DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

MODELLING ACOUSTIC WAVE PROPAGATIONS INSIDE ROOMS

by
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July, 2005
İZMİR

MODELLING ACOUSTIC WAVE PROPAGATIONS INSIDE ROOMS

**A Thesis Submitted to the
Graduate School of Natural and Applied Sciences of
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the Degree of Master of Science in Mathematics**

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M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"MODELLING ACOUSTIC WAVE PROPAGATIONS INSIDE ROOMS"** completed by **ÜNSAL ATASOY** under supervision of **PROF. DR. VALERY G. YAKHNO** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Ünsal ATASOY

MODELLING ACOUSTIC WAVE PROPAGATION INSIDE ROOMS

ABSTRACT

Modeling and simulating acoustic waves are studied in this thesis. Mathematical models are based on initial boundary value problems with Dirichlet, Neumann and mixed boundary conditions. Sources of acoustic waves are pulse point sources. These sources are described by the Dirac delta function. The methods of partial differential equations, generalized functions, computer tools (symbolic calculation, graph package) are actively used in the thesis.

Keywords: acoustic system, acoustic equation, initial boundary value problem, modeling, simulation, wave propagation.

ODALARDA AKUSTİK DALGA YAYILIMLARININ MODELLENMESİ

ÖZ

Bu tezde akustik dalgaların modellemesi ve simulasyonu çalışıldı. Matematiksel modelin temeli, Dirichlet, Neumann ve karışık sınır koşullu başlangıç sınır değer problemidir. Akustik dalgaların kaynağı noktasal kaynaktır. Bu kaynaklar Dirac delta fonksiyonu olarak tanımlanırlar. Kısmi difransiyel denklemler, genelleştrilmiş fonksiyonlar metodu, bilgisayar araçları(sembolik hesaplamalar, grafik paketleri) tezde aktif olarak kullanılmıştır.

Anahtar sözcükler: akustik sistem, akustik denklem, başlangıç sınır değer problemi, modelleme, simulasyon, dalga yayılımı.

CONTENTS

CHAPTER ONE
INTRODUCTION

1.1 Goal and Methods

The phenomena of acoustic waves are the main object of the study in different applied sciences (Kagawa, Tsuchiya, Fujioka & Takeuchi, 1999; Muijers, Herman & Bussink, 1998; Korany, Blauert & Alim, 2001; Özgür, Öziş, & Alpkoçak, 2004a, 2004b). Nowadays there exists adequate mathematical models of acoustic wave propagations from one hand and modern methods and computer facilities from the other hand. An impressive large number of researchers are trying to apply the recent development of theoretical issues and computer facilities to visualize (simulate) invisible acoustic waves to study acoustic phenomenon in details.(Tsuchiya & Kagawa, Y. 2001; Tsuji, Tsuchiya & Kagawa, 2002; Campo, Rissone, & Toderi, 2000)

Faithful mathematical models for acoustic waves are described by a partial differential equation system called as the acoustic system.

This acoustic system is very often reducible to one partial differential equation called the acoustic equation. This thesis study is related to the initial boundary value problems(IBVPs) for the acoustic equation. The goal of the thesis is to

1. obtain explicit formulae for the solutions of IBVPs;
2. adjust these formulae for the case of a pulse point acoustic source described by the Dirac delta function;
3. using obtained formulae and modern computer tools to simulate acoustic waves in a homogeneous parallelepiped (an empty parallelepiped room) and a spherical domain (an empty spherical room);
4. present the results of simulation as 3-D graphs and animated movies;

1.2 Acoustic System

Let us consider the following system of partial differential equations (Cohen, 2002):

$$\frac{1}{\kappa(x)}\frac{\partial u(x,t)}{\partial t} = \nabla_x \cdot v(x,t) + F(x,t), \tag{1.2.1}$$

$$\rho(x)\frac{\partial v(x,t)}{\partial t} = \nabla_x u(x,t), \tag{1.2.2}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t \in \mathbb{R}$; $v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ is the vector-function of variables x, t and $u(x,t)$, $F(x,t)$ are scalar functions of variables x, t; $\kappa(x)$, $\rho(x)$ are scalar functions of x. The physical meaning of these functions is the following: $v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ is the velocity of the fluid motion and $u(x,t)$ is the acoustic pressure field, $\rho(x)$ is the fluid density, $\kappa(x)$ is the uncompressibility of the fluid. The velocity of the sound propagation is defined by the following formula $c(x) = \sqrt{\dfrac{\kappa(x)}{\rho(x)}}$.

The system (1.2.1), (1.2.2) is called the ***acoustic system.***

1.3 Reduction of Acoustic System to Acoustic Equation

The acoustic system (1.2.1), (1.2.2) can be reduced to the *acoustic equation* in the following way. Differentiating the both sides of the equation (1.2.1) with respect to t and using the following equality

$$\frac{\partial}{\partial t}\big(\nabla_x \cdot v(x,t)\big) = \nabla_x \cdot \big(\frac{\partial v(x,t)}{\partial t}\big)$$

we find

$$\frac{\partial}{\partial t}\big(\frac{1}{\kappa(x)}\frac{\partial u(x,t)}{\partial t}\big) \;=\; \nabla_x \cdot \big(\frac{\partial v(x,t)}{\partial t}\big) + \frac{\partial}{\partial t}\big(F(x,t)\big).$$

Using the equation (1.2.2) the last equation can be written as follows

$$\frac{1}{\kappa(x)}\frac{\partial^2 u(x,t)}{\partial t^2} = \nabla_x \cdot \big(\frac{1}{\rho(x)}\nabla_x u(x,t)\big) + \frac{\partial}{\partial t}\big(F(x,t)\big).$$

The first term in the right-hand side of the last equation can be presented in the form

$$\nabla_x \cdot \big(\frac{1}{\rho(x)}\nabla_x u(x,t)\big) = \frac{1}{\rho(x)}\Delta_x u(x,t) + \nabla_x\big(\frac{1}{\rho(x)}\big)\nabla_x u(x,t).$$

Hence, we find

$$\frac{1}{\kappa(x)}\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{1}{\rho(x)}\Delta_x u(x,t) + \nabla_x\big(\frac{1}{\rho(x)}\big)\nabla_x u(x,t) + f(x,t), \qquad (1.3.1)$$

where $\Delta_x = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, $\nabla_x = \big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\big)$, $f(x,t) = \frac{\partial}{\partial t}F(x,t)$

The equation (1.3.1) is called the ***acoustic wave equation.***

The main object of this study is the following Initial Boundary Value problems (IBVPs) for this acoustic equation in a parallelepiped or a sphere with Dirichlet, Neumann, Robin (Mixed) boundary conditions.

1.4 Statements of IBVPs for Acoustic Wave Equation

Let $D \subset \mathbb{R}^3$ be a bounded domain, $x \in D$ and $t \geq 0$. Let us consider

$$\frac{1}{\kappa(x)}\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{1}{\rho(x)}\Delta_x u(x,t) + \nabla_x(\frac{1}{\rho(x)})\nabla_x u(x,t) + f(x,t), \quad (1.4.1)$$

$$u(x,0) = \phi(x), \quad \frac{\partial u(x,t)}{\partial t}\Big|_{t=0} = \psi(x), \quad (1.4.2)$$

$$\big(\alpha_1 u(x,t) + \alpha_2 \frac{\partial u(x,t)}{\partial t}\big)|_{\partial D} = 0, \quad (1.4.3)$$

where ∂D is the boundary of the domain D, $f(x,t)$ is given scalar function of variables x, t; $\kappa(x)$, $\rho(x)$, $\phi(x)$, $\psi(x)$ are given scalar functions of x. This problem (1.4.1) - (1.4.3) is called ***initial boundary value problem for the acoustic equation.*** The initial boundary value problem (1.4.1) - (1.4.3) will be studied for the following boundary conditions (Cohen, 2002):

1. Dirichlet boundary condition : $\alpha_1 \neq 0$, $\alpha_2 = 0$,
2. Neumann boundary condition : $\alpha_1 = 0$, $\alpha_2 \neq 0$
3. Mixed (or Robin) boundary condition : $\alpha_1 \neq 0$, $\alpha_2 \neq 0$

1.5 Pulse Point Acoustic Source

If a source of the wave propagation is acting in a moment, in order to model this physical situation we will use pulse point sources. The mathematical model to represent this pulse point sources is the Dirac delta function which was originally introduced by Paul Dirac, a well known physicist of the first half of the C20.

The function $f(x,t)$ in the equation (1.4.1) is the internal source in the room and the equations (1.4.2) are the initial conditions.

In our study we use the Dirac delta function in the following cases of the acoustic wave equation in a parallelepiped,

1. $\phi(x) = 0$, $\psi(x) = 0$, $f = \delta(x - x_1)\delta(x - x_2)\delta(x - x_3)\delta(t - t_0)$
2. $\phi(x) = \delta(x - x_1)\delta(x - x_2)\delta(x - x_3)$, $\psi(x) = 0$, $f = 0$
3. $\phi(x) = 0$, $\psi(x) = \delta(x - x_1)\delta(x - x_2)\delta(x - x_3)$, $f = 0$

   where $\delta(\cdot)$ is the Dirac delta function.

   Here (x_1, x_2, x_3) is the location of the source and t_0 is the time when the source acts.

CHAPTER TWO
IBVP FOR ACOUSTIC EQUATION IN A PARALLELEPIPED WITH DIRICHLET BOUNDARY CONDITION

Let $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \le x \le b_1, 0 \le y \le b_2, 0 \le z \le b_3\}$ be a parallelepiped from $\mathbb{R}^3$. We assume that $\rho(x)$ and $\kappa(x)$ have constant values inside the parallelepiped D. In this case we will have a homogeneous parallelepiped (*medium*). In this section IBVP for acoustic(wave) equation with different boundary conditions in a homogeneous parallelepiped will be studied.

2.1 IBVP for a Homogeneous Acoustic Equation with Dirichlet Boundary Condition

Let $(x, y, z) \in \mathbb{R}^3$, $t \in \mathbb{R}$, $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \le x \le b_1, 0 \le y \le b_2, 0 \le z \le b_3\}$ be a homogeneous parallelepiped. Let us consider the IBVP for the wave equation in the parallelepiped D:

$$\frac{1}{c^2}\frac{\partial^2 u(x,y,z,t)}{\partial t^2} = \Delta_{x,y,z} u(x,y,z,t), \quad (x,y,z) \in D, \quad t \ge 0, \tag{2.1.1}$$

$$u(x,y,z,0) = \phi(x,y,z), \tag{2.1.2}$$

$$\left.\frac{\partial u(x,y,z,t)}{\partial t}\right|_{t=0} = \psi(x,y,z), \tag{2.1.3}$$

$$u(x,y,z,t)|_{\partial D} = 0, \tag{2.1.4}$$

where ∂D is the boundary of D. Here $\phi(x, y, z)$, $\psi(x, y, z)$ are given functions in $D \cup \partial D$, c is a positive constant. IBVP consists of finding $u(x, y, z, t)$ satisfying (2.1.1), (2.1.2), (2.1.3), (2.1.4).
We will use the separation of variables:

$$u(x,y,z,t) = T(t)V(x,y,z). \tag{2.1.5}$$

Substituting (2.1.5) into (2.1.1) we find

$$\frac{1}{c^2}T''(t)V(x,y,z) = T(t)\Delta_{x,y,z}V(x,y,z).$$

Dividing both sides of the last equation by $T(t)V(x,y,z)$, we have

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{\Delta_{x,y,z}V(x,y,z)}{V(x,y,z)}.$$

Let us fix t and let (x,y,z) vary in the parallelepiped D. Since t and (x,y,z) are independent variables then the last equality takes place if and only if the left hand side and the right hand side of this equation are equal to the same constant which we denote as $-\lambda$ (Kyte, 1997).

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{\Delta_{x,y,z}V(x,y,z)}{V(x,y,z)} = -\lambda.$$

As a result of it we have the following equations:

$$\Delta_{x,y,z}V(x,y,z) + \lambda V(x,y,z) = 0, \quad (x,y,z) \in D, \tag{2.1.6}$$

$$T''(t) + c^2\lambda T(t) = 0, \quad t > 0. \tag{2.1.7}$$

Substituting (2.1.5) into (2.1.4) we obtain

$$V(x,y,z)|_{\partial D} = 0. \tag{2.1.8}$$

This means that $V(x,y,z)$ has to be a solution of the following eigenvalue-eigenfunction problem (Sturm-Liouville problem):

$$\Delta_{x,y,z}V(x,y,z) + \lambda V(x,y,z) = 0, \quad (x,y,z) \in D, \tag{2.1.9}$$

$$V(x,y,z)|_{\partial D} = 0. \tag{2.1.10}$$

For the solution of the eigenvalue-eigenfunction problem we will use the method of separation of variables. Setting

$$V(x,y,z) = X(x)Y(y)Z(z) \tag{2.1.11}$$

and substituting (2.1.11) into (2.1.9) and dividing both sides of the equation (2.1.9) by $X(x)Y(y)Z(z)$, we have

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} + \lambda = 0.$$

Thus

$$\frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} + \lambda = -\frac{X''(x)}{X(x)}.$$

The left-hand side of the last equation depends on variables y, z, right-hand side depends on x only. Since the variable x, y, z are independent variables, the last equation holds if and only if the both sides of this equation are equal to the same constant which we denote as α

$$\frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} + \lambda = -\frac{X''(x)}{X(x)} = \alpha.$$

Hence

$$\frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} + \lambda = \alpha.$$

Repeating the same argument to the last equation we find

$$\begin{aligned} \frac{Z''(z)}{Z(z)} + \lambda - \alpha &= -\frac{Y''(y)}{Y(y)}, \\ \frac{Z''(z)}{Z(z)} + \lambda - \alpha &= -\frac{Y''(y)}{Y(y)} = \beta, \end{aligned}$$

where α, β, γ are some constants such that

$$\begin{aligned} \lambda - \alpha - \beta &= \frac{Z''(z)}{Z(z)} = \gamma, \\ \lambda &= \alpha + \beta + \gamma. \end{aligned}$$

Also, substituting (2.1.11) into (2.1.10) we have three Sturm-Liouville equations for the ordinary differential equations:

$$X''(x) + \alpha X(x) = 0, \quad X(0) = X(b_1) = 0, \tag{2.1.12}$$

$$Y''(x) + \beta Y(x) = 0, \quad Y(0) = Y(b_2) = 0, \tag{2.1.13}$$

$$Z''(x) + \gamma Z(x) = 0, \quad Z(0) = Z(b_3) = 0. \tag{2.1.14}$$

The solutions of these eigenvalue - eigenfunction problem (2.1.12), (2.1.13), (2.1.14) are given by

$$\begin{aligned}
\alpha_i = \left(\frac{i\pi}{b_1}\right)^2, \quad X_i(x) &= \sqrt{\frac{2}{b_1}} \sin \sqrt{\alpha_i} x, \quad i = 1, 2, \ldots; \\
\beta_j = \left(\frac{j\pi}{b_2}\right)^2, \quad Y_j(y) &= \sqrt{\frac{2}{b_2}} \sin \sqrt{\beta_j} y, j = 1, 2, \ldots; \\
\gamma_k = \left(\frac{k\pi}{b_3}\right)^2, \quad Z_k(z) &= \sqrt{\frac{2}{b_3}} \sin \sqrt{\gamma_k} z, \quad k = 1, 2, \ldots;
\end{aligned}$$

(see Appendix A.1 for detailed explanation).
As a result of this we get the solution of eigenvalue - eigenfunction problem (2.1.9), (2.1.10) as follows:

$$\lambda_{ijk} = \left(\frac{i\pi}{b_1}\right)^2 + \left(\frac{j\pi}{b_2}\right)^2 + \left(\frac{k\pi}{b_3}\right)^2, \tag{2.1.15}$$

$$V_{ijk} = \sqrt{\frac{8}{b_1 b_2 b_3}} \sin \sqrt{\alpha_i} x \sin \sqrt{\beta_j} y \sin \sqrt{\gamma_k} z, \tag{2.1.16}$$

$$i, j, k = 1, 2, \ldots$$

The following remark will help to construct the Fourier series.

Remark 2.1.1. The system of eigenfunctions $\{V_{ijk}\}_{i,j,k=1,2,\ldots}$ is a complete orthonormal basis in the space $L_2(D)$, $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq b_1, 0 \leq y \leq b_2, 0 \leq z \leq b_3\}$. This means that each function $\varphi(x, y, z)$ from $L_2(D)$ (or generalized function [3]), may be written in the form of Fourier Series expansion which is convergent in $L_2(D)$:

$$\varphi(x, y, z) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \varphi_{ijk} V_{ijk}(x, y, z),$$

where

$$\varphi_{ijk} = \int_0^{b_1} \int_0^{b_2} \int_0^{b_3} \varphi(x, y, z) V_{ijk}(x, y, z) dz dy dx.$$

Remark 2.1.2 (Fourier Series Expansion of $\delta(x, y, z)$ function). We can generalize the idea of Fourier expansion of a classical function for the Dirac delta function $\delta(x - x_0, y - y_0, z - z_0)$, where (x_0, y_0, z_0) is a fixed point in D. It may be written in form of Fourier Series expansion as follows:

$$\delta(x - x_0, y - y_0, z - z_0) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \delta_{ijk} V_{ijk}(x, y, z), \tag{2.1.17}$$

where

$$\begin{aligned}\delta_{ijk} &= \int_0^{b_1}\int_0^{b_2}\int_0^{b_3} \delta(x-x_0, y-y_0, z-z_0) V_{ijk}(x,y,z)dzdydx \\ &= V_{ijk}(x_0, y_0, z_0).\end{aligned}$$

Let us consider the equation (2.1.7) for $\lambda = \lambda_{ijk}$ for $i, j, k = 1, 2, \ldots$. This equation may be written in the form

$$T''_{ijk}(t) + c^2\lambda_{ijk}T_{ijk}(t) = 0, \quad i, j, k = 1, 2, \ldots \tag{2.1.18}$$

A general solution of (2.1.18) is given by

$$T_{ijk}(t) = A_{ijk}\cos(c\sqrt{\lambda_{ijk}}t) + B_{ijk}\sin(c\sqrt{\lambda_{ijk}}t),\ t > 0, \tag{2.1.19}$$

where A_{ijk}, B_{ijk} are arbitrary constants for $i, j, k = 1, 2, \ldots$. Let us consider function $u(x, y, z, t)$ defined by

$$\begin{aligned}u(x,y,z,t) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\Big(A_{ijk}\cos(c\sqrt{\lambda_{ijk}}t) \\ &+ B_{ijk}\sin(c\sqrt{\lambda_{ijk}}t)\Big)V_{ijk}(x,y,z),\end{aligned} \tag{2.1.20}$$

where A_{ijk}, B_{ijk} are arbitrary constants for $i, j, k = 1, 2, \ldots$. This function satisfies (2.1.1), (2.1.4). The main goal consists in finding A_{ijk}, B_{ijk} for which the initial data (2.1.2), (2.1.3) are hold. Using the orthonormal functions $V_{ijk}(x, y, z)$, we can write $\phi(x, y, z)$ and $\psi(x, y, z)$ in the form of the Fourier series expansion as follows:

$$\phi(x,y,z) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\phi_{ijk}V_{ijk}(x,y,z), \tag{2.1.21}$$

$$\psi(x,y,z) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\psi_{ijk}V_{ijk}(x,y,z), \tag{2.1.22}$$

where

$$\begin{aligned}\phi_{ijk} &= \int_0^{b_1}\int_0^{b_2}\int_0^{b_3} \phi(x,y,z)V_{ijk}(x,y,z)dzdydx, \\ \psi_{ijk} &= \int_0^{b_1}\int_0^{b_2}\int_0^{b_3} \psi(x,y,z)V_{ijk}(x,y,z)dzdydx.\end{aligned}$$

Using the initial conditions (2.1.2), (2.1.3) and (2.1.20) – (2.1.22) we can find A_{ijk} and B_{ijk} by

$$\begin{aligned} A_{ijk} &= \phi_{ijk}, \\ B_{ijk} &= \frac{\psi_{ijk}}{c\sqrt{\lambda_{ijk}}},\ i,j,k = 1,2,\dots \end{aligned}$$

Substituting found values of A_{ijk} and B_{ijk} into (2.1.20), we have

$$\begin{aligned} u(x,y,z,t) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\big(\phi_{ijk}\cos(c\sqrt{\lambda_{ijk}}t) \\ &+ \frac{\psi_{ijk}}{c\sqrt{\lambda_{ijk}}}\sin(c\sqrt{\lambda_{ijk}}t)\big)V_{ijk}(x,y,z), \end{aligned} \tag{2.1.23}$$

where

$$\begin{aligned} \phi_{ijk} &= \int_0^{b_1}\int_0^{b_2}\int_0^{b_3}\phi(x,y,z)V_{ijk}(x,y,z)dzdydx, \\ \psi_{ijk} &= \int_0^{b_1}\int_0^{b_2}\int_0^{b_3}\psi(x,y,z)V_{ijk}(x,y,z)dzdydx, \\ V_{ijk} &= X_i(x)Y_j(y)Z_k(z) = \sqrt{\frac{8}{b_1b_2b_3}}\sin\sqrt{\alpha_i}x\sin\sqrt{\beta_j}y\sin\sqrt{\gamma_k}z, \\ \lambda_{ijk} &= \alpha_i+\beta_j+\gamma_k = \big(\frac{i\pi}{b_1}\big)^2+\big(\frac{j\pi}{b_2}\big)^2+\big(\frac{k\pi}{b_3}\big)^2, \\ & i,j,k = 1,2,\dots \end{aligned}$$

This function $u(x,y,z,t)$ satisfies (2.1.1)–(2.1.4). Therefore this function is a solution of the problem (2.1.1)–(2.1.4).

