

**THE PANDEMIC FUSION SYSTEM FOR
ENDOMORPHISM ALGEBRAS OF
 P -PERMUTATION MODULES**

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Andi Nika
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We certify that we have read this thesis and that in our opinion it is fully adequate,
in scope and in quality, as a thesis for the degree of Master of Science.

Laurence John Barker(Advisor)

Matthew Justin Karcher Gelvin

Semra Pamuk

Approved for the Graduate School of Engineering and Science:

Ezhan Kardeşan
Director of the Graduate School

ABSTRACT

THE PANDEMIC FUSION SYSTEM FOR ENDOMORPHISM ALGEBRAS OF P -PERMUTATION MODULES

Andi Nika

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Advisor: Laurence John Barker

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During the 1980's Puig developed a new approach to modular representation theory, introducing new p -local invariants and thereby extending Green's work on G -algebras. We investigate the Puig category, commenting on its local structure and then introduce a new notion, namely pandemic fusion, which extends the Puig's axioms globally on the G -algebra. Finally we give a sketch of the proof on the existence of some p -permutation $\mathbb{F}G$ -module realizing the minimal pandemic fusion system.

Keywords: pandemic fusion, G -algebra, p -permutation module.

ÖZET

P-PERMÜTASYON MODÜLLERİNİN ANDOMORFİ CEBİRLERİ İÇİN PANDEMİK FÜZYON SİSTEMİ

Andi Nika

Matematik, Yüksek Lisans

Tez Danışmanı: Laurence Barker

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Puig 80'li yıllarda yeni p -yerel değişmezler tanımlayarak modüler temsil teorisine yeni bir yaklaşım getirdi ve böylece Green'in G -cebirleri üzerindeki çalışmasını genelledi. Bu tezde Puig kategorisi üzerindeki yerel koşulları yorumlayarak Puig'in aksiomlarını G -cebirleri üzerinde global olarak genişleten pandemik füzyon kavramını tanımladık. Son olarak, minimal pandemik füzyon kavramı için bir p -permütasyon modülünün varlığının ispatının bir taslağını verdik.

Anahtar sözcükler: pandemik füzyon, G -cebiri, p -permütasyon modülü.

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Chapter 1

Introduction

In his 1981 paper [1] Puig introduced the notion of *pointed groups*, initializing a new approach to modular representation theory. Briefly, a G -algebra is an algebra upon which a finite group G acts via automorphisms. Given a G -algebra A , a pointed group on A is a pair (H, α) , where H is a subgroup of G and α is an equivalence class of primitive idempotents of A^H (the subalgebra of H -fixed points). These pairs admit a partial order relation, that of subgroups of finite groups. Following Puig, a pointed group (P, γ) is said to be *local* if γ does not lie inside the kernel of a certain map called the *Brauer map*.

In this thesis we will try to describe the local pointed groups on a special kind of G -algebras, namely the *endomorphism algebras of p -permutation $\mathbb{F}G$ -modules*, where \mathbb{F} is an algebraically closed field of prime characteristic p . One

of the reasons for focusing on these is that we can use Higman’s criterion and the Mackey decomposition formula to describe the local pointed groups on these G -algebras.

Furthermore, we will introduce a new notion which we believe can be developed further, namely the *pandemic fusion system*. Our motivation comes from [2], where Puig introduced local fusions. Given pointed groups (H, β) and (K, γ) on an interior G -algebra A , an A -fusion from (K, γ) to (H, β) is a group exomorphism $\Phi : K \rightarrow H$ for which any $\phi \in \Phi$ is a group monomorphism and there is an exomorphism $F_\phi : A_\gamma \rightarrow \text{Res}_\phi(A_\beta)$ of interior K -algebras fulfilling

$$\text{Res}_1^K(F_\gamma) = \text{Res}_1^H(F_\beta) \circ \text{Res}_1^K(F_\phi)$$

An interior G -algebra is a pair (A, ψ) , where A is an \mathbb{F} -algebra and $\psi : G \rightarrow A^\times$ is a group homomorphism from G to the group of units of A . At first sight the definition seems “heavy”. Intuitively, what the condition is telling us is to regard the maps F_γ , F_β and F_ϕ as \mathbb{F} -algebra homomorphisms. See Chapter 4 for the definitions of the notations and terminology here. We will give an equivalent definition of local fusions in order to understand their properties.

The local fusion category (or the Puig system) actually serves a useful purpose when it comes to fusion systems. For the fusion system $\mathcal{F}_P(G)$, where G is a finite group and P is a Sylow p -subgroup of G , the morphisms are those group homomorphisms $\psi : Q \rightarrow R$ between subgroups Q and R of P for which there is an element $x \in G$ such that $\psi(u) = x u$, for all $u \in Q$. Puig’s idea concerns an enlargement of this category.

For interior G -algebras we can consider every element of G as an element of A . Puig’s local fusions are actually the morphisms of the above-mentioned enlargement of the fusion system of G . These fusions are “local” in the sense that they use information from the local pointed groups on A .

Our pandemic fusion system gets rid of the locality constraint. We consider the case when A is the endomorphism algebra of some p -permutation $\mathbb{F}G$ -module. A p -permutation module M is an $\mathbb{F}G$ -module such that for every p -subgroup P

of G , there exists a P -invariant \mathbb{F} -basis of M . Moreover we define a refinement of this pandemic fusion system, namely the *G -pandemic category* and prove its existence in Theorem 6.4.1. We also show that this category is indeed larger than the fusion system of the group and it is contained in the local fusion category.

The thesis is organized in 6 chapters. In Chapter 2 we present general results of representation theory and idempotents which we will use later on.

Chapter 3 is focused on modules. We prove some known results such as the Mackey decomposition formula and other properties of modules. Then we discuss permutation modules and their basic properties. We end the chapter with Green's indecomposability theorem which we will use to prove Proposition 5.3.2 which describes the local pointed groups on the endomorphism algebra of a certain $\mathbb{F}G$ -module.

In Chapter 4 we define G -algebras and some of their basic properties. Then we focus on subalgebras of fixed elements and subsequently define the Brauer homomorphism using the relative trace map. We prove basic properties of the relative trace map and the trace map version of the Mackey decomposition formula. We define special types of exomorphisms called embeddings and prove some useful results with them. In the last section of this chapter we apply induction on interior G -algebras and construct such a structure.

In Chapter 5 we formally define pointed groups and exhibit their most prominent properties. After that we define defect groups and prove their existence. The last section of this chapter is devoted to Proposition 5.3.2.

The last chapter contains most of our original work. First, we define the fusion system of a group and then discuss the local fusion category, which we call the *Puig system*. Subsequently we introduce the pandemic fusion system and prove some basic results that follow from the definition. Then we focus on the pandemic fusion system on a Sylow p -subgroup and give an example (6.3.3) where we show the strict containment of the group fusion system in the pandemic fusion system.

In the last section we introduce the minimal pandemic fusion system and prove our main theorem (Theorem 6.4.1), which implies the existence of an $\mathbb{F}G$ -module realizing this category.



Chapter 2

Preliminaries

2.1 Foundations of representation theory

Throughout this thesis we will be working with finite-dimensional algebras over an algebraically closed field \mathbb{F} of prime characteristic p . The main results in this section refer to [3], [4] and [5]

Definition 2.1.1 *An \mathbb{F} -algebra A is a ring endowed with an \mathbb{F} -action where \mathbb{F} is embedded in the center of A . In other words, there is a ring homomorphism $\phi : \mathbb{F} \rightarrow A$ such that $\text{Im}(\phi) \subseteq Z(A)$.*

Example 2.1.2 Let G be a finite group. The group algebra defined as

$$\mathbb{F}G := \left\{ \sum_{g \in G} \alpha_g g : \alpha_g \in \mathbb{F}, g \in G \right\}$$

is an \mathbb{F} -algebra with $\dim_{\mathbb{F}} \mathbb{F}G = |G|$. It has an \mathbb{F} -basis consisting the elements of G .

Example 2.1.3 Let V be an $\mathbb{F}G$ -module. The \mathbb{F} -endomorphisms of V form an \mathbb{F} -algebra, $\text{End}_{\mathbb{F}}(V)$, with respect to addition and composition of module homomorphisms.

These two are the most prominent examples of what we will later define to be a G -algebra, which will be our primary focus. There is a rich theory behind these concepts that we will use to investigate these structures. In particular, there is a striking notion that makes understanding the nature of them a lot easier, namely *semisimplicity*. If we have an \mathbb{F} -algebra A , it would be very convenient if we could completely break it up into smaller algebras. Let us first define the notion of semisimplicity for modules.

Definition 2.1.4 Let A be an \mathbb{F} -algebra and let M be a non-zero A -module.

1. M is said to be a simple (or irreducible) A -module if there are no A -submodules of M other than $\{0\}$ and M .
2. M is said to be semisimple (or completely reducible) if it can be written as a direct sum of simple A -submodules.

Following [4] we give one of the equivalent definitions of semisimple algebras.

Definition 2.1.5 An algebra A is said to be semisimple if all (left) A -modules are semisimple.

Semisimple algebras inherit a much more richer structure than we have already mentioned. They not only can be broken up into smaller pieces, but the structure of these pieces is one that we are already familiar with. This is what our next theorem [3] will exhibit. For an algebra A , we define the regular module to be A regarded as an A -module and denote it by ${}_A A$.

Theorem 2.1.6 (*Artin-Wedderburn*) *Let A be a semisimple \mathbb{F} -algebra. Then A can be written as a direct sum of matrix algebras over \mathbb{F} . Specifically, if:*

$${}_A A \cong V_1^{n_1} \oplus \dots \oplus V_r^{n_r}$$

where $\{V_1, \dots, V_r\}$ is a complete list of non-isomorphic simple A -modules occurring with multiplicities n_1, \dots, n_r in the regular module ${}_A A$, then

$$A \cong M_{n_1}(\mathbb{F}) \oplus \dots \oplus M_{n_r}(\mathbb{F})$$

See [3] for a proof. We will use modules to better understand the structure of algebras and, as we mentioned before, we would like to work with semisimple objects, as they are algebraically “aesthetic”. Fortunately, semisimplicity in the case of group algebras has a very elegant criterion, providing us with plenty of examples.

Theorem 2.1.7 (*Maschke*) *Let G be a finite group with $|G|$ prime to p (the characteristic of \mathbb{F}). Then every finite dimensional $\mathbb{F}G$ -module is semisimple.*

See [3] for a proof. With such “power” in our hands, we can now get essential information about $\mathbb{F}G$ -modules. It suffices to describe the simple $\mathbb{F}G$ -modules and then the arbitrary ones will be just direct sums of these.

Remark 2.1.8 *The condition $S \cong A/I$ characterizes a bijective correspondence between the isomorphism classes of simple A -modules S and maximal ideals I of A .*

Proof: Given a maximal ideal I of A , then A/I can be regarded as a simple A -module. Conversely, if S is any given simple A -module and $x \in S \setminus \{0\}$, then the A -homomorphism $S \rightarrow A$ specified by $a \mapsto ax$ is surjective with kernel some maximal ideal I of A . **QED**

Definition 2.1.9 *An ideal I of A is said to be nilpotent if $I^n = 0$, for some positive integer n .*

Remark 2.1.10 *Let A be an algebra over \mathbb{F} . Then A contains a unique maximal nilpotent ideal called the Jacobson radical and denoted by $J(A)$.*

Sketch of proof: Let I_1 and I_2 be nilpotent ideals of A . Let n_1, n_2 be positive integers such that $I_1^{n_1} = I_2^{n_2} = 0$. We have:

$$(I_1 + I_2)^{n_1+n_2} \subseteq I_1^{n_1} + I_2^{n_2} = 0$$

The result follows. **QED**

2.2 Idempotents and points

An efficient tool for breaking up algebras and understanding their structure is the systematic use of idempotents, so it is important to understand their nature and behavior. We follow [5].

Definition 2.2.1 *An element e of A is said to be an idempotent if $e^2 = e$. We denote the set of idempotents of A by $\text{Ipot}(A)$.*

Two examples of idempotents in an algebra are 1 and 0. Two idempotents e and f are called *orthogonal* if $ef = fe = 0$. In particular, any idempotent e is orthogonal to $1 - e$.

Definition 2.2.2 A non-zero idempotent e in A is said to be *primitive* if it cannot be written as the sum of two non-zero orthogonal idempotents in A .

We can decompose any idempotent of A as a finite sum of mutually orthogonal idempotents. This decomposition is said to be *primitive* if the decomposition summands are also primitive. In particular, if $e = \sum_{i \in I} e_i$ where I is a finite set of pairwise orthogonal primitive idempotents, then for every $e_i \in I$, $ee_i = e_i e = e_i$. Consequently, $e_i = ee_i e$. Conversely, if f is an idempotent such that $f = efe$, then f appears in some decomposition of e , since $e = f + (e - f)$ is an orthogonal decomposition. Moreover, we have $1_A = \sum_{j \in J} e_j$ and if the decomposition is primitive, then it is unique up to conjugation. In this case A can be decomposed as follows as a direct sum of ideals:

$$A = \bigoplus_{j \in J} Ae_j$$

Furthermore, $e \in \text{Ipot}(A)$ is primitive if and only if Ae is indecomposable as an ideal of A .

There is an interesting relation between equivalence classes of idempotents and isomorphism classes of A -modules. This is why it is suitable to organize idempotents into classes as follows. We let A^\times denote the group of units of A .

