FIRAT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
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ON SOME GENERALIZED SEQUENCE SPACES BASED ON LUCAS BAND MATRIX AND MODULUS FUNCTIONS

Mustafa Ismael HATIM

Doctoral Thesis

DEPARTMENT OF MATHEMATICS
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DECLARATION

I hereby declare that I wrote this [Doctoral thesis] titled On Some Generalized Sequence Spaces Based on Lucas Band Matrix and Modulus Functions” inconsistent with the thesis writing guide of the Graduate School of Natural and Applied Sciences, Firat University. I also declare that all information in it is correct, that I acted according to scientific ethics in producing and presenting the findings, cited all the references I used, and express all institutions or organizations, or persons who supported the thesis financially. I have never used the data and information I provide here in order to get a degree in any way.

February 2024

Mustafa Ismael HATIM

PREFACE

First of all, I would like to thank God for giving me the help, energy and courage to complete my study. And I am pleasured to show my great thank presentation to my supervisor " Prof. Dr. Çiğdem BEKTAŞ" for her great efforts and valuable time who spent in supervising my all researches and thesis in all details and accurately. I also would like to present my acknowledgements to my family for encouraging me to succeed in all stages of my study. Finally, I want to say thanks to everyone that helped me in preparing this thesis.

Mustafa Ismael HATIM

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In this thesis, we first mention some fundamental definitions and basic theorems, then we give and examine some basic concepts such as g-density, g-statistical convergence, and g-statistical boundedness, where g is a modulus function. Later, we establish the concept of g^k-density of a natural numbers subset B, then by using a lacunary sequence $\theta = (k_r)$, we extend it to put the definition of $S_\theta(g^k)$-statistical convergence, where g^k is the composition of g in k times, also we give some inclusion relationships between the sets of $S_\theta(g^k)$-statistically convergent sequences. After that, we focus on g-strong Cesaro summability, then by using $\theta = (k_r)$ we introduce the concept of $N_\theta(g^k)$-strong convergence and determine the connections between the sets of $N_\theta(g^k)$-strongly convergent sequences. Moreover, by using a sequence of modulus functions G, we establish the concept of G^k-density of B, where G^k is a sequence consisting of the compositions of every modulus of G in k times. Then by using $\theta = (k_r)$ and with the help of the Lucas transform $\hat{E}(r,t)$, we introduce the concepts of $S_\theta(\hat{E}(r,t), G^k)$-statistical convergence and $N_\theta(\hat{E}(r,t), G^k)$-strong convergence including the relation between the sets of $N_\theta(\hat{E}(r,t), G^k)$-strongly convergent sequences and $S_\theta(\hat{E}(r,t), G^k)$-statistically convergent sequences. In addition, we generalize two sequence spaces based on the Lucas band matrix and a sequence of modulus functions such as $\ell_p(G^k, v, \hat{E}(r,t))$ and $\ell_\infty(G^k, v, \hat{E}(r,t))$, then we discuss a geometrical property such as the Gurarii's modulus of convexity of $\ell_p(G^k, v, \hat{E}(r,t))$.

Keywords: Sequence space, modulus function, lacunary sequence, g-statistical convergence, g-strong Cesàro summability, Lucas numbers, Lucas band matrix.

ÖZET

Modülüs Fonksiyonlarına ve Lucas Band Matrisine Bağlı Bazı Genelleştirilmiş Dizi Uzayları

Mustafa Ismael HATIM

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Bu tezde, ilk olarak bazı temel tanım ve teoremlere vereceğiz sonra g bir modülüs fonksiyonu olmak üzere, g-yoğunluk, g-istatistiksel yakınsaklık ve g-istatistiksel sınırlılık gibi bazı temel kavramları vereceğiz ve inceleyeceğiz. Daha sonra, bir $\theta = (k_r)$ lacunary dizisini kullanarak bir B doğal sayı alt kümesinin g^k-yoğunluğu kavramını vereceğiz ve onu g^k, g ın k kez bileşeni olmak üzere $S_\theta(g^k)$-istatistiksel yakınsaklık tanımına genişleteceğiz, aynı zamanda $S_\theta(g^k)$-istatistiksel yakınsak dizilerin cümleleri arasındaki bazı içerme bağıntılarını vereceğiz. Bundan sonra, g-kuvvetli Cesàro toplanabilirliğe odaklanacağız ve $\theta = (k_r)$ yi kullanarak $N_\theta(g^k)$-kuvvetli yakınsaklık kavramını göstereceğiz ve $N_\theta(g^k)$-kuvvetli yakınsak dizilerin cümleleri arasındaki bağıntıları tanımlayacağız. Dahası, G modülüs fonksiyonlarının bir dizisini kullanacak, G^k her G modülüsünün k kez bileşenlerinden oluşan bir dizi olmak üzere B nın G^k-yoğunluk kavramını inşa edeceğiz. Daha sonra, $\theta = (k_r)$ yi kullanarak ve $\hat{E}(r,t)$ Lucas dönüşümü yardımı ile $N_\theta(\hat{E}(r,t), G^k)$-kuvvetli yakınsak dizilerin ve $S_\theta(\hat{E}(r,t), G^k)$-istatistiksel yakınsak dizilerin cümleleri arasındaki bağıntıyı içeren $S_\theta(\hat{E}(r,t), G^k)$-istatistiksel yakınsaklık ve $N_\theta(\hat{E}(r,t), G^k)$-kuvvetli yakınsaklık kavramlarını göstereceğiz. İlaveten, $\ell_p(G^k, v, \hat{E}(r,t))$ ve $\ell_\infty(G^k, v, \hat{E}(r,t))$ gibi modülüs fonksiyonlarının bir dizisine ve Lucas band matrisine bağlı iki dizi uzayını genelleştireceğiz, ve daha sonra $\ell_p(G^k, v, \hat{E}(r,t))$ nın Gurarii modülüs konveksliği gibi geometrik bir özelliğini tartışacağız.

Anahtar Kelimeler: Dizi uzayı, modülüs fonksiyonu, lacunary dizisi, g-istatistiksel yakınsaklık, g-kuvvetli Cesaro toplanabiliklik, Lucas sayıları, Lucas band matrisi.

LIST OF SYMBOLS

Symbols	
$\mathbb{N}$	: Natural numbers.
$\mathbb{R}$	: Real numbers.
$\mathbb{R}^+$	: Non-negative real numbers.
$\mathbb{C}$	: Complex numbers.
ω	: All sequences.
ℓ_∞	: Bounded sequences.
c	: Convergent sequences.
c_0	: Null sequences.
ℓ_p	: Absolutely p-summable sequences.
$\delta(B)$	: Natural density of the set B.
$\delta^g(B)$	: g-density of the set B.
$\delta^{g^k}(B)$	: g^k-density of the set B.
$\delta^{G^k}(B)$	: G^k-density of the set B.
S^g	: g-statistically convergent sequences.
BS^g	: g-statistically bounded sequences.
S_θ	: Lacunary statistically convergent sequences.
$S_\theta(g^k)$	: Lacunary statistically convergent sequences according to g^k.
N_θ	: Lacunary strongly convergent sequences.
$N_\theta(g^k)$	: Lacunary strongly convergent sequences according to g^k.
$S_\theta(\hat{E}(r,t),G^k)$	: Lacunary statistically $\hat{E}(r,t)$-convergent sequences according to G^k.
$N_\theta(\hat{E}(r,t),G^k)$	: Lacunary strongly $\hat{E}(r,t)$-convergent sequences according to G^k.
w	: Strongly Cesàro summable sequences.
w_0	: Strongly Cesàro summable null sequences.
w^g	: g-strongly Cesàro summable sequences.
w_0^g	: g-strongly Cesàro summable null sequences.
$w(\hat{E}(r,t),G^k)$	: Strongly Cesàro $\hat{E}(r,t)$-summable sequences according to G^k.

1. INTRODUCTION

It is known that the set of even natural numbers and the set of natural numbers have the same cardinality. In other meaning, there is no difference in the size or number of elements between the two sets. Based on our intuitive feeling that the total number of the elements of the natural numbers set appears to be half of the integers. This intuition is derived into a mathematical idea called natural density (or, asymptotic density) [1]. The natural density is considered a method to determine what proportion of the natural number set is large.

Let $B \subset \mathbb{N}$. Then the value $\delta(B)$ is referred to as a natural density of B and is expressed by

$$\delta(B) = \lim_{n\to\infty} \frac{1}{n} |\{i \le n : i \in B\}|$$

where $|\{i \le n : i \in B\}|$ represents the number of elements of B not more than n.

Statistical convergence is a type of convergence that, forms a technical standpoint, is based on the natural density of subsets of the natural numbers. The thought of statistical convergence appeared in the first edition of a monograph by Zygmund [2] published in 1935 in Warsaw. Its definition was instated by Steinhaus [3] and Fast [4] and later reinstated by Schoenberg [5] independently. Concerning the afterward work of Fridy [6] and Salat [7], this way of thinking has become an important area of study in the summability theory. Through the past few decades, the idea of statistical convergence was investigated under various names in the theory of Fourier analysis, Banach spaces, locally convex spaces, summability theory, neural networks, trigonometric series, measure theory, number theory, and some other fields. Later, it was more studied from the viewpoint of sequence spaces and connected to the summability theory by many authors (see [8-11]).

The principle of modulus function has been introduced in a study by Nakano [12]. Many Mathematicians have defined and talked about sequence spaces in their studies by using a modulus function.

In 2014, Aizpuru et al. [13] described another idea of density, and as a result, a new nonmatrix convergence principle was obtained. This was done with the help of an unbounded modulus function. Later, Bhardwaj et al. [14] have used the way of Aizpuru et al. [13] to identify a new idea of g-statistical bounded, which is essentially an extension of the idea of statistical convergence. In addition, by using a composite modulus function in k times (or, simply g^k), Altin et al. [15] have constructed a new sequence space and obtained some new results in a short part of their study. Recently, with the help of a composite modulus function g^k, Hatim and Bektas [16] have identified some new concepts in summability theory.

From Freedman et al. [17] an increasing sequence $\theta = (k_r)$ of non-negative integer numbers is implied by a lacunary sequence such that $\theta = 0$ at $r = 0$ and the sequence h_r that consists of the first backward difference terms of θ, diverges to ∞ as r tends to ∞.

Many authors have used a lacunary sequence to establish new concepts in summability theory and introduced some sequence spaces during their studies (see in [18-23]).

A matrix of infinite dimensions (or, infinite matrix) $A = (a_{mi})$ is a double sequence consisting of real or complex numbers that is stated by a function A mapping the set of natural numbers $\mathbb{N} \times \mathbb{N}$ to the field of complex numbers $\mathbb{C}$ (or $\mathbb{R}$). The complex value a_{mi} is called the entry of the matrix in the m^{th} row and i^{th} column and it indicates the function value at $(m; i) \in \mathbb{N} \times \mathbb{N}$. The corresponding theorems for infinite matrices are rarely obtained from those theorems that have been established for finite i-square matrices, by letting i tends to infinity. Treating infinite matrices is quite different compared to finite matrices. There are several factors that cause it, (see in [24]).

The most common linear operator that can be determined between two sequence spaces is frequently referred to as an infinite matrix. An infinite matrix A is known as a matrix transformation from X into Y if $A: X \to Y$ represents a map that $A\zeta = \{(A\zeta)_m\} \in Y$ exists, for every $\zeta \in X$ [24]. More information on matrix transformation is given in chapter five.

The idea of a matrix domain for an infinite matrix A in a sequence space X is expressed as follows

$$X_A = \{\zeta = (\zeta_i) \in \omega : A\zeta \in X\} \tag{1.1}$$

that represents a sequence space [24].

In recent years, many mathematicians such as Mursaleen and Noman [25, 26], Karakas [27], Candan and Kara [28], Basar and Altay [29], Savas et al [30], and some others have constructed some sequence spaces using the matrix domain for an infinite triangle matrix.

For the first time, the difference sequence spaces were presented by Kizmaz [31] as the form of $X(\Delta) = \{\zeta \in \omega : \zeta_i - \zeta_{i+1} \in X\}$, $X = \ell_\infty, c, c_0$. After that, these sequence spaces have been generalized by Et and Colak [32] such as $X(\Delta^r) = \{\zeta \in \omega : \Delta^r \zeta \in X\}$, $X = \ell_\infty, c, c_0$. Later, Mursaleen [33], Bektas and Colak [34], Bektas et al. [35], Polat et. Al [36], Colak and Et [37], Altin [38] and some other authors have analyzed the difference sequence spaces in their studies. Recently, Kirisci and Basar [39] have introduced and investigated the difference sequence spaces $\hat{X} = \{\zeta \in \omega : B(r,t)\zeta \in X\}$ for $1 \leq p < \infty$ and $X = \ell_\infty, c, c_0, \ell_p$, where $B(r,t) = (t\zeta_{i-1} + r\zeta_i)$; $r, t \neq 0$.

2. PRELIMINARIES

2.1. Fundamental Definitions and Theorems

In this subsection, we mentioned some fundamental definitions and basic theorems which are related to this study.

Definition 2.1.1 [40] A vector space, often known as a linear space, refers to a non-empty set X containing two operations

$$(\zeta, \xi) \to \zeta + \xi \text{ from } X \times X \text{ into } X \text{ is named the addition}$$

$$(\lambda, \zeta) \to \lambda.\zeta \text{ from } K \times X \text{ into } X \text{ is named the multiplication scalars}$$

such that the conditions listed below are fulfilled for every $\zeta, \xi, \eta \in X$ and $\alpha, \beta \in K$:

(1) $\zeta + \xi = \xi + \zeta$,
(2) $(\zeta + \xi) + \eta = \zeta + (\xi + \eta)$,
(3) There is a zero vector $\theta \in X$ such that $\zeta + \theta = \zeta$,
(4) For each $\zeta \in X$, there is $(-\zeta) \in X$ is called the negative of ζ, such that $\zeta + (-\zeta) = \theta$,
(5) $\alpha(\beta\zeta) = (\alpha\beta)\zeta$,
(6) $(\alpha + \beta)\zeta = \alpha\zeta + \beta\zeta$,
(7) $\alpha(\zeta + \xi) = \alpha\zeta + \alpha\xi$,
(8) $1.\zeta = \zeta$.

Elements in X are known as vectors. X is classified as a real linear space if $K = \mathbb{R}$ and a complex linear space if $K = \mathbb{C}$.

Definition 2.1.2 [41] Assume that X is a linear space. A subset M in X is considered to be a linear subspace of X if $\alpha\zeta + \beta\xi \in M$ for all $\zeta, \xi \in M$ and α, β are scalars.

Definition 2.1.3 [24] Assume that X is a linear space on the field $\mathbb{R}$ or $\mathbb{C}$, with $\|\cdot\|: X \to \mathbb{R}^{+}$ is a function and $p > 0$. We define $(X, \|\cdot\|)$ as a p-normed linear space and $\|\cdot\|$ as a p-norm on X, if the following axioms hold true for all $\zeta, \xi \in X$ and for every scalar λ:

(N.1) $\|\zeta\| = 0$ if and only if $\zeta = \theta$,
(N.2) $\|\lambda\zeta\| = |\lambda|^{p}.\|\zeta\|$,
(N.3) $\|\zeta + \xi\| \leq \|\zeta\| + \|\xi\|$.

Remark 2.1.4 From the above definition, replacing (N.2) by $\|\lambda\zeta\| = |\lambda|.\|\zeta\|$, then $(X, \|\cdot\|)$ is named a normed linear space, and N.1, N.2, and N.3 are named the norm axioms. Additionally, in that case, N.2 is known as the absolute homogeneity and N.3 is called triangle inequality.

Definition 2.1.5 [41, 42] Assume that $(X, \|\cdot\|)$ is a normed linear space:

(a) A sequence (ζ_i) in $(X, \|\cdot\|)$ is considered to be a Cauchy sequence if, for every $\varepsilon > 0$, there is a natural value N (depending on ε) such that for every $i, n \geq N, \|\zeta_i - \zeta_n\| < \varepsilon$. Or, simply, $\|\zeta_i - \zeta_n\| \to 0$ as $i, n \to \infty$.

(b) A sequence (ζ_i) in $(X, \|\cdot\|)$ is considered to converge or to be convergent if there exists an $\zeta \in X$ such that $\|\zeta_i - \zeta\| \to 0$ as $i \to \infty$, ζ is named the limit of (ζ_i) and it is simply written by $\lim_{i\to\infty} \zeta_i = \zeta$.

Theorem 2.1.6 [42, 43] Assume that $(X, \|\cdot\|)$ is a normed linear space. Then the statements listed below are held:

(a) For any (ζ_i) in X, if $\lim_{i\to\infty} \zeta_i = \zeta$, then ζ is unique.

(b) For any (ζ_i) in X, if $\lim_{i\to\infty} \zeta_i = \zeta$, then ζ is considered as the limit of every subsequence of ζ_i.

(c) If (ζ_i) is a convergent sequence in X, then it becomes a Cauchy. But the converse of that is generally not held true.

Definition 2.1.7 [41, 44]

(a) A normed linear space $(X, \|\cdot\|)$ is considered to be complete if every Cauchy sequence in X converges to a limit. Furthermore, completeness means that if $\|\zeta_i - \zeta_n\| \to 0$ as $i, n \to \infty$, where $\zeta_i \in X$ then there has $\zeta \in X$ such that $\|\zeta_i - \zeta\| \to 0$ as $n \to \infty$.

(b) A complete normed linear space X is defined to be a Banach space.