2.2 IBVP for a Non-homogeneous Acoustic Equation with Dirichlet Boundary Condition

Let $(x,y,z)\in\mathbb{R}^3$, $t\in\mathbb{R}$, $D=\{(x,y,z)\in\mathbb{R}^3 : 0\le x\le b_1, 0\le y\le b_2, 0\le z\le b_3\}$ be a homogeneous parallelepiped. Let us consider the IBVP for the wave equation in the parallelepiped D:

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial^2 u(x,y,z,t)}{\partial t^2} &= \Delta_{x,y,z}u(x,y,z,t) \\
&\quad +F(x,y,z,t), \quad (x,y,z)\in D, t>0, && (2.2.1)\\
u(x,y,z,0) &= \phi(x,y,z), && (2.2.2)\\
\frac{\partial u(x,y,z,t)}{\partial t}\Big|_{t=0} &= \psi(x,y,z), && (2.2.3)\\
u(x,y,z,t)|_{\partial D} &= 0, && (2.2.4)
\end{aligned}$$

where ∂D is the boundary of D. Here $F(x,y,z,t)$, $\phi(x,y,z)$, $\psi(x,y,z)$ are given functions, c is a positive constant. IBVP consists of finding $u(x,y,z,t)$ satisfying (2.2.1)–(2.2.4).

Consider the system of eigenfunctions which is a solution of the eigenvalue-eigenfunction problem for the Laplace operator in a parallelepiped with the Dirichlet boundary condition. These are functions $\{V_{ijk}(x,y,z)\}_{i,j,k=1,2,\dots}$ satisfying (2.1.9) and (2.1.10). Using this system of functions, we will find the solution in the form:

$$u(x,y,z,t)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}T_{ijk}(t)V_{ijk}(x,y,z), \tag{2.2.5}$$

where $T_{ijk}(t)$ are unknown Fourier coefficients of $u(x,y,z,t)$. We have

$$F(x,y,z,t)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}F_{ijk}(t)V_{ijk}(x,y,z), \tag{2.2.6}$$

where

$$F_{ijk}=\int_0^{b_1}\int_0^{b_2}\int_0^{b_3}F(x,y,z,t)V_{ijk}(x,y,z)dzdydx,\ i,j,k=1,2,\dots \tag{2.2.7}$$

Substituting (2.2.6) and (2.2.5) into (2.2.1) we will get

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{1}{c^2}T''_{ijk}(t)V_{ijk}(x,y,z)-T_{ijk}(t)\Delta V_{ijk}(x,y,z)-F_{ijk}(t)V_{ijk}(x,y,z)\right)=0.$$

And using (2.1.9) we get

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{1}{c^2}T''_{ijk}(t)V_{ijk}(x,y,z)+T_{ijk}(t)\lambda_{ijk}V_{ijk}(x,y,z)-F_{ijk}(t)V_{ijk}(x,y,z)\right)=0.$$

Hence

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{1}{c^2}T''_{ijk}(t)+\lambda_{ijk}T_{ijk}(t)-F_{ijk}(t)\right)V_{ijk}(x,y,z)=0.$$

Since the function system $\{V_{ijk}(x,y,z)\}_{i,j,k=1,2,3,\dots}$ are normalized orthogonal system, the previous equation implies a second order ordinary differential equation as follows

$$\frac{1}{c^2}T''_{ijk}(t)+\lambda_{ijk}T_{ijk}(t)-F_{ijk}(t)=0,\quad t>0.$$

And using (2.2.2), (2.2.3) and (2.1.21), (2.1.22) we get

$$T_{ijk}(0)=\phi_{ijk},$$
$$T'_{ijk}(0)=\psi_{ijk}.$$

So we have the following system for $T_{ijk}(t)$, $i,j,k=1,2,3,\dots$

$$\frac{1}{c^2}T''_{ijk}(t)+\lambda_{ijk}T_{ijk}(t)-F_{ijk}(t)=0,\quad t>0, \tag{2.2.8}$$
$$T_{ijk}(0)=\phi_{ijk},\quad T'_{ijk}(0)=\psi_{ijk}. \tag{2.2.9}$$

Remark 2.2.1. The function $u(x,y,z,t)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}T_{ijk}(t)V_{ijk}(x,y,z)$ is a solution for (2.2.1)–(2.2.4) if and only if $T_{ijk}(t)$ satisfy (2.2.8) and (2.2.9) for $i,j,k=1,2,3,\dots$.

The solution of (2.2.8), (2.2.9) for $i,j,k=1,2,3,\dots$ is given by

$$T_{ijk}(t)=\phi_{ijk}\cos(c\sqrt{\lambda_{ijk}}t)+\frac{\psi_{ijk}}{c\sqrt{\lambda_{ijk}}}\sin(c\sqrt{\lambda_{ijk}}t)$$
$$+\frac{c}{\sqrt{\lambda_{ijk}}}\int_0^t F_{ijk}(\tau)\sin(c\sqrt{\lambda_{ijk}}(t-\tau))d\tau, \tag{2.2.10}$$

(See Appendix A.1).

Thus

$$u(x,y,z,t)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}T_{ijk}(t)V_{ijk}(x,y,z) \tag{2.2.11}$$

is the solution of the problem (2.2.1)–(2.2.4), where $T_{ijk}(t)$ is defined in (2.2.10) and $V_{ijk}(x,y,z)$ is defined in (2.1.16) for $i,j,k=1,2,3,\dots$.

2.3 Examples of simulations of acoustic waves in 3-D rooms with soft walls

This subsection deals with examples of modeling and simulations of acoustic waves in 3-D rooms with soft walls. IBVP for the wave equation with the Dirichlet boundary condition in a parallelepiped is the mathematical model of the wave propagation in 3-D rooms with soft walls. We took a pulse point source concentrated in different positions in the room: in the center, near the wall, in the corner of the room. For each of these cases acoustic waves were modeled and simulated. The room is a parallelepiped $[0,6]\times[0,12]\times[0,3]$ and the speed of sound is equal to 1 for all examples.

2.3.1 Example 1:

The mathematical model of acoustic wave is given by (2.1.1) – (2.1.4), where

$$\begin{aligned}\phi(x,y,z) &= 0,\\ \psi(x,y,z) &= \delta(x-x_0)\delta(y-y_0)\delta(z-z_0),\ (x_0,y_0,z_0)\in D,\end{aligned}$$

where $\delta(x)$ is Dirac delta function. Using formulas for $\phi(x,y,z)$, $\psi(x,y,z)$ and the properties of the Dirac-delta function the solution of (2.1.1) - (2.1.4) is given by the formula:

$$u(x,y,z,t)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\Big(\frac{\psi_{ijk}}{c\sqrt{\lambda_{ijk}}}\sin(c\sqrt{\lambda_{ijk}}t)\Big)V_{ijk}(x,y,z),\quad t\geq 0,\qquad (2.3.1)$$

where

$$\begin{aligned}\psi_{ijk} &= \int_0^{b_1}\int_0^{b_2}\int_0^{b_3}\psi(x,y,z)V_{ijk}(x,y,z)dzdydx\\ &= \int_0^{b_1}\int_0^{b_2}\int_0^{b_3}\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)V_{ijk}(x,y,z)dzdydx\\ &= V_{ijk}(x_0,y_0,z_0),\end{aligned}$$

λ_{ijk} and $V_{ijk}(x,y,z)$ are defined in (2.1.15), (2.1.16) for $i,j,k=1,2,3,\ldots$.

For the given data of Example 1, we used Mathematica codes in order to find the solution and to animate when $x_0 = 3$, $y_0 = 6$, $z_0 = 2$, $c = 1$. The exact solution (2.3.1) is given by triple series containing infinite number of terms. We found an approximate solution by means of the formula

$$u(x,y,z,t) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\Big(\frac{\psi_{ijk}}{c\sqrt{\lambda_{ijk}}}\sin(c\sqrt{\lambda_{ijk}}t)\Big)V_{ijk}(x,y,z). \qquad (2.3.2)$$

where $n = 20$. This number $n = 20$ we choose from the theoretical evaluation of the general term of the seris (2.3.2) and the numerical experiments.

2.3.1.1 Commands in Mathematica for Example 1:

```
<< Graphics`Animation`

bx = 6; by = 12; bz = 3; (dimesion of the parallelepiped)

x0 = 3;y0=6; z0 = 2; (the position of the source where it is
located)

n = 20; (number of terms in the sum)

sc = 1; (the speed of ropagation)

(eigenvalues)

a[kk_] = (kk*Pi/bx)^2; b[kk_] = (kk*Pi/by)^2; c[kk_] = (kk*Pi/bz)^2;

(computing the eigenfunctions and Fourier coefficients in this loop)

For[i = 1, i <= n, i++,
 For[j = 1, j <= n, j++,
  For[k = 1, k <= n,k++,
   Lambda[i, j, k] = a[i] + b[j] + c[k] ;
    V[i, j, k]= Sqrt[8/(bx*by*bz)]*Sin[Sqrt[a[i]]*x]*Sin[Sqrt[b[j]]*y]
                *Sin[Sqrt[c[k]]*z];
    Apsi[i, j, k] =
    ReplaceAll[ReplaceAll[ReplaceAll[V[i, j, k], x -> x0], y -> y0],
     z -> z0];
```

```
     T[i, j, k] = (Apsi[i, j, k]/(sc*Sqrt[Lambda[i, j, k]]))*
      Sin[sc*Sqrt[Lambda[i, j, k]]*t];
]; ]; ]
```

(computing the sum)

```
son := Sum[Sum[Sum[T[i, j, k]*V[i, j, k], {i, 1, n}], {j, 1, n}],
{k, 1, n}]
```

(Fixing parameter z = 1)

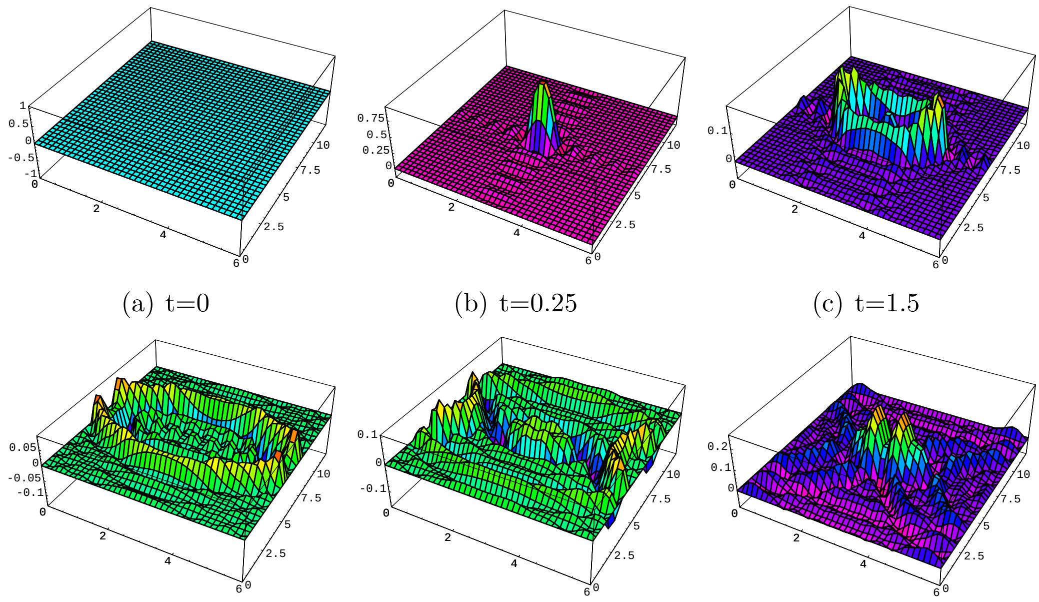
```
son2 = ReplaceAll[son, z -> 1];
```

(Animating the picture with respect to time variable with step size 0.25)

```
Animate[DensityPlot[son2, {x, 0, 6}, {y, 0, 12},
    ColorFunction -> (Hue[1 - #] &), PlotRange -> All,
    AspectRatio -> Automatic, Mesh -> False, PlotPoints -> 40],
    {t, 0, 10, 0.5}]
```

2.3.1.2 Results of the simulation by Formula (2.3.2)

Using the formula (2.3.2) for $n = 20$ and Mathematica codes we simulated the acoustic waves arising from a pulse point source which is located in the center of the room. The results of this simulation are presented on the figure 2.1 as screen shots for different times.



(a) t=0 (b) t=0.25 (c) t=1.5

(d) t=3 (e) t=4.25 (f) t=6

Figure 2.1: Example 1:The source in the center of the room with soft walls

The horizontal axes x and y show the length and width of the room. Vertical axis z is the acoustic pressure field. In figure 2.1 (a) there is no fluctuation of acoustic field. We can see on the figure 2.1 (b) the acoustic field arising from the pulse point source concentrated at the center of the room. There is also small fluctuations which stem from the numerical error of the series (2.3.2). In figure 2.1 (c) the circular wave front is seen and in figure 2.1 (d) wave front touches the boundary, in the figure 2.1 (e) reflection from the boundary and in last figure 2.1 (f) we can see the interaction between the refracted waves.

2.3.2 Example 2:

For the given functions

$$\begin{aligned} \phi(x,y,z) &= \delta(x-x_0)\delta(y-y_0)\delta(z-z_0),\ (x_0,y_0,z_0)\in D, \\ \psi(x,y,z) &= 0, \end{aligned}$$

the solution of (2.1.1) – (2.1.4) can be written in the following form:

$$u(x,y,z,t)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\phi_{ijk}\cos(c\sqrt{\lambda_{ijk}}t)V_{ijk}(x,y,z), \quad t\geq 0, \tag{2.3.3}$$

where

$$\begin{aligned} \phi_{ijk} &= \int_0^{b_1}\int_0^{b_2}\int_0^{b_3}\phi(x,y,z)V_{ijk}(x,y,z)dzdydx \\ &= \int_0^{b_1}\int_0^{b_2}\int_0^{b_3}\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)V_{ijk}(x,y,z)dzdydx \\ &= V_{ijk}(x_0,y_0,z_0). \end{aligned}$$

λ_{ijk} and $V_{ijk}(x,y,z)$ are defined in (2.1.15), (2.1.16) for $i,j,k=1,2,3,\ldots$.

For the given data of Example 2, we used following Mathematica codes in order to find the solution and to animate when $x_0=3$, $y_0=6$, $z_0=2$, $c=1$. The exact solution (2.3.1) is given by triple series containing infinite number of terms. We found an approximate solution by means of the formula

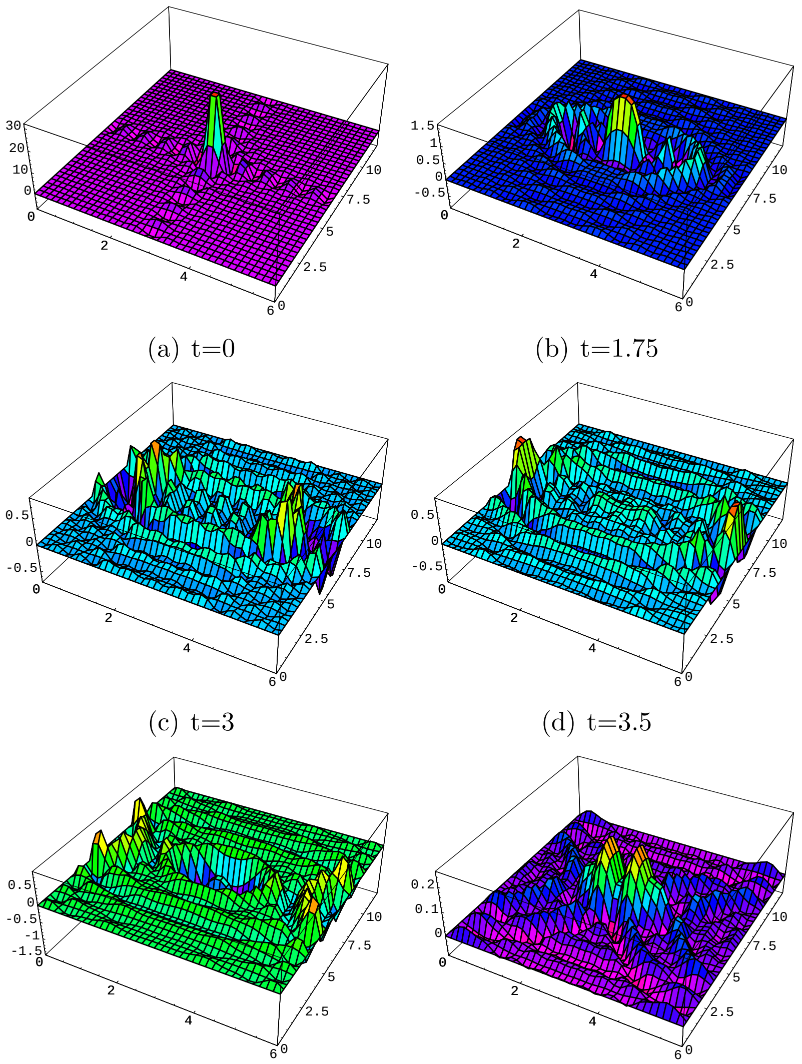
$$u(x,y,z,t)=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\phi_{ijk}\cos(c\sqrt{\lambda_{ijk}}t)V_{ijk}(x,y,z), \quad t\geq 0, \tag{2.3.4}$$

where $n=20$.

2.3.2.1 Commands in Mathematica for Example 2

```
<< Graphics`Animation`

bx = 6; by = 12; bz = 3;

x0 = 3; y0=6; z0 = 2; n = 20; sc = 1;

a[kk_] = (kk*Pi/bx)^2; b[kk_] = (kk*Pi/by)^2; c[kk_] = (kk*Pi/bz)^2;

For[i = 1, i <= n, i++,
 For[j = 1, j <= n, j++,
  For[k = 1, k <= n,k++,
   Lambda[i, j, k] = a[i] + b[j] + c[k] ;
    V[i, j, k]= Sqrt[8/(bx*by*bz)]*Sin[Sqrt[a[i]]*x]*Sin[Sqrt[b[j]]*y]
                *Sin[Sqrt[c[k]]*z];
    Aphi[i, j, k] =
    ReplaceAll[ReplaceAll[ReplaceAll[V[i, j, k], x -> x0], y -> y0],
     z -> z0];
     T[i, j, k] = Aphi[i, j, k]*Cos[sc*Sqrt[Lambda[i, j, k]]*t];;
]; ]; ]

son := Sum[Sum[Sum[T[i, j, k]*V[i, j, k], {i, 1, n}], {j, 1, n}],
{k, 1, n}]

son2 = ReplaceAll[son, z -> 1];

(Animating the picture with respect to time variable with step size
0.25 )

Animate[DensityPlot[son2, {x, 0, 6}, {y, 0, 12},
    ColorFunction -> (Hue[1 - #] &), PlotRange -> All,
    AspectRatio -> Automatic, Mesh -> False, PlotPoints -> 40],
     {t, 0, 10, 0.25}]
```

2.3.2.2 Results of simulation by Formula (2.3.4)



(a) t=0 (b) t=1.75

(c) t=3 (d) t=3.5

(e) t=4 (f) t=6

Figure 2.2: Example 2:The source in the center of the room with soft walls

The horizontal and vertical axes are the same as defined before. In the figure 2.2 (a) we have the wave field which arises from the pulse point source described by the function $\phi(x, y, z) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$. In the figure 2.2 (b) the circular wave front is seen, in the figure 2.2 (c) the wave front touches the boundary, in figure 2.2 (d) reflection starts from the boundary and we can see clearly in figure 2.2 (e) the reflection of the acoustic wave from the boundary, in the last figure 2.2 (f) the interaction of the wave fronts are arise.

2.3.3 Example 3:

Let us consider the problem (2.2.1) - (2.2.4). The functions $\phi(x,y,z)$, $\psi(x,y,z)$, $F(x,y,z,t)$ are given by

$$\begin{aligned} \phi(x,y,z) &= 0, \\ \psi(x,y,z) &= 0, \\ F(x,y,z,t) &= \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\delta(t),\ (x_0,y_0,z_0) \in D,\ t \geq 0. \end{aligned}$$

Using formulas for $\phi(x,y,z)$, $\psi(x,y,z)$, $F(x,y,z,t)$ and the properties of the Dirac delta function we can write the solution of the problem (2.2.1) – (2.2.4) in the following form

$$u(x,y,z,t) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{c}{\sqrt{\lambda_{ijk}}} \sin(c\sqrt{\lambda_{ijk}}t) V_{ijk}(x_0,y_0,z_0) V_{ijk}(x,y,z), \quad (2.3.5)$$

and λ_{ijk} and $V_{ijk}(x,y,z)$ are defined in (2.1.15), (2.1.16). For this example we consider the different positions of (x_0,y_0,z_0) (the point in which the pulse point source is concentrated). These positions of (x_0,y_0,z_0) are given in the following table:

	(x_0,y_0,z_0)
Case 1	$(3,6,2)$
Case 2	$(3,11,2)$
Case 3	$(1,11,2)$

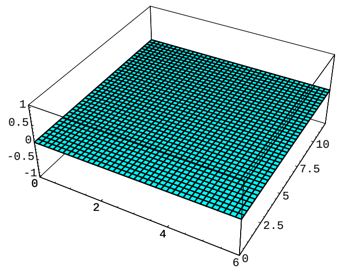
Table 2.1: Positions of Sources in a room with soft walls

2.3.3.1 Commands in Mathematica for Example 3

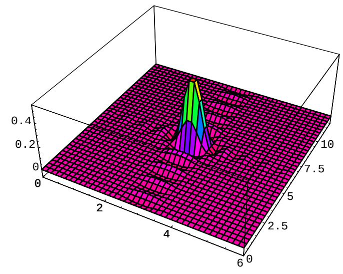
```
<< Graphics`Animation` bx = 6; by = 12; bz = 3; x0 = 3; y0=6; z0=2;
n = 20; sc = 1;

a[kk_] = (kk*Pi/bx)^2; b[kk_] = (kk*Pi/by)^2; c[kk_] = (kk*Pi/bz)^2;

For[i = 1, i <= n, i++,
 For[j = 1, j <= n, j++,
  For[k = 1, k <= n,k++,
   Lambda[i, j, k] = a[i] + b[j] + c[k] ;
    V[i, j, k]= Sqrt[8/(bx*by*bz)]*Sin[Sqrt[a[i]]*x]*Sin[Sqrt[b[j]]*y]
                *Sin[Sqrt[c[k]]*z];
      PV[i, j, k] =ReplaceAll[ReplaceAll[ReplaceAll[V[i, j, k], x -> x0],
       y -> y0],z -> z0];
    T[i, j, k] = (sc/Sqrt[Lambda[i, j, k]])*Sin[sc*Sqrt[Lambda[i,
j, k]]*t]* PV[i, j, k]; ]; ]; ]

son := Sum[Sum[Sum[T[i, j, k]*V[i, j, k], {i, 1, n}], {j, 1, n}],
{k, 1, n}] son2 = ReplaceAll[son, z -> 1];
```

2.3.3.2 Result of the simulation by formula(2.3.5)

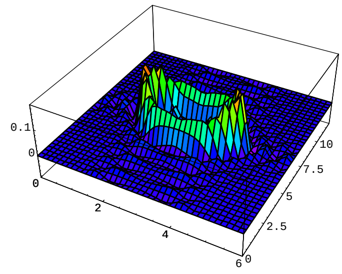
The result of the simulation of acoustic field corresponding to $x_0 = 3, y_0 = 6, z_0 = 2$ (the position of the source is the center) is presented on the figure 2.3 (a)-(f).



