DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON COATOMIC MODULES

by
OSMAN CAN HATİPOĞLU

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İZMİR

ON COATOMIC MODULES

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We have read the thesis entitled **"ON COATOMIC MODULES"** completed by **OSMAN CAN HATİPOĞLU** under supervision of **Assistant Prof. GÖKHAN BİLHAN** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Assist. Prof. Dr. Gökhan BİLHAN

Supervisor

. . . .

Prof. Dr. Refail ALİZADE

Jury Member

. . . .

Assist. Prof. Dr. Engin MERMUT

Jury Member

Prof. Dr. Cahit HELVACI
Director
Graduate School of Natural and Applied Sciences

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Osman Can Hatipoğlu[3]

[1]Ast. Prof. Dr. Gökhan Bilhan, Dokuz Eylül Üniversitesi, Fen-Edebiyat Fakültesi, Matematik Bölümü, 35160, Buca/İZMİR, TURKEY. e-mail: gokhan.bilhan@deu.edu.tr

[2]Engin Büyükaşık, İzmir Yüksek Teknoloji Enstitüsü, Fen Fakültesi, Matematik Bölümü, Gülbahçe Köyü, 35430, Urla/İZMİR, TURKEY. e-mail: enginbuyukasik@iyte.edu.tr

[3]Osman Can Hatipoğlu, Dokuz Eylül Üniversitesi, Fen-Edebiyat Fakültesi, Matematik Bölümü, 35160, Buca/İZMİR, TURKEY. e-mail: can.hatipoglu@deu.edu.tr

ON COATOMIC MODULES

ABSTRACT

This thesis attempts to observe whether some of the facts known for coatomic modules are still valid for a generalization of coatomic modules. A module M is finitely coatomic if every finitely generated submodule of M is contained in a maximal submodule of M. It is proved that several classes of modules are contained in the class of finitely coatomic modules. Namely, it is proved that if M is finitely supplemented with small radical then M is finitely coatomic. Similarly, f-semiperfect modules with small radical are also finitely coatomic. We call a module finitely semisimple if every finitely generated submodule of it is a direct summand of it. It is proved that f-semisimple modules are also finitely coatomic.

Keywords: Rings, modules, supplemented modules, semiperfect modules, coatomic modules, finitely coatomic modules.

KOATOMİK MODÜLLER ÜZERİNE

ÖZ

Bu tezde koatomik modüller için geçerli olan bazı durumların koatomik modüllerin bir genellemesi altında da geçerli olup olmadığı incelenmeye çalışıldı. Bir M modülünün her sonlu üretilmiş alt modülü M modülünün bir maksimal alt modülü tarafından içeriliyorsa M sonlu koatomiktir. Eğer M sonlu tümlenen ve radikali küçük olan bir modülse, M sonlu koatomiktir. Ayrıca sonlu yarı basit modüller de sonlu koatomiktir. Sonlu yarı kusursuz modüller eğer radikalleri küçükse sonlu koatomiktir.

Anahtar sözcükler: Halkalar, modüller, tümlenen modüller, yarı kusursuz modüller, koatomik modüller, sonlu koatomik modüller.

CONTENTS

CHAPTER ONE
INTRODUCTION

It is well-known that for a finitely generated module M every proper submodule of M is contained in a maximal submodule of M (Kasch (1982), Theorem 2.3.11). As an attempt to generalize this property of finitely generated modules we have coatomic modules. So, Zöschinger calls a module M coatomic if every proper submodule of M is contained in a maximal submodule of M (Zöschinger (1980)). Semisimple modules and local modules are some examples of coatomic modules. In a series of papers Zöschinger developed much of the properties of coatomic modules: In Zöschinger (1980) he gave many ring theoretic results on coatomic modules, and he proved that over a commutative Noetherian ring every submodule of a coatomic module is coatomic. In Zöschinger (1992) he connected coatomic modules with primary decomposition theory. Coatomic modules also appear in Kasch (1982) page 239 and Azumaya (1974). Several contributions were made to the theory of coatomic modules after Zöschinger. See Koşan & Harmancı (2005), Güngöroğlu (1998), Güngöroğlu & Harmancı (1999), Güngöroğlu & Harmancı (2000).

In this thesis, we mainly attempt to generalize coatomic modules. First, in Chapter 2 we develop basic tools of modules and rings which we will use in Chapter 3. In Chapter 3 we gather some results on coatomic modules. We prove that a module M is semisimple if and only if M is coatomic and every maximal submodule of M is a direct summand, which is given as an exercise in Kasch (1982). We also show that local modules are coatomic. We give an example of a coatomic module M which has a submodule which is not coatomic. Then we introduce finitely coatomic modules as a generalization of coatomic modules. In the remaining part of the thesis we deal with finitely coatomic modules. Namely, a module M is called finitely coatomic (or f-coatomic for short) if every finitely generated proper submodule of M is contained in a maximal submodule of M. First we try to establish basic properties of f-coatomic modules. Unlike the coatomic modules, we obtain that the radical of an f-coatomic module does not have to be small. By following the papers Güngöroğlu (1998) and Güngöroğlu &

Harmancı (1999), we try to see if some of the results valid for coatomic modules are still valid for f-coatomic modules. A module M is called finitely semisimple (f-semisimple for short) if every finitely generated submodule of M is a direct summand of M (Bilhan (2006)). We prove that every f-semisimple module is f-coatomic. We provide an example to show that the converse of this result is not true. Regarding supplemented modules, we prove that every f-supplemented module M with $\mathrm{Rad}(M) \ll M$ is f-coatomic. Similarly, it is proved that an f-semiperfect module M is f-coatomic if $\mathrm{Rad}(M) \ll M$.

CHAPTER TWO
RINGS AND MODULES

In this chapter, we will give the basics about rings and modules. We will give the definitions and results which we will use in Chapter 3. We start with rings and ideals:

2.1 Rings and Their Homomorphisms

Definition 2.1.1. A *ring* is a set R with two binary operations $+$ and $\cdot$, called addition and multiplication, respectively, such that the following properties are satisfied:

1. Addition is associative: for all $r, s, t \in R$ we have $r + (s + t) = (r + s) + t$.
2. Addition is commutative: that is for all $r, s \in R$, $r + s = s + r$
3. There is an element denoted by 0_R such that $r + 0_R = 0_R + r = r, \forall r \in R$. 0_R is called the zero element of the ring.
4. Every element has an additive inverse, that is, for every $r \in R$ there is an element $-r \in R$ such that $r + (-r) = (-r) + r = 0_R$.
5. Multiplication is associative: For every $r, s, t \in R$ we have $r \cdot (s \cdot t) = (r \cdot s) \cdot t$.
6. The left and right distributive laws hold: For all $r, s, t \in R$,
   $r \cdot (s + t) = r \cdot s + r \cdot t$ and $(r + s) \cdot t = r \cdot t + s \cdot t$.

Note on notation: For simplicity, we will denote $a \cdot b$ by just ab, as long as there is no chance of ambiguity. Also, 0_R will be written as 0.
In every ring the following properties are satisfied:

Theorem 2.1.1. *(Fraleigh (2003), Theorem 18.8) If R is a ring with additive identity 0, then for any $a, b \in R$ we have*

1. $0a = a0 = 0$,

2. $a(-b) = (-a)b = -(ab)$,

3. $(-a)(-b) = ab$.

Definition 2.1.2. A ring R is called *commutative* if $rs = sr$ for every $r, s \in R$. Also, a ring in which there is a multiplicative identity 1_R such that $1_R r = r 1_R = r$ for all $r \in R$ is called a *ring with identity.* This multiplicative identity is called *unity.* We will denote the unity of a ring by 1 unless there is no ambiguity.

Some rings satisfy certain multiplicative properties. Namely, a commutative ring R is called a *field* if every nonzero element has a multiplicative inverse. That is, for every $r \in R$, there exists $s \in R$ such that $rs = 1$. Also, R is called an *integral domain* if it has no divisors of zero, which means that, whenever $rs = 0$ for some $r, s \in R$ then either $r = 0$ or $s = 0$. Throughout our work, by a *ring*, we will always mean a *ring with identity.*

Definition 2.1.3. A subset S of a ring R is called a *subring* if it is a ring with the operations of R, and $1_R = 1_S$ in case R has identity.

A list of some examples of rings is:

1. The set $\mathbb{Z}$ of integers is a commutative ring with usual addition and multiplication. Indeed, it is a subring of the field of rational numbers $\mathbb{Q}$.

2. The set of complex numbers is a field with a subfield the set of real numbers.

3. For $n \geq 2$ the set $M_n(R)$ of all $n \times n$ matrices with coefficients in a ring R is a noncommutative ring with matrix addition and multiplication.

After these definitions and examples we give the necessary and sufficient conditions to be a subring:

Proposition 2.1.1. *The Subring Criterion. Let R be a ring and S be a subset of R. Then S is a subring of R if and only if for every $a, b \in S$:*

(i) $a - b \in S$;

(ii) $ab \in S$.