Definition 2.1.8 [41, 42] Assume that X is a normed linear space and (e_i) is a sequence of elements in X. We claim that (e_i) is a Schauder basis, or simply a basis, for X if for every element ζ in X, there has a unique sequence (λ_i) of scalars such that

$$\left\| \zeta - \sum_{n=1}^{i} \lambda_n e_n \right\| \to 0 \; as \; i \to \infty.$$

The series

$$\sum_{n=1}^{\infty} \lambda_n e_n$$

that corresponds to the sum of ζ, is referred to as the expansion of ζ based on (e_i), and it is expressed in a form as follows

$$\zeta = \sum_{n=1}^{\infty} \lambda_n e_n.$$

Definition 2.1.9 [41] Consider two normed linear spaces, X and Y, that extend on the same field of scalars, then

(a) A transformation (or operator, map) $T: X \to Y$ is considered to be linear, if

$$T(\zeta + \xi) = T(\zeta) + T(\xi) \text{ and } T(\lambda\zeta) = \lambda T(\zeta)$$

for every $\zeta, \xi \in X$ and for all scalars λ.

(b) A linear transformation $T: X \to Y$ is considered to be continuous at $\zeta \in X$ if

$$\zeta_i \to \zeta \implies T(\zeta_i) \to T(\zeta) \; as \; i \to \infty,$$

if T is continuous at every point in X, then T is claimed to be continuous.

(c) A linear transformation $T: X \to Y$ is considered to be bounded if there has $K > 0$ such that

$$\|T(\zeta)\| \leq K\|\zeta\|$$

for every $\zeta \in X$.

Definition 2.1.10 [24] Let $(X, \|\cdot\|)$ be a normed linear space, then the unit ball and unit sphere of X, respectively, are introduced as follows

$$B_X = \{\zeta \in X: \|\zeta\| < 1\} \text{ and } S_X = \{\zeta \in X: \|\zeta\| = 1\}.$$

Definition 2.1.11 Assume that X is a normed linear space, and B_X and S_X are the unit ball and the unit sphere of X, respectively. The thought of modulus of convexity has been defined by Clarkson [45] and Gurarii [46], respectively, as follows:

$$\delta_X(\varepsilon) = inf\left\{1 - \frac{\|\zeta + \xi\|}{2} : \zeta, \xi \in S_X, \|\zeta - \xi\| = \varepsilon\right\}, \quad \varepsilon \in [0,2]$$

and

$$\gamma_X(\varepsilon) = inf\left\{1 - \inf_{h\in[0,1]} \|h\zeta + (1-h)\xi\| : \zeta, \xi \in S_X, \|\zeta - \xi\| = \varepsilon\right\}, \quad \varepsilon \in [0,2].$$

Remark 2.1.12 The case of $0 < \gamma_X(\varepsilon) < 1$ means that X becomes uniformly convex, while the choice of $\gamma_X(\varepsilon) \leq 1$ means that X becomes strictly convex.

Definition 2.1.13 [24, 41] By ω, we mean the space of all sequences with complex terms, i.e.,

$$\omega = \{\zeta = (\zeta_i): \zeta_i \in \mathbb{C} \text{ for all } i \in \mathbb{N}\}.$$

Each element $\zeta = (\zeta_i)$ of ω is a sequence of the form $\zeta = (\eta_i + j\mu_i)_{i=1}^{\infty}$, where η_i and μ_i are both real sequences and j corresponds to the square root of (-1). A straightforward verification can be made that ω becomes a linear space by using the usual co-ordinatewise addition and scalar multiplication of sequences, which are clearly expressed as follows

$$\zeta + \xi = (\zeta_i) + (\xi_i) = (\zeta_i + \xi_i) \text{ and } \alpha\zeta = \alpha(\zeta_i) = (\alpha\zeta_i)$$

respectively, where $\zeta = (\zeta_i), \xi = (\xi_i) \in \omega$ and $\alpha \in \mathbb{C}$.

Definition 2.1.14 [24, 41] The spaces of all bounded, null, and convergent sequences are indicated by ℓ_∞, c_0 and c, respectively, and given as below:

$$\ell_\infty = \left\{\zeta = (\zeta_i) \in \omega: \sup_{i\in\mathbb{N}}|\zeta_i| < \infty\right\},$$

$$c_0 = \left\{\zeta = (\zeta_i) \in \omega: \lim_{i\to\infty}|\zeta_i| = 0\right\}.$$

$$c = \left\{\zeta = (\zeta_i) \in \omega: \lim_{i\to\infty}|\zeta_i - \ell| = 0 \; for\ some\ \ell \in \mathbb{C}\right\},$$

It is clear that c_0 becomes a Banach space with the norm $\|\zeta\|_0 = \max_{i\in\mathbb{N}}|\zeta_i|$ for every $\zeta = (\zeta_i) \in c_0$. As well as ℓ_∞ and c become Banach spaces with the norm $\|\zeta\|_\infty = \sup_{i\in\mathbb{N}}|\zeta_i|$ for every $\zeta = (\zeta_i) \in \ell_\infty$ or c.

Definition 2.1.15 [24, 41] The space of all sequences (ζ_i) such that $\sum_{i=1}^{\infty}|\zeta_i|^p$ converges, is indicated by ℓ_p and given as follows

$$\ell_p = \left\{\zeta = (\zeta_i) \in \omega: \sum_{i=1}^{\infty}|\zeta_i|^p < \infty\right\}, (0 < p < \infty).$$

This space ℓ_p is known as the space of absolutely p-summable sequences. The norm $\|\cdot\|_p$ on ℓ_p is described as follows

$$\|\zeta\|_p = \begin{cases} \sum_{i=1}^{\infty}|\zeta_i|^p, & 0 < p < 1, \\ \left(\sum_{i=1}^{\infty}|\zeta_i|^p\right)^{\frac{1}{p}}, & 1 \le p < \infty, \end{cases}$$

for every $\zeta = (\zeta_i) \in \ell_p$. It seems that ℓ_p with $\|\cdot\|_p$ becomes a p-normed linear space for $0 < p < 1$ whereas it is a Banach space for $1 \le p < \infty$.

Definition 2.1.16 [24] The spaces of bounded and convergent series are indicated by bs and cs, respectively, and given as follows

$$bs = \left\{\zeta = (\zeta_i) \in \omega: \sup_{n\in\mathbb{N}}\left|\sum_{i=1}^{n}\zeta_i\right| < \infty\right\},$$

$$cs = \left\{\zeta = (\zeta_i) \in \omega: \lim_{n\to\infty}\left|\sum_{i=1}^{n}\zeta_i - \ell\right| = 0 \text{ for some } \ell \in \mathbb{C}\right\}.$$

These sequence sets become Banach spaces with the norm $\|\cdot\|_{bs}$ defined as follows

$$\|\zeta\|_{bs} = \sup_{n\in\mathbb{N}} \left|\sum_{i=1}^{n} \zeta_i\right| < \infty$$

for every $\zeta = (\zeta_i) \in bs$ or cs.

Definition 2.1.17 [47] A Banach space X is considered to be a BK-space if there is a function $P_i(\zeta) = \zeta_i$ such that $P_i: X \to \mathbb{C}$ is continuous for all $i \in \mathbb{N}$.

As a simple instance, the sequence sets ℓ_∞, c, c_0 and ℓ_p become BK spaces using the norms $\|\cdot\|_\infty$ and $\|\cdot\|_p$, respectively, for $1 \leq p < \infty$.

2.2. Some Inequalities

In this subsection, we give some popular inequalities which are used in some different steps of our study.

(a) [40] Let $\alpha, \beta \in \mathbb{C}$ and $0 < p \leq 1$, then

$$|\alpha + \beta|^p \leq |\alpha|^p + |\beta|^p. \tag{2.1}$$

Note that in the case $p = 1$ the above inequality is known as triangle inequality.

(b) [24, 41] Minkowski's inequality: Let $1 \leq p < \infty$ and $(\zeta_i), (\xi_i) \in \ell_p$, then

$$\left(\sum_{i=1}^{\infty} |\zeta_i + \xi_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} |\zeta_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{\frac{1}{p}} \tag{2.2}$$

3. DENSITIES AND THEIR APPLICATIONS

The concepts and results that are given in the next two subsections are known and widely used in the literature, their sources, which are the references, are generally given for the definitions and omitted for the others.

3.1. Basic Densities

In this subpart, the concept of the natural density of a subset B of natural numbers and some properties of this density are mentioned. After that, the idea of g-density of B is given and then the connections between these two types of densities on a subset of natural numbers are talked about.

Definition 3.1.1 [48] Suppose $\mathbb{N}$ is the set of all natural numbers and B is a subset of $\mathbb{N}$. The natural density of B is indicated by $\delta(B)$ and expressed as the following

$$\delta(B) = \lim_{n\to\infty} \frac{1}{n}|B(n)|,$$

where $B(n) = B \cap \{1,2,\dots,n\} = \{i \leq n: i \in B\}$ and $|B(n)|$ indicates the cardinality of the set $B(n)$. The natural density comes from the common value of $\overline{\delta}(B)$ and $\underline{\delta}(B)$, that is, $\overline{\delta}(B) = \underline{\delta}(B)$, where $\overline{\delta}(B)$ and $\underline{\delta}(B)$ indicate the upper natural density and the lower natural density of B, and given as follows

$$\overline{\delta}(B) = \limsup_{n\to\infty} \frac{1}{n}|B(n)|$$

and

$$\underline{\delta}(B) = \liminf_{n\to\infty} \frac{1}{n}|B(n)|.$$

It is obvious that $0 \leq \delta(B) \leq 1$ for any subset $B \subseteq \mathbb{N}$ if $\delta(B)$ exists.

Propositions 3.1.2 Let B and C be subsets in $\mathbb{N}$, if $\delta(B)$ and $\delta(C)$ exist, then the characteristics below are held.

(1) $\delta(\mathbb{N}) = 1$.

(2) $\delta(\phi) = 0$.

(3) For every finite set B, $\delta(B) = 0$.

(4) For every $m \in \mathbb{N}$, $\delta(mB) = \frac{1}{m}\delta(B)$.

(5) If $B \subseteq C \subseteq \mathbb{N}$, then $\delta(B) \leq \delta(C)$.

(6) $\delta(\mathbb{N}\backslash B) = 1 - \delta(B)$.

(7) $\delta(B + c) = \delta(B)$, where c represents a constant in $\mathbb{N}$ and $B + c = \{a + c: a \in B\}$.

Proof. The proof of some parts is only given. The other parts are easily verified

(1) It can be obviously seen that

$$\delta(\mathbb{N}) = \lim_{n\to\infty}\frac{1}{n}|\mathbb{N}(n)| = \lim_{n\to\infty}\frac{1}{n}\ |\mathbb{N}\cap\{1,2,\dots,n\}| = \lim_{n\to\infty}\frac{1}{n}n = 1.$$

(3) Suppose B is a finite subset in $\mathbb{N}$ and let $|B| = i_0$ (i_0 represents a constant in $\mathbb{N}$), so that $|B(n)| \le i_0$ for any $n \in \mathbb{N}$. Then we have

$$\delta(B) = \lim_{n\to\infty}\frac{1}{n}|B(n)| \le \lim_{n\to\infty}\frac{i_0}{n} = 0.$$

Hence $\delta(B) = 0$.

(5) Since $B \subseteq C$, then $B(n) \subseteq C(n)$, and so $|B(n)| \le |C(n)|$. Then we have

$$\delta(B) = \lim_{n\to\infty}\frac{1}{n}|B(n)| \le \lim_{n\to\infty}\frac{1}{n}|C(n)| = \delta(C).$$

Hence $\delta(B) \le \delta(C)$.

Examples 3.1.3 Consider the following subsets $B, C, E, F \subseteq \mathbb{N}$:

$$B = \{i: i = m^2, m \in \mathbb{N}\} = \{1, 4, 9, 16, \dots\},$$

$$C = \{i: i = 3m, m \in \mathbb{N}\} = \{3, 6, 9, 12, \dots\},$$

$$E = \{i: i = 3^m, m \in \mathbb{N}\} = \{3, 9, 27, 81, \dots\},$$

$$F = \{i: 5 < i \le 15\} = \{6, 7, 8, 9, 10, \dots, 15\}.$$

According to the above knowledge, we might easily find the natural density of each given set.

Now for the subset B, we see that $i = m^2$, then $m_i = \sqrt{i}$ and so $|B(n)| = [m_n] = [\sqrt{n}]$. Then

$$\delta(B) = \lim_{n\to\infty}\frac{1}{n}|B(n)| = \lim_{n\to\infty}\frac{1}{n}[\sqrt{n}] = 0\ .$$

For the subset C, we see that $i = 3m$, then $m_i = \frac{i}{3}$, and so $|C(n)| = [m_n] = \left[\frac{n}{3}\right]$. Then

$$\delta(C) = \lim_{n\to\infty}\frac{1}{n}|C(n)| = \lim_{n\to\infty}\frac{1}{n}\left[\frac{n}{3}\right] = \frac{1}{3}.$$

For the subset E, we see that $i = 3^m$, then $m_i = \frac{\log i}{\log 3}$ and so $|E(n)| = [m_n] = \left[\frac{\log n}{\log 3}\right]$. Then

$$\delta(E) = \lim_{n\to\infty}\frac{1}{n}|E(n)| = \lim_{n\to\infty}\frac{1}{n}\left[\frac{\log n}{\log 3}\right] = 0.$$

For the subset F, we see that $5 < i \le 15$, then $|F(n)| = 10$. We have

$$\delta(E) = \lim_{n\to\infty}\frac{1}{n}|F(n)| \le \lim_{n\to\infty}\frac{10}{n} = 0.$$

Definition 3.1.4 [49] We remember that a function g from $\mathbb{R}^+\cup\{0\}$ into $\mathbb{R}^+\cup\{0\}$ is indeed a modulus function or a modulus if the next properties are satisfied: *(i)* $g(u) = 0$ if and only if $u = 0$, *(ii)* $g(u_1 + u_2) \le g(u_1) + g(u_2)$ for every $u_1, u_2 \in \mathbb{R}^+ \cup \{0\}$, *(iii)* g is increasing, and *(iv)* g is continuous on the right side of zero.

Since $|g(u_1) - g(u_2)| \leq g(|u_1 - u_2|)$, as a consequence of the fourth property *(iv)* that g becomes continuous on $\mathbb{R}^+ \cup \{0\}$. It is conceivable for a modulus to either be bounded or not. For instance, the function $g(u) = u/(u+1)$ is a bounded modulus, but $h(u) = u$ is an unbounded modulus function. Additionally, from the second property *(ii)*, we have $g(mu) \leq mg(u)$ and so $\frac{1}{m} g(u) \leq g\left(\frac{u}{m}\right)$ for every $m \in \mathbb{N}$.

Definition 3.1.5 [50] Assume that g is an unbounded modulus function and $B \subseteq \mathbb{N}$. The number $\delta^g(B)$ is named to be the g-density of the set B and expressed as follows

$$\delta^g(B) = \lim_{n\to\infty} \frac{1}{g(n)} g(|\{i \leq n : i \in B\}|)$$

in a case, the above limit exists.

Remark 3.1.6 If we take $g(u) = u$ then the g-density returns to the natural density. It is obvious that $\delta(B) + \delta(\mathbb{N}\backslash B) = 1$ is held in the case of natural density, for any $B \subseteq \mathbb{N}$. However, in g-density, this is different, i.e., $\delta^g(B) + \delta^g(\mathbb{N}\backslash B) = 1$ may not be held. This fact is demonstrated in the instance below.

Example 3.1.7 Let us put $B = \{i : i = 2m, m \in \mathbb{N}\} = \{2, 4, 6, \dots\}$ and $g(u) = \log(u+1)$. Then $\delta^g(B) = \delta^g(B\backslash\mathbb{N}) = 1$. Since $\frac{n}{2} - 1 \leq |\{i \leq n : i \in B\}| \leq \frac{n}{2}$ for any $n \in \mathbb{N}$ and g is a modulus function, then we can write

$$\lim_{n\to\infty} \frac{1}{g(n)} g\left(\frac{n}{2} - 1\right) \leq \lim_{n\to\infty} \frac{1}{g(n)} g(\{i \leq n : i \in B\}) \leq \lim_{n\to\infty} \frac{1}{g(n)} g\left(\frac{n}{2}\right).$$

And so

$$\lim_{n\to\infty} \frac{1}{\log(n+1)} \log\left(\frac{n}{2}\right) \leq \lim_{n\to\infty} \frac{1}{g(n)} g(\{i \leq n : i \in B\}) \leq \lim_{n\to\infty} \frac{1}{\log(n+1)} \log\left(\frac{n}{2} + 1\right)$$

$$1 \leq \lim_{n\to\infty} \frac{1}{g(n)} g(\{i \leq n : i \in B\}) \leq 1$$

Hence, we see that $\delta^g(B) = 1$. In addition, using the fact $\frac{n+1}{2} - 1 \leq |\{i \leq n : i \in B\}| \leq \frac{n+1}{2}$ we have $\delta^g(B) = 1$.

Remark 3.1.8 For any unbounded modulus g and a subset $B \subseteq \mathbb{N}$, if $\delta^g(B) = 0$, then $\delta(B) = 0$. In fact, if $\delta^g(B) = 0$ then $\lim_{n\to\infty} \frac{1}{g(n)} g(|\{i \leq n : i \in B\}|) = 0$. Now for each $p \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then we have

$$g(|\{i \leq n : i \in B\}|) \leq \frac{1}{p} g(n) \leq g\left(\frac{1}{p} n\right).$$

Since g is increasing, in this manner, we obtain

$$\frac{1}{n}|\{i \le n: i \in B\}| \le \frac{1}{p}$$

for every $n \ge n_0$. Hence, we get $\delta(B) = 0$.

The opposite is not necessarily true in all situations. For that, we give the instance below to demonstrate the case.

Example 3.1.9 Given $g(u) = \log(u + 1)$ and set $B = \{i: i = m^2, m \in \mathbb{N}\}$. Then $B(n) = \{i \le n: i = m^2, m \in \mathbb{N}\}$ so that $|B(n)| = \sqrt{n}$. Then, by calculating $\delta^g(B)$, we see that $\delta^g(B) = 1/2$, whereas $\delta(B) = 0$.

Lemma 3.1.10 Assume that g represents an unbounded modulus function and $B \subseteq \mathbb{N}$. If $\lim_{n\to\infty} \frac{g(n)}{n} > 0$, then $\delta(B) = 0$ implies $\delta^g(B) = 0$.

