DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

INITIAL BOUNDARY VALUE PROBLEM FOR THE ELASTICITY SYSTEM IN THE SPHERICAL DOMAIN

by
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İZMİR

INITIAL BOUNDARY VALUE PROBLEM FOR THE ELASTICITY SYSTEM IN THE SPHERICAL DOMAIN

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M. Sc. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"INITIAL BOUNDARY VALUE PROBLEM FOR THE ELASTICITY SYSTEM IN THE SPHERICAL DOMAIN"** completed by **MELTEM TOPÇUOĞLU** under supervision of **PROF. DR. VALERY G. YAKHNO** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Meltem TOPÇUOĞLU

INITIAL BOUNDARY VALUE PROBLEM
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ABSTRACT

Lame's system of elasticity for homogeneous isotropic media is considered in this thesis. The main objects of the study are Initial Value and Initial Boundary Value Problems. Using scalar and vector potentials the Initial Value Problem (IVP) for Lame's system is reduced to IVP for scalar and vector wave equations. An exact formula for the solution is obtained. Using Fourier series expansion method and spherical and generalized functions theory, formulas for the solutions of Initial Boundary Value Problems (IBVP) are obtained. These formulas are adjusted for pulse point source, explosion, pulse rotation center. Simulations of elastic waves are created by these formulas.

Keywords: Elasticity system; IVP; IBVP; generalized solution; simulation; waves.

KÜRESEL BÖLGEDE ELASTİK SİSTEMİN SINIR DEĞER PROBLEMLERİ

ÖZ

Bu tezde, homojen izotropik ortamda elastikiyetin Lame sistemi ele alındı. Çalışmanın ana teması, başlangıç değer ve sınır değer problemleridir. Skalar ve vektör potensiyelleri kullanılarak Lame sistemin başlangıç değer problemi (BDP), skalar ve vektör dalga denklemlerinin başlangıç değer problemlerine indirgendi. Çözümün kesin formülü elde edildi. Fourier serileri açılım methodu, küresel ve genelleştirilmiş fonksiyonlar teorisi kullanılarak başlangıç sınır değer problemlerinin (BSDP) çözümleri elde edildi. Bu formuller nokta etkili, patlama, merkezde donme etkili kaynaklar için uygulandı. Bu formullerle elastik dalgaların simulasyonları yapıldı.

Anahtar sözcükler: Elastik sistem; BDP; BSDP; genelleştirilmiş çözüm; simulasyon; dalgalar.

CONTENTS

CHAPTER ONE
INTRODUCTION

Phenomena of elastic waves are the main object for the study in different applied sciences:(Geophysics, materials, solids and structures sciences, etc.) [(Ting, 1996), (Ting & Barnett & Wu, 1990), (Fedorov, 1968), (Yakhno, 1998), (Yakhno & Merazhov, 2000), (Yakhno & Akmaz, 2005), (Yakhno & Akmaz, 2005), (Yakhno, 2005)].

Modern development of the theory and methods for solving initial value and boundary value problems from one hand [(Boyce & DiPrima, 1992), (Cohen, 2002), (Cohen & Heikkola & Joly & Neittaan, 2003), (Dieulesaint & Royer, 1980), (Goldberg, 1992), (Monk, 2003), (Ramo & Whinnery & Duzer, 1994) and the computer facilities from another hand (Beltzer, 1990), (Pavlovic, 2003), (Pavlovic & Sapouuntzakis, 1986)] give the great chance to study more about invisible phenomena of the elastic wave propagation. The visualization of the elastic wave propagation is the main important item of this study.

The main object of the thesis study is the Lame's system of elasticity. One problem of the thesis is the Initial Value Problem (IVP) for this system which is studied for $x = (x_1, x_2, x_3) \in \mathbf{R}^3, \quad t \in \mathbf{R}$ variables. Another problem is Initial Boundary Value Problem (IBVP) for the system of isotropic elasticity which is studied in a spherical domain.

The main aim of the study is to

- obtain explicit formula for the solution of IVPs and IBVPs,
- adjust these formulas for pulse point sources (pulse directional force, explosion, pulse rotation center),
- simulate elastic waves using these formulas and modern computer tools,
- obtain 3-D graphs and animated movies of processes of the wave propagations,
- analyze the obtained simulation results.

The theory and methods of partial differential equations, generalized functions, ordinary differential equations, spectral analysis, symbolic commutation, `Mathematica`, graphic facilities are actively used in the thesis.

CHAPTER TWO
SCALAR AND VECTOR POTENTIALS

The initial value problem for the Lame system of elasticity for homogeneous isotropic medium is solved in this section. The Lame's system describes the propagation of seismic waves in the Earth. There are two types of waves in isotropic elastic media: longitudinal and transverse waves. The main aim of this section is modeling and simulation of the elastic wave propagation in unbounded 3-D isotropic elastic media on the basis of Lame's system of partial differential equations and modern computer facilities and tools. Using the vector analysis technique and the theory of partial differential equations and generalized functions theory explicit formula for a fundamental solution of the initial value problem for the Lame's vector operator was obtained. Using this formula we made modeling and simulations of elastic waves arising from pulse directional forces by means of `Mathematica`

2.1 Helmholtz's Theorem

Let $\mathbf{Z}(\mathbf{x})$ be a vector field whose components $Z_1(\mathbf{x}), Z_2(\mathbf{x}), Z_3(\mathbf{x})$ are continuous over $\mathbf{R}^3$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$. Then

$$\mathbf{Z} = \nabla_x X + \mathrm{curl}_x \mathbf{Y},$$

for appropriate X and $\mathbf{Y}$ on $\mathbf{R}^3$.

We know that $\mathrm{div}_x \mathrm{curl}_x \mathbf{Y}{=}0$ $(\nabla_x \cdot (\nabla_x \times \mathbf{Y}) = 0)$ and $\mathrm{curl}_x \mathrm{grad}_x X{=}0$, $(\nabla_x \times (\nabla_x X) = 0)$. Recalling the Helmholtz's theorem, we can write any vector field as the sum the gradient of a scalar potential and curl_x of a vector potential:

$$\mathbf{Z} = \nabla_x X + \mathrm{curl}_x \mathbf{Y}.$$

Note that we may assume $\nabla_x \cdot \mathbf{Y} = 0$ without loss of generality, since the curl_x operator discards any components of $\mathbf{Y}$ with non-zero divergence.

Representing the field in the above manner, we note the two vector identities:

$$\nabla_x \times (\nabla_x X) = 0,$$

and

$$\nabla_x \cdot (\nabla_x \times \mathbf{Y}) = 0.$$

Hence the portion of the displacement field represented by the gradient of the scalar potential X has the curl_x equal to zero, expressing no rotation. Similarly the portion of the field represented by the curl_x of the vector potential $\mathbf{Y}$ has a divergence equal to zero, expressing no volume charge.
Assume

$$\triangle_x \mathbf{V} = \mathbf{Z}(x).$$

It implies

$$\triangle_x \mathbf{V} = \mathbf{Z} = \nabla_x X + \mathrm{curl}_x \mathbf{Y}.$$

We have the following property

$$\triangle_x \mathbf{V} = \nabla_x \mathrm{div}_x \mathbf{V} - \mathrm{curl}_x \mathrm{curl}_x \mathbf{V}.$$

Using this property we obtain $X = \mathrm{div}_x \mathbf{V}$ and $\mathbf{Y} = -\mathrm{curl}_x \mathbf{V}$.
Also

$$\triangle_x \mathbf{V} = \triangle_x (G * \mathbf{Z}) = \triangle_x G * \mathbf{Z} = \delta * \mathbf{Z} = \mathbf{Z}$$

$$G * \mathbf{Z} = \int_{\mathbf{R}^3} G(x-\xi)\mathbf{Z}(\xi)\mathrm{d}\xi,$$

where $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$. Therefore

$$\mathbf{V} = -\frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{\mathbf{Z}(\xi)}{|x-\xi|}\mathrm{d}\xi.$$

Thus we obtain

$$X = \mathrm{div}_x \mathbf{V} = -\frac{1}{4\pi} \mathrm{div}_x \int_{\mathbf{R}^3} \frac{\mathbf{Z}(\xi)}{|x-\xi|}\mathrm{d}\xi,$$

$$\mathbf{Y} = \mathrm{curl}_x \mathbf{V} = \frac{1}{4\pi} \mathrm{curl}_x \int_{\mathbf{R}^3} \frac{\mathbf{Z}(\xi)}{|x-\xi|}\mathrm{d}\xi.$$

2.2 Reduction of Lame System to the Wave Equation

Consider the Lame system

$$\rho \frac{\partial^2 \mathbf{U}}{\partial t^2} = (\lambda + 2\mu)\nabla_x \mathrm{div}_x \mathbf{U} - \mu \mathrm{curl}_x \mathrm{curl}_x \mathbf{U} + \mathbf{f}(x,t) \ , \tag{2.2.1}$$

where $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3, t \in \mathbf{R}$,
$\mathbf{U} = (U_1, U_2, U_3)$ is displacement vector, $U_j = U_j(\mathbf{x}, t)$, j=1,2,3,...;
$\mathbf{f} = (f_1, f_2, f_3)$ is a vector of external force, $f_j = f_j(\mathbf{x}, t)$, j=1,2,3,...;
ρ is the density of elastic medium,
λ, μ are Lame parameters(physical constants),
$\rho > 0, \quad \lambda + 2\mu > 0, \quad \mu > 0$ are positive constants.
This system describes elastic propagation (seismic wave propagation) in the Earth.

Remark 2.2.1. *If $\boldsymbol{f}$ is given then using the potential approach we can obtain $\boldsymbol{f}$ in the form $\boldsymbol{f} = \nabla_x g + curl_x \boldsymbol{F}$, where*

$$g = div_x \boldsymbol{V} = -\frac{1}{4\pi} div_x \int_{\mathbf{R}^3} \frac{\boldsymbol{f}(\xi)}{|x-\xi|} \mathrm{d}\xi \ ,$$

$$\boldsymbol{F} = curl_x \boldsymbol{V} = \frac{1}{4\pi} curl_x \int_{\mathbf{R}^3} \frac{\boldsymbol{f}(\xi)}{|x-\xi|} \mathrm{d}\xi \ ,$$

and $\boldsymbol{f} = \triangle_x \boldsymbol{V}$.

Lemma 2.2.1. *Let $\phi(x,t)$ be a solution of the scalar wave equation*

$$\rho \frac{\partial^2 \phi}{\partial t^2} = (\lambda + 2\mu)\triangle_x \phi + g \ ,$$

and $\boldsymbol{A}(x,t)$ is a solution of the vector wave equation

$$\rho \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \mu \triangle_x \boldsymbol{A} + \boldsymbol{F} \ ,$$

then the vector function $\boldsymbol{U} = \nabla_x \phi + curl_x \boldsymbol{A}$ is a solution of (2.2.1).
$\phi(x,t)$ is called the scalar potential and $\boldsymbol{A}(x,t)$ is called the vector potential.

Proof. Consider the Lame System (2.2.1). Substituting $\mathbf{f} = \nabla_x g + \mathrm{curl}_x \mathbf{F}$ and $\mathbf{U} = \nabla_x \phi + \mathrm{curl}_x \mathbf{A}$ to the last equation we obtain

$$\rho\left(\frac{\partial^2 (\nabla_x \phi)}{\partial t^2} + \frac{\partial^2 (\mathrm{curl}_x \mathbf{A}(x,t))}{\partial t^2}\right) = (\lambda + 2\mu)\nabla_x \mathrm{div}_x [\nabla_x \phi + \mathrm{curl}_x \mathbf{A}]$$

$$-\mu \mathrm{curl}_x \mathrm{curl}_x (\nabla_x \phi + \mathrm{curl}_x \mathbf{A}) + \nabla_x g + \mathrm{curl}_x \mathbf{F} \ . \qquad (2.2.2)$$

Using the following formulas

$$\frac{\partial^2 (\nabla_x \phi)}{\partial t^2} = \nabla_x (\frac{\partial^2 \phi}{\partial t^2}) \ ,$$

$$\frac{\partial^2 (\mathrm{curl}_x \mathbf{A}(x,t))}{\partial t^2} = \mathrm{curl}_x \frac{\partial^2 \mathbf{A}(x,t)}{\partial t^2} \ ,$$

$$\begin{aligned}\nabla_x \mathrm{div}_x [\nabla_x \phi + \mathrm{curl}_x \mathbf{A}] &= \nabla_x \mathrm{div}_x (\nabla_x \phi) + \nabla_x \mathrm{div}_x (\mathrm{curl}_x \mathbf{A}) \\ &= \nabla_x (\mathrm{div}_x \nabla_x \phi) + \nabla_x (\mathrm{div}_x \mathrm{curl}_x \mathbf{A}) \\ &= \nabla_x (\triangle_x \phi) \ ,\end{aligned}$$

$$\begin{aligned}\mu \mathrm{curl}_x \mathrm{curl}_x [\nabla_x \phi + \mathrm{curl}_x \mathbf{A}] &= \mu \mathrm{curl}_x \mathrm{curl}_x \nabla_x \phi + \mu \mathrm{curl}_x \mathrm{curl}_x \mathbf{A} \\ &= \mu \mathrm{curl}_x (\mathrm{curl}_x \nabla_x \phi) + \mu (\mathrm{curl}_x \mathrm{curl}_x \mathbf{A}) \\ &= \mu \mathrm{curl}_x (\nabla_x \mathrm{div}_x \mathbf{A} - \triangle_x \mathbf{A}) \\ &= \mu (\mathrm{curl}_x \nabla_x) \mathrm{div}_x \mathbf{A} - \mu \mathrm{curl}_x (\triangle_x \mathbf{A}) \\ &= -\mathrm{curl}_x (\mu \triangle_x \mathbf{A}) \ ,\end{aligned}$$

the vector equation (2.2.2) can be written in the form

$$\nabla_x (\rho \frac{\partial^2 \phi}{\partial t^2} - (\lambda + 2\mu) \triangle_x \phi - g) + \mathrm{curl}_x (\rho \frac{\partial^2 \mathbf{A}}{\partial t^2} - \mu \triangle_x \mathbf{A} - \mathbf{F}) = 0 \ .$$

Since $\phi(x,t)$ is a solution of the scalar wave equation

$$\rho \frac{\partial^2 \phi}{\partial t^2} = (\lambda + 2\mu) \triangle_x \phi + g \ ,$$

and $\mathbf{A}(x,t)$ is a solution of the vector wave equation

$$\rho \frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu \triangle_x \mathbf{A} + \mathbf{F} \ ,$$

then the vector function $\mathbf{U} = \nabla_x \phi + \mathrm{curl}_x \mathbf{A}$ is a solution of the Lame system. $\square$

For the solution of the Lame System we need to solve two wave equations

$$\rho \frac{\partial^2 \phi}{\partial t^2} = (\lambda + 2\mu) \triangle_x \phi + g \ ,$$

$$\rho\frac{\partial^2\mathbf{A}}{\partial t^2} = \mu\triangle_x\mathbf{A} + \mathbf{F} \ ,$$

or

$$\frac{1}{a_p^2}\frac{\partial^2\phi}{\partial t^2} = \triangle_x\phi + \frac{1}{\lambda+2\mu}g \ ,$$

$$\frac{1}{a_s^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \triangle_x\mathbf{A} + \frac{1}{\mu}\mathbf{F} \ ,$$

where

$$a_p^2 = \frac{\lambda+2\mu}{\rho}, \qquad a_s^2 = \frac{\mu}{\rho} \ ,$$

a_p is the velocity of longitudinal wave,
a_s is the velocity of transversal wave.

Definition 2.2.1. (Fundamental Solution)
A matrix $\mathcal{G}(x,t) = [G_{ij}(x,t)]_{3x3}$ whose $j-$th column

$$\mathbf{G}^j(x,t) = \begin{pmatrix} G_{1j}(x,t) \\ G_{2j}(x,t) \\ G_{3j}(x,t) \end{pmatrix}$$

satisfies

$$L\mathbf{G}^j(x,t) = \mathbf{e}^j\delta(x,t),$$

$$\mathbf{G}^j(x,t)\Big|_{t<0} = 0,$$

$j = 1,2,3$; where $\mathbf{e}^1 = (1,0,0)$, $\mathbf{e}^2 = (0,1,0)$, $\mathbf{e}^3 = (0,0,1)$ is the fundamental solution of the Cauchy problem for the Lame operator L.

Remark 2.2.2. *Let* $\boldsymbol{G}^j = \nabla_x\phi + curl_x\boldsymbol{A}$. *We need to present* $\boldsymbol{e}^j\delta(x,t)$ *in the following form:*

$$\boldsymbol{e}^j\delta(x,t) = \nabla_x g + curl_x\boldsymbol{F}.$$

Since

$$\delta(x) = \triangle_x\Big(-\frac{1}{4\pi|x|}\Big),$$

we have

$$\boldsymbol{e}^j\delta(x)\delta(t) = \triangle_x\Big(-\frac{\boldsymbol{e}^j}{4\pi|x|}\Big)\delta(t),$$

$$= \nabla_x div_x\Big(-\frac{\boldsymbol{e}^j}{4\pi|x|}\Big)\delta(t) - curl_x curl_x\Big(-\frac{\boldsymbol{e}^j}{4\pi|x|}\Big)\delta(t),$$

$$= \nabla_x div_x\Big(-\frac{\boldsymbol{e}^j}{4\pi|x|}\Big)\delta(t) + curl_x curl_x\Big(\frac{\boldsymbol{e}^j}{4\pi|x|}\Big)\delta(t),$$

$$= \nabla_x div_x\Big(-\frac{\boldsymbol{e}^j}{4\pi|x|}\delta(t)\Big) + curl_x curl_x\Big(\frac{\boldsymbol{e}^j}{4\pi|x|}\delta(t)\Big).$$

If $\boldsymbol{f} = \nabla_x g + curl_x \boldsymbol{F}$, *then*

$$g = -\frac{1}{4\pi} div_x\Big(\frac{\boldsymbol{e}^j}{|x|}\delta(t)\Big),$$

$$\boldsymbol{F} = \frac{1}{4\pi} curl_x\Big(\frac{\boldsymbol{e}^j}{|x|}\delta(t)\Big).$$

Lemma 2.2.2. *Let* $\boldsymbol{V}_a$ *be a solution of*

$$\frac{1}{a^2}\frac{\partial^2 \boldsymbol{V}_a}{\partial t^2} - \triangle_x \boldsymbol{V}_a = \frac{\boldsymbol{e}^j}{|x|}\delta(t) \ ,$$

$$\boldsymbol{V}_a(x,t)|_{t<0} = 0 \ ,$$

then

$$\phi(x,t) = -\frac{1}{4\pi\rho c_1^2} div_x \boldsymbol{V}_{c_1}(x,t),$$

$$\boldsymbol{A}(x,t) = \frac{1}{4\pi\rho c_2^2} curl_x \boldsymbol{V}_{c_2}(x,t),$$

are solutions of

$$\frac{1}{c_1^2}\frac{\partial^2 \phi}{\partial t^2} - \triangle_x \phi = \frac{1}{\rho c_1^2} g \ ,$$

$$\phi(x,t)\Big|_{t<0} = 0 \ ,$$

and

$$\frac{1}{c_2^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \triangle_x \boldsymbol{A} = \frac{1}{\rho c_2^2}\boldsymbol{F} \ ,$$

$$\boldsymbol{A}(x,t)\Big|_{t<0} = 0 \ ,$$

where

$$g = -\frac{1}{4\pi} div_x\Big(\frac{\boldsymbol{e}^j}{|x|}\delta(t)\Big),$$

$$\boldsymbol{F} = \frac{1}{4\pi} curl_x\Big(\frac{\boldsymbol{e}^j}{|x|}\delta(t)\Big).$$

Proof. Let $\mathbf{V}_{c_1}$ be a solution of

$$\frac{1}{c_1^2}\frac{\partial^2 \mathbf{V}_{c_1}}{\partial t^2} - \triangle_x \mathbf{V}_{c_1} = \frac{\mathbf{e}^j}{|x|}\delta(t) \ ,$$

$$\mathbf{V}_{c_1}(x,t)|_{t<0} = 0 \ .$$

Applying the operator $(-\dfrac{1}{4\pi\rho c_1^2}\mathrm{div}_x)$ to the above equation we have

$$-\frac{1}{4\pi\rho c_1^4}\mathrm{div}_x(\frac{\partial^2 \mathbf{V}_{c_1}}{\partial t^2}) + \frac{1}{4\pi\rho c_1^2}\mathrm{div}_x(\triangle_x \mathbf{V}_{c_1}) = -\frac{1}{4\pi\rho c_1^2}\mathrm{div}_x(\frac{\mathbf{e}^j}{|x|}\delta(t)) \ ,$$

$$\Big(-\frac{1}{4\pi\rho c_1^2}\mathrm{div}_x \mathbf{V}_{c_1}\Big)\Big|_{t<0} = 0.$$

Since

$$\mathrm{div}_x(\frac{\partial^2 \mathbf{V}_{c_1}}{\partial t^2}) = \frac{\partial^2}{\partial t^2}(\mathrm{div}_x \mathbf{V}_{c_1}) \ ,$$

and

$$\begin{aligned}\mathrm{div}_x(\triangle_x \mathbf{V}_{c_1}) &= \mathrm{div}_x(\nabla_x \mathrm{div}_x \mathbf{V}_{c_1} - \mathrm{curl}_x \mathrm{curl}_x \mathbf{V}_{c_1}) \\ &= (\mathrm{div}_x \nabla_x)\mathrm{div}_x \mathbf{V}_{c_1} \\ &= \triangle_x(\mathrm{div}_x \mathbf{V}_{c_1}) \ ,\end{aligned}$$

we have

$$\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\Big(-\frac{\mathrm{div}_x \mathbf{V}_{c_1}(x,t)}{4\pi\rho c_1^2}\Big) - \triangle_x\Big(-\frac{\mathrm{div}_x \mathbf{V}_{c_1}(x,t)}{4\pi\rho c_1^2}\Big) = \frac{1}{\rho c_1^2}\Bigg(-\frac{1}{4\pi}\mathrm{div}_x\Big(\frac{\mathbf{e}^j\delta(t)}{|x|}\Big)\Bigg),$$

$$\Big(-\frac{1}{4\pi\rho c_1^2}\mathrm{div}_x \mathbf{V}_{c_1}(x,t)\Big)\Big|_{t<0} = 0 \ .$$

Using this equality we can say that

$$\phi(x,t) = -\frac{1}{4\pi\rho c_1^2}\mathrm{div}_x \mathbf{V}_{c_1}(x,t)$$

is a solution of

$$\frac{1}{c_1^2}\frac{\partial^2 \phi}{\partial t^2} - \triangle_x \phi = \frac{1}{\rho c_1^2} g \ , \ x \in \mathbf{R}^3, \ t \in \mathbf{R},$$

$$\phi(x,t)\Big|_{t<0} = 0 \ ,$$

where

$$g = -\frac{1}{4\pi}\mathrm{div}_x\Big(\frac{\mathbf{e}^j}{|x|}\delta(t)\Big).$$

Let $a = c_2$. Then applying the operator $(\dfrac{1}{4\pi\rho c_2^2}\mathrm{curl}_x)$ to equations of the

Lemma 2.2.2 we have

$$\frac{1}{4\pi\rho c_2^4}\text{curl}_x(\frac{\partial^2\mathbf{V}_{c_2}}{\partial t^2})-\frac{1}{4\pi\rho c_2^2}\text{curl}_x(\triangle_x\mathbf{V}_{c_2})=\frac{1}{4\pi\rho c_2^2}\text{curl}_x(\frac{\mathbf{e}^j}{|x|}\delta(t))\ \ ,$$

$$\Big(\frac{1}{4\pi\rho c_2^2}\text{curl}_x\mathbf{V}_{c_2}\Big)\Big|_{t<0}=0.$$

Since

$$\text{curl}_x(\frac{\partial^2\mathbf{V}_{c_2}}{\partial t^2})=\frac{\partial^2}{\partial t^2}(\text{curl}_x\mathbf{V}_{c_2})\ \ ,$$

and

$$\begin{aligned}\text{curl}_x(\triangle_x\mathbf{V}_{c_2}) &= \text{curl}_x(\nabla_x\text{div}_x\mathbf{V}_{c_2}-\text{curl}_x\text{curl}_x\mathbf{V}_{c_2})\\ &=(\text{curl}_x\nabla_x)\text{div}_x\mathbf{V}_{c_2}-(\text{curl}_x\text{curl}_x)\text{curl}_x\mathbf{V}_{c_2}\\ &=-(\nabla_x\text{div}_x-\triangle_x)\text{curl}_x\mathbf{V}_{c_2}\\ &=-\nabla_x(\text{div}_x\text{curl}_x\mathbf{V}_{c_2})+\triangle_x(\text{curl}_x\mathbf{V}_{c_2})\\ &=\triangle_x(\text{curl}_x\mathbf{V}_{c_2})\ \ ,\end{aligned}$$

we have

$$\frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\Big(\frac{\text{curl}_x\mathbf{V}_{c_2}(x,t)}{4\pi\rho c_2^2}\Big)-\triangle_x\Big(\frac{\text{curl}_x\mathbf{V}_{c_2}(x,t)}{4\pi\rho c_2^2}\Big)=\frac{1}{\rho c_2^2}\left(\frac{1}{4\pi}\text{curl}_x\Big(\frac{\mathbf{e}^j\delta(t)}{|x|}\Big)\right)\ \ ,$$

$$\Big(\frac{1}{4\pi\rho c_2^2}\text{curl}_x\mathbf{V}_{c_2}(x,t)\Big)\Big|_{t<0}=0\ \ .$$

Therefore we can say that

$$\mathbf{A}(x,t)=\frac{1}{4\pi\rho c_2^2}\text{curl}_x\mathbf{V}_{c_2}(x,t)$$

is a solution of

$$\frac{1}{c_2^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\triangle_x\mathbf{A}=\frac{1}{\rho c_2^2}\mathbf{F}\ \ ,$$

$$\mathbf{A}(x,t)\Big|_{t<0}=0\ \ ,$$

where

$$\mathbf{F}=\frac{1}{4\pi}\text{curl}_x\Big(\frac{\mathbf{e}^j}{|x|}\delta(t)\Big).$$

□

2.3 Constructing the Solution of the Generalized Cauchy Problem for the Wave Equation

Let us consider the following generalized Cauchy problem:

$$\frac{1}{a^2}\frac{\partial^2 \mathbf{V}_a}{\partial t^2} - \triangle_x \mathbf{V}_a = \frac{\mathbf{e}^j}{|x|}\delta(t), \quad \mathbf{x} \in \mathbf{R}^3, t \in \mathbf{R} \tag{2.3.1}$$

$$\mathbf{V}_a(x,t)\Big|_{t<0} = 0 \ , \tag{2.3.2}$$

where a is a given positive number. A solution of (2.3.1) and (2.3.2) may be found using the formula

$$\mathbf{V}_a(x,t) = G * \left[\frac{\mathbf{e}^j}{|x|}\delta(t)\right] \ ,$$

where

$$G(x,t) = \frac{a}{2}\theta(t)\frac{\delta(a^2t^2 - |x|^2)}{\pi} \ ,$$

so

$$\mathbf{V}_a(x,t) = \int_{R^4} G(x-\zeta, t-\tau)\frac{\mathbf{e}^j}{|\zeta|}\delta(\tau)\mathrm{d}\tau\mathrm{d}\zeta = \int_{\mathbf{R}^3} G(x-\zeta, t)\frac{\mathbf{e}^j}{|\zeta|}\mathrm{d}\zeta$$

$$= \int_{\mathbf{R}^3}\theta(t)\frac{1}{4\pi}\frac{\delta(t-\frac{|x-\zeta|}{a})}{|x-\zeta|}\frac{\mathbf{e}^j}{|\zeta|}\mathrm{d}\zeta = \frac{\theta(t)\mathbf{e}^j}{4\pi}\int_{\mathbf{R}^3}\frac{\delta(t-\frac{|x-\zeta|}{a})}{|x-\zeta||\zeta|}\mathrm{d}\zeta.$$

Remark 2.3.1. *Let* $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in R^3$, $x = (x_1, x_2, x_3) \in R^3$,

$$\zeta = x + r\upsilon, \quad \upsilon = (\upsilon_1, \upsilon_2, \upsilon_3),$$

$$r = |\zeta - x|, \quad \upsilon = \frac{\zeta - x}{|\zeta - x|},$$

where

$$\upsilon_1 = \cos\varphi\sin\theta, \quad \upsilon_2 = \sin\varphi\sin\theta, \quad \upsilon_3 = \cos\theta,$$

$$\mathrm{d}\zeta = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi.$$

Then

$$\mathbf{V}_a(x,t) = \frac{\theta(t)\mathbf{e}^j}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{1}{|x+r\upsilon|}\frac{1}{r}\delta(t-\frac{r}{a})r^2\sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$$

$$= \frac{\theta(t)\mathbf{e}^j}{4\pi}\int_0^{2\pi}\int_0^{\pi}\Big(\int_0^{\infty}\frac{a}{|x+r\upsilon|}\delta(r-at)r\mathrm{d}r\Big)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi$$

$$= \frac{\theta(t)\mathbf{e}^j a^2 t}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{1}{|x+at\upsilon|}\sin\theta\mathrm{d}\theta\mathrm{d}\varphi. \tag{2.3.3}$$

Remark 2.3.2. *Let* $x \in \mathbf{R}^3$

$$|x+at\upsilon| = \sqrt{|x|^2+a^2t^2+2|x|at\cos\theta}.$$

Then

$$\frac{1}{\sqrt{|x|^2+a^2t^2+2|x|at\cos\theta}} = \begin{cases} \dfrac{1}{at}\displaystyle\sum_{n=0}^{\infty}\Big(\frac{|x|}{at}\Big)^n P_n(-\cos\theta), & |x|<at, \\ \\ \dfrac{1}{|x|}\displaystyle\sum_{n=0}^{\infty}\Big(\frac{at}{|x|}\Big)^n P_n(-\cos\theta), & |x|>at. \end{cases}$$

Remark 2.3.3. *(See, Appendix C) The Legendre's polynomials* $P_n(-\cos\theta)$ $n = 0,1,2,...$ *satisfy the following property*

$$\int_0^{\pi}P_n(-\cos\theta)\sin\theta\mathrm{d}\theta = \int_{-1}^{1}P_n(z)dz$$

$$= \int_{-1}^{1}P_n(z)P_0(z)dz$$

$$= \begin{cases} 2, & n=0 \\ \\ 0, & n\neq 0. \end{cases}$$

Using the Remark 2.3.2 and (2.3.3) we find

$$
\mathbf{V}_a(\theta,\varphi) = \begin{cases} \dfrac{\theta(t)\mathbf{e}^j a^2 t}{4\pi a t} \displaystyle\sum_{n=0}^{\infty} \int_0^{2\pi} \int_0^{\pi} \Big(\frac{|x|}{at}\Big)^n P_n(-\cos\theta)\sin\theta \mathrm{d}\theta\mathrm{d}\varphi, & |x| < at, \\ \\ \dfrac{\theta(t)\mathbf{e}^j a^2 t}{4\pi} \displaystyle\sum_{n=0}^{\infty} \int_0^{2\pi} \int_0^{\pi} \Big(\frac{at}{|x|}\Big)^n \frac{1}{|x|} P_n(-\cos\theta)\sin\theta \mathrm{d}\theta\mathrm{d}\varphi, & |x| > at. \end{cases} \tag{2.3.4}
$$

Using Remark 2.3.3 we find from (2.3.4)

$$
\mathbf{V}_a(\theta,\varphi) = \begin{cases} \dfrac{\theta(t)\mathbf{e}^j a}{4\pi} \displaystyle\int_0^{2\pi} 2\mathrm{d}\varphi = \theta(t)\mathbf{e}^j a, & |x| < at, \\ \\ \dfrac{\theta(t)\mathbf{e}^j a^2 t}{4\pi} \displaystyle\int_0^{2\pi} \frac{2}{|x|}\mathrm{d}\varphi = \frac{1}{|x|}\theta(t)\mathbf{e}^j a^2 t, & |x| > at. \end{cases} \tag{2.3.5}
$$

The formula (2.3.5) may be written as

$$
\begin{aligned} \mathbf{V}_a(x,t) &= \theta(t)\mathbf{e}^j a\theta(at-|x|) + \frac{\theta(t)\mathbf{e}^j a^2 t}{|x|}\theta(|x|-at) \\ &= \theta(t)\mathbf{e}^j a\Big[\theta(at-|x|) + \frac{at}{|x|}\theta(|x|-at)\Big]. \end{aligned}
$$

Therefore using the Lemma 2.2.2 we find

$$
\phi(x,t) = -\frac{\theta(t)}{4\pi\rho c_1}\mathrm{div}_x\Big[\mathbf{e}^j\Big(\theta(c_1 t-|x|) + \frac{c_1 t}{|x|}\theta(|x|-c_1 t)\Big)\Big],
$$

$$
\mathbf{A}(x,t) = \frac{\theta(t)}{4\pi\rho c_2}\mathrm{curl}_x\Big[\mathbf{e}^j\Big(\theta(c_2 t-|x|) + \frac{c_2 t}{|x|}\theta(|x|-c_2 t)\Big)\Big].
$$

As a result the fundamental solution's components are given by

$$
\begin{aligned} \mathbf{G}^j = \frac{\theta(t)}{4\pi\rho}\{&-\frac{1}{c_1}\nabla_x\mathrm{div}_x\Big[\mathbf{e}^j\Big(\theta(c_1 t-|x|) + \frac{c_1 t}{|x|}\theta(|x|-c_1 t)\Big)\Big] \\ &+\frac{1}{c_2}\mathrm{curl}_x\mathrm{curl}_x\Big[\mathbf{e}^j\Big(\theta(c_2 t-|x|) + \frac{c_2 t}{|x|}\theta(|x|-c_2 t)\Big)\Big]\}. \end{aligned} \tag{2.3.6}
$$

Therefore the fundamental solution is the matrix

$$
\mathcal{G}(x,t) = (\mathbf{G}^1(x,t), \mathbf{G}^2(x,t), \mathbf{G}^3(x,t)),
$$

whose components are vectors

$$\mathbf{G}^j(x,t) = (G_1^j(x,t), G_2^j(x,t), G_3^j(x,t)), \quad j = 1,2,3.$$

2.4 Mathematica Commands for Finding the Components of the Fundamental Solution

In this subsection we use the formula (2.3.6) in programming tool `Mathematica` for the simulation of the wave propagation in elastic isotropic media.
`Mathematica` commands for finding $G_1^1(x,t)$, $G_2^1(x,t)$, $G_3^1(x,t)$ are listed below.

```
INPUT: mu,lambda,rho,varepsilon;
k = UnitStep[t]/(4*Pi*rho);
c1 = Sqrt[(lambda +2*mu)/rho];
c2 = Sqrt[mu/rho];
r = x^2 + y^2 + z^2;
a = -1/r^(3/2);
aa= (x*y)/r^(3/2);
aaa=(x*z)/r^(3/2);
b1 =(-3*t/r^(1/2)) + 1/(c1) + t/(x^2*r^(-1/2)) -1/(c1*x^2*r^(-2));
b2=(-3*t/r^(1/2))+1/(c2)+t/(y^2*r^(-1/2))-1/(c2*y^2*r^(-2));
b3=(-3*t/r^(1/2))+1/(c2)+t/(z^2*r^(-1/2))-1/(c2*z^2*r^(-2));
bb = -3*t/r^(1/2) + 1/(c);
bb1 = ReplaceAll[bb, {c -> c1}];
bb2 = ReplaceAll[bb, {c -> c2}];
bbb=-3*t/r^(1/2) + 1/(c);
bbb1 = ReplaceAll[bbb, {c -> c1}];
bbb2=ReplaceAll[bbb, {c -> c2}];
d = -c*t + Sqrt[r];
d1 = ReplaceAll[d, {c -> c1}];
d2 = ReplaceAll[d, {c -> c2}];
f =(1/(2*Sqrt[Pi*varepsilon]))*Exp[(-d^2)/(4*varepsilon)];
f1 = ReplaceAll[f, {c -> c1}];
f2= ReplaceAll[f, {c -> c2}];
gx = D[f, x];
gy = D[f, y];
gz = D[f,z];
```

```
gx1 = ReplaceAll[gx, {c -> c1}];
gx2 = ReplaceAll[gx, {c -> c2}];
gy1 = ReplaceAll[gy, {c -> c1}];
gy2 = ReplaceAll[gy,{c ->c2}];
gz1 = ReplaceAll[gz, {c -> c1}];
gz2 = ReplaceAll[gz, {c -> c2}];
h = t-1/(c*r^(-1/2));
h1 =ReplaceAll[h, {c -> c1}];
h2=ReplaceAll[h,{c->c2}];
G[1,1]=a*k*(x^2*b1*f1+x^2*(3*t/r-t/x^2)*UnitStep[d1]+ x^2*h1*gx1+
y^2*b2*f2+y^2*(3*t/r-t/y^2)*UnitStep[d2]+y^2*h2*gy2
+z^2*b3*f2+z^2*(3*t/r-t/z^2)*UnitStep[d2]+z^2*h2*gz2);
G[1,2] = aa*k*(bb2*f2 + (3/r)*UnitStep[d2] + h2*gx2 - bb1*f1
                              -(3/r)*UnitStep[d1] -h1*gy1);
G[1,3] = aaa*k*(bbb2*f2 + (3/r)*UnitStep[d2] + h2*gx2 - bbb1*f1
                              -(3/r)*UnitStep[d1] - h1*gz1);
OUTPUT: G[1,1], G[1,2], G[1,3]  ;
```

Here `G[1,1], G[1,2], G[1,3]` are $G_1^1(\mathbf{x},t)$, $G_2^1(\mathbf{x},t)$, $G_3^1(\mathbf{x},t)$

Other components of $\mathcal{G}(\mathbf{x},t)$ can be calculated by the similar `Mathematica` commands. Using Mathematica codes we simulated the elastic wave propagation arising from pulse directional forces. The results of the simulation are presented in the figures (2.1)-(2.3).

