DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

LEAST SQUARES METHOD FOR SOLVING DIFFERENTIAL EQUATIONS USING BEZIER CONTROL POINTS

by
Meltem EVRENOSOĞLU

July, 2005
İZMİR

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M.Sc. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"LEAST SQUARES METHOD FOR SOLVING DIFFERENTIAL EQUATIONS USING BÉZIER CONTROL POINTS"** completed by **MELTEM EVRENOSOĞLU** under supervision of **PROF. DR. ŞENNUR SOMALI** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Meltem EVRENOSOĞLU

LEAST SQUARES METHOD FOR SOLVING DIFFERENTIAL EQUATIONS USING BÉZIER CONTROL POINTS

ABSTRACT

This thesis investigates the use of the control points of the Bernstein-Bézier form for numerically solving the singular two-point boundary value problem in linear case

$$(t^{\alpha}u'(t))' = g(t), \qquad 0 < t \leq 1,$$

$$u(0) = a, \qquad u(1) = b,$$

where $0 < \alpha < 1$, a, b are finite constants, $g(t)$ is continuous for $t \in [0, 1]$. The least squares scheme based on division of interval is proposed and the convergence of the method applied to the singular two-point boundary value problem is analyzed.

Keywords: Bernstein-Bézier form, Least squares method, Singular two-point boundary value problem.

BÉZİER KONTROL NOKTALARI KULLANIMI İLE DİFERANSİYEL DENKLEMLERİN EN KÜÇÜK KARELER YÖNTEMİ İLE ÇÖZÜMÜ

ÖZ

Bu çalışmada, iki noktalı tekil doğrusal sınır-değer probleminin

$$(t^{\alpha}u'(t))' = g(t), \qquad 0 < t \leq 1,$$

$$u(0) = a, \qquad u(1) = b,$$

sayısal çözümü için Bernstein-Bézier eğrilerinin kontrol noktalarının kullanımı incelenmiştir. [0,1] aralığını parçalara bölerek, en küçük kareler yönteminin kullanılması esas alınmış ve yöntemin yakınsaklığı tekil sınır-değer problemini için analiz edilmiştir.

Anahtar sözcükler: Bernstein-Bézier form, En küçük kareler metodu, İki noktalı tekil sınır değer problemi.

CONTENTS

CHAPTER ONE
INTRODUCTION

1.1 Introduction

In the numerical solution of differential equations, polynomial or piecewise polynomial functions are often used to represent the approximate solution (Lai & Wenston, 2000). In particular, B-splines have become a popular tool for solving differential equations (de Boor, 1978; Shariff & Moser 1998). While the numerical analysis community was investigating splines as basis functions for solving differential equations, B-splines have become an industry standard in computer aided design(CAD) (Farin, 1997). In the field of CAD, curves and surfaces are commonly represented through the use of Bézier and B-spline curves and surfaces. For both Bézier and B-spline representations, the curve or surface shape is outlined by a ”control polygon” or ”control mesh” that is formed by connecting the control points.

This thesis investigates the use of the control points of Bézier representations for finding approximate solutions to the singular two-point boundary value problem

$$tu''(t) + \alpha u'(t) = t^{1-\alpha} g(t), \qquad 0 < t \leq 1, \tag{1.1.1}$$

$$u(0) = a, \qquad u(1) = b,$$

in which $0 < \alpha < 1$ and a, b are finite constants. We assume for $t \in [0, 1]$, $g(t)$ is continuous. Numerical methods for solving the singular two-point boundary value problem have been considered in many years. In general, standard three point finite difference scheme (Jamet, 1970), Rayleight-Ritz-Galerkin method (Ciarlet, Natterer & Varga, 1970), Galerkin approximation (Ciarlet *et al.*, 1970), collacation method (Reddien, 1973; Reddien & Schumaker, 1976) and iterated defect correction methods (Ertem & Somalı, 1998) are used.

In this thesis, we consider the least squares approach to the singular two-point boundary value problem (1.1.1) using degree 3 orthonormal Bézier curves based on the Bézier control points by using similar consideration as in (Zheng, Sederberg & Johnson, 2004).

This thesis is organized as follows : Section 2.1 briefly reviews some important properties of Bézier curves. Section 2.2 proposes a new method for solving the singular two-point boundary value problem that are based on the control points of the Bézier representation. In Section 2.3, the convergence of the proposed scheme for two-point boundary value problems is analyzed. In Chapter 3, numerical examples with their graphs are given by illustrating the convergence theory.