Proof. Since g represents a modulus function and $|\{i \le n: i \in \mathbb{N}\}|$ is a natural value, we have $g(|\{i \le n: i \in \mathbb{N}\}|) \le |\{i \le n: i \in \mathbb{N}\}|\, g(1)$. Then it can be written as

$$\frac{1}{n}|\{i \le n: i \in B\}| \ge \frac{g(n)}{n}\frac{1}{g(n)} g(|\{i \le n: i \in B\}|)\frac{1}{g(1)}$$

Giving the limits into both sides as $n \to \infty$, we possess that $\delta^g(B) = 0$, since $\lim_{n\to\infty} \frac{g(n)}{n} > 0$.

3.2. Statistical Convergence and Statistical Boundedness with Respect to a Modulus Function

In this subsection, the concepts of g-statistically convergence, g-statistically boundedness, and some results related to these two ideas are provided.

Definition 3.2.1 [50] Assume that g is an unbounded modulus function. Then a sequence (ζ_i) in $\mathbb{R}$ (or $\mathbb{C}$) is referred to as g-statistically convergent (or S^g-convergent) to ζ if the set $\{i \le n: |\zeta_i - \zeta| \ge \varepsilon\}$ possesses g-density zero, for every $\varepsilon > 0$, i.e.,

$$\lim_{n\to\infty} \frac{1}{g(n)} g(|\{i \le n: |\zeta_i - \zeta| \ge \varepsilon\}|) = 0,$$

and this is written as $S^g - \lim \zeta_i = \zeta$ or $\zeta_i \to \zeta(S^g)$. In this study, the collection of all S^g-convergent sequences is indicated by S^g. We also indicated the collection of all g-statistically null sequences by S_0^g. For any unbounded modulus g, we definitely have $S_0^g \subset S^g$. In the case of $g(u) = u$ the collection S^g will return to S (S is the collection of all statistically convergent sequences).

Lemma 3.2.2 The S^g-limit of any g-statistically convergent sequence is considered to be unique.

Lemma 3.2.3 Let the functions g and h be unbounded modulus. If $S^g - \lim \zeta_i = \zeta_a$ and $S^h - \lim \zeta_i = \zeta_b$, then $\zeta_a = \zeta_b$.

Theorem 3.2.4 Every convergent sequence represents a g-statistically convergent, i.e., $c \subset S^g$ for any unbounded modulus function.

Proof. Assume that (ζ_i) is a convergent sequence, then $\zeta_i \to \zeta$ as $i \to \infty$. Given any $\varepsilon > 0$, then there is $n \in \mathbb{N}$ such that $|\zeta_i - \zeta| < \varepsilon$ for all $i \geq n$. So, there is a finite set $\{i \in \mathbb{N}: |\zeta_i - \zeta| \geq \varepsilon \}$ and hence $\delta^g(\{i \in \mathbb{N}: |\zeta_i - \zeta| \geq \varepsilon \}) = 0$, this is sufficient to fulfill the proof. □

The contrary to the above theorem does not necessarily hold true, the following instance demonstrates this fact.

Example 3.2.5 Define the sequence (ζ_i) as

$$\zeta_i = \begin{cases} i, & i = n^2 \\ 0, & i \neq n^2 \end{cases} \quad n = 1,2,3, \ldots$$

and put $g(u) = u^p, 0 < p \leq 1$. Clearly $(\zeta_i) \notin c$, but for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{g(n)} g(|\{i \leq n: |\zeta_i - 0| \geq \varepsilon\}|) \leq \lim_{n\to\infty} \frac{g(\sqrt{n})}{g(n)} = 0.$$

Hence, (ζ_i) is S^g-convergent.

Remark 3.2.6 It is well known that each subsequence in a convergent sequence also becomes convergent, but this case does not necessarily hold true for g-statistical convergence, this says that a subsequence of an g-statistically convergent sequence may not be g-statistically convergent. For instance, if we put $g(u) = u$ and put the sequence (ζ_i) as follows

$$\zeta_i = \begin{cases} i, & i = n^2 \\ \frac{1}{i}, & i \neq n^2 \end{cases} \quad n = 1,2,3, \ldots,$$

or, clearly $(\zeta_i) = \left(1, \frac{1}{2}, \frac{1}{3}, 4, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, 9, \ldots\right)$, then we see that $S^g - \lim \zeta_i = 0$, although $(1,4,9, \ldots)$ is contained as a subsequence in (ζ_i) and not to be S^g-convergent.

Definition 3.2.7 [50] Assume that g is an unbounded modulus function. Then a sequence (ζ_i) in $\mathbb{R}$ (or $\mathbb{C}$) is referred to as g-statistically bounded (or S^g-bounded) if there has a real number $M > 0$ such that $\delta^g(\{i \in \mathbb{N}: |\zeta_i| > M\}) = 0$, i.e.,

$$\lim_{n\to\infty} \frac{1}{g(n)} g(|\{i \leq n: |\zeta_i| > M\}|) = 0.$$

We use BS^g to indicate the collection of all S^g-bounded sequences. Note that in the case of $g(u) = u$ the collection BS^g will return to BS (BS is the collection of all statistically bounded sequences).

Theorem 3.2.8 Every g-statistically bounded sequence becomes statistically bounded, that is, $BS^g \subset BS$ for every unbounded modulus g.

The proof follows the assertion $\delta^g(B) = 0$ implies $\delta(B) = 0$ for any $B \subset \mathbb{N}$ and any unbounded modulus function g. Using this fact if $(\zeta_i) \in BS^g$, then we have $\delta^g(\{i \in \mathbb{N}: |\zeta_i| > M\}) = 0$ for some enough large real number $M > 0$. Now $\delta^g(\{i \in \mathbb{N}: |\zeta_i| > M\}) = 0$ implies $\delta(\{i \in \mathbb{N}: |\zeta_i| > M\}) = 0$ and this means that $(\zeta_i) \in BS$.

Note that the contrary to the above theorem does not necessarily hold true, the following instance demonstrates this fact.

Example 3.2.9 Let $g(u) = \log(u + 1)$ and put the sequence (ζ_i) as follows

$$\zeta_i = \begin{cases} i, & i = n^2 \\ 0, & i \neq n^2 \end{cases} \quad n = 1,2,3, \dots,$$

or, clearly $(\zeta_i) = (1,0,0,4,0,0,0,0,9, \dots)$. Then for any number $M > 0$, we possess $\{i \in \mathbb{N}: |\zeta_i| > M\} = \{1,4,9, \dots\}$, in which each element is a square positive integer number. Since $\delta^g(\{1,4,9, \dots\}) = 1/2 \neq 0$ but $\delta(\{1,4,9, \dots\}) = 0$, then $(\zeta_i) \in BS - BS^g$.

Theorem 3.2.10 Every bounded sequence becomes g-statistically bounded, that is, $\ell_\infty \subset BS^g$ for every unbounded modulus g, however, the opposite of that does not necessarily hold true.

Proof. Assume that g is an unbounded modulus function and let $(\zeta_i) \in \ell_\infty$. Then there has real number $M > 0$ such that $|\zeta_i| \leq M$, for all $i \in \mathbb{N}$, that is, $\delta^g(\{i \in \mathbb{N}: |\zeta_i| > M\}) = \delta^g(\emptyset) = 0$. So (ζ_i) is g-statistically bounded. For the opposite part, the sequence of Example 3.2.9 serves the purpose if we put $g(u) = u$.

Theorem 3.2.11 Suppose g is any unbounded modulus, then $S^g \subset BS^g$.

Proof. Assume that the sequence (ζ_i) is S^g-convergent to ζ. Given $\varepsilon > 0$ and define the sets $B(n) = \{i \leq n: |\zeta_i - \zeta| > \varepsilon\}$ and $C(n) = \{i \leq n: |\zeta_i| > |\zeta| + \varepsilon\}$. Then $C(n) \subset B(n)$ and so $|C(n)| \leq |B(n)|$. We obtain that $\delta^g(C) \leq \delta^g(B)$ and so that $\delta^g(B) = 0$ implies $\delta^g(C) = 0$. As a consequence, (ζ_i) is S^g-bounded.

However, the opposite of the above inclusion does not necessarily hold true. For instance, if we put $g(u) = u$ and define the sequence $(\zeta_i) = (1,2,1,2,\dots)$, then it is clear to see that $(\zeta_i) \in BS^g - S^g$.

Corollary 3.2.12 From the above theorems, for any unbounded modulus g, we have

(i) If $\lim_{u\to\infty} \frac{g(u)}{u} > 0$, then a sequence is S^g-bounded if and only if it is S-bounded.

(ii) Every bounded sequence is S-bounded, i.e., $\ell_\infty \subset BS$.

(iii) Every S^g-convergent sequence is S-bounded, i.e., $S^g \subset BS$.

Lemma 3.2.13 [15] For any modulus function g. The function $g^k = g \circ g \circ \dots \circ g$ (k times) also becomes a modulus for each $k \in \mathbb{N}$. This notation is implemented throughout the next parts.

3.3. Lacunary Statistical Convergence According to a Modulus Function

A lacunary sequence $\theta = (k_r)$ implies an increasing sequence of non-negative integer numbers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. And the periods are given by $\theta = (k_r)$ are indicated by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ may be shortened by q_r. These notations are used throughout some parts of our study.

Definition 3.3.1 Assume that g is an unbounded modulus function and $B \subseteq \mathbb{N}$. The number $\delta^{g^k}(B)$ is introduced as follows

$$\delta^{g^k}(B) = \lim_{n\to\infty} \frac{1}{g^k(n)} g^k(|\{i \le n : i \in B\}|)$$

in the case the above limit exists, and it is referred to as g^k-density of B.

Remark 3.3.2 If $g(u) = u$ for every $u \in \mathbb{R}^+ \cup \{0\}$, is taken then the idea of g^k-density is returned to the natural density, as well as, in the case of $k = 1$ then from g^k-density we shall have g-density of a subset of natural numbers.

Definition 3.3.3 [51] Suppose that $\theta = (k_r)$ is a lacunary sequence. A sequence (ζ_i) of numbers is referred to as lacunary statistically convergent (or S_θ-convergent) to ζ, if

$$\lim_{r\to\infty} \frac{1}{h_r} |\{i \in I_r : |\zeta_i - \zeta| \ge \varepsilon\}| = 0,$$

for each $\varepsilon > 0$. In this situation, it is written that $\zeta_i \to \zeta(S_\theta)$, or $S_\theta - \lim \zeta_i = \zeta$. The collection of all S_θ-convergent sequences is written by S_θ.

Definition 3.3.4 Suppose that $\theta = (k_r)$ is a lacunary sequence and let g be considered an unbounded modulus function. Let $\zeta \in \mathbb{C}$ be a number, then the sequence (ζ_i) in $\mathbb{C}$ is defined to be lacunary statistically convergent to ζ according to g^k (or $S_\theta(g^k)$-statistically convergent), if

$$\lim_{r\to\infty} \frac{1}{g^k(h_r)} g^k(|\{i \in I_r : |\zeta_i - \zeta| \geq \varepsilon\}|) = 0.$$

for every $\varepsilon > 0$. In this manner, we refer it to as $\zeta_i \to \zeta(S_\theta(g^k))$ or $S_\theta(g^k) - \lim \zeta_i = \zeta$. The collection of all $S_\theta(g^k)$-convergent sequences is indicated by $S_\theta(g^k)$. That is

$$S_\theta(g^k) = \left\{(\zeta_i) : \lim_{r\to\infty} \frac{1}{g^k(h_r)} g^k(|i \in I_r : |\zeta_i - \zeta| \geq \varepsilon|\ ) = 0 \text{ for some } \zeta \in \mathbb{C}\right\}.$$

Remark 3.3.5 In the case of $g(u) = u$, the thoughts of $S_\theta(g^k)$-statistically convergence and S_θ-convergence are the same. This means that $S_\theta(g^k)$ will become S_θ. As well as, if $k = 1$ then $g^k(u) = g(u)$ and so from $S_\theta(g^k)$, we have the sequence space $S_\theta(g)$.

Theorem 3.3.6 Suppose that $\theta = (k_r)$ is a lacunary sequence and g is an unbounded modulus function. Let (ζ_i), (ξ_i) be sequences of complex numbers, then we have the following cases:

(i) If $S_\theta(g^k) - \lim \zeta_i = \zeta$ and $\alpha \in \mathbb{C}$, then $S_\theta(g^k) - \lim(\alpha\zeta_i) = \alpha\zeta$.
(ii) If $S_\theta(g^k) - \lim \zeta_i = \zeta$ and $S_\theta(g^k) - \lim \xi_i = \xi$, then $S_\theta(g^k) - \lim(\zeta_i + \xi_i) = \zeta + \xi$.

The proof is clearly done, so we leave it here.

Theorem 3.3.7 Suppose $\theta = (k_r)$ is a lacunary sequence and let g be considered an unbounded modulus function. If a sequence is convergent, then it is $S_\theta(g^k)$-statistically convergent, i.e., $c \subset S_\theta(g^k)$.

Proof. Assume that (ζ_i) is a convergent sequence. Given $\varepsilon > 0$, then $\{i \in \mathbb{N} : |\zeta_i - \zeta| \geq \varepsilon\}$ represents a finite set, and so $|\{i \in \mathbb{N} : |\zeta_i - \zeta| \geq \varepsilon\}|$ becomes a positive integer. Since $\{i \in I_r : |\zeta_i - \zeta| \geq \varepsilon\} \subseteq \{i \in \mathbb{N} : |\zeta_i - \zeta| \geq \varepsilon\}$, so we may write

$$\frac{1}{g^k(h_r)} g^k(|\{i \in I_r : |\zeta_i - \zeta| \geq \varepsilon\}|) \leq \frac{1}{g^k(h_r)} g^k(|\{i \in \mathbb{N} : |\zeta_i - \zeta| \geq \varepsilon\}|).$$

Then adding the limits on both given sides as r tends to ∞, we obtained that (ζ_i) is $S_\theta(g^k)$-statistically convergent sequence. □

Theorem 3.3.8 Suppose $\theta = (k_r)$ is a lacunary sequence and let g be considered an unbounded modulus function.

(i) If $g(u) \le u$ and $\lim_{u\to\infty} \frac{g^k(u)}{u} > 0$, then every S_θ-convergent sequence implies an $S_\theta(g^k)$-statistically convergent, i.e., $S_\theta \subset S_\theta(g^k)$.

(ii) If $g(u) \ge u$ and $\lim_{u\to\infty} \frac{u}{g^k(u)} > 0$, then every $S_\theta(g^k)$-statistically convergent sequence implies an S_θ-convergent, i.e., $S_\theta(g^k) \subset S_\theta$.

Proof. (i) Since g is increasing and $g(u) \le u$, then

$$g^k(u) \le g^{k-1}(u) \le \cdots \le g(u) \le u. \tag{3.1}$$

Let $(\zeta_i) \in S_\theta$ and given $\varepsilon > 0$. According to (3.1), we have

$$\begin{aligned}\frac{1}{h_r}|\{i \in I_r : |\zeta_i - \zeta| \ge \varepsilon\}| &\ge \frac{1}{h_r} g^k(|\{i \in I_r : |\zeta_i - \zeta| \ge \varepsilon\}|) \\ &= \frac{1}{g^k(h_r)} g^k(|\{i \in I_r : |\zeta_i - \zeta| \ge \varepsilon\}|) \frac{g^k(h_r)}{h_r}.\end{aligned}$$

Then adding the limits into both given sides as r tends to ∞, we obtain $(\zeta_i) \in S_\theta(g^k)$, since $\lim_{r\to\infty} \frac{g^k(h_r)}{h_r} > 0$. □

(ii) Since g is increasing and $g(u) \ge u$, then

$$g^k(u) \ge g^{k-1}(u) \ge \cdots \ge g(u) \ge u. \tag{3.2}$$

Let $(\zeta_i) \in S_\theta(g^k)$ and $\varepsilon > 0$ be given. Then according to (3.2), the next steps can be followed. □

Corollary 3.3.9 Assume that $\theta = (k_r)$ is a lacunary sequence and let g be considered an unbounded modulus function with $k_1 < k_2$. The cases below are possessed.

(i) If $g(u) \le u$ and $\lim_{u\to\infty} \frac{g^{k_2}(u)}{u} > 0$ then $S_\theta(g^{k_1}) \subset S_\theta(g^{k_2})$.

(ii) If $g(u) \ge u$ and $\lim_{u\to\infty} \frac{u}{g^{k_2}(u)} > 0$ then $S_\theta(g^{k_2}) \subset S_\theta(g^{k_1})$.

The proof of (i) and (ii) can be done with the help of (3.1) and (3.2), respectively.

Theorem 3.3.10 Assume that $\theta = (k_r)$ is a lacunary sequence and let g and h be considered two unbounded modulus functions with $g(u) \le h(u)$ for every $u \in \mathbb{R}^+ \cup \{0\}$.

(i) If $\lim_{u\to\infty} \frac{g^{k+1}(u)}{g^k(h(u))} > 0$, then $S_\theta(g^k \circ h) \subset S_\theta(g^{k+1})$.

(ii) If $\lim_{u\to\infty} \frac{h^k(g(u))}{h^{k+1}(u)} > 0$, then $S_\theta(h^{k+1}) \subset S_\theta(h^k \circ g)$.

Since g and h are increasing and $g(u) \le h(u)$ for every $u \in \mathbb{R}^+ \cup \{0\}$, then

$$g^{k+1}(u) \le g^k\big(h(u)\big). \tag{3.3}$$

and

$$h^k\big(g(u)\big) \le h^{k+1}(u). \tag{3.4}$$

The proof of (i) and (ii) follow from (3.3) and (3.4), respectively.