In these figures the horizontal axes are x_1, x_2 the vertical axis is values of one of components of $\mathbf{G}^1$. In figures (2.1)-(2.3) the three dimensional graphs correspond to the following values

$$\lambda = 4,\ \rho = 2.203,\ \mu = 3.12,$$

$$c_1 = \sqrt{\frac{(\lambda + 2\mu)}{\rho}},\ c_2 = \sqrt{\frac{\mu}{\rho}},$$

where c_1, c_2 are longitudinal and transverse speeds of elastic waves.
Analyzing figures (2.1)-(2.3) we can see arising the elastic waves on figures (2.1a), (2.1b), (2.2a), (2.2b), (2.3a), (2.3b). Two different waves with different speeds we can see on figures 2.1(c,d,e,f), 2.2(c,d,e,f), 2.3(c,d,e,f). Analyzing the formula

(2.3.6) we find two terms

$$\mathbf{U}^{jp} = \frac{\theta(t)}{4\pi\rho}\{-\frac{1}{c_1}\nabla_x \mathrm{div}_x\Big[\mathbf{e}^j\Big(\theta(c_1 t - |x|) + \frac{c_1 t}{|x|}\theta(|x| - c_1 t)\Big)\Big]\}, \tag{2.4.1}$$

$$\mathbf{U}^{js} = \frac{\theta(t)}{4\pi\rho}\{\frac{1}{c_2}\mathrm{curl}_x \mathrm{curl}_x\Big[\mathbf{e}^j\Big(\theta(c_2 t - |x|) + \frac{c_2 t}{|x|}\theta(|x| - c_2 t)\Big)\Big]\}. \tag{2.4.2}$$

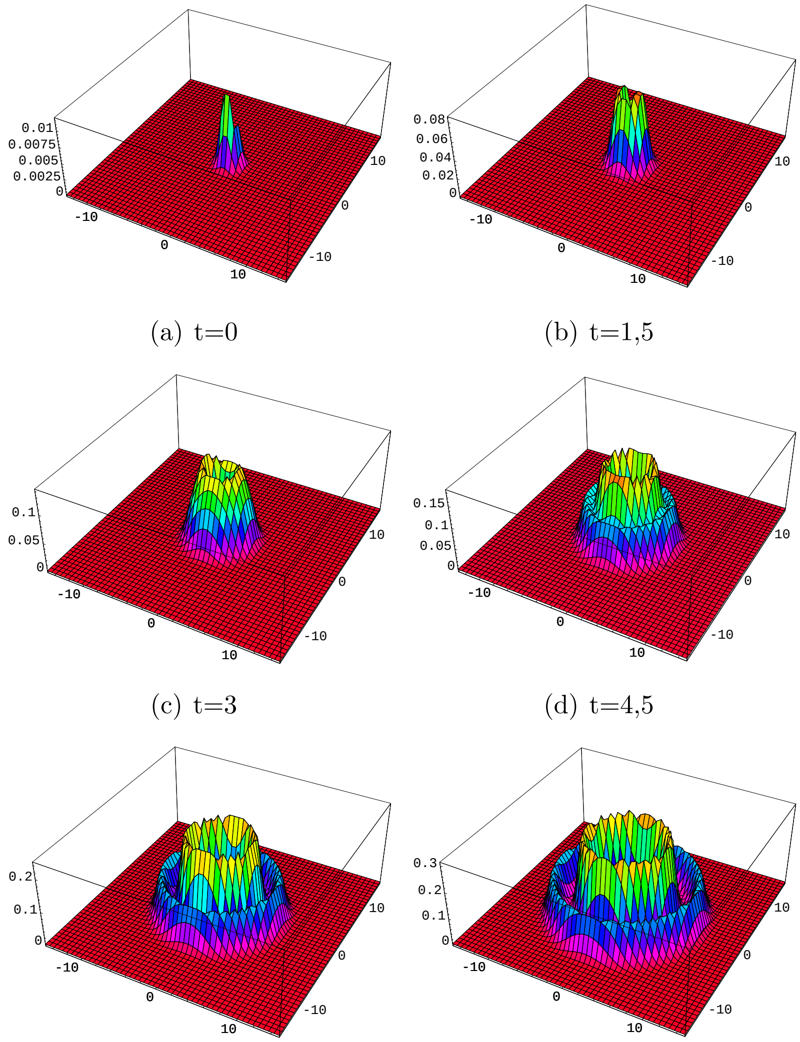
The term $\mathbf{U}^{jp}$ depends on c_1 and does not depend on c_2, the term $\mathbf{U}^{js}$ depends on c_2 and does not depend on c_1. These terms give an analytic presentation of longitudinal and transverse waves respectively. Using formulas (2.4.1),(2.4.2) and `Mathematica` commands similar listed above, we have simulated longitudinal and transverse waves separately. The result of simulation of the second components U^{1p}, U^{1s} are shown on figures 2.4(b), 2.4(c) for

$$\lambda = 4,\ \rho = 2.203,\ \mu = 3.12,$$

$$c_1 = 2.155,\ c_2 = 1.190,\ t = 6.$$

The figure 2.4(a) corresponds to the second component of $\mathbf{G}^1 = \mathbf{U}^{1p} + \mathbf{U}^{1s}$ with the same values of ρ, μ, λ, and t. The result of the simulation of longitudinal and transverse waves separately and their superposition $\mathbf{G}^1$ as well give us understanding which parts of Figures of 2.1 - 2.3 corresponding to longitudinal and transverse waves. In the Figures 2.5(a) - 2.5(b) we simulated $\mathbf{G}^1_1(x_1, x_2, 1, 6)$ for $\mu = 1$, $\lambda = 5$, $\lambda = 1$ respectively. These graphs show us if the difference between speeds increases then the distance between fronts of longitudinal and transverse waves increases also.

Remark 2.4.1. *Modeling and simulations of wave propagations by formulas for* G^m_n, $m = 2, 3$, $n = 1, 2, 3$ *are similar to* G^1_n, $n = 1, 2, 3$.

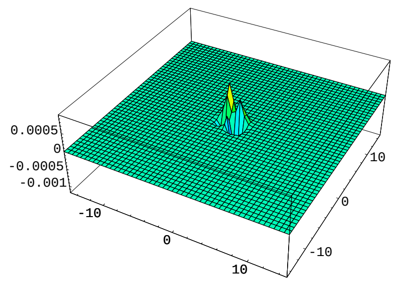


(a) t=0 (b) t=1,5

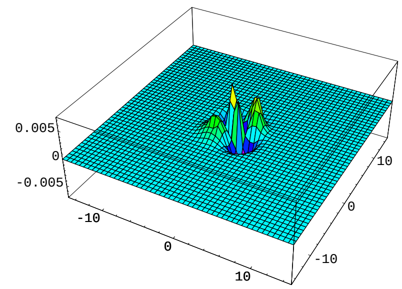
(c) t=3 (d) t=4,5

(e) t=6 (f) t=7,5

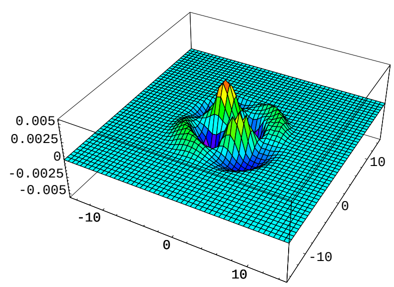
Figure 2.1: The element of the fundamental solution $G_1^1(\mathbf{x},\mathrm{t})$, $\mathbf{e}_1 = (1,0,0)$.



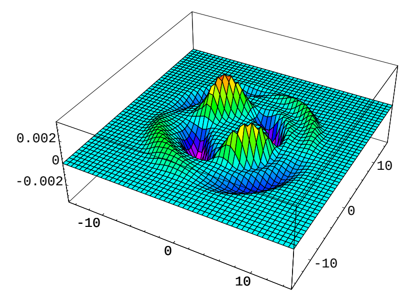
(a) t=0



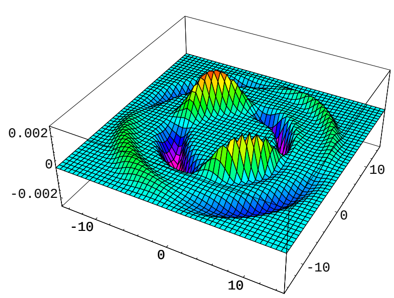
(b) t=1,5



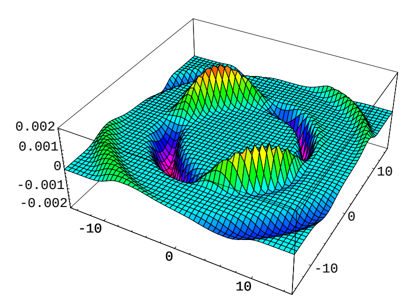
(c) t=3



(d) t=4,5

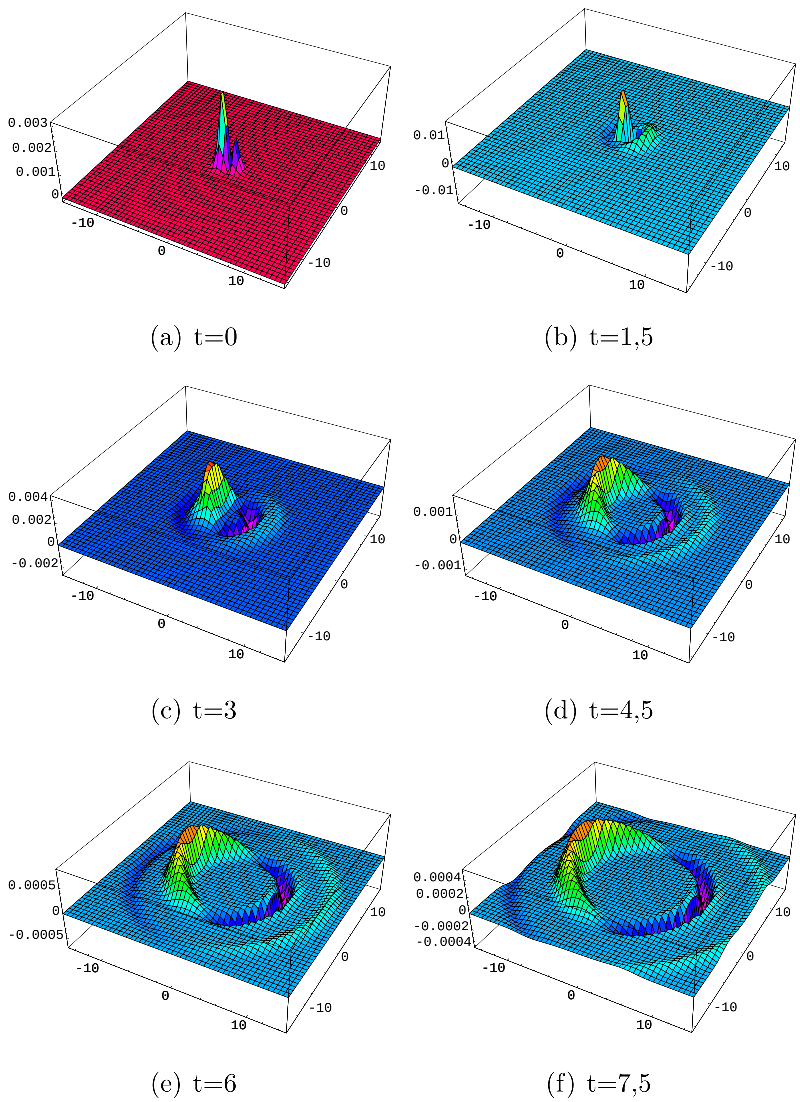


(e) t=6



(f) t=7,5

Figure 2.2: The element of the fundamental solution $G_2^1(\mathbf{x},\mathrm{t})$, $\mathbf{e}_2 = (0, 1, 0)$.

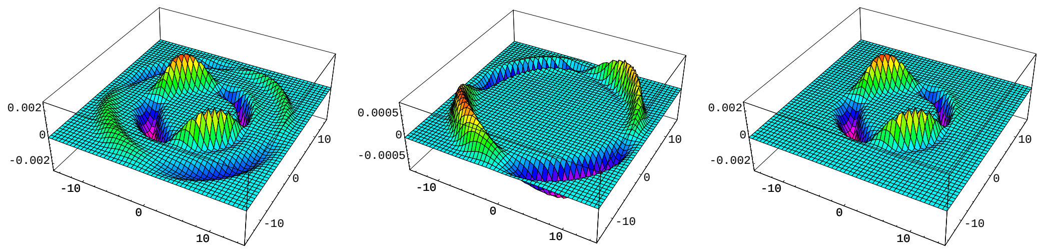


(a) t=0 (b) t=1,5

(c) t=3 (d) t=4,5

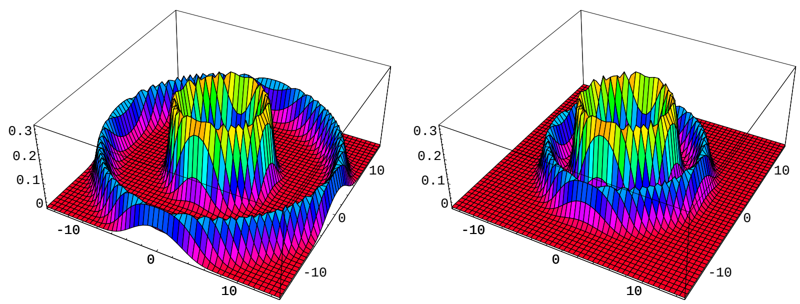
(e) t=6 (f) t=7,5

Figure 2.3: The element of the fundamental solution $G_3^1(\mathbf{x},\mathrm{t})$, $\mathbf{e}_3 = (0, 0, 1)$.



(a) t=6 (b) t=6 (c) t=6

Figure 2.4: The element of the fundamental solution $G_2^1(\mathbf{x},\mathrm{t})$.



(a) λ=5,$\mu = 1$ (b) λ=1,$\mu = 1$

Figure 2.5: The element of the fundamental solution $G_1^1(\mathbf{x},\mathrm{t})$.

2.5 Conclusion of the Chapter Two

- An explicit formula for the fundamental solution of initial value problem for Lame's system is constructed.
- The formula for the fundamental solution of initial value problem for Lame's system is used for modeling and simulation of the wave propagation in isotropic elastic media.
- Results of simulations of displacement components of elastic wave fields are presented by 3-D pictures and animated movies.
- Analysis of obtained formula and results of simulations is described.

CHAPTER THREE
INITIAL BOUNDARY VALUE PROBLEM FOR LAME SYSTEM

The initial boundary value problem for the Lame system of elasticity for a homogeneous isotropic medium was considered in this section. The initial boundary value problem for Lame's system reduced to initial boundary value problem for scalar wave equation and initial boundary value problem for vector wave equation in spherical coordinates. Using the vector analysis technique, spherical coordinates and Fourier series expansion method, formulas for the solutions of the initial boundary value problems for the Lame's system were obtained. Using this formulas and `Mathematica` codes we made simulations of elastic waves with different sources.

3.1 Reduction of Initial Boundary Value Problem for the Lame System to the Initial Boundary Value Problem for System of Wave Equations

Consider in the sphere $|\mathbf{x}| < r_0,\ r_0 \in \mathbf{R}$ is a constant, $t \in \mathbf{R}$,
$\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$ the following system

$$\rho \frac{\partial^2 \mathbf{U}}{\partial t^2} = (\lambda + 2\mu)\nabla_x div_x \mathbf{U} - \mu curl_x curl_x \mathbf{U} + \mathbf{f}(\mathbf{x}, t), \tag{3.1.1}$$

subject to the initial conditions

$$\mathbf{U}(\mathbf{x}, t)|_{t=0} = \varphi(\mathbf{x}), \tag{3.1.2}$$

$$\frac{\partial \mathbf{U}}{\partial t}(\mathbf{x}, t)|_{t=0} = \psi(\mathbf{x}), \tag{3.1.3}$$

and the boundary condition

$$\mathbf{U}(\mathbf{x}, t)|_{|\mathbf{x}|=r_0} = 0, \tag{3.1.4}$$

where

ρ is the density of elastic medium,

λ, μ are Lame parameters(physical constants),

$\rho > 0, \quad \lambda + 2\mu > 0, \quad \mu > 0$ are positive constants.
$\mathbf{f}(\mathbf{x}, t), \ \varphi(\mathbf{x}), \ \psi(\mathbf{x})$ are given functions, $\mathbf{U}(\mathbf{x}, t)$ is unknown function.

Our aim is to reduce IBVP for Lame system to IBVP for system of wave equations using Helmholtz theorem as we did in previous chapter.

Let $\mathbf{f}$ of the Lame system have the form $\mathbf{f} = \nabla_x g + curl_x \mathbf{F}$ then the vector function $\mathbf{U} = \nabla_x \phi + curl_x \mathbf{A}$ satisfies the Lame system since $\phi(\mathbf{x}, t)$ is solution of the wave equation

$$\rho \frac{\partial^2 \phi}{\partial t^2} = (\lambda + 2\mu) \triangle_x \phi + g,$$

and $\mathbf{A}(\mathbf{x}, t)$ is a solution of the wave equation

$$\rho \frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu \triangle_x \mathbf{A} + \mathbf{F}.$$

Firstly, let us consider the initial conditions. If $\varphi(\mathbf{x}) = \nabla_x \tilde{\phi} + curl_x \tilde{\mathbf{A}}$ then substituting $\varphi(\mathbf{x})$ and $\mathbf{U} = \nabla_x \phi + curl_x \mathbf{A}$ into (3.1.2) we obtain,

$$\phi(\mathbf{x}, t)|_{t=0} = \tilde{\phi}(\mathbf{x}), \tag{3.1.5}$$

$$\mathbf{A}(\mathbf{x}, t)|_{t=0} = \tilde{\mathbf{A}}(\mathbf{x}), \tag{3.1.6}$$

where

$$\tilde{\phi}(\mathbf{x}) = -\frac{1}{4\pi} div_x \int_D \frac{\varphi(\xi)}{|x - \xi|} \mathrm{d}\xi,$$

$$\tilde{\mathbf{A}}(\mathbf{x}) = \frac{1}{4\pi} curl_x \int_D \frac{\varphi(\xi)}{|x - \xi|} \mathrm{d}\xi.$$

$D = \{x \in \mathbf{R}^3 : |\mathbf{x}| < r_0\}$.
If $\psi(\mathbf{x}) = \nabla_x \tilde{\tilde{\phi}} + curl_x \tilde{\tilde{\mathbf{A}}}$ then substituting $\psi(\mathbf{x})$ and $\mathbf{U} = \nabla_x \phi + curl_x \mathbf{A}$ into (3.1.3) we obtain,

$$\frac{\partial \phi}{\partial t}(\mathbf{x}, t)|_{t=0} = \tilde{\tilde{\phi}}(\mathbf{x}), \tag{3.1.7}$$

$$\frac{\partial \mathbf{A}}{\partial t}(\mathbf{x}, t)|_{t=0} = \tilde{\tilde{A}}(\mathbf{x}), \tag{3.1.8}$$

where

$$\tilde{\tilde{\phi}}(\mathbf{x}) = -\frac{1}{4\pi} div_x \int_D \frac{\psi(\xi)}{|x-\xi|} \mathrm{d}\xi,$$

$$\tilde{\tilde{\mathbf{A}}}(\mathbf{x}) = \frac{1}{4\pi} curl_x \int_D \frac{\psi(\xi)}{|x-\xi|} \mathrm{d}\xi.$$

Now, let us analyze the boundary condition. The boundary condition is

$$\mathbf{U}(\mathbf{x},t)|_{|\mathbf{x}|=r_0} = 0.$$

Substituting $\mathbf{U} = \nabla_x \phi + curl_x \mathbf{A}$ into equation (3.1.4) we obtain

$$\Big(\nabla_x \phi(\mathbf{x},t) + curl_x \mathbf{A}(\mathbf{x},t)\Big)|_{|\mathbf{x}|=r_0} = 0,$$

or

$$\begin{cases} \Big[\frac{\partial \phi}{\partial x_1} + \Big(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\Big)\Big]\Big|_{|\mathbf{x}|=r_0} &= 0\ , \\ \Big[\frac{\partial \phi}{\partial x_2} + \Big(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\Big)\Big]\Big|_{|\mathbf{x}|=r_0} &= 0\ , \\ \Big[\frac{\partial \phi}{\partial x_3} + \Big(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\Big)\Big]\Big|_{|\mathbf{x}|=r_0} &= 0\ , \end{cases}$$

Using spherical coordinates $x_1 = r\cos\varphi\sin\theta$, $x_2 = r\sin\varphi\sin\theta$, $x_3 = r\cos\theta$, we can define

$$\phi(r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta, t) = \bar{\phi}(r,\theta,\varphi,t),$$

$$A_i(r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta, t) = \bar{A}_i(r,\theta,\varphi,t)\ ,\quad i = 1,2,3.$$

Using these notations we find from boundary conditions that (See, Appendix A)

$$\Big[\frac{\partial \phi}{\partial x_1} + \Big(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\Big)\Big]\Big|_{|\mathbf{x}|=r_0} = \Bigg[\cos\varphi\sin\theta\frac{\partial\bar{\phi}}{\partial r} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial\bar{\phi}}{\partial\varphi} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial\bar{\phi}}{\partial\theta}$$

$$+\Bigg(\sin\varphi\sin\theta\frac{\partial\bar{A}_3}{\partial r} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial\bar{A}_3}{\partial\varphi} + \frac{1}{r}\sin\varphi\cos\theta\frac{\partial\bar{A}_3}{\partial\theta}$$

$$-\cos\theta\frac{\partial\bar{A}_2}{\partial r} + \frac{1}{r}\sin\theta\frac{\partial\bar{A}_2}{\partial\theta}\Bigg)\Bigg]\Bigg|_{|\mathbf{x}|=r_0} = 0, \qquad (3.1.9)$$

$$\left[\frac{\partial \phi}{\partial x_2} + \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right)\right]\Big|_{|\mathbf{x}|=r_0} = \Bigg[\sin\varphi\sin\theta\frac{\partial\bar{\phi}}{\partial r} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial\bar{\phi}}{\partial\varphi} + \frac{1}{r}\sin\varphi\cos\theta\frac{\partial\bar{\phi}}{\partial\theta}$$

$$+\Bigg(\cos\theta\frac{\partial\bar{A}_1}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial\bar{A}_1}{\partial\theta} - \cos\varphi\sin\theta\frac{\partial\bar{A}_3}{\partial r}$$

$$+\frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial\bar{A}_3}{\partial\varphi} - \frac{1}{r}\cos\theta\cos\varphi\frac{\partial\bar{A}_3}{\partial\theta}\Bigg)\Bigg]\Bigg|_{|\mathbf{x}|=r_0} = 0, \qquad (3.1.10)$$

$$\left[\frac{\partial \phi}{\partial x_3} + \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)\right]\Big|_{|\mathbf{x}|=r_0} = \Bigg[\cos\theta\frac{\partial\bar{\phi}}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial\bar{\phi}}{\partial\theta}$$

$$+\Bigg(\cos\varphi\sin\theta\frac{\partial\bar{A}_2}{\partial r} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial\bar{A}_2}{\partial\varphi} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial\bar{A}_2}{\partial\theta}$$

$$-\sin\varphi\sin\theta\frac{\partial\bar{A}_1}{\partial r} - \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial\bar{A}_1}{\partial\varphi} - \frac{1}{r}\sin\varphi\cos\theta\frac{\partial\bar{A}_1}{\partial\theta}\Bigg)\Bigg]\Bigg|_{|\mathbf{x}|=r_0} = 0. \qquad (3.1.11)$$

As a result, IBVP for Lame system defined in (3.1.1)-(3.1.4) is reduced to the IBVP for system of wave equations (3.1.5)-(3.1.11).

3.1.1 Eigenvalue Problem for the Laplace Operator in a Sphere

Consider the Laplace equation in a sphere

$$\triangle_{\mathbf{x}} V(\mathbf{x}) + \lambda V(\mathbf{x}) = 0, \qquad \mathbf{x} = (x_1, x_2, x_3), \qquad |x| < r_0, \qquad (3.1.12)$$

subject to the condition

$$V(\mathbf{x})\Big|_{|x|<r_0} = 0. \qquad (3.1.13)$$

Our goal is to find eigenvalues and eigenfunctions of (3.1.12) and (3.1.13). The problem (3.1.12) and (3.1.13) may be written in spherical coordinates,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\triangle_{\theta,\varphi}V+\lambda V=0, \tag{3.1.14}$$

$$0<r<r_0, \quad \varphi\in\mathbf{R}, \quad \theta\in(0,\pi)$$

$$\left|V(r,\theta,\varphi)|_{\substack{\theta=0\\ \theta=\pi}}\right|<\infty, \quad V(r,\theta,\varphi)=V(r,\theta,\varphi+2\pi), \tag{3.1.15}$$

$$\left|V(r,\theta,\varphi)|_{r=0}\right|<\infty, \quad V(r,\theta,\varphi)|_{r=r_0}=0. \tag{3.1.16}$$

The solution of the problem (3.1.14),(3.1.15),(3.1.16) we seek in the form,

$$V(r,\theta,\varphi)=R(r)Y(\theta,\varphi). \tag{3.1.17}$$

Substituting (3.1.17) into (3.1.14) and dividing obtained relation by $R(r)Y(\theta,\varphi)$, we have,

$$\frac{\frac{d}{dr}(r^2\frac{dR(r)}{dr})}{R(r)}+\lambda r^2+\frac{\triangle_{\theta,\varphi}Y(\theta,\varphi)}{Y(\theta,\varphi)}=0, \tag{3.1.18}$$

$$\frac{\frac{d}{dr}(r^2\frac{dR(r)}{dr})}{R(r)}+\lambda r^2=-\frac{\triangle_{\theta,\varphi}Y(\theta,\varphi)}{Y(\theta,\varphi)}, \tag{3.1.19}$$

where $\triangle_{\theta,\varphi}$ is the Beltrami operator.

The righthand side of (3.1.19) is a function of θ,φ the left hand side of (3.1.19) is a function of r. The variables r,θ,φ are independent. Therefore the relation (3.1.19) is possible if and only if both sides are equal to the same constant μ.

Hence

$$\triangle_{\theta,\varphi}Y(\theta,\varphi)+\mu Y(\theta,\varphi)=0, \quad 0<\theta<\pi, \quad \varphi\in\mathbf{R}, \tag{3.1.20}$$

and

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)+\left(\lambda-\frac{\mu}{r^2}\right)R(r)=0, \quad 0<r<r_0, \tag{3.1.21}$$

substituting (3.1.17) into (3.1.15),(3.1.16) we find,

$$\left|Y(\theta,\varphi)|_{\substack{\theta=0\\ \theta=\pi}}\right|<\infty, \quad Y(\theta,\varphi)=Y(\theta,\varphi+2\pi), \tag{3.1.22}$$

$$\left|R(r)|_{r=0}\right|<\infty, \quad R(r)\Big|_{r=r_0}=0. \tag{3.1.23}$$

(3.1.20) and (3.1.22) are Sturm Liouville problem for Spherical functions. (See, Appendix F)

$\mu_n = n(n+1),\ n = 0, 1, 2, ...$ are eigenvalues,
$Y_n^{(m)}(\theta, \varphi),\ m = 0, \pm 1, \pm 2, ..., \pm n;\ n = 0, 1, 2, ...$ are eigenfunctions.
Consider the eigenvalue problem (3.1.21),(3.1.23) for $\mu_n = n(n{+}1)$. This problem may be written in terms of new function

$$Z(r) = \sqrt{r}R(r).$$

Using formulas

$$R(r) = \frac{Z(r)}{\sqrt{r}},$$

$$R'(r) = \frac{Z'(r)}{\sqrt{r}} - \frac{1}{2r^{3/2}}Z(r),$$

$$R''(r) = \frac{Z''(r)}{\sqrt{r}} - \frac{Z'(r)}{r^{3/2}} + \frac{3Z(r)}{4r^{5/2}},$$

the equation (3.1.21) may be written,

$$\underbrace{\frac{Z''(r)}{\sqrt{r}} - \frac{Z'(r)}{r^{3/2}} + \frac{3Z(r)}{4r^{5/2}}}_{R''(r)} + \underbrace{\frac{2Z'(r)}{r^{3/2}} - \frac{1}{r^{5/2}}Z(r)}_{\frac{2}{r}R'(r)} + \lambda\frac{Z(r)}{r^{1/2}} - \frac{n(n+1)}{r^{5/2}}Z(r) = 0. \tag{3.1.24}$$

Note that the coefficients of $\dfrac{Z(r)}{r^{5/2}}$ may be written in the form,

$$\frac{3}{4} - 1 - n(n+1) = -\Big[n(n+1) + \frac{1}{4}\Big] = -\Big[n^2 + n + \frac{1}{4}\Big] = -\Big(n + \frac{1}{2}\Big)^2.$$

Hence (3.1.24) has the form

$$\frac{Z''(r)}{\sqrt{r}} + \frac{Z'(r)}{r^{3/2}} + \left(\frac{\lambda}{r^{1/2}} - \frac{(n+\frac{1}{2})^2}{r^{5/2}}\right)Z(r) = 0,$$

$$Z''(r) + \frac{1}{r}Z'(r) + \left(\lambda - \frac{(n+\frac{1}{2})^2}{r^2}\right)Z(r) = 0, \quad 0 < r < r_0. \tag{3.1.25}$$

Condition (3.1.23) may be written in the term of $Z(r)$ as

$$\Big|Z(r)|_{r=0}\Big| < \infty, \quad Z(r)\Big|_{r=r_0} = 0. \tag{3.1.26}$$

As the result we wrote eigenvalue problem (3.1.21),(3.1.23) in the form (3.1.25),(3.1.26). The last problem is the eigenvalue problem for the Bessel operator (See, Appendix B). The solution of this problem is given by,

$\lambda=\lambda_{kn}=\left(\frac{\mu_k^{(n+\frac{1}{2})}}{r_0}\right)^2$, $k=1,2,...$ are eigenvalues where

$\mu_k^{(n+\frac{1}{2})}$, $k=1,2,...;$ $n=0,1,2,...$ are roots of the equation $J_{n+\frac{1}{2}}(\mu)=0$,

$$Z_{kn}(r)=c_{kn}J_{n+\frac{1}{2}}\left(\frac{r\mu_k^{(n+\frac{1}{2})}}{r_0}\right),$$

$k=1,2,...;$ $n=0,1,2,...$ are eigenfunctions where

$$c_{kn}^2=\frac{2}{r_0^2\left(J'_{n+\frac{1}{2}}\left(\mu_k^{(n+\frac{1}{2})}\right)\right)^2}.$$

Therefore the solution of the original problem (3.1.14),(3.1.16) may be written as
$\lambda=\lambda_{kn}=\left(\frac{\mu_k^{(n+\frac{1}{2})}}{r_0}\right)^2$, $k=1,2,...;$ $n=0,1,2,...$ are eigenvalues,
$V_{knm}(r,\theta,\varphi)=R_{kn}(r)Y_n^{(m)}(\theta,\varphi)$, $k=1,2,...;$ $n=0,1,2,...;$
$m=0,\pm1,\pm2,...,\pm n$ are eigenfunctions. Here $R_{kn}(r)$, $k=1,2,...;$
$n=0,1,2,...$ are defined by

$$R_{kn}(r)=\frac{Z_{kn}(r)}{\sqrt{r}}=\frac{\sqrt{2}}{r_0\left(J'_{n+\frac{1}{2}}\left(\mu_k^{(n+\frac{1}{2})}\right)\right)}\frac{1}{\sqrt{r}}J_{n+\frac{1}{2}}\left(\frac{r\mu_k^{(n+\frac{1}{2})}}{r_0}\right),$$

so

$$V_{knm}(r,\theta,\varphi)=\frac{\sqrt{2}}{r_0\left(J'_{n+\frac{1}{2}}\left(\mu_k^{(n+\frac{1}{2})}\right)\right)}\frac{1}{\sqrt{r}}J_{n+\frac{1}{2}}\left(\frac{r\mu_k^{(n+\frac{1}{2})}}{r_0}\right)Y_n^{(m)}(\theta,\varphi). \tag{3.1.27}$$

Remark 3.1.1. *The system of eigenfunctions* $\{V_{knm}(r,\theta,\varphi)\}$, $k=1,2,...;$ $n=0,1,2,...;$ $m=0,\pm1,\pm2,...,\pm n$ *is complete orthonormal system in the space* $L_{2,r^2\sin\theta}(D)$, D=$(0,r_0),(0,\pi),(0,2\pi)$.

Here $L_{2,r^2\sin\theta}(D)$, *is the space of square integrable functions with the weight* $r^2\sin\theta$ *over the domain D, i.e.*

$$L_{2,r^2\sin\theta}(D)=\{f(r,\theta,\varphi):\int\int\int_D|f(r,\theta,\varphi)|^2r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi<\infty\}.$$

Moreover, every function $h(r,\theta,\varphi)$ *from* $L_{2,r^2\sin\theta}(D)$ *has the Fourier series* *expansion of the form,*

$$h(r,\theta,\varphi)=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}h_{knm}V_{knm}(r,\theta,\varphi),$$

where the fourier coefficients are defined by

$$h_{knm}=\int_0^{r_0}\int_0^{2\pi}\int_0^{\pi}h(r,\theta,\varphi)V_{knm}(r,\theta,\varphi)r^2\sin\theta \mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r$$

3.1.2 Initial Boundary Value Problem for Wave Equation with Dirichlet Boundary Condition

Consider in the sphere $|x|<r_0,\ r_0\in\mathbf{R}$ is a constant $\mathbf{x}=(x_1,x_2,x_3)\in\mathbf{R}^3$ for $t>0$ the following equation

$$\frac{1}{a^2}\frac{\partial^2\mathbf{U}}{\partial t^2}-\triangle\mathbf{U}=\mathbf{f}(\mathbf{x},t), \tag{3.1.28}$$

subject to the conditions

$$\mathbf{U}\Big|_{t=0}=\mathbf{g}(\mathbf{x}),\quad \frac{\partial\mathbf{U}}{\partial t}\Big|_{t=0}=\mathbf{h}(\mathbf{x}), \tag{3.1.29}$$

and the Dirichlet boundary condition

$$\mathbf{U}\Big|_{|\mathbf{x}|=r_0}=0. \tag{3.1.30}$$

$\mathbf{f}(\mathbf{x},t),\ \mathbf{g}(\mathbf{x}),\ \mathbf{h}(\mathbf{x})$ are given functions from the space L_2, $\mathbf{U}(\mathbf{x},t)$ is unknown function. Our aim is to find the function $\mathbf{U}(\mathbf{x},t)$. Problem (3.1.28)-(3.1.30) may be written in spherical coordinates as follows,

$$\frac{1}{a^2}\frac{\partial^2\bar{\mathbf{U}}(r,\theta,\varphi,t)}{\partial t^2}-\triangle_{r,\theta,\varphi}\bar{\mathbf{U}}(r,\theta,\varphi,t)=\bar{\mathbf{f}}(r,\theta,\varphi,t), \tag{3.1.31}$$

$$\bar{\mathbf{U}}(r,\theta,\varphi,t)\Big|_{t=0}=\bar{\mathbf{g}}(r,\theta,\varphi),\quad \frac{\partial\bar{\mathbf{U}}(r,\theta,\varphi,t)}{\partial t}\Big|_{t=0}=\bar{\mathbf{h}}(r,\theta,\varphi), \tag{3.1.32}$$

$$\Big|\bar{\mathbf{U}}(r,\theta,\varphi,t)\Big|_{r\to+0}<\infty;\qquad \bar{\mathbf{U}}(r,\theta,\varphi,t)\Big|_{r=r_0}=0, \tag{3.1.33}$$

$$\Big|\bar{\mathbf{U}}(r,\theta,\varphi,t)|_{\substack{\theta=0\\ \theta=\pi}}\Big|<\infty,\qquad \bar{\mathbf{U}}(r,\theta,\varphi,t)=\bar{\mathbf{U}}(r,\theta,\varphi+2\pi,t), \tag{3.1.34}$$

$\bar{\mathbf{f}}(r,\theta,\varphi,t)$, $\bar{\mathbf{g}}(r,\theta,\varphi)$, $\bar{\mathbf{h}}(r,\theta,\varphi)$ are given functions from the space L_2, $\bar{\mathbf{U}}(r,\theta,\varphi,t)$ is unknown function. Our aim is to find the function $\bar{\mathbf{U}}(r,\theta,\varphi,t)$ that satisfies (3.1.31)-(3.1.34).