Definition 2.2.3 Given $e, f \in \text{Ipot}(A)$, we say e is *associate* to f and denote by $e \sim f$, if there are elements x and y in A such that $e = xy$ and $f = yx$.

Remark 2.2.4 Being associate is an equivalence relation in the set of idempotents in A .

Proof: Let $a := exf \in eAf$ and $b := fye \in fAe$ [5]. We have: $ab = exfy e = exyxe = e$ and similarly, $ba = f$. Thus, if g is another idempotent of A such that $f \sim g$, then there are elements $c \in fAg$ and $d \in gAf$ such that $cd = f$ and $dc = g$. Hence $acdb = afb = ab = e$ and $dbacdfc = dc = g$. **QED**

Definition 2.2.5 *The equivalence classes of primitive idempotents in A are called the points of A . We denote the set of points of A by $\mathcal{P}(A)$.*

We usually denote points of A by greek letters (α, β, \dots) . The notion of a point goes back to [1]; throughout this project they will play a fundamental role. The next theorem summarizes the importance of these conjugacy classes of primitive idempotents.

Theorem 2.2.6 *Let A be an \mathbb{F} -algebra. There is a bijective correspondence $(\alpha \leftrightarrow \mathfrak{m}_\alpha \leftrightarrow V(\alpha))$ between the points α of $\mathcal{P}(A)$, the maximal ideals \mathfrak{m}_α of A and the isomorphism classes of simple A -modules $V(\alpha)$, such that $e \notin \mathfrak{m}_\alpha$, for any $e \in \alpha$ and $eV(\alpha) \neq 0$, for every $e \in \alpha$. Moreover, $V(\alpha) \cong Ae/J(A)e$, for $e \in \alpha$.*

Refer to [6] for a proof. Now if $\phi : A \rightarrow B$ is an \mathbb{F} -algebra homomorphism (by that we mean that ϕ is \mathbb{F} -linear and that $\phi(ab) = \phi(a)\phi(b)$, for any $a, b \in A$), clearly $\phi(1_A) \in \text{Ipot}(B)$, but in general we do not require $\phi(1_A) = 1_B$ in which case the homomorphism is called *unitary*. If e is primitive in A , then $\phi(e)$ may not be primitive in B .

Definition 2.2.7 *An algebra A is said to be primitive if it contains only two idempotents.*

An example of a primitive algebra is eAe , where A is an algebra over \mathbb{F} and e is a primitive idempotent of A . There is a very elegant and important result which is useful when working with idempotents.

Lemma 2.2.8 (Rosenberg) *Let A be an \mathbb{F} -algebra, $e \in \text{Ipot}(A)$ be primitive and $\mathcal{I} = \{I_1, \dots, I_n\}$ be a family of ideals of A . If $e \in \sum_{i=1}^n I_i$, then $e \in I_j$, for some $j \in \{1, \dots, n\}$.*

Proof: Clearly eI_1e, \dots, eI_ne are ideals of eAe such that:

$$e = e^3 \in eI_1e + \dots + eI_ne$$

Now $e \notin J(eAe)$, so $eI_je \not\subseteq J(eAe)$ for some $j \in \{1, \dots, n\}$. Since eAe is a primitive algebra over \mathbb{F} , then eI_je contains a unit of eAe by Fitting's lemma (see page 25 of [5]). Thus, $e \in eAe = eI_je \subseteq I_j$, and we are done. **QED**

Above we saw how we could decompose 1_A as a finite sum of primitive idempotents. Let J be such a decomposition. Now, for every $\alpha \in \mathcal{P}(A)$, we can consider the intersection $J \cap \alpha$ which would be finite at most. So we can express our decomposition of 1_A as:

$$1_A = \sum_{\alpha \in \mathcal{P}(A)} \sum_{i=1}^{m_\alpha} e_i$$

where m_α is the number of occurrences of α in J and each $e_i \in \mathcal{P}(A)$. We call m_α the *multiplicity* of the point α . For the next result we refer to [5].

Theorem 2.2.9 *Let e and f be associate idempotents in A and write*

$$e = \sum_{\alpha \in \mathcal{P}(A)} \sum_{i=1}^{k_\alpha} e_i^\alpha, \quad f = \sum_{\beta \in \mathcal{P}(A)} \sum_{j=1}^{l_\beta} f_j^\beta$$

with pairwise orthogonal primitive idempotents e_i^α and f_j^β in A where $e_i^\alpha \in \alpha$ and $f_j^\beta \in \beta$, for $\alpha, \beta \in \mathcal{P}(A)$, $i \in \{1, \dots, k_\alpha\}$, $j \in \{1, \dots, l_\beta\}$. Then $k_\alpha = l_\beta$ for $\alpha \in \mathcal{P}(A)$ and there is a unit $u \in A^\times$ such that ${}^u e_i^\alpha = f_i^\alpha$.

Corollary 2.2.10 *Two idempotents e and f in A are associate if and only if $f = {}^u e$ for a unit u in A .*

Proof: Suppose $e \sim f$. Then, by Theorem 2.2.9 we have $f = {}^u e$ for a unit $u \in A^\times$. Conversely, if e and f are two idempotents in A for which there exists some unit $u \in A^\times$ satisfying $f = {}^u e$, then $e \sim f$ in A , since $(eu^{-1})u = e$. **QED**

As a consequence of Corollary 2.2.10, the points of A are actually the conjugacy classes of primitive idempotents of A . This is the way Puig originally defined

points [1]. From now on, we will regard points both as equivalence classes and conjugacy classes of primitive idempotents in relation to the context.



Chapter 3

Modules

3.1 Miscellaneous notions and results

An FG-module is defined as an \mathbb{F} -module endowed with a G -action. Every module can be decomposed into indecomposable pieces and this decomposition is unique. This is the content of the theorem below. See [6] for a proof. As mentioned in the last chapter, A is a finite-dimensional algebra over \mathbb{F} . We refer to [3], [5] and [7].

Theorem 3.1.1 (*Krull-Schmidt*) *Let A be an \mathbb{F} -algebra and M be an A -module. Then M can be written as a direct sum of finitely many indecomposable A -modules. Furthermore, this decomposition is unique, that is, if $M \cong \bigoplus_{i \in \Omega} M_i$ and $M \cong$*

$\bigoplus_{j \in \Delta} N_j$ are two decompositions of M , then there exists a bijection $\sigma : \Omega \rightarrow \Delta$ such that $M_i \cong N_{\sigma(j)}$, for each $i \in \Omega$.

Our focus in this project will be on a specific type of module, namely *p-permutation modules*, but we will talk about them later. First, let us recall some common concepts.

Let U and V be two A -modules. We define $\text{Hom}_A(U, V)$ to be the \mathbb{F} -vector space of all A -module homomorphisms from U to V . In particular, we write $\text{End}_A(V)$ for the *endomorphism algebra* of all A -endomorphisms of V over \mathbb{F} . Now, if U is an $\mathbb{F}G$ -module and $H \leq G$, we define the *restriction* of U to H to be an $\mathbb{F}H$ -module, thus forgetting about the whole G -action, and denote it by $\text{Res}_H^G(U)$. More generally, if $\phi : A \rightarrow B$ is an algebra homomorphism, and U is a B -module, then $\text{Res}_\phi(U) := \phi(1_A)U$ is a B -module.

There is a dual concept of restriction called induction. If U is an $\mathbb{F}H$ -module, where $H \leq G$, we define the *induced* module from H to G as $\text{Ind}_H^G(U) := \mathbb{F}G \otimes_{\mathbb{F}H} U$ where the G -action comes from the $\mathbb{F}G$ -module structure:

$$g \left(\sum_{g \in G} \alpha_g g \otimes V \right) = \left(g \sum_{g \in G} \alpha_g g \right) \otimes V$$

Using these two powerful concepts, we can manipulate modules and better understand their structure. We may also regard this induced module in a different way.

Proposition 3.1.2 *Let $H \leq G$ and U be an $\mathbb{F}H$ -module. Then:*

$$\text{Ind}_H^G(U) = \bigoplus_{i=1}^{|G:H|} g_i \otimes U$$

where g_i are representatives of left cosets of H in G and $g_i \otimes U = \{g_i \otimes u : u \in U\} \subseteq \mathbb{F}G_{\mathbb{F}H} \otimes U$. We have $g_i \otimes U \cong U$ as \mathbb{F} -modules and if U is free, then

$$\dim_{\mathbb{F}} \text{Ind}_H^G(U) = |G : H| \dim_{\mathbb{F}} U$$

Moreover, the subspaces $g_i \otimes U$ are permuted by the action of G which is transitive and if $g_i \in H$, then $\text{Stab}_G(g_i \otimes U) = H$.

Proof: We have $\mathbb{F}G_{\mathbb{F}H} \cong \bigoplus_{i=1}^{|G:H|} g_i \mathbb{F}H \cong \mathbb{F}H^{|G:H|}$. This is because $(\sum_{g \in G} g)H = \bigoplus_{i=1}^{|G:H|} g_i H$ and each $g_i \mathbb{F}H$ is an $\mathbb{F}H$ -submodule of $\mathbb{F}G$, isomorphic to $\mathbb{F}H_{\mathbb{F}H}$ (the regular module) via the isomorphism $g_i h \mapsto h$, for all $h \in H$. So we obtain:

$$\mathbb{F}G \otimes_{\mathbb{F}H} U = \left(\bigoplus_{i=1}^{|G:H|} g_i \mathbb{F}H \right) \otimes_{\mathbb{F}H} U = \bigoplus_{i=1}^{|G:H|} (g_i \mathbb{F}H \otimes_{\mathbb{F}H} U) = \bigoplus_{i=1}^{|G:H|} g_i \otimes U$$

Moreover, $g_i \mathbb{F}H \otimes_{\mathbb{F}H} U \cong \mathbb{F}H \otimes_{\mathbb{F}H} U \cong U$ as \mathbb{F} -modules. Next we show that the G permutes these \mathbb{F} -submodules. We use the fact that if $x \in G$, we have $xg_i = g_j h$, for some $h \in H$. So, for $u \in U$, we have $x(g_i \otimes u) = xg_i \otimes u = g_j h \otimes u = g_j \otimes hu$, therefore $x(g_i \otimes U) \subseteq g_j \otimes U$. Similarly, $x^{-1}g_j \otimes U \subseteq g_i \otimes U$, so $g_j \otimes U = xx^{-1}(g_j \otimes U) \subseteq x(g_i \otimes U)$. The transitivity holds since given two subspaces $g_i \otimes U$ and $g_j \otimes U$, we have $(g_j g_i^{-1})g_i U = g_j U$. Now for the stabilizer of $g_i \otimes U$ where $g_i \in H$ we have two cases. If $x \in H$, then $x(g_i \otimes U) = g_i(g_i^{-1}xg_i) \otimes U = g_i \otimes U$. If $x \notin H$, then $x \in g_j H$, for some $j \neq i$. So $x(g_i \otimes U) = g_i \otimes U$. Therefore $\text{Stab}_G(g_i \otimes U) = H$. **QED**

Proposition 3.1.3 *Let M be an $\mathbb{F}G$ -module and let N be an \mathbb{F} -submodule of M such that $M \cong \bigoplus_{g \in G} gN$. Then, if $H = \text{Stab}_G(N)$, we have $M \cong \text{Ind}_H^G(N)$.*

Proof: Define the \mathbb{F} -module homomorphism

$$\begin{aligned} \text{Ind}_H^G(N) = \mathbb{F}G \otimes_{\mathbb{F}H} N &\longrightarrow M \\ g \otimes n &\longmapsto gn \end{aligned}$$

It is in fact an $\mathbb{F}G$ -module homomorphism. We have $H = \text{Stab}_G(N)$ and G permutes the \mathbb{F} -submodules gN of M transitively, therefore these submodules correspond bijectively with the cosets of H in G . Thus, we have:

$$\mathbb{F}G \otimes_{\mathbb{F}H} N \cong \bigoplus_{i=1}^{|G:H|} g_i \otimes N \quad \text{and} \quad M \cong \bigoplus_{i=1}^{|G:H|} gN$$

Note that on each summand the map $g \otimes n \mapsto gn$ is an isomorphism. The result follows. **QED**

Example 3.1.4 Let \mathbb{F}_G denote the trivial $\mathbb{F}G$ -module, meaning that the G -action on it is trivial. We observe that

$$\text{Ind}_1^G(\mathbb{F}_G) \cong \mathbb{F}G \otimes_{\mathbb{F}} \mathbb{F}_G \cong \mathbb{F}G_{\mathbb{F}G}$$

so the regular module is induced from the trivial group.

There is an interesting result concerning the trivial $\mathbb{F}G$ -modules when G is a p -group.

Proposition 3.1.5 Let P be a p -group. Then \mathbb{F}_P is the only simple $\mathbb{F}P$ -module up to isomorphism.

Proof: Let V be a simple $\mathbb{F}P$ -module and let $0 \neq v \in V$. Define U as the \mathbb{F}_P -vector subspace of V generated by elements of the form gv , for $g \in P$. Then U is invariant under the action of P by construction. We can decompose U as a disjoint union of orbits. The orbit of an element $\{u\}$ is trivial if $u \in U^P$. So U^P is the union of all the orbits with one element. If the orbit of u is not trivial, then the stabilizer Q of u is a proper subgroup of P and this orbit has $|P : Q|$ elements. Since P is a p -group, U is a disjoint union of U^P and orbits of cardinality divisible by p . We have $p \mid |U|$ since U is a vector space over \mathbb{F}_P , therefore $p \mid |U^P|$. Since $0 \in U^P$, then U^P must contain at least one other element v . The one-dimensional \mathbb{F} -subspace generated by v is an $\mathbb{F}P$ -submodule of V , hence the whole V , since V is simple. Therefore V is one-dimensional and is isomorphic to the trivial $\mathbb{F}P$ -module. **QED**

Now we introduce certain properties relating induction and restriction, which are very useful when working with modules.