Definition 3.3.11 Suppose that $\theta = (k_r)$ is a lacunary sequence and let g be considered an unbounded modulus function. Let $\zeta \in \mathbb{C}$ be a number, then the sequence (ζ_i) in $\mathbb{C}$ is defined to be lacunary statistically convergent to ζ according to $g^{k,m}$ (or $S_\theta(g^{k,m})$-convergent), if

$$\lim_{r\to\infty} \frac{1}{g^m(h_r)} g^k(|\{i \in I_r : |\zeta_i - \zeta| \ge \varepsilon\}|) = 0.$$

Then, we refer it to as $\zeta_i \to \zeta(S_\theta(g^{k,m}))$ or $S_\theta(g^{k,m}) - \lim \zeta_i = \zeta$. The collection of all $S_\theta(g^{k,m})$-convergent sequences is indicated by $S_\theta(g^{k,m})$. That is

$$S_\theta\big(g^{k,m}\big) = \left\{(\zeta_i) : \lim_{r\to\infty} \frac{1}{g^m(h_r)} g^k(|\{i \in I_r : |\zeta_i - \zeta| \ge \varepsilon\}|) = 0 \text{ for some } \zeta \in \mathbb{C}\right\}.$$

Notice that in the case of $g(u) = u$, the thoughts of $S_\theta(g^{k,m})$-convergence and S_θ-convergence are the same. This means that $S_\theta(g^{k,m})$ will become S_θ. As well as in the case of $k = m$ we see that $S_\theta(g^{k,m}) = S_\theta(g^k)$.

Theorem 3.3.12 Assume $\theta = (k_r)$ is a lacunary sequence and g is any unbounded modulus function. Let $k_1, k_2, m_1, m_2 \in \mathbb{N}$ with $k_1 < k_2$ and $m_1 < m_2$. If $g(u) \le u$, then

(i) $S_\theta\big(g^{k_1,m}\big) \subset S_\theta\big(g^{k_2,m}\big)$ for each $m \in \mathbb{N}$,

(ii) $S_\theta\big(g^{k,m_2}\big) \subset S_\theta\big(g^{k,m_1}\big)$ for each $k \in \mathbb{N}$.

Proof. (i) Let $(\zeta_i) \in S_\theta(g^{k_1,m})$ and given $\varepsilon > 0$. Then according to (3.1), we may write

$$\frac{g^{k_2}(|\{i \in I_r : |\zeta_i - \zeta| \ge \varepsilon\}|)}{g^m(h_r)} \le \frac{g^{k_1}(|\{i \in I_r : |\zeta_i - \zeta| \ge \varepsilon\}|)}{g^m(h_r)}$$

for every $r \in \mathbb{N}$. Adding the limits into both given sides as r tends to ∞, we have

$$\lim_{r\to\infty} \frac{g^{k_2}(|\{i \in I_r : |\zeta_i - \zeta| \ge \varepsilon\}|)}{g^m(h_r)} = 0.$$

So, we obtain that $(\zeta_i) \in S_\theta(g^{k_2,m})$. □

(ii) Let $(\zeta_i) \in S_\theta(g^{k,m_2})$. Then according to (3.1), we may write

$$\frac{g^k(|\{i \in I_r: |\zeta_i - \zeta| \geq \varepsilon\}|)}{g^{m_1}(h_r)} \leq \frac{g^k(|\{i \in I_r: |\zeta_i - \zeta| \geq \varepsilon\}|)}{g^{m_2}(h_r)}$$

for every $r \in \mathbb{N}$. Adding the limits into both given sides as r tends to ∞, we have

$$\lim_{r\to\infty} \frac{g^k(|\{i \in I_r: |\zeta_i - \zeta| \geq \varepsilon\}|)}{g^{m_1}(h_r)} = 0.$$

So, we obtain that $(\zeta_i) \in S_\theta(g^{k,m_1})$. Hence the proof. □

Theorem 3.3.13 Assume $\theta = (k_r)$ is a lacunary sequence and g is any unbounded modulus function. Let $k_1, k_2, m_1, m_2 \in \mathbb{N}$ with $k_1 < k_2$ and $m_1 < m_2$. If $g(u) \geq u$, then

(i) $S_\theta(g^{k_2,m}) \subset S_\theta(g^{k_1,m})$ for each $m \in \mathbb{N}$,

(ii) $S_\theta(g^{k,m_1}) \subset S_\theta(g^{k,m_2})$ for each $k \in \mathbb{N}$.

Then the proof is done by using (3.2) in the same way of Theorem 3.3.12.

Theorem 3.3.14 Assume $\theta = (k_r)$ is a lacunary sequence and g is any unbounded modulus function with $m > k$.

(i) If $g(u) \leq u$, then $S_\theta(g^{k,m}) \subset S_\theta(g^{m,k})$.

(ii) If $g(u) \geq u$, then $S_\theta(g^{m,k}) \subset S_\theta(g^{k,m})$.

Proof. (i) Let $(\zeta_i) \in S_\theta(g^{k,m})$. Since $g(u) \leq u$ and $m > k$ then according to (3.1) we may state

$$\frac{g^k(|\{i \in I_r: |\zeta_i - \zeta| \geq \varepsilon\}|)}{g^m(h_r)} \geq \frac{g^m(|\{i \in I_r: |\zeta_i - \zeta| \geq \varepsilon\}|)}{g^m(h_r)}$$

$$\geq \frac{g^m(|\{i \in I_r: |\zeta_i - \zeta| \geq \varepsilon\}|)}{g^k(h_r)}$$

Adding the limits into both given sides as r tends to ∞, we have

$$\lim_{r\to\infty} \frac{g^m(|\{i \in I_r: |\zeta_i - \zeta| \geq \varepsilon\}|)}{g^k(h_r)} = 0.$$

So, we possess that $(\zeta_i) \in S_\theta(g^{m,k})$. The proof of (ii) is similarly done by using (3.2). □

Corollary 3.3.15 Consider $\theta = (k_r)$ is a lacunary sequence and g is any unbounded modulus function with $m > k$. The results below are concluded.

(i) If $g(u) \leq u$, then $S_\theta\left(g^{k,m}\right) \subset S_\theta(g^k)$.

(ii) If $g(u) \geq u$, then $S_\theta(g^k) \subset S_\theta\left(g^{k,m}\right)$.

The proof of (i) and (ii) can be done by using (3.1) and (3.2), respectively, and we omit them.

Theorem 3.3.16 Assume $\theta = (k_r)$ is a lacunary sequence and g is any unbounded modulus function. If $\lim_{u\to\infty} \frac{g^k(u)}{g^m(u)} > 0$ exists, then $S_\theta(g^k) = S_\theta(g^{k,m})$.

Proof. Let $(\zeta_i) \in S_\theta(g^{k,m})$, then we may write

$$\frac{g^k(|\{i \in I_r : |\zeta_i - \zeta| \geq \varepsilon\}|)}{g^m(h_r)} = \frac{g^k(|\{i \in I_r : |\zeta_i - \zeta| \geq \varepsilon\}|)}{g^k(h_r)} \frac{g^k(h_r)}{g^m(h_r)}.$$

Adding the limits on both given sides as r tends to ∞, we have

$$\lim_{r\to\infty} \frac{g^k(|\{i \in I_r : |\zeta_i - \zeta| \geq \varepsilon\}|)}{g^k(h_r)} \frac{g^k(h_r)}{g^m(h_r)} = 0,$$

since $\lim_{u\to\infty} \frac{g^k(h_r)}{g^m(h_r)} > 0$ exists. So, we obtain that $(\zeta_i) \in S_\theta(g^k)$. The proof on the other hand is similarly done. Hence the proof. □

Corollary 3.3.17 Given a lacunary sequence $\theta = (k_r)$ and let g be considered an unbounded modulus function. If $\lim_{u\to\infty} \frac{g^k(u)}{g^m(u)} > 0$ exists, then $S_\theta(g^{k,m}) = S_\theta(g^{m,k})$.

The proof follows from Theorem 3.3.16.

4. CESARO SUMMABILITY

4.1. Strong Cesàro Summability with Respect to a Modulus Function

A generalization on strong Cesàro summability was made by using a modulus function, for that, another concept has been concluded. Later, the connections between the sets of g-strong Cesàro summable sequences were determined [51].

Definition 4.1.1 [52] Assume that g is an unbounded modulus function. Then a sequence (ζ_i) of real (or complex) numbers is named g-strongly Cesàro summable (or w^g-summable) to ζ, if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} g(|\zeta_i-\zeta|)=0.$$

In this manner, $\zeta_i \to \zeta(w^g)$ is used, and the collection of all sequences of g-strongly Cesàro summable is indicated by w^g. The collection of sequences in which $\zeta = 0$ in the definition of w^g is indicated by w_0^g. In other forms, here are the definitions of the spaces w^g and w_0^g as follows

$$w_0^g=\left\{(\zeta_i)\colon \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} g(|\zeta_i|)=0\right\},$$

$$w^g=\left\{(\zeta_i)\colon \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} g(|\zeta_i-\zeta|)=0 \text{ for some number } \zeta\right\}.$$

Note that in the case of the identity modulus $g(u)=u$ the g-strong Cesàro summability will be reduced to the strong Cesàro summability, and so the sequence spaces w_0^g and w^g are become w_0 and w, respectively.

Theorem 4.1.2 [53] Assume g is any modulus function. Then

(i) $w \subset w^g$

(ii) $w_0 \subset w_0^g$

Proof. For the first part, let $(\xi_i) \in w$ be taken, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} g(|\xi_i-\xi|)=0$$

for some ξ. Given $\varepsilon > 0$ and take δ with $0<\delta<1$ such that $g(t)<\varepsilon$ for $t\in(0,\delta]$. Let $\zeta_i = |\xi_i-\xi|$ and consider

$$\sum_{i=1}^{n} g(\zeta_i) = \sum_{\substack{i=1 \\ \zeta_i \le \delta}}^{n} g(\zeta_i) + \sum_{\substack{i=1 \\ \zeta_i > \delta}}^{n} g(\zeta_i).$$

Since $g(\zeta_i) < \varepsilon$ for $\zeta_i \le \delta$, then $\sum_{i=1}^{n} g(\zeta_i) < \varepsilon n$ and also for $\zeta_i > \delta$, we have $\zeta_i < \zeta_i/\delta < 1 + [\zeta_i/\delta]$, where $[c]$ intends the integer component of the real number c. Since g is a modulus, then $g(\zeta_i) \le g(1 + [\zeta_i/\delta]) \le g(1)(1 + [\zeta_i/\delta]) \le 2g(1)(\zeta_i/\delta)$, so we obtain

$$\sum_{\substack{i=1 \\ \zeta_i > \delta}}^{n} g(\zeta_i) \le \frac{2g(1)}{\delta} \sum_{\substack{i=1 \\ \zeta_i > \delta}}^{n} \zeta_i.$$

Therefore,

$$\frac{1}{n}\sum_{i=1}^{n} g(|\xi_i - \xi|) \le \varepsilon + \frac{2g(1)}{\delta}\frac{1}{n} \sum_{\substack{i=1 \\ |\xi_i - \xi| > \delta}}^{n} |\xi_i - \xi| \le \varepsilon + \frac{2g(1)}{\delta}\frac{1}{n}\sum_{i=1}^{n} |\xi_i - \xi|.$$

This implies that $(\xi_i) \in w^g$. Hence the proof. □

The proof of the second part is the same as the first part when $\xi = 0$ is taken and so it is omitted.

Lemma 4.1.3 [54] For any modulus g, then $\lim_{u\to\infty} \frac{g(u)}{u} = \beta$ exists and $\lim_{u\to\infty} \frac{g(u)}{u} = \inf\left\{\frac{g(u)}{u} : u > 0\right\}$.

Theorem 4.1.4 [53] Assume g is any modulus. If $\lim_{u\to\infty} \frac{g(u)}{u} > 0$ then $w^g \subset w$.

Proof. Using Lemma 4.1.3 there is $\beta > 0$ such that $\beta = \inf\left\{\frac{g(u)}{u} : u > 0\right\}$. So it gives us $\beta u \le g(u)$ and then $u \le \frac{1}{\beta} g(u)$ for all $u \ge 0$ and so

$$\frac{1}{n}\sum_{i=1}^{n} |\zeta_i - \zeta| \le \frac{1}{\beta}\frac{1}{n}\sum_{i=1}^{n} g(|\zeta_i - \zeta|).$$

This concludes that $(\zeta_i) \in w^g$ implies $(\zeta_i) \in w$. Hence the proof. □

Corollary 4.1.5 Assume g is any modulus. If $\lim_{u\to\infty} \frac{g(u)}{u} > 0$ then $w^g = w$.

Theorem 4.1.6 [51] Assume g and h are modulus functions. If

$$\sup_{u\in(0,\infty)} \frac{g(u)}{h(u)} < \infty,$$

then we have $w^h \subset w^g$.

Proof. Assume that $\alpha = \sup_{u\in(0,\infty)} \frac{g(u)}{h(u)} < \infty$ then $\frac{g(u)}{h(u)} \leq \alpha$ and so that $g(u) \leq \alpha h(u)$ for every $u \in \mathbb{R}^+ \cup \{0\}$. Now it is obvious that $\alpha > 0$ and let $(\zeta_i) \in w^h$, then we may write

$$\frac{1}{n}\sum_{i=1}^{n} g(|\zeta_i - \zeta|) \leq \frac{\alpha}{n}\sum_{i=1}^{n} h(|\zeta_i - \zeta|).$$

Providing the limits into both given sides as $n \to \infty$, we get that $(\zeta_i) \in w^g$. Hence the proof. □

Note that the above inclusion may be strict, for this, the instance below demonstrates the strictness of the inclusion.

Example 4.1.7 Given the sequence (ζ_i) which is defined as

$$\zeta_i = \begin{cases} i, & i = m^3 \\ 0, & i \neq m^3 \end{cases} \quad m \in \mathbb{N}$$

and the modulus $g(u) = \frac{u}{u+1}$ and $h(u) = u$. Now $\sup_{u\in(0,\infty)} \frac{g(u)}{h(u)} = 1 < \infty$ and since

$$\frac{1}{n}\sum_{i=1}^{n} g(|\zeta_i|) = \frac{1}{n}\sum_{\substack{i=1\\ i=m^3}}^{n} g(i) + \frac{1}{n}\sum_{\substack{i=1\\ i\neq m^3}}^{n} g(0) = \frac{1}{n}\sum_{\substack{i=1\\ i=m^3}}^{n} \frac{i}{i+1} < \frac{\sqrt[3]{n}}{n} \to 0 \text{ as } n \to \infty,$$

then we have $\zeta \in w^g$. But since

$$\frac{1}{n}\sum_{i=1}^{n} h(|\zeta_i|) = \frac{1}{n}\sum_{\substack{i=1\\ i=m^3}}^{n} h(i) + \frac{1}{n}\sum_{\substack{i=1\\ i\neq m^3}}^{n} g(0) \geq \frac{1}{n}\left[\frac{([\sqrt[3]{n}] - 1)([\sqrt[3]{n}])}{2}\right]^2 \to \infty \; as \; n \to \infty,$$

we obtain $\zeta \notin w^h$, where $[c]$ indicates the integral part of the real number c. Hence $\zeta \in w^g - w^h$ and the inclusion is strict.

4.2. Lacunary Strong Convergence According to a Modulus Function

With the use of a modulus function g and a lacunary sequence $\theta = (k_r)$, the idea of lacunary strong convergence according to g^k is introduced. After that, the relation between this idea and statistical convergence is investigated. Also, the connection between the sets of lacunary strongly convergent sequences according to g^k is talked about.

Definition 4.2.1 Suppose that $\theta = (k_r)$ is a lacunary sequence. The space N_θ of lacunary strongly convergent sequences has been created by Freedman [17] as follows

$$N_\theta = \left\{ (\zeta_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} |\zeta_i - \zeta| = 0 \text{ for some } \zeta \right\}.$$

In this manner, if $(\zeta_i) \in N_\theta$ then (ζ_i) is considered to be lacunary strongly convergent to ζ, and $\zeta_i \to \zeta(N_\theta)$ or $N_\theta - \lim \zeta_i = \zeta$ is refered. Notice that this space is indeed a BK space with the norm $\|\cdot\|_\theta$, where for $\eta = (\eta_i) \in N_\theta$ it is expressed as follows

$$\|\eta\|_\theta = \sup_r \frac{1}{h_r} \sum_{i\in I_r} |\eta_i|.$$

Definition 4.2.2 Suppose that $\theta = (k_r)$ is a lacunary sequence and let g be considered a modulus function. Let $\zeta \in \mathbb{C}$ be a number, then the sequence (ζ_i) in $\mathbb{C}$ is defined to be lacunary strongly convergent to ζ according to g^k (or $N_\theta(g^k)$-strongly convergent), if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} g^k(|\zeta_i - \zeta|) = 0.$$

Then, we refer it to as $\zeta_i \to \zeta(N_\theta(g^k))$ or $N_\theta(g^k) - \lim \zeta_i = \zeta$. We indicate the collection of all $N_\theta(g^k)$-strongly convergent sequences by $N_\theta(g^k)$. That is

$$N_\theta(g^k) = \left\{ (\zeta_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} g^k(|\zeta_i - \zeta|) = 0 \text{ for some } \zeta \in \mathbb{C} \right\}.$$

Notice that in this definition the function g is not required to be an unbounded modulus. If we put $g(u) = u$, then the $N_\theta(g^k)$-strong convergence is reduced to the N_θ-strong convergence. $N_\theta^0(g^k)$ is indicated as the collection of all sequences in which $\zeta = 0$ in the definition of $N_\theta(g^k)$. It's clear to see that if $k = 1$, then the spaces $N_\theta(g^k)$ and $N_\theta^0(g^k)$ will respectively become the same as $N_\theta(g)$ and $N_\theta^0(g)$ of Pehlivan and Fisher [23]. In the particular case of $\theta =$

(2^r) and $g(u) = u$, we see that $N_\theta(g^k) = |\sigma_1|$, where $|\sigma_1| = w$ is the set of strongly Cesaro summable sequences.