Remark 3.1.2. *Using the completeness and orthogonality of eigenfunctions* $\{V_{knm}(r,\theta,\varphi)\}$, $k=1,2,...;$ $n=0,1,2,...;$ $m=0,\pm1,\pm2,...,\pm n$ *in the space* $L_{2,r^2\sin\theta}(D)$, *the functions* $\bar{g}(r,\theta,\varphi), \bar{h}(r,\theta,\varphi), \bar{f}(r,\theta,\varphi,t)$ *may be presented in the form of the Fourier series*

$$\bar{g}(r,\theta,\varphi)=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} g_{knm}V_{knm}(r,\theta,\varphi),$$

$$\bar{h}(r,\theta,\varphi)=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} h_{knm}V_{knm}(r,\theta,\varphi),$$

$$\bar{f}(r,\theta,\varphi,t)=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_{knm}(t)V_{knm}(r,\theta,\varphi),$$

$$g_{knm}=\int_0^{r_0}\int_0^{2\pi}\int_0^{\pi}\bar{g}(r,\theta,\varphi)V_{knm}(r,\theta,\varphi)r^2\sin\theta \mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r,$$

$$h_{knm}=\int_0^{r_0}\int_0^{2\pi}\int_0^{\pi}\bar{h}(r,\theta,\varphi)V_{knm}(r,\theta,\varphi)r^2\sin\theta \mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r,$$

$$f_{knm}(t)=\int_0^{r_0}\int_0^{2\pi}\int_0^{\pi}\bar{f}(r,\theta,\varphi,t)V_{knm}(r,\theta,\varphi)r^2\sin\theta \mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r.$$

A solution of the problem (3.1.31)-(3.1.34) we seek in the form

$$\bar{\mathbf{U}}(r,\theta,\varphi,t)=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} T_{knm}(t)V_{knm}(r,\theta,\varphi), \tag{3.1.35}$$

where $T_{knm}(t)$ are unknown coefficients depending on t and $\{V_{knm}(r,\theta,\varphi)\}$ $k=1,2,...;$ $n=0,1,2,...;$ $m=0,\pm1,\pm2,...,\pm n$ are eigenfunctions of the Laplace operator in the sphere. Substituting (3.1.35) into (3.1.31) we can find

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(\frac{1}{a^2}T''_{knm}(t)V_{knm}(r,\theta,\varphi)-T_{knm}(t)\triangle V_{knm}(r,\theta,\varphi)\right.$$

$$\left.-f_{knm}(t)V_{knm}(r,\theta,\varphi)\right)=0,$$

from the Sturm-Liouville problem we know that $-\triangle V_{knm} = \lambda_{kn} V_{knm}$ and using previous remark we obtain

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(\frac{1}{a^2}T''_{knm}(t) + \lambda_{knm}T_{knm}(t) - f_{knm}(t)\right)V_{knm}(r,\theta,\varphi) = 0.$$

Using the orthonormality of $\{V_{knm}(r,\theta,\varphi)\}$, we have

$$T''_{knm}(t) + a^2\lambda_{kn}T_{knm}(t) - a^2 f_{knm}(t) = 0, \tag{3.1.36}$$

$k = 1, 2, ...;$ $n = 0, 1, 2, ...;$ $m = 0, \pm1, \pm2, ..., \pm n.$ Substituting (3.1.35) into (3.1.32) we obtain

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} T_{knm}(0)V_{knm}(r,\theta,\varphi) = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} g_{knm}V_{knm}(r,\theta,\varphi),$$

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} T'_{knm}(0)V_{knm}(r,\theta,\varphi) = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} h_{knm}V_{knm}(r,\theta,\varphi),$$

$$T_{knm}(0) = g_{knm},\ T'_{knm}(0) = h_{knm}, \tag{3.1.37}$$

$k = 1, 2, ...;$ $n = 0, 1, 2, ...;$ $m = 0, \pm1, \pm2, ..., \pm n.$ As a result we obtain the Cauchy problem (3.1.36),(3.1.37) for each indices $k,\ n,\ m.$

Remark 3.1.3. *Let us consider the Cauchy problem for ordinary differential equation,*

$$T''(t) + w^2T(t) = f(t), \tag{3.1.38}$$

$$T(0) = \vartheta,\ T'(0) = \psi. \tag{3.1.39}$$

Here ϑ, ψ *are given constants,* $f(t)$ *is given function. The solution of (3.1.38), (3.1.39) is given by the formula,*

$$T(t) = \vartheta\cos(wt) + \frac{\psi}{w}\sin(wt) + \frac{1}{w}\int_0^t f(\tau)\sin[w(t-\tau)]\mathrm{d}\tau. \tag{3.1.40}$$

Proof. From calculus we know that

$$\frac{d}{dt}\Big(\int_{\alpha(t)}^{\beta(t)} g(x,t)\mathrm{d}x\Big) = g(\beta(t),t)\cdot\beta'(t) - g(\alpha(t),t)\cdot\alpha'(t) + \int_{\alpha(t)}^{\beta(t)}\frac{\partial}{\partial t}g(x,t)\mathrm{d}t,$$

$\alpha \in C^1$ $\beta \in C^1,$ $\frac{\partial}{\partial t}g(x,t) \in C$

So,

$$T'(t) = -w\vartheta \sin(wt) + \frac{w\psi}{w}\cos(wt) + \frac{w}{w}\int_0^t f(\tau)\cos[w(t-\tau)]\mathrm{d}\tau,$$

$$T''(t) = -w^2\vartheta\cos(wt) - \frac{w^2\psi}{w}\sin(wt) + f(t) - \frac{1}{w}\int_0^t f(\tau)\sin[w(t-\tau)]\mathrm{d}\tau,$$

$$T''(t) = -w^2 T(t) + f(t),$$

$$T''(t) + w^2 T(t) = f(t).$$

Hence $T(t)$ which is defined by (3.1.40) satisfies (3.1.38). Substituting (3.1.40) in (3.1.39) we obtain $T(0) = \vartheta$, and $T'(0) = \psi$.

□

A solution of the Cauchy problem (3.1.36),(3.1.37) is given by the formula

$$T_{knm}(t) = g_{knm}\cos(a\sqrt{\lambda_{kn}}t) + \frac{h_{knm}}{a\sqrt{\lambda_{kn}}}\sin(a\sqrt{\lambda_{kn}}t)$$

$$+\frac{a}{\sqrt{\lambda_{knm}}}\int_0^t f_{knm}(\tau)\sin[a\sqrt{\lambda_{knm}}(t-\tau)]\mathrm{d}\tau. \tag{3.1.41}$$

Hence the solution of the original problem (3.1.31)-(3.1.34) is given by the formula (3.1.35) in which $T_{knm}(t)$ are defined by (3.1.41) where $\lambda = \lambda_{kn} = \left(\frac{\mu_k^{(n+\frac{1}{2})}}{r_0}\right)^2$, $k = 1, 2, ...;$ $n = 0, 1, 2, ...$ are eigenvalues and $\mu_k^{(n+\frac{1}{2})}$, $k = 1, 2, ...$ are roots of the equation $J_{n+\frac{1}{2}}(\mu) = 0$, $V_{knm}(r, \theta, \varphi)$, $k = 1, 2, ...;$ $n = 0, 1, 2, ...;$ $m = 0, \pm 1, \pm 2, ..., \pm n$ are eigenfunctions of the Laplace operator in the sphere of the radius r_0 given by the formula (3.1.27).

3.1.3 Eigenvalue Problem for the Laplace Operator in a Sphere for Function Depending r, t

Consider the following problem

$$\triangle_r R(r) + \lambda R(r) = 0, \quad 0 < r < r_0, \tag{3.1.42}$$

$$\left|R(r)|_{r=0}\right| < \infty, \quad \frac{\partial}{\partial r}R(r)|_{r=r_0} = 0. \tag{3.1.43}$$

where $\triangle_r R(r) = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\Big(r^2\dfrac{\partial R(r)}{\partial r}\Big)$.

Thus (3.1.42),(3.1.43) can be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial R(r)}{\partial r}\Big) + \lambda R(r) = 0, \quad 0 < r < r_0, \tag{3.1.44}$$

$$\Big|R(r)|_{r=0}\Big| < \infty, \quad \frac{\partial}{\partial r}R(r)|_{r=r_0} = 0. \tag{3.1.45}$$

Our aim is to find eigenvalues and eigenfunctions of (3.1.44)-(3.1.45). This problem may be written in terms of new function

$$Z(r) = \sqrt{r}R(r).$$

Using formulas

$$R(r) = \frac{Z(r)}{\sqrt{r}},$$

$$R'(r) = \frac{Z'(r)}{\sqrt{r}} - \frac{1}{2r_0^{3/2}}Z(r),$$

$$R''(r) = \frac{Z''(r)}{\sqrt{r}} - \frac{Z'(r)}{r^{3/2}} + \frac{3Z(r)}{4r^{5/2}},$$

the equation (3.1.44) may be written,

$$\underbrace{\frac{Z''(r)}{\sqrt{r}} - \frac{Z'(r)}{r^{3/2}} + \frac{3Z(r)}{4r^{5/2}}}_{R''(r)} + \underbrace{\frac{2Z'(r)}{r^{3/2}} - \frac{1}{r^{5/2}}Z(r)}_{\frac{2}{r}R'(r)} + \lambda\frac{Z(r)}{\sqrt{r}} = 0. \tag{3.1.46}$$

Hence (3.1.46) has the form

$$\frac{Z''(r)}{\sqrt{r}} + \frac{Z'(r)}{r^{3/2}} + \left(\frac{\lambda}{\sqrt{r}} - \frac{\frac{1}{4}}{r^{5/2}}\right)Z(r) = 0,$$

$$Z''(r) + \frac{1}{r}Z'(r) + \left(\lambda - \frac{1}{4r^2}\right)Z(r) = 0, \quad 0 < r < r_0. \tag{3.1.47}$$

Condition (3.1.45) may be written in the term of $Z(r)$ as

$$\Big|Z(r)|_{r=0}\Big| < \infty, \quad \Big|\frac{Z'(r)}{\sqrt{r}} - \frac{1}{2r^{3/2}}Z(r)\Big|_{r=r_0} = 0. \tag{3.1.48}$$

As the result we write eigenvalue problem (3.1.44),(3.1.45) in the form (3.1.47),(3.1.48). The last problem is the eigenvalue problem for the Bessel operator. The solution of this problem is given by,

$$\lambda = \lambda_k = \left(\frac{\mu_k^{(\frac{1}{2})}}{r_0} \right)^2, \ k = 1, 2, ...$$

are eigenvalues

$$Z_k(r) = c_k J_{\frac{1}{2}} \left(\frac{r \mu_k^{(\frac{1}{2})}}{r_0} \right),$$

$k = 1, 2, ...$ are eigenfunctions where $\mu_k^{(\frac{1}{2})}$, $k = 1, 2, ...$ are roots of the equation

$$J'_{\frac{1}{2}}(\mu)\mu - \frac{1}{2} J_{\frac{1}{2}}(\mu) = 0.$$

Here (See, Appendix B),

$$c_k^2 = \frac{1}{\| J_{\frac{1}{2}}(\sqrt{\lambda_k} x) \|^2},$$

$$\| J_{\frac{1}{2}}(\sqrt{\lambda_k} x) \|^2 = \int_0^{r_0} \left[J_{\frac{1}{2}}(\sqrt{\lambda_k} x) \right]^2 x \mathrm{d}x$$

$$= \frac{r_0^2}{2} \left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k} r_0) \right]^2 + \left(1 - \frac{1}{4 r_0^2 \lambda_k} \right) \left[J_{\frac{1}{2}}(\sqrt{\lambda_k} r_0) \right]^2 \right].$$

Therefore the solution of the problem (3.1.44)-(3.1.45) may be written as

$$\lambda = \lambda_k = \left(\frac{\mu_k^{(\frac{1}{2})}}{r_0} \right)^2, \ k = 1, 2, ...$$

are eigenvalues,

$$R_k(r) = \frac{Z(r)}{\sqrt{r}}, \ k = 1, 2, ...,$$

$$R_k(r) = \frac{\sqrt{2}}{r_0 \left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k} r_0) \right]^2 + \left(1 - \frac{1}{4 r_0^2 \lambda_k} \right) \left[J_{\frac{1}{2}}(\sqrt{\lambda_k} r_0) \right]^2 \right]^{\frac{1}{2}}} \frac{1}{\sqrt{r}} J_{\frac{1}{2}} \left(\frac{r \mu_k^{(\frac{1}{2})}}{r_0} \right),$$

are eigenfunctions.

Remark 3.1.4. *The system of eigenfunctions* $\{R_k(r)\}_{k=1,2...}$ *is complete orthonormal system in the space* $L_{2,r^2}(D)$, D=$(0, r_0)$. *Here* $L_{2,r^2}(D)$ *is the space* *of square integrable functions with the weight* r^2, *over the domain D, i.e.*

$$L_{2,r^2}(D) = \{f(r) : \int_D |f(r)|^2 r^2 \mathrm{d}r < \infty\}.$$

Moreover, every function $h(r)$ *from* $L_{2,r^2}(D)$ *has the Fourier series expansion of the form,*

$$h(r) = \sum_{k=1}^{\infty} h_k R_k(r),$$

where the Fourier coefficients are defined by

$$h_k = \int_0^{r_0} h(r) R_k(r) r^2 \mathrm{d}r.$$

3.1.4 Initial Boundary Value Problem for Wave Equation with Neumann Boundary Condition In Sphere

Consider the following problem

$$\frac{1}{a^2}\frac{\partial^2 \mathbf{U}}{\partial t^2} - \triangle \mathbf{U} = \mathbf{f}(\mathbf{x}, t), \tag{3.1.49}$$

subject to the conditions

$$\mathbf{U}\Big|_{t=0} = \mathbf{g}(\mathbf{x}), \quad \frac{\partial \mathbf{U}}{\partial t}\Big|_{t=0} = \mathbf{h}(\mathbf{x}), \tag{3.1.50}$$

and the Dirichlet boundary condition

$$\frac{\partial \mathbf{U}}{\partial n}\Big|_{|\mathbf{x}|=r_0} = 0. \tag{3.1.51}$$

$\mathbf{f}(\mathbf{x}, t)$, $\mathbf{g}(\mathbf{x})$, $\mathbf{h}(\mathbf{x})$ are given functions from the space L_2, $\mathbf{U}(\mathbf{x}, t)$ is unknown function. Our aim is to find the function $\mathbf{U}(\mathbf{x}, t)$.

Let $\mathbf{U}$ depend on r, t in spherical coordinate system. Therefore problem (3.1.49)-(3.1.51) may be written in spherical coordinates as follows, (See, Appendix A)

$$\frac{1}{a^2}\frac{\partial^2 \bar{\mathbf{U}}(r, t)}{\partial t^2} - \triangle_r \bar{\mathbf{U}}(r, t) = \bar{\mathbf{f}}(r, t), \tag{3.1.52}$$

$$\bar{\mathbf{U}}(r,t)\Big|_{t=0} = \bar{\mathbf{g}}(r), \quad \frac{\partial \bar{\mathbf{U}}(r,t)}{\partial t}\Big|_{t=0} = \bar{\mathbf{h}}(r), \tag{3.1.53}$$

$$\Big|\bar{\mathbf{U}}(r,t)\Big|_{r\to+0} < \infty; \tag{3.1.54}$$

$$\frac{\partial}{\partial r}\bar{\mathbf{U}}(r,t)\Big|_{r=r_0} = 0, \tag{3.1.55}$$

$\bar{\mathbf{f}}(r,t)$, $\bar{\mathbf{g}}(r)$, $\bar{\mathbf{h}}(r)$ are given functions from the space L_2, $\bar{\mathbf{U}}(r,t)$ is unknown function. Our aim is to find the function $\bar{\mathbf{U}}(r,t)$ that satisfies (3.1.52)-(3.1.55).

Remark 3.1.5. *Using the completeness and orthogonality of eigenfunctions* $\{R_k(r)\}_{k=1,2...}$ *in the space* $L_{2,r^2}(D)$, *the functions* $\bar{g}(r), \bar{h}(r), \bar{f}(r,t)$ *may be presented in the form of the Fourier series*

$$\bar{g}(r) = \sum_{k=1}^{\infty} g_k R_k(r), \quad \bar{h}(r) = \sum_{k=1}^{\infty} h_k R_k(r), \quad \bar{f}(r,t) = \sum_{k=1}^{\infty} f_k(t) R_k(r),$$

$$g_k = \int_0^{r_0} \bar{g}(r) R_k(r) r^2 \mathrm{d}r,$$

$$h_k = \int_0^{r_0} \bar{h}(r) R_k(r) r^2 \mathrm{d}r,$$

$$f_k(t) = \int_0^{r_0} \bar{f}(r,t) R_k(r) r^2 \mathrm{d}r.$$

A solution of the problem (3.1.52),(3.1.53),(3.1.54),(3.1.55) we seek in the form

$$\bar{\mathbf{U}}(r,t) = \sum_{k=1}^{\infty} T_k(t) R_k(r), \tag{3.1.56}$$

where $T_k(t)$ are unknown coefficients depending on t and $\{R_k(r)\}$, $k = 1, 2, ...$ are eigenfunctions of the operator $\triangle_r$ in the sphere that has the formula

$$R_k(r) = \frac{\sqrt{2}}{r_0\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k} r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k} r_0)\right]^2\right]^{\frac{1}{2}}} \frac{1}{\sqrt{r}} J_{\frac{1}{2}}\left(\frac{r\mu_k^{(\frac{1}{2})}}{r_0}\right). \tag{3.1.57}$$

Substituting (3.1.56)into (3.1.52) we find

$$\sum_{k=1}^{\infty}\left(\frac{1}{a^2}T_k''(t)R_k(r)-T_k(t)\triangle_r R_k(r)-f_k(t)R_k(r)\right)=0,$$

from the Sturm-Liouville Problem we know that $-\triangle_r R_k = \lambda_k R_k$ and using previous remark we obtain

$$\sum_{k=1}^{\infty}\left(\frac{1}{a^2}T_k''(t)+\lambda_k T_k(t)-f_k(t)\right)R_k(r)=0.$$

Using the orthonormality of $\{R_k(r)\}$, we have

$$T_k''(t)+a^2\lambda_k T_k(t)-a^2 f_k(t)=0, \tag{3.1.58}$$

$k=1,2,...$ substituting (3.1.56) into (3.1.53) we obtain

$$\sum_{k=1}^{\infty}T_k(0)R_k(r)=\sum_{k=1}^{\infty}g_k R_k(r),$$

$$\sum_{k=1}^{\infty}T_k'(0)R_k(r)=\sum_{k=1}^{\infty}h_k R_k(r),$$

$$T_k(0)=g_k,\ T_{knm}'(0)=h_k, \tag{3.1.59}$$

$k=1,2,...$ As a result we obtain the Cauchy problem (3.1.58),(3.1.59) for each indice k. A solution of Cauchy problem (3.1.58),(3.1.59) is given by the formula that we proved before

$$T_k(t)=g_k\cos(a\sqrt{\lambda_k}t)+\frac{h_k}{a\sqrt{\lambda_k}}\sin(a\sqrt{\lambda_k}t)+\frac{a}{\sqrt{\lambda_k}}\int_0^t f_k(\tau)\sin[a\sqrt{\lambda_k}(t-\tau)]d\tau. \tag{3.1.60}$$

Hence the solution of the original problem (3.1.52),(3.1.53),(3.1.54),(3.1.55) is given by the formula (3.1.56) in which $T_k(t)$ are defined by (3.1.60) where $\lambda=\lambda_k=\left(\frac{\mu_k^{(\frac{1}{2})}}{r_0}\right)^2$, $k=1,2,...$ are eigenvalues and $\mu_k^{(\frac{1}{2})}$, $k=1,2,...$ are roots of the equation $J_{\frac{1}{2}}'(\mu)\mu-\frac{1}{2}J_{\frac{1}{2}}(\mu)=0$, $R_k(r)$, $k=1,2,...$ are eigenfunctions of the Laplace operator in the sphere of the radius r_0 defined in (3.1.57).

3.2 Examples of Initial Boundary Value Problem for Lame System Depending on Variables r, t with Different Sources

3.2.1 The Source of the Compression Center $f(\mathbf{x}, t) = \nabla_x \delta(\mathbf{x})\delta(t)$

Consider in the sphere $|\mathbf{x}| < r_0$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$, $t \in \mathbf{R}$ Lame system

$$\rho \frac{\partial^2 \mathbf{U}}{\partial t^2} = (\lambda + 2\mu)\nabla_x div_x \mathbf{U} - \mu curl_x curl_x \mathbf{U} + \mathbf{f}(\mathbf{x}, t), \tag{3.2.1}$$

subject to the initial conditions

$$\mathbf{U}(\mathbf{x}, t)|_{t=0} = 0, \tag{3.2.2}$$

$$\frac{\partial \mathbf{U}}{\partial t}(\mathbf{x}, t)|_{t=0} = 0, \tag{3.2.3}$$

and the boundary condition

$$\mathbf{U}(\mathbf{x}, t)|_{|\mathbf{x}|=r_0} = 0, \tag{3.2.4}$$

where
$\mathbf{f}(\mathbf{x}, t)$ is given function from the space L_2 defined as $\mathbf{f} = \nabla_x \delta(\mathbf{x})\delta(t)$,
ρ is the density of elastic medium,
λ, μ are Lame parameters(physical constants),
$\rho > 0$, $\lambda + 2\mu > 0$, $\mu > 0$ are positive constants. $\mathbf{U}(\mathbf{x}, t)$ is unknown function. Our aim is to find the function $\mathbf{U}(\mathbf{x}, t)$ satisfies (3.2.1)-(3.2.4).

Let $\mathbf{f}$ of the Lame system have the form $\mathbf{f} = \nabla_x g + curl_x \mathbf{F}$. We will find the solution of (3.2.1)-(3.2.4) in the form $\mathbf{U}(\mathbf{x}, t) = \nabla_x \phi + curl_x \mathbf{A}$.

Let $\mathbf{U}$ depend on r, t in spherical coordinate system. Lame system will be reduced to two wave equations. For $\overline{\mathbf{U}}(r, t) = \nabla \bar{\phi} + curl \overline{\mathbf{A}}$ boundary condition in spherical coordinate system defined in (3.1.9)-(3.1.11) becomes

$$\left[\cos\varphi \sin\theta \frac{\partial \bar{\phi}}{\partial r} + \sin\varphi \sin\theta \frac{\partial \bar{A}_3}{\partial r} - \cos\theta \frac{\partial \bar{A}_2}{\partial r}\right]\Big|_{r=r_0} = 0,$$

$$\left[\sin\varphi\sin\theta\frac{\partial\bar{\phi}}{\partial r}+\cos\theta\frac{\partial\bar{A}_1}{\partial r}-\cos\varphi\sin\theta\frac{\partial\bar{A}_3}{\partial r}\right]\Big|_{r=r_0}=0,$$

$$\left[\cos\theta\frac{\partial\bar{\phi}}{\partial r}+\cos\varphi\sin\theta\frac{\partial\bar{A}_2}{\partial r}-\sin\varphi\sin\theta\frac{\partial\bar{A}_1}{\partial r}\right]\Big|_{r=r_0}=0.$$

In matrix form

$$\begin{pmatrix} \cos\varphi\sin\theta & 0 & -\cos\theta & \sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \cos\theta & 0 & -\cos\varphi\sin\theta \\ \cos\theta & -\sin\varphi\sin\theta & \cos\varphi\sin\theta & 0 \end{pmatrix}\cdot\begin{pmatrix} \frac{\partial\bar{\phi}}{\partial r} \\ \frac{\partial\bar{A}_1}{\partial r} \\ \frac{\partial\bar{A}_2}{\partial r} \\ \frac{\partial\bar{A}_3}{\partial r} \end{pmatrix}_{r=r_0}=0, \tag{3.2.5}$$

Choosing $\frac{\partial\bar{\phi}}{\partial r}\big|_{r=r_0}=0$ (3.2.5) becomes

$$\begin{pmatrix} 0 & -\cos\theta & \sin\varphi\sin\theta \\ \cos\theta & 0 & -\cos\varphi\sin\theta \\ -\sin\varphi\sin\theta & \cos\varphi\sin\theta & 0 \end{pmatrix}\cdot\begin{pmatrix} \frac{\partial\bar{A}_1}{\partial r} \\ \frac{\partial\bar{A}_2}{\partial r} \\ \frac{\partial\bar{A}_3}{\partial r} \end{pmatrix}_{r=r_0}=0$$

Let

$$\mathcal{B}=\begin{pmatrix} 0 & -\cos\theta & \sin\varphi\sin\theta \\ \cos\theta & 0 & -\cos\varphi\sin\theta \\ -\sin\varphi\sin\theta & \cos\varphi\sin\theta & 0 \end{pmatrix}$$

Since $\det\mathcal{B}=0$ let's chose $\frac{\partial\bar{A}_3}{\partial r}\big|_{r=r_0}=0$ then

$$-\cos\theta \frac{\partial \bar{A}_2}{\partial r}\Big|_{r=r_0} = 0, \quad \text{if} \quad \theta \neq \frac{\pi}{2}(2k+1), \quad k \in \mathbf{N},$$

$$\frac{\partial \bar{A}_2}{\partial r}\Big|_{r=r_0} = 0,$$

and

$$\cos\theta \frac{\partial \bar{A}_1}{\partial r}\Big|_{r=r_0} = 0, \quad \text{if} \quad \theta \neq \frac{\pi}{2}(2k+1), \quad k \in \mathbf{N},$$

$$\frac{\partial \bar{A}_1}{\partial r}\Big|_{r=r_0} = 0.$$

Therefore, we obtain $\frac{\partial \bar{\phi}}{\partial r}\Big|_{r=r_0} = 0$ and $\frac{\partial \bar{A}_i}{\partial r}\Big|_{r=r_0} = 0, \quad i = 1, 2, 3.$

If $\mathbf{f}$ assumed in the form $\mathbf{f} = \nabla_x g + curl_x \mathbf{F}$ then $g = \delta(x)\delta(t), \ \mathbf{F} = 0$. Using the properties of the Dirac delta function that are (Barton, 1989)

$$\delta(\mathbf{x}) = \frac{(-1)^{n-1}2}{w_n(n-1)!}\delta^{(n-1)}(r),$$

where w_n is the area of the unit sphere and

$$r^2\delta''(r) = (-1)^2 2!\delta(r) = 2\delta(r),$$

since for the sphere $(n = 3)$ area of the unit sphere in $\mathbf{R}^3$ is $w_3 = 4\pi$ then

$$\delta(\mathbf{x})\delta(t) = \frac{(-1)^{3-1}}{w_3(3-1)!}2\delta^{(3-1)}(r)\delta(t) = \frac{1}{4\pi}\delta''(r) = \frac{1}{2\pi r^2}\delta(r)\delta(t).$$

The initial boundary value problem for the scalar wave equation becomes

$$\frac{1}{a_p^2}\frac{\partial^2 \bar{\phi}}{\partial t^2} = \triangle_r \bar{\phi} + \bar{g}(r,t), \tag{3.2.6}$$

$$\bar{\phi}(r,t)|_{t=0} = 0, \quad \frac{\partial \bar{\phi}}{\partial t}(r,t)|_{t=0} = 0, \tag{3.2.7}$$

$$|\bar{\phi}(r,t)|_{r=0} < \infty, \quad \frac{\partial \bar{\phi}(r,t)}{\partial r}\Big|_{r=r_0} = 0, \tag{3.2.8}$$

where

$$a_p^2 = \frac{\lambda + 2\mu}{\rho},$$

and

$$\bar{g}(r,t) = \frac{1}{(\lambda + 2\mu)2\pi r^2}\delta(r)\delta(t).$$

The initial boundary value problem with the Neumann boundary condition for the vector wave equation is

$$\frac{1}{a_s^2}\frac{\partial^2 \overline{\mathbf{A}}}{\partial t^2} = \triangle_r \overline{\mathbf{A}},$$

$$\overline{\mathbf{A}}(r,t)|_{t=0} = 0, \quad \frac{\partial \overline{\mathbf{A}}}{\partial t}(r,t)|_{t=0} = 0,$$

$$\left|\overline{\mathbf{A}}(r,t)|_{r=0}\right| < \infty, \quad \frac{\partial \overline{\mathbf{A}}(r,t)}{\partial r}\Big|_{r=r_0} = 0,$$

where

$$a_s^2 = \frac{\mu}{\rho},$$

It can be easily found that the solution of the initial boundary value problem for the vector wave equation is zero because $\bar{\mathbf{A}}(r,t) = 0$ satisfies the problem and by uniqueness theorem this is the unique solution. So $\mathbf{A}(\mathbf{x},t) = 0$ is. Thus, $curl_x \mathbf{A} = 0$. To solve the initial boundary value problem for the scalar wave equation we must find the Fourier coefficient

$$\bar{g}_k(t) = \int_0^{r_0} \frac{\sqrt{2}r^2\delta(r)\delta(t)J_{\frac{1}{2}}\left(\frac{r\mu_k^{(\frac{1}{2})}}{r_0}\right)\mathrm{d}r}{(\lambda+2\mu)2\pi r^2\sqrt{r}r_0\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]^{\frac{1}{2}}}$$

$$= \frac{\sqrt{2}\delta(t)}{(\lambda+2\mu)2\pi r_0\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]^{\frac{1}{2}}}$$

$$\times \int_0^{r_0} \frac{\delta(r)J_{\frac{1}{2}}\left(\frac{r\mu_k^{(\frac{1}{2})}}{r_0}\right)}{\sqrt{r}}\mathrm{d}r. \qquad (3.2.9)$$

The Bessel functions has the following equality (Arfken, 1968)

$$J_\alpha(\mathbf{x}) = \frac{2}{\sqrt{\pi}(\alpha-\frac{1}{2})!}\Big(\frac{x}{2}\Big)^\alpha \int_0^{\pi/2} \cos(x\sin\theta)\cos^{2\alpha}\theta \mathrm{d}\theta, \tag{3.2.10}$$

so for

$$\alpha = \frac{1}{2} \ \ and \ \ x = \mu_k^{(\frac{1}{2})}(\frac{r}{r_0})$$

we have

$$J_{\frac{1}{2}}\Big(\mu_k^{(\frac{1}{2})}(\frac{r}{r_0})\Big) = \frac{2}{\sqrt{\pi}}\Big(\frac{r\mu_k^{(\frac{1}{2})}}{2r_0}\Big)^{\frac{1}{2}} \int_0^{\pi/2} \cos(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\sin\theta)\cos\theta \mathrm{d}\theta. \tag{3.2.11}$$

Property 3.2.1. *$\delta(r)f(r) = f(0)\delta(r)$ for any continuous function f(r).*

Property 3.2.2. $\displaystyle\int_0^{r_0} \delta(r)\mathrm{d}r = 1.$

Using (3.2.11) and Properties (3.2.1),(3.2.2) of $\delta-$function (Barton, 1989) we obtain from (3.2.9)

$$\bar{g}_k(t) = \frac{\sqrt{2}\delta(t)}{(\lambda+2\mu)2\pi r_0\Bigg[\Big[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\Big]^2 + \Big(1-\frac{1}{4r_0^2\lambda_k}\Big)\Big[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\Big]^2\Bigg]^{\frac{1}{2}}}$$

$$\times\int_0^{r_0} \frac{\delta(r)}{\sqrt{r}}\frac{2}{\sqrt{\pi}}\Big(\frac{r\mu_k^{(\frac{1}{2})}}{2r_0}\Big)^{\frac{1}{2}} \int_0^{\pi/2} \cos(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\sin\theta)\cos\theta \mathrm{d}\theta \mathrm{d}r.$$

$$\bar{g}_k(t) = \frac{2\sqrt{\mu_k^{(\frac{1}{2})}}\delta(t)}{(\lambda+2\mu)2\pi r_0\sqrt{r_0\pi}\Bigg[\Big[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\Big]^2 + \Big(1-\frac{1}{4r_0^2\lambda_k}\Big)\Big[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\Big]^2\Bigg]^{\frac{1}{2}}}.$$

Then

$$T_k(t) = \frac{a_p}{\sqrt{\lambda_k}}\int_0^t g_k(\tau)\sin[a_p\sqrt{\lambda_k}(t-\tau)]\mathrm{d}\tau,$$

becomes

$$T_k(t)=\int_0^t \frac{a_p\sqrt{\mu_k^{(\frac{1}{2})}}\delta(\tau)\sin[a_p\sqrt{\lambda_k}(t-\tau)]\mathrm{d}\tau}{\sqrt{\lambda_k}(\lambda+2\mu)\pi r_0\sqrt{r_0\pi}\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2+\left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]^{\frac{1}{2}}},$$

$$T_k(t)=\frac{a_p\sqrt{\mu_k^{(\frac{1}{2})}}\sin[a_p\sqrt{\lambda_k}t]}{\sqrt{\lambda_k}(\lambda+2\mu)\pi r_0\sqrt{r_0\pi}\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2+\left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]^{\frac{1}{2}}},$$

where $\sqrt{\lambda_k}=\dfrac{\mu_k^{(\frac{1}{2})}}{r_0}$ and

$$R_k(r)=\frac{\sqrt{2}}{r_0\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2+\left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]^{\frac{1}{2}}}\frac{1}{\sqrt{r}}J_{\frac{1}{2}}\left(\frac{r\mu_k^{(\frac{1}{2})}}{r_0}\right).$$

A solution of the initial boundary value problem for the scalar wave equation is in the form

$$\bar{\phi}(r,t)=\sum_{k=1}^{\infty}T_k(t)R_k(r),$$

where

$$T_k(t)=\frac{a_p\sqrt{\mu_k^{(\frac{1}{2})}}\sin[a_p\sqrt{\lambda_k}t]}{\sqrt{\lambda_k}(\lambda+2\mu)\pi r_0\sqrt{r_0\pi}\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2+\left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]^{\frac{1}{2}}},$$

$\lambda_k=\left(\frac{\mu_k^{(\frac{1}{2})}}{r_0}\right)^2$,

$\mu_k^{(\frac{1}{2})}, k=1,2,...$ are roots of the Bessel equation $J'_{\frac{1}{2}}(\mu)\mu-\dfrac{1}{2}J_{\frac{1}{2}}(\mu)=0$ and the eigenfunctions are

$$R_k(r)=\frac{\sqrt{2}}{r_0\sqrt{r}\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2+\left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]^{\frac{1}{2}}}J_{\frac{1}{2}}\left(\frac{r\mu_k^{(\frac{1}{2})}}{r_0}\right).$$

$$\bar{\phi}(r,t)=\sum_{k=1}^{\infty}\frac{2a_p\mu_k^{(\frac{1}{2})}\sin[a_p\sqrt{\lambda_k}t]}{\sqrt{\lambda_k}(\lambda+2\mu)\pi^2 r_0^3\sqrt{2}\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2+\left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}$$

$$\times\int_0^{\pi/2}\cos(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\sin\theta)\cos\theta\mathrm{d}\theta,$$

where $a_p^2=\dfrac{\lambda+2\mu}{\rho}$.

Remark 3.2.1.

$$\int_0^{\pi/2}\cos(\mu_k^{(\frac{1}{2})}\sin\theta\frac{r}{r_0})\cos\theta\, d\theta=\frac{r_0}{r\mu_k^{(\frac{1}{2})}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\right).$$

Proof. Let $x=\mu_k^{(\frac{1}{2})}\sin\theta\frac{r}{r_0}$ then $\mathrm{d}x=\mu_k^{(\frac{1}{2})}\cos\theta\frac{r}{r_0}\mathrm{d}\theta$ then

$$\int_0^{\pi/2}\cos(\mu_k^{(\frac{1}{2})}\sin\theta\frac{r}{r_0})\cos\theta\mathrm{d}\theta=\int_0^{\mu_k^{(\frac{1}{2})}\frac{r}{r_0}}\cos x\frac{r_0\mathrm{d}x}{\mu_k^{(\frac{1}{2})}r}$$

$$=\frac{r_0}{r\mu_k^{(\frac{1}{2})}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\right).$$

□

Now using Remark (3.2.1) we obtain

$$\bar{\phi}(r,t)=\sum_{k=1}^{\infty}\frac{2a_p\mu_k^{(\frac{1}{2})}\sin[a_p\sqrt{\lambda_k}t]}{\sqrt{\lambda_k}(\lambda+2\mu)\pi^2 r_0^3\sqrt{2}\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2+\left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}$$

$$\times\frac{r_0}{r\mu_k^{(\frac{1}{2})}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\right).$$

Using the coordinate transforms

$$\begin{aligned} r &= \sqrt{x_1^2+x_2^2+x_3^2}, \\ \theta &= \arctan(\sqrt{x_1^2+x_2^2}/x_3), \\ \varphi &= \arctan(x_2/x_1), \end{aligned}$$

we obtain

$$\phi(\mathbf{x},t)=\sum_{k=1}^{\infty}\frac{2a_p\sin[a_p\sqrt{\lambda_k}t]}{\sqrt{\lambda_k}(\lambda+2\mu)\pi^2r_0^2\sqrt{2}\Bigg[\Big[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\Big]^2+\Big(1-\frac{1}{4r_0^2\lambda_k}\Big)\Big[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\Big]^2\Bigg]}$$

$$\times\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big),$$

is the solution of the initial boundary value problem (3.2.6)-(3.2.8). Before we found that the solution of the initial boundary value problem for the vector wave equation is zero. So we will find the solution of the Cauchy problem for the Lame system in the form

$$U(x_1,x_2,x_3,t)=\nabla_x\phi+curl_xA.$$

Now we will find

$$\nabla_x\phi=\Big(\frac{\partial\phi}{\partial x_1},\frac{\partial\phi}{\partial x_2},\frac{\partial\phi}{\partial x_3}\Big).$$

$$\frac{\partial\phi}{\partial x_i}=\sum_{k=1}^{\infty}\frac{2a_p\sin[a_p\sqrt{\lambda_k}t]}{\sqrt{\lambda_k}(\lambda+2\mu)\pi^2r_0^2\sqrt{2}\Bigg[\Big[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\Big]^2+\Big(1-\frac{1}{4r_0^2\lambda_k}\Big)\Big[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\Big]^2\Bigg]}$$

$$\times\frac{\partial}{\partial x_i}\Big(\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big),$$

$$\frac{\partial\phi}{\partial x_i}=\sum_{k=1}^{\infty}\frac{2a_p\sin[a_p\sqrt{\lambda_k}t]}{\sqrt{\lambda_k}(\lambda+2\mu)\pi^2r_0^2\sqrt{2}\Bigg[\Big[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\Big]^2+\Big(1-\frac{1}{4r_0^2\lambda_k}\Big)\Big[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\Big]^2\Bigg]}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}x_i}{r_0(x_1^2+x_2^2+x_3^2)}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)$$

$$-\frac{x_i}{\sqrt{(x_1^2+x_2^2+x_3^2)^3}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big].$$

If

$$\nabla_x \phi = \Big((\nabla\phi)_{x_1}, (\nabla\phi)_{x_2}, (\nabla\phi)_{x_3} \Big)$$

then

$$(\nabla\phi)_{x_1} = \sum_{k=1}^{\infty} \frac{2a_p \sin[a_p\sqrt{\lambda_k}t]x_1}{(x_1^2+x_2^2+x_3^2)\sqrt{\lambda_k}(\lambda+2\mu)\pi^2 r_0^2\sqrt{2}}$$

$$\times \frac{1}{\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)$$

$$-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\Big].$$

$$(\nabla\phi)_{x_2} = \sum_{k=1}^{\infty} \frac{2a_p \sin[a_p\sqrt{\lambda_k}t]x_2}{(x_1^2+x_2^2+x_3^2)\sqrt{\lambda_k}(\lambda+2\mu)\pi^2 r_0^2\sqrt{2}}$$

$$\times \frac{1}{\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)$$

$$-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\Big].$$

$$(\nabla\phi)_{x_3} = \sum_{k=1}^{\infty} \frac{2a_p \sin[a_p\sqrt{\lambda_k}t]x_3}{(x_1^2+x_2^2+x_3^2)\sqrt{\lambda_k}(\lambda+2\mu)\pi^2 r_0^2\sqrt{2}}$$

$$\times \frac{1}{\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)$$

$$-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\Big].$$

Since

$$U(x_1,x_2,x_3,t) = \nabla_x\phi + curl_x A,$$

and $curl_x\mathbf{A} = 0$ then

$$U(x_1,x_2,x_3,t) = \Big(U_1(x_1,x_2,x_3,t), U_2(x_1,x_2,x_3,t), U_3(x_1,x_2,x_3,t)\Big).$$

$$U_1(\mathbf{x},t) = \sum_{k=1}^{\infty} \frac{2a_p \sin[a_p\sqrt{\lambda_k}t]x_1}{(x_1^2+x_2^2+x_3^2)\sqrt{2\lambda_k}(\lambda+2\mu)\pi^2 r_0^2}$$

$$\times\Bigg[\frac{\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)}{\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2+\left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}$$

$$-\frac{\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)}{\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2+\left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}\Bigg] \qquad (3.2.12)$$

$$U_2(\mathbf{x},t) = \sum_{k=1}^{\infty} \frac{2a_p \sin[a_p\sqrt{\lambda_k}t]x_2}{(x_1^2+x_2^2+x_3^2)\sqrt{2\lambda_k}(\lambda+2\mu)\pi^2 r_0^2}$$

$$\times\Bigg[\frac{\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)}{\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2+\left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}$$

$$-\frac{\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)}{\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2+\left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}\Bigg]. \qquad (3.2.13)$$

$$U_3(\mathbf{x},t)=\sum_{k=1}^{\infty}\frac{2a_p\sin[a_p\sqrt{\lambda_k}t]x_3}{(x_1^2+x_2^2+x_3^2)\sqrt{2\lambda_k}(\lambda+2\mu)\pi^2r_0^2}$$

$$\times\left[\frac{\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)}{\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2+\left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}\right.$$

$$\left.-\frac{\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)}{\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2+\left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}\right]. \tag{3.2.14}$$

where $a_p^2=\dfrac{\lambda+2\mu}{\rho}$; $\lambda_k=\left(\dfrac{\mu_k^{(\frac{1}{2})}}{r_0}\right)^2$ and
$\mu_k^{(\frac{1}{2})}$, $k=1,2,...$ are roots of the equation $J'_{\frac{1}{2}}(\mu)\mu-\dfrac{1}{2}J_{\frac{1}{2}}(\mu)=0.$

Solution of the problem defined in (3.2.1)-(3.2.4) is given by the formulas (3.2.12)-(3.2.14)

3.2.1.1 Mathematica Commands for Finding the Components of the Solution

In this subsection we use the formulas that we obtained for the components of $\mathbf{U}(\mathbf{x},t)$ and `Mathematica` for the simulation of the wave propagation in spherical domain.
`Mathematica` commands for finding $\mathbf{U}_1(\mathbf{x},t)$, $\mathbf{U}_2(\mathbf{x},t)$, $\mathbf{U}_3(\mathbf{x},t)$ are listed below.

```
<< NumericalMath‘BesselZeros‘
<< Graphics‘Graphics3D‘
<<Graphics‘ParametricPlot3D‘
INPUT: Nt,rho,lambda,mu,radius;
speed = Sqrt[(lambda + 2*mu)/rho];
T = radius/speed;
Clear[aa]
tt = 170;
```

```
aa = 2*x*D[BesselJ[1/2, x], x];
bb = BesselJ[1/2, x];
setaa = {};
For[i = 1, i <= tt, i++,
  a[i] = x /. FindRoot[aa == bb, {x, i}];
  setaa = Append[setaa, a[i]];]
setaa = Sort[setaa];
k = 2;
b[1] = setaa[[1]];
For[j = 2, j <= tt, j++,
  If[setaa[[j]] != b[k -1],
    b[k - 1] = Module[{}, k = k + 1; setaa[[j]]];   ] ]
muk = b;
Do[l[i] = (muk[i]/radius)^2;
    f[i] = D[BesselJ[1/2, x]] /. x -> muk[i];
    g[i] = f[i]*f[i];
    ff[i] = BesselJ[1/2, x] /. x -> muk[i];
    gg[i] = ff[i]*ff[i];
  h[i] = 1/(muk[i]*(lambda + 2*mu)*Pi*Pi*
            radius*(g[i] + (1 - 1/(4*radius^2*l[i]))*gg[i]));
    p[i] = 2*speed*Sin[speed*Sqrt[l[i]]*t];
    zz[i] = muk[i]/radius;
    kk[i] = zz[i]*Cos[zz[i]*r] - Sin[zz[i]*r]*(1/r);
    , {i, 1, k - 1}];
x = r*Cos[varphi]*Sin[theta];
y = r*Sin[varphi]*Sin[theta];
z = r*Cos[theta];
f1 = Sum[p[ii]*h[ii]*kk[ii]*(x/r^2), {ii, 1, k - 1}];
f2 = Sum[p[ii]*h[ii]*kk[ii]*(y/r^2), {ii, 1, k - 1}];
f3 = Sum[p[ii]*h[ii]*kk[ii]*(z/r^2), {ii, 1, k - 1}];
RTfi1 = ReplaceAll[f1, varphi -> Pi/4];
RTfi2 = ReplaceAll[f2, varphi -> Pi/4];
RTfi3 = ReplaceAll[f3, varphi -> Pi/4];
```

OUTPUT: U_1, U_2, U_3."

