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LAPLACIAN EIGENVALUES OF THRESHOLD GRAPHS IN GRAPH THEORY

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June 2023

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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Farah Basim Salim AL-MAHDI

ABSTRACT

LAPLACIAN EIGENVALUES OF THRESHOLD GRAPHS IN GRAPH THEORY

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The main source used in the preparation of this thesis is the graduate textbook "Graphs and Matrices", Bapat (2014). Essentially, what we do is to study the eleventh chapter of the mentioned textbook to understand and explain the “Laplacian eigenvalues of threshold graphs”, as can be understood from the title of the thesis. But of course, no part of the aforementioned book has been quoted exactly, a study has been put forward with our own words and our own sentences; almost every proof has been written in more detail, and parts of the book that were left to the reader have been explained completely and the subject has been presented more understandably. In addition to these, the sources listed in the references were also consulted. To summarize in outline: In the first chapter, basic notions are given about the majorization and related fundamental facts are proved. In the first section of the second chapter, threshold graphs are defined, and a characterization of threshold graphs according to the Laplacian eigenvalues is stated and proved; in the second section, the concept of Laplacian integral graph is introduced, and as an example, cographs are defined, which determine a class of Laplacian integral graphs containing the class of threshold graphs, also a characterization of spectral integral variation is stated and proved. In the third chapter, the last chapter before the conclusions and recommendation chapter, a brief literature review on the subject of the thesis is presented mainly by using the notes at the end of the eleventh chapter of the mentioned book.

2023, 38 pages

Keywords: Majorization, Threshold graphs, Laplacian eigenvalues, Spectral integral variation

ÖZET

GRAF TEORİSİNDE EŞİK GRAFLARININ LAPLACE ÖZDEĞERLERİ

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Matematik, Yüksek Lisans
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Bu tezin hazırlanmasında kullanılan başlıca kaynak, Bapat (2014)'in "Graphs and Matrices" başlıklı lisansüstü kitabıdır. Esas itibariyle bizim yaptığımız, tezin başlığından da anlaşılacağı üzere, "eşik graflarının Laplace özdeğerlerini anlamak ve anlatmak için, söz konusu kitabın on birinci bölümünün çalışılmasından ibarettir. Fakat tabi ki mevzubahis kitabın herhangi bir kısmı aynen alıntılanmamış, kendi sözcüklerimiz ve kendi cümlelerimizle bir çalışma ortaya konulmuştur ve hemen her ispat ayrıntılı bir şekilde yazılmış ve kitabın okuyucuya bırakılan bölümleri eksiksiz bir şekilde açıklanarak konu daha anlaşılır bir şekilde sunulmuştur. Bunlara ek olarak, referanslar kısmında listelenmiş olan kaynaklara da başvurulmuştur. Ana hatlarıyla özetlemek gerekirse: Birinci bölümde majorizasyonla ilgili temel kavramlar verilmiş ve ilgili temel sonuçlar ispat edilmiştir. İkinci bölümün ilk alt bölümünde, eşik grafları tanımlanmış ve eşik graflarının Laplace özdeğerlerine göre karakterizasyonu ifade ve ispat edilmiştir; ikinci alt bölümünde, Laplace integral graf kavramı verilmiş ve örnek olarak, eşik grafları sınıfını içeren, Laplace integral graflarının bir sınıfını belirleyen kograflar tanımlanmıştır, ayrıca spektral integral varyasyonun bir karakterizasyonu da ifade ve ispat edilmiştir. Tezin üçüncü, sonuç ve öneriler bölümünden önceki son bölümünde ise, esas olarak adı geçen kitabın on birinci bölümünün sonunda yer alan notlardan yararlanılarak, tezin konusuna ilişkin kısa bir literatür taraması sunulmuştur.

2023, 38 sayfa

Anahtar Kelimeler: Majorizasyon, Eşik graflar, Laplace özdeğerler, Spektral integral varyasyon

PREFACE AND ACKNOWLEDGEMENTS

This thesis is a review study, and the main source used in its preparation is "Graphs and Matrices", Bapat (2014). In order for this book to be the main source for the thesis, our advisor has communicated with the author Ravindra B. Bapat via e-mail, and he has received the following reply: “I am pleased that you and your students found my book useful. You may use the book for the thesis, subject to copyright laws. There is a close connection between graph theory and matrix theory. Techniques in matrix theory can be fruitfully used in graph theory. But this connection is ignored to a large extent by graph-theorists. Perhaps the reason is that graph theory is introduced as a part of combinatorics and not as algebra/linear algebra. There are many examples of linear algebra techniques in graph theory. The recent proof of the sensitivity conjecture by Huang is a brilliant example of such applications. These were some of my motivations in writing the book. It seems that the book is well-received, particularly in China, Iran, etc. I wish your students best luck and success in their future endeavours in graphs and matrices.”

After the permission of the author, the mentioned book, which is one of the excellent books prepared at the graduate level in the intersection of linear algebra and graph theory, has been adopted as the main source of the thesis. Therefore, almost all definitions and theorems in the thesis have been taken from Bapat (2014), and instead of citing each one separately, it was found more convenient to state this situation here.

I would like to dedicate my dissertation work to my family. A special feeling of gratitude to my loving parents and husband, whose words of encouragement and push for tenacity ring in my ears.
And thank my thesis advisor, Asst. Prof. Dr. Celalettin KAYA, for his patience, guidance, and understanding.

Farah Basim Salim AL-MAHDI
Çankırı-2023

CONTENTS

LIST OF SYMBOLS

Γ	A graph
$A(\Gamma)$	Adjacency matrix of a graph Γ
$\Gamma + \Omega$	Disjoint union or sum of the graphs Γ and Ω
$E(\Gamma)$	Edge set of a graph Γ
$\Gamma \vee \Omega$	Join of graphs Γ and Ω
$L(\Gamma)$, L	Laplacian matrix of a graph Γ
(A)	Null space of a matrix A
$c_A(x)$	The characteristic polynomial of a square matrix A
Γ^c	The complement of a graph Γ
K_n	The complete graph on n vertices
$\rho(i)$, $deg(i)$	The degree of a vertex I
$\mu_i(\Gamma)$	The i th largest eigenvalue of $L(\Gamma)$ of a graph Γ
$\Delta(\Gamma)$, Δ	The maximum vertex degree of a graph Γ
$\delta(\Gamma)$, δ	The minimum vertex degree of a graph Γ
$0_{m\times n}$	The $m\times n$ matrix with all entries equal to 0
$\overline{1}_n$, $\overline{1}$	The $n\times 1$ vector with all coordinates equal to 1
$[n]$	The set $\{1, 2,\ldots, n\}$
$N_\Gamma(i)$, $N(i)$	The set of neighbors of a vertex $i \in V(\Gamma)$ of a graph Γ
J_n, J	The square matrix of order n with all entries equal to 1
$tr(A)$	The trace of a square matrix $A = [a_{ij}]$, *i.e.*, $tr(A) = \sum_{i=1}^{n} a_{ij}$
$\overline{0}$	The zero vector of appropriate order
x^T	Transpose of an $n \times 1$ victor x
$B(\Gamma)$, B	(Vertex - edge) incidence matrix of a graph Γ
$V(\Gamma)$, V	Vertex set of a graph Γ
$i \sim j$	Vertices i and j are adjacent

LIST OF ABBREVIATIONS

DSM	Doubly stochastic matrix
iff	If and only if
IM	Incidence matrix
LI	Laplacian integral
LM	Laplacian matrix
PM	Permutation matrix
s.t.	Such that
TG	Threshold graph
Wlog	Without loss of generality

1. MAJORIZATION

In the chapter, we review some fundamental facts about majorization, which we need in the sequel of the thesis.

Definition 1.1 *Let* $a = [a_1, a_2, \dots, a_n]^T \in \mathbb{R}^n$, and let $a_{[1]} \geq a_{[2]} \geq \dots \geq a_{[n]}$ be a nonincreasing sequence of the components of the vector a. That is:
$a_{[1]} = max\{a_1, a_2, \dots, a_n\}$, and $a_{[j]} = max(\{a_1, a_2, \dots, a_n\} \backslash \{a_{[1]}, a_{[2]}, \dots, a_{[j-1]}\})$ for each $j = 2, 3, \dots, n$. Let $b = [\, b_1, b_2, \dots, b_n]^T \in \mathbb{R}^n$. If we have:
i) $\sum_{j=1}^{k} a_{[j]} \leq \sum_{j=1}^{k} b_{[j]}$, for each $k = 1, 2, \dots, n-1$,
ii) $\sum_{j=1}^{n} a_{[j]} \leq \sum_{j=1}^{n} b_{[j]}$,
then we say that a is "majorized" by b, or b "majorizes" a, and it is denoted by $a < b$. If $a < b$, we frequently say that $a_1, a_2, \dots, a_n$ are majorized by $b_1, b_2, \dots, b_n$. If c and d are $1 \times n$ real vectors, and if $c^T < d^T$, then we say that c is majorized by d, and it is also denoted by $c < d$.

Note 1.2 Let $a = [a_1, a_2, \dots, a_n]^T \in \mathbb{R}^n$, and let $\bar{a} = \frac{a_1 + a_2 + \dots + a_n}{n}$ be the arithmetic mean of the components of the vector a. Also let b = $[\bar{a}, \bar{a}, \dots, \bar{a}]^T$. Then $<$ a :
First of all, $\sum_{j=1}^{n} b_{[j]} = \sum_{j=1}^{n} \bar{a} = n \; \bar{a} = a_1 + a_2 + \dots + a_n = \sum_{j=1}^{n} a_{[j]}$.
∴ The condition (ii) in the previous definition is satisfied.
Secondly, $\bar{a} \leq a_{[1]}$: Suppose not. That is, $\bar{a} > a_{[1]}$.
Then, since $a_{[1]} \geq a_j$ for each $j = 1, \dots, n$, $\bar{a} > a_{[j]}$ for each $j = 1, \dots, n$.
⇒ $n\bar{a} > a_{[1]} + a_{[2]} + \dots + a_{[n]} = a_1 + a_2 + \dots + a_n$, a contradiction.

Now, we can show that the condition (i) of the previous definition is also satisfied:
Suppose to the contrary that there is some $k \in \{2, 3, \dots, n-1\}$ s.t. $\sum_{j=1}^{k} b_{[j]} > \sum_{j=1}^{k} a_{[j]}$.
Let m be the smallest such k. That is, $\sum_{j=1}^{m-1} b_{[j]} \leq \sum_{j=1}^{m-1} a_{[j]}$, but $\sum_{j=1}^{m} b_{[j]} > \sum_{j=1}^{m} a_{[j]}$.
⇒ In particular, $b_{[m]} = \bar{a} > a_{[m]}$.
⇒Since $a_{[m]} \geq a_{[j]}$ for each $j = m, m+1, \dots, n$, $\bar{a} > a_{[j]}$ for each $j = m, m+1, \dots, n$.
⇒ Since $\sum_{j=1}^{m} b_{[j]} > \sum_{j=1}^{m} a_{[j]}$, $\sum_{j=1}^{m} b_{[j]} + (n-m)\,\bar{a} > \sum_{j=1}^{m} a_{[j]} + \sum_{j=m+1}^{n} a_{[j]}$.

$\Rightarrow \sum_{j=1}^{n} b_{[j]} > \sum_{j=1}^{n} a_{[j]} = a_{[1]} + \dots + a_{[n]} = a_1 + \dots + a_n$

$\Rightarrow n\bar{a} > a_1 + a_2 + \dots + a_n$, a contradiction.

∴ The condition (i) of the previous definition must also be satisfied. As a result,

$[\bar{a}, \bar{a}, \dots, \bar{a}]^T \prec [a_1 + a_2 + \dots + a_n]^T$.

Definition 1.3 Let $M = [m_{ij}]$ be a square matrix of order n. If $m_{ij} \geq 0$ for all $i, j = 1, 2, \dots, n$, and each row sum and each column sum equal 1, then M is called a "doubly stochastic matrix (DSM)".

Theorem 1.4 (Hardy-Littlewood-Polya Theorem) (Hardy, Littlewood, Polya (1934, 1952)) *Let* $a = [a_1, a_2, \dots, a_3]^T$, $b = [b_1, b_2, \dots, b_n]^T \in \mathbb{R}^n$. Then:

$$a \prec b \Leftrightarrow \text{There is a DSM } M_{n\times n} = [m_{ij}] \text{ s.t. } a = Mb.$$

Proof.

($\Leftarrow$) Suppose that $a = Mb$ for some DSM M_{nxn}. Then :

$a_j = [Mb]_j = M_{j\,th\,row}.\, b = m_{j1}b_1 + m_{j2}b_2 + \dots + m_{jn}b_n = \sum_{l=1}^{n} m_{jl}b_l$.

$\therefore \sum_{j=1}^{n} a_j = \sum_{j=1}^{n}(\sum_{l=1}^{n} m_{jl}b_l) = \sum_{l=1}^{n}\sum_{j=1}^{n} m_{jl}b_l = \sum_{l=1}^{n} b_l(\sum_{j=1}^{n} m_{jl}) = \sum_{l=1}^{n} b_l$,

because $\sum_{j=1}^{n} m_{jl} = 1$

∴ Condition (ii) of the first definition is satisfied.

Now, let $k \in \{1, 2, \dots, n-1\}$ be a fixed integer.