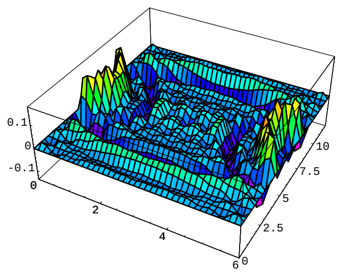
(a) t=0



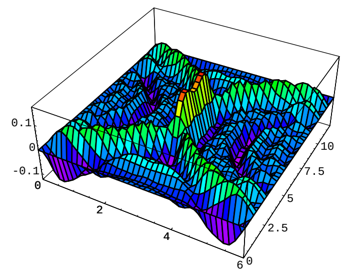
(b) t=1.75



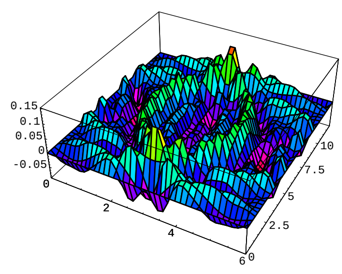
(c) t=3



(d) t=4



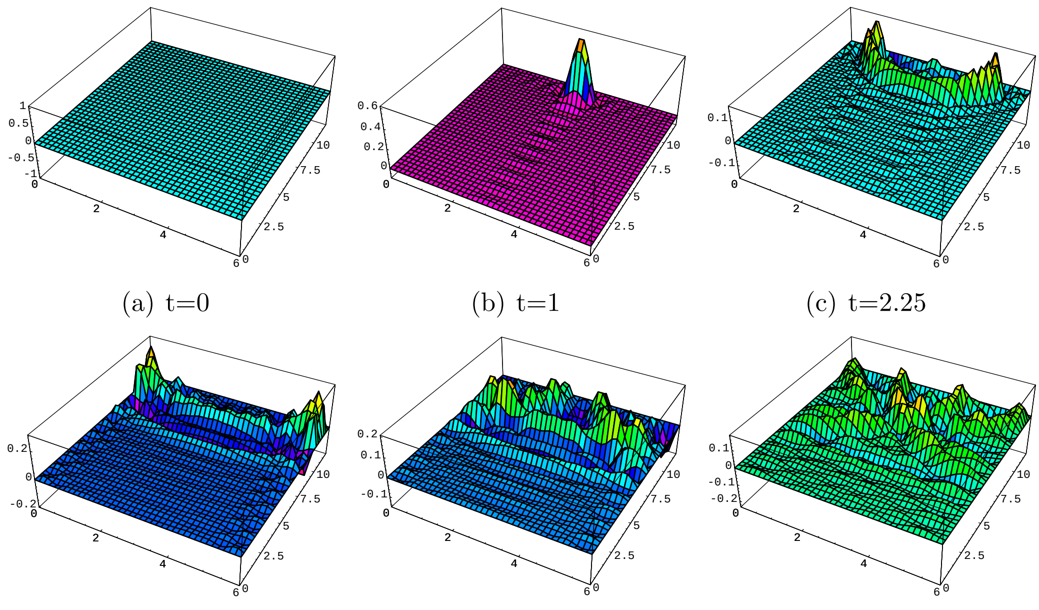
(e) t=6



(f) t=8

Figure 2.3: Case 1 of Example 3: The source in the center of the room

The result of the simulation of acoustic field corresponding to $x_0 = 3, y_0 = 11, z_0 = 2$ (the position of the source is near the wall) is presented on the figure 2.4 (a)-(f).

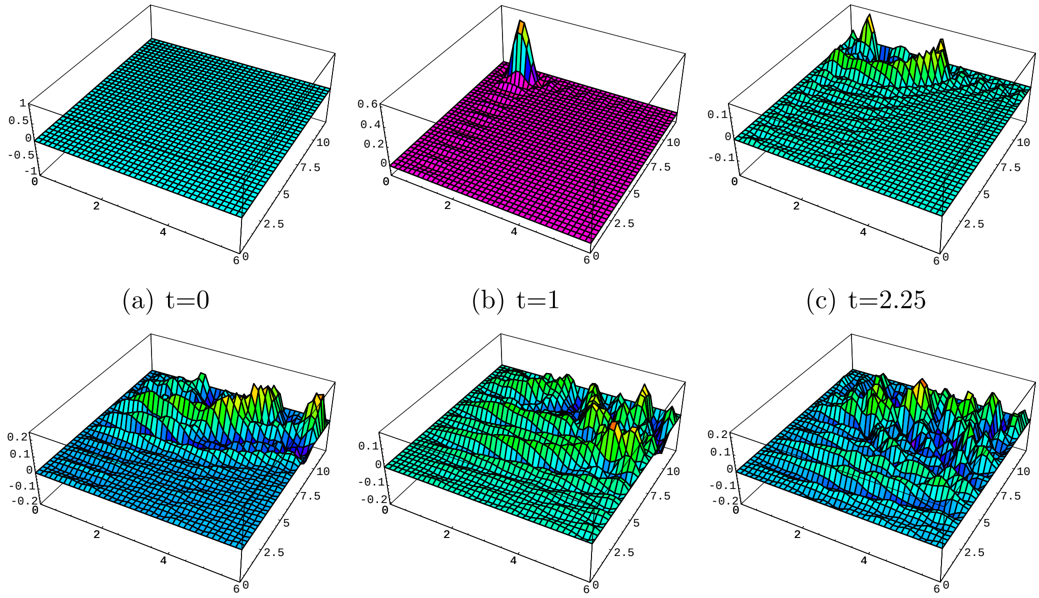


(a) t=0 (b) t=1 (c) t=2.25

(d) t=4.5 (e) t=6 (f) t=7.5

Figure 2.4: Case 2 of Example 3: The source near the wall of the room

The result of the simulation of acoustic field corresponding to $x_0 = 1, y_0 = 11, z_0 = 2$ (the position of the source is near the corner) is presented on the figure 2.5 (a)-(f).



(a) t=0 (b) t=1 (c) t=2.25

(d) t=6 (e) t=8 (f) t=10

Figure 2.5: Case 3 of Example 3: The source in the corner of the room

Analyzing these pictures we can see arising the acoustic waves, wave propagations, refraction of waves from walls, interaction of refracted waves.

2.4 Conclusion of the Chapter Two

1. Exact and approximate formulae for the solution of IBVP with Dirichlet boundary condition for the acoustic (wave) equation in a parallelepiped were obtained.

2. These formulae were adjusted to cases when initial data or inhomogeneous term are the Dirac delta function with a support in a point which is inside the parallelepiped. The pulse point source is modeled by these initial data and in homogeneous term.

3. Obtained formulae for generalized solutions of IBVP for the acoustic equation were used for modeling and simulations of acoustic waves in an empty room with soft walls arising from the pulse point sources.

4. Results of acoustic fields simulation are presented by 3-D pictures and animated movies.

5. Obtained formulae and results of the simulation were analyzed.

CHAPTER THREE
IBVP FOR ACOUSTIC EQUATION IN A PARALLELEPIPED WITH NEUMANN BOUNDARY CONDITION

3.1 IBVP for a Homogeneous Acoustic Equation with the Neumann Boundary Condition

Let $(x, y, z) \in \mathbb{R}^3$, $t \in \mathbb{R}$, $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq b_1, 0 \leq y \leq b_2, 0 \leq z \leq b_3\}$ be a homogeneous parallelepiped. Let us consider the IBVP for the wave equation in the parallelepiped D,

$$\frac{1}{c^2}\frac{\partial^2 u(x,y,z,t)}{\partial t^2} = \Delta_{x,y,z} u(x,y,z,t), \quad (x,y,z) \in D, \quad t > 0, \tag{3.1.1}$$

$$u(x,y,z,0) = \phi(x,y,z), \tag{3.1.2}$$

$$\frac{\partial u(x,y,z,t)}{\partial t}\bigg|_{t=0} = \psi(x,y,z), \tag{3.1.3}$$

$$\frac{\partial u(x,y,z,t)}{\partial n}|_{\partial D} = 0, \tag{3.1.4}$$

where ∂D is the boundary of D. Here $\phi(x, y, z)$, $\psi(x, y, z)$ are given functions in $D \cup \partial D$, c is a positive constat. IBVP consists of finding $u(x, y, z, t)$ satisfying (3.1.1) - (3.1.4).

We will use the same method which is applied for the problem (2.1.1)-(2.1.4) to solve the problem (3.1.1)-(3.1.4). We will use separation of variables:

$$u(x,y,z,t) = T(t)V(x,y,z). \tag{3.1.5}$$

Using the same technics which is done before we get:

$$\Delta_{x,y,z} V(x,y,z) + \lambda V(x,y,z) = 0, \tag{3.1.6}$$

$$T''(t) + c^2 \lambda T(t) = 0. \tag{3.1.7}$$

Substituting (3.1.5) into (3.1.4) we obtain

$$\frac{\partial V(x,y,z)}{\partial n}|_{\partial D} = 0. \qquad (3.1.8)$$

This means that $V(x,y,z)$ has to be a solution of the following eigenvalue-eigenfunction problem:

$$\begin{aligned} \Delta_{x,y,z} V(x,y,z) + \lambda V(x,y,z) &= 0, & (3.1.9) \\ \frac{\partial V(x,y,z)}{\partial n}|_{\partial D} = 0. & & (3.1.10) \end{aligned}$$

For the solution of eigenvalue-eigenfunction problem we will use the method of separation of variable. Setting

$$V(x,y,z) = X(x)Y(y)Z(z). \qquad (3.1.11)$$

Substituting (3.1.11) into (3.1.9) and (3.1.10) we have three eigenvalue-eigenfunction problem:

$$X''(x) + \mu X(x) = 0,\ X'(0) = X'(b_1) = 0, \qquad (3.1.12)$$

$$Y''(x) + \beta Y(x) = 0,\ Y'(0) = Y'(b_2) = 0, \qquad (3.1.13)$$

$$Z''(x) + \gamma Z(x) = 0,\ Z'(0) = Z'(b_3) = 0. \qquad (3.1.14)$$

We can get the eigenvalues and corresponding to them eigenfunctions for the systems (3.1.12), (3.1.13), (3.1.14) as follows;

$$\begin{aligned} \alpha_i = (\frac{i\pi}{b_1})^2; \quad X_i(x) &= \sqrt{\frac{2}{b_1}} \cos\sqrt{\alpha_i}x, \quad i = 0,1,2,\ldots \\ \beta_j = (\frac{j\pi}{b_2})^2; \quad Y_j(y) &= \sqrt{\frac{2}{b_2}} \cos\sqrt{\beta_j}y, \quad j = 0,1,2,\ldots \\ \gamma_k = (\frac{k\pi}{b_3})^2; \quad Z_k(z) &= \sqrt{\frac{2}{b_3}} \cos\sqrt{\gamma_k}z, \quad k = 0,1,2,\ldots \end{aligned}$$

(See Appendix A.2).

Thus we get the solution of the eigenvalue-eigenfunction problem (3.1.9) and (3.1.10) as follows:

$$V_{ijk} = X_i(x)Y_j(y)Z_k(z) = \sqrt{\frac{8}{b_1 b_2 b_3}} \cos\sqrt{\alpha_i}x \cos\sqrt{\beta_j}y \cos\sqrt{\gamma_k}z \qquad (3.1.15)$$

and corresponding eigenvalues

$$\lambda_{ijk} = \big(\frac{i\pi}{b_1}\big)^2 + \big(\frac{j\pi}{b_2}\big)^2 + \big(\frac{k\pi}{b_3}\big)^2; \quad i, j, k = 0, 1, 2, \ldots \tag{3.1.16}$$

Now, we can find the function $T(t)$. Substitute λ_{ijk} in to (3.1.7) we get

$$T''_{ijk}(t) + c^2\lambda_{ijk}T_{ijk}(t) = 0. \tag{3.1.17}$$

Hence (3.1.17) for $i^2 + j^2 + k^2 \neq 0$ has the following solution

$$T_{ijk}(t) = A_{ijk}\cos{(c\sqrt{\lambda_{ijk}}t)} + B_{ijk}\sin{(c\sqrt{\lambda_{ijk}}t)}. \tag{3.1.18}$$

Otherwise the solution is

$$T_{000}(t) = A_{000}t + B_{000}. \tag{3.1.19}$$

We have $u_{ijk}(x, y, z, t) = T_{ijk}(t)V_{ijk}(x, y, z)$ for each $i, j, k = 1, 2, \ldots$ are solutions of (3.1.1) and (3.1.4). Hence by the superposition principle we have the solution

$$u(x, y, z, t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} T_{ijk}(t)V_{ijk}(x, y, z). \tag{3.1.20}$$

Using the initial conditions (3.1.2), (3.1.3) and (3.1.25) we can find A_{ijk} and B_{ijk}

$$T_{ijk}(0) = \phi_{ijk}, \tag{3.1.21}$$
$$T'_{ijk}(0) = \psi_{ijk}. \tag{3.1.22}$$

So we have for $i^2 + j^2 + k^2 \neq 0$,$i, j, k = 0, 1, \ldots$

$$A_{ijk} = \phi_{ijk}, \quad B_{ijk} = \frac{\psi_{ijk}}{c\sqrt{\lambda_{ijk}}}, \tag{3.1.23}$$

and

$$A_{000} = \phi_{000}, \quad B_{000} = \psi_{000}, \tag{3.1.24}$$

where

$$\phi_{ijk} = \int_0^{b_1}\int_0^{b_2}\int_0^{b_3} \phi(x,y,z)V_{ijk}(x,y,z)dzdydx,$$
$$\psi_{ijk} = \int_0^{b_1}\int_0^{b_2}\int_0^{b_3} \psi(x,y,z)V_{ijk}(x,y,z)dzdydx.$$

So we conclude that we obtain the solution of our problem (3.1.1)–(3.1.4)

$$u(x,y,z,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} T_{ijk}(t)V_{ijk}(x,y,z), \tag{3.1.25}$$

where $T_{ijk}(t)$ are defined in (3.1.18), (3.1.19) and $V_{ijk}(x,y,z)$ are defined (3.1.15).

3.2 IBVP for a Non-homogeneous Acoustic Equation with the Neumann Boundary Condition

Let $(x,y,z) \in \mathbb{R}^3$, $t \in \mathbb{R}$, $D = \{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le b_1, 0 \le y \le b_2, 0 \le z \le b_3\}$ be a homogeneous parallelepiped. Let us consider the IBVP for the wave equation in the parallelepiped D,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \Delta_{x,y,z}u + F(x,y,z,t), \quad (x,y,z) \in D, \quad t > 0, \tag{3.2.1}$$
$$u(x,y,z,0) = \phi(x,y,z), \tag{3.2.2}$$
$$\left.\frac{\partial u(x,y,z,t)}{\partial t}\right|_{t=0} = \psi(x,y,z), \tag{3.2.3}$$
$$\frac{\partial u(x,y,z,t)}{\partial n}|_{\partial D} = 0, \tag{3.2.4}$$

where ∂D is the boundary of D. Here $F(x,y,z,t)$, $\phi(x,y,z)$, $\psi(x,y,z)$ are given functions in $D \cup \partial D$, c is a positive constat. IBVP consists of finding $u(x,y,z,t)$ satisfying (3.2.1) - (3.2.4).

Using the same techniques of the Chapter two for non-homogenous problem with the boundary condition $u(x,y,z,t)|_{\partial D} = 0$ and the homogenous problem for the Neumann boundary condition we will get the following result;

$$
\begin{aligned}
T_{ijk}(t) = \phi_{ijk} \cos(c\sqrt{\lambda_{ijk}}t) + \frac{\psi_{ijk}}{c\sqrt{\lambda_{ijk}}} \sin(c\sqrt{\lambda_{ijk}}t) + \\
+\frac{c}{\sqrt{\lambda_{ijk}}} \int_0^t \sin(c\sqrt{\lambda_{ijk}}(t-\tau))F_{ijk}(\tau)d\tau, \\
i^2 + j^2 + k^2 \neq 0; \quad i, j, k = 0, 1, \ldots
\end{aligned} \tag{3.2.5}
$$

$$
T_{000}(t) = \phi_{000} + \psi_{000}t + c^2 \int_0^t \int_0^\xi F_{000}(\tau)d\tau d\xi. \tag{3.2.6}
$$

Therefore

$$
u(x, y, z, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} T_{ijk}(t)V_{ijk}(x, y, z) \tag{3.2.7}
$$

is the solution of (3.2.1)–(3.2.4), where $T_{ijk}(t)$ is defined in (3.2.6) and $V_{ijk}(x, y, z)$ is defined in (3.1.15) for $i, j, k = 0, 1, 2, \ldots$

3.3 Examples of Simulations of Acoustic Waves in 3-D Rooms with Hard Walls

This subsection deals with examples of modeling and simulations of acoustic waves in 3-D rooms with hard walls. IBVP for the wave equation with the Neumann boundary condition in a parallelepiped is the mathematical model of the wave propagation in 3-D rooms with soft walls. We took a pulse point source concentrated in different positions in the room: in the center, near the wall, in the corner of the room. For each of these cases acoustic waves were modeled and simulated. The room is a parallelepiped $[0,6] \times [0,12] \times [0,3]$ and the speed of sound is equal to 1 for all examples.

3.3.1 Example 1

The mathematical model of the acoustic wave is given by (3.2.1)–(3.2.4), where

$$\begin{aligned} \phi(x,y,z) &= 0, \\ \psi(x,y,z) &= 0, \\ F(x,y,z,t) &= \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\delta(t),\ (x_0,y_0,z_0) \in D,\ t \geq 0, \end{aligned}$$

where $\delta(x)$ is the Dirac delta function. Using formulae for $\phi(x,y,z)$, $\psi(x,y,z)$, $F(x,y,z,t)$ and the properties of the Dirac delta function the solution of (3.1.1)–(3.1.4) can be found by the following formula

$$u(x,y,z,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} T_{ijk}(t)V_{ijk}(x,y,z), \tag{3.3.1}$$

where

$$\begin{aligned} &T_{ijk}(t) = \frac{c}{\sqrt{\lambda_{ijk}}}\int_0^t \sin\left(c\sqrt{\lambda_{ijk}}(t-\tau)\right)F_{ijk}(\tau)d\tau, \\ &i^2+j^2+k^2 \neq 0,\ i,j,k = 0,1,2,\dots \\ &T_{000}(t) = c^2\int_0^t\int_0^{\xi} F_{000}(\tau)d\tau d\xi, \end{aligned}$$

and

$$F_{ijk}(t) = V_{ijk}(x_0, y_0, z_0)\delta(t),$$
$$F_{000}(t) = 0.$$

Hence

$$T_{ijk}(t) = \frac{c}{\sqrt{\lambda_{ijk}}} sin(c\sqrt{\lambda_{ijk}}t)V_{ijk}(x_0, y_0, z_0),$$
$$i^2 + j^2 + k^2 \neq 0, i, j, k = 0, 1, 2, \ldots$$
$$T_{000}(t) = 0,$$

and λ_{ijk} and $V_{ijk}(x, y, z)$ are defined in (3.1.16), (3.1.15). For this example we consider the different positions of (x_0, y_0, z_0) (the point in which the pulse point source is concentrated). These positions of (x_0, y_0, z_0) are given in the following table:

	(x_0, y_0, z_0)
Case 1	$(3, 6, 2)$
Case 2	$(3, 11, 2)$
Case 3	$(1, 11, 2)$

Table 3.2: Positions of sources in a room with hard walls

For the given data of the Example 1, we used Mathematica codes in order to find the solution and to animate for each cases. The exact solution (3.3.1) is given by triple series containing infinite number of terms. We found an approximate solution by means of the formula

$$u(x, y, z, t) = \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} T_{ijk}(t)V_{ijk}(x, y, z), \qquad (3.3.2)$$

where $n = 20$. This number n we choose from the theoretical evaluation of the general term of the series (3.3.2) and the numerical experiments. We will consider each case for the function (3.3.2).