In this subsection IBVP for the Lame system with Dirichlet boundary condition and with the external force of the form $\mathbf{f}(\mathbf{x},t) = \nabla\delta(x)\delta(t)$ is reduced to IBVP for the scalar wave equation and IBVP for the vector wave equation with Neumann boundary condition in spherical coordinates. These problems are solved and simulated in spherical domain. We use vector analysis technique, generalized functions theory and Fourier series expansion method to obtain the formula for the solution of the considered IBVP problem. Using this formula the simulation of elastic waves are done. As a result solution of the problem defined in (3.2.1)-(3.2.4) is (3.2.12)-(3.2.14). For simulation first 25 terms are added. So (3.2.12)-(3.2.14) becomes

$$U_1(\mathbf{x},t) = \sum_{k=1}^{25} \frac{2a_p \sin[a_p\sqrt{\lambda_k}t]x_1}{(x_1^2+x_2^2+x_3^2)\sqrt{2\lambda_k}(\lambda+2\mu)\pi^2 r_0^2}$$

$$\times \frac{1}{\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]} \Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)$$

$$-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\Big]. \tag{3.2.15}$$

$$U_2(\mathbf{x},t) = \sum_{k=1}^{25} \frac{2a_p \sin[a_p\sqrt{\lambda_k}t]x_2}{(x_1^2+x_2^2+x_3^2)\sqrt{2\lambda_k}(\lambda+2\mu)\pi^2 r_0^2}$$

$$\times \frac{1}{\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]} \Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)$$

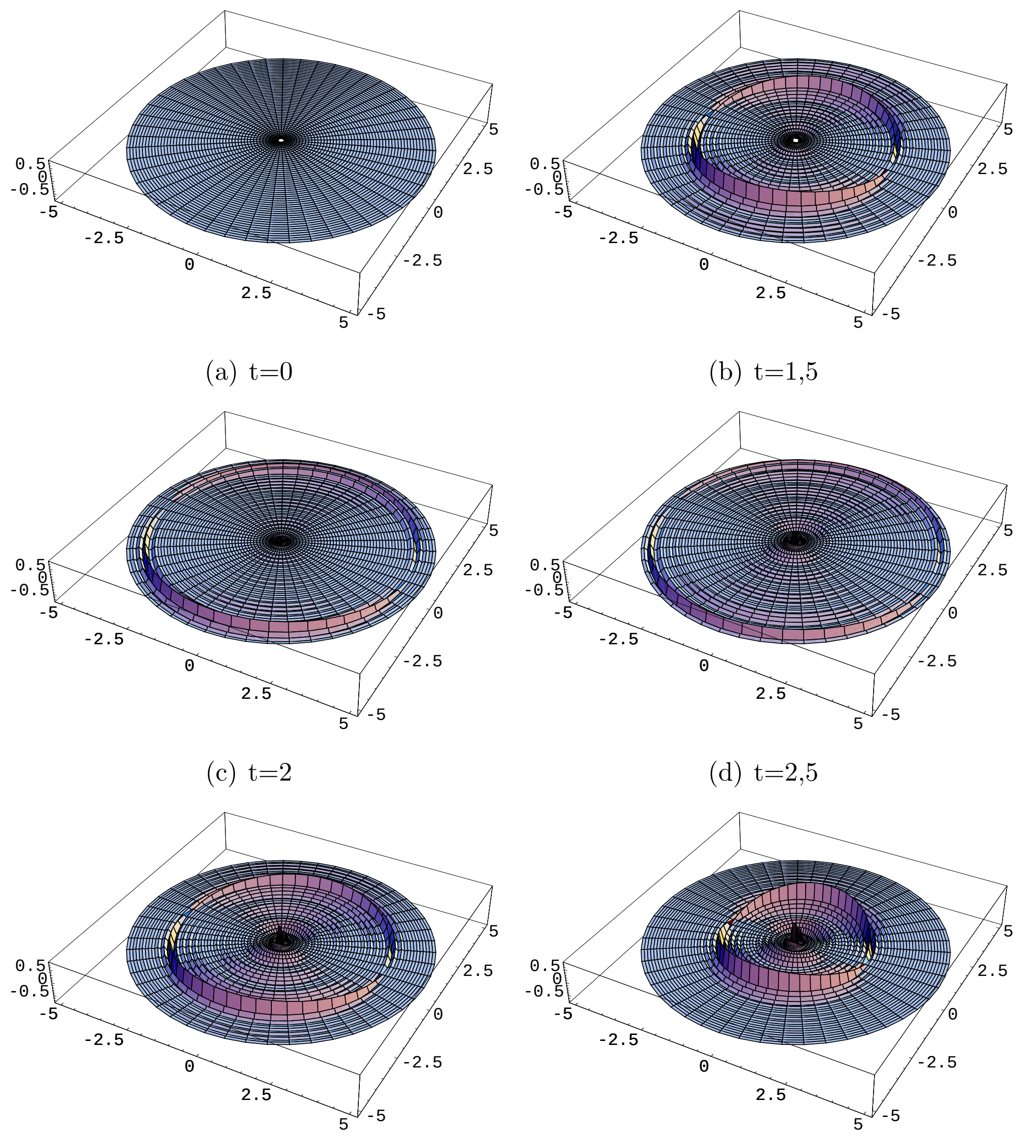
$$-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\Big]. \tag{3.2.16}$$

$$U_3(\mathbf{x},t)=\sum_{k=1}^{25}\frac{2a_p\sin[a_p\sqrt{\lambda_k}t]x_3}{(x_1^2+x_2^2+x_3^2)\sqrt{2\lambda_k}(\lambda+2\mu)\pi^2r_0^2}$$

$$\times\frac{1}{\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2+\left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)$$

$$-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\Big]. \tag{3.2.17}$$

For simulation of these components, the following values are used $\lambda=4;\ \rho=2,203;\ \mu=3,12;\ r_0=5.$ Here ρ is the density of elastic medium, λ,μ are Lame parameters(physical constants), $\lambda+2\mu>0,\ \rho>0,\ \mu>0$ are positive constants, r_0 is the radius of sphere and $a_p=\sqrt{\dfrac{\lambda+2\mu}{\rho}},\quad a_s=\sqrt{\dfrac{\mu}{\rho}},$ where a_p,a_s are longitudinal and transversal speeds of elastic waves.

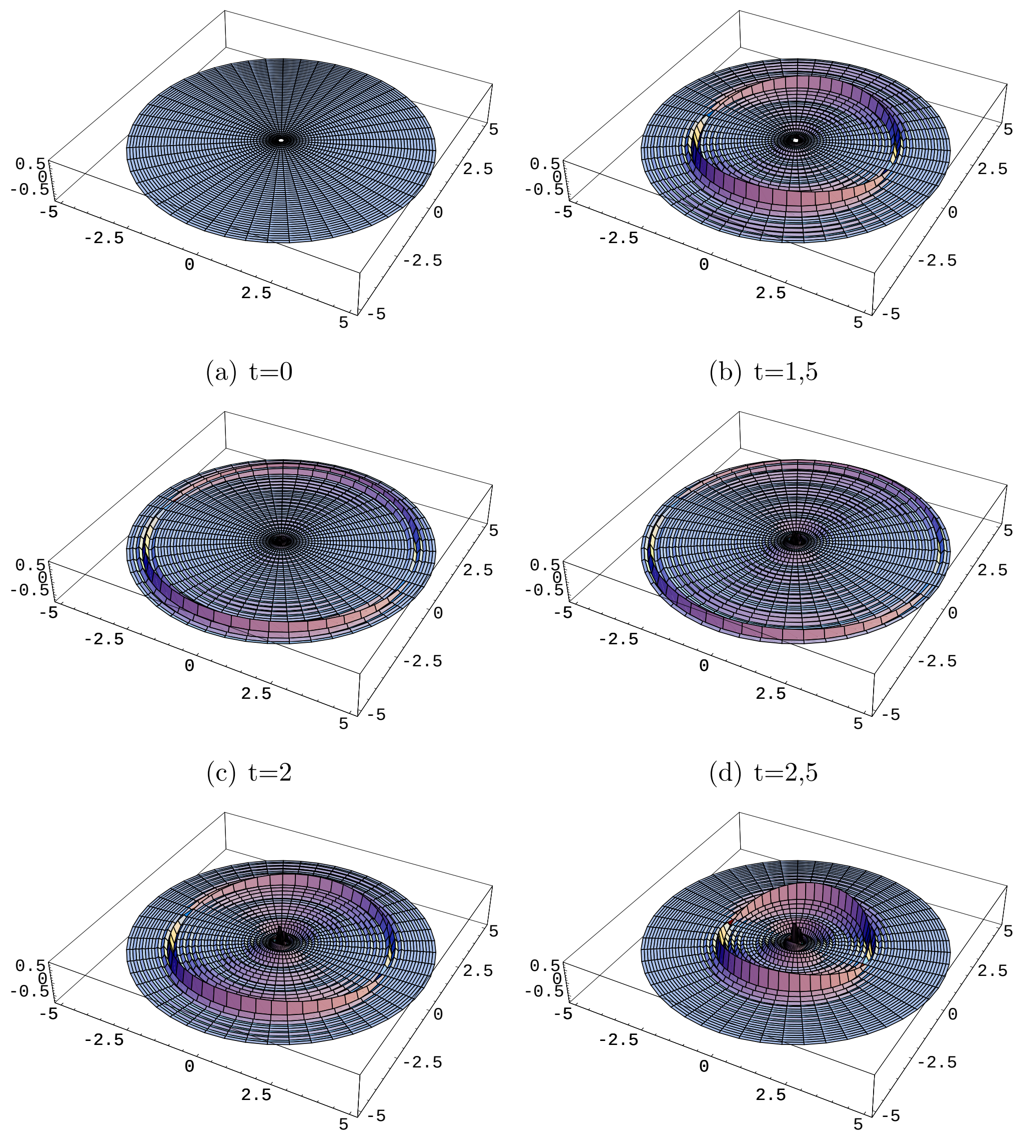


(a) t=0 (b) t=1,5

(c) t=2 (d) t=2,5

(e) t=3 (f) t=3,5

Figure 3.6: The first component of the displacement vector $\mathbf{U}_1(\mathbf{x},t)$



(a) t=0

(b) t=1,5

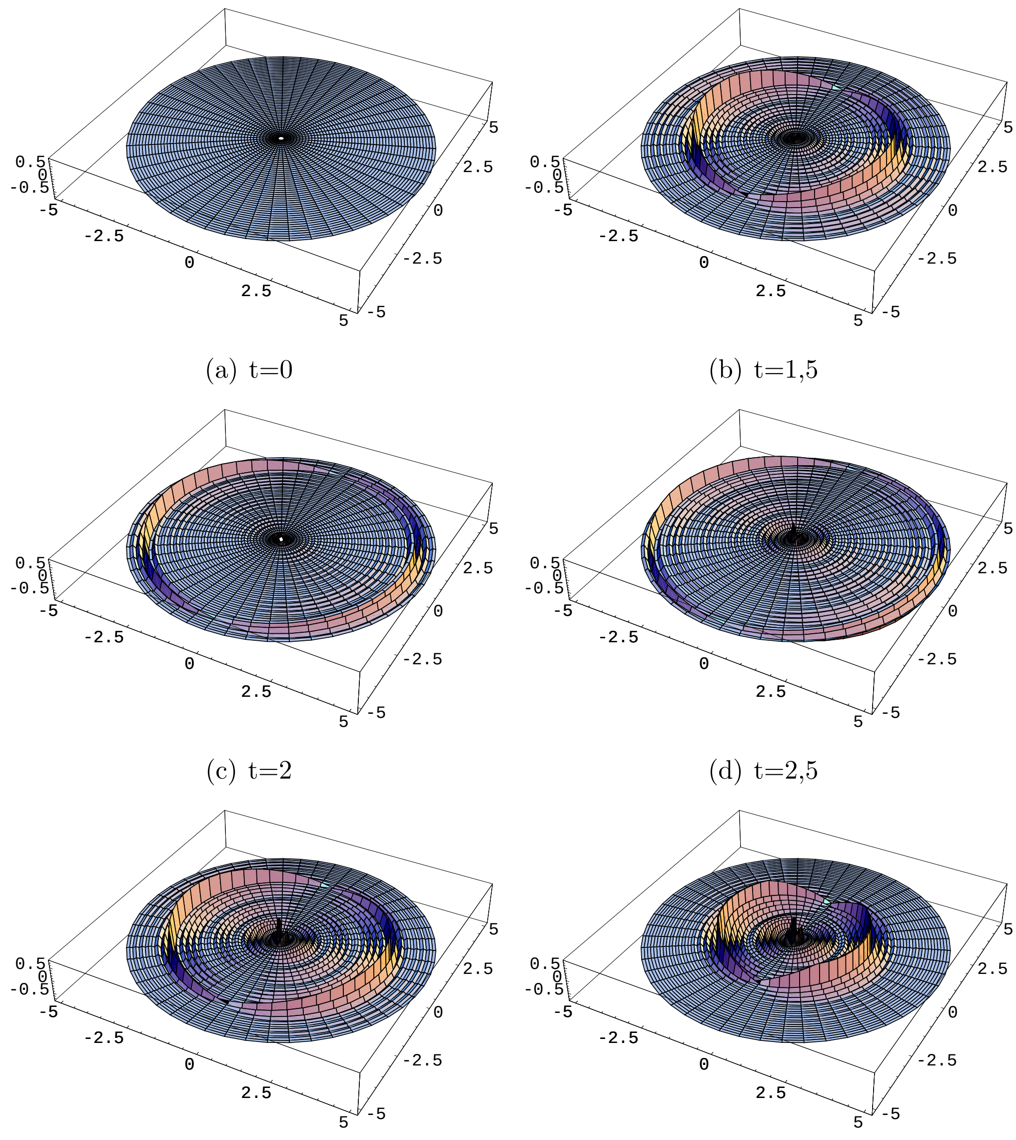
(c) t=2

(d) t=2,5

(e) t=3

(f) t=3,5

Figure 3.7: The second component of the displacement vector $\mathbf{U}_2(\mathbf{x},t)$



(a) t=0 (b) t=1,5

(c) t=2 (d) t=2,5

(e) t=3 (f) t=3,5

Figure 3.8: The third component of the displacement vector $\mathbf{U}_3(\mathbf{x},t)$

In Figure (3.6)-Figure (3.8), components of vector function $\mathbf{U}(\mathbf{x},t)$ that are $\mathbf{U}_1(\mathbf{x},t)$, $\mathbf{U}_2(\mathbf{x},t)$, $\mathbf{U}_3(\mathbf{x},t)$ are simulated. The horizontal axes are r, θ and the vertical axis corresponds to the component of $\mathbf{U}(\mathbf{x},t)$. The variable φ taken as $\varphi = \pi/4$. Since the IBVP for vector wave equation has a zero solution for external force $\mathbf{f}(\mathbf{x},t) = \nabla\delta(x)\delta(t)$ there exists only one wave that occurs from the solution of the IBVP for scalar wave equation.

3.2.2 The Source of the Rotation Center $f(\mathbf{x},t) = curl_x[\mathbf{e}\delta(\mathbf{x})\delta(t)]$

Now let us consider the following Cauchy problem for the Lame system:

$$\rho\frac{\partial^2\mathbf{U}}{\partial t^2} = (\lambda+2\mu)\nabla_x div_x\mathbf{U} - \mu curl_x curl_x\mathbf{U} + \mathbf{f}(\mathbf{x},t), \tag{3.2.18}$$

subject to the initial conditions

$$\mathbf{U}(\mathbf{x},t)|_{t=0} = 0, \quad \frac{\partial\mathbf{U}}{\partial t}(\mathbf{x},t)|_{t=0} = 0, \tag{3.2.19}$$

and the boundary condition

$$\mathbf{U}(\mathbf{x},t)|_{|\mathbf{x}|=r_0} = 0, \tag{3.2.20}$$

where $D = \{x \in \mathbf{R}^3 : |\mathbf{x}| < r_0\}$ and let vector f is given in the form

$$\mathbf{f}(\mathbf{x},t) = curl_x[\mathbf{e}\delta(\mathbf{x})\delta(t)],$$

where δ is the Dirac delta function and $\mathbf{e}$ is any vector function.

Our goal is to find the function $\mathbf{U}(\mathbf{x},t)$ satisfies (3.2.18)-(3.2.20).

If we assume $\mathbf{f}$ in the form

$$\mathbf{f} = \nabla_x g + curl_x\mathbf{F},$$

using potential approach we obtain

$$g = 0,$$

and

$$\mathbf{F} = \mathbf{e}\delta(\mathbf{x})\delta(t).$$

Let $\mathbf{U}$ depend on r, t in spherical coordinate system. Lame system will be reduced to two wave equations. For $\bar{\phi}(r,t)$ boundary condition in spherical coordinate system is $\frac{\partial\bar{\phi}(r,t)}{\partial r}\Big|_{r=r_0} = 0$ and for $\bar{\mathbf{A}}(r,t)$ is $\frac{\partial\bar{\mathbf{A}}(r,t)}{\partial r}\Big|_{r=r_0} = 0$. In spherical coordinates (Barton, 1989)

$$\mathbf{e}\delta(\mathbf{x})\delta(t) = \mathbf{e}\frac{1}{4\pi}\delta''(r) = \frac{\mathbf{e}}{2\pi r^2}\delta(r)\delta(t).$$

Hence we obtain IBVP for the scalar wave equation and IBVP for the vector wave equation.

The initial boundary value problem with Neumann boundary condition for the scalar wave equation is (See, Appendix A)

$$\frac{1}{a_p^2}\frac{\partial^2\bar{\phi}}{\partial t^2} = \triangle_r\bar{\phi},$$

$$\bar{\phi}(r,t)|_{t=0} = 0, \quad \frac{\partial\bar{\phi}}{\partial t}(r,t)|_{t=0} = 0,$$

$$\left|\bar{\phi}(r,t)|_{r=0}\right| < \infty, \quad \frac{\partial\bar{\phi}(r,t)}{\partial r}|_{r=r_0} = 0,$$

where

$$a_p^2 = \frac{\lambda + 2\mu}{\rho},$$

It can be easily found that the solution of the initial boundary value problem for the scalar wave equation is zero because $\phi(\mathbf{x},t) = 0$ satisfies the problem and by uniqueness theorem this is the unique solution. So $\nabla_x\phi = 0$.

The initial boundary value problem with Neumann boundary condition for the vector wave equation is

$$\frac{1}{a_s^2}\frac{\partial^2\bar{\mathbf{A}}}{\partial t^2} = \triangle_r\bar{\mathbf{A}} + \bar{\mathbf{F}}(r,t),$$

$$\bar{\mathbf{A}}(r,t)|_{t=0} = 0, \quad \frac{\partial\bar{\mathbf{A}}}{\partial t}(r,t)|_{t=0} = 0,$$

$$\left|\bar{\mathbf{A}}(r,t)|_{r=0}\right| < \infty, \quad \frac{\partial\bar{\mathbf{A}}(r,t)}{\partial r}|_{r=r_0} = 0,$$

where

$$a_s^2 = \frac{\mu}{\rho},$$

and

$$\bar{\mathbf{F}}(r,t) = \frac{\mathbf{e}}{\mu 2\pi r^2}\delta(r)\delta(t).$$

To solve the initial boundary value problem for the vector wave equation we must find the Fourier coefficient

$$\bar{\mathbf{F}}_k(t) = \int_0^{r_0} \frac{\mathbf{e}\delta(r)\delta(t)r^2\sqrt{2}J_{\frac{1}{2}}\left(\frac{r\mu_k^{(\frac{1}{2})}}{r_0}\right)\mathrm{d}r}{\mu 2\pi r^2\sqrt{r}r_0\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]^{\frac{1}{2}}},$$

$$\bar{\mathbf{F}}_k(t) = \frac{\mathbf{e}\sqrt{2}\delta(t)}{\mu 2\pi r_0\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]^{\frac{1}{2}}}$$

$$\times\int_0^{r_0}\frac{\delta(r)J_{\frac{1}{2}}\left(\frac{r\mu_k^{(\frac{1}{2})}}{r_0}\right)}{\sqrt{r}}\mathrm{d}r.$$

Using Remark (3.2.10) for

$$\alpha = \frac{1}{2} \ \ \textit{and} \ \ x = \mu_k^{(\frac{1}{2})}(\frac{r}{r_0}),$$

and using the properties of $\delta-$function from property (3.2.1) and (3.2.2) we obtain

$$\bar{\mathbf{F}}_k(t) = \frac{\mathbf{e}\sqrt{2}\delta(t)}{\mu 2\pi r_0\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1-\frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]^{\frac{1}{2}}}$$

$$\times\int_0^{r_0}\frac{\delta(r)}{\sqrt{r}}\frac{2}{\sqrt{\pi}}\Big(\frac{r\mu_k^{(\frac{1}{2})}}{2r_0}\Big)^{\frac{1}{2}}\int_0^{\pi/2}\cos(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\sin\theta)\cos\theta\mathrm{d}\theta\mathrm{d}r.$$

$$\bar{\mathbf{F}}_k(t) = \frac{\mathbf{e}\sqrt{\mu_k^{(\frac{1}{2})}}\delta(t)}{\mu\pi r_0\sqrt{r_0\pi}\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]^{\frac{1}{2}}}.$$

Then

$$T_k(t) = \frac{a_s}{\sqrt{\lambda_k}}\int_0^t \mathbf{F}_k(\tau)\sin[a_s\sqrt{\lambda_k}(t-\tau)]\mathrm{d}\tau,$$

$$T_k(t) = \frac{a_s}{\sqrt{\lambda_k}}\int_0^t \frac{2\mathbf{e}\sqrt{\mu_k^{(\frac{1}{2})}}\delta(\tau)\sin[a_s\sqrt{\lambda_k}(t-\tau)]\mathrm{d}\tau}{\mu 2\pi r_0\sqrt{r_0\pi}\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]^{\frac{1}{2}}},$$

$$T_k(t) = \frac{a_s}{\sqrt{\lambda_k}}\frac{2\mathbf{e}\sqrt{\mu_k^{(\frac{1}{2})}}\sin[a_s\sqrt{\lambda_k}t]}{\mu 2\pi r_0\sqrt{r_0\pi}\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]^{\frac{1}{2}}},$$

where

$$\sqrt{\lambda_k} = \frac{\mu_k^{(\frac{1}{2})}}{r_0},$$

and

$$R_k(r) = \frac{\sqrt{2}}{r_0\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]^{\frac{1}{2}}}\frac{1}{\sqrt{r}}J_{\frac{1}{2}}\left(\frac{r\mu_k^{(\frac{1}{2})}}{r_0}\right).$$

A solution of the initial boundary value problem for the vector wave equation is in the form

$$\bar{\mathbf{A}}(r,t) = \sum_{k=1}^{\infty} T_k(t)R_k(r),$$

where

$$T_k(t) = \frac{a_s}{\sqrt{\lambda_k}} \frac{\mathbf{e}\sqrt{\mu_k^{(\frac{1}{2})}}\sin[a_s\sqrt{\lambda_k}t]}{\mu\pi r_0\sqrt{r_0\pi}\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]^{\frac{1}{2}}},$$

$\lambda_k = \left(\frac{\mu_k^{(\frac{1}{2})}}{r_0}\right)^2,$

$\mu_k^{(\frac{1}{2})}, k = 1, 2, ...$ are roots of the Bessel equation $J'_{\frac{1}{2}}(\mu)\mu - \frac{1}{2}J_{\frac{1}{2}}(\mu) = 0$ and the eigenfunctions are

$$R_k(r) = \frac{\sqrt{2}}{r_0\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]^{\frac{1}{2}}} \frac{1}{\sqrt{r}} J_{\frac{1}{2}}\left(\frac{r\mu_k^{(\frac{1}{2})}}{r_0}\right).$$

$$\bar{\mathbf{A}}(r,t) = \sum_{k=1}^{\infty} \frac{2\mathbf{e}a_s\mu_k^{(\frac{1}{2})}\sin[a_s\sqrt{\lambda_k}t]}{\sqrt{\lambda_k}\mu\pi^2 r_0^3\sqrt{2}\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}$$

$$\times \int_0^{\pi/2} \cos(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\sin\theta)\cos\theta \mathrm{d}\theta,$$

where

$$a_s^2 = \frac{\mu}{\rho}.$$

Using Remark (3.2.1) we obtain

$$\bar{\mathbf{A}}(r,t) = \sum_{k=1}^{\infty} \frac{2\mathbf{e}a_s\mu_k^{(\frac{1}{2})}\sin[a_s\sqrt{\lambda_k}t]}{\sqrt{\lambda_k}\mu\pi^2 r_0^3\sqrt{2}\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}$$

$$\times \frac{r_0}{r\mu_k^{(\frac{1}{2})}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\right).$$

Using the coordinate transforms

$$\begin{aligned} r &= \sqrt{x_1^2+x_2^2+x_3^2}, \\ \theta &= \arctan(\sqrt{x_1^2+x_2^2}/x_3), \\ \varphi &= \arctan(x_2/x_1), \end{aligned}$$

we obtain

$$\mathbf{A}(\mathbf{x},t) = \sum_{k=1}^{\infty} \frac{2\mathbf{e}a_s \sin[a_s\sqrt{\lambda_k}t]}{\sqrt{\lambda_k}\mu\pi^2 r_0^2\sqrt{2}\Big[\Big[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\Big]^2 + \Big(1-\frac{1}{4r_0^2\lambda_k}\Big)\Big[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\Big]^2\Big]}$$

$$\times \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}} \sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big),$$

is solution of the initial boundary value problem for the vector wave equation. Before we found that the solution of the initial boundary value problem for the vector wave equation is zero. So we will find the solution of the Cauchy problem for the Lame system in the form

$$U(\mathbf{x},t) = \nabla_x\phi + curl_x A.$$

Now we will find

$$curl_x\mathbf{A} = \Big((curl_x\mathbf{A})_1, (curl_x\mathbf{A})_2, (curl_x\mathbf{A})_3\Big)$$

where

$$(curl_x\mathbf{A})_1 = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3},$$

$$(curl_x\mathbf{A})_1 = \sum_{k=1}^{\infty} \frac{a_s}{\sqrt{\lambda_k}} \frac{2\sin(a_s\sqrt{\lambda_k}t)\{e_3x_2 - e_2x_3\}}{\mu\pi^2\Big[\Big[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\Big]^2 + \Big(1-\frac{1}{4r_0^2\lambda_k}\Big)\Big[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\Big]^2\Big]}$$

$$\times \frac{1}{r_0^3(x_1^2+x_2^2+x_3^2)}\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)$$

$$-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big].$$

$$(curl_x\mathbf{A})_2 = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1},$$

$$(curl_x\mathbf{A})_2 = \sum_{k=1}^{\infty} \frac{a_s}{\sqrt{\lambda_k}} \frac{2\sin(a_s\sqrt{\lambda_k}t)\{e_1x_3 - e_3x_1\}}{\mu\pi^2\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}$$

$$\times \frac{1}{r_0^3(x_1^2+x_2^2+x_3^2)}\left[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\right.$$

$$\left. - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\right].$$

$$(curl_x\mathbf{A})_3 = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2},$$

$$(curl_x\mathbf{A})_3 = \sum_{k=1}^{\infty} \frac{a_s}{\sqrt{\lambda_k}} \frac{2\sin(a_s\sqrt{\lambda_k}t)\{e_2x_1 - e_1x_2\}}{\mu\pi^2\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}$$

$$\times \frac{1}{r_0^3(x_1^2+x_2^2+x_3^2)}\left[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\right.$$

$$\left. - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\right].$$

Since

$$U(\mathbf{x},t) = \nabla_x\phi + curl_x\mathbf{A},$$

and

$$\nabla\phi = 0,$$

then

$$U(\mathbf{x},t) = \Big(U_1(\mathbf{x},t), U_2(\mathbf{x},t), U_3(\mathbf{x},t)\Big).$$

$$U_1(\mathbf{x},t) = \sum_{k=1}^{\infty} \frac{a_s}{\sqrt{\lambda_k}} \frac{2\sin(a_s\sqrt{\lambda_k}t)\{e_3x_2 - e_2x_3\}}{\mu\pi^2\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}$$

$$\times \frac{1}{r_0^3(x_1^2+x_2^2+x_3^2)}\left[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\right.$$

$$\left.-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\right], \tag{3.2.21}$$

$$U_2(\mathbf{x},t) = \sum_{k=1}^{\infty} \frac{a_s}{\sqrt{\lambda_k}} \frac{2\sin(a_s\sqrt{\lambda_k}t)\{e_1x_3 - e_3x_1\}}{\mu\pi^2\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}$$

$$\times \frac{1}{r_0^3(x_1^2+x_2^2+x_3^2)}\left[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\right.$$

$$\left.-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\right], \tag{3.2.22}$$

$$U_3(\mathbf{x},t) = \sum_{k=1}^{\infty} \frac{a_s}{\sqrt{\lambda_k}} \frac{2\sin(a_s\sqrt{\lambda_k}t)\{e_2x_1 - e_1x_2\}}{\mu\pi^2\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}$$

$$\times \frac{1}{r_0^3(x_1^2+x_2^2+x_3^2)}\left[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\right.$$

$$\left.-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\right], \tag{3.2.23}$$

where $a_s^2 = \dfrac{\mu}{\rho}$; $\lambda_k = \left(\dfrac{\mu_k^{(\frac{1}{2})}}{r_0}\right)^2$;
$\mu_k^{(\frac{1}{2})}$; $k = 1, 2, ...$ are roots of the Bessel equation $J'_{\frac{1}{2}}(\mu)\mu - \frac{1}{2}J_{\frac{1}{2}}(\mu) = 0$.

Solution of the problem defined in (3.2.18)-(3.2.20) is given by the formulas (3.2.21)-(3.2.23)

3.2.2.1 Mathematica Commands for Finding the Components of the Solution

In this subsection we use the formulas that we obtained for the components of $\mathbf{U}(\mathbf{x}, t)$ and `Mathematica` for the simulation of the wave propagation in spherical domain.
`Mathematica` commands for finding $\mathbf{U}_1(\mathbf{x}, t)$, $\mathbf{U}_2(\mathbf{x}, t)$, $\mathbf{U}_3(\mathbf{x}, t)$ are listed below.

```
<< NumericalMath`BesselZeros`
<<Graphics`Graphics3D`
<<Graphics`ParametricPlot3D`

INPUT: Nt,rho,lambda,mu,radius;
T = radius/as;
as = Sqrt[mu/rho];
Clear[aa]
tt = 170;
aa = 2*x*D[BesselJ[1/2, x], x];
bb = BesselJ[1/2, x];
setaa = {}; For[i = 1, i <= tt, i++,
a[i]=x/.FindRoot[aa == bb, {x, i}];
  setaa = Append[setaa, a[i]]]
setaa = Sort[setaa];
k = 2;
b[1] = setaa[[1]];
For[j = 2, j <= tt, j++,
  If[setaa[[j]] != b[k - 1],
    b[k - 1] = Module[{}, k = k + 1; setaa[[j]]] ]  ]
muk = b;
```

```
Do[l[i] = (muk[i]/radius)^2;
    f[i] = D[BesselJ[1/2, x]] /. x -> muk[i];
    g[i] = f[i]*f[i];
    ff[i] = BesselJ[1/2, x] /. x -> muk[i];
    gg[i] = ff[i]*ff[i];
    h[i] = 1/(muk[i]*(mu)*Pi*Pi*radius*(g[i]
          + (1 - 1/(4*radius^2*l[i]))*gg[i]));
    p[i] = 2*as*Sin[as*Sqrt[l[i]]*t];
    zz[i] = muk[i]/radius;
    kk[i] = zz[i]*Cos[zz[i]*r] - Sin[zz[i]*r]*(1/r);
                                        , {i, 1, k - 1}];

x1 = r*Cos[varp]*Sin[the];
x2 = r*Sin[varp]*Sin[the];
x3 = r*Cos[the];

A1 = Sum[p[ii]*h[ii]*kk[ii]*(x1/r^2), {ii, 1, k - 1}];
A2 = Sum[p[ii]*h[ii]*kk[ii]*(x2/r^2), {ii, 1, k - 1}];
A3 = Sum[p[ii]*h[ii]*kk[ii]*(x3/r^2), {ii, 1, k - 1}];

RTcurlU1=ReplaceAll[A1, varp -> Pi/4];
RTcurlU2=ReplaceAll[A2, varp -> Pi/4];
RTcurlU3=ReplaceAll[A3, varp -> Pi/4];
```

OUTPUT: U_1, U_2, U_3.

