FREQUENCY INDEPENDENT EVALUATION OF HIGHLY OSCILLATORY INTEGRALS

by

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ABSTRACT

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Oscillatory integrals appear in many applications and therefore, several different quadrature rules for their accurate and efficient evaluation have been proposed in the literature. The efficiency of these methods naturally depend on the degree of smoothness of the integrands.

In this thesis, we present a survey for the paper [1] by Domínguez *et al.*, which is recently exhibited in the literature, about composite algorithm which eases the conditions of the Filon-Clenshaw-Curtis rule and enables the use of the algorithm with finitely many algebraic and logarithmic singularities or oscillators with a finite number of stationary points. Our survey includes the complete error analysis in [1], which is in the setting of appropriate Sobolev spaces of 2π-periodic functions. We support the results of this paper via providing some numerical experiments.

ÖZET

YÜKSEK FREKANSLI İNTEGRALLERİN FREKANSTAN BAĞIMSIZ OLARAK HESAPLANMASI

Yüksek frekanslı integraller pek çok uygulamada ortaya çıkmaktadır ve bu nedenle, bu integrallerin kesin ve etkili şekilde hesaplanmarı için literatürde çeşitli kuadratür yöntemleri bulunmaktadır. Doğal olarak, bu metotların verimlilikleri integrali hesaplanan fonksiyonun düzgünlüğüne bağlıdır.

Bu tezde, Domínguez ve diğerleri tarafından yazılmış olan ve yakın zamanda literatürde yer almış olan, Filon-Clenshaw-Curtis kuadratür kuralının koşullarının ne şekilde hafifletilebileceği üzerine olan ve ayrıca algoritmayı sonlu sayıda cebirsel veya logaritmik tekillik noktası olan fonksiyonlara veya sonlu sayıda durgunluk noktası olan osilatörlere uygulanabilir hale getiren [1] araştırması sunulmaktadır. Ayrıca bu çalışma, araştırmada bulunan, uygun şekilde seçilmiş 2π periyotlu fonksiyonların oluşturduğu Sobolev uzaylarını içeren tamamlanmış hata analizini de içermektedir. Bununla birlikte, araştırmanın sonuçları bazı numerik deneylerle desteklenmektedir.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

C^∞	Infinitely many times differentiable
$C^\infty[a,b]$	function that are infinitely differentiable on $[a,b]$
$\mathbb{R}$	Real numbers
C^m_β	Subspace of Sobolev space of 2π periodic function with the norm $\|.\|_{H^m}$
$ci(x)$	Special function defined at 3.14
$\hat{f}(k)$	K^{th} Fourier constant of the function f
$f_c(x)$	The function $f(cos(x))$
$f_k(x)$	The function $f(x)\exp(ikx)$
${}_1F_1(a;b;z)$	The confluent hypergeometric function
H^m	The space of Sobolev space of 2π periodic functions
$I_k(f)$	The oscillatory integral with linear oscillator and oscillation parameter k over $[-1,1]$
$I_k^{[a,b]}(f)$	The oscillatory integral with linear oscillator and oscillation parameter k over $[a,b]$
$I_k^{[a,b]}(f,g)$	The oscillatory integral with oscillator g and oscillation parameter k over $[a,b]$
$I_{k,N}(f)$	The approximation for $I_k(f)$ with N quadrature points
i	Imaginary unit
$\mathbb{K}_n$	The set of integers between $-n/2$ and $n/2$ including $n/2$, if n is even
$L^1[a,b]$	The space of integrable functions over $[a,b]$
$L^2[a,b]$	The space of square integrable functions over $[a,b]$
$Q_N f$	The N^{th} degree polynomial interpolant of f at the points $\cos(j\pi/N)$
$\mathbb{R}$	The set of real numbers
$S_n f$	The truncated Fourier series of f with $2n+1$ elements
$si(x)$	Special function defined at 3.13
$T_n(t)$	Trigonometric Chebyshev polynomial for $\|t\| \leq 1$
$\mathbb{T}_n$	A special set of functions defined at the proof of Lemma A.2

$U_n(t)$	Chebyshev polynomials of the second kind
$\mathbb{Z}_n$	The set of integers between $-n$ and n including n
$\alpha_{n,N}(f)$	The n^{th} coefficient of $Q_N f$
$\gamma_n(k)$	The integral defined at (5.2)
$\Gamma(z)$	The gamma function
$\underline{\mu}$	A special notation for integers defined for $\|.\|_{H^m}$
ρ_n	The oscillatory linear integral of n^{th} Chebyshev polynomial of second kind with oscillation parameter k
$\sum''$	Finite sum whose first and last terms are to be halved
$\omega_n(k)$	The oscillatory linear integral of n^{th} trigonometric Chebyshev polynomial of first kind with oscillation parameter k
$\lvert.\rvert_{H^m_\omega}[a,b]$	A weighted seminorm for functions from H^m
$\lvert.\rvert_{H^m_\omega}$	The weighted seminorm $\lvert.\rvert_{H^m_\omega}[a,b]$ with $[a,b]=[-1,1]$
$\lVert.\rVert_{H^m}$	The norm for H^m of 2π periodic functions
$\lVert.\rVert_{m,\beta}$	A norm for composite algorithm
$\gg$	Notation for much more greater

LIST OF ACRONYMS/ABBREVIATIONS

FCC	Filon-Clenshaw-Curtis

1. INTRODUCTION

Highly oscillatory integrals of the form

$$\int_a^b f(t)\exp(ikg(t))dt \qquad (1.1)$$

for $f \in L^1[a,b]$ and $k > 0$ appear in many applications, such as boundary integral methods for high-frequency wave scattering (see e.g. [2–7]). Therefore, several different approaches for the accurate evaluation of these integrals have been proposed in the literature.

Indeed, standard integral approximation methods depend on integrating the polynomial interpolation of the function instead of integrating the function itself. If we apply these classical methods to (1.1), they need a fixed number of quadrature points per wavelength which is proportional to $1/k$. Hence, the number of quadrature points these rules require to keep a fixed error increases linearly with k which results the increase in the number of calculations with increasing k. Therefore, instead of these standard methods, alternative approaches have been considered in the literature.

These methods can be classified into three main categories consisting of Asymptotic methods, Levin type methods and Filon type methods. Asymptotic methods depend on applying integration by parts repeatedly. Even if they need very small number of operations, they do not generally converge for fixed k. Levin-type methods are another family of methods used for evaluating of oscillatory integrals of the form (1.1). Basically, they depend on the approximation of a function F satisfying $F'(t) + ikg'(t)F(t) = g(t)$ and this naturally requires f and g to be sufficiently differentiable [8]. However, since error terms behave asymptotically like $O(k^r)$ for some $r \geq 1$, (see [9–12]), they converge very slowly for moderate values of k [13]. Filon-type methods are based on interchanging the function f with its N^{th} degree polynomial interpolant $Q_N f$. The number of operations they need is independent of k and give high accuracy for fixed k. However, when f has some singularities in $[a,b]$ or g has some stationary points, to bound the convergence rate of the quadrature rule becomes problematic.

In this thesis, we follow the papers [1] and [14], and the book [15] to study a modified version of Filon-type quadrature methods known as Filon-Clenshaw-Curtis quadrature rule for highly oscillatory integrals of type (1.1) for functions f with finitely many singularities and functions g with a finite number of stationary points. We present a complete error analysis in the setting of appropriate Sobolev spaces and in addition to it, we submit numerical implementations.

This thesis is organized as follows. In Chapter 2, we restrict ourselves to the case of smooth f and consider the case of linear oscillators, namely $g(x) = x$, on the interval $[-1, 1]$. In §2.1, we present the quadrature rule with an error analysis. In §2.2 we extend the quadrature rule and the error analysis to arbitrary intervals $[a, b]$. In chapter 3, we study the case of f with finitely many algebraic or logarithmic singularities. In §3.1 we treat the integrals over $[0, 1]$ with singularity of f at $x = 0$. Then, in §section:relate[a,b][0,1] we extend the method to functions f with finitely many algebraic or logarithmic singularities in an arbitrary $[a, b]$. In chapter 4, we generalize the developments in Chapter 2 and 3 to the case of nonlinear oscillator. §4.1 concerns the case of smooth f and g having no singular point in an generic interval $[a, b]$, and §section:comnon corresponds to f with singularities and g with stationary points. In Chapter 5, we provide some implementation details and a stable algorithm for the computation of quadrature weights. Finally, in Chapter 6, we present a variety of numerical experiments for both the linear and the nonlinear oscillators and verify the theoretical results.

2. FILON-CLENSHAW-CURTIS RULE FOR LINEAR OSCILLATORS

Here we describe the Filon-Clenshaw-Curtis (FCC) quadrature rule for the accurate approximation of oscillatory integrals of the form

$$I_k^{[a,b]}(f) = \int_a^b f(t)\exp(ikt)dt \tag{2.1}$$

where we assume that the function f is smooth aside from finitely many algebraic or logarithmic singularities in the interval $[a,b]$, and $k>0$.

To this end, we first assume that the function f is smooth and, in this setting, discuss the FCC rule for $I_k(f) = I_k^{[-1,1]}(f)$ in §2.1. Next, in section §2.2, we show that an affine change of variables allows one to modify the FCC method so as to cover the case of the oscillatory integrals $I_k^{[a,b]}(f)$.

In what follows, we say that $I_k(f)$ is oscillatory when $k \geq 1/2$ and non-oscillatory when $0 < k < 1/2$.

2.1. Smooth functions on the interval $[-1,1]$

Here we study the FCC rule for the accurate approximation of integrals

$$I_k(f) = \int_{-1}^1 f(t)\exp(ikt)dt \tag{2.2}$$

assuming that the function f is smooth.

When $I_k(f)$ is oscillatory, for a given natural number N, the basic FCC rule approximates $I_k(f)$ by replacing the function f with its algebraic polynomial interpolant $Q_N f$ at the Clenshaw-Curtis quadrature points $\cos(j\pi/N)$, $j = 0,1,...,N$, and writing this interpolant as a linear combination of Chebyshev polynomials of the first kind

$$T_n(t) = \cos(n \arccos(t)). \tag{2.3}$$

In more precise terms, setting

$$\omega_n(k) = \int_{-1}^{1} T_n(t) \exp(ikt) dt \tag{2.4}$$

and

$$\alpha_{n,N}(f) = \frac{2}{N} \sum_{p=0}^{N}{}'' \cos(\frac{np\pi}{N}) f(\cos(\frac{p\pi}{N})), \tag{2.5}$$

we have the approximation

$$\begin{aligned} I_k(f) = \int_{-1}^{1} f(t) \exp(ikt) dt &\approx \int_{-1}^{1} (Q_N f)(t) \exp(ikt) dt \\ &= \int_{-1}^{1} \sum_{n=0}^{N}{}'' \alpha_{n,N}(f) T_n(t) \exp(ikt) dt \\ &= \sum_{n=0}^{N}{}'' \alpha_{n,N}(f) \int_{-1}^{1} T_n(t) \exp(ikt) dt, \end{aligned}$$

and the FCC quadrature rule read as

$$I_k(f) \approx I_{k,N}(f) = \sum_{n=0}^{N}{}'' \alpha_{n,N}(f)\, \omega_n(k) \tag{2.6}$$

where $\sum''$ means that the first and the last summands are halved.

When $I_k(f)$ is non-oscillatory, we directly apply the standard Clenshaw-Curtis quadrature method [16] to the function $f_k(t) = f(t) \exp(ikt)$ which yields the quadrature rule

$$I_k(f) = \int_{-1}^{1} f(t)\exp(ikt)dt \approx \int_{-1}^{1} (Q_N f_k)(t)dt$$
$$= \int_{-1}^{1} \sum_{n=0}^{N}{}'' \alpha_{n,N}(f_k) T_n(t)dt$$
$$= \sum_{n=0}^{N}{}'' \alpha_{n,N}(f_k) \int_{-1}^{1} T_n(t)dt$$

where $T_n(t)$ and $\alpha_{n,N}$ are as in (2.3) and (2.5).

Let us note that

$$\int_{-1}^{1} T_n(t)dt = \int_{-1}^{1} \cos(n \arccos(t))dt = \begin{cases} \frac{2}{1-n^2}, & \text{n is even,} \\ 0, & \text{n is odd.} \end{cases}$$

Therefore, for the implementation of the FCC rule, we need to compute only the quadrature weights $\omega_n(k)$ in (2.4) for $n = 0, 1, ..., N$. Deferring this to §5.1, we now discuss the convergence of the FCC rule.

For the error analysis, we will use the Sobolev space H^m of 2π-periodic functions with the norm [15]

$$\|f\|_{H^m}^2 = |\hat{f}(0)|^2 + \sum_{\mu \in \mathbb{Z}\setminus\{0\}} |\mu|^{2m} |\hat{f}(\mu)|^2, \quad \hat{f}(\mu) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\exp(-i\mu\theta)d\theta \tag{2.7}$$

along with the truncated Fourier series

$$(S_n\phi)(\theta) = \sum_{\mu=-n}^{n} \hat{\phi}(\mu)\exp(i\mu\theta) \tag{2.8}$$

for $\phi \in H^0 = L^2[-\pi,\pi]$, which converges to $\phi \in H^0$ as $n \to \infty$. Observe that if ϕ is an even function, we have

$$(S_n\phi)(\theta) = \hat{\phi}(0) + 2\sum_{\mu=1}^{n} \hat{\phi}(\mu)\cos(\mu\theta) \quad n \geq 1, \quad \hat{\phi}(\mu) = \frac{1}{\pi}\int_0^\pi \phi(\theta)\cos(\mu\theta)d\theta. \quad (2.9)$$

We also use the notation

$$f_c(t) = f(\cos(t)), \qquad \mathbb{Z}_n = \{\mu \in \mathbb{Z} : -n < \mu \leq n\}, \quad \rho(r) = \begin{cases} r, & r \in [0,1], \\ \frac{5r}{2} - \frac{3}{2}, & r \in (1,2] \end{cases}$$

and, for $\mu \in \mathbb{Z}$,

$$\underline{\mu} = \begin{cases} 1, & \mu = 0, \\ |\mu|, & \mu \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Also, note that with this notation, we can write (2.9) as

$$\|f\|^2_{H^m} = \sum_{\mu \in \mathbb{Z}} \underline{\mu}^{2m} |\hat{f}(\mu)|^2.$$

In the next theorem, we consider $I_k(f)$ as oscillatory for any $k > 0$.

Theorem 2.1 ([14]). *For $r = 0, 1, 2$ and $\nu \geq \nu_0 > \max\{\frac{1}{2}, \rho(r)\}$, there is a positive constant C_{ν_0} such that for any positive k*

$$|I_k(f) - I_{k,N}(f)| \leq C_{\nu_0} \left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{\nu-\rho(r)} \|f_c\|_{H^\nu}.$$

provided that f is smooth.

Proof. In Theorem A.3, setting

$$u = f_c, \quad \gamma_m = \left(1 + 2\sum_{j=1}^{\infty} \frac{1}{j^{2m}}\right)^{1/2} = C_{\nu_0,m}, \quad n = N, \quad m = \nu,$$

we obtain

$$\|e_N\|_{H^\lambda} \leq C_{\nu_0,m} N^{\lambda-\nu} \|f_c\|_{H^\nu} \tag{2.10}$$

where e_N is the 2π-periodic even error function

$$e_N = f_c - (Q_N f)_c. \tag{2.11}$$

Via the cosine transform and Cauchy-Schwartz inequality, we have

$$\begin{aligned} |I_k(f) - I_{k,N}(f)| &= \left| \int_{-1}^{1} (f - Q_N f)(s) \exp(iks) ds \right| \\ &\leq \int_0^\pi |e_N(\theta)| \, d\theta \leq \sqrt{\pi} \|e_N\|_{H^0} \end{aligned}$$

and, by (2.10),

$$|I_k(f) - I_{k,N}(f)| \leq C_{\nu_0} N^{-\nu} \|f_c\|_{H^\nu}$$

which gives the result for $r = 0$. Further, since the Chebyshev points include the end points ± 1, we get

$$e_N(0) = e_N(\pi) = 0.$$

Note that, by integration by parts

$$\begin{aligned}
\hat{e'_N}(\mu) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e'_N(\theta)\exp(-i\mu\theta)d\theta \\
&= \frac{1}{2\pi}\left[e_N(\pi)\exp(-i\mu\pi) - e_N(-\pi)\exp(i\mu\pi) + i\mu\int_{-\pi}^{\pi} e_N(\theta)\exp(-i\mu\theta)d\theta\right] \\
&= \frac{1}{2\pi}i\mu\int_{-\pi}^{\pi} e_N(\theta)\exp(-i\mu\theta)d\theta = i\mu\hat{e_N}(\mu).
\end{aligned}$$

Then, we have

$$\left|\hat{e'_N}(\mu)\right| \leq \underline{\mu}^2 \left|\hat{e_N}(\mu)\right|^2$$

which gives

$$\|e'_N\|_{H^\lambda} \leq \|e_N\|_{H^\lambda} \tag{2.12}$$

Also, by integration by parts,

$$\begin{aligned}
|I_k(f) - I_{k,N}(f)| &= \left|\int_0^{\pi} e_N(\theta)\exp(ik\cos(\theta))\sin(\theta)d\theta\right| \\
&\leq \left|-\left[e_N(\pi)(\frac{1}{ik})\exp(ik\cos(\pi)) - e_N(0)(\frac{1}{ik})\exp(ik\cos(0))\right]\right| \\
&\quad + \left|\frac{1}{ik}\int_0^{\pi} e'_N(\theta)\exp(ik\cos(\theta))d\theta\right| \\
&\leq \frac{1}{k}\int_0^{\pi} |e'_N(\theta)|\, d\theta \leq \frac{1}{k}\sqrt{\pi}\|e'_N\|_{H^0}.
\end{aligned}$$