CHAPTER TWO
BÉZIER CURVES

2.1 Bézier Curves

We define the Bernstein polynomials

$$B_i^n\left(\frac{t-c}{d-c}\right) = \binom{n}{i}\left(\frac{d-t}{d-c}\right)^{n-i}\left(\frac{t-c}{d-c}\right)^{i}, \tag{2.1.1}$$

over the interval $[c, d]$. It follows from Bernstein polynomials that any polynomial $p(t)$ of degree n can be expressed as,

$$p(t) = \sum_{i=0}^{n} p_i B_i^n\left(\frac{t-c}{d-c}\right), \qquad t \in [c, d], \tag{2.1.2}$$

which is called Bézier form. If $p(t)$ is vector-valued polynomial, $p(t)$ is a parametric Bézier curve and the Bézier coefficients p_i are the control points. The control polygon of a Bézier curve consists of the line segments connecting p_i and p_{i+1} , $i = 0, \ldots, n-1$. If $p(t)$ is a scalar-valued polynomial, we call the function $y = p(t)$ an explicit Bézier curve given by $(t, p(t))$. The control points of an explicit curve are $(i/n, p_i)$ since $\sum\limits_{i=0}^{n} \frac{i}{n} B_i^n(t) = t$. In this study, for convenience we also call p_i the Bézier control points.

Bézier curves have many properties involving the control points. Some of them are:

i) <u>*Convex Hull Property*</u> : Any point on the Bézier curve lies inside the convex-hull of the control polygon.

ii)<u>*End Point Interpolation*</u> : It is clear from (2.1.2) that,

$$p(c) = p_0, \qquad p(d) = p_n.$$

Hence Bézier curves for $p_0, p_1, \ldots, p_n$ interpolates $p(t)$ at both end points of the interval $[c, d]$. {This is easily verified by writing down the equation for the cases $t = c$ and at $t = d$}. Many operations on the Bézier curve can be performed using control points.

iii)<u>*Derivatives*</u> : The derivatives of the Bézier curve (2.1.2) can be written

$$\frac{d^k p(t)}{dt^k} = \frac{n(n-1)\cdots(n-k+1)}{(d-c)^k} \sum_{i=0}^{n-k} \triangle^k \, p_i B_i^{n-k}\left(\frac{t-c}{d-c}\right), \tag{2.1.3}$$

where $\triangle$ is the forward difference operator defined by $\triangle \ p_i = p_{i+1} - p_i$ and $\triangle^{k+1} \ p_i = \triangle \ (\triangle^k \ p_i)$. In particular ,

$$\frac{d^k p(c)}{dt^k} = \triangle^k \ p_0, \qquad \frac{d^k p(d)}{dt^k} = \triangle^k \ p_{n-k}. \tag{2.1.4}$$

These formulas can be used to set constraints on Bézier segments to make them connect with C^k continuity and hence form a Bézier spline.

2.2 Least Squares Approximations for Solving Singular Two-Point Boundary Value Problem

Consider the following boundary value problem:

$$Lu(t) = \varphi(t)u''(t) + \psi(t)u'(t) = f(t), \qquad t \in [0, 1], \tag{2.2.1}$$

$$u(0) = a, \qquad u(1) = b,$$

where L is the second order linear differential operator with polynomial coefficients $\varphi(t) = t$ and $\psi(t) = \alpha$ and $f(t) = t^{1-\alpha}g(t)$. Our goal is to find a polynomial or a piecewise polynomial function $u(t)$ such that it satisfies the boundary conditions and minimizes the residual function $Lu(t)-f(t)$ in L_2 norm.

We propose to represent the approximate solution in Bézier form. The choice of the Bézier form rather than the B-spline form is due to the fact that the Bézier form is easier symbolically in carry out the operations of multiplication, composition and degree elevation than the B-spline form. We choose the residual function in the form of linear combinations of orthonormal degree 3 Bézier polynomials to make the integral operation easy.