Now we can give the definiton of a ring homomorphism:

Definition 2.1.4. Let R, S be rings. The mapping $f : R \to S$ is called a *ring homomorphism* if it satisfies the following:

(i) $f(a + b) = f(a) + f(b)$ for all $a, b \in R$;

(ii) $f(ab) = f(a)f(b)$, for all $a, b \in R$;

(iii) $f(1_R) = 1_S$.

Special names are given to homomorphisms which satisfy certain properties. An onto homomorphism is called an *epimorphism*, and a one-to-one homomorphism is called a *monomorphism*. A one-to-one and onto ring homomorphism is called an *isomorphism*. If there is an isomorphism between two rings R and S we say that R and S are *isomorphic* and denote it by $R \cong S$.

Proposition 2.1.2. *(Fraleigh (2003), Theorem 26.3) Let* $f : R \to S$ *be a homomorphism. Then* $\operatorname{Im} f$*, the image of* f*, is a subring of* S*.*

2.2 Ideals and Factor Rings

We go on developing the necessary tools for our work. Usage of ideals to develop ring theory is of great importance. In this section we will give the fundamental properties of ideals.

Definition 2.2.1. Let R be a ring. We say that the subset I of R is a *left ideal* of R if the following are satisfied:

(i) $I \neq \emptyset$;

(ii) whenever $a, b \in I$, then $a + b \in I$;

(iii) whenever $a \in I$ and $r \in R$, then $ra \in I$ also.

Similarly a *right ideal* of a ring can be defined by changing the left multiplication in the definition with right multiplication. If I is both left and right ideal, we say that I is a two sided ideal. Clearly, for a commutative ring, left and right ideals coincide. By an ideal we will always mean a *two sided* ideal.

Definition 2.2.2. An ideal I of a commutative ring R is called *finitely generated* if $I = \{r_1a_1 + r_2a_2 + \ldots + r_na_n : r_i \in R \text{ for all } i\}$ for some finite number of elements $a_1, \ldots, a_n$ of I. In this case we say that I is generated by $a_1, \ldots, a_n$ or that $\{a_1, \ldots, a_n\}$ is a generating set for I. If the ideal I is generated by a single element we say that I is a *principal ideal.* If every ideal of an integral domain R is principal, then R is called a *principal ideal domain (PID).*

The *kernel* of a homomorphism $f : R \to S$ is the set

$$\operatorname{Ker} f := \{r \in R : f(r) = 0\}.$$

The kernel of a homomorphism is an ideal of its domain. We can tell whether a homomorphism is a monomorphism or not by looking at its kernel:

Proposition 2.2.1. *(Anderson & Fuller (1992), Proposition 1.3.) Let R and S be rings, let $f : R \to S$ be a ring homomorphism. Then f is a monomorphism if and only if* $\operatorname{Ker} f = \{0\}$

Suppose that I is a proper ideal of a ring R. The relation defined by

$$a \equiv b (\operatorname{mod} I) \iff a - b \in I$$

determines an equivalence relation on R. The congruence class of an element a is defined by

$$a + I = \{a + x : x \in I\}$$

and is called a *coset* of the element a, and the set R/I of all cosets of I is a ring with operations defined by

$$(a + I) + (b + I) = (a + b) + I \text{ and } (a + I)(b + I) = ab + I.$$

Additive and multiplicative identities are

$$0 + I \text{ and } 1 + I.$$

The ring R/I is called the *factor ring of* R *modulo* I. Further the map $\sigma : R \to R/I$ defined by $r \mapsto r + I$ is an epimorphism with kernel I, called the *natural* or *canonical* epimorphism.

2.2.1 *Ideals of Factor Rings*

Let I be an ideal of a ring R. Consider the factor ring R/I. If J is an ideal of R such that $I \subset J$, then $J/I = \{a + I : a \in J\}$ is an ideal of R/I. Conversely, every ideal of R/I is of the form K/I for a unique ideal K of R.

2.2.2 *Operations on Ideals*

For ideals A, B of a ring R, we define the *sum* and *product* of ideals as follows:

$$A + B = \{a + b : a \in A, b \in B\}$$

$$AB = \left\{ \sum_{i=1}^{k} a_i b_i : k \in \mathbb{N}, \text{ and for every } i = 1, 2, \ldots, k, a_i \in A, b_i \in B \right\}$$

It is easy to check that the resulting sets are also ideals of R.

Theorem 2.2.1. *The Isomorphism Theorems For Rings. Let R, S be rings and I, J be ideals of R.*

1. *Let $f : R \to S$ be a ring homomorphism. Then there is an isomorphism $\bar{f} : R/\operatorname{Ker} f \to \operatorname{Im} f$.*

2. *If $I \subset J$, then there is an isomorphism*

$$(R/I)/(J/I) \cong R/J.$$

3. There is an isomorphism

$$(I + J)/J \cong I/(I \cap J).$$

2.2.3 *Maximal Ideals*

Definition 2.2.3. We say that an ideal M of a ring R is a *maximal* ideal, if

(i) $M \subsetneq R$

(ii) if $M \subsetneq I \subseteq R$, then $I = R$ for every ideal I of R.

Using Zorn's lemma we are able to say that every ring has a maximal ideal. But first let us state Zorn's lemma:

Lemma 2.2.1. *(Zorn's Lemma) Let $(P, \preceq)$ be a partially ordered set. If every non-empty totally ordered subset of P has an upper bound in P, then P has at least one maximal element.*

Those readers who wish to know more about Zorn's lemma or partially ordered sets may refer for example to Rotman (2002). This much of ring theory is sufficient for understanding the remaining parts of the text, therefore we close this part with the following two results. We will turn back to rings and give more definitions and results from time to time, but for further information on rings, curious reader may have a look at Rotman (2002). Now we can say the existence of maximal ideals are guaranteed:

Proposition 2.2.2. *(Wisbauer (1991), §2.6) Let R be a non-trivial ring with identity. Then R has at least one maximal ideal.*

Applying the above proposition to factor rings, we get the following:

Corollary 2.2.1. *Let R be a ring with identity and let I be a proper ideal of R. Then there exists a maximal ideal M of R containing I.*

From now on, we give the results on modules necessary for our work. Briefly, an R-module can be considered as the generalization of the notion of vector space in the sense scalars are allowed to be taken from a ring R instead of a field.

2.3 Modules, Submodules, Factor Modules and Module Homomorphisms

Although modules are in fact considered as a pair (M, λ), where M is an additive abelian group and λ is a map from R to the set of endomorphisms of M; we find the following definition more common and simple:

Definition 2.3.1. Let R be a ring (with unity 1). A *left R-module* is an additive abelian group M together with a mapping $R \times M \to M$, which we call a *scalar multiplication*, denoted by

$$(r, m) \mapsto rm$$

such that the following properties hold: for all $m, n \in M$ and $r, s, \in R$;

1. $r(m + n) = rm + rn$,
2. $(r + s)m = rm + sm$,
3. $(rs)m = r(sm)$,.

If, in addition, for every $m \in M$ we have $1m = m$ M is called a *unitary left R-module*. If M is a left R-module, we denote it by ${}_RM$.

Note that one can obtain the *right R-module* definition by applying the scalar multiplications from the right. For commutative rings, two notions of left and right R-module coincide. In our work, all modules will be unitary left R-modules.

Example 2.3.1. Here is a list of some elementary examples of modules:

1. As we indicated at the beginning, every vector space over a field F is an F-module.

2. Every abelian group is a $\mathbb{Z}$-module, where $\mathbb{Z}$ is the set of integers. Hence abelian groups can be generalized via module theory.

3. Every ring R is a module over itself.

A *submodule* of an R-module M is a subgroup N of M which is closed under scalar multiplication, i.e., $rn \in N$ for all $r \in R$, $n \in N$. Clearly the $\{0\}$ and the module M itself are submodules of M. They are called *trivial submodules* of M. We call a submodule N of M a *proper submodule* of M if $N \subsetneqq M$.

When a ring R is considered as a left module over itself, its submodules are precisely the left ideals of R.

Given any two R-modules M_1, M_2, we can always produce a new module, which we call the *sum* of M_1, M_2, containing both M_1 and M_2. This is done by defining

$$M_1 + M_2 = \{m_1 + m_2 : m_1 \in M_1, m_2 \in M_2\}.$$

Also, for an infinite family $\{M_\lambda\}_{\lambda \in \Lambda}$ of submodules of M, we define the sum as

$$\sum_{\lambda \in \Lambda} M_\lambda = \left\{ \sum_{k=1}^{r} m_{\lambda k} : r \in \mathbb{N}, \text{ and for } k = 1, 2, \ldots, r, \lambda_k \in \Lambda, m_{\lambda_k} \in M_{\lambda_k} \right\}$$

This is a submodule of M and so is the intersection $\bigcap_{\lambda \in \Lambda} M_\lambda$. It is worth noting that $\bigcap_{\lambda \in \Lambda} M_\lambda$ is the largest submodule of M which is contained in all M_λ, and $\sum_{\lambda \in \Lambda} M_\lambda$ is the smallest submodule which contains all M_λ (Wisbauer (1991), §6.2).