In this subsection IBVP for the Lame system with Dirichlet boundary condition and with the external force of the form $\mathbf{f}(\mathbf{x}, t) = curl\Big[\mathbf{e}\delta(x)\delta(t)\Big]$ is reduced to IBVP for the scalar wave equation and IBVP for the vector wave equation with Neumann boundary condition in spherical coordinates. These problems are solved and simulated in spherical domain. We use vector analysis technique, generalized functions theory and Fourier series expansion method to obtain the formula for the solution of the considered IBVP problem. Using this formula the simulation of elastic waves are done. As a result solution of the problem defined in (3.2.18)-(3.2.20) is (3.2.21)-(3.2.23). For simulation first 25 terms are added. So (3.2.21)-(3.2.23) becomes

$$U_1(\mathbf{x},t) = \sum_{k=1}^{25} \frac{a_s}{\sqrt{\lambda_k}} \frac{2\sin(a_s\sqrt{\lambda_k}t)\{e_3x_2 - e_2x_3\}}{\mu\pi^2\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}$$

$$\times \frac{1}{r_0^3(x_1^2+x_2^2+x_3^2)}\Bigg[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)$$

$$-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\Bigg], \tag{3.2.24}$$

$$U_2(\mathbf{x},t) = \sum_{k=1}^{25} \frac{a_s}{\sqrt{\lambda_k}} \frac{2\sin(a_s\sqrt{\lambda_k}t)\{e_1x_3 - e_3x_1\}}{\mu\pi^2\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}$$

$$\times \frac{1}{r_0^3(x_1^2+x_2^2+x_3^2)}\Bigg[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)$$

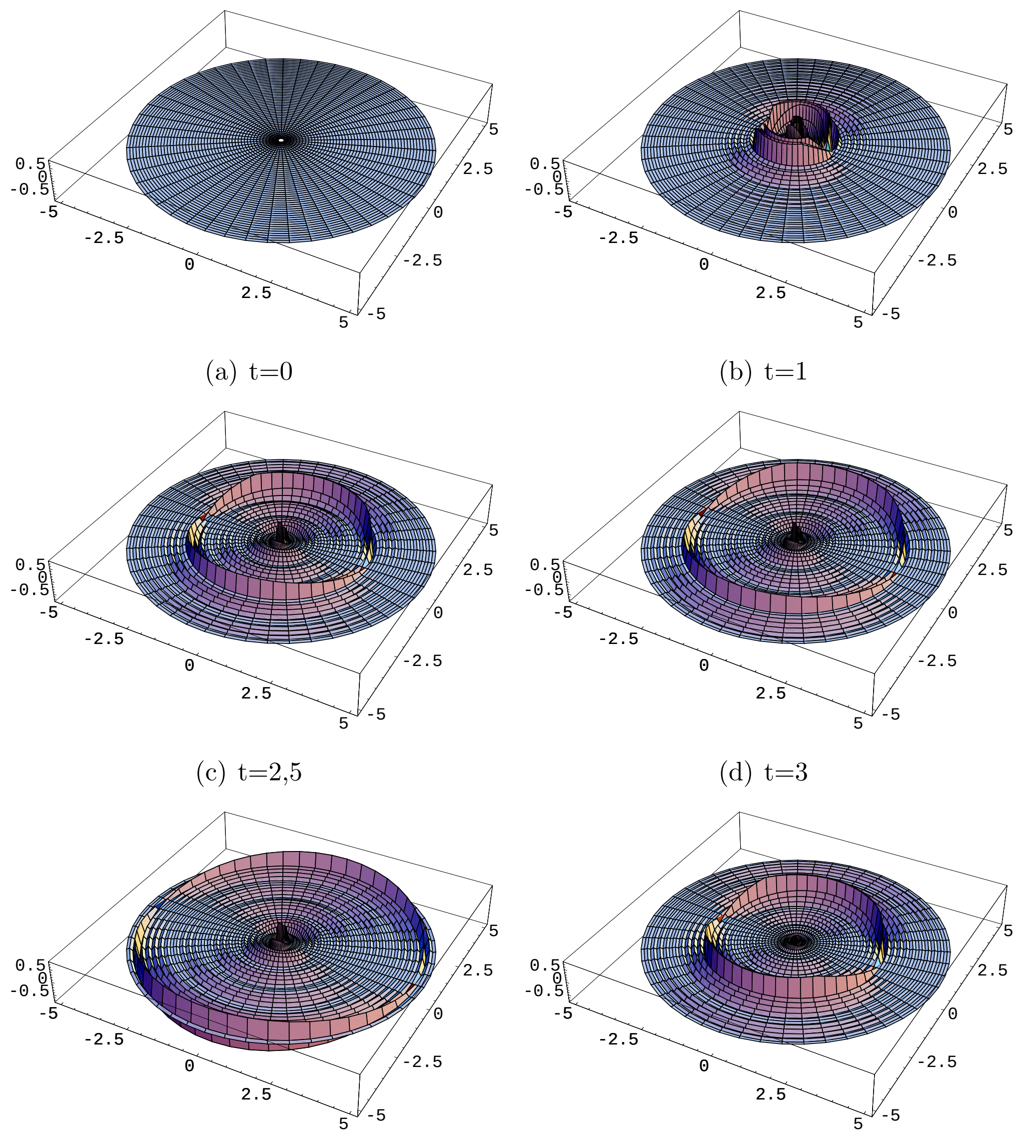
$$-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\Bigg], \tag{3.2.25}$$

$$U_3(\mathbf{x},t) = \sum_{k=1}^{25} \frac{a_s}{\sqrt{\lambda_k}} \frac{2\sin(a_s\sqrt{\lambda_k}t)\{e_2x_1 - e_1x_2\}}{\mu\pi^2\left[\left[J'_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2 + \left(1 - \frac{1}{4r_0^2\lambda_k}\right)\left[J_{\frac{1}{2}}(\sqrt{\lambda_k}r_0)\right]^2\right]}$$

$$\times \frac{1}{r_0^3(x_1^2+x_2^2+x_3^2)}\Bigg[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)$$

$$-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\Bigg], \tag{3.2.26}$$

For simulation of these components, the following values are used $\lambda = 4$; $\rho = 2,203$; $\mu = 3,12$; $r_0 = 5$; $e_1 = 1$; $e_2 = 2$; $e_3 = 4$. Here ρ is the density of elastic medium, λ, μ are Lame parameters(physical constants), $\rho > 0$, $\lambda + 2\mu > 0$, $\mu > 0$ are positive constants, r_0 is the radius of sphere and $a_p = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$, $a_s = \sqrt{\dfrac{\mu}{\rho}}$ where a_p, a_s are longitudinal and transversal speeds of elastic waves.

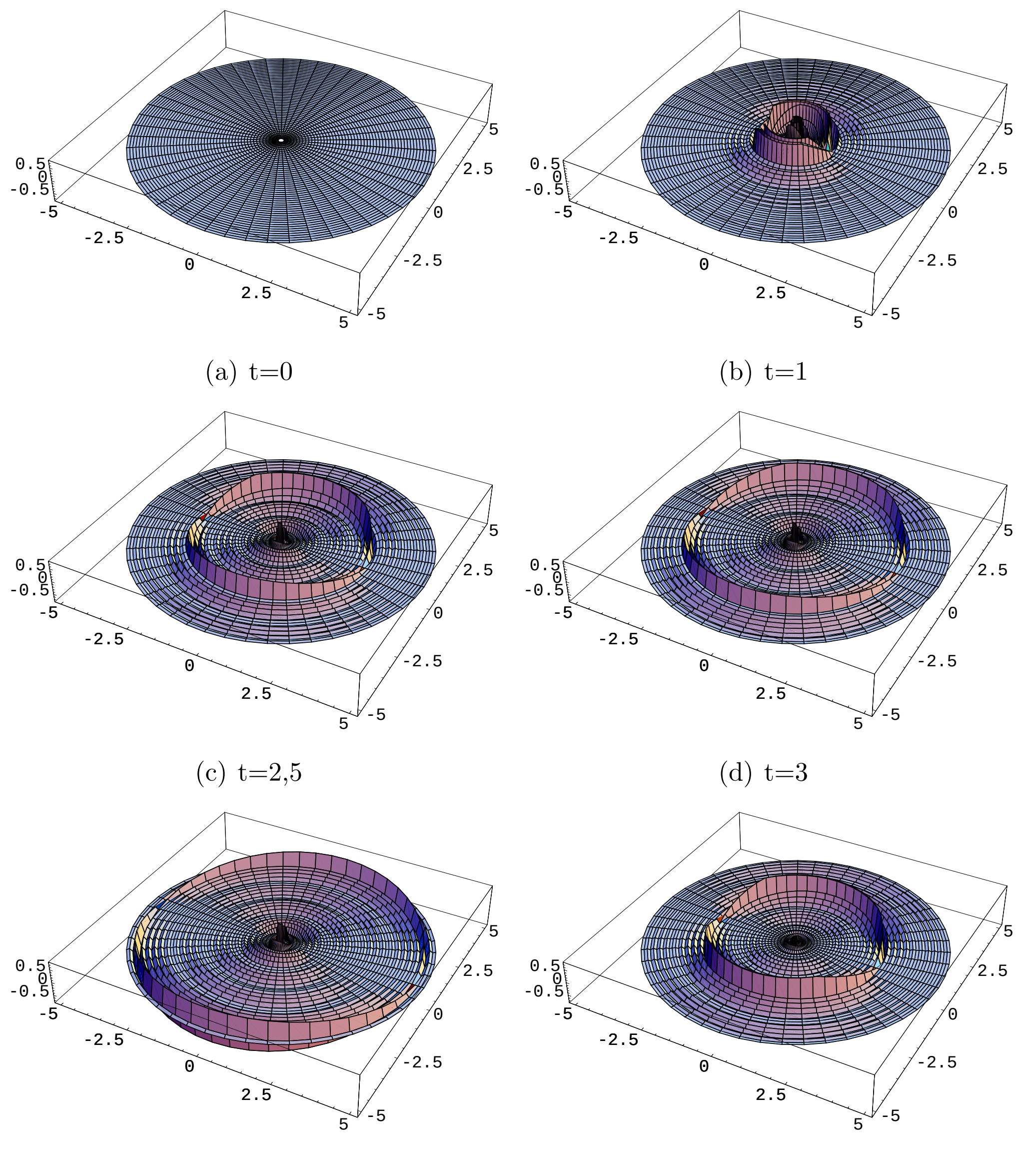


(a) t=0 (b) t=1

(c) t=2,5 (d) t=3

(e) t=4,5 (f) t=6

Figure 3.9: The first component of the displacement vector $\mathbf{U}_1(\mathbf{x},\mathrm{t})$, $e_1 = 1$

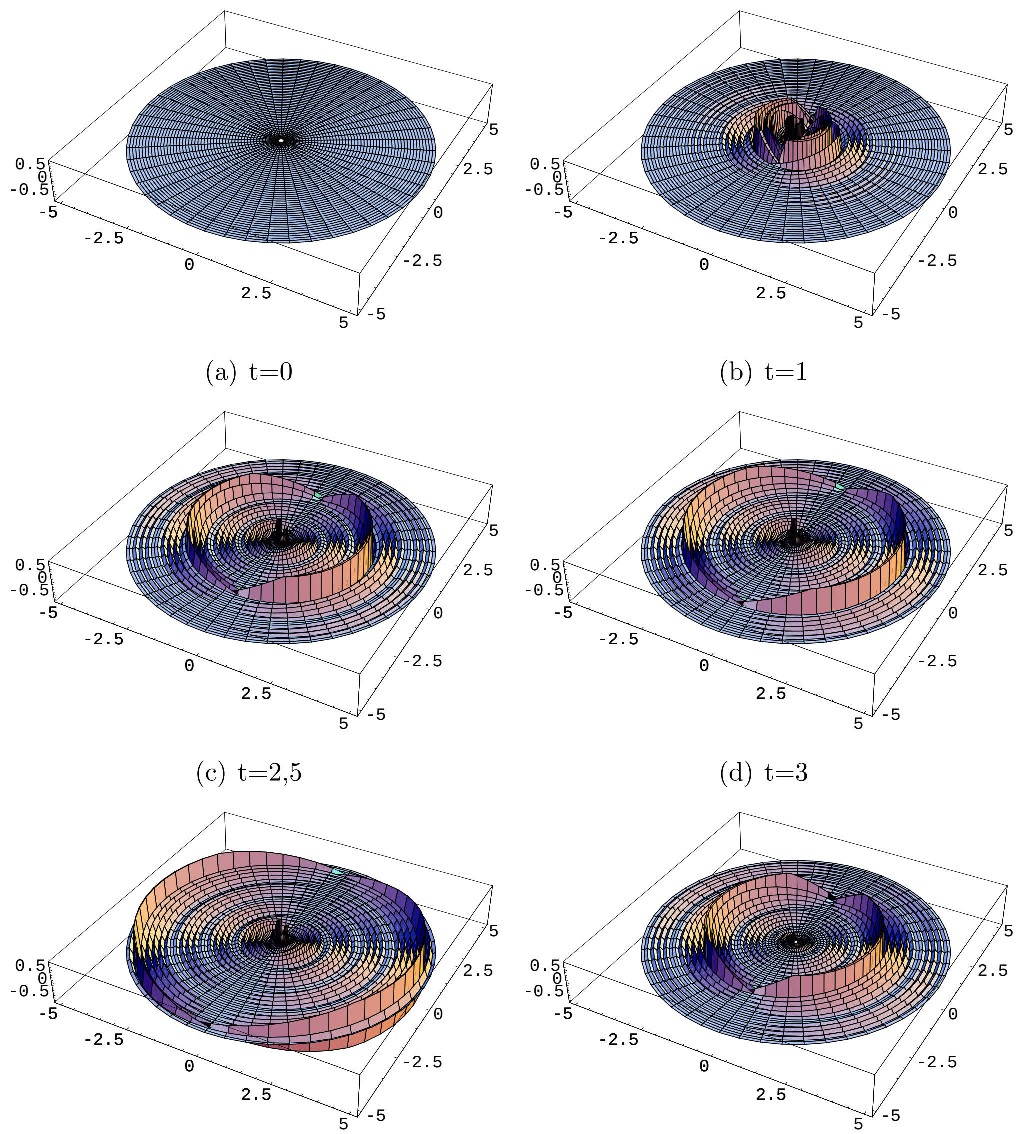


(a) t=0 (b) t=1

(c) t=2,5 (d) t=3

(e) t=4,5 (f) t=6

Figure 3.10: The second component of the displacement vector $\mathbf{U}_2(\mathbf{x},\mathrm{t})$, $e_2 = 2$



(a) t=0

(b) t=1

(c) t=2,5

(d) t=3

(e) t=4,5

(f) t=6

Figure 3.11: The third component of the displacement vector $\mathbf{U}_3(\mathbf{x},\mathrm{t})$, $e_3 = 4$

In Figure (3.9)-Figure (3.11), components of vector function $\mathbf{U}(\mathbf{x},t)$ that are $\mathbf{U}_1(\mathbf{x},t)$ $\mathbf{U}_2(\mathbf{x},t)$ $\mathbf{U}_3(\mathbf{x},t)$ are simulated. The horizontal axes are r, θ and the vertical axis corresponds to the component of $\mathbf{U}(\mathbf{x},t)$. The variable φ taken as $\varphi = \pi/4$. Since the IBVP for scalar wave equation has a zero solution for external force $\mathbf{f}(\mathbf{x},t) = \nabla\delta(x)\delta(t)$, there exists only one wave that occurs from the solution of the IBVP for vector wave equation.

3.3 Examples of Initial Boundary Value Problem for the Lame System with Dirichlet Boundary Condition

3.3.1 The Source of the Compression Center $f(\mathbf{x},t) = \nabla_x\delta(\mathbf{x})\delta(t)$

Now let us consider the following Cauchy problem for the Lame system:

$$\rho\frac{\partial^2\mathbf{U}}{\partial t^2} = (\lambda+2\mu)\nabla_x div_x\mathbf{U} - \mu curl_x curl_x\mathbf{U} + \mathbf{f}(\mathbf{x},t), \tag{3.3.1}$$

subject to the initial conditions

$$\mathbf{U}(\mathbf{x},t)|_{t=0} = 0, \tag{3.3.2}$$

$$\frac{\partial\mathbf{U}}{\partial t}(\mathbf{x},t)|_{t=0} = 0, \tag{3.3.3}$$

and the boundary condition

$$\mathbf{U}(\mathbf{x},t)|_{|\mathbf{x}|=r_0} = 0, \tag{3.3.4}$$

where $D = \{x \in \mathbf{R}^3 : |\mathbf{x}| < r_0\}$ and let vector f is given in the form

$$\mathbf{f}(\mathbf{x},t) = \nabla_x\delta(\mathbf{x})\delta(t),$$

where δ is the Dirac delta function. Our goal is to find the function $\mathbf{U}(\mathbf{x},t)$ satisfies (3.3.1)-(3.3.4).

If we assume

$$\mathbf{f} = \nabla_x g + curl_x\mathbf{F},$$

then we obtain

$$g = \delta(\mathbf{x})\delta(t),$$

and

$$\mathbf{F} = 0.$$

If $a_pT < r_0$ and $a_sT < r_0$ the source of the center will not reach to the boundary so there will not be any reflection. And also for the source with the distance r_1 from the center will not reach to the boundary in the case $a_pT < (r_0 - r_1)$ and $a_sT < (r_0 - r_1)$. This means there is no reflection in this case also. Thus solutions of initial boundary value problem for the Lame system do not depend on boundary values in the cases we discussed above. This means that we can take arbitrary boundary conditions. Assume that we have

$$\phi(\mathbf{x},t)|_{|\mathbf{x}|=r_0} = 0 \text{ and } \mathbf{A}(\mathbf{x},t)|_{|\mathbf{x}|=r_0} = 0.$$

So we obtain initial boundary value problem for $\phi(\mathbf{x},t)$

$$\frac{1}{a_p^2}\frac{\partial^2\phi}{\partial t^2} = \triangle_x\phi + \frac{1}{\lambda+2\mu}g, \qquad a_p^2 = \frac{\lambda+2\mu}{\rho}, \tag{3.3.5}$$

$$\phi(\mathbf{x},t)|_{t=0} = 0, \ \ \frac{\partial\phi}{\partial t}(\mathbf{x},t)|_{t=0} = 0, \tag{3.3.6}$$

$$\phi(\mathbf{x},t)|_{|\mathbf{x}|=r_0} = 0, \tag{3.3.7}$$

and initial boundary value problem for $\mathbf{A}(\mathbf{x},t)$

$$\frac{1}{a_s^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \triangle_x\mathbf{A} + \frac{1}{\mu}\mathbf{F}, \qquad a_s^2 = \frac{\mu}{\rho}, \tag{3.3.8}$$

$$\mathbf{A}(\mathbf{x},t)|_{t=0} = 0, \ \ \frac{\partial\mathbf{A}}{\partial t}(\mathbf{x},t)|_{t=0} = 0, \tag{3.3.9}$$

$$\mathbf{A}(\mathbf{x},t)|_{|\mathbf{x}|=r_0} = 0. \tag{3.3.10}$$

Solution of the initial boundary value problem for the vector wave equation is zero because $\mathbf{A}(\mathbf{x},t) = 0$ satisfies the problem and by uniqueness theorem this is the unique solution. So $curl_x\mathbf{A} = 0$.

Now we are going to solve the initial boundary value problem for scalar wave equation in the sphere so let us firstly write the equation in spherical coordinates. Substituting $x_1 = r\cos\varphi\sin\theta,\ x_2 = r\sin\varphi\sin\theta,\ x_3 = r\cos\theta$ into $\phi(x_1, x_2, x_3, t)$ we have

$$\phi(r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta, t) = \bar{\phi}(r, \theta, \varphi, t).$$

We obtain new function $\bar{\phi}(r, \theta, \varphi, t)$. Using the properties of the Dirac delta function that are (Barton, 1989)

$$\delta(\mathbf{x}) = \frac{(-1)^{n-1}2}{w_n(n-1)!}\delta^{(n-1)}(r),$$

where w_n is the area of the unit sphere and

$$r^2\delta''(r) = (-1)^2 2!\delta(r) = 2\delta(r),$$

since for the sphere $(n = 3)$ area of the unit sphere in $\mathbf{R}^3$ is $w_3 = 4\pi$ then

$$\delta(\mathbf{x}) = \frac{(-1)^{3-1}}{w_3(3-1)!}2\delta^{(3-1)}(r) = \frac{1}{4\pi}\delta''(r) = \frac{1}{2\pi r^2}\delta(r),$$

the initial boundary value problem for the scalar wave equation becomes

$$\frac{1}{a_p^2}\frac{\partial^2\bar{\phi}}{\partial t^2} = \triangle_{r,\theta,\varphi}\bar{\phi} + \bar{g}(r, \theta, \varphi, t),$$

$$\bar{\phi}(r, \theta, \varphi, t)|_{t=0} = 0, \quad \frac{\partial\bar{\phi}}{\partial t}(r, \theta, \varphi, t)|_{t=0} = 0,$$

$$|\bar{\phi}(r, \theta, \varphi, t)|_{\theta=0} < \infty, \quad \bar{\phi}(r, \theta, \varphi + \pi, t) = \bar{\phi}(r, \theta, \varphi, t),$$

$$|\bar{\phi}(r, \theta, \varphi, t)|_{r=0} < \infty, \quad \bar{\phi}(r, \theta, \varphi, t)|_{r=r_0} = 0,$$

where

$$a_p^2 = \frac{\lambda + 2\mu}{\rho},$$

and

$$\bar{g}(r, \theta, \varphi, t) = \frac{1}{(\lambda + 2\mu)2\pi r^2}\delta(r)\delta(t).$$

To solve the initial boundary value problem for the scalar wave equation we must find the Fourier coefficient

$$\bar{g}_{knm}(t)=\int_0^{r_0}\int_0^{2\pi}\int_0^{\pi}\frac{\delta(r)\delta(t)}{(\lambda+2\mu)2\pi r^2}r^2\sin\theta\frac{\sqrt{2}Y_n^{(m)}(\theta,\varphi)}{r_0\sqrt{r}\Big[J'_{n+\frac{1}{2}}(\mu_k^{(n+\frac{1}{2})})\Big]}$$
$$\times J_{n+\frac{1}{2}}\Big(\mu_k^{(n+\frac{1}{2})}\frac{r}{r_0}\Big)\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r,$$

$$\bar{g}_{knm}(t)=\int_0^{r_0}\int_0^{2\pi}\int_0^{\pi}\frac{\delta(r)\delta(t)}{(\lambda+2\mu)2\pi}\frac{\sqrt{2}Y_n^{(m)}(\theta,\varphi)}{r_0\sqrt{r}\Big[J'_{n+\frac{1}{2}}(\mu_k^{(n+\frac{1}{2})})\Big]}$$
$$\times J_{n+\frac{1}{2}}\Big(\mu_k^{(n+\frac{1}{2})}\frac{r}{r_0}\Big)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r,$$

$$\bar{g}_{knm}(t)=\frac{\delta(t)\sqrt{2}}{r_0(\lambda+2\mu)2\pi}\Big(\int_0^{r_0}\frac{\delta(r)}{\sqrt{r}\Big[J'_{n+\frac{1}{2}}(\mu_k^{(n+\frac{1}{2})})\Big]}J_{n+\frac{1}{2}}\Big(\mu_k^{(n+\frac{1}{2})}\frac{r}{r_0}\Big)\mathrm{d}r\Big)$$
$$\times\Big(\int_0^{2\pi}\int_0^{\pi}\sin\theta Y_n^{(m)}(\theta,\varphi)\mathrm{d}\theta\mathrm{d}\varphi\Big).$$

Remark 3.3.1. *(See, Appendix F)*

$$\int_0^{2\pi}\int_0^{\pi}\sin\theta Y_n^{(m)}(\theta,\varphi)\,d\theta\,d\varphi=\begin{cases}4\pi, & n=0\quad\wedge\quad m=0\\ 0, & n\neq 0\quad\vee\quad m\neq 0.\end{cases}$$

If we substitute this relation to Fourier coefficient we have

$$\bar{g}_{k00}(t)=\frac{2\sqrt{2}\delta(t)}{r_0(\lambda+2\mu)}\int_0^{r_0}\frac{\delta(r)}{\sqrt{r}\Big[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\Big]}J_{\frac{1}{2}}\Big(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\Big)\mathrm{d}r,$$

when $n=0,\ m=0$. Bessel functions has the following equality (Arfken, 1968)

$$J_\alpha(\mathbf{x})=\frac{2}{\sqrt{\pi}(\alpha-\frac{1}{2})!}\Big(\frac{x}{2}\Big)^\alpha\int_0^{\pi/2}\cos(x\sin\theta)\cos^{2\alpha}\theta\mathrm{d}\theta,$$

so for

$$\alpha=\frac{1}{2}\ \textit{and}\ x=\mu_k^{(\frac{1}{2})}(\frac{r}{r_0})$$

we have

$$J_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})}(\frac{r}{r_0})) = \frac{2}{\sqrt{\pi}}\Big(\frac{r\mu_k^{(\frac{1}{2})}}{2r_0}\Big)^{\frac{1}{2}}\int_0^{\pi/2}\cos(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\sin\theta)\cos\theta \mathrm{d}\theta,$$

and using the following properties of $\delta-$function (3.2.1) and (3.2.2) we obtain

$$\bar{g}_{k00}(t) = \frac{2\sqrt{2}\delta(t)}{r_0(\lambda+2\mu)}\int_0^{r_0}\frac{\delta(r)}{\sqrt{r}\Big[J_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\Big]}\Big[\frac{2}{\sqrt{\pi}(\frac{1}{2}-\frac{1}{2})!}\Big(\frac{\mu_k^{(\frac{1}{2})}r}{2r_0}\Big)^{1/2}$$

$$\times\int_0^{\pi/2}\cos(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\sin\theta)\cos\theta \mathrm{d}\theta\Big]\mathrm{d}r.$$

Using Remark (3.2.1) we obtain

$$\bar{g}_{k00}(t) = \frac{4\delta(t)\sqrt{\mu_k^{(\frac{1}{2})}}}{(\lambda+2\mu)r_0\sqrt{r_0}\sqrt{\pi}J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})}.$$

Since we obtain a non-zero solution for g_{knm} when $n=0$ and $m=0$ then we will find T_{knm} and V_{knm} for $n=0$ and $m=0$

$$T_{knm}(t) = \frac{a_p}{\sqrt{\lambda_{kn}}}\int_0^t \frac{4\delta(\tau)\sqrt{\mu_k^{(\frac{1}{2})}}}{r_0\sqrt{r_0}(\lambda+2\mu)\sqrt{\pi}J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})}\sin[a_p\sqrt{\lambda_{kn}}(t-\tau)]\mathrm{d}\tau,$$

$$T_{k00}(t) = \frac{a_p}{\sqrt{\lambda_{k0}}}\frac{4\sqrt{\mu_k^{(\frac{1}{2})}}}{r_0\sqrt{r_0}(\lambda+2\mu)\sqrt{\pi}J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})}\sin[a_p\sqrt{\lambda_{k0}}t],$$

where $\sqrt{\lambda_{k0}} = \dfrac{\mu_k^{(\frac{1}{2})}}{r_0}$ and

$$V_{k00}(r,\theta,\varphi) = \frac{\sqrt{2}}{r_0\sqrt{r}\Big[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\Big]}\Big[\frac{2}{\sqrt{\pi}(\frac{1}{2}-\frac{1}{2})!}\Big(\frac{\mu_k^{(\frac{1}{2})}r}{2r_0}\Big)^{1/2}$$

$$\times\int_0^{\pi/2}\cos(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\sin\theta)\cos\theta \mathrm{d}\theta\Big]$$

$$= \frac{2\sqrt{\mu_k^{(\frac{1}{2})}}}{r_0\sqrt{r_0}\sqrt{\pi}\Big[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\Big]}\int_0^{\pi/2}\cos(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\sin\theta)\cos\theta \mathrm{d}\theta,$$

since (See, Appendix F)

$$Y_0^{(0)}(\theta,\varphi) = 1,$$

and using Remark (3.2.1)

$$V_{k00}(r,\theta,\varphi) = \frac{2\sqrt{\mu_k^{(\frac{1}{2})}}}{r_0\sqrt{r_0}\sqrt{\pi}\left[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\right]}\frac{r_0}{r\mu_k^{(\frac{1}{2})}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\right),$$

$$V_{k00}(r,\theta,\varphi) = \frac{2}{r\sqrt{\mu_k^{(\frac{1}{2})}}\sqrt{r_0\pi}\left[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\right]}\sin\left(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\right).$$

A solution of the initial boundary value problem for the scalar wave equation is in the form

$$\bar{\phi}(r,\theta,\varphi,t) = \sum_{k=1}^{\infty} T_{k00}(t)V_{k00}(r,\theta,\varphi),$$

where

$$T_{k00}(t) = \frac{a_p}{\sqrt{\lambda_{k0}}}\frac{4\sqrt{\mu_k^{(\frac{1}{2})}}}{r_0\sqrt{r_0}(\lambda+2\mu)\sqrt{\pi}J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})}\sin[a_p\sqrt{\lambda_{k0}}t],$$

$\lambda = \lambda_{k0} = \left(\frac{\mu_k^{(\frac{1}{2})}}{r_0}\right)^2$, $k = 1, 2, ...$ are eigenvalues

$\mu_k^{(\frac{1}{2})}, k = 1, 2, ...$ are roots of the Bessel equation $J_{\frac{1}{2}}(\mu) = 0$ and the eigenfunctions of the Laplace operator in the sphere is

$$V_{k00}(r,\theta,\varphi) = \frac{2}{r\sqrt{\mu_k^{(\frac{1}{2})}}\sqrt{r_0\pi}\left[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\right]}\sin\left(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\right).$$

Hence

$$\bar{\phi}(r,\theta,\varphi,t) = \sum_{k=1}^{\infty}\frac{8a_p\sin[a_p\sqrt{\lambda_{k0}}t]\sin\left(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\right)}{\sqrt{\lambda_{k0}}rr_0^2(\lambda+2\mu)\pi\left[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\right]^2},$$

where $a_p^2 = \dfrac{\lambda+2\mu}{\rho}$.

Using the coordinate transforms

$$\begin{aligned} r &= \sqrt{x_1^2+x_2^2+x_3^2}, \\ \theta &= \arctan(\sqrt{x_1^2+x_2^2}/x_3), \\ \varphi &= \arctan(x_2/x_1), \end{aligned}$$

we obtain

$$\phi(x_1, x_2, x_3, t) = \sum_{k=1}^{\infty} \frac{8a_p \sin[a_p\sqrt{\lambda_{k0}}t]}{\sqrt{\lambda_{k0}} r_0^2(\lambda + 2\mu)\pi \Big[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\Big]^2}$$

$$\times \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \sin\Big(\mu_k^{(\frac{1}{2})} \frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{r_0}\Big),$$

is solution of the initial boundary value problem for the scalar wave equation (3.3.5)-(3.3.7). Before we found that the solution of the initial boundary value problem for the vector wave equation is zero. So we will find the solution of the Cauchy problem for the Lame system in the form

$$U(x_1, x_2, x_3, t) = \nabla_x \phi + curl_x A.$$

Now we will find

$$\nabla_x \phi = \Big(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3}\Big).$$

$$\frac{\partial \phi}{\partial x_i} = \sum_{k=1}^{\infty} \frac{8a_p \sin[a_p\sqrt{\lambda_{k0}}t]}{\sqrt{\lambda_{k0}} r_0^2(\lambda + 2\mu)\pi \Big[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\Big]^2} \frac{\partial}{\partial x_i}\Big[\frac{\sin\Big(\mu_k^{(\frac{1}{2})} \frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\Big],$$

$$\frac{\partial \phi}{\partial x_i} = \sum_{k=1}^{\infty} \frac{8a_p \sin[a_p\sqrt{\lambda_{k0}}t]}{\sqrt{\lambda_{k0}} r_0^2(\lambda + 2\mu)\pi \Big[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\Big]^2}$$

$$\times \Big[\frac{\mu_k^{(\frac{1}{2})} x_i}{r_0(x_1^2 + x_2^2 + x_3^2)} \cos\Big(\mu_k^{(\frac{1}{2})} \frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{r_0}\Big)$$

$$- \frac{x_i}{\sqrt{(x_1^2 + x_2^2 + x_3^2)^3}} \sin\Big(\mu_k^{(\frac{1}{2})} \frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{r_0}\Big)\Big].$$

If we denote

$$\nabla_x \phi = \Big((\nabla\phi)_{x_1}, (\nabla\phi)_{x_2}, (\nabla\phi)_{x_3}\Big)$$

then

$$(\nabla\phi)_{x_1} = \sum_{k=1}^{\infty} \frac{a_p}{\sqrt{\lambda_{k0}}} \frac{8\sin(a_p\sqrt{\lambda_{k0}}t)x_1}{(\lambda+2\mu)\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big) - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big].$$

$$(\nabla\phi)_{x_2} = \sum_{k=1}^{\infty} \frac{a_p}{\sqrt{\lambda_{k0}}} \frac{8\sin(a_p\sqrt{\lambda_{k0}}t)x_2}{(\lambda+2\mu)\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big) - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big].$$

$$(\nabla\phi)_{x_3} = \sum_{k=1}^{\infty} \frac{a_p}{\sqrt{\lambda_{k0}}} \frac{8\sin(a_p\sqrt{\lambda_{k0}}t)x_3}{(\lambda+2\mu)\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big) - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big].$$

Since

$$U(x_1,x_2,x_3,t) = \nabla_x\phi + curl_x A,$$

and $curl_x\mathbf{A} = 0$ then

$$U(x_1,x_2,x_3,t) = \Big(U_1(x_1,x_2,x_3,t), U_2(x_1,x_2,x_3,t), U_3(x_1,x_2,x_3,t)\Big).$$

$$U_1(\mathbf{x},t) = \sum_{k=1}^{\infty} \frac{a_p}{\sqrt{\lambda_{k0}}} \frac{8\sin(a_p\sqrt{\lambda_{k0}}t)x_1}{(\lambda+2\mu)\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big) - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big]. \tag{3.3.11}$$

$$U_2(\mathbf{x},t)=\sum_{k=1}^{\infty}\frac{a_p}{\sqrt{\lambda_{k0}}}\frac{8\sin(a_p\sqrt{\lambda_{k0}}t)x_2}{(\lambda+2\mu)\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\Big]. \quad (3.3.12)$$

$$U_3(\mathbf{x},t)=\sum_{k=1}^{\infty}\frac{a_p}{\sqrt{\lambda_{k0}}}\frac{8\sin(a_p\sqrt{\lambda_{k0}}t)x_3}{(\lambda+2\mu)\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right)\Big]. \quad (3.3.13)$$

where $a_p^2=\dfrac{\lambda+2\mu}{\rho}$;
$\lambda_{k0}=\left(\mu_k^{(\frac{1}{2})}\dfrac{r}{r_0}\right)^2$ are eigenvalues and
$\mu_k^{(\frac{1}{2})}$; $k=1,2,...$ are roots of the Bessel equation $J_{\frac{1}{2}}(\mu)=0$.

Solution of the problem defined in (3.3.1)-(3.3.4) is given by the formulas (3.3.11)-(3.3.13)

3.3.1.1 Mathematica Commands for Finding the Components of the Solution

In this subsection we use the formulas that we obtained for the components of $\mathbf{U}(\mathbf{x},t)$ and `Mathematica` for the simulation of the wave propagation in spherical domain.
`Mathematica` commands for finding $\mathbf{U}_1(\mathbf{x},t)$, $\mathbf{U}_2(\mathbf{x},t)$, $\mathbf{U}_3(\mathbf{x},t)$ are listed below.

```
<< NumericalMath‘BesselZeros‘
<<Graphics‘Graphics3D‘
<<Graphics‘ParametricPlot3D‘
INPUT: Nt,rho,lambda,mu,radius;
```

```
ap = Sqrt[(lambda + 2*mu)/rho];
T = radius/ap;
muk = BesselJZeros[1/2, Nt];
l = (muk/radius)^2;
f = D[BesselJ[1/2, x], x] /. x -> muk;
g =f*f;
h = 1/((lambda + 2*mu)*Pi*muk*radius*g);
p =8*ap*Sin[ap*Sqrt[l]*t];
z = muk/radius;
k = z*Cos[z*r] -Sin[z*r]*(1/r);
RQTfi1 =Sum[p[[i]]*k[[i]]*h[[i]]
                   *(Sin[theta]*Cos[varphi]/r), {i, 1, Nt}];
RQTfi2 =Sum[p[[i]]*k[[i]]*h[[i]]
                   *(Sin[theta]*Sin[varphi]/r), {i, 1, Nt}];
RQTfi3 =Sum[p[[i]]*k[[i]]*h[[i]]*(Cos[theta]/r), {i, 1, Nt}];
RQTgradU1 = ReplaceAll[RQTfi1, varphi -> Pi/4];
RQTgradU2 = ReplaceAll[RQTfi2, varphi -> Pi/4];
RQTgradU3 = ReplaceAll[RQTfi3, varphi -> Pi/4];
```

OUTPUT: U_1, U_2, U_3.

In this subsection IBVP for the Lame system with Dirichlet boundary condition and with the external force of the form $\mathbf{f}(\mathbf{x}, t) = \nabla\delta(x)\delta(t)$ is reduced to IBVP for the scalar wave equation and IBVP for the vector wave equation with Dirichlet boundary condition in spherical coordinates. These problems are solved and simulated in spherical domain. We use vector analysis technique, generalized functions theory and Fourier series expansion method to obtain the formula for the solution of the considered IBVP problem. Using this formula the simulation of elastic waves are done. As a result solution of the problem defined in (3.3.1)-(3.3.4) is (3.3.11)-(3.3.13). For simulation first 25 terms are added. So (3.3.11)-(3.3.13) becomes

$$U_1(\mathbf{x}, t) = \sum_{k=1}^{25} \frac{a_p}{\sqrt{\lambda_{k0}}} \frac{8\sin(a_p\sqrt{\lambda_{k0}}t)x_1}{(\lambda+2\mu)\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big) - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big]. \tag{3.3.14}$$

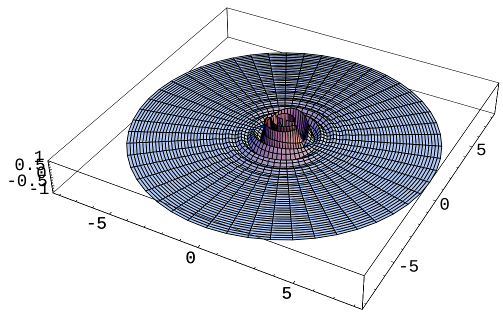
$$U_2(\mathbf{x},t) = \sum_{k=1}^{25} \frac{a_p}{\sqrt{\lambda_{k0}}} \frac{8\sin(a_p\sqrt{\lambda_{k0}}t)x_2}{(\lambda+2\mu)\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big) - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big]. \tag{3.3.15}$$

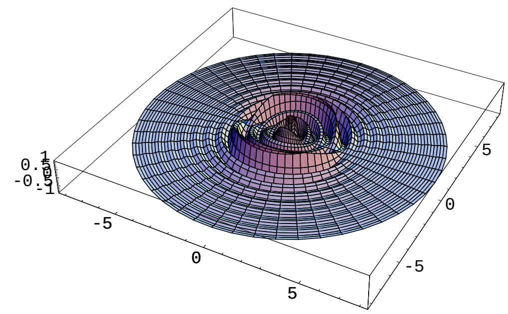
$$U_3(\mathbf{x},t) = \sum_{k=1}^{25} \frac{a_p}{\sqrt{\lambda_{k0}}} \frac{8\sin(a_p\sqrt{\lambda_{k0}}t)x_3}{(\lambda+2\mu)\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big) - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big]. \tag{3.3.16}$$

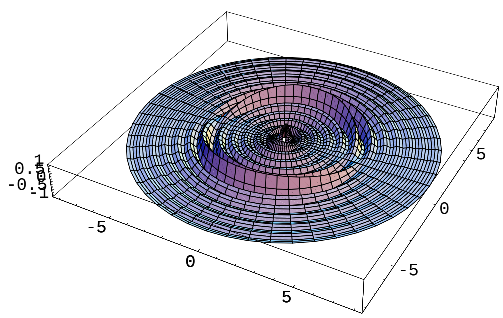
For simulation of these components, the following values are used $\lambda = 4$; $\rho = 2,203$; $\mu = 3,12$; $r_0 = 8$. Here ρ is the density of elastic medium, λ, μ are Lame parameters(physical constants), $\rho > 0$, $\mu > 0, \lambda + 2\mu > 0$ are positive constants, r_0 is the radius of sphere and $a_p = \sqrt{\dfrac{\lambda+2\mu}{\rho}}$, $a_s = \sqrt{\dfrac{\mu}{\rho}}$, where a_p, a_s are longitudinal and transversal speeds of elastic waves.