Theorem 4.2.3 Consider a lacunary sequence $\theta = (k_r)$ and let g be a modulus function. Then the sequence sets $N_\theta(g^k)$ and $N_\theta^0(g^k)$ are linear spaces.

The proof can be easily done by using these inequalities: $|\alpha| \leq A_\alpha$ and $|\beta| \leq B_\beta$ where $\alpha, \beta \in \mathbb{C}$, and A_α and B_β are positive integers, and we leave it.

Theorem 4.2.4 Given a lacunary sequence $\theta = (k_r)$ and let g be considered a modulus function. If $g(u) \leq u$ for every $u \in \mathbb{R}^+ \cup \{0\}$, then every lacunary strongly convergent sequence implies $N_\theta(g^k)$-strongly convergent, i.e., $N_\theta \subset N_\theta(g^k)$.

Proof. Let $(\zeta_i) \in N_\theta$. Since g is increasing and $g(u) \leq u$ for every $u \in \mathbb{R}^+ \cup \{0\}$, then according to (3.1), we may write

$$\frac{1}{h_r}\sum_{i\in I_r} g^k(|\zeta_i - \zeta|) \leq \frac{1}{h_r}\sum_{i\in I_r} g^{k-1}(|\zeta_i - \zeta|) \leq \cdots \leq \frac{1}{h_r}\sum_{i\in I_r} g(|\zeta_i - \zeta|) \leq \frac{1}{h_r}\sum_{i\in I_r} |\zeta_i - \zeta|.$$

That is,

$$\frac{1}{h_r}\sum_{i\in I_r} g^k(|\zeta_i - \zeta|) \leq \frac{1}{h_r}\sum_{i\in I_r} |\zeta_i - \zeta|.$$

Then adding the limits into both given sides as r tends to ∞, we have

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{i\in I_r} g^k(|\zeta_i - \zeta|) = 0.$$

Therefore, we obtain that $(\zeta_i) \in N_\theta(g^k)$. Hence the proof. □

The above inclusion relationship may be strict, for this, the instance, which is mentioned below, clarifies the strictness of the inclusion.

Suppose $\theta = (k_r)$ is a given lacunary sequence. Define (γ_i) as γ_i is to be $1, 2, \ldots, [\sqrt{h_r}]$ at the first $[\sqrt{h_r}]$ integers in I_r, and $\zeta_i = 0$ otherwise, where $[c]$ indicates the integral component of the real number c. Now let $(\zeta_i) = (\gamma_i)$ and put $g(u) = \frac{u}{u+1}$ then $g(u) \leq u$ for every $u \in \mathbb{R}^+ \cup \{0\}$ and $g^k(u) = \frac{u}{ku+1}$. For that, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} g^k(|\zeta_i|) = \lim_{r\to\infty}\frac{1}{h_r}\frac{[\sqrt{h_r}][\sqrt{h_r}]}{(k[\sqrt{h_r}]+1)} = 0,$$

but

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}|\zeta_i| = \lim_{r\to\infty}\frac{1}{h_r}[\sqrt{h_r}][\sqrt{h_r}] = 1,$$

hence $(\zeta_i) \in N_\theta(g^k) - N_\theta$.

Corollary 4.2.5 Suppose that $\theta = (k_r)$ is a given lacunary sequence and let g be considered a modulus function. If $g(u) \le u$ for every $u \in \mathbb{R}^+ \cup \{0\}$, then

$$N_\theta \subset N_\theta(g) \subset \cdots \subset N_\theta(g^{k-1}) \subset N_\theta(g^k)$$

Theorem 4.2.6 Consider a lacunary sequence $\theta = (k_r)$ and let g be a modulus function. If $g(u) \ge u$ for every $u \in \mathbb{R}^+ \cup \{0\}$, then every $N_\theta(g^k)$-strongly convergent sequence implies lacunary strongly convergent, i.e., $N_\theta(g^k) \subset N_\theta$.

Proof. Let $(\zeta_i) \in N_\theta(g^k)$. Since g is increasing and $g(u) \ge u$ for every $u \in \mathbb{R}^+ \cup \{0\}$, then according to (3.2), we may write

$$\frac{1}{h_r}\sum_{i\in I_r} g^k(|\zeta_i - \zeta|) \ge \frac{1}{h_r}\sum_{i\in I_r} g^{k-1}(|\zeta_i - \zeta|) \ge \cdots \ge \frac{1}{h_r}\sum_{i\in I_r} g(|\zeta_i - \zeta|) \ge \frac{1}{h_r}\sum_{i\in I_r}|\zeta_i - \zeta|,$$

that is

$$\frac{1}{h_r}\sum_{i\in I_r} g^k(|\zeta_i - \zeta|) \ge \frac{1}{h_r}\sum_{i\in I_r}|\zeta_i - \zeta|.$$

Then adding the limits into both given sides as r tends to ∞, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}|\zeta_i - \zeta| = 0.$$

Therefore, we obtain that $(\zeta_i) \in N_\theta$. Hence the proof. □

Corollary 4.2.7 Suppose that $\theta = (k_r)$ is a given lacunary sequence and let g be considered a modulus function. If $g(u) \ge u$ for every $u \in \mathbb{R}^+ \cup \{0\}$, then

$$N_\theta(g^k) \subset N_\theta(g^{k-1}) \subset \cdots \subset N_\theta(g) \subset N_\theta$$

Corollary 4.2.8 Suppose that $\theta = (k_r)$ is a given lacunary sequence and let g be a modulus function with $k < n$.

(i) If $g(u) \leq u$ for every $u \in \mathbb{R}^+ \cup \{0\}$, then $N_\theta(g^k) \subset N_\theta(g^n)$.

(ii) If $g(u) \geq u$ for every $u \in \mathbb{R}^+ \cup \{0\}$, then $N_\theta(g^n) \subset N_\theta(g^k)$.

The proof of (i) and (ii) is done by using (3.1) and (3.2), respectively.

Theorem 4.2.9 Consider a lacunary sequence $\theta = (k_r)$ and let g and h be considered two modulus functions. If $g(u) \leq h(u)$ for every $u \in \mathbb{R}^+ \cup \{0\}$, then

(i) $N_\theta(g^k \circ h) \subset N_\theta(g^{k+1})$.

(ii) $N_\theta(h^{k+1}) \subset N_\theta(h^k \circ g)$.

Proof. (i) Let $(\zeta_i) \in N_\theta(g^k \circ h)$. Since $g(u) \leq h(u)$ for every $u \in \mathbb{R}^+ \cup \{0\}$, then according to (3.3), we may write

$$\frac{1}{h_r}\sum_{i\in I_r} g^{k+1}(|\zeta_i - \zeta|) \leq \frac{1}{h_r}\sum_{i\in I_r} g^k\big(h(|\zeta_i - \zeta|)\big),$$

after, adding the limits into both given sides as r tends to ∞, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} g^{k+1}(|\zeta_i - \zeta|) = 0.$$

Therefore, we conclude that $(\zeta_i) \in N_\theta(g^{k+1})$. The proof of (ii) is followed by (3.4).

Theorem 4.2.10 Consider a lacunary sequence $\theta = (k_r)$ and let g be given as a modulus function. If a sequence (ζ_i) is $N_\theta(g^k)$-strongly convergent to ζ, then it is S_θ-convergent to ζ, i.e., $N_\theta(g^k) \subset S_\theta$.

Proof. Assume that $(\zeta_i) \in N_\theta(g^k)$ and let $\varepsilon > 0$ be given, then we may write

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r} g^k(|\zeta_i - \zeta|) &= \frac{1}{h_r}\sum_{\substack{i\in I_r\\ |\zeta_i-\zeta|\geq\varepsilon}} g^k(|\zeta_i - \zeta|) + \frac{1}{h_r}\sum_{\substack{i\in I_r\\ |\zeta_i-\zeta|<\varepsilon}} g^k(|\zeta_i - \zeta|)\\ &\geq \frac{1}{h_r}\sum_{\substack{i\in I_r\\ |\zeta_i-\zeta|\geq\varepsilon}} g^k(|\zeta_i - \zeta|)\\ &\geq \frac{1}{h_r}|\{i \in I_r :\ |\zeta_i - \zeta| \geq \varepsilon\}|\ g^k(\varepsilon).\end{aligned}$$

Then adding the limits into both given sides as r tends to ∞, we get that $(\zeta_i) \in S_\theta$. Hence the proof. □

The following instance shows the strictness of the above inclusion relationship.

Example 4.2.11 Suppose $\theta = (k_r)$ is a given lacunary sequence. Choose $(\zeta_i) = (\gamma_i)$, where γ_i is given as in the above instance, and put $g(u) = 2u$ as a modulus function, then $g^k(u) = 2^k u$. Now for any given $\varepsilon > 0$, we have

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{i \in I_r : \left|\zeta_i\right| \geq \varepsilon\right\}\right| = \lim_{r\to\infty} \frac{\left[\sqrt{h_r}\right]}{h_r} = 0,$$

so, $(\zeta_i) \in S_\theta$. But since

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} g^k\left(\left|\zeta_i\right|\right) = \lim_{r\to\infty} \frac{2^k\left[\sqrt{h_r}\right]\left[\sqrt{h_r}\right]}{h_r} = 2^k,$$

then $\left(\zeta_i\right) \notin N_\theta(g^k)$. For that, the inclusion strict is demonstrated.

Theorem 4.2.12 Consider a lacunary sequence $\theta = (k_r)$ and let g be given as a modulus function. If a sequence (ζ_i) is bounded and S_θ-convergent to ζ, then it is $N_\theta(g^k)$-strongly convergent to ζ, i.e., $\ell_\infty \cap S_\theta \subset N_\theta(g^k)$.

Proof. Assume that (ζ_i) is a bounded sequence and $S_\theta - \lim \zeta_i = \zeta$ with a given $\varepsilon > 0$. Then there is some $H > 0$ such that $|\zeta_i - \zeta| \leq H$ for every i. Now we may write

$$\frac{1}{h_r}\sum_{i\in I_r} g^k(|\zeta_i - \zeta|) = \frac{1}{h_r} \sum_{\substack{i\in I_r \\ |\zeta_i - \zeta| \geq \varepsilon}} g^k(|\zeta_i - \zeta|) + \frac{1}{h_r} \sum_{\substack{i\in I_r \\ |\zeta_i - \zeta| < \varepsilon}} g^k(|\zeta_i - \zeta|)$$

$$\leq \frac{1}{h_r} \sum_{\substack{i\in I_r \\ ||\zeta_i - \zeta|| \geq \varepsilon}} g^k(H) + \frac{1}{h_r} \sum_{\substack{i\in I_r \\ |\zeta_i - \zeta| < \varepsilon}} g^k(\varepsilon)$$

$$\leq g^k(H) \frac{1}{h_r} |\{i \in I_r : |\zeta_i - \zeta| \geq \varepsilon\}| + \frac{h_r}{h_r} g^k(\varepsilon).$$

Then, by adding the limits into both given sides as r tends to ∞, we possess that the sequence (ζ_i) is $N_\theta(g^k)$-strongly convergent to ζ. Hence the proof. □

Corollary 4.2.13 Assume that $\theta = (k_r)$ is a given lacunary sequence and let g be considered a modulus function. Then from Theorem 4.2.10 and Theorem 4.2.12, we get the result $S_\theta \cap \ell_\infty = N_\theta(g^k) \cap \ell_\infty$.

Theorem 4.2.14 Consider a lacunary sequence $\theta = (k_r)$ and let g be given as a modulus function. Then $N_\theta(g^k) - \lim \zeta_i = \zeta$ is considered to be unique.

Proof. Suppose that $N_\theta(g^k) - \lim \zeta_i = \zeta_a$ and $N_\theta(g^k) - \lim \zeta_i = \zeta_b$. Then

$$\frac{1}{h_r}\sum_{i\in I_r} g^k(|\zeta_a - \zeta_b|) = \frac{1}{h_r}\sum_{i\in I_r} g^k(|(\zeta_i - \zeta_a) - (\zeta_i - \zeta_b)|)$$

$$\leq \frac{1}{h_r}\sum_{i\in I_r} g^k(|\zeta_i - \zeta_a| + |\zeta_i - \zeta_b|)$$

$$\leq \frac{1}{h_r}\sum_{i\in I_r} g^k(\zeta_i - \zeta_a) + \frac{1}{h_r}\sum_{i\in I_r} g^k\left(|\zeta_i - \zeta_b|\right).$$

Then adding the limits into both given sides as r tends to ∞, we have

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{i\in I_r} g^k(|\zeta_a - \zeta_b|) = 0.$$

So, we obtain that $\zeta_a = \zeta_b$. Hence the proof. □

5. MATRICES AND LUCAS NUMBERS

5.1. Matrix Transformations

An infinite matrix is sometimes given as the most general linear operator between two sequence spaces. So, the theory of matrix transformations is generally being of great interest in the study of sequence spaces. The investigation of the general theory of matrix transformations was initiated by specific findings in the field of summability theory.

By $A = (a_{mi})$, we assume an infinite matrix consisting of real or complex numbers a_{mi}, where $m, i \in \mathbb{N}$, and $\zeta = (\zeta_i)$ is an arbitrary sequence in ω. Then, we acquire the sequence $A\zeta$ (referred to as the A-transform of ζ) by the standard matrix product

$$A\zeta = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0i} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1i} & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2i} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \cdots \\ a_{m0} & a_{m1} & a_{m2} & \cdots & a_{mi} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_i \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} a_{00}\zeta_0 + a_{01}\zeta_1 + a_{02}\zeta_2 + \cdots + a_{0i}\zeta_i + \cdots \\ a_{10}\zeta_0 + a_{11}\zeta_1 + a_{12}\zeta_2 + \cdots + a_{1i}\zeta_i + \cdots \\ a_{20}\zeta_0 + a_{21}\zeta_1 + a_{22}\zeta_2 + \cdots + a_{2i}\zeta_i + \cdots \\ \vdots \\ a_{m0}\zeta_0 + a_{m1}\zeta_1 + a_{m2}\zeta_2 + \cdots + a_{mi}\zeta_i + \cdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_i a_{0i}\zeta_i \\ \sum_i a_{1i}\zeta_i \\ \sum_i a_{2i}\zeta_i \\ \vdots \\ \sum_i a_{mi}\zeta_i \\ \vdots \end{pmatrix}.$$

So, in this manner, the sequence ζ is transformed into the sequence $A\zeta = \{(A\zeta)_m\}$ with

$$(A\zeta)_m = \sum_i a_{mi}\zeta_i \quad (m \in \mathbb{N}) \tag{5.1}$$

such that the series $\sum_i a_{mi}\zeta_i$ converges for each $m \in \mathbb{N}$ [24].

Given two arbitrary sequence spaces X and Y. A is considered as a matrix mapping (transformation) from X into Y if $A\zeta = \{(A\zeta)_m\} \in Y$ exists for every $\zeta = (\zeta_i) \in X$ and we express it by $A: X \to Y$. The collection of all matrices A from X into Y is indicated by $(X: Y)$. In this manner, $A \in (X: Y)$ if and only if the series $\sum_i a_{mi}\zeta_i$ in (5.1) converges for each $m \in \mathbb{N}$ and for all $\zeta \in X$, and we have $A\zeta = \{(A\zeta)_m\} \in Y$ for all $\zeta \in X$.

Matrix transformations play a main role in summability theory of divergent sequences and series. A simple instance of this, is the Cesáro method C_1 of order one which is the most popular way of summability and is considered by a matrix $C_1 = (c_{mi})$ as follows

$$c_{mi} = \begin{cases} \dfrac{1}{m+1} & 0 \le i \le m \\ 0 & i > m \end{cases}$$

The Cesáro transform (or C_1-transform) of a sequence $(\zeta_i) \in \omega$ becomes the sequence (ξ_m) that is defined as follows

$$\xi_m = (C_1\zeta)_m = \frac{1}{m+1}\sum_{i=1}^{m} \zeta_i \text{ for all } m \in \mathbb{N}.$$

Consider $(\zeta_i) = \{(-1)^i\}_{i\in\mathbb{N}}$ which is clearly bounded but divergent, then

$$(\xi_m) = \left\{\frac{1+(-1)^m}{2(m+1)}\right\}$$

forms a null sequence. Or, clearly since

$$0 \le (C_1\zeta)_m \le \frac{1}{m+1}$$

for every $m \in \mathbb{N}$, hence $(C_1\zeta)_m \to 0$ as $m \to \infty$. This indicates that the divergent sequence ζ is C_1-summable to zero [24].

5.2. Lucas Band Matrix

The sequence $(L_i)_{i\in\mathbb{N}}$ is considered as Lucas numbers 1, 3, 4, 7, 11, 18, 29, ... which is known as the series of lame whose general form is provided by the Fibonacci recurrence relation with different initial conditions as follows

$$L_i = L_{i-1} + L_{i-2}, i \ge 2 \text{ and } L_0 = 2, L_1 = 1,$$

where the general form of the Fibonacci sequence is given by;

$$F_i = F_{i-1} + F_{i-2}, i \ge 2 \text{ and } F_0 = F_1 = 1,$$

so, extremely as the Fibonacci numbers, each Lucas number is considered to be the sum of its two immediate previous terms [55, 56]. The connections between Fibonacci and Lucas numbers can be determined in some equalities, some of them are shown as follows

$$L_{i-1} + L_{i+1} = 5F_i \text{ and } F_{i-1} + F_{i+1} = L_i,$$

these hold for $i=1$, and for $i=2$, and therefore, after addition, also for $i=3, i=4$ and hence for any integer i. As a consequence of the above equalities any formula contains Fibonacci and Lucas numbers can be expressed in terms just one of these types; however, this is not always convenient, for that, there are some other connections between Fibonacci and Lucas numbers (see in [56]). There are various interesting properties and applications for Lucas numbers. The golden ratio $\alpha=(1+\sqrt{5})/2$ is an outcome of Lucas numbers, and it has many applications [55, 56].

