

**PERFORMANCE MEASUREMENT IN MULTI OBJECTIVE
COMBINATORIAL OPTIMIZATION**

**A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY**

BY

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**IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
INDUSTRIAL ENGINEERING**

SEPTEMBER 2007

Approval of the thesis:

**PERFORMANCE MEASUREMENT IN MULTI OBJECTIVE
COMBINATORIAL OPTIMIZATION**

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ABSTRACT

PERFORMANCE MEASUREMENT IN MULTI OBJECTIVE COMBINATORIAL OPTIMIZATION

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September 2007, 96 pages

In this study we address the problem of measuring the quality of different sets of nondominated solutions obtained by different approaches in multi objective combinatorial optimization (MOCO). We propose a new measure that quantitatively compares the sets of nondominated solutions, without needing an efficient frontier. We develop the measure for bi-criteria and more than two criteria cases separately. Rather than considering only the supported solutions in the evaluation, the measure captures both supported and unsupported solutions through utilizing weighted Tchebycheff function characteristics. We also adapt this method for determining the neighborhood relations on the weight space for both bi-criteria and more than two criteria cases. We check the consistency of the neighborhood assumption on the objective space with the neighborhood relations on the weight space by this measure and obtain highly good results.

Keywords: Multi objective combinatorial optimization, performance measurement

ÖZ

ÇOK KRİTERLİ BİRLEŞİMSEL OPTİMİZASYON İÇİN PERFORMANS ÖLÇÜMÜ

Bozkurt, Bilge

Yüksek Lisans, Endüstri Mühendisliği Bölümü

Tez Yöneticisi: Prof. Dr. Murat Köksalan

Eylül 2007, 96 sayfa

Bu çalışma ile çok kriterli birleşimsel optimizasyonda farklı yaklaşımlarla elde edilmiş hakim çözüm kümelerinin kalitesinin ölçülenmesi sorunu ele alınmıştır. Hakim çözüm kümelerini herhangi bir verimli sınıra gerek kalmadan sayısal olarak karşılaştıran yeni bir ölçüt önerilmiştir. Ölçüt iki kriterli ve daha çok kriterli durumlar için ayrı ayrı geliştirilmiştir. Ölçüt sadece desteklenmiş çözümleri değerlendirmeye dahil etmek yerine desteklenmiş ve desteklenmemiş çözümlerin her ikisini ağırlıklandırılmış Tchebycheff fonksiyonunun kullanımı sayesinde değerlendirmeye dahil etmektedir. Ayrıca, bu ölçüt iki kriterli ve daha çok kriterli durumların her ikisi için ağırlık uzayındaki komşuluk ilişkilerinin belirlenmesi için uyarlanmıştır. Bu ölçüt ile amaç uzayındaki komşuluk varsayımının ağırlık uzayındaki komşuluk ilişkileri ile olan tutarlılığı kontrol edilmiş ve yüksek oranda iyi sonuçlar elde edilmiştir.

Anahtar kelimeler: Çok kriterli birleşimsel optimizasyon, performans ölçümü

To My Family

ACKNOWLEDGEMENTS

First of all I would like to present deepest gratitude to my supervisor Prof. Dr. Murat Köksalan for his patient guidance, encouragements and advice throughout my research. It has been a great pleasure for me to work under his supervision.

I wish to express my sincere thanks to my manager and SOURCE Technology Services Ltd. for supporting my academic work.

I also would like to thank to my friends for their great help and moral support throughout the study.

Finally, most special thanks go to my family for their endless love, patience and support, nothing would be possible without you.

TABLE OF CONTENTS

ABSTRACT.....	iv
ÖZ.....	v
ACKNOWLEDGEMENTS.....	vii
TABLE OF CONTENTS.....	viii
LIST OF TABLES.....	x
LIST OF FIGURES.....	xi
CHAPTER	
1. INTRODUCTION.....	1
2. LITERATURE REVIEW.....	3
3. THE IPF MEASURE – BI-CRITERIA COMBINATORIAL OPTIMIZATION..	8
3.1 Some Definitions.....	8
3.2 IPF Measure with a Weighted Linear Utility Function: Bi-Criteria Case.....	10
3.3 IPF measure with Tchebycheff Utility Function: Bi-criteria Case.....	14
4. THE IPF MEASURE – GENERAL CASE FOR $p (\geq 3)$ CRITERIA.....	28
4.1 IPF Measure with Weighted Linear Utility Function: p Criteria Case.....	28
4.2 IPF Measure with Weighted Tchebycheff Utility Function: More Than Two Criteria Case.....	29
4.2.1 Determining W_j^i s:.....	32
4.2.2 Determining W^k s:.....	33
4.2.3 Calculating IPF(z^i):.....	37
4.3 Neighborhood Assumption: More Than Two Criteria Case.....	41
5. CONCLUSIONS.....	53
REFERENCES.....	55
APPENDICES	
A. THE WEIGHT RANGE AND IPF CALCULATIONS FOR CHAPTER 3	
EXAMPLE 2.....	58

B. THE WEIGHT RANGE AND IPF CALCULATIONS FOR CHAPTER 3	
EXAMPLE 3 APPROXIMATION 1.....	61
C. THE WEIGHT RANGE AND IPF CALCULATIONS FOR CHAPTER 3	
EXAMPLE 3 APPROXIMATION 2.....	63
D. PARTITION OF THE WEIGHT REGIONS FOR CHAPTER 4.....	65
E. PARTITION OF W^i REGIONS FOR CHAPTER 4.....	68
F. PARTITION OF W^i REGIONS FOR INTEGRAL CALCULATIONS FOR	
CHAPTER 4.....	71
G. IPF CALCULATIONS FOR CHAPTER 4 EXAMPLE 4.....	75
H. NEIGHBORHOOD CONSTRAINTS FOR CHAPTER 4 EXAMPLE 6.....	92

LIST OF TABLES

Table 1. Five nondominated solutions for Example 1	12
Table 2. Eight nondominated solutions for Example 2.....	21
Table 3. The results of IPF(Z) calculations.....	22
Table 4. Four supported solutions of approximation 1	24
Table 5. Four dominated solutions of approximation 2	24
Table 6. The results of IPF(Z) calculations for approximation 1.....	26
Table 7. The results of IPF(Z) calculations for approximation 2.....	26
Table 8. Four nondominated solutions for Example 4.....	31
Table 9. Four nondominated solutions for Example 5.....	43
Table 10. Seven nondominated solutions for Example 6.....	49
Table 11. The neighborhood relations of the solutions.....	49
Table 12. Related points and equations for W^1	72
Table 13. Related points and equations for W^3	73
Table 14. Related points and equations for W^4	74

LIST OF FIGURES

Figure 1. Demonstration of alternatives in the objective space	9
Figure 2. Graphical representation of z^i 's in the objective space	12
Figure 3. Graphical representation of z^i 's with Tchebycheff function	16
Figure 4. The illustration of the three nondominated solutions	19
Figure 5. Graphical representation of z^i 's and respective α ranges	22
Figure 6. Graphical representation of z^i 's for two approximations	25
Figure 7. Partition of the weight space between criteria for z^2	33
Figure 8. Partition of the weight space between alternatives	37
Figure 9. Partition of the W^2 region	38
Figure 10. Partition of the W^2 region for the integral calculations	39
Figure 11. Graphical representation of the constraints	47
Figure 12. Partition of the weight space between alternatives	48
Figure 13. Graphical representation of the constraints	51
Figure 14. Partition of the weight space between alternatives	51
Figure 15. Partition of the weight space between criteria for z^1	65
Figure 16. Partition of the weight space between criteria for z^3	66
Figure 17. Partition of the weight space between criteria for z^4	67
Figure 18. Partition of the W^1 region	68
Figure 19. Partition of the W^3 region	69
Figure 20. Partition of the W^4 region	70
Figure 21. Partition of the W^1 region for the integral calculations	71
Figure 22. Partition of the W^3 region for the integral calculations	72
Figure 23. Partition of the W^4 region for the integral calculations	74

CHAPTER 1

INTRODUCTION

The problem of comparing different sets of nondominated solutions or their approximations obtained by different approaches in multi objective combinatorial optimization (MOCO) problems is of interest for researchers especially in recent years. The idea is to try to measure how well different sets represent the set of all nondominated solutions. However evaluating a set of nondominated solutions is not straightforward.

In MOCO since it is NP hard to obtain a nondominated solution, a lot of heuristic methods are developed and commonly used. For the purpose of evaluating the heuristics, some evaluation techniques utilize the distance of the approximation to the efficient frontier, the number of the solutions provided by the approximation, or the uniformity of the distribution of the solutions. Some other techniques try to utilize the decision maker's (DM's) utility functions. Therefore, some measures may be good in measuring certain properties but overlook other properties. Some other measures may capture certain other aspects. Therefore there is a need for measures that quantitatively compare and evaluate the performance of the approximations considering all the aspects.

In this study, we develop a measure that aims to capture all important aspects and that differentiates between all types of solutions. Our method is in the group of measures which utilize the DM's utility function. The measure is based on the assumption that DM has a Tchebycheff utility function. By this assumption, we aim to include all the nondominated points of the approximation in our evaluation. We develop the methodology for bi-criteria and more than two criteria cases, separately. In the bi-criteria part of our study, Integrated Preference Functional (IPF) measure

with linear utility function assumption, which is proposed by Carlyle *et al.* (2003), is extended with Tchebycheff utility function assumption. Then we generalize our measure for more than two criteria cases and discuss the neighborhood relations between the solutions.

In chapter 2, we review the literature on performance measurement techniques in MOCO, the aspects they concentrate on, and their real life applications and the results they achieved.

In chapter 3, we describe our measure for bi-criteria MOCO problems, and discuss it with examples, theorems and proofs.

In chapter 4, we generalize our measure for more than two criteria problems and discuss the neighborhood relations of the alternatives and the consistency of our neighborhood assumption with examples, theorems and proofs.

In chapter 5, we discuss the conclusions and directions for further research.

CHAPTER 2

LITERATURE REVIEW

In this chapter, we review the literature on approaches to compare the performance of approximate algorithms.

Hansen and Jaszkiwicz (1998) propose qualitative and quantitative comparison methods to the approximations. Qualitative comparison is based on outperformance relations for comparing pairs of approximations under very weak assumptions about the decision-maker's preferences. Quantitative comparison is based on probability and expected values based on DM's utility function. The methods suggested can be applied whether the efficient frontier is known or not. R_1 measure reflects the probability that one approximation is better than the other approximation. R_2 measure expresses the difference (or the ratio) of expected values of the utility function. The choice of the set of utility functions is not restricted in R_1 and R_2 measures. However since the calculation of these measures is not straightforward, numerical integration is recommended for the use of these measures.

Jaszkiwicz, Hapke and Kominek (2001) provide an experimental study on a real life multiobjective combinatorial optimization (MOCO) problem for the purpose of comparing the performances of four Multiobjective Evolutionary Algorithms (MOEAs). While evaluating the performances of MOEAs, they use two performance metrics from the literature: "Coverage measure" and "R measure" which is proposed by Hansen and Jaszkiwicz (1998).

Daniels (1992) proposes an approach for evaluating heuristic solution quality by determining the error value between the efficient solution and the approximate solution. The problem is solved using an available optimization approach and a

heuristic, then the resulting set (E) of m efficient solutions, and set (H) of n heuristic solutions is compared based on the proposed error value calculation. A linear weighted sum value function is assumed. The algorithm which is based on the maximum and average relative heuristic error calculation is developed for bi-criterion heuristics. Daniels (1992) suggested using that algorithm for multi-criteria heuristics, with a few modifications. For example, in order to determine the maximum heuristic error in multi-criteria case, an LP should be solved for each pair of efficient and heuristic solutions. However, the major shortcoming of this approach is that, it requires the knowledge of the efficient set.

Sayın (2000) suggests three quality attributes: coverage, uniformity and cardinality for measuring the quality of discrete representations in multi objective mathematical programming (MOMP).

Three characteristics suggested in evaluating a discrete representation of the efficient set are:

1. Coverage: The representation should cover all portions of the efficient set.
2. Uniformity: The points should be uniformly distributed. The distances between the points should be similar.
3. Cardinality: The representation should have sufficient number of points.

Sayın (2000) proposes a method for the coverage error and the uniformity level determination. The coverage error and an upper bound on it are computed by solving a max min type mathematical programming, and the uniformity level is computed by solving a minimization type mathematical programming problem. Sayın (2000) states that the formulations are practically applicable when the multiobjective linear programming problem (MOLP) is considered.

Jaszkiewicz (2000) introduces a measure called effectiveness index (EI) for the purpose of comparing the effectiveness of the approaches. EI is defined by the ratio of the times needed for a multiple objective metaheuristic (MOMH) and a single objective metaheuristic (SOMH) to reach the same quality of the solutions. With an

experimental study on traveling salesperson problem (TSP), single objective genetic local search (GLS) and multiple objective genetic local search (MOGLS) metaheuristics are compared. It is indicated that EI will not only be used to compare MOMH and SOMH but also be used to compare MOMHs, by referencing a SOMH.

Jaszkiewicz (2004) proposes a technique to compare computational efforts needed by a MOMH and a SOMH to generate approximately pareto-optimal (efficient) solutions of the same quality. A computational experiment on the multiple objective multiple constraints knapsack problem (MOMCKP) is designed and three multiple objective evolutionary algorithms (EA) are compared with a standard single objective EA. It is assumed that SOMH is applied to the optimization of achievement scalarizing functions. The average qualities of the solutions generated by SOMH and MOMH are defined based on the average value of the achievement scalarizing functions. SOMH and MOMH are run to the same quality of the solutions and then computational efforts (running times) are compared by using an Efficiency Index. Jaszkiewicz (2004) states that “Efficiency Index” mainly depends on the number of objectives.

Ehrgott and Gandibleux (2001) propose to use defined lower and upper bound sets in multi objective optimization, with the idea that the bounds provide valuable information for MOCO. Ideal point and nadir point, which are generally accepted bounds in multi objective optimization, may be difficult to obtain for MOCO problems and may generally be far away from the efficient points. The lower (L) and upper bound sets (U) are obtained by solving weighted sum problems, which may be solvable in reasonable time. L and U are obtained by the first phase of the two phases procedure described by Ulungu and Teghem (1994) for bi-criteria. A bi-criteria binary knapsack problem is used to measure the quality of the proposed bound sets. It's demonstrated that the bound sets are both easier to compute than ideal and nadir points, and they are closer to the efficient points.

Deb (2001, p. 306-324) provides performance metrics used in the literature for evaluating the quality of the solution sets obtained from the Evolutionary Algorithms (EAs). The performance metrics are investigated in three groups: “Metrics evaluating closeness to the pareto-optimal front”, “Metrics evaluating diversity among nondominated solutions”, “Metrics evaluating closeness and diversity”. When evaluating the performance of EAs, use of at least two performance metrics is suggested in order to consider more aspects of the evaluation.

Ehrgott and Gandibleux (2000) provide a general framework for the MOCO concept and the summary of the available literature. They illustrate the MOCO solution techniques in the literature problem by problem. They indicated the points which are subject to more research in MOCO.

Zitzler *et al.*(2003) propose an approach for examining and discussing the existing quality comparison techniques. They group and analyze the quality comparison techniques into two parts: “Unary quality indicators”, which evaluate an approximation by judging a value to it, and “binary indicators”, which evaluate the approximations in pairs. They suggest using “binary indicators”.

Van Veldhuizen and Lamont (2000) propose an experimental methodology to compare the Multiobjective Evolutionary Algorithm (MOEA) performance. They compare the performances of four MOEAs in the experiments they conduct. For the comparison they use some MOEA performance metrics: “Final generational distance metric”, “spacing metric” and “overall nondominated vector generation metric”.

Fonseca V., Fonseca C., and Hall (2001) propose a stochastic approach for the purpose of evaluating the performance of multiobjective metaheuristics like MOEAs, simulated annealing (SA), and tabu search (TS). They suggest using the cumulative distribution function as a mean measure of the approximation set, in order to see the location of the approximation.

Kim, Gel, Carlyle and Fowler (2000) propose a new measure called Integrated Convex Preference (ICP) for bi-criteria MOCO problems. In this quantitative comparison measure, they utilize DM's utility function, which is assumed to be a linear utility function.

Fowler, Kim, Carlyle, Gel and Horng (2005) propose two versions of ICP measure, one with the uniform weight density function and other with the triangular density function. They also develop and suggest to use the scaled version of the measures. They evaluated the performance of ICP with an experimental study on comparing the performances of two genetic algorithms on bi-criteria parallel machine scheduling problem.

Zitzler and Thiele (1999) present an experimental study on a 0/1 knapsack problem for the purpose of quantitatively comparing four MOEAs. They propose two measures while comparing MOEAs: "Size of the space covered" and "Coverage of two sets". One compares the approximations by comparing the sizes of the covered spaces by the approximations; the other compares the pairs of solution sets in terms of dominance relations between the solution sets.

Zitzler, Deb and Thiele (2000) provide a methodology for the comparison of EAs. They use the performance measure "Coverage of two sets", which is presented by Zitzler and Thiele (1999), in their study. For the further researches they suggest to use three more performance metrics for more detailed investigations: μ_1 , μ_2 , and μ_3 . These metrics basically deal with average distance to the efficient frontier, the distribution of the approximation and the spread of the approximation respectively.

CHAPTER 3

THE IPF MEASURE – BI-CRITERIA COMBINATORIAL OPTIMIZATION

In this chapter, first we look at an existing performance measurement tool in MOCO. Then, we develop our approach for bi-criteria MOCO problems with illustrative examples. Finally, we analyze the neighborhood relations of the solutions both in the objective space and the weight space.

3.1 Some Definitions

The MOCO problem is formulated as:

$$\text{“min” } \{z_1(x), z_2(x), \dots, z_p(x)\}$$

$$\text{s.t. } x \in X,$$

where x is the decision variable vector, X is the set of feasible solutions, p is the number of objectives, z_j is the j^{th} objective function. The distinguished property of MOCO problem is that the decision variables are discrete, not continuous. The quotation marks are used because the minimization of a vector is not a well defined mathematical operation.

Definition: For a minimization problem, a feasible solution $x \in X$ is efficient if there does not exist $x' \in X$ such that $z_j(x') \leq z_j(x)$ for all $j=1, \dots, p$ and $z_k(x') < z_k(x)$ for at least one k . If there exists such x' then x is inefficient. If x is efficient then $z(x) = \{z_1(x), z_2(x), \dots, z_p(x)\}$ is nondominated. If x is inefficient then $z(x)$ is dominated.

Definition: The efficient solutions, which are not optimal for any weighted sum of the objectives, are defined as nonsupported efficient solutions. The rest of the efficient solutions are defined as supported efficient solutions, which are not dominated by a convex combination of other solutions. Since nonsupported solutions are still efficient, they are important when considering a MOCO problem.

For example, for a MOCO problem with two objectives to be minimized, let us have six alternative solutions, where z^i is the i^{th} alternative solution in the objective space. The graphical representation of the alternatives in the objective space is seen in Figure 1. The classification of the solutions seen in Figure 1 will be as follows: z^1, z^2, z^3, z^6 are supported efficient solutions and z^5 is unsupported efficient solution, whereas z^4 is inefficient solution.

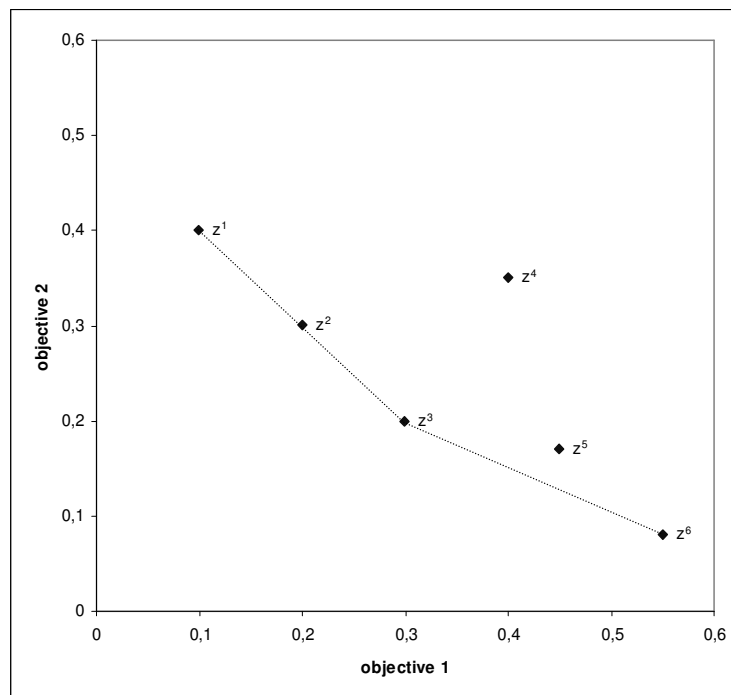


Figure 1. Demonstration of alternatives in the objective space

Definition: The solution $z = \sum_{i=1}^n \mu_i z_i$, where $\sum_{i=1}^n \mu_i = 1$ and $\mu_i \geq 0$ for all i , is called a convex combination of vectors z_1, \dots, z_n , where z_i is the i^{th} solution in the objective space, and μ_i is the weight of the i^{th} solution in the objective space.

Definition: For a minimization problem z_k is said to be convex dominated by z_i if

$\sum_{i \neq k} \mu_i z_i \leq z_k$, where $\sum_{i \neq k} \mu_i = 1$ and $\mu_i \geq 0$ for all i , where z_i is the i^{th} solution in the objective space, and μ_i is the weight of the i^{th} solution in the objective space.

Definition: Two solutions are said to be adjacent efficient if any of their convex combinations are not convex-dominated.

3.2 IPF Measure with a Weighted Linear Utility Function: Bi-Criteria Case

Carlyle *et al.* (2003) proposed IPF (Integrated Preference Functional) quantitative measure, which uses DM's utility function. DM's utility function is assumed to be a convex combination of the objectives.

The aggregate utility function is represented as $g(x; \alpha)$, where α is the weight vector, x is the decision variable vector. The uncertainty of DM's preference for the weight of each objective is represented by the probability density function $h(\alpha)$. Weight density function $h(\alpha)$ is assumed to be a uniform density function. A uniform density function, since all weights are equally likely, implies no knowledge about the specifics of the DM's utility function. Although the below IPF measure can be interpreted as the expected utility of the DM for the nondominated solutions at hand, it can also be interpreted differently. When $h(\alpha)$ is uniformly distributed, IPF is just a measure that shows how closely the set of nondominated solutions at hand represents the efficient frontier. In this sense it can be used to compare different sets

of nondominated solutions in order to see which set represents the efficient frontier better.

Because of weighted linear utility function assumption $g(x_g(\alpha); \alpha)$ is expressed as below:

$$g(x_g(\alpha); \alpha) = \alpha z_1(x_g(\alpha)) + (1 - \alpha) z_2(x_g(\alpha))$$

where:

$x_g(\alpha)$: g^{th} supported efficient solution in the decision space, which has the best linear utility value for weight α .

$z_i(x_g(\alpha))$: i^{th} objective function value corresponding to solution $x_g(\alpha)$.

Then, $IPF(Z)$ can be expressed as:

$$IPF(Z) = \int_{\alpha \in A} h(\alpha) g(x_g(\alpha); \alpha) d(\alpha) = \sum_{z \in Z} \left[\int_{\alpha \in x_g^{-1}(z)} h(\alpha) g(x; \alpha) d(\alpha) \right] = \sum_{z \in Z} IPF(z)$$

Different solutions are best for different α values. Assume solutions $x_1(\alpha)$, $x_2(\alpha), \dots$ are the best solutions for α values in ranges $[\alpha_0, \alpha_1), [\alpha_1, \alpha_2), \dots$ respectively. Then the IPF value can be calculated as:

$$\begin{aligned} IPF(Z) &= \int_{\alpha_0=0}^{\alpha_1} h(\alpha) [\alpha z_1(x_1(\alpha)) + (1 - \alpha) z_2(x_1(\alpha))] d(\alpha) \\ &+ \int_{\alpha_1}^{\alpha_2} h(\alpha) [\alpha z_1(x_2(\alpha)) + (1 - \alpha) z_2(x_2(\alpha))] d(\alpha) + \dots \\ &= \sum_i \left[z_1(x_i) \int_{\alpha_{i-1}}^{\alpha_i} \alpha h(\alpha) d(\alpha) + z_2(x_i) \int_{\alpha_{i-1}}^{\alpha_i} (1 - \alpha) h(\alpha) d(\alpha) \right] \end{aligned} \quad (1)$$

The calculation method of α ranges $[\alpha_0, \alpha_1), [\alpha_1, \alpha_2), \dots$ is explained with an example below:

Example 1: Let us have five efficient solutions shown in Table 1 for a minimization bi-criteria problem with linear utility function assumption. z^i denotes the i^{th} nondominated solution in the objective space.

Table 1. Five nondominated solutions for Example 1

z^i	1	2	3	4	5
z_1^i	0.3	2.5	3	6	10
z_2^i	10	6	3	2.5	0.3

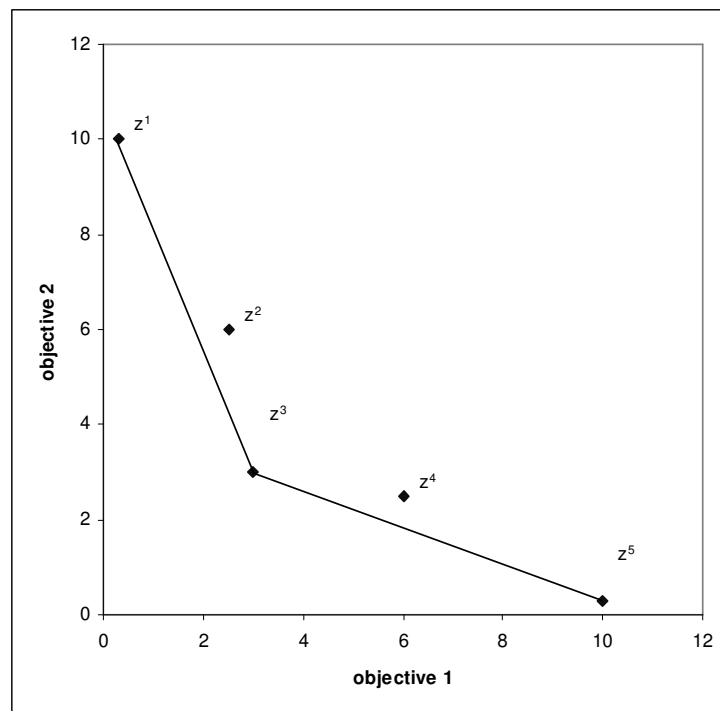


Figure 2. Graphical representation of z^i 's in the objective space

As shown in Figure 2, z^2 and z^4 are non-supported points, therefore they are not considered while determining the α ranges.