First of all, for simplicity of the notation, we can assume that $a_1 \geq a_2 \geq \dots \geq a_n$ and $b_1 \geq b_2 \geq \dots \geq b_n$:

Let σ (respectively, τ) be the permutation of the set $[n] = \{1, 2, \dots, n\}$ s.t. if we apply σ (respectively, τ) to the components of a (respectively, b), then we obtain the ordering $a_{[1]} \geq a_{[2]} \geq \dots \geq a_{[n]}$ (respectively, $b_{[1]} \geq b_{[2]} \geq \dots \geq b_{[n]}$). Now, if we apply σ (respectively, τ) to the rows (respectively, columns) of the matrix M, and denote the obtained matrix with $\widetilde{M}$, then the equation $a = Mb$ becomes the equation $\widetilde{a} = \widetilde{M}\,\tilde{b}$, where $\widetilde{a} = [a_{[1]}, a_{[2]}, \dots, a_{[n]}]^T$ and $\tilde{b} = [b_{[1]}, b_{[2]}, \dots, b_{[n]}]^T$. Obviously, the resulting matrix $\widetilde{M}$ is still a DSM. As a result, the assumptions $a_1 \geq a_2 \geq \dots \geq a_n$ and $b_1 \geq b_2 \geq \dots \geq b_n$ do not violate the generalization.

Now, let $s_l = \sum_{i=1}^{k} m_{jl}$ for each $l = 1, 2, \dots, n$.

Then: $\sum_{l=1}^{n} s_l = \sum_{l=1}^{n} \sum_{i=1}^{k} m_{jl} = \sum_{i=1}^{k} \sum_{l=1}^{n} m_{jl} = \sum_{i=1}^{k} 1 = k$.

And, $\sum_{j=1}^{k} a_j \leq \sum_{j=1}^{k} b_j$, or equivalently, $\sum_{j=1}^{k} (a_j - b_j) \leq 0$:

$$\sum_{j=1}^{k}(a_j - b_j) = \sum_{j=1}^{k} a_j - \sum_{j=1}^{k} b_j = \sum_{j=1}^{k}\left(\sum_{l=1}^{n} m_{jl}\, b_l\right) - \sum_{j=1}^{k} b_j$$

$$= \sum_{l=1}^{n} \sum_{j=1}^{k} m_{jl}\, b_l - \sum_{j=1}^{k} b_j$$

$$= \sum_{l=1}^{n} b_l\left(\sum_{j=1}^{k} m_{jl}\right) - \sum_{j=1}^{k} b_j$$

$$= \sum_{l=1}^{n} b_l s_l - \sum_{j=1}^{k} b_j + b_k \left(k - \sum_{l=1}^{n} s_l\right)$$

$$= \sum_{l=1}^{n} b_l s_l + \sum_{l=k+1}^{n} b_l s_l - \sum_{l=1}^{k} b_l + b_k k - \sum_{l=1}^{k} b_k s_l - \sum_{l=k+1}^{n} b_k s_l$$

$$= \sum_{l=1}^{k} b_l (s_l - 1) - \sum_{l=1}^{k} b_l (s_l - 1) + \sum_{l=k+1}^{n} (b_l - b_k) s_l$$

$$= \sum_{l=1}^{k} (b_l - b_k)(s_l - 1) + \sum_{l=k+1}^{n} (b_l - b_k) s_l \leq 0,$$

because $b_l \geq b_k$ for $l = 1, 2, \dots, k$ and $b_l \leq b_k$ for $l = k+1, k+2, \dots, n$, and also $0 \leq s_l \leq 1$.

∴ Condition (i) of the first definition is also satisfied.

As a result, $a < b$.

(⇒) (This part of the proof is taken from Zhang (2011).)

To prove this necessity part, we first need a definition.

A 2×2 matrix of the form $= \begin{bmatrix} t & 1-t \\ 1-t & t \end{bmatrix}$, where $t \in [0, 1]$, is called a "T-transform". Let M be an $n \times n$ matrix. If M is obtained from the $n \times n$ identity matrix I by interchanging a 2×2 principal submatrix of I with a 2×2 T-transform T, then M is also called an "(n×n) T-transform".

First, a T-transform $M_{n \times n} = [m_{ij}]$ is a DSM:

Since $0 \leq t \leq 1, 0 \leq 1 - t \leq 1$.

∴ $m_{ij} \geq 0$ for all $i, j = 1, 2, \dots, n$. Let M be obtained from I by interchanging its principal submatrix determined by i th and j th rows, and i the and j th columns. Then each row sum and each column sum of M is equal to 1. (Let $k \neq i, j$. Then row k and column k contain only one nonzero entry which is 1. And for $k \in \{i, j\}$, each of the sum of elements in row k and in column k is equal to $t + (1 - t) = 1$.) Therefore, M is a DSM.

Second, obviously, $M = tI + (1-t)\ P$, where $t \in [0,1]$ and P is a permutation matrix corresponding to the interchanging columns i and j of the identity matrix. (If each row and each column of a square matrix P contains exactly one nonzero entry which is equal to l, then P is called a "permutation matrix (PM)".)

Note that if $t = 0$, then $M = 0I + (1-0)\ P = P$, that is, M is a PM.

Third, it is a well-known fact from algebra that a permutation $\sigma: [n] \to [n]$ can be expressed as a product (function composition) of permutations, each of which corresponds to a single interchange. Therefore, a PM can be expressed as a product (matrix product) of T-transforms.

Now, we can prove the necessity part by induction on n:

$n = 1$: In this case, $a, b \in \mathbb{R}$. Therefore, $a < b \Rightarrow a = b$, by the definition of majorization. Thus, since $M = [1]_{1\times 1}$ is a DSM and trivially $a = Mb$, the statement is true.

∴ Suppose $n > 1$ and the statement holds for $n - 1$.

(i) If $a_1 = b_1$, Then since $a < b$, we have $[a_2, a_3, \dots \dots, a_n]^T < [b_2, b_3, \dots \dots, b_n]^T$. Then by the induction assumption, there are T-transforms $M_1, M_2, \dots, M_k$, each has order $n-1$ s.t. $[a_2, a_3, \dots, a_n]^T = M_1\ M^2 \dots\ M_k\ [b_2, b_3, \dots \dots, b_n]^T$.

Let $N_i = \begin{bmatrix} 1 & \bar{0}^T \\ 0 & M_i \end{bmatrix}$ for each $i = 1, 2, \dots, k$. Then, each of N_i is also a T-transform, and by block multiplication, $[a_1, a_2, \dots, a_n]^T = N_1 N_2 \dots\ N_k\ [b_1, b_2, \dots \dots, b_n]^T$.

(ii) If $a_i = b_j$ for some i and j, then a permutation σ and τ can be applied to a and b, respectively, s.t. a_i and b_j are the first coordinates of the obtained vectors, say $\tilde{a}$ and $\tilde{b}$, respectively. Let P_σ and P_τ be the corresponding PMs. Then $\widetilde{a} = P_\sigma\ a$ and $\widetilde{b} = P_\tau\ b$. Then, since applying permutations does not charge the majorization relation (that is, $a < b \Leftrightarrow \widetilde{a} < \widetilde{b}$), by the case proved in the previous paragraph, there are T-transforms $S_1, S_2, \dots, S_l$ s.t. $\widetilde{a} = S_1 S_2 \dots S_l \widetilde{b}$. Therefore $P_\sigma\ a = S_1\ S_2 \dots\ S_l P_\tau b$. Thus, $a = P_\sigma^{-1}\ S_1\ S_2 \dots\ S_l\ P_\tau\ b$. (We know that every PM is invertible.)

As a result, since each PM is a product of some T-transformations, $a = T_1\ T_2 \dots T_k b$ for some T-transforms $T_1, T_2, \dots, T_k$.

(iii) Suppose that $a_i \neq b_j$ for all i and j. Wlog, suppose also that the vectors a and b are in non-increasing order.

First, we prove a general fact: Let $x = [x_1, x_2, \ldots, x_n]^T$, y $= [y_1, y_2, \ldots, y_n]^T$ be two vectors whose coordinates are in non-increasing order. Suppose that $x < y$. Then, there is some $m \in \{1, 2, \ldots, n\}$ s.t. $y_m \geq x_m \geq y_{m+1}$: Suppose not. Then, $y_m < x_m$ or $x_m < y_{m+1}$ for all $m = 1, 2, \ldots, n$. Since $x < y$, by the definition of majorization, "$y_m < x_m$ for each m" is not possible. Therefore, the case "$x_m < y_m + 1$ for each m" must hold. Then, $x_1 < y_2, x_2 < y_3, \ldots, x_{n-1} < y_n$.

$\therefore \sum_{j=1}^{n-1} x_j < \sum_{j=2}^{n} y_j$

$\therefore$ Since $\sum_{j=1}^{n} x_j$=$\sum_{j=1}^{n} y_j$ by the definition of majorization, $x_n > y_1$. On the other hand, since the vector a is in non-increasing order, $x_n \leq x_1$. And, again by the definition of majorization, $x_1 \leq y_1$. Therefore, $x_n \leq y_1$, which contradicts with $x_n > y_1$.

As a result, the statement is true, that is, $y_m \geq x_m \geq y_{m+1}$ for some $m \in \{1, 2, \ldots, n\}$.

Now, by the fact proved in the previous paragraph, there is some $m \in [1, 2, \ldots, n\}$ s.t. $b_m \geq a_m \geq b_{m+1}$. But since $a_i \neq b_j$ for all i and j, this means that $b_m > a_m > b_{m+1}$ some m.

Therefore, $a_m = tb_m + (1 - t)\, b_{m+1}$ for some m.

Let M_0 be the T-trasfom $\begin{bmatrix} t & 1-t \\ 1-t & t \end{bmatrix}$ corresponding to the rows m and $m + 1$, and columns m and $m + 1$. And let $c = M_0 b$. Then, for each $j \neq m,\ m + 1$, $c_j = b_j$, and:

$c_m = t\, b_m + (1 - t) b_{m+1} = a_m$,

$c_{m+1} = (1 - t)\, b_m + t b_{m+1} = b_m + b_{m+1} - (t b_m + (1 - t) b_{m+1})$

$= b_m + b_{m+1} - a_m$.

Now, $a < c$:

For $k \neq m$, that is, for $k < m$ or $k > m$, since c_m+ $c_{m+1} = b_m + b_{m+1}$ (for the case $k > m$), $\sum_{j=1}^{k} a_j \leq \sum_{j=1}^{k} b_j = \sum_{j=1}^{k} c_j \leq \sum_{j=1}^{k} c_{[j]}$.

And for $k = m$, $\sum_{j=1}^{m} a_j \leq \sum_{j=1}^{m-1} b_j + a_m$= $\sum_{j=1}^{m-1} c_j$+$c_m = \sum_{j=1}^{m} c_j \leq \sum_{j=1}^{m} c_{[j]}$.

$\therefore$ For each $k = 1, 2, \ldots \ldots, n - 1$, we have: $\sum_{j=1}^{k} a_j \leq \sum_{j=1}^{k} c_{[j]}$.

Also, for $k = n$, we have:

$$\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j = \sum_{\substack{j=1 \\ j \neq m, m+1}}^{n} b_j + b_m + b_{m+1} = \sum_{\substack{j=1 \\ j \neq m, m+1}}^{n} c_j + c_m + c_{m+1} = \sum_{j=1}^{n} c_j$$

$\therefore\ a < c$.

Finally, since the vectors a and c have the same components, namely, $a_m = c_m$, we have $a = U_1 U_2 \ldots U_r\, c$ for some T-transforms $U_1, U_2, \ldots, U_r$ by the above case (ii). Therefore, since $c = M_0 b$, $a = U_1 U_2 \ldots U_r M_0 b$, where all $U_1, U_2, \ldots, U_r$ and M_0 are T-transforms.
As a result, in all three cases above, we showed that the vector a is obtained from the vector b by a finite sequence of T-transforms. Since a product of T-transforms is a DSM, the statement is proved. ■

A significant corollary of the previous result is the following theorem:

Theorem 1.5 Let $M_{n\times n} = [m_{ij}]$ be a symmetric matrix, and let the eigenvalues of M be $\mu_1, \mu_2, \ldots, \mu_n$. Then we have:

$$m = [m_{11}, m_{22}, \ldots, m_{nn}] < \mu = [\mu_1, \mu_2, \ldots, \mu_n].$$

Proof. First, since M is symmetric, there is an orthogonal matrix $Q = [q_{ij}]$ s.t.
$M = Q\ diag\,(\mu_1, \mu_2, \ldots, \mu_n)\, Q^T$ by the spectral theorem.

$$\therefore m_{jj} = (Q\ diag(\mu_1, \mu_2, \ldots, \mu_n)\, Q^T)_{jj}$$
$$= (Q\ diag(\mu_1, \mu_2, \ldots, \mu_n))_{j \text{ th row}}\, (Q^T)_{j \text{ th column}}$$
$$= [\, q_{j1}\,\mu_1, q_{j2}\mu_2, \ldots, q_{jn}\mu_n]\,[\, q_{j1}, q_{j2}, \ldots, q_{jn}]^T$$
$$= \sum_{k=1}^{n} (q_{jk}\,\mu_k)\, q_{jk} = \sum_{k=1}^{n} q_{jk}^2 \mu_k,\ j = 1, 2, \ldots, n.$$

Now, since Q is an orthogonal matrix, the columns (and the transpose of the rows) of Q form an orthonormal basis of $\mathbb{R}^n$. In particular, each column (and each row) has a unit length.
∴ The matrix $R_{n\times n}$ with (i, j) entry equals q_{ij}^2 is a DSM.
∴ Since $m^T = [m_{11}, m_{22}, \ldots, m_{nn}]^T = R\, [\mu_1, \mu_2, \ldots, \mu_n]^T$, $m^T < \mu^T$.
∴ $m < \mu$. ■

Definition 1.6 Let Γ be a graph with $V(\Gamma) = [n]$ and $E(\Gamma) = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\}$. Let $L(\Gamma) = [l_{ij}]$ be the square matrix of order n defined as follows: If $i \neq j$, $l_{ij} = -1$ if the vertices i and j are adjacent and $l_{ij} = 0$ otherwise; if $i = j$, $l_{ii} = \rho_i$, where $\rho_i = deg(i)$. Then $L(\Gamma)$ is called the "Laplacian matrix (LM)" of Γ.