In the see of the above information, by using the Lucas sequence and two real numbers r and t such that $r, t \neq 0$, the Lucas band matrix $\hat{E}(r,t)=(\hat{E}_{im}(r,t))$ is established as follows:

$$\hat{E}_{im}(r,t)=\begin{cases} t\dfrac{L_i}{L_{i-1}} & (m=i-1) \\ r\dfrac{L_{i-1}}{L_i} & (m=i) \\ 0 & (m>i \text{ or } 0\leq m<i-1) \end{cases} \tag{5.2}$$

The Lucas band matrix is a double band matrix and it plays a great role in summability theory. It might be used to define the Lucas transform (or, $\hat{E}$-transform) [57]. The $\hat{E}$-transform for a sequence $\zeta=(\zeta_i)$ is formed by using (5.2) as below:

$$\hat{E}_i(r,t)(\zeta)=r\frac{L_{i-1}}{L_i}\zeta_i+t\frac{L_i}{L_{i-1}}\zeta_{i-1}, \qquad i\geq 1. \tag{5.3}$$

The assertions: (i) $\hat{E}_i(r,t)(\zeta+\xi)=\hat{E}_i(r,t)(\zeta)+\hat{E}_i(r,t)(\xi)$, (ii) $\hat{E}_i(r,t)(\zeta-\xi)=\hat{E}_i(r,t)(\zeta)-\hat{E}_i(r,t)(\xi)$ and (iii) $\hat{E}_i(r,t)(\alpha\zeta)=\alpha\hat{E}_i(r,t)(\zeta)$ for $i\geq 1$ are some properties, the $\hat{E}$-transform contains them, where $\zeta=(\zeta_i)$ and $\xi=(\xi_i)$ are two arbitrary sequences of complex numbers and α is a fixed in $\mathbb{C}$. These properties can be easily verified. If we consider the first one (i), then with the help of (5.3) and for $i\geq 1$, we may write

$$\begin{aligned}\hat{E}_i(r,t)(\zeta+\xi)&=r\frac{L_{i-1}}{L_i}(\zeta_i+\xi_i)+t\frac{L_i}{L_{i-1}}(\zeta_{i-1}+\xi_{i-1})\\ &=r\frac{L_{i-1}}{L_i}\zeta_i+r\frac{L_{i-1}}{L_i}\xi_i+t\frac{L_i}{L_{i-1}}\zeta_{i-1}+t\frac{L_i}{L_{i-1}}\xi_{i-1}\\ &=r\frac{L_{i-1}}{L_i}\zeta_i+t\frac{L_i}{L_{i-1}}\zeta_{i-1}+r\frac{L_{i-1}}{L_i}\xi_i+t\frac{L_i}{L_{i-1}}\xi_{i-1}\\ &=\hat{E}_i(r,t)(\zeta)+\hat{E}_i(r,t)(\xi).\end{aligned}$$

The $\hat{E}$-transform will be the main object in identifying some new concepts and creating some new sequence spaces in the next parts of this study.

6. LACUNARY STATISTICAL CONVERGENCE AND LACUNARY STRONG CONVERGENCE BY USING LUCAS BAND MATRIX AND MODULUS FUNCTIONS

Assume that $\mathbb{G}$ is the set of sequences of modulus functions $G = (g_i)$ such that $\lim_{u\to 0^+} \sup_i g_i(u) = 0$. The sequence of modulus functions identified by G is indicated by $G = (g_i) \in \mathbb{G}$. We write $G^k = (g_i^k) = \{g_1^k, g_2^k, \ldots\}$ $(k \in \mathbb{N})$, and let $v = (v_i)$ be considered as a sequence of strictly positive real numbers. We use these notations throughout the next.

6.1. Lacunary Statistical $\widehat{E}(r,t)$-Convergence According to G^k

In this subsection, a sequence of unbounded modulus functions G is first used to introduce the idea of G^k-density of a natural numbers' subset, then with the help of the Lucas transform $\widehat{E}(r,t)$ and a lacunary sequence $\theta = (k_r)$, this definition is extended into establishing the concept of lacunary statistical $\widehat{E}(r,t)$-convergence according to G^k. Also, some connections between new concepts and some others are found.

Definition 6.1.1 Assume that $G = (g_r)$ is a sequence of unbounded modulus functions in $\mathbb{G}$. The number $\delta^{G^k}(B)$ of a subset $B \subset \mathbb{N}$ is introduced by

$$\delta^{G^k}(B) = \lim_{r\to\infty} \frac{1}{g_r^k(r)} g_r^k(|\{i \le r : i \in B\}|)$$

in the case the above limit exists, and it is named to be G^k-density of B.

Remark 6.1.2 If $g_r(u) = u$ is taken then the concept of G^k-density is returned to the natural density, as well as, if $g_r(u) = g(u)$ then from G^k-density we shall have g^k-density, for all $r \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$.

Definition 6.1.3 Suppose that $\theta = (k_r)$ is a lacunary sequence and let $G = (g_r)$ be considered a sequence of unbounded modulus functions in $\mathbb{G}$. Let $s \in \mathbb{C}$ be a number, then the sequence $\zeta = (\zeta_i)$ in $\mathbb{C}$ is defined to be lacunary statistically $\hat{E}(r,t)$-convergent to s according to G^k (or, $S_\theta(\hat{E}(r,t), G^k)$-statistically convergent), if

$$\lim_{r\to\infty} \frac{1}{g_r^k(h_r)} g_r^k\left(\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \ge \varepsilon\right\}\right|\right) = 0.$$

In this manner, we shall write $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^k))$ or $S_\theta(\hat{E}(r,t), G^k) - \lim \zeta_i = s$. The collection of all $S_\theta(\hat{E}(r,t), G^k)$-statistically convergent sequences is indicated by $S_\theta(\hat{E}(r,t), G^k)$. That is

$$S_\theta\big(\hat{E}(r,t), G^k\big) = \left\{\zeta = (\zeta_i): \lim_{r\to\infty} \frac{1}{g_r^k(h_r)} g_r^k\big(\big|\{i \in I_r: \big|\hat{E}_i(r,t)(\zeta) - s\big| \geq \varepsilon\}\big|\big) = 0 \text{ for } s \in \mathbb{C}\right\}.$$

Notice that in the case of $g_r(u) = u$, the concept of $S_\theta(\hat{E}(r,t), G^k)$-statistically convergent will reduce to $S_\theta(\hat{E}(r,t))$-statistically convergent, as well as, if $k = 1$ then $g_r^k(u) = g_r(u)$ and so from $S_\theta(\hat{E}(r,t), G^k)$, we have the sequence space $S_\theta(\hat{E}(r,t), G)$, for all $r \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$. $S_\theta^0(\hat{E}(r,t), G^k)$ is indicated as the collection of all sequences in which $s = 0$ in the definition of $S_\theta(\hat{E}(r,t), G^k)$.

Theorem 6.1.4 Assume that $\theta = (k_r)$ is a lacunary sequence and G is considered to be a sequence of unbounded modulus functions in $\mathbb{G}$. Let (ζ_i), (ξ_i) be sequences of complex numbers, then cases below are possessed.

(i) If $S_\theta(\hat{E}(r,t), G^k) - \lim \zeta_i = s$ and $\alpha \in \mathbb{C}$, then $S_\theta(\hat{E}(r,t), G^k) - \lim(\alpha\zeta_i) = \alpha s$.

(ii) If $S_\theta(\hat{E}(r,t), G^k) - \lim \zeta_i = s_1$ and $S_\theta(\hat{E}(r,t), G^k) - \lim \xi_i = s_2$, then $S_\theta(\hat{E}(r,t), G^k) - \lim(\zeta_i + \xi_i) = s_1 + s_2$.

Proof. (i) The proof is clearly done for $\alpha = 0$. Now let $\alpha \neq 0$ and we suppose $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^k))$, then we may write

$$\frac{1}{g_r^k(h_r)} g_r^k\big(\big|\{i \in I_r: \big|\hat{E}_i(r,t)(\alpha\zeta) - \alpha s\big| \geq \varepsilon\}\big|\big)$$
$$= \frac{1}{g_r^k(h_r)} g_r^k\big(\big|\{i \in I_r: |\alpha|\big|\hat{E}_i(r,t)(\zeta) - s\big| \geq \varepsilon\}\big|\big)$$
$$= \frac{1}{g_r^k(h_r)} g_r^k\left(\left|\left\{i \in I_r: \big|\hat{E}_i(r,t)(\zeta) - s\big| \geq \frac{\varepsilon}{|\alpha|}\right\}\right|\right).$$

Thus

$$\frac{1}{g_r^k(h_r)} g_r^k\big(\big|\{i \in I_r: \big|\hat{E}_i(r,t)(\alpha\zeta) - \alpha s\big| \geq \varepsilon\}\big|\big) \to 0 \; as \; r \to \infty.$$

This implies that $(\alpha\zeta_i) \to \alpha s(S_\theta(\hat{E}(r,t), G^k))$, or clearly $S_\theta(\hat{E}(r,t), G^k) - \lim(\alpha\zeta_i) = \alpha s$.

(ii) Suppose that $\zeta_i \to s_1(S_\theta(\hat{E}(r,t), G^k))$ and $\xi_i \to s_2(S_\theta(\hat{E}(r,t), G^k))$, then we may write

$$\frac{1}{g_r^k(h_r)} g_r^k\left(\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta+\xi) - (s_1+s_2)\right| \geq \varepsilon\right\}\right|\right)$$
$$= \frac{1}{g_r^k(h_r)} g_r^k\left(\left|\left\{i \in I_r : \left|\left(\hat{E}_i(r,t)(\zeta) - s_1\right) + \left(\hat{E}_i(r,t)(\xi) - s_2\right)\right| \geq \varepsilon\right\}\right|\right)$$
$$\leq \frac{1}{g_r^k(h_r)} g_r^k\left(\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s_1\right| \geq \frac{\varepsilon}{2}\right\}\right| + \left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\xi) - s_2\right| \geq \frac{\varepsilon}{2}\right\}\right|\right)$$
$$\leq \frac{1}{g_r^k(h_r)} g_r^k\left(\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s_1\right| \geq \frac{\varepsilon}{2}\right\}\right|\right) + \frac{1}{g_r^k(h_r)} g_r^k\left(\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\xi) - s_2\right| \geq \frac{\varepsilon}{2}\right\}\right|\right).$$

Thus

$$\frac{1}{g_r^k(h_r)} g_r^k\left(\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta+\xi) - (s_1+s_2)\right| \geq \varepsilon\right\}\right|\right) \to 0 \text{ as } r \to \infty.$$

This concludes that $(\zeta_i + \xi_i) \to (s_1 + s_2)(S_\theta(\hat{E}(r,t), G^k))$, or clearly $S_\theta(\hat{E}(r,t), G^k) - \lim(\zeta_i + \xi_i) = s_1 + s_2$. Hence the proof. □

Theorem 6.1.5 Assume that $\theta = (k_r)$ and $\vartheta = (s_r)$ are lacunary sequences with $I_r \subset J_r$, and let $G = (g_r)$ be given as a sequence of any unbounded modulus functions in $\mathbb{G}$. If $\lim_{r\to\infty} sup\left(g_r^k(\ell_r)/g_r^k(h_r)\right) < \infty$, then $\zeta_i \to s(S_\vartheta(\hat{E}(r,t), G^k))$ implies $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^k))$.

Proof. Assume that $I_r \subset J_r$, and $\varepsilon > 0$ is given. Let $\zeta_i \to s(S_\vartheta(\hat{E}(r,t), G^k))$, then we have

$$\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\} \subseteq \left\{i \in J_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}.$$

Since $\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|$ and $\left|\left\{i \in J_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|$ are positive integer values and (g_r) is a sequence consisting of modulus functions for all $r \in \mathbb{N}$, then we may write

$$\frac{1}{g_r^k(h_r)} g_r^k\left(\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|\right) \leq \frac{1}{g_r^k(h_r)} g_r^k\left(\left|\left\{i \in J_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|\right)$$
$$= \frac{g_r^k(\ell_r)}{g_r^k(h_r)} \frac{1}{g_r^k(\ell_r)} g_r^k\left(\left|\left\{i \in J_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|\right).$$

Thus

$$\frac{1}{g_r^k(h_r)} g_r^k\left(\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|\right) \to 0 \text{ as } r \to \infty,$$

since $\lim_{r\to\infty} sup(g_r^k(\ell_r)/g_r^k(h_r)) < \infty$. This concludes that $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^k))$. Hence the proof. □

Theorem 6.1.6 Assume that $\theta = (k_r)$ and $\vartheta = (s_r)$ are lacunary sequences with $I_r \subset J_r$, and let $G = (g_r)$ be given as a sequence of any unbounded modulus functions in $\mathbb{G}$. If $\lim_{r\to\infty}(g_r^k(\ell_r - h_r)/g_r^k(\ell_r)) = 0$, then $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^k))$ implies $\zeta_i \to s(S_\vartheta(\hat{E}(r,t), G^k))$.

Proof. Assume that $I_r \subset J_r$ and $\varepsilon > 0$ is given. Let $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^k))$. Since (g_r) is a sequence consisting of modulus functions for all $r \in \mathbb{N}$, then we have

$$\frac{1}{g_r^k(\ell_r)} g_r^k\left(\left|\left\{i \in J_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|\right)$$

$$= \frac{1}{g_r^k(\ell_r)} g_r^k\Big(\left|\left\{i \in (s_{r-1}, k_{r-1}] : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|$$

$$+\left|\left\{i \in (k_r, s_r] : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right| + \left|\left\{i \in (k_{r-1}, k_r] : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|\Big)$$

$$\leq \frac{1}{g_r^k(\ell_r)} g_r^k\left((k_{r-1} - s_{r-1}) + (s_r - k_r) + \left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|\right)$$

$$= \frac{1}{g_r^k(\ell_r)} g_r^k\left((s_r - s_{r-1}) - (k_r - k_{r-1}) + \left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|\right)$$

$$= \frac{1}{g_r^k(\ell_r)} g_r^k\left(\ell_r - h_r + \left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|\right)$$

$$\leq \frac{1}{g_r^k(\ell_r)} g_r^k(\ell_r - h_r) + \frac{1}{g_r^k(h_r)} g_r^k\left(\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|\right)$$

Thus

$$\frac{1}{g_r^k(\ell_r)} g_r^k\left(\left|\left\{i \in J_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|\right) \to 0 \text{ as } r \to \infty,$$

since $\lim_{r\to\infty}(g_r^k(\ell_r - h_r)/g_r^k(\ell_r)) = 0$. This concludes that $\zeta_i \to s(S_\vartheta(\hat{E}(r,t), G^k))$. Hence the proof. □

Theorem 6.1.7 Assume that $\theta = (k_r)$ is a lacunary sequence and $G = (g_r)$ is given as a sequence of unbounded modulus functions in $\mathbb{G}$.

(i) If $g_r(u) \leq u$ and $\lim_{u\to\infty} \frac{g_r^k(u)}{u} > 0$ for all $r \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then $\zeta_i \to s(S_\theta(\hat{E}(r,t)))$ implies $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^k))$.

(ii) If $g_r(u) \geq u$ and $\lim_{u\to\infty} \frac{u}{g_r^k(u)} > 0$ for all $r \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^k))$ implies $\zeta_i \to s(S_\theta(\hat{E}(r,t)))$.

Proof. (**i**) Since (g_r) is increasing for each $r \in \mathbb{N}$ and $g_r(u) \leq u$ for all $r \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then the next inequality is followed

$$g_r^k(u) \leq g_r^{k-1}(u) \leq \cdots \leq g_r(u) \leq u. \tag{6.1}$$

Let $\zeta_i \to s(S_\theta(\hat{E}(r,t)))$ and $\varepsilon > 0$ be given, then according to (6.1), we may write

$$\frac{1}{h_r}\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right| \geq \frac{1}{h_r} g_r^k\left(\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|\right)$$

$$= \frac{1}{g_r^k(h_r)} g_r^k\left(\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|\right) \frac{g_r^k(h_r)}{h_r}$$

Thus

$$\frac{1}{g_r^k(h_r)} g_r^k\left(\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|\right) \frac{g_r^k(h_r)}{h_r} \to 0 \; as \; r \to \infty.$$

This concludes that $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^k))$. □

(**ii**) Since $g_r(u) \geq u$ for all $r \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then the following inequality is followed

$$g_r^k(u) \geq g_r^{k-1}(u) \geq \cdots \geq g_r(u) \geq u. \tag{6.2}$$

Let $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^k))$ and $\varepsilon > 0$ be given, then according to (6.2), we may have

$$\frac{1}{g_r^k(h_r)} g_r^k\left(\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|\right) \geq \frac{1}{g_r^k(h_r)}\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right|$$

$$= \frac{1}{h_r}\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right| \frac{h_r}{g_r^k(h_r)}$$

Thus

$$\frac{1}{h_r}\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(\zeta) - s\right| \geq \varepsilon\right\}\right| \frac{h_r}{g_r^k(h_r)} \to 0 \text{ as } r \to \infty,$$

This concludes that $\zeta_i \to s(S_\theta(\hat{E}(r,t)))$. □

Corollary 6.1.8 Assume that $\theta = (k_r)$ is a lacunary sequence and $G = (g_r)$ is given as a sequence of unbounded modulus functions in $\mathbb{G}$ with $k_1 < k_2$. Then from the above theorem, we possess the results below.

(i) If $g_r(u) \leq u$ and $\lim_{u\to\infty} \frac{g_r^{k_2}(u)}{u} > 0$ for all $r \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^{k_1}))$ implies $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^{k_2}))$.

(ii) If $g_r(u) \geq u$ and $\lim_{u\to\infty} \frac{u}{g_r^{k_2}(u)} > 0$ for all $r \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^{k_2}))$ implies $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^{k_1}))$.

The proof of (i) and (ii) can be done by using (6.1) and (6.2), respectively.

Theorem 6.1.9 Assume that $\theta = (k_r)$ is a lacunary sequence and let $G = (g_r)$ and $H = (h_r)$ be given as two sequences of unbounded modulus functions in $\mathbb{G}$ with $g_r(u) \leq h_r(u)$ for all $r \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$.