It is known that $g(x_g(\alpha); \alpha)$ has the equation below:

$$g(x_g(\alpha); \alpha) = \alpha z_1(x_g(\alpha)) + (1 - \alpha) z_2(x_g(\alpha))$$

Using this equation α ranges making each supported point optimal is found:

α range that makes z^1 optimal:

$$0.3\alpha + (1 - \alpha)10 \leq 3\alpha + (1 - \alpha)3 \text{ (comparing with } z^3)$$

$$0.72 \leq \alpha < 1$$

α range that makes z^3 optimal:

$$3\alpha + (1 - \alpha)3 \leq 0.3\alpha + (1 - \alpha)10 \text{ (comparing with } z^1)$$

$$3\alpha + (1 - \alpha)3 < 10\alpha + (1 - \alpha)0.3 \text{ (comparing with } z^5)$$

$$0.28 \leq \alpha < 0.72$$

α range that makes z^5 optimal:

$$10\alpha + (1 - \alpha)0.3 < 3\alpha + (1 - \alpha)3 \text{ (comparing with } z^3)$$

$$0 \leq \alpha < 0.28$$

The method consists of three steps:

1. Find all supported points and adjacent supported points;
2. Find the weight ranges for which each supported point is optimal, by solving a series of linear inequalities;
3. Calculate the IPF by solving the equation (1) with integrals.

IPF can be interpreted as the expected utility value that is obtained from a set of approximate solutions. Therefore for a minimization problem, IPF value will not increase while the approximation gets better for instance by adding a supported point to the approximation.

In this method the only computational issue is to solve a series of integrals, which is not difficult in bi-criteria case. However this becomes computationally very difficult

in multi-criteria case. Additionally, since the approach concentrates on the weighted linear utility function, it works only for the supported points. However, the unsupported efficient points are also important when comparing the approximations.

3.3 IPF measure with Tchebycheff Utility Function: Bi-criteria Case

When evaluating the approximations, unsupported efficient points will be very important and should not be missed. In order to include the unsupported efficient points in the evaluation, we build our approach on Tchebycheff utility function assumption. We focus on minimization problems in our examples throughout the chapters. However in maximization problems the calculations will be very similar.

Let us recall weighted Tchebycheff metrics (see Steuer, 1986 p. 422-425 for details):

p : number of criteria

z^{**} : ideal criterion vector

$z = (z_1, \dots, z_p)$

$$\|z^{**} - z\|_{\infty}^{\alpha} = \max_{i=1, \dots, p} \{\lambda_i |z_i^{**} - z_i|\}$$

$\lambda = [\lambda_1, \dots, \lambda_p]$, $\lambda_i \geq 0$ ($i=1, \dots, p$), is a weight vector.

The required weights that place z at the shortest weighted Tchebycheff distance from z^{**} are defined by Steuer (1986 p. 422-425) as:

$$\lambda_i = \begin{cases} \frac{1}{(z_i^{**} - z_i) \left[\sum_{i=1}^p \frac{1}{(z_i^{**} - z_i)} \right]^{-1}} & \text{if } z_i \neq z_i^{**} \text{ for all } i \\ 1 & \text{if } z_i = z_i^{**} \\ 0 & \text{if } z_i \neq z_i^{**} \text{ but } \exists j \ni z_j = z_j^{**} \end{cases}$$

Weighted Tchebycheff utility functions can be defined as:

$$u_{\infty}(z, z^{**}, \lambda) = \left\{ \max_i \{ \lambda_i (z_i^{**} - z_i) \} \right\} \text{ for a maximization problem.}$$

Considering a minimization problem, with Tchebycheff utility function assumption, the formulation of $g(z^i; \alpha)$ function becomes:

$$g(z^i; \alpha) = \max \{ \alpha_1 (z_1^i - z_1^{**}), \alpha_2 (z_2^i - z_2^{**}) \}$$

Weight density function $h(\alpha)$ is assumed to be a uniform density function. Since it is uniform density function we do not include $h(\alpha)$ in IPF equation.

When we put $g(z^i; \alpha)$ into the IPF equation and using the property $\alpha_1 + \alpha_2 = 1$, the IPF measure becomes:

$$\begin{aligned} IPF(z^i) &= \int_{\alpha_L^i}^{\alpha_U^i} \max \{ \alpha (z_1^i - z_1^{**}), (1 - \alpha) (z_2^i - z_2^{**}) \} d\alpha \\ &= \int_{\alpha_L^i}^{\alpha_b^i} (1 - \alpha) (z_2^i - z_2^{**}) d\alpha + \int_{\alpha_b^i}^{\alpha_U^i} \alpha (z_1^i - z_1^{**}) d\alpha \end{aligned}$$

where (α_L^i, α_U^i) denote the weight interval where z^i is optimal,

$$z^i = (z_1^i, z_2^i).$$

Then,

$$IPF(Z) = \sum_{z \in Z} IPF(z^i)$$

Now let us present how we derive the weight ranges for each alternative, α_L^i and α_U^i below:

The derivation of weight ranges (α_L^i and α_U^i) for each alternative:

Consider the solutions z^1 and z^2 given in Figure 3.

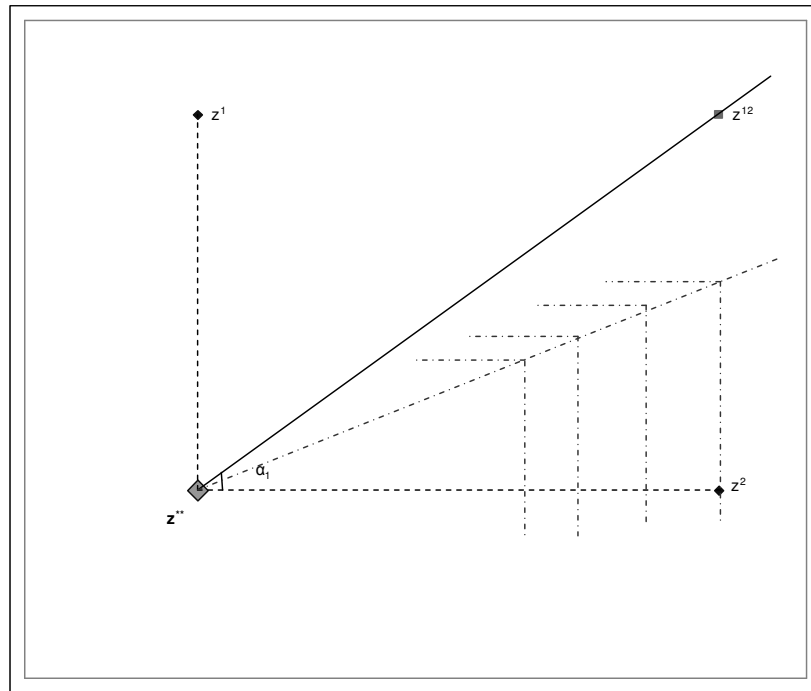


Figure 3. Graphical representation of z^i 's with Tchebycheff function

Without loss of generality assume :

$$z_1^2 > z_1^1,$$

$$z_2^1 > z_2^2,$$

$$z^{**} = (z_1^1, z_2^2),$$

Then, $z^{12} = (z_1^2, z_2^1)$ which is the corner point that is used to define the weight ranges between alternatives,

α_1 is expressed as:

$$\alpha_1 = \frac{1}{z_1^2 - z_1^1} \left[\frac{1}{\left(\frac{1}{z_1^2 - z_1^1} + \frac{1}{z_2^1 - z_2^2} \right)} \right] = \frac{(z_2^1 - z_2^2)}{[z_1^2 - z_1^1 + z_2^1 - z_2^2]}$$

Then the weight range that makes z^2 better than z^1 is determined as follows:

$$\max\{\alpha(z_1^2 - z_1^{**}), (1-\alpha)(z_2^2 - z_2^{**})\} < \max\{\alpha(z_1^1 - z_1^{**}), (1-\alpha)(z_2^1 - z_2^{**})\}$$

$$\max\{\alpha(z_1^2 - z_1^1), (1-\alpha)(z_2^2 - z_2^2)\} < \max\{\alpha(z_1^1 - z_1^1), (1-\alpha)(z_2^1 - z_2^2)\}$$

$$\alpha(z_1^2 - z_1^1) < (1-\alpha)(z_2^1 - z_2^2)$$

$$\alpha[z_1^2 - z_1^1 + z_2^1 - z_2^2] < (z_2^1 - z_2^2)$$

$$\alpha < \alpha_1. \quad \blacksquare$$

By this derivation we show that between two alternatives the weight range is divided into two parts, in which α_1 forms the boundary of the α ranges between two alternatives.

The weight region α_1 used in this derivation, corresponds to the α_L^1 , which is the same as α_U^2 , in our general notation. The detailed calculations of weight ranges for more than two alternatives can be seen in Example 2. When there are more than two alternatives, neighborhood issue comes into question. The neighborhood relation of the alternatives is an important issue to consider while determining the weight ranges. The alternatives closest to each other in any criterion are accepted as neighbors throughout the studies in the examples.

Definition: The nondominated solutions that are closest to each other in any criterion are neighbors in the objective space.

Theorem 1: Let $p=2$. Then, a nondominated solution can have at most two neighbors.

Proof: Let $n_{ik}=\{j^s\}$: the set of criteria for which solutions i and k has the neighborhood relations (alternative i and k has the neighborhood relations in criterion j from upper/lower side. s can be $+$ or $-$, $+$ defining the neighborhood is from positive direction, $-$ defining the neighborhood is from negative direction.)

Since $p=2$, all the possible neighborhood relations are:

$$n_{ik}=\{1^+, 2^-\}$$

$$n_{ik}=\{1^-, 2^+\}$$

$$n_{ik}=\{1^+, 2^+\}$$

$$n_{ik}=\{1^-, 2^-\}$$

The last two neighborhood relations $n_{ik}=\{1^+, 2^+\}$ and $n_{ik}=\{1^-, 2^-\}$ can not exist in bi-criteria MOCO problems. Because, according to the nondominance definition, if one solution is better in one criterion, the other will be better in the other criterion. Otherwise, if one solution is better in both criteria (as it is in cases $n_{ik}=\{1^+, 2^+\}$ and $n_{ik}=\{1^-, 2^-\}$), the other solution will be dominated. Hence, in bi-criteria MOCO problems, only $n_{ik}=\{1^+, 2^-\}$ and $n_{ik}=\{1^-, 2^+\}$ neighborhood relations are valid, which implies that one nondominated solution can have at most two neighbors. ■

Theorem 2: Let $p=2$. Assume without loss of generality that z^i is not a neighbor of z^k in the objective space. Then z^i is not a neighbor of z^k in the weight space either.

Proof: Let us consider a minimization problem, in which it is given that z^1, z^2 and z^3 are the nondominated solutions such that z^3 is not a neighbor of z^1 in the objective space, whereas z^2 is a neighbor of z^1 . The illustration of the nondominated solutions is given in Figure 4.

$z^i=(z_1^i, z_2^i)$ represents i^{th} alternative in the objective space,

$z^{**} = (z_1^{**}, z_2^{**})$ represents the ideal point,

Since z^2 is a neighbor of z^1 whereas z^3 is not, we can assume:

$$z_1^3 > z_1^2 > z_1^1$$

$$z_2^1 > z_2^2 > z_2^3$$

$$z^{**} = (z_1^1, z_2^3)$$

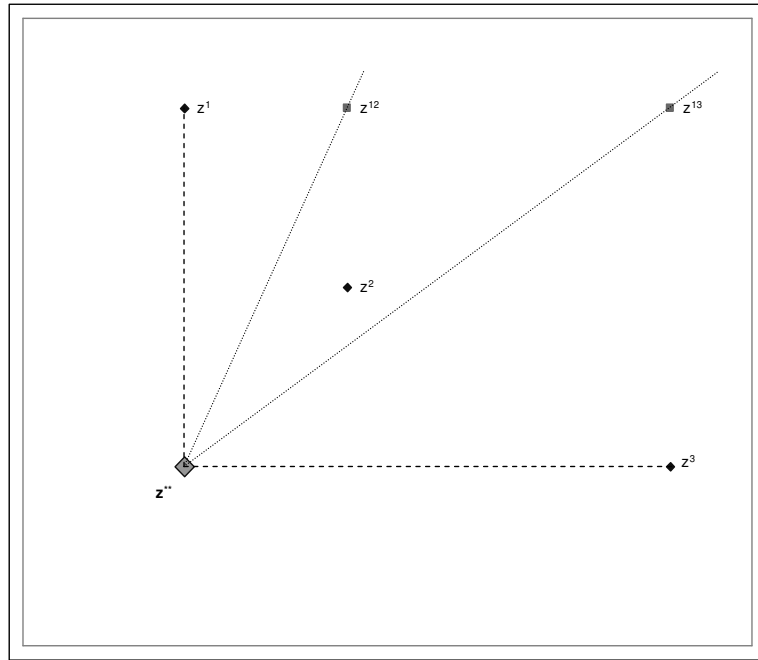


Figure 4. The illustration of the three nondominated solutions

Let us determine and compare the weight ranges where z^1 is better than z^2 and z^1 is better than z^3 .

The weight range that makes z^1 better than z^2 :

Consider $\max\{\alpha(z_1^1 - z_1^{**}), (1-\alpha)(z_2^1 - z_2^{**})\} < \max\{\alpha(z_1^2 - z_1^{**}), (1-\alpha)(z_2^2 - z_2^{**})\}$.

By substituting (z_1^1, z_2^3) for (z_1^{**}, z_2^{**}) we have,

$(1-\alpha)(z_2^1 - z_2^3) < \max\{\alpha(z_1^2 - z_1^1), (1-\alpha)(z_2^2 - z_2^3)\}$. The right hand side results in two cases.

Case 1.

$$\max\{\alpha(z_1^2 - z_1^1), (1-\alpha)(z_2^2 - z_2^3)\} = (1-\alpha)(z_2^2 - z_2^3).$$

Since $z_2^1 > z_2^2 > z_2^3$, we have $(z_2^1 - z_2^3) > (z_2^2 - z_2^3)$,

which also implies $(1-\alpha)(z_2^1 - z_2^3) > (1-\alpha)(z_2^2 - z_2^3)$.

Thus, for the weights which satisfy the equation,

$$\max\{\alpha(z_1^2 - z_1^1), (1-\alpha)(z_2^2 - z_2^3)\} = (1-\alpha)(z_2^2 - z_2^3), z^1 \text{ will not be better than } z^2.$$

Case 2.

$$\max\{\alpha(z_1^2 - z_1^1), (1-\alpha)(z_2^2 - z_2^3)\} = \alpha(z_1^2 - z_1^1).$$

It implies that $(1-\alpha)(z_2^1 - z_2^3) < \alpha(z_1^2 - z_1^1)$.

Thus, z^1 is better than z^2 for the weights satisfying $(1-\alpha)(z_2^1 - z_2^3) < \alpha(z_1^2 - z_1^1)$.

The weight range that makes z^1 better than z^3 :

$$\max\{\alpha(z_1^1 - z_1^{**}), (1-\alpha)(z_2^1 - z_2^{**})\} < \max\{\alpha(z_1^3 - z_1^{**}), (1-\alpha)(z_2^3 - z_2^{**})\}$$

substituting (z_1^1, z_2^3) for (z_1^{**}, z_2^{**}) we have,

$$(1-\alpha)(z_2^1 - z_2^3) < \alpha(z_1^3 - z_1^1)$$

Thus, z^1 is better than z^3 for the weights satisfying $(1-\alpha)(z_2^1 - z_2^3) < \alpha(z_1^3 - z_1^1)$.

When the constraints obtained above are compared:

$$(1-\alpha)(z_2^1 - z_2^3) < \alpha(z_1^2 - z_1^1)$$

$$(1-\alpha)(z_2^1 - z_2^3) < \alpha(z_1^3 - z_1^1)$$

It is known that $z_1^3 > z_1^2 > z_1^1$ and $z_2^1 > z_2^2 > z_2^3$.

Then we have $\alpha(z_1^2 - z_1^1) < \alpha(z_1^3 - z_1^1)$ which makes the constraint defining the weight region between z^1 and z^3 redundant. Thus, we can say that z^3 is not a neighbor

of z^1 in the weight space, either. In other words, z^1 and z^3 do not have a common weight in the weight space. However, we see that z^2 is a neighbor of z^1 in the weight space also. Hence, in bi-criteria MOCO problems, only neighbors have a common weight. ■

Example 2: For a bi-criteria minimization problem, let's say the approximation has 8 nondominated points given in Table 2. The graphical representation of the solutions can be seen in Figure 5.

Table 2. Eight nondominated solutions for Example 2

z^i	1	2	3	4	5	6	7	8
z_1^i	0.229	0.371	0.457	0.486	0.514	0.629	0.714	1.000
z_2^i	1.000	0.561	0.540	0.439	0.252	0.222	0.166	0.001

The ideal point is determined as $z^{**} = (0.228, 0)$.

(α_L^i, α_U^i) : the weight interval where z^i is optimal

α_b^i : the weight of the specific diagonal which results in z^i (the specific weight that places z^i at the shortest weighted Tchebycheff distance from z^{**})

The output of IPF(Z) calculations are shown in Table 3.

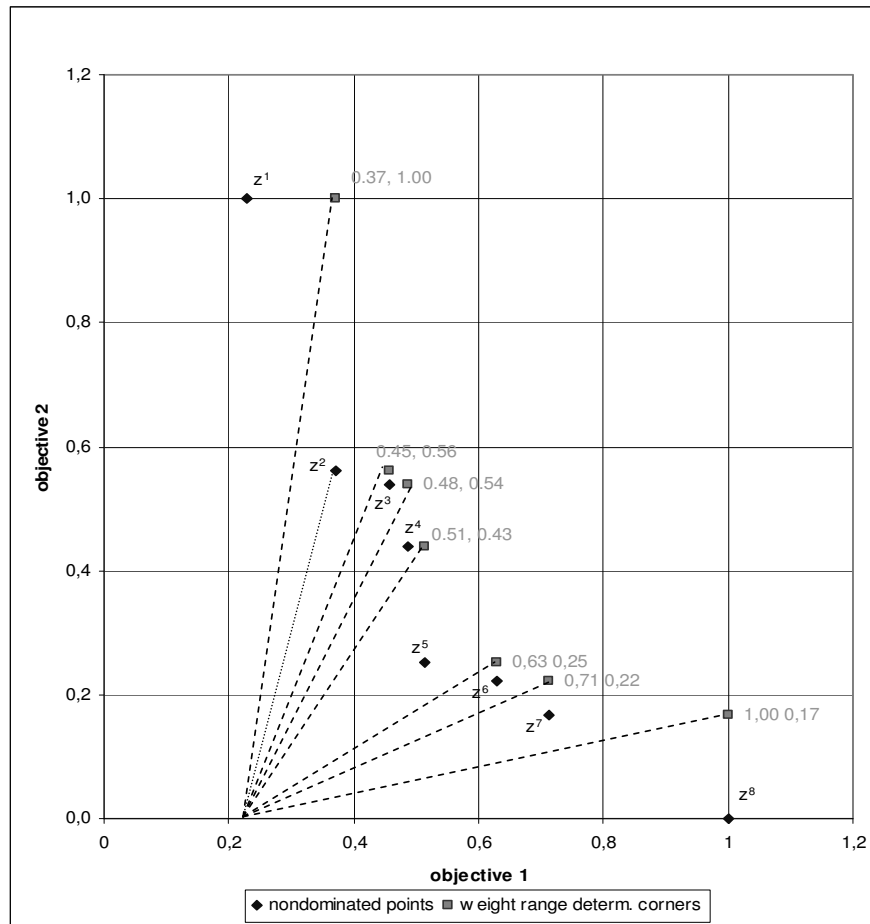


Figure 5. Graphical representation of z^i s and respective α ranges

Table 3. The results of IPF(Z) calculations

z^i	α_L^i	α_b^i	α_U^i	$\alpha_L^i \leq \alpha \leq \alpha_b^i$ $\int (1-\alpha)z_2^i d\alpha$	$\alpha_b^i \leq \alpha \leq \alpha_U^i$ $\int \alpha(z_1^i - 0.228) d\alpha$	IPF $\{z^i\}$
1	0.8748	0.9999	1	0.0078	0	0.0078
2	0.7101	0.7968	0.8748	0.0119	0.0093	0.0213
3	0.6766	0.7022	0.7101	0.0043	0.0013	0.0056
4	0.6055	0.6298	0.6766	0.0040	0.0079	0.0119
5	0.3859	0.4684	0.6055	0.0119	0.0210	0.0329
6	0.3135	0.3563	0.3859	0.0063	0.0044	0.0107
7	0.1769	0.2546	0.3135	0.0101	0.0081	0.0182
8	0	0.0012	0.1769	0	0.0121	0.0121

IPF(Z) = 0.1205

The detailed calculations of α_L^i , α_b^i and $IPF(z^i)$ for z^1 and z^2 are shown below, the detailed calculations for the other alternatives can be seen in the Appendix A.

$$\alpha_L^1 = \frac{1}{(0.371-0.228)} \left[\frac{1}{(0.371-0.228)} + \frac{1}{1} \right]^{-1} = 0.8748$$

$$\alpha_L^2 = \frac{1}{(0.457-0.228)} \left[\frac{1}{(0.457-0.228)} + \frac{1}{0.561} \right]^{-1} = 0.7101$$

$$\alpha_b^1 = \frac{1}{(0.229-0.228)} \left[\frac{1}{(0.229-0.228)} + \frac{1}{1} \right]^{-1} = 0.9999$$

$$\alpha_b^2 = \frac{1}{(0.371-0.228)} \left[\frac{1}{(0.371-0.228)} + \frac{1}{0.561} \right]^{-1} = 0.7968$$

$$\begin{aligned} IPF(z^i) &= \int_{\alpha_L^i}^{\alpha_U^i} \max\{\alpha(z_1^i - z_1^{**}), (1-\alpha)(z_2^i - z_2^{**})\} d\alpha \\ &= \int_{\alpha_L^i}^{\alpha_b^i} (1-\alpha)(z_2^i - z_2^{**}) d\alpha + \int_{\alpha_b^i}^{\alpha_U^i} \alpha(z_2^i - z_2^{**}) d\alpha \end{aligned}$$

$$\begin{aligned} IPF\{z^1\} &= \begin{cases} \int_{0.8748}^{0.9999} (1-\alpha) d\alpha = (0.9999 - 0.8748) - \frac{\alpha^2}{2} \Big|_{0.8748}^{0.9999} \\ + \int_{0.9999}^1 \alpha(0.229 - 0.228) d\alpha = 0.001 \frac{\alpha^2}{2} \Big|_{0.9999}^1 \end{cases} \\ &= 0.0078 + 0 = 0.0078 \end{aligned}$$

$$\begin{aligned} IPF\{z^2\} &= \begin{cases} \int_{0.7101}^{0.7968} (1-\alpha)(0.561) d\alpha = (0.7968 - 0.7101)(0.561) - (0.561) \frac{\alpha^2}{2} \Big|_{0.7101}^{0.7968} \\ + \int_{0.7968}^{0.8748} \alpha(0.371 - 0.228) d\alpha = (0.143) \frac{\alpha^2}{2} \Big|_{0.7968}^{0.8748} \end{cases} \\ &= 0.0119 + 0.0093 = 0.0213 \end{aligned}$$

While the set of solutions represents the efficient frontier better, the IPF value becomes smaller for both minimization and the maximization problems. We do not need to have efficient frontier to compare the sets, but if we have, the efficient frontier will have the minimum IPF value, which can be accepted as the lower bound on IPF value. There is no upper bound for IPF value.

Let us have two more sets of solutions obtained by two different heuristics. First set consists of four supported solutions of the nondominated set shown in Table 2, the other set consists of four dominated solutions, which is dominated by the first set. Then, we can see the comparison of three sets of solutions by using IPF.

Example 3: For a bi-criteria minimization problem, let us have two solution sets. The sets of solutions for two approximations are given in Table 4 and Table 5. The graphical representation of the solutions can be seen in Figure 6.

Table 4. Four supported solutions of approximation 1

z^i	1	2	3	4
z_1^i	0.229	0.371	0.514	1.000
z_2^i	1.000	0.561	0.252	0.001

Table 5. Four dominated solutions of approximation 2

z^i	1	2	3	4
z_1^i	0.259	0.401	0.544	1.03
z_2^i	1.12	0.591	0.282	0.031

The ideal point is still $z^{**} = (0.228, 0)$.

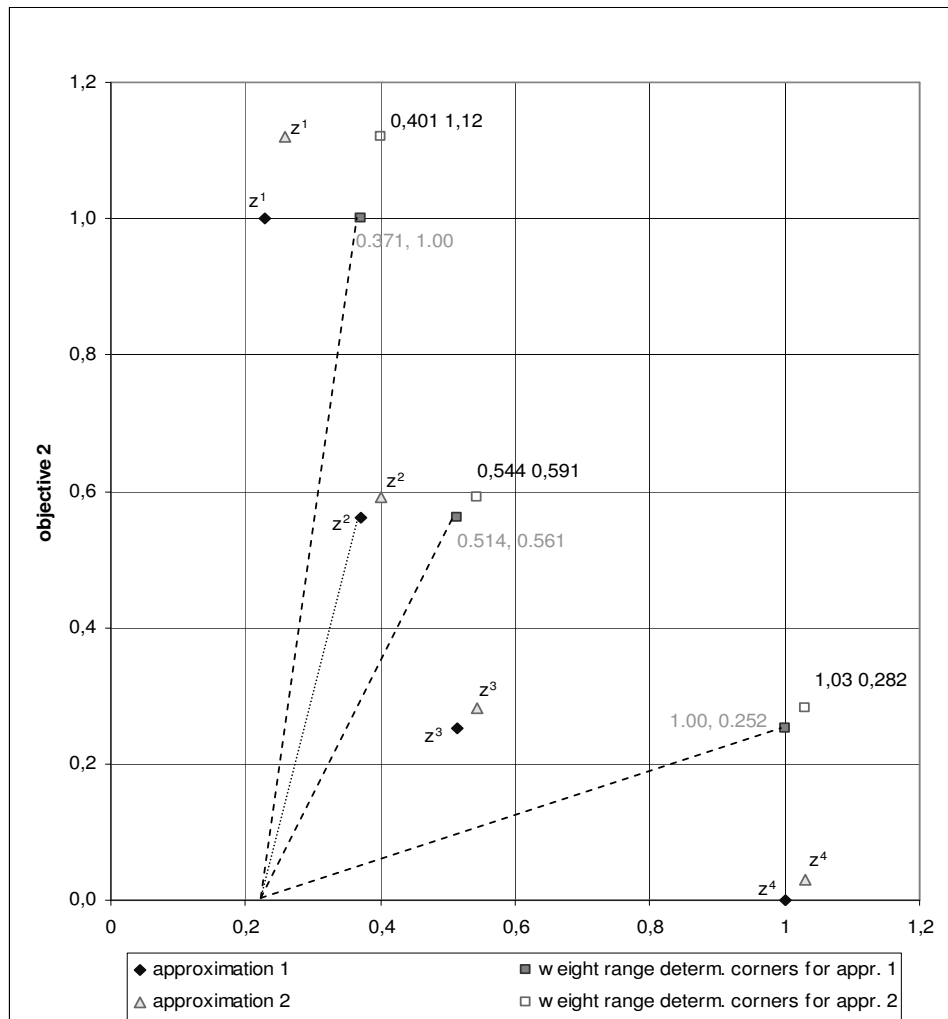


Figure 6. Graphical representation of z^i 's for two approximations

Each solution in approximation 2 is dominated by one solution in approximation 1, which can easily be seen from the Figure 6 also. Therefore, we expect to have IPF value for approximation 1 smaller than approximation 2. While the quality of the set of solutions gets better, the IPF value becomes smaller, for both minimization and the maximization problems.