Therefore, using (2.10) and (2.12), we obtain

$$\begin{aligned}
|I_k(f) - I_{k,N}(f)| &\leq \frac{1}{k}\sqrt{\pi}\|e_N\|_{H^0} \\
&\leq \frac{1}{k}\sqrt{\pi}\|e_N\|_{H^1} \leq C_{\nu_0,m}\frac{1}{k}N^{1-\nu}\|f_c\|_{H^\nu}
\end{aligned}$$

which is the result for $r = 1$. For $r = 2$, we have

$$e'_N(\theta) = -(f - Q_N f)'(\cos(\theta))(\sin(\theta)).$$

Thus,

$$e'_N(0) = e'_N(\pi) = 0.$$

Let

$$\varphi_N(\theta) = \frac{e'_N(\theta)}{\sin(\theta)} = -(f - Q_N f)'(\cos(\theta)).$$

Then, for $u = \varphi_N(\theta)$ and $dv = \exp(ik\cos(\theta))\sin(\theta)d\theta$,

$$\begin{aligned}
I_k(f) - I_{k,N}(f) &= \frac{1}{ik}\int_0^{\pi} \varphi_N(\theta)\sin(\theta)\exp(ik\cos(\theta))d\theta \\
&= \frac{1}{ik}\left[\left(\frac{-1}{ik}\right)\varphi_N(\pi)\exp(ik\cos(\pi)) - \left(\frac{-1}{ik}\right)\varphi_N(0)\exp(ik\cos(0))\right] \\
&\quad + \frac{1}{ik}\left[\frac{1}{ik}\int_0^{\pi} \varphi'_N(\theta)\exp(ik\cos(\theta))d\theta\right] \\
&= \frac{1}{k^2}\left[\varphi_N(\pi)\exp(ik\cos(\pi)) - \varphi_N(0)\exp(ik\cos(0))\right] \\
&\quad - \frac{1}{k^2}\left[\int_0^{\pi} \varphi'_N(\theta)\exp(ik\cos(\theta))d\theta\right] \\
&= \frac{1}{k^2}\left[E_1 - E_2\right].
\end{aligned}$$

Concerning E_1, we note that

$$\lim_{\theta\to 0}\varphi_N(\theta) = \lim_{\theta\to 0}\frac{e'_N(\theta)}{\sin(\theta)} = \lim_{\theta\to 0}\frac{e''_N(\theta)}{\cos(\theta)} = e''_N(0)$$

and

$$\lim_{\theta\to \pi}\varphi_N(\theta) = \lim_{\theta\to \pi}\frac{e'_N(\theta)}{\sin(\theta)} = \lim_{\theta\to \pi}\frac{e''_N(\theta)}{\cos(\theta)} = -e''_N(\pi).$$

and therefore

$$\varphi_N(0) = e_N''(0) \quad \text{and} \quad \varphi_N(\pi) = -e_N''(\pi). \tag{2.13}$$

Thus, via using the Sobolev Embedding Theorem [17–20], we have

$$\begin{aligned}|E_1| &= |\varphi_N(\pi)\exp(ik\cos(\pi)) - \varphi_N(0)\exp(ik\cos(0))| \\ &\leq |\varphi_N(\pi)||\exp(-ik)| + |\varphi_N(0)||\exp(ik)| \\ &= |e_N''(\pi)| + |e_N''(0)| \leq C\|e_N\|_{H^3}\end{aligned}$$

and, by (2.10),

$$|E_1| \leq C_{\nu_0}\frac{1}{N^{\nu-3}}\|f_c\|_{H^\nu} \leq C_{\nu_0}\frac{1}{N^{\nu-7/2}}\|f_c\|_{H^\nu}$$

for all $\nu \geq \nu_0 \geq \max\{\frac{1}{2}, \rho(2)\} \geq 3$ [14]. To estimate E_2, we write e_N as a cosine series

$$e_N(\theta) = \frac{1}{2}\hat{e}_N(0) + \sum_{m=1}^{\infty}\hat{e}_N(m)\cos(m\theta)$$

where

$$\hat{e}_N(m) = \begin{cases} \frac{1}{\pi}\int_0^\pi e_N(\theta)d\theta, & m = 0, \\ \frac{2}{\pi}\int_0^\pi e_N(\theta)\cos(m\theta)d\theta, & m \geq 1. \end{cases}$$

Since, by (2.11), e_N consists of subtraction of composition of smooth functions, its derivative is smooth and also we have $e_N(-\pi) = e_N(\pi)$, so we can differentiate its Fourier series term by term. Hence

$$\varphi_N(\theta) = \frac{e_N'(\theta)}{\sin(\theta)} = -\sum_{m=1}^{\infty} m\hat{e}_N(m)\frac{\sin(m\theta)}{\sin(\theta)}.$$

Here we note that $\sigma(x) = \frac{\sin(x)}{x} \in C^\infty$ and for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\left(\sigma(x) - \frac{2}{\pi}\right)' = 0 \implies x = \tan(x).$$

Further, for $g(x) = \tan(x) - x$, then we need to find the zero of $g(x)$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have $g'(x) = \sec^2(x) - 1$ so that

$$g'(x) = 0 \implies \sec^2(x) = 1 \implies x = 2n\pi, \quad n \in \mathbb{Z},$$

and

$$\lim_{x \to -\frac{\pi}{2}} g(x) = -\infty, \quad \lim_{x \to \frac{\pi}{2}} g(x) = \infty.$$

Therefore, there is only one critical point, $x = 0$ of g in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, which has image $g(0) = 0$ on that interval. Then, since we have $\lim_{x\to 0} \sigma(x)/x - (2/\pi) > 0$, we get $\frac{2}{\pi} \leq \sigma(\theta)$ for $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Hence, for $\theta \in \left(0, \frac{\pi}{2}\right]$ and $m \geq 1$, we get

$$\begin{aligned}
\left|\left(\frac{\sin(m\theta)}{\sin(\theta)}\right)'\right| &= \left|\left(\frac{\frac{\sin(m\theta)}{m\theta}}{\frac{\sin(\theta)}{m\theta}}\right)'\right| \\
&= \left|m^2 \frac{\sigma'(m\theta)}{\sigma(\theta)} - m \frac{\sigma(m\theta)\sigma'(\theta)}{\sigma^2(\theta)}\right| \\
&\leq m^2 \left|\frac{\sigma'(m\theta)}{\sigma(\theta)}\right| + m \left|\frac{\sigma(m\theta)\sigma'(\theta)}{\sigma^2(\theta)}\right| \\
&\leq m^2 \left|\frac{\sigma'(m\theta)}{\sigma(\theta)}\right| + m^2 \left|\frac{\sigma(m\theta)\sigma'(\theta)}{\sigma^2(\theta)}\right| \\
&\leq m^2 \left|\frac{\sigma'(m\theta)}{\frac{2}{\pi}}\right| + m^2 \left|\frac{\sigma(m\theta)\sigma'(\theta)}{\left(\frac{2}{\pi}\right)^2}\right| \\
&= m^2 \left[\frac{\pi}{2}|\sigma'(m\theta)| + \left(\frac{\pi}{2}\right)^2 |\sigma(m\theta)\sigma'(\theta)|\right] = \widetilde{C} m^2
\end{aligned}$$

for a constant $\widetilde{C}$. Moreover, the identity

$$\frac{\sin(m\theta)}{\sin(\theta)} = \frac{(-1)^m \sin(m\pi - m\theta)}{(-1)\sin(\pi - \theta)} = (-1)^{m-1} \frac{\sin(m(\pi - \theta))}{\sin(\pi - \theta)}$$

allows us to extend the last result to $\theta \in [0, \pi]$. Accordingly

$$\begin{aligned} |E_2| &= \left| \int_0^{\pi} \varphi_N'(\theta) \exp(ik\cos(\theta)) d\theta \right| \\ &\leq \int_0^{\pi} |\varphi_N'(\theta)| |\exp(ik\cos(\theta))| d\theta \\ &= \int_0^{\pi} |\varphi_N'(\theta)| d\theta \\ &\leq \int_0^{\pi} \|\varphi_N'(\theta)\|_{L^\infty} d\theta \\ &= \pi \|\varphi_N'(\theta)\|_{L^\infty} \\ &= \pi \| \left(-\sum_{m=1}^{\infty} m \hat{e}_N(m) \frac{\sin(m\theta)}{\sin(\theta)} \right)' \|_{L^\infty}. \end{aligned}$$

Since φ_N' is the ratio of two smooth functions, by (2.13), we obtain

$$\begin{aligned} |E_2| &\leq \pi \sum_{m=1}^{\infty} m \, |\hat{e}_N(m)| \, \| \left(\frac{\sin(m\theta)}{\sin(\theta)} \right) \|_{L^\infty} \\ &\leq C \sum_{m=1}^{\infty} m^3 \, |\hat{e}_N(m)| \end{aligned} \tag{2.14}$$

for some constant C. In addition to these results, interpreting the sum $\sum_{m=1}^{N} m^6$ and $\sum_{m=N+1}^{\infty} (1/m^{1+\alpha})$ $(\alpha > 0)$ as lower Riemann sums corresponding to the functions $f(x) = x^6$ and $f(x) = 1/x^{1+\alpha}$, we have the estimates

$$\sum_{m=1}^{N} m^6 < \int_1^{N+1} x^6 \, dx = \frac{(N+1)^7}{7} - \frac{1}{7} < \frac{(N+1)^7}{7} \tag{2.15}$$

and

$$\sum_{m=N+1}^{\infty} \frac{1}{m^{1+\alpha}} < \int_N^{\infty} \frac{1}{x^{1+\alpha}} \, dx = \frac{1}{\alpha N^{\alpha}}, \quad (\alpha > 0). \tag{2.16}$$

Therefore, splitting the sum in (2.14) to $m \leq N$ and $m \geq N+1$, we obtain, via Cauchy-Schwartz inequality and the definition of $\|.\|_{H^\nu}$,

$$
\begin{aligned}
|E_2| &\leq C\left[\sum_{m=1}^{N} m^3\,|\hat{e}_N(m)| + \sum_{m=N+1}^{\infty} m^3\,|\hat{e}_N(m)|\right] \\
&\leq C\left[\left[\sum_{m=1}^{N} m^6\right]^{1/2}\left[\sum_{m=1}^{N} |\hat{e}_N(m)|^2\right]^{1/2}\right] \\
&\quad + C\left[\left[\sum_{m=N+1}^{\infty} \frac{1}{m^{2\nu-6}}\right]^{1/2}\left[\sum_{m=N+1}^{\infty} m^{2\nu}\,|\hat{e}_N(m)|^2\right]^{1/2}\right] \\
&\leq C\left[\left[\sum_{m=1}^{N} m^6\right]^{1/2}\left[\sum_{m=1}^{N} |\hat{e}_N(m)|^2\right]^{1/2}\right] \\
&\quad + C\left[\left[\sum_{m=N+1}^{\infty} \frac{1}{m^{2\nu-6}}\right]^{1/2}\left[\sum_{m=N+1}^{\infty} m^{2\nu}\,|\hat{e}_N(m)|^2\right]^{1/2}\right].
\end{aligned}
$$

Since $r=2$, we have $\nu \geq \nu_0 > 7/2$. Thus, if we apply the inequalities in (2.15) and (2.16) to these two sums respectively, we get

$$
\begin{aligned}
|E_2| &\leq C\left[\left(\frac{(N+1)^7}{7}\right)^{1/2}\|e_N\|_{H^0} + \left(\frac{1}{(2\nu-7)N^{2\nu-7}}\right)^{1/2}\|e_N\|_{H^\nu}\right] \\
&\leq C\left[\left(\frac{1}{N}\right)^{-7/2}\|e_N\|_{H^0} + \left(\frac{1}{N^{2\nu-7}}\right)^{1/2}\|e_N\|_{H^\nu}\right] \\
&= C\left[\left(\frac{1}{N}\right)^{-7/2}\|e_N\|_{H^0} + \left(\frac{1}{N}\right)^{\nu-7/2}\|e_N\|_{H^\nu}\right] \\
&\leq C_{\nu_0}\left(\frac{1}{N}\right)^{\nu-7/2}\|f_c\|_{H^\nu}
\end{aligned}
$$

(here note that the constants differ from line to line). Combining the estimates for E_1 and E_2, we therefore obtain

$$
|I_k(f) - I_{k,N}(f)| \leq \frac{1}{k^2}\left(|E_1| + |E_2|\right) \leq 2C_{\nu_0}\frac{1}{k^2}\left(\frac{1}{N}\right)^{\nu-7/2}\|f_c\|_{H^\nu}
$$

and this completes the proof. □

Theorem 2.2 ([1]). *There is a positive constant C such that for any $r \in [0,2]$ and for all integers $m > \max\{\frac{1}{2}, \rho(r)\}$*

$$|I_k(f) - I_{k,N}(f)| \leq C \left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} \|f_c\|_{H^m}, \quad k \geq 1, \quad N \geq 1. \tag{2.17}$$

provided $f_c \in H^m$.

Proof. We have the results for $k \geq \frac{1}{2}$ and $r = 0, 1, 2$ in Theorem 2.1. Also, for $\theta \in [0,1]$, we have

$$|I_k(f) - I_{k,N}(f)| = |I_k(f) - I_{k,N}(f)|^{\theta} |I_k(f) - I_{k,N}(f)|^{1-\theta} \tag{2.18}$$

and applying Theorem 2.1 for $r = 1$ and $r = 2$ to the first and the second terms in (2.18) respectively, we get, for some constants C_1 and C_2,

$$\begin{aligned} |I_k(f) - I_{k,N}(f)| &= |I_k(f) - I_{k,N}(f)|^{\theta} |I_k(f) - I_{k,N}(f)|^{1-\theta} \\ &\leq C_1^{\theta} \left(\frac{1}{k}\right)^{\theta} \left(\frac{1}{N}\right)^{(m-1)\theta} C_2^{(1-\theta)} \left(\frac{1}{k}\right)^{2-2\theta} \left(\frac{1}{N}\right)^{(m-7/2)(1-\theta)} \|f_c\|_{H^m} \\ &= C \left(\frac{1}{k}\right)^{2-\theta} \left(\frac{1}{N}\right)^{m-\frac{7}{2}+\frac{5\theta}{2}} \|f_c\|_{H^m} \end{aligned}$$

where $C = C_1^{\theta} C_2^{(1-\theta)}$. Then setting $\theta = 2 - r$, which forces $r \in [1,2]$ we have

$$\begin{aligned} |I_k(f) - I_{k,N}(f)| &\leq C \left(\frac{1}{k}\right)^{2-\theta} \left(\frac{1}{N}\right)^{m-\frac{7}{2}+\frac{5\theta}{2}} \|f_c\|_{H^m} \\ &= C \left(\frac{1}{k}\right)^{2-(2-r)} \left(\frac{1}{N}\right)^{m-\frac{7}{2}+\frac{5(2-r)}{2}} \|f_c\|_{H^m} \\ &= C \left(\frac{1}{k}\right)^{r} \left(\frac{1}{N}\right)^{m+\frac{3}{2}-\frac{5r}{2}} \|f_c\|_{H^m} \\ &= C \left(\frac{1}{k}\right)^{r} \left(\frac{1}{N}\right)^{m-\rho(r)} \|f_c\|_{H^m}. \end{aligned}$$

This proves (2.17) for $r \in [1,2]$ and $k \geq 1$. For $r \in [0,1]$, we, again, apply 2.1 but now for $r = 0$ and $r = 1$ to the first and the second terms in (2.18) respectively and we get

$$\begin{aligned}|I_k(f) - I_{k,N}(f)| &\leq C_1^{\theta} \left(\frac{1}{N}\right)^{m\theta} C_2^{(1-\theta)} \left(\frac{1}{k}\right)^{1-\theta} \left(\frac{1}{N}\right)^{(m-1)(1-\theta)} \|f_c\|_{H^m} \\ &= C \left(\frac{1}{k}\right)^{1-\theta} \left(\frac{1}{N}\right)^{m-(1-\theta)} \|f_c\|_{H^m}\end{aligned}$$

and setting $\theta = 1 - r$, which forces $r \in [0,1]$, entails

$$|I_k(f) - I_{k,N}(f)| \leq C \left(\frac{1}{k}\right)^{r} \left(\frac{1}{N}\right)^{m-r} \|f_c\|_{H^m} = C \left(\frac{1}{k}\right)^{r} \left(\frac{1}{N}\right)^{m-\rho(r)} \|f_c\|_{H^m}$$

For $k < 1/2$, it is the classical Clenshaw-Curtis rule for which the case $r = 0$ follows by Theorem 2.1 [21]; the same argument applies to the case $r \in (0,2)$ to yield estimate (2.17) without the $(1/k)^r$ term. So, the result follows. □

We know use the following two results to get estimates for f.