(i) If $\lim_{u\to\infty} \frac{g_r^{k+1}(u)}{g_r^k(h_r(u))} > 0$, then $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^k \circ H))$ implies $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^{k+1}))$.

(ii) If $\lim_{u\to\infty} \frac{h_r^k(g_r(u))}{h_r^{k+1}(u)} > 0$, then $\zeta_i \to s(S_\theta(\hat{E}(r,t), H^{k+1}))$ implies $\zeta_i \to s(S_\theta(\hat{E}(r,t), H^k \circ G))$.

Proof. Since g_r and h_r are increasing for each $r \in \mathbb{N}$, and $g_r(u) \leq h_r(u)$ for all $r \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then the inequalities below are concluded.

$$g_r^{k+1}(u) \leq g_r^k\big(h_r(u)\big) \tag{6.3}$$

and

$$h_r^k\big(g_r(u)\big) \leq h_r^{k+1}(u) \tag{6.4}$$

For the first part, we suppose $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^k \circ H))$. Then, according to (6.3), we may write

$$\frac{g_r^k\Big(h_r(|\{i \in I_r : |\widehat{E}_i(r,t)(\zeta) - s| \geq \varepsilon\}|)\Big)}{g_r^k\big(h_r(h_r)\big)} \geq \frac{g_r^{k+1}(|\{i \in I_r : |\widehat{E}_i(r,t)(\zeta) - s| \geq \varepsilon\}|)}{g_r^k\big(h_r(h_r)\big)}$$

$$= \frac{g_r^{k+1}(|\{i \in I_r : |\widehat{E}_i(r,t)(\zeta) - s| \geq \varepsilon\}|)}{g_r^{k+1}(h_r)} \frac{g_r^{k+1}(h_r)}{g_r^k\big(h_r(h_r)\big)}$$

Providing the limits into both given sides as $r \to \infty$, it implies that $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^{k+1}))$ since $0 < \lim_{u\to\infty} \frac{g_r^{k+1}(h_r)}{g_r^k(h_r(h_r))} < \infty$. For the second part, we assume $\zeta_i \to s(S_\theta(\hat{E}(r,t), H^{k+1}))$. Then, according to (6.4), it follows that $\zeta_i \to s(S_\theta(\hat{E}(r,t), H^k \circ G))$ since $0 < \lim_{u\to\infty} \frac{h_r^k(g_r(h_r))}{h_r^{k+1}(h_r)} < \infty$. □

Theorem 6.1.10 Assume that $\theta = (k_r)$ is a lacunary sequence and $G = (g_r)$ is a sequence of unbounded modulus functions in $\mathbb{G}$. Then $\zeta_i \to s(S_\theta^0(G^k))$ implies $\zeta_i \to s(S_\theta^0(\hat{E}(r,t), G^k))$, i.e., $S_\theta^0(G^k) \subset S_\theta^0(\hat{E}(r,t), G^k)$.

Proof. From the Lucas sequence, we see that $(L_{i-1}/L_i) \leq 2$ and $(L_i/L_{i-1}) \leq 3$ ($\forall\, i \in \mathbb{N}$). Then by using (5.3), we may have

$$\left|\hat{E}_i(r,t)(\zeta)\right| = \left|r\frac{L_{i-1}}{L_i}\zeta_i + t\frac{L_i}{L_{i-1}}\zeta_{i-1}\right|$$

$$\leq 6|2r\zeta_i + 3t\zeta_{i-1}|$$

$$\leq 6^2\, max\{|r|,|t|\}\,(|\zeta_i| + |\zeta_{i-1}|)$$

so that

$$\left|\hat{E}_i(r,t)(\zeta)\right| \leq 36\, max\{|r|,|t|\}\,(|\zeta_i| + |\zeta_{i-1}|) \tag{6.5}$$

Using (6.5), we may have

$$\left\{i \in I_r: \left|\hat{E}_i(r,t)(\zeta)\right| \geq \varepsilon\right\} \subset \{i \in I_r: 36\, max\{|r|,|t|\}\,(|\zeta_i| + |\zeta_{i-1}|) \geq \varepsilon\}$$

So that

$$\left|\left\{i \in I_r: \left|\hat{E}_i(r,t)(\zeta)\right| \geq \varepsilon\right\}\right| \leq |\{i \in I_r: 36\, max\{|r|,|t|\}\,(|\zeta_i| + |\zeta_{i-1}|) \geq \varepsilon\}|$$

$$\leq \left|\left\{i \in I_r: 36\, max\{|r|,|t|\}\,|\zeta_i| \geq \frac{\varepsilon}{2}\right\}\right| + \left|\left\{i \in I_r: 36\, max\{|r|,|t|\}\,|\zeta_{i-1}| \geq \frac{\varepsilon}{2}\right\}\right|$$

$$= \left|\left\{i \in I_r: |\zeta_i| \geq \frac{\varepsilon}{72\, max\{|r|,|t|\}}\right\}\right| + \left|\left\{i \in I_r: |\zeta_{i-1}| \geq \frac{\varepsilon}{72\, max\{|r|,|t|\}}\right\}\right|.$$

Now suppose $\zeta = (\zeta_i) \in S_\theta^0(G^k)$. Since g_r is increasing for each $r \in \mathbb{N}$, then we have

$$\frac{g_r^k\left(\left|\left\{i \in I_r: \left|\hat{E}_i(r,t)(\zeta)\right| \geq \varepsilon\right\}\right|\right)}{g_r^k(h_r)}$$

$$\leq \frac{g_r^k\left(\left|\left\{i \in I_r: |\zeta_i| \geq \frac{\varepsilon}{72\,max\{|r|,|t|\}}\right\}\right| + \left|\left\{i \in I_r: |\zeta_{i-1}| \geq \frac{\varepsilon}{72\,max\{|r|,|t|\}}\right\}\right|\right)}{g_r^k(h_r)}$$

$$\leq \frac{g_r^k\left(\left|\left\{i \in I_r: |\zeta_i| \geq \frac{\varepsilon}{72\,max\{|r|,|t|\}}\right\}\right|\right)}{g_r^k(h_r)} + \frac{g_r^k\left(\left|\left\{i \in I_r: |\zeta_{i-1}| \geq \frac{\varepsilon}{72\,max\{|r|,|t|\}}\right\}\right|\right)}{g_r^k(h_r)}.$$

Thus

$$\frac{g_r^k\left(\left|\left\{i \in I_r: \left|\hat{E}_i(r,t)(\zeta)\right| \geq \varepsilon\right\}\right|\right)}{g_r^k(h_r)} \to 0 \; as \; r \to \infty.$$

This concludes that $\zeta_i \to s(S_\theta^0(\hat{E}(r,t), G^k))$ and so that $(\zeta_i) \in S_\theta^0(\hat{E}(r,t), G^k)$. Hence the proof. □

6.2. Lacunary Strong $\hat{E}(r,t)$-Convergence According to F^k

In this subsection, a lacunary sequence $\theta = (k_r)$, the Lucas transform $\hat{E}(r,t)$ and a sequence of modulus functions $G = (g_i)$ are used to introduce some new concepts in summability theory. In addition, that, the relations between these concepts are determined.

Definition 6.2.1 Suppose that $\theta = (k_r)$ is a lacunary sequence and let $G = (g_i)$ be given as a sequence of modulus functions in $\mathbb{G}$. Let s be a number in $\mathbb{C}$, then the sequence (ζ_i) in $\mathbb{C}$ is defined to be lacunary strongly $\hat{E}(r,t)$-convergent to s according to G^k (or $N_\theta(\hat{E}(r,t), G^k)$-strongly convergent), if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta) - s\right|\right) = 0.$$

In this manner, we refer it to as $\zeta_i \to s(N_\theta(\hat{E}(r,t), G^k))$ or $N_\theta(\hat{E}(r,t), G^k) - \lim \zeta_i = s$. The collection of all $N_\theta(\hat{E}(r,t), G^k)$-strongly convergent sequences is indicated by $N_\theta(\hat{E}(r,t), G^k)$. That is

$$N_\theta\left(\hat{E}(r,t), G^k\right) = \left\{\zeta = (\zeta_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta) - s\right|\right) = 0 \text{ for some } s \in \mathbb{C}\right\}.$$

The collection of all sequences in which $s = 0$ in the definition of $N_\theta(\hat{E}(r,t), g^k)$ is indicated by $N_\theta^0(\hat{E}(r,t), G^k)$. That is

$$N_\theta^0\left(\hat{E}(r,t), G^k\right) = \left\{\zeta = (\zeta_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right) = 0\right\}.$$

Notice that in this definition the functions in G are not required to be unbounded modulus. If we put $g_i(u) = u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then the $N_\theta(\hat{E}(r,t), G^k)$-strong convergence is reduced to the $N_\theta(\hat{E}(r,t))$-strong convergence and it is so for the $N_\theta^0(\hat{E}(r,t), G^k)$-strong convergence. Also, it's clear to see that if $k = 1$, then from the spaces $N_\theta(\hat{E}(r,t), G^k)$ and $N_\theta^0(\hat{E}(r,t), G^k)$ we have $N_\theta(\hat{E}(r,t), G)$ and $N_\theta^0(\hat{E}(r,t), G)$, respectively. As well as if $g_i(u) = g(u)$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then instead of $N_\theta(\hat{E}(r,t), G^k)$ and $N_\theta^0(\hat{E}(r,t), G^k)$ we shall write $N_\theta(\hat{E}(r,t), g^k)$ and $N_\theta^0(\hat{E}(r,t), g^k)$, respectively.

Theorem 6.2.2 Consider a lacunary sequence $\theta = (k_r)$ and let $G = (g_i)$ be given as a sequence of modulus functions in $\mathbb{G}$. Then the sets $N_\theta(\hat{E}(r,t), G^k)$ and $N_\theta^0(\hat{E}(r,t), G^k)$ are linear spaces.

Proof. We here only consider $N_\theta(\hat{E}(r,t), G^k)$. Suppose $\zeta_i \to s_1(N_\theta(\hat{E}(r,t), G^k))$ and $\xi_i \to s_2(N_\theta(\hat{E}(r,t), G^k))$ as $i \to \infty$, and let $\alpha, \beta \in \mathbb{C}$. Then there are natural numbers N_α and M_β such that $|\alpha| \le N_\alpha$ and $|\beta| \le M_\beta$. Since g_i is a modulus for each $i \in \mathbb{N}$, we may have

$$\frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\alpha\zeta + \beta\xi) - (\alpha s_1 + \beta s_2)\right|\right)$$

$$= \frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(\left|\alpha\hat{E}_i(r,t)(\zeta) - s_1 + \beta\hat{E}_i(r,t)(\xi) - s_2\right|\right)$$

$$\le \frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(|\alpha|\left|\hat{E}_i(r,t)(\zeta) - s_1\right| + |\beta|\left|\hat{E}_i(r,t)(\xi) - s_2\right|\right)$$

$$\le N_\alpha \frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta) - s_1\right|\right) + M_\beta \frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\xi) - s_2\right|\right).$$

Then adding the limits into both given sides as r tends to ∞, we have

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\alpha\zeta + \beta\xi) - (\alpha s_1 + \beta s_2)\right|\right) = 0.$$

This implies that $(\zeta_i + \xi_i) \to (s_1 + s_2)$ as $i \to \infty$. Therefore $N_\theta(\hat{E}(r,t), G^k)$ is indeed a linear space. □

Theorem 6.2.3 Consider a lacunary sequence $\theta = (k_r)$ and let $G = (g_i)$ be given as a sequence of modulus functions in $\mathbb{G}$. Then every $N_\theta^0(G^k)$-strongly convergent sequence implies $N_\theta^0(\hat{E}(r,t), G^k)$-strongly convergent, i.e, $N_\theta^0(G^k) \subset N_\theta^0(\hat{E}(r,t), G^k)$.

Proof. From (6.5), we have

$$\left|\hat{E}_i(r,t)(\zeta)\right| \leq 36\, max\{|r|,|t|\}\left(\left|\zeta_i\right| + \left|\zeta_{i-1}\right|\right)$$

Let $\zeta = (\zeta_i)$ be an $N_\theta^0(G^k)$-strongly convergent sequence. Since g_i is increasing for each $i \in \mathbb{N}$, then we may write

$$\frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right) \leq 36\, max\{T_r,T_t\}\frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(\left|\zeta_i\right| + \left|\zeta_{i-1}\right|\right)$$

$$\leq 36\, max\{T_r,T_t\}\left(\frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(\left|\zeta_i\right|\right) + \frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(\left|\zeta_{i-1}\right|\right)\right)$$

where T_r and T_t are natural values such that $|r| \leq T_r$ and $|t| \leq T_t$. Thus, we have

$$\frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right) \to 0 \text{ as } r \to \infty.$$

This concludes that $\zeta \in N_\theta^0(\hat{E}(r,t),G^k)$. Hence the proof. □

Corollary 6.2.4 Consider a lacunary sequence $\theta = (k_r)$ and let $G = (g_i)$ be given as a sequence of modulus functions in $\mathbb{G}$. We have the results below

(i) If $g_i(u) \leq u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then $N_\theta^0 \subset N_\theta^0(\hat{E}(r,t),G^k)$.

(ii) If $g_i(u) \geq u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then $N_\theta^0(G^k) \subset N_\theta^0(\hat{E}(r,t))$.

(iii) If $g_i(u) = u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then $N_\theta^0 \subset N_\theta^0(\hat{E}(r,t))$.

Theorem 6.2.5 Consider the lacunary sequences $\theta = (k_r)$ and $\vartheta = (s_r)$ such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, and let $G = (g_i)$ be given as a sequence of modulus functions in $\mathbb{G}$. If $\lim_{r\to\infty}\frac{\ell_r}{h_r} = 1$ and $(\zeta_i) \in \ell_\infty(\hat{E}(r,t))$ then $N_\theta(\hat{E}(r,t),G^k) \subset N_\vartheta(\hat{E}(r,t),G^k)$, where

$$\ell_\infty\left(\hat{E}(r,t)\right) = \left\{(\zeta_i) : \sup_i\left|\hat{E}_i(r,t)(\zeta)\right| < \infty\right\}.$$

Proof. Let $(\zeta_i) \in \ell_\infty(\hat{E}(r,t)) \cap N_\theta(\hat{E}(r,t),G^k)$. Then $(\hat{E}_i(r,t)(\zeta))$ is a bounded sequence, so there has some $H > 0$ such that $\left|\hat{E}_i(r,t)(\zeta) - s\right| \leq H$ for all $i \in \mathbb{N}$. Since $I_r \subseteq J_r$ and $h_r \leq \ell_r$ for all $r \in \mathbb{N}$, we are going to write

$$\frac{1}{\ell_r}\sum_{i\in J_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta) - s\right|\right) = \frac{1}{\ell_r}\sum_{i\in J_r - I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta) - s\right|\right) + \frac{1}{\ell_r}\sum_{i\in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta) - s\right|\right)$$

$$\leq \left(\frac{\ell_r - h_r}{\ell_r}\right)\sup_i g_i^k(H) + \frac{1}{\ell_r}\sum_{i\in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta) - s\right|\right)$$

$$\leq \left(\frac{\ell_r}{h_r} - 1\right) \sup_i g_i^k(H) + \frac{1}{h_r} \sum_{i \in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right).$$

Then adding the limits into both given sides as r tends to ∞, we have

$$\lim_{r\to\infty} \frac{1}{\ell_r} \sum_{i \in J_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta) - s\right|\right) = 0,$$

since $\lim_{r\to\infty} \frac{\ell_r}{h_r} = 1$. Therefore, we obtain that $(\zeta_i) \in N_\vartheta(\hat{E}(r,t), G^k)$. Hence the proof. □

Corollary 6.2.6 Assume that $G = (g_i)$ is given as a sequence of modulus functions in $\mathbb{G}$ and $k < n$. If $g_i(u) \leq u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then from Theorem 6.2.5, we obtain the following results.

(i) $N_\theta\left(\hat{E}(r,t)\right) \subset N_\vartheta\left(\hat{E}(r,t), G^k\right)$.

(ii) $N_\theta\left(\hat{E}(r,t), G^k\right) \subset N_\vartheta\left(\hat{E}(r,t), G^n\right)$.

Corollary 6.2.7 Assume that $G = (g_i)$ is given as a sequence of modulus functions in $\mathbb{G}$ and $k < n$. If $g_i(u) \geq u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then from Theorem 6.2.5, we obtain the following results.

(i) $N_\theta\left(\hat{E}(r,t), G^k\right) \subset N_\vartheta\left(\hat{E}(r,t)\right)$.

(ii) $N_\theta\left(\hat{E}(r,t), G^n\right) \subset N_\vartheta\left(\hat{E}(r,t), G^k\right)$.

Theorem 6.2.8 Consider the lacunary sequences $\theta = (k_r)$ and $\vartheta = (s_r)$ such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, and let $G = (g_i)$ be given as a sequence of modulus functions in $\mathbb{G}$. If $\lim_{r\to\infty} sup \frac{\ell_r}{h_r} < \infty$, then $N_\vartheta(\hat{E}(r,t), G^k) \subset N_\theta(\hat{E}(r,t), G^k)$.

Proof. Let $(\zeta_i) \in N_\vartheta(\hat{E}(r,t), G^k)$. Since $I_r \subseteq J_r$ we may write

$$\frac{1}{\ell_r} \sum_{i \in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta) - s\right|\right) \leq \frac{1}{\ell_r} \sum_{i \in J_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta) - s\right|\right)$$

and then

$$\frac{1}{h_r} \sum_{i \in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta) - s\right|\right) \leq \frac{\ell_r}{h_r} \frac{1}{\ell_r} \sum_{i \in J_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta) - s\right|\right).$$

Adding the limits into both given sides as r tends to ∞, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)-s\right|\right)=0.$$

Therefore, we obtain that $(\zeta_i)\in N_\theta(\hat{E}(r,t),G^k)$. Hence the proof. □

Corollary 6.2.9 Assume that $G=(g_i)$ is given as a sequence of modulus functions in $\mathbb{G}$ and $k<n$. If $g_i(u)\le u$ for all $i\in\mathbb{N}$ and for every $u\in\mathbb{R}^+\cup\{0\}$, then from Theorem 6.2.8, we obtain the following results.