We divide the interval [0,1] uniformly $[t_{i-1},t_i]$, $t_i = ih$ where $h = \frac{1}{n}$, $i = 1, 2, \ldots, n$, and define the degree 3 approximate solution in the interval $[t_{i-1}, t_i]$ by;

$$u_i(t) = \sum_{j=0}^{3} b_j^i B_j^3 \left(\frac{t - t_{i-1}}{h} \right), \qquad i = 1, 2, \ldots, n.$$

Our aim is to determine the control points b_j^i such that the error between the approximate solution and the exact solution stays minimum. In the first step, we substitute the approximate solution $u_i(t)$ into differential equation of the singular boundary value problem (2.2.1),

$$\varphi(t)u_i''(t) + \psi(t)u_i'(t) = f(t), \qquad t \in [t_{i-1}, t_i],$$

for $i = 1, 2, \ldots, n$ and we obtain n residual functions,

$$R_i(t) = Lu_i(t) - f(t), \qquad t \in [t_{i-1}, t_i], \qquad i = 1, \ldots, n.$$

From the differentiation of Bézier curves, we have

$$\frac{d^k u_i(t_{i-1})}{dt^k} = \triangle^k\, b_0^i, \qquad \frac{d^k u_i(t_i)}{dt^k} = \triangle^k\, b_{3-k}^i.$$

Since a second order differential equation requires C^2 continuity on $[t_{i-1}, t_i]$, then

$$\frac{d^j u_i(t_i)}{dt^j} = \frac{d^j u_{i+1}(t_i)}{dt^j},$$

holds for $j = 0, 1$ and 2. Thus the control points b^i_j must satisfy

$$\triangle^p\, b^i_{3-p} = \triangle^p\, b^{i+1}_0, \qquad p = 0, 1, 2. \tag{2.2.2}$$

So we can write some of our unknowns as a linear combinations of the others as follows

$$b^{i+1}_0 = b^i_3 \tag{2.2.3}$$

$$b^{i+1}_1 = 2b^i_3 - b^i_2 \tag{2.2.4}$$

$$b^{i+1}_2 = 4b^i_3 - 4b^i_2 + b^i_1 \tag{2.2.5}$$

by (2.2.2) and $b^1_0 = a$ and $b^n_3 = b$ with the $n+1$ unknowns b^1_1, b^1_2, b^1_3, b^2_3,..., b^{n-1}_3. These continuity conditions together with boundary conditions are added into the minimization problem to form a constrained optimization problem.

To make the integral equation easy we will write $Lu_i(t)$ as the linear combinations of the elements of orthonormal set $\{\mathbb{B}^3_0(t), \mathbb{B}^3_1(t), \mathbb{B}^3_2(t), \mathbb{B}^3_3(t)\}$ for $t \in [t_{i-1}, t_i]$, where

$$\mathbb{B}^3_0\left(\frac{t - t_{i-1}}{h}\right) = \sqrt{\frac{7}{h}} B^3_0\left(\frac{t - t_{i-1}}{h}\right),$$

$$\mathbb{B}^3_1\left(\frac{t - t_{i-1}}{h}\right) = \sqrt{\frac{20}{h}}\left[B^3_1\left(\frac{t - t_{i-1}}{h}\right) - \frac{1}{2} B^3_0\left(\frac{t - t_{i-1}}{h}\right)\right],$$

$$\mathbb{B}_2^3\left(\frac{t-t_{i-1}}{h}\right)=\sqrt{\frac{100}{3h}}\left[B_2^3\left(\frac{t-t_{i-1}}{h}\right)-B_1^3\left(\frac{t-t_{i-1}}{h}\right)+\frac{3}{10}B_0^3\left(\frac{t-t_{i-1}}{h}\right)\right],$$

and

$$\mathbb{B}_3^3\left(\frac{t-t_{i-1}}{h}\right)=\sqrt{\frac{16}{h}}\left[B_3^3\left(\frac{t-t_{i-1}}{h}\right)-\frac{3}{2}B_2^3\left(\frac{t-t_{i-1}}{h}\right)\right.$$

$$\left.+B_1^3\left(\frac{t-t_{i-1}}{h}\right)-\frac{1}{4}B_0^3\left(\frac{t-t_{i-1}}{h}\right)\right]$$

This orthonormal set is computed using the symbolic computation program Mathematica.