The output of IPF(Z) calculations for two approximations are shown in Table 6 and Table 7.

Table 6. The results of IPF(Z) calculations for approximation 1

z^i	α_L^i	α_b^i	α_U^i	$\alpha_L^i \leq \alpha \leq \alpha_b^i$ $\int (1-\alpha)z_2^i d\alpha$	$\alpha_b^i \leq \alpha \leq \alpha_U^i$ $\int \alpha(z_1^i - 0.228) d\alpha$	IPF $\{z^i\}$
1	0.8748	0.9999	1	0.0078	0	0.0078
2	0.6623	0.7968	0.8748	0.0204	0.0093	0.0297
3	0.2461	0.4684	0.6623	0.0360	0.0314	0.0674
4	0	0.0013	0.2461	0	0.0234	0.0234
IPF(Z) = 0.1283						

Table 7. The results of IPF(Z) calculations for approximation 2

z^i	α_L^i	α_b^i	α_U^i	$\alpha_L^i \leq \alpha \leq \alpha_b^i$ $\int (1-\alpha)z_2^i d\alpha$	$\alpha_b^i \leq \alpha \leq \alpha_U^i$ $\int \alpha(z_1^i - 0.228) d\alpha$	IPF $\{z^i\}$
1	0.8662	0.9731	1	0.0096	0.0008	0.0104
2	0.6516	0.7736	0.8662	0.0207	0.0132	0.0339
3	0.26	0.4716	0.6516	0.0379	0.0319	0.0698
4	0	0.0372	0.26	0.0011	0.0266	0.0277
IPF(Z) = 0.1418						

The detailed calculations of α_L^i , α_b^i and $IPF(z^i)$ for the approximations 1 and 2 can be seen in the Appendix B and Appendix C, respectively.

As we expected, the IPF value of the approximation 1 appeared to be smaller than that of approximation 2, which brings the same result with the visual comparison method. Since the solutions in the approximation 2 are dominated by the solutions of approximation 1, we can say that the quality of the approximation 1 is better than the approximation 2. Moreover, the solutions in approximation 1 are actually the supported points of the nondominated set which is provided in Table 2 in Example 2. The IPF value of that set was found to be 0.1205 which is smaller than the IPF value of the approximation 1. This is also expected because IPF measure captures not only supported solutions but also nonsupported efficient solutions. Since the

approximation given in Table 2 contains not only the set of supported solutions of approximation 1, but also four additional nondominated solutions, its quality is better than that of approximation 1. Accordingly, the IPF value of the approximation given in Table 2 is smaller than that of approximation 1.

CHAPTER 4

THE IPF MEASURE – GENERAL CASE FOR $p (\geq 3)$ CRITERIA

In this chapter, first we look at an existing performance measurement tool in MOCO for p -criteria cases. Then we develop our measure for more than two criteria MOCO problems. We point out the shortcuts of the method. Then, we utilize our measure for determining the neighborhood relations of the solutions in the weight space. We examine the neighborhood relations both in the objective space and the weight space with the results obtained for more than two criteria cases.

4.1 IPF Measure with Weighted Linear Utility Function: p Criteria Case

The original IPF measure with weighted linear utility function is extended by Kim *et al.*(2006) for $p (\geq 3)$ objective combinatorial optimization problems. IPF measure in multi objective case needs calculating multi dimensional integrations. The integration of $g(x;\alpha)$ over the weight regions becomes computing the volumes of p -convex polytopes. Therefore, instead of calculating the k -dimensional integration, calculating the volumes of p -convex polytopes is suggested. As in the bi-criteria case, utility function is assumed to be a convex combination of the objectives, $h(\alpha)$ is assumed to be a uniform density function, and the method considers supported points only. Hence the unsupported efficient points do not contribute to the IPF measure at all.

The method consists of five steps:

1. Find all supported points and adjacent supported points;
2. Find the weight ranges for each supported point to be optimal, by solving a series of linear inequalities;

3. For each supported point, form p-convex polytope by adding one dimension for value function to the region of weights dimensions. The p-convex polytope is represented as a system of linear inequalities;
4. Calculate the volumes of p-convex polytopes;
5. Calculate IPF by summing the volumes of p-convex polytopes for supported points.

Constructing a p-convex polytope and calculating its volume is not straight forward. According to the experiments conducted by Kim *et al.*(2006), the computational effort for exact IPF method increases linearly with increasing number of supported points and increases exponentially with increasing number of objectives. Therefore, Monte Carlo estimation is suggested by Kim *et al.*(2006) for p-criteria case because of the computational effort needed for the exact IPF method. The reasonable error ratio and the number of random weight vectors are recommended according to the experimental results of Monte Carlo estimation.

4.2 IPF Measure with Weighted Tchebycheff Utility Function: More Than Two Criteria Case

While extending our measure to more than two criteria case, the most important issues will be determining the weight regions for each nondominated solution to be optimal, and within that region determining the weight sub-regions where each criterion represents that nondominated point when comparing with the other alternatives.

Dell and Karwan (1990), develop an interactive procedure locally approximating the DM's utility function with a weighted Tchebycheff function. Based on pairwise preferences of the DM, they restrict the weight space. As it is not known which criterion is at maximum weighted distance from the ideal point, they obtain weight spaces conditional upon different criteria. This leads to a number of disjoint convex weight regions that might contain the weights that represent the DM's preferences. In

our study, we made use of their weight space restriction approach while determining the weight regions for each nondominated solution.

Therefore, we have three sub-problems to solve:

1. Partition the weight region for each alternative z^i such that in each partition a different criterion determines the Tchebycheff distance of z^i from z^{**} .
2. Determine the weight regions where each nondominated point is optimal,
3. Calculate the IPF by solving integrals over the weight regions determined.

The weight regions are determined by solving series of linear equalities. While considering these sub-problems our neighborhood assumption is valid here also. It is assumed that the alternatives closest in every criterion are neighbors. In the succeeding parts the neighborhood issue will be discussed in detail.

Before proceeding, let us give the notation:

p : the number of criteria,

W_j^i : the weight set for which criterion j determines the weighted Tchebycheff distance of z^i from z^{**} .

W_{jr}^{ki} : the weight set for which z^k has the smaller weighted Tchebycheff distance to z^{**} than z^i , when criteria j and r determine the weighted Tchebycheff distance of z^k and z^i respectively. There are p^2 combinations for each pair of alternatives z^k and z^i .

$$W_{jr}^{ki} = \{ \alpha: \alpha_{j,z_j^k} \leq \alpha_{r,z_r^i}, \alpha \in W, \alpha \in W_j^k \cap W_r^i \}.$$

$J^{ki} : J^{ki} = \{ j : z_j^k \leq z_j^i \}$ denote the index set of criteria for which z^k is at least as good as z^i . Then $J^{ki} \cup J^{ik} = J$ where J is the index set of all criteria.

W^{ki} : the weight set for which the weighted Tchebycheff distance of z^k from z^{**} is smaller than that of z^i .

$$W^{ki} = \bigcup_{j \in J^{ik}, r \in J^{ki}} W_{jr}^{ki} \cup W_j^k.$$

W^k : the weight set that makes the alternative k (z^k) the best alternative, i.e. the weight set that makes z^k at the minimum weighted Tchebycheff distance from z^{**} .

$$W^k = \bigcap_{i \neq k} W^{ki}.$$

Example 4: Let's say we have an approximation with three criteria to be minimized and four nondominated alternatives. The non-dominated alternatives can be seen in Table 8.

Table 8. Four nondominated solutions for Example 4

z^i	1	2	3	4
z_1^i	0.1	2.1	3.1	4.1
z_2^i	2.0	1.6	1.1	0.1
z_3^i	0.1	1.1	2.1	3.1

Let $z^{**} = (0,0,0)$, then $z_j^i - z_j^{**} = z_j^i$ for all i, j .

4.2.1 Determining W_j^i s:

For each alternative i , the set of weights W_j^i for which criterion j determines the weighted Tchebycheff distance of z^i from z^{**} can be calculated as follows:

$$\begin{aligned}W_1^1 &= \{\alpha : 0.1\alpha_1 \geq 2\alpha_2, 0.1\alpha_1 \geq 0.1(1-\alpha_1-\alpha_2)\} \\ &= \{\alpha : 0.1\alpha_1 - 2\alpha_2 \geq 0, 0.2\alpha_1 + 0.1\alpha_2 \geq 0.1\}\} \\ W_2^1 &= \{\alpha : 0.1\alpha_1 \leq 2\alpha_2, 2\alpha_2 \geq 0.1(1-\alpha_1-\alpha_2)\} \\ &= \{\alpha : 0.1\alpha_1 - 2\alpha_2 \leq 0, 0.1\alpha_1 + 2.1\alpha_2 \geq 0.1\}\} \\ W_3^1 &= \{\alpha : 0.2\alpha_1 + 0.1\alpha_2 \leq 0.1, 0.1\alpha_1 + 2.1\alpha_2 \leq 0.1\}\} \\ W_1^2 &= \{\alpha : 2.1\alpha_1 - 1.6\alpha_2 \geq 0, 3.2\alpha_1 + 1.1\alpha_2 \geq 1.1\}\} \\ W_2^2 &= \{\alpha : 2.1\alpha_1 - 1.6\alpha_2 \leq 0, 1.1\alpha_1 + 2.7\alpha_2 \geq 1.1\}\} \\ W_3^2 &= \{\alpha : 3.2\alpha_1 + 1.1\alpha_2 \leq 1.1, 1.1\alpha_1 + 2.7\alpha_2 \leq 1.1\}\} \\ W_1^3 &= \{\alpha : 3.1\alpha_1 - 1.1\alpha_2 \geq 0, 5.1\alpha_1 + 2.1\alpha_2 \geq 2.1\}\} \\ W_2^3 &= \{\alpha : 3.1\alpha_1 - 1.1\alpha_2 \leq 0, 2.1\alpha_1 + 3.2\alpha_2 \geq 2.1\}\} \\ W_3^3 &= \{\alpha : 5.1\alpha_1 + 2.1\alpha_2 \leq 2.1, 2.1\alpha_1 + 3.2\alpha_2 \leq 2.1\}\} \\ W_1^4 &= \{\alpha : 4.1\alpha_1 - 0.1\alpha_2 \geq 0, 7.1\alpha_1 + 3.1\alpha_2 \geq 3.1\}\} \\ W_2^4 &= \{\alpha : 4.1\alpha_1 - 0.1\alpha_2 \leq 0, 3.1\alpha_1 + 3.2\alpha_2 \geq 3.1\}\} \\ W_3^4 &= \{\alpha : 7.1\alpha_1 + 3.1\alpha_2 \leq 3.1, 3.1\alpha_1 + 3.2\alpha_2 \leq 3.1\}\}\end{aligned}$$

The weight regions of alternative 2, W_j^2 can be seen in Figure 7 where $\alpha_3=1-\alpha_1-\alpha_2$.

The weight regions of the other alternatives are available in Figures 15 to 17 in Appendix D.

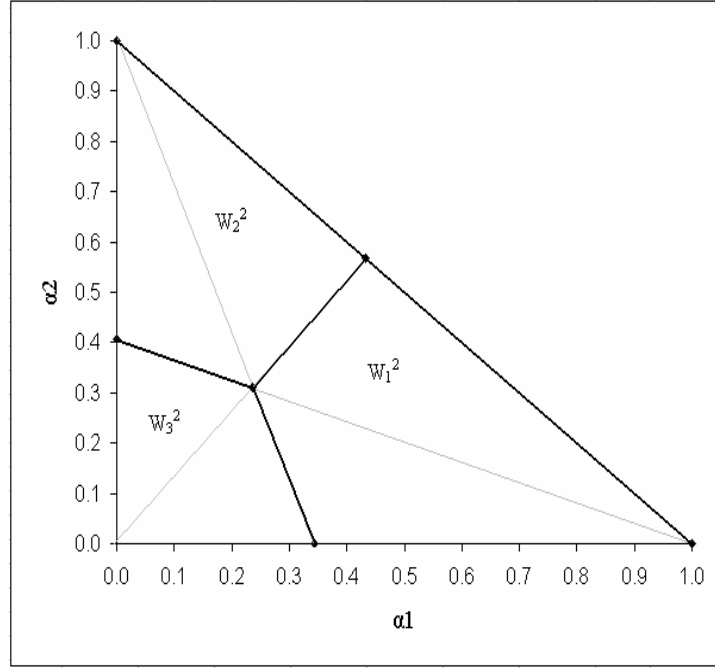


Figure 7. Partition of the weight space between criteria for z^2

Up to now, we have solved our first sub-problem and determined for each alternative W_j^i s. Then the next step is to determine the weight regions where each alternative is optimal, i.e. determining W^k s.

4.2.2 Determining W^k s:

$$W^k = \{\alpha : u_\alpha(z^k) \leq u_\alpha(z^i), i \neq k\} = \{\alpha : \max_j (\alpha_j z_j^k) \leq \max_j (\alpha_j z_j^i), i \neq k\}$$

Comparison of alternatives z^1 and z^2 :

$$u_\alpha(z^1) = u_\alpha(z^2)$$

$$u_\alpha(z^1) = \text{Max}\{0.1\alpha_1, 2\alpha_2, 0.1\alpha_3\} = \text{Max}\{2.1\alpha_1, 1.6\alpha_2, 1.1\alpha_3\} = u_\alpha(z^2)$$

Let $0.1\alpha_1$ determine the weighted Tchebycheff distance of z^1 from z^{**} , then

$$0.1\alpha_1 \geq 2\alpha_2 \text{ and } 0.1\alpha_1 \geq 0.1\alpha_3. \text{ Since } 0.1\alpha_1 < 2.1\alpha_1, u_\alpha(z^1) < u_\alpha(z^2).$$

A similar situation is valid if $0.1 \alpha_3$ determines the weighted Tchebycheff distance of z^1 from z^{**} , and $u_\alpha(z^1) < u_\alpha(z^2)$ in this case as well.

We can talk about the weight region boundaries for alternatives 1 and 2, only if $2 \alpha_2$ determines the weighted Tchebycheff distance of z^1 from z^{**} . In this case,

$$2 \alpha_2 = \text{Max}\{2.1 \alpha_1, 1.6 \alpha_2, 1.1 \alpha_3\}$$

$$2 \alpha_2 = 2.1 \alpha_1 \text{ for } 2.1 \alpha_1 \geq 1.1 \alpha_3 \text{ (i.e. } 3.2 \alpha_1 + 1.1 \alpha_2 \geq 1.1 \text{ substituting } 1 - \alpha_1 - \alpha_2 \text{ for } \alpha_3)$$

$$2 \alpha_2 = 1.1 \alpha_3 \text{ for } 2.1 \alpha_1 < 1.1 \alpha_3 \text{ (i.e. } 1.1 \alpha_1 + 3.1 \alpha_2 = 1.1 \text{ for } 3.2 \alpha_1 + 1.1 \alpha_2 < 1.1 \text{ substituting } 1 - \alpha_1 - \alpha_2 \text{ for } \alpha_3)$$

While determining the weight region boundaries for alternatives 1 and 2, we only need to consider these two constraints above (Actually, these two constraints determine the weight sets W_{21}^{12} and W_{23}^{12} , respectively). There is no need to consider W_2^2 here, since in that situation $u_\alpha(z^2)$ will be smaller than $u_\alpha(z^1)$. The important point that should not be ignored is that the constraints are not valid throughout the weight regions. For example $2 \alpha_2 = 2.1 \alpha_1$ constraint is valid only for the weights satisfying the equation $3.2 \alpha_1 + 1.1 \alpha_2 \geq 1.1$, which will be very important while calculating the IPF.

Observation 1: In three criteria case we see that we have two equations to consider for each pair of alternatives. That is, there are two weight equations for which the two alternatives have equal Tchebycheff distances from the ideal point. Similarly, in four criteria case we either have 3 equations or 4 equations. We have 3 equations when one alternative is better than the other in three criteria, and 4 equations when one is better than the other in two criteria.

Theorem 3: Let $p=3$. Assume without loss of generality that z^k is better than z^i in two criteria, and z^i is better than z^k in one criterion. Then only 5 of the $p^2=9$ weight space regions of W_{jr}^{ki} $j=1,\dots,3$ $r=1,\dots,3$ need to be considered.

Proof: Considering a minimization MOCO problem, let $z^{**}=(0,0,0)$ without loss of generality, then $z_j^i - z_j^{**} = z_j^i$ for all i, j .

$$u_\alpha(z^k) = \text{Max}\{z_1^k \alpha_1, z_2^k \alpha_2, z_3^k \alpha_3\} = \text{Max}\{z_1^i \alpha_1, z_2^i \alpha_2, z_3^i \alpha_3\} = u_\alpha(z^i)$$

Without loss of generality assume

$$z_1^k < z_1^i$$

$$z_3^k < z_3^i$$

$$z_2^k > z_2^i$$

When z_1^k determines the weighted Tchebycheff distance of z^k from z^{**} (W_1^k), then since $z_1^k < z_1^i$, $u_\alpha(z^k) < u_\alpha(z^i)$. A similar situation is valid when z_3^k determines the weighted Tchebycheff distance of z^k from z^{**} (W_3^k). Thus, we have two regions from W_j^k at hand (W_1^k and W_3^k). Similarly we have one region from W_2^i . Therefore there is no need to consider W_{12}^{ki} , W_{13}^{ki} , W_{31}^{ki} , W_{32}^{ki} . The remaining regions to consider are W_{21}^{ki} and W_{23}^{ki} defined by the equations below:

$$u_\alpha(z^k) = z_2^k \alpha_2 = \text{Max}\{z_1^i \alpha_1, z_2^i \alpha_2, z_3^i \alpha_3\} = u_\alpha(z^i)$$

$$z_2^k \alpha_2 = z_1^i \alpha_1 \text{ for } z_1^i \alpha_1 \geq z_3^i \alpha_3,$$

$$z_2^k \alpha_2 = z_3^i \alpha_3 \text{ for } z_1^i \alpha_1 < z_3^i \alpha_3.$$

Hence, two regions come from W_j^k , one region comes from W_j^i , and two regions come from W_{jr}^{ki} . ■

Comparison of alternatives z^2 and z^3 :

$$u_{\alpha}(z^2) = u_{\alpha}(z^3)$$

$$\text{Max}\{2.1\alpha_1, 1.6\alpha_2, 1.1\alpha_3\} = \text{Max}\{3.1\alpha_1, 1.1\alpha_2, 2.1\alpha_3\}$$

Since $2.1\alpha_1 < 3.1\alpha_1$ and $1.1\alpha_3 < 2.1\alpha_3$, the boundaries of the weight regions are formed only when $1.6\alpha_2$ determines the weighted Tchebycheff distance of z^2 from z^{**} .

$$1.6\alpha_2 = \text{Max}\{3.1\alpha_1, 1.1\alpha_2, 2.1\alpha_3\}$$

$$1.6\alpha_2 = 3.1\alpha_1 \text{ for } 3.1\alpha_1 \geq 2.1\alpha_3 \text{ (i.e. } 5.1\alpha_1 + 2.1\alpha_2 \geq 2.1 \text{ substituting } 1 - \alpha_1 - \alpha_2 \text{ for } \alpha_3)$$

$$1.6\alpha_2 = 2.1\alpha_3 \text{ for } 3.1\alpha_1 < 2.1\alpha_3 \text{ (i.e. } 2.1\alpha_1 + 3.7\alpha_2 = 2.1 \text{ for } 5.1\alpha_1 + 2.1\alpha_2 < 2.1 \text{ substituting } 1 - \alpha_1 - \alpha_2 \text{ for } \alpha_3)$$

Comparison of alternatives z^3 and z^4 :

$$u_{\alpha}(z^3) = u_{\alpha}(z^4)$$

$$\text{Max}\{3.1\alpha_1, 1.1\alpha_2, 2.1\alpha_3\} = \text{Max}\{4.1\alpha_1, 0.1\alpha_2, 3.1\alpha_3\}$$

Since $3.1\alpha_1 < 4.1\alpha_1$ and $2.1\alpha_3 < 3.1\alpha_3$, the boundaries of the weight regions are formed only when $1.1\alpha_2$ determines the weighted Tchebycheff distance of z^3 from z^{**} .

$$1.1\alpha_2 = \text{Max}\{4.1\alpha_1, 0.1\alpha_2, 3.1\alpha_3\}$$

$$1.1\alpha_2 = 4.1\alpha_1 \text{ for } 4.1\alpha_1 \geq 3.1\alpha_3 \text{ (} 7.1\alpha_1 + 3.1\alpha_2 \geq 3.1)$$

$$1.1\alpha_2 = 3.1\alpha_3 \text{ for } 4.1\alpha_1 < 3.1\alpha_3 \text{ (} 3.1\alpha_1 + 4.2\alpha_2 = 3.1 \text{ for } 7.1\alpha_1 + 3.1\alpha_2 < 3.1)$$

In this step, we have determined the weight region constraints between all alternatives, that is to say, we obtained W^k s. The graphical representation of W^k s can be seen in Figure 8. It can be seen from the figure that W^k s are not convex.

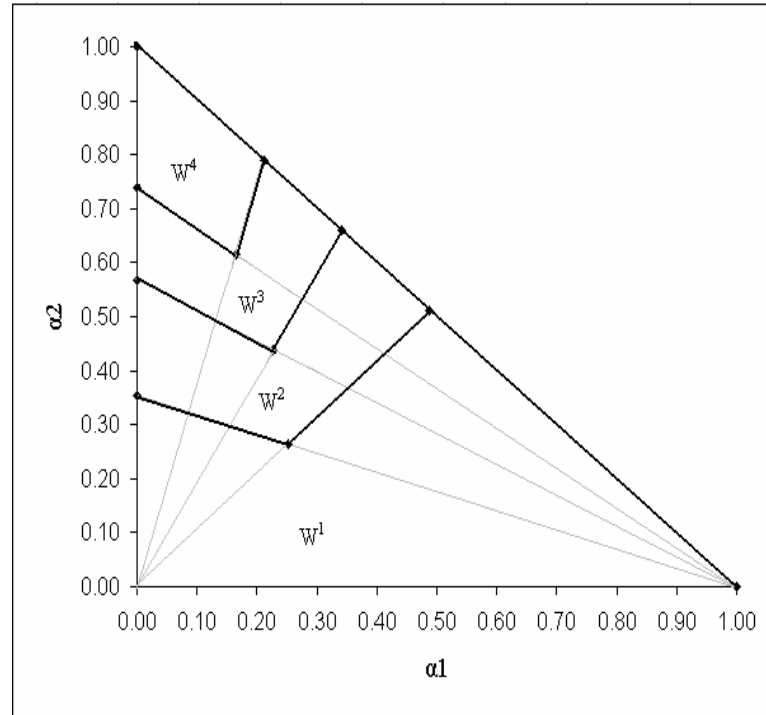


Figure 8. Partition of the weight space between alternatives

Up to now, we have determined W_j^i s and W^k s. In other words we have solved sub-problems 1 and 2. Now the next and the last step is to calculate the IPF, by solving integrals over weight regions.

4.2.3 Calculating IPF(z^i):

By overlaying Figure 7 on Figure 8, we can divide W^2 region into three parts as shown in Figure 9. By the intersection of W_j^2 and W^2 , we obtain one sub-region for each criterion determining the weighted Tchebycheff distance of z^2 , in the region of

W^2 . These sub-regions can be represented as $W^2 \cap W_j^2$. Similarly, for all alternatives the sub-regions are represented as $W^i \cap W_j^i$. The partition of the W^i regions for the other alternatives are available in Figures 18 to 20 in Appendix E. Obtaining those sub-regions is important since without knowing which criterion determines the weighted Tchebycheff distance for each alternative, IPF measure can not be formulated.

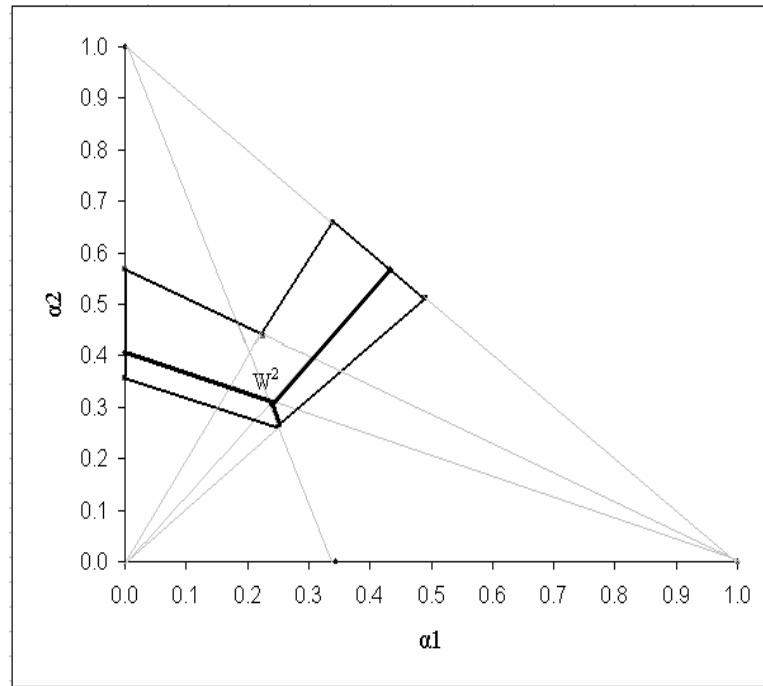


Figure 9. Partition of the W^2 region

IPF(z^i) can be formulated as:

$$\text{IPF}(z^i) = \iint_{W^i \cap W_1^i} \alpha_1 z_1^i d\alpha_2 d\alpha_1 + \iint_{W^i \cap W_2^i} \alpha_2 z_2^i d\alpha_2 d\alpha_1 + \iint_{W^i \cap W_3^i} (1 - \alpha_1 - \alpha_2) z_3^i d\alpha_2 d\alpha_1$$

In order to calculate $IPF(z^i)$, the regions obtained from $W^i \cap W_j^i$ should also be divided into smaller parts needed for the calculation of the partitions of the integrals. For z^2 those partitions for the integral calculations are demonstrated in Figure 10. For the other alternatives the partitions for the integral calculations are available in Figures 21 to 23 in Appendix F.