Proposition 3.1.6 Let $H \leq K \leq G$ and let U be an $\mathbb{F}H$ -module and V be an $\mathbb{F}G$ -module. We have:

1. (Transitivity of induction): There is an isomorphism of $\mathbb{F}G$ -modules

$$\text{Ind}_K^G(\text{Ind}_H^K(U)) \cong \text{Ind}_H^G(U)$$

2. (Transitivity of restriction): There is an isomorphism of $\mathbb{F}H$ -modules

$$\text{Res}_H^K(\text{Res}_K^G(V)) \cong \text{Res}_H^G(V)$$

3. (Frobenius reciprocity):

$$\text{Hom}_{\mathbb{F}G}(\text{Ind}_H^G(U), V) \cong \text{Hom}_{\mathbb{F}H}(U, \text{Res}_H^G(V))$$

Proof: We have $\text{Ind}_K^G(\text{Ind}_H^K(U)) = \mathbb{F}G \otimes_{\mathbb{F}K} \mathbb{F}K \otimes_{\mathbb{F}H} U$ and $\text{Ind}_H^G(U) = \mathbb{F}G \otimes_{\mathbb{F}H} U$. The map:

$$\begin{aligned} f : \text{Ind}_K^G(\text{Ind}_H^K(U)) &\longrightarrow \text{Ind}_H^G(U) \\ g \otimes k \otimes u &\longmapsto gk \otimes u \end{aligned}$$

has inverse $f^{-1} : g \otimes u \mapsto g \otimes 1 \otimes u$, where $k \in \mathbb{F}K$, $g \in \mathbb{F}G$ and $u \in U$, so part 1 holds. Part 2 is easy.

Now for the third part, given $f : \mathbb{F}G \otimes_{\mathbb{F}H} U \rightarrow V$, we define $f' : U \rightarrow \text{Res}_H^G(V)$ by $f'(u) = f(1 \otimes u)$. Given $h' : U \rightarrow \text{Res}_H^G(V)$, we define $h : \mathbb{F}G \otimes_{\mathbb{F}H} U \rightarrow V$, by $h(g \otimes u) = gh'(u)$. It is easy to check that the maps $f \mapsto f'$ and $h' \mapsto h$ are mutual inverses. **QED**

There is another elegant property relating induction and restriction due to Mackey, which will be very useful in our calculations. Before introducing it, let us recall the notion of double cosets and prove a rather technical result about them.

Let $H \leq G \geq K$ and let $x \in G$. A *double coset* of H and K in G is a set $HxK = \{h x k : h \in H, k \in K\}$. We denote the set of double cosets of H and K in G by $H \backslash G / K$.

Remark 3.1.7 *Any two double cosets are either disjoint or equal.*

We refer to [7] for the next result.

Proposition 3.1.8 *Let $H \leq G \geq K$ and let $x \in G$. Then, we have:*

$$|HxK| = \frac{|H| |K|}{|H \cap {}^xK|} = \frac{|H| |K|}{|H^x \cap K|}$$

Proof: We can consider G/K as a left H -set, so HxK will be the union of those cosets of K lying in the orbit of xK in G/K . Now since $\text{Stab}_H(xK) = H \cap {}^xK$, we obtain

$$|H| |K| = |HxK| |H \cap {}^xK|$$

which follows from the orbit-stabilizer theorem. For the second equality we have:

$$|HxK| = |H^xK| = \frac{|H| |{}^xK|}{|H \cap {}^xK|} = \frac{|H| |K|}{|(H \cap {}^xK)^x|} = \frac{|H| |K|}{|H^x \cap K|}$$

and we are done. **QED**

As we will see, the number of double cosets will be very useful when we want to find the multiplicity of certain indecomposable summands of some modules by using the next result.

Theorem 3.1.9 (*Mackey decomposition formula*) *Let $H \leq G \geq K$ and let V be an $\mathbb{F}K$ -module. Then*

$$\text{Res}_H^G(\text{Ind}_K^G(V)) \cong \bigoplus_{HgK \subseteq [H \backslash G / K]} \text{Ind}_{H \cap {}^gK}^H({}^g(\text{Res}_{H^g \cap K}^K(V)))$$

as $\mathbb{F}H$ -modules.

Proof: We can write $\text{Ind}_K^G(V) \cong \bigoplus_{k \in [G/K]} k \otimes V$. Now let $g \in G$. As we saw in Proposition 3.1.8, the double coset HgK is a union of those cosets of K lying in

the H -orbit of gK . As k runs over the elements of HgK , the subspaces $k \otimes V$ are transitively permuted by H . Now $\text{Stab}_G(1 \otimes V) = K$, since V itself is K -stable. So

$$\text{Stab}_H(g \otimes V) = H \cap {}^gK$$

Now, by Proposition 3.1.3 we obtain

$$\text{Res}_H^G(\text{Ind}_K^G(V)) \cong \bigoplus_{HgK \subseteq [H \backslash G / K]} \text{Ind}_{H \cap {}^gK}^H(g \otimes V)$$

Plainly, $g \otimes V \cong {}^g(\text{Res}_{H \cap {}^gK}^K(V))$.

QED

3.2 Permutation modules

We now discuss objects with remarkable properties, which are fundamental to our work. We follow [6] and [8] for the definitions and basic properties of them.

Definition 3.2.1 *An $\mathbb{F}G$ -module is said to be a permutation $\mathbb{F}G$ -module if it has a G -invariant \mathbb{F} -basis.*

Let M be such a module and let Ω be an \mathbb{F} -basis for M . We can obviously look at Ω as a G -set. If we decompose Ω as a disjoint union of G -orbits, we obtain a direct sum decomposition of $\mathbb{F}\Omega$ as an $\mathbb{F}G$ -module. So without loss of generality, we may assume Ω to be a transitive G -set, in which case we would have $\mathbb{F}\Omega \cong \text{Ind}_H^G(\mathbb{F}_H)$, where H is the stabilizer of some $y \in G$ and \mathbb{F}_H is the trivial $\mathbb{F}H$ -module. Therefore, a permutation $\mathbb{F}G$ -module is isomorphic to a direct sum of modules of the form $\text{Ind}_H^G(\mathbb{F}_H)$, for various $H \leq G$. Conversely, $\text{Ind}_H^G(\mathbb{F}_H)$ is a permutation $\mathbb{F}G$ -module with invariant basis $\{g \otimes 1_{\mathbb{F}} \mid g \in [G/H]\}$.

Now we introduce a more general notion.

Definition 3.2.2 Let M be an $\mathbb{F}G$ -module. M is said to be a p -permutation $\mathbb{F}G$ -module if it has a P -invariant \mathbb{F} -basis, for every p -subgroup P of G .

Remark 3.2.3 Let M be an $\mathbb{F}G$ -module and let S be a Sylow p -subgroup of G . M is a p -permutation $\mathbb{F}G$ -module if and only if $\text{Res}_S^G(M)$ is a permutation $\mathbb{F}S$ -module.

Proposition 3.2.4 Let $H \leq G$, M and M' be two p -permutation $\mathbb{F}G$ -modules and N be a p -permutation $\mathbb{F}H$ -module. We have:

1. $M \oplus M'$ and $M \otimes M'$ are p -permutation $\mathbb{F}G$ -modules.
2. $\text{Res}_H^G(M)$ is a p -permutation $\mathbb{F}H$ -module and $\text{Ind}_H^G(N)$ is a p -permutation $\mathbb{F}G$ -module.
3. Any direct summand of a p -permutation $\mathbb{F}G$ -module is a p -permutation $\mathbb{F}G$ -module.

Proof: 1. Let Ω_1 and Ω_2 be \mathbb{F} -bases for M and M' respectively. Then $\Omega_1 \oplus \Omega_2$ and $\Omega_1 \otimes \Omega_2$ will be \mathbb{F} -bases for $M \oplus M'$ and $M \otimes M'$ respectively which are obviously G -stable and the result follows.

2. We write $\text{Ind}_H^G(N) \cong \bigoplus_{g \in [G/H]} g \otimes N$ and since N is a p -permutation $\mathbb{F}H$ -module, it has a P -stable basis Δ for all p -subgroup $P \leq H$, therefore $\text{Ind}_H^G(N)$ is a p -permutation $\mathbb{F}G$ -module. Trivially, $\text{Res}_H^G(M)$ is also a p -permutation $\mathbb{F}H$ -module.

3. In order to prove this result we need to use the next theorem, the proof [6] of which we omit due to excessive technicality.

Theorem 3.2.5 (Green's indecomposability) Let P be a finite p -group. Then the p -permutation $\mathbb{F}P$ -modules coincide with the permutation $\mathbb{F}P$ -modules and the indecomposable permutation $\mathbb{F}P$ -modules are precisely the $\mathbb{F}P$ -modules isomorphic to $\text{Ind}_Q^P(\mathbb{F}_Q)$ where $Q \leq P$.

Now let us prove the third assertion of the proposition. Let S be a Sylow p -subgroup of G . Since $\text{Res}_S^G(M)$ is a permutation $\mathbb{F}G$ -module by Remark 3.2.3 we can write

$$\text{Res}_S^G(M) \cong \bigoplus_i \text{Ind}_{Q_i}^S(\mathbb{F}_{Q_i})$$

By Theorem 3.2.5 these composition factors are indecomposable, and by Krull-Schmidt theorem, any direct summand of M is a direct sum of these factors. The result follows. **QED**

Chapter 4

G -algebras

4.1 Introduction

Now it is time to bring in our most important objects. For the results of this chapter we refer to [6] and [5].

Definition 4.1.1 *Let G be a finite group. A G -algebra A is an \mathbb{F} -algebra endowed with a G -action, that is to say there is a group homomorphism $\phi : G \rightarrow \text{Aut}(A)$.*

It was J. Green who first introduced this notion [9] back in 1968. What we are actually interested in is a refinement of G -algebras due to L. Puig, which turns out to be more interesting.

Definition 4.1.2 We define an interior G -algebra to be a pair (A, ϕ) , where A is a G -algebra and $\phi : G \rightarrow A^\times$ is a group homomorphism.

An interior G -algebra gives rise to a G -algebra since $A^\times \rightarrow \text{Aut}(A)$, $a \mapsto \text{Inn}(a)$ is a group homomorphism. So G acts on A via the interior automorphism $\text{Inn}(a)$, hence the term *interior*. For simplicity reasons, we omit ϕ and write:

$${}^g a := \phi(g)a$$

Example 4.1.3 As we mentioned before (example 1.1.2), the group algebra $\mathbb{F}G$ is actually a G -algebra. Moreover, the canonical inclusion $G \rightarrow \mathbb{F}G^\times$ implies that $\mathbb{F}G$ is actually an interior G -algebra.

Example 4.1.4 Let M be a finitely-generated $\mathbb{F}G$ -module and let $A = \text{End}_{\mathbb{F}}(M)$. Then, since M can also be regarded as a representation of G by $\rho : G \rightarrow \text{Aut}_{\mathbb{F}}(M) = A^\times$, we can deduce that $\text{End}_{\mathbb{F}}(M)$ is an interior G -algebra.

We shall usually work with specific modules such as $\mathbb{F}G$ -lattices. We define an $\mathbb{F}G$ -lattice to be an $\mathbb{F}G$ -module which is free as an \mathbb{F} -module. In this case $\text{End}_{\mathbb{F}}(M) = \text{Mat}(n, \mathbb{F})$, where $n = \dim_{\mathbb{F}} M$.

4.2 Subalgebras of fixed elements and the Brauer homomorphism

If A is a G -algebra, we define a G -subalgebra of A to be a subalgebra B of A such that ${}^g b \in B$ for all $b \in B$, $g \in G$. One of the most important examples of subalgebras of A is that of the H -fixed elements for some subgroup H of G .

Let G be a finite group and $H \leq G$. It is usually important to consider the H -fixed elements whenever we are working with a G -action on some object. If A is a G -algebra, we denote the set of H -fixed elements by:

$$A^H := \{a \in A : {}^h a = a, \text{ for all } h \in H\}$$

Example 4.2.1 *If $A = \text{End}_{\mathbb{F}}(M)$ for some $\mathbb{F}G$ -lattice M and $H \leq G$, then $A^H = \text{End}_{\mathbb{F}H}(M)$. Indeed, an endomorphism $\phi \in A$ is H -fixed if and only if it commutes with every $h \in H$, that is to say ϕ is an $\mathbb{F}H$ -linear endomorphism of M .*

Note that for $a \in A^H$, ${}^{ghg^{-1}}(a) = a$. In general, $A^{gHg^{-1}} = {}^g(A^H)$, from which it follows that if $g \in N_G(H)$, then ${}^g(A^H) = A^H$, so we can regard A^H as an $N_G(H)$ -algebra over \mathbb{F} , or better still, an $N_G(H)/H$ -algebra over \mathbb{F} , since H acts trivially on A^H .

For some subgroup K of H , clearly $A^H \subseteq A^K$. Moreover there is a mapping from A^K to A^H , namely the *relative trace map* defined as

$$\text{tr}_K^H : A^K \rightarrow A^H, \quad \text{tr}_K^H(a) = \sum_{h \in [H/K]} {}^h a$$

where $[H/K]$ is a set of representatives of left cosets of K in H and $a \in A^K$. Since h runs over all left cosets of K , the choice of h doesn't really affect ${}^h a$, so we obtain a well-defined map. The following result exhibits basic properties of the relative trace map.