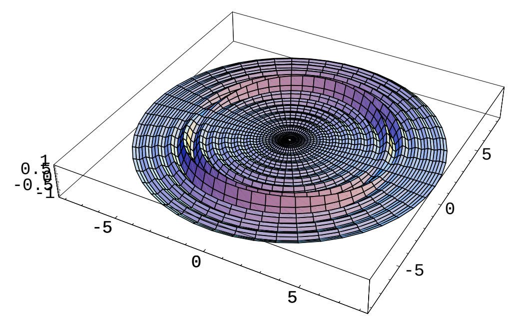
(a) t=0,3



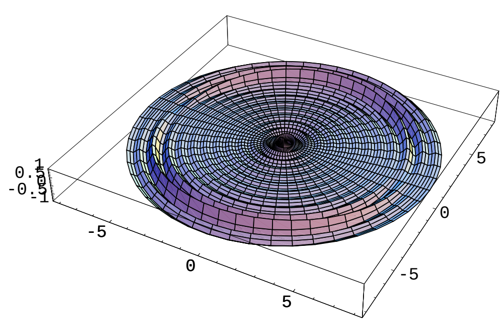
(b) t=1,2



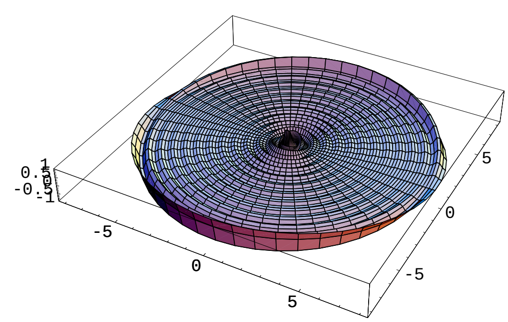
(c) t=1,8



(d) t=2,4

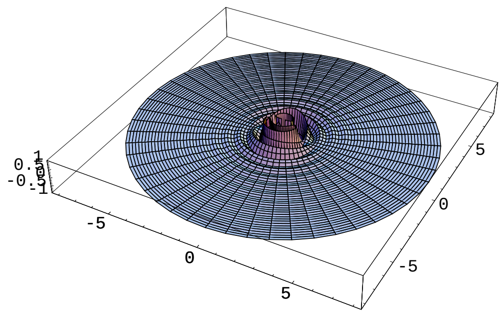


(e) t=3

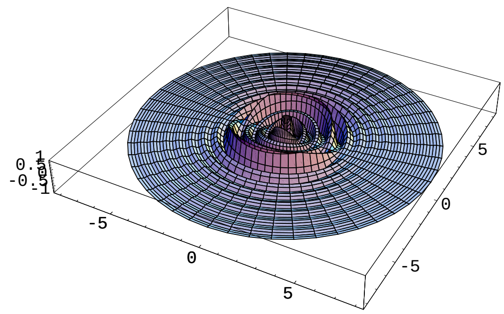


(f) t=3,6

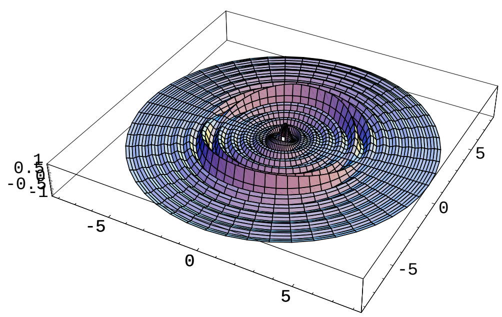
Figure 3.12: The first component of the displacement vector $\mathbf{U}_1(\mathbf{x},\mathrm{t})$



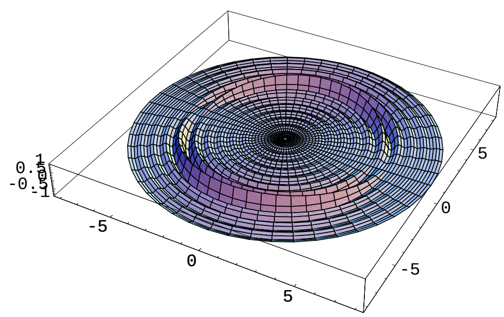
(a) t=0.3



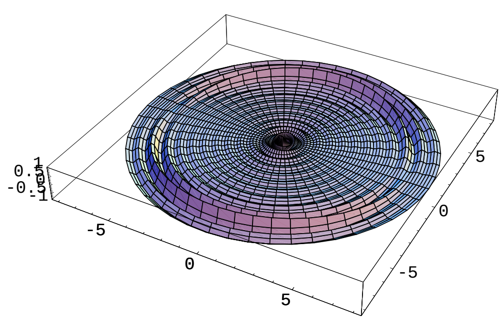
(b) t=1,2



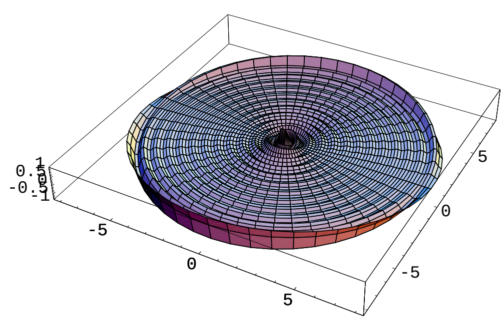
(c) t=1,8



(d) t=2,4

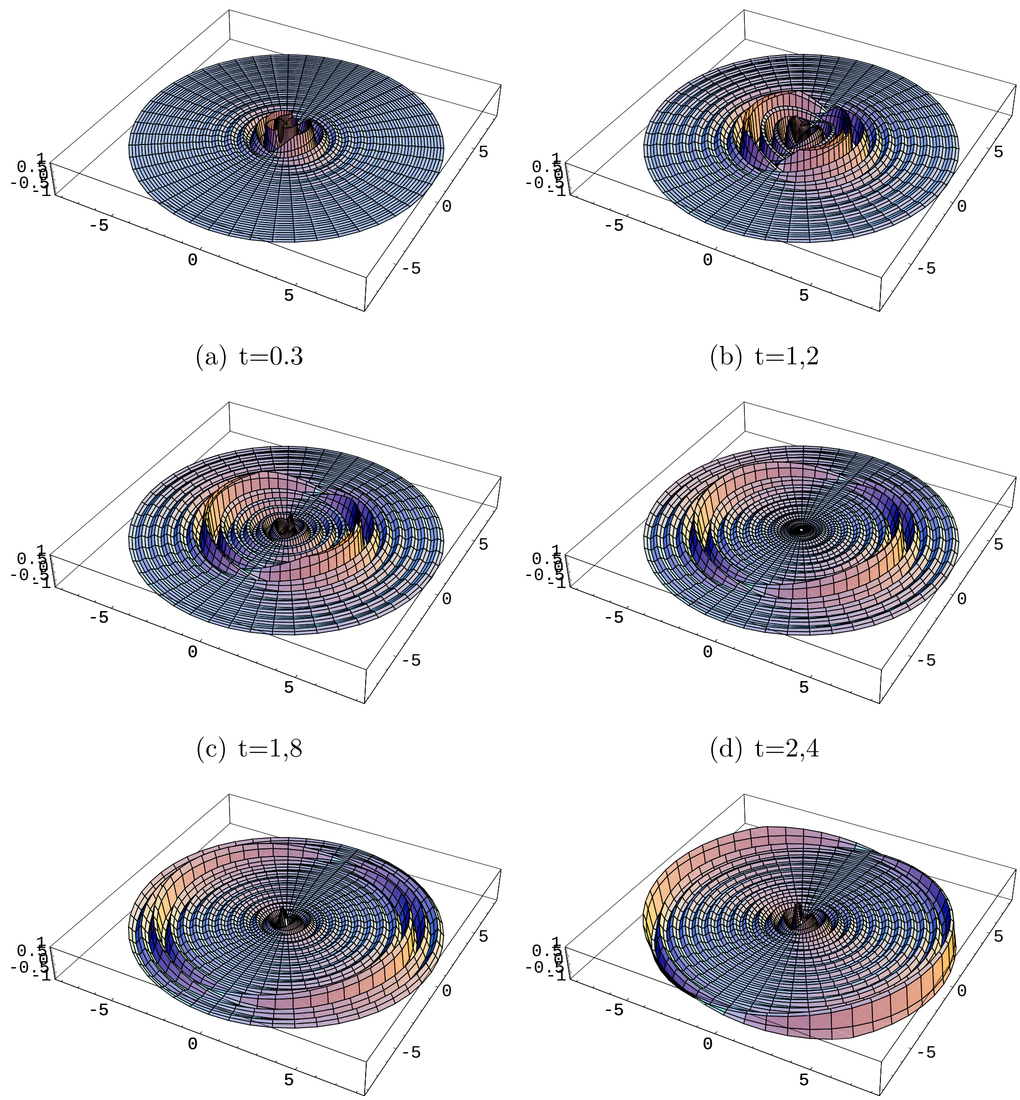


(e) t=3



(f) t=3,6

Figure 3.13: The second component of the displacement vector $\mathbf{U}_2(\mathbf{x},\mathrm{t})$



(a) t=0.3 (b) t=1,2

(c) t=1,8 (d) t=2,4

(e) t=3 (f) t=3,6

Figure 3.14: The third component of the displacement vector $\mathbf{U}_3(\mathbf{x},\mathrm{t})$

In Figure (3.12)-Figure (3.14), components of vector function $\mathbf{U}(\mathbf{x},t)$ that are $\mathbf{U}_1(\mathbf{x},t)$, $\mathbf{U}_2(\mathbf{x},t)$, $\mathbf{U}_3(\mathbf{x},t)$ are simulated. The horizontal axes are r, θ and the vertical axis corresponds to the component of $\mathbf{U}(\mathbf{x},t)$. The variable φ taken as $\varphi = \pi/4$. Since the IBVP for vector wave equation has a zero solution for external force $\mathbf{f}(\mathbf{x},t) = \nabla\delta(x)\delta(t)$, there exists only one wave that occurs from the solution of the IBVP for scalar wave equation.

3.3.2 The Source of the Rotation Center $f(\mathbf{x},t) = curl_x[\mathbf{e}\delta(\mathbf{x})\delta(t)]$

Now let us consider the following Cauchy problem for the Lame system:

$$\rho\frac{\partial^2\mathbf{U}}{\partial t^2} = (\lambda+2\mu)\nabla_x div_x\mathbf{U} - \mu curl_x curl_x\mathbf{U} + \mathbf{f}(\mathbf{x},t), \tag{3.3.17}$$

subject to the initial conditions

$$\mathbf{U}(\mathbf{x},t)|_{t=0} = 0, \tag{3.3.18}$$

$$\frac{\partial\mathbf{U}}{\partial t}(\mathbf{x},t)|_{t=0} = 0, \tag{3.3.19}$$

and the boundary condition

$$\mathbf{U}(\mathbf{x},t)|_{|\mathbf{x}|=r_0} = 0, \tag{3.3.20}$$

where $D = \{x \in \mathbf{R}^3 : |\mathbf{x}| < r_0\}$ and let vector f is given in the form

$$\mathbf{f}(\mathbf{x},t) = curl_x[\mathbf{e}\delta(\mathbf{x})\delta(t)],$$

where δ is the Dirac delta function and $\mathbf{e}$ is any vector function. Our goal is to find the function $\mathbf{U}(\mathbf{x},t)$ satisfies (3.3.17)-(3.3.20).

If we assume $\mathbf{f}$ in the form

$$\mathbf{f} = \nabla_x g + curl_x\mathbf{F},$$

using potential approach we obtain

$$g = 0,$$

and

$$\mathbf{F} = \mathbf{e}\delta(\mathbf{x})\delta(t).$$

Hence we obtain two wave equation from the given Lame system. The initial boundary value problem for $\phi(\mathbf{x}, t)$ is

$$\frac{1}{a_p^2}\frac{\partial^2 \phi}{\partial t^2} = \triangle_x \phi,$$

$$\phi(\mathbf{x}, t)|_{t=0} = 0, \quad \frac{\partial \phi}{\partial t}(\mathbf{x}, t)|_{t=0} = 0,$$

$$\phi(\mathbf{x}, t)|_{|\mathbf{x}|=r_0} = 0,$$

where $a_p^2 = \dfrac{\lambda + 2\mu}{\rho}$.

And initial boundary value problem for $\mathbf{A}(\mathbf{x}, t)$ is

$$\frac{1}{a_s^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \triangle_x \mathbf{A} + \frac{1}{\mu}\mathbf{F},$$

$$\mathbf{A}(\mathbf{x}, t)|_{t=0} = 0, \quad \frac{\partial \mathbf{A}}{\partial t}(\mathbf{x}, t)|_{t=0} = 0,$$

$$\mathbf{A}(\mathbf{x}, t)|_{|\mathbf{x}|=r_0} = 0,$$

where

$$a_s^2 = \frac{\mu}{\rho} \qquad \textit{and} \qquad \mathbf{F} = \mathbf{e}\delta(\mathbf{x})\delta(t).$$

It can be easily found that the solution of the initial boundary value problem for the scalar wave equation is zero because $\phi(\mathbf{x}, t) = 0$ satisfies the problem and by uniqueness theorem this is the unique solution. So $\nabla_x \phi = 0$.

Now we are going to solve the initial boundary value problem for the vector wave equation in the sphere so let us firstly write the equation in spherical coordinates. Substituting

$$x_1 = r\cos\varphi\sin\theta,\ x_2 = r\sin\varphi\sin\theta,\ x_3 = r\cos\theta,$$

into $\mathbf{A}(x_1,x_2,x_3,t)$ we have

$$\mathbf{A}(r\cos\varphi\sin\theta,\ r\sin\varphi\sin\theta,\ r\cos\theta)=\bar{\mathbf{A}}(r,\theta,\varphi,t)$$

and using the properties of the Dirac delta function that are (Barton, 1989)

$$\delta(\mathbf{x})=\frac{(-1)^{n-1}2}{w_n(n-1)!}\delta^{(n-1)}(r),$$

where w_n is the area of unit sphere and

$$r^2\delta''(r)=(-1)^2 2!\delta(r)=2\delta(r),$$

since for the sphere $(n=3)$ area of the unit sphere in $\mathbf{R}^3$ is $w_3=4\pi$ then

$$\delta(\mathbf{x})=\frac{(-1)^{3-1}}{w_3(3-1)!}2\delta^{(3-1)}(r)=\frac{1}{4\pi}\delta''(r)=\frac{1}{2\pi r^2}\delta(r),$$

the initial boundary value problem for the vector wave equation becomes

$$\frac{1}{a_s^2}\frac{\partial^2\bar{\mathbf{A}}}{\partial t^2}=\triangle_{r,\theta,\varphi}\bar{\mathbf{A}}+\bar{\mathbf{F}},$$

$$\bar{\mathbf{A}}(r,\theta,\varphi,t)|_{t=0}=0,\quad \frac{\partial\bar{\mathbf{A}}}{\partial t}(r,\theta,\varphi,t)|_{t=0}=0,$$

$$|\bar{\mathbf{A}}(r,\theta,\varphi,t)|_{\theta=0}<\infty,\quad \bar{\mathbf{A}}(r,\theta,\varphi+\pi,t)=\bar{\mathbf{A}}(r,\theta,\varphi,t),$$

$$|\bar{\mathbf{A}}(r,\theta,\varphi,t)|_{r=0}<\infty,\quad \bar{\mathbf{A}}(r,\theta,\varphi,t)|_{r=r_0}=0,$$

where $a_s^2=\dfrac{\mu}{\rho}$ and

$$\bar{\mathbf{F}}=\frac{1}{\mu}\mathbf{e}\frac{1}{2\pi r^2}\delta(r)\delta(t).$$

Now we will find the Fourier coefficient

$$\bar{\mathbf{F}}_{knm}(t)=\int_0^{r_0}\int_0^{2\pi}\int_0^{\pi}\frac{\mathbf{e}\delta(r)\delta(t)}{2\pi\mu r^2}r^2\sin\theta\Big[\frac{\sqrt{2}Y_n^{(m)}(\theta,\varphi)}{r_0\sqrt{r}\Big[J'_{n+\frac{1}{2}}(\mu_k^{(n+\frac{1}{2})})\Big]}$$

$$\times J_{n+\frac{1}{2}}\Big(\mu_k^{(n+\frac{1}{2})}\frac{r}{r_0}\Big)\Big]\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r,$$

$$\bar{\mathbf{F}}_{knm}(t) = \frac{\mathbf{e}\delta(t)\sqrt{2}}{2\mu\pi r_0}\int_0^{r_0}\int_0^{2\pi}\int_0^{\pi}\frac{\delta(r)\sin\theta}{\sqrt{r}}\mathrm{Y}_n^{(m)}(\theta,\varphi)\frac{J_{n+\frac{1}{2}}\left(\mu_k^{(n+\frac{1}{2})}\frac{r}{r_0}\right)}{J'_{n+\frac{1}{2}}(\mu_k^{(n+\frac{1}{2})})}\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r,$$

$$\bar{\mathbf{F}}_{knm}(t) = \frac{\mathbf{e}\delta(t)\sqrt{2}}{2\mu\pi r_0}\int_0^{r_0}\frac{\delta(r)}{\sqrt{r}}\frac{J_{n+\frac{1}{2}}\left(\mu_k^{(n+\frac{1}{2})}\frac{r}{r_0}\right)}{\left[J'_{n+\frac{1}{2}}(\mu_k^{(n+\frac{1}{2})})\right]}\mathrm{d}r$$

$$\times\Big(\int_0^{2\pi}\int_0^{\pi}\sin\theta Y_n^{(m)}(\theta,\varphi)\mathrm{d}\theta\mathrm{d}\varphi\Big).$$

Using Remark (3.3.1) we have

$$\bar{\mathbf{F}}_{k00}(t) = \frac{2\sqrt{2}\mathbf{e}\delta(t)}{r_0\mu}\frac{1}{\left[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\right]}\int_0^{r_0}\frac{\delta(r)}{\sqrt{r}}J_{\frac{1}{2}}\Big(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\Big)\mathrm{d}r,$$

and using the equation (3.2.10) for

$$\alpha = \frac{1}{2} \;\; and \;\; x = \mu_k^{(\frac{1}{2})}(\frac{r}{r_0}),$$

and using the following properties of δ-function in property (3.2.1) and (3.2.2) we obtain

$$\bar{\mathbf{F}}_{k00}(t) = \frac{2\sqrt{2}\mathbf{e}\delta(t)}{r_0\mu}\frac{1}{\left[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\right]}\int_0^{r_0}\frac{\delta(r)}{\sqrt{r}}\Big[\frac{2}{\sqrt{\pi}(\frac{1}{2}-\frac{1}{2})!}\Big(\frac{\mu_k^{(\frac{1}{2})}r}{2r_0}\Big)^{1/2}$$

$$\times\int_0^{\pi/2}\cos(\mu_k^{(\frac{1}{2})}\sin\theta\frac{r}{r_0})\cos\theta\mathrm{d}\theta\Big]\mathrm{d}r$$

$$= \frac{4\mathbf{e}\delta(t)\sqrt{\mu_k^{(\frac{1}{2})}}}{r_0\sqrt{r_0}\mu\sqrt{\pi}J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})}.$$

Since we obtain a non-zero solution for F_{knm} when $n=0$ and $m=0$ then we will find T_{knm} and V_{knm} for $n=0$ and $m=0$

$$T_{knm}(t) = \frac{a_s}{\sqrt{\lambda_{kn}}}\int_0^t\frac{4\mathbf{e}\delta(\tau)\sqrt{\mu_k^{(\frac{1}{2})}}}{r_0\sqrt{r_0}\mu\sqrt{\pi}J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})}\sin[a_s\sqrt{\lambda_{kn}}(t-\tau)]\mathrm{d}\tau,$$

$$T_{k00}(t) = \frac{a_s}{\sqrt{\lambda_{k0}}} \int_0^t \frac{4\mathbf{e}\delta(\tau)\sqrt{\mu_k^{(\frac{1}{2})}}}{r_0\sqrt{r_0}\mu\sqrt{\pi}J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})} \sin[a_s\sqrt{\lambda_{k0}}(t-\tau)]\mathrm{d}\tau$$

$$= \frac{a_s}{\sqrt{\lambda_{k0}}} \frac{4\mathbf{e}\sqrt{\mu_k^{(\frac{1}{2})}}}{r_0\sqrt{r_0}\mu\sqrt{\pi}J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})} \sin[a_s\sqrt{\lambda_{k0}}t],$$

and since $Y_0^{(0)}(\theta,\varphi) = 1$ then

$$V_{k00}(r,\theta,\varphi) = \frac{\sqrt{2}}{r_0\sqrt{r}\Big[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\Big]} \Big[\frac{2}{\sqrt{\pi}(\frac{1}{2}-\frac{1}{2})!}\Big(\frac{\mu_k^{(\frac{1}{2})}r}{2r_0}\Big)^{1/2}$$

$$\times \int_0^{\pi/2} \cos(\mu_k^{(\frac{1}{2})}\sin\theta\frac{r}{r_0})\cos\theta\mathrm{d}\theta\Big]$$

$$= \frac{2\sqrt{\mu_k^{(\frac{1}{2})}}}{r_0\sqrt{r_0}\sqrt{\pi}\Big[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\Big]} \int_0^{\pi/2} \cos(\mu_k^{(\frac{1}{2})}\sin\theta\frac{r}{r_0})\cos\theta\mathrm{d}\theta$$

Now using Remark (3.2.1) we obtain

$$V_{k00}(r,\theta,\varphi) = \frac{2\sqrt{\mu_k^{(\frac{1}{2})}}}{r_0\sqrt{r_0}\sqrt{\pi}\Big[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\Big]} \frac{r_0}{r\mu_k^{(\frac{1}{2})}} \sin\Big(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\Big),$$

$$V_{k00}(r,\theta,\varphi) = \frac{2}{r\sqrt{r_0}\sqrt{\pi}\sqrt{\mu_k^{(\frac{1}{2})}}} \frac{\sin\Big(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\Big)}{\Big[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\Big]}.$$

A solution of the initial boundary value problem for the vector wave equation is in the form

$$\bar{\mathbf{A}}(r,\theta,\varphi,t) = \sum_{k=1}^{\infty} T_{k00}(t)V_{k00}(r,\theta,\varphi),$$

where

$$T_{k00}(t) = \frac{a_s}{\sqrt{\lambda_{k0}}} \frac{4\mathbf{e}\sqrt{\mu_k^{(\frac{1}{2})}}}{r_0\sqrt{r_0}\mu\sqrt{\pi}J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})} \sin[a_s\sqrt{\lambda_{k0}}t],$$

$\lambda = \lambda_{k0} = \left(\dfrac{\mu_k^{(\frac{1}{2})}}{r_0}\right)^2$, $k = 1, 2, ...$ are eigenvalues,

$\mu_k^{(\frac{1}{2})}, k = 1, 2, ...$ are roots of the Bessel equation $J_{\frac{1}{2}}(\mu) = 0$ and the eigenfunctions of the Laplace operator in the sphere is

$$V_{k00}(r,\theta,\varphi)=\frac{2}{r\sqrt{r_0\pi}\left[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\right]}\frac{1}{\mu_k^{(\frac{1}{2})}}\frac{1}{\sqrt{\mu_k^{(\frac{1}{2})}}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\right).$$

Hence

$$\bar{\mathbf{A}}(r,\theta,\varphi,t)=\sum_{k=1}^{\infty}\frac{a_s}{\sqrt{\lambda_{k0}}}\frac{8\mathbf{e}}{r_0^2 r\mu\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2}\sin[a_s\sqrt{\lambda_{k0}}t]\sin\left(\mu_k^{(\frac{1}{2})}\frac{r}{r_0}\right).$$

Using the coordinate transforms

$$\begin{aligned} r &= \sqrt{x_1^2+x_2^2+x_3^2},\\ \theta &= \arctan(\sqrt{x_1^2+x_2^2}/x_3),\\ \varphi &= \arctan(x_2/x_1), \end{aligned}$$

we obtain

$$\mathbf{A}(x_1,x_2,x_3,t)=\sum_{k=1}^{\infty}\frac{a_s}{\sqrt{\lambda_{k0}}}\frac{8\mathbf{e}}{r_0^2(\sqrt{x_1^2+x_2^2+x_3^2})\mu\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2}$$

$$\times\sin[a_s\sqrt{\lambda_{k0}}t]\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right),$$

is solution of the initial boundary value problem for the vector wave equation. Before we found that the solution of the initial boundary value problem for the scalar wave equation is zero. So we will find the solution of the Cauchy problem for the Lame system in the form

$$U(x_1,x_2,x_3,t)=\nabla_x\phi+curl_x\mathbf{A}.$$

Now we will find

$$curl_x\mathbf{A}=\Big((curl_x\mathbf{A})_1,(curl_x\mathbf{A})_2,(curl_x\mathbf{A})_3\Big)$$

where

$$(curl_x\mathbf{A})_1=\frac{\partial A_3}{\partial x_2}-\frac{\partial A_2}{\partial x_3},$$

$$(curl_x\mathbf{A})_1 = \sum_{k=1}^{\infty} \frac{a_s}{\sqrt{\lambda_{k0}}} \frac{8\sin(a_s\sqrt{\lambda_{k0}}t)\{e_3x_2 - e_2x_3\}}{\mu\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big) - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big].$$

$$(curl_x\mathbf{A})_2 = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1},$$

$$(curl_x\mathbf{A})_2 = \sum_{k=1}^{\infty} \frac{a_s}{\sqrt{\lambda_{k0}}} \frac{8\sin(a_s\sqrt{\lambda_{k0}}t)\{e_1x_3 - e_3x_1\}}{\mu\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big) - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big].$$

$$(curl_x\mathbf{A})_3 = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2},$$

$$(curl_x\mathbf{A})_3 = \sum_{k=1}^{\infty} \frac{a_s}{\sqrt{\lambda_{k0}}} \frac{8\sin(a_s\sqrt{\lambda_{k0}}t)\{e_2x_1 - e_1x_2\}}{\mu\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big) - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big].$$

Since

$$U(x_1,x_2,x_3,t) = \nabla_x\phi + curl_x\mathbf{A},$$

and

$$\nabla\phi = 0,$$

then

$$U(x_1,x_2,x_3,t) = \Big(U_1(x_1,x_2,x_3,t), U_2(x_1,x_2,x_3,t), U_3(x_1,x_2,x_3,t)\Big).$$

$$U_1(x_1,x_2,x_3,t)=\sum_{k=1}^{\infty}\frac{a_s}{\sqrt{\lambda_{k0}}}\frac{8\sin(a_s\sqrt{\lambda_{k0}}t)\{e_3x_2-e_2x_3\}}{\mu\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big],\tag{3.3.21}$$

$$U_2(x_1,x_2,x_3,t)=\sum_{k=1}^{\infty}\frac{a_s}{\sqrt{\lambda_{k0}}}\frac{8\sin(a_s\sqrt{\lambda_{k0}}t)\{e_1x_3-e_3x_1\}}{\mu\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big],\tag{3.3.22}$$

$$U_3(x_1,x_2,x_3,t)=\sum_{k=1}^{\infty}\frac{a_s}{\sqrt{\lambda_{k0}}}\frac{8\sin(a_s\sqrt{\lambda_{k0}}t)\{e_2x_1-e_1x_2\}}{\mu\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big].\tag{3.3.23}$$

where $a_s^2=\dfrac{\mu}{\rho}$;

$\lambda_{k0}=\Big(\dfrac{\mu_k^{(\frac{1}{2})}}{r_0}\Big)^2$ are eigenvalues and

$\mu_k^{(\frac{1}{2})}$; $k=1,2,...$ are roots of the Bessel equation $J_{\frac{1}{2}}(\mu)=0$.

Solution of the problem defined in (3.3.17)-(3.3.20) is given by the formulas (3.3.21)-(3.3.23)

3.3.2.1 Mathematica Commands for Finding the Components of the Solution

In this subsection we use the formulas that we obtained for the components of $\mathbf{U}(\mathbf{x}, t)$ and `Mathematica` for the simulation of the wave propagation in spherical domain.
`Mathematica` commands for finding $\mathbf{U}_1(\mathbf{x}, t)$, $\mathbf{U}_2(\mathbf{x}, t)$, $\mathbf{U}_3(\mathbf{x}, t)$ are listed below.

```
<< NumericalMath`BesselZeros`
<<Graphics`Graphics3D`
<<Graphics`ParametricPlot3D`
INPUT: Nt,rho,lambda,mu,radius,e1, e2,e3;
as = Sqrt[mu/rho];
T = radius/as;
muk = BesselJZeros[1/2, Nt];
l = (muk/radius)^2;
f = D[BesselJ[1/2, x], x] /. x -> muk;
g = f*f;
h =1/(mu*Pi*muk*radius*g);
p = 8*as*Sin[as*Sqrt[l]*t];
z = muk/radius;
k = z*Cos[z*r] - Sin[z*r]*(1/r);
a = r*Cos[varphi]*Sin[theta];
b =r*Sin[varphi]*Sin[theta];
c = r*Cos[theta];
m1 = (e3*b - e2*c)/r^2;
m2 = (e1*c - e3*a)/r^2;
m3 = (e2*a - e1*b)/r^2;
RQTA1 = Sum[p[[i]]*k[[i]]*h[[i]]*m1, {i, 1, Nt}];
RQTA2 = Sum[p[[i]]*k[[i]]*h[[i]]*m2, {i, 1, Nt}];
RQTA3 = Sum[p[[i]]*k[[i]]*h[[i]]*m3, {i, 1, Nt}];
RQTU1 =ReplaceAll[RQTA1, varphi -> Pi/4];
RQTU2 =ReplaceAll[RQTA2, varphi -> Pi/4];
RQTU3 =ReplaceAll[RQTA3, varphi -> Pi/4];
```

`OUTPUT`: U_1, U_2, U_3.

In this subsection IBVP for the Lame system with Dirichlet boundary condition and with the external force of the form $\mathbf{f}(\mathbf{x},t) = curl\Big[\mathbf{e}\delta(x)\delta(t)\Big]$ is reduced to IBVP for the scalar wave equation and IBVP for the vector wave equation with Dirichlet boundary condition in spherical coordinates. These problems are solved and simulated in spherical domain. We use vector analysis technique, generalized functions theory and Fourier series expansion method to obtain the formula for the solution of the considered IBVP problem. Using this formula the simulation of elastic waves are done. As a result solution of the problem defined in (3.3.17)-(3.3.20) is (3.3.21)-(3.3.23). For simulation first 25 terms are added. So (3.3.21)-(3.3.23) becomes

$$U_1(x_1,x_2,x_3,t) = \sum_{k=1}^{\infty} \frac{a_s}{\sqrt{\lambda_{k0}}} \frac{8\sin(a_s\sqrt{\lambda_{k0}}t)\{e_3x_2 - e_2x_3\}}{\mu\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big) - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big], \tag{3.3.24}$$

$$U_2(x_1,x_2,x_3,t) = \sum_{k=1}^{\infty} \frac{a_s}{\sqrt{\lambda_{k0}}} \frac{8\sin(a_s\sqrt{\lambda_{k0}}t)\{e_1x_3 - e_3x_1\}}{\mu\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2(x_1^2+x_2^2+x_3^2)}$$

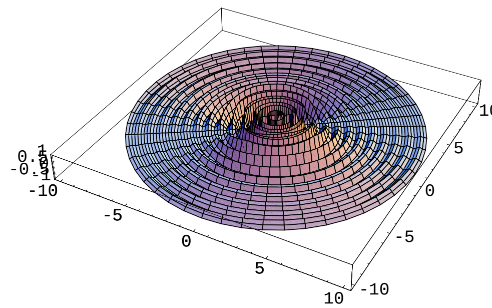
$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big) - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big], \tag{3.3.25}$$

$$U_3(x_1,x_2,x_3,t) = \sum_{k=1}^{\infty} \frac{a_s}{\sqrt{\lambda_{k0}}} \frac{8\sin(a_s\sqrt{\lambda_{k0}}t)\{e_2x_1 - e_1x_2\}}{\mu\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2(x_1^2+x_2^2+x_3^2)}$$

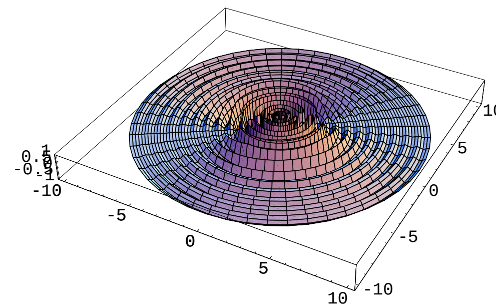
$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big) - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big]. \tag{3.3.26}$$

For simulation of these components, the following values are used $\lambda = 4$;

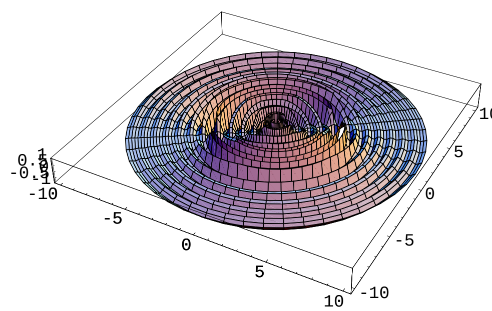
$\rho = 2,203;$ $\mu = 3,12;$ $r_0 = 10;$ $e_1 = 1;$ $e_2 = 2;$ $e_3 = 4.$ Here ρ is the density of elastic medium, λ, μ are Lame parameters(physical constants), $\rho > 0,$ $\lambda + 2\mu > 0,$ $\mu > 0$ are positive constants, r_0 is the radius of sphere and $a_p = \sqrt{\dfrac{\lambda + 2\mu}{\rho}},$ $a_s = \sqrt{\dfrac{\mu}{\rho}},$ where a_p, a_s are longitudinal and transversal speeds of elastic waves.