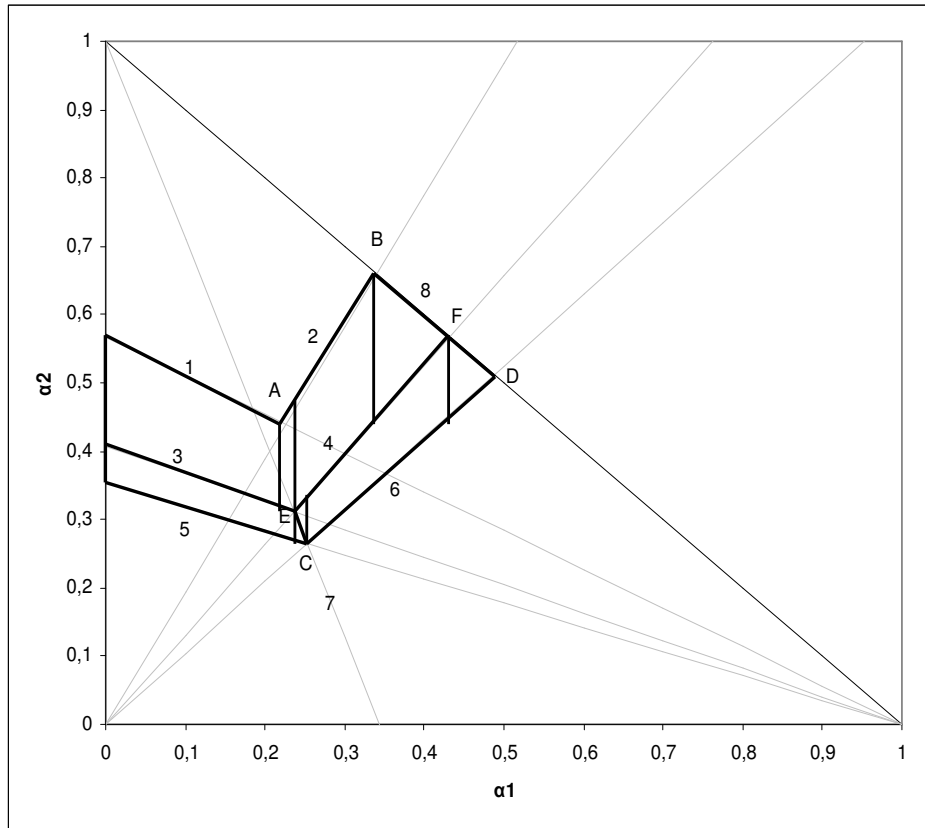


Figure 10. Partition of the W^2 region for the integral calculations

The points and the equations shown in Figure 10 are needed for the integral calculations. The respective points and the equations for W^2 region are given below. For the other alternatives the related points and the equations are available in Tables 12 to 14 in Appendix F.

$$A=(0.227, 0.439)$$

$$B=(0.340, 0.660)$$

$$C=(0.252, 0.266)$$

$$D=(0.488, 0.512)$$

$$E=(0.237, 0.3109)$$

$$F=(0.4324, 0.5676)$$

$$(1) \alpha_2 = \frac{2.1 - 2.1\alpha_1}{3.7}$$

$$(2) \alpha_2 = \frac{3.1\alpha_1}{1.6}$$

$$(3) \alpha_2 = \frac{1.1 - 1.1\alpha_1}{2.7}$$

$$(4) \alpha_2 = \frac{2.1\alpha_1}{1.6}$$

$$(5) \alpha_2 = \frac{1.1 - 1.1\alpha_1}{3.1}$$

$$(6) \alpha_2 = \frac{2.1\alpha_1}{2}$$

$$(7) \alpha_2 = \frac{1.1 - 3.2\alpha_1}{1.1}$$

The calculation of the third criterion portion of $IPF(z^2)$ is demonstrated below:

$$\begin{aligned} & \iint_{W^2 \cap W_3^2} (1 - \alpha_1 - \alpha_2) z_3^2 d\alpha_2 d\alpha_1 \\ &= \int_0^{0.237} \int_{(1.1-1.1\alpha_1)/3.1}^{(1.1-1.1\alpha_1)/2.7} 1.1(1 - \alpha_1 - \alpha_2) d\alpha_2 d\alpha_1 + \int_{0.237}^{0.252} \int_{(1.1-1.1\alpha_1)/3.1}^{(1.1-3.2\alpha_1)/1.1} 1.1(1 - \alpha_1 - \alpha_2) d\alpha_2 d\alpha_1 \end{aligned}$$

$$=0.013$$

The first and the second criterion portions are calculated in a similar manner, and we obtained $IPF(z^2) = 0.082$.

Similarly, we obtained $IPF(z^1)=0.1074$, $IPF(z^3)=0.0459$, $IPF(z^4)=0.0227$,

Then $IPF(Z)= 0.1074+0.082+0.0459+0.0227=0.258$

The detailed IPF calculations for all the alternatives are available in Appendix G.

4.3 Neighborhood Assumption: More Than Two Criteria Case

As mentioned before, we assume that the alternatives that are closest in any criterion are neighbors. Therefore these alternatives are expected to share weight space consistent with the neighborhood relations. In this section we discuss the consistency of our neighborhood assumption with the examples studied before.

Theorem 4: Let $p=3$. Assume without loss of generality that z^i is not a neighbor of z^k in the objective space. Then z^i is not a neighbor of z^k in the weight space either.

Proof: Considering a maximization problem, let us consider 3 alternatives such that z^2 is not a neighbor of z^1 in the objective space, whereas z^3 is a neighbor of z^1 in the objective space.

$$z^1 = (z_1^1, z_2^1, z_3^1)$$

$$z^2 = (z_1^2, z_2^2, z_3^2)$$

$$z^3 = (z_1^3, z_2^3, z_3^3)$$

$$z^{**} = (z_1^{**}, z_2^{**}, z_3^{**})$$

Let

N_k : the set of neighbors of alternative k,

W_{jr}^{ki} : the weight set for which z^k has the smaller weighted Tchebycheff distance than z^i , when the criterion j and r determines the weighted Tchebycheff distance of z^k and z^i respectively.

$$N_1=\{z^3\}$$

Comparison of alternatives z^1 and z^2 :

$$u_{\alpha}(z^1) = u_{\alpha}(z^2)$$

$$\text{Max}\{(z_1^{**} - z_1^1)\alpha_1, (z_2^{**} - z_2^1)\alpha_2, (z_3^{**} - z_3^1)\alpha_3\}$$

$$= \text{Max}\{(z_1^{**} - z_1^2)\alpha_1, (z_2^{**} - z_2^2)\alpha_2, (z_3^{**} - z_3^2)\alpha_3\}$$

Then W_{jr}^{12} and W_{jr}^{13} constraints are:

$$W_{jr}^{12} \text{ constraint: } (z_j^{**} - z_j^1)\alpha_j \leq (z_r^{**} - z_r^2)\alpha_r$$

$$\alpha_j \leq [(z_r^{**} - z_r^2)/(z_j^{**} - z_j^1)]\alpha_r$$

$$W_{jr}^{13} \text{ constraint: } (z_j^{**} - z_j^1)\alpha_j \leq (z_r^{**} - z_r^3)\alpha_r$$

$$\alpha_j \leq [(z_r^{**} - z_r^3)/(z_j^{**} - z_j^1)]\alpha_r$$

Comparing $[(z_r^{**} - z_r^2)/(z_j^{**} - z_j^1)]$ and $[(z_r^{**} - z_r^3)/(z_j^{**} - z_j^1)]$,

$$(z_r^{**} - z_r^2) > (z_r^{**} - z_r^1)$$

$$(z_r^{**} - z_r^3) > (z_r^{**} - z_r^1)$$

The following inequalities are obtained,

$$z_r^2 < z_r^1$$

$$z_r^3 < z_r^1$$

Since z^3 is closer to z^1 in criterion r, it can be said that $z_r^1 > z_r^3 > z_r^2$.

Then it is obtained that,

$$(z_r^{**} - z_r^2) > (z_r^{**} - z_r^3).$$

Since in a maximization problem $(z_j^{**} - z_j^1)$ is always positive the equation below can be written:

$$[(z_r^{**} - z_r^2)/(z_j^{**} - z_j^1)] > [(z_r^{**} - z_r^3)/(z_j^{**} - z_j^1)]$$

This equation makes the constraint $(z_j^{**} - z_j^1) \alpha_j \leq (z_r^{**} - z_r^2) \alpha_r$ redundant.

The proof for the minimization problem is similar. ■

Therefore with this theorem, we show that we do not need to consider and compare all pairs of alternatives while determining the neighborhood relations in the weight space. That is to say, we do not need to compare all pairs of solutions with each other (i.e. $n(n-1)/2$ comparisons). Instead, considering the solutions that are neighbors in the objective space is sufficient for defining the neighborhood relations in the weight space and determining the weight space partition of the alternatives.

Theorem 5: Let $p \geq 3$. An alternative z^i may have at most $2p$ neighbors in the objective space.

Proof: Since z^i has at most two closest solutions in every criterion, there will be at most $2p$ such solutions. Note that the same solution z^k may be closest to z^i in more than one criterion. In such cases we will have less than $2p$ neighbors in the objective space. ■

Example 5: Let's say we have an approximation with three criteria to be maximized and four nondominated alternatives. The nondominated solutions can be seen in Table 9.

Table 9. Four nondominated solutions for Example 5

z^i	1	2	3	4
z_1^i	1	3	5	2
z_2^i	0.1	2	0.5	1
z_3^i	6	3	4	5

Let $z^{**}=(5.1, 2.1, 6.1)$,

Let N_k denote the set of neighbors of alternative k ,

$n_{ik}=\{j^s\}$ denote the set of criteria for which alternatives i and k have the neighborhood relations (alternatives i and k have the neighborhood relations in criterion j from upper/lower side. s can be $+$ or $-$, $+$ defining the neighborhood is from positive direction, $-$ defining the neighborhood is from negative direction.)

Let us define the neighborhood relations in this example:

$$n_{12}=\{ \}$$

$$n_{13}=\{2^+\}$$

$$n_{14}=\{1^+, 3^-\}$$

$$n_{23}=\{1^+, 3^+\}$$

$$n_{24}=\{1^-, 2^-\}$$

$$n_{34}=\{2^+, 3^+\}$$

We need to calculate W^k s and compare all pairs of alternatives in order to determine the neighborhood relations by observing the share of weight space between the alternatives. We need not compare the alternatives that are not neighbors (i.e. z^1 and z^2) due to Theorem 4.

Remember we defined W^k as below:

$$W^k = \{ \alpha : u_\alpha(z^k) \leq u_\alpha(z^i), i \neq k \} = \{ \alpha : \max_j (\alpha_j z_j^k) \leq \max_j (\alpha_j z_j^i), i \neq k \}$$

Comparison of alternatives z^1 and z^3 :

$$u_\alpha(z^1) = u_\alpha(z^3)$$

$$\text{Max}\{4.1 \alpha_1, 2 \alpha_2, 0.1 \alpha_3\} = \text{Max}\{0.1 \alpha_1, 1.6 \alpha_2, 2.1 \alpha_3\}$$

$$\text{Max}\{4.1 \alpha_1, 2 \alpha_2, 0.1 \alpha_3\} = 2.1 \alpha_3$$

$$4.1 \alpha_1 = 2.1 \alpha_3 \text{ for } 4.1 \alpha_1 \geq 2 \alpha_2$$

$$2 \alpha_2 = 2.1 \alpha_3 \text{ for } 4.1 \alpha_1 < 2 \alpha_2$$

$$2.1 \alpha_2 = 2.1 - 6.2 \alpha_1 \text{ for } 4.1 \alpha_1 \geq 2 \alpha_2$$

$$4.1 \alpha_2 = 2.1 - 2.1 \alpha_1 \text{ for } 4.1 \alpha_1 < 2 \alpha_2$$

Comparison of alternatives z^1 and z^4 :

$$u_\alpha(z^1) = u_\alpha(z^4)$$

$$\text{Max}\{4.1 \alpha_1, 2 \alpha_2, 0.1 \alpha_3\} = \text{Max}\{3.1 \alpha_1, 1.1 \alpha_2, 1.1 \alpha_3\}$$

$$\text{Max}\{4.1 \alpha_1, 2 \alpha_2, 0.1 \alpha_3\} = 1.1 \alpha_3$$

$$4.1 \alpha_1 = 1.1 \alpha_3 \text{ for } 4.1 \alpha_1 \geq 2 \alpha_2$$

$$2 \alpha_2 = 1.1 \alpha_3 \text{ for } 4.1 \alpha_1 < 2 \alpha_2$$

$$1.1 \alpha_2 = 1.1 - 5.2 \alpha_1 \text{ for } 4.1 \alpha_1 \geq 2 \alpha_2$$

$$3.1 \alpha_2 = 1.1 - 1.1 \alpha_1 \text{ for } 4.1 \alpha_1 < 2 \alpha_2$$

Comparison of alternatives z^2 and z^3 :

$$u_\alpha(z^2) = u_\alpha(z^3)$$

$$\text{Max}\{2.1 \alpha_1, 0.1 \alpha_2, 3.1 \alpha_3\} = \text{Max}\{0.1 \alpha_1, 1.6 \alpha_2, 2.1 \alpha_3\}$$

$$\text{Max}\{2.1 \alpha_1, 0.1 \alpha_2, 3.1 \alpha_3\} = 1.6 \alpha_2$$

$$2.1 \alpha_1 = 1.6 \alpha_2 \text{ for } 2.1 \alpha_1 \geq 3.1 \alpha_3$$

$$3.1 \alpha_3 = 1.6 \alpha_2 \text{ for } 2.1 \alpha_1 < 3.1 \alpha_3$$

$$1.6 \alpha_2 = 2.1 \alpha_1 \text{ for } 2.1 \alpha_1 \geq 3.1 \alpha_3$$

$$4.7 \alpha_2 = 3.1 - 3.1 \alpha_1 \text{ for } 2.1 \alpha_1 < 3.1 \alpha_3$$

Comparison of alternatives z^2 and z^4 :

$$u_{\alpha}(z^2) = u_{\alpha}(z^4)$$

$$\text{Max}\{2.1 \alpha_1, 0.1 \alpha_2, 3.1 \alpha_3\} = \text{Max}\{3.1 \alpha_1, 1.1 \alpha_2, 1.1 \alpha_3\}$$

$$3.1 \alpha_3 = \text{Max}\{3.1 \alpha_1, 1.1 \alpha_2, 1.1 \alpha_3\}$$

$$3.1 \alpha_3 = 3.1 \alpha_1 \text{ for } 3.1 \alpha_1 \geq 1.1 \alpha_2$$

$$3.1 \alpha_3 = 1.1 \alpha_2 \text{ for } 3.1 \alpha_1 < 1.1 \alpha_2$$

$$3.1 \alpha_2 = 3.1 - 6.2 \alpha_1 \text{ for } 3.1 \alpha_1 \geq 1.1 \alpha_2$$

$$4.2 \alpha_2 = 3.1 - 3.1 \alpha_1 \text{ for } 3.1 \alpha_1 < 1.1 \alpha_2$$

Comparison of alternatives z^3 and z^4 :

$$u_{\alpha}(z^3) = u_{\alpha}(z^4)$$

$$\text{Max}\{0.1 \alpha_1, 1.6 \alpha_2, 2.1 \alpha_3\} = \text{Max}\{3.1 \alpha_1, 1.1 \alpha_2, 1.1 \alpha_3\}$$

$$\text{Max}\{0.1 \alpha_1, 1.6 \alpha_2, 2.1 \alpha_3\} = 3.1 \alpha_1$$

$$1.6 \alpha_2 = 3.1 \alpha_1 \text{ for } 1.6 \alpha_2 \geq 2.1 \alpha_3$$

$$2.1 \alpha_3 = 3.1 \alpha_1 \text{ for } 1.6 \alpha_2 < 2.1 \alpha_3$$

$$1.6 \alpha_2 = 3.1 \alpha_1 \text{ for } 1.6 \alpha_2 \geq 2.1 \alpha_3$$

$$2.1 \alpha_2 = 2.1 - 5.2 \alpha_1 \text{ for } 1.6 \alpha_2 < 2.1 \alpha_3$$

The graphical representation of the constraints obtained above can be seen in Figure 11. The constraints in Figure 11 are not valid for the whole weight space. They are valid for certain conditions. For example while comparing the alternatives z^3 and z^4 in Example 5 above, $1.6 \alpha_2 = 3.1 \alpha_1$ equation is valid for the condition $1.6 \alpha_2 \geq 2.1 \alpha_3$ and $2.1 \alpha_3 = 3.1 \alpha_1$ equation is valid for the condition $1.6 \alpha_2 < 2.1 \alpha_3$. We first draw

all the constraints and their validity conditions, which can be seen in Figure 11, and then by dropping the invalid parts of the constraints we partition the weight space between the alternatives. According to the constraints, the resulting partition of weight space between alternatives is shown in Figure 12.

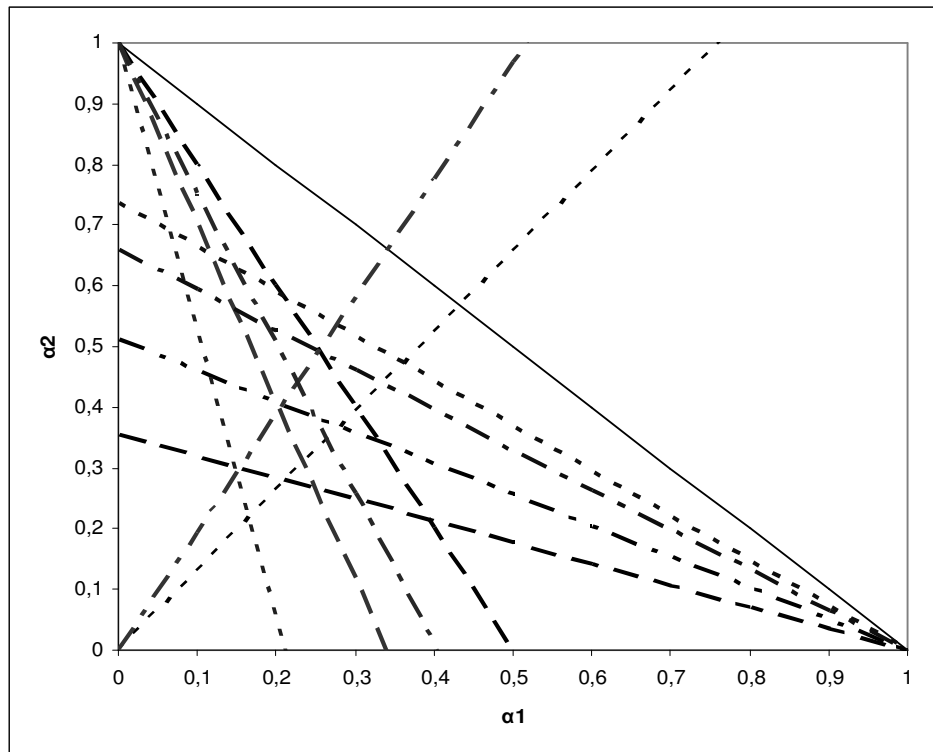


Figure 11. Graphical representation of the constraints

According to Figure 12, the set of neighbor solutions are as follows:

$$N_1 = \{4\}$$

$$N_2 = \{3,4\}$$

$$N_3 = \{2,4\}$$

$$N_4 = \{1,2,3\}$$

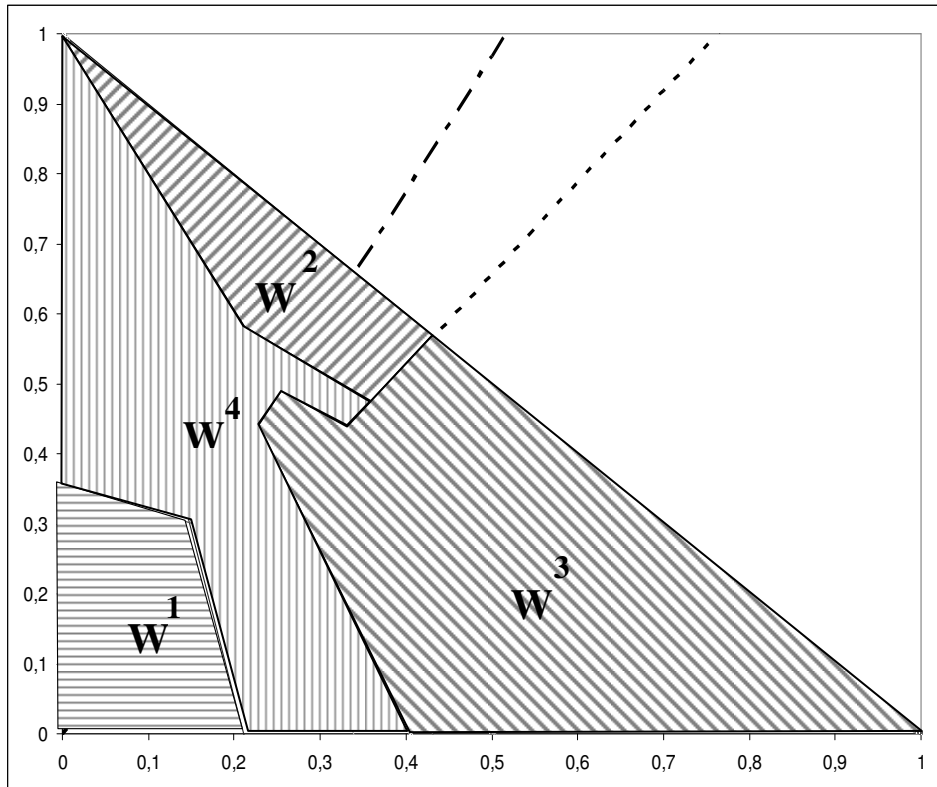


Figure 12. Partition of the weight space between alternatives

If we compare the neighborhood relation sets n_{ik} defined at the beginning, with the results obtained at the end, it is seen that all the assumptions are true except the neighborhood relation between z^1 and z^3 . We defined the relation as $n_{13}=\{2^+\}$. However both in criteria 1 and 3, z^4 exists between z^1 and z^3 in the weight space. For some reason the neighborhood relation of z^1 and z^3 seen in the objective space does not exist in the weight space.

Example 6: Let us have an approximation with three criteria to be maximized and seven nondominated alternatives, shown in Table 10. In this example we try to observe six (2p) neighbors of a solution. We organized the nondominated alternatives such that alternative 4 has neighborhood relation with the other six alternatives.

Table 10. Seven nondominated solutions for Example 6

z^i	1	2	3	4	5	6	7
z_1^i	1	3	5	2	0.6	1.5	2.5
z_2^i	0.2	2	0.5	1	0.7	0.1	3
z_3^i	6	3	4	5	6.5	6.2	3.5

Let $z^{**}=(5.1, 3.1, 6.6)$,

The neighborhood relations of the nondominated solutions in the objective space is shown in Table 11. While calculating W^k s in order to determine the neighborhood relations in the weight space, we need not compare the alternatives that are not neighbors (i.e. z^1 and z^2 , z^1 and z^7 , z^2 and z^5 , z^2 and z^6 , z^3 and z^6 , z^5 and z^7 , z^6 and z^7) due to Theorem 4.

Table 11. The neighborhood relations of the solutions

z^i	1	2	3	4	5	6
n_{i2}	{}					
n_{i3}	{2 ⁺ }	{1 ⁺ }				
n_{i4}	{3 ⁻ }	{2 ⁻ }	{3 ⁺ }			
n_{i5}	{1 ⁻ }	{}	{2 ⁺ }	{2 ⁻ }		
n_{i6}	{1 ⁺ ,2 ⁻ ,3 ⁺ }	{}	{}	{1 ⁻ }	{3 ⁻ }	
n_{i7}	{}	{1 ⁻ ,2 ⁺ ,3 ⁺ }	{3 ⁻ }	{1 ⁺ }	{}	{}

As can be seen from Table 11, the neighborhood relations of z^4 with the other alternatives are $n_{14}=\{3^-\}$, $n_{24}=\{2^-\}$, $n_{34}=\{3^+\}$, $n_{45}=\{2^-\}$, $n_{46}=\{1^-\}$, $n_{47}=\{1^+\}$.

Comparison of alternatives z^1 and z^3 :

$$u_\alpha(z^1) = u_\alpha(z^3)$$

$$\text{Max}\{4.1 \alpha_1, 2.9 \alpha_2, 0.6 \alpha_3\} = \text{Max}\{0.1 \alpha_1, 2.6 \alpha_2, 2.6 \alpha_3\}$$

$$\text{Max}\{4.1 \alpha_1, 2.9 \alpha_2, 0.6 \alpha_3\} = 2.6 \alpha_3$$

$$4.1 \alpha_1 = 2.6 \alpha_3 \text{ for } 4.1 \alpha_1 \geq 2.9 \alpha_2$$

$$2.9 \alpha_2 = 2.6 \alpha_3 \text{ for } 4.1 \alpha_1 < 2.9 \alpha_2$$

$$2.6 \alpha_2 = 2.6 - 6.7 \alpha_1 \text{ for } 4.1 \alpha_1 \geq 2.9 \alpha_2$$

$$5.5 \alpha_2 = 2.6 - 2.6 \alpha_1 \text{ for } 4.1 \alpha_1 < 2.9 \alpha_2$$

Comparison of alternatives z^1 and z^4 :

$$u_\alpha(z^1) = u_\alpha(z^4)$$

$$\text{Max}\{4.1 \alpha_1, 2.9 \alpha_2, 0.6 \alpha_3\} = \text{Max}\{3.1 \alpha_1, 2.1 \alpha_2, 1.6 \alpha_3\}$$

$$\text{Max}\{4.1 \alpha_1, 2.9 \alpha_2, 0.6 \alpha_3\} = 1.6 \alpha_3$$

$$4.1 \alpha_1 = 1.6 \alpha_3 \text{ for } 4.1 \alpha_1 \geq 2.9 \alpha_2$$

$$2.9 \alpha_2 = 1.6 \alpha_3 \text{ for } 4.1 \alpha_1 < 2.9 \alpha_2$$

$$1.6 \alpha_2 = 1.6 - 5.7 \alpha_1 \text{ for } 4.1 \alpha_1 \geq 2.9 \alpha_2$$

$$4.5 \alpha_2 = 1.6 - 1.6 \alpha_1 \text{ for } 4.1 \alpha_1 < 2.9 \alpha_2$$

The detailed calculations for the rest of the constraints are available in Appendix H.

The graphical representation of the constraints obtained above can be seen in Figure 13. Those constraints corresponds to all equations. When we take out the redundant constraints, the resulting division of weight space between alternatives is shown in Figure 14.