Corollary 1.7 Let Γ be a graph with $V(\Gamma) = [n]$. Suppose that $\zeta_1, \zeta_2, \ldots, \zeta_n$ are the eigenvalues of $L(\Gamma)$, and $\rho_1, \rho_2, \ldots, \rho_n$ are the degrees of vertices. Then we have:

$$\rho = [\rho_1, \rho_2, \ldots, \rho_n] < \zeta = [\zeta_1, \zeta_2, \ldots, \zeta_n].$$

Proof. Since $L(\Gamma)$ is symmetric with diagonal entries ρ_i, and eigenvalues ζ_i, $i = 1, 2, \ldots, n$, $\rho < \zeta$ by the previous theorem. ■

Now, we study with vectors $s = [s_1, s_2, \ldots, s_n]^T \in \mathbb{Z}^n$, that is, with vectors whose components are all integers.

Definition 1.8 Let $s_1, s_2, \ldots, s_n \in \mathbb{Z}$, and assume that $s_k > s_l$ for some $1 \leq k \neq l \leq n$. Then, define:

$$s'_k = s_k - 1,\ s'_l = s_l + 1, \text{ and } s'_m = s_m \text{ for all } m \neq k, l.$$

Then s'_1, s'_2, ..., s'_n are said to be gotten from $s_1, s_2, \ldots, s_n$ by a "transfer (from k to l)". Let $s, t \in \mathbb{Z}^n$. If the components of t are gotten from the components of s by a transfer, then t is said to be gotten from s by a "transfer".

Theorem 1.9 Let $s = [s_1, s_2, \ldots, s_n]^T, t = [t_1, t_2, \ldots, t_n]^T \in \mathbb{Z}^n$ Then:

$$s < t \Leftrightarrow s \text{ is gotten from } t \text{ by a finite sequence of transfers.}$$

Proof.

(⇐) Basic Step: Assume that s is gotten from t by only one transfer. Then, by definition of a transfer, s and t differ at only two components, say i and j, s.t. $t_i > t_j$ and $s_i = t_i - 1, s_j = t_j + 1$. Now, let $k \in \{1, 2, \ldots, n-1\}$.

Case 1. $t_i = t_i + 1 \Rightarrow s_i = t_i - 1 = t_j,\ s_j = t_j + 1 = t_i$.

∴ The set of largest k components of s and t are the same.

∴ $\sum_{l=1}^{k} s_{[l]} = \sum_{l=1}^{k} t_{[l]}$.

Case 2. $t_i > t_j + 2 \Rightarrow s_i = t_i - 1 > t_j + 1,\ s_j = t_j + 1 < t_i - 1 \Rightarrow s_j < t_i - 1 = s_i$.

If the set of largest k components of s does not contain s_i (and thus it does not contain s_j), then $\sum_{l=1}^{k} s_{[l]} = \sum_{l=1}^{k} t_{m_l} \leq \sum_{l=1}^{k} t_{[l]}$, where $s_{[l]} = t_{m_l}$ for each $l \in \{1, 2, \dots, k-1\}$.

If the set of largest k components of s does contain s_i, but does not contain s_j, then $\sum_{l=1}^{k} s_{[l]} = \sum_{\substack{l=1 \\ l \neq m_i}}^{k} s_{[l]} + s_i = \sum_{\substack{l=1 \\ l \neq m_i}}^{k} s_{[l]} + (t_i - 1) \leq \sum_{\substack{l=1 \\ l \neq i}}^{k} t_{m_l} + t_i - 1 \leq \sum_{\substack{l=1 \\ l \neq i}}^{k} t_{m_l} + t_i \leq \sum_{l=1}^{k} t_{[l]}$, where $s_{[m_i]} = s_i$ and $s_{[l]} = s_{m_l} = t_{m_l}$ for each each $l \in (\{1, 2, \dots, k-1\} \setminus \{m_i\})$. (Note that $t_{m_l} \neq t_i$ for each $l \neq i$ and this is needed for the last inequality.)

If the set of largest k components of s does contain s_j (and thus it does also contain s_i), then $\sum_{l=1}^{k} s_{[l]} = \sum_{\substack{l=1 \\ l \neq m_i, m_j}}^{k} s_{[l]} + s_i + s_j = \sum_{\substack{l=1 \\ l \neq m_i, m_j}}^{k} s_{[l]} + (t_i - 1) + (t_j + 1)$

$= \sum_{\substack{l=1 \\ l \neq i,j}}^{k} t_{m_l} + t_i + t_j \leq \sum_{l=1}^{k} t_{[l]}$, where $s_{[m_i]} = s_i$, $s_{[m_j]} = s_j$ and $s_{[l]} = s_{m_l} = t_{m_l}$ for each $l \in (\{1, 2, \dots, k\} \setminus \{m_i, m_j\})$. (Note that $t_{m_l} \neq t_i$ and $t_{m_l} \neq t_j$ for each $l \neq i, j$, and this is needed for the last inequality.)

Case 3. $t_i = t_j + 2 \Rightarrow s_i = t_i - 1,\ s_j = t_j + 1 \Rightarrow$ Since $t_i - 1 = t_j + 1, s_i = s_j$. With exactly the same manner as in Case 2, it can be shown easily that $\sum_{l=1}^{k} s_{[l]} \leq \sum_{l=1}^{k} t_{[l]}$. Therefore, in any case, we have: $\sum_{l=1}^{k} s_{[l]} \leq \sum_{l=1}^{k} t_{[l]}$.

Also, for $k = n$, we have:

$$\sum_{l=1}^{n} s_l = \sum_{\substack{l=1 \\ l \neq i,j}}^{n} s_l + s_i + s_j = \sum_{\substack{l=1 \\ l \neq i,j}}^{n} t_l + (t_i - 1) + (t_j + 1) = \sum_{l=1}^{n} t_l.$$

As a result, $s < t$.

∴ If s is gotten from t by a finite number of transfers, then it can be seen easily by applying the Basic Step finite number of times that $s < t$.

(⇒) Suppose that $s < t$, $s \neq t$. And Wlog suppose also that $s_1 \geq s_2 \geq \cdots \geq s_n$ and $t_1 \geq t_2 \geq \cdots \geq t_n$. (We can assume this because the majorization relation is independent of the order of components.)

First, since $s < t$, for each $k = 1, 2, \dots, n-1$, $\sum_{i=1}^{k} s_i = \sum_{i=1}^{k} s_{[i]} \leq \sum_{i=1}^{k} t_{[i]} = \sum_{i=1}^{k} t_i$, and since $s \neq t$, the above inequality is strict for some $k \in \{1, 2, \dots, n-1\}$. Let l be the greatest k s.t. $\sum_{i=1}^{l} s_i < \sum_{i=1}^{l} t_i$. Then $\sum_{i=1}^{l+1} s_i = \sum_{i=1}^{l+1} t_i$ by the maximality of l.

$\therefore s_{l+1} > t_{l+1}$.

On the other hand, since $\sum_{i=1}^{l} s_i < \sum_{i=1}^{l} t_i$, there exists a greatest $p \leq l$ s.t. $s_p < t_p$.

$\therefore s_i \geq t_i$ for $p < i \leq l$ if $p < l$. But since l is the greatest integer s.t. $\sum_{i=1}^{l} s_i < \sum_{i=1}^{l} t_i$, $s_i > t_i$ for $p < i \leq l$ is not possible.

$\therefore s_i = t_i$ for $p < i \leq l$ if $p < l$.

$\therefore t_p > s_p \geq s_{l+1} > t_{l+1}$. (Since $p < l$, $s_p \geq s_{l+1}$.)

$\therefore$ Since all the components of vectors are integers, we have:

$t_p \geq s_p + 1 \geq s_{l+1} + 1 \geq (t_{l+1} + 1) + 1 = t_{l+1} + 2$.

$\therefore t_p \geq t_{l+1} + 2$.

$\therefore t_p - 1 \geq t_{l+1} + 1$.

Now, let t' be gotten from t by the following transfer:

$t'_p = t_p - 1,\ t_{l+1} = t'_{l+1} + 1,\ t'_k = t_k$ for $k \neq p, l+1$.

Then, by the Basic Step above, $t' < t$. And, $s < t'$:

i) Let $k \in \{1, 2, \dots, n-1\}$.

If $k \geq l+1$, then we have:

$\sum_{i=1}^{k} s_i \leq \sum_{i=1}^{k} t_i = \sum_{\substack{i=1 \\ i \neq p, l+1}}^{k} t_i + (t_p - 1) + (t_{l+1} + 1) = \sum_{i=1}^{k} t'_i = \sum_{i=1}^{k} t'_{[i]}$.

(Note that since $t_1 \geq t_2 \geq \cdots \geq t_p \geq \cdots \geq t_{l+1} \geq \cdots \geq t_n$, and since $t_p - 1 \geq t_{l+1} + 1$, the first greatest l+1 elements of t' and those of t are the same. We use this fact in the last equation.)

If $k < p$, then we have:

$\sum_{i=1}^{k} s_i \leq \sum_{i=1}^{k} t_i = \sum_{i=1}^{k} t'_i = \sum_{i=1}^{k} t'_{[i]}$.

(Note that the first greatest $p-1$ elements of t' are $t'_1 = t_1 \geq t'_2 = t_2 \geq \dots \geq t'_p = t_p$. We use this fact in the last equation.)

If $p \leq k \leq l$, then we have:

$\sum_{i=1}^{k} s_i = \sum_{i=1}^{p-1} s_i + s_p + s_{p+1} + \cdots + s_l \leq \sum_{i=1}^{p-1} t_i + (t_p - 1) + t_{p+1} + \cdots + t_l$

(Note that since $s_p < t_p, s_p \leq t_p - 1$, and also note that $s_i = t_i$ for $p < i \leq l$ if $p < l$.)

$\Rightarrow \sum_{i=1}^{k} s_i \leq \sum_{\substack{i=1 \\ i \neq p}}^{k} t_i + t_p - 1 \leq \sum_{i=1}^{k} t'_{[i]}$.

Therefore, in any case, $\sum_{i=1}^{k} s_i \leq \sum_{i=1}^{k} t'_{[i]}$.

ii) $\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} t_i = \sum_{\substack{i=1 \\ i \neq p, l+1}}^{n} t_i + t_p + t_{l+1} = \sum_{\substack{i=1 \\ i \neq p, l+1}}^{n} t'_i + (t_p - 1) + (t_{l+1} + 1)$

$$= \sum_{\substack{i=1 \\ i \neq p, l+1}}^{n} t'_i + t'_p + t'_{l+1} = \sum_{i=1}^{n} t'_i.$$

As a result, $s < t'$.

Therefore, since $s < t'$ and $t' < t$, we have $s < t' < t$.

Also note that for the vectors s and t, we have:

$s_p < t_p$ and $s_{l+1} > t_{l+1}$.

$\Rightarrow s_p \leq t_p - 1$ and $s_{l+1} \geq t_{l+1} + 1$.

$\Rightarrow s_p \leq t'_p$ and $s_{l+1} \geq t'_{l+1}$.

∴ After applying the same procedure finite number of steps, that is after applying a finite sequence of transfers, we obtain s from t. ■

Definition 1.10 Let $s_1, s_2, \ldots, s_n \in \mathbb{Z}^+ \cup \{0\}$. Define

$$s_l^* = \left|\{s_j : s_j \geq l\}\right|, l = 1, 2, \ldots, m = max\{s_1, s_2, \ldots, s_n\}.$$

Therefore, s_l^* is the size of the set consisting of all $s_j \geq l$. The sequence $s_1^*, s_2^*, \ldots, s_m^*$, where $m = max\{s_1, s_2, \ldots, s_n\}$, is called the ″conjugate sequence″ of the sequence $s_1, s_2, \ldots, s_n$.

Definition 1.11 Let $s_i \in \mathbb{Z}^+ \cup \{0\}$ for $i = 1, 2, \ldots, n$, and let $s_1 \geq s_2 \geq \cdots \geq s_n$. The ″Ferrers diagram″ of $s_1, s_2, \ldots, s_n$ is the diagram consisting of $s_1 + s_2 + \cdots + s_n$ square boxes s.t. all rows are left-justified and l th row comprises of s_l square boxes. If some $s_l = 0$, then remove the l th row.

Note 1.12 $s_l^* = \left|\{s_j : s_j \geq l\}\right|$ is equal to the number of square boxes in the l th column of the Ferrers diagram of the nonnegative integers $s_1 \geq s_2 \geq \cdots \geq s_n$. Therefore, since counting the square boxes row by row, and respectively, column by column, gives the same number, we have:

$$\sum_{l=1}^{n} s_l = \sum_{l=1}^{m} s_l^* .$$

Now, we can prove the following theorem:

Theorem 1.13 (Gale-Ryser Theorem) (Gale (1957), Ryser (1957)) let $a_i, b_j \in \mathbb{Z}^+ \cup \{0\}$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, s.t. $n \geq a_1 \geq a_2 \geq \cdots \geq a_m$, $b_1 \geq b_2 \geq \cdots \geq b_n$ and $\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_j$. Then there is a (0, 1)-matrix $M_{m\times n}$ with the i th row sum (respectively, the j th column sum) is a_i (respectively, b_j) for each $i = 1, 2, \dots, m$ (respectively, $j = 1, 2, \dots, n$) iff $a_1^*, a_2^*, \dots, a_n^*$ majorizes $b_1, b_2, \dots, b_n$.

Proof.

(⇒) Let $M = [m_{ij}]$ be a (0-1)-matrix stated as in the theorem. Wlog, we can suppose that $b_1 \geq b_2 \geq \cdots \geq b_n$ (by permuting the columns of M if it is needed).

