < \infty, \\ u_t(0,x) = -\ e^{-x^3}, -\infty < x < \infty. \end{cases} \tag{3.38}$$

First, applying the Fourier transform method, we reach

$$F\left\{u_{tt}(t,x)\right\} = F\left\{u_{xx}(t,x)\right\} - F\left\{(9x^4 - 6x)e^{-x^3}\right\} + F_xF_y\left\{e^{-x^3}\right\} \tag{3.39}$$

or

$$F\left\{u_{tt}(t,x)\right\} - F\left\{u_{xx}(t,x)\right\} = -F\left\{(e^{-x^3})''\right\} + F\left\{e^{-x^3}\right\}. \tag{3.40}$$

From that it follows

$$u_{tt}(t,w) - (iw)^2 u(t,w) = -(iw)^2 F\left\{e^{-x^3}\right\} e^{-t} + F\left\{e^{-x^3}\right\} e^{-t} \tag{3.41}$$

or

$$u_t(t,w) + w^2 u(t,w) = w^2 F\left\{e^{-x^3}\right\} e^{-t} + F\left\{e^{-x^3}\right\} e^{-t}. \tag{3.42}$$

We assume the solution as a sum in the form

$$u(t,w) = u_c(t,w) + u_p(t,w). \tag{3.43}$$

Here $u_c(t,w)$ denotes the complementary solution and $u_p(t,w)$ denotes the particular solution. For the complementary solution, we consider the differential equation

$$u_c(t,w) : u_{tt}(t,w) + w^2 u(t,w) = 0. \tag{3.44}$$

The auxiliary equation is

$$m^2 + w^2 = 0, \tag{3.45}$$

and

$$m = \pm iw. \tag{3.46}$$

Hence, the complementary solution is

$$u_c(t,w) = c_1 \cos wt + c_2 \sin wt. \tag{3.47}$$

Now, we will obtain the particular solution. Using the method of undetermined coefficients, we assume the particular solution as

$$u_p(t,w) = AF\left\{e^{-x^3}\right\}e^{-t}. \tag{3.48}$$

Substituting this solution into (3.42), we have

$$AF\left\{e^{-x^3}\right\}e^{-t} + w^2AF\left\{e^{-x^3}\right\}e^{-t} = w^2F\left\{e^{-x^3}\right\}e^{-t} + F\left\{e^{-x^3}\right\}e^{-t}. \tag{3.49}$$

Therefore,

$$(w^2+1)A\,F\left\{e^{-x^3}\right\}e^{-t} = (w^2+1)A\,F\left\{e^{-x^3}\right\}e^{-t}. \tag{3.50}$$

So, we find the constant

$$A = 1 \tag{3.51}$$

and the particular solution

$$u_p(t,w) = F\left\{e^{-x^3}\right\}e^{-t}. \tag{3.52}$$

Thereby, the general solution is

$$u(t,w) = c_1\cos wt + c_2\sin wt + F\left\{e^{-x^3}\right\}e^{-t}. \tag{3.53}$$

Using the boundary condition

$$u(0,x) = e^{-x^3}\ , -\infty < x < \infty, \tag{3.54}$$

we achieve

$$u(0,w) = c_1 + F\left\{e^{-x^3}\right\}. \tag{3.55}$$

Using the other boundary condition

$$u_t(0,x) = -\,e^{-x^3}, -\infty < x < \infty, \tag{3.56}$$

we get

$$u_t(0,w) = c_2w - F\left\{e^{-x^3}\right\}. \tag{3.57}$$

Solving (3.55) and (3.57), we obtain $c_1 = c_2 = 0$. Thus the complementary solution is $u_c\left(t,w\right) = 0$. Also, the particular solution is

$$u_p\left(t,w\right) = F\left\{e^{-x^3}\right\}e^{-t}. \tag{3.58}$$

Substituting $u_c\left(t,w\right)$ and $u_p\left(t,w\right)$ in (3.43), we obtain

$$u\left(t,w\right) = 0 + F\left\{e^{-x^3}\right\}e^{-t}. \tag{3.59}$$

Finally, applying the inverse Fourier transform,

$$u(t,x) = F^{-1}\left\{F\left\{e^{-x^3}\right\}\right\}e^{-t}, \tag{3.60}$$

we reach the solution as

$$u(t,x) = e^{-x^3-t}. \tag{3.61}$$

Therefore, $u(t,x) = e^{-x^3-t}$ is the solution of the problem (3.38) obtained by the Fourier transform method.