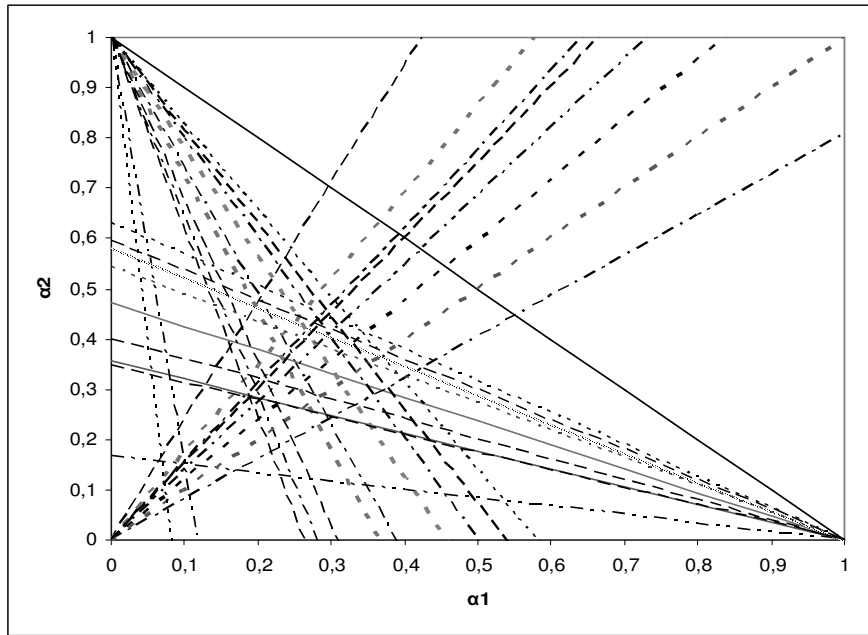


Figure 13. Graphical representation of the constraints

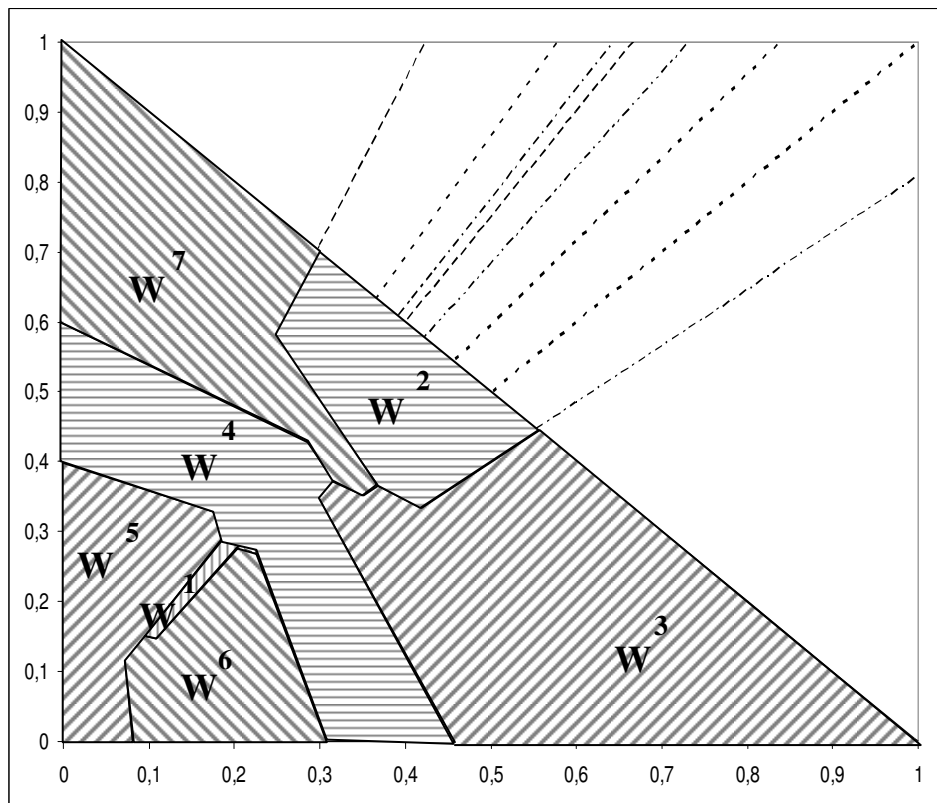


Figure 14. Partition of the weight space between alternatives

According to Figure 14, the set of neighbors of z^4 are as follows:

$$N_4 = \{1, 3, 5, 6, 7\}$$

If we compare the neighborhood relation sets n_{ik} defined at the beginning, with the results obtained at the end, it is seen that all the assumptions are true except the neighborhood relation of z^4 with z^2 . We defined the relation as $n_{24} = \{2\}$. However it is seen from the weight space graph that alternatives 3 and 7 exist between z^4 and z^2 . For some reason the neighborhood relation of z^2 and z^4 seen in the objective space does not exist in the weight space. Therefore, if the solutions are neighbors in the objective space, then they may be neighbors in the weight space (necessary condition). However, some neighbors in the objective space may not be neighbors in the weight space (not sufficient).

CHAPTER 5

CONCLUSIONS

In this study, we have focused on the performance measurement in MOCO problems. We developed a new performance measurement tool for the purpose of evaluating the approximate solutions quantitatively. Before developing the tools, we reviewed the literature on the performance measures proposed for MOCO problems with bi-criteria and more than two criteria. We developed a measure, which compares the approximations by calculating and comparing the resulting expected utility of the DM. Therefore our measure does not need the knowledge of the efficient frontier.

We divided our study into two parts: performance measurement in bi-criteria MOCO problems, and in more than two criteria MOCO problems. For bi-criteria case, we first analyzed the original IPF measure, which is proposed by Carlyle *et al.* (2003), and we used weighted Tchebycheff utility function characteristics, for the purpose of the improvement of the measure. We developed a new measure that captures the supported and unsupported solutions, unlike the original IPF measure that only captures supported solutions.

Once we constructed our measure for the bi-criteria case, we moved one step forward and generalized the measure for MOCO problems with more than two criteria. In this part of the study, we made use of weight space restriction method proposed by Dell and Karwan (1990), in order to identify the weight space for each non-dominated alternative in the approximation. Besides, when the number of criteria is three or more, the neighborhood becomes an important issue. Throughout the study the solutions closest in any criteria are assumed to be neighbors. We also analyzed the neighborhood relations on the weight space, which is also produced by our method. The results on the neighborhood relations, obtained from the examples, are used to

check the consistency of our neighborhood assumption. It is seen that our assumption is highly consistent with the results.

In this study we showed that in order to be neighbors in the weight space, it is necessary to be neighbors in the objective space, but this is not sufficient. That is, some neighbors in the objective space may turn out not to be neighbors in the weight space. As a further research, it may be useful to search for a necessary and sufficient condition to be neighbors in the weight space.

The measure can be used in practice for bi-criteria MOCO problems. However, when MOCO problems especially with four or more criteria are considered, the calculations needed for the measure will be more complex and time consuming. For these types of problems a simplified version of the measure can be developed.

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APPENDIX A

THE WEIGHT RANGE AND IPF CALCULATIONS FOR CHAPTER 3 EXAMPLE 2

$$\alpha_L^3 = \frac{1}{(0.486 - 0.228)} \left[\frac{1}{(0.486 - 0.228)} + \frac{1}{0.54} \right]^{-1} = 0.6766$$

$$\alpha_L^4 = \frac{1}{(0.514 - 0.228)} \left[\frac{1}{(0.514 - 0.228)} + \frac{1}{0.439} \right]^{-1} = 0.6055$$

$$\alpha_L^5 = \frac{1}{(0.629 - 0.228)} \left[\frac{1}{(0.629 - 0.228)} + \frac{1}{0.252} \right]^{-1} = 0.3859$$

$$\alpha_L^6 = \frac{1}{(0.714 - 0.228)} \left[\frac{1}{(0.714 - 0.228)} + \frac{1}{0.222} \right]^{-1} = 0.3136$$

$$\alpha_L^7 = \frac{1}{(1 - 0.228)} \left[\frac{1}{(1 - 0.228)} + \frac{1}{0.166} \right]^{-1} = 0.1769$$

$$\alpha_b^3 = \frac{1}{(0.457 - 0.228)} \left[\frac{1}{(0.457 - 0.228)} + \frac{1}{0.54} \right]^{-1} = 0.7022$$

$$\alpha_b^4 = \frac{1}{(0.486 - 0.228)} \left[\frac{1}{(0.486 - 0.228)} + \frac{1}{0.439} \right]^{-1} = 0.6298$$

$$\alpha_b^5 = \frac{1}{(0.514 - 0.228)} \left[\frac{1}{(0.514 - 0.228)} + \frac{1}{0.252} \right]^{-1} = 0.4684$$

$$\alpha_b^6 = \frac{1}{(0.629 - 0.228)} \left[\frac{1}{(0.629 - 0.228)} + \frac{1}{0.222} \right]^{-1} = 0.3563$$

$$\alpha_b^7 = \frac{1}{(0.714 - 0.228)} \left[\frac{1}{(0.714 - 0.228)} + \frac{1}{0.166} \right]^{-1} = 0.2546$$

$$\alpha_b^8 = \frac{1}{(1-0.228)} \left[\frac{1}{(1-0.228)} + \frac{1}{0.001} \right]^{-1} = 0.0012$$

$$\begin{aligned} & \int_{0.6766}^{0.7022} (1-\alpha)(0.54)d\alpha = (0.7022 - 0.6766)(0.54) - (0.54) \frac{\alpha^2}{2} \Big|_{0.6766}^{0.7022} \\ \text{IPF}\{z^3\} = & + \int_{0.7022}^{0.7101} \alpha(0.457 - 0.228)d\alpha = (0.229) \frac{\alpha^2}{2} \Big|_{0.7022}^{0.7101} \end{aligned}$$

$$= 0.0043 + 0.0013 = 0.0056$$

$$\begin{aligned} & \int_{0.6055}^{0.6298} (1-\alpha)(0.439)d\alpha = (0.6298 - 0.6055)(0.439) - (0.439) \frac{\alpha^2}{2} \Big|_{0.6055}^{0.6298} \\ \text{IPF}\{z^4\} = & + \int_{0.6298}^{0.6766} \alpha(0.486 - 0.228)d\alpha = (0.258) \frac{\alpha^2}{2} \Big|_{0.6298}^{0.6766} \end{aligned}$$

$$= 0.0040 + 0.0079 = 0.0119$$

$$\begin{aligned} & \int_{0.3859}^{0.4684} (1-\alpha)(0.252)d\alpha = (0.4684 - 0.3859)(0.252) - (0.252) \frac{\alpha^2}{2} \Big|_{0.3859}^{0.4684} \\ \text{IPF}\{z^5\} = & + \int_{0.4684}^{0.6055} \alpha(0.514 - 0.228)d\alpha = (0.286) \frac{\alpha^2}{2} \Big|_{0.4684}^{0.6055} \end{aligned}$$

$$= 0.0119 + 0.0210 = 0.0329$$

$$\begin{aligned} & \int_{0.3135}^{0.3563} (1-\alpha)(0.222)d\alpha = (0.3563 - 0.3135)(0.222) - (0.222) \frac{\alpha^2}{2} \Big|_{0.3135}^{0.3563} \\ \text{IPF}\{z^6\} = & + \int_{0.3563}^{0.3859} \alpha(0.629 - 0.228)d\alpha = (0.401) \frac{\alpha^2}{2} \Big|_{0.3563}^{0.3859} \end{aligned}$$

$$= 0.0063 + 0.0044 = 0.0107$$

$$\int_{0.1769}^{0.2546} (1-\alpha)(0.166)d\alpha = (0.2546 - 0.1769)(0.166) - (0.166)\frac{\alpha^2}{2}\Big|_{0.1769}^{0.2546}$$

$$\text{IPF}\{z^7\} = + \int_{0.2546}^{0.3135} \alpha(0.714 - 0.228)d\alpha = (0.486)\frac{\alpha^2}{2}\Big|_{0.2546}^{0.3135}$$

$$= 0.0101 + 0.0081 = 0.0182$$

$$\int_0^{0.0012} (1-\alpha)(0.001)d\alpha = (0.0012 - 0)(0.001) - (0.001)\frac{\alpha^2}{2}\Big|_0^{0.0012}$$

$$\text{IPF}\{z^8\} = + \int_{0.0012}^{0.1769} \alpha(1 - 0.228)d\alpha = (0.772)\frac{\alpha^2}{2}\Big|_{0.0012}^{0.1769}$$

$$= 0 + 0.0121 = 0.0121$$

APPENDIX B

THE WEIGHT RANGE AND IPF CALCULATIONS FOR CHAPTER 3 EXAMPLE 3 APPROXIMATION 1

$$\alpha_L^1 = \frac{1}{(0.371-0.228)} \left[\frac{1}{(0.371-0.228)} + \frac{1}{1} \right]^{-1} = 0.8749$$

$$\alpha_L^2 = \frac{1}{(0.514-0.228)} \left[\frac{1}{(0.514-0.228)} + \frac{1}{0.561} \right]^{-1} = 0.6623$$

$$\alpha_L^3 = \frac{1}{(1-0.228)} \left[\frac{1}{(1-0.228)} + \frac{1}{0.252} \right]^{-1} = 0.2461$$

$$\alpha_b^1 = \frac{1}{(0.229-0.228)} \left[\frac{1}{(0.229-0.228)} + \frac{1}{1} \right]^{-1} = 0.999$$

$$\alpha_b^2 = \frac{1}{(0.371-0.228)} \left[\frac{1}{(0.371-0.228)} + \frac{1}{0.561} \right]^{-1} = 0.7969$$

$$\alpha_b^3 = \frac{1}{(0.514-0.228)} \left[\frac{1}{(0.514-0.228)} + \frac{1}{0.252} \right]^{-1} = 0.4684$$

$$\alpha_b^4 = \frac{1}{(1-0.228)} \left[\frac{1}{(1-0.228)} + \frac{1}{0.001} \right]^{-1} = 0.0013$$

$$\int_{0.8749}^{0.999} (1-\alpha)(1)d\alpha = (0.999-0.8749) - \frac{\alpha^2}{2} \Big|_{0.8749}^{0.999}$$

$$\text{IPF}\{z'\} = + \int_{0.999}^1 \alpha(0.229-0.228)d\alpha = (0.001) \frac{\alpha^2}{2} \Big|_{0.999}^1$$

$$= 0.0078 + 0 = 0.0078$$

$$\int_{0.6623}^{0.7969} (1-\alpha)(0.561)d\alpha = (0.7969-0.6623)(0.561) - (0.561)\frac{\alpha^2}{2} \Big|_{0.6623}^{0.7969}$$

$$\text{IPF}\{z^2\} = + \int_{0.7969}^{0.8749} \alpha(0.371-0.228)d\alpha = (0.143)\frac{\alpha^2}{2} \Big|_{0.7969}^{0.8749}$$

$$= 0.0204 + 0.0033 = 0.0297$$

$$\int_{0.2461}^{0.4684} (1-\alpha)(0.252)d\alpha = (0.4684-0.2461)(0.252) - (0.252)\frac{\alpha^2}{2} \Big|_{0.2461}^{0.4684}$$

$$\text{IPF}\{z^3\} = + \int_{0.4684}^{0.6623} \alpha(0.514-0.228)d\alpha = (0.286)\frac{\alpha^2}{2} \Big|_{0.4684}^{0.6623}$$

$$= 0.036 + 0.0314 = 0.0674$$

$$\int_0^{0.0013} (1-\alpha)(0.001)d\alpha = (0.0013)(0.001) - (0.001)\frac{\alpha^2}{2} \Big|_0^{0.0013}$$

$$\text{IPF}\{z^4\} = + \int_{0.0013}^{0.2461} \alpha(1-0.228)d\alpha = (0.772)\frac{\alpha^2}{2} \Big|_{0.0013}^{0.2461}$$

$$= 0 + 0.0234 = 0.0234$$

APPENDIX C

THE WEIGHT RANGE AND IPF CALCULATIONS FOR CHAPTER 3 EXAMPLE 3 APPROXIMATION 2

$$\alpha_L^1 = \frac{1}{(0.401-0.228)} \left[\frac{1}{(0.401-0.228)} + \frac{1}{1.12} \right]^{-1} = 0.8662$$

$$\alpha_L^2 = \frac{1}{(0.544-0.228)} \left[\frac{1}{(0.544-0.228)} + \frac{1}{0.591} \right]^{-1} = 0.6516$$

$$\alpha_L^3 = \frac{1}{(1.03-0.228)} \left[\frac{1}{(1.03-0.228)} + \frac{1}{0.282} \right]^{-1} = 0.26$$

$$\alpha_b^1 = \frac{1}{(0.259-0.228)} \left[\frac{1}{(0.259-0.228)} + \frac{1}{1.12} \right]^{-1} = 0.9731$$

$$\alpha_b^2 = \frac{1}{(0.401-0.228)} \left[\frac{1}{(0.401-0.228)} + \frac{1}{0.591} \right]^{-1} = 0.7736$$

$$\alpha_b^3 = \frac{1}{(0.544-0.228)} \left[\frac{1}{(0.544-0.228)} + \frac{1}{0.282} \right]^{-1} = 0.4716$$

$$\alpha_b^4 = \frac{1}{(1.03-0.228)} \left[\frac{1}{(1.03-0.228)} + \frac{1}{0.031} \right]^{-1} = 0.0372$$

$$\int_{0.8662}^{0.9731} (1-\alpha)(1.12)d\alpha = (0.9731-0.8662)(1.12) - (1.12) \frac{\alpha^2}{2} \Big|_{0.8662}^{0.9731}$$

$$\text{IPF}\{z^j\} = + \int_{0.9731}^1 \alpha(0.259-0.228)d\alpha = (0.031) \frac{\alpha^2}{2} \Big|_{0.9731}^1$$

$$= 0.0096 + 0.0008 = 0.0104$$

$$\int_{0.6516}^{0.7736} (1-\alpha)(0.591)d\alpha = (0.7736-0.6516)(0.591) - (0.591)\frac{\alpha^2}{2} \Big|_{0.6516}^{0.7736}$$

$$\text{IPF}\{z^2\} = + \int_{0.7736}^{0.8662} \alpha(0.401-0.228)d\alpha = (0.173)\frac{\alpha^2}{2} \Big|_{0.7736}^{0.8662}$$

$$= 0.0207 + 0.0131 = 0.0339$$

$$\int_{0.26}^{0.4716} (1-\alpha)(0.282)d\alpha = (0.4716-0.26)(0.282) - (0.282)\frac{\alpha^2}{2} \Big|_{0.26}^{0.4716}$$

$$\text{IPF}\{z^3\} = + \int_{0.4716}^{0.6516} \alpha(0.544-0.228)d\alpha = (0.316)\frac{\alpha^2}{2} \Big|_{0.4716}^{0.6516}$$

$$= 0.0378 + 0.0319 = 0.0698$$

$$\int_0^{0.0372} (1-\alpha)(0.031)d\alpha = (0.0372)(0.031) - (0.031)\frac{\alpha^2}{2} \Big|_0^{0.0372}$$

$$\text{IPF}\{z^4\} = + \int_{0.0372}^{0.26} \alpha(1.03-0.228)d\alpha = (0.802)\frac{\alpha^2}{2} \Big|_{0.0372}^{0.26}$$