Now, assume that there are i_0, j_0 s.t. $a_{i_0 j_0} = 0$ and $a_{i_0(j_0+1)} = 1$. Then define the $m \times n$ matrix $N = [n_{ij}]$ as follows:

$n_{i_0 j_0} = 1$, $n_{i_0(j_0+1)} = 0$, and $n_{ij} = m_{ij}\ for\ i \neq i_0, j \neq j_0$.

Denote the j th column sum of N by b_j' for each $j = 1, 2, \dots, n$. Then, we obviously have:

$b_j' = b_j + 1$, $b_{j+1}' = b_j - 1$, and $b_l' = b_l$ for $l \neq j,\ j+1$.

∴ $b_1, b_2, \dots, b_n$ is gotten from $b_1', b_2', \dots, b_n'$ by a transfer from $j_0 + 1$ to j_0.

∴ $b_1', b_2', \dots, b_n'$ majorizes $b_1, b_2, \dots, b_n$ by the previous theorem.

After repeating this procedure a finite number of times, we get a matrix $\widetilde{M}_{m\times n}$ s.t. the i th row sum of $\widetilde{M}$ is a_i for each $i = 1, 2, \dots, m$; and, i th row of $\widetilde{M}$ is of the form $(1, 1, \dots, 1, 0, 0, \dots, 0)$, where there are r_i 1 s and $n - r_i$ 0 s; and also, the column sums, say $\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n$, majorize $b_1, b_2, \dots, b_n$.

Finally, by using the Ferrers diagram, we see that $\tilde{b}_i = a_i^*$ for each $i = 1, 2, \dots, n$. Therefore, $a_1^*, a_2^*, \dots, a_n^*$ majorizes $b_1, b_2, \dots, b_n$.

(⇐) (This part of the proof is taken from Marshall and Olkin (1979).)

We construct a required matrix M explicitly. We start with row 1 and successively with the residual rows, scatter ones in the matrix in the following way:

In the row i, put a one in the column 1 if < b_1 ones have already been put in column 1 and $a_i \geq 1$; put a one to the column 2 if < b_2 ones have already been put in the column 2 and < a_i ones have been put in the row i. Generally, put a one in the column j (as (i, j)-entry) if < b_j ones have been put in the column j and rows $1, 2, \ldots, i-1$; and put a one in the row i (as the (i, j)-entry) if < a_i ones have been put in row i and columns $1, 2, \ldots, j-1$.

Of course, we need to show that this construction can be accomplished every time. This can be proved by induction on m (the number of rows):

$m = 1$: Let $a_1 = k$. Then: $a_1^* = a_2^* = \cdots = a_k^* = 1$, and $a_{k+1}^* = a_{k+2}^* = \cdots = a_n^* = 0$. And since $(b_1, b_2, \ldots, b_n) < (a_1^*, a_2^*, \ldots, a_n^*) = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$, $b_i = 0$ or 1 for each $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} b_i = \sum_{i=1}^{n} a_i^* = k$. Therefore, since $b_1 \geq b_2 \geq \ldots \geq b_n$,

$$b_1 = b_2 = \cdots = b_k = 1 \text{ and } b_{k+1} = b_{k+2} = \cdots = b_n = 0.$$

As a result, $M = [1, 1, \ldots, 1, 0, 0, \ldots, 0]$ is the matrix satisfying the required condition. (The first k entry of M is 1 and the remaining entries are all zero.)

Now, assume that this construction can be accomplished with $m-1$ rows every time. Let $r_1 = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$, where the first a_1 coordinates are 1 and the last $n - a_1$ coordinates are 0, be the first row of M. To accomplish the construction, we need $(m-1) \times n$ $(0-1)$ matrix with row sums $a_2, a_3, \ldots, a_m$ and column sums

$$b_1 - 1, b_2 - 1, \ldots, b_{a_1} - 1, b_{a_1+1}, b_{a_1+2}, \ldots, b_n.$$

Now, to use the induction assumption, we must show that the hypotheses of the theorem are satisfied:

i) $n \geq a_1 \geq a_2 \geq \cdots \geq a_m \Rightarrow n \geq a_2 \geq a_3 \geq \cdots \geq a_m$.

And say $\tilde{a}_k = a_{k+1}$ for each k =1, 2, ..., m−1.

ii) $b_1 \geq \cdots \geq b_n \Rightarrow b_1 - 1 \geq b_2 - 1 \geq \ldots \geq b_{a_1} - 1$ and $b_{a_1+1} \geq b_{a_1+2} \geq \cdots b_n$.

(We know that $b_{a_1} \geq b_{a_1+1}$. But if $b_{a_1} = b_{a_1+1}$, then $b_{a_1} - 1 < b_{a_1+1}$.)

Therefore, we may need to reorder these numbers to get the non-increasing sequence of them. But instead of such a reordering, we can proceed with these numbers. In fact, the non-increasing assumption given in the statement is just for the simplicity of the notation, and thus it can be ignored.

iii) $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \Rightarrow \sum_{i=2}^{m} a_i = (\sum_{j=1}^{n} b_j) - a_1 = \sum_{j=1}^{a_1}(b_j - 1) + \sum_{j=a_1+1}^{n} b_j$.

iv) The following majorization relation must hold:

$$(b_1 - 1, b_2 - 1, \dots, b_{a_1} - 1, b_{a_1+1}, \dots, b_n) < (\tilde{a}_1^*, \tilde{a}_2^*, \dots, \tilde{a}_n^*).$$

Now, for each $i = 1, 2, \dots, a_1$, we have:

$$\tilde{a}_i^* = |\{\tilde{a}_j : \tilde{a}_j \geq i\}| = a_i^* - 1,$$

and for each $i = a_1 + 1, a_1 + 2, \dots, n$, we have:

$$\tilde{a}_i^* = a_i^*.$$

For $1 \leq i \leq a_1$, the sequence $s_1 : \tilde{a}_1 = a_2 \geq \tilde{a}_2 = a_3 \geq \cdots \geq \tilde{a}_{m-1} = a_m$ dose not contain a_1 and $a_1 \geq a_2$. Therefor, for $1 \leq i \leq a_1$ "the number of terms in the sequence $s_2 : a_1 \geq a_2 \geq \cdots \geq a_m$ which are $\geq i$." Is equal to the "the number of terms in the sequence s_1: $a_2 \geq a_3 \geq \cdots \geq a_m$ which are $\geq i + 1$".

$\therefore \tilde{a}_i^* = a_i^* - 1$ for $1 \leq i \leq a_1$.

For $a_1 + 1 \leq i \leq n$, "the number of tums in the sequence s_1 and s_2 which one $\geq i$" is equal.

$\therefore \tilde{a}_i^* = a_i^*$ for ${}_1 + 1 \leq i \leq n$.

$\therefore (\tilde{a}_1^*, \tilde{a}_2^*, \dots, \tilde{a}_n^*) = (a_1^* - 1, a_2^* - 1, \dots, a_{a_1}^* - 1, a_{a_1}^* + 1, \dots, a_n^*)$.

$\therefore$ We must show that the following majorization relation holds:

$$(b_1 - 1, \dots, b_{a_1} - 1, b_{a_1+1}, \dots, b_n) < (a_1^* - 1, \dots, a_{a_1}^* - 1, a_{a_{1+1}}^*, \dots, a_n^*).$$

And this majorization relation holds by the lemma given after this theorem.

Therefore, all the hypotheses of the theorem are satisfied.

$\therefore$ By the induction assumption, there is an $(m - l) \times n(0,1) -$ matrix, say N, s.t. the i th row sum of N is a_{i+1} for each $i = 1, 2, \dots, m - 1$; and the j th column sum of N is $b_j - 1$ for each $j = 1, 2, \dots, a_1$ and is b_j for each $j = a_1 + 1, a_1 + 2, \dots, n$.

As a result, the block matrix $M = \begin{bmatrix} r_1 \\ N \end{bmatrix}$ is a matrix with the i th row sum a_i for $i = 1, 2, \dots, m$ and the j th column sum b_j for $j = 1, 2, \dots, n$. ■

Lemma 1.14 (Fulkerson and Ryser (1962).) Let $x_1 \geq x_2 \geq \cdots \geq x_n,\ y_1 \geq y_2 \geq \cdots \geq y_n$ be integers. Reduce components. in positions $k_1, k_2, \ldots, k_m$ by 1 in the vector $x = (x_1, x_2, \ldots, x_n)$, and call the resulting vector by z. Similarly, reduce components in positions $l_1, l_2, \ldots, l_m$. by 1 in the vector $y = (y_1, y_2, \ldots, y_n)$, and call the resulting vector by w. If $k_1 \leq l_1, k_2 \leq l_2, \ldots, k_m \leq l_m$, and if $x < y$, then $z < w$.

Proof. (This proof is taken from Marshall and Olkin (1979).)

We prove the theorem for $m = 1$, and the general case can be obtained by the repeated application of the $m = 1$ case.

For simplicity of the notation, let $k = k_1, l = l_1$, and let e_j be the vector with 1 in the *j* th coordinate and zeros elsewhere.

Under this setting, we want to prove the following:

"If $k \leq l$, and if $x < y$, then $z = x - e_k < y - e_l = w$".

First of all, the components of the vectors z and w need not be in non-increasing order. But if we choose $k' \geq k$ and $l' \geq l$ s.t.

$$x_k = x_{k+1} = \cdots = x_{k'-1} = x_{k'} \text{ and either } x_{k'} > x_{k'+1} \text{ or } k' = n,$$

$$y_l = y_{l+1} = \cdots = y_{l'-1} = y_{l'} \text{ and either } y_{l'} > y_{l'+1} \text{ or } l' = n,$$

then the vectors $x - e_{k'}$, and $y - e_{l'}$ have the components reordered decreasingly with the vectors $x - e_k$ and $y - e_l$, respectively.

∴ Since reordering of components does not have any effect on the majorization relation, instead of showing $z = x - e_k < y - e_l = w$, it is equivalent and more convenient to show that

$$u = x - e_{k'} < y - e_{l'} = v.$$

i) If $m \leq min\,\{k', l'\}$, then: $\sum_{i=a}^{m} u_i = \sum_{i=1}^{m} x_i \leq \sum_{i=1}^{m} y_i = \sum_{i=1}^{m} v_i$.

ii) If $m \geq max\,\{k', l'\}$, then: $\sum_{i=1}^{m} u_i = (\sum_{i=1}^{m} x_i) - 1 \leq (\sum_{i=1}^{m} y_i) - 1 = \sum_{i=1}^{m} v_i$.

iii) If $k' \leq l'$ and $k' \leq m < l'$, then:

$$\sum_{i=1}^{m} u_i = (\sum_{i=1}^{m} x_i) - 1 \leq (\sum_{i=1}^{m} y_i) - 1 < \sum_{i=1}^{m} y_i = \sum_{i=1}^{m} v_i.$$

iv) If $k' > l'$ and $l' \leq m < k'$, then again $\sum_{i=1}^{m} u_i \leq \sum_{i=1}^{m} v_i$:

First, since $\sum_{i=1}^{m} u_i = \sum_{i=1}^{m} x_i$ and $\sum_{i=1}^{m} v_i = (\sum_{i=1}^{m} y_i) - 1$,

$\sum_{i=1}^{m} u_i \leq \sum_{i=1}^{m} v_i \Leftrightarrow \sum_{i=1}^{m} x_i \leq (\sum_{i=1}^{m} y_i) - 1 \Leftrightarrow \sum_{i=1}^{m} x_i < \sum_{i=1}^{m} y_i$.

∴ It is equivalent to show that $\sum_{i=1}^{m} x_i < \sum_{i=1}^{m} y_i$:

Subcase (a). If $x_{m+1} > y_{m+1}$, then:

$\sum_{i=1}^{m}(y_i - x_i) > (\sum_{i=1}^{m}(y_i - x_i)) + (y_{m+1} - x_{m+1}) = (\sum_{i=1}^{m+1}(y_i - x_i)) \geq 0$

(The last inequality holds, because $x < y$.)

$\therefore \sum_{i=1}^{m}(y_i - x_i) > 0 \Leftrightarrow \sum_{i=1}^{m} x_i < \sum_{i=1}^{m} y_i$.

Subcase (b). $\therefore$ The only remaining case is $x_{m+1} < y_{m+1}$:

Since $k \leq l \leq l' \leq m < k'$, we have:

$x_{l'} = x_{l'+1} = \cdots = x_m = x_{m+1} \leq y_{m+1} \leq y_m \leq \cdots \leq y_{l'+1} < y_{l'}$.

$\therefore 0 < \sum_{i=l'}^{m}(y_i - x_i)$ $(x_{l'} < y_{l'}, x_{l'+1} \leq y_{l'+1}, \ldots, x_m \leq y_m.)$

$\leq \sum_{i=l'}^{m}(y_i - x_i) - \sum_{i=l'}^{n}(y_i - x_i)$

$(x < y \Rightarrow \sum_{i=1}^{s}(y_i - x_i) \geq 0$ for any $1 \leq s \leq n - 1$ and $\sum_{i=1}^{n}(y_i - x_i) = 0$

$\Rightarrow \sum_{i=1}^{l'-1}(y_i - x_i) + \sum_{i=l'}^{n}(y_i - x_i) = 0$

$\Rightarrow$ Since $\sum_{i=1}^{l'-1}(y_i - x_i) \geq 0$, $\sum_{i=l'}^{n}(y_i - x_i) \leq 0$.)

$= -\sum_{i=m+1}^{n}(y_i - x_i)$.