Lemma 2.3 ([1]). *If $(f')_c \in L^1[-\pi,\pi]$, we have*

$$\hat{f}_c(\mu) = \frac{1}{2\mu}\left[(\hat{f}_c')(\mu-1) - (\hat{f}_c')(\mu+1)\right] \quad \textit{for} \quad \mu \in \mathbb{Z} \setminus \{0\}.$$

Proof. Since f_c is even, an integration by parts entails

$$\begin{aligned}
\hat{f}_c(\mu) &= \frac{1}{\pi}\int_0^{\pi} f_c(\theta)\cos(\mu\theta)d\theta \\
&= \frac{1}{\pi}\left[\left[f_c(\pi)\frac{\sin(\mu\pi)}{\mu} - f_c(0)\frac{\sin(\mu 0)}{\mu}\right] - \frac{1}{\mu}\int_0^{\pi}(f_c)'(\theta)\sin(\mu\theta)d\theta\right] \\
&= -\frac{1}{\pi\mu}\int_0^{\pi}(f_c)'(\theta)\sin(\mu\theta)d\theta \\
&= \frac{1}{\pi\mu}\int_0^{\pi}(f)'(\cos(\theta))\sin(\mu\theta)\sin(\theta)d\theta \\
&= \frac{1}{2\pi\mu}\int_0^{\pi}(f)'_c(\theta)\left[\cos((\mu-1)\theta) - \cos((\mu+1)\theta)\right]d\theta \\
&= \frac{1}{2\mu}\left[(\hat{f}'_c)(\mu-1) - (\hat{f}'_c)(\mu+1)\right]
\end{aligned}$$

as desired. □

Lemma 2.4 ([1]). *For any $0 \leq m \leq N+1$, there is a positive constant $\rho_{m,N}$ such that*

$$\|(I - S_N)f_c\|_{H^m} \leq \sigma_{m,N}\|(f^{(m)})_c\|_{H^0}.$$

Proof. Since S_N is the orthogonal projection of H^0 onto $\text{span}\{\exp(ij\theta) : 0 \leq |j| \leq N\}$ [15], the result is immediate for $m = 0$. We can therefore assume $m \geq 1$. Since f_c is even, from (2.9), we have for all nonnegative J,

$$(I - S_J)f_c = 2\sum_{\mu \geq J+1}\hat{f}_c(\mu)\cos(\mu\theta) \quad \text{and} \quad \|(I - S_J)f_c\|^2_{H^m} = 2\sum_{\mu \geq J+1}\mu^{2m}|\hat{f}_c(\mu)|^2.$$

Thus, from Lemma 2.3, we obtain

$$\begin{aligned}
\|(I - S_J)f_c\|^2_{H^m} &\leq \frac{1}{2}\sum_{\mu \geq J+1}\mu^{2m-2}|(\hat{f'})_c(\mu-1) - (\hat{f'})_c(\mu+1)|^2 \\
&\leq \sum_{\mu \geq J+1}\mu^{2m-2}|(\hat{f'})_c(\mu-1)|^2 + \sum_{\mu \geq J+1}\mu^{2m-2}|(\hat{f'})_c(\mu+1)|^2 \\
&= \sum_{\mu \geq J}(\mu+1)^{2m-2}|(\hat{f'})_c(\mu)|^2 + \sum_{\mu \geq J+2}(\mu-1)^{2m-2}|(\hat{f'})_c(\mu)|^2
\end{aligned}$$

so that

$$\|(I - S_J)f_c\|_{H^m}^2 \leq 2\sum_{\mu \geq J}(\mu+1)^{2m-2}|(\hat{f'})_c(\mu)|^2. \tag{2.19}$$

Accordingly, for $J \geq 1$,

$$\begin{aligned}\|(I - S_J)f_c\|_{H^m}^2 &\leq \left(\frac{J+1}{J}\right)^{2m-2} 2\sum_{\mu \geq J}\mu^{2m-2}\left|(\hat{f'})_c(\mu)\right|^2 \\ &= \left(\frac{J+1}{J}\right)^{2m-2} \|(I - S_{J-1})(f')_c\|_{H^{m-1}}^2\end{aligned} \tag{2.20}$$

Using (2.20) $m-1$ times gives

$$\|(I - S_N)f_c\|_{H^m} \leq \sigma_{m,N}\|(I - S_{N-m+1})(f^{(m-1)})_c\|_{H^1} \tag{2.21}$$

with

$$\sigma_{m,N} = \left(\frac{N+1}{N}\right)^{m-1}\left(\frac{N}{N-1}\right)^{m-2} \cdots \left(\frac{N-m+3}{N-m+2}\right)$$

Now, if $m < N+1$, we can use (2.20) one more time in (2.21) together with the fact that S_{N-m} is the orthogonal projection on H^0 to get the desired result. If $m = N+1$, (2.21) becomes

$$\|(I - S_N)f_c\|_{H^m} \leq \sigma_{m,N}\frac{N-m+2}{N-m+1}\|(I - S_0)f_c^{(m-2)}\|_{H^0}$$

and we can deduce the required result from the fact that

$$\|(I - S_0)(f^{(m-1)})_c\|_{H^1} \leq \|(f^{(m)})_c\|_{H^0}$$

which is obtained from (2.19) with $m = 1$. □

Now, we will use Lemmas 2.3 and 2.4 and improve Theorem 2.2. To this end, we introduce the weighted seminorm

$$|f|_{H_\omega^m[a,b]} = \left[\int_a^b \frac{|f^{(m)}(x)|^2}{\sqrt{(b-x)(x-a)}}\,dx\right]^{1/2}$$

and we will write $|.|_{H_\omega^m}$ when $[a,b]=[-1,1]$. Observe that

$$|f|_{H_\omega^m} = \left[\int_{-1}^1 \frac{|f^{(m)}(x)|^2}{\sqrt{1-x^2}}\,dx\right]^{1/2}$$

and with the change of variables $x=\cos(\theta)$, we have

$$\begin{aligned}
|f|_{H_\omega^m} &= \left[\int_\pi^0 \frac{|(f^{(m)})_c(\theta)|^2}{\sqrt{1-\cos^2(\theta)}}(-\sin(\theta))d\theta\right]^{1/2} \\
&= \left[\int_0^\pi \frac{|(f^{(m)})_c(\theta)|^2}{\sin(\theta)}\sin(\theta)d\theta\right]^{1/2} \\
&= \left[\int_0^\pi |(f^{(m)})_c(\theta)|^2 d\theta\right]^{1/2} = \sqrt{\pi}\|(f^{(m)})_c\|_{H^0}. \qquad (2.22)
\end{aligned}$$

Then we have the following result:

Theorem 2.5 ([1]). *For $r\in[0,2]$ and $0\le m\le N+1$, if $|f|_{H_\omega^m}<\infty$, then there is a constant $\sigma'_{m,N}$ such that*

$$|I_k(f)-I_{k,N}(f)| \le \sigma'_{m,N}\left(\frac{1}{k}\right)^r\left(\frac{1}{N}\right)^{m-\rho(r)}|f|_{H_\omega^m}.$$

Moreover $\sigma'_{m,N}=C\sigma_{m,N}$ for C independent of m and N.

Proof. Since $|f|_{H_\omega^m}<\infty$, we have $f\in L^2[-1,1]$ and we can define an algebraic polynomial P of degree N by

$$P(x) = \hat{f}_c(0) + 2\sum_{n=1}^N \hat{f}_c(n)T_n(x).$$

In fact, from (2.9), we have

$$\begin{aligned} P_c(\theta) &= \hat{f}_c(0) + 2\sum_{n=1}^{N} \hat{f}_c(n) T_n(\cos(\theta)) \\ &= \hat{f}_c(0) + 2\sum_{n=1}^{N} \hat{f}_c(n) \cos(n \arccos(\cos(\theta))) \\ &= \hat{f}_c(0) + 2\sum_{n=1}^{N} \hat{f}_c(n) \cos(n\theta)) = (S_N f)_c(\theta). \end{aligned}$$

Since $I_{k,N}$ is exact for any polynomial of degree N, we have, via Theorem 2.2,

$$\begin{aligned} |I_k(f) - I_{k,N}(f)| &= |I_k(f-P) - I_{k,N}(f-P)| \\ &\leq C\left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} \|(f-P)_c\|_{H^m} \\ &= C\left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} \|(I-S_N)f_c\|_{H^m}. \end{aligned}$$

Therefore, from Lemma 2.4, we get

$$\begin{aligned} |I_k(f) - I_{k,N}(f)| &\leq C\left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} \|(I-S_N)f_c\|_{H^m} \\ &\leq C\rho_{m,N}\left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} \|(f^{(m)})_c\|_{H^0} \\ &= C\rho_{m,N}\left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} \left[\frac{1}{\pi}\int_0^\pi |(f^{(m)})_c(\theta)|^2 d\theta\right]^{1/2} \end{aligned}$$

Thus, if we use (2.22), we obtain

$$|I_k(f) - I_{k,N}(f)| \leq C\rho_{m,N}\left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} \frac{1}{\sqrt{\pi}}|f|_{H^m_\omega}$$

completing the proof. □

2.2. Smooth functions on the interval $[a, b]$

In this section, we will get the estimate for

$$I_k^{[a,b]}(f) = \int_a^b f(t)\exp(ikt)dt.$$

for a smooth function f. To this end, we use the change of variables

$$t = c + h\tau, \quad \tau \in [-1, 1] \quad \text{for} \quad c = \frac{b+a}{2}, \quad h = \frac{b-a}{2}.$$

to compute

$$\begin{aligned} I_k^{[a,b]}(f) &= \int_a^b f(t)\exp(ikt)dt \qquad (2.23)\\ &= \int_{-1}^1 f(c+h\tau)\exp(ik(c+h\tau))hd\tau \\ &= h\exp(ikc)\int_{-1}^1 \tilde{f}(\tau)\exp(i\tilde{k}\tau)d\tau \\ &= h\exp(i\tilde{k}c)I_{\tilde{k}}(\tilde{f}) \qquad (2.24) \end{aligned}$$

where

$$\tilde{f}(t) = f(c+ht), \quad \tilde{k} = kh,$$

and we use the rule in §2.1 for approximating $I_{\tilde{k}}(\tilde{f})$. i.e. We have the approximation

$$I_{k,N}^{[a,b]}(f) = h\exp(ikc)I_{\tilde{k},N}(\tilde{f}). \qquad (2.25)$$

In the next theorem, we have the extension of Theorem 2.5 to the case of integration over $[a, b]$.

Theorem 2.6 ([1]). *For $r \in [0,2]$, $0 \le m \le N+1$ we have*

$$\left| I_k^{[a,b]}(f) - I_{k,N}^{[a,b]}(f) \right| \le \sigma'_{m,N} \left(\frac{1}{k}\right)^r h^{m+1-r} \left(\frac{1}{N}\right)^{m-\rho(r)} |f|_{H_\omega^m[a,b]}$$

provided $|f|_{H_\omega^m[a,b]} < \infty$.

Proof. From (2.23) and (2.24) and Theorem 2.5, we get

$$\begin{aligned}
\left| I_k^{[a,b]}(f) - I_{k,N}^{[a,b]}(f) \right| &= h|I_{\tilde{k}}(\tilde{f}) - I_{\tilde{k},N}(\tilde{f})| \\
&\le \sigma'_{m,N} h \left(\frac{1}{\tilde{k}}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} |\tilde{f}|_{H_\omega^m} \\
&= \sigma'_{m,N} h \left(\frac{1}{kh}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} |\tilde{f}|_{H_\omega^m} \\
&= \sigma'_{m,N} h^{1-r} \left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} |\tilde{f}|_{H_\omega^m}
\end{aligned}$$

and, clearly,

$$\tilde{f}^{(m)}(t) = h^m f^{(m)}(c+ht)$$

so that

$$|\tilde{f}|_{H_\omega^m}^2 = h^{2m} \int_{-1}^{1} \frac{|f^{(m)}(c+ht)|^2}{\sqrt{1-t^2}} dt = h^{2m} |f|_{H_\omega^m[a,b]}^2.$$

Therefore

$$\begin{aligned}
\left| I_k^{[a,b]}(f) - I_{k,N}^{[a,b]}(f) \right| &\le \sigma'_{m,N} h^{1-r} \left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} |\tilde{f}|_{H_\omega^m} \\
&= \sigma'_{m,N} h^{m+1-r} \left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} |f|_{H_\omega^m[a,b]}.
\end{aligned}$$

□

One of the most important consequences of this theorem is that, for fixed N, the convergence is obtained as $h \to 0$ (which shows up at the case of composite algorithm). Thus, for $m = N + 1$, we have the following corollary of Theorem 2.6:

Corollary 2.7 ([1]). *For $r \in [0, 2]$, for each $N \geq 1$, there is a positive constant $c_N = C\sigma'_{N+1,N}(1/N)^{N+1-\rho(r)}$, such that*

$$\left| I_k^{[a,b]}(f) - I_{k,N}^{[a,b]}(f) \right| \leq c_N \left(\frac{1}{k} \right)^r h^{N+2-r} \max_{x \in [a,b]} |f^{(N+1)}(x)|$$

for $f \in C^{N+1}[a, b]$.

3. COMPOSITE ALGORITHM FOR LINEAR OSCILLATORS

Until now, we considered the case where f does not have any singularity. Now we will examine the case where f has some algebraic or logarithmic singularities in $[a, b]$. We will assume firstly that we are in the case of integral over $[0, 1]$ with a unique singularity at $x = 0$. After that, we will study the case of finitely many singularities in $[a, b]$.

To get the composite algorithm, we introduce three norms and three subspaces of $L^1[0, 1]$:

- For $\beta \in (0, 1)$,

$$\|\nu\|_{m,\beta} = \max\{\sup_{x\in[0,1]} |\nu(x)|, \sup_{x\in(0,1]} |x^{j-\beta}\nu^{(j)}(x)| : j = 1, 2, ..., m\}$$

  and the space $C^m_\beta[0, 1]$ of functions ν such that $\|\nu\|_{m,\beta} < \infty$.

- For $\beta \in (-1, 0)$,

$$\|\nu\|_{m,\beta} = \max\{\sup_{x\in(0,1]} |x^{j-\beta}\nu^{(j)}(x)| : j = 0, 1, 2, ..., m\}$$

  and the space $C^m_\beta[0, 1]$ of functions ν such that $\|\nu\|_{m,\beta} < \infty$.

- For $\beta = 0$,

$$\|\nu\|_{m,0} = \max\{\sup_{x\in(0,1]} |(|\log x| + 1)^{-1}\nu(x)|, \sup_{x\in[0,1]} |x^j\nu^{(j)}(x)| : j = 0, 1, 2, ..., m\}$$

  and the associated space is $C^m_0[0, 1]$.

3.1. Singular functions on $[0,1]$

Throughout this section, we will assume that f has a unique algebraic or logarithmic singularity at $x = 0$.

When $f \in C^m_\beta[0,1]$ for $\beta \in (-1,1)$, we use a mesh graded toward the singularity to divide the interval into subintervals [22]. The right choice of mesh gives a uniform error estimate on subintervals. The mesh

$$\{x_j = \left(\frac{j}{M}\right)^q : j = 0, 1, ..., M\} \tag{3.1}$$

where $q \geq 1$ is well-known to give optimal approximation of functions with singularity by fixed order piecewise polynomials [1]. Now we write

$$I_k^{[0,1]}(f) = \left[\tilde{I}_k^{[x_0,x_1]}(f) + \sum_{j=2}^{M} I_k^{[x_{j-1},x_j]}(f)\right]. \tag{3.2}$$

and note that, for $2 \leq j \leq M$, $I_k^{[x_{j-1},x_j]}(f)$ is an oscillatory integral without any singularities. Thus, the method described in §2.2 can be applied to each one of them. This reduces the problem to the approximation of $I_k^{[x_0,x_1]}(f)$, which will depend on whether $\beta \leq 0$ or $\beta > 0$. To this end, we define

$$\tilde{I}_k^{[x_0,x_1]}(f) = \begin{cases} I_{k,1}^{[x_0,x_1]}(f), & \text{if} \quad \beta \in (0,1), \\ 0, & \text{if} \quad \beta \in (-1,0] \end{cases}$$

where

$$I_{k,1}^{[x_0,x_1]}(f) = \begin{cases} \int_{x_0}^{x_1} \left(Q_1^{[x_0,x_1]} f\right)(x) \exp(ikx)\, dx, & \text{if} \quad \frac{x_1 - x_0}{2} k \geq \frac{1}{2}, \\ \int_{x_0}^{x_1} \left(Q_1^{[x_0,x_1]} f_k\right)(x)\, dx, & \text{if} \quad \frac{x_1 - x_0}{2} k < \frac{1}{2} \end{cases}$$

and, for a function g,

$$\left(Q_1^{[x_0,x_1]}g\right)(x) = \frac{g(x_1) - g(x_0)}{x_1 - x_0}(x - x_0) + g(x_0).$$

is the linear algebraic interpolant of g at the points x_0 and x_1. Then we have the approximation

$$I_{k,N,M,q}(f) = \tilde{I}_k^{[x_0,x_1]}(f) + \sum_{j=2}^{M} I_{k,N}^{[x_{j-1},x_j]}(f)$$

for $I_k^{[0,1]}(f)$ with singularity at $x = 0$.

For the error analysis, we have

$$\begin{aligned}
\left|I_k^{[0,1]}(f) - I_{k,N,M,q}^{[0,1]}(f)\right| &= \left|I_k^{[x_0,x_1]}(f) + \sum_{j=2}^{M} I_k^{[x_{j-1},x_j]}(f) - \tilde{I}_k^{[x_0,x_1]}(f) - \sum_{j=2}^{M} I_{k,N}^{[x_{j-1},x_j]}(f)\right| \\
&\leq \left|I_k^{[x_0,x_1]}(f) - \tilde{I}_k^{[x_0,x_1]}(f)\right| + \sum_{j=2}^{M} \left|I_k^{[x_{j-1},x_j]}(f) - I_{k,N}^{[x_{j-1},x_j]}(f)\right| \\
&\leq |\tilde{e}_1| + \sum_{j=2}^{M} |e_j|.
\end{aligned}$$

Thus, we need estimations for $\tilde{e}_1$ and e_j's. For this aim, we have two results.

Lemma 3.1 ([1]). *For $\beta \in (-1, 0)$ and any $f \in C_\beta^1[0, 1]$ there exists a positive constant C_β such that for $\epsilon \in (0, 1]$, we have*

$$\left|I_k^{[0,\epsilon]}(f)\right| \leq C_\beta \epsilon^{1+\beta-s} \left(\frac{1}{k}\right)^s \|f\|_{1,\beta} \tag{3.3}$$

where $s \in [0, 1+\beta]$. Furthermore, for $f \in C_0^1[0, 1]$ there is positive $C_0 > 0$ such that for $\epsilon \in (0, 1]$, we have

$$\left|I_k^{[0,\epsilon]}(f)\right| \leq C_0(\epsilon + \epsilon|\log(\epsilon)|)^{1-s} \left(\frac{1 + \log(k)}{k}\right)^s \|f\|_{1,0}$$

where $s \in [0,1]$.