Proposition 2.3.1. *Modular law. (Wisbauer (1991), §6.2) If H, K, L are submodules of an R-module M and $K \subset H$, then*

$$H \cap (K + L) = K + (H \cap L).$$

Now we define the module homomorphisms:

Definition 2.3.2. Let R be a ring and M, N be R-modules. A function $f : M \to N$ is called an *R-homomorphism* if, for all $m_1, m_2 \in M$ and for all $r \in R$,

(i) $f(m_1 + m_2) = f(m_1) + f(m_2)$;

(ii) $f(rm_1) = rf(m_1)$.

We see no need to list the definitions of *R-epimorphism, R-monomorphism*, and *R-isomorphism* since they are similar to the corresponding definitions for ring homomorphisms. Also the definitions of *kernel* and *image* of a homomorphism are similar to the ones we gave in chapter one. For a module homomorphism $f : M \to N$, as one may expect, $\ker f$ is a submodule of M and $\operatorname{Im} f$ is a submodule of N. Note that we will just write homomorphism instead of R-homomorphism.

Definition 2.3.3. Let M be an R-module and N be a submodule of M. Then the set of cosets

$$M/N = \{x + N : x \in M\}$$

is a left R-module if we define the addition and scalar multiplication as

$$(x + N) + (y + N) = (x + y) + N, \, r(x + N) = rx + N.$$

This new module is called the *factor module of M modulo N*. The map $\pi : M \to M/N$ defined by $m \mapsto (m + N)$ is an epimorphism called the *natural* or *canonical epimorphism.*

It is wise to give the isomorphism theorems now:

Theorem 2.3.1. *Isomorphism Theorems. (Anderson & Fuller (1992), Corollary 3.7) Let M and N be R-modules.*

1. *If $f : M \to N$ is an epimorphism with* $\ker f = K$*, then there is a unique isomorphism*

$$\eta : M/K \to N$$

such that

$$\eta(m + K) = f(m)$$

for all $m \in M$.

2. *If K and L are submodules of M such that $K \subseteq L$ then*

$$(M/K)/(L/K) \cong M/L.$$

3. *If H and K are submodules of M then*

$$(H+K)/K \cong H/(H \cap K)$$

The next theorem characterizes the submodules of factor modules:

Theorem 2.3.2. *Correspondence Theorem. (Anderson & Fuller (1992), Proposition 2.9.) Let T be a submodule of a left R-module M. Then there is an isomorphism between the set of submodules of M/T and the set of submodules of M which contains T. That is, the submodules of M/T are precisely all factor modules N/T where N is a submodule of M which contains T.*

2.4 Generating Sets, Finitely Generated Modules, and Maximal Submodules

Let M be an R-module. A subset N of M is called a *generating set* of M if

$$M = RN = \left\{ \sum_{i=1}^{k} r_i n_i : k \in \mathbb{N}, \text{ and for } i = 1, 2, \ldots, k, r_i \in R, n_i \in N \right\}.$$

If this is the case we say that N generates M or that M is generated by N.

If M has finite generating set, then we say that M is *finitely generated*. In particular, if M is generated by a single element, then M is called *cyclic*. In this case $M = Ra$ for some element a of M.

Let N be a submodule of M. If the factor module M/N is finitely generated then the submodule N of M is called a *cofinite* submodule of M.

We will use the following properties without mentioning in our work. Proofs can be found in Wisbauer (1991, §6.6):

Lemma 2.4.1. *Let $f : M \to N$ be a module homomorphism, and L be a generating set of M. Then*

1. *$f(L)$ is a generating set of* $\operatorname{Im} f$.

2. *If M is finitely generated, then* $\operatorname{Im} f$ *is also finitely generated.*

Every ring R is a cyclic module over itself. A proper submodule N of a module M is called *maximal* if N is not contained in any proper submodule of M. That is, if $N \subsetneqq K$ then $K = M$. Factor modules of finitely generated modules are also finitely generated. To see this, consider the natural epimorphism $\sigma : M \to M/N$, where M is a finitely generated module and N is a submodule of M. Then by Lemma 2.4.1 (2), it follows that M/N is also finitely generated.

2.5 Exact Sequences

Let K, L, M be R-modules. Consider the sequence $K \xrightarrow{f} M \xrightarrow{g} L$ where f and g are module homomorphisms. We say that this sequence is *exact at M* if $\operatorname{Im} f = \ker g$. Generally, a sequence of homomorphisms

$$\cdots \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n \xrightarrow{f_{n+1}} M_{n+1} \longrightarrow \cdots$$

is exact if it is exact at each M_i, that is, if $\operatorname{Im} f_n = \ker f_{n+1}$ for all n. The next result follows from the definition:

Proposition 2.5.1. *(Wisbauer (1991), §7.14) Let M, N be modules and $f : M \to N$ be a homomorphism. Then*

(i) $0 \longrightarrow M \xrightarrow{f} N$ *is exact if and only if f is a monomorphism;*

(ii) $M \xrightarrow{f} N \longrightarrow 0$ *is exact if and only if f is an epimorphism;*

(iii) $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$ *is exact if and only if f is an isomorphism.*

More generally, an exact sequence of the form

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is called a *short exact sequence*. It can be derived from the above proposition that in such an exact sequence f is a monomorphism and g is an epimorphism. By isomorphism theorems, one can see that $K \cong \operatorname{Im} f$ and that $M/\operatorname{Im} f \cong N$. Thus, in an exact sequence generally K is regarded as a submodule of M and N is regarded as a factor module of M.

2.6 Direct Sums

While a considerable amount of module theory deals with decomposing a module into smaller parts, either by additive decompositions or residue class decomposition, we also may want to construct new modules from the modules we have at hand. As we have mentioned before, given any number of modules we can create a larger module containing all of the given modules. This is possible with the so-called notion of products and in this section we briefly give this notion.

2.6.1 *Products, Coproducts, and External Direct Sums*

Let $\{M_i : i \in I\}$ be a family of left R-modules, where I is a nonempty index set. Consider the set theoretic cartesian product $\prod_{i \in I} M_i$ of these modules. Then this product of the family $\{M_i\}$ becomes a left R-module in the following way: let $(m_i), (n_i) \in \prod_{i \in I} M_i, r \in R$. Addition is defined componentwise by $(m_i) + (n_i) = (m_i + n_i)$ and scalar multiplication is defined by: $r(m_i) = (rm_i)$. This componentwise addition and scalar multiplication makes sense because each M_i is a left R-module alone.

Definition 2.6.1. We say that the element $(m_i)_{i \in I} \in \prod_{i \in I} M_i$ *has finite support* if the set $\{i \in I : m_i \neq 0\}$ is finite.

Now consider the set of all elements of $\prod_{i \in I} M_i$ with finite support. This subset is actually a submodule of $\prod_{i \in I} M_i$ (Anderson & Fuller (1992), §6).

Definition 2.6.2. Let $\{M_i : i \in I\}$ be a family of left R-modules. Then the left

R-module $\prod_{i\in I} M_i$ is called the *direct product* of the family $\{M_i : i \in I\}$. The submodule of all elements with finite support of $\prod_{i\in I} M_i$ is called the *external direct sum* of the family $\{M_i : i \in I\}$, denoted $\coprod_{i\in I} M_i$. External direct sum of the family $\{M_i : i \in I\}$ is also denoted by $\bigoplus_{i\in I} M_i$, $\bigoplus_I M_i$, or $\bigoplus M_i$. One can see that in case the index set I is finite, product and coproduct of the family $\{M_i : i \in I\}$ coincide. That is, $\prod_I M_i = \coprod_I M_i$ when I is finite. Also, if $M_i = M$ for all $i \in I$ then we write

$$M^{(I)} = \oplus_I M$$

for the external direct sum of card$I$ copies of M.

We should say a few words about categorical meanings of what we do here. Our definition of product and external direct sum actually agree with the categorical notions of product and coproduct in the category of all left R-modules (R-$\mathcal{M}od$), respectively. That is, products correspond to products, and external direct sums correspond to coproducts in the category of all left R-modules (Anderson & Fuller (1992), §6).

2.6.2 *Internal Direct Sums*

We give the definition of internal direct sum in a separate section, because this notion is a little different from that of products and coproducts. Here we deal with sums of submodules of a given module. So, let $\{M_i : i \in I\}$ be a family of submodules of a module M. M is said to be the *internal direct sum* of the family $\{M_i : i \in I\}$ if

1. $M = \sum_{i\in I} M_i$ and
2. $M_j \bigcap (\sum_{i\neq j} M_i) = \{0\}$ for all $j \in I$.

In this case we write $M = \bigoplus_{i\in I} M_i$. This should not lead to any confusion that we use the same notation for external direct sums, because as we have said, in case of a direct sum we have submodules of a module as summands, while in external direct sum we have any family of modules as summands. For simplicity we will use the phrase *direct sum* for internal direct sum. We will use direct sums, both external and internal, frequently in our work.