We now write $Lu_i(t)$ in terms of orthonormal polynomials

$$Lu_i(t)=\sum_{j=0}^{3}c_j^i\mathbb{B}_j^3\left(\frac{t-t_{i-1}}{h}\right), \tag{2.2.6}$$

where c_j^i are linear combinations of the unknowns $b_1^1,\ b_2^1,\ b_3^1,\ b_3^2,\ldots,\ b_3^{n-1}$. Substituting $Lu_i(t)$ into the residual functions $R_i(t)$, we get $R_i(t)=Lu_i(t)-f(t)$. The least squares method is employed which reduces the problem $\min\|R(t)\|_2^2$ into a problem of solving the following system of linear equations

$$\frac{\partial}{\partial b_1^1}\left[\int_{t_0}^{t_1}(R_1(t))^2dt+\int_{t_1}^{t_2}(R_2(t))^2dt+\cdots+\int_{t_{n-1}}^{1}(R_n(t))^2dt\right]=0$$

$$\frac{\partial}{\partial b_2^1}\left[\int_{t_0}^{t_1}(R_1(t))^2dt+\int_{t_1}^{t_2}(R_2(t))^2dt+\cdots+\int_{t_{n-1}}^{1}(R_n(t))^2dt\right]=0$$

$$\frac{\partial}{\partial b_3^l}\left[\int_{t_0}^{t_1}(R_1(t))^2dt+\int_{t_1}^{t_2}(R_2(t))^2dt+\cdots+\int_{t_{n-1}}^{1}(R_n(t))^2dt\right]=0,$$

where $l=1,\ldots,n-1,\quad R(t)=R_i(t)\quad \text{for}\quad t\in[t_{i-1},t_i],\quad i=1,\ldots,n$

Taking the derivatives of the equations and using the properties of the orthonormality of Bézier polynomials $\mathbb{B}$, we have

$$[c_0^1\frac{dc_0^1}{db_1^1}+\cdots+c_3^1\frac{dc_3^1}{db_1^1}+\cdots+c_0^n\frac{dc_0^n}{db_1^1}+\cdots+c_3^n\frac{dc_3^n}{db_1^1}]=\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\sum_{j=0}^{3}c_j^i\frac{dc_j^i}{db_1^1}\mathbb{B}_i^3(\frac{t-t_i}{h})\right)f(t)dt$$

$$[c_0^1\frac{dc_0^1}{db_2^1}+\cdots+c_3^1\frac{dc_3^1}{db_2^1}+\cdots+c_0^n\frac{dc_0^n}{db_2^1}+\cdots+c_3^n\frac{dc_3^n}{db_2^1}]=\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\sum_{j=0}^{3}c_j^i\frac{dc_j^i}{db_2^1}\mathbb{B}_i^3(\frac{t-t_i}{h})\right)f(t)dt$$

and

$$[c_0^1\frac{dc_0^1}{db_3^l}+\cdots+c_3^1\frac{dc_3^1}{db_3^l}+\cdots+c_0^n\frac{dc_0^n}{db_3^l}+\cdots+c_3^n\frac{dc_3^n}{db_3^l}=\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\sum_{j=0}^{3}c_j^i\frac{dc_j^i}{db_3^l}\mathbb{B}_i^3(\frac{t-t_i}{h})\right)f(t)dt,$$

for $l=1,\ldots,n-1$.

Hence we have $n+1$ equations with $n+1$ unknowns. Solving the system for unknowns b_1^1, b_2^1, b_3^1, b_3^2,..., b_3^{n-1} and substituting the control points into the approximate solutions, we obtain an approximate piecewise polynomial to the solution of the singular boundary value problem (2.2.1).

2.3 Convergence Analysis

In this section, we analyze the convergence of the least squares method based on Bézier curves applied to the singular boundary value problem (1.1.1) by using theorem in (Zheng *et al.*, 2004).

Theorem 2.3.1. *Let v(t) be the approximate solution of the two-point boundary value problem (1.1.1) obtained by the least squares method for solving differential equation using Bézier control points. If (1.1.1) has unique solution u(t) and u(t) is smooth enough so that the cubic spline Tu(t) interpolating u(t) converges to u(t) in the order* $O(h^q)$, $(q > 2)$, *where h is the maximal width of all subintervals, then v(t) converges to u(t) as* $h \to 0$.

Proof. We first impose a uniform partition $\Pi_h = U[t_i, t_{i+1}]$ on the interval [0,1] as $t_i = ih$, where $i = 0, \ldots, n+1$, $h = \frac{1}{n+1}$. Let $I_h u$ be the cubic spline over Π_h interpolating $u(t)$. Then for an arbitrary small positive number $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|Lu(t) - L(I_h u)\|_\infty \leq \varepsilon \tag{2.3.1}$$

provided that $h < \delta$.