Proof. Note that

$$\|f\|_{1,\beta} = \max\{\sup_{x\in(0,1]} |x^{-\beta} f(x)|, \sup_{x\in(0,1]} |x^{(1-\beta)} f'(x)|\}$$

so that

$$|f(x)| = x^{\beta} \left|x^{-\beta} f(x)\right| \leq x^{\beta} \|f\|_{1,\beta}, \qquad x \geq 0$$

Thus, for $\beta \in (-1,0)$, we have

$$\begin{aligned} |I_k^{[0,\epsilon]}(f)| &= \left|\int_0^{\epsilon} f(x)\exp(ikx)\,dx\right| \leq \int_0^{\epsilon} |f(x)|\,|\exp(ikx)|\;dx \\ &= \int_0^{\epsilon} |f(x)|\;dx = \int_0^{\epsilon} x^{\beta}\left|x^{-\beta} f(x)\right| dx \\ &\leq \left[\int_0^{\epsilon} x^{\beta}\,dx\right] \|f\|_{1,\beta} = \frac{1}{1+\beta}\epsilon^{1+\beta}\|f\|_{1,\beta} \\ &= \frac{1}{1-|\beta|}\epsilon^{1+\beta}\|f\|_{1,\beta}. \end{aligned}$$

This implies for $\epsilon k \leq 1$,

$$\begin{aligned} |I_k^{[0,\epsilon]}(f)| &\leq \frac{1}{1-|\beta|}\epsilon^{1+\beta-s}\epsilon^{s}\|f\|_{1,\beta} \\ &\leq \frac{1}{1-|\beta|}\epsilon^{1+\beta-s}\left(\frac{1}{k}\right)^{s}\|f\|_{1,\beta} \end{aligned}$$

which is the desired result. Therefore, we can assume that $\epsilon > \frac{1}{k}$ and write

$$I_k^{[0,\epsilon]}(f) = I_k^{[0,1/k]}(f) + I_k^{[1/k,\epsilon]}(f).$$

For the first term, we have

$$\begin{aligned}
\left|I_k^{[0,1/k]}(f)\right| &= \left|\int_0^{1/k} f(x)\exp(ikx)\,dx\right| \\
&\leq \int_0^{1/k} |f(x)|\,dx \\
&= \int_0^{1/k} x^{\beta}\left|x^{-\beta}f(x)\right|\,dx \\
&\leq \int_0^{1/k} x^{\beta}\|f\|_{1,\beta}\,dx \\
&= \frac{1}{1+\beta}\left(\frac{1}{k}\right)^{\beta+1}\|f\|_{1,\beta} \\
&\leq \frac{1}{1-|\beta|}\left(\frac{1}{k}\right)^{\beta+1}\|f\|_{1,\beta}.
\end{aligned}$$

For the second term, by integration by parts, we have

$$\begin{aligned}
\left|I_k^{[1/k,\epsilon]}(f)\right| &= \left|\int_{1/k}^{\epsilon} f(x)\exp(ikx)\,dx\right| \\
&= \left|\frac{1}{ik}f(\epsilon)\exp(ik\epsilon) - \frac{1}{ik}f(\frac{1}{k})\exp(\frac{ik}{k}) - \frac{1}{ik}\int_{1/k}^{\epsilon} f'(x)\exp(ikx)\,dx\right| \\
&\leq \frac{1}{k}\left|f(\epsilon)\exp(ik\epsilon) - f(\frac{1}{k})\exp(ik\frac{1}{k})\right| + \frac{1}{k}\int_{1/k}^{\epsilon}|f'(x)|\,dx \\
&\leq \frac{1}{k}\left|f(\epsilon)\exp(ik\epsilon) - f(\frac{1}{k})\exp(ik\frac{1}{k})\right| + \int_{1/k}^{\epsilon} x^{\beta-1}\left|x^{1-\beta}f'(x)\right|\,dx \\
&\leq \frac{1}{k}\left|f(\epsilon)\exp(ik\epsilon) - f(\frac{1}{k})\exp(ik\frac{1}{k})\right| + \int_{1/k}^{\epsilon} x^{\beta-1}\,dx\|f\|_{1,\beta} \\
&\leq \frac{1}{k}\left|f(\epsilon)\exp(ik\epsilon) - f(\frac{1}{k})\exp(ik\frac{1}{k})\right| + \frac{1}{|\beta|}\left[\epsilon^{\beta} - \left(\frac{1}{k}\right)^{\beta}\right]\|f\|_{1,\beta} \\
&\leq \frac{1}{k}\left[\epsilon^{\beta}\|f\|_{1,\beta} + \left(\frac{1}{k}\right)^{\beta}\|f\|_{1,\beta}\right] + \frac{1}{|\beta|}\left[\epsilon^{\beta} + \left(\frac{1}{k}\right)^{\beta}\right]\|f\|_{1,\beta} \\
&= \left[\frac{1}{k} + \frac{1}{|\beta|}\right]\left[\epsilon^{\beta} + \left(\frac{1}{k}\right)^{\beta}\right]\|f\|_{1,\beta}.
\end{aligned}$$

Since $\beta < 0$ and $\epsilon > 1/k$, we therefore have

$$\leq \frac{1}{k}\left[1+\frac{1}{|\beta|}\right]\left[\epsilon^{\beta}+\left(\frac{1}{k}\right)^{\beta}\right]\|f\|_{1,\beta} \leq \frac{1}{k}\left[1+\frac{1}{|\beta|}\right]2\left(\frac{1}{k}\right)^{\beta}\|f\|_{1,\beta}$$
$$= 2\left[1+\frac{1}{|\beta|}\right]\left(\frac{1}{k}\right)^{\beta+1}\|f\|_{1,\beta}.$$

Combining these two estimates, we get

$$\begin{aligned} I_k^{[0,\epsilon]}(f) &= I_k^{[0,1/k]}(f) + I_k^{[1/k,\epsilon]}(f) \\ &\leq \frac{1}{1-|\beta|}\left(\frac{1}{k}\right)^{\beta+1}\|f\|_{1,\beta} + 2\left[1+\frac{1}{|\beta|}\right]\left(\frac{1}{k}\right)^{\beta+1}\|f\|_{1,\beta} \\ &= \left[\frac{|\beta| + 2(1-|\beta|^2)}{(1-|\beta|)|\beta|}\right]\left(\frac{1}{k}\right)^{1+\beta}\|f\|_{1,\beta} \\ &= \left[\frac{|\beta| + 2(1-|\beta|^2)}{(1-|\beta|)|\beta|}\right]\left(\frac{1}{k}\right)^{1+\beta-s}\left(\frac{1}{k}\right)^{s}\|f\|_{1,\beta} \\ &\leq \left[\frac{|\beta| + 2(1-|\beta|^2)}{(1-|\beta|)|\beta|}\right]\epsilon^{1+\beta-s}\left(\frac{1}{k}\right)^{s}\|f\|_{1,\beta}. \end{aligned}$$

This completes the proof of the first inequality.

For the second one, let us note that

$$\|f\|_{1,0} = \max\{\sup_{x\in(0,1]} |(|\log(x)|+1)^{-1}f(x)|, \sup_{x\in[0,1]} |xf'(x)|\}$$

and

$$|f(x)| \leq (|\log(x)|+1)\,\|f\|_{1,0}$$

which gives

$$|f(x)| \leq (1-\log(x))\,\|f\|_{1,0} \quad \text{for} \quad x \in (0,1].$$

Thus, we have

$$\begin{aligned}
\left|I_k^{[0,\epsilon]}(f)\right| &= \left|\int_0^\epsilon f(x)\exp(ikx)dx\right| \\
&\leq \int_0^\epsilon |f(x)|\, dx \\
&\leq \|f\|_{1,0} \int_0^\epsilon (|\log(x)| + 1)\, dx \\
&\leq \|f\|_{1,0} \int_0^\epsilon (1 - \log(x))\, dx \\
&= [2\epsilon + \epsilon\, |\log(\epsilon)|]\, \|f\|_{1,0} \\
&= [2\epsilon + \epsilon\, |\log(\epsilon)|]^{1-s}\, [2\epsilon + \epsilon\, |\log(\epsilon)|]^{s}\, \|f\|_{1,0}
\end{aligned}$$

Therefore if $\epsilon k \leq 1$, we have

$$\left|I_k^{[0,\epsilon]}(f)\right| \leq [2\epsilon + \epsilon\, |\log(\epsilon)|]^{1-s} \left[\frac{2 + \log(k)}{k}\right]^s \|f\|_{1,0}$$

which is the result we desire. Thus, we can assume, as above, $\epsilon k > 1$. Then we have

$$\begin{aligned}
\left|I_k^{[0,1/k]}(f)\right| &\leq \int_0^{1/k} |f(x)|\, dx \\
&\leq \|f\|_{1,0} \int_0^{1/k} (1 - \log(x))dx \\
&= \left[\frac{2 + \left|\log\left(\frac{1}{k}\right)\right|}{k}\right] \|f\|_{1,0}
\end{aligned}$$

and, by integration by parts, we have

$$\begin{aligned}
\left|I_k^{[1/k,\epsilon]}(f)\right| &\leq \left|\frac{1}{ik}\left[f(\epsilon)\exp(ik\epsilon) - f\left(\frac{1}{k}\right)\exp\left(ik\frac{1}{k}\right)\right]\right| + \frac{1}{k}\int_{1/k}^{\epsilon} |f'(x)|\, dx \\
&= \left|\frac{1}{ik}\left[f(\epsilon)\exp(ik\epsilon) - f\left(\frac{1}{k}\right)\exp\left(ik\frac{1}{k}\right)\right]\right| + \frac{1}{k}\left[\log(\epsilon) + \log(k)\right] \|f\|_{1,0} \\
&\leq \frac{1}{k}\left[1 + |\log(\epsilon)| + 1 + |\log(1/k)|\right] \|f\|_{1,0} + \frac{1}{k}\left[\log(\epsilon) + \log(k)\right] \|f\|_{1,0} \\
&= \left[\frac{2 + 2\left|\log\left(\frac{1}{k}\right)\right|}{k}\right] \|f\|_{1,0}.
\end{aligned}$$

For $g(x) = 4x - x\log(x)$, we have

$$g'(x) = 1 - \log(x).$$

Thus, we obtain $g'(x) > 0$ for $x \in (0, 1]$. Since $\epsilon > 1/k$, we have

$$4\epsilon + 3\epsilon\left|\log(\epsilon)\right| = 4\epsilon - 3\epsilon\log(\epsilon) > \frac{4 + 3\log(k)}{k} = \frac{4 - 3\log(1/k)}{k}$$

which enables us to write

$$\begin{aligned}
\left|I_k^{[0,\epsilon]}(f)\right| &= \left|I_k^{[0,1/k]}(f) + I_k^{[1/k,\epsilon]}(f)\right| \\
&\leq \left(\left[\frac{2 + \left|\log\left(\frac{1}{k}\right)\right|}{k}\right] + \left[\frac{2 + 2\left|\log\left(\frac{1}{k}\right)\right|}{k}\right]\right)\|f\|_{1,0} \\
&= \left[\frac{4 + 3\left|\log\left(\frac{1}{k}\right)\right|}{k}\right]^{1-s}\left[\frac{4 + 3\left|\log\left(\frac{1}{k}\right)\right|}{k}\right]^{s}\|f\|_{1,0} \\
&\leq \left[4\epsilon + 3\epsilon\left|\log(\epsilon)\right|\right]^{1-s}\left[\frac{4 + 3\left|\log\left(\frac{1}{k}\right)\right|}{k}\right]^{s}\|f\|_{1,0} \\
&\leq 4\left[\epsilon + \epsilon\left|\log(\epsilon)\right|\right]^{1-s}\left[\frac{1 + \log(k)}{k}\right]^{s}\|f\|_{1,0}
\end{aligned}$$

which proves the result. □

Lemma 3.2 ([1]). *For $\beta \in (0, 1)$ and $f \in C_\beta^2[0, 1]$, there is a positive constant C_β such that for $\epsilon \in (0, 1]$ and $s \in [0, 1]$, we have*

$$\left|I_k^{[0,\epsilon]}(f)\right| \leq C_\beta\epsilon^{1-s}\left(\frac{1}{k}\right)^{s}\|f\|_{1,\beta} \tag{3.4}$$

Moreover, if $s \in [1, 1 + \beta]$,

$$\begin{aligned}
\left|I_k^{[0,\epsilon]}(f)\right| &\leq \frac{1}{k}\left[|f(0)| + |f(\epsilon)|\right] + C_\beta\epsilon^{1+\beta-s}\left(\frac{1}{k}\right)^{s}\|f'\|_{1,\beta-1} \\
&\leq \frac{2}{k}\|f\|_{0,\beta} + C_\beta\epsilon^{1+\beta-s}\left(\frac{1}{k}\right)^{s}\|f\|_{2,\beta-1}.
\end{aligned} \tag{3.5}$$

Proof. Concerning (3.4), we have

$$\left|I_k^{[0,\epsilon]}(f)\right| = \left|\int_0^\epsilon f(x)\exp(ikx)\,dx\right| \le \int_0^\epsilon |f(x)|\,dx \tag{3.6}$$
$$\le \|f\|_{0,\beta}\int_0^\epsilon dx$$
$$\le \epsilon\|f\|_{0,\beta}. \tag{3.7}$$

By integration by parts, we also have

$$\begin{aligned}
\left|I_k^{[0,\epsilon]}(f)\right| &= \left|\int_0^\epsilon f(x)\exp(ikx)\,dx\right| \\
&= \left|\frac{1}{ik}f(\epsilon)\exp(ik\epsilon) - \frac{1}{ik}f(0)\exp(ik0) - \frac{1}{ik}\int_0^\epsilon f'(x)\exp(ikx)\,dx\right| \\
&\le \frac{1}{k}\left[|f(\epsilon)| + |f(0)|\right] + \frac{1}{k}\left|\int_0^\epsilon f'(x)\exp(ikx)\,dx\right| \qquad (3.8) \\
&\le \frac{2}{k}\|f\|_{0,\beta} + \frac{1}{k}\int_0^\epsilon |f'(x)|\,dx \\
&\le \frac{2}{k}\|f\|_{0,\beta} + \frac{1}{k}\int_0^\epsilon x^{\beta-1}\left|x^{1-\beta}f'(x)\right|\,dx \\
&\le \frac{2}{k}\|f\|_{0,\beta} + \frac{1}{k}\frac{1}{\beta}\epsilon^\beta\|f\|_{1,\beta} \\
&\le \frac{2}{k}\|f\|_{1,\beta} + \frac{1}{k}\frac{1}{\beta}\epsilon^\beta\|f\|_{1,\beta} \\
&\le \left(2 + \frac{1}{\beta}\right)\frac{1}{k}\|f\|_{1,\beta}.
\end{aligned}$$

Then interpolation of this last result with (3.7) we have

$$\begin{aligned}
\left|I_k^{[0,\epsilon]}(f)\right| &= \left|I_k^{[0,\epsilon]}(f)\right|^{1-s}\left|I_k^{[0,\epsilon]}(f)\right|^s \\
&\le (\epsilon\|f\|_{1,\beta})^{1-s}\left[\left(2 + \frac{1}{\beta}\right)\frac{1}{k}\|f\|_{1,\beta}\right]^s
\end{aligned}$$

and taking $C_\beta = 2 + (1/\beta)$, we obtain

$$\left|I_k^{[0,\epsilon]}(f)\right| \le C_\beta^s\epsilon^{1-s}\left(\frac{1}{k}\right)^s\|f\|_{1,\beta} \le C_\beta\epsilon^{1-s}\left(\frac{1}{k}\right)^s\|f\|_{1,\beta}$$

which gives the first desired inequality. As for (3.5), firstly note that if $f \in C^2_\beta[0,1]$, then $f' \in C^1_{\beta-1}[0,1]$. Therefore, by Lemma 3.1, we have

$$\begin{aligned} \left| \int_0^\epsilon f'(x) \exp(ikx)\, dx \right| &\leq C_{\beta-1} \epsilon^{\beta - s'} \left(\frac{1}{k} \right)^{s'} \|f'\|_{1,\beta-1} \\ &\leq C_{\beta-1} \epsilon^{\beta - s'} \left(\frac{1}{k} \right)^{s'} \|f\|_{2,\beta} \end{aligned}$$

for any $s' \in [0, \epsilon]$. Then substituting this last result to (3.8) and taking $s = 1 + s'$, we get the result. □

To show the sharpness of the estimates in Lemmas 3.1 and 3.2 for $k \to \infty$ let us consider the family functions of the form

$$f_\beta(x) = \begin{cases} x^\beta, & \beta \in (-1,1) \setminus \{0\}, \\ \log(x), & \beta = 0. \end{cases}$$

For these types of functions, we note the following lower bounds confirming the sharpness of the estimates in Lemmas 3.1 and 3.2.