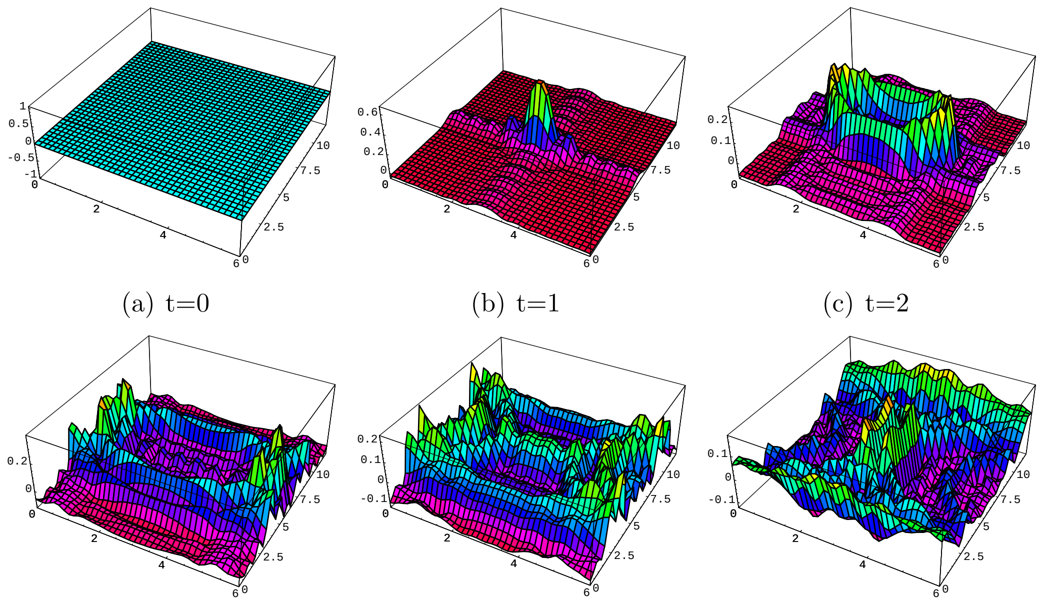
3.3.1.1 Commands in Mathematica for Example 1

```
<< Graphics‘Animation‘
bx = 6; by = 12; bz = 3;
x0 = 3; y0=6; z0=2;
n = 20; sc = 1;

a[kk_] = (kk*Pi/bx)^2; b[kk_] = (kk*Pi/by)^2; c[kk_] = (kk*Pi/bz)^2;

For[i = 1, i <= n, i++,
 For[j = 1, j <= n, j++,
  For[k = 1, k <= n,k++,
   Lambda[i, j, k] = a[i] + b[j] + c[k] ;
      V[i, j, k] =
      Sqrt[8/(bx*by*bz)]*Cos[Sqrt[a[i]]*x]*Cos[Sqrt[b[j]]*y]*
      Cos[Sqrt[c[k]]*z]
   PV[i, j, k] =ReplaceAll[ReplaceAll[ReplaceAll[V[i, j, k], x -> x0],
       y -> y0],z -> z0];
  If[i == 0 && j == 0 && k == 0, (T[i, j, k] = 0),
    (T[i, j, k] = (sc/Sqrt[Lambda[i, j, k]])*Sin[sc*Sqrt[Lambda[i,
j, k]]*t]* PV[i, j, k]); ]; ]; ]

son := Sum[Sum[Sum[T[i, j, k]*V[i, j, k], {i, 0, n}], {j, 0, n}],
{k, 0, n}]
son2 = ReplaceAll[son, z -> 1];

Animate[DensityPlot[son2, {x, 0, 6}, {y, 0, 12},
    ColorFunction -> (Hue[1 - #] &), PlotRange -> All,
    AspectRatio -> Automatic, Mesh -> False, PlotPoints -> 40],
    {t, 0, 10, 0.25}]
```

3.3.2 *Simulation Examples*

3.3.2.1 *The result of the simulation by formula* (3.3.2) *for Case 1*

Using the formula (3.3.2) for n = 20 and Mathematica codes we simulated the acoustic waves arising from the pulse point source which is located in the center of the room. The results of this simulation are presented on the following figures as screen shots for different times.



(a) t=0 (b) t=1 (c) t=2

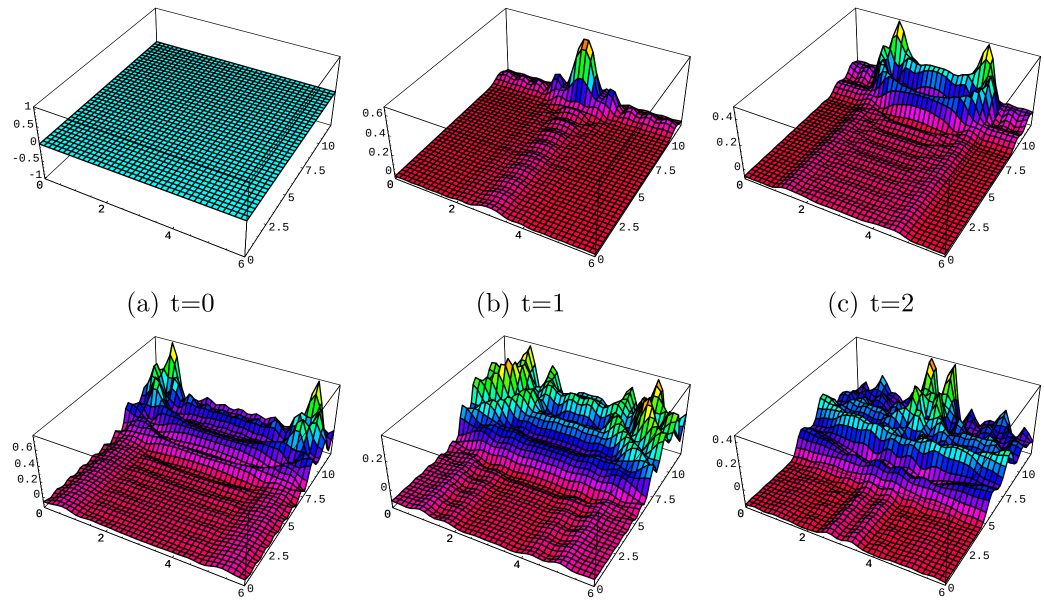
(d) t=4 (e) t=5 (f) t=6.5

Figure 3.6: Example 1: Source in the center of the room with hard walls

The axes are defined as before in the previous chapter. In figure 3.6 (a) there is no any fluctuation of the acoustic field. We can see on the figure 3.6 (b) the acoustic field arising from the pulse point source concentrated at the center of the room. There is also small fluctuations which stem from the numerical error of the series (3.3.2). In figure 3.6 (c) the circular wave front is seen and in figure 3.6 (d) the wave front touches the boundary, in the figure 3.6 (e) there is the reflection from the boundary and in last figure 3.6 (f) we can see the interaction between refracted waves.

3.3.2.2 The result of the simulation by formula (3.3.2) *for Case 2*

The result of the simulation of the acoustic field corresponding to Case 2 (the position of the source is near the wall) is presented on the figure 3.7 (a)-(f).



(a) t=0 (b) t=1 (c) t=2

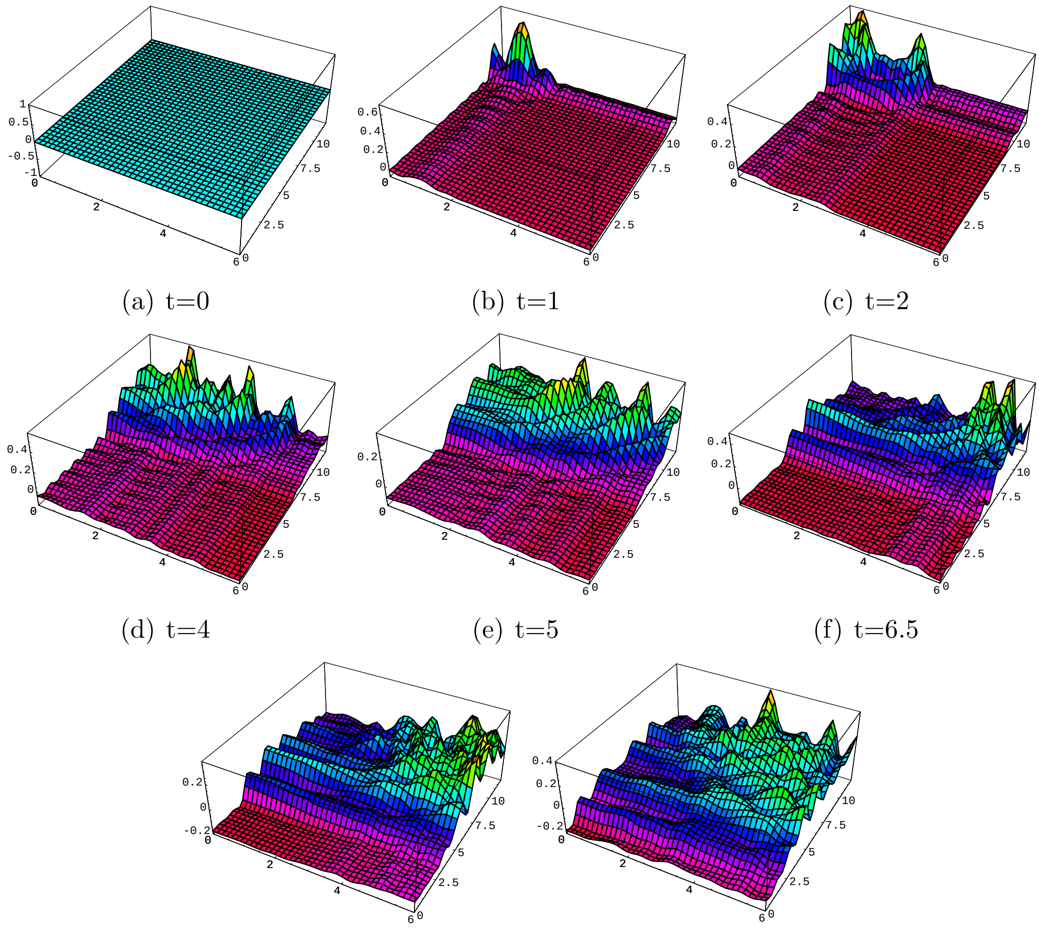
(d) t=4 (e) t=5 (f) t=6.5

Figure 3.7: Example 1: Source near the wall of the room with hard walls

In figure 3.7 (a) there is no any fluctuation of the acoustic field due to the initial condition $\phi(x, y, z) = 0$. We can see on the figure 3.7 (b) the acoustic field arising from the pulse point source near the wall of the room. There is also small fluctuations which stem from the numerical error of the series (3.3.2). In the figure 3.7 (c) the circular wave front and the reflection from the nearest wall are seen and in figure 3.6 (d) the wave front touches the other boundaries, in the figure 3.7 (e) there is the reflection from those boundaries and in last figure 3.7 (f) we can see the interaction between refracted waves.

3.3.2.3 The result of the simulation by formula (3.3.2) *for Case 3*

The result of the simulation of the acoustic field corresponding to Case 3 (the position of the source is near the corner) is presented on the figure 3.8 (a)-(g).



(a) t=0 (b) t=1 (c) t=2

(d) t=4 (e) t=5 (f) t=6.5

(g) t=8 (h) t=10

Figure 3.8: Example 1: Source near the corner of the room with hard walls

In the figure 3.8 (a) there is no any fluctuation. We can see on the figure 3.8 (b) the acoustic field arising from the pulse point source near the corner of the room. In the figure 3.8 (c) the circular wave front and the reflection from the nearest walls are seen and in the figure 3.8 (d)-(e) the wave front is getting bigger, in the figure 3.8 (f) wave front touches the other wall and in the last two figures 3.8 (g)-(h) we can see interaction between refracted waves.

3.4 Conclusion of the Chapter Three

1. Formulae for exact and approximate the solution of IBVP with Neumann boundary conditions for the acoustic (wave) equation in a parallelepiped were obtained.

2. These formulae were adjusted to cases when initial data or inhomogeneous term are the Dirac delta function with a support in a point which is inside the parallelepiped. The pulse point source is modeled by these initial data and inhomogeneous term.

3. Obtained formulae for generalized solutions of IBVP for the acoustic equation were used for modeling and simulations of acoustic waves in a room with hard walls arising from the pulse point sources.

4. Results of acoustic fields simulation are presented by 3-D pictures and animated movies.

5. Obtained formulae and results of simulation were analyzed. The different behavior of the boundary due to the Neumann boundary condition was seen.

CHAPTER FOUR
IBVP FOR THE ACOUSTIC EQUATION IN A PARALLELEPIPED WITH THE MIXED BOUNDARY CONDITION

4.1 IBVP for a Homogeneous Acoustic Equation with the Mixed Boundary Condition

Let $(x, y, z) \in \mathbb{R}^3$, $t \in \mathbb{R}$, $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq b_1, 0 \leq y \leq b_2, 0 \leq z \leq b_3\}$ be a homogeneous parallelepiped. Let us consider the IBVP for the wave equation in the parallelepiped D:

$$\frac{1}{c^2}\frac{\partial^2 u(x,y,z,t)}{\partial t^2} = \Delta_{x,y,z}u(x,y,z,t), \quad (x,y,z) \in D,\ t > 0 \tag{4.1.1}$$

$$u(x,y,z,0) = \phi(x,y,z), \tag{4.1.2}$$

$$\frac{\partial u(x,y,z,t)}{\partial t}\Big|_{t=0} = \psi(x,y,z), \tag{4.1.3}$$

$$(\frac{\partial u}{\partial n} + \alpha u)|_{\partial D} = 0, \tag{4.1.4}$$

where ∂D is the boundary of D, $\phi(x, y, z)$ is the initial displacement and $\psi(x, y, z)$ is the initial velocity. Here $\phi(x, y, z)$, $\psi(x, y, z)$ are given functions $D \cup \partial D$ and D is defined as:

$$D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq b_1, 0 \leq y \leq b_2, 0 \leq z \leq b_3\}.$$

Using Fourier Series Expansion Method we will find $u(x, y, z, t)$ which satisfies (4.1.1) – (4.1.4). We will use the same method which is applied for solving the problem (2.1.1)–(2.1.4) and (3.1.1)–(3.1.4) to solve (4.1.1) – (4.1.4). We use the separation of variables. Setting

$$u(x,y,z,t) = T(t)V(x,y,z) \tag{4.1.5}$$

and substituting (4.1.5) into (4.1.1) we find

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{\Delta_{x,y,z}V(x,y,z)}{V(x,y,z)} = -\lambda.$$

So we get the following equations for $T(t)$ and $V(x,y,z)$

$$\Delta_{x,y,z}V(x,y,z) + \lambda V(x,y,z) = 0, \tag{4.1.6}$$

$$T''(t) + c^2\lambda T(t) = 0. \tag{4.1.7}$$

Our aim is to construct *Strum-Liouville equations*;
Substituting (4.1.5) into (4.1.4) we have

$$(\frac{\partial u}{\partial n} + \alpha u)|_{\partial D} = (\nabla u \cdot \overrightarrow{n} + \alpha u)|_{\partial D} = 0$$

and so

$$(\nabla V(x,y,z)T(t) \cdot \overrightarrow{n} + \alpha V(x,y,z)T(t))|_{\partial D} = 0,$$

$$T(t)(\nabla V(x,y,z) \cdot \overrightarrow{n} + \alpha V(x,y,z))|_{\partial D} = 0,$$

$$(\nabla V(x,y,z) \cdot \overrightarrow{n} + \alpha V(x,y,z))|_{\partial D} = 0. \tag{4.1.8}$$

Equations (4.1.6) and (4.1.8) we will write as an Strum Liouville problem

$$\Delta_{x,y,z}V(x,y,z) + \lambda V(x,y,z) = 0, \tag{4.1.9}$$

$$(\nabla V(x,y,z) \cdot \overrightarrow{n} + \alpha V(x,y,z))|_{\partial D} = 0. \tag{4.1.10}$$

Our aim is to find eigenvalues and corresponding to them eigenfunctions.
Using the method of separation of variables, we find

$$V(x,y,z) = X(x)Y(y)Z(z). \tag{4.1.11}$$

Substituting (4.1.11) into (4.1.9) (4.1.10) we get

$$X''(x) + \mu X(x) = 0,\ X'(0) - \alpha X(0) = 0,\ X'(b_1) + \alpha X(b_1) = 0, \tag{4.1.12}$$

$$Y''(x) + \beta Y(x) = 0,\ Y'(0) - \alpha Y(0) = 0,\ Y'(b_2) + \alpha Y(b_2) = 0, \tag{4.1.13}$$

$$Z''(x) + \gamma Z(x) = 0,\ Z'(0) - \alpha Z(0) = 0,\ Z'(b_3) + \alpha Z(b_3) = 0. \tag{4.1.14}$$

Using Appendix A.3 we can get the eigenvalues and correspondingly eigenfunctions for the systems (4.1.12), (4.1.13), (4.1.14) as follows:

$$X_i(x) \;=\; C_i \sin(\zeta_i x + \varphi_i^x), \quad \mu_i = \zeta_i^2, \quad \varphi_i^x = \arctan(\frac{\zeta_i}{\alpha}), \quad i = 1, 2, \ldots,$$

where ζ_i for each $i = 1, 2, \ldots$ is the solution of

$$\cot(xb_1) = \frac{1}{2}\left(\frac{x}{\alpha} - \frac{\alpha}{x}\right), \quad \alpha > 0,\, l > 0 \quad fixed$$

and

$$C_i = 2\sqrt{\frac{\zeta_i}{2b_1\zeta_i + sin[2\varphi_i^x] - sin[2(b_1\zeta_i + \varphi_i^x)]}} \quad i = 1, 2, \ldots, \tag{4.1.15}$$

$$Y_j(y) \;=\; C_j \sin(\eta_j x + \varphi_j^y), \quad \beta_j = \eta_j^2, \quad \varphi_j^y = \arctan(\frac{\eta_j}{\alpha}), \quad j = 1, 2, \ldots$$

where η_j for each $j = 1, 2, \ldots$ is the solution of

$$\cot(xb_2) = \frac{1}{2}\left(\frac{x}{\alpha} - \frac{\alpha}{x}\right), \quad \alpha > 0,\, l > 0 \quad fixed$$

and

$$C_j = 2\sqrt{\frac{\eta_j}{2b_2\eta_j + sin[2\varphi_j^y] - sin[2(b_2\eta_j + \varphi_j^y)]}} \quad j = 1, 2, \ldots, \tag{4.1.16}$$

$$Z_k(z) \;=\; C_k \sin(\upsilon_k x + \varphi_k^z), \quad \gamma_k = \upsilon_k^2, \quad \varphi_k^z = \arctan(\frac{\upsilon_k}{\alpha}), \quad k = 1, 2, \ldots$$

where υ_j for each $k = 1, 2, \ldots$ is the solution of

$$\cot(xb_3) = \frac{1}{2}\left(\frac{x}{\alpha} - \frac{\alpha}{x}\right), \quad \alpha > 0,\, l > 0 \quad fixed$$

and

$$C_k = 2\sqrt{\frac{\upsilon_k}{2b_3\upsilon_k + sin[2\varphi_k^z] - sin[2(b_3\upsilon_k + \varphi_k^z)]}} \quad k = 1, 2, \ldots \tag{4.1.17}$$

So we get the eigenvalues and eigenfunctions of (4.1.9) and (4.1.10) as follows:

$$\begin{aligned} V_{ijk} &= X_i(x)Y_j(y)Z_k(z) \\ &= C_iC_jC_k \sin(\zeta_i x + \varphi_i^x) \sin(\eta_j x + \varphi_j^y) \sin(\upsilon_k x + \varphi_k^z) \end{aligned} \tag{4.1.18}$$

and corresponding eigenvalues

$$\lambda_{ijk} = \zeta_i^2 + \eta_j^2 + \upsilon_k^2 \quad i, j, k = 1, 2, \ldots, \tag{4.1.19}$$

where C_i, C_j, C_k are defined in (4.1.15),(4.1.16),(4.1.17).
We get the solution of the equation (4.1.7) using the equation (2.1.19) from the previous section. We get

$$T_{ijk}(t) = A_{ijk} \cos(c\sqrt{\lambda_{ijk}}t) + B_{ijk} \sin(c\sqrt{\lambda_{ijk}}t),, \ i, j, k = 1, 2, \ldots \tag{4.1.20}$$

Hence $u_{ijk}(x, y, z, t) = T_{ijk}(t)V_{ijk}(x, y, z)$ are solutions of (4.1.1) and (4.1.4) for each $i, j, k = 1, 2, \ldots$ Hence by the superposition principle we have the solution:

$$u(x, y, z, t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} u_{ijk}(x, y, z, t),$$

$$u(x, y, z, t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \big(A_{ijk}\cos(c\sqrt{\lambda_{ijk}}t) + B_{ijk}\sin(c\sqrt{\lambda_{ijk}}t)\big)V_{ijk}(x, y, z). \tag{4.1.21}$$

It remains to determine the Fourier coefficients A_{ijk}, B_{ijk}. Using the orthonormal functions $V_{ijk}(x, y, z)$, we can write $\phi(x, y, z)$ and $\psi(x, y, z)$ in the form of Fourier series expansion:

$$\phi(x, y, z) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \phi_{ijk}V_{ijk}(x, y, z), \tag{4.1.22}$$

$$\psi(x, y, z) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \psi_{ijk}V_{ijk}(x, y, z), \tag{4.1.23}$$

where

$$\begin{aligned} \phi_{ijk} &= \int_0^{b_1}\int_0^{b_2}\int_0^{b_3} \phi(x, y, z)V_{ijk}(x, y, z)dzdydx, \\ \psi_{ijk} &= \int_0^{b_1}\int_0^{b_2}\int_0^{b_3} \psi(x, y, z)V_{ijk}(x, y, z)dzdydx. \end{aligned}$$

Using the initial conditions (4.1.2), (4.1.3) and (4.1.22), (4.1.23) we can find A_{ijk} and B_{ijk}

$$A_{ijk} = \phi_{ijk}$$
$$B_{ijk} = \frac{\psi_{ijk}}{c\sqrt{\lambda_{ijk}}}$$

Substituting A_{ijk} and B_{ijk} into (4.1.21), we can conclude that

$$u(x,y,z,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\big(\phi_{ijk}\cos(c\sqrt{\lambda_{ijk}}t) + \frac{\psi_{ijk}}{c\sqrt{\lambda_{ijk}}}\sin(c\sqrt{\lambda_{ijk}}t)\big)V_{ijk}(x,y,z) \tag{4.1.24}$$

is the solution of the problem (4.1.1) - (4.1.4).