3.4.2 Fourier Transform Solutions of Two Dimensional PDEs

In the present section we will study the two dimensional PDE. Consider the following two dimensional hyperbolic problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - (9x^4 - 6x)e^{-x^3-y^3-t} - (9y^4 - 6y)e^{-x^3-y^3-t} + e^{-x^3-y^3-t}, \\ 0 < t < 1,\ -\infty < x < \infty,\ -\infty < y < \infty, \\ u(0,x,y) = e^{-x^3-y^3},\ -\infty < x < \infty,\ -\infty < y < \infty, \\ u_t(0,x,y) = -\,e^{-x^3-y^3},\ -\infty < x < \infty,\ -\infty < y < \infty. \end{cases} \tag{3.62}$$

First, applying the Fourier transform method, we get

$$\begin{aligned} F_xF_y\left\{u_{tt}(t,x,y)\right\} = {} & F_xF_y\left\{u_{xx}(t,x,y)\right\} + F_xF_y\left\{u_{yy}(t,x,y)\right\} \\ & - F_xF_y\left\{(9x^4 - 6x)e^{-x^3-y^3}\right\} \\ & - F_xF_y\left\{(9y^4 - 6y)e^{-x^3-y^3}\right\} + F_xF_y\left\{e^{-x^3-y^3}\right\} \end{aligned} \tag{3.63}$$

or

$$\begin{aligned} & F_xF_y\left\{u_{tt}(t,x,y)\right\} - F_xF_y\left\{u_{xx}(t,x,y)\right\} - F_xF_y\left\{u_{yy}(t,x,y)\right\} \\ & = -F_xF_y\left\{(e^{-x^3})''e^{-y^3}\right\} - F_xF_y\left\{(e^{-y^3})''e^{-x^3}\right\} + F_xF_y\left\{e^{-x^3-y^3}\right\}. \end{aligned} \tag{3.64}$$

From that it follows

$$\begin{aligned} & u_{tt}(t,w,s) - (iw)^2u(t,w,s) - (is)^2u(t,w,s) \\ & = -(iw)^2F_xF_y\left\{e^{-x^3-y^3}\right\}e^{-t} - (is)^2F_xF_y\left\{e^{-x^3-y^3}\right\}e^{-t} \\ & + F_xF_y\left\{e^{-x^3-y^3}\right\}e^{-t} \end{aligned} \tag{3.65}$$

or

$$\begin{aligned} & u_{tt}(t,w,s) + (w^2 + s^2)u(t,w,s) \\ & = (w^2 + s^2)F_xF_y\left\{e^{-x^3-y^3}\right\}e^{-t} + F_xF_y\left\{e^{-x^3-y^3}\right\}e^{-t}. \end{aligned} \tag{3.66}$$

Let the solution be in the form

$$u(t,w,s) = u_c(t,w,s) + u_p(t,w,s). \tag{3.67}$$

Here $u_c(t,w,s)$ denotes the complementary solution and $u_p(t,w,s)$ denotes the particular solution. For complementary solution, the auxiliary equation is

$$u_c(t,w,s) : m^2 + (w^2 + s^2) = 0, \tag{3.68}$$

and

$$m = \pm i\sqrt{(w^2 + s^2)}, \tag{3.69}$$

So, the complementary solution is

$$u_c(t,w,s) = c_1 \cos\sqrt{(w^2 + s^2)}t + c_2 \sin\sqrt{(w^2 + s^2)}t. \tag{3.70}$$