Lemma 3.3 ([1]). *For $\beta \in (-1,1)$, there is a positive constant $A_\beta > 0$ such that for all k sufficiently large, for $\beta \neq 0$*

$$k^{\min\{1+\beta,1\}} \left| I_k^{[0,1]}(f_\beta) \right| \geq A_\beta,$$

and for $\beta = 0$

$$\frac{k}{\log(k)} \left| I_k^{[0,1]}(f_0) \right| \geq A_0.$$

Proof. For the first inequality, we recall

$$\int_0^1 x^{a-1} \sin(bx) dx = \frac{-i}{2a} \left[{}_1F_1(a; a+1; ib) - {}_1F_1(a; a+1; -ib) \right] \tag{3.9}$$

for $b > 0$, $Re(a) > -1$ and $a \neq 0$ [23, (3.761.1)], and

$$\int_0^1 x^{a-1}\cos(bx)dx = \frac{1}{2a}\left[{}_1F_1(a; a+1; ib) + {}_1F_1(a; a+1; -ib)\right] \tag{3.10}$$

for $b > 0$ and $Re(a) > 0$ where ${}_1F_1(a; b; z)$ is the confluent hypergeometric function [23, (3.761.6)]. Therefore

$$\begin{aligned}
I_k^{[0,1]}(f_\beta) &= \int_0^1 f_\beta(t)\exp(ikt)dt = \int_0^1 f_\beta(t)\cos(kt)dt + i\int_0^1 f_\beta(t)\sin(kt)dt \\
&= \int_0^1 t^\beta\cos(kt)dt + i\int_0^1 t^\beta \sin(kt)dt \\
&= \frac{1}{2(\beta+1)}\left[{}_1F_1(\beta+1; \beta+2; ik) + {}_1F_1(\beta+1; \beta+2; -ik)\right] \\
&\quad + i\frac{-i}{2(\beta+1)}\left[{}_1F_1(\beta+1; \beta+2; ik) - {}_1F_1(\beta+1; \beta+2; -ik)\right] \\
&= \frac{1}{\beta+1}{}_1F_1(\beta+1; \beta+2; ik).
\end{aligned}$$

${}_1F_1(a; b; z)$ has the following asymptotic behavior

$${}_1F_1(a; b; z) = \Gamma(b)\left(\frac{\exp(i\pi a)z^{-a}}{\Gamma(b-a)} + \frac{\exp(z)z^{a-b}}{\Gamma(a)}\right)(1 + O(|z|^{-1}))$$

for $-\pi/2 < \arg(z) < 3\pi/2$ as $|z| \to \infty$ [24, (13.5.1)], and therefore

$$\begin{aligned}
I_k^{[0,1]}(f_\beta) &= \frac{1}{\beta+1}{}_1F_1(\beta+1; \beta+2; ik) \\
&= \frac{1}{\beta+1}\Gamma(\beta+2)\left(\frac{\exp(i\pi(\beta+1))(ik)^{-(\beta+1)}}{\Gamma(1)} + \frac{\exp(ik)(ik)^{-1}}{\Gamma(\beta+1)}\right)\left(1 + O(|ik|^{-1})\right) \\
&= \Gamma(\beta+1)\left(\frac{\exp(i\pi(\beta+1))(ik)^{-(\beta+1)}}{\Gamma(1)} + \frac{\exp(ik)(ik)^{-1}}{\Gamma(\beta+1)}\right)\left(1 + O(|ik|^{-1})\right) \\
&= \frac{1}{ik}\left(\frac{\Gamma(\beta+1)\exp(i\pi(\beta+1))}{(ik)^\beta} + \exp(ik)\right)(1 + O(|z|^{-1}))
\end{aligned}$$

where we used that

$$\Gamma(1+z) = z\Gamma(z)$$

Thus, for sufficiently large k and $\beta \in (-1, 0)$,

$$\begin{aligned}\left|I_k^{[0,1]}(f_\beta)\right| &\geq |\Gamma(\beta+1)|\,|\exp(i\pi(\beta+1))|\,\left|(ik)^{-\beta-1}\right| - |\exp(ik)|\,\left|(ik)^{-1}\right| \\ &= \Gamma(\beta+1)k^{-\beta-1} - k^{-1} \geq \frac{1}{4}\Gamma(\beta+1)k^{-\beta-1}\end{aligned}$$

and for $\beta \in (0, 1)$, with similar arguments,

$$\left|I_k^{[0,1]}(f_\beta)\right| \geq k^{-1} - \Gamma(\beta+1)k^{-\beta-1} \geq \frac{1}{4}k^{-1}$$

which proves the first inequality.

To prove the second inequality, let us recall that [23, (4.381.1)]

$$\int_0^1 \log(x)\sin(ax)dx = -\frac{1}{a}\left[\gamma + \log(a) - ci(a)\right], \quad a > 0 \tag{3.11}$$

and [23, (4.381.2)]

$$\int_0^1 \log(x)\cos(ax)dx = -\frac{1}{a}\left(si(a) + \frac{\pi}{2}\right) \tag{3.12}$$

where

$$si(x) = -\frac{\pi}{2} + \int_0^x \frac{\sin(t)}{t}dt, \tag{3.13}$$

$$ci(x) = \gamma + \log(x) + \int_0^x \frac{\cos(t)-1}{t}dt \tag{3.14}$$

and $\gamma \approx 0,5772$ is Euler-Mascheroni constant. This yield

$$\begin{aligned}
I_k^{[0,1]}(f_0) &= \int_0^1 f_0(t)\exp(ikt)dt \\
&= \int_0^1 \log(t)\exp(ikt)dt \\
&= \int_0^1 \log(t)\cos(kt)dt + i\int_0^1 \log(t)\sin(kt)dt \\
&= \left(-\frac{1}{k}\right)\left(si(k)+\frac{\pi}{2}\right) + i\left(-\frac{1}{k}\right)(\gamma + \log(k) + ci(k)) \\
&= \left(-\frac{1}{k}\right)\left[\frac{\pi}{2} + i\gamma + i\log(k)\right] + \frac{i}{k}\left[ci(k) + isi(k)\right]
\end{aligned}$$

Also [24, (5.2.3)] and in [24, (5.2.9)] give

$$\begin{aligned}
si(z) &= -h(z)\cos(z) - g(z)\sin(z), \\
ci(z) &= h(z)\sin(z) - g(z)\cos(z)
\end{aligned}$$

where

$$h(z) \sim \frac{1}{z}\left(1 - \frac{2!}{z^2} + \frac{4!}{z^4} - \frac{6!}{z^6}...\right), \quad g(z) \sim \frac{1}{z^2}\left(1 - \frac{3!}{z^2} + \frac{5!}{z^4} - \frac{7!}{z^6}...\right)$$

as $|z| \to \infty$ for $|\arg(z)| < \pi$ so that

$$si(k) = O\left(\frac{1}{k}\right), ci(k) = O\left(\frac{1}{k}\right)$$

as $k \to \infty$. Therefore we have the asymptotic behavior

$$I_k^{[0,1]}(f_0) = -\frac{1}{k}\left(\frac{\pi}{2} + i\gamma + i\log(k)\right) + O\left(\frac{1}{k^2}\right).$$

as $k \to \infty$. Hence, for sufficiently large k,

$$\left|I_k^{[0,1]}(f_0)\right| \geq \frac{\log(k)}{k}$$

which gives the second inequality. □

The estimation of e_j's is now given in the following lemma.

Lemma 3.4 ([1]). *For $f \in C_\beta^{N+1}[0,1]$, $\beta \in (-1,1)$, $r \geq 0$ and choosing*

$$q > (N+1-r)/(\beta+1-r), \quad \textit{for} \quad r < 1+\beta$$

there is a constant C which depends on N, β and q such that

$$\sum_{j=2}^{M} |e_j| \leq C \left(\frac{1}{k}\right)^r \left(\frac{1}{M}\right)^{N+1-r} \|f\|_{N+1,\beta}.$$

Proof. Here, C is a constant which may depend on N, β and q but its value may differ from line to line. By Corollary 2.7 and recalling the notation $h_j = (x_j - x_{j-1})/2$, we have

$$\begin{aligned}\sum_{j=2}^{M} |e_j| &\leq C \left(\frac{1}{k}\right)^r \sum_{j=2}^{M} h_j^{N+2-r} \max_{x\in[x_{j-1},x_j]} \left|f^{(N+1)}(x)\right| \\ &\leq C \left(\frac{1}{k}\right)^r \left[\sum_{j=2}^{M} h_j^{N+2-r} x_{j-1}^{\beta-N-1}\right] \|f\|_{N+1,\beta}.\end{aligned}$$

The mean value theorem applied to $g(x) = (x/M)^q$ over $[j-1,j]$, $j = 2,3,...,M$, gives that

$$h_j \leq C\frac{1}{M}\left(\frac{j}{M}\right)^{q-1} = C\frac{1}{M}\left(\frac{j}{j-1}\right)^{q-1}\left(\frac{j-1}{M}\right)^{q-1} \leq \frac{C}{M}\left(\frac{j-1}{M}\right)^{q-1}$$

and thus

$$h_j^{N+2-r} x_{j-1}^{\beta-N-1} \leq C\left(\frac{1}{M}\right)^{N+2-r}\left(\frac{j-1}{M}\right)^{\alpha}$$

where $\alpha = q(\beta+1-r)-(N+2-r)$. Therefore, we get

$$\sum_{j=2}^{M} |e_j| \leq C \left(\frac{1}{k}\right)^r \left(\frac{1}{M}\right)^{N+1-r} \left[\sum_{j=2}^{M} \frac{1}{M}\left(\frac{j-1}{M}\right)^{\alpha}\right] \|f\|_{N+1,\beta}.$$

Here the sum in the right hand side is a Riemann sum for $\int_0^1 x^\alpha\, dx$ which is finite for $\alpha > -1$ which is ensured by the hypothesis the lemma. □

Next we give the estimation for $\tilde{e}_1$.

Lemma 3.5 ([1]). *For the same hypothesis of Lemma 3.4, there is a constant C which depends on β and q such that, for $N \geq 1$*

$$|\tilde{e}_1| \leq C \left(\frac{1}{k}\right)^r \left(\frac{1}{M}\right)^{N+1-r} \begin{cases} \|f\|_{2,\beta}, & \text{for} \quad \beta \in (-1,1)\setminus\{0\}, \\ (1+\log(k))^r(\log(M))^{1-r}\|f\|_{2,0}, & \text{for} \quad \beta = 0. \end{cases}$$

Proof. Note that during the whole proof, we will use the fact that $x_0 = (0/M)^q = 0$. We will consider two cases for $\beta \in (-1,1)$.

For $\beta \in (0,1)$, we have

$$\begin{aligned} (Q_1^{[0,x_1]} f)' &= \left|\frac{f(x_1)-f(0)}{x_1}\right| \\ &= \left|\frac{1}{x_1}\int_0^{x_1} f'(x)\, dx\right| \\ &\leq \frac{1}{x_1}\int_0^{x_1} |f'(x)|\, dx \\ &= \frac{1}{x_1}\int_0^{x_1} x^{\beta-1}|x^{1-\beta} f'(x)|\, dx \\ &\leq \frac{1}{x_1}\int_0^{x_1} x^{\beta-1}\, dx \|f\|_{1,\beta} \\ &= \frac{1}{\beta} x_1^{\beta-1}\|f\|_{1,\beta}. \end{aligned} \tag{3.15}$$

Also, for any $t \in [0, x_1]$ and $f \in C^1_\beta[0,1]$

$$\begin{aligned}\left|f(t) - Q_1^{[0,x_1]}f(t)\right| &= \left|\int_0^t \left(f - Q_1^{[0,x_1]}f\right)'(x)\,dx\right| \leq \int_0^t \left|\left(f - Q_1^{[0,x_1]}f\right)'(x)\right| dx \\ &\leq \int_0^{x_1} |f'(x)| + \left|\left(Q_1^{[0,x_1]}f\right)'(x)\right| dx \\ &= \int_0^{x_1} |f'(x)|\,dx + \int_0^{x_1} \left|\left(Q_1^{[0,x_1]}f\right)'(x)\right| dx \\ &= \int_0^{x_1} x^{\beta-1}\left|x^{1-\beta}f'(x)\right| dx + \left|\left(Q_1^{[0,x_1]}f\right)'\right| \int_0^{x_1} 1\,dx \\ &\leq \int_0^{x_1} x^{\beta-1}\,dx\|f\|_{1,\beta} + x_1\left|\left(Q_1^{[0,x_1]}f\right)'\right| \\ &\leq \frac{1}{\beta}x_1^\beta\|f\|_{1,\beta} + x_1\frac{1}{\beta}x_1^{\beta-1}\|f\|_{1,\beta} = \frac{2}{\beta}x_1^\beta\|f\|_{1,\beta}. \qquad (3.16)\end{aligned}$$

Then from (3.2) and (3.16), for $f \in C^1_\beta[0,1]$,

$$|\tilde{e}_1| = \left|\int_0^{x_1} \left(f(x) - Q_1^{[0,x_1]}f(x)\right)\exp(ikx)\,dx\right| \leq \frac{2}{\beta}x_1^{1+\beta}\|f\|_{1,\beta}. \qquad (3.17)$$

On the other hand, using integration by parts for the integral in (3.17), we have

$$\begin{aligned}\tilde{e}_1 &= \int_0^{x_1} \left(f(x) - Q_1^{[0,x_1]}f(x)\right)\exp(ikx)\,dx \qquad (3.18) \\ &= \frac{1}{ik}\left[\left(f - Q_1^{[0,x_1]}f\right)(x_1)\exp(ikx_1) - \left(f - Q_1^{[0,x_1]}f\right)(0)\exp(ik0)\right] \\ &\quad - \frac{1}{ik}\int_0^{x_1} \left(f(x) - Q_1^{[0,x_1]}f(x)\right)'\exp(ikx)\,dx \\ &= -\frac{1}{ik}\int_0^{x_1} \left(f(x) - Q_1^{[0,x_1]}f(x)\right)'\exp(ikx)\,dx.\end{aligned}$$

Since $f' \in C^N_{\beta-1}$, we can use Lemma 3.1 replacing β with $\beta - 1$ and choosing s as β to obtain

$$\left|\int_0^{x_1} f'(x)\exp(ikx)\,dx\right| \leq C_\beta\left(\frac{1}{k}\right)^\beta\|f\|_{1,\beta-1} \leq C_\beta\left(\frac{1}{k}\right)^\beta\|f\|_{2,\beta}. \qquad (3.19)$$

Also, since $\left(Q_1^{[0,x_1]}f\right)'$ is constant, we have

$$\begin{aligned}\left|\int_0^{x_1}\left(Q_1^{[0,x_1]}f\right)'\exp(ikx)\,dx\right| &= \left|\left(Q_1^{[0,x_1]}f\right)'\right|\left|\int_0^{x_1}\exp(ikx)\,dx\right| \\ &\leq \frac{x_1^{\beta-1}}{\beta k}\|f\|_{1,\beta}. \qquad (3.20)\end{aligned}$$

Therefore, combining (3.19) and (3.20) with (3.18), we get

$$\begin{aligned}|\tilde{e}_1| &\leq C'_\beta\left[\left(\frac{1}{k}\right)^{1+\beta}+\frac{x_1^{\beta-1}}{k^2}\right]\|f\|_{2,\beta} \\ &\leq C'_\beta\left(\frac{1}{k}\right)^{1+\beta}\left[1+(x_1k)^{\beta-1}\right]\|f\|_{2,\beta}. \qquad (3.21)\end{aligned}$$

Interpolating (3.17) and (3.21) gives

$$\begin{aligned}|\tilde{e}_1| &= |\tilde{e}_1|^\theta\,|\tilde{e}_1|^{1-\theta} \\ &\leq \left(\frac{2}{\beta}\right)^\theta x_1^{(1+\beta)\theta}\|f\|_{1,\beta}^\theta\left(C'_\beta\right)^{1-\theta}\left(\frac{1}{k}\right)^{(1+\beta)(1-\theta)}\left[1+(x_1k)^{\beta-1}\right]^{1-\theta}\|f\|_{2,\beta}^{1-\theta}.\end{aligned}$$

Also, since $x_1k \geq 1$, we get

$$|\tilde{e}_1| \leq C''_\beta\left(\frac{1}{k}\right)^{(1+\beta)(1-\theta)}x_1^{(1+\beta)\theta}\|f\|_{2,\beta}.$$

By taking $\theta = (1+\beta-r)/(1+\beta)$, we get

$$|\tilde{e}_1| \leq C''_\beta\left(\frac{1}{k}\right)^r x_1^{1+\beta-r}\|f\|_{2,\beta} \leq C''_\beta\left(\frac{1}{k}\right)^r\left(\frac{1}{M}\right)^{N+1-r}\|f\|_{2,\beta}.$$

which is the desired result for $x_1k \geq 1$. If $x_1k < 1$, replacing f by f_k in (3.15), (3.16) and (3.17) we obtain

$$|\tilde{e}_1| \leq \frac{2}{\beta}x_1^{1+\beta}\|f_k\|_{1,\beta} \leq \frac{2}{\beta}\left(\frac{1}{k}\right)^r x_1^{1+\beta-r}\|f\|_{1,\beta} \leq \frac{2}{\beta}\left(\frac{1}{k}\right)^r\left(\frac{1}{M}\right)^{N+1-r}\|f\|_{1,\beta}$$

which proves the result for $\beta \in (0,1)$. For $\beta \in (-1,0]$, the integral over $[0,x_1]$ is approximated by 0. Thus the result holds by Lemma 3.1 with $\epsilon = x_1$ and $s = r$. □

Therefore from Lemmas 3.4 and 3.5, we directly get the following main result for the error associated with the composite algorithm.