A submodule N of a module M is called a *direct summand* of M if there exists a submodule K of M such that $M = N \oplus K$. Trivially for any module M, the zero submodule and M itself are direct summands.

2.7 Socle and Radical of a Module

2.7.1 *Simple, Semisimple, Small, and Large Modules*

Definition 2.7.1. A module M is called *simple* if it has no non-trivial submodules. From this we derive the notion of semisimplicty. Namely, a module is called *semisimple* if it is a direct sum of simple modules.

Theorem 2.7.1. *(Wisbauer (1991), §20.1) Let $\{N_\lambda\}_\Lambda$ be a family of simple submodules of an R-module M with $\sum_\Lambda N_\lambda = M$. Then:*
For every submodule $K \subset M$, there is an index set $\Lambda_K \subset \Lambda$ such that

$$M = K \oplus (\bigoplus_{\Lambda_K} N_\lambda)$$

Setting $K = 0$ in the above theorem we see that to be semisimple, it is enough to be a sum of simple modules.

Definition 2.7.2. A submodule K of an R-module M is called *essential* or *large* if for every nonzero submodule N of M, $K \cap N \neq 0$. If K is a large submodule of M then we write $K \trianglelefteq M$. K is called *small* or *superflous* if for every submodule N of M, $K + N = M$ implies $N = M$. We denote small submodules by $K \ll M$.

Theorem 2.7.2. *(Wisbauer (1991), §20.2) For an R-module M, the following properties are equivalent:*

(a) M is a sum of simple submodules;

(b) M is a direct sum of simple submodules (= M is semisimple);

(c) Every submodule of M is a direct summand;

(d) M contains no proper essential submodules.

The following result will prove useful in our work:

Lemma 2.7.1. *(Wisbauer (1991), §19.6.) Let K be a small submodule of an R-module N. Then N is finitely generated if and only if N/K is finitely generated.*

2.7.2 *Characterization of Socle and Radical*

Definition 2.7.3. Let M be an R-module. The *socle* of M is defined to be the sum of all simple submodules of M, denoted by $\mathrm{Soc}(M)$ or $\mathrm{Soc}\, M$. If M has no simple submodule, then we set $\mathrm{Soc}(M) = 0$.

We now list the fundamental properties that characterize the socle of a module:

Proposition 2.7.1. *(Wisbauer (1991), §21.1) For an R-module M, we have*

$$\begin{aligned}\mathrm{Soc}(M) &= \bigcap\{L \subset M : L \text{ is an essential submodule of } M\}\\ &= \sum\{K \subset M : K \text{ is a simple submodule of } M\}.\end{aligned}$$

Proposition 2.7.2. *(Wisbauer (1991), §21.2) Let M be an R-module. Then:*

1. *For any homomorphism $f : M \to N$, we have $f(\mathrm{Soc}(M)) \subset \mathrm{Soc}(N)$.*

2. *For any submodule $K \subset M$, we have $\mathrm{Soc}(K) = K \cap \mathrm{Soc}(M)$.*

3. *$\mathrm{Soc}(M) \trianglelefteq M$ if and only if $\mathrm{Soc}(K) \neq 0$ for every non-zero submodule $K \subset M$.*

4. *$\mathrm{Soc}(\oplus_\Lambda M_\lambda) = \oplus_\Lambda \mathrm{Soc}(M_\lambda)$.*

Dual to the socle of a module we have the radical of a module:

Definition 2.7.4. Let M be an R-module. We define the *radical* of M as the intersection of all maximal submodules of M. We denote the radical of M by $\operatorname{Rad}(M)$ or $\operatorname{Rad} M$. If M has no maximal submodule we set $\operatorname{Rad}(M) = M$.

As we did for the socle, we list the properties of radical now.

Proposition 2.7.3. *(Wisbauer (1991), §21.5) For an R-module M, we have*

$$\begin{aligned} \operatorname{Rad}(M) &= \bigcap\{K \subset M : K \text{ is a maximal submodule of } M\} \\ &= \sum\{L \subset M : L \text{ is small in } M\}. \end{aligned}$$

Proposition 2.7.4. *(Wisbauer (1991), §21.6) Let M be an R-module.*

1. *For a homomorphism $f : M \to N$ we have*

   (i) $f(\operatorname{Rad}(M)) \subset \operatorname{Rad}(N)$,

   (ii) $\operatorname{Rad}(M/\operatorname{Rad}(M)) = 0$,

   (iii) $f(\operatorname{Rad}(M)) = \operatorname{Rad}(f(M))$ *if* $\operatorname{Ker} f \subset \operatorname{Rad}(M)$.

2. *If $M = \oplus_\Lambda M_\lambda$, then*

   (i) $\operatorname{Rad}(M) = \oplus_\Lambda \operatorname{Rad}(M_\lambda)$ *and*

   (ii) $M/\operatorname{Rad}(M) \cong \oplus_\Lambda M_\lambda/\operatorname{Rad}(M_\lambda)$.

2.8 Projective and Injective Modules

2.8.1 *Commutative Diagrams*

Let A, B, C, D be R-modules. Suppose $f : A \to B$, $g : B \to D$, $h : A \to C$, and $k : C \to D$ be homomorphisms. A *diagram* of these modules and homomorphisms is

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{k} & D \end{array}$$

This diagram is *commutative* if $gf = kh$, where fg denotes the usual composition of maps.

2.8.2 *Projective and injective modules*

Theorem 2.8.1. *(Kasch (1982), Theorem 5.3.1)*

(a) The following are equivalent for a module ${}_RQ$*:*

1. *Every monomorphism*

$$\xi : Q \to B$$

splits, i.e. $\mathrm{Im}(\xi)$ *is a direct summand in* B*.*

2. *For every monomorphism* $\alpha : A \to B$ *and for every homomorphism* $\phi : A \to Q$ *there is a homomorphism* $k : B \to Q$ *with* $\phi = k\alpha$*, that is,*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle \phi}\downarrow & \swarrow {\scriptstyle k} & \\ Q & & \end{array}$$

is commutative.

(b) The following are equivalent for a module P_R*:*

1. *Every epimorphism*

$$\xi : B \to P$$

splits, i.e. $\ker(\xi)$ *is a direct summand in* B*.*

2. *For every epimorphism* $\beta : B \to C$ *and every homomorphism* $\psi : P \to C$ *there is a homomorphism* $\lambda : P \to B$ *with* $\psi = \beta\lambda$*, that is,*

$$\begin{array}{ccc} & & P \\ & {\scriptstyle \lambda}\swarrow & \downarrow{\scriptstyle \psi} \\ B & \xrightarrow{\beta} & C \end{array}$$

is commutative.

Definition 2.8.1. A module Q satisfying one of the equivalent conditions of Theorem 2.8.1(a) is called an *injective* module. A module P which satisfies one of the conditions of Theorem 2.8.1(b) is called a *projective* module.

We now define injective hulls and projective covers:

Definition 2.8.2. Let M be an R-module. A monomorphism $\eta : M \to Q$ is called an *injective hull* of M if Q is injective and η is an essential monomorphism, i.e. $\mathrm{Im}(\eta)$ is essential in Q. We denote the injective hull of a module M by $E(M)$. An epimorphism $\xi : P \to M$ is called a *projective cover* of M if P is projective and ξ is a small epimorphism, i.e. $\ker(\xi) \ll P$.

These are the final results of this chapter. We close this chapter here, because we have developed enough terminology and theory to follow the third chapter of our work. We will give new terminology and theory as we proceed in chapter three.

CHAPTER THREE
COATOMIC AND FINITELY COATOMIC MODULES

In this final chapter, we deal with coatomic and finitely coatomic modules and their relations with some other modules. Throughout R will be a ring with unity, and all modules will be unitary left R-modules. We start with characteristics of coatomic modules.

3.1 Coatomic Modules

Before giving the definitions, let us talk about the motivating idea for the coatomic modules. The following result, which is a result of Zorn's lemma, is the starting point of coatomic modules:

Theorem 3.1.1. *(Kasch (1982), Theorem 2.3.11) If the R-module M is finitely generated then every proper submodule of M is contained in a maximal submodule of M.*

Applying the above theorem for the zero submodule of a finitely generated module, we have:

Corollary 3.1.1. *Every finitely generated module $M \neq 0$ has a maximal submodule.*

This noteworthy property of finitely generated modules leads us to the question of finding modules with the same property. So, as a generalization of finitely generated modules we have the following definition:

Definition 3.1.1. (Zöschinger (1980)) Let M be an R-module. We say that M is a *coatomic* module if every proper submodule of M is contained in a maximal submodule of M.