Proposition 4.2.2 *Let L, K and H be subgroups of G such that $L \leq K \leq H$. Then we have:*

1. $\text{tr}_K^H(a) = |H : K| a$, for $a \in A^H$
2. $\text{tr}_K^H(\text{tr}_L^K(a)) = \text{tr}_L^H(a)$, for $a \in A^L$
3. ${}^g(\text{tr}_K^H(a)) = \text{tr}_K^{{}^g H}({}^g a)$, for $a \in A^K$

4. $\text{tr}_K^H(ab) = a \text{tr}_K^H(b)$ and $\text{tr}_K^H(ba) = \text{tr}_K^H(b)a$, for $a \in A^H$, $b \in A^K$

Proof:

1. Since ${}^h a = a$, for all $h \in H$ we have:

$$\text{tr}_K^H(a) = \sum_{h \in [H/K]} {}^h a = \sum_{h \in [H/K]} a = |H : K| a$$

2. Let h run over representatives of left cosets of K in H and h' over those of L in K . Then for $a \in A^L$ we have:

$$\begin{aligned} \text{tr}_K^H(\text{tr}_L^K(a)) &= \sum_{h \in [H/K]} ({}^h (\sum_{h' \in [K/L]} {}^{h'} a)) \\ &= \sum_{h \in [H/K]} \sum_{h' \in [K/L]} {}^{hh'} a = \sum_{f \in [H/L]} f a = \text{tr}_L^H(a) \end{aligned}$$

3. Consider the cosets ${}^g(hK)$ of ${}^g K$ in ${}^g H$. Since ${}^g a \in A^{gK}$ we have:

$$\text{tr}_K^{gH}({}^g a) = \sum_{h \in [H/K]} {}^{gh}({}^g a) = {}^g \left(\sum_{h \in [H/K]} ({}^h a) \right) = {}^g(\text{tr}_K^H(a))$$

4. For $a \in A^H$ and $b \in A^K$ we have:

$$\text{tr}_K^H(ab) = \sum_{h \in [H/K]} {}^h(ab) = \sum_{h \in [H/K]} {}^h a {}^h b = a \sum_{h \in [H/K]} {}^h b = a \text{tr}_K^H(b)$$

We argue similarly for the other equation. **QED**

Now we present the *relative trace* map version of the Mackey decomposition formula.

Proposition 4.2.3 (*Mackey*) *Let K and L be two subgroups of H and let $a \in A^K$. Then:*

$$\text{tr}_K^H(a) = \sum_{LhK \subseteq [L \backslash H / K]} \text{tr}_{L \cap {}^h K}^L({}^h a)$$

where $[L \backslash H / K]$ represents the set of double cosets $KhL \subseteq H$.

Proof: We can decompose H/K into L -orbits:

$$H/K = \bigcup_{h \in [L \backslash H/K]} L(hK)$$

Note that for any $x \in L \cap {}^h K$, we have $xhK \in hK$, so $L \cap {}^h K$ is the stabilizer of hK in H/K . Thus:

$$\mathrm{tr}_K^H(a) = \sum_{h \in [L \backslash H/K]} \sum_{g \in [L/L \cap {}^h K]} {}^g h a = \sum_{h \in [L \backslash H/K]} \mathrm{tr}_{L \cap {}^h K}^L({}^h a)$$

QED

Observe that the image of A^K in A^H under the trace map is a two-sided ideal of A^H by the fourth property of Proposition 4.2.2. We denote that ideal by $A_K^H := \mathrm{tr}_K^H(A^K)$. Moreover we denote the sum of these ideals for each proper subgroup of H by $A_{<H}^H := \sum_{K < H} \mathrm{tr}_K^H(A^K)$ which is also an ideal in A^H and for $g \in G$ we have:

$$\begin{aligned} {}^g(A_{<H}^H) &= \sum_{K < H} {}^g(\mathrm{tr}_K^H(A^K)) = \sum_{K < H} \mathrm{tr}_{gK}^{gH}({}^g(A^K)) \\ &= \sum_{K < H} \mathrm{tr}_{gK}^{gH}(A^{gK}) = \sum_{K < H} A_{gK}^{gH} = A_{<gH}^{gH} \end{aligned}$$

which means that $A_{<H}^H$ inherits the structure of an $N_G(H)/H$ -ideal of the $N_G(H)/H$ -algebra A^H over \mathbb{F} . Hence $A(H) := A^H/A_{<H}^H$ is obviously an $N_G(H)/H$ -algebra over \mathbb{F} . For a subgroup H of G , we define

$$\mathrm{br}_H : A^H \longrightarrow A(H), \quad a \mapsto a + A_{<H}^H$$

to be the *Brauer homomorphism* on A with respect to H .

Remark 4.2.4 *Let H be a subgroup of G . If H is not a p -group, then $A(H) = 0$.*

Proof: Let $K \in \mathrm{Syl}_p(H)$. Then for $a \in A^H$, $a = \mathrm{tr}_K^H(|H : K|^{-1}a)$, by the first property of Proposition 4.2.2, so $A^H = A_K^H \Rightarrow A^H = A_{<H}^H$, hence $A(H) = 0$.

QED

In general, if H is a p -group, $A_K^H \subsetneq A^H$, since $|H : K| \equiv 0 \pmod{p}$, for every $K < H$, meaning that $A(H) \neq \emptyset$.

4.3 Embeddings in relation to direct summands of modules

We will now consider homomorphisms of G -algebras and certain equivalence classes of them which exhibit important features.

Definition 4.3.1 *Let A and B be two G -algebras. A G -algebra homomorphism $\phi : A \rightarrow B$ is an algebra homomorphism satisfying ${}^g\phi(a) = \phi({}^ga)$, for all $g \in G, a \in A$.*

Now if $\phi : A \rightarrow B$ is a homomorphism of G -algebras it is convenient to consider the composition of ϕ with some inner automorphism of A as equivalent to ϕ . This can be easily seen to be an equivalence class. We define an *exomorphism* from A to B to be such an equivalence class. If $a \in A^\times$ we write $\text{Inn}(a)(x) = {}^ax$. In general, if $a \in A^\times$, it does not mean that $\phi(a)$ will also be a unit in B . In order to obtain a unit we add $1_B - \phi(1_A)$ to $\phi(a)$. Hence $\phi(a) + 1_B - \phi(1_A)$ is indeed a unit in B with $\phi(a^{-1}) + 1_B - \phi(1_A)$ as its inverse. Now we have:

$$\phi \cdot \text{Inn}(a) = \text{Inn}(\phi(a) + 1_B - \phi(1_A)) \cdot \phi$$

We denote by Φ the exomorphism containing ϕ . We can describe this exomorphism to be $\Phi = \{\text{Inn}(b) \cdot \phi \mid b \in B^\times\}$. It can be shown that the composition of two exomorphisms is also an exomorphism. If A is an interior G -algebra, by definition we have $\text{Inn}(a)(g1_A) = g1_A$, that is $a({}^{g1_A}) = a$ so $a \in (A^G)^\times$. Conversely, any $a \in (A^G)^\times$ defines an inner automorphism of A . So the inner automorphisms are contained in $(A^G)^\times$. Now we define a special type of exomorphisms.

Definition 4.3.2 An exomorphism $\Phi : A \rightarrow B$ is called an embedding, provided there is some $\phi \in \Phi$ which is injective and $\text{Im}(\phi) = \phi(1_A)B\phi(1_A)$.

We prove two important results that will be needed later. If $\Phi : A \rightarrow B$ is an exomorphism of G -algebras and $H \leq G$, define $\text{Res}_H^G(\Phi) : \text{Res}_H^G(A) \rightarrow \text{Res}_H^G(B)$ to be the restriction of Φ to H , thus yielding an exomorphism of H -algebras.

Proposition 4.3.3 Let $\Phi : A \rightarrow B$ and $\Phi' : A \rightarrow B$ be two exomorphisms of interior G -algebras. If $\text{Res}_H^G(\Phi) = \text{Res}_H^G(\Phi')$ for some $H \leq G$, then $\Phi = \Phi'$.

Proof: Let $f \in \Phi$ and $f' \in \Phi'$. Also, let $i = f(1_A)$ and $i' = f'(1_A)$. Now since $\text{Res}_H^G(\Phi) = \text{Res}_H^G(\Phi')$, there is some $b \in (B^H)^\times$ such that $f'(a) = bf(a)b^{-1}$, for all $a \in A$. In particular, $f'(1_Ag) = bf(1_Ag)b^{-1} \Rightarrow i'g = gi' = b(ig)b^{-1}$ from which we get $gi'b = big$. Letting $g = 1$, we obtain $i'b = bi$. Thus, $bi \in (A^G)^\times$. Similarly, $b^{-1}i' \in (B^G)^\times$.

Furthermore, $(bi)(b^{-1}i') = i'^2 = i'$ and $(b^{-1}i')(bi) = i^2 = i$, so by definition $i \sim i'$ which allows us to substitute f' with another representative in Φ' such that $f(1_A) = f'(1_A) = i$. Now we apply a maneuver mentioned above. Let $c = f(b) + (1_B - f(1_A)) = bi + (1_B - i)$. Its inverse is $c^{-1} = b^{-1}i + (1_B - i)$. Obviously $c \in (B^G)^\times$. Now we have

$$cf(a)c^{-1} = cf(1_Aa1_A)c^{-1} = cif(a)ic^{-1} = bif(a)ib^{-1} = bf(a)b^{-1} = f'(a)$$

Therefore $f' = \text{Inn}(c)f$, so f and f' belong to the same exomorphism. **QED**

Now we consider the special case where the interior G -algebra is the endomorphism algebra of some $\mathbb{F}G$ -module.

Lemma 4.3.4 Two $\mathbb{F}G$ -modules M and N are isomorphic if and only if $\text{End}_{\mathbb{F}}(M) \cong \text{End}_{\mathbb{F}}(N)$ as G -algebras.

Proof: If $M \cong N$, the isomorphism is obvious. Now let $A = \text{End}_{\mathbb{F}}(M)$ and $B = \text{End}_{\mathbb{F}}(N)$ and assume $A \cong B$ as interior G -algebras. Clearly A is a matrix algebra over \mathbb{F} and M can be identified with Ai , where i is a primitive idempotent of A . Now let $f : A \xrightarrow{\sim} B$ be an isomorphism of interior G -algebras and let $f(i) = j$. Then N can be identified with Bj and clearly the restriction of f to Ai induces an $\mathbb{F}G$ -module isomorphism. This is an isomorphism of $\mathbb{F}G$ -modules since f is an isomorphism of interior G -algebras, so that we have $f(ga) = gf(a)$, for all $g \in G$ and $a \in A$. **QED**

Lemma 4.3.5 *Let M be an $\mathbb{F}G$ -module and let $i \in \text{Ipot}(\text{End}_{\mathbb{F}G}(M))$. Then there is an isomorphism of G -algebra $\text{End}_{\mathbb{F}}(iM) \cong i\text{End}_{\mathbb{F}}(M)i$.*

Proof: Let π be the projection $M \rightarrow iM$ and let ρ be the inclusion $iM \rightarrow M$. Define

$$\begin{aligned} f : i\text{End}_{\mathbb{F}}(M)i &\longrightarrow \text{End}_{\mathbb{F}}(iM) \\ \phi &\longmapsto \pi\phi\rho \end{aligned}$$

It is easy to check that f is an \mathbb{F} -algebra homomorphism. Since $f(gi) = \pi gi\rho = g\pi i\rho = g1_{iM}$ it is also a G -algebra homomorphism. The inverse of f is the map:

$$\begin{aligned} \text{End}_{\mathbb{F}}(iM) &\longrightarrow i\text{End}_{\mathbb{F}}(M)i \\ \psi &\longmapsto \rho\psi\pi \end{aligned}$$

Since $\rho\pi = i$ and $\pi\rho = 1_{iM}$ we have $\pi\rho\psi\pi\rho = \psi$ and $\rho\pi\phi\rho\pi = \phi$, so f is an isomorphism of interior G -algebras. **QED**

Proposition 4.3.6 *Let M and N be two $\mathbb{F}G$ -modules. There exists an embedding $\Phi : \text{End}_{\mathbb{F}}(M) \rightarrow \text{End}_{\mathbb{F}}(N)$ if and only if M is isomorphic to a direct summand of N (we write $M|N$).*

Proof: If $M|N$, then there exists $i \in \text{Ipot}(\text{End}_{\mathbb{F}G}(N))$ such that $M \cong iN$. Then $\text{End}_{\mathbb{F}}(M) \cong \text{End}_{\mathbb{F}}(iN) \cong i\text{End}_{\mathbb{F}}(N)i$ by Lemma 4.3.5 and this isomorphism, followed by the inclusion $i\text{End}_{\mathbb{F}}(N)i \rightarrow \text{End}_{\mathbb{F}}(N)$ induces an embedding $\text{End}_{\mathbb{F}}(M) \rightarrow \text{End}_{\mathbb{F}}(N)$.

Conversely, let $\Phi : \text{End}_{\mathbb{F}}(M) \rightarrow \text{End}_{\mathbb{F}}(N)$ be an embedding and let $f \in \Phi$. Also, let $f(1_M) = i \in \text{End}_{\mathbb{F}G}(N)$. By definition of embeddings and Lemma 4.3.5, we get

$$\text{End}_{\mathbb{F}}(M) \cong i\text{End}_{\mathbb{F}}(N)i \cong \text{End}_{\mathbb{F}}(iN)$$

Lemma 4.3.4 implies that $M \cong iN|N$.