Let $R(I_h u) = L(I_h u) - f(t)$ be residual. Express $R(I_h u)$ in $[t_i, t_{i+1}]$ as

$$R(I_h u) = \sum_{j=0}^{3} d_j^i(b_1^1, b_2^1, b_3^1, \ldots, b_3^{n-1}) \mathbb{B}_j^3 \left(\frac{t - t_i}{h}\right) - f(t)$$

Then from (2.3.1), we have

$$\|R(I_h u)\|_\infty^2 = \int_{t_i}^{t_{i+1}} \left(\sum_{j=0}^{3} d_j^i \mathbb{B}_j^3 \left(\frac{t - t_i}{h}\right) - f(t) \right)^2 dt$$

$$= \sum_{j=0}^{3} \left(d_j^i(b_1^1, b_2^1, b_3^1, \ldots, b_3^{n-1}) \right)^2 - 2 \int_{t_i}^{t_{i+1}} \left(\sum_{j=0}^{3} d_j^i \mathbb{B}_j^3 \left(\frac{t - t_i}{h}\right) \right) f(t) dt \leq \varepsilon^2$$

Suppose $v(t)$ is the approximate solution by the least squares method in solving differential equation using Bézier control points with respect to the Π_h partition, and denote the residual $[t_i, t_{i+1}]$ by

$$R(v(t)) = \sum_{j=0}^{3} c_j^i(b_1^1, b_2^1, b_3^1, \dots, b_3^{n-1})\mathbb{B}_j^3\left(\frac{t-t_i}{h}\right) - f(t).$$

We now estimate the Sobolev norm of the difference of the approximate solution $v(t)$ and the exact solution $u(t)$

$$\|u(t) - v(t)\|_{H^2[0,1]}^2 = \int_0^1 \sum_{j=0}^{2} \left|\frac{d^j v(t)}{dt^j} - \frac{d^j u(t)}{dt^j}\right|^2 dt$$

$$\leq C\|R(u(t) - v(t))\|_2^2 = C\int_0^1 (R(v(t)))^2 dt,$$

C is a constant. Writing the last inequality explicitly we obtain

$$\leq C\sum_{i=0}^{n+1}\left(\sum_{j=0}^{3}(c_j^i)^2 - 2\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}}\left(\sum_{j=0}^{3} c_j^i\mathbb{B}_j^3\left(\frac{t-t_i}{h}\right)\right)f(t)dt\right)$$

and

$$\leq C\sum_{i=0}^{n+1}\left(\sum_{j=0}^{3}(d_j^i)^2 - 2\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}}\left(\sum_{j=0}^{3} d_j^i\mathbb{B}_j^3\left(\frac{t-t_i}{h}\right)\right)f(t)dt\right)$$

Since the least squares method for solving differential equations using Bézier control points minimizes the residuals of the all degree 3 approximate solutions, the approximate solution $v(t)$'s is smaller than that of approximate solution $I_h u$.

Namely, we obtain

$$\|u(t) - v(t)\|^2_{H^2[0,1]} \leq C\|R(I_h u)\|^2_\infty \leq C\varepsilon^2.$$

Hence

$$\|u(t) - v(t)\|_\infty < K\varepsilon,$$

where $K = \sqrt{C}$. □

CHAPTER THREE
NUMERICAL EXAMPLES

3.1 Numerical Results

To illustrate the least squares method for solving differential equations using Bézier control points based on division of interval applied to the second order two-point boundary value problem, we use the problem

$$(t^{\alpha}u')' = t^{5+2\alpha}\ln(t), \qquad 0 < t < 1, \tag{3.1.1}$$

with the boundary conditions

$$u(0) = 1, \qquad u(1) = -\frac{\alpha}{1-\alpha} - \frac{13+3\alpha}{(6+2\alpha)^2(7+\alpha)^2}$$

considered in Twizell (Twizell, 1987). The exact solution is

$$u(t) = 1 - \frac{t^{1-\alpha}}{1-\alpha} + \frac{t^{7+\alpha}lnt}{(6+2\alpha)(7+\alpha)} - \frac{(13+3\alpha)t^{7+\alpha}}{(6+2\alpha)^2(7+\alpha)^2}.$$

The following graphs show the exact solutions and the approximate cubic Bézier polynomial solutions of (3.1.1) for various numbers of intervals and α.