An equivalent condition is given in the next result:

Proposition 3.1.1. *Let M be a module. Then M is coatomic if and only if for every submodule N of M,* $\text{Rad}(M/N) = M/N$ *implies* $M/N = 0$*.*

Proof. Suppose that M is coatomic. Assume that $\text{Rad}(M/N) = M/N$ for a submodule N of M. Suppose for the contrary that $M/N \neq 0$, that is, N is a proper submodule of M. But by assumption the module M/N does not contain a maximal submodule, a contradiction. Conversely, suppose for the contrary that M is not coatomic. Then for a proper submodule K of M there is no maximal submodule of M containing K. Then the factor module M/K does not have a maximal submodule, so $\text{Rad}(M/K) = M/K$ and by assumption this implies that $M/K = 0$, contradicting the fact that K is a proper submodule of M. □

A general property of coatomic modules is the following:

Proposition 3.1.2. *Coatomic modules have small radicals.*

Proof. Let M be a coatomic module. Suppose for the contrary that $M = \text{Rad}(M) + K$ for some proper submodule K of M. Since M is coatomic, K is contained in a maximal submodule, say N, of M. It follows from the definition of radical that $M = \text{Rad}(M) + K \subseteq N$, and hence $N = M$, contradiction to the fact that N is a maximal submodule of M. So, for every proper submodule K of M, $\text{Rad}(M) + K \neq M$. Thus, $\text{Rad}(M) \ll M$. □

From this lemma and Theorem 3.1.1 we derive the following:

Proposition 3.1.3. *Let M be a finitely generated module. Then* $\text{Rad}(M) \ll M$*.*

Now we will show that other than finitely generated modules, semisimple modules also have this property. First we need the following result:

Lemma 3.1.1. *(Alizade & Bilhan & Smith (2001), Lemma 2.7.) The following are equivalent for a module M:*

1. *Every cofinite submodule of M is a direct summand of M.*

2. *Every maximal submodule of M is a direct summand of M.*

3. $\frac{M}{\mathrm{Soc}(M)}$ *does not contain a maximal submodule.*

Proof. (1) $\Rightarrow$ (2) Let N be a maximal submodule of M. Then the factor module M/N is a simple module, so it is finitely generated and hence N is a cofinite submodule of M. Thus (2) follows by assumption.
(2) $\Rightarrow$ (3) Let N be a maximal submodule of M. By assumption there exists a submodule N' of M such that $N \oplus N' = M$. By the isomorphism theorem, we have $M/N \cong N'$ and so N' is a simple submodule of M. Since the sum is direct, $N \cap N' = 0$ and hence $N' \nsubseteq N$. This means that for every maximal submodule of M, there is a simple submodule of M which is not contained in that maximal submodule. It follows from the characterization of the socle that there is no maximal submodule of M containing $\mathrm{Soc}(M)$. Therefore $\mathrm{Soc}(M)$ does not contain a maximal submodule.
(3) $\Rightarrow$ (1) Let N be a cofinite submodule of M. Then $N + \mathrm{Soc}(M)$ is also cofinite. By assumption $M/\mathrm{Soc}(M)$ does not contain a maximal so it must be $M = N + \mathrm{Soc}(M)$. It follows that $M = N \oplus N'$ for some submodule $N' \subseteq \mathrm{Soc}(M)$ of M. This proves (1). $\square$

The following proposition which also shows semisimple modules are coatomic appears as an exercise in Kasch (1982) but it has no proof in any place:

Proposition 3.1.4. *(Kasch (1982), Page 239, Exercise 9(c)) Let M be a module. M is semisimple if and only if M is coatomic and every maximal submodule of M is direct summand.*

Proof. ($\Rightarrow$) Suppose that M is semisimple. Then by Theorem 2.7.2 every submodule of M is a direct summand. Since M is semisimple, there exists a family $\{M_i : i \in I\}$ of simple submodules of M such that $M = K \oplus (\oplus_{i \in I} M_i)$. Since K is a proper submodule of M, $I \neq \varnothing$. Then there is an isomorphism $M/K \cong \oplus_{i \in I} M_i$. But clearly, for any $k \in I$, $K \bigoplus_{i \in I - \{k\}} M_i$ is a maximal submodule of $K \oplus_{i \in I} M_i$ containing K because each M_i is simple. So M is coatomic.
($\Leftarrow$) Suppose for the contrary that $\mathrm{Soc}(M) \neq M$. Since M is coatomic, the proper

submodule $\mathrm{Soc}(M)$ is contained in a maximal submodule of M, say K:

$$\mathrm{Soc}(M) \subseteq K \subseteq M.$$

By assumption, every maximal submodule of M is a direct summand of M. So, $M = K \oplus S$ for some submodule S of M. Since K is maximal in M, M/K is simple. Thus $S \cong M/K$ is simple. But then $S \subseteq \mathrm{Soc}(M)$. So $0 \neq S \subseteq (\mathrm{Soc}(M)) \cap S \subseteq K \cap S = 0$, contradiction. Thus we must have $M = \mathrm{Soc}(M)$, so M is semisimple. □

After this result, since every ring is a coatomic module over itself, we can say that a ring R is semisimple if and only if every left ideal of R is a direct summand of R. We will now provide more examples of coatomic modules.

Definition 3.1.2. We say that a module M is *local* if it has a largest submodule, that is a proper submodule which contains all other proper submodules.

Proposition 3.1.5. *Local modules are coatomic.*

Proof. Let M be a local module. Indeed, the largest submodule of M is just $\mathrm{Rad}(M)$ since $\mathrm{Rad}(M)$ is defined to be the sum of all small submodules of M. From the definition, every proper submodule is contained in $\mathrm{Rad}(M)$ and M is coatomic. □

Lemma 3.1.2. *(Güngöroğlu (1998), Lemma 3) Let M be an R-module. Then:*

1. *Every factor module of a coatomic module M is coatomic.*
2. *Every extension of a coatomic module by a coatomic module is coatomic.*

Generally, a submodule of a coatomic module does not have to be coatomic, as the following example shows:

Example 3.1.1. (Büyükaşık (2005), Example 1.6.2) Consider the ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, c \in \mathbb{Z}, b \in \mathbb{Q} \right\}$$

Then ${}_RR$ is coatomic, because it is finitely generated. Consider the submodule

$$M = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$$

The left R-module structure of M is completely determined by the left $\mathbb{Z}$-module structure of $\mathbb{Q}$. Then M is not coatomic since $\mathbb{Q}$ is not coatomic: $\mathbb{Z} \subseteq \mathbb{Q}$ is a proper submodule and $\mathbb{Q}$ has no maximal submodule (Kasch (1982), p. 25).

For a coatomic R-module M, a sufficient condition for every submodule of M to be coatomic is that R is a commutative Noetherian ring. We give the definitions first:

Definition 3.1.3. Let M be an R-module. A set Γ of submodules of M is said to satisfy the *ascending chain condition* if for every chain

$$L_1 \subseteq L_2 \subseteq \ldots \subseteq L_n \subseteq \ldots$$

in Γ, there is an n with $L_{n+i} = L_n$ for all $i = 1, 2, \ldots$
The set Γ is said to satisfy the *descending chain condition* if for every chain

$$L_1 \supseteq L_2 \supseteq \ldots \supseteq L_n \supseteq \ldots$$

in Γ there exists an n with $L_{n+i} = L_n$ for all $i = 1, 2, \ldots$. A module is said to be *Noetherian* if the set of submodules of M satisfies the ascending chain condition. Similarly, a ring R is called *Noetherian* if the set of ideals of R satisfy the ascending chain condition. M is called *Artinian* if the set of submodules of M satisfy the descending chain condition.

Proposition 3.1.6. *(Anderson & Fuller (1992), Proposition 10.9) For a module M the following statements are equivalent:*

(a) M is Noetherian;

(b) Every submodule of M is finitely generated;

(c) Every nonempty set of submodules of M has a maximal element.

Now we can state our result:

Theorem 3.1.2. *(Zöschinger (1980), Lemma 1.1) Let M be a coatomic module over a commutative Noetherian ring. Then every submodule of M is coatomic.*

Lemma 3.1.3. *(Güngöroğlu (1998), Corollary 5) Let $M = \oplus_{i=1}^{n} M_i$ be a finite direct sum of submodules M_i $(i = 1, 2, \ldots, n)$. Then M is coatomic if and only if each M_i $(i = 1, 2, \ldots, n)$ is coatomic.*

Definition 3.1.4. A module M is called *semiperfect* if every factor module of M has a projective cover.

Regarding this we have:

Proposition 3.1.7. *(By Wisbauer (1991), 42.5) The following are equivalent for a projective R-module M:*

(a) M is semiperfect;

(b) M is coatomic, and

(α) Every simple factor module of M has a projective cover, or

(β) Every maximal and every cyclic submodule has a supplement in M;

(c) M is a direct sum of local modules and $\text{Rad}(M) \ll M$.

3.2 Finitely Coatomic Modules

We derived coatomic modules as a generalization of finitely generated modules. With further generalizing, we introduce the following definition:

Definition 3.2.1. A module M is called *finitely coatomic*, or *f-coatomic* for short, if every finitely generated proper submodule of M is contained in a maximal submodule of M.