$= \sum_{i=1}^{m}(y_i - x_i)$ $(\sum_{i=1}^{m}(y_i - x_i) + \sum_{i=m+1}^{n}(y_i - x_i) = 0 \Longrightarrow \sum_{i=1}^{m}(y_i - x_i)$

$= -\sum_{i=m+1}^{n}(y_i - x_i)$.

$\therefore \sum_{i=1}^{m}(y_i - x_i) > 0 \Leftrightarrow \sum_{i=1}^{m} x_i < \sum_{i=1}^{m} y_i$. ■

Corollary 1.15 (A corollary of the previous theorem)

Let Γ be a graph with $V(\Gamma) = [n]$, and let $deg(j) = \rho_j$ for each $j = 1, 2, \ldots, n$. Then, $\rho_1^*, \rho_2^*, \ldots, \rho_n^*$ majorizes $\rho_1, \rho_2, \ldots, \rho_n$.

Proof. $A(\Gamma)$ is a (0, 1)-matrix s.t. both *i* th row sum and *i* th column sum of $A(\Gamma)$ is p_i for each $i = 1, 2, \ldots, n$.

$\therefore$ By the previous theorem, $\rho_1^*, \rho_2^*, \ldots, \rho_n^*$ majorizes $\rho_1, \rho_2, \ldots, \rho_n$. ■

2. LAPLACIAN EIGENVALUES OF THRESHOLD GRAPHS

2.1 Threshold Graphs

Threshold graphs are encountered in various areas and their structures are very interesting. In this main chapter of the thesis, we study the basic properties of these graphs and their Laplacian eigenvalues.

Definition 2.1 Let Γ be a graph with $V(\Gamma) = [n]$, and let $j \in V(\Gamma)$. Then j is called "dominating" if $jl \in E(\Gamma)$ for every $l \in [n]\backslash\{j\}$.

Definition 2.2 Let Γ be a graph with $V(\Gamma) = [n]$. Assume that Γ is constructed recursively as follows:

Begin with K_1.

Then apply the following (i) or (ii) process finitely many times in any order: Let Γ be denote the present graph at each step, and let K_1 denote a new vertex not in $V(\Gamma)$.

i) $\Gamma + K_1$.

ii) $\Gamma \vee K_1$.

Then Γ is called a "threshold graph (TG)".

Now, let Γ be a given graph. How can we understand whether Γ is a TG or not? Is there a recursive process or an algorithmic procedure to determine whether $\Gamma$is a TG or not? The answer is affirmative:

Case 1. Γ is connected.

The first necessary condition for Γ to be a TG is to have a dominant $v \in V(\Gamma)$. Then the second necessary condition is that $\Gamma\backslash\{v\}$ has only one nontrivial component, say $\Omega \subseteq \Gamma\backslash\{v\}$ (and, there may be some trivial components). Moreover, Γ is a TG iff Ω is a TG.

Case 2. Γ is disconnected.

The first necessary condition for Γ to be a TG graph is that Γ has only one nontrivial component, say $\Omega \subseteq \Gamma$ (and, there may be some other trivial components). Moreover, Γ is a TG iff Ω is a TG.

Now, to prove the main result of this section, we first state and prove two lemmas:

Lemma 2.3 Let Γ be a graph with $V(\Gamma) = [n]$. Assume that the eigenvalues of $L(\Gamma)$ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$.Then for any$x \in \mathbb{R}$, $L + xJ$ has eigenvalues: $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$ and nx.

Proof. First of all, since L is symmetric, it is orthogonally diagonalizable. That is, there is an orthogonal matrix Q s.t. each of the columns of Q is an eigenvector of L.

Wlog, assume that the last column of Q is $\left[\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right]^T$. (By the definition of L, $L\overline{1} = 0 = 0\,\overline{1}$. Therefore, $\overline{1}$ is an eigenvector of L belonging to the eigenvalue 0. And since $||\overline{1}|| = \sqrt{n}$, $\frac{1}{\sqrt{n}}\,\overline{1}$ a unit eigenvector of L corresponding to the eigenvalue 0.) Then:

$$Q^T L Q = diag\,(\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n = 0).$$

Now, let $Q = [c_1, c_2, \dots, c_n]$, where c_j is the j th column of Q. Then, by definition of an orthogonal matrix, $||c_i|| = 1$ for any $i = 1, 2, \dots, n$, and $c_i.c_j = 0$ for any $i \neq j \in \{1, 2, \dots, n\}$. In particular the vector $\overline{1}$ is orthogonal to each column of Q except for the last column. Therefore, by the usual matrix product, we have:

$$JQ = \begin{bmatrix} 0 & 0 & \dots & 0 & \sqrt{n} \\ 0 & 0 & \dots & 0 & \sqrt{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \sqrt{n} \end{bmatrix}$$

$\therefore$ $Q^T L Q = diag\,(0, 0, \dots, 0, n)$ by the usual matrix product.

$\therefore$ $Q^T(L + xJ)Q = Q^T L Q + x\ Q^T J\,Q$

$= diag\,(\mu_1, \mu_2, \dots, \mu_{n-1}, 0) + x\, diag\,(0, 0, \dots 0, n)$

$= diag\,(\mu_1, \mu_2, \dots, \mu_{n-1}, nx)$.

$\therefore$ The eigenvalues of $L + xJ$: $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$ and nx. ■

Lemma 2.4 Let Γ be a graph with $V(\Gamma) = [n]$. Let $deg\ (j) = \rho_j$, for each $j = 1, 2, \ldots, n$, and assume that $n-1 = \rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. Let $\Omega = \Gamma \backslash \{1\}$. Then one of the eigenvalues of $L(\Gamma)$ is n. Moreover, assume that the eigenvalues of $L(\Gamma)$ are μ_2, $\mu_3, \ldots, \mu_{n-1}, n, 0$. Then, $\mu_2 - 1, \mu_3 - 1, \ldots, \mu_{n-1} - 1$, 0 are the eigenvalues of $L(\Omega)$.

Proof. First, by definition of Laplacian, $L(\Gamma) = \begin{bmatrix} n-1 & -\bar{1}^T \\ -\bar{1} & L(\Omega) + I_{n-1} \end{bmatrix}$, where $\bar{1} = [1, 1, \ldots, 1]^T \in \mathbb{R}^{n-1}$. (Since $\Omega = \Gamma \backslash \{1\}$, and since $deg_\Gamma\ (1) = n-1$, $deg_\Omega\ (j)\ deg_\Gamma\ (j)$ -1 for each j = 2, 3, .., n. Therefore, we add I_{n-1} to the block matrix corresponding to $L(\Omega)$.)

$\therefore L(\Gamma) + J_n = \begin{bmatrix} n & \bar{0}^T \\ \bar{0} & L(\Omega) + I_{n-1} + J_{n-1} \end{bmatrix}$, where J_n is the $n \times n$ square matrix with all entries equal to 1, and $\bar{0} = [0, 0, \ldots, 0]^T \in \mathbb{R}^{n-1}$.

Now, by the previous lemma, since the eigenvalues of $L(\Gamma)$ are $\mu_2, \mu_3, \ldots, \mu_{n-1}, n, 0$, the eigenvalues of $L(\Gamma) + J_n$ are $\mu_2, \mu_3, \ldots, \mu_{n-1}$ and n with multiplicity 2.

$\therefore$ Since $L(\Gamma) + J_n = \begin{bmatrix} n & \bar{0}^T \\ \bar{0} & L(\Omega) + I_{n-1} + J_{n-1} \end{bmatrix}$, the eigenvalues of $L(\Omega) + I_{n-1} + J_{n-1}$ are $\mu_2, \mu_3, \ldots, \mu_{n-1}, n$.

$\therefore$ The eigenvalues of $L(\Omega) + J_{n-1}$ are $\mu_2 - 1, \mu_3 - 1, \ldots, \mu_{n-1} - 1, n-1$.

($det\ (\mu I_{n-1} - (L(\Omega) + J_{n-1} + I_{n-1}))$ = det $((\mu - 1)I_{n-1} - (L(\Omega) + J_{n-1}))$.

$\therefore \mu$ is an eigenvalue of $L(\Omega) + J_{n-1} + I_{n-1}$ iff $\mu - 1$ is an eigenvalue of $L(\Omega) + J_{n-1}$.)

$\therefore$ By the previous lemma again, the eigenvalues of $L(\Omega)$ are $\mu_2 - 1, \ldots, \mu_{n-1} - 1, 0$. ■

Theorem 2.5 Let Γ be a graph with $V(\Gamma) = [n]$. Let $deg(j) = \rho_j$, for each $j = 1$, $2, \ldots, n$. Then the eigenvalues of $L(\Gamma)$ are $\rho_1^*, \rho_2^*, \ldots, \rho_n^*$ iff Γ is a TG.

Proof.

(⇐) Proof can be done by induction on the number of vertices.

$n = 1$: First of all, since we study only simple graphs, $L(\Gamma) = [0]$. And the only eigenvalue of the zero matrix is 0. Also, $deg(1) = 0 = \rho_1$ and $\rho_1^* = |\{\rho_j : \rho_j \geq 1\}| = 0$, because there is only are ρ_j which is ρ_1 and $\rho_1 = 0$. As a result, the only eigenvalue of $L(\Gamma)$ is $0 = \rho_1^*$.

Suppose that the result holds for all TGs with the number of vertices $\leq n-1$.

Now, let Γ be a TG with $|V(\Gamma)| = n$.

First of all, it is enough to demonstrate the conclusion for a connected case, because each vertex of degree zero appends a 0 both to the sequence of degrees of Γ and to the list of eigenvalues of $L(\Gamma)$, simply because $0^* = 0$.

∴ Wlog, suppose that Γ is connected.

Now, by definition of being a TG, Γ has a vertex with degree $n-1$, say $deg(1) = n-1$.

Let $\Omega = \Gamma \backslash \{1\}$.

Let $\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n = 0$ be the eigenvalues of $L(\Gamma)$. Then, by the previous lemma, one of $\mu_j = n$ for some $j \in \{1, 2, \dots, n-1\}$, say $\mu_1 = n$.

And by the previous lemma, $\mu_2 - 1, \mu_3 - 1, \dots, \mu_{n-1} - 1, 0$ are the eigenvalues of $L(\Omega)$.

As mentioned above, if k vertices each of which has degree 0, are added to a graph, then k 0s are appended both to the sequence of degrees of the graph and the list of eigenvalues of the Laplacian of the graph.

∴ If the statement of the theorem holds for a graph, then it also holds if we add some vertices, each of which has degree 0, to that graph. As a result, since Ω has only one nontrivial component, which is threshold, and there may be some other trivial components; the eigenvalues of $L(\Omega)$, namely, $\mu_2 - 1, \mu_3 - 1, \dots, \mu_{n-1} - 1, 0$, satisfy the statement of the theorem by the induction hypothesis. That is, since ρ_2-1, ρ_3-1,…, $\rho_n - 1$ is the sequence of degrees of the vertices in Ω, $(\rho_j - 1)^* = \mu_j - 1$ for each $j = 2, 3, \dots, n-1$, $(\rho_n - 1)^* = 0$. On the other hand, since $\rho_1 = n-1$ (that is, each vertex is adjacent to the vertex 1, and thus $\rho_j \geq 1$ for each $j = 1, 2, \dots, n$), $\rho_1^* = |\{\rho_j : \rho_j \geq 1\}| = n$. Therefore, since $\mu_1 = n$, $\rho_1^* = \mu_1$. Finally, for clearness and simplicity of the notation, say $\tau_{j-1} = \rho_j - 1$ for each $j = 2, 3, \dots, n-1$. Then we have:

$$
\begin{aligned}
(\rho_j - 1)^* = \tau_{j-1}^* &= |\{\tau_l : \tau_l \geq j-1\}| \\
&= |\{\rho_{l+1} - 1 : \rho_{l+1} - 1 \geq j-1\}| \\
&= |\{\rho_{l+1} - 1 : \rho_{l+1} \geq j\}| \\
&= |\{\rho_k - 1 : \rho_k \geq j\}| \\
&= |\{\rho_k : \rho_k \geq j\}|
\end{aligned}
$$

The last equation is true because the matter is not the set, the matter is only the number of elements in the set.

$\therefore (\rho_j - 1)^* = |\{\rho_k\colon \rho_k \geq j\}|$ for the sequence $\rho_2 - 1, \rho_3 - 1, \dots, \rho_{n-1} - 1, 0$, for each $j = 2, 3, \dots, n$.

$\therefore$ For the sequence $n - 1 = \rho_1, \rho_2, \rho_3, \dots, \rho_{n-1}, \rho_n = 0$, we have:

$$\rho_j^* = |\{ \rho_l\colon \rho_l \geq j\}| = (\rho_j - 1)^* + 1, \text{for } j = 2, 3, \dots, n,$$

because ρ_1 is the largest possible number in the sequence $\rho_1, \rho_2, \dots, \rho_n = 0$, and thus $\rho_l \geq j$ implies $\rho_1 \geq j$. ($\therefore$ We add +1 to $(\rho_j - 1)^*$ to find ρ_j^*.)

$\therefore \rho_j^* = (\rho_j - 1)^* + 1 = (\mu_j - 1) + 1 = \mu_j$ for each $j = 2, 3, \dots, n$.

And we know that $\rho_1^* = n = \mu_1$.

As a result, $\rho_j^* = \mu_j$ for each $j = 1, 2, \dots, n$.

($\Rightarrow$)(This part of the proof is taken from Merris (1994).)

The proof can be done easily by induction on the number of vertices *n* of Γ, and by using the recursive definition of a TG.

Step 1. First of all, if $\Gamma = K_n^c$, then Γ is a TG by definition.