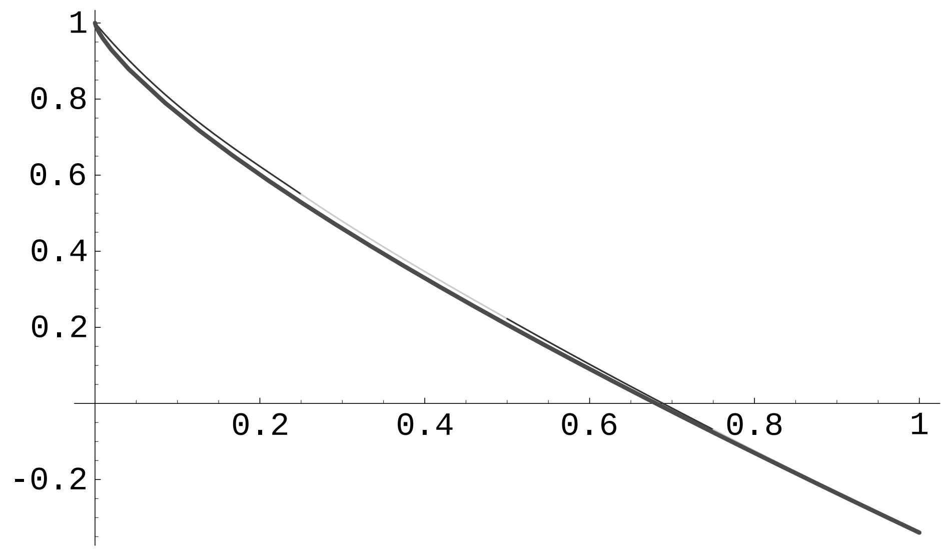

Figure 3.1: The dashed curve is the approximate solution for $\alpha = 0,25$ with 4 intervals and the thick curve is the exact solution. The maximal error is less then or equal to 0,0237087.

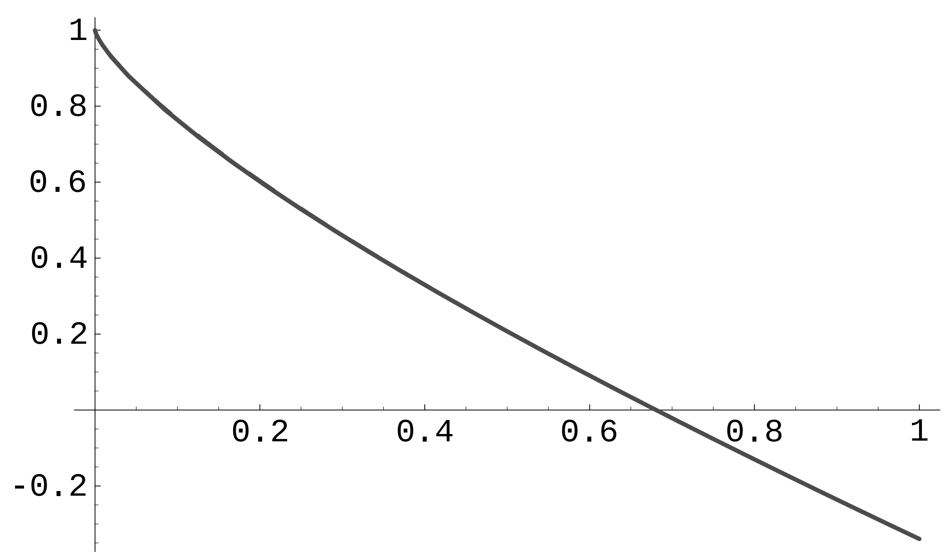

Figure 3.2: The dashed curve is the approximate solution for $\alpha = 0,25$ with 32 intervals and the thick curve is the exact solution. The maximal error is less then or equal to 0,00698.