After this definition we will see an example of a module which is f-coatomic but not coatomic, so our definition will make sense.

Example 3.2.1. Consider the $\mathbb{Z}$-module $K = \bigoplus_{\mathbb{N}} M_n$, where $M_n = \mathbb{Z}$ for all $n \in \mathbb{N}$. K is not coatomic by (Güngöroğlu & Harmancı (1999), Example 2.8). Let L be a finitely generated submodule of K. Then $L \subseteq \oplus_{i=1}^{n} M_i$ for some $n \in \mathbb{N}$. So, L is contained in a maximal submodule N of $\oplus_{i=1}^{n} M_i$ because $\mathbb{Z}$ is coatomic and finite direct sum of coatomic modules are coatomic. Clearly then $N \oplus (\oplus_{i>n} M_i)$ is a maximal submodule of M containing L. Therefore K is f-coatomic.

We have the following equivalent condition as we had for coatomic modules:

Proposition 3.2.1. *The following are equivalent for a module M:*

(a) M is f-coatomic;

(b) For every finitely generated submodule N of M, $\mathrm{Rad}(M/N) = M/N$ *implies* $M/N = 0$.

Proof. (a)⇒(b) Let M be an f-coatomic module and let $\mathrm{Rad}(M/N) = M/N$ for a finitely generated submodule N of M. Suppose to the contrary that $M/N \neq 0$, so N is a proper submodule of M. But $\mathrm{Rad}(M/N) = M/N$ means that there is no maximal submodule of M containing N, a contradiction.
(b)⇒(a) This time suppose that M is not f-coatomic. Then there is a finitely generated proper submodule N of M such that N is not contained in a maximal submodule of M. This implies that $\mathrm{Rad}(M/N) = M/N$, and by assumption $M/N = 0$, contradiction since N is a proper submodule of M. □

Proposition 3.2.2. *For every small submodule N of M, if the factor module M/N is f-coatomic then M is f-coatomic.*

Proof. Suppose that M/N is f-coatomic where N is a small submodule of M. Let K be a finitely generated submodule of M. If $N \subseteq K$ we are done. If not, consider the factor module $K + N/N$. It is finitely generated since
$K + N/N \cong K/K \cap N$. Then $K + N/N$ is contained in a maximal submodule of M/N. By the correspondence theorem $K + N$ and hence K is contained in a maximal submodule of M. Thus M is f-coatomic. □

Unfortunately, it is not true that every submodule and every factor module of an f-coatomic module are again f-coatomic as the following examples indicates:

Example 3.2.2. Consider the $\mathbb{Z}$-module K of Example 3.2.1. Let $M = K \oplus \mathbb{Q}$. M is still f-coatomic, but neither the factor module $M/K \cong \mathbb{Q}$ nor the submodule $\mathbb{Q}$ is f-coatomic.

The following two propositions give conditions for which the factor module of an f-coatomic module is again f-coatomic:

Proposition 3.2.3. *Let M be an f-coatomic module. Then every factor module of M by a finitely generated submodule is also f-coatomic.*

Proof. Let M be an f-coatomic module, and N be a finitely generated submodule of M. Consider the factor module M/N. Let K/N be a finitely generated proper submodule of M/N. Since N is finitely generated, it follows that K is also finitely generated. Since M is f-coatomic, $K \subseteq M_0$ for a maximal submodule M_0 of M. Hence M_0/N is the required maximal submodule of M/N containing K/N. □

Proposition 3.2.4. *Let M be an f-coatomic module and N a submodule of M. The factor module M/N is f-coatomic if $N \subseteq \mathrm{Rad}(M)$.*

Proof. Let M be an f-coatomic module and let N be a submodule of M such that $N \subseteq \mathrm{Rad}(M)$. Consider a finitely generated submodule K/N of M/N. This means that $Rk_1 + Rk_2 + \ldots + Rk_s + N = K$ for some $s \in \mathbb{Z}^+$ and $k_1, \ldots, k_s \in K$. $Rk_1 + \ldots + Rk_s \subseteq L$ for a maximal submodule L of M, because it is finitely generated and M is f-coatomic. $N \subseteq \mathrm{Rad}(M) \subseteq L$ is always true. Thus $K \subseteq L$ and hence $K/N \subseteq L/N$ where L/N is a maximal submodule of M/N. □

Unlike the coatomic modules, radical of an f-coatomic module does not have to be small:

Example 3.2.3. Again consider the $\mathbb{Z}$-module $M = K \oplus \mathbb{Q}$ of Example 3.2.2. We have $\mathrm{Rad}(M) = \mathrm{Rad}(K \oplus \mathbb{Q}) = \mathrm{Rad}(\mathbb{Z}^{\mathbb{N}} \oplus \mathbb{Q}) = \mathrm{Rad}(\mathbb{Q})$ by Proposition 2.7.4(2(i)). Since $\mathbb{Q}$ has no maximal submodules, $\mathrm{Rad}(Q) = \mathbb{Q}$ and thus, M is an f-coatomic module with $\mathrm{Rad}(M)$ is not small in M.

We now deal with sums and direct sums of finitely coatomic modules.

Proposition 3.2.5. *Let M be an f-coatomic module. Suppose $U + V = M$ for some submodules U, V of M where U is finitely generated and $U \cap V \ll V$. Then V is also f-coatomic.*

Proof. Let N be a finitely generated proper submodule of V. $N + U \cap V/U \cap V$ is also finitely generated. It is proper since $U \cap V \ll V$. M/U is also f-coatomic by Proposition 3.2.3. By the isomorphism $M/U \cong V/U \cap V$ it follows that $V/U \cap V$ is f-coatomic too. Thus $N + U \cap V/U \cap V$ is contained in a maximal submodule of $V/U \cap V$, so N is also contained in a maximal submodule of V. Therefore V is f-coatomic. □

Example 3.2.4. This example shows that direct summands of an f-coatomic module do not have to be f-coatomic. Consider the module $M = K \oplus \mathbb{Q}$ of Example 3.2.2. M is f-coatomic but $\mathbb{Q}$ is not.

From this point we start deriving relations between coatomic modules and other modules.

3.3 Finitely coatomic modules over discrete valuation rings

Definition 3.3.1. A commutative ring R is called a *valuation ring* if its ideals are totally ordered by inclusion. If R is a noetherian valuation domain we say that R is a *discrete valuation ring (DVR)* (Büyükaşık (2005), p.14).

Theorem 3.3.1. *(by Dummit & Foote (2004), p.757, Theorem 7.) The following properties are equivalent for a ring R:*

1. *R is a discrete valuation ring;*

2. *R is a P.I.D. with a unique maximal ideal P.*

Definition 3.3.2. A module M is said to be *divisible* if for every non-zero $r \in R$, $rM = \{rm \mid m \in M\} = M$ holds.

The following result is important for modules over discrete valuation rings and can be obtained from Alizade & Bilhan & Smith (2001) Lemma 4.4:

Lemma 3.3.1. *(by Alizade & Bilhan & Smith (2001), Lemma 4.4) Let R be a discrete valuation ring. The following are equivalent for an R-module M:*

(i) *M is injective.*

(ii) *M is divisible.*

(iii) *$M = PM$ for every maximal ideal P of R.*

(iv) *M does not contain a maximal submodule.*

Now we have:

Theorem 3.3.2. *Let R be a discrete valuation ring. An R-module M is f-coatomic if and only if every divisible factor module M/N, where N is finitely generated, is zero.*

Proof. ($\Rightarrow$) Let M be an f-coatomic module. Let $N \subseteq M$ be a finitely generated submodule of M for which M/N is divisible. Then since M/N divisible, by Lemma 3.3.1 $M/N = P(M/N)$ where P is the unique maximal ideal of R. Since R is a discrete valuation ring, $P(M/N) = \text{Rad}(M/N)$. So that $\text{Rad}(M/N) = M/N$ and then by Proposition 3.2.1 $M/N = 0$.
($\Leftarrow$) Assume that M is such that every divisible factor module M/N where N is finitely generated is zero. Let N be a finitely generated submodule of M and suppose that $\text{Rad}(M/N) = M/N$. Since $\text{Rad}(M/N) = P(M/N)$ and R is a discrete valuation ring, M/N is injective and hence it is divisible. Thus $M/N = 0$ by assumption and M is f-coatomic. □

3.4 Relations with supplemented modules

An interesting class of modules is the supplemented modules. For two submodules U, V of a module M, we say that V is a *supplement* of U in M if $U + V = M$ and $U + K \neq M$ for any $K \subset V$. This is equivalently given as:

Proposition 3.4.1. *(Wisbauer (1991), p. 348) Let U, V be submodules of a module M. V is a supplement of U in M if and only if $U+V = M$ and $U \cap V \ll V$.*

If every submodule of a module M has a supplement in M then M is called *supplemented*. Similarly, if every cofinite submodule of M has a supplement in M then M is called *cofinitely supplemented* (see Alizade & Bilhan & Smith (2001)). We immediately have:

Theorem 3.4.1. *(Wisbauer (1991), 41.5.) Let M be an R-module. Then M is a sum of hollow submodules and* $\mathrm{Rad}(M) \ll M$ *if and only if M is coatomic and*

(i) every maximal submodule has a supplement in M, or

(ii) every cofinite submodule has a supplement in M.