If $\Gamma \neq K_n^c$, then there exists, say $m > 1$ vertices of Γ with vertex degree > 0. Let Ω be the subgraph of Γ induced by these m vertices. Then:

$$L(\Gamma) = \begin{bmatrix} L(\Omega) & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & 0_{n-m} \end{bmatrix}.$$

$\therefore$ For each $i = 1, 2, \dots, m$ for the following first two equations and for each $i = 1, 2, \dots, m - 1$ for the following last equation, we have:

$\mu_i\,(\Gamma) = \mu_i\,(\Omega), \rho_i\,(\Gamma) = \rho_i\,(\Omega), \rho_i^*\,(\Gamma) = \rho_i^*\,(\Omega)$,

where $\mu_i\,(\Gamma), \rho_i(\Gamma)$, and $\rho_i^*(\Gamma)$ are the *i* th largest eigenvalue, the *i* th largest degree (in the degree sequence), and the *i* th conjugate element (in the conjugate sequence) of Γ. Similarly, $\mu_i(\Omega), \rho_i(\Omega)$, and $\rho_i^*(\Omega)$ denote the same notions for Ω.

Step 2. Now, $\rho_1^*(\Omega) = m$. ($\rho_1^*(\Omega) = |\{\rho_i(\Omega) | \rho_i(\Omega) \geq 1\}| = m$, because every vertex of Ω has a degree > 0, i.e., has a degree ≥ 1.) Therefore, since $\mu_1(\Gamma) = \rho_1^*(\Gamma)$ by hypothesis and $\mu_1(\Omega) = \mu_1(\Gamma)$, $\mu_1(\Omega) = \rho_1^*(\Omega) = \rho_1^*(\Gamma) = m$.

Step 3. Furthermore, for any graph Λ,

$$\mu_i(\Lambda) = n - \mu_{n-i}(\Lambda^c), 1 \le i < n \ (***):$$

$L(\Lambda) + L(\Lambda^c) = nI_n - J_n$ by the definition of LM.

Therefore, since $L(\Lambda)$ commutes with itself, with I_n, and with J_n, and since $L(\Lambda^c) = nI_n - J_n - L(\Lambda), L(\Lambda)$ commutes with $L(\Lambda^c)$. Then it is a well-known fact from Linear Algebra that the matrices $L(\Lambda)$ and $L(\Lambda^c)$ are simultaneously triangularizable. That is, there exists an invertible matrix, P s.t. $P^{-1}L(\Lambda)\,P$ and $P^{-1}L(\Lambda^c)\,P$ are both triangular matrices. Now, the eigenvalues of a triangular matrix are the main diagonal entries of that matrix. Therefore, we have:

$P^{-1}(L(\Lambda) + L(\Lambda^c))\,P = P^{-1}\,(nI_n - J_n)\,P$

$\Rightarrow P^{-1}\,L\,(\Lambda)\,P + P^{-1}\,L\,(\Lambda^c)\,P = nI_n - P^{-1}J_n\,P.$

On the other hand, the eigenvalues of $nI_n - P^{-1}J_n\,P$ are n with multiplicity $n-1$ and 0 with multiplicity 1. (First, since J_n and $P^{-1}J_n\,P$ are similar matrices, they have the same eigenvalues. J_n is a symmetric matrix and rank(J_n) = 1. Therefore, there is only one nonzero eigenvalue of J_n. Thus, since the sum of all the eigenvalues is equal to the trace, and since ($n-1$) of the eigenvalues of J_n is 0, the unique nonzero eigenvalue of J_n must be $n = tr(J_n)$. Therefore, the eigenvalues of J_n are 0 with multiplicity $n-1$ and n with multiplicity 1. As a result, the eigenvalues of $nI_n - J_n$ are n with multiplicity $n-1$ and $n - n = 0$ with multiplicity 1: $det(I_n - (nI_n - J_n)) = det\,((x-n)I_n - (-J_n))$. Now, $det(xI_n - (-J_n)) = 0 \Leftrightarrow x = 0$ with multiplicity $n-1$ and $x = -n$ with multiplicity 1. Therefore, $det\,((x-n)\,I_n - (-J_n)) = 0 \Leftrightarrow (x-n) = 0$ with multiplicity $n-1$ and $(x-n) = -n$ with multiplicity 1 $\Leftrightarrow x = n$ with multiplicity $n-1$ and $x = 0$ with multiplicity 1.)

∴ The eigenvalues of $P^{-1}\,L\,(\Lambda)\,P + P^{-1}\,L\,(\Lambda^c)\,P$ are n with multiplicity $n-1$ and 0 with multiplicity 1.

∴ Since the eigenvalues of $P^{-1}\,L(\Gamma)\,P + P^{-1}\,L(\Gamma^c)\,P$, which are the same as the eigenvalues of $L(\Gamma) + L(\Gamma^c)$, are the main diagonal entries, we have:

$$\mu_i\,(\Gamma) + \mu_{n-i}\,(\Gamma^c) = n \quad \text{for each} \quad 1 \le i < n. \qquad (*)$$

(Since $(P^{-1}L(\Gamma)\ P)_{ii}$ + $(P^{-1}L(\Gamma^c)\ P)_{ii}$= n for each $i = 1, 2, \dots, n-1$, the constant n; if one of them is large, then the other must be small. That is: $\mu_1\ (\Gamma) + \mu_{n-1}\ (\Gamma^c) = n$, $\mu_2\ (\Gamma) + \mu_{n-2}\ (\Gamma^c) = n, \dots, \mu_{n-1}\ (\Gamma) + \mu_1\ (\Gamma^c) = 0$.)
In addition, we know that the remaining eigenvalue of $L\ (\Gamma)$ is 0 with multiplicity 1 and $\overline{1}$ is one of the corresponding eigenvectors.

Step 4. Now, if we apply (*) to Ω, then we get:
$\mu_i\ (\Omega) + \mu_{m-i}(\Omega^c) = m$ for each $1 \leq i < m$.
∴ Since $\mu_1(\Omega^c) = m$, $\mu_{p-1}\ (\Omega) = 0$.
∴ Ω^c is a disconnected graph.
∴ Since a graph and its complement can not be both disconnected, Ω must be connected.
∴ $\mu_{m-1}(\Omega) \neq 0$.
∴ Since $\mu_{m-1}\ (\Omega) = \rho^*_{m-1}\ (\Omega)$ by the hypothesis, $\rho^*_{m-1}\ (\Omega) \neq 0$.
∴ Since $\rho^*_{m-1}\ (\Omega) =|\{\rho_i\ (\Omega) :\ \rho_i\ (\Omega) \geq\ m-1\}|$, and since $\Delta(\Omega) \leq\ m-1$ (because Ω contain m vertices), $\Delta(\Omega) = m-1$.
∴ $\rho_1(\Omega) = m-1$.

Step 5. Now, suppose that there are s vertices of degree $m-1$ in Ω.
Then for a uniquely determined subgraph Λ of Ω, we have :
$\Omega = \Lambda\, V K_s$.

Step 6. ∴ $\mu_{s+i}\ (\Omega) = \mu_i\ (\Lambda) + s,\ 1 \leq\ i < t$, where $t = m - s$ is $|V(\Lambda)|$:
First of all, for any two graphs φ and ψ we have $(\varphi\ V\ \psi) = (\varphi^c + \psi^c)^c$:
$V(\varphi\ V\ \psi\ ) = V(\varphi) \cup V(\psi\ ) = V((\varphi^c + \psi^c)^c\ )$.
$e = \{\ \alpha, \beta\} \in E\ ((\varphi^c + \psi^c)^c\ ) \Leftrightarrow e = \{\ \alpha\ , \beta\ \} \notin E\ (\varphi^c + \psi^c)$
$\Leftrightarrow$ $e \notin E(\varphi^c)$ or $e \notin E\ (\psi^c)$ or "one of the end vertices of e, say α, is in $V(\varphi^c) = V(\varphi)$ and β is in $V(\psi^c) = V(\psi)$".
$\Leftrightarrow$ $e \in E(\varphi)$ or $e \in E(\psi)$ or "$\alpha \in V(\varphi)$ and $\beta \in V(\psi)$ ".
$\Leftrightarrow e\ \in E(\varphi\ V\ \psi)$
∴ $(\varphi\ V\ \psi) = (\varphi^c + \psi^c)^c$.

Now, by using the equation given in Step 3, we can prove the required equation:

$\mu_{s+i}(\Omega) = \mu_{s+i}\ (\Lambda V K_s) = \mu_{s+i}((\Lambda^c + K_s^c)^c\ )$

$= m - \mu_{m-(s+i)}(\Lambda^c + K_s^c)$ ($\Omega = (\Lambda^c + K_s^c)^c$ has m vertices.)

$= m - \mu_{m-(s-i)}\ (\Lambda^c)$ (K_s^c is s isolated vertices. Thus, the corresponding block of K_s^c in $L(\Omega)$ is the $s \times s$ zero matrix.)

$= m - ((m - s\ ) - \mu_{(m-s)\ -\ (m-s-i)}(\Lambda))$ (Λ has $m - s$ vertices)

$=$ s + $\mu_i(\Lambda)$.

$\therefore\ \mu_{s+i}\ (\Omega) = \mu_i(\Lambda) + s$, where $1 \leq i < t = m - s = |V\ (\Lambda)|$.

(Note that $1 \leq i < (m - s) \Rightarrow -(m - s) < -i \leq\ -1$

$\Rightarrow 0 < m - s - i \leq m - s - 1$

$\Rightarrow 1 \leq m - s - i \leq m - s - 1 = t - 1 \leq m - 1 < m$,

i.e, $1 \leq m - s - i < m$. Therefore, we can apply $(***)$ in the first case above. Similarly, $1 \leq m - s - i < m - s$, because $1 \leq i < m - s$. Therefore we can apply $(***)$ in the second case above.)

Step 7. $\rho^*_{s+i}(\Omega) =\ \rho^*_i(\Lambda) + s, 1 \leq\ i < t$:

$\rho^*_{s+i}\ (\Omega) = |\{\rho_j(\Omega) \colon \rho_j(\Omega) \geq s + i\}| = |\{\rho_j((\Lambda V K_s) \colon \rho_j((\Lambda V K_s) \geq\ s + i\}|$

$= |\{\rho_j(\Lambda V K_s) \colon \rho_j(\Lambda) + s \geq s + i\ \}| + s$

(Note that $v \in V(\Lambda V K_s) \Rightarrow v \in V(\Lambda)\ or\ v \in V(K_s\ )$.

$v \in V(\Lambda) \Rightarrow\ \rho_{\Lambda V K_s}(v) = \rho(v)_\Lambda + s$, because each vertex of Λ is adjacent to every vertex of K_s.

$v \in V(K_s) \Rightarrow\ \rho_{\Lambda V K_s}(v) = \rho_{K_s}(v) + |V(\Lambda\ )|$, because each vertex of K_s is adjacent to every vertex of Λ.

$\therefore\ v \in V(\ K_s) \Rightarrow \rho_{\Lambda V K_s}(v) = (s -\ 1) + (m\ -\ s) = m - 1 \geq\ s + i$ for each $1 \leq i < t$, because $1 \leq i < t = m - s\ \Rightarrow s + 1 \leq s + i < (m - s)\ ts = m \Rightarrow\ s + i < m - 1$.

Also, note that $\left|\{\rho_j(\Lambda V K_s) \colon \rho_j(\Lambda) \geq\ i\ \}\right| = \left|\{\ \rho_j(\Lambda) \colon \rho_j(\Lambda) \geq i\ \}\right|$, because only the order of the set does matter. In both cases, we count *j's* for which $\rho_j(\Lambda) \geq i$.)

Step 8. $\therefore$ Since $\rho^*_{s+i}(\Omega) = \rho^*_{s+i}(\Gamma) = \mu_{s+i}(\Gamma) = \mu_{s+i}(\Omega)$

(the second equation holds by hypothesis, and the other two equations hold by Step 1), both $\mu_{s+i}(\Omega) = \mu_i(\Lambda) + s$ (Step 6) and $\rho^*_{s+i}(\Omega)$=$\rho^*_i(\Lambda) + s$ (Step 7) imply that $\rho^*_i(\Lambda) = \mu_i(\Lambda) + s$.

$\therefore\ \rho^*_i(\Lambda) = \mu_i(\Lambda)$ for each $1 \le i < t = |V(\Lambda)|$.

(Note that $1 \le i < t = m - s \Rightarrow s + 1 \le s + i < m$. Therefore, the equations proved in Step 1 can be applied.)

Step 9. Since $\rho^*_i(\Lambda) = \mu_i(\Lambda)$ for each $1 \le i < t = |V(\Lambda)|$, and since $t = |V(\Lambda)| = m - s < m \le n,\ \mathrm{t} < \mathrm{n}$.

$\therefore$ Since Λ satisfies the hypothesis of the theorem, Λ is a TG by induction assumption.

$\therefore$ Since $\Omega = \Lambda V K_s$, Ω is a TG by the recursive definition of TGs.

$\therefore$ Since $\Gamma = \Omega + K_{n-m}, \Gamma$ is a TG again by the recursive definition of TGs.

(Note that we can not apply induction assumption to the graph Ω, because $|V(\Omega)| = m = n = V(\Gamma)$ is possible.) ■

2.2 Spectral Integral Variation

Definition 2.6 Let Γ be a graph and let $\mu_1, \mu_2, \dots, \mu_n$ be the eigenvalues of Γ. If $\mu_j \in \mathbb{Z}$ for each $j = 1, 2, \dots, n$, then Γ is called a ″Laplacian integral (LI)″ graph.

From the last theorem of the previous section, a TG is LI. But there are more, that is, the set of LI graphs properly contains the set of TGs. Now, we define a new type of graph, which is LI, and TG belongs to the set of this type of graphs.