Theorem 3.6 ([1]). *Under the hypothesis of Lemma 3.4, there is a constant C that depends on N,β and q such that*

$$\left| I_k^{[0,1]}(f) - I_{k,N,M,q}^{[0,1]}(f) \right| \leq C \left(\frac{1}{k}\right)^r \left(\frac{1}{M}\right)^{N+1-r} \begin{cases} \|f\|_{N+1,\beta}, & \textit{for} \quad \beta \in (-1,1) \setminus \{0\}, \\ (1+\log(k))^r \left(\log(M)\right)^{1-r} \|f\|_{N+1,0}, & \textit{for} \quad \beta = 0. \end{cases} \tag{3.22}$$

3.2. Relating $I_k^{[a,b]}(f)$ with $I_k^{[0,1]}(f)$

Now, we will study the case of integral over $[a,b]$ with finitely many singularities at the points $\xi_1, ..., \xi_\ell$ with $a \leq \xi_1 < \xi_2 < ... < \xi_\ell \leq b$, for $\ell \geq 1$. In this case, we firstly take the points

$$c_i \in (\xi_{i-1}, \xi_i), \quad i = 2, 3, .., \ell$$

and we divide the interval $[a,b]$ into subintervals

$$\begin{aligned} &[a, \xi_1], \\ &[\xi_{j-1}, c_j] \quad \text{and} \quad [c_j, \xi_j] \quad j = 2, 3, ..., \ell \\ &[\xi_\ell, b] \end{aligned}$$

(in case $\xi_1 = a$, the interval $[a, \xi_1]$ is not included; similarly when $\xi_\ell = b$, $[\xi_\ell, b]$ is excluded). Therefore, we have subintervals each of which has a unique point of singularity which appears at one of the endpoints. Hence, we can map each subinterval to $[0, 1]$ in an affine way such that the singularity appears at $x = 0$. Then we can use the method described in §3.1 for each of these subintervals. Thus, we get a quadrature rule for the integral over $[a, b]$ with finitely many algebraic or logarithmic singularities.

4. FILON-CLENSHAW-CURTIS RULE FOR NONLINEAR OSCILLATORS

We generalize the composite FCC rule discussed in the chapter 3 for the efficient evaluation of more general oscillatory integrals of the form

$$I_k^{[a,b]}(f,g) = \int_a^b f(x)\exp(ikg(x))dx.$$

where $g \in C^\infty[a,b]$. To this end, in §4.1, we consider the case where f is smooth and g has no stationary points. Then, in §4.2, we treat the case where the function f may have finitely many algebraic or logarithmic singularities and the function g is smooth and has finitely many stationary points in the interval $[a,b]$. Finally, in section §4.3, we present an efficient method for the calculation of the quadrature weights $\omega_n(k)$.

4.1. Relating $I_k^{[a,b]}(f,g)$ with $I_k^{[a,b]}(f)$

We assume f has no singularities and $g'(x) \neq 0$ for any $x \in [a,b]$. Considering first the case that $g'(x) > 0$ for every $x \in [a,b]$, we use the change of variables

$$\tau = g(t) \tag{4.1}$$

to write

$$\begin{aligned} I_k^{[a,b]}(f,g) &= \int_a^b f(t)\exp(ikg(t))dt \\ &= \int_{g(a)}^{g(b)} f(g^{-1}(\tau))\exp(ik\tau)\frac{1}{g'(g^{-1}(\tau))}d\tau \\ &= \int_{g(a)}^{g(b)} F(\tau)\exp(ik\tau)d\tau \\ &= I_k^{[g(a),g(b)]}(F) \end{aligned}$$

where

$$F(\tau) = f(g^{-1}(\tau))\frac{1}{g'(g^{-1}(\tau))}. \tag{4.2}$$

This shows that we approximate $I_k^{[a,b]}(f,g)$ by $I_{k,N}^{[g(a),g(b)]}(F)$.

When $g'(x) < 0$ for every $x \in [a,b]$, we use the change of variables in (4.1) to write

$$\begin{aligned} I_k^{[a,b]}(f,g) &= \int_a^b f(t)\exp(ikg(t))dt \\ &= \int_{g(a)}^{g(b)} f(g^{-1}(\tau))\exp(ik\tau)\frac{1}{g'(g^{-1}(\tau))}d\tau \\ &= -\int_{g(b)}^{g(a)} f(g^{-1}(\tau))\exp(ik\tau)\frac{1}{g'(g^{-1}(\tau))}d\tau \\ &= \int_{g(b)}^{g(a)} F(\tau)\exp(ik\tau)d\tau \end{aligned}$$

where

$$F(\tau) = -f(g^{-1}(\tau))\frac{1}{g'(g^{-1}(\tau))}.$$

Hence, again, we can approximate $I_k^{[a,b]}(f,g)$ by $I_{k,N}^{[g(b),g(a)]}(F)$.

4.2. Composite algorithm for nonlinear oscillators

We assume the function f has finitely many singularities, the function g has finitely many stationary points in the interval $[a,b]$ and the singularities of f and the stationary points of g do not coincide. Then we divide $[a,b]$ into subintervals such that each one has either a unique singular point or a unique stationary point as one of its endpoints. Then, for each subinterval, the change of variables in (4.1) gives that the unique singular point ξ_j of f will appear as a singular point of F at the point $g(\xi_j)$. Thus, we can use the composite rule described in Chapter 3 to approximate the integral for each subinterval with a singular point. Hence, we can consider the case of a smooth function f with a smooth function g that has a unique stationary point at the point a. We may assume without loss of generality that

$$g'(a) = g''(a) = ... = g^{(n)}(a) = 0, \quad g^{(n+1)}(a) > 0 \tag{4.3}$$

for some $n \geq 1$. Then the change of variable in (4.1) gives a singularity to F at the point $g(a)$. Thus, we have

$$\begin{aligned} I_k^{[a,b]}(f,g) &= \int_a^b f(t)\exp(ikg(t))dt \\ &= \int_{g(a)}^{g(b)} F(\tau)\exp(ik\tau)d\tau. \end{aligned} \tag{4.4}$$

For understanding the regularity of the function F, we make use of Faà di Bruno's formula for the derivatives of composition of two functions φ, Ψ which read as

$$(\varphi \circ \Psi)^{(m)} = \sum \frac{m!}{b_1!b_2!...b_m!}\left(\varphi^{(\ell)} \circ \Psi\right)\left(\prod_{j=1}^{m}\left(\frac{\Psi^{(j)}}{j!}\right)^{b_j}\right), \quad m \in \mathbb{N} \tag{4.5}$$

where the sum is taken over all solutions in nonnegative integers $b_1, b_2, ..., b_m$ of $\sum_{j=1}^{m} jb_j = m$ and $\ell = b_1 + ... + b_m$ [25].

Lemma 4.1 ([1]). *Let $g \in C^\infty[a,b]$ be as in* (4.3) *with a unique stationary point at a. Then for all $p \in \mathbb{N}$, there exists a positive constant C_p such that*

$$\left|\left(g^{-1}\right)^{(p)}(\tau)\right| \leq C_p|\tau - g(a)|^{\alpha-p}. \tag{4.6}$$

Proof. Condition (4.3) implies that g is increasing in $[a, b]$. By Taylor's theorem with integral remainder, we have

$$\begin{aligned} g(x) &= g(a) + g'(a)(x-a) + g''(a)\frac{(x-a)^2}{2!} + \ldots + g^{(n)}(a)\frac{(x-a)^n}{n!} + R(x) \\ &= g(a) + R(x) \end{aligned}$$

where

$$R(x) = \frac{1}{n!}\int_a^x (x-t)^n g^{(n+1)}(t)dt.$$

With the change of variables

$$y = \frac{t-a}{x-a}$$

we have

$$\begin{aligned} R(x) &= \frac{1}{n!}\int_a^x (x-t)^n g^{(n+1)}(t)dt \\ &= \frac{1}{n!}\int_0^1 (x-(y(x-a)+a))^n\, g^{(n+1)}(y(x-a)+a)(x-a)dy \\ &= \frac{1}{n!}(x-a)\int_0^1 (x-yx+ya-a)^n\, g^{(n+1)}(y(x-a)+a)dy \\ &= \frac{1}{n!}(x-a)\int_0^1 ((x-a)(1-y))^n\, g^{(n+1)}(y(x-a)+a)dy \\ &= \frac{1}{n!}(x-a)^{n+1}\int_0^1 (1-y)^n\, g^{(n+1)}(y(x-a)+a)dy \\ &= (x-a)^{n+1}T(x) \end{aligned}$$

where

$$T(x) = \frac{1}{n!}\int_0^1 (1-y)^n\, g^{(n+1)}(y(x-a)+a)dy.$$

Now, setting $\alpha = 1/(n+1)$, we define

$$h(x) = (g(x) - g(a))^{\alpha} = (R(x))^{\alpha} = \left((x-a)^{n+1}T(x)\right)^{\alpha} = (x-a)\,(T(x))^{\alpha}\,. \tag{4.7}$$

Note that, for any $x \in (a,b]$, $R(x) > 0$ and $T(x) > 0$. Also, since we have $g \in C^{\infty}[a,b]$, $T \in C^{\infty}[a,b]$ and $T(a) = \frac{1}{(n+1)!}g^{(n+1)}(a) > 0.$. Therefore $h \in C^{\infty}[a,b]$ and for $x \in (a,b]$,

$$h'(x) = \alpha\,(g(x) - g(a))^{\alpha-1}\,g'(x) > 0.$$

Moreover, since also $h'(a) = (T(a))^{\alpha}$, it follows that $h'(x) > 0$ for $x \in [a,b]$. Therefore, $h : [a,b] \to \mathbb{R}$ is invertible and $(h)^{-1} \in C^{\infty}[h(a), h(b)]$. Thus, we have, via setting $\tau = g(x)$,

$$g^{-1}(\tau) = x = h^{-1}\left((\tau - g(a))^{\alpha}\right). \tag{4.8}$$

To prove (4.6) , we use (4.5) with $\varphi = h^{-1}$ and $\Psi(\tau) = (\tau - g(a))^{\alpha}$ to compare the derivatives of the right hand side of (4.8) to obtain

$$\begin{aligned}&\left(h^{-1}\left(((\tau - g(a))^{\alpha}\right)\right)^{(p)}\\ &\qquad = \sum \frac{p!}{b_1!b_2!...b_m!}\left((h^{-1})^{(\ell)}\left((\tau - g(a))^{\alpha}\right)\right)\left(\prod_{j=1}^{p}\left(\frac{d^j}{d\tau^j}\frac{(\tau - g(a))^{\alpha}}{j!}\right)^{b_j}\right)\end{aligned} \tag{4.9}$$

where b_j's and ℓ are as in Faà di Bruno's formula (4.5). Since h^{-1} is smooth, each of the functions $(h^{-1})^{(\ell)}\left((\tau - g(a))^{\alpha}\right)$ is bounded on $[a,b]$.

For the product appearing in (4.9), we have

$$\prod_{j=1}^{p}\left(\frac{d^j}{d\tau^j}((\tau - g(a))^{\alpha})\right)^{b_j} = C\,|\tau - g(a)|^{(\alpha-1)b_1+(\alpha-2)b_2+...+(\alpha-p)b_p} \tag{4.10}$$

for a constant C. Here the power of the term in the absolute value is $\alpha k - p$. Since $\alpha k - p \geq \alpha - p$, we get the result. $\square$

For the regularity of the function F, we have the following result:

Theorem 4.2 ([1]). *For f and g in $C^\infty[a,b]$ with a unique stationary point a of g of order n, for $p \in \mathbb{N}$, there is a positive constant C_p such that, for $\alpha = 1/(n+1)$*

$$\left|F^{(p)}(\tau)\right| \leq C_p \left|\tau - g(x_0)\right|^{\alpha-p-1}, \quad \tau \in (g(a), g(b)].$$

Proof. For

$$F(x) = \left(f \circ g^{-1}\right)(x) \left(g^{-1}(x)\right)',$$

by Leibniz rule for derivative of a product of two functions, $F^{(p)}(x)$ is a linear combination of

$$\left(f \circ g^{-1}\right)^{(j)} \left(g^{-1}\right)^{(p-j+1)}, \quad j = 0, 1, ..., p. \tag{4.11}$$

Further by Faà di Bruno's formula arguing as in the paragraph around Equation (4.9), we see that $\left|(f \circ g^{-1})^{(j)}\right|$ is bounded by

$$C_2 \left|\prod_{n=1}^{j} \left[\left(g^{-1}\right)^{(n)}\right]^{b_n}\right| \tag{4.12}$$

for some constant C_2 where b_n's are as in (4.5). As in the paragraph around (4.10), this lead to the conclusion that $\left|(f \circ g^{-1})^{(j)}\right|$ is bounded by $C_j \left|\tau - g(a)\right|^{\alpha-j}$ for some constant C_j. Then, each term in (4.11), except $j = 0$, can be estimated by

$$\tilde{C} \left|\tau - g(x_0)\right|^{\alpha-j} \left|\tau - g(x_0)\right|^{\alpha-p+j-1} = C_p \left|\tau - g(x_0)\right|^{2\alpha-p-1}.$$

For $j = 0$, the bound is

$$C_p \, |\tau - g(x_0)|^{\alpha-p-1}$$

where C_p is a generic constant. Thus, the result follows. □

5. IMPLEMENTATION DETAILS

In this chapter, we present the implementation details related with the FCC-algorithm. In this connection, in §5.1, we demonstrate a fast numerically stable algorithm for the computation of quadrature weights $\omega_n(k)$ in (2.4). Also, in §5.2, we address a numerical stability issue that arises in computing the function F in (4.2).

5.1. Calculation of $\omega_n(k)$

Here, we demonstrate an efficient algorithm for the computation of [14]

$$\omega_n(k) = \int_{-1}^{1} T_n(t)\exp(ikt)dt, \quad n = 0, 1, ...N.$$

To this end, we note for the Chebyshev polynomials of the second kind, $U_n(t) = \frac{1}{n+1}T'_{n+1}(t)$ [24], that

$$\begin{aligned}
U_n(t) - U_{n-2}(t) &= \frac{1}{n+1}T'_{n+1}(t) - \frac{1}{n-1}T'_{n-1}(t) \\
&= \frac{1}{n+1}\frac{n+1}{\sqrt{1-t^2}}\sin\left((n+1)\arccos(t)\right) \\
&\quad - \frac{1}{n-1}\frac{n-1}{\sqrt{1-t^2}}\sin\left((n-1)\arccos(t)\right) \\
&= \frac{1}{\sqrt{1-t^2}}\left[\cos(n\arccos(t))\sin(\arccos(t))\right] \\
&\quad + \frac{1}{\sqrt{1-t^2}}\left[\cos(n\arccos(t))\sin(\arccos(t))\right] \\
&= \frac{1}{\sqrt{1-t^2}}\left[2T_n(t)\sqrt{1-t^2}\right] = 2T_n(t)
\end{aligned}$$

Thus, for $n \geq 2$

$$\begin{aligned}
2\omega_n(k) &= \int_{-1}^{1} 2T_n(t)\exp(ikt)dt = \int_{-1}^{1}\left[U_n(t) - U_{n-2}(t)\exp(ikt)dt\right] \\
&= \int_{-1}^{1} U_n(t)\exp(ikt)dt - \int_{-1}^{1} U_{n-2}(t)\exp(ikt)dt.
\end{aligned}$$

Then if we define

$$\rho_{n+1}(k) = \int_{-1}^{1} U_n(t)\exp(ikt)dt,$$

we get

$$2\omega_n(k) = \rho_{n+1}(k) - \rho_{n-1}(k), \quad n \geq 2. \tag{5.1}$$

Also, by integration by parts, we have

$$\begin{aligned}
\omega_n(k) &= \int_{-1}^{1} T_n(t)\exp(ikt)dt \\
&= \left[\frac{T_n(1)\exp(ik1)}{ik} - \frac{T_n(-1)\exp(-ik)}{ik}\right] - \frac{1}{ik}\int_{-1}^{1} T_n'(t)\exp(ikt)dt \\
&= \frac{\exp(ik)}{ik} - \frac{(-1)^n\exp(-ik)}{ik} - \frac{1}{ik}\int_{-1}^{1} T_n'(t)\exp(ikt)dt \\
&= \gamma_n(k) - \frac{1}{ik}\int_{-1}^{1} T_n'(t)\exp(ikt)dt \\
&= \gamma_n(k) - \frac{n}{ik}\frac{1}{n}\int_{-1}^{1} T_n'(t)\exp(ikt)dt
\end{aligned}$$

where

$$\gamma_n(k) = \begin{cases} \frac{2\cos(k)}{ik}, & \text{n is odd,} \\ \frac{2\sin(k)}{k}, & \text{n is even.} \end{cases} \tag{5.2}$$

This gives

$$\omega_n(k) = \gamma_n(k) - \frac{n}{ik}\rho_n(k), \quad n \geq 1. \tag{5.3}$$

Combining (5.1) and (5.3), we therefore get

$$2\gamma_n(k) - \frac{2n}{ik}\rho_n(k) = \rho_{n+1}(k) - \rho_{n-1}(k), \quad n \geq 2. \tag{5.4}$$

For $n = 0$,

$$\begin{aligned} w_0(k) &= \int_{-1}^{1} T_0(t)\exp(ikt)dt \\ &= \int_{-1}^{1} \cos(0\arccos(t))\exp(ikt)dt \\ &= \int_{-1}^{1} \exp(ikt)dt \\ &= \frac{\exp(ik)-\exp(-ik)}{ik} \\ &= \frac{2\sin(k)}{k} \\ &= \gamma_0(k). \end{aligned}$$

Also, we have

$$U_0(t) = T_1'(t) = \frac{1}{\sqrt{1-t^2}}\sin(\arccos(t)) = 1$$

and

$$\begin{aligned} U_1(t) = \frac{1}{2}T_2'(t) &= \frac{1}{2}\frac{2}{\sqrt{1-t^2}}\sin(2\arccos(t)) \\ &= \frac{1}{\sqrt{1-t^2}} 2\sin(\arccos(t))\cos(\arccos(t)) = 2t. \end{aligned}$$

Therefore,

$$\rho_1(k) = \int_{-1}^{1} U_0(t)\exp(ikt)dt = \int_{-1}^{1} \exp(ikt)dt = \gamma_0(k)$$

and

$$\begin{aligned} \rho_2(t) &= \int_{-1}^{1} U_1(t)\exp(ikt)dt = \int_{-1}^{1} 2t\exp(ikt)dt \\ &= 2\left[\left[\frac{\exp(ik)}{ik} - \frac{(-1)\exp(-ik)}{ik}\right] - \frac{1}{ik}\int_{-1}^{1}\exp(ikt)dt\right] \\ &= 2\left[\frac{2\cos(ik)}{ik} - \frac{1}{ik}\gamma_0(k)\right] = 2\gamma_1(k) - \frac{2}{ik}\gamma_0(k). \end{aligned}$$

This gives a recursive algorithm for the computation of $\omega_n(k)$ for $n = 0, 1, ..., N$. However, this algorithm becomes unstable when $n \geq k$. To resolve this issue, we add a second phase to the algorithm where we do the following modification. When $N > k$, we define

$$n_0 = \min_{m \geq k}\{m \in \mathbb{N}\}$$

and fix an integer M such that $M \geq \max\{n_0/2, N/2\}$. The recursion (5.4) implies that the vector

$$X_M(k) = \begin{bmatrix}\rho_{n_0}(k) & \rho_{n_0+1}(k) & \dots & \rho_{2M-1}(k)\end{bmatrix}^T \tag{5.5}$$

is the solution of the linear system

$$A_M(k)x = b_M(k) \tag{5.6}$$

where

$$A_M(k) = \begin{bmatrix} \frac{2n_0}{ik} & 1 & & & & \\ -1 & \frac{2(n_0+1)}{ik} & 1 & & & \\ & -1 & \frac{2(n_0+2)}{ik} & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots \\ & & & & -1 & \frac{2(2M-1)}{ik} \end{bmatrix}, \quad b_M(k) = \begin{bmatrix} 2\gamma_{n_0}(k) + \rho_{n_0-1} \\ 2\gamma_{n_0+1}(k) \\ 2\gamma_{n_0+2}(k) \\ \vdots \\ 2\gamma_{2M-1}(k) - \rho_{2M}(k) \end{bmatrix} \tag{5.7}$$

In solving the linear system, we use Oliver's algorithm [26] which is the standard (numerically stable) algorithm for the solution of tridiagonal linear systems. Aside from $\rho_{n_0-1}(k)$ and $\rho_{2M}(k)$, all the quantities appearing in $A_M(k)$ and $b_M(k)$ are explicit (see (5.2)). On the other hand, $\rho_{n_0-1}(k)$ was already computed in the first phase of the algorithm. Concerning $\rho_{2M}(k)$, we have the following result.