QED

4.4 Induction of interior G -algebras and further results

Let A be an interior H -algebra where $H \leq G$. We can induce A to G to obtain the interior G -algebra $\text{Ind}_H^G(A) := \mathbb{F}G \otimes_{\mathbb{F}H} A \otimes_{\mathbb{F}H} \mathbb{F}G$. Note that $\text{Ind}_H^G(A)$ has an $(\mathbb{F}G, \mathbb{F}G)$ -bimodule structure. We define the multiplication in $\text{Ind}_H^G(A)$ as follows: for $x, x', y, y' \in [G/H]$ and $a, a' \in A$ we have:

$$(x \otimes a \otimes y)(x' \otimes a' \otimes y') = \begin{cases} x \otimes ay'x'a' \otimes y' & \text{if } yx' \in H \\ 0 & \text{otherwise} \end{cases} \quad (4.4.1)$$

It is clear that $1_{\text{Ind}_H^G(A)} = \sum_{g \in [G/H]} g \otimes 1_A \otimes g^{-1}$. Now we want to put an interior G -algebra structure on $\text{Ind}_H^G(A)$.

Define

$$\phi : G \longrightarrow \text{Ind}_H^G(A)^\times, \quad g \longmapsto \sum_{f \in [G/H]} gf \otimes 1_A \otimes f^{-1}$$

For $g, g' \in G$ we have:

$$\phi(g)\phi(g') = \sum_{f \in [G/H]} gf \otimes 1_A \otimes f^{-1} \sum_{f' \in [G/H]} g'f' \otimes 1_A \otimes (f')^{-1}$$

For $f \in [G/H]$, there is a unique $f' \in [G/H]$ such that $f^{-1}g'f' \in H$, so:

$$\begin{aligned} \phi(g)\phi(g') &= \sum_{f \in [G/H]} gf \otimes f^{-1}g'f' \otimes (f')^{-1} \\ &= \sum_{f' \in [G/H]} gg'f' \otimes 1_A \otimes (f')^{-1} \\ &= \phi(gg') \end{aligned}$$

Note that we changed representatives from f to f' . That is because $f \mapsto f'$ defines a permutation induced by multiplication on the left by $(g')^{-1}$. We have thus constructed $\text{Ind}_H^G(A)$ as an interior G -algebra.

Another way of looking at the unity element is

$$1_{\text{Ind}_H^G(A)} = \text{tr}_H^G(1 \otimes 1_A \otimes 1) \quad (4.4.2)$$

Now let us jump to our case of interest.

Proposition 4.4.1 *Let M be an $\mathbb{F}H$ -module and $H \leq G$. There is an isomorphism of G -algebras:*

$$\text{End}_{\mathbb{F}}(\text{Ind}_H^G(M)) \cong \text{Ind}_H^G(\text{End}_{\mathbb{F}}(M))$$

Proof: We have $\text{Ind}_H^G(M) = \sum_{g \in [G/H]} g \otimes M$, so $\text{End}_{\mathbb{F}}(\text{Ind}_H^G(M))$ has dimension $|G : H|$ over $\text{End}_{\mathbb{F}}(M)$. On the other hand, if we form an $n \times n$ matrix, where $n = |G : H|$ and index the entries with pairs of a transversal $[G/H]$, we can define an \mathbb{F} -linear isomorphism

$$\phi : \text{Ind}_H^G(\text{End}_{\mathbb{F}}(M)) \longrightarrow M_n(\text{End}_{\mathbb{F}}(M))$$

by extending linearly the map sending $f \otimes b \otimes g^{-1} \mapsto [b]$, the elementary matrix with b in the (f, g) -entry. Observe that multiplication is preserved, so it is indeed an \mathbb{F} -linear isomorphism. So we get equal dimensions over \mathbb{F} .

Now we define an \mathbb{F} -linear action of $\text{Ind}_H^G(\text{End}_{\mathbb{F}}(M))$ on $\text{Ind}_H^G(M)$. For $f \in \text{End}_{\mathbb{F}}(M)$, $g, h, k \in [G/H]$ and $m \in M$:

$$(g \otimes f \otimes h^{-1})(k \otimes m) = \begin{cases} g \otimes f(h^{-1}km) & \text{if } h^{-1}k \in H \\ 0 & \text{otherwise} \end{cases} \quad (4.4.3)$$

This action induces an \mathbb{F} -linear homomorphism mapping $g \otimes f \otimes h^{-1}$ to that endomorphism of $\text{Ind}_H^G(M)$ which sends $h \otimes M$ to $g \otimes M$ via f and is zero on the other summands of $\text{Ind}_H^G(M)$. Thus we have an isomorphism of \mathbb{F} -algebras.

Moreover $\phi(g1) = \phi(\sum_{t \in [G/H]} gt \otimes 1_M \otimes t^{-1}) = g1_{\text{Ind}_H^G(M)}$ so the action preserves the structure of interior G -algebras. **QED**

We end this section with an important result which we will need later. Refer to [6] for a proof.

Theorem 4.4.2 *Let A be an interior G -algebra, $H \leq G$ and let $j \in \text{Ipot}(A^H)$. Then there exists an embedding $\mathfrak{F} : A \rightarrow \text{Ind}_H^G(jAj)$ if and only if there are $a', a'' \in A^H$ such that $1_A = \text{tr}_H^G(a'ja'')$.*

Chapter 5

Pointed groups and defect theory

5.1 Pointed groups

Our next fundamental object will be that of pointed groups. In the last section we examined some features of the subalgebra of fixed elements. Again, let A be a G -algebra over \mathbb{F} . Now let $H \leq G$ and let α be a point on A^H . Following [5], we define the pair (H, α) to be a *pointed group* and we denote it by H_α . Basically what we are doing is considering some extra structure added to the group structure. Pointed groups satisfy the partial order relation the same way finite groups do. The group action on these objects is defined as follows: If H_α is a pointed group on A , then for $g \in G$ we have ${}^g(H_\alpha) := ({}^gH)_{({}^g\alpha)}$.

We define the normalizer of H_α to be $N_G(H_\alpha) := \{g \in G : {}^g(H_\alpha) = H_\alpha\}$.

Now we can regard A^H as a $N_G(H_\alpha)/H$ -algebra over \mathbb{F} since H acts trivially on it. Moreover, let \mathfrak{m}_α be the unique maximal ideal not containing α . We can also consider it as an $N_G(H_\alpha)/H$ -ideal. Subsequently, there is another context in which we can look at the Brauer homomorphism. Letting $A(H_\alpha) := A^H/\mathfrak{m}_\alpha$ we define

$$\text{br}_\alpha := A^H \longrightarrow A(H_\alpha), \quad a \mapsto a + \mathfrak{m}_\alpha$$

This map is an algebra epimorphism with kernel \mathfrak{m}_α . So if $i \in \alpha$, then $\text{br}_\alpha(i) \neq 0$. Moreover, the image of the ideal $A^H \alpha A^H$ under this map is an ideal in $A(H_\alpha)$ and since this is a simple algebra we have $\text{br}_\alpha(A^H \alpha A^H) = A(H_\alpha)$.

Proposition 5.1.1 *Let $K \leq H$ be two finite groups and let β and α be two points on A^K and A^H respectively. Then the following assertions are equivalent.*

1. *There exists $i \in \alpha$ and $j \in \beta$ such that $jAj \subseteq iAi$.*
2. *For any $e \in \alpha$ there is an $f \in \beta$ such that $fAf \subseteq eAe$.*
3. $\text{br}_\beta(\alpha) \neq 0$.

Proof: (1 \Rightarrow 2): Let $i \in \alpha, j \in \beta$ such that $jAj \subseteq iAi$. If $e \in \alpha$, then $e \sim i \Rightarrow$ there exists some unit x in A^H such that $e = xi$ by Corollary 2.2.10. Since $A^H \subseteq A^K$, then $x \in A^K$ so we let $f := xj$ and obtain $fAf = xjAjx \subseteq xiAix = eAe$.

(2 \Rightarrow 3): If $fAf \subseteq eAe$, then $e, f \in A^K$ and we have $\text{br}_\beta(e)\text{br}_\beta(f) = \text{br}_\beta(e) = \text{br}_\beta(f) \neq 0$, since $ef = f$. So $\text{br}_\beta(e) \neq 0$ and in particular $\text{br}_\beta(\alpha) \neq 0$.

(3 \Rightarrow 1): If $\text{br}_\beta(\alpha) \neq 0$, then there is some $e \in \alpha$ such that $\text{br}_\beta(e) \neq 0$ and e will also live in A^K so we can choose a primitive decomposition of it with idempotents of A^K . Let $e = i_1 + \dots + i_k$ be such a decomposition. Then we have

$$0 \neq \text{br}_\beta(e) = \text{br}_\beta(i_1 + \dots + i_k) = \text{br}_\beta(i_1) + \dots + \text{br}_\beta(i_k)$$

which means that there is some $t \in \{1, \dots, k\}$ such that $\text{br}_\beta(i_t) \neq 0$. It follows that $i_t A i_t = i_t A i_t \subseteq i A i$. **QED**

If the above conditions are satisfied, then K_β is a *pointed subgroup* of H_α and we write $K_\beta \leq H_\alpha$. So we can regard pointed groups as refinements of finite groups and similarly for the inclusion relation between them.

Corollary 5.1.2 *K_β is a pointed subgroup of H_α if $K \leq H$ and for $i \in \alpha$ there is some $j \in \beta$ such that $ij = j = ji$.*

Now if H_α is a pointed group, we can construct the primitive algebra $A_\alpha := iAi$ for some $i \in \alpha$, which clearly is an H -algebra since α is H -fixed. The algebra A_α is independent of the choice of i (up to isomorphism), since if j is another idempotent of α , then there exists some unit a of A^H such that $aj = i$, so $jAj \cong iAi$. If we consider the unique maximal ideal \mathfrak{m}_α not containing α and the quotient algebra $A(H_\alpha)$ called the *multiplicity algebra* of H_α , we have $A(H_\alpha) \cong \text{End}_{\mathbb{F}}(V(\alpha))$ where $V(\alpha) = A/\mathfrak{m}_\alpha$ is called the *multiplicity module* of H_α .

Lemma 5.1.3 *Let H_α and K_β be two pointed groups on A such that $K_\beta \leq H_\alpha$ and $|H| \leq |K|$. Then $K_\beta = H_\alpha$.*

Proof: Let $i \in \alpha$ and $j \in \beta$ such that $jAj \subseteq iAi$. By definition, $K \leq H$, so $K = H$ as groups since $|H| \leq |K|$. Consequently, α and β lie in A^H and $j = iji$ implies that $j \in iA^H i$. Now since $iA^H i$ is primitive, we conclude that $i = j \Rightarrow \alpha = \beta$. **QED**

Now we introduce a refinement of the pointed groups which exhibit special properties.

Proposition 5.1.4 *Let Q_δ be a pointed group on A . The following are equivalent.*

1. $\delta \not\subseteq A_R^Q$, for any $R < Q$.
2. $\text{br}_Q(\delta) \neq 0$

3. $\text{br}_\delta(A_R^Q) = 0$, for every $R < Q$.

Proof: (1 \Rightarrow 2): Suppose $\text{br}_Q(\delta) = 0$. Then for all $i \in \delta$ we have $\text{br}_Q(i) = 0$, so that

$$i \in A_{<Q}^Q = \sum_{R < Q} \text{tr}_R^Q(A^R) = \sum_{R < Q} A_R^Q$$

as a sum of ideals in A^Q . Now by Rosenberg's lemma $i \in A_R^Q$ for some $R < Q$, contradicting our condition.

(2 \Rightarrow 3): Let $i \in \delta$ such that $\text{br}_Q(i) \neq 0$ and let $R < Q$. Now $i \notin \ker \text{br}_\delta$ by definition. On the other hand $i \notin A_R^Q$, so applying Rosenberg's lemma we have that $i \notin A_R^Q + \ker \text{br}_\delta \Rightarrow \text{br}_\delta(i) \notin \text{br}_\delta(A_R^Q)$. Since $\text{br}_\delta(i) \in A(Q_\delta)$, we have $\text{br}_\delta(A_R^Q) \neq A(Q_\delta)$, thus $\text{br}_\delta(A_R^Q) = 0$, knowing that $A(Q_\delta)$ is simple.

(3 \Rightarrow 1): For every proper subgroup R of Q we have $\text{br}_\delta(A_R^Q) = 0$ and since $\text{br}_\delta(\delta) \neq 0$, we conclude that $\delta \not\subseteq A_R^Q$. **QED**

A pointed group Q_δ satisfying the above properties is said to be *local*. We also say δ is a *local point* of Q on A .

We now introduce another relation between pointed groups. Given two pointed groups H_α and K_β on A , we say H_α is *projective relative to* K_β and write $H_\alpha \text{pr} K_\beta$ if $K \leq H$ and $\alpha \subseteq \text{tr}_K^H(A^K \beta A^K)$. Note that this is equivalent to requiring that $A^H \alpha A^H \subseteq \text{tr}_H^K(A^K \beta A^K)$. So it is enough for some $i \in \alpha$ to be in $\text{tr}_H^K(A^K \beta A^K)$. We can simplify further.