Definition 2.7 Let Γ be a graph. Γ is said to be a ″cograph″ if it is obtained recursively by obeying the following principles:

a) K_1, i.e., a single vertex is a cograph.

b) If Ω is a cograph, then $\overline{\Omega}$ is a cograph.

c) If Ω and Λ are two cographs s.t. $V(\Omega) \cap V(\Lambda) = \emptyset$, then $\Omega + \Lambda$ is a graph.

Proposition 2.8 If Γ is a cograph, then it is LI.

Proof. We use the recursive definition of cographs and the following two facts:

1) Ω and Λ are LI $\Rightarrow$ $\Omega + \Lambda$ is LI:

Since $L(\Omega + \Lambda) = \begin{bmatrix} L(\Omega) & 0 \\ 0 & L(\Omega) \end{bmatrix}$, the set the of eigenvalues $L(\Omega + \Lambda)$ is the union of the set of eigenvalues of $L(\Omega)$ and $L(\Lambda)$, counting with multiply .

Now, since Ω and Λ are LI their eigenvalues are all integers.

∴ The eigenvalues of $L(\Omega + \Lambda)$ are integers.

∴ $L(\Omega + \Lambda)$ is LI .

2) Γ is LI $\Rightarrow$ $\bar{\Gamma}$ is LI:

We know from Step 3 of the proof of the last theorem of the previous section that for any graph Ω, we have:

$$\mu^*_{n-i}(\Omega^c) = n - \mu_i\ (\Omega) \text{ for each } 1 \leq i < n, \text{ where } n = |V(\Omega)|\ .$$

And, for any graph Ω, $\mu_n(\Omega) = 0$:

Step 1. Let $L(\Omega) = [\, n \,]$ and $E(\Omega) = \{e_1, e_2, \dots, e_m\}$. Assume that each edge of Ω is given a direction. Let $B(\Omega) = B = [b_{ij}]$ be the $n \times m$ matrix, whose rows (respectively, columns) are indexed by $V(\Omega)$ (respectively, $E(\Omega)$) s.t.

$$b_{ij} = \begin{cases} 0, & if \quad i \notin e_j = \{v, w\}; \\ 1, & if\ i\ \in e_j = \{v, w\} \text{ is the initial vertex of } e_j; \\ -1, & if\ i\ \in e_j = \{v, w\} \text{ is the final vertex of } e_j; \end{cases}$$

Then, $B(\Omega) = B$ is called the "incidence matrix (IM)" of Ω.

Step 2. Let Ω be a connected graph. Then rank $B(\Omega) = n - 1$:

Let $v = [v_1, v_2, \dots, v_n]^T \in \mathbb{R}^n$ be s.t. $v^T\mathrm{B} = \bar{0}$. Thus, by definition of B, if $i{\sim}j$ ($i\,.\,j \in V(\Omega)$), then $v_i - v_j = 0$, (respectively, $v_j - v_i = 0$) if i (respectively, j) is the initial vertex of the edge $e = \{i\,, j\}$.

Therefore, if there is an $v_a v_b$ - path between the vertexes v_a and v_b, then $v_a = v_b$:

Let $P = i_1 = v_a, i_2, \dots, i_{m-1}, i_m = v_b$ be a $v_a\, v_b - p\ a$ th. Then $i_1 \sim i_2, i_2 \sim i_3, \dots, i_{m-1} \sim i_m$. Therefore, $v_a = v_{i_1} = v_{i_2}$, $v_{i_2} = v_{i_3}, \dots, v_{i_{m-1}} = v_{i_m} = v_b$. Thus, $v_a = v_b$.

∴ Since there is an ij − path in Ω for any two vertices $i, j \in V(\Omega)$ (because Ω is connected), all the components of v must be equal to each other.

∴ dim ($\mathcal{N}(B)$) ≤ 1. (dim(left $\mathcal{N}$)) = dim (right $\mathcal{N}$) = dim ($\mathcal{N}$).)

∴ rank $(B) \geq n-1$.

On the other hand, by definition of B, the sum of entries in each column (in fact, there are only two non-zero entries, one of them is 1 and the other is -1) is zero. Therefore, the sum of all the rows of B is $[0, 0, \ldots, 0]$. Thus, the rows of B are linearly independent.

∴ rank $(B) \leq n-1$.

∴ rank $(B) = n-1$.

Step 3. If Ω has k components, then rank $(B\,(\Omega)) = n-k$:

Let $\Omega_1,\ \Omega_2, \ldots, \Omega_k$ be the components of Ω. Then, after relabeling the elements of $V(\Omega)$ and $E(\Omega)$ if necessary, IM, of Ω is the following block diagonal matrix:

$$B(\Omega) = \begin{bmatrix} B\,(\Omega_1) & 0 & \ldots & 0 \\ 0 & B\,(\Omega_2) & \ldots & 0 \\ & 0 & & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & B\,(\Omega_k) \end{bmatrix}$$

Now, for each $i = 1, 2, \ldots, k$, since Ω_i is connected, rank $B(\Omega_i) = |V\,(\Omega_i)| - 1$ by the previous step.

∴ rank($B(\Omega)$) = rank($B(\Omega_1)$) + rank ($B(\Omega_2)$) + ... + rank ($B(\Omega_k)$)

= $(|V(\Omega_1)| - 1) + (|V(\Omega_2)| - 1) + \ldots + (|V(\Omega_k)| - 1)$

= $(|V(\Omega_1)| + |V(\Omega_2)| + \ldots + |V(\Omega_k)|) - k$

= $n - k$.

Step 4. For any matrix M, (rank (MM^T)) = rank ($M^T M$) = rank M:

$\mathcal{N}(M^T M) = \mathcal{N}(M)$:

$(\subseteq): x \in \mathcal{N}(M^T M) \Rightarrow M^T M x = 0 \Rightarrow x^T\ M^T M_x = 0 \Rightarrow (M_n)^T (M_n) = 0$

$\Rightarrow \left||Mx|\right| = 0 \Rightarrow Mx = \bar{0} \Rightarrow x \in \mathcal{N}(M)$.

$(\supseteq)\ x \in \mathcal{N}(M) \Rightarrow Mx = 0 \Rightarrow M^T M x = 0 \Rightarrow x \in \mathcal{N}(M^T M)$.

∴ $\mathcal{N}(M^T M) = \mathcal{N}(M)$.

∴ nullity ($M^T M$) = nullity (M).

∴ Since rank + nullity = The number of columns of M, a fixed number, rank ($M^T M$) = rank (M).

Step 5. $L(\Omega) = B(\Omega)B(\Omega)^T$:

Let $L(\Omega) = L = [l_{ij}]$, $B(\Omega) = B = [b_{ij}]$ and $B(\Omega)^T = B^T = [b_{ij}^T]$. Then:

$i = j$: $l_{ii} = \rho_i = deg\ (i)$ by definition of L. And$(BB^T)_{ii} = \sum_{k=l}^{n} b_{ik}\ b_{ki}^T = \sum_{k=1}^{n} b_{ik}\ b_{ik}$ $= \sum_{k=1}^{n}(b_{ik})^2 = \sum_{k\sim i}(b_{ik})^2 = \sum_{k\sim i}(\pm 1)^2 = \sum_{k\sim i} 1 = \rho_i = deg\ (i)$.

$\therefore\ l_{ii} = (BB^T)_{ii}$.

$i \neq j$: $l_{ij} = \begin{cases} 0,\ if & i \nsim j; \\ -1, if & i \sim j. \end{cases}$ And $(BB^T)_{ij} = \sum_{k=1}^{n} b_{ik}\ b_{kj}^T = \sum_{k=1}^{n} b_{ik}\ b_{jk}$ $= \sum_{\substack{i\in e_k=\{v,\ w\} \\ j\in e_k=\{v,\ w\}}} b_{ik}b_{jk}$, because if $i \notin e_k = \{v, w\}$ or if $j \notin e_k = \{v, w\}$, then $b_{ik} = 0$ or $b_{jk} = 0$, respectively. Then, $b_{ik}b_{jk} = 0$.

Therefore, if $i \nsim j$, i.e., if there is no edge with end vertices i and j, then there is no non-zero term in the sum. Thus, $(BB^T)_{ij}$ = 0. And if $i{\sim}j$, i.e., if there is an edge. $e = \{i, j\}$, then there is only one non-zero term in the sum which corresponds to the term, say $k = m$, that is, e_m = {i, j}. In this case, "$b_{ik} = 1$ and $b_{jk} = -1$" or "$b_{ik} = -1$ and b_{jk} =1", and thus $(BB^T)_{ij} = b_{im}b_{jm} = -1$.

$\therefore l_{ij} = (BB^T)_{ij}$.

As a result, $L = BB^T$.

Step 6. $rank\ \big(L(\Omega)\big) = rank(B(\Omega)B(\Omega)^T)$ (Step 5)

$= rank\big(B(\Omega)\big)$ (Step 4)

$\leq n - 1$ (Step 3).

∴ nullity $(L(\Omega)) \geq 1$.

∴ There is a non-zero vector in the null space of $L(\Omega)$.

∴ 0 is the eigenvalue of $L(\Omega)$, that is, $\mu_n(\Omega)$ = 0.

(Note that $L(\Omega) = B(\Omega)B(\Omega)^T \Rightarrow L(G)$ is (symmetric) and positive semi-definite matrix ⇒ Each eigenvalue of $L(\Omega) \geq 0$.)

Finally: Γ is $LI \Rightarrow \mu_i(\Gamma) \in \mathbb{Z}$ for each $1 \leq i \leq n$.

⇒ Since $\mu_{n-i}(\Gamma^c) = n - \mu_i(\Gamma)$ for each $1 \leq i \leq n$, $\mu_{n-i}(\Gamma^c) \in \mathbb{Z}$ for each $1 \leq i \leq n$.

⇒ Since also $\mu_n(\Gamma^c) = 0 \in \mathbb{Z}$, $\mu_i(\Gamma^c) \in \mathbb{Z}$ for each $1 \leq i \leq n$.

⇒Γ^c is LI.

Now, by using these two facts, we can prove the proposition:

First of all, K_1 is trivially LI. Let Ω and Λ be two cographs. Assume that both are LI. Then the cograph $\bar{\Omega}$ is LI by Fact (2) above, and the cograph $\Omega + \Lambda$ is LI by Fact (1) above.

∴ Recursively obtained each cograph is also LI.

As a result, if Γ is a cograph, then it is LI. ■

Proposition 2.9 If Γ is a TG, then it is a cograph.

Proof. Wlog, we can assume that Γ is connected, because if Γ is disconnected, then it has only one nontrivial component, and (if exists) the remaining components are all trivial, i.e., each remaining component comprises of a single vertex; thus, they are all cographs, and disjoint union cographs is a cograph.

Now, proof can be done by induction on $|V(\Gamma)| = n$:

$n = 1$: $\Gamma = K_1 \Rightarrow \Gamma$ is a cograph.

Assume that the proposition is true for all graphs with the number of vertices $\leq n - 1$.

Let Γ be a connected TG with $|V(\Gamma)| = n > 1$.

Since Γ is a nontrivial TG, it has a dominating vertex, say $v \in V(\Gamma)$. Let $\Omega = \Gamma \backslash \{v\}$.

Then, Ω is a TG with $|V(\Omega)| = n - 1$. Therefore, by the induction hypothesis, Ω is a cograph. Thus, $\bar{\Omega}$ is a cograph.

$\therefore \bar{\Omega} + \{v\}$ is a cograph.

$\therefore \overline{\Omega + \{v\}} = \bar{\bar{\Omega}} \vee \{v\} = \Omega \vee \{v\} = \Gamma$ is a cograph.

As a result, the proposition is proved by induction. ■

Note 2.10 The converse of the previous proposition does not hold. That is, there are cographs that are not threshold. For example, C_4 is a cograph ($K_1$is a cograph, by part (a) of the definition of cograph. $4K_1 = K_1 + K_1 + K_1 + K_1$ is a cograph part (c) of the definition of cograph. Thus, $4K_1 = C_4$ is a cograph by part (b) of the definition of cograph); but C_4 is not a TG. (C_4 is connected, but does not have a dominating vertex.)

Now, Let Γ be a graph, and e be an edge not in Γ. Then, we want to understand the variation in the eigenvalues of $L(\Gamma)$ if we add e to Γ. To study this variation, we need a result from Linear Algebra about the consequence of the rank 1 perturbation process on the eigenvalues of A, where A is a square matrix of order n and $A^T = A$. (A square matrix $B = A + xy^T$, where $x, y \in \mathbb{R}^n$, is called a "rank 1 perturbation" of A.)
First, we study two lemmas which are needed to prove the mentioned result.

Lemma 2.11 Let M be a square matrix of order n, and $M^T = M$. Partition M as follows:

$$M = \begin{bmatrix} m_{11} & y^T \\ y & M(1|1) \end{bmatrix}.$$

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the eigenvalues of M. Assume that the set of eigenvalues of M (1|1) is a subset of the set of eigenvalues of M. Then y is the zero vector.