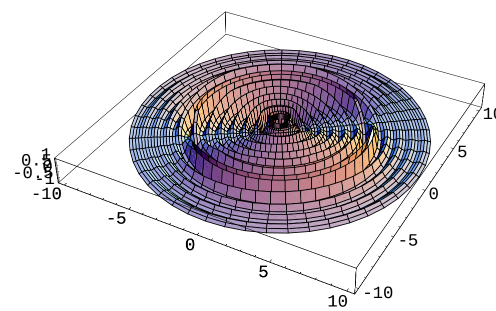
(a) t=0.3



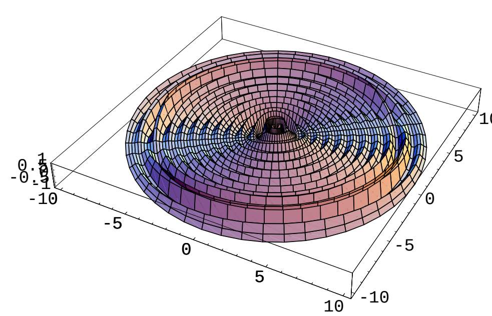
(b) t=0,9



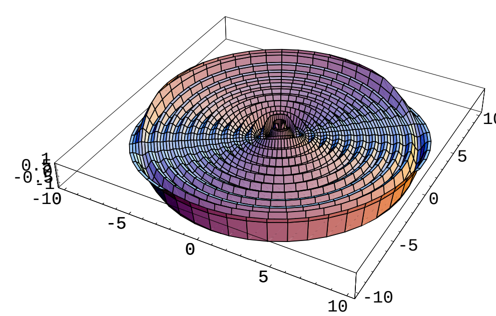
(c) t=2,1



(d) t=3

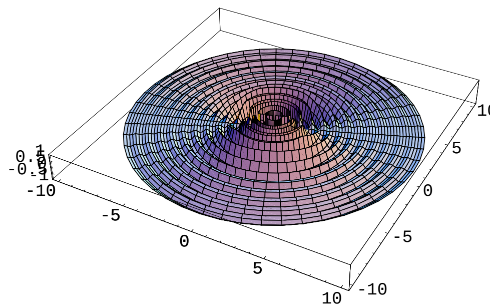


(e) t=4,2

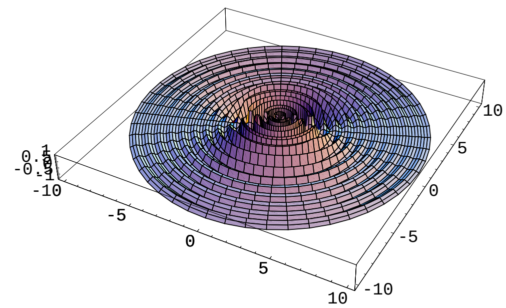


(f) t=4,8

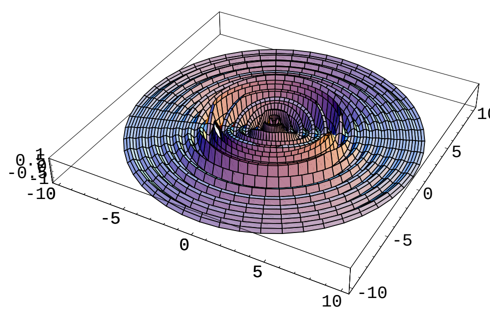
Figure 3.15: The first component of the displacement vector $\mathbf{U}_1(\mathbf{x},\mathrm{t})$, $e_1 = 1$



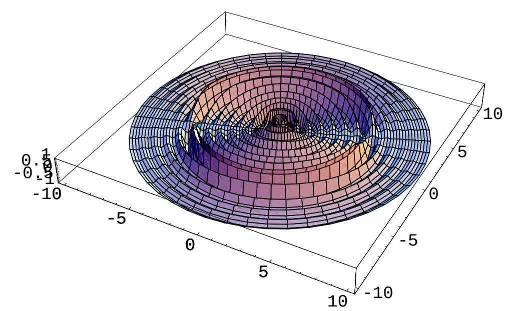
(a) t=0.3



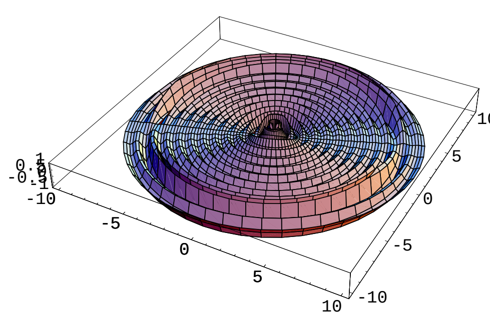
(b) t=0,9



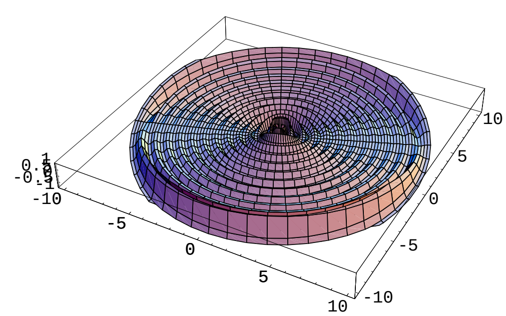
(c) t=2,1



(d) t=3

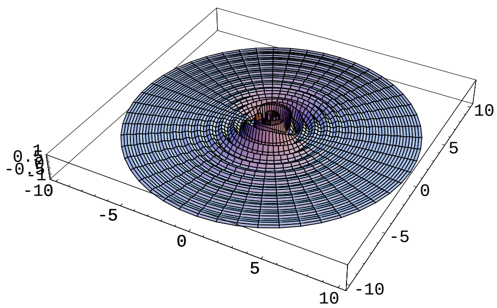


(e) t=4,2

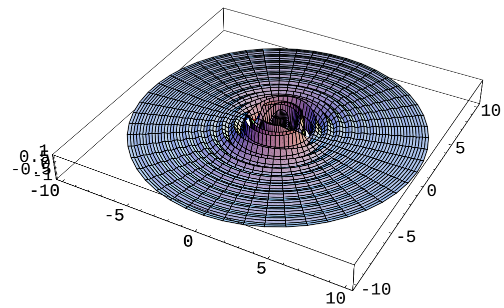


(f) t=4,8

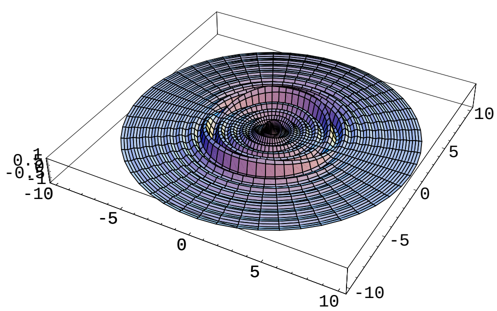
Figure 3.16: The second component of the displacement vector $\mathbf{U}_2(\mathbf{x},\mathrm{t})$, $e_2 = 2$



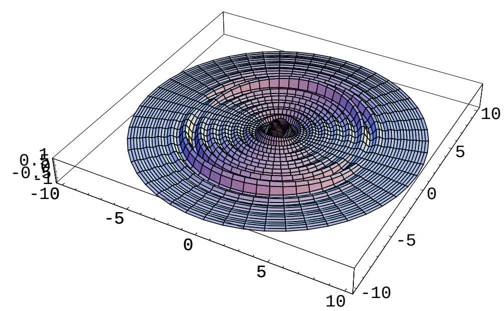
(a) t=0.3



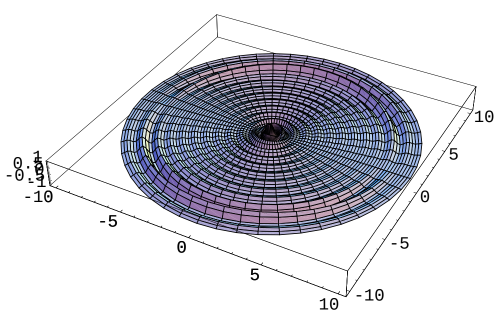
(b) t=0,9



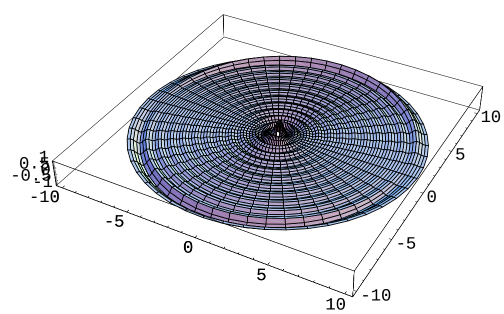
(c) t=2,1



(d) t=3



(e) t=4,2



(f) t=4,8

Figure 3.17: The third component of the displacement vector $\mathbf{U}_3(\mathbf{x},\mathrm{t})$, $e_3 = 4$

In Figure (3.15)-Figure (3.17), components of vector function $\mathbf{U}(\mathbf{x},t)$ that are $\mathbf{U}_1(\mathbf{x},t)$ $\mathbf{U}_2(\mathbf{x},t)$ $\mathbf{U}_3(\mathbf{x},t)$ are simulated. The horizontal axes are r, θ and the vertical axis corresponds to the component of $\mathbf{U}(\mathbf{x},t)$. The variable φ taken as $\varphi=\pi/4$. Since the IBVP for scalar wave equation has a zero solution for external force $\mathbf{f}(\mathbf{x},t)=\nabla\delta(x)\delta(t)$, there exists only one wave that occurs from the solution of the IBVP for vector wave equation.

3.3.3 $f(\mathbf{x},t)=\nabla_x\delta(\mathbf{x})\delta(t)+curl_x[\mathbf{e}\delta(\mathbf{x})\delta(t)]$

Now let us consider the following Cauchy problem for the Lame system:

$$\rho\frac{\partial^2\mathbf{U}}{\partial t^2}=(\lambda+2\mu)\nabla_x div_x\mathbf{U}-\mu curl_x curl_x\mathbf{U}+\mathbf{f}(\mathbf{x},t), \tag{3.3.27}$$

subject to the initial conditions

$$\mathbf{U}(\mathbf{x},t)|_{t=0}=0, \tag{3.3.28}$$

$$\frac{\partial\mathbf{U}}{\partial t}(\mathbf{x},t)|_{t=0}=0, \tag{3.3.29}$$

and the boundary condition

$$\mathbf{U}(\mathbf{x},t)|_{|\mathbf{x}|=r_0}=0, \tag{3.3.30}$$

where $D=\{x\in\mathbf{R}^3:|\mathbf{x}|<r_0\}$ and let vector f is given in the form

$$\mathbf{f}(\mathbf{x},t)=\nabla_x\delta(\mathbf{x})\delta(t)+curl_x[e\delta(\mathbf{x})\delta(t)],$$

where δ is the Dirac delta function and $\mathbf{e}$ is any vector function. Our goal is to find the function $\mathbf{U}(\mathbf{x},t)$ satisfies (3.3.27)-(3.3.30).

When we assume f in the following form

$$\mathbf{f}=\nabla_x g+curl_x\mathbf{F},$$

we obtain

$$g=\delta(\mathbf{x})\delta(t),$$

and

$$\mathbf{F}=\mathbf{e}\delta(\mathbf{x})\delta(t).$$

Hence we obtain two wave equation from the given Lame system. The initial boundary value problem for scalar function $\phi(\mathbf{x},t)$ is

$$\frac{1}{a_p^2}\frac{\partial^2\phi}{\partial t^2} = \triangle_x\phi + \frac{1}{\lambda+2\mu}g,$$

$$\phi(\mathbf{x},t)|_{t=0} = 0, \quad \frac{\partial\phi}{\partial t}(\mathbf{x},t)|_{t=0} = 0,$$
$$\phi(\mathbf{x},t)|_{|\mathbf{x}|=r_0} = 0,$$

where $a_p^2 = \dfrac{\lambda+2\mu}{\rho}$.

And initial boundary value problem for $\mathbf{A}(\mathbf{x},t)$ is

$$\frac{1}{a_s^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \triangle_x\mathbf{A} + \frac{1}{\mu}\mathbf{F},$$

$$\mathbf{A}(\mathbf{x},t)|_{t=0} = 0, \quad \frac{\partial\mathbf{A}}{\partial t}(\mathbf{x},t)|_{t=0} = 0,$$
$$\mathbf{A}(\mathbf{x},t)|_{|\mathbf{x}|=r_0} = 0,$$

where

$$a_s^2 = \frac{\mu}{\rho} \qquad \textit{and} \qquad \mathbf{F} = \mathbf{e}\delta(\mathbf{x})\delta(t).$$

We solved these initial boundary value problems in previous examples and obtain following formulas

$$\phi(x_1,x_2,x_3,t) = \sum_{k=1}^{\infty}\frac{a_p}{\sqrt{\lambda_{k0}}}\frac{8}{r_0^2(\lambda+2\mu)\pi\left[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})\right]^2}\sin[a_p\sqrt{\lambda_{k0}}t]$$
$$\times\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right),$$

is solution of the initial boundary value problem for the scalar wave equation and

$$\mathbf{A}(x_1,x_2,x_3,t) = \sum_{k=1}^{\infty}\frac{a_s}{\sqrt{\lambda_{k0}}}\frac{8\mathbf{e}}{r_0^2(\sqrt{x_1^2+x_2^2+x_3^2})\mu\pi[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2}$$
$$\times\sin[a_s\sqrt{\lambda_{k0}}t]\sin\left(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\right),$$

is solution of the initial boundary value problem for the vector wave equation. Since

$$U(x_1, x_2, x_3, t) = \nabla_x \phi + curl_x \mathbf{A},$$

then

$$U_1(x_1, x_2, x_3, t) = \sum_{k=1}^{\infty} \frac{8}{\pi\sqrt{\lambda_{k0}}[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2 (x_1^2 + x_2^2 + x_3^2)}$$

$$\times \Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0} \cos\left(\mu_k^{(\frac{1}{2})} \frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{r_0}\right) - \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \sin\left(\mu_k^{(\frac{1}{2})} \frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{r_0}\right)\Big]$$

$$\times \Big[\frac{a_p \mu_k^{(\frac{1}{2})} \sin(a_p\sqrt{\lambda_{k0}}t) x_1}{\lambda + 2\mu} + \frac{a_s \sin(a_s\sqrt{\lambda_{k0}}t)\{e_3 x_2 - e_2 x_3\}}{\mu}\Big] \tag{3.3.31}$$

$$U_2(x_1, x_2, x_3, t) = \sum_{k=1}^{\infty} \frac{8}{\pi\sqrt{\lambda_{k0}}[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2 (x_1^2 + x_2^2 + x_3^2)}$$

$$\times \Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0} \cos\left(\mu_k^{(\frac{1}{2})} \frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{r_0}\right) - \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \sin\left(\mu_k^{(\frac{1}{2})} \frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{r_0}\right)\Big]$$

$$\times \Big[\frac{a_p \mu_k^{(\frac{1}{2})} \sin(a_p\sqrt{\lambda_{k0}}t) x_2}{\lambda + 2\mu} + \frac{a_s \sin(a_s\sqrt{\lambda_{k0}}t)\{e_1 x_3 - e_3 x_1\}}{\mu}\Big] \tag{3.3.32}$$

$$U_3(x_1, x_2, x_3, t) = \sum_{k=1}^{\infty} \frac{8}{\pi\sqrt{\lambda_{k0}}[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2 (x_1^2 + x_2^2 + x_3^2)}$$

$$\times \Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0} \cos\left(\mu_k^{(\frac{1}{2})} \frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{r_0}\right) - \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \sin\left(\mu_k^{(\frac{1}{2})} \frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{r_0}\right)\Big]$$

$$\times \Big[\frac{a_p \mu_k^{(\frac{1}{2})} \sin(a_p\sqrt{\lambda_{k0}}t) x_3}{\lambda + 2\mu} + \frac{a_s \sin(a_s\sqrt{\lambda_{k0}}t)\{e_2 x_1 - e_1 x_2\}}{\mu}\Big] \tag{3.3.33}$$

where $a_p^2 = \dfrac{\lambda + 2\mu}{\rho}$; $a_s^2 = \dfrac{\mu}{\rho}$;

$\lambda_{k0} = \left(\frac{\mu_k^{(\frac{1}{2})}}{r_0}\right)^2$ are eigenvalues and
$\mu_k^{(\frac{1}{2})}$; $k = 1, 2, ...$ are roots of the Bessel equation $J_{\frac{1}{2}}(\mu) = 0$.

Solution of the problem defined in (3.3.27)-(3.3.30) is given by the formulas (3.3.31)-(3.3.33)

3.3.3.1 Mathematica Commands for Finding the Components of the Solution

In this subsection we use the formulas that we obtained for the components of $\mathbf{U}(\mathbf{x}, t)$ and `Mathematica` for the simulation of the wave propagation in spherical domain.
`Mathematica` commands for finding $\mathbf{U}_1(\mathbf{x}, t)$, $\mathbf{U}_2(\mathbf{x}, t)$, $\mathbf{U}_3(\mathbf{x}, t)$ are listed below.

```
<< NumericalMath‘BesselZeros‘
<<Graphics‘Graphics3D‘
<<Graphics‘ParametricPlot3D‘
INPUT: Nt,rho,lambda,mu,radius,e1, e2,e3;
ap = Sqrt[(lambda + 2*mu)/rho];
as = Sqrt[mu/rho];
T1 = radius/ap;
T2 = radius/as;
muk =BesselJZeros[1/2, Nt];
l =(muk/radius)^2;
f = D[BesselJ[1/2, x], x] /. x -> muk;
g = f*f;
h1 = 1/((lambda + 2*mu)*Pi*muk*radius*g);
h2 = 1/(mu*Pi*muk*radius*g);
p1 = 8*ap*Sin[ap*Sqrt[l]*t];
p2 = 8*as*Sin[as*Sqrt[l]*t];
z =muk/radius;
k = z*Cos[z*r] - Sin[z*r]*(1/r);
a =r*Cos[varphi]*Sin[theta];
b =r*Sin[varphi]*Sin[theta];
c =r*Cos[theta];
```

```
m1 = (e3*b - e2*c)/r^2;
m2 = (e1*c - e3*a)/r^2;
m3 = (e2*a - e1*b)/r^2;
RQTfi1 =Sum[p[[i]]*k[[i]]*h[[i]]*(a/r^2), {i, 1, Nt}];
RQTfi2 =Sum[p[[i]]*k[[i]]*h[[i]]*(b/r^2), {i, 1, Nt}];
RQTfi3 =Sum[p[[i]]*k[[i]]*h[[i]]*(c/r^2), {i, 1, Nt}];
RQTA1 = Sum[p[[i]]*k[[i]]*h[[i]]*m1, {i, 1, Nt}];
RQTA2 = Sum[p[[i]]*k[[i]]*h[[i]]*m2, {i, 1, Nt}];
RQTA3 = Sum[p[[i]]*k[[i]]*h[[i]]*m3, {i, 1, Nt}];
U1 = RQTfi1 + RQTA1;
U2 = RQTfi2 + RQTA2;
U3 = RQTfi3 + RQTA3;
RQTU1 =ReplaceAll[U1, varphi -> Pi/4];
RQTU2 =ReplaceAll[U2, varphi -> Pi/4];
RQTU3 =ReplaceAll[U3, varphi -> Pi/4];
```

`OUTPUT:` U_1, U_2, U_3.

In this subsection IBVP for the Lame system with Dirichlet boundary condition and with the external force of the form $\mathbf{f}(\mathbf{x},t) = \nabla\delta(x)\delta(t) + curl\Big[\mathbf{e}\delta(x)\delta(t)\Big]$ is reduced to IBVP for the scalar wave equation and IBVP for the vector wave equation with Dirichlet boundary condition in spherical coordinates. These problems are solved in previous subsections and in this subsection solutions are simulated in spherical domain. As a result solution of the problem defined in (3.3.27)-(3.3.30) is (3.3.31)-(3.3.33). For simulation first 25 terms are added. So (3.3.31)-(3.3.33) becomes

$$U(x_1,x_2,x_3,t) = \nabla_x\phi + curl_x\mathbf{A},$$

then

$$U_1(x_1,x_2,x_3,t) = \sum_{k=1}^{25}\frac{8}{\pi\sqrt{\lambda_{k0}}[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2 r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big) - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big]$$

$$\times\Big[\frac{a_p\mu_k^{(\frac{1}{2})}\sin(a_p\sqrt{\lambda_{k0}}t)x_1}{\lambda+2\mu}+\frac{a_s\sin(a_s\sqrt{\lambda_{k0}}t)\{e_3x_2-e_2x_3\}}{\mu}\Big], \tag{3.3.34}$$

$$U_2(x_1,x_2,x_3,t)=\sum_{k=1}^{25}\frac{8}{\pi\sqrt{\lambda_{k0}}[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big]$$

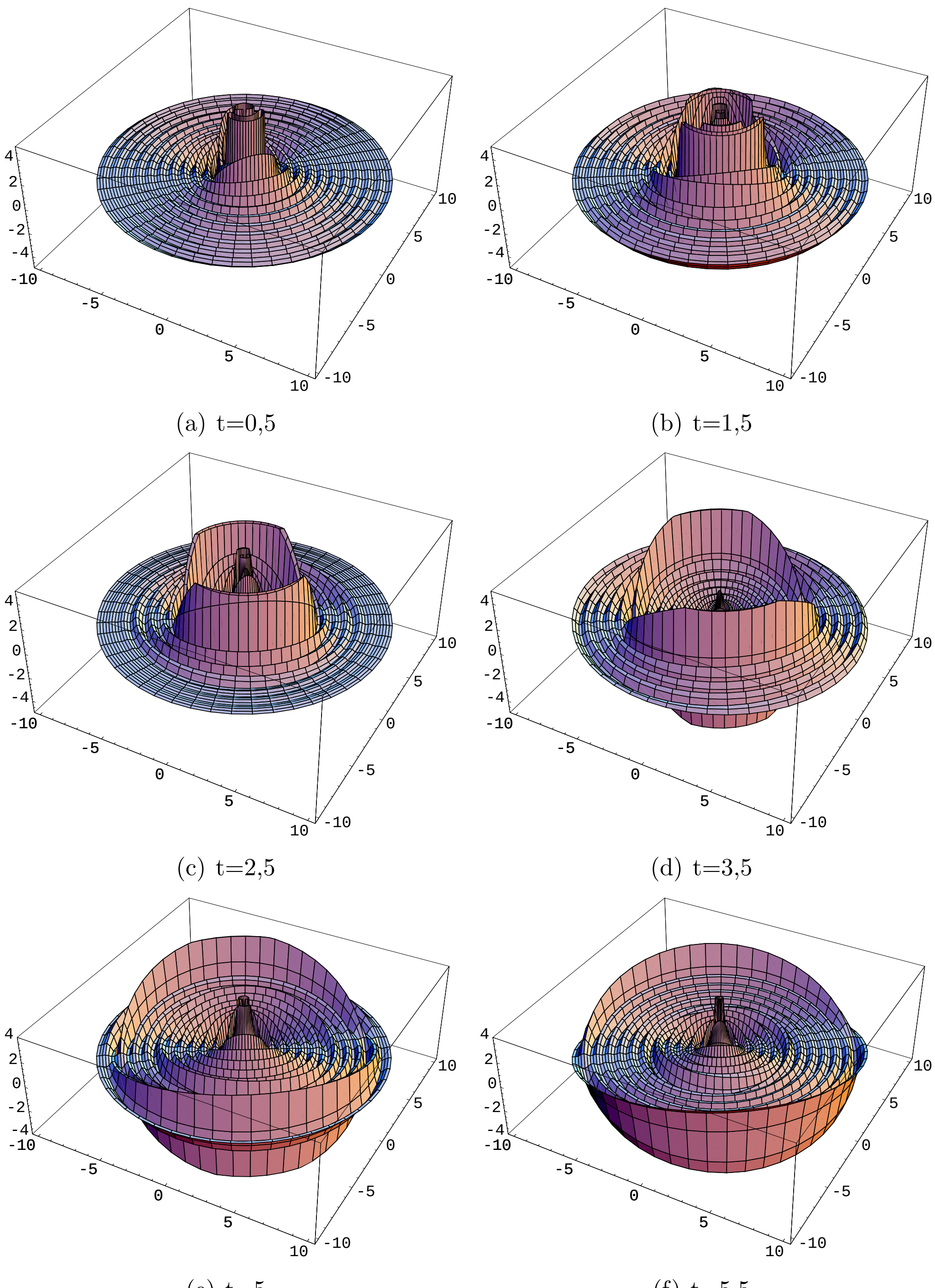
$$\times\Big[\frac{a_p\mu_k^{(\frac{1}{2})}\sin(a_p\sqrt{\lambda_{k0}}t)x_2}{\lambda+2\mu}+\frac{a_s\sin(a_s\sqrt{\lambda_{k0}}t)\{e_1x_3-e_3x_1\}}{\mu}\Big], \tag{3.3.35}$$

$$U_3(x_1,x_2,x_3,t)=\sum_{k=1}^{25}\frac{8}{\pi\sqrt{\lambda_{k0}}[J'_{\frac{1}{2}}(\mu_k^{(\frac{1}{2})})]^2r_0^2(x_1^2+x_2^2+x_3^2)}$$

$$\times\Big[\frac{\mu_k^{(\frac{1}{2})}}{r_0}\cos\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\Big(\mu_k^{(\frac{1}{2})}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{r_0}\Big)\Big]$$

$$\times\Big[\frac{a_p\mu_k^{(\frac{1}{2})}\sin(a_p\sqrt{\lambda_{k0}}t)x_3}{\lambda+2\mu}+\frac{a_s\sin(a_s\sqrt{\lambda_{k0}}t)\{e_2x_1-e_1x_2\}}{\mu}\Big]. \tag{3.3.36}$$

For simulation of these components, the following values are used $\lambda = 2$, $\rho = 1$, $\mu = 3$, $r_0 = 10$, $e_1 = 1$, $e_2 = 2$, $e_3 = 4$. Here ρ is the density of elastic medium, λ, μ are Lame parameters(physical constants), $\rho > 0$, $\lambda + 2\mu > 0$, $\mu > 0$ are positive constants, r_0 is the radius of sphere and $a_p = \sqrt{\dfrac{\lambda+2\mu}{\rho}}$, $a_s = \sqrt{\dfrac{\mu}{\rho}}$, where a_p, a_s are longitudinal and transversal speeds of elastic waves.

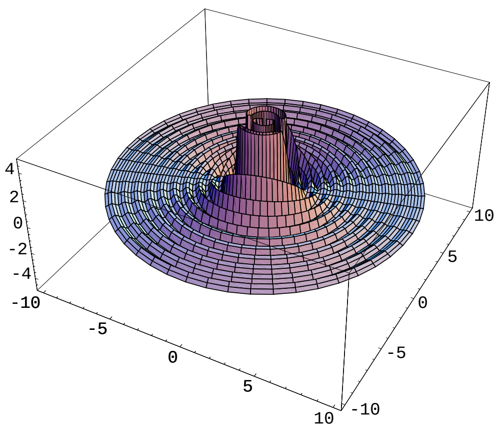


(a) t=0,5 (b) t=1,5

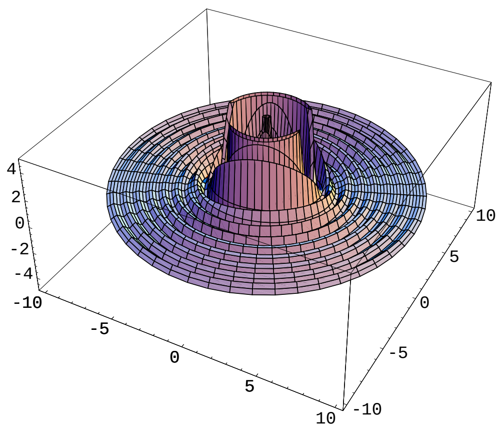
(c) t=2,5 (d) t=3,5

(e) t=5 (f) t=5,5

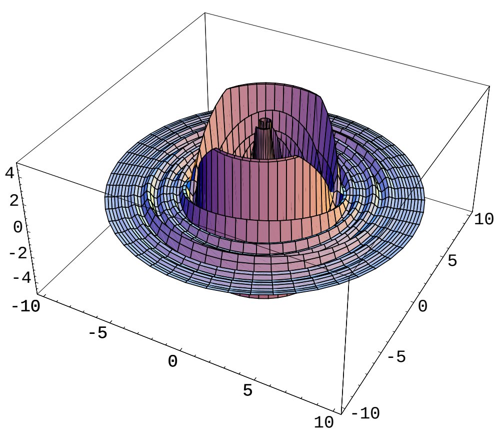
Figure 3.18: The first component of the displacement vector $\mathbf{U}_1(\mathbf{x},\mathrm{t})$, $e_1 = 1$



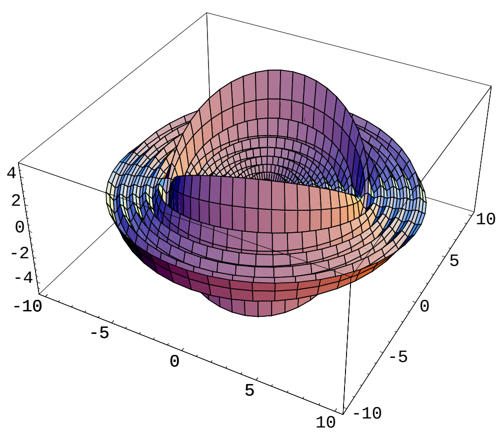
(a) t=0,5



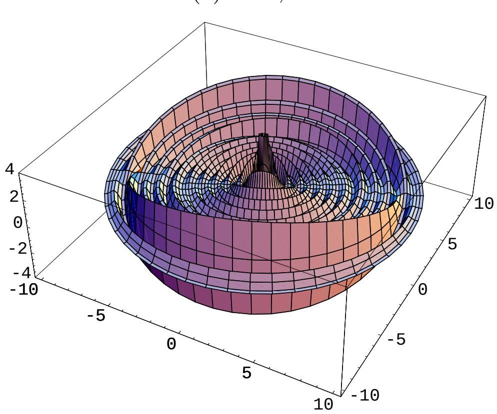
(b) t=1,5



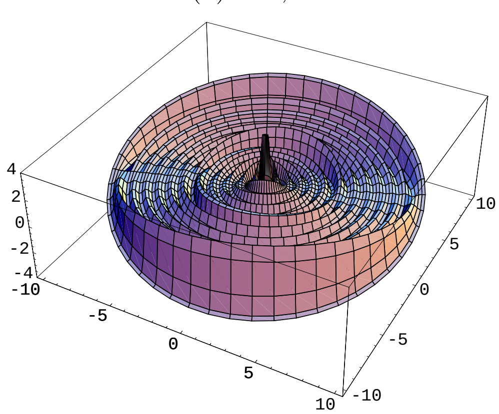
(c) t=2,5



(d) t=3,5

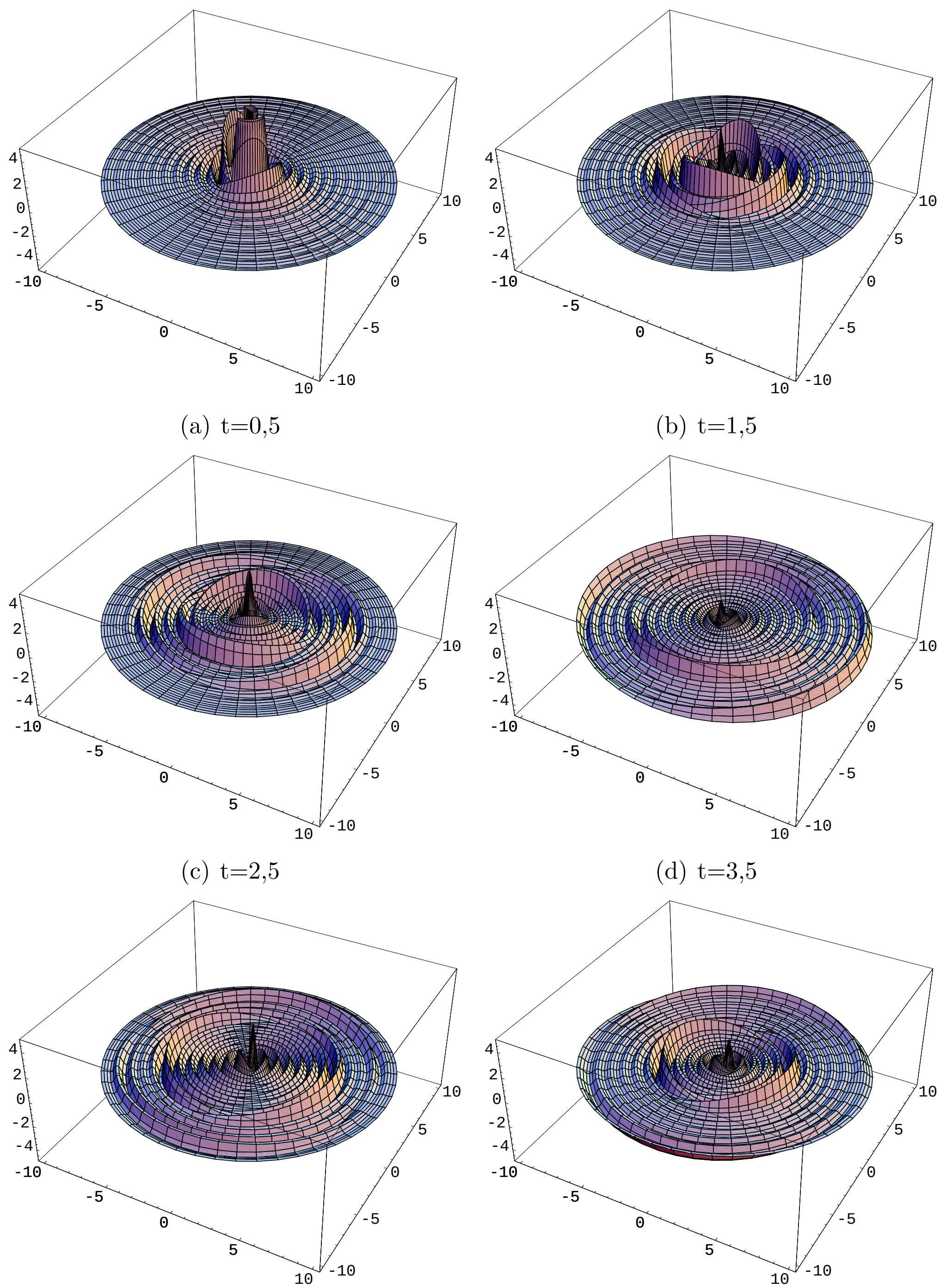


(e) t=5



(f) t=5,5

Figure 3.19: The second component of the displacement vector $\mathbf{U}_2(\mathbf{x},\mathrm{t})$, $e_2 = 2$



(a) t=0,5

(b) t=1,5

(c) t=2,5

(d) t=3,5

(e) t=5

(f) t=5,5

Figure 3.20: The third component of the displacement vector $\mathbf{U}_3(\mathbf{x},\mathrm{t})$, $e_3 = 4$

In Figure (3.18)-Figure (3.20), components of vector function $\mathbf{U}(\mathbf{x}, t)$ that are $\mathbf{U}_1(\mathbf{x}, t)$ $\mathbf{U}_2(\mathbf{x}, t)$ $\mathbf{U}_3(\mathbf{x}, t)$ are simulated. The horizontal axes are r, θ and the vertical axis corresponds to the component of $\mathbf{U}(\mathbf{x}, t)$. The variable φ taken as $\varphi = \pi/4$. There exists two waves that occurs from the solution of the IBVP for scalar wave equation and the solution of the IBVP for vector wave equation.

3.4 Conclusion of the Chapter Three

The main results of this chapter are the following

- Initial boundary value problems for Lame's system is reduced to two wave equations.
- Initial boundary value problems for scalar wave equation and initial boundary value problems for vector wave equation with Dirichlet boundary condition are solved with different sources and solutions for initial boundary value problems of Lame system in a spherical domain is obtained. These solutions are used for simulation of the wave propagation in isotropic elastic media.
- Initial boundary value problems for scalar wave equation and initial boundary value problems for vector wave equation with Neumann boundary condition are solved with different sources when the displacement vector depend on (r, t) in spherical coordinate system and solutions for initial boundary value problems of Lame system in spherical domain is obtained. These solutions are used for simulation of the wave propagation in isotropic elastic media.

CHAPTER FOUR
CONCLUSION

The main result of this thesis are the following

- The explicit formula for the fundamental solution of the initial value problem for Lame's system was constructed.
- Formulas for the solution of initial boundary value problems for Lame's system in spherical domains were obtained by the Fourier series expansion method.
- These formulas were adjusted for pulse point sources (pulse directional force, explosion, pulse rotation center)
- The formulas for the solutions of initial value problem and initial boundary value problems for Lame's system were used for modeling and simulations of wave propagations in isotropic elastic media.
- Results of simulations of displacement components of elastic wave fields were presented by 3-D pictures and animated movies.
- Analysis of obtained formulas and results of simulations were described.

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Appendix A

Changing the Coordinates

Let $\phi(\mathbf{x}, t), \quad \mathbf{x} \in \mathbf{R}^3, \quad t \in \mathbf{R}$ be a function. In spherical coordinates using $x_1 = r\cos\varphi\sin\theta$, $x_2 = r\sin\varphi\sin\theta$, $x_3 = r\cos\theta$ the function $\phi(\mathbf{x}, t)$ can be written as $\phi(r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta, t)$. Let us denote this function as $\bar{\phi}(r, \theta, \varphi, t)$. Our aim is to write $\dfrac{\partial\phi}{\partial x_i}$, $i = 1, 2, 3$, in terms of $\bar{\phi}(r, \theta, \varphi, t)$.

$$\frac{\partial\bar{\phi}}{\partial r} = \frac{\partial\phi}{\partial x_1}\frac{\partial x_1}{\partial r} + \frac{\partial\phi}{\partial x_2}\frac{\partial x_2}{\partial r} + \frac{\partial\phi}{\partial x_3}\frac{\partial x_3}{\partial r},$$

$$\frac{\partial\bar{\phi}}{\partial \varphi} = \frac{\partial\phi}{\partial x_1}\frac{\partial x_1}{\partial \varphi} + \frac{\partial\phi}{\partial x_2}\frac{\partial x_2}{\partial \varphi} + \frac{\partial\phi}{\partial x_3}\frac{\partial x_3}{\partial \varphi},$$

$$\frac{\partial\bar{\phi}}{\partial \theta} = \frac{\partial\phi}{\partial x_1}\frac{\partial x_1}{\partial \theta} + \frac{\partial\phi}{\partial x_2}\frac{\partial x_2}{\partial \theta} + \frac{\partial\phi}{\partial x_3}\frac{\partial x_3}{\partial \theta},$$

or in matrix form

$$\begin{pmatrix} \dfrac{\partial\bar{\phi}}{\partial r} \\[2ex] \dfrac{\partial\bar{\phi}}{\partial \varphi} \\[2ex] \dfrac{\partial\bar{\phi}}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x_1}{\partial r} & \dfrac{\partial x_2}{\partial r} & \dfrac{\partial x_3}{\partial r} \\[2ex] \dfrac{\partial x_1}{\partial \varphi} & \dfrac{\partial x_2}{\partial \varphi} & \dfrac{\partial x_3}{\partial \varphi} \\[2ex] \dfrac{\partial x_1}{\partial \theta} & \dfrac{\partial x_2}{\partial \theta} & \dfrac{\partial x_3}{\partial \theta} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial\phi}{\partial x_1} \\[2ex] \dfrac{\partial\phi}{\partial x_2} \\[2ex] \dfrac{\partial\phi}{\partial x_3} \end{pmatrix}$$

$$\frac{\partial x_1}{\partial r} = \cos\varphi\sin\theta, \qquad \frac{\partial x_2}{\partial r} = \sin\varphi\sin\theta, \qquad \frac{\partial x_3}{\partial r} = \cos\theta,$$

$$\frac{\partial x_1}{\partial \varphi} = -r\sin\varphi\sin\theta, \qquad \frac{\partial x_2}{\partial \varphi} = r\cos\varphi\sin\theta, \qquad \frac{\partial x_3}{\partial \varphi} = 0,$$

$$\frac{\partial x_1}{\partial \theta} = r\cos\varphi\cos\theta, \qquad \frac{\partial x_2}{\partial \theta} = r\sin\varphi\cos\theta, \qquad \frac{\partial x_3}{\partial \theta} = -r\sin\theta,$$

If we substitute all these calculations in to the matrix we obtain

$$\begin{pmatrix} \frac{\partial\bar{\phi}}{\partial r} \\ \frac{\partial\bar{\phi}}{\partial \varphi} \\ \frac{\partial\bar{\phi}}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ -r\sin\varphi\sin\theta & r\cos\varphi\sin\theta & 0 \\ r\cos\varphi\cos\theta & r\sin\varphi\cos\theta & -r\sin\theta \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial\phi}{\partial x_1} \\ \frac{\partial\phi}{\partial x_2} \\ \frac{\partial\phi}{\partial x_3} \end{pmatrix}.$$

$$\begin{pmatrix} \frac{\partial\bar{\phi}}{\partial r} \\ \frac{\partial\bar{\phi}}{\partial \varphi} \\ \frac{\partial\bar{\phi}}{\partial \theta} \end{pmatrix} = \mathcal{A} \cdot \begin{pmatrix} \frac{\partial\phi}{\partial x_1} \\ \frac{\partial\phi}{\partial x_2} \\ \frac{\partial\phi}{\partial x_3} \end{pmatrix}.$$

where

$$\mathcal{A} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ -r\sin\varphi\sin\theta & r\cos\varphi\sin\theta & 0 \\ r\cos\varphi\cos\theta & r\sin\varphi\cos\theta & -r\sin\theta \end{pmatrix}.$$

The inverse of $\mathcal{A}$ can easily found as

$$\mathcal{A}^{-1} = \begin{pmatrix} \cos\varphi\sin\theta & -\frac{1}{r}\frac{\sin\varphi}{\sin\theta} & \frac{1}{r}\cos\theta\cos\varphi \\ \sin\varphi\sin\theta & \frac{1}{r}\frac{\cos\varphi}{\sin\theta} & \frac{1}{r}\sin\varphi\cos\theta \\ \cos\theta & 0 & -\frac{1}{r}\sin\theta \end{pmatrix}.$$

Hence

$$\frac{\partial\phi}{\partial x_1} = (\cos\varphi\sin\theta)\frac{\partial\bar{\phi}}{\partial r} - \Big(\frac{1}{r}\frac{\sin\varphi}{\sin\theta}\Big)\frac{\partial\bar{\phi}}{\partial\varphi} + \Big(\frac{1}{r}\cos\theta\cos\varphi\Big)\frac{\partial\bar{\phi}}{\partial\theta},$$

$$\frac{\partial\phi}{\partial x_2} = (\sin\varphi\sin\theta)\frac{\partial\bar{\phi}}{\partial r} + \Big(\frac{1}{r}\frac{\cos\varphi}{\sin\theta}\Big)\frac{\partial\bar{\phi}}{\partial\varphi} + \Big(\frac{1}{r}\sin\varphi\cos\theta\Big)\frac{\partial\bar{\phi}}{\partial\theta},$$

$$\frac{\partial\phi}{\partial x_3} = (\cos\theta)\frac{\partial\bar{\phi}}{\partial r} - \Big(\frac{1}{r}\sin\theta\Big)\frac{\partial\bar{\phi}}{\partial\theta}.$$

Easily we can show $\nabla\phi$ after these calculations.

$$\nabla\phi = (\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3})$$

$$\begin{aligned} \nabla_{r,\theta,\varphi}\phi = &\Big[(\cos\varphi\sin\theta)\frac{\partial\bar{\phi}}{\partial r} - \Big(\frac{1}{r}\frac{\sin\varphi}{\sin\theta}\Big)\frac{\partial\bar{\phi}}{\partial\varphi} + \Big(\frac{1}{r}\cos\theta\cos\varphi\Big)\frac{\partial\bar{\phi}}{\partial\theta}\Big]e_1 \\ &+\Big[(\sin\varphi\sin\theta)\frac{\partial\bar{\phi}}{\partial r} + \Big(\frac{1}{r}\frac{\cos\varphi}{\sin\theta}\Big)\frac{\partial\bar{\phi}}{\partial\varphi} + \Big(\frac{1}{r}\sin\varphi\cos\theta\Big)\frac{\partial\bar{\phi}}{\partial\theta}\Big]e_2 \\ &+\Big[(\cos\theta)\frac{\partial\bar{\phi}}{\partial r} - \Big(\frac{1}{r}\sin\theta\Big)\frac{\partial\bar{\phi}}{\partial\theta}\Big]e_3. \end{aligned}$$

In spherical coordinates Neumann boundary condition can be found as follows

$$\frac{\partial U}{\partial n} = \nabla_x U \cdot \vec{n}(x) = \nabla_{r,\theta,\varphi}\bar{U} \cdot \frac{\mathbf{x}}{|\mathbf{x}|}. \qquad \text{(A.0.1)}$$

The unit vectors in the spherical coordinate system can be expressed in terms of the spherical coordinates and unit vectors of the rectangular coordinate system. One of the unit vector is l_r that can be formulated as follows

$$l_r = \frac{x}{|x|} = \frac{x_1e_1 + x_2e_2 + x_3e_3}{r} = e_1\sin\theta\cos\varphi + e_2\sin\theta\sin\varphi + e_3\cos\theta,$$

so (A.0.1) can be written as

$$\frac{\partial U}{\partial n} = \frac{\partial \bar{U}}{\partial r},$$

since

$$e_i e_j = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Appendix B

Bessel Function and Properties

Consider the differential equation

$$y''(x) + \frac{1}{x}y'(x) + \Big(1 - \frac{\gamma^2}{x^2}\Big)y(x) = 0, \quad x > 0. \tag{B.0.1}$$

γ is a real parameter. The differential equation B.0.1 is called Bessel's differential equation. A general solution of B.0.1 is given by

$$y(x) = c_1 J_\gamma(x) + c_2 N_\gamma(x),$$

where c_1, c_2 are arbitrary constants. The solution $J_\gamma(x)$ which has a finite limit as x approaches to zero, is called Bessel function and the solution $N_\gamma(x)$ which has no finite limit as x approaches to zero, is called Neumann function.