Lemma 5.1.5 *Let H_α and K_β be two pointed groups on A and let $i \in \alpha$ and $j \in \beta$. Given $K \leq H$, then $H_\alpha \text{pr} K_\beta$ if and only if there exist elements $a, b \in A^K$ such that $i = \text{tr}_K^H(ajb)$.*

Proof: If $i = \text{tr}_H^K(ajb)$, then $i \in \text{tr}_K^H(A^K \beta A^K)$. Conversely, if $H_\alpha \text{pr} K_\beta$, then $i = \sum_{k=1}^n \text{tr}_K^H(a_k j b_k)$, for some $n \in \mathbb{N}$ and $a_k, b_k \in A^K$. Multiplying on both sides

by i we get:

$$i = \sum_{k=1}^n \text{tr}_K^H(ia_kjb_ki)$$

since $i \in A^H$. Now i is a primitive idempotent, meaning that iA^Hi is a primitive algebra with i as a unity element. Thus, there is some $r \in \{1, \dots, n\}$ such that $\text{tr}_K^H(ia_rjb_r i)$ is invertible in A^H , so we obtain

$$i = \text{tr}_K^H(ia_kjb_ki)c = \text{tr}_K^H(ia_kjb_kic)$$

where $c \in iA^Hi$. We are done. **QED**

A pointed group H_α is said to be *projective relative to K* if it is projective relative to K for some $\beta \in \mathcal{P}(A^K)$.

Lemma 5.1.6 *A pointed group H_α is projective relative to K if and only if $K \leq H$ and $\alpha \subseteq A_K^H$*

Proof: If $H_\alpha pr K_\beta$, then clearly $K \leq H$ and $\alpha \subseteq A_K^H$. Conversely, suppose $K \leq H$ and $\alpha \subseteq A_K^H$. Choose a primitive decomposition of 1_{A^K} and multiply it on both sides by A^K . We obtain $A^K = \sum_{\beta \in \mathcal{P}(A^K)} A^K \beta A^K$. Therefore

$$\text{tr}_K^H(A^K) = A_K^H = \sum_{\beta \in \mathcal{P}(A^K)} \text{tr}_K^H(A^K \beta A^K)$$

Let $i \in \alpha$. By Rosenberg's lemma we have that $i \in \text{tr}_K^H(A^K \beta A^K)$, for some β and so $\alpha \subseteq \text{tr}_K^H(A^K \beta A^K)$. **QED**

Remark 5.1.7 *Let Q_δ be a pointed group on A . Q_δ is local if and only if it is minimal with respect to the pr relation.*

Proof: Let Q_δ be local. Suppose there exists some proper subgroup R of Q such that $Q_\delta pr R_\gamma$, for a point $\gamma \in \mathcal{P}(A^R)$. Let $i \in \delta$ and $j \in \gamma$. Then, by Lemma 5.1.5, $i = \text{tr}_R^Q(ajb)$, for $a, b \in A^R$. Therefore, $i \in \text{tr}_R^Q(A^R \gamma A^R) \subseteq A_R^Q$, contradicting the fact that $\delta \not\subseteq A_R^Q$, for any $R < Q$. The converse follows from Lemma 5.1.6 and Rosenberg's lemma. **QED**

5.2 Defect theory

In this section we talk about special types of local pointed groups referring to [5]. We begin with an important result.

Proposition 5.2.1 *Let H_α and K_β be two pointed groups on A such that $H_\alpha \text{pr} K_\beta$. Furthermore, let Q_δ be a local pointed subgroup of H_α . Then $Q_\delta \leq {}^h(K_\beta)$ for some $h \in H$.*

Proof: Proposition 5.1.1 implies that $\text{br}_\delta(\alpha) \neq 0$. $H_\alpha \text{pr} K_\beta \Leftrightarrow A^H \alpha A^H \subseteq \text{tr}_K^H(A^K \beta A^K)$ and by applying Mackey's decomposition formula (Proposition 4.2.3) we obtain:

$$\begin{aligned} 0 \neq \text{br}_\delta(\alpha) &\subseteq \text{br}_\delta(A^H \alpha A^H) \subseteq \text{br}_\delta(\text{tr}_K^H(A^K \beta A^K)) \\ &\subseteq \sum_{h \in H} \text{br}_\delta(\text{tr}_{Q \cap {}^h K}^Q(A^{Q \cap {}^h K} ({}^h \beta) A^{Q \cap {}^h K})) \\ &= \sum_{h \in H, Q \subseteq {}^h K} \text{br}_\delta(A^Q ({}^h \beta) A^Q) \end{aligned}$$

so there is some $h \in H$ such that $Q \leq {}^h K$ and $\text{br}_\delta({}^h \beta) \neq 0$, that is $Q_\delta \leq {}^h(K_\beta)$.

QED

Definition 5.2.2 *Let H_α be a pointed group on A . A local pointed subgroup Q_δ of H_α is said to be a defect pointed subgroup of H_α if $H_\alpha \text{pr} Q_\delta$.*

It is not clear that such a pointed group exists, so we show its existence.

Proposition 5.2.3 *Let H_α be a pointed group on A . Then a defect pointed subgroup of H_α exists.*

Proof: Let $i \in \alpha$. Choose $Q \leq H$ minimal subject to $i \in \text{tr}_Q^H(A^Q) = A_Q^H$. Since $A^H \subseteq A^Q$ we have $i \in A^Q$. Let J be a primitive decomposition of i with pairwise orthogonal idempotents of A^Q . By using property 4 in Proposition 4.2.2 we have:

$$i = i^2 \in i \text{tr}_Q^H(A^Q) = \text{tr}_Q^H(iA^Q) \subseteq \text{tr}_Q^H(A^Q i A^Q) \subseteq \sum_{j \in J} \text{tr}_Q^H(A^Q j A^Q)$$

Since $\text{tr}_Q^H(A^Q j A^Q)$ are ideals of A^Q , by using Rosenberg's lemma we have $i \in \text{tr}_Q^H(A^Q j' A^Q)$ for some $j' \in J$. Letting δ be the point of Q containing j' we get $Q_\delta \leq H_\alpha$ since $j' A j' = i j' A j' i \subseteq i A i$ and $H_\alpha \text{pr} Q_\delta$.

Moreover, Q_δ is local, for if not, there would exist some $R < Q$ such that $\delta \subseteq A_R^Q \Rightarrow j' \in A_R^Q$, implying that $i \in \text{tr}_Q^H(A^Q j' A^Q) \subseteq \text{tr}_Q^H(A_R^Q) = A_R^H$ contradicting the minimality of Q . **QED**

We can think of defect groups in a different way provided in our next result.

Proposition 5.2.4 *Let H_α and Q_δ be pointed groups on A . The following are equivalent.*

1. Q_δ is a defect pointed subgroup of H_α .
2. Q_δ is minimal subject to the pr relation.
3. Q_δ is maximal among the local pointed subgroups of H_α .

Proof: (1 \Rightarrow 2): If Q_δ is a defect pointed subgroup of H_α , then by definition $H_\alpha \text{pr} Q_\delta$. Now suppose there is some $R_\gamma \leq Q_\delta$ such that $Q_\delta \text{pr} R_\gamma$. Then by Proposition 5.2.1 there is some $h \in H$ such that $Q_\delta \leq {}^h(R_\gamma)$ and since $|R| \leq |Q|$, Lemma 5.1.2 implies that $R_\gamma = Q_\delta$.

(2 \Rightarrow 1): Let Q_δ be minimal subject to the relation pr , i.e. Q_δ is minimal such that $A^H \alpha A^H \subseteq \text{tr}_Q^H(A^Q \delta A^Q)$. Now let R_γ be a defect pointed subgroup of H_α .

By Proposition 5.2.1 there is some $h \in H$ such that $R_\gamma \leq {}^h(Q_\delta) \Rightarrow {}^{h^{-1}}(R_\gamma) \leq Q_\delta$.
We have:

$$A^H \alpha A^H = {}^{h^{-1}}(A^H \alpha A^H) \subseteq {}^{h^{-1}}(\text{tr}_R^H(A^R \gamma A^R)) = \text{tr}_{R^h}^H(A^{R^h} ({}^{h^{-1}}\gamma) A^{R^h})$$

By minimality of Q_δ we have $Q_\delta = {}^{h^{-1}}(R_\gamma)$, which implies that Q_δ is a defect pointed subgroup of H_α .

(1 \Rightarrow 3): This is an easy application of Proposition 5.2.1 and Lemma 5.1.3.

(3 \Rightarrow 1): Let Q_δ be a maximal local pointed subgroup of H_α and let P_γ be a defect pointed subgroup of H_α . Again, by Proposition 5.2.1 there is some $h \in H$ such that $Q_\delta \leq {}^h(P_\gamma)$ and maximality of Q_δ implies $Q_\delta = {}^h(P_\gamma)$. Particularly:

$$\begin{aligned} A^H \alpha A^H &= {}^h(A^H \alpha A^H) = {}^h(\text{tr}_P^H(A^P \gamma A^P)) \\ &= \text{tr}_{P^h}^H(A^{P^h} ({}^h\gamma) A^{P^h}) = \text{tr}_Q^H(A^Q \delta A^Q) \end{aligned}$$

meaning that Q_δ is indeed a defect pointed subgroup of H_α . **QED**

Corollary 5.2.5 *Let A be a primitive G -algebra. All defect pointed groups on A are conjugate under G .*

Proof: Let $\alpha = \{1_A\}$ be the unique point of A^G . Any pointed group on A is contained in G_α . The result follows from Proposition 5.2.1 and Proposition 5.2.4.

QED

Remark 5.2.6 *Local pointed groups are generalizations of p -subgroups of G and defect groups are generalizations of Sylow p -subgroups of G .*

We end this section with a different characterization of defect groups in a special case.

Definition 5.2.7 Let $A = \text{End}_{\mathbb{F}}(M)$ for some indecomposable $\mathbb{F}G$ -module M . Obviously A is primitive. In this case the unique defect group (up to conjugation) is called the vertex of M . Moreover, if Q is a vertex of M and if $j \in A^Q$ belongs to a point of P , then the indecomposable $\mathbb{F}Q$ -module jQ is called a source of M .

Below we give an equivalent definition of the vertex of a module.

Definition 5.2.8 Let M be an indecomposable $\mathbb{F}G$ -module. A vertex of M is a minimal subgroup P of G subject to the condition that M is a direct summand of $\text{Ind}_P^G(\text{Res}_P^G(M))$ which we denote by $M \mid \text{Ind}_P^G(\text{Res}_P^G(M))$. If P is a vertex of M , then there exists an indecomposable $\mathbb{F}P$ -module S , such that $S \mid \text{Ind}_P^G(S)$. Any such module S is called a P -source of M . We say (P, S) is a vertex-source pair of M .

Remark 5.2.9 The pair (P, S) exists and is unique up to conjugation by elements of G . The existence and uniqueness follows from Proposition 5.2.3 and Corollary 5.2.5.

5.3 A special case

Now we show what the local pointed groups on the endomorphism algebra of a p -permutation $\mathbb{F}G$ -module are. First let us define the *relative projectivity* for G -algebras. Let $H \leq G$ and A be a G -algebra. A is said to be *projective relative to H* if $\text{tr}_H^G : A^H \rightarrow A^G$ is surjective. In other words, A is *relative projective to H* if $1_A \in A_H^G$, since A_H^G is an ideal of A^G .

Proposition 5.3.1 (*Higman's criterion*) Let M be an $\mathbb{F}G$ -module and let $H \leq G$. The following are equivalent:

1. $\text{End}_{\mathbb{F}}(M)$ is projective relative to H .
2. M is isomorphic to a direct summand of $\text{Ind}_H^G \text{Res}_H^G(M)$.

Proof: $\text{End}_{\mathbb{F}}(M)$ is a projective relative to H i.e. $1_{\text{End}_{\mathbb{F}}(M)} = \text{tr}_H^G(1 \otimes 1_{\text{End}_{\mathbb{F}} \text{Res}_H^G(M)} \otimes 1)$, so by Theorem 4.4.2 there exists an embedding

$$\text{End}_{\mathbb{F}}(M) \longrightarrow \text{Ind}_H^G(\text{End}_{\mathbb{F}}(\text{Res}_H^G(M))) \cong \text{End}_{\mathbb{F}}(\text{Ind}_H^G \text{Res}_H^G(M))$$

by Proposition 4.4.1. This is equivalent to $M | \text{Ind}_H^G \text{Res}_H^G(M)$ by Proposition 4.3.6.
QED

Let M be an $\mathbb{F}G$ -module and let $A = \text{End}_{\mathbb{F}}(M)$. Recall that we can think of M in a different way, that is, as a representation of G over \mathbb{F} . So there is a group homomorphism $\rho : G \rightarrow \text{Aut}_{\mathbb{F}}(M) = A^{\times}$, making A an interior G -algebra. Now if $H \leq G$, an endomorphism $\phi \in \text{End}_{\mathbb{F}}(M)$ is H -fixed if and only if it commutes with the H -action, which means that $A^H = \text{End}_{\mathbb{F}H}(M)$. If $i \in \text{Ipot}(A^H)$, we can think of i as the projection onto a direct summand of $\text{Res}_H^G(M)$. Moreover, i is primitive if and only if M is indecomposable, which holds if and only if A is primitive. It follows by Theorem 2.2.5 that two direct summands iM and jM of $\text{Res}_H^G(M)$ are isomorphic if and only if $i \sim j$. For $i \in \alpha$ we have $iAi \cong \text{End}_{\mathbb{F}}(iM)$. If M is an indecomposable $\mathbb{F}G$ -module, the interior G -algebra $A = \text{End}_{\mathbb{F}}(M)$ is primitive and in this case, the unique defect group on A is exactly the vertex of M . As a notation, we write $M_{\alpha} := iM$.