4.2 IBVP for a Non-homogeneous Acoustic Equation with the Mixed Boundary Condition

Let $(x, y, z) \in \mathbb{R}^3$, $t \in \mathbb{R}$, $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq b_1, 0 \leq y \leq b_2, 0 \leq z \leq b_3\}$ be a homogeneous parallelepiped. Let us consider the IBVP for the wave equation in the parallelepiped D:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \Delta_{x,y,z}u + F(x, y, z, t), \quad (x, y, z) \in D, \quad t > 0, \tag{4.2.1}$$

$$u(x, y, z, 0) = \phi(x, y, z), \tag{4.2.2}$$

$$\frac{\partial u(x, y, z, t)}{\partial t}\Big|_{t=0} = \psi(x, y, z), \tag{4.2.3}$$

$$(\frac{\partial u}{\partial n} + \alpha u)|_{\partial D} = 0, \tag{4.2.4}$$

where ∂D is the boundary of D. Here $F(x, y, z, t)$, $\phi(x, y, z)$, $\psi(x, y, z)$ are given functions in $D \cup \partial D$, c is a positive constat. IBVP consists of finding $u(x, y, z, t)$ satisfying (4.2.1) – (4.2.4).

Using the same techniques from the previous sections for non-homogenous problem with the Dirichlet and Neumann boundary conditions and homogenous problem for mixed boundary condition we will get the following result;

$$T_{ijk}(t) = \phi_{ijk}\cos(c\sqrt{\lambda_{ijk}}t) + \frac{\psi_{ijk}}{c\sqrt{\lambda_{ijk}}}\sin(c\sqrt{\lambda_{ijk}}t) + $$
$$+\frac{c}{\sqrt{\lambda_{ijk}}}\int_0^t F_{ijk}(\tau)\sin(c\sqrt{\lambda_{ijk}}(t-\tau))d\tau \tag{4.2.5}$$

and

$$u(x, y, z, t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} T_{ijk}(t)V_{ijk}(x, y, z) \tag{4.2.6}$$

is the solution of (4.2.1) - (4.2.4), where $T_{ijk}(t)$ is defined in (4.2.5) and $V_{ijk}(x, y, z)$ is defined in (4.1.18).

4.3 Examples of Simulations of Acoustic Waves in 3-D Rooms with the Mixed Boundary

This subsection deals with examples of modeling and simulations of acoustic waves in 3-D rooms with mixed type of walls. IBVP for the wave equation with the Mixed boundary condition in a parallelepiped is the mathematical model of the wave propagation in 3-D rooms. We took a pulse point source concentrated in different positions in the room: in the center, near the wall, in the corner of the room. For each of these cases acoustic waves were modeled and simulated. The room is a parallelepiped $[0,6] \times [0,12] \times [0,3]$ and the speed of sound is equal to 1 for all examples.

4.3.1 Example 1

The mathematical model of acoustic wave is given by (4.2.1)–(4.2.4), where

$$\begin{aligned} \phi(x,y,z) &= 0, \\ \psi(x,y,z) &= 0, \\ F(x,y,z,t) &= \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\delta(t),\ (x_0,y_0,z_0) \in D,\ t \geq 0, \end{aligned}$$

where $\delta(x)$ is the Dirac delta function. Using formulae for $\phi(x,y,z)$, $\psi(x,y,z)$, $F(x,y,z,t)$ and the properties of the Dirac delta function the solution of (3.1.1)–(3.1.4) can be found by the following formula

$$u(x,y,z,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} T_{ijk}(t)V_{ijk}(x,y,z), \tag{4.3.1}$$

where

$$T_{ijk}(t) = \frac{c}{\sqrt{\lambda_{ijk}}}\int_0^t \sin\left(c\sqrt{\lambda_{ijk}}(t-\tau)\right)F_{ijk}(\tau)d\tau, \quad i,j,k = 1,2,\ldots$$

and

$$F_{ijk}(t) = V_{ijk}(x_0,y_0,z_0)\delta(t).$$

Hence

$$T_{ijk}(t) \quad = \quad \frac{c}{\sqrt{\lambda_{ijk}}} sin(c\sqrt{\lambda_{ijk}}t)V_{ijk}(x_0, y_0, z_0), \quad i, j, k = 1, 2, \ldots,$$

where λ_{ijk} and $V_{ijk}(x, y, z)$ are defined in (4.1.19), (4.1.18). For this example we consider the different positions of (x_0, y_0, z_0) (the point in which the pulse point source is concentrated). These positions of (x_0, y_0, z_0) are given in the following table: For the given data of the Example 1, we used Mathematica codes in order to

	(x_0, y_0, z_0)
Case 1	$(3, 6, 2)$
Case 2	$(3, 11, 2)$
Case 3	$(1, 11, 2)$

Table 4.3: Positions of sources in a room with the mixed boundary

find the solution and to animate for each cases. The exact solution (4.3.1) is given by triple series containing infinite number of terms. We found an approximate solution by means of the formula

$$u(x, y, z, t) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk}(t)V_{ijk}(x, y, z), \tag{4.3.2}$$

where $n = 20$. This number n we choose from the theoretical evaluation of the general term of the series (4.3.2) and the numerical experiments. We will consider the each case for the function (4.3.2).

4.3.1.1 Commands in Mathematica for Example 1

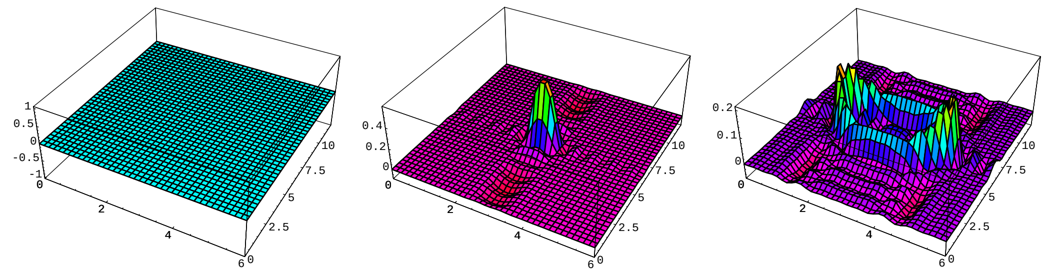
```
 << Graphics`Animation`
bx = 6; by = 12; bz = 3;

x0 = 3; y0=6; z0=2;

sc = 1; Alpha = 5; n = 20;

a[kk_] = (kk*Pi/bx)^2; b[kk_] = (kk*Pi/by)^2; c[kk_] = (kk*Pi/bz)^2;


For[i = 0, i <= n, i++,
  R[i] = FindRoot[
      Cot[r[i]*bx] == (1/2)*(r[i]/Alpha - Alpha/r[i]), {r[i],
        0.001 + i*Pi/bx}];
  e1[i] = r[i] /. R[i];
  Nrm[i] = (1/
          Integrate[(Cos[e1[i]*x + ArcCot[-e1[i]/Alpha]])^2,
          {x, 0, bx}])^(1/2);
  X[i] = Nrm[i]*Cos[e1[i]*x + ArcCot[-e1[i]/Alpha]];
  ]

For[i = 0, i <= n, i++, R[i] = FindRoot[
    Cot[r[i]*by] == (1/2)*(r[i]/Alpha - Alpha/r[i]), {r[i],
      0.001 + i*Pi/by}];
e2[i] = r[i] /. R[i]; Nrm[i] = (1/
          Integrate[(Cos[e2[i]*y + ArcCot[-e2[i]/Alpha]])^2,
          {y, 0, by}])^(1/2);
Y[i] = Nrm[i]*Cos[e2[i]*y + ArcCot[-e2[i]/Alpha]]; ]

For[i = 0, i <= n, i++,
  R[i] = FindRoot[
      Cot[r[i]*bz] == (1/2)*(r[i]/Alpha - Alpha/r[i]), {r[i],
        0.001 + i*Pi/bz}];
  e3[i] = r[i] /. R[i];
  Nrm[i] = (1/Integrate[(Cos[e3[i]*z + ArcCot[-e3[i]/Alpha]])^2,
         {z, 0, bz}])^(1/2);
  Z[i] = Nrm[i]*Cos[e3[i]*z + ArcCot[-e3[i]/Alpha]];
```

```
  ]

For[i = 0, i <= n, i++,
  For[j = 0, j <= n, j++,
      For[k = 0, k <= n, k++,
          Lambda[i, j, k] = (e1[i]^2) + (e2[j]^2) + (e3[k]^2) ;
          V[i, j, k] = X[i]*Y[j]*Z[k];

          PV[i, j, k] =
            ReplaceAll[ReplaceAll[ReplaceAll[V[i, j, k], x -> x0],
             y -> y0], z -> z0];

          T[i, j, k] = (sc/Sqrt[Lambda[i, j, k]])*
              Sin[sc*Sqrt[Lambda[i, j, k]]*t]*PV[i, j, k];
          ];
      ];
  ]

son := Sum[Sum[Sum[T[i, j, k]*V[i, j, k], {i, 0, n}], {j, 0, n}],
{k, 0, n}] son2 = ReplaceAll[son, z -> 1];

Animate[Plot3D[son2, {x, 0, 6}, {y, 0, 12}, ColorFunction -> (Hue[1
- #] &),
    PlotRange -> All, AspectRatio -> Automatic, PlotPoints -> 40],
    {t, 0, 10, 0.1}]
```

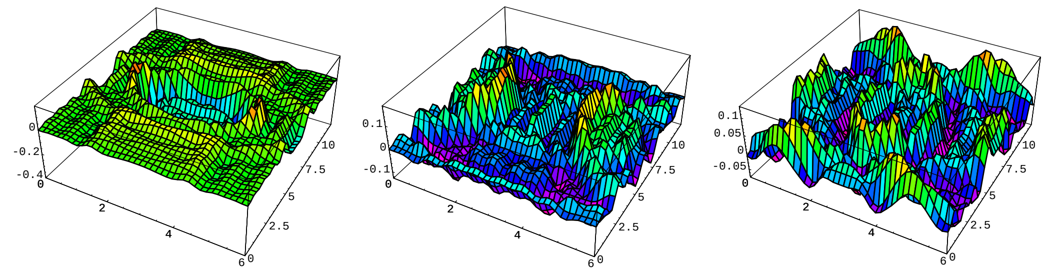
4.3.2 Simulation Examples

4.3.2.1 The result of the simulation by formula (4.3.2) *for Case 1*

Using the formula (4.3.2) for $n = 20$ and Mathematica codes we simulated the acoustic waves arising from a pulse point source which is located in the center of the room. The results of this simulation are presented on the following figures as screen shots for different times.



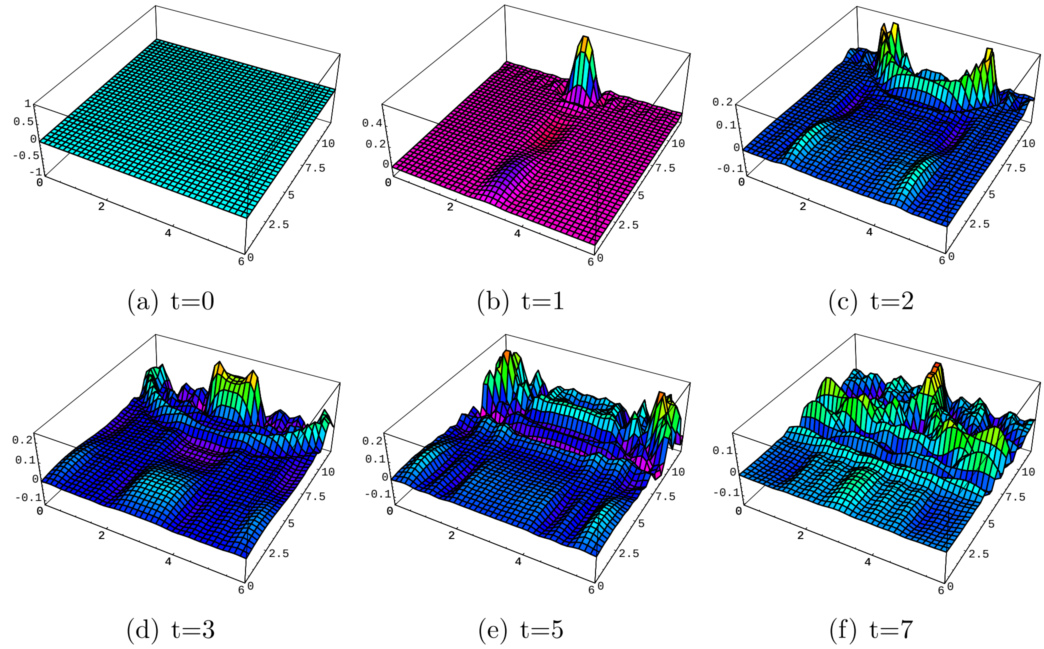
(a) t=0 (b) t=1 (c) t=2



(d) t=3 (e) t=6 (f) t=8

Figure 4.9: Example 1: Source in the center of the room

The axes are defined as before in the previous chapter. In figure 4.9 (a) there is no fluctuation of the acoustic field. We can see on the figure 4.9 (b) the acoustic field arising from the pulse point source concentrated at the center of the room. There is also small fluctuations which stem from the numerical error of the series (4.3.2). In figure 4.9 (c) the circular wave front is seen and in the figure 4.9 (d) the wave front touches the boundary, in the figure 4.9 (e) there is the reflection from the boundary and in last figure 4.9 (f) we can see the interaction between the refracted waves.



(a) t=0 (b) t=1 (c) t=2

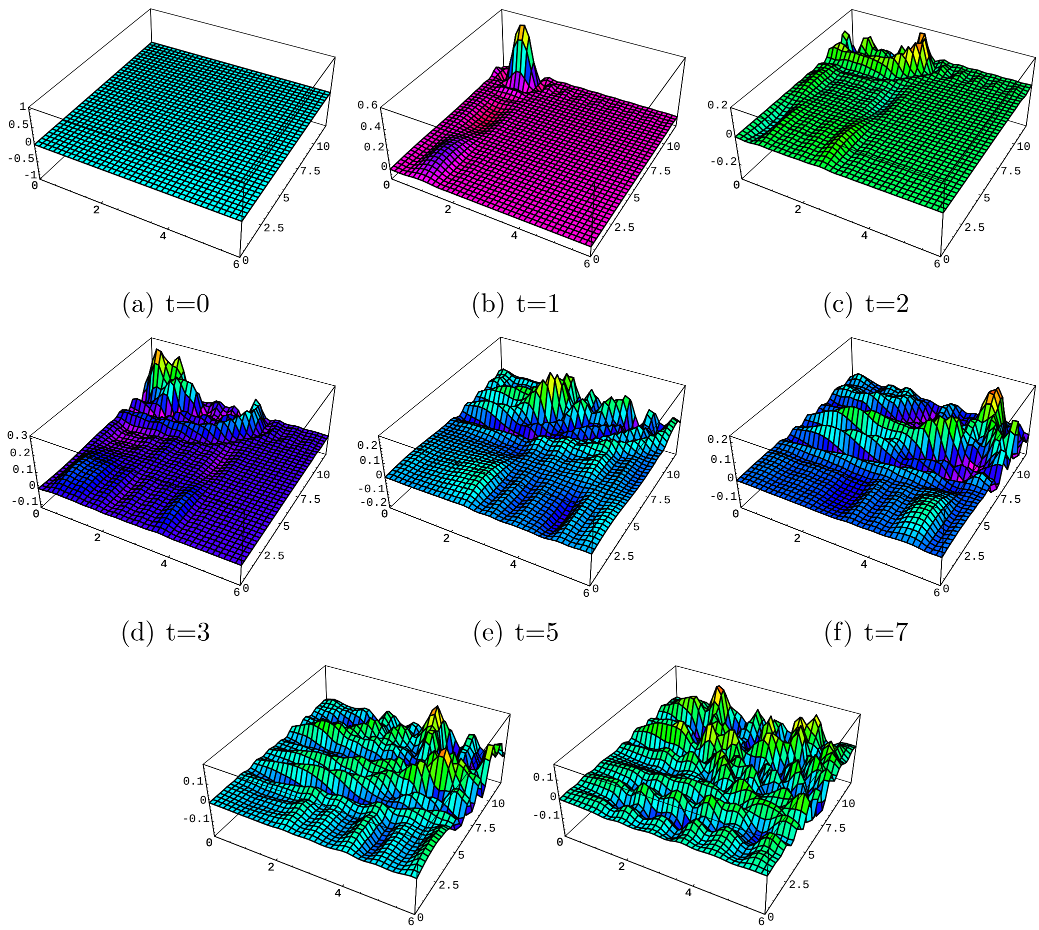
(d) t=3 (e) t=5 (f) t=7

Figure 4.10: Example 1: Source near the wall of the room

4.3.2.2 The result of the simulation by formula (4.3.2) *for Case 2*

In figure 4.10 (a) there is no any fluctuation of the acoustic field due to the initial condition $\phi(x, y, z) = 0$. We can see on the figure 4.10 (b) the acoustic field arising from the pulse point source near the wall of the room. There is also small fluctuations which stem from the numerical error of the series (4.3.2). In the figure 4.10 (c) the circular wave front and the reflection from the nearest wall are seen and in figure 4.10 (d) the wave front touches the other boundaries, in the figure 4.10 (e) there is the reflection from those boundaries and in last figure 4.10 (f) we can see the interaction between refracted waves.

4.3.2.3 The result of the simulation by formula (4.3.2) *for Case 3*



(a) t=0 (b) t=1 (c) t=2

(d) t=3 (e) t=5 (f) t=7

(g) t=8 (h) t=10

Figure 4.11: Example 1: Source near the corner of the room

In the figure 4.11 (a) there is no any fluctuation. We can see on the figure 4.11 (b) the acoustic field arising from the pulse point source near the corner of the room. There is also small fluctuations which stem from the numerical error of the series (4.3.2). In the figure 4.11 (c) the circular wave front and the reflection from the nearest walls are seen and in the figure 4.11 (d)-(e) the wave front is getting bigger, in the figure 4.11 (f) wave front touches the other wall and in the last two figures 4.11 (g)-(h) we can see interaction between refracted waves.

4.4 Conclusion of the Chapter Four

1. Formulae for exact and approximate solution of IBVP with the Mixed boundary condition for the acoustic (wave) equation in a parallelepiped were obtained.

2. These formulae were adjusted to cases when initial data or inhomogeneous term are the Dirac delta function with a support in a point which is inside the parallelepiped. The pulse point source is modeled by these initial data and inhomogeneous term.

3. Obtained formulae for generalized solutions of IBVP for the acoustic equation were used for modeling and simulations of the acoustic waves arising from the pulse point sources.

4. Results of acoustic fields simulation are presented by 3-D pictures and animated movies.

5. Obtained formulae and results of the simulation were analyzed. The behavior of the boundary due to the Mixed boundary condition was seen.

CHAPTER FIVE
GREEN'S FUNCTION FOR INITIAL BOUNDARY VALUE PROBLEM FOR THE ACOUSTIC EQUATION

5.1 Generalized IBVP for the Wave Equation

Let $x = (x_1, x_2, x_3)$, $x \in D \subset \mathbb{R}^3$, $t \in \mathbb{R}$, $\Delta_x = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator; $\Box_c = (\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta_x)$ is the wave operator, c is a given positive constant.

Definition 5.1.1. The space $\mathfrak{D}'(U)$ of distributions on U is defined to be the continuous linear dual of $C_c^\infty(U)$; i.e., the space of continuous linear functionals $C_c^\infty(U) \to \mathbb{C}$.

Definition 5.1.2 (Generalized IBVP for the wave equation)**.** Let D be a bounded domain

$$\Box_c u(x,t) = f(x,t), \quad t > 0, \quad x \in D, \tag{5.1.1}$$
$$u(x,+0) = g(x), \quad u_t(x,+0) = h(x), \quad x \in D, \tag{5.1.2}$$
$$u(x,t)|_{x\in D} = 0, \tag{5.1.3}$$

where $f(x,t) \in \mathfrak{D}'(D \times (0,\infty))$, $g(x), h(x) \in \mathfrak{D}'(D)$ are given generalized functions. The problem (5.1.1)-(5.1.3) is called *a generalized IBVP for the wave equation with the Dirichlet boundary condition.*

Lemma 5.1.3. *Let $u(x,t)$ be a solution of the problem* (5.1.1)-(5.1.3) *then*

$$v(x,t) = \theta(t)u(x,t)$$

is a solution of the following problem

$$\Box_c v(x,t) = F(x,t), \quad t > 0, \quad x \in D, \tag{5.1.4}$$
$$v(x,t)|_{t<0} = 0, \tag{5.1.5}$$
$$v(x,t)|_{x\in\partial D} = 0, \tag{5.1.6}$$

where

$$F(x,t) = \theta(t)f(x,t) + \frac{1}{c^2}\delta'(t)g(x) + \frac{1}{c^2}\delta(t)h(x),$$

$\theta(t)$ is the Heaviside function, $\delta(t)$ is the Dirac delta function.