B.1 Eigenvalue Problem for Bessel Function

$$Ly = -(xy'(x))' + \frac{\gamma^2}{x}y(x),$$

is called Bessel's operator, γ is a parameter, $\gamma \in \mathrm{R}$, $x > 0$.

$$Ly = \lambda x y(x), \quad x \in (0, b)$$

$$-(xy'(x))' + \frac{\gamma^2}{x}y(x) = \lambda xy(x), \quad x \in (0,b) \tag{B.1.1}$$

$$\Big|y(0)\Big| < \infty, \quad y'(b)\cos\beta + y(b)\sin\beta = 0 \tag{B.1.2}$$

(B.1.1),(B.1.2)is eigenvalue problem for Bessel's operator.

Consider the equation (B.1.1),

$$-(xy'(x))' + \frac{\gamma^2}{x}y(x) = \lambda xy(x),$$

$$-xy''(x) - y'(x) + \frac{\gamma^2}{x}y(x) - \lambda xy(x) = 0.$$

Dividing both sides by $(-x)$ we have,

$$y''(x) + \frac{1}{x}y'(x) - \frac{\gamma^2}{x^2}y(x) + \lambda y(x) = 0,$$

$$y''(x) + \frac{1}{x}y'(x) + \Big(\lambda - \frac{\gamma^2}{x^2}y(x)\Big)y(x) = 0. \tag{B.1.3}$$

Last equation can be reducible to Bessel's equation. Let $x = \frac{t}{\sqrt{\lambda}}$

$y(x) = y(\frac{t}{\sqrt{\lambda}}) = \tilde{y}(t)$ then

$$y'(x) = \sqrt{\lambda}\tilde{y}'(t),$$

$$y''(x) = \lambda\tilde{y}''(t).$$

Substituting these into B.1.3 we obtain following Bessel differential equation.

$$\tilde{y}''(t) + \frac{1}{t}\tilde{y}'(x) + \Big(1 - \frac{\gamma^2}{t^2}\tilde{y}(t)\Big)y(x) = 0, \quad t > 0. \tag{B.1.4}$$

A general solution of B.1.4 is

$$\tilde{y}(t) = c_1 J_\gamma(t) + c_2 N_\gamma(t).$$

Thus, a general solution of (B.1.3) is given by

$$y(x) = c_1 J_\gamma(\sqrt{\lambda}x) + c_2 N_\gamma(\sqrt{\lambda}x). \tag{B.1.5}$$

c_1, c_2 are arbitrary constants, $J_\gamma(x)$, $N_\gamma(x)$ are Bessel's and Neumann functions. A general solution of (B.1.1) is given by (B.1.5). The function of the form (B.1.5) will satisfy (B.1.1) and $|y(0)| < \infty$ if and only if $c_2 = 0$. So $y(x) = c_1 J_\gamma(\sqrt{\lambda}x)$ where c_1 is an arbitrary constant satisfies (B.1.1) and $|y(0)| < \infty$. Second boundary condition of (B.1.2) may be written in the form,

$$c_1\Big[\sqrt{\lambda}J'_\gamma(\sqrt{\lambda}b)\cos\beta + J_\gamma(\sqrt{\lambda}b)\sin\beta\Big] = 0 \tag{B.1.6}$$

If $c_1 = 0$ we have $y(x) = 0$ but zero solution is not interesting for us. If $c_1 \neq 0$ then $\lambda_n,\ n = 1, 2, ...$ which are roots of (B.1.6) are eigenvalues and $y(x) = c_1 J_\gamma(\sqrt{\lambda_n}x)$ $n = 1, 2, ...$ are eigenfunctions of (B.1.1),(B.1.2) corresponding to λ_n, $n = 1, 2, ...$.

B.2 Orthogonality of Bessel Functions

Lemma B.2.1. *Let* λ_n *and* λ_m *be eigenvalues of B.1.1 and B.1.2 and* $y_n(x) = J_\gamma(\sqrt{\lambda_n}x)$ $y_m(x) = J_\gamma(\sqrt{\lambda_m}x)$ *be eigenfunctions relative to* λ_n *and* λ_m *respectively. Then if* $n \neq m$

$$\int_0^b x y_m(x) y_n(x) \mathrm{d}x = \int_0^b x J_\gamma(\sqrt{\lambda_n}x) J_\gamma(\sqrt{\lambda_m}x) \mathrm{d}x = 0.$$

B.3 Norm of Bessel Functions

Lemma B.3.1. *Let* λ_n *be eigenvalue of (B.1.1) and (B.1.2) and* $y_n(x) = J_\gamma(\sqrt{\lambda_n}x)$ *be eigenfunction relative to* λ_n*. Then if*

$$\|y_n(x)\|^2 = \int_0^b x y_n^2(x) \mathrm{d}x = \int_0^b x \Big[J_\gamma(\sqrt{\lambda_m}x)\Big]^2 \mathrm{d}x$$

$$= \frac{b^2}{2}\Big\{[J'_\gamma(\sqrt{\lambda_n}b)]^2 + \Big(1 - \frac{\gamma^2}{b^2\lambda_n}\Big)[J_\gamma(\sqrt{\lambda_n}b)]^2\Big\}.$$

Appendix C

Legendre Functions

Consider the function

$$\Psi(\delta, x) = \frac{1}{\sqrt{1 + \delta^2 - 2x\delta}} \,,$$

where $\forall \delta \in [0, 1)$, $\forall x \in [-1, 1]$. This function is continuously differentiable function on $\delta \in [0, 1)$, $x \in [-1, 1]$ and analytic function with respect to $\delta \in [0, 1)$, this means that this function

$$\Psi(\delta, x) = \sum_{n=0}^{\infty} P_n(x)\delta^n,$$

where

$$P_n(x) = \frac{1}{n!}\frac{\partial \Psi(\delta, x)}{\partial \delta^n}\Big|_{\delta=0}.$$

Legendre functions can also be represented as follows

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big[(x^2 - 1)^n\Big], \quad n = 0, 1, 2, ...$$

This formula is called Rodrigues formula.

C.1 Orthogonality of Legendre Functions

Lemma C.1.1.

$$\int_{-1}^{1} P_m(x)P_n(x)\mathrm{d}x = 0, \quad n \neq m.$$

C.2 Norm of Legendre Functions

Lemma C.2.1.

$$\|P_n(x)\|^2 = \int_{-1}^{1} P_n^2(x)\mathrm{d}x = \frac{2}{2n+1}, \quad n = 0, 1, 2...$$

C.3 Fourier Series Expansion for Legendre Functions

Lemma C.3.1. *If* $f(x) \in \mathrm{C}[-1,1]$ $f(x) = \sum_{n=0}^{\infty} f_k \frac{P_n(x)}{\|P_n(x)\|}$ *where*

$$f_k = \int_{-1}^{1} f(x)\frac{P_n(x)}{\|P_n(x)\|}\mathrm{d}x.$$

If $f(x) \in \mathrm{L}_2[-1,1]$

$$\lim_{N\to\infty} \int_{-1}^{1} \Big[f(x) - \sum_{n=0}^{N} f_k \frac{P_n(x)}{\|P_n(x)\|}\Big]^2 \mathrm{d}x = 0.$$

C.4 Eigenvalue Problem for Legendre Function

Consider the following differential operator,

$$Ly = -\frac{d}{\mathrm{d}x}\Big[(1-x^2)\frac{dy}{\mathrm{d}x}\Big], \quad x \in [-1,1]. \tag{C.4.1}$$

This operator is called Legendre's operator.

$$Ly(x) = \lambda y(x), \quad x \in [-1,1], \tag{C.4.2}$$

$$\left|y(-1)\right| < \infty, \quad \left|y(1)\right| < \infty. \tag{C.4.3}$$

Equations (C.4.2), (C.4.3) are called Sturm-Liouville problem for Legendre function. $\lambda = \lambda_n = n(n+1)$ $n = 0, 1, 2, ...$ are eigenvalues and $y_n(x) = P_n(x)$, $n = 0, 1, 2, ...$ are eigenfunctions of Sturm-Liouville problem for Legendre function (C.4.2), (C.4.3).

Appendix D

Associated Legendre Function and Properties

$$P_n^{(m)}(x) = (1-x^2)^{\frac{m}{2}} \frac{\mathrm{d}^m y}{\mathrm{d}x^m} P_n(x), \quad m = 0, 1, 2, ..., n; \quad n = 0, 1, 2, ...$$

are called Associated Legendre function.

D.1 Orthogonality of Associated Legendre Functions

Lemma D.1.1.

$$\int_{-1}^{1} P_n^{(m)}(x) P_k^{(m)}(x) \mathrm{d}x = 0, \quad n \neq k.$$

D.2 Norm of Associated Legendre Functions

Lemma D.2.1.

$$\|P_n^{(m)}(x)\|^2 = \int_{-1}^{1} \left[P_n^{(m)}(x)\right]^2 \mathrm{d}x = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}, \; m = 0, 1, 2..., n; \quad n = 0, 1, 2...$$

D.3 Fourier Series Expansion for Associated Legendre Functions

Lemma D.3.1. *If* $f(x) \in \mathrm{C}[-1,1]$ $f(x) = \sum_{n=m}^{\infty} f_k \frac{P_n^{(m)}(x)}{\|P_n^{(m)}(x)\|}$ *where*

$$f_n^{(m)} = \int_{-1}^{1} f(x) \frac{P_n^{(m)}(x)}{\|P_n^{(m)}(x)\|} \mathrm{d}x$$

If $f(x) \in \mathrm{L}_2[-1,1]$

$$\lim_{N\to\infty} \int_{-1}^{1} \Big[f(x) - \sum_{n=m}^{N} f_k^{(m)} \frac{P_n^{(m)}(x)}{\|P_n^{(m)}(x)\|} \Big]^2 \mathrm{d}x = 0.$$

D.4 Eigenvalue Problem for Associated Legendre Function

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\Big] + \Big(\lambda - \frac{m^2}{1-x^2}\Big) y(x) = 0, \quad x \in (-1,1), \tag{D.4.1}$$

m is fixed.

$$\Big|y(-1)\Big| < \infty, \quad \Big|y(1)\Big| < \infty. \tag{D.4.2}$$

Aim is to find eigenvalues and eigenfunctions of this problem. Let $y(x) = (1-x^2)^{\frac{m}{2}} V(x)$. Then the following equations are valid.

Lemma D.4.1.

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\Big] = (1-x^2)^{\frac{m}{2}} \Big((1-x^2)V''(x) - 2(m+1)xV'(x) - mV(x) + \frac{m^2 x^2}{1-x^2} V(x) \Big)$$

Lemma D.4.2.

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\Big] + \lambda y - \frac{m^2}{1-x^2} y(x) = (1-x^2)^{\frac{m}{2}} \Big[(1-x^2)V''(x) - 2(m+1)xV'(x) + \Big(\lambda - m(m+1)\Big) V(x) \Big]$$

Corollary D.4.3. $(\lambda, y(x))$ *is a solution of (D.4.1),(D.4.2) if and only if* $(\lambda, V(x))$ *is a solution of the following Sturm- Liouville problem*

$$(1-x^2)V''(x) - 2(m+1)xV'(x) + (\lambda - m(m+1))V(x) = 0, \tag{D.4.3}$$

$$\Big|V(-1)\Big| < \infty,\ \Big|V(1)\Big| < \infty. \tag{D.4.4}$$

Lemma D.4.4. *Let* $(\lambda, Y(x))$ *be a solution of*

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\Big] + \lambda Y(x) = 0,\ x \in [-1,1], \tag{D.4.5}$$

$$\Big|Y(-1)\Big| < \infty,\ \Big|Y(1)\Big| < \infty. \tag{D.4.6}$$

if and only if $(\lambda, V(x))$ *is a solution of (D.4.3),(D.4.4) where* $V(x) = \dfrac{d^m Y(x)}{\mathrm{d}x^m}$.

Corollary D.4.5. $\lambda = \lambda_n = n(n+1),\ n = 0,1,2,...$ *are eigenvalues and* $Y_n(x) = P_n(x),\ n = 0,1,2,...$ *are eigenfunctions of (D.4.5), (D.4.6). Hence* $\lambda = \lambda_n = n(n+1),\ n = 0,1,2,...$ *are eigenvalues and* $V(x) = \dfrac{d^m}{\mathrm{d}x^m}P_n(x)$, $n = 0,1,2,...$ *are eigenfunctions of (D.4.3), (D.4.4). The solution of (D.4.1), (D.4.2) is given by* $\lambda = \lambda_n = n(n+1)$ *are eigenvalues and* $y_n(x) = (1-x^2)^{\frac{m}{2}}\dfrac{d^m P_n(x)}{\mathrm{d}x^m}$ $n = m, m+1, m+2, ...$ *are eigenfunctions corresponding to* λ_n.

Appendix E

Sturm Liouville Problem I

Consider the following Sturm-Liouville problem

$$\Psi''(\varphi) + \lambda\Psi(\varphi) = 0, \quad \varphi \in \mathbf{R} \tag{E.0.1}$$

$$\Psi(0) = \Psi(2\pi), \quad \Psi'(0) = \Psi'(2\pi) \tag{E.0.2}$$

Our aim is to find eigenvalues and eigenfunctions. $m^2 + \lambda = 0$ is characteristic equation for (E.0.1). $k_1 = \sqrt{-\lambda}$, $k_2 = -\sqrt{-\lambda}$ are roots of characteristic equation so

$$\Psi(\varphi) = c_1 e^{\sqrt{-\lambda}\varphi} + c_2 e^{-\sqrt{-\lambda}\varphi}, \tag{E.0.3}$$

is a general solution in complex form. c_1 and c_2 are arbitrary complex numbers. Substituting (E.0.3) into (E.0.2) we obtain

$$\Psi(0) = \Psi(2\pi) \Leftrightarrow c_1 + c_2 = c_1 e^{\sqrt{-\lambda}2\pi} + c_2 e^{-\sqrt{-\lambda}2\pi}$$

$$c_1(1 - e^{\sqrt{-\lambda}2\pi}) + c_2(1 - e^{-\sqrt{-\lambda}2\pi}) = 0.$$

and since

$$\Psi'(\varphi) = \sqrt{\lambda}c_1 e^{\sqrt{-\lambda}\varphi} - \sqrt{-\lambda}c_2 e^{-\sqrt{-\lambda}\varphi}$$

we obtain

$$\Psi'(0) = \Psi'(2\pi) \Leftrightarrow c_1 - c_2 = c_1 e^{\sqrt{-\lambda}2\pi} - c_2 e^{-\sqrt{-\lambda}2\pi},$$

$$c_1(1 - e^{\sqrt{\lambda}2\pi}) - c_2(1 - e^{-\sqrt{-\lambda}2\pi} = 0.$$

$$\det(\lambda) = \begin{vmatrix} 1-e^{\sqrt{-\lambda}2\pi} & 1-e^{\sqrt{-\lambda}2\pi} \\ 1-e^{\sqrt{-\lambda}2\pi} & -(1-e^{\sqrt{-\lambda}2\pi}) \end{vmatrix} = 0$$

$$\det(\lambda) = (1-e^{\sqrt{-\lambda}2\pi})\Big(-(1-e^{\sqrt{-\lambda}2\pi})\Big) - (1-e^{\sqrt{-\lambda}2\pi})(1-e^{\sqrt{-\lambda}2\pi}) = 0,$$

$$\det(\lambda) = -2(1-e^{\sqrt{-\lambda}2\pi})(1-e^{-\sqrt{-\lambda}2\pi}) = 0,$$

since $1 = e^{2\pi mi}$ then

$$\det(\lambda) = -2(e^{2\pi mi}-e^{\sqrt{-\lambda}2\pi})(e^{2\pi mi}-e^{-\sqrt{-\lambda}2\pi}) = 0.$$

Therefore $\lambda_m = m^2, \quad m = 0,1,2,...$ are eigenvalues of (E.0.1) and (E.0.2). $\lambda_m = m^2$ can not be negative. If $\lambda = 0$ then

$$\Psi''(\varphi) = 0 \Leftrightarrow \Psi(\varphi) = c_1\varphi + c_2,$$

$$\Psi(0) = \Psi(2\pi) \Leftrightarrow c_1 = 0,$$

$$\Psi'(0) = \Psi'(2\pi) \Leftrightarrow c_1 = c_1,$$

$\Psi_0(\varphi) = c_2 \neq 0$ is eigenfunction corresponding to $\lambda_0 = 0$.

$$\int_0^{2\pi} \Psi_0^2(\varphi)\mathrm{d}\varphi = 1,$$

$$\int_0^{2\pi} c_2^2(\varphi)\mathrm{d}\varphi = 1 \Leftrightarrow c_2 = \sqrt{\frac{1}{2\pi}}.$$

For $\lambda_0 = 0$ we have $\Psi_0(\varphi) = \sqrt{\frac{1}{2\pi}}$ is eigenfunction.

If $\lambda_m = m^2, \quad m = 0,1,2,...$ then $k^2 + m^2 = 0$ is characteristic equation for (E.0.1). $k_1 = im, \quad k_2 = -im$ are roots of characteristic equation so

$$\Psi(\varphi) = c_1\cos\varphi + c_2\sin\varphi, \tag{E.0.4}$$

is a general solution. c_1, and c_2 are arbitrary numbers. Substituting (E.0.4) into (E.0.2) we obtain

$$\Psi(0) = \Psi(2\pi) \Leftrightarrow c_1 = c_2,$$

and

$$\Psi'(0) = \Psi'(2\pi) \Leftrightarrow c_1 m = c_2 m.$$

This means that for any c_1, and c_2 the function (E.0.4) is eigenfunction.

Consider

$$c_2 = 0, \quad \bar{\Psi}_m(\varphi) = c_1 \cos(m\varphi),$$

$$c_1 = 0, \quad \bar{\bar{\Psi}}_m(\varphi) = c_1 \sin(m\varphi).$$

For $\lambda_m = m^2, \quad m = 0, 1, 2, ...$ there exist two linearly independent eigenfunctions,

$$\bar{\Psi}_m(\varphi) = c_1 \cos(m\varphi),$$

$$\bar{\bar{\Psi}}_m(\varphi) = c_1 \sin(m\varphi),$$

$$c_1 = c_2 = \sqrt{\frac{1}{\pi}},$$

solution of Sturm Liouville problem I is, $\lambda_m = m^2, \quad m = 0, 1, 2, ...$ are eigenvalues and

$$\Psi_0(\varphi) = \sqrt{\frac{1}{2\pi}},$$

$$\bar{\Psi}_m(\varphi) = \sqrt{\frac{1}{\pi}} \cos(m\varphi),$$

$$\bar{\bar{\Psi}}_m(\varphi) = \sqrt{\frac{1}{\pi}} \sin(m\varphi),$$

are linearly independent eigenfunctions.

E.1 Sturm Liouville Problem II

Consider the following Sturm-Liouville problem

$$\Psi''(\varphi) + \lambda\Psi(\varphi) = 0, \quad \varphi \in \mathbf{R}, \tag{E.1.1}$$

$$\Psi(\varphi) = \Psi(\varphi + 2\pi). \tag{E.1.2}$$

Lemma E.1.1. *Sturm Liouville problem I is equivalent to Sturm Liouville problem II. This means* (λ, Ψ) *is a solution of Sturm Liouville problem II if and only if* (λ, Ψ) *is a solution of Sturm Liouville problem I.*

So, solution of Sturm Liouville problem II is $\lambda_m = m^2, \quad m = 0, 1, 2, ...$ are eigenvalues and

$$\Psi_0(\varphi) = \sqrt{\frac{1}{2\pi}},$$

$$\bar{\Psi}_m(\varphi) = \sqrt{\frac{1}{\pi}}\cos(m\varphi),$$

$$\bar{\bar{\Psi}}_m(\varphi) = \sqrt{\frac{1}{\pi}}\sin(m\varphi),$$

are linearly independent eigenfunctions.

Appendix F

Spherical Functions

Let $P_n(x)$,$n = 0, 1, 2, ...$ be Legendre functions and

$$P_n^{(k)}(x) = (1-x)^{\frac{k}{2}} \frac{d^k}{dx^k} P_n(x) \ ,$$

$k = 1, 2, ...;\ n = 0, 1, 2, ...$ be associated Legendre functions. Consider the following set of functions

$$Y_n^{(0)}(\theta, \varphi) = P_n(\cos\theta)$$

$$Y_n^{(-1)}(\theta, \varphi) = P_n^{(1)}(\cos\theta)\cos(\varphi), \qquad Y_n^{(1)}(\theta, \varphi) = P_n^{(1)}(\cos\theta)\sin(\varphi)$$

....

$$Y_n^{(-k)}(\theta, \varphi) = P_n^{(k)}(\cos\theta)\cos(k\varphi), \qquad Y_n^{(k)}(\theta, \varphi) = P_n^{(k)}(\cos\theta)\sin(k\varphi)$$

....

$$Y_n^{(-n)}(\theta, \varphi) = P_n^{(n)}(\cos\theta)\cos(n\varphi), \qquad Y_n^{(n)}(\theta, \varphi) = P_n^{(n)}(\cos\theta)\sin(n\varphi)$$

$k = 1, 2, ...;\ n = 0, 1, 2, ...;\ \theta \in [0, \pi);\ \varphi \in [0, 2\pi)$. These (2n+1) functions are called spherical functions.
Property:

$$Y_0^{(0)}(\theta, \varphi) = P_0(\cos\theta) = \frac{1}{0!}\Psi(\delta, \cos\theta)\Big|_{\delta=0} = \Psi(0, \cos\theta) = 1.$$

Property:

$$\int_0^{2\pi}\int_0^{\pi} \sin\theta Y_n^{(m)}(\theta,\varphi)\mathrm{d}\theta\mathrm{d}\varphi = \begin{cases} 4\pi, & n=0 \quad \wedge \quad m=0 \\ 0, & n\neq 0 \quad \vee \quad m\neq 0. \end{cases}$$

Proof. Case: $m=0$

$$\int_0^{2\pi}\int_0^{\pi} Y_n^{(m)}(\theta,\varphi)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi} P_n(\cos\theta)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi$$

$$= \int_0^{2\pi}\Big(\int_0^{\pi} P_n(\cos\theta)\sin\theta\mathrm{d}\theta\Big)\mathrm{d}\varphi$$

$$= \int_0^{2\pi}\Big(\int_{-1}^{1} P_n(x)P_0(x)dx\Big)\mathrm{d}\varphi$$

$$= \int_0^{2\pi} 2\mathrm{d}\varphi.$$

Using the properties of Spherical Functions we obtain

$$\begin{cases} 4\pi, & n=0 \\ 0, & n\neq 0. \end{cases}$$

Case: $m>0$

$$\int_0^{2\pi}\int_0^{\pi} Y_n^{(m)}(\theta,\varphi)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi} P_n^{(m)}(\cos\theta)\sin m\varphi\sin\theta\mathrm{d}\theta\mathrm{d}\varphi$$

$$= \int_0^{2\pi}\sin(m\theta)\Big(\int_0^{\pi} P_n^{(m)}(\cos\theta)\sin\theta\mathrm{d}\theta\Big)\mathrm{d}\varphi = 0,$$

since

$$\int_0^{2\pi}\sin(m\varphi)\mathrm{d}\varphi = 0.$$

Case: $m<0$

$$\int_0^{2\pi}\int_0^{\pi} Y_n^{(m)}(\theta,\varphi)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi} P_n^{(-m)}(\cos\theta)\cos m\varphi \sin\theta \mathrm{d}\theta\mathrm{d}\varphi$$

$$= \int_0^{2\pi} \cos(m\varphi)\Big(\int_0^{\pi} P_n^{(m)}(\cos\theta)\sin\theta\mathrm{d}\theta\Big)\mathrm{d}\varphi = 0,$$

since

$$\int_0^{2\pi}\cos(m\varphi)\mathrm{d}\varphi = 0.$$

proof is complete.

□

F.1 Orthogonality of Spherical Functions

Lemma F.1.1.

$$\int_0^{2\pi}\int_0^{\pi} Y_n^{(m)}(\theta,\varphi)P_n^{(k)}(\theta,\varphi)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi = 0, \quad m \neq k.$$

F.2 Norm of Spherical Functions

Lemma F.2.1.

$$\|Y_n^{(k)}(\theta,\varphi)\|^2 = \int_0^{2\pi}\int_0^{\pi}\Big[Y_n^{(k)}(\theta,\varphi)\Big]^2\sin\theta\mathrm{d}\theta\mathrm{d}\varphi = \begin{cases} \dfrac{2\pi}{2n+1}\dfrac{(n+|k|)!}{(n-|k|)!}, & k \neq 0 \\ \dfrac{4\pi}{2n+1}, & k = 0. \end{cases}$$

F.3 Fourier Series Expansion for Spherical Functions

Lemma F.3.1. *The system of eigenfunctions* $\{Y_n^{(m)}(\theta,\varphi)\},\ n = 0,1,2,...;\ \ m = 0,\pm1,\pm2,...,\pm n$ *is complete orthonormal system in the space* $L_{2,\sin\theta}(D),\ D{=}(0,\pi)\times(0,2\pi)$. *Here* $L_{2,\sin\theta}(D),$ *is the space of square integrable functions with the weight* $\sin\theta$ *over the domain* D*, i.e.*

$$L_{2,\sin\theta}(D) = \{f(\theta,\varphi) : \int\int_D |f(\theta,\varphi)|^2\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\}.$$

Moreover, every function $h(\theta,\varphi)$ *from* $L_{2,\sin\theta}(D)$ *has the Fourier series expansion of the form,*

$$h(\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}h_{nm}Y_n^{(m)}(\theta,\varphi),$$

where the Fourier coefficients are defined by

$$h_{nm}=\int_0^{\pi}\int_0^{2\pi}h(\theta,\varphi)Y_n^{(m)}(\theta,\varphi)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi.$$

F.4 Eigenvalue Problem for Spherical Functions

$$\left.\begin{array}{l}\triangle_{\theta,\varphi}Y(\theta,\varphi)+\alpha Y(\theta,\varphi)=0,\qquad 0<\theta<\pi,\quad \varphi\in\mathbf{R}\\ \Big|Y(0,\varphi)\Big|<\infty,\Big|Y(\pi,\varphi)\Big|<\infty\\ Y(\theta,\varphi)=Y(\theta,\varphi+2\pi).\end{array}\right\}\tag{F.4.1}$$

This problem is to find eigenvalues and eigenfunctions which satisfy all relations (F.4.1). This problem we will solve by the method of the separation of variables. The function $Y(\theta,\varphi)$ we find in the form

$$Y(\theta,\varphi)=\Theta(\theta)\Phi(\varphi).\tag{F.4.2}$$

Substituting (F.4.2) in the relations (F.4.1) we get,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Big(\Theta(\theta)\Phi(\varphi)\Big)}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\Big(\Theta(\theta)\Phi(\varphi)\Big)}{\partial\varphi^2}+\alpha\Big(\Theta(\theta)\Phi(\varphi)\Big)=0,$$

multiplying both sides by $\sin^2\theta$ we have,

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Big(\Theta(\theta)\Phi(\varphi)\Big)}{\partial\theta}\right)+\frac{\partial^2\Big(\Theta(\theta)\Phi(\varphi)\Big)}{\partial\varphi^2}+\alpha\Big(\Theta(\theta)\Phi(\varphi)\Big)\sin^2(\theta)=0,$$

dividing both sides by $\Theta(\theta)\Phi(\varphi)$ we have,

$$\frac{\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)}{\Theta(\theta)}+\alpha\sin^2(\theta)=-\frac{\Phi''(\varphi)}{\Phi(\varphi)},$$

since $\left|Y(0,\varphi)\right|<\infty$, $\left|Y(\pi,\varphi)\right|<\infty$ then $\left|\Theta(0)\right|<\infty$, $\left|\Theta(\pi)\right|<\infty$ and since then $Y(\theta,\varphi)=Y(\theta,\varphi+2\pi)$ then $\Phi(\varphi)=\Phi(\varphi+2\pi)$. So we obtain the following system

$$\frac{\sin(\theta)\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Theta(\theta)}{\partial\theta}\right)}{\Theta(\theta)}+\alpha\sin^2(\theta)=-\frac{\Phi''(\varphi)}{\Phi(\varphi)},\quad 0<\theta<\pi,\quad \varphi\in\mathrm{R},\tag{F.4.3}$$

$$\left|\Theta(0)\right|<\infty,\quad \left|\Theta(\pi)\right|<\infty,\tag{F.4.4}$$

$$\Phi(\varphi)=\Phi(\varphi+2\pi),\tag{F.4.5}$$

The left hand side of (F.4.3) depends on θ and right hand side is the function depending on φ . The variables θ and φ are independent. Hence the solution of the form (F.4.2) will exist if and only if the left hand side and right hand side are equal to the same constant so we find that the problem (F.4.3)-(F.4.5) may be reduced to two following eigenvalue problems,

$$\left.\begin{array}{l}\Phi''(\varphi)+\gamma\Phi(\varphi)=0,\quad \varphi\in\mathbf{R}\\ \Phi(\varphi)=\Phi(\varphi+2\pi),\end{array}\right\}\tag{F.4.6}$$

and

$$\left.\begin{array}{l}\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial\Theta(\theta)}{\partial\theta}\right)+\left(\alpha-\dfrac{\gamma}{\sin^2(\theta)}\right)\Theta(\theta)=0,\quad 0<\theta<\pi,\\ \left|\Theta(0)\right|<\infty,\ \left|\Theta(\pi)\right|<\infty,\end{array}\right\}\tag{F.4.7}$$

The solution of the problem (F.4.6) is given by formulas,[see, appendix]
$\gamma_m=m^2,\ m=0,1,2,...$ are eigenvalues,
$\Phi_0(\varphi)=\sqrt{\dfrac{1}{2\pi}}$ is eigenfunction corresponding to $\gamma_0=0$

$$\left.\begin{array}{l}\overline{\Phi}_m(\varphi)=\sqrt{\dfrac{1}{\pi}}\cos(m\varphi),\\ \overline{\overline{\Phi}}_m(\varphi)=\sqrt{\dfrac{1}{\pi}}\sin(m\varphi),\end{array}\right\}$$

are two linearly independent eigenfunctions corresponding to the eigenvalue γ_m. Now we need to solve problem (F.4.7) for γ_m, for each $m=0,1,2,....$ For the solution of this problem we have to rewrite relations (F.4.7) in the term of new variable $t=\cos\theta$, and new function $X(t)$ which is defined as $X(\cos\theta)=\Theta(\theta)$. The problem (F.4.7) may be written in the terms of the variable $t=\cos\theta$ and function $X(t) = X(\cos\theta) = \Theta(\theta)$ as follows

$$\left.\begin{aligned} &\frac{d}{dx}\Big[(1-x^2)\frac{dX}{dx}\Big] + \Big(\alpha - \frac{m^2}{1-x^2}\Big)X(x) = 0, \\ &\quad x \in [-1,1], \quad m = 1,2,...,n, \\ &\Big|X(-1)\Big| < \infty, \quad \Big|X(1)\Big| < \infty. \end{aligned}\right\} \tag{F.4.8}$$

Let

$$t = \cos\theta,$$

$$\Theta(\theta) = X(t) = X(\cos\theta),$$

$$\Theta'(\theta) = X'(t) = X'(\cos\theta)(-\sin\theta),$$

if we substitute these to problem (F.4.7) we have

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(-\sin\theta\sin\theta X'(\cos\theta)\Big) + \Big(\alpha - \frac{\gamma}{\sin^2(\theta)}\Big)X(t) = 0,$$

$$-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big((1-\cos^2\theta)X'(\cos\theta)\Big) + \Big(\alpha - \frac{\gamma}{\sin^2(\theta)}\Big)X(t) = 0,$$

$$-\frac{1}{\sin\theta}\Big\{\frac{\partial}{\partial t}\Big((1-t^2)X'(t)\Big)\Big\}(-\sin\theta) + \Big(\alpha - \frac{\gamma}{1-t^2}\Big)X(t) = 0,$$

$$\frac{\partial}{\partial t}\Big((1-t^2)X'(t)\Big) + \Big(\alpha - \frac{\gamma}{1-t^2}\Big)X(t) = 0,$$

since $\gamma = m^2$ then we have the following equality

$$\frac{\partial}{\partial t}\Big((1-t^2)X'(t)\Big) + \Big(\alpha - \frac{m^2}{1-t^2}\Big)X(t) = 0.$$

We now that

$$\Big|\Theta(0)\Big| < \infty, \quad \Big|\Theta(\pi)\Big| < \infty,$$

then since $\Theta(0) = X(1)$ and $\Theta(\pi) = X(-1)$, $-1 < t < 1$. Thus we have

$$\Big|X(1)\Big| < \infty, \quad \Big|X(-1)\Big| < \infty.$$

This shows that we can write (F.4.7) in the form (F.4.8).

The problem (F.4.8) is the eigenvalue problem for associated Legendre equation. The solution of the problem (F.4.8) may be written as

$$\alpha_n = n(n+1), \; n = 0,1,2,...$$

are eigenvalues

$$X_{nm}(t)=\sqrt{\frac{(2n+1)}{2}\frac{(n-m)!}{(n+m)!}}P_n^{(m)}(t),\ m=0,1,...,n$$

are eigenfunctions where $P_n^{(m)}(t),\quad m=0,1,2,...,n;\quad n=m,m+1,\ldots,$ are associated Legendre polynomials.[see, appendices]
Hence the solution of the problem (F.4.7) is given by

$$\alpha_n=n(n+1),\quad n=m,m+1,m+2,...$$

are eigenvalues;

$$\Theta_{nm}(\theta)=\sqrt{\frac{(2n+1)}{2}\frac{(n-m)!}{(n+m)!}}P_n^{(m)}(\cos\theta),\ m=0,1,...,n$$

are eigenfunctions are eigenfunctions.

A solution of problem (F.4.1) is given by

$$\alpha=\alpha_n=n(n+1),\ n=m,m+1,m+2,...$$

are eigenvalues;

$$Y_n^{(0)}(\theta,\varphi)=\Theta_{n0}(\theta)\Phi_0(\varphi),$$

$$Y_n^{(m)}(\theta,\varphi)=\Theta_{nm}(\theta)\overline{\Phi}_m(\varphi),$$

$$Y_n^{(-m)}(\theta,\varphi)=\Theta_{nm}(\theta)\overline{\overline{\Phi}}_m(\varphi),$$

$m=1,2,...,n$ are $2n+1$ linearly independent eigenfunctions corresponding to the same eigenvalue $\alpha_n=n(n+1),\ n=m,m+1,m+2,...$
A solution of eigenvalue problem for Beltrami operator is

$$\alpha=\alpha_n=n(n+1),\ n=0,1,2,...$$

are eigenvalues $Y_n^{(m)}(\theta,\varphi),\ m=\pm1,\pm2,...,\pm n;\ n=0,1,2,...$ are eigenfunctions corresponding to α_n.