Proof. First, by the assumption, the eigenvalues of $M\ (1|1)$ are
$\alpha_1, \alpha_2, \dots, \alpha_{t-1}, \alpha_{t+1}, \dots, \alpha_n$ for some $t \in \{1, 2, \dots, n\}$.
Then we have:

$$tr(M) - tr\big(M(1|1)\big) = \alpha_t.$$

And since the trace of the square of a matrix A is equal to the sum of the squares of eigenvalues of A, we also have:

$$tr(M^2) - tr(M(1|1)^2) = \alpha_t^2.$$

On the other hand, from the statement of the lemma, we have:

$$tr(M) - tr\big(M(1|1)\big) = m_{11}.$$

And since $M^2 = \begin{bmatrix} m_{11}^2 + y^T y & m_{11}y^T + y^T M(1|1) \\ ym_{11} + M(1|1)y & yy^T + M(1|1)^2 \end{bmatrix}$, we have:

$$\begin{aligned} tr(M^2) &= m_{11}^2 + y^T y + tr(yy^T + M(1|1)^2) \\ &= m_{11}^2 + y^T y + tr(yy^T) + tr(M\ (1|1)^2) \\ &= m_{11}^2 + y^T y + tr(y^T y) + tr(M(1|1)^2) \\ &= m_{11}^2 + 2y^T y + tr(M(1|1)^2). \end{aligned}$$

$\therefore\ tr(M^2) - tr(M(1|1)^2) = m_{11}^2 + 2y^T y.$

$\therefore\ m_{11} = \alpha_t,\ m_{11}^2 + 2y^T y = \alpha_t^2.$

$\therefore y^T y = 0 \Rightarrow\ ||y||^2 = 0.$

$\therefore y = 0.$ ■

Lemma 2.12 By the same notation of the previous lemma, let $0 \neq \delta \in \mathbb{R}$, and let

$$N = \begin{bmatrix} m_{11} + \delta & y^T \\ y & M\,(1|1) \end{bmatrix}$$

be a block matrix.

Assume that the eigenvalues of M (respectively, N) are $\alpha_1, \alpha_2, \dots, \alpha_n$ (respectively, $\alpha_1, \alpha_2, \dots, \alpha_{t-1}, \alpha_t + \delta, \alpha_{t+1}, \dots, \alpha_n$ for some $t \in \{1, 2, \dots, n\}$). Then y is the zero vector.

Proof. First of all, we have:

$c_M(\mu) = det(\mu I - M) = (\mu - \alpha_1)(\mu - \alpha_2) \dots (\mu - \alpha_n),$

$c_M(\mu) = det(\mu I - N) = (\mu - \alpha_1) \dots (\mu - \alpha_{t-1})(\mu - \alpha_t - \delta)(\mu - \alpha_{t+1}) \dots (\mu - \alpha_n).$

$\therefore\ c_N(\mu) = c_M(\mu) - \delta(\mu - \alpha_1) \dots (\mu - \alpha_{t-1})(\mu - \alpha_{t+1}) \dots (\mu - \alpha_n).$

On the other hand, from the statement of the lemma, we have:

$$c_N(\mu) = \begin{vmatrix} \mu - m_{11} - \delta & -y^T \\ -y & \mu I - M(1|1) \end{vmatrix} = \begin{vmatrix} \mu - m_{11} & -y^T \\ -y & \mu I - M(1|1) \end{vmatrix} + \begin{vmatrix} -\delta & 0^T \\ -y & \mu I - M(1|1) \end{vmatrix}$$

$$= det(\mu I - M) - \delta\, det(\mu I - M(1|1))$$

$$= c_N(\mu) - \delta\, det\big(\mu I - M(1|1)\big).$$

$\therefore \delta(\mu - \alpha_1) \dots (\mu - \alpha_{t-1})(\mu - \alpha_{t+1}) \dots (\mu - \alpha_n) = \delta\, det(\mu I - M(1|1)).$

$\therefore$ Since $\delta \neq 0$ by assumption, we have:

$$det(\mu I - M(1|1)) = (\mu - \alpha_1) \dots (\mu - \alpha_{t-1})(\mu - \alpha_{t+1}) \dots (\mu - \alpha_n).$$

$\therefore$ The eigenvalues of $M(1|1)$ comprises of $n - 1$ eigenvalues of M.

$\therefore$ By the previous lemma, y is the zero matrix. ■

Theorem 2.13 Let M and N be square matrices of order n s.t. $M^T = M$ and $N^T = N$, rank$(N) = 1$. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ (respectively, $0 \neq \delta, 0, \dots, 0$) be the eigenvalues of M (respectiely, N). Then matrix $M + N$ has the eigenvalues $\alpha_1, \dots, \alpha_{t-1}, \alpha_t\ \delta, \alpha_{t+1}, \dots, \alpha_n$ for some $t \in \{1, 2, \dots, n\}$ iff M and N commute.

Proof.

($\Leftarrow$) Since M and N are commuting symmetric matrices, there is an orthogonal matrix P s.t.

$$PMP^T = diag(\alpha_1, \alpha_2, \dots, \alpha_n),\ PNP^T = diag(0, \dots, 0, \delta, 0, \dots 0),$$

where δ is the *t* th diagonal entry of the matrix PNP^T for some $t \in \{1, 2, \dots, n\}$.

Then we have:

$$P(M+N)P^T = PMP^T + PNP^T = diag(\alpha_1, \dots,\ \alpha_{t-1}, \alpha_t + \delta\ ,\ \alpha_{t+1}, \dots, \alpha_n),$$

for some $t\ \in\ \{1, 2, \dots, n\}$.

$\therefore$ The eigenvalues of $M+N$ are $\alpha_1, \dots,\ \alpha_{t-1}, \alpha_t + \delta\ ,\ \alpha_{t+1}, \dots, \alpha_n$ for some $t \in \{1, 2, \dots, n\}$.

($\Rightarrow$) Wlog, we can suppose that $N = diag\ (\delta, 0, \dots, 0)$ (Since N is symmetric, there is an orthogonal matrix P s.t. $PNP^T = diag\ (\delta, 0, \dots, 0)$. And since (orthogonally) similar matrices have the same eigenvalues, both M and PMP^T, also both $M+N$ and $P(M+N)P^T$ have the same eigenvalues. Therefore, instead of studying with M and N, we can study with PMP^T and PNP^T. And thus we may suppose that $N = diag(\delta, 0, \dots, 0)$.)

Let $\quad M = \begin{bmatrix} m_{11} & y^T \\ y & M(1|1) \end{bmatrix}$.

Then $\quad M+N = \begin{bmatrix} m_{11} + \delta & y^T \\ y & M\,(1|1) \end{bmatrix}$.

Now, the eigenvalues of M (respectively, $M+N$) are $\alpha_1, \alpha_2, \dots, \alpha_n$ (respectively, $\alpha_1, \dots,\ \alpha_{t-1}, \alpha_t + \delta\ ,\ \alpha_{t+1}, \dots, \alpha_n$) for some $t \in \{1, 2, \dots, n\}$ by the hypothesis.

$\therefore$ By the previous lemma, y is the zero vector.

$\therefore M = \begin{bmatrix} m_{11} & 0^T \\ 0 & M(1|1) \end{bmatrix}$.

$\therefore$ Since $N = diag\ (\delta, 0, \dots, 0), M$ and N commute. ∎

Now, we apply the previous theorem to Laplacian matrices.

Let Γ be a graph with $V(\Gamma) = [n]$, and suppose that $\{i, j\} \notin E(\Gamma)$. Let $\Omega = \Gamma + \{i, j\}$. Then:

$$L(\Omega) = L(\Gamma) + e_{ij}e_{ij}^T,$$

where $e_{ij} \in \mathbb{R}^n$ is the vector whose *i* th component is 1, *j* th component is -1, and all the remaining components are zero.

∴ By the related interlacing result, if $\alpha_1, \alpha_2, \ldots, \alpha_n = 0$ (respectively, $\beta_1,\ \beta_2, \ldots, \beta_n = 0$) are the eigenvalues of $L(\Gamma)$ (respectively, L(Ω)), then we have:

$$\beta_1 \geq \alpha_1 \geq \ \beta_2 \geq \alpha_2 \geq \cdots \geq \alpha_{n-1} \geq \ \beta_n \geq \alpha_n.$$

Also note that:

$$tr(L(\Omega)) = tr(L(\Gamma) + e_{ij}\, e_{ij}^T) = tr(L(\Gamma)) + tr(\, e_{ij}\, e_{ij}^T) = tr(L(\Gamma)) + 2. \qquad (*)$$

Assume that Γ is LI. Also assume in consideration of (*) above that one of the following holds:

a) Either $\beta_{j_m} = \ \alpha_{i_m}$ for $m = 1, 2, \ldots, n-1$, and $\beta_{j_n} = \ \alpha_{i_n} + 2$;

b) Or $\beta_{j_m} = \ \alpha_{i_m}$ for $m = 1, 2, \ldots, n-2$, and $\beta_{j_{n-1}} = \ \alpha_{i_{n-1}} + 1$.

Then, in consideration of (*) above, Ω is also LI.

Definition 2.14 If case (a) (respectively, (b)) occurs, then it is said that "spectral integral variation happens in 1 (respectively, 2) position(s)."

The following theorem characterizes case (a).

Theorem 2.15 Let Γ be a graph with $V(\Gamma) = [n]$ and $\{i,j\} \notin E(\Gamma)$. Let $\Omega = \Gamma + \{i,j\}$. Then, $n-1$ eigenvalues of $L(\Gamma)$ and $L(\Omega)$ concur iff $N(i) = N(j)$.

Proof. First, we showed above that $(\Omega) = L(\Gamma) + e_{ij}\, e_{ij}^T$. By the previous theorem, $n-1$ eigenvalues of $L(\Gamma)$ and $L(\Omega)$ concur.

⇔ $L(\Gamma)\ \ e_{ij}\, e_{ij}^T = e_{ij}\, e_{ij}^T\ \ L(\Gamma)$.

⇔ (By block multiplication) $(c_i - c_j)\, e_{ij}^T = e_{ij}\, (r_i - \ r_j)$, where c_k (respectively, r_k) is the k th column (respectively, k th row) of $L(\Gamma)$ for $k = \ i, j$.

⇔ $(r_i^T$ - $r_j^T)\, e_{ij}^T = e_{ij}\, (r_i - \ r_j)$ (because $L(\Gamma)$ is a symmetric matrix).

⇔ $\left(r_i - \ r_j\right)^T e_{ij}^T = e_{ij}\, \left(r_i - \ r_j\right)$.

⇔ $(e_{ij}\, (r_i - \ r_j))^T = e_{ij}\left(r_i - \ r_j\right)$.

⇔ (By block multiplication) since

$$e_{ij}\, \left(r_i - \ r_j\right) = \left[0, \ldots, 0, \left(r_i - \ r_j\right)^T, 0, \ldots, 0, \left(r_j - \ r_i\right)^T, 0, \ldots, 0\right]^T,$$

$$r_i - \ r_j = \left[0, \ldots, 0, l_{ii} - l_{ji}, 0, \ldots, 0, l_{ji} - l_{ii}, 0, \ldots, 0\right], \text{ where } L(\Gamma) = [l_{ji}].$$

$\Leftrightarrow (r_i - r_j)_k = l_{ik} - l_{jk} = 0$ for each $k \neq i, j$ and $(r_i - r_j)_i = l_{ii} - l_{ji}$, $(r_i - r_j)_j = l_{ji} - l_{ii}$

$\Leftrightarrow l_{ik} = l_{jk}$ for each $k \neq i, j$ and $l_{ii} - l_{ji} = l_{ii} - l_{ji}$, $l_{ij} - l_{jj} = l_{ji} - l_{ii}$ (Note that since $\{i, j\} \notin E(\Gamma)$, $l_{ij} = l_{ji} = 0$.)

$\Leftrightarrow l_{ik} = l_{jk}$ for each $k \neq i, j$ and $l_{ii} = l_{jj}$.

$\Leftrightarrow$ The *i* th and *j* th rows of $L(G)$ are identical.

$\Leftrightarrow N(i) = N(j)$. ■

Corollary 2.16 Let Γ be a graph with $V(\Gamma) = [n]$ and $\{i, j\} \notin E(\Gamma)$, also assume that $N(i) = N(j)$. And let $\Omega = \Gamma + \{i, j\}$. Then Γ is LI iff Ω is LI.

Proof. First of all, since $N(i) = N(j)$, $(n-1)$ eigenvalues of $L(\Gamma)$ and $L(\Omega)$ concur by the previous theorem. Therefore, Γ is LI iff these concurrent eigenvalues and the remaining eigenvalue of $L(\Gamma)$ are all integers iff (Since $tr(\Omega) = tr(\Gamma) + 2$,) these concurrent eigenvalues and the remaining eigenvalue of $L(\Omega)$ are all integers iff Ω is LI.
■

3. LITERATURE REVIEW

First of all, as it is known, there are lots of good resources about graph theory and matrix theory. One of the best textbooks written on graph theory is West (2002). For matrix theory, we recommend Zhang (2011), or a more detailed textbook Horn and Johnson (2012). Bapat (2014) is a perfect book combining these two areas of mathematics.

About majorization, a well-known classic resource is Marshall and Olkin (1979); and about threshold graphs, a comprehensive textbook is Mahadev and Peled (1995).

The first and the second sections of chapter two predicate on Merris (1994) and So (1999), respectively.

A construction of an infinite class of Laplacian integral graphs which are not cographs was given in Grone and Merris (2008).

One of the well-known conjectures about the Laplacian spectrum of a graph was asserted by Grone and Merris (1994), and this conjecture claimed that the conjugate of a degree sequence of a graph majorizes the Laplacian characteristic values of this graph. That conjecture is solved by Bai (2011).

4. CONCLUSIONS AND RECOMMENDATION

This thesis is a review study, it does not contain any original results. As stated before, the main source used in the preparation of the thesis is Bapat (2014).

The topic of Laplacian eigenvalues of threshold graphs is one of the fundamental topics in graph theory. In this thesis, we introduced the basic notions and theorems about majorization, threshold graphs, and Laplacian eigenvalues, and we surveyed some of the recent results in this area.

The purpose of this thesis is to introduce this rich and active research area at a basic level and to present some of the recent results and developments in a very compact form. In other words, our purpose is not to solve an open problem in this area or to obtain an original result about this area. But we aim to present a survey or a review/tutorial source about Laplacian eigenvalues of threshold graphs for non-experts in this area. In other words, we aim our thesis to be an introduction source related to this area. Therefore, we hope this thesis will be useful for students and researchers who want to learn the basics of this pleasurable and influential area of mathematics.

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