Now, we will consider the particular solution. Using the method of undetermined coefficients, we assume the particular solution as

$$u_p(t,w,s) = AF_xF_y\left\{e^{-x^3-y^3}\right\}e^{-t}. \tag{3.71}$$

Substituting this solution into (3.66), we have

$$\begin{aligned} &AF_xF_y\left\{e^{-x^3-y^3}\right\}e^{-t} + (w^2+s^2)AF_xF_y\left\{e^{-x^3-y^3}\right\}e^{-t} \\ &= (w^2+s^2)F_xF_y\left\{e^{-x^3-y^3}\right\}e^{-t} + F_xF_y\left\{e^{-x^3-y^3}\right\}e^{-t}. \end{aligned} \tag{3.72}$$

Therefore,

$$(w^2+s^2+1)AF_xF_y\left\{e^{-x^3-y^3}\right\}e^{-t} = (w^2+s^2+1)AF_xF_y\left\{e^{-x^3-y^3}\right\}e^{-t}. \tag{3.73}$$

So, we find the constant

$$A = 1, \tag{3.74}$$

and the particular solution

$$u_p(t,w,s) = F_xF_y\left\{e^{-x^3-y^3}\right\}e^{-t}. \tag{3.75}$$

Thereby, the general solution is

$$u(t,w,s) = c_1 \cos\sqrt{(w^2+s^2)}t + c_2 \sin\sqrt{(w^2+s^2)}t + F_xF_y\left\{e^{-x^3-y^3}\right\}e^{-t}. \tag{3.76}$$

Using the initial condition

$$u(0,x,y)=e^{-x^3-y^3},\ -\infty<x<\infty,\ -\infty<y<\infty, \tag{3.77}$$

we get

$$u(0,w,s)=c_1+F_xF_y\left\{e^{-x^3-y^3}\right\}. \tag{3.78}$$

If we use the other initial condition

$$u_t(0,x,y)=-e^{-x^3-y^3},\ -\infty<x<\infty,\ -\infty<y<\infty, \tag{3.79}$$

we get

$$u_t(0,w,s)=c_2\sqrt{(w^2+s^2)}-F_xF_y\left\{e^{-x^3-y^3}\right\}. \tag{3.80}$$

Solving (3.78) and (3.80), we obtain $c_1=c_2=0$. Thus the complementary solution is $u_c(t,w,s)=0$ and the particular solution is

$$u_p(t,w,s)=F_xF_y\left\{e^{-x^3-y^3}\right\}e^{-t}. \tag{3.81}$$

Substituting $u_c(t,w,s)$ and $u_p(t,w,s)$ in (3.67), we get

$$u(t,w,s)=0+F_xF_y\left\{e^{-x^3-y^3}\right\}e^{-t}. \tag{3.82}$$

Finally, implementing the inverse Fourier transform,

$$u(t,x,y)=F_y^{-1}F_x^{-1}\left\{F_xF_y\left\{e^{-x^3-y^3}\right\}\right\}e^{-t}, \tag{3.83}$$

we get the solution as

$$u(t,x,y)=e^{-x^3-y^3-t}. \tag{3.84}$$

Therefore, $u(t,x,y)=e^{-x^3-y^3-t}$ is the solution of (3.62) obtained by the Fourier transform method.

4

RESULTS AND DISCUSSION

In this thesis analytical solutions of one and two dimensional hyperbolic partial differential equations are obtained by using Fourier series and Fourier transform methods. In particular, the study of two dimensional hyperbolic PDE's is of great interest. It is presented that Fourier series method is applicable in the solutions of the BVPs in two dimensional case. However, in the Fourier transform method no boundary condition is necessary, the initial conditions are enough to solve the problem with the space variable domain $(-\infty, \infty)$ which made the Fourier transformation method advantageous. There are many researchers studying intensively in the field of unconditionally stable difference schemes (see [14] [15] [16] [17] [18] [19] [20] [21], and references therein).

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Contact Information: gonul.burcuuu@gmail.com

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1. O. Yildirim, B. Gonul, "On the analytical and numerical solution of the two dimensional hyperbolic partial differential equations," 3rd International Congress on Statistics Mathematics and Analytical Methods, 12-13 Mart, 2020, Istanbul.