Proof. We will prove this lemma by the direct substitution. Since $v(x,t) = \theta(t)u(x,t)$ we have

$$\begin{aligned} v_t(x,t) &= \frac{d\theta(t)}{dt}u(x,t) + \theta(t)\frac{\partial u(x,t)}{\partial t} \\ &= \delta(t)u(x,t) + \theta(t)\frac{\partial u(x,t)}{\partial t} \\ &= \delta(t)u(x,0) + \theta(t)\frac{\partial u(x,t)}{\partial t}. \end{aligned}$$

Hence

$$v_{tt}(x,t) = \delta'(t)u(x,0) + \delta(t)\frac{\partial u(x,0)}{\partial t} + \theta(t)\frac{\partial^2 u(x,t)}{\partial t^2}.$$

Using the conditions (5.1.2) the last equation can be written in the following form

$$v_{tt}(x,t) = \delta'(t)u(x,0) + \delta(t)\frac{\partial u(x,0)}{\partial t} + \theta(t)\frac{\partial^2 u(x,t)}{\partial t^2}.$$

As a result of it we find

$$\Delta_x v(x,t) = \Delta_x(\theta(t)u(x,t)) = \theta(t)\Delta_x u(x,t).$$

Thus

$$\begin{aligned} \Box_c v(x,t) &= \frac{1}{c^2}\frac{\partial^2 v(x,t)}{\partial t^2} - \Delta_x v(x,t) \\ &= \frac{1}{c^2}\delta'(t)u(x,0) + \frac{1}{c^2}\delta(t)\frac{\partial u(x,0)}{\partial t} + \theta(t)\left(\frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta_x u(x,t)\right) \\ &= \frac{1}{c^2}\delta'(t)u(x,0) + \frac{1}{c^2}\delta(t)\frac{\partial u(x,0)}{\partial t} + \theta(t)f(x,t) \\ &= F(x,t). \end{aligned} \qquad (5.1.7)$$

We showed that the equation (5.1.4) is satisfied. Also we have

$$v(x,t)|_{t<0} = (\theta(t)u(x,t))\,|_{t<0} = 0,$$

and

$$v(x,t)|_{x\in D} = (\theta(t)u(x,t))\,|_{x\in D} = \theta(t)(u(x,t)|_{x\in D}) = 0. \qquad (5.1.8)$$

So we showed that $v(x,t)$ satisfies also (5.1.5), (5.1.6). This completes the proof. □

5.2 Green's Function for IBVP for the Wave Equation With the Dirichlet Boundary Condition

Definition 5.2.1. Let $\xi \in D$, $\tau \in \mathbb{R}$ be parameters. A generalized function $G(x,t|\xi,\tau) \in \mathfrak{D}'(D\times\mathbb{R})$ is called Green's function of the IBVP for the wave operator $\Box_c$ if

$$\Box_c G(x,t|\xi,\tau) = \delta(x-\xi,t-\tau), \quad (x,t)\in D\times\mathbb{R}, \qquad (5.2.1)$$

$$G(x,t|\xi,\tau)|_{t<\tau} = 0, \quad (x,t)\in D\times\mathbb{R}, \qquad (5.2.2)$$

$$G(x,t|\xi,\tau)|_{x\in\partial D} = 0, \quad t\in\mathbb{R}. \qquad (5.2.3)$$

Lemma 5.2.2. *Let $G(x,t|\xi,\tau)$ be Green's function of the IBVP for the wave operator $\Box_c$ and $F(x,t) \in \mathfrak{D}'(D\times\mathbb{R})$ is an arbitrary generalized function such as $F(x,t)|_{t<0}$. Then*

$$v(x,t) = \int_0^\infty \int_0^{b_1}\int_0^{b_2}\int_0^{b_3} G(x,t|\xi,\tau)F(\xi,\tau)d\xi d\tau \qquad (5.2.4)$$

is a solution of the generalized IBVP (5.1.4)- (5.1.6).

Proof. Applying the operator $\Box_c$ to (5.2.4) we find

$$\begin{aligned} \Box_c v(x,t) &= \int_{-\infty}^\infty \int_0^{b_1}\int_0^{b_2}\int_0^{b_3} \Box_c G(x,t|\xi,\tau)F(\xi,\tau)d\xi d\tau \\ &= \int_{-\infty}^\infty \int_0^{b_1}\int_0^{b_2}\int_0^{b_3} \delta(x-\xi,t-\tau)F(\xi,\tau)d\xi d\tau \\ &= F(x,t). \end{aligned} \qquad (5.2.5)$$

Hence (5.1.4) is satisfied. We find by the direct substitution into (5.2.4) for $t<0$

and $x \in D$

$$\begin{aligned} v(x,t)|_{t<0} &= \int_0^\infty \int_0^{b_1} \int_0^{b_2} \int_0^{b_3} G(x,t|\xi,\tau)|_{t<0} F(\xi,\tau) d\xi d\tau = 0, \\ v(x,t)|_{x\in D} &= \int_0^\infty \int_0^{b_1} \int_0^{b_2} \int_0^{b_3} G(x,t|\xi,\tau)|_{x\in D} F(\xi,\tau) d\xi d\tau = 0. \end{aligned}$$

This completes the proof. □

5.3 Construction of Green's Function by a Fourier Series Expansion

Let $\xi = (\xi_1, \xi_2, \xi_3)$ be a parameter from D, τ be a parameter from $(0,\infty)$, $D = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \le x_1 \le b_1, 0 \le x_2 \le b_2, 0 \le x_3 \le b_3\}$ be a parallelepiped. Let us consider the IBVP in the parallelepiped D:

$$\Box_c^{x,t} G(x,t|\xi,\tau) = \delta(x-\xi, t-\tau), \quad (x,t) \in D \times (0,\infty), \tag{5.3.1}$$

$$G(x,t|\xi,\tau)|_{t<\tau} = 0, \quad 0 < \tau < \infty, \tag{5.3.2}$$

$$G(x,t|\xi,\tau)|_{x\in\partial D} = 0, \quad t \in \mathbb{R}, \tag{5.3.3}$$

where $\Box_c^{x,t} = (\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta_x)$.

The solution of (5.3.1)–(5.3.3) can be found by the Fourier series expansion method. Using $\bar{t} = t - \tau$ we can write the problem (5.3.1)-(5.3.3) as follows:

$$\Box_c^{x,\bar{t}} G = \delta(x-\xi, \bar{t}), \quad x \in D, \bar{t} \in (0,\infty), \tag{5.3.4}$$

$$G|_{\bar{t}<0} = 0, \quad x \in D, \tag{5.3.5}$$

$$G|_{x\in\partial D} = 0, \quad t > 0. \tag{5.3.6}$$

Using the result reasoning which we applied for the IBVP for a non-homogeneous acoustic equation with the Dirichlet boundary condition we can write the solution of (5.3.4)-(5.3.6) as follows

$$G = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{c}{\sqrt{\lambda_{ijk}}} \left(\int_0^{\bar{t}} \delta_{ijk} \sin(c\sqrt{\lambda_{ijk}}(\bar{t}-\nu)) d\nu \right) V_{ijk}(x,y,z), \tag{5.3.7}$$

where λ_{ijk} and $V_{ijk}(x_1, x_2, x_3)$ are defined in (2.1.15), (2.1.16) and

$$\begin{aligned} \delta_{ijk} &= \int_0^{b_1}\int_0^{b_2}\int_0^{b_3} \delta(x-\xi)\delta(t)V_{ijk}(x_1,x_2,x_3)dx_1dx_2dx_3 \\ &= V_{ijk}(\xi_1,\xi_2,\xi_3)\delta(\bar{t}). \end{aligned} \tag{5.3.8}$$

Using (5.3.8) the equation (5.3.7) can be written as

$$G = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{c}{\sqrt{\lambda_{ijk}}}\theta(\bar{t})\sin(\sqrt{\lambda_{ijk}}\bar{t})V_{ijk}(\xi_1,\xi_2,\xi_3)V_{ijk}(x_1,x_2,x_3), \tag{5.3.9}$$

where $\theta(\bar{t})$ is the Heaviside function. Replacing $\bar{t} = t - \tau$ we have

$$G = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{c}{\sqrt{\lambda_{ijk}}}\theta(t-\tau)\sin(\sqrt{\lambda_{ijk}}(t-\tau))V_{ijk}(\xi)V_{ijk}(x).$$

This function is the solution of the generalized problem (5.3.1)-(5.3.3).

5.4 Commands for Green's Function Finding

```
bx = 6; by = 12; bz = 3;

n = 35; sc = 1; tau=0;

a[kk_] = (kk*Pi/bx)^2; b[kk_] = (kk*Pi/by)^2; c[kk_] = (kk*Pi/bz)^2;

For[i = 1, i <= n, i++,
 For[j = 1, j <= n, j++,
  For[k = 1, k <=n, k++,
 Lambda[i, j, k] = a[i] + b[j] + c[k] ;
 V[i, j, k]= Sqrt[8/(bx*by*bz)]*Sin[Sqrt[a[i]]*x]*Sin[Sqrt[b[j]]*y]
                *Sin[Sqrt[c[k]]*z];
 d[i, j, k] =ReplaceAll[ReplaceAll[ReplaceAll[V[i, j, k], x -> x0],
  y -> y0], z -> z0];
 T[i, j, k] = (d[i, j,k]/(sc*Sqrt[Lambda[i, j, k]]))*
      Sin[sc*Sqrt[Lambda[i, j, k]]*(t-tau)];
]; ]; ]

G := Sum[Sum[Sum[T[i, j, k]*V[i, j, k], {i, 1, n}], {j, 1, n}], {k,
1, n}]
```

5.5 Examples of the Simulation Using Green's Function

This section deals with modeling and the simulation of acoustic waves by Green's function. We get the formula for the Green's function as follows

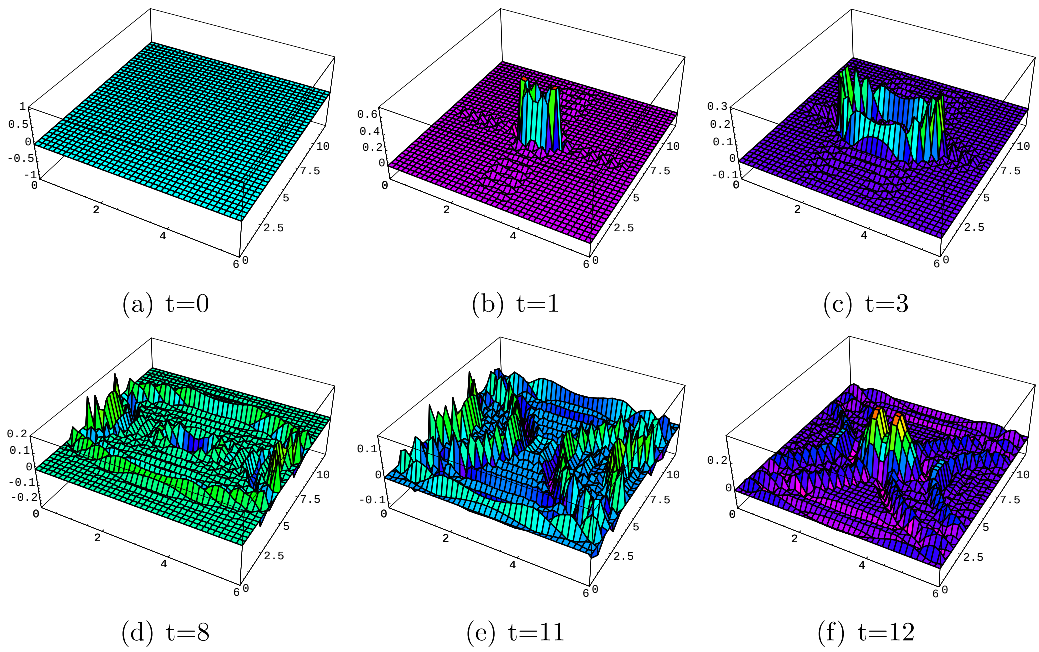
$$G = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{c}{\sqrt{\lambda_{ijk}}}\theta(t-\tau)\sin(\sqrt{\lambda_{ijk}}(t-\tau))V_{ijk}(\xi)V_{ijk}(x), \qquad (5.5.1)$$

where

$$\theta(t-\tau) = \begin{cases} 0, & t < \tau \\ 1, & t \geq \tau \end{cases}$$

$\xi = (\xi_1, \xi_2, \xi_3)$ is the position of the pulse point source, $x = (x_1, x_2, x_3)$ is the space variable, τ is the time when the pulse point source starts to act.

5.5.1 Result of the simulation by formula (5.5.1)



(a) t=0 (b) t=1 (c) t=3

(d) t=8 (e) t=11 (f) t=12

Figure 5.12: Example 1: Source in the center of the room

Using the formula (5.5.1) for $n = 35$ and Mathematica codes we simulated Green's function for $(\xi_1, \xi_2, \xi_3) = (3, 6, 2)$ and $\tau = 0$. The results of the simulation are presented on the figure 5.7.

CHAPTER SIX
IBVPs FOR THE ACOUSTIC EQUATION IN A HOMOGENEOUS SPHERE

6.1 IBVP in a Sphere with the Dirichlet Boundary Condition

Let $S = \{(r, \theta, \varphi) : 0 \leq r \leq r_0,\ 0 \leq \theta < \pi,\ 0 \leq \varphi < 2\pi\}$ be a sphere from $\mathbb{R}^3$. Let us consider the IBVP for the wave equation in the sphere S

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} &= \Delta_{r,\theta,\varphi} u + F(r,\theta,\varphi,t), && (6.1.1)\\ & \quad 0 < r < r_0, 0 < \theta < \pi, \varphi \in \mathbb{R}, \quad t > 0, \\ u(r,\theta,\varphi,0) &= \phi(r,\theta,\varphi), && (6.1.2)\\ \left.\frac{\partial u(r,\theta,\varphi,t)}{\partial t}\right|_{t=0} &= \psi(r,\theta,\varphi), && (6.1.3)\\ |u(r,\theta,\varphi,t)_{r\to+0}| < \infty, & \quad |u(r,\theta,\varphi,t)_{\theta\to+0,\theta\to\pi}| < \infty, \\ u(r,\theta,\varphi,t) &= u(r,\theta,\varphi+2\pi,t), && (6.1.4)\\ u(r,\theta,\varphi,t)|_{\partial S} &= 0, && (6.1.5) \end{aligned}$$

where ∂S is the boundary of S, ϕ is the initial displacement and ψ is the initial velocity. Here $F(r,\theta,\varphi,t)$, $\phi(r,\theta,\varphi)$, $\psi(r,\theta,\varphi)$ are given functions.

We will find $u(r,\theta,\varphi,t)$ which satisfies (6.1.1), (6.1.5) using Fourier series expansion method.

Remark 6.1.1. The Laplace operator in spherical coordinates is given by

$$\Delta_{r,\theta,\varphi} u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}. \qquad (6.1.6)$$

Firstly, we will consider the homogenous equation: $F(r,\theta,\varphi,t) = 0$.
We use the separation of variables. Let

$$u(r,\theta,\varphi,t) = T(t)V(r,\theta,\varphi). \qquad (6.1.7)$$

Substituting (6.1.7) into (6.1.1) we find

$$\frac{1}{c^2}T''(t)V(r,\theta,\varphi) = T(t)\Delta_{r,\theta,\varphi}V(r,\theta,\varphi).$$

After dividing both sides of the last equation by $T(t)V(r,\theta,\varphi)$ we have

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{\Delta_{r,\theta,\varphi}V(r,\theta,\varphi)}{V(r,\theta,\varphi)}.$$

Let us fix t and let (r,θ,φ) varies in the sphere S. Since t and (r,θ,φ) are independent variables then the last equality takes place if and only if the left hand side and the right hand side of this equation are equal to the same constant which we denote as $-\lambda$ (Kyte, Puri & Schaferlotter, 1997). This means

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{\Delta_{r,\theta,\varphi}V(r,\theta,\varphi)}{V(r,\theta,\varphi)} = -\lambda.$$

So we get the following equations for $T(t)$ and $V(r,\theta,\varphi)$:

$$\Delta_{r,\theta,\varphi}V(r,\theta,\varphi) + \lambda V(r,\theta,\varphi) = 0, \tag{6.1.8}$$

$$T''(t) + c^2\lambda T(t) = 0. \tag{6.1.9}$$

The boundary condition (6.1.5)

$$u(r,\theta,\varphi,t)|_{\partial S} = u(r,\theta,\varphi,t)|_{r=R_0} = T(t)V(r,\theta,\varphi)|_{r=r_0} = 0,\ \forall t \geq 0$$

implies

$$V(r,\theta,\varphi)|_{r=r_0} = 0. \tag{6.1.10}$$

As a result we find

$$\Delta_{r,\theta,\varphi}V(r,\theta,\varphi) + \lambda V(r,\theta,\varphi) = 0, \quad (r,\theta,\varphi) \in S, \tag{6.1.11}$$

$$|V(r,\theta,\varphi)|_{\theta=0,\theta=\pi} < \infty, \quad V(r,\theta,\varphi) = V(r,\theta,\varphi+\pi), \tag{6.1.12}$$

$$|V(r,\theta,\varphi)|_{r=0}| < \infty, \quad V(r,\theta,\varphi)|_{r=r_0} = 0. \tag{6.1.13}$$

We will try to find non-zero solution for the equation (6.1.12), (6.1.13) for the specific values of λ which are called as eigenvalues. This problem is called eigenvalue-eigenfunction problem. For this eigenvalue-eigenfunction problem, we will use again separation of variables, i.e.

$$V(r,\theta,\varphi) = R(r)Y(\theta,\varphi). \tag{6.1.14}$$

Substituting (6.1.14) into (6.1.11) and dividing by $R(r)Y(\theta,\varphi)$ we get the following equation

$$\frac{\frac{d}{dr}(r^2\frac{dR(r)}{dr})}{R(r)} + \lambda r^2 + \frac{\Delta_{\theta,\varphi}Y(\theta,\varphi)}{Y(\theta,\varphi)} = 0, \tag{6.1.15}$$

or

$$\frac{\frac{d}{dr}(r^2\frac{dR(r)}{dr})}{R(r)} + \lambda r^2 = -\frac{\Delta_{\theta,\varphi}Y(\theta,\varphi)}{Y(\theta,\varphi)} = \mu. \tag{6.1.16}$$

So we have

$$\Delta_{\theta,\varphi}Y(\theta,\varphi) + \mu Y(\theta,\varphi) = 0, \tag{6.1.17}$$

and

$$\frac{\frac{d}{dr}(r^2\frac{dR(r)}{dr})}{R(r)} + (\lambda - \frac{\mu}{r^2})R(r) = 0. \tag{6.1.18}$$

Substituting (6.1.14) into (6.1.12) and (6.1.13) we get

$$|Y(\theta,\varphi)|_{\theta=0,\theta=\pi}| < \infty, \quad Y(\theta,\varphi) = Y(\theta,\varphi+\pi), \tag{6.1.19}$$

$$|R(0)| < \infty, \quad R(r_0) = 0. \tag{6.1.20}$$

The solution of the eigenvalue eigenfunction problem (6.1.17) and (6.1.19) is the following functions

$$Y_n^{(m)}(\theta,\varphi), \quad m = 0, \mp 1, ..., \mp n, \quad n = 0, 1, \ldots \tag{6.1.21}$$

which corresponds to eigenvalues

$$\mu_n = n(n+1), \quad n = 0, 1, \ldots$$

The functions $Y_n^{(m)}(\theta,\varphi)$ are called spherical harmonic and these functions defined by

$$Y_n^{(m)}(\theta,\varphi) = \sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}}P_n^{(m)}(\cos\theta) \times \begin{cases} \sqrt{\frac{1}{\pi}}\sin m\varphi & , \quad m<0 \\ \sqrt{\frac{1}{2\pi}} & , \quad m=0 \\ \sqrt{\frac{1}{\pi}}\cos m\varphi & , \quad m>0 \end{cases}.$$

Here $P_n^{(m)}(\cos\theta)$ are the associated Legendre polynomials which defined as

$$P_n^{(m)}(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m P_n(x)}{dx^m},$$

where $P_n(x)$ is known as Legendre polynomial which is the solution of the following ordinary differential equation

$$(1-x^2)y''(x) - 2xy'(x) + n(n+1)y(x) = 0.$$

Consider the problem (6.1.18) and (6.1.20) for $\mu_n = n(n+1)$

$$\frac{\frac{d}{dr}(r^2\frac{dR(r)}{dr})}{R(r)} + (\lambda - \frac{\mu_n}{r^2})R(r) = 0. \tag{6.1.22}$$

Substituting $R(r) = \dfrac{Z(r)}{\sqrt{r}}$ into (6.1.22) we reduce equation (6.1.22) to the following

$$Z''(r) + \frac{Z'(r)}{r} + (\lambda - \frac{(n+1/2)^2}{r^2})Z(r) = 0. \tag{6.1.23}$$

Conditions (6.1.20) may be written as

$$|Z(0)| < \infty, \quad Z(r_0) = 0. \tag{6.1.24}$$

The eigenvalue problem (6.1.23), (6.1.24) is the eigenvalue problem for the Bessel operator. The solution of this problem is the following:

$$\lambda = \lambda_{kn} = (\frac{\mu_k^{(n+\frac{1}{2})}}{r_0})^2, \quad k = 1,2,3,\ldots \tag{6.1.25}$$

are eigenvalues, where $\mu_k^{(n+\frac{1}{2})}$ for $k = 0,1,\ldots$ are roots of $J_{n+\frac{1}{2}}(\mu) = 0$;

$$Z_{kn}(r) = C_{kn}J_{n+\frac{1}{2}}(\mu_k^{(n+\frac{1}{2})}\frac{r}{r_0}), \quad k = 1,2,3 \tag{6.1.26}$$

are eigenfunctions corresponding to λ_{kn}, where

$$C_{kn} = \frac{\sqrt{2}}{r_0 J'_{n+\frac{1}{2}}(\mu_k^{(n+\frac{1}{2})})}, \; k = 1, 2, 3, \ldots, \tag{6.1.27}$$

Therefore the solution of the problem (6.1.11)–(6.1.13) can be written as

$$V_{knm}(r, \theta, \varphi) = Y_n^{(m)}(\theta, \varphi) R_{kn}(r), \tag{6.1.28}$$
$$m = 0, \mp 1, ..., \mp n \quad n = 0, 1, \ldots, \quad k = 1, 2, 3, \ldots$$

where $R_{kn}(r) = \dfrac{Z_{kn}(r)}{\sqrt{r}}$.