$$= 0.0011 + 0.0266 = 0.0277$$

APPENDIX D

PARTITION OF THE WEIGHT REGIONS FOR CHAPTER 4

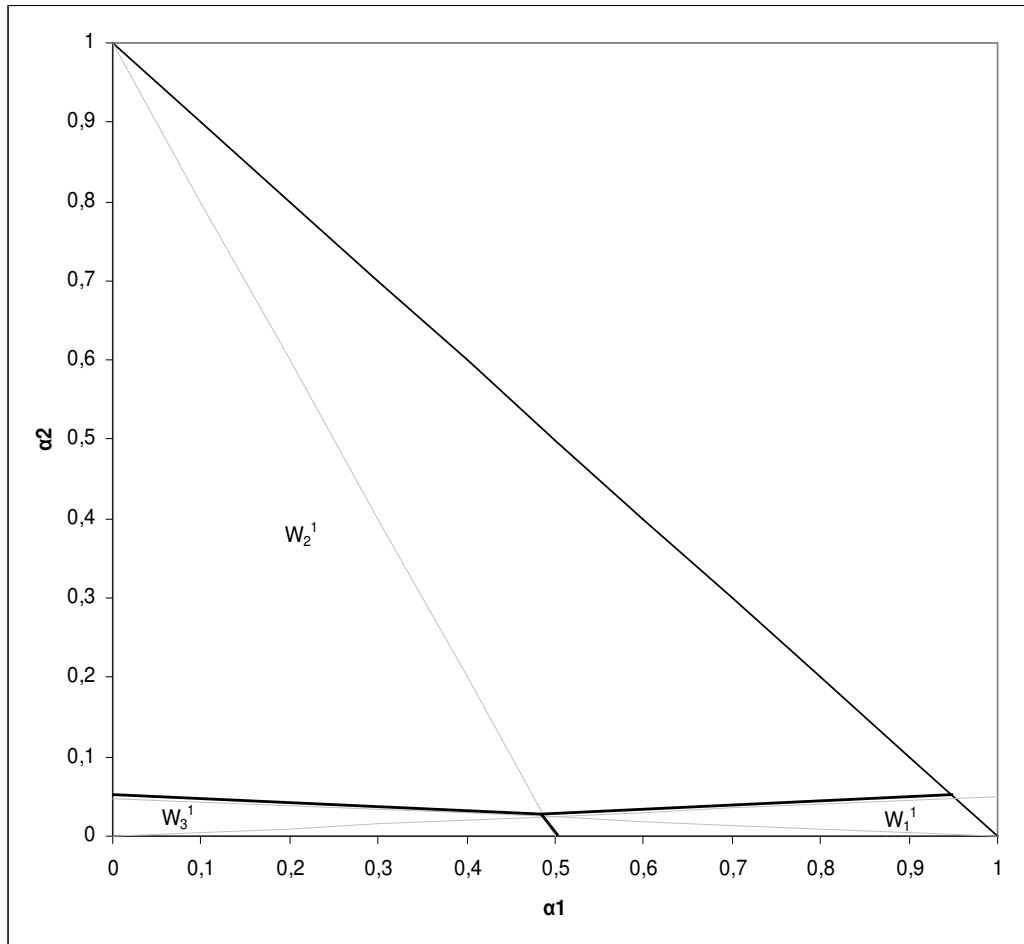


Figure 15. Partition of the weight space between criteria for z^1

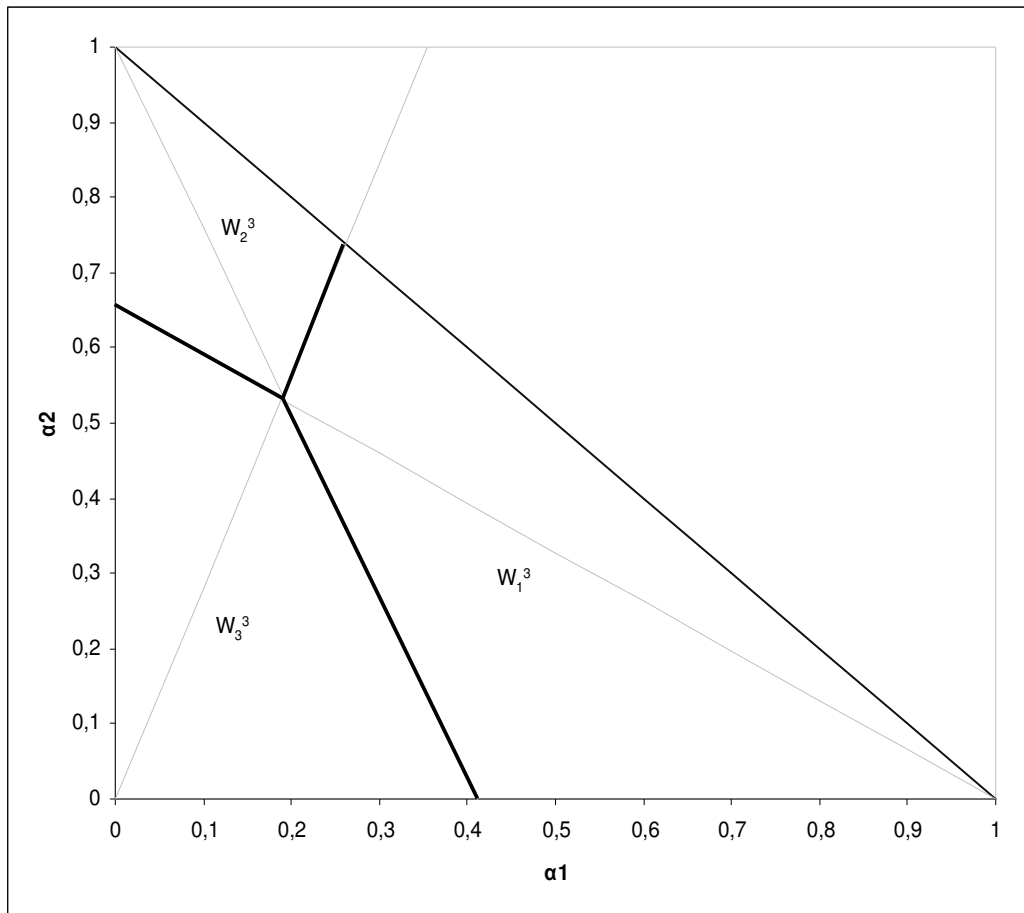


Figure 16. Partition of the weight space between criteria for z^3

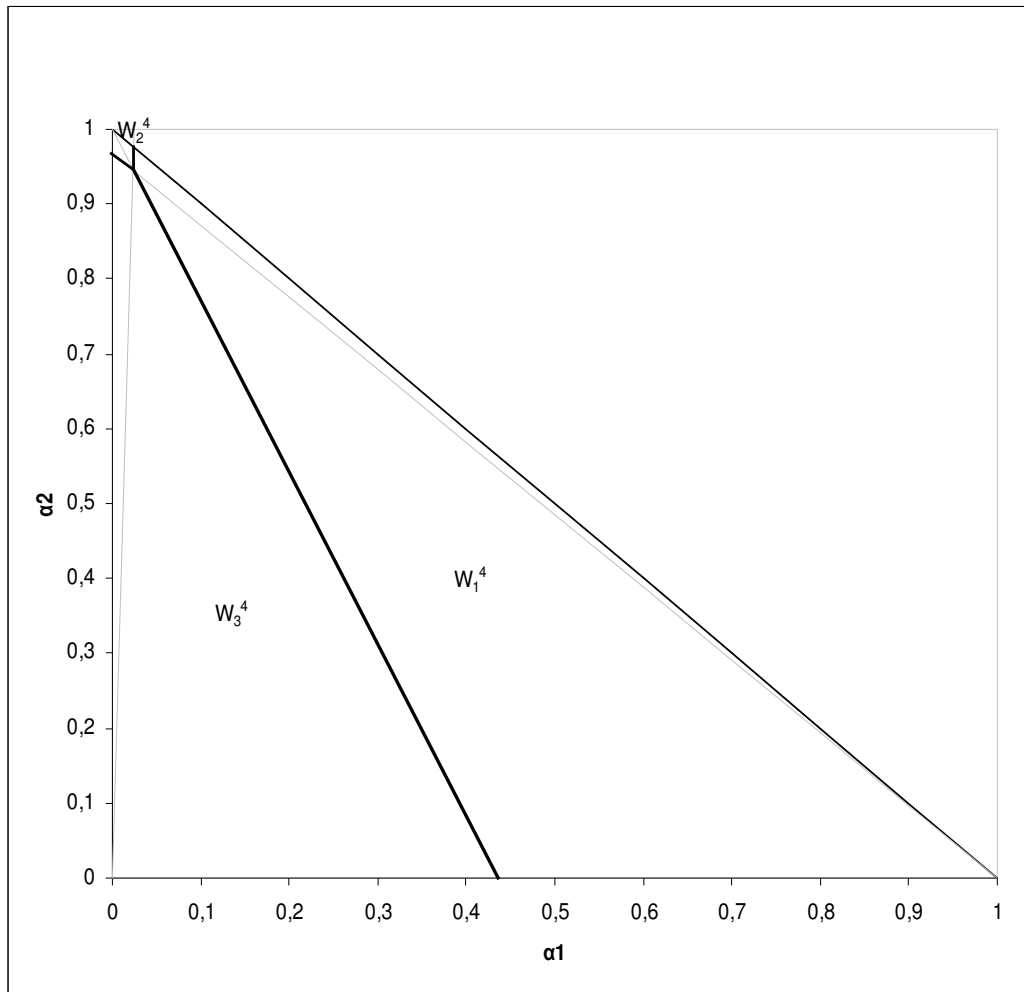


Figure 17. Partition of the weight space between criteria for z^4

APPENDIX E

PARTITION OF W^i REGIONS FOR CHAPTER 4

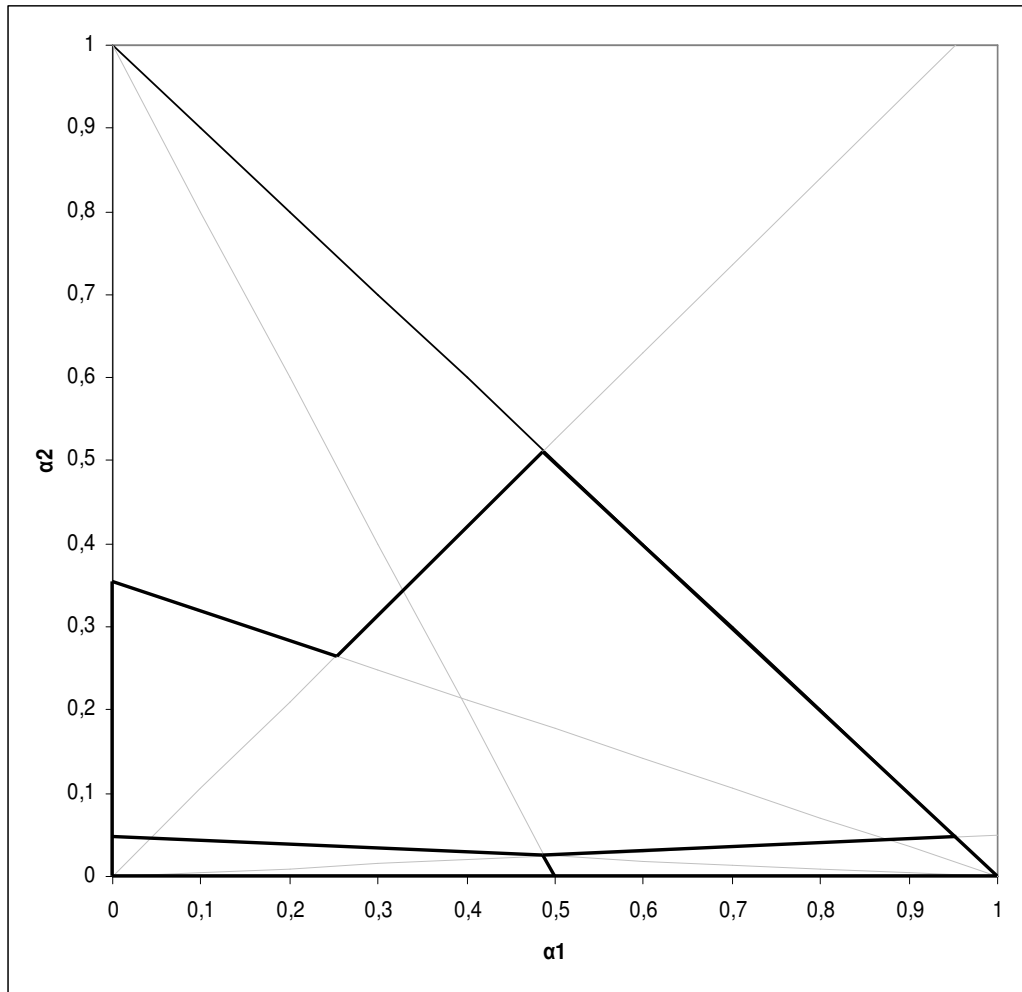


Figure 18. Partition of the W^1 region

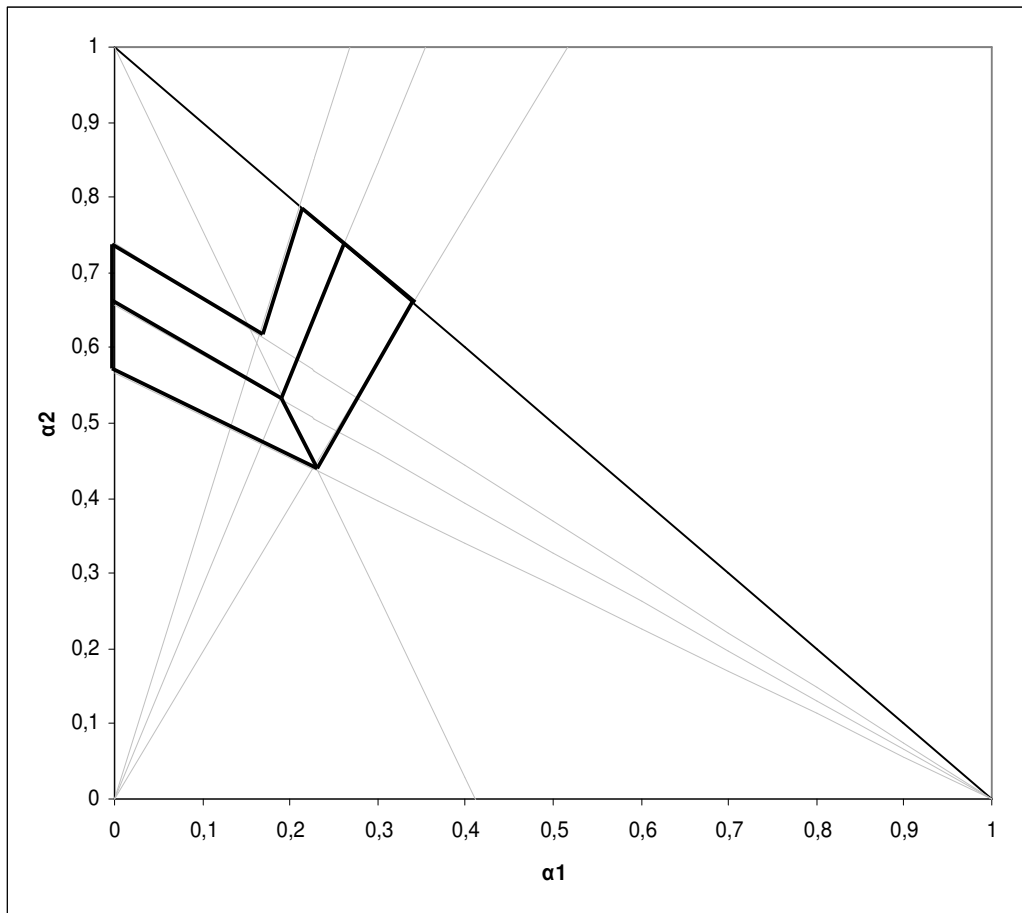


Figure 19. Partition of the W^3 region

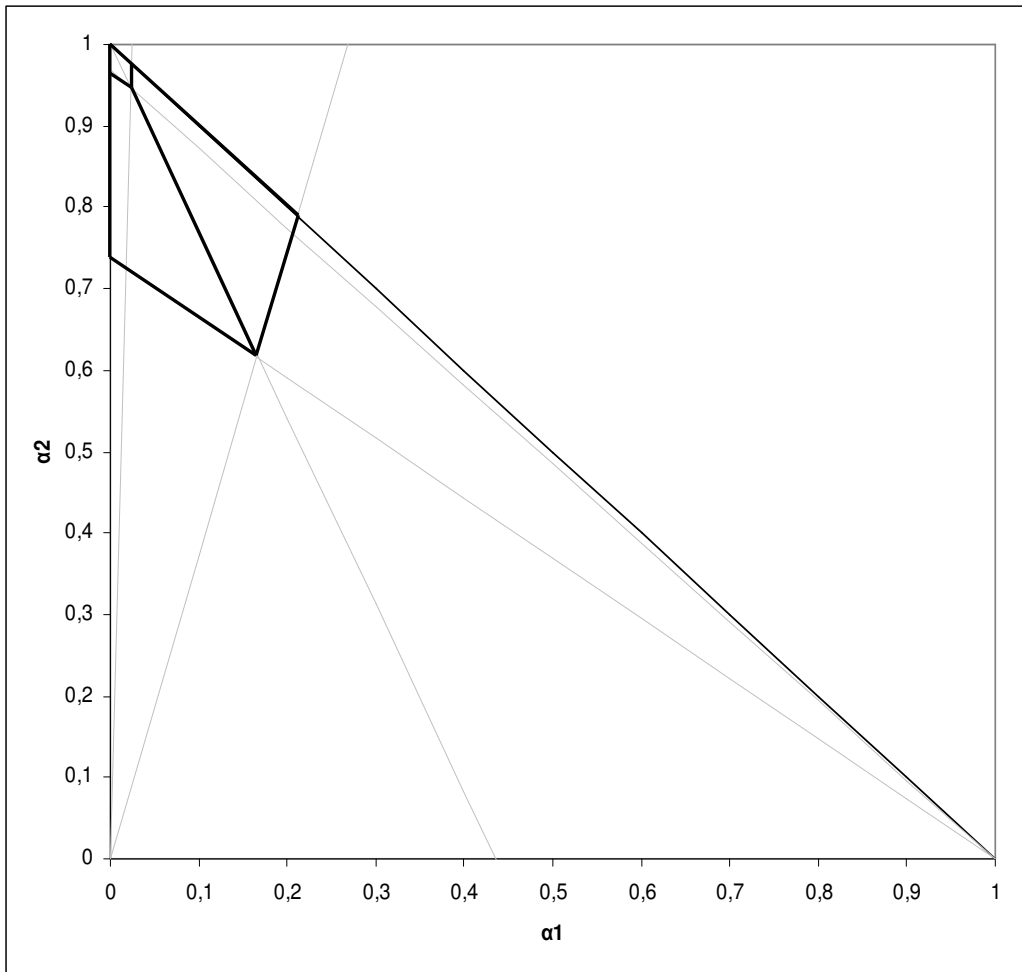


Figure 20. Partition of the W^4 region

APPENDIX F

PARTITION OF W^1 REGIONS FOR INTEGRAL CALCULATIONS FOR CHAPTER 4

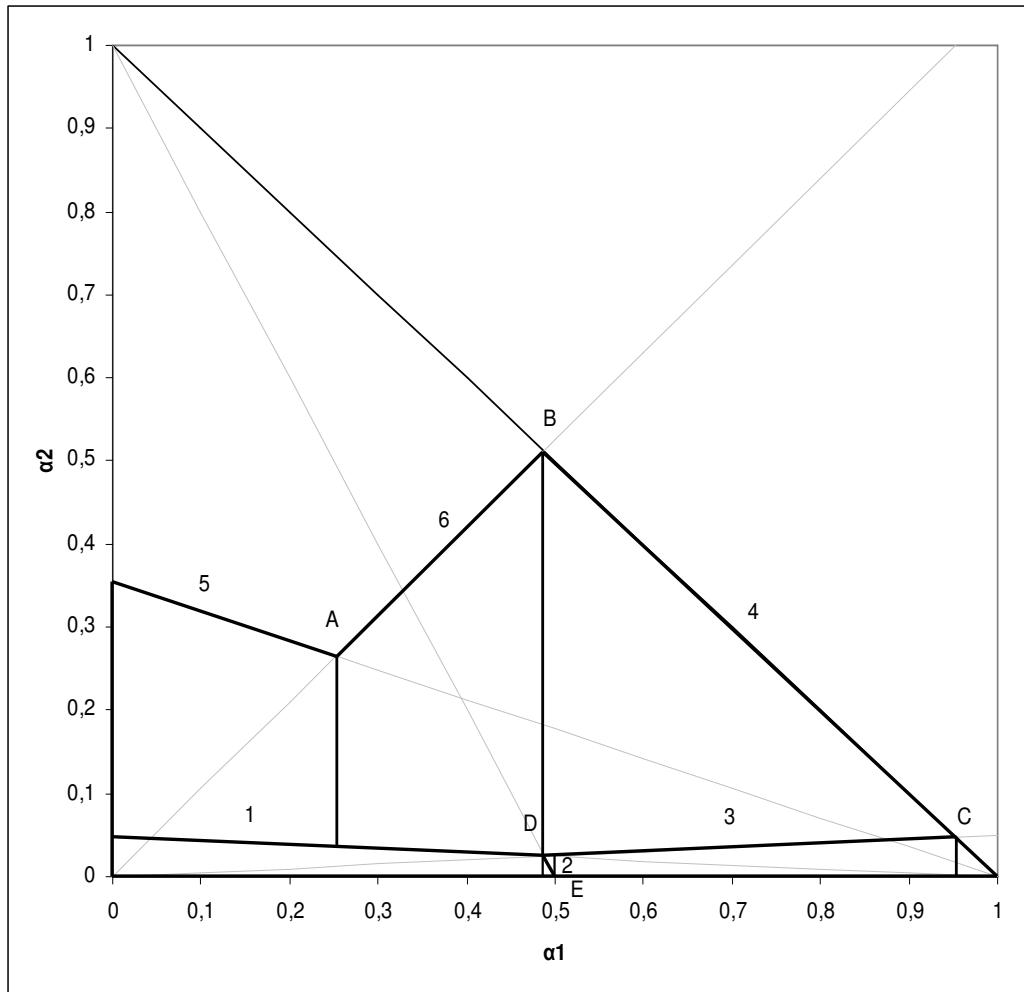


Figure 21. Partition of the W^1 region for the integral calculations

Table 12. Related points and equations for W^1

Points	Equations
A=(0.2525, 0.2652)	$\alpha_2 = \frac{1-\alpha_1}{21}$ (1)
B=(0.4878, 0.5121)	$\alpha_2 = 1-2\alpha_1$ (2)
C=(0.9523, 0.0476)	$\alpha_2 = 1-0.05\alpha_1$ (3)
D=(0.4878, 0.0243)	$\alpha_2 = 1-\alpha_1$ (4)
E=(0.5, 0)	$\alpha_2 = \frac{1.1(1-\alpha_1)}{3.1}$ (5)
	$\alpha_2 = \frac{2.1}{2}\alpha_1$ (6)

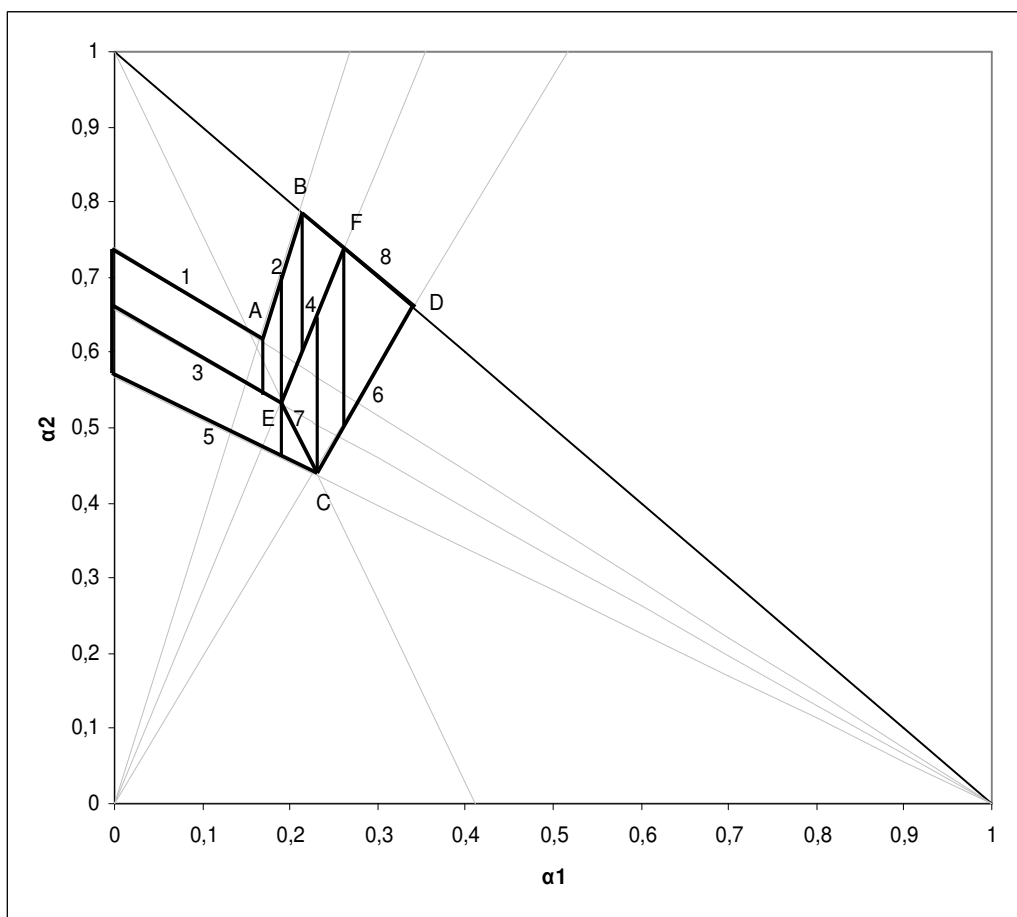


Figure 22. Partition of the W^3 region for the integral calculations

Table 13. Related points and equations for W^3

Points	Equations
A=(0.1652, 0.6161)	$\alpha_2 = \frac{3.1 - 3.1\alpha_1}{4.2}$ (1)
B=(0.2115, 0.7884)	$\alpha_2 = \frac{4.1}{1.1}\alpha_1$ (2)
C=(0.2265, 0.4389)	$\alpha_2 = \frac{2.1 - 2.1\alpha_1}{3.2}$ (3)
D=(0.3404, 0.6595)	$\alpha_2 = \frac{1.1\alpha_1}{3.1}$ (4)
E=(0.1888, 0.5322)	$\alpha_2 = \frac{2.1 - 2.1\alpha_1}{3.7}$ (5)
F=(0.2619, 0.7380)	$\alpha_2 = \frac{3.1}{1.6}\alpha_1$ (6)
	$\alpha_2 = \frac{2.1 - 5.1\alpha_1}{2.1}$ (7)
	$\alpha_2 = 1 - \alpha_1$ (8)

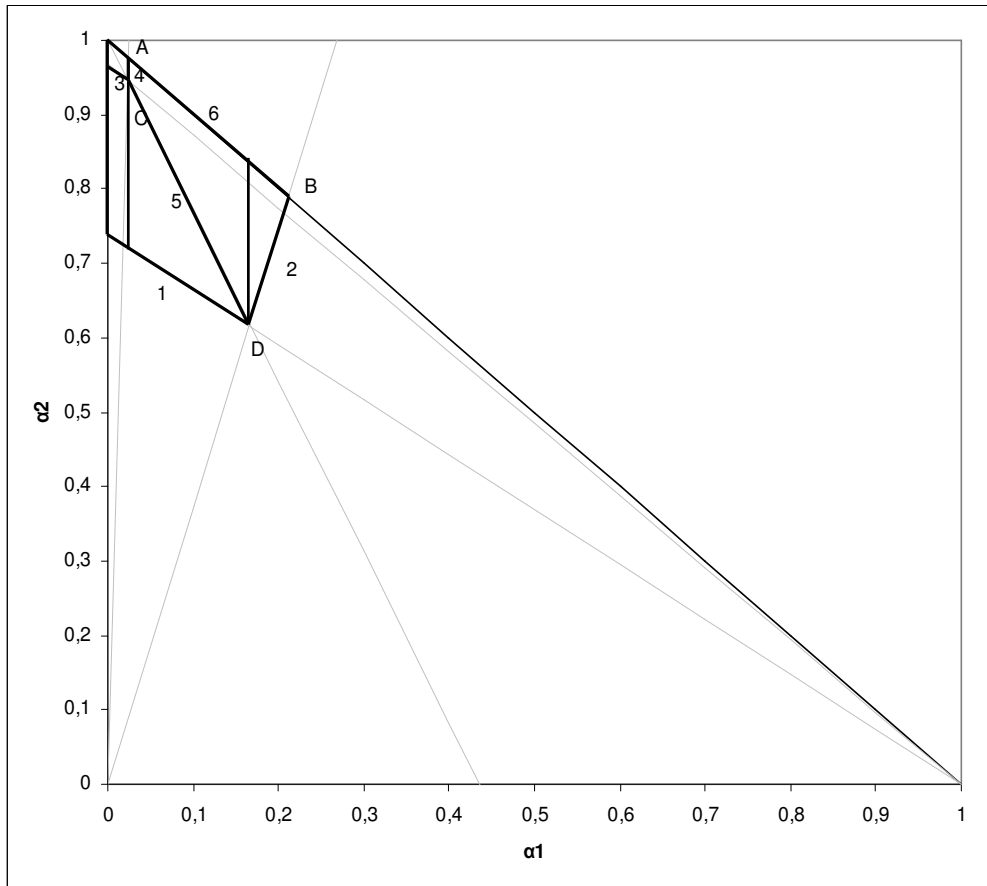


Figure 23. Partition of the W^4 region for the integral calculations

Table 14. Related points and equations for W^4

Points	Equations
A=(0.0238, 0.9761)	$\alpha_2 = \frac{1.1(1-\alpha_1)}{4.2}$ (1)
B=(0.2115, 0.7884)	$\alpha_2 = \frac{4.1}{1.1} \alpha_1$ (2)
C=(0.0231, 0.9463)	$\alpha_2 = \frac{3.1(1-\alpha_1)}{3.2}$ (3)
D=(0.1661, 0.6193)	$\alpha_2 = 41\alpha_1$ (4)
	$\alpha_2 = 1 - \frac{7.1\alpha_1}{3.1}$ (5)
	$\alpha_2 = 1 - \alpha_1$ (6)

APPENDIX G

IPF CALCULATIONS FOR CHAPTER 4 EXAMPLE 4

The Detailed IPF Calculations for z^1

First Criterion Portion of IPF(z^1):

$$\int_{0.4878}^{0.500} \int_{1-2\omega_1}^{0.1\omega_1/2} 0.1\omega_1 d\omega_2 d\omega_1 + \int_{0.500}^{0.9524} \int_0^{0.1\omega_1/2} 0.1\omega_1 d\omega_2 d\omega_1 + \int_{0.9524}^1 \int_0^{1-\omega_1} 0.1\omega_1 d\omega_2 d\omega_1$$

Second Criterion Portion of IPF(z^1):

$$\int_0^{0.2526} \int_{(1-\omega_1)/21}^{\frac{1.1}{3.1}(1-\omega_1)} 2\omega_2 d\omega_2 d\omega_1 + \int_{0.2526}^{0.4878} \int_{(1-\omega_1)/21}^{\frac{2.1}{2}\omega_1} 2\omega_2 d\omega_2 d\omega_1 + \int_{0.4878}^{0.9524} \int_{(1-\omega_1)/21}^{1-\omega_1} 2\omega_2 d\omega_2 d\omega_1$$

$$+ \int_{0.4878}^{0.9524} \int_{0.1\omega_1/2}^{\frac{2.1}{2}\omega_1} 2\omega_2 d\omega_2 d\omega_1$$

Third Criterion Portion of IPF(z^1):

$$\int_0^{0.4878} \int_0^{(1-\omega_1)/21} 0.1(1-\omega_1-\omega_2)d\omega_2 d\omega_1 + \int_{0.4878}^{0.500} \int_0^{1-2\omega_1} 0.1(1-\omega_1-\omega_2)d\omega_2 d\omega_1$$

F1(First partial of the first criterion portion):

$$\int_{0.4878}^{0.5000} 0.1\omega_1\omega_2 \Big|_{1-2\omega_1}^{0.1\omega_1/2} d\omega_1 = \int_{0.4878}^{0.5000} 0.1\omega_1 \left(\frac{0.1}{2}\omega_1 - 1 + 2\omega_1 \right) d\omega_1$$

$$= \int_{0.4878}^{0.5000} \left(\frac{0.1^2}{2}\omega_1^2 - 1\omega_1 + 0.2\omega_1^2 \right) d\omega_1$$

$$= \frac{0.205}{3}\omega_1^3 - \frac{0.1}{2}\omega_1^2 \Big|_{0.4878}^{0.5000} = 7.560e-6$$

F2(Second partial of the first criterion portion):

$$\int_{0.5000}^{0.9524} 0.1\omega_1\omega_2 \Big|_0^{0.1\omega_1/2} d\omega_1 = \int_{0.5000}^{0.9524} 0.1\omega_1 \left(\frac{0.1}{2}\omega_1 - 0 \right) d\omega_1 = \frac{0.1^2 \omega_1^3}{6} \Big|_{0.5000}^{0.9524} = 0.001231$$

F3(Third partial of the first criterion portion):

$$\begin{aligned} \int_{0.9524}^1 0.1\omega_1\omega_2 \Big|_0^{1-\omega_1} d\omega_1 &= \int_{0.9524}^1 0.1\omega_1(1-\omega_1-0) d\omega_1 \\ &= \frac{0.1}{2}\omega_1^2 - \frac{0.1}{3}\omega_1^3 \Big|_{0.9524}^1 = 0.0001098 \end{aligned}$$

S1(First partial of the second criterion portion):

$$\begin{aligned} \int_0^{0.2526} \frac{2\omega_2^2}{2} \Big|_{(1-\omega_1)/21}^{\frac{1.1}{3.1}(1-\omega_1)} d\omega_1 &= \int_0^{0.2526} \left(\frac{11}{31} \right)^2 (1-\omega_1)^2 - \frac{(1-\omega_1)^2}{21^2} d\omega_1 \\ &= \int_0^{0.2526} 0.1236(1-2\omega_1+\omega_1^2) d\omega_1 \\ &= 0.1236 \left(\omega_1 - \omega_1^2 + \frac{\omega_1^3}{3} \right) \Big|_0^{0.2526} = 0.02401 \end{aligned}$$