Theorem 3.4.2. *Let M be a module and U, V be submodules of M such that V is a supplement of U in M. Consider the statements*

(a) M/U is f-coatomic.

(b) V is f-coatomic.

We have in general $(a) \Rightarrow (b)$ and if $U \cap V$ is finitely generated then $(b) \Rightarrow (a)$ is also true.

Proof. $(\Rightarrow)$ If N is a finitely generated proper submodule of V, then $N+U\cap V/U\cap V$ is a finitely generated submodule of $V/U\cap V$. It is proper because $U \cap V \ll V$ since V is a supplement of U in M. Since $V/U\cap V \cong M/U$ and M/U is f-coatomic, it follows that $V/U \cap V$ is f-coatomic too. Thus $N + U \cap V/U \cap V$ is contained in a maximal submodule of $V/U \cap V$. Thus N is contained in a maximal submodule of V. Therefore V is f-coatomic.
$(\Leftarrow)$ Suppose that V is f-coatomic. We have $M/U \cong V/U \cap V$ and since $V \cap U$ is finitely generated we have by Proposition 3.2.3 M/U is f-coatomic. □

Lemma 3.4.1. *(Wisbauer (1991), 41.1. (6)) Let U, V be submodules of an R-module M. Assume V to be a supplement of U in M. If* $\mathrm{Rad}(M) \ll M$ *then U is contained in a maximal submodule of M.*

Definition 3.4.1. A module M is called *finitely supplemented* or *f-supplemented* if every finitely generated submodule of M has a supplement in M.

Regarding finitely supplemented module we have:

Proposition 3.4.2. *Let M be an f-supplemented module such that* $\mathrm{Rad}(M) \ll M$. *Then M is f-coatomic.*

Proof. A finitely generated submodule U of M has a supplement V in M and since $\mathrm{Rad}(M) \ll M$ result follows by Lemma 3.4.1. □

Definition 3.4.2. A module M is called *f-semiperfect* if for every finitely generated submodule N of M, the factor module M/N has a projective cover.

What we have to do with f-semiperfect modules are the following:

Lemma 3.4.2. *(Wisbauer (1991), §19.2) An epimorphism $f : M \to N$ in R-$\mathcal{M}od$ is small if and only if every (mono) morphism $h : L \to M$ in R-$\mathcal{M}od$ with fh epic is epic.*

Lemma 3.4.3. *(by Wisbauer (1991), 42.8(2(ii))) Projective cover of an f-semiperfect module is again f-semiperfect.*

Proof. Let M be an f-semiperfect module, and $\pi : P \to M$ be a projective cover of M. Let U be a finitely generated submodule of P. We have the following diagram

$$\begin{array}{ccc} P & \xrightarrow{\pi} & N \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ P/U & \xrightarrow{g} & N/\pi(U) \end{array}$$

where g is epic with small kernel. Since $\pi(U)$ is finitely generated and N is f-semiperfect there is a projective cover $\pi' : Q \to N/\pi(U)$. Since Q is projective there exists an $h : Q \to P/U$ such that $gh = \pi'$ and $\mathrm{Ker}(h) \ll Q$ since $\mathrm{Ker}(h) \subseteq \mathrm{Ker}(\pi')$ and (Q, π') is a projective cover. Hence (Q, h) is a projective cover of P/U. □

Lemma 3.4.4. *(Wisbauer (1991), 42.8(2(iii))) Projective cover of an f-semiperfect module is f-supplemented.*

Proof. Let M be an f-semiperfect module and U be a finitely generated submodule of M. Them the factor module M/U has a projective cover, say (P, π). Then the following diagram with canonical epimorphism σ

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow{\scriptstyle \sigma} & & \\ P & \xrightarrow{\pi} & M/U & \longrightarrow & 0 \end{array}$$

can be completed by an $f : M \to P$. f is surjective by Lemma 3.4.2, hence it splits. Therefore there exists a $g : P \to M$ with $fg = id_P$ and hence $\pi = \pi fg = pg$. From this we derive $U + g(P) = M$, $g(P)$ is a projective cover of M/U and consequently $U \cap g(P) \ll g(P)$. So, $g(P)$ is a supplement of U in M and therefore M is f-supplemented. □

Lemma 3.4.5. *(Wisbauer (1991), §41.3, (2)(i)) Let M be an f-supplemented module. If $L \subseteq M$ is a finitely generated or a small submodule, then M/L is also f-supplemented.*

Theorem 3.4.3. *An f-semiperfect module M is f-coatomic if* $\mathrm{Rad}(M) \ll M$.

Proof. Suppose that M is an f-semiperfect module, and $\mathrm{Rad}(M) \ll M$. Let (P, π) be a projective cover of M. From Lemma 3.4.4 P is f-supplemented. Since projective covers have small kernels, by Lemma 3.4.5, $P/\operatorname{Ker}(\pi)$ is also f-supplemented. Thus M is also f-supplemented because of the isomorphism $M \cong P/\operatorname{Ker}(\pi)$. The result follows from Proposition 3.4.2. □

3.5 Finitely semisimple modules and finitely coatomic modules

In this section we derive some relations between f-coatomic modules and finitely semisimple modules which are, in a way, a generalization of semisimple modules. Remember we proved that in a semisimple module every submodule is a direct summand. Finitely semisimple modules arose out of this property:

Definition 3.5.1. (Bilhan (2006)) A module M is called *finitely semisimple* or *f-semisimple* if every finitely generated submodule of M is a direct summand of M.

By the modular law we have:

Lemma 3.5.1. *(Bilhan (2006), Lemma 1.3.) Every submodule of an f-semisimple module is f-semisimple.*

Proof. Let M be an f-semisimple module and let N be a submodule of M. For a finitely generated submodule K of N we have $M = K \oplus S$ for some submodule S of M. By using the modular law we obtain $N = K \oplus N \cap S$. So K is also a direct summand of N and hence N is f-semisimple. □

Lemma 3.5.2. *(by Bilhan (2006), Lemma 2.8.) f-semisimple modules have zero radical.*

Proof. Let M be an f-semisimple module and A be an ideal of R. We will show that M/AM has zero radical. Suppose that M is not equal to AM. Let m be an element of M which is not an element of AM. Since Rm is finitely generated, there is a maximal submodule of Rm which contains Am. Also there is a submodule K of M such that $M = Rm \oplus K$. Then $N = L \oplus K$ is a maximal submodule of M such that $AM = Am \oplus AK$ is contained in N. But m does not belong to N. Thus AM is an intersection of maximal submodules of M. Taking $A = 0$, we obtain the particular case where $\text{Rad}(M) = 0$. □

Theorem 3.5.1. *Every f-semisimple module is f-coatomic.*

Proof. Let M be an f-semisimple module and N be a finitely generated submodule of M. Suppose that $\text{Rad}(M/N) = M/N$. Since M is f-semisimple and N is finitely generated, there exists a submodule K of M such that $M = N \oplus K$. Since every submodule of an f-semisimple module is again f-semisimple, K is f-semisimple and as a result of the isomorphism $K \cong M/N$, M/N is also f-semisimple. It follows from the fact that f-semisimple modules have zero radical that $\text{Rad}(M/N) = M/N = 0$. Hence M is f-coatomic. □

The module M of Example 3.1.3 this time shows that the converse of above result is not true:

Example 3.5.1. Consider the module $M_K \oplus \mathbb{Q}$ of Example 3.2.2. M is f-coatomic, with $\text{Rad}(M) = \mathbb{Q}$. Thus $\text{Rad}(M) \neq 0$ and so M is not f-semisimple.

Corollary 3.5.1. *Every submodule of an f-semisimple module is f-coatomic.*

Proof. This follows from Lemma 3.5.1 and Theorem 3.5.1. □

Definition 3.5.2. Let M be a module and U, V be submodules of M. We say that V is a *weak supplement* of U in M if $M = U + V$ and $U \cap V \ll M$. M is called *weakly supplemented* if every submodule of M has a weak supplement. Similar for supplemented modules, we say that M is *finitely weak supplemented* if every finitely generated submodule of M has a weak supplement in M.