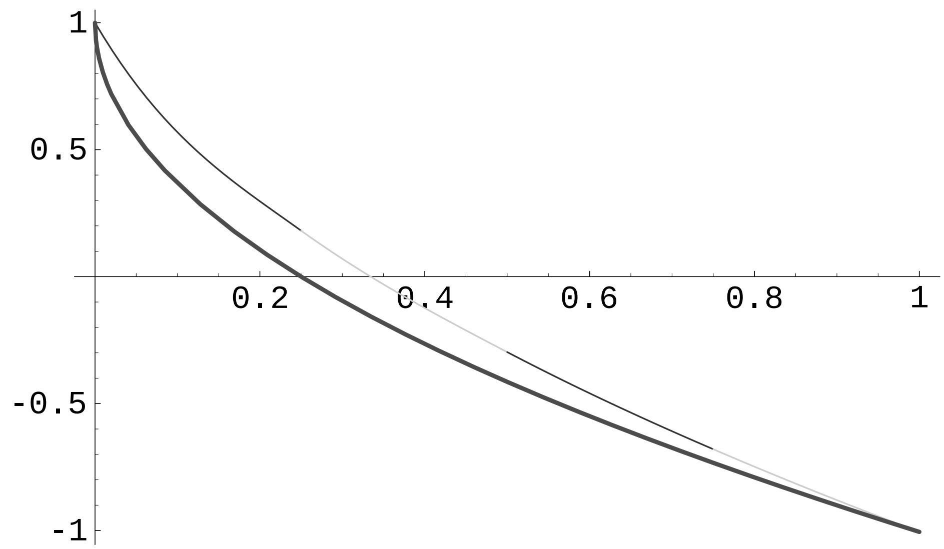

Figure 3.3: The dashed curve is the approximate solution for $\alpha = 0,50$ with 4 intervals and the thick curve is the exact solution. The maximal error is less then or equal to 0,204898.

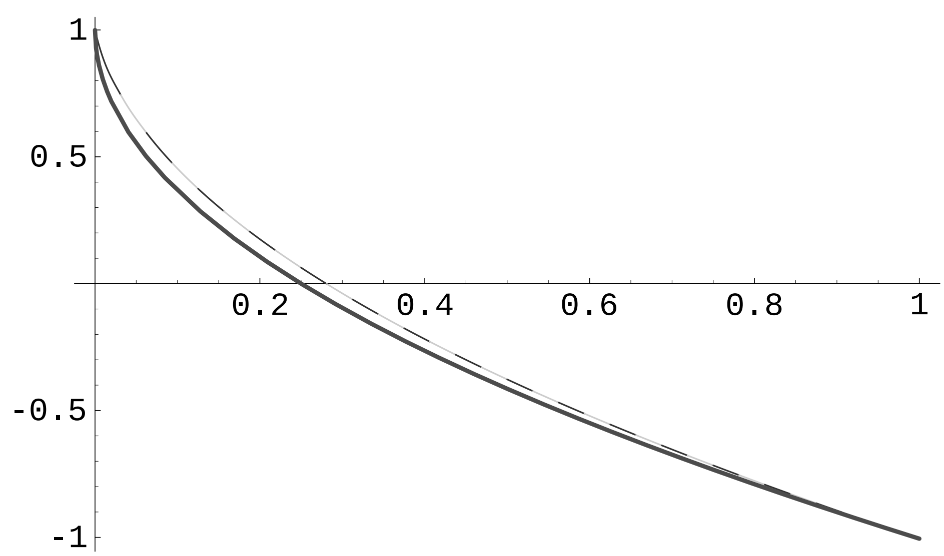

Figure 3.4: The dashed curve is the approximate solution for $\alpha = 0,50$ with 32 intervals and the thick curve is the exact solution. The maximal error is less then or equal to 0,0975186.