Theorem 5.1 ([14]). *If M is an integer with $M \geq k$ and*

$$p_0(\theta) = \frac{1}{(2M - k\sin(\theta))}, \quad p_r(\theta) = p_0(\theta)\frac{d}{d\theta}p_{r-1}(\theta), \quad r = 1, 2, ...,$$

then

$$\rho_{2M}(k) = 2i\left[\sum_{r=0}^{J}(-1)^r p_{2r}(0)\sin(k) + \sum_{r=0}^{J}(-1)^r p_{2r+1}(0)\cos(k)\right] + R_J(M,k)$$

where

$$|R_J(M,k)| \leq C_J k M^{-2J-4}$$

and C_J is a constant which is independent of M and k.

The proof for Theorem 5.1 can be found in [14]. The general algorithm for computing $\omega_n(k)$ for $n = 0, 1, ..., N$ is now given in Figure 5.1. We have to mention that, when compared with the second phase, the first phase of the algorithm is faster as it does not require the solution of a linear system. However, the sizes of the tridiagonal linear systems (5.1) appearing in the implementation of the second phase are usually bounded roughly by 15, and Oliver's algorithm does not require an extensive amount of time for linear systems of this size. Perhaps more importantly, the proof of the stability of the second phase as presented in [14] depends heavily on the fact that the tridiagonal matrix $A_M(k)$ is in fact diagonally dominant when $n \geq k$.

5.2. Resolving a numerical stability issue

In §4.1, we considered $I_k^{[a,b]}(f,g)$ where the only stationary point of g is at $x = a$ in the interval $[a,b]$. We assume that the order of the stationary point is $n \geq 1$. Recall

For $0 \leq n \leq N+1$:

Define

$$\gamma_n(k) = \begin{cases} \frac{2\cos(k)}{ik}, & \text{if n is odd,} \\ \frac{2\sin(k)}{k}, & \text{if n is even.} \end{cases}$$

$n_0 = \min_{x \geq k}\{x \in \mathbb{N}\}$

For $0 \leq n \leq \min\{N, k\}$:

(i) $\rho_1(k) = \gamma_0(k)$

(ii) $\rho_2(k) = 2\gamma_1(k) - \frac{2}{ik}\gamma_0(k)$

(iii) $\rho_{n+1}(k) = 2\gamma_n(k) - \frac{2n}{ik}\rho_n(k) + \rho_{n-1}(k), \quad n = 2, 3, ..., \min\{N, k\} - 1$

(i) $\omega_0(k) = \rho_1(k)$

(ii) $\omega_n(k) = \gamma_n(k) - \frac{n}{ik}\rho_n(k), \quad n = 1, 2, ..., \min\{N, k\}$

For $n_0 \leq n \leq N$:

Take $M \geq \min_{m \in \mathbb{N}}\{m \geq \frac{N+1}{2}\}$ and compute the value $\rho_{2M}(k)$ as in Theorem 5.1.

Construct $A_M(k)$ and $b_M(k)$ as in (5.7) and solve the tridiagonal system

$$A_M(k)X_M(k) = b_M(k)$$

Define

$\rho_{n_0+j-1}(k) = (X_M(k))\,(j), \quad j = 1, 2, ..., 2M - n_0$

$\omega_n(k) = \gamma_n(k) - \frac{n}{ik}\rho_n(k), \quad n = n_0, n_0 + 1, ..., N$

Figure 5.1. An efficient algorithm for computing ω_n.

from (4.4) that

$$I_k^{[a,b]} = \int_{g(a)}^{g(b)} F(\tau)\exp(ik\tau)d\tau$$

where F consists of some compositions and products of f,g and g'. When we transform each of these integrals to integrals over $[0,1]$ to apply the composite algorithm, we have

$$\int_{g(a)}^{g(b)} F(\tau)\exp(ik\tau)d\tau$$
$$= \int_0^1 F\left([g(b)-g(a)]\,t+g(a)\right)\exp(ik\left[(g(b)-g(a))t+g(a)\right])\left(g(b)-g(a)\right)dt$$

which requires the computation of

$$F\left([g(b)-g(a)]\,t+g(a)\right)$$

for with very small values of t which occurs during the use of mesh in (3.1). This condition may cause rounding errors when $g(a) \gg [g(b)-g(a)]t$, and this, in return, may cause inaccuracies in the computation of $F\left([g(b)-g(a)]\,t+g(a)\right)$. Equation (4.2) implies that this computational problem is equivalent to computing

$$x = g^{-1}([g(b)-g(a)]\,t+g(a)) \tag{5.8}$$

for very small values of t. Setting

$$\chi = [g(b)-g(a)]\,t,$$

we see that computing x in (5.8) is equivalent to solving the equation

$$g(x)-g(a) = \chi,$$

and from (4.7), this is equivalent to solving

$$[T(x)]^{\alpha}(x-a) = \chi^{\alpha}. \tag{5.9}$$

Hence, if we define

$$H(x,\chi) = [T(x)]^{\alpha}(x-a) - \chi^{\alpha},$$

then we have to solve for x in the equation

$$H(x,\chi) = 0. \tag{5.10}$$

In this connection, we note from the proof of Lemma 4.1 that $T(a) > 0$, and therefore

$$H(a,0) = 0 \neq H_x(a,0).$$

This implies, by the implicit function theorem, that there exists a neighborhood of $\chi = 0$ in which x is a smooth function of χ^{α}, and there is a positive constant C_1 such that

$$|x-a| \leq C_1\chi^{\alpha}$$

for sufficiently small χ. Moreover, from (5.10), x is a solution of

$$x = a + \left(\frac{\chi}{T(x)}\right)^{\alpha} = K(x). \tag{5.11}$$

Since $T(a) > 0$ and T is smooth in a neighborhood of $x = a$, the derivative of K is bounded in an compact neighborhood of a, which, in turn, implies that $K(x)$ is Lipschitz in a neighborhood a and its Lipschitz constant is $C\chi^{\alpha}$ for some constant C. Therefore for small enough χ, the fixed point algorithm converges to x, and we can deduce that we can choose x as

$$\tilde{x} = K(a)$$

with error

$$|\tilde{x} - x| = |K(\tilde{x}) - K(x)| \leq C\chi^{\alpha}|x - \chi| \leq CC_1\epsilon^{2\alpha} = O(\chi^{2\alpha}).$$

Therefore, when

$$g(a) \gg (g(b) - g(a))t,$$

we use $\tilde{x} = K(a)$ to deal with the roundoff error for K as in (5.11) with an error of $O(([g(b) - g(x_0)]t)^{2\alpha})$.

6. NUMERICAL EXPERIMENTS

In this chapter, we present numerical experiments depicting the performance of the FCC-algorithm for the evaluation of highly oscillatory integrals $I_k^{[a,b]}(f)$ and $I_k^{[a,b]}(f,g)$. Experiments relating to the linear oscillator, namely $I_k^{[0,1]}(f)$, are given in §6.1 and those related with the non-linear oscillator, namely $I_k^{[0,1]}(f,g)$, appear in §6.2.

In all experiments, we take $[0,1]$ as the interval of integration, and we choose the functions f and g so that the exact values of $I_k^{[0,1]}(f)$ and $I_k^{[0,1]}(f,g)$ can be expressed in terms of some special functions (for which highly accurate evaluations are computed using matlab). In Tables $[6.1-6.8]$, we display the absolute errors for various different values of the degree of interpolation N, number of subintervals M, and for increasing values of k

6.1. The linear oscillator $I_k^{[a,b]}(f)$

Here we display numerical results for the evaluation of

$$I_k^{[0,1]}(f_\beta) = \int_0^1 f_\beta(x)\exp(ikx)dx$$

for various different values of $\beta \in (-1,1)$, where

$$f_\beta(x) = \begin{cases} x^\beta, & \beta \in (-1,1)\setminus\{0\}, \\ \log(x), & \beta = 0. \end{cases}$$

6.1.1. Increasing the degree of interpolation and the number of intervals

In tables $[6.1-6.3]$, we display the absolute error for increasing number of N and M and for different values values of $\beta \in (-1,1)$ and $k = 1000$. From Theorem 3.6 with $r = 0$, the error should converge to zero with the order $O(M^{-(N+1)})$ provided that $q > (N+1)/(\beta+1)$ for $\beta \neq 0$, and there is an extra $\log(M)$ term in the order of convergence when $\beta = 0$. Here we choose $q = (N+1)/(\beta+1) + 0.1$. We can see the error decreases as predicted by Theorem 3.6 for increasing values of N and M, and for the values $\beta = 1/2$, $\beta = 0$ and $\beta = 1/4$.

Table 6.1. Absolute error for increasing values of N and M, and $\beta = 1/2$.

$\beta = 1/2$	N=4	N=6	N=8
M=8	$5.9e^{-6}$	$5.2e^{-8}$	$1.7e^{-9}$
M=16	$9.4e^{-8}$	$5.7e^{-10}$	$6.6e^{-12}$
M=32	$2.9e^{-9}$	$2.0e^{-12}$	$1.0e^{-14}$
M=64	$8.3e^{-11}$	$2.2e^{-14}$	$1.4e^{-16}$

Table 6.2. Absolute error for increasing values of N and M, and $\beta = 0$.

$\beta = 0$	N=4	N=6	N=8
M=8	$2.7e^{-4}$	$7.9e^{-6}$	$1.0e^{-6}$
M=16	$1.0e^{-5}$	$7.2e^{-8}$	$2.2e^{-9}$
M=32	$4.0e^{-7}$	$7.4e^{-10}$	$3.0e^{-12}$
M=64	$1.4e^{-8}$	$3.7e^{-12}$	$3.4e^{-15}$

6.1.2. Increasing the oscillation

Here we take $M = 10$, $N = 3$ and $q = 12$, and test the performance of the algorithm for increasing values of k, and for various different values of β. The results are given in Table 6.4. Recall from Theorem 3.6, the composite algorithm should converge with order $O(k^{-r})$ as $k \to \infty$ for $r < (q(\beta+1) - N - 1)/(q-1)$ which is coming from the inequality $q > \frac{N+1-r}{\beta+1-r}$, which appears as an assumption in Theorem 3.6. As $\beta \to 1^{-}$, the inequality $0 \leq r \leq 1 + \beta$ allows r to increase and therefore the term $(1/k)^r$ appearing in (3.22), in Theorem 3.6 becomes more effective as k increases. This is clearly visible in Table 6.4.

Table 6.3. Absolute error for increasing values of N and M, and $\beta = -1/4$.

$\beta = -1/4$	N=4	N=6	N=8
M=8	$4.5e^{-5}$	$1.7e^{-5}$	$6.0e^{-6}$
M=16	$2.6e^{-6}$	$7.8e^{-8}$	$2.0e^{-8}$
M=32	$1.9e^{-8}$	$9.2e^{-10}$	$1.0e^{-11}$
M=64	$2.3e^{-9}$	$3.9e^{-12}$	$2.9e^{-14}$

Table 6.4. Absolute error for increasing values of k and β.

	$\beta = 1/8$	$\beta = 1/4$	$\beta = 1/2$	$\beta = 3/4$
$k = 10^3$	$4.9e^{-6}$	$4.0e^{-6}$	$1.2e^{-6}$	$2.2e^{-7}$
$k = 10^4$	$4.6e^{-7}$	$2.7e^{-7}$	$4.5e^{-8}$	$4.5e^{-9}$
$k = 10^5$	$5.7e^{-8}$	$2.6e^{-8}$	$2.3e^{-9}$	$1.1e^{-10}$
$k = 10^6$	$1.2e^{-8}$	$3.8e^{-9}$	$1.8e^{-10}$	$4.9e^{-12}$
$k = 10^7$	$1.3e^{-9}$	$2.5e^{-10}$	$4.4e^{-12}$	$7.1e^{-14}$

6.1.3. Increasing the oscillation for singular functions

Here we take $M = 12$ and $N = 3$ and $\beta = 0$ so that, $f_\beta = \log(x)$ and test the algorithm for increasing values of k, and for various different values of $q > \frac{N+1-r}{\beta+1-r}$. As q gets larger, the dependence of the convergence to k becomes more visible.

Table 6.5. Absolute error for increasing values of k and q for $\beta = 0$.

$\log(x)$	$q = 4$	$q = 8$	$q = 12$	$q = 16$
$k = 10^1$	$5.5e^{-4}$	$1.3e^{-4}$	$1.0e^{-3}$	$3.6e^{-3}$
$k = 10^2$	$5.2e^{-4}$	$5.2e^{-5}$	$2.1e^{-4}$	$3.7e^{-4}$
$k = 10^3$	$5.2e^{-4}$	$3.1e^{-5}$	$3.8e^{-5}$	$1.0e^{-4}$
$k = 10^4$	$5.0e^{-4}$	$6.7e^{-6}$	$7.0e^{-6}$	$8.4e^{-6}$
$k = 10^5$	$1.4e^{-4}$	$9.1e^{-7}$	$1.1e^{-6}$	$1.9e^{-6}$
$k = 10^6$	$2.0e^{-5}$	$4.0e^{-7}$	$2.0e^{-7}$	$2.5e^{-7}$
$k = 10^7$	$1.9e^{-6}$	$1.3e^{-7}$	$5.2e^{-8}$	$8.5e^{-8}$

6.2. The nonlinear oscillator $I_k^{[a,b]}(f, g)$

In this section, we show numerical results demonstrating the performance of FCC rule for the evaluation of

$$I_k^{[0,1]}(f, g) = \int_0^1 f(x) \exp(ikg_d(x))dx$$

where

$$f(x) = \sin(x) \quad \text{and} \quad g_d(x) = x^d, \quad d = 1, 2, ...$$

The choice of the function g_d enables us to test the FCC rule as the singularity becomes stronger with increasing d. When we apply the change of variables in (4.1), we get

$$F(\tau) = \frac{\sin(\tau^{\frac{1}{d}})}{d\tau^{1-\frac{1}{d}}}$$

which has a singularity at $\tau = 0$ for $d \geq 3$. Thus, we need to determine the vale of β for use in the composite algorithm. To this end, we apply Theorem 4.2 to deduce that, for any $p \in \mathbb{N}$,

$$\left|F^{(p)}(\tau)\right| \leq C_p \left|\tau - g(0)\right|^{\frac{1}{d-1+1}-p-1}.$$

This implies

$$\begin{aligned}\left|\tau^{p-\beta} F^{(p)}(\tau)\right| &\leq C_p \tau^{p-\beta} \left|\tau - g(0)\right|^{\frac{1}{d}-p-1} \\ &= C_p \tau^{p-\beta} \tau^{\frac{1}{d}-p-1} \\ &= C_p \tau^{\frac{1-d}{d}-\beta}\end{aligned}$$

so that, with $\beta = \frac{1-d}{d}$, we have $F \in C^m_\beta[0,1]$. Since $\beta = \frac{1-d}{d} \to -1^+$ as $d \to \infty$, this allows us to test the quadrature rule for stronger singularities. Note that the error bound (3.3) in Lemma 3.1 blows up as $\beta \to -1^+$.

Finally, let us note that for $d = 1$, $I_k^{[0,1]}(f,g) = I_k^{[0,1]}(f)$, and the singularity of the function F is removable for $d = 2$. Thus, we expect to get better approximations for $d = 1$ and $d = 2$ compared larger values of d.

6.2.1. Increasing the number of intervals

Here we take $k = 1000$, $N = 4$, $\beta = (1-d)/d$, $q = (N+1)/(\beta+1) + 0.1$ and in Table 6.6, we display the absolute error as M and d increases. As we anticipated, the algorithm delivers smaller errors for $d = 1$ and $d = 2$. For larger values of d, the approximation get better with increasing M, however, as the singularity becomes stronger with increasing d, convergence is attained for larger values of M when d is large as predicted in Theorem 3.6.