S2(Second partial of the second criterion portion):

$$\begin{aligned} \int_{0.2526}^{0.4878} \omega_2^2 \Big|_{(1-\omega_1)/21}^{\frac{2.1}{2}\omega_1} d\omega_1 &= \int_{0.2526}^{0.4878} \left[\left(\frac{2.1}{2} \right)^2 \omega_1^2 - \frac{(1-\omega_1)^2}{21^2} \right] d\omega_1 \\ &= \int_{0.2526}^{0.4878} \left[\left(\frac{2.1}{2} \right)^2 \omega_1^2 - \frac{(1-2\omega_1+\omega_1)^2}{21^2} \right] d\omega_1 \\ &= \int_{0.2526}^{0.4878} \left[\left(\frac{2.1}{2} \right)^2 \omega_1^2 - \frac{1}{21^2} + \frac{2\omega_1}{21^2} - \frac{\omega_1^2}{21^2} \right] d\omega_1 \\ &= \int_{0.2526}^{0.4878} \left[\left(\frac{2.1}{2} \right)^2 \omega_1^2 - \frac{1}{21^2} + \frac{2\omega_1}{21^2} - \frac{\omega_1^2}{21^2} \right] d\omega_1 \\ &= \frac{2.1^2}{12} \omega_1^3 - \frac{1}{21^2} \omega_1 + \frac{2\omega_1}{2 \times 21^2} - \frac{\omega_1^3}{3 \times 21^2} \Big|_{0.2526}^{0.4878} \\ &= 0.03652 \end{aligned}$$

S3(Third partial of the second criterion portion):

$$\begin{aligned}
 \int_{0.4878}^{0.4878} \omega_2^2 \Big|_{(1-\omega_1)/21}^{1-\omega_1} d\omega_1 &= \int_{0.4878}^{0.4878} (1-\omega_1)^2 - \frac{(1-\omega_1)^2}{21^2} d\omega_1 \\
 &= 0.9977 \left(\omega_1 - \omega_1^2 + \frac{\omega_1^3}{3} \right) \Big|_{0.4878}^{0.4878} \\
 &= 0
 \end{aligned}$$

S4(Fourth partial of the second criterion portion):

$$\begin{aligned}
 \int_{0.4878}^{0.9524} \omega_2^2 \Big|_{0.1\omega_1/2}^{\frac{2.1}{2}\omega} d\omega_1 &= \int_{0.4878}^{0.9524} (1-\omega_1)^2 - \frac{0.1^2}{4} \omega_1^2 d\omega_1 \\
 &= \int_{0.4878}^{0.9524} 1 - 2\omega_1 + \omega_1^2 - \frac{0.01}{4} \omega_1^2 d\omega_1 \\
 &= \omega_1 - \omega_1^2 + \frac{\omega_1^3}{3} \Big|_{0.4878}^{0.9524} = 0.04413
 \end{aligned}$$

T1(First partial of the third criterion portion):

$$\begin{aligned}
 \int_0^{0.4878} 0.1 \left(\omega_2 - \omega_1 \omega_2 - \frac{\omega_2^2}{2} \right) \Big|_0^{(1-\omega_1)/21} d\omega_1 &= \int_0^{0.4878} 0.1 \left(\frac{1-\omega_1}{21} - \omega_1 \frac{1-\omega_1}{21} - \frac{\left(\frac{1-\omega_1}{21} \right)^2}{2} \right) d\omega_1 \\
 &= \int_0^{0.4878} 0.1 \left[\frac{1-\omega_1}{21} (1-\omega_1) - \frac{(1-\omega_1)^2}{2 \times 21^2} \right] d\omega_1 \\
 &= \int_0^{0.4878} 0.1 \left[(1-\omega_1)^2 \left(\frac{1}{21} - \frac{1}{2 \times 21^2} \right) \right] d\omega_1 \\
 &= 0.1 \times 0.04648 \times \left(\omega_1 - \omega_1^2 + \frac{\omega_1^3}{3} \right) \Big|_0^{0.4878} = 0.001341
 \end{aligned}$$

T2(Second partial of the third criterion portion):

$$\begin{aligned}
& \int_{0.4878}^{0.500} 0.1 \left[\omega_2 (1 - \omega_1) - \frac{\omega_2^2}{2} \right]_0^{1-2\omega_1} d\omega_1 \\
&= \int_{0.4878}^{0.500} 0.1 \left[(1 - 2\omega_1)(1 - \omega_1) - \frac{(1 - 2\omega_1)^2}{2} \right] d\omega_1 \\
&= \int_{0.4878}^{0.500} 0.1 \left(1 - \omega_1 - 2\omega_1 + 2\omega_1^2 - \frac{1 - 4\omega_1 + 4\omega_1^2}{2} \right) d\omega_1 \\
&= \int_{0.4878}^{0.500} 0.1 (0.5 - 3\omega_1 + 2\omega_1 + 2\omega_1^2 - 2\omega_1^2) d\omega_1 = \int_{0.4878}^{0.500} 0.1 (0.5 - \omega_1) d\omega_1 \\
&= 0.1 \left(0.5\omega_1 - \frac{\omega_1^2}{2} \right) \Big|_{0.4878}^{0.5000} = 0.05 (\omega_1 - \omega_1^2) \Big|_{0.4878}^{0.5000} = 7.4336e - 6
\end{aligned}$$

The Detailed IPF Calculations for z^2

First Criterion Portion of IPF(z^2):

$$\int_{0.266}^{0.2526} \int_{\frac{1.1-3.2\omega_1}{1.1}}^{\frac{2.1\omega_1}{1.6}} 2.1\omega_1 d\omega_2 d\omega_1 + \int_{0.2526}^{0.4324} \int_{\frac{2.1\omega_1}{2}}^{\frac{2.1\omega_1}{1.6}} 2.1\omega_1 d\omega_2 d\omega_1 + \int_{0.4324}^{0.488} \int_{\frac{2.1\omega_1}{2}}^{1-\omega_1} 2.1\omega_1 d\omega_2 d\omega_1$$

Second Criterion Portion of IPF(z^2):

$$\begin{aligned}
& \int_0^{0.227} \int_{\frac{1.1-1.1\omega_1}{2.7}}^{\frac{2.1-2.1\omega_1}{3.7}} 1.6\omega_2 d\omega_2 d\omega_1 + \int_{0.227}^{0.2369} \int_{\frac{1.1-1.1\omega_1}{2.7}}^{\frac{3.1\omega_1}{1.6}} 1.6\omega_2 d\omega_2 d\omega_1 \\
&+ \int_{0.2368}^{0.340} \int_{\frac{2.1\omega_1}{1.6}}^{\frac{3.1\omega_1}{1.6}} 1.6\omega_2 d\omega_2 d\omega_1 + \int_{0.340}^{0.4324} \int_{\frac{2.1\omega_1}{1.6}}^{1-\omega_1} 1.6\omega_2 d\omega_2 d\omega_1
\end{aligned}$$

Third Criterion Portion of IPF(z^2):

$$\int_0^{0.2369} \int_{\frac{1.1-1.1\omega_1}{3.1}}^{\frac{1.1-1.1\omega_1}{2.7}} 1.1(1-\omega_1-\omega_2) d\omega_2 d\omega_1 + \int_{0.2369}^{0.2526} \int_{\frac{1.1-1.1\omega_1}{3.1}}^{\frac{1.1-3.2\omega_1}{1.1}} 1.1(1-\omega_1-\omega_2) d\omega_2 d\omega_1$$

$$\sum (F_1, F_2, F_3, S_1, S_2, S_3, S_4, T_1, T_2) = 0.082$$

F1(First partial of the first criterion portion):

$$\begin{aligned} & \int_{0.2369}^{0.2526} 2.1\omega_1 \omega_2 \Big|_{1.1-3.2\omega_1/1.1}^{2.1\omega_1/1.6} d\omega_1 \\ &= \int_{0.2369}^{0.2526} 2.1\omega_1 \left(\frac{2.1\omega_1}{1.6} - \frac{1.1-3.2\omega_2}{1.1} \right) d\omega_1 \\ &= \int_{0.2369}^{0.2526} \left(\frac{2.1^2}{1.6} \omega_1^2 - 2.1\omega_1 + \frac{2.1 \times 3.2}{1.1} \omega_1^2 \right) d\omega_1 \\ &= \int_{0.2369}^{0.2526} (2.7563\omega_1^2 - 2.1\omega_1 + 6.109\omega_1^2) d\omega_1 \\ &= \frac{2.75}{3} \omega_1^3 - \frac{2.1}{2} \omega_1^2 + \frac{6.109}{3} \omega_1^3 \Big|_{0.2369}^{0.2526} = 0.002705 \end{aligned}$$

F2(Second partial of the first criterion portion):

$$\begin{aligned} & \int_{0.2526}^{0.4324} 2.1\omega_1 \omega_2 \Big|_{2.1\omega_1/2}^{2.1\omega_1/1.6} d\omega_1 \\ &= \int_{0.2526}^{0.4324} 2.1\omega_1 \left(\frac{2.1\omega_1}{1.6} - \frac{2.1\omega_1}{2} \right) d\omega_1 \\ &= \int_{0.2526}^{0.4324} 0.55125\omega_1^2 d\omega_1 \\ &= \frac{0.55125}{3} \omega_1^3 \Big|_{0.2526}^{0.4324} = 0.01190 \end{aligned}$$

F3(Third partial of the first criterion portion):

$$\begin{aligned}
& \int_{0.4324}^{0.488} 2.1 \omega_1 \omega_2 \Big|_{2.1\omega_1/2}^{1-\omega_1} d\omega_1 \\
&= \int_{0.4324}^{0.488} 2.1 \omega_1 \left(1 - \omega_1 - \frac{2.1}{2} \omega_1\right) d\omega_1 \\
&= \int_{0.4324}^{0.488} \left(2.1 \omega_1 - 2.1 \times 2.05 \omega_1^2\right) d\omega_1 \\
&= \frac{2.1}{2} \omega_1^2 - \frac{2.1 \times 2.05}{3} \omega_1^3 \Big|_{0.432}^{0.488} = 0.002976
\end{aligned}$$

S1(First partial of the second criterion portion):

$$\begin{aligned}
& \int_0^{0.227} \frac{1.6}{2} \omega_2^2 \Big|_{1.1-1.1\omega_1/2.7}^{2.1-2.1\omega_1/3.7} \\
&= \int_0^{0.227} \frac{1.6}{2} \left(\left(\frac{2.1-2.1\omega_1}{3.7} \right)^2 - \left(\frac{1.1-1.1\omega_1}{2.7} \right)^2 \right) d\omega_1 \\
&= \int_0^{0.227} \frac{1.6}{2} \left(\frac{2.1^2 - 2(2.1)^2 \omega_1}{3.7^2} - \frac{1.1^2}{2.7^2} + \frac{2(1.1)^2 \omega_1}{2.7^2} - \frac{1.1^2 \omega_1^2}{2.7^2} \right) d\omega_1 \\
&= \int_0^{0.227} \frac{1.6}{2} \left(0.156 - 0.312\omega_1 + 0.156\omega_1^2 \right) d\omega_1 \\
&= \int_0^{0.227} \left(0.125 - 0.250\omega_1 + 0.125\omega_1^2 \right) d\omega_1 \\
&= 0.125\omega_1 - \frac{0.250}{2} \omega_1^2 + \frac{0.125}{3} \omega_1^3 \Big|_0^{0.227} \\
&= 0.028 - 0.006 + 0.0015 \\
&= 0.02338
\end{aligned}$$

S2(Second partial of the second criterion portion):

$$\int_{0.227}^{0.2369} \frac{1.6}{2} \omega_2^2 \Big|_{1.1-1.1\omega_1/2.7}^{3.1\omega_1/1.6}$$

$$\begin{aligned}
&= \int_{0.227}^{0.2369} \frac{1.6}{2} \left(\frac{3.1^2}{1.6^2} \omega_1^2 - \left(\frac{1.1 - 1.1\omega_1}{2.7} \right)^2 \right) d\omega_1 \\
&= \int_{0.227}^{0.2369} 0.8 \left(3.754\omega_1^2 - 0.166 + 0.332\omega_1 - 0.166\omega_1^2 \right) d\omega_1 \\
&= \int_{0.227}^{0.2369} \left(2.87\omega_1^2 - 0.133 + 0.266\omega_1 \right) d\omega_1 \\
&= 2.87\omega_1^3 - 0.133\omega_1 + \frac{0.266}{2}\omega_1^2 \Big|_{0.227}^{0.237} \\
&= 0.0015 - 0.0013 + 0.0003 \\
&= 0.00052
\end{aligned}$$

S3(Third partial of the second criterion portion):

$$\begin{aligned}
&\int_{0.2369}^{0.340} \frac{1.6}{2} \omega_2^2 \Big|_{\frac{2.1}{1.6}\omega_1}^{\frac{3.1}{1.6}\omega_1} \\
&= \int_{0.2369}^{0.340} 0.8 \left(\left(\frac{3.1}{1.6} \omega_1 \right)^2 - \left(\frac{2.1}{1.6} \omega_1 \right)^2 \right) d\omega_1 \\
&= \int_{0.2369}^{0.340} 1.625\omega_1^2 d\omega_1 \\
&= \frac{1.625}{3} \omega_1^3 \Big|_{0.236}^{0.340} \\
&= 0.02129
\end{aligned}$$

S4(Fourth partial of the second criterion portion):

$$\begin{aligned}
&\int_{0.340}^{0.4324} \frac{1.6}{2} \omega_2^2 \Big|_{\frac{2.1}{1.6}\omega_1}^{1-\omega_1} \\
&= \int_{0.340}^{0.4324} 0.8 \left(1 - 2\omega_1 + \omega_1^2 - \left(\frac{2.1}{1.6} \omega_1 \right)^2 \right) d\omega_1
\end{aligned}$$

$$\begin{aligned}
&= \int_{0.340}^{0.4324} (0.8 - 1.6\omega_1 + 0.578\omega_1^2) d\omega_1 \\
&= 0.8\omega_1 - \frac{1.6}{2}\omega_1^2 - \frac{0.578}{3}\omega_1^3 \Big|_{0.340}^{0.432} \\
&= 0.074 - 0.057 - 0.008 \\
&= 0.008819
\end{aligned}$$

T1(First partial of the third criterion portion):

$$\begin{aligned}
&1.1 \int_0^{0.2369} \left(\omega_2 - \omega_1\omega_2 - \frac{\omega_2^2}{2} \right) \Big|_{\frac{1.1-1.1\omega_1}{3.1}}^{\frac{1.1-1.1\omega_1}{2.7}} \\
&= 1.1 \int_0^{0.2369} \left(\frac{1.1-1.1\omega_1}{2.7} - \frac{1.1-1.1\omega_1}{3.1} \right) - \left(\frac{1.1-1.1\omega_1^2}{2.7} - \frac{1.1\omega_1-1.1\omega_1^2}{3.1} \right) - \\
&\quad \left[\left(\frac{1.1^2 - 2(1.1)^2\omega_1 + 1.1^2\omega_1^2}{2(2.7)^2} \right) - \left(\frac{1.1^2 - 2(1.1)^2\omega_1 + 1.1^2\omega_1^2}{2(3.1)^2} \right) \right] \\
&= 1.1 \int_0^{0.2369} \left(\frac{1.1}{2.7} - \frac{1.1}{3.1} - \frac{1}{2} \left(\frac{1.1}{2.7} \right)^2 + \frac{1}{2} \left(\frac{1.1}{3.1} \right)^2 \right. \\
&\quad \left. - \left(\frac{1.1}{2.7} - \frac{1.1}{3.1} - \frac{1.1}{2.7} + \frac{1.1}{3.1} + \left(\frac{1.1}{2.7} \right)^2 - \left(\frac{1.1}{3.1} \right)^2 \right) \omega_1 \right. \\
&\quad \left. + \left(\frac{1.1}{2.7} - \frac{1.1}{3.1} - \left(\frac{1.1}{2.7} \right)^2 \frac{1}{2} + \left(\frac{1.1}{3.1} \right)^2 \frac{1}{2} \right) \omega_1^2 \right) \\
&= 1.1 \int_0^{0.2369} (0.0263 + 0.4\omega_1 + 0.0263\omega_1^2) d\omega_1 \\
&= 1.1 \left(0.0263\omega_1 + \frac{0.04}{2}\omega_1^2 + \frac{0.0263}{3}\omega_1^3 \right) \Big|_0^{0.237} \\
&= 0.009706
\end{aligned}$$

T2(Second partial of the third criterion portion):

$$\begin{aligned}
& 1.1 \int_{0.2369}^{0.2526} \left(\omega_2 - \omega_1 \omega_2 - \frac{\omega_2^2}{2} \right) \Big|_{\frac{1.1-3.2\omega_1}{3.1}}^{\frac{1.1-3.2\omega_1}{1.1}} \\
&= 1.1 \int_{0.2369}^{0.2526} \left(\frac{1.1-3.2\omega_1}{1.1} - \frac{1.1-3.2\omega_1}{3.1} \right) - \left[\left(\frac{1.1\omega_1-3.2\omega_1^2}{1.1} - 1.1\omega_1 - \frac{1.1\omega_1^2}{3.1} \right) \right. \\
&\quad \left. \left[\left(\frac{1.1^2-2(1.1)(3.2)\omega_1+3.2^2\omega_1^2}{2(1.1)^2} \right) - \left(\frac{1.1^2-2(1.1)^2\omega_1+1.1^2\omega_1^2}{2(3.1)^2} \right) \right] \right. \\
&= 1.1 \int_{0.2369}^{0.2526} \left[\left(1 - \frac{1.1}{3.1} - \frac{1}{2} + \frac{1}{2} \left(\frac{1.1}{3.1} \right)^2 \right) - \left(\frac{3.2}{1.1} - \frac{1.1}{3.1} + 1 - 1.1 - \frac{3.2}{1.1} + \left(\frac{1.1}{3.1} \right)^2 \right) \right] \omega_1 \\
&\quad + \left[\left(\frac{3.2}{1.1} - \frac{1.1}{3.1} - \frac{1}{2} \left(\frac{3.2}{1.1} \right)^2 + \frac{1}{2} \left(\frac{1.1}{3.1} \right)^2 \right) \omega_1^2 \right] d\omega_1 \\
&= 1.1 \int_{0.2369}^{0.2526} (0.208 + 0.329\omega_1 - 1.61\omega_1^2) d\omega_1 \\
&= 1.1 \left(0.208\omega_1 + \frac{0.329}{2}\omega_1^2 - \frac{1.61}{3}\omega_1^3 \right) \Big|_{0.237}^{0.252} \\
&= 0.003315
\end{aligned}$$

The Detailed IPF Calculations for z^3

First Criterion Portion of IPF(z^3):

$$\int_{\frac{0.1889}{5.1}}^{\frac{0.2266}{1.1}} \int_{\frac{2.1-2.1\omega_1}{5.1}}^{\frac{3.1\omega_1}{1.1}} 3.1\omega_1 d\omega_2 d\omega_1 + \int_{\frac{0.2266}{1.6}}^{\frac{0.2619}{1.1}} \int_{\frac{3.1\omega_1}{1.6}}^{\frac{3.1\omega_1}{1.1}} 3.1\omega_1 d\omega_2 d\omega_1 + \int_{\frac{0.2619}{1.6}}^{\frac{0.3404}{1.6}} \int_{\frac{3.1\omega_1}{1.6}}^{1-\omega_1} 3.1\omega_1 d\omega_2 d\omega_1$$

Second Criterion Portion of IPF(z^3):

$$\begin{aligned} & \int_0^{0.1653} \int_{\frac{2.1-2.1\omega_1}{3.2}}^{\frac{3.1-3.1\omega_1}{4.2}} 1.1\omega_2 d\omega_2 d\omega_1 + \int_0^{0.5323} \int_{\frac{2.1-2.1\omega_1}{3.2}}^{\frac{4.1\omega_1}{1.1}} 1.1\omega_2 d\omega_2 d\omega_1 \\ & + \int_0^{0.2115} \int_{\frac{3.1\omega_1}{1.1}}^{\frac{4.1\omega_1}{1.1}} 1.1\omega_2 d\omega_2 d\omega_1 + \int_0^{0.2619} \int_{\frac{3.1\omega_1}{1.1}}^{\frac{2.1-2.1\omega_1}{3.2}} 1.1\omega_2 d\omega_2 d\omega_1 \end{aligned}$$

Third Criterion Portion of IPF(z^3):

$$\int_0^{0.1889} \int_{\frac{2.1-2.1\omega_1}{3.7}}^{\frac{2.1-2.1\omega_1}{3.2}} 2.1(1-\omega_1-\omega_2) d\omega_2 d\omega_1 + \int_0^{0.2266} \int_{\frac{2.1-2.1\omega_1}{3.7}}^{\frac{2.1-5.1\omega_1}{2.1}} 2.1(1-\omega_1-\omega_2)\omega_2 d\omega_2 d\omega_1$$

F1:

$$\begin{aligned} & \int_{0.1889}^{0.2266} 3.1\omega_1 \omega_2 \Big|_{\frac{2.1-2.1\omega_1}{5.1}}^{\frac{3.1\omega_1}{1.1}} d\omega_1 = \int_{0.1889}^{0.2266} 3.1\omega_1 \left(\frac{3.1\omega_1}{1.1} - \frac{2.1-5.1\omega_1}{2.1} \right) d\omega_1 \\ & = \int_{0.1889}^{0.2266} \left[3.1\omega_1 \left(\frac{3.1\omega_1}{1.1} \right) - \frac{3.1\omega_1(2.1-5.1\omega_1)}{2.1} \right] d\omega_1 \\ & = \int_{0.1889}^{0.2266} \left[\frac{3.1 \times 3.1\omega_1}{1.1} - \frac{3.1 \times 2.1\omega_1}{2.1} - \frac{3.1\omega_1^2 \times 5.1}{2.1} \right] d\omega_1 \\ & = \frac{3.1^2 \omega_1^3}{3.3} - \frac{3.1\omega_1^2}{2} + \frac{3.1 \times 5.1\omega_1^3}{2.1 \times 3} \Big|_{0.1889}^{0.2266} \\ & = -0.01651 - (-0.01876) = 0.002254 \end{aligned}$$

F2:

$$\begin{aligned}
\int_{0.2266}^{0.2619} 3.1\omega_1\omega_2 \Big|_{3.1\omega_1/1.6}^{3.1\omega_1/1.1} d\omega_1 &= \int_{0.2266}^{0.2619} \left[3.1\omega_1 \left(\frac{3.1}{1.1} \omega_1 \right) - 3.1\omega_1 \left(\frac{3.1}{1.6} \omega_1 \right) \right] d\omega_1 \\
&= \int_{0.2266}^{0.2619} \left[\frac{3.1^2 \omega_1^2}{1.1} - \frac{3.1^2 \omega_1^2}{1.6} \right] d\omega_1 \\
&= \frac{3.1^2 \omega_1^3 \left(\frac{1}{1.1} - \frac{1}{1.6} \right)}{3} \Big|_{0.2266}^{0.2619} = 0.005765
\end{aligned}$$

F3:

$$\begin{aligned}
\int_{0.2619}^{0.3404} 3.1\omega_1\omega_2 \Big|_{3.1\omega_1/1.6}^{1-\omega_1} d\omega_1 &= \int_{0.2619}^{0.3404} \left[3.1\omega_1(1-\omega_1) - 3.1\omega_1 \left(\frac{3.1}{1.6} \omega_1 \right) \right] d\omega_1 \\
&= \frac{3.1\omega_1^2}{2} - \frac{3.1\omega_1^3}{3} - \frac{3.1^2 \omega_1^3}{3 \times 1.6} \Big|_{0.2619}^{0.3404} = 0.008087
\end{aligned}$$

S1:

$$\begin{aligned}
\int_0^{0.1653} \frac{1.1\omega_2^2}{2} \Big|_{2.1-2.1\omega_1/3.2}^{3.1-3.1\omega_1/4.2} d\omega_1 &= \int_0^{0.1653} \frac{1.1}{2} \left[\left(\frac{3.1-3.1\omega_1}{4.2} \right)^2 \left(\frac{2.1-2.1\omega_1}{3.2} \right)^2 \right] d\omega_1 \\
&= \int_0^{0.1653} \frac{1.1}{2} \left[\frac{3.1^2 - 3.1^2 2\omega_1 + 3.1^2 \omega_1^2}{4.2^2} - \frac{2.1^2 - 2.1^2 \omega_1 + 2.1^2 \omega_1^2}{3.2^2} \right] d\omega_1 \\
&= \int_0^{0.1653} \frac{1.1}{2} \left[\frac{3.1^2}{4.2^2} - \frac{2 \times 3.1^2 \omega_1}{4.2^2} + \frac{3.1^2 \omega_1^2}{4.2^2} - \frac{2.1^2}{3.2^2} + \frac{2 \times 2.1^2 \omega_1}{3.2^2} - \frac{2.1^2 \omega_1^2}{3.2^2} \right] d\omega_1 \\
&= \int_0^{0.1653} \frac{1.1}{2} \left[0.1141 - 0.2282\omega_1 + 0.1141\omega_1^2 \right] d\omega_1 \\
&= \left[\frac{1.1}{2} 0.1141\omega_1 - \frac{1.1}{2} \frac{0.2282}{2} \omega_1^2 + \frac{1.1}{2} \frac{0.1141}{3} \omega_1^3 \right] \Big|_0^{0.1653} \\
&= 0.008754
\end{aligned}$$

S2:

$$\begin{aligned}
\int_{0.1653}^{0.5323} \frac{11}{2} \omega_2^2 \Big|_{2.1-2.1\omega_1/3.2}^{4.1\omega_1/1.1} d\omega_1 &= \int_{0.1653}^{0.5323} \frac{11}{2} \left(\frac{4.1^2 \omega_1^2}{1.1^2} - \frac{(2.1-2.1\omega_1^2)}{3.2^2} \right) d\omega_1 \\
&= \int_{0.1653}^{0.5323} \frac{11}{2} \left(\frac{4.1^2 \omega_1^2}{1.1^2} - \frac{2.1^2 - 2 \times 2.1^2 \omega_1 + 2.1^2 \omega_1^2}{3.2^2} \right) d\omega_1 \\
&= \int_{0.1653}^{0.5323} \frac{11}{2} \left(\frac{4.1^2 \omega_1^2}{1.1^2} - \frac{2.1^2}{3.2^2} + \frac{2 \times 2.1^2 \omega_1}{3.2^2} - \frac{2.1^2 \omega_1^2}{3.2^2} \right) d\omega_1 \\
&= \frac{11}{2} \left(\frac{13.462 \omega_1^3}{3} - \frac{2.1^2}{3.2^2} \omega_1 + \frac{2 \times 2.1^2}{2 \times 3.2^2} \omega_1^2 \right) \Big|_{0.1653}^{0.5323} \\
&= \left(2.468 \omega_1^3 - 0.2369 \omega_1 + 0.2369 \omega_1^2 \right) \Big|_{0.1653}^{0.5323} \\
&= -0.01966 - (-0.02154) = 0.001876
\end{aligned}$$