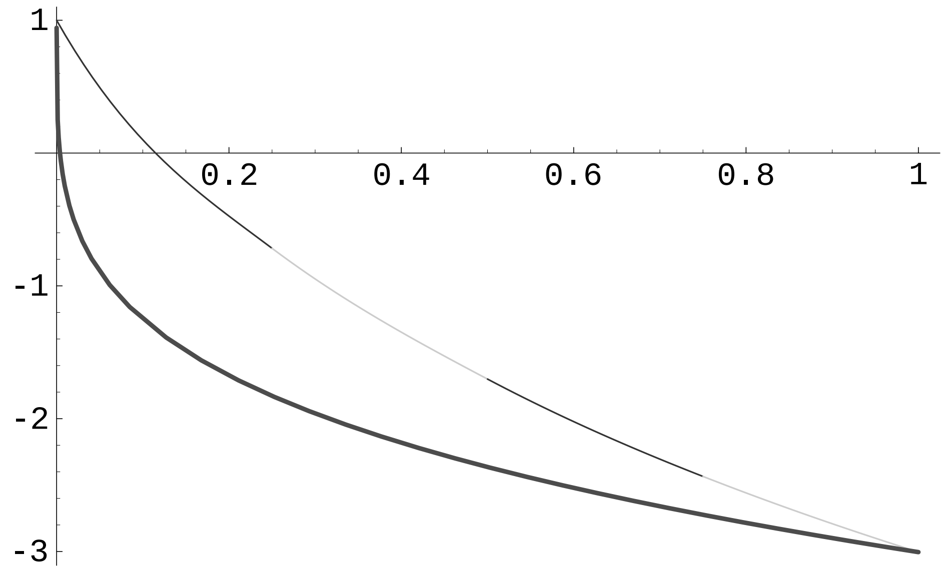

Figure 3.5: The dashed curve is the approximate solution for $\alpha = 0,75$ with 4 intervals and the thick curve is the exact solution. The maximal error is less then or equal to 1,38624.

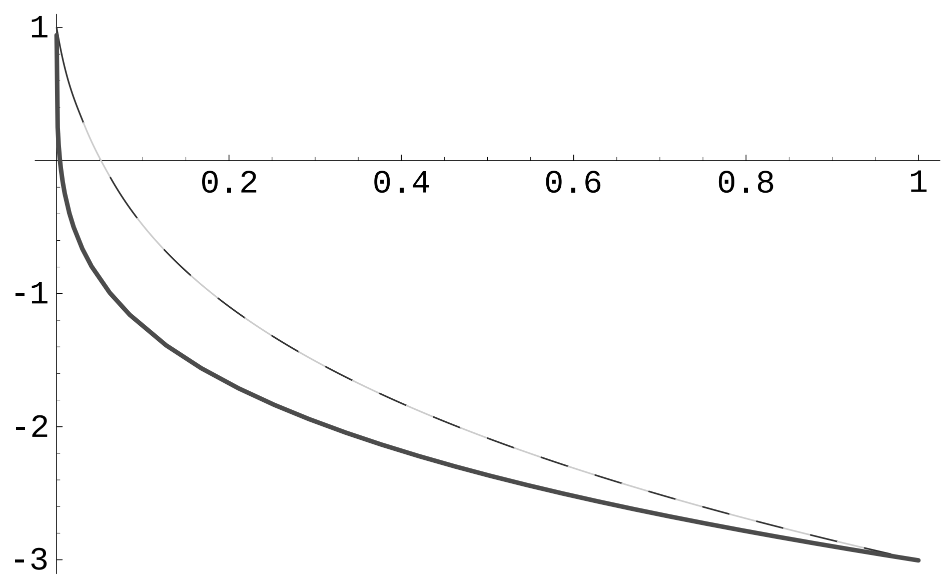

Figure 3.6: The dashed curve is the approximate solution for $\alpha = 0,75$ with 32 intervals and the thick curve is the exact solution. The maximal error is less then or equal to 0,974622.

REFERENCES

Ciarlet, P. G., Natterer, F.,& Varga R. S. (1970). Numerical methods of higher order accuracy for singular nonlinear boundary value problems. *Numer. Math., 15*, 87-99.

de Boor, C. (1978). *A practical guide to splines* . New York: Springer.

Ertem, S.,& Somalı, Ş. (1998). The iterated defect correction methods for singular two-point boundary value problems. *Intern. J. Math., 69*, 331-349.

Farin, G. (1997). *Curves and surfaces for computer aided geometric desing , A practical guide* (4th ed.). New York: Academic Press.

Jamet, P. (1970). On the convergence of finite difference approximations to one-dimensional singular boundary value problems. *Numer. Math., 14*, 355-378.

Lai, M.J.,& Wenston, P. (2000). Bivariate spline method for numerical solution of Navier-Stokes equations over polygons in stream function formulation. *Numer. Methods PDE, 16*, 147-183.

Reddien, G. W. (1973). Projection methods and singular two point boundary value problems. *Numer. Math., 21*, 193-205.

Reddien, G. W.,& Schumaker, L. L. (1976). On a collacation method for singular two point boundary value problems. *Numer. Math., 25*, 427-432.

Shariff, K.,& Moser, R. D. (1998). Two-dimensional mesh embedding for B-spline methods. *J. Comput. Phys., 145*, 471-488.

Twizell, E. H. (1987). One-, two- and three-grid extrapolation methods for a class of singular two-point boundary problems. *Communications in Applied Numerical Methods, 3*, 195-199.

Zheng, J., Sederberg, T. W.,& Johnson, R.W. (2004). Least squares methods for solving differential equations using Bézier control points. *Applied Numerical Mathematics, 48*, 237-252.