Remark 6.1.2. The sequences of eigenfunction $\{V_{knm}(r, \theta, \varphi)\}$ for $m = 0, \mp 1, ..., \mp n$, $n = 0, 1, \ldots$, $k = 1, 2, \ldots$ is complete orthonormal system in the space $L_{2, r^2 \sin\theta}(S)$ which is the space of square integrable functions with the weight function $r^2 \sin\theta$ over the sphere S.

Consider the equation (6.1.9)

$$T''_{knm}(t) + c^2 \lambda_{kn} T(t)_{knm} = 0, \tag{6.1.29}$$
$$m = 0, \mp 1, ..., \mp n \quad n = 0, 1, \ldots, \quad k = 1, 2, 3, \ldots$$

We will find our solution in the form

$$u(r, \theta, \varphi, t) = \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} T_{knm}(t) V_{knm}(r, \theta, \varphi). \tag{6.1.30}$$

Using (6.1.2) and (6.1.3) we have

$$T_{knm}(0) = \phi_{knm},$$
$$T'_{knm}(0) = \psi_{knm},$$

where

$$\phi_{knm} = \int_0^{r_0} \int_0^{2\pi} \int_0^{\pi} \phi(r, \theta, \varphi) V_{knm}(r, \theta, \varphi) r^2 \sin\theta dr d\theta d\varphi, \tag{6.1.31}$$

$$\psi_{knm} = \int_0^{r_0} \int_0^{2\pi} \int_0^{\pi} \psi(r, \theta, \varphi) V_{knm}(r, \theta, \varphi) r^2 \sin\theta dr d\theta d\varphi. \tag{6.1.32}$$

So we have the following problem:

$$T''_{knm}(t) + c^2\lambda_{kn}T(t)_{knm} = 0, \tag{6.1.33}$$

$$T_{knm}(0) = \phi_{knm} \quad T'_{knm}(0) = \psi_{knm}, \tag{6.1.34}$$

$$m = 0, \mp 1, ..., \mp n \, n = 0, 1, \ldots, \; k = 1, 2, 3, \ldots$$

The solution for (6.1.33), (6.1.34) is given by

$$T_{knm}(t) = \phi_{knm}\cos(c\sqrt{\lambda_{kn}}t) + \frac{\psi_{knm}}{c\sqrt{\lambda_{kn}}}\sin(c\sqrt{\lambda_{kn}}t), \tag{6.1.35}$$

$$m = 0, \mp 1, ..., \mp n \quad n = 0, 1, \ldots, \quad k = 1, 2, 3, \ldots$$

So we have the solution of the problem (6.1.1)–(6.1.5)

$$u(r, \theta, \varphi, t) = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(\phi_{knm}\cos(c\sqrt{\lambda_{kn}}t) + \frac{\psi_{knm}}{c\sqrt{\lambda_{kn}}}\sin(c\sqrt{\lambda_{kn}}t)\right)V_{knm}(r, \theta, \varphi). \tag{6.1.36}$$

Consider the non-homogenous equation: $F(r, \theta, \varphi, t) \neq 0$.
Using the same discussion of initial value problem in a parallelepiped with the Dirichlet boundary condition for $F(r, \theta, \varphi, t) \neq 0$ we will get the following result

$$\begin{aligned} T_{knm}(t) = \phi_{knm}\cos(c\sqrt{\lambda_{kn}}t) + \frac{\psi_{knm}}{c\sqrt{\lambda_{kn}}}\sin(c\sqrt{\lambda_{kn}}t) + \\ + \frac{c}{\sqrt{\lambda_{kn}}}\int_0^t \sin(c\sqrt{\lambda_{kn}}(t-\tau))F_{knm}(\tau)d\tau \end{aligned} \tag{6.1.37}$$

$$m = 0, \mp 1, ..., \mp n \quad n = 0, 1, \ldots, \quad k = 1, 2, 3, \ldots$$

and the solution of (6.1.1)–(6.1.5) is given by

$$u(r, \theta, \varphi, t) = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} T_{knm}(t)V_{knm}(r, \theta, \varphi), \tag{6.1.38}$$

where $T_{knm}(t)$ is defined in (6.1.37) and λ_{kn} and $V_{knm}(r, \theta, \varphi)$ are defined respectively in (6.1.25), (6.1.28).

6.2 Examples of the Simulation for the Acoustic Wave Propagation in a Spherical Domain

This subsection deals with examples of modeling and simulations of acoustic waves in 3-D spherical rooms with the Dirichlet boundary condition.We took a pulse point source concentrated in different positions in the room: in the center, near the wall. For each of these cases acoustic waves were modeled and simulated. The room is a sphere with radius 3 and the speed of sound is equal to 1 for all examples.

6.2.1 Example 1

The mathematical model of the acoustic wave propagation in the spherical domain S is described in equations (6.1.1) – (6.1.5), where

$$\begin{aligned} \phi(r,\theta,\varphi) &= 0, \\ \psi(r,\theta,\varphi) &= 0, \\ F(r,\theta,\varphi,t) &= \delta(r-r_0)\delta(\theta-\theta_0)\delta(\varphi-\varphi_0)\delta(t),\ (x_0,y_0,z_0)\in D,\ t\geq 0. \end{aligned}$$

Using formulae for $\phi(r,\theta,\varphi)$, $\psi(r,\theta,\varphi)$, $F(r,\theta,\varphi,t)$ and the properties of the Dirac delta function the solution of (6.1.1)–(6.1.5) can be found by the following formula

$$u(r,\theta,\varphi,t)=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}T_{knm}(t)V_{knm}(r,\theta,\varphi),$$

where

$$T_{knm}(t)=\frac{c}{\sqrt{\lambda_{kn}}}V_{knm}(r_0,\theta_0,\varphi_0)r_0^2\sin\theta_0\sin\left(c\sqrt{\lambda_{kn}}t\right), \tag{6.2.1}$$

λ_{kn} and $V_{knm}(r,\theta,\varphi)$ are defined in equation (6.1.25) and (6.1.28) respectively.

For the given data of the Example 1 and for fixed values $\theta_0=\frac{\pi}{2}$, $\varphi_0=\pi$, $r=1.5$ we used Mathematica codes in order to find the solution and to animate the solution. The exact solution (6.2.1) is given by triple series containing infinite number of terms. We found an approximate solution by means of the formulae

$$u(x, y, z, t) = \sum_{k=1}^{Nt} \sum_{n=0}^{Nt} \sum_{m=-n}^{n} T_{knm}(t) V_{knm}(r, \theta, \varphi), \qquad (6.2.2)$$

where $Nt = 7$. This number Nt we choose from the theoretical evaluation of the general term of the series (6.2.2) and the numerical experiments.

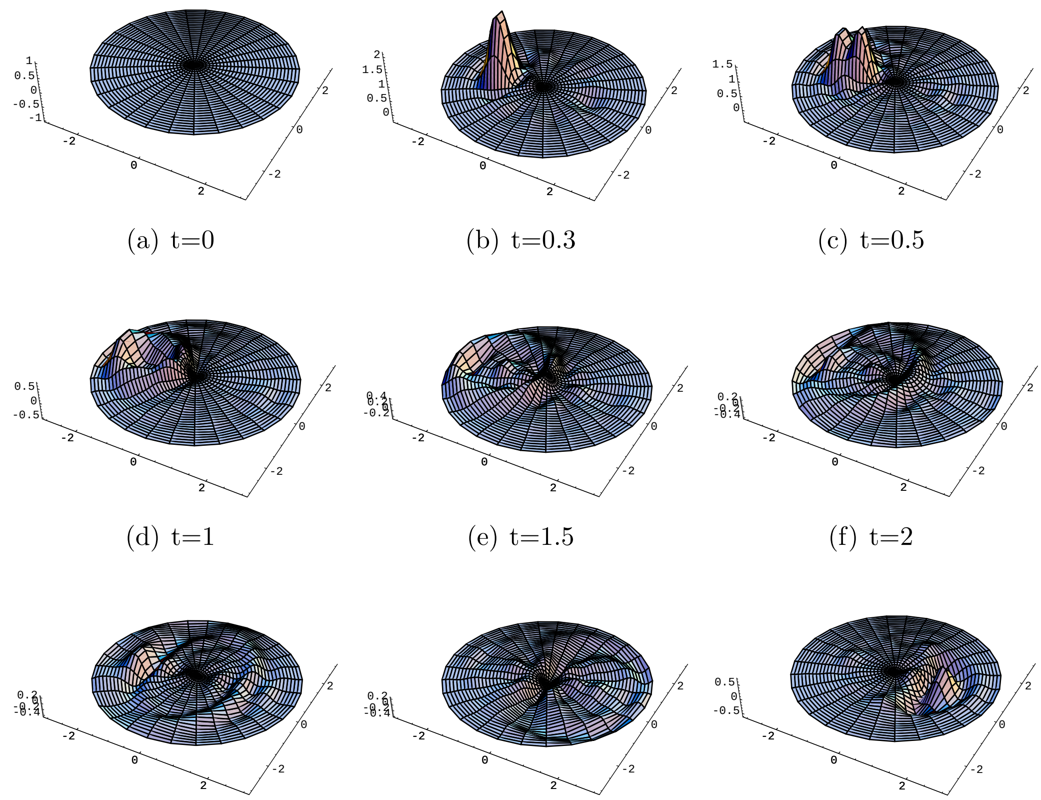
6.2.1.1 *Commands in Mathematica for Example 1:*

```
<< Graphics`ParametricPlot3D`

<< Graphics`Animation`

<< NumericalMath`BesselZeros`

c = 1; R = 3; r0 = 1.5; Theta0 = Pi/2; Phi0 = Pi; Nt = 7;

For[n = 0, n <= Nt , n++,
  Mu[n] = BesselJZeros[n + 1/2, Nt]
  ]

For[k = 1, k <= Nt , k++,
  For[n = 0, n <= Nt , n++,
Lambda[k,n] = (Mu[n][[k]]/R)^2; \!\(\(A[k, n] =
      ReplaceAll[\[PartialD]\_x BesselJ[n + 1/2, x],
        x -> Mu[n][[k]]];\)\)
Cc[k, n] = Abs[Sqrt[2]/(R*A[k, n])]; Z[k, n] = Cc[k, n]*
ReplaceAll[BesselJ[n + 1/2, x], x -> (r/R)*Mu[n][[k]]]; ];
  ]

For[n = 0, n <= Nt, n++,
  For[m = -n, m <= n, m++,
      yfun1[n, m] =
 Sqrt[((2*n + 1)/2)*((n - m)!/(n + m)!)]*LegendreP[n, m, Cos[Theta]];
      Y[n, m] =Which[m < 0, yfun1[n, m]*Sqrt[1/Pi]*Sin[m*Phi], m > 0,
          yfun1[n, m]*Sqrt[1/Pi]*Cos[m*Phi], m == 0,
          yfun1[n, m]*Sqrt[1/(2*Pi)]]
```

```
    ];]

(*--Eigen Function V[k, n, m]--*)
 For[k = 1, k <= Nt, k++,
  For[n = 0, n <= Nt , n++,
      For[m = -n, m <= n, m++,
          V[k, n, m] = Y[n, m]*(Z[k, n]/Sqrt[r])
          ];];]

For[k = 1, k <= Nt , k++,
  For[n = 0, n <= Nt , n++,
      For[m = -n, m <= n, m++,
          T[k, n, m] = (c/Sqrt[Lambda[k, n]])*
                ReplaceAll[ReplaceAll[ReplaceAll[V[k, n, m],
                      r -> r0], Theta -> Theta0], Phi -> Phi0]*
                r0^2*Sin[Theta0]*Sin[c*Sqrt[Lambda[k, n]]*t];
          ];];]

son = 0;
For[k = 1, k <= Nt , k++,
  For[n = 0, n <= Nt , n++,
      For[m = -n, m <= n, m++,
          son = son + T[k, n, m]*V[k, n, m]
          ];];]

son2 = ReplaceAll[son, Theta -> Pi/2];

For [im = 0, im <= 15, im = im + 0.05,
  sonn[im] := ReplaceAll[son2, t -> im];
  grph[im] = CylindricalPlot3D[ sonn[im], {r, 0.01, 3}, {\[Phi], 0, 2Pi},
      Axes -> False, Boxed -> False, PlotRange -> All];
  ]
```

6.2.1.2 Results of the Simulations by the formula (6.2.2)

Using the formula (6.2.2) for $Nt = 7$ and Mathematica codes we simulated acoustic waves arising from pulse point source which is located in the middle of the room between center and the boundary of the room. The results of this



(a) t=0 (b) t=0.3 (c) t=0.5

(d) t=1 (e) t=1.5 (f) t=2

(g) t=3 (h) t=4.5 (i) t=5.5

Figure 6.13: Example 1: Source between the center and the boundary

simulation are presented on the figures as screen shots for different times.

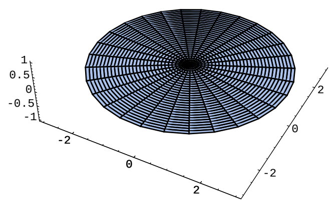
The axes are defined as before in the previous chapter. In figure 6.13 (a) there is no fluctuation of the acoustic field. We can see on the figure 6.13 (b) the acoustic field arising from the pulse point source concentrated at $(\theta_0, \varphi_0, r_0) = (\frac{\pi}{2}, \pi, 1.5)$. There is also small fluctuations which stem from the numerical error of the series (6.2.2). In figure 6.13 (c) the circular wave front is seen and in the figure 6.13 (d) the wave front touches the boundary in the figure 6.13 (e) the wave start to reflect, in the figure 6.13 (f) reflection is seen , in the figure 6.13 (g) the wave completely reflected, in the figure 6.13 (h) reflected waves reach the opposite wall, in the figure 6.13 (i) reflection from the opposite walls is seen.

6.2.2 Example 2

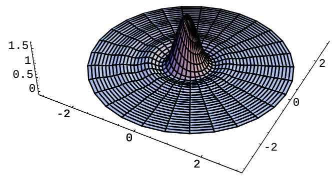
For the given data of the Example 1 and for fixed values $\theta_0 = \dfrac{\pi}{2}$, $\varphi_0 = \pi$, $r = 0$ we used Mathematica codes of Example 1 in order to animate the solution. The exact solution (6.2.1) is given by triple series containing infinite number of terms. We found an approximate solution by means of the formula

$$u(x, y, z, t) = \sum_{k=1}^{Nt} \sum_{n=0}^{Nt} \sum_{m=-n}^{n} T_{knm}(t) V_{knm}(r, \theta, \varphi), \tag{6.2.3}$$

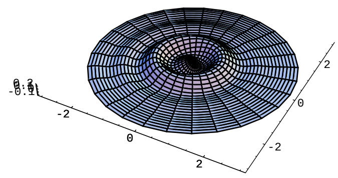
where $Nt = 7$. This number Nt we choose from the theoretical evaluation of the general term of the series (6.2.3) and the numerical experiments.



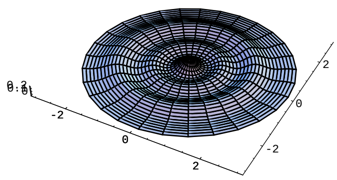
(a) t=0



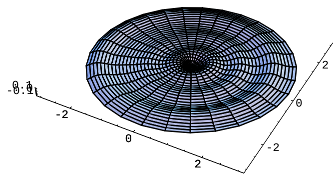
(b) t=0.3



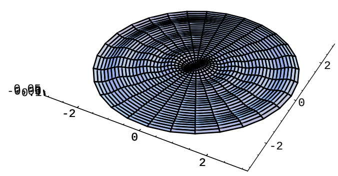
(c) t=1



(d) t=2

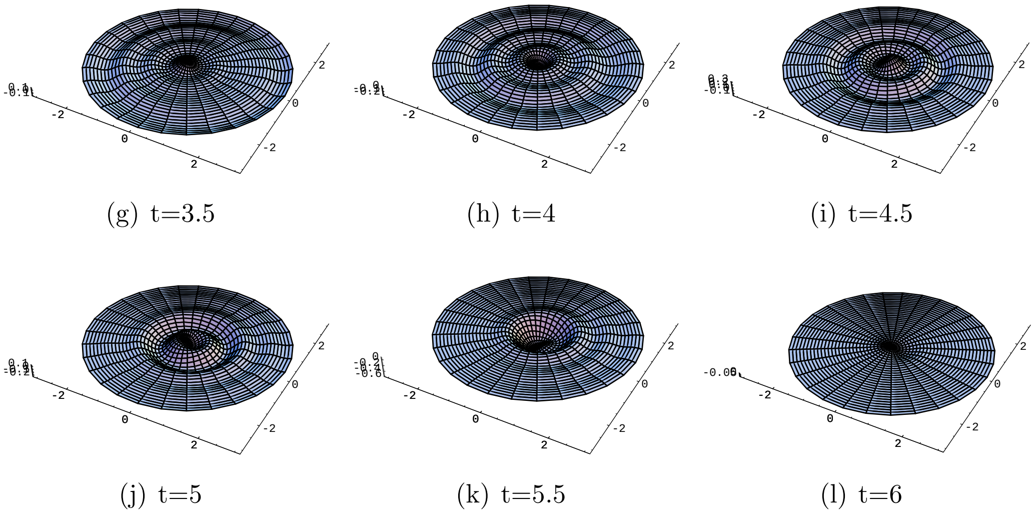


(e) t=2.5



(f) t=3

In figure 6.14 (a) there is no fluctuation of the acoustic field. We can see on the figure 6.14 (b) the acoustic field arising from the pulse point source concentrated at $(\theta_0, \varphi_0, r_0) = (\dfrac{\pi}{2}, \pi, 0)$. In figure 6.14 (c)-(d) the circular wave front is seen and in the figure 6.14 (e) the wave front touches the boundary, in the figure 6.14 (f) the wave start to reflect, in the figure 6.14 (g) reflection is seen , in the figure 6.14 (h) the wave completely reflected, in the figure 6.14 (i)-(k) waves gathering in the center of the room, in the figure 6.14 (i) the interface of waves is seen.



(g) t=3.5 (h) t=4 (i) t=4.5

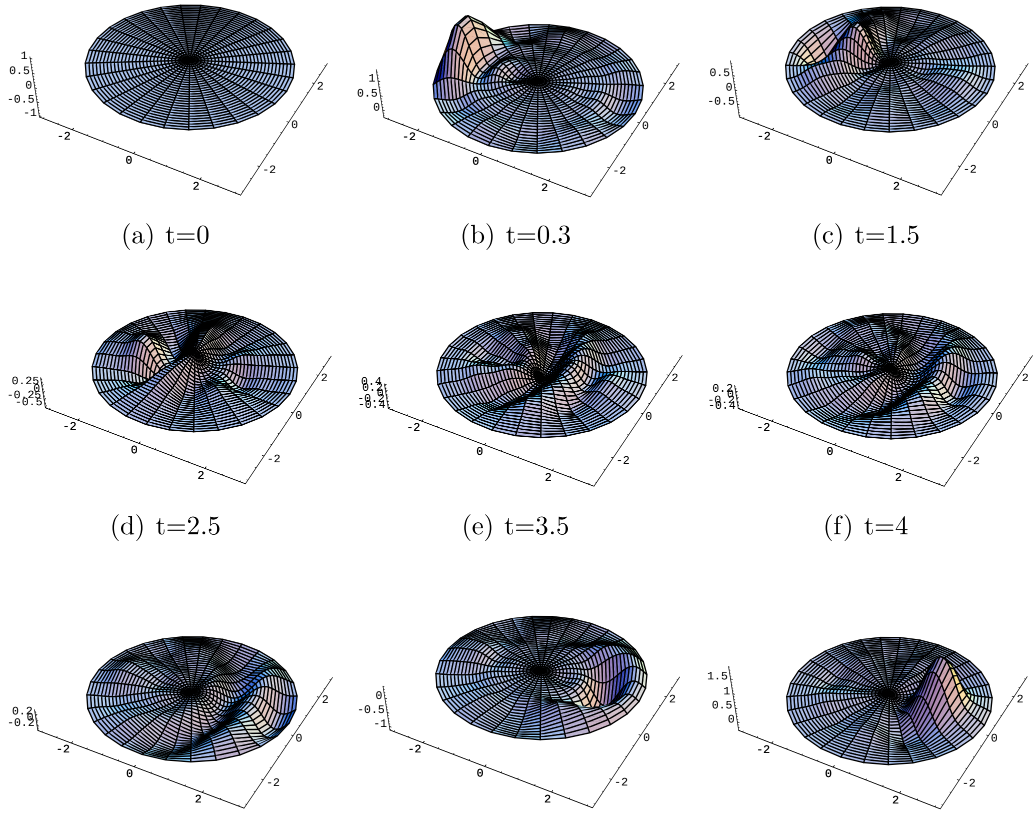
(j) t=5 (k) t=5.5 (l) t=6

Figure 6.14: Example 1: Source in the center of the room

6.2.3 Example 3

For the given data of the Example 1 and for fixed values $\theta_0 = \dfrac{\pi}{2}$, $\varphi_0 = \pi$, $r = 2.5$ we used Mathematica codes of Example 1 in order to animate the solution.