Theorem 3.5.2. *Let M be a module with $\operatorname{Rad}(M)$ finitely generated. Then $M/\operatorname{Rad}(M)$ is f-semisimple and every submodule of $\operatorname{Rad}(M)$ is f-coatomic if and only if M is finitely weak supplemented and every submodule of M is f-coatomic.*

Proof. (⇒) We will first show that every submodule of M is f-coatomic. For this we let U be a submodule of M and N be a finitely generated submodule of U. Assume that $\operatorname{Rad}(U/N) = U/N$. We have by isomorphism theorem

$$(U + \operatorname{Rad}(M))/(N + \operatorname{Rad}(M)) = (U + (N + \operatorname{Rad}(M)))/(N + \operatorname{Rad}(M))$$

$$\cong U/U \cap (N + \operatorname{Rad}(M)) \cong U/(N + (U \cap \operatorname{Rad}(M)))$$

We know by assumption that U/N does not have any maximal submodule. We claim that $(U + \operatorname{Rad}(M))/(N + \operatorname{Rad}(M))$ does not have a maximal submodule too. Suppose to the contrary that $(U + \operatorname{Rad}(M))/(N + \operatorname{Rad}(M))$ has a maximal submodule. This means by the above isomorphism that there is a maximal submodule of U containing $N + (U \cap \operatorname{Rad}(M))$. But U also contains N. This is not possible since $\operatorname{Rad}(U/N) = U/N$. Thus we have for $U_1 = U + \operatorname{Rad}(M)$ and $N_1 = N + \operatorname{Rad}(M)$ we have $\operatorname{Rad}(U_1/N_1) = U_1/N_1$. Since $M/\operatorname{Rad}(M)$ is f-semisimple, every finitely generated submodule is a direct summand. So for $N_1/\operatorname{Rad}(M)$ there is $K_1/\operatorname{Rad}(M)$ such that $M = N_1 + K_1$ and $N_1 \cap K_1 = \operatorname{Rad}(M)$. Since $U_1/N_1 = \operatorname{Rad}(U_1/N_1)$ $\subseteq \operatorname{Rad}(M/N_1) \cong \operatorname{Rad}(K_1/\operatorname{Rad}(M)) \subseteq \operatorname{Rad}(M/\operatorname{Rad}(M)) = 0$ then $U_1 = N_1$. Hence $U + \operatorname{Rad}(M) = N + \operatorname{Rad}(M)$, and so $U = N + (U \cap \operatorname{Rad}(M))$. From $\operatorname{Rad}(U/N) = U/N = \frac{N+(U\cap\operatorname{Rad}(M))}{N} \cong \frac{U\cap\operatorname{Rad}(M)}{N\cap\operatorname{Rad}(M)}$ it follows that $\frac{U\cap\operatorname{Rad}(M)}{N\cap\operatorname{Rad}(M)} = \operatorname{Rad}(\frac{U\cap\operatorname{Rad}(M)}{N\cap\operatorname{Rad}(M)})$ and since each submodule of $\operatorname{Rad}(M)$ is f-coatomic we have $U \cap \operatorname{Rad}(M) = N \cap \operatorname{Rad}(M)$, so $U = N + (U \cap \operatorname{Rad}(M)) \subseteq N$ hence $U = N$. Therefore U is f-coatomic as desired. To show that M is a finitely weak supplemented module, let U be a finitely generated submodule of M. Then $U + \operatorname{Rad}(M)/\operatorname{Rad}(M)$ is a finitely generated submodule of $M/\operatorname{Rad}(M)$ and so there exists $V/\operatorname{Rad}(M) \subseteq M/\operatorname{Rad}(M)$ such that $U + \operatorname{Rad}(M) + V = M$ and $(U + \operatorname{Rad}(M)) \cap V = \operatorname{Rad}(M)$. So $U \cap V + \operatorname{Rad}(M) = \operatorname{Rad}(M)$ and $U \cap V \subseteq \operatorname{Rad}(M) \ll M$. Thus it follows that $U + V = M$ and that $U \cap V \ll M$. That is, V is a weak supplement of U in M and hence M is finitely weak supplemented.

($\Leftarrow$) Conversely, suppose that M is finitely weak supplemented and every submodule of M is f-coatomic. Then $\operatorname{Rad}(M)$ is automatically f-coatomic. It remains to show that $M/\operatorname{Rad}(M)$ is f-semisimple. Let $U/\operatorname{Rad}(M)$ be a finitely generated submodule of $M/\operatorname{Rad}(M)$. Since $\operatorname{Rad}(M)$ is finitely generated then U is finitely generated and so by assumption there exists a submodule V of M such that $U + V = M$ and $U \cap V \ll M$. That is, $U \cap V \subseteq \operatorname{Rad}(M)$. Then it is clear that $\frac{U}{\operatorname{Rad}(M)} \oplus \frac{V+\operatorname{Rad}(M)}{\operatorname{Rad}(M)} = \frac{M}{\operatorname{Rad}(M)}$. Therefore $M/\operatorname{Rad}(M)$ is f-semisimple and the proof is over. □

A mixture of f-semisimple and finitely weak supplemented module will be given after the following technical lemma:

Lemma 3.5.3. *(Güngöroğlu (1998), Lemma 9) Let M be a module and suppose that $M = U + V$ for some submodules U, V of M with $U \cap V \ll M$ and $\operatorname{Rad}(\frac{V}{V\cap U}) = \frac{V}{V\cap U}$. Then $V \subseteq \operatorname{Rad}(M)$.*

Proof. Since $U \cap V \ll M$ then $V \cap U \subseteq \operatorname{Rad}(M)$. If $\operatorname{Rad}(M) = M$, then $V \subseteq M \subseteq \operatorname{Rad}(M)$ and there is nothing to show. If not, then we need show that V is contained in every maximal submodule of M. So let L be a maximal submodule of M. From $V \cap U \subseteq L$ and $V \cap U \subseteq V$ it follows that $V \cap U \subseteq V \cap L$. Suppose to the contrary that $V \nsubseteq L$. Since $\operatorname{Rad}(M) \subseteq L$ then we have $U \cap V \subseteq L$. Also, since V is not contained in L we have $M = V + L$ and $M/L \cong V/V \cap L$. Hence $V \cap L$ is maximal in V. But this is a contradiction with the assumption $V/V \cap U = \operatorname{Rad}(V/V \cap U)$. Therefore $V \subseteq \operatorname{Rad}(M)$. □

Proposition 3.5.1. *Let M be a module such that $\operatorname{Rad}(M) \ll M$. Then M is f-coatomic if it satisfies one of the following:*

(1) $M/\operatorname{Rad}(M)$ is f-semisimple.

(2) M is finitely weak supplemented.

Proof. (1) Suppose that $M/\operatorname{Rad}(M)$ is f-semisimple. Hence $M/\operatorname{Rad}(M)$ is f-coatomic by Proposition 3.5.1. By Proposition 3.2.2 M is f-coatomic.
(2)Suppose that M is finitely weak supplemented and let U be a finitely generated submodule of M such that $M/U = \operatorname{Rad}(M/U)$. Since M is weakly supplemented, $M = U + V$ and $U \cap V \ll M$. Since $\frac{V}{U \cap V} = \frac{M}{U} (= \operatorname{Rad}(\frac{M}{U}))$, by Lemma 3.5.3, we have $V \subseteq \operatorname{Rad}(M)$. Thus $M = U + V = U + \operatorname{Rad}(M)$ and so $M = U$. Therefore M is f-coatomic. □

3.6 Other modules

A module M is called *cosemisimple* if every submodule of M is an intersection of maximal submodules of M. Similar to this notion, let us consider a module M with the property that every finitely generated submodule of M is an intersection of maximal submodules of M. We then prove:

Lemma 3.6.1. *Let M be a module. Every finitely generated submodule of M is an intersection of maximal submodules of M if and only if for every finitely generated submodule N of M $\operatorname{Rad}(M/N) = 0$.*

Proof. Let N be a finitely generated submodule of M and suppose that N is an intersection of maximal submodules of M, $N = \cap M_i$, M_i is a maximal submodule of M for all $i \in I$. The intersection of all maximal submodules of M containing N gives the $\operatorname{Rad}(M/N)$. Thus $\operatorname{Rad}(M/N) = N/N = 0$. Conversely, if for a finitely generated submodule N of M we have $\operatorname{Rad}(M/N) = 0$ then this means the intersection of all maximal submodules of M containing N is equal to N. So the proof is completed. □

Proposition 3.6.1. *Let M be a module such that every finitely generated submodule of M is an intersection of maximal submodules of M. Then M is f-coatomic.*

Proof. Let M be with the given property. If N is a finitely generated proper submodule of M such that $\operatorname{Rad}(M/N) = M/N$ then since N is an intersection of maximal submodules of M, Lemma 3.6.1 we have $\text{Rad}(M/N) = 0$. So M is f-coatomic. □

3.7 Conclusion

During this work, as a generalization of coatomic modules, we introduced f-coatomic modules. We observed that some of the results obtained for coatomic modules can be generalized to f-coatomic modules. For instance, we generalized the statement *every semisimple module is coatomic* as *every f-semisimple module is f-coatomic*. Some cases needed some restrictions; for example we required that $\text{Rad}(M)$ to be small in order to make an f-supplemented module or an f-semiperfect module an f-coatomic module. We also provided examples for certain cases, an example to show that not every f-coatomic module is f-semisimple, and, probably one of the most notable cases, an example to show that radical of an f-coatomic module does not have to be small unlike the coatomic case.

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