Table 6.6. Absolute error for increasing values of M and d.

	$M = 8$	$M = 16$	$M = 32$	$M = 64$
$d = 1$	$3.4e^{-10}$	$3.5e^{-12}$	$3.1e^{-14}$	$8.8e^{-13}$
$d = 2$	$3.5e^{-10}$	$6.3e^{-12}$	$1.7e^{-11}$	$8.3e^{-11}$
$d = 3$	$3.1e^{-4}$	$1.2e^{-5}$	$7.3e^{-7}$	$2.1e^{-8}$
$d = 4$	$7.5e^{-4}$	$1.1e^{-4}$	$1.8e^{-6}$	$2.2e^{-7}$
$d = 5$	$2.1e^{-3}$	$1.6e^{-4}$	$1.3e^{-5}$	$2.1e^{-7}$
$d = 6$	$3.9e^{-2}$	$6.0e^{-4}$	$2.5e^{-5}$	$1.6e^{-6}$
$d = 7$	$3.3e^{-1}$	$4.3e^{-3}$	$1.1e^{-4}$	$4.2e^{-6}$
$d = 8$	4.6	$7.0e^{-3}$	$5.6e^{-5}$	$7.9e^{-6}$
$d = 9$	124.9	$1.2e^{-1}$	$4.0e^{-4}$	$1.2e^{-5}$
$d = 10$	3833.4	3.5	$3.4e^{-3}$	$1.4e^{-5}$

6.2.2. Increasing the degree of interpolation

Here we take $k = 100$, $q = 12$, $M = 8$, and display the absolute error in Table 6.7 as N and d increases. As in §6.2.1, the algorithm delivers better approximations when d is smaller. For larger values of d, even if we get convergence, it is very slow for increasing values of N.

6.2.3. Increasing the oscillation

Finally, here we take $N = 4$, $M = 8$ and $q = 12$, and test the algorithm for increasing values of k. The results are depicted in Table 6.8. As in 6.2.2, we observe better performance when $d = 1$ and $d = 2$. However, for other values of d, the error does not decay with increasing k but it remains stable.

Table 6.7. Absolute error for increasing values of N and d.

	$N = 4$	$N = 6$	$N = 8$	$N = 10$
$d = 1$	$2.1e^{-8}$	$1.5e^{-11}$	$6.1e^{-15}$	$1.2e^{-17}$
$d = 2$	$6.8e^{-12}$	$7.3e^{-12}$	$7.3e^{-12}$	$7.3e^{-12}$
$d = 3$	$1.6e^{-4}$	$4.4e^{-5}$	$7.5e^{-6}$	$2.8e^{-6}$
$d = 4$	$3.7e^{-4}$	$6.9e^{-5}$	$3.2e^{-5}$	$9.0e^{-6}$
$d = 5$	$1.4e^{-3}$	$2.9e^{-4}$	$2.1e^{-4}$	$9.4e^{-5}$
$d = 6$	$5.8e^{-3}$	$1.8e^{-3}$	$9.1e^{-4}$	$4.5e^{-4}$
$d = 7$	$1.6e^{-2}$	$5.4e^{-3}$	$2.6e^{-3}$	$1.3e^{-3}$
$d = 8$	$3.3e^{-2}$	$1.2e^{-2}$	$5.5e^{-3}$	$2.7e^{-3}$
$d = 9$	$5.8e^{-2}$	$2.1e^{-2}$	$9.6e^{-3}$	$4.7e^{-3}$
$d = 10$	$9.0e^{-2}$	$3.2e^{-2}$	$1.5e^{-2}$	$7.2e^{-3}$

Table 6.8. Absolute error for increasing values of k.

	$k = 10^0$	$k = 10^1$	$k = 10^2$	$k = 10^3$
$d = 1$	$1.4e^{-6}$	$2.1e^{-5}$	$2.1e^{-8}$	$1.2e^{-10}$
$d = 2$	$2.4e^{-8}$	$1.1e^{-9}$	$6.8e^{-12}$	$7.2e^{-12}$
$d = 3$	$8.6e^{-5}$	$6.4e^{-4}$	$1.6e^{-4}$	$5.5e^{-5}$
$d = 4$	$8.6e^{-6}$	$8.9e^{-4}$	$3.7e^{-4}$	$3.2e^{-4}$
$d = 5$	$1.3e^{-3}$	$4.5e^{-4}$	$1.4e^{-3}$	$1.7e^{-3}$
$d = 6$	$5.8e^{-3}$	$4.8e^{-3}$	$5.8e^{-3}$	$6.2e^{-3}$
$d = 7$	$1.6e^{-2}$	$1.5e^{-2}$	$1.6e^{-2}$	$1.6e^{-2}$
$d = 8$	$3.3e^{-2}$	$3.2e^{-2}$	$3.3e^{-2}$	$3.4e^{-2}$
$d = 9$	$5.8e^{-2}$	$5.7e^{-2}$	$5.8e^{-2}$	$5.9e^{-2}$
$d = 10$	$9.0e^{-2}$	$8.9e^{-2}$	$9.0e^{-2}$	$9.1e^{-2}$

7. CONCLUSION

In this thesis, we presented a survey on the FCC algorithm for evaluation of oscillatory integrals of the form

$$I_k^{[a,b]}(f,g) = \int_a^b f(x) \exp(ikg(x))dx$$

where f has finitely many algebraic or logarithmic singularities and g has a finite number of stationary points. We also gave an error analysis which is supported by numerical experiments. It can be seen from the numerical results that the algorithm works effectively for singularities and stationary points of low degree. However, when the degree of stationary points increases, even if the algorithm stays stable, it becomes ineffective (see §6.2.2 and §6.2.3).

An algorithm for highly oscillatory integrals with high order stationary points is still an unsolved question and so it remains as a possible research direction for this area.

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APPENDIX A: AUXILIARY RESULTS

We have some notation for some auxiliary results.

$$\mathbb{K}_n = \{x \in \mathbb{Z} : -n/2 < x \leq n/2\}, \quad P_n u = \sum_{\mu \in K_n} \hat{u}(\mu) \exp(ik\mu) \tag{A.1}$$

and

$$\varphi_{n,j}(t) = \frac{1}{n} \sum_{\nu \in \mathbb{K}_n} \exp(ik(t - \frac{j}{n})), \quad Q_n u = \sum_{j=0}^{n-1} u\left(\frac{j}{n}\right) \varphi_{n,j}(t).$$

In other words, $Q_n u$ is the interpolation projection of 2π periodic function u. With these notation, in the convergence analysis of the FCC method, we make use of the following auxiliary results.

Theorem A.1 ([15]). *For $m \in \mathbb{R}$, if $u \in H^m$, then we have*

$$\|u - P_n u\|_{H^\lambda} \leq \left(\frac{n}{2}\right)^{\lambda - m} \|u\|_{H^m}, \quad (\lambda \leq m)$$

where $P_n u$ is as in (A.1)*.*

Proof. Expanding u into its Fourier series, we estimate

$$\begin{aligned} \|u - P_n u\|_{H^\lambda} &= \left(\sum_{\mu \in \mathbb{Z} \setminus \mathbb{K}_n} \underline{\mu}^{2\lambda} |\hat{u}(\mu)|^2\right)^{1/2} \leq \left(\sum_{|\mu| \geq n/2} \underline{\mu}^{2(\lambda - m)} \underline{\mu}^{2m} |\hat{u}(\mu)|^2\right)^{1/2} \\ &\leq \left(\frac{n}{2}\right)^{\lambda - m} \left(\sum_{|\mu| \geq n} \underline{\mu}^{2m} |\hat{u}(\mu)|^2\right)^{1/2} \\ &\leq \left(\frac{n}{2}\right)^{\lambda - m} \|u\|_{H^m}. \end{aligned}$$

□

Lemma A.2 ([15]). *For $u \in H^m$, $m > \frac{1}{2}$, we have*

$$(Q_n u)(t) = \sum_{p \in \mathbb{K}_n} \left[\sum_{\ell \in \mathbb{Z}} \hat{u}(p + n\ell) \right] \exp(ipt).$$

Proof. Since $m > 1/2$, we have

$$\sum_{\mu \in \mathbb{Z}} |\hat{u}(\mu)|^2 \leq \sum_{\mu \in \mathbb{Z}} \underline{\mu} \, |\hat{u}(\mu)|^2 \leq \sum_{\mu \in \mathbb{Z}} \underline{\mu}^{2m} \, |\hat{u}(\mu)|^2 = \|u\|_{H^m}^2 < \infty$$

which gives us that $\sum_{\mu \in \mathbb{Z}} \hat{u}(\mu)$ is an absolutely convergent series. Since, for $l \in \mathbb{Z}$, e^{ipt} and $e^{i(p+nl)t}$ have the same values at the grid points $\frac{2\pi j}{n}$ $(j = 0, 1, 2, ..., n-1)$, we have

$$Q_n e^{i(p+nl)t} = Q_n e^{ipt} = e^{ipt}.$$

Thus, rearranging the Fourier series of u, we have

$$\begin{aligned} Q_n u &= Q_n \sum_{\mu \in \mathbb{Z}} \hat{u}(\mu) \exp(i\mu t) \\ &= Q_n \sum_{p \in \mathbb{K}_n} \sum_{l \in \mathbb{Z}} \hat{u}(p + nl) \exp(i(p + nl)t). \end{aligned}$$

Since $Q_n \in L(H^\mu, H^\lambda)$ for $\mu > 1/2$, $\lambda \in \mathbb{R}$ [15, p. 242], we obtain

$$Q_n u = \sum_{p \in \mathbb{K}_n} \left[\sum_{l \in \mathbb{Z}} \hat{u}(p + nl) \right] \exp(ipt)$$

which is the desired result. □

Using Theorem A.1 and Lemma A.2, we obtain the following estimate

Theorem A.3 ([15]). *For $u \in H^m$, $m > 1/2$, there is a constant γ_m such that*

$$\|u - Q_n u\|_{H^\lambda} \leq \gamma_m \left(\frac{n}{2} \right)^{\lambda - m} \|u\|_{H^m}, \quad (0 \leq \lambda \leq m).$$

Proof. Since $m > 1/2$, we can use Lemma A.2. Setting $A_p = \left(\sum_{j\in\mathbb{Z}} \hat{u}(p+jn)\right)\exp(ipt)$, we have

$$\begin{aligned}
(\widehat{Q_n u - P_n u})(q) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(Q_n u - P_n u)(t)\exp(-iqt)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{p\in\mathbb{K}_n} A_p - \sum_{p\in\mathbb{K}_n}\hat{u}(p)\exp(ipt)\right]\exp(-iqt)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{p\in\mathbb{K}_n}\left[\sum_{j\in\mathbb{Z}}(\hat{u}(p+jn)-\hat{u}(p))\right]\exp(ipt)\exp(-iqt)dt \\
&= \frac{1}{2\pi}\sum_{p\in\mathbb{K}_n}\left[\sum_{j\in\mathbb{Z}}(\hat{u}(p+jn)-\hat{u}(p))\right]\int_{-\pi}^{\pi}\exp(i(p-q)t)dt.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
(\widehat{Q_n u - P_n u})(q) &= \frac{1}{2\pi}\sum_{p\in\mathbb{K}_n}\left[\sum_{j\in\mathbb{Z}\setminus\{0\}}\hat{u}(p+jn)\right]\begin{cases}2\pi, & \text{if } q = p,\\ 0, & \text{if } q \neq p\end{cases} \\
&= \begin{cases}\sum_{j\in\mathbb{Z}\setminus\{0\}}\hat{u}(q+jn), & \text{if } q \in \mathbb{K}_n,\\ 0, & \text{if } q \in \mathbb{Z}\setminus\mathbb{K}_n\end{cases}
\end{aligned}$$

which enables us to write

$$\|Q_n u - P_n u\|_{H^\lambda}^2 = \sum_{p\in\mathbb{K}_n}\underline{p}^{2\lambda}\left|\sum_{j\in\mathbb{Z}\setminus\{0\}}\hat{u}(p+jn)\right|^2. \tag{A.2}$$

Since $\{\exp(ikt) : k \in \mathbb{Z}\}$ is a set of orthogonal functions with respect to the L^2 inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt,$$

we have

$$\begin{aligned}
\|u - Q_n u\|_{H^\lambda}^2 &= \|\sum_{p\in\mathbb{Z}} \hat{u}(p)\exp(ipt) - \sum_{p\in\mathbb{K}_n} A_p\|_{H^\lambda}^2 \\
&= \|\sum_{p\in\mathbb{Z}\backslash\mathbb{K}_n} \hat{u}(p)\exp(ipt) + \sum_{p\in\mathbb{K}_n} \hat{u}(p)\exp(ipt) - \sum_{p\in\mathbb{K}_n} A_p\|_{H^\lambda}^2 \\
&= \|\sum_{p\in\mathbb{Z}\backslash\mathbb{K}_n} \hat{u}(p)\exp(ipt) + \sum_{p\in\mathbb{K}_n}\left[\hat{u}(p) - \sum_{j\in\mathbb{Z}} \hat{u}(p+nj)\right]\exp(ipt)\|_{H^\lambda}^2 \\
&= \|\sum_{p\in\mathbb{Z}\backslash\mathbb{K}_n} \hat{u}(p)\exp(ipt)\|_{H^\lambda}^2 + \|\sum_{p\in\mathbb{K}_n}\left[\hat{u}(p) - \sum_{j\in\mathbb{Z}} \hat{u}(p+nj)\right]\exp(ipt)\|_{H^\lambda}^2.
\end{aligned}$$

Using A.2, we can therefore conclude

$$\|u - Q_n u\|_{H^\lambda}^2 = \|u - P_n u\|_{H^\lambda}^2 + \|P_n u - Q_n u\|_{H^\lambda}^2.$$

From Theorem A.1, we have an estimation for the first summand. Thus, we need to estimate the second summand. To this end, from (A.2), we estimate

$$\begin{aligned}
\|Q_n u - P_n u\|_{H^\lambda}^2 &= \sum_{p\in\mathbb{K}_n} \underline{p}^{2\lambda}\left|\sum_{j\in\mathbb{Z}\backslash\{0\}} \hat{u}(p+jn)\right|^2 \\
&\leq \sum_{p\in\mathbb{K}_n}\left[\sum_{j\in\mathbb{Z}\backslash\{0\}} \frac{\underline{p}^\lambda}{|p+jn|^m}|p+jn|^m|\hat{u}(p+jn)|\right]^2 \qquad \text{(A.3)} \\
&= \sum_{p\in\mathbb{K}_n}\left[\sum_{j\in\mathbb{Z}\backslash\{0\}} \frac{\underline{p}^\lambda}{|p+jn|^m}|p+jn|^m|\hat{u}(p+jn)|\right]^2
\end{aligned}$$

and, from the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
\|Q_n u - P_n u\|_{H^\lambda}^2 &\leq \sum_{p\in\mathbb{K}_n}\left[\left(\sum_{j\in\mathbb{Z}\backslash\{0\}} \frac{\underline{p}^{2\lambda}}{|p+jn|^{2m}}\right)\left(\sum_{j\in\mathbb{Z}\backslash\{0\}} |p+jn|^{2m}|\hat{u}(p+jn)|^2\right)\right] \\
&\leq \epsilon_n^2\|u\|_{H^m}^2
\end{aligned}$$

where

$$\begin{aligned}
\epsilon_n^2 &= \max_{p\in\mathbb{K}_n} \sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{\underline{p}^{2\lambda}}{|p+jn|^{2m}} \\
&\leq \max_{p\in\mathbb{K}_n} \sum_{j\in\mathbb{Z}\setminus\{0\}} \left(\frac{n}{2}\right)^{2\lambda} |p+jn|^{-2m} \\
&= \left(\frac{n}{2}\right)^{2\lambda} \max_{p\in\mathbb{K}_n} \sum_{j\in\mathbb{Z}\setminus\{0\}} \left(\frac{n}{2}\right)^{-2m} \left|\frac{2p}{n}+2j\right|^{-2m} \\
&\leq \left(\frac{n}{2}\right)^{2(\lambda-m)} \left(\sum_{j=-\infty}^{-1} |1+2j|^{-2m} + \sum_{j=1}^{\infty} |2j|^{-2m}\right) \\
&= \left(\frac{n}{2}\right)^{2(\lambda-m)} \sum_{j=1}^{\infty} j^{-2m}.
\end{aligned}$$

Therefore,

$$\|Q_n u - P_n u\|_{H^\lambda}^2 \leq \left(\frac{n}{2}\right)^{2(\lambda-m)} \sum_{j=1}^{\infty} (\frac{1}{j})^{2m} \|u\|_{H^\mu}^2. \tag{A.4}$$

Theorem A.1 combined with A.4 implies

$$\begin{aligned}
\|u - Q_n u\|_{H^\lambda}^2 &= \|P_n u - Q_n u\|_{H^\lambda}^2 + \|u - P_n u\|_{H^\lambda}^2 \\
&\leq \left(\frac{n}{2}\right)^{2(\lambda-m)} \sum_{j=1}^{\infty} (\frac{1}{j})^{2m} \|u\|_{H^m}^2 + \left(\frac{n}{2}\right)^{2(\lambda-m)} \|u\|_{H^m}^2 \\
&= \left(\frac{n}{2}\right)^{2(\lambda-m)} \left(\sum_{j=1}^{\infty} (\frac{1}{j})^{2m} + 1\right) \|u\|_{H^m}^2 \\
&= \gamma_m^2 \left(\frac{n}{2}\right)^{2(\lambda-m)} \|u\|_{H^m}^2
\end{aligned}$$

and this completes the proof. □