In figure 6.15 (a) there is no fluctuation of the acoustic field. We can see on the figure 6.15 (b) the acoustic field arising from the pulse point source concentrated at $(\theta_0, \varphi_0, r_0) = (\dfrac{\pi}{2}, \pi, 2.5)$. In the figure 6.15 (c) first reflection from the nearest wall is seen, in the figure (d) the circular wave front is seen and in the figure 6.15 (e)-(f) the wave fronts reaches the boundary, in the figure 6.15 (g) wave front touches the boundary, in the figure 6.15 (h) the wave start to reflect, in the figure 6.15 (i) the reflection is seen.



(a) t=0 (b) t=0.3 (c) t=1.5

(d) t=2.5 (e) t=3.5 (f) t=4

(g) t=4.5 (h) t=6.5 (i) t=7.5

Figure 6.15: Example 1: Source near the boundary of the room

CONCLUSION

The main results of the thesis are the following:

1. Formulae for exact and approximate solutions of IBVPs (with Dirichlet, Neumann and Mixed boundary condition) for the acoustic equation in a parallelepiped and IBVP with the Dirichlet boundary condition for acoustic equation in a sphere were obtained.
2. These formulae were adjusted to cases when the source is the pulse point source.
3. Obtained formulae for generalized solutions of IBVPs for the acoustic equation were used for modeling of acoustic waves in homogeneous acoustic media arising from the pulse point sources.
4. Using obtained formulae and Mathematica codes the simulation of the acoustic waves were made.
5. Results of acoustic fields simulation are presented by 3-D pictures and animated movies.

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Appendix A

STURM-LIOUVILLE PROBLEMS (SLP) FOR AN ORDINARY DIFFERENTIAL EQUATION WITH DIFFERENT BOUNDARY CONDITION

1.1 SLP with Dirichlet Boundary Condition

Consider the following Strum-Liouville Problem (Pain, 1993),

$$\begin{aligned} y''(x) + \lambda y(x) &= 0, \quad x \in (0, l) && (1.1.1) \\ y(0) = y(l) &= 0, && (1.1.2) \end{aligned}$$

where l is a given positive number, λ is a parameter.

The general solution of (1.1.1) depends in a fundamental manner on whether

$$\lambda < 0, \quad \lambda = 0 \quad or \quad \lambda > 0.$$

1. *The case* $\lambda < 0$. A general solution of the equation (1.1.1) can be written in the form

$$y(x) = C_1 \exp(-\sqrt{-\lambda}x) + C_2 \exp(\sqrt{-\lambda}x). \qquad (1.1.3)$$

Substituting (1.1.3) to (1.1.2) we will find:

$$\begin{aligned} y(0) &= C_1 + C_2 = 0, \\ y(l) &= C_1 \exp(-\sqrt{-\lambda}l) + C_2 \exp(\sqrt{-\lambda}l) = 0. \end{aligned}$$

Since the determinant of the this system is different from zero for $\lambda > 0$ the this system has only zero solution. So we conclude for $\lambda < 0$ there is no eigenvalues and eigenfunctions.

2. *The case* $\lambda = 0$. Here we have the general solution of (1.1.1) in the form

$$y(x) = C_1 x + C_2. \qquad (1.1.4)$$

Substituting (1.1.4) into (1.1.2) we will find that $y(0) = C_2 = 0$ so $y(x) = C_1 x$ and impose $y(l) = 0$. We have $y(l) = C_1 l = 0$, since $l \neq 0$ implies $C_1 = 0$. This shows that for $\lambda = 0$ we get just the zero solution. So we conclude for $\lambda = 0$ is not an eigenvalue, so there is no eigenfunction for $\lambda = 0$.

3. *The case* $\lambda > 0$. Here the general solution of the equation (1.1.1) can be written in the form

$$y(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x). \qquad (1.1.5)$$

Substituting this general solution to (1.1.2) we will find: $y(0) = C_1 = 0$ implies $C_1 = 0$ and $y(x) = C_2 \sin(\sqrt{\lambda} x)$. Imposing $y(l) = 0$ we will get: $y(l) = C_2 \sin(\sqrt{\lambda} l) = 0$. Choosing $C_2 \neq 0$ we get $\sin(\sqrt{\lambda} l) = 0$ implies $\sqrt{\lambda} l = k\pi$ for $k = 1, 2, \dots$ and $\lambda_k = (\frac{k\pi}{l})^2$ for $k = 1, 2, \dots$ So we have eigenvalues λ_k and corresponding eigenfunctions

$y_k(x) = C_2 \sin(\sqrt{\lambda_k} x)$ for $k = 1, 2, \dots$ We can normalize this orthogonal functions by choosing $C_2 = \sqrt{\frac{2}{l}}$

Conclusion: There are eigenvalues and correspondingly eigenfunction of the problem (1.1.1),(1.1.2) respectively as follows;

$$\lambda_k = (\frac{k\pi}{l})^2, \qquad y_k(x) = \sqrt{\frac{2}{l}} \sin \sqrt{\lambda_k} x, \quad k = 1, 2, \dots$$

1.2 SLP with Neumann Boundary Condition

Consider the following Sturm-Liouville Problem (Pain, 1993),

$$\begin{aligned} y''(x) + \lambda y(x) &= 0, \quad x \in (0, l) &(1.2.1)\\ y'(0) = y'(l) &= 0, &(1.2.2) \end{aligned}$$

where l is a given positive number, λ is a parameter.

The general solution of (1.2.1) depends in a fundamental manner on whether

$$\lambda < 0, \quad \lambda = 0 \quad or \quad \lambda > 0.$$

1. *The case* $\lambda < 0$. A general solution of the equation (1.2.1) can be written in the form

$$y(x) = C_1 \exp(-\sqrt{-\lambda} x) + C_2 \exp(\sqrt{-\lambda} x). \tag{1.2.3}$$

   Substituting (1.2.3) to (1.2.2) we will find:

$$\begin{aligned} y'(0) &= C_1 + C_2 = 0,\\ y'(l) &= C_1 \exp(-\sqrt{-\lambda} l) + C_2 \exp(\sqrt{-\lambda} l) = 0. \end{aligned}$$

   Since the determinant of the this system is differen from zero for $\lambda > 0$ the this system has only zero solution. So we conclude for $\lambda < 0$ there is no eigenvalues and eigenfunctions.

2. *The case* $\lambda = 0$. Here we have the general solution of (1.2.1) in the form

$$y(x) = C_1 x + C_2. \tag{1.2.4}$$

   Substituting (1.2.4) into (1.2.2) we will find that $y'(0) = C_1 = 0$ and $y'(l) = C_1 = 0$. This shows that for $\lambda = 0$ we get just a solution if we choose $C_2 \neq 0$. So $y(x) = C_2$. So we conclude $\lambda = 0$ is an eigenvalue and $y(x) = C_2$ is corresponding eigenfunction to $\lambda = 0$.

3. *The case* $\lambda > 0$. Here the general solution of the equation (1.1.1) can be

written in the form

$$y(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x). \qquad (1.2.5)$$

Substituting this general solution to (1.1.2) we will find: $y'(0) = \sqrt{\lambda}C_2 = 0$ implies $C_1 = 0$ and $y(x) = C_1 \cos(\sqrt{\lambda}x)$. Imposing $y(l) = 0$ we will get: $y(l) = C_1\sqrt{\lambda}\sin(\sqrt{\lambda}l) = 0$. Choosing $C_1 \neq 0$ we get $\sin(\sqrt{\lambda}l) = 0$ implies $\sqrt{\lambda}l = k\pi$ for $k = 1, 2, \ldots$ and $\lambda_k = (\frac{k\pi}{l})^2$ for $k = 1, 2, \ldots$ So we have eigenvalues λ_k and corresponding eigenfunctions

$y_k(x) = C_1 \cos(\sqrt{\lambda_k}x)$ for $k = 1, 2, \ldots$ We can normalize this orthogonal functions by choosing $C_1 = \sqrt{\dfrac{2}{l}}$

Conclusion: There are eigenvalues and correspondingly eigenfunction of the problem (1.1.1),(1.1.2) respectively as follows:

$$\lambda_k = (\frac{k\pi}{l})^2, \qquad y_k(x) = \sqrt{\frac{2}{l}} \cos \sqrt{\lambda_k}x, \quad k = 0, 1, 2, \ldots$$

1.3 SLP with Mixed Boundary Condition

Consider the following Sturm-Liouville Problem (Pain, 1993),

$$y''(x) + \lambda y(x) = 0, \quad x \in (0, l) \tag{1.3.1}$$

$$y'(0) - \alpha y(0) = 0, \tag{1.3.2}$$

$$y'(l) + \alpha y(l) = 0, \tag{1.3.3}$$

where α and l are given positive constants, λ is a parameter.

The general solution of this system depends in a fundamental manner on whether

$$\lambda < 0, \quad \lambda = 0 \quad or \quad \lambda > 0.$$

1. *The case* $\lambda < 0$. A general solution of the equation (1.3.1) can be written in the form

   $$y(x) = C_1 e^{(-\sqrt{-\lambda} x)} + C_2 e^{(\sqrt{-\lambda} x)}$$

   Set $-\lambda = \vartheta^2$. then;

   $$y(x) = C_1 e^{-\vartheta x} + C_2 e^{\vartheta x}, \tag{1.3.4}$$

   $$y'(x) = -C_1 \vartheta e^{-\vartheta x} + C_2 \vartheta e^{\vartheta x}. \tag{1.3.5}$$

   Substituting these into (1.3.2), (1.3.3) we will find;

   $$\begin{aligned} y'(0) - \alpha y(0) &= -C_1\vartheta + C_2\vartheta - \alpha(C_1 + C_2) = 0, \\ y'(l) + \alpha y(l) &= -C_1\vartheta e^{-\vartheta l} + C_2\vartheta e^{\vartheta l} + \alpha(C_1 e^{-\vartheta l} + C_2 e^{\vartheta l}) = 0 \end{aligned}$$

   We need to solve this system for C_1, C_2:

   $$\begin{aligned} -C_1\vartheta + C_2\vartheta - \alpha(C_1 + C_2) &= 0, \\ -C_1\vartheta e^{-\vartheta l} + C_2\vartheta e^{\vartheta l} + \alpha(C_1 e^{-\vartheta l} + C_2 e^{\vartheta l}) &= 0. \end{aligned}$$

   Grouping for C_1, C_2:

   $$\begin{aligned} C_1(-\vartheta - \alpha) + C_2(\vartheta - \alpha) &= 0, \\ C_1(-\vartheta + \alpha)e^{-\vartheta l} + C_2(\vartheta + \alpha)e^{\vartheta l} &= 0. \end{aligned}$$

To find a non-zero solution we must have;

$$\begin{aligned}
\begin{vmatrix} (-\vartheta-\alpha) & (\vartheta-\alpha) \\ (-\vartheta+\alpha)e^{-\vartheta l} & (\vartheta+\alpha)e^{\vartheta l} \end{vmatrix} &= 0, \\
(-\vartheta-\alpha)(\vartheta+\alpha)e^{\vartheta l} - (\vartheta-\alpha)(-\vartheta+\alpha)e^{-\vartheta l} &= 0, \\
-(\vartheta+\alpha)^2 e^{\vartheta l} + (\vartheta-\alpha)^2 e^{-\vartheta l} &= 0, \\
(\vartheta+\alpha)^2 e^{\vartheta l} &= (\vartheta-\alpha)^2 e^{-\vartheta l}, \\
\frac{(\vartheta+\alpha)^2}{(\vartheta-\alpha)^2} &= e^{-2\vartheta l}, \\
\left|\frac{(\vartheta+\alpha)}{(\vartheta-\alpha)}\right| &= e^{-\vartheta l}.
\end{aligned}$$

But the latter equation is true if $\vartheta = 0$. This implies $\lambda = 0$. It means for $\lambda < 0$ we have no eigenvalues and correspondingly no eigenfunctions.

2. *The case* $\lambda = 0$. Here the general solution of the equation (1.3.1) can be written in the form

$$y(x) = C_1 x + C_2$$

and

$$y'(x) = C_1.$$

Substituting these into (1.3.2), (1.3.3) we will find;

$$\begin{aligned}
y'(0) - \alpha y(0) &= C_1 - \alpha C_2 = 0, \\
y'(l) + \alpha y(l) &= C_1 + \alpha(C_1 l + C_2) = 0.
\end{aligned}$$

So we need to solve:

$$\begin{aligned}
&C_1 - \alpha C_2 = 0, \\
&C_1 + \alpha(C_1 l + C_2) = 0
\end{aligned}$$

But we have the following non-zero determinant

$$\begin{vmatrix} 1 & -\alpha \\ 1+\alpha l & \alpha \end{vmatrix} = \alpha + (\alpha + \alpha^2 l) = \alpha(2 + \alpha l) \neq 0.$$

Since $\alpha > 0$ and $l > 0$. We conclude that $\lambda = 0$ is not an eigenvalue and there is no eigenfunction for $\lambda = 0$.

3. *The case* $\lambda > 0$. Here the general solution of the equation (1.3.1) can be written in the form

$$y(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x).$$

Set $\lambda = \vartheta^2$ then we get:

$$\begin{aligned} y(x) &= C_1 \cos(\vartheta x) + C_2 \sin(\vartheta x), \\ y'(x) &= -C_1\vartheta \sin(\vartheta x) + C_2\vartheta \cos(\vartheta x). \end{aligned}$$

Substituting these into (1.3.2), (1.3.3) we will find:

$$\begin{aligned} y'(0) - \alpha y(0) &= C_2\vartheta - \alpha C_1 = 0, \\ y'(l) + \alpha y(l) &= -C_1\vartheta \sin(\vartheta l) + C_2\vartheta \cos(\vartheta l) + \alpha(C_1 \cos(\vartheta l) + C_2 \sin(\vartheta l)) = 0. \end{aligned}$$

Hence we need to find:

$$\begin{vmatrix} -\alpha & \vartheta \\ -\vartheta \sin(\vartheta l) + \alpha \cos(\vartheta l) & \vartheta \cos(\vartheta l) + \alpha \sin(\vartheta l) \end{vmatrix} = 0.$$

So,

$$\begin{aligned} -\alpha(\vartheta \cos(\vartheta l) + \alpha \sin(\vartheta l)) - \vartheta(-\vartheta \sin(\vartheta l) + \alpha \cos(\vartheta l)) &= 0, \\ -\alpha\vartheta \cos(\vartheta l) - \alpha^2 \sin(\vartheta l) + \vartheta^2 \sin(\vartheta l) - \vartheta\alpha \cos(\vartheta l) &= 0, \\ -2\alpha\vartheta \cos(\vartheta l) + (\vartheta^2 - \alpha^2) \sin(\vartheta l) &= 0. \end{aligned}$$

Hence after rearranging we have;

$$\begin{aligned} \frac{\cos(\vartheta l)}{\sin(\vartheta l)} &= \frac{\vartheta^2 - \alpha^2}{2\alpha\vartheta}, \\ \frac{\cos(\vartheta l)}{\sin(\vartheta l)} &= \frac{1}{2}\left(\frac{\vartheta}{\alpha} - \frac{\alpha}{\vartheta}\right), \\ \cot(\vartheta l) &= \frac{1}{2}\left(\frac{\vartheta}{\alpha} - \frac{\alpha}{\vartheta}\right). \end{aligned}$$

The latter equation implies that there are infinitely many ϑ_k, $k = 1, 2, \ldots$ which satisfies the equation. So correspondingly there are infinitely many eigenvalues of the problem for $\lambda > 0$. Since, it was $\lambda = \vartheta^2$. So we have eigenfunctions

$$y_k(x) = C_1 \cos(\vartheta_k x) + C_2 \sin(\vartheta_k x).$$

Choose

$$C_1 = \frac{\vartheta_k}{\alpha} C_k, \quad C_2 = C_k.$$

We get

$$\begin{aligned} y_k(x) &= \frac{\vartheta_k}{\alpha} C_k \cos(\vartheta_k x) + C_k \sin(\vartheta_k x) \\ &= C_k(\frac{\vartheta_k}{\alpha} \cos(\vartheta_k x) + \sin(\vartheta_k x)) \\ &= C_k(\sin(\vartheta_k x + \varphi_k)), \end{aligned}$$

where

$$\varphi_k = \arctan(\frac{\vartheta_k}{\alpha}), \quad k = 1, 2, \ldots$$

Conclusion: We have eigenfunctions

$$y_k(x) = C_k(\sin(\vartheta_k x + \varphi_k), \quad k = 1, 2, \ldots$$

and correspondingly the eigenvalues $\lambda_k = \vartheta_k^2$, where ϑ_k, $k = 1, 2, \ldots$ are a solutions of

$$\cot(\vartheta_k l) = \frac{1}{2}\left(\frac{\vartheta_k}{\alpha} - \frac{\alpha}{\vartheta_k}\right),$$

and C_k is chosen as follows

$$\int_0^l y_k^2(x)dx = \int_0^l C_k^2 \sin^2(\vartheta_k x + \varphi_k)dx = C_k^2 \int_0^l \sin^2(\vartheta_k x + \varphi_k)dx = 1.$$

Hence

$$C_k = \frac{1}{(\int_0^l \sin^2(\vartheta_k x + \varphi_k)dx)^{1/2}}.$$

So

$$\begin{aligned}
\int_0^l \sin^2(\vartheta_k x + \varphi_k)dx &= \int_0^l \frac{1 - \cos 2(\vartheta_k x + \varphi_k)}{2}dx \\
&= \frac{x}{2}\bigg|_0^l - \frac{sin2(\vartheta_k x + \varphi_k)}{4\vartheta_k}\bigg|_0^l \\
&= \frac{l}{2} - \frac{sin2(lx + \varphi_k)}{4\vartheta_k} + \frac{sin2\varphi_k}{4\vartheta_k} \\
&= \frac{2l\vartheta_k - sin2(lx + \varphi_k) + sin2\varphi_k}{4\vartheta_k}.
\end{aligned}$$

Hence

$$C_k = \sqrt{\frac{4\vartheta_k}{2l\vartheta_k - sin2(lx + \varphi_k) + sin2\varphi_k}}, \quad k = 1, 2, \ldots$$

Appendix B

INITIAL VALUE PROBLEM FOR AN ORDINARY DIFFERENTIAL EQUATION

Consider the following system;

$$y''(x) + w^2 y(x) = f(x), \tag{2.0.1}$$

$$y(0) = 0, \quad y'(0) = 0. \tag{2.0.2}$$

The fundamental solution $\varepsilon(x)$ for the problem (2.0.1) and (2.0.2) must satisfy

$$\varepsilon''(x) + w^2 \varepsilon(x) = \delta(x). \tag{2.0.3}$$

See Appendix C.
We have $\varepsilon(x) = \theta(x) Z(x)$ where $Z(x)$ is the solution of

$$Z''(x) + w^2 Z(x) = 0$$

$$Z(0) = 0 \quad Z'(0) = 1$$

Thus $Z(x) = \dfrac{1}{w} \sin(wx)$.

Using Lemma 3.0.3 we can conclude that $\varepsilon(x) * f(x)$ is the solution of (2.0.1),(2.0.2).

Hence

$$\begin{aligned} \varepsilon(x) * f(x) &= \int_{-\infty}^{\infty} \varepsilon(x-\tau) f(\tau) d\tau \\ &= \int_{-\infty}^{\infty} \theta(x-\tau) \frac{1}{w} \sin(w(x-\tau)) f(\tau) d\tau \\ &= \frac{1}{w} \int_{0}^{t} \sin(w(x-\tau)) f(\tau) d\tau. \end{aligned}$$

Appendix C

ELEMENTS OF GENERALIZED FUNCTION THEORY

Definition 3.0.1. Let $\mathfrak{L}(x,D) = \sum_{|\alpha|=0}^{m} a_\alpha(x) D^\alpha$ be a linear differential operator, where $x = (x_1, ..., x_n)$, $\alpha = (\alpha_1, ..., \alpha_n)$, $|\alpha| = \alpha_1 + ... + \alpha_n$, $D^\alpha = \dfrac{\partial^{|}\alpha|}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$. The fundamental solution of $\mathfrak{L}(x,D)$ is a generalized function $\varepsilon(x)$ satisfying $\mathfrak{L}(x,D)\varepsilon(x) = \delta(x)$.

Lemma 3.0.2. *$\mathfrak{L}(x,D)\varepsilon(x) = \delta(x)$ has a solution $\varepsilon(x) = \theta(x)Z(x)$, where $Z(x)$ is a solution of the following problem:*

$$\begin{aligned} &\mathfrak{L}(x,D)Z(x) = 0, \\ &Z(0) = Z'(0) = ... = Z^{m-1}(0) = 0; Z^m(0) = 1. \end{aligned}$$

Lemma 3.0.3. *Let $\mathfrak{L}(x,D)$ be a linear operator and $\varepsilon(x)$ be a fundamental solution of $\mathfrak{L}(x,D)$, $f(x)$ be a generalized function such that $\varepsilon(x) * f(x)$ exists. Then $u(x) = (\varepsilon * f)_{(x)}$ is a generalized solution of $\mathfrak{L}(x,D)$ and this solution is unique in the class of generalized functions for which a convolution with $\varepsilon(x)$ exists.*