Proposition 5.3.2 *Let P be a p -subgroup of G . Let $M = \text{Ind}_P^G(\mathbb{F}_P)$ be a p -permutation $\mathbb{F}G$ -module and let $A = \text{End}_{\mathbb{F}}(M)$. The subgroups of G that have a local point on A are precisely the p -subgroups Q of G such that $Q \leq {}^g P$, for $g \in G$. Moreover, if Q is such a group, there exists a unique local point δ of Q on A .*

Proof: Assume that Q_δ is a local pointed group on A . This is equivalent to saying that A_δ is not *projective relative* to any proper subgroup of Q . By Higman's criterion (Proposition 5.3.1), that means $M_\delta \nmid \text{Ind}_R^Q \text{Res}_R^Q(M_\delta)$, for any subgroup R of Q . Thus, by definition Q is a vertex of M_δ . Now we apply the Mackey decomposition formula (Theorem 3.1.9):

$$\begin{aligned} \text{Res}_Q^G \text{Ind}_P^G(\mathbb{F}_P) &\cong \bigoplus_{QgP \subseteq [Q \backslash G / P]} \text{Ind}_{Q \cap {}^g P}^Q ({}^g(\text{Res}_{Q \cap {}^g P}^P(\mathbb{F}_P))) \\ &\cong \bigoplus_{QgP \subseteq [Q \backslash G / P]} \text{Ind}_{Q \cap {}^g P}^Q(\mathbb{F}_{Q \cap {}^g P}) \end{aligned}$$

By Green's indecomposability (Theorem 3.2.6) and Lemma 3.2.5, these summands are all indecomposable. The summand $\text{Ind}_{Q \cap {}^g P}^Q(\mathbb{F}_{Q \cap {}^g P})$ has vertex $Q \cap {}^g P$, which coincides with Q if and only if $Q \leq {}^g P$. The point δ is unique because the trivial $\mathbb{F}Q$ -module \mathbb{F}_Q is the only indecomposable direct summand of $\text{Res}_Q^G(M)$ with vertex Q .

Conversely, the same decomposition shows that given any p -subgroup Q of G satisfying $Q \leq {}^g P$, for some $g \in G$, then $\text{Res}_Q^G(M)$ has isomorphically unique indecomposable direct summand $\text{Ind}_Q^Q(\mathbb{F}_Q) \cong \mathbb{F}_Q$ with vertex Q . **QED**

Chapter 6

Local categories and the pandemic fusion system

6.1 The Puig system

Following [10], we define a notion that encodes the p -local information about a finite group G .

Definition 6.1.1 *Let S be a Sylow p -subgroup of G . The fusion system of G on S is the category $\mathcal{F}_S(G)$, where the objects are the subgroups of S and the morphisms are the group homomorphisms induced by G -conjugation. In particular, if $Q \leq$*

$S \geq R$, we set:

$$\text{Hom}_{\mathcal{F}_S(G)}(Q, R) = \text{Hom}_G(Q, R)$$

the set of those group homomorphisms $\phi : Q \rightarrow R$ for which there is some $g \in G$ such that $\phi(q) = {}^gq$, for all $q \in Q$. The composition of morphisms is the usual composition of group homomorphisms.

We will deal with special types of fusion systems in this section. Let A be an interior G -algebra with $\phi : G \rightarrow A^\times$. If $\psi : H \rightarrow G$ is a group homomorphism, denote by $\text{Res}_\psi(A)$ the interior H -algebra defined by the group homomorphism $\phi \circ \psi : H \rightarrow A^\times$. When $H \leq G$ and ψ is the inclusion $H \hookrightarrow G$, then $\text{Res}_\psi(A)$ is just $\text{Res}_H^G(A)$. If K_β and H_α are two local pointed groups on A , denote by $F_\alpha : A_\alpha \rightarrow A$ and $F_\beta : A_\beta \rightarrow A$ the embeddings containing the canonical inclusions.

We have talked about the notion of exomorphisms between algebras in the previous chapters. Analogously, for two groups H and G , we define a group exomorphism $\Psi : H \rightarrow G$ to be the set of all homomorphisms $\psi : H \rightarrow G$ obtained by composing ψ with all the inner automorphisms of H and G . In [2] Puig defined A -fusions to be certain types of exomorphisms, what he called G -exomorphisms, having additional properties.

Definition 6.1.2 *Let H_α and K_β be two pointed groups on A . A G -exomorphism $\Phi : K_\beta \rightarrow H_\alpha$ is a group exomorphism $\Phi : K \rightarrow H$ such that there is $x \in G$ satisfying $(K_\beta)^x \leq H_\alpha$ and $\phi(y) = y^x$, for all $y \in K$ and $\phi \in \Phi$.*

Any $x \in G$ satisfying $(K_\beta)^x \leq H_\alpha$ and $\phi(y) = y^x$, for all $y \in K$, induces an exomorphism $F_x : A_\beta \rightarrow \text{Res}_\phi(A_\alpha)$ of interior K -algebras. Indeed, if $i \in \alpha$, then there is some $j \in \beta$ such that $ij^xi = j^x$, meaning $(jAj)^x \subseteq iAi$. The required exomorphism is just the exomorphism containing the inclusion $f_x : jAj \rightarrow iAi$, $a \mapsto a^x$.

Definition 6.1.3 *Let H_α and K_β be two local pointed groups on A . An A -fusion $\Phi : K_\beta \rightarrow H_\alpha$ is a group exomorphism $\Phi : K \rightarrow H$ such that any $\phi \in \Phi$ is*

into and there is an exomorphism $F_\phi : A_\beta \rightarrow \text{Res}_\phi(A_\alpha)$ of interior K -algebras satisfying

$$\text{Res}_1^K(F_\beta) = \text{Res}_1^H(F_\alpha) \circ \text{Res}_1^K(F_\phi)$$

The diagram below gives a nice picture of what is going on in the above definition.

$$\begin{array}{ccc} A_\beta & \xrightarrow{F_\beta} & A \\ & \searrow F_\phi & \nearrow F_\alpha \\ & A_\alpha & \end{array}$$

So in order for Φ to be an A -fusion, the diagram must commute. In other words, let $a \in A$ and let $x \in G$ be the unit that satisfies the condition. Then $ja \in A_\beta$ and F_ϕ sends ja to $j^x\phi(a)$ in A_α . Moreover, ja maps into A via f_β and $j^x\phi(a)$ maps into A via f_α . In order for the diagram to commute we must have: $(ja)^x = j^x\phi(a)$ in A . Here x plays the role of ϕ . Moreover, the condition

$$\text{Res}_1^K(F_\beta) = \text{Res}_1^H(F_\alpha) \circ \text{Res}_1^K(F_\phi)$$

means that if we have equality between algebra exomorphisms, then they will commute as G -algebra exomorphisms as well, by Proposition 4.3.3. If we forget about G and consider the group of units in A instead, we come up with a larger category. Let us summarize these conditions in a more refined definition.

Definition 6.1.4 *The Puig system denoted by $\mathcal{L}_{A^\times}(A)$ is the category with objects local pointed groups on A , and for K_β and H_α , the set of morphisms is the set of all group monomorphisms $\phi : K \rightarrow H$, such that, for $i \in \alpha$ and $j \in \beta$, there exists a unit $x \in A^\times$ satisfying the following conditions.*

1. $\phi(k) \cdot {}^xj = {}^xj \cdot \phi(k)$, for all $k \in K$.
2. ${}^x(k \cdot j) = \phi(k) \cdot {}^xj$, for all $k \in K$.

3. $xj = i^xji$

Now let us prove that $\mathcal{L}_{A^\times}(A)$ is a category. Let Q_δ , P_γ and R_ϵ be pointed groups on A . Let $\phi : Q_\delta \rightarrow P_\gamma$, choose $j \in \delta$ and $i \in \gamma$ and let $x \in A^\times$ satisfying the conditions. Also let $\psi : P_\gamma \rightarrow R_\epsilon$, choose $k \in \epsilon$ and let $y \in A^\times$ satisfying the conditions for P_γ and R_ϵ . Then $yx \in A^\times$ is a unit satisfying the conditions for Q_δ and R_ϵ . Indeed, for all $u \in Q$ we have:

1. $\psi\phi(u)^{yxj} = \psi(\phi(u))^y(xj) = {}^{yxj}\psi\phi(u)$
2. $\psi\phi(u)^{yxj} = \psi\phi(u)^y(i^xj) = (\psi\phi(u)^yi)^{(yxj)} = {}^y(\phi(u)i)^{(yxj)} = {}^y(\phi(u)^xj) = {}^{yx}(uj)$
3. ${}^{yx}j = {}^y(i^xj) = k^{yx}j$, where ${}^yi = k^yik$ and ${}^xj = i^xji$.
Similarly, ${}^{yx}j = {}^{yx}jk$. Thus, ${}^{yx}j = k^{yx}jk$.

In trying to understand the purpose of these conditions, we note that the first condition implies that ${}^xj \in A^{\phi(K)}$ and that ${}^xjA^xj$ can be given an interior K -algebra structure by restriction along ϕ written $\text{Res}_\phi({}^xjA^xj)$. Here k is mapped to $\phi(k)^xj \in {}^xjA^xj$ and conjugation by x yields an isomorphism of interior K -algebras:

$$\text{Conj}(x) : jAj \longrightarrow \text{Res}_\phi({}^xjA^xj)$$

Thus we have briefly commented on the nature of the second condition. Now xj is a primitive idempotent of $A^{\phi(K)}$, hence ${}^x\beta$ is a point of $A^{\phi(K)}$. Subsequently, the third condition implies that xj appears in a decomposition of i in $A^{\phi(K)}$, meaning that $\phi(K)_{({}^x\beta)} \leq H_\alpha$.

6.2 The pandemic fusion system

Now we introduce a new notion, namely that of the *pandemic fusion system* which, in a sense that will be explained below, is the global analogue of the Puig

system.

Let A be an interior G -algebra over \mathbb{F} . Given $K \leq G \geq H$ and a group monomorphism $\phi : K \rightarrow H$, we define the diagonal subgroup of $G \times G$ to be:

$$\Delta(\phi) = \{(\phi(k), k) : \text{for all } k \in K\}$$

Definition 6.2.1 *A transporting unit for ϕ in A is an element of the set:*

$$A^{\times, \Delta(\phi)} = \{a \in A^\times : \forall k \in K, \phi(k)ak^{-1} = a\}$$

When a transporting unit for ϕ exists, we call ϕ a pandemic fusion to H from K in A .

If we let $\sigma_A : \mathbb{F}G \rightarrow A$ to be the representation of G , then we understand that $ak = a\sigma_A(k)$. So if a is a transporting unit we have $\sigma_A(\phi(k)) = {}^a(\sigma_A(k))$. Note that our transporting unit is not restricted to some primitive algebra, but affords ϕ in the whole algebra.

Definition 6.2.2 *The pandemic fusion system denoted by $\mathcal{P}[A]$ is the category whose objects are the subgroups of G and for $H \leq G \geq K$, the set of morphisms is the set of pandemic fusions from K to H in A :*

$$\mathcal{P}[A](K, H) = \{\phi \in \text{mongrp}(K, H) : A^{\times, \Delta(\phi)} \neq \emptyset\}$$

where $\text{mongrp}(K, H)$ is the set of group monomorphisms from K to H .

Proposition 6.2.3 *Let K_β and H_α be two pointed groups on A . Let ϕ be a pandemic fusion in A such that $\phi : K \rightarrow K$ is an automorphism. Then, given any $a \in A^{\times, \Delta(\phi)}$, the set ${}^a\beta$ is a point of $\phi(K)$ on A . Furthermore, ${}^a\beta$ depends only on ϕ and β , not on the choice of a .*

Proof: First, given $x \in A^K$, then ${}^ax \in A^{\phi(K)}$. Indeed

$$\phi({}^ax)a({}^ax)^{-1} = a\phi(x)a^{-1}a(ax^{-1}a^{-1}) = a\phi(x)ax^{-1}a^{-1} = aaa^{-1} = a$$

Moreover, given a primitive idempotent $j \in \beta$, we have ${}^a j \in A^{\phi(K)}$ as a primitive idempotent.

Suppose not. Let ${}^a j = j_1 + \dots + j_n$ be a primitive decomposition of ${}^a j$ in $A^{\phi(K)}$. We have $j = j_1^a + \dots + j_n^a$, which contradicts the fact that j is primitive.

Now let j' be another idempotent of β such that $j' = {}^b j$, for some unit b . Then ${}^a j' = {}^{aba^{-1}}({}^a j)$, so ${}^a \beta$ is indeed a point of $\phi(K)$ on A .

Let a' be another transporting unit for ϕ . It is easy to check that $a'a^{-1} \in A^{\Delta(\phi), \times}$.