S3:

$$\begin{aligned}
\int_{0.1889}^{0.2115} \frac{11}{2} \omega_2^2 \Big|_{3.1\omega_1/1.1}^{4.1\omega_1/1.1} d\omega_1 &= \int_{0.1889}^{0.2115} \frac{11}{2} \left(\frac{4.1^2 \omega_1^2}{1.1^2} - \frac{3.1^2 \omega_1^2}{1.1^2} \right) d\omega_1 \\
&= \int_{0.1889}^{0.2115} \left(\frac{4.1^2 \omega_1^2}{2 \times 1.1} - \frac{3.1^2 \omega_1^2}{2 \times 1.1} \right) d\omega_1 = \frac{(4.1^2 - 3.1^2) \omega_1^3}{2 \times 1.1 \times 3} \Big|_{0.1889}^{0.2115} \\
&= 0.002976
\end{aligned}$$

S4:

$$\begin{aligned}
\int_{0.2115}^{0.2619} \frac{1.1}{2} \omega_2^2 \Big|_{3.1\omega_1/1.1}^{2.1-2.1\omega_1/3.2} d\omega_1 &= \int_{0.2115}^{0.2619} \frac{1.1}{2} \left[(1-\omega_1)^2 - \frac{3.1^2}{1.1^2} \omega_1^2 \right] d\omega_1 \\
&= \int_{0.2115}^{0.2619} \frac{1.1}{2} \left[1 - 2\omega_1 + \omega_1^2 - \frac{3.1^2 \omega_1^2}{1.1^2} \omega_1^2 \right] d\omega_1 \\
&= \int_{0.2115}^{0.2619} \left(\frac{1.1}{2} - 1.1\omega_1 + \frac{1.1}{2} \omega_1^2 - \frac{3.1^2 \omega_1^2}{2 \times 1.1} \right) d\omega_1 \\
&= \frac{1.1}{2} \omega_1 - \frac{1.1}{2} \omega_1^2 + \frac{1.1^2 \omega_1^3}{6 \times 1.1} - \frac{3.1^2 \omega_1^3}{6 \times 1.1} \Big|_{0.2115}^{0.2619} = 0.003769
\end{aligned}$$

T1:

$$\begin{aligned}
& \int_0^{0.1889} \left(2.1\omega_2 - 2.1\omega_1\omega_2 - \frac{2.1}{2}\omega_2^2 \right) \Big|_{2.1-2.1\omega_1/3.7}^{2.1-2.1\omega_1/3.2} d\omega_1 = \\
& = \int_0^{0.1889} \left((2.1-2.1\omega_1)\omega_2 - \frac{2.1}{2}\omega_2^2 \right) \Big|_{2.1-2.1\omega_1/3.7}^{2.1-2.1\omega_1/3.2} d\omega_1 \\
& = \int_0^{0.1889} (2.1-2.1\omega_1) - \frac{2.1}{2} \left(\frac{2.1-2.1\omega_1}{3.2} \right)^2 - (2.1-2.1\omega_1) + \frac{2.1}{2} \left(\frac{2.1-2.1\omega_1}{3.7} \right)^2 d\omega_1 \\
& = \int_0^{0.1889} \frac{2.1^2(1-\omega_1)^2}{3.2} - \frac{2.1}{2} \frac{2.1^3(1-\omega_1)^2}{3.2^2} - \frac{2.1^2(1-\omega_1)^2}{3.7} + \frac{2.1}{2} \frac{2.1^3(1-\omega_1)^2}{3.7^2} d\omega_1 \\
& = \int_0^{0.1889} (0.1862(1-\omega_1)^2 - 0.1140(1-\omega_1^2)) d\omega_1 \\
& = \int_0^{0.1889} 0.07227(1-\omega_1)^2 d\omega_1 = 0.07227\omega_1 - (0.07227)\omega_1^2 + \frac{0.07227\omega_1^3}{3} \Big|_0^{0.1889} \\
& = 0.01124
\end{aligned}$$

T2:

$$\begin{aligned}
& \int_{0.1889}^{0.2266} 2.1 \left(\omega_2 - \omega_1\omega_2 - \frac{\omega_2^2}{2} \right) \Big|_{2.1-2.1\omega_1/3.7}^{2.1-5.1\omega_1/2.1} d\omega_1 \\
& = \int_{0.1889}^{0.2266} 2.1 \left(\omega_2(1-\omega_1) - \frac{\omega_2^2}{2} \right) \Big|_{2.1-2.1\omega_1/3.7}^{2.1-5.1\omega_1/2.1} d\omega_1 \\
& = \int_{0.1889}^{0.2266} 2.1 \left[\frac{2.1-5.1\omega_1}{2.1} (1-\omega_1) - \frac{(2.1-5.1\omega_1)}{2(2.1)^2} - \frac{(2.1-2.1\omega_1)}{3.7} (1-\omega_1) \right. \\
& \quad \left. + \frac{(2.1-2.1\omega_1)^2}{2(3.7)^2} \right] d\omega_1 \\
& = \int_{0.1889}^{0.2266} (2.1-5.1\omega_1)(1-\omega_1) - \frac{(2.1-5.1\omega_1^2)}{2 \times 2.1} - \frac{2.1^2(1-\omega_1^2)}{3.7} + \frac{2.1^3(1-\omega_1^2)}{2(3.7)^2} d\omega_1 \\
& = \int_{0.1889}^{0.2266} (2.1-5.1\omega_1) \left(1-\omega_1 - \frac{2.1-5.1\omega_1}{2 \times 2.1} \right) - \frac{2.1^2(1-\omega_1)^2}{3.7} + \frac{2.1^3(1-\omega_1)^2}{2(3.7)^2} d\omega_1
\end{aligned}$$

$$\begin{aligned}
&= \int_{0.1889}^{0.2266} 2.1 - 2.1\omega_1 - 5.1\omega_1 + 5.1\omega_1^2 - \frac{2.1^2 - 2(2.1)(5.1)\omega_1 + 5.1^2\omega_1^2}{2 \times 2.1} \\
&\quad - \frac{2.1^2(1-\omega_1)^2}{3.7} + \frac{2.1^3(1-\omega_1)^2}{2(3.7)^2} d\omega_1 \\
&= \int_{0.1889}^{0.2266} 1.05 - 7.2\omega_1 + 5.1\omega_1 + 5.1\omega_1^2 - \frac{5.1^2\omega_1^2}{2(2.1)^2} - 0.8537(1-\omega_1^2) d\omega_1 \\
&= \int_{0.1889}^{0.2266} 1.05 - 2.1\omega_1 - 1.0929\omega_1^2 - 0.8537 + 2 \times 0.8537\omega_1 - 0.8537\omega_1^2 d\omega_1 \\
&= 0.1963\omega_1 - \frac{0.3927\omega_1^2}{2} - \frac{1.9465\omega_1^3}{3} \Big|_{0.1889}^{0.2266} = 0.001152
\end{aligned}$$

The Detailed IPF Calculations for z^4

First Criterion Portion of IPF(z^4):

$$\int_{0.0231}^{0.02381} \int_{1-\frac{7.1\omega_1}{3.1}}^{4.1\omega_1} 4.1\omega_1 d\omega_2 d\omega_1 + \int_{0.002381}^{0.1662} \int_{1-\frac{7.1\omega_1}{3.1}}^{1-\omega_1} 4.1\omega_1 d\omega_2 d\omega_1 + \int_{0.1662}^{0.2115} \int_{\frac{4.1\omega_1}{1.1}}^{1-\omega_1} 4.1\omega_1 d\omega_2 d\omega_1$$

Second Criterion Portion of IPF(z^4):

$$\int_0^{0.02308} \int_{\frac{3.1}{3.2}(1-\omega_1)}^{\frac{3.1-3.1\omega_1}{4.2}} 0.1\omega_2 d\omega_2 d\omega_1 + \int_{0.02308}^{0.02381} \int_{4.1\omega_1}^{1-\omega_1} 0.1\omega_2 d\omega_2 d\omega_1$$

Third Criterion Portion of IPF(z^4):

$$\int_0^{0.2308} \int_{\frac{3.1}{4.2}(1-\omega_1)}^{\frac{3.1}{3.2}(1-\omega_1)} 3.1(1-\omega_1-\omega_2) d\omega_2 d\omega_1 + \int_{0.02308}^{0.1662} \int_{\frac{3.1}{4.2}(1-\omega_1)}^{1-\frac{7.1}{3.1}\omega_1} 3.1(1-\omega_1-\omega_2) d\omega_2 d\omega_1$$

F1:

$$\begin{aligned}
\int_{0.0231}^{0.02381} 4.1\omega_1\omega_2 \Big|_{1-7.1\omega_1/3.1}^{41\omega_1} d\omega_1 &= \int_{0.0231}^{0.02381} 4.1\omega_1 \left(41\omega_1 - 1 + \frac{71}{31}\omega_1 \right) d\omega_1 \\
&= \frac{4.1 \times 41\omega_1^3}{3} - \frac{4.1\omega_1^2}{2} + \frac{4.1 \times 71\omega_1^3}{3 \times 31} \Big|_{0.0231}^{0.02381} \\
&= 59.163\omega_1^3 - \frac{4.1}{2}\omega_1^2 \Big|_{0.0231}^{0.02381} = 0.00000105299
\end{aligned}$$

F2:

$$\begin{aligned}
\int_{0.002381}^{0.1662} 4.1\omega_1\omega_2 \Big|_{1-7.1\omega_1/3.1}^{1-\omega_1} d\omega_1 &= \int_{0.002381}^{0.1662} 4.1\omega_1 \left[(1-\omega_1) - 1 + \frac{71}{31}\omega_1^3 \right] \frac{\omega_1}{31} d\omega_1 \\
&= \int_{0.002381}^{0.1662} \frac{(4.1)40\omega_1^2}{31} d\omega_1 = \frac{(4.1)40}{3 \times 31} \omega_1^3 \Big|_{0.002381}^{0.1662} = 0.008069
\end{aligned}$$

F3:

$$\begin{aligned}
\int_{0.1662}^{0.2115} 4.1\omega_1\omega_2 \Big|_{4.1\omega_1/1.1}^{1-\omega_1} d\omega_1 &= \int_{0.1662}^{0.2115} 4.1\omega_1 \left(1 - \omega_1 - \frac{41}{11}\omega_1 \right) d\omega_1 \\
&= \frac{4.1\omega_1^2}{2} - \frac{4.1 \times 52\omega_1^3}{3 \times 11} \Big|_{0.1662}^{0.2115} = 0.003615
\end{aligned}$$

S1:

$$\begin{aligned}
\int_0^{0.02308} \frac{0.1\omega_2^2}{2} \Big|_{3.1(1-\omega_1)/3.2}^{3.1-3.1\omega_1/4.2} d\omega_1 &= \int_0^{0.02308} \frac{0.1}{2} \left[(1-\omega_1)^2 - \left(\frac{31}{32} \right)^2 (1-\omega_1)^2 \right] d\omega_1 \\
&= \int_0^{0.02308} \frac{0.1}{2} \left[(1-\omega_1)^2 0.06152 \right] d\omega_1 = \int_0^{0.02308} 0.003076 (1-2\omega_1 + \omega_1^2) d\omega_1 \\
&= 0.003076 \left(\omega_1 - 2\omega_1 + \frac{\omega_1^3}{3} \right) \Big|_0^{0.02308} = 0.00006938
\end{aligned}$$

S2:

$$\begin{aligned}
& \int_{0.02308}^{0.02381} \frac{0.1}{2} \omega_2^2 \Big|_{41\omega_1}^{1-\omega_1} d\omega_1 = \int_{0.02308}^{0.02381} \frac{0.1}{2} [(1-\omega_1)^2 - 41^2 \omega_1^2] d\omega_1 \\
& = \int_{0.02308}^{0.02381} \frac{0.1}{2} (1 - 2\omega_1 + \omega_1^2 - 41^2 \omega_1^2) d\omega_1 = \int_{0.02308}^{0.02381} \frac{0.1}{2} (1 - 2\omega_1 - 1680\omega_1^2) d\omega_1 \\
& = \frac{0.1}{2} \left(\omega_1 - \omega_1^2 - \frac{1680}{3} \omega_1^3 \right) \Big|_{0.02308}^{0.02381} = 0.000001072
\end{aligned}$$

T1:

$$\begin{aligned}
& \int_0^{0.2308} 3.1 \left[(1-\omega_1)\omega_2 - \frac{\omega_2^2}{2} \right]_{3.1(1-\omega_1)/4.2}^{3.1(1-\omega_1)/3.2} d\omega_1 \\
& = \int_0^{0.2308} 3.1 \left[(1-\omega_1) \frac{31}{32} (1-\omega_1) - \frac{1}{2} \left(\frac{31}{32} \right)^2 (1-\omega_1)^2 \right. \\
& \quad \left. - (1-\omega_1) \frac{31}{42} (1-\omega_1) + \frac{1}{2} \left(\frac{31}{42} \right)^2 (1-\omega_1)^2 \right] d\omega_1 \\
& = \int_0^{0.2308} 3.1 \left[\left(\frac{31}{32} - \frac{31^2}{2 \times 32^2} - \frac{31}{42} + \frac{31^2}{2 \times 42^2} \right) (1-\omega_1)^2 \right] d\omega_1 \\
& = \int_0^{0.2308} 3.1 [0.03381(1-\omega_1)^2] d\omega_1 \\
& = 3.1 \times 0.03381 \left(\omega_1 - \omega_1^2 + \frac{\omega_1^3}{3} \right) \Big|_0^{0.2308} = 0.002363
\end{aligned}$$

T2:

$$\begin{aligned}
& \int_{0.02308}^{0.1662} 3.1 \left[(1-\omega_1)\omega_2 - \frac{\omega_2^2}{2} \right]_{3.1(1-\omega_1)/4.2}^{1-7.1\omega_1/3.1} d\omega_1 \\
& = \int_{0.02308}^{0.1662} 3.1 \left[(1-\omega_1) \left(1 - \frac{71}{31} \omega_1 \right) - \frac{\left(1 - \frac{71}{31} \omega_1 \right)^2}{2} - (1-\omega_1) \left(\frac{31}{42} (1-\omega_1) \right) \right. \\
& \quad \left. + \frac{\left(\frac{31}{42} (1-\omega_1) \right)^2}{2} \right] d\omega_1
\end{aligned}$$

$$\begin{aligned}
&= \int_{0.02308}^{0.1662} 3.1 \left[(1-\omega_1) \left(\frac{31-71\omega_1}{31} - \frac{31-31\omega_1}{42} \right) + \frac{\left(\frac{31-31\omega_1}{42} \right)^2 - \left(\frac{31-71\omega_1}{31} \right)^2}{2} \right] d\omega_1 \\
&= \int_{0.02308}^{0.1662} 3.1 \left[(1-\omega_1) \left(1 - \frac{71}{31}\omega_1 - \frac{31}{42} + \frac{31\omega_1}{42} \right) + \frac{(31-31\omega_1)^2}{2 \times 42^2} - \frac{(31-71\omega_1)^2}{2 \times 31^2} \right] d\omega_1 \\
&= \int_{0.02308}^{0.1662} 3.1 \left[(1-\omega_1)(0.2619 - 1.5522\omega_1) + \frac{31^2(1-\omega_1)^2}{2 \times 42^2} - \frac{31^2 - 2 \times 31 \times 71\omega_1 + 71^2\omega_1^2}{2 \times 31^2} \right] d\omega_1 \\
&= \int_{0.02308}^{0.1662} 3.1 \left[0.2619 - 1.5522\omega_1 - 0.2619\omega_1 + 1.5522\omega_1^2 + 0.2724(1-\omega_1)^2 - \frac{1}{2} + \frac{71\omega_1}{31} - \frac{71^2\omega_1^2}{2 \times 31^2} \right] d\omega_1 \\
&= \int_{0.02308}^{0.1662} 3.1 \left[0.2619 - 1.5522\omega_1 - 0.2619\omega_1 + 1.5522\omega_1^2 + 0.2723 - 2 \times 0.2739\omega_1 + 0.2724\omega_1^2 - \frac{1}{2} + \frac{71}{31}\omega_1 - \frac{71^2}{2 \times 31^2}\omega_1^2 \right] d\omega_1 \\
&= 3.1 \left(0.0343\omega_1 - \frac{0.06859}{2}\omega_1^2 - \frac{0.7981}{3}\omega_1^3 \right) \Big|_{0.02308}^{0.1662} \\
&= 0.008560
\end{aligned}$$

APPENDIX H

NEIGHBORHOOD CONSTRAINTS FOR CHAPTER 4 EXAMPLE 6

Comparison of alternatives z^1 and z^5 :

$$u_{\alpha}(z^1) = u_{\alpha}(z^5)$$

$$\text{Max}\{4.1 \alpha_1, 2.9 \alpha_2, 0.6 \alpha_3\} = \text{Max}\{4.5 \alpha_1, 2.4 \alpha_2, 0.1 \alpha_3\}$$

$$\text{Max}\{4.1 \alpha_1, 2.9 \alpha_2, 0.6 \alpha_3\} = 4.5 \alpha_1$$

$$2.9 \alpha_2 = 4.5 \alpha_1 \text{ for } 2.9 \alpha_2 \geq 0.6 \alpha_3$$

$$0.6 \alpha_3 = 4.5 \alpha_1 \text{ for } 2.9 \alpha_2 < 0.6 \alpha_3$$

$$2.9 \alpha_2 = 4.5 \alpha_1 \text{ for } 3.5 \alpha_2 \geq 0.6 - 0.6 \alpha_1$$

$$0.6 \alpha_2 = 0.6 - 5.1 \alpha_1 \text{ for } 3.5 \alpha_2 < 0.6 - 0.6 \alpha_1$$

Comparison of alternatives z^1 and z^6 :

$$u_{\alpha}(z^1) = u_{\alpha}(z^6)$$

$$\text{Max}\{4.1 \alpha_1, 2.9 \alpha_2, 0.6 \alpha_3\} = \text{Max}\{3.6 \alpha_1, 3 \alpha_2, 0.4 \alpha_3\}$$

$$\text{Max}\{4.1 \alpha_1, 2.9 \alpha_2, 0.6 \alpha_3\} = 3 \alpha_2$$

$$4.1 \alpha_1 = 3 \alpha_2 \text{ for } 4.1 \alpha_1 \geq 0.6 \alpha_3$$

$$0.6 \alpha_3 = 3 \alpha_2 \text{ for } 4.1 \alpha_1 < 0.6 \alpha_3$$

$$3 \alpha_2 = 4.1 \alpha_1 \text{ for } 0.6 \alpha_2 \geq 0.6 - 4.7 \alpha_1$$

$$3.6 \alpha_2 = 0.6 - 0.6 \alpha_1 \text{ for } 0.6 \alpha_2 < 0.6 - 4.7 \alpha_1$$

Comparison of alternatives z^2 and z^3 :

$$u_{\alpha}(z^2) = u_{\alpha}(z^3)$$

$$\text{Max}\{2.1 \alpha_1, 1.1 \alpha_2, 3.6 \alpha_3\} = \text{Max}\{0.1 \alpha_1, 2.6 \alpha_2, 2.6 \alpha_3\}$$

$$\text{Max}\{2.1 \alpha_1, 1.1 \alpha_2, 3.6 \alpha_3\} = 2.6 \alpha_2$$

$$2.6 \alpha_2 = 2.1 \alpha_1 \text{ for } 3.6 \alpha_2 \geq 3.6 - 5.7 \alpha_1$$

$$6.2 \alpha_2 = 3.6 - 3.6 \alpha_1 \text{ for } 3.6 \alpha_2 < 3.6 - 5.7 \alpha_1$$

Comparison of alternatives z^2 and z^4 :

$$u_{\alpha}(z^2) = u_{\alpha}(z^4)$$

$$\text{Max}\{2.1 \alpha_1, 1.1 \alpha_2, 3.6 \alpha_3\} = \text{Max}\{3.1 \alpha_1, 2.1 \alpha_2, 1.6 \alpha_3\}$$

$$3.6 \alpha_3 = \text{Max}\{3.1 \alpha_1, 2.1 \alpha_2, 1.6 \alpha_3\}$$

$$3.6 \alpha_2 = 3.6 - 6.7 \alpha_1 \text{ for } 3.1 \alpha_1 \geq 2.1 \alpha_2$$

$$5.7 \alpha_2 = 3.6 - 3.6 \alpha_1 \text{ for } 3.1 \alpha_1 < 2.1 \alpha_2$$

Comparison of alternatives z^2 and z^7 :

$$u_{\alpha}(z^2) = u_{\alpha}(z^7)$$

$$\text{Max}\{2.1 \alpha_1, 1.1 \alpha_2, 3.6 \alpha_3\} = \text{Max}\{2.6 \alpha_1, 0.1 \alpha_2, 3.1 \alpha_3\}$$

$$\text{Max}\{2.1 \alpha_1, 1.1 \alpha_2, 3.6 \alpha_3\} = 2.6 \alpha_1$$

$$1.1 \alpha_2 = 2.6 \alpha_1 \text{ for } 4.7 \alpha_2 \geq 3.6 - 3.6 \alpha_1$$

$$3.6 \alpha_2 = 3.6 - 6.2 \alpha_1 \text{ for } 4.7 \alpha_2 < 3.6 - 3.6 \alpha_1$$

Comparison of alternatives z^3 and z^4 :

$$u_{\alpha}(z^3) = u_{\alpha}(z^4)$$

$$\text{Max}\{0.1 \alpha_1, 2.6 \alpha_2, 2.6 \alpha_3\} = \text{Max}\{3.1 \alpha_1, 2.1 \alpha_2, 1.6 \alpha_3\}$$

$$\text{Max}\{0.1 \alpha_1, 2.6 \alpha_2, 2.6 \alpha_3\} = 3.1 \alpha_1$$

$$2.6 \alpha_2 = 3.1 \alpha_1 \text{ for } 5.2 \alpha_2 \geq 2.6 - 2.6 \alpha_1$$

$$2.6 \alpha_2 = 2.6 - 5.7 \alpha_1 \text{ for } 5.2 \alpha_2 < 2.6 - 2.6 \alpha_1$$

Comparison of alternatives z^3 and z^5 :

$$u_{\alpha}(z^3) = u_{\alpha}(z^5)$$

$$\text{Max}\{0.1 \alpha_1, 2.6 \alpha_2, 2.6 \alpha_3\} = \text{Max}\{4.5 \alpha_1, 2.4 \alpha_2, 0.1 \alpha_3\}$$

$$\text{Max}\{0.1 \alpha_1, 2.6 \alpha_2, 2.6 \alpha_3\} = 4.5 \alpha_1$$

$$2.6 \alpha_2 = 4.5 \alpha_1 \text{ for } 5.2 \alpha_2 \geq 2.6 - 2.6 \alpha_1$$

$$2.6 \alpha_2 = 2.6 - 7.1 \alpha_1 \text{ for } 5.2 \alpha_2 < 2.6 - 2.6 \alpha_1$$

Comparison of alternatives z^3 and z^7 :

$$u_{\alpha}(z^3) = u_{\alpha}(z^7)$$

$$\text{Max}\{0.1 \alpha_1, 2.6 \alpha_2, 2.6 \alpha_3\} = \text{Max}\{2.6 \alpha_1, 0.1 \alpha_2, 3.1 \alpha_3\}$$

$$2.6 \alpha_2 = \text{Max}\{2.6 \alpha_1, 0.1 \alpha_2, 3.1 \alpha_3\}$$

$$\alpha_2 = \alpha_1 \text{ for } 3.1 \alpha_2 \geq 3.1 - 5.7 \alpha_1$$

$$5.7 \alpha_2 = 3.1 - 3.1 \alpha_1 \text{ for } 3.1 \alpha_2 < 3.1 - 5.7 \alpha_1$$

Comparison of alternatives z^4 and z^5 :

$$u_{\alpha}(z^4) = u_{\alpha}(z^5)$$

$$\text{Max}\{3.1 \alpha_1, 2.1 \alpha_2, 1.6 \alpha_3\} = \text{Max}\{4.5 \alpha_1, 2.4 \alpha_2, 0.1 \alpha_3\}$$

$$1.6 \alpha_3 = \text{Max}\{4.5 \alpha_1, 2.4 \alpha_2, 0.1 \alpha_3\}$$

$$1.6 \alpha_2 = 1.6 - 6.1 \alpha_1 \text{ for } 4.5 \alpha_1 \geq 2.4 \alpha_2$$

$$4 \alpha_2 = 1.6 - 1.6 \alpha_1 \text{ for } 4.5 \alpha_1 < 2.4 \alpha_2$$

Comparison of alternatives z^4 and z^6 :

$$u_{\alpha}(z^4) = u_{\alpha}(z^6)$$

$$\text{Max}\{3.1 \alpha_1, 2.1 \alpha_2, 1.6 \alpha_3\} = \text{Max}\{3.6 \alpha_1, 3 \alpha_2, 0.4 \alpha_3\}$$

$$1.6 \alpha_3 = \text{Max}\{3.6 \alpha_1, 3 \alpha_2, 0.4 \alpha_3\}$$

$$1.6 \alpha_2 = 1.6 - 5.2 \alpha_1 \text{ for } 3.6 \alpha_1 \geq 3 \alpha_2$$

$$4.6 \alpha_2 = 1.6 - 1.6 \alpha_1 \text{ for } 3.6 \alpha_1 < 3 \alpha_2$$

Comparison of alternatives z^4 and z^7 :

$$u_{\alpha}(z^4) = u_{\alpha}(z^7)$$

$$\text{Max}\{3.1 \alpha_1, 2.1 \alpha_2, 1.6 \alpha_3\} = \text{Max}\{2.6 \alpha_1, 0.1 \alpha_2, 3.1 \alpha_3\}$$

$$\text{Max}\{3.1 \alpha_1, 2.1 \alpha_2, 1.6 \alpha_3\} = 3.1 \alpha_3$$

$$3.1 \alpha_2 = 3.1 - 6.2 \alpha_1 \text{ for } 3.1 \alpha_1 \geq 2.1 \alpha_2$$

$$5.2 \alpha_2 = 3.1 - 3.1 \alpha_1 \text{ for } 3.1 \alpha_1 < 2.1 \alpha_2$$

Comparison of alternatives z^5 and z^6 :

$$u_{\alpha}(z^5) = u_{\alpha}(z^6)$$

$$\text{Max}\{4.5\alpha_1, 2.4\alpha_2, 0.1\alpha_3\} = \text{Max}\{3.6\alpha_1, 3\alpha_2, 0.4\alpha_3\}$$

$$4.5\alpha_1 = \text{Max}\{3.6\alpha_1, 3\alpha_2, 0.4\alpha_3\}$$

$$3\alpha_2 = 4.5\alpha_1 \text{ for } 3.4\alpha_2 \geq 0.4 - 0.4\alpha_1$$

$$0.4\alpha_2 = 0.4 - 4.9\alpha_1 \text{ for } 3.4\alpha_2 < 0.4 - 0.4\alpha_1$$