So we have ${}^{a'}\beta \ni {}^{a'}j = {}^{a'a^{-1}a}j \in {}^a\beta \Rightarrow {}^{a'}\beta = {}^a\beta$. **QED**

We are interested in the case that the interior G -algebra is the endomorphism algebra of some p -permutation $\mathbb{F}G$ -module. Given an $\mathbb{F}G$ -module M , we define:

$$E(M) = \text{End}_{\mathbb{F}}(M)$$

Proposition 6.2.4 *Given an $\mathbb{F}G$ -module M , subgroups $K \leq G \geq H$ and a group monomorphism $\phi : K \rightarrow H$, then $\phi \in \mathcal{P}[E(M)](K, H)$ if and only if:*

$$\text{Res}_K^G(M) \cong \text{Res}_{\phi}(\text{Res}_H^G(M))$$

Proof: Suppose $u : \text{Res}_{\phi}(\text{Res}_H^G(M)) \rightarrow \text{Res}_K^G(M)$ is an isomorphism. We have: $\phi(k)u(m) = u(km)$, for all $k \in K$ and $m \in M$. So u is a transporting unit for ϕ . Reversing the argument yields the converse. **QED**

Corollary 6.2.5 *Let M be an $\mathbb{F}G$ -module and $P \leq S \geq Q$. Let $\phi : P \rightarrow Q$ be a group monomorphism. Then $\phi \in \mathcal{P}[E(M)](P, Q)$ if and only if $m(\text{Res}_P^G(U), M) = m(\text{Res}_{\phi}(\text{Res}_Q^G(U), M))$ for every indecomposable direct summand U of M .*

Proof: The proof is an easy application of Proposition 6.2.4. **QED**

A refinement of $\mathcal{P}[A]$ is $\mathcal{P}^{\bullet}[A]$ whose objects instead are the local pointed groups on A and given H_{α} and K_{β} on A , then $\mathcal{P}^{\bullet}[A](K_{\beta}, H_{\alpha})$ is the set of morphisms $\phi \in \mathcal{P}[A](K, H)$ such that $\phi(K_{\beta}) \leq H_{\alpha}$. The composition operation is

the usual composition of group monomorphisms.

Proposition 6.2.6 *Let M be a p -permutation $\mathbb{F}G$ -module. Let P and Q be p -subgroups of G . Suppose that there exist local points γ of P and δ of Q on $E(M)$. Then:*

$$\mathcal{P}[E(M)](P, Q) = \mathcal{P}^\bullet[E(M)](P_\gamma, Q_\delta)$$

Proof: Let $A = E(M)$. Plainly $\mathcal{P}^\bullet[E(M)](P_\gamma, Q_\delta) \leq \mathcal{P}[E(M)](P, Q)$. Conversely, given $\phi \in \mathcal{P}[E(M)](P, Q)$, then $\mathbb{F}_P | \text{Res}_\phi(\mathbb{F}_Q)$ by Remark 6.2.4 and Higman's criterion. Thus, by Proposition 4.3.6 there exists an embedding $\text{End}_{\mathbb{F}}(\mathbb{F}_P) \rightarrow \text{Res}_\phi(\text{End}_{\mathbb{F}}(\mathbb{F}_Q))$, meaning that $\phi(P_\gamma) \leq Q_\delta$. **QED**

6.3 The pandemic fusion system on a Sylow p -subgroup

Let S be a Sylow p -subgroup of G . Given a p -permutation $\mathbb{F}G$ -module M , let $\mathcal{P}_S[E(M)]$ be the full subcategory of $\mathcal{P}[E(M)]$ with objects the subgroups of S . Thus $\mathcal{P}_S[E(M)]$ is a fusion system on S .

Proposition 6.3.1 *Given p -permutation $\mathbb{F}G$ -modules M_1 and M_2 , then:*

$$\mathcal{P}_S[E(M_1)] \cap \mathcal{P}_S[E(M_2)] \leq \mathcal{P}_S[E(M_1 \oplus M_2)]$$

Proof: Write $M = M_1 \oplus M_2$. Let $P \leq S \geq Q$ and $\phi \in \mathcal{P}_S[E(M_1)] \cap \mathcal{P}_S[E(M_2)]$. Let a_1 and a_2 be the transporting units for ϕ in $E(M_1)$ and $E(M_2)$ respectively. Embedding $E(M_1)$ and $E(M_2)$ into $E(M)$ canonically, then $a_1 + a_2$ is a transporting unit for ϕ in $E(M)$. **QED**

Remark 6.3.2 *Given a p -permutation $\mathbb{F}G$ -module M , then*

$$\mathcal{F}_S[G] \leq \mathcal{P}_S[E(M)] \leq \mathcal{L}_{E(M)^\times}(E(M))$$

Proof: For the left inequality let $\sigma_M : \mathbb{F}G \rightarrow E(M)$ be the representation of G . For $K \leq S \geq H$ and $g \in G$ satisfying ${}^gK \leq H$, it is easy to check that the unit $\sigma_M(g) \in E(M)^\times$ is a transporting unit for the monomorphism $c[g] : K \rightarrow H$.

For the right inequality, let β and α be the unique local points of K and H respectively on A and let $\phi : K \rightarrow H$ be a pandemic fusion. Let $a \in A^{\times, \Delta(\phi)}$ be a transporting unit affording ϕ . Choosing $i \in \alpha$ and $j \in \beta$, we will show that a is a unit satisfying the conditions in Definition 6.1.4. Indeed, for all $k \in K$ we have:

1. $\phi(k)^{a_j} = {}^a j \phi(k)$, since ${}^a j \in A^{\phi(K)}$ by Proposition 6.2.3.
2. ${}^a(kj) = akja^{-1} = \phi(k)aja^{-1} = \phi(k)^{a_j}$
3. By Proposition 6.2.5 $\phi(K_\beta)$ is a local pointed subgroup of H_α , so by definition ${}^a(jAj) \subseteq iAi$, meaning that ${}^a j = i^a j i$.

QED

Let us give an example illustrating that the pandemic fusion system can be larger than the group fusion system.

Example 6.3.3 *Let $M = \text{Ind}_{C_2}^{A_4}(\mathbb{F}_{C_2})$ be a 2-permutation $\mathbb{F}A_4$ -module. The objects in $\mathcal{P}[E(M)]$ are the 2-subgroups of A_4 , namely $\{1\}$, three copies C_2^1, C_2^2, C_2^3 of the cyclic group of order 2 and V_4 . Now let us find all the morphisms.*

The trivial morphisms from $\{1\}$ to these objects are pandemic fusions in $\mathcal{P}[E(M)]$. Indeed, for any object P , letting $a \in A^P$ we have $xax^{-1} = a$, for all $x \in P$. Next, if $\sigma_M : \mathbb{F}A_4 \rightarrow E(A_4)$ is the representation of A_4 over \mathbb{F} , then

we let $\sigma_M(g)$ be the transporting unit affording $c[g]$, for every $c[g] : C_2 \rightarrow C_2$. Since G acts transitively on $\{C_2^1, C_2^2, C_2^3\}$, all the isomorphisms having the form $C_2^i \rightarrow C_2^j$, $i, j \in \{1, 2, 3\}$ are in $\mathcal{P}[E(M)]$. Similarly, all the monomorphisms having the form $C_2^i \rightarrow V_4$, $i \in \{1, 2, 3\}$ are in $\mathcal{P}[E(M)]$.

Now for V_4 the situation is a bit different. In fact $\text{Aut}(V_4) \cong S_3$ and there are 3 of these automorphisms which are not in A_4 , so they are not A_4 -fusions. Nevertheless, they are pandemic fusions. Indeed, using Mackey decomposition formula, we have:

$$\begin{aligned} \text{Res}_{V_4}^{A_4} \text{Ind}_{C_2}^{A_4}(\mathbb{F}_{C_2}) &\cong \bigoplus_{V_4 g C_2 \subseteq [V_4 \setminus A_4 / C_2]} \text{Ind}_{V_4 \cap g C_2}^{V_4}(\mathbb{F}_{C_2}) \\ &\cong \text{Ind}_{C_2^1}^{V_4}(\mathbb{F}_{C_2}) \oplus \text{Ind}_{C_2^2}^{V_4}(\mathbb{F}_{C_2}) \oplus \text{Ind}_{C_2^3}^{V_4}(\mathbb{F}_{C_2}) \end{aligned}$$

Since we have equal multiplicities, $\text{Res}_{V_4}^{A_4}(M) \cong \text{Res}_\phi \text{Res}_{V_4}^{A_4}(M)$, where $\phi : V_4 \rightarrow V_4$ is an automorphism. Therefore, all the automorphisms of V_4 are pandemic fusions in $\mathcal{P}[E(M)]$. Note, this shows that the pandemic fusion system can indeed be larger than the group fusion system.

6.4 The minimal pandemic fusion system

The two previous results lead us to introduce the following p -local invariant of G . Let S be a Sylow p -subgroup of G . As a fusion system on S we define:

$$\mathcal{P}_S[G] = \bigcap_M \mathcal{P}_S[E(M)]$$

where M runs over all p -permutation $\mathbb{F}G$ -modules. We call $\mathcal{P}_S[G]$ the *minimal pandemic fusion system* of G .

We show that there exists some p -permutation $\mathbb{F}G$ -module M that realizes $\mathcal{P}_S[G]$ in the sense that $\mathcal{P}_S[G] = \mathcal{P}_S[E(M)]$.

Theorem 6.4.1 *Let M_1 and M_2 be two p -permutation $\mathbb{F}G$ -modules. There exist integers r_1 and r_2 such that:*

$$\mathcal{P}_S[E(M_1)] \cap \mathcal{P}_S[E(M_2)] = \mathcal{P}_S[E(r_1M_1 \oplus r_2M_2)]$$

Proof: Let $\mathcal{U} = \{U_1, U_2, \dots, U_n\}$ be a list of indecomposable $\mathbb{F}G$ -modules appearing in the decomposition of M_1 and M_2 . More specifically let $M_1 \cong a_1U_1 \oplus a_2U_2 \oplus \dots \oplus a_nU_n$ and $M_2 \cong b_1U_1 \oplus b_2U_2 \oplus \dots \oplus b_nU_n$, where $a_1, \dots, a_n, b_1, \dots, b_n$ are non-negative integers representing the multiplicities of each summand. It is clear from Proposition 6.3.1 that $\mathcal{P}_S[E(M_1)] \cap \mathcal{P}_S[E(M_2)] \leq \mathcal{P}_S[E(r_1M_1 \oplus r_2M_2)]$, for any r_1, r_2 .

Conversely, we need to find integers r_1, r_2 for which, if $\phi \in \mathcal{P}_S[E(r_1M_1 \oplus r_2M_2)]$, then $\phi \in \mathcal{P}_S[E(M_1)] \cap \mathcal{P}_S[E(M_2)]$. This is equivalent to requiring that if $\phi \notin \mathcal{P}_S[E(M_1)] \cap \mathcal{P}_S[E(M_2)]$, then $\phi \notin \mathcal{P}_S[E(r_1M_1 \oplus r_2M_2)]$.

Applying Corollary 6.2.5, we see that if we can find integers r_1 and r_2 such that if $m(U_i, M_1) \neq m(U_j, M_1)$ or $m(U_i, M_2) \neq m(U_j, M_2)$, we have $m(U_i, r_1M_1 \oplus r_2M_2) \neq m(U_j, r_1M_1 \oplus r_2M_2)$ for any $i, j \in \{1, 2, \dots, n\}$, then we are done.

Now $m(U_i, r_1M_1 \oplus r_2M_2) = r_1m(U_i, M_1) + r_2m(U_i, M_2) = r_1a_i + r_2b_i$ and similarly $m(U_j, r_1M_1 \oplus r_2M_2) = r_1a_j + r_2b_j$. So we must have: $r_1a_i + r_2b_i \neq r_1a_j + r_2b_j \Rightarrow$

$$\frac{r_1}{r_2} \neq \frac{b_j - b_i}{a_i - a_j}$$

for any $i, j \in \{1, 2, \dots, n\}$. Since there are such positive integers r_1 and r_2 satisfying the above condition, we conclude that the module $M := r_1M_1 \oplus r_2M_2$ realizes $\mathcal{P}_S[E(M_1)] \cap \mathcal{P}_S[E(M_2)]$. **QED**

Corollary 6.4.2 *There exists a p -permutation $\mathbb{F}G$ -module M , realizing $\mathcal{P}_S[G]$.*

Proof: Since there are finitely many isomorphism classes of indecomposable p -permutation $\mathbb{F}G$ -modules say $\{M_1, M_2, \dots, M_n\}$, we only need to apply the above

proposition to find integers r_1, r_2, \dots, r_n and then let $M \cong \bigoplus_{i=1}^n M_i$.

QED



Bibliography

- [1] L. Puig, “Pointed groups and construction of characters,” *Mathematische Zeitschrift*, vol. 176, no. 2, pp. 265–292, 1981.
- [2] L. Puig, “Local fusions in block source algebras,” *Journal of Algebra*, vol. 104, no. 2, pp. 358–369, 1986.
- [3] P. Webb, “A course in finite group representation theory,” vol. 161, 2016.
- [4] T.-Y. Lam, “A first course in noncommutative rings,” vol. 131, 2013.
- [5] B. Külshammer, “Lectures on block theory,” vol. 161, 1991.
- [6] J. Thévenaz, “G-algebras and modular representation theory,” 1995.
- [7] J. L. Alperin and R. B. Bell, “Groups and representations,” vol. 162, 2012.
- [8] M. Broué, “On scott modules and p-permutation modules: an approach through the brauer morphism,” *Proceedings of the American Mathematical Society*, pp. 401–408, 1985.
- [9] J. A. Green, “Some remarks on defect groups,” *Mathematische Zeitschrift*, vol. 107, no. 2, pp. 133–150, 1968.
- [10] M. Linckelmann, “Introduction to fusion systems,” *Group representation theory*, pp. 79–113, 2007.