(i) $N_\vartheta\left(\hat{E}(r,t)\right)\subset N_\theta\left(\hat{E}(r,t),G^k\right)$.

(ii) $N_\vartheta\left(\hat{E}(r,t),G^k\right)\subset N_\theta\left(\hat{E}(r,t),G^n\right)$.

Corollary 6.2.10 Assume that $G=(g_i)$ is given as a sequence of modulus functions in $\mathbb{G}$ and $k<n$. If $g_i(u)\ge u$ for all $i\in\mathbb{N}$ and for every $u\in\mathbb{R}^+\cup\{0\}$, then from Theorem 6.2.8, we obtain the following results.

(i) $N_\vartheta\left(\hat{E}(r,t),G^k\right)\subset N_\theta\left(\hat{E}(r,t)\right)$.

(ii) $N_\vartheta\left(\hat{E}(r,t),G^n\right)\subset N_\theta\left(\hat{E}(r,t),G^k\right)$.

Theorem 6.2.11 Consider a lacunary sequence $\theta=(k_r)$, and let $G=(g_i)$ and $H=(h_i)$ be given as two sequences of modulus functions in $\mathbb{G}$. If $\sup_{u,i}\frac{g_i(u)}{h_i(u)}<\infty$, then $N_\theta(\hat{E}(r,t),G^k\circ H)\subset N_\theta(\hat{E}(r,t),G^{k+1})$.

Proof. Let $\alpha\in\mathbb{C}$. Then there is a natural number T_α such that $|\alpha|\le T_\alpha$. Now suppose $0<\alpha=\sup_{u,i}\frac{g_i(u)}{h_i(u)}<\infty$, then $\alpha\ge\frac{g_i(u)}{h_i(u)}$ and so that $g_i(u)\le\alpha h_i(u)$ for all $i\in\mathbb{N}$ and for every $u\in\mathbb{R}^+\cup\{0\}$. Since (g_i) is increasing for each $i\in\mathbb{N}$, then we get

$$g_i^{k+1}(u)\le g_i^k\left(\alpha h_i(u)\right)\le T_\alpha g_i^k\left(h_i(u)\right). \tag{6.6}$$

Now if $(\zeta_i)\in N_\theta(\hat{E}(r,t),G^k\circ H)$, then according to (6.6), we shall possess

$$\frac{1}{h_r}\sum_{i\in I_r} g_i^{k+1}\left(\left|\hat{E}_i(r,t)(\zeta)-s\right|\right)\le T_\alpha\frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(h_i\left(\left|\hat{E}_i(r,t)(\zeta)-s\right|\right)\right)\to 0\ as\ r\to\infty.$$

This concludes that $(\zeta_i)\in N_\theta(\hat{E}(r,t),G^k\circ H)$ implies $(\zeta_i)\in N_\theta(\hat{E}(r,t),G^{k+1})$. Hence the proof. □

Theorem 6.2.11 Consider a lacunary sequence $\theta = (k_r)$ and let $G = (g_i)$ and $H = (h_i)$ be given as two sequences of modulus functions in $\mathbb{G}$. If $\inf_{u,i} \frac{g_i(u)}{h_i(u)} > 0$, then $N_\theta(\hat{E}(r,t), G^{k+1}) \subset N_\theta(\hat{E}(r,t), G^k \circ H)$.

Proof. Let $b \in \mathbb{C}$ with $b \neq 0$. Then there has a natural value $T_{b^{-1}}$ such that $|b^{-1}| \leq T_{b^{-1}}$. Now suppose $\beta = \inf_{u,i} \frac{g_i(u)}{h_i(u)} > 0$ then $\beta \leq \frac{g_i(u)}{h_i(u)}$ and so that $h_i(u) \leq \frac{1}{\beta} g_i(u)$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$. Since (g_i) is increasing for each $i \in \mathbb{N}$ and $\beta > 0$, then we get

$$g_i^k\big(h_i(u)\big) \leq g_i^k\big(\beta^{-1} g_i(u)\big) \leq T_{\beta^{-1}} g_i^{k+1}(u). \tag{6.7}$$

Now let $(\zeta_i) \in N_\theta(\hat{E}(r,t), G^{k+1})$, then according to (6.7), we shall possess

$$\frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(h_i\big(\big|\hat{E}_i(r,t)(\zeta) - s\big|\big)\right) \leq T_{\beta^{-1}} \frac{1}{h_r}\sum_{i\in I_r} g_i^{k+1}\big(\big|\hat{E}_i(r,t)(\zeta) - s\big|\big) \to 0 \; as \; r \to \infty.$$

This concludes that $(\zeta_i) \in N_\theta(\hat{E}(r,t), G^{k+1})$ implies $(\zeta_i) \in N_\theta(\hat{E}(r,t), G^k \circ H)$. Hence the proof. □

Corollary 6.2.12 Consider a lacunary sequence $\theta = (k_r)$, and let $G = (g_i)$ and $H = (h_i)$ be given as two sequences of modulus functions in $\mathbb{G}$.

(i) If $\sup_{u,i} \frac{g_i(u)}{h_i(u)} < \infty$, then $N_\theta(\hat{E}(r,t), H^{k+1}) \subset N_\theta(\hat{E}(r,t), H^k \circ G)$.

(ii) If $\inf_{u,i} \frac{g_i(u)}{h_i(u)} > 0$, then $N_\theta(\hat{E}(r,t), H^k \circ G) \subset N_\theta(\hat{E}(r,t), H^{k+1})$.

Theorem 6.2.13 Consider a lacunary sequence $\theta = (k_r)$, and let $G = (g_i)$ be given as a sequence of modulus functions in $\mathbb{G}$. If $\lim_{u\to\infty}(g_i^k(u)/u) > 0$ for all $i \in \mathbb{N}$ and there is a positive integer d such that $g_i(uv) \geq d g_i(u) g_i(v)$ for all $i \in \mathbb{N}$ and for every $u, v \in \mathbb{R}^+ \cup \{0\}$. Then $\zeta_i \to s(N_\theta(\hat{E}(r,t), G^k))$ implies $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^k))$.

Proof. Suppose that $\zeta_i \to s(N_\theta(\hat{E}(r,t), G^k))$, and $\varepsilon > 0$ is given. Let $\sum_A$ and $\sum_B$ be defined on $\big|\hat{E}_i(r,t)(\zeta) - s\big| \geq \varepsilon$ and $\big|\hat{E}_i(r,t)(\zeta) - s\big| < \varepsilon$, respectively. Then from the definition of modulus functions, we may write

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r} g_i^k\big(\big|\hat{E}_i(r,t)(\zeta) - s\big|\big) &= \frac{1}{h_r}\sum_{i\in I_r\backslash A} g_i^k\big(\big|\hat{E}_i(r,t)(\zeta) - s\big|\big) + \frac{1}{h_r}\sum_{i\in I_r\backslash B} g_i^k\big(\big|\hat{E}_i(r,t)(\zeta) - s\big|\big) \\ &\geq \frac{1}{h_r}\sum_{i\in I_r\backslash A} g_i^k\big(\big|\hat{E}_i(r,t)(\zeta) - s\big|\big)\end{aligned}$$

$$\geq \frac{1}{h_r} g_r^k \left(\sum_{i \in I_r \setminus A} \left| \hat{E}_i(r,t)(\zeta) - s \right| \right)$$

$$\geq \frac{1}{h_r} g_r^k \left(\left| \left\{ i \in I_r : \left| \hat{E}_i(r,t)(\zeta) - s \right| \geq \varepsilon \right\} \right| \varepsilon \right)$$

$$\geq \frac{D}{h_r} g_r^k \left(\left| \left\{ i \in I_r : \left| \hat{E}_i(r,t)(\zeta) - s \right| \geq \varepsilon \right\} \right| \right) g_r^k(\varepsilon),$$

where D is a positive constant. Since $\left| \left\{ i \in I_r : \left| \hat{E}_i(r,t)(\zeta) - s \right| \geq \varepsilon \right\} \right|$ is a natural value, then

$$\frac{1}{h_r} \sum_{i \in I_r} g_i^k \left(\left| \hat{E}_i(r,t)(\zeta) - s \right| \right) \geq \frac{D}{h_r} g_r^k \left(\left| \left\{ i \in I_r : \left| \hat{E}_i(r,t)(\zeta) - s \right| \geq \varepsilon \right\} \right| \right) \frac{\inf_r f_r^k(\varepsilon)}{\inf_r f_r^k(1)}$$

$$= \frac{g_r^k \left(\left| \left\{ i \in I_r : \left| \hat{E}_i(r,t)(\zeta) - s \right| \geq \varepsilon \right\} \right| \right)}{g_r^k(h_r)} \frac{g_r^k(h_r)}{h_r} \frac{\inf_r g_r^k(\varepsilon)}{\inf_r g_r^k(1)} D.$$

Giving the limits on both sides as $r \to \infty$, it concludes that $\zeta_i \to s(S_\theta(\hat{E}(r,t), G^k))$. Hence the proof. □

Theorem 6.2.14 Consider a lacunary sequence $\theta = (k_r)$ and let $G = (g_i)$ be given as a sequence of modulus functions in $\mathbb{G}$.

(i) If $\liminf q_r > 1$ then $\zeta_i \to s(w(\hat{E}(r,t), G^k))$ implies $\zeta_i \to s(N_\theta(\hat{E}(r,t), G^k))$,

(ii) If $\limsup q_r < \infty$, then $\zeta_i \to s(N_\theta(\hat{E}(r,t), G^k))$ implies $\zeta_i \to s(w(\hat{E}(r,t), G^k))$,

where

$$w\left(\hat{E}(r,t), G^k\right) = \left\{ (\zeta_i) : \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} g_i^k \left(\left| \hat{E}_i(r,t)(\zeta) - s \right| \right) = 0 \text{ for some } s \in \mathbb{C} \right\}.$$

Proof. (i) Since $\liminf q_r > 1$ then there is $\delta > 0$ such that

$$q_r = \frac{k_r}{k_{r-1}} \geq 1 + \delta$$

for sufficiently large r, then we have that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta} \text{ and } \frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$$

Now assume that $\zeta_i \to s(w(\hat{E}(r,t), G^k))$, then we may write

$$\frac{1}{k_r} \sum_{i=1}^{k_r} g_i^k \left(\hat{E}_i(r,t)(\zeta) - s \right) \geq \frac{1}{k_r} \sum_{i \in I_r} g_i^k \left(\left| \hat{E}_i(r,t)(\zeta) - s \right| \right)$$

$$= \frac{h_r}{k_r}\frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)-s\right|\right)$$

$$\geq \frac{1+\delta}{\delta}\frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)-s\right|\right)$$

Thus

$$\frac{1+\delta}{\delta}\frac{1}{h_r}\sum_{i\in I_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)-s\right|\right) \to 0 \text{ as } r\to\infty.$$

This concludes that $\zeta_i \to s(N_\theta(\hat{E}(r,t),G^k))$.

(ii) Since $\limsup q_r < \infty$ then there has a positive number K, say $K = \sup q_r$ so that $q_r < K$ for every $r \in \mathbb{N}$. Assume that $\zeta_i \to s(N_\theta(\hat{E}(r,t),G^k))$ and given $\varepsilon > 0$. There has n_0 such that for every $n > n_0$, we shall have

$$T_n = \frac{1}{h_n}\sum_{i\in I_n} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)-s\right|\right) < \varepsilon.$$

Also, a positive number S can be found such that $T_n \leq S$ for all n. Let m be any integer such that $m \in (k_{r-1}, k_r]$. Now we may write

$$\frac{1}{m}\sum_{i=1}^{m} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)-s\right|\right) \leq \frac{1}{k_r}\sum_{i=1}^{k_r} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)-s\right|\right)$$

$$\leq \frac{1}{k_{r-1}}\left(\sum_{n=1}^{n_0} + \sum_{n=n_0+1}^{k_r}\right)\sum_{i\in I_n} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)-s\right|\right)$$

$$= \frac{1}{k_{r-1}}\sum_{n=1}^{n_0}\sum_{i\in I_n} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)-s\right|\right) + \frac{1}{k_{r-1}}\sum_{n=n_0+1}^{k_r}\sum_{i\in I_n} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)-s\right|\right)$$

$$\leq \frac{1}{k_{r-1}}\sum_{n=1}^{n_0}\sum_{i\in I_n} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)-s\right|\right) + \frac{1}{k_{r-1}}\left(k_r - k_{n_0}\right)\varepsilon$$

$$= \frac{1}{k_{r-1}}\left(h_1T_1 + h_2T_2 + \cdots + h_{n_0}T_{n_0}\right) + \frac{1}{k_{r-1}}\left(k_r - k_{n_0}\right)\varepsilon$$

$$\leq \frac{1}{k_{r-1}}\left(\sup_{i\in[i,n_0]} T_i\, k_{n_0}\right) + \varepsilon K < \frac{1}{k_{r-1}}k_{n_0}S + \varepsilon K.$$

Thus

$$\frac{1}{m}\sum_{i=1}^{m} g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)-s\right|\right) \to 0 \text{ as } r\to\infty.$$

This implies that $\zeta_i \to s(w(\hat{E}(r,t),G^k))$. Hence the proof. □

7. SOME LUCAS DIFFERENCE SEQUENCE SPACES

7.1. On Some Banach Sequence Spaces Based on Lucas Band Matrix

In this subsection, two Banach sequence spaces based on the Lucas band matrix, and some inclusion relationships are studied. The most results of this subsection are taken from [57].

Definition 7.1.1 Consider the Lucas band matrix (5.2) and the Lucas transform (5.3). The Lucas difference sequence spaces $\ell_p(\hat{E}(r,t))$ and $\ell_\infty(\hat{E}(r,t))$ are introduced by Karakas [57] as follows

$$\ell_p\left(\hat{E}(r,t)\right) = \left\{\zeta = (\zeta_i) \in \omega \colon \sum_i \left|\hat{E}_i(r,t)(\zeta)\right|^p < \infty\right\}$$

and

$$\ell_\infty\left(\hat{E}(r,t)\right) = \left\{\zeta = (\zeta_i) \in \omega \colon \sup_i \left|\hat{E}_i(r,t)(\zeta)\right| < \infty\right\}.$$

Theorem 7.1.2 The sequence sets $\ell_p(\hat{E}(r,t))$ and $\ell_\infty(\hat{E}(r,t))$ are normed linear spaces for $1 \le p < \infty$, respectively, normed by

$$\|\zeta\|_{\ell_p(\hat{E}(r,t))} = \left(\sum_{i=1}^{\infty} \left|\hat{E}_i(r,t)(\zeta)\right|^p\right)^{\frac{1}{p}}$$

and

$$\|\zeta\|_{\ell_\infty(\hat{E}(r,t))} = \sup_i \left|\hat{E}_i(r,t)(\zeta)\right|.$$

Proof. We here do the proofing only on $\ell_p(\hat{E}(r,t))$. Assume that $\zeta = (\zeta_i), \xi = (\xi_i) \in \ell_p(\hat{E}(r,t))$. The norm axioms N_1 and N_2 are obviously held. N_3 follows from Minkowski's inequality (2.2) as follows

$$\begin{aligned}
\|\zeta + \xi\|_{\ell_p(\hat{E}(r,t))} &= \left(\sum_{i=1}^{\infty} \left|\hat{E}_i(r,t)(\zeta + \xi)\right|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^{\infty} \left|\hat{E}_i(r,t)(\zeta) + \hat{E}_i(r,t)(\xi)\right|^p\right)^{\frac{1}{p}} \\
&\le \left(\sum_{i=1}^{\infty} \left|\hat{E}_i(r,t)(\zeta)\right|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} \left|\hat{E}_i(r,t)(\xi)\right|^p\right)^{\frac{1}{p}} \\
&= \|\zeta\|_{\ell_p(\hat{E}(r,t))} + \|\xi\|_{\ell_p(\hat{E}(r,t))}. \ \square
\end{aligned}$$

Remark 7.1.3 Note that the given sequence spaces $\ell_p(\hat{E}(r,t))$ and $\ell_\infty(\hat{E}(r,t))$ become of non-absolute type, since $\|\zeta\|_{\ell_p(\hat{E}(r,t))} \neq \||\zeta|\|_{\ell_p(\hat{E}(r,t))}$ and $\|\zeta\|_{\ell_\infty(\hat{E}(r,t))} \neq \||\zeta|\|_{\ell_\infty(\hat{E}(r,t))}$.

Remark 7.1.4 It seems that the spaces $\ell_p(\hat{E}(r,t))$ and $\ell_\infty(\hat{E}(r,t))$ can be redefined with the help of (1.1) as follows

$$\ell_p(\hat{E}(r,t)) = (\ell_p)_{\hat{E}(r,t)} \text{ and } \ell_\infty(\hat{E}(r,t)) = (\ell_\infty)_{\hat{E}(r,t)}.$$

Consequently, we are allowed to write.

$$\|\zeta\|_{\ell_p(\hat{E}(r,t))} = \left\|\hat{E}(r,t)(\zeta)\right\|_p \text{ and } \|\zeta\|_{\ell_\infty(\hat{E}(r,t))} = \left\|\hat{E}(r,t)(\zeta)\right\|_\infty,$$

where $\|\cdot\|_p$ and $\|\cdot\|_\infty$ are standard norms on ℓ_p and ℓ_∞, respectively.

Theorem 7.1.5 For $1 \le p < \infty$, the sequence sets $\ell_p(\hat{E}(r,t))$ and $\ell_\infty(\hat{E}(r,t))$ are Banach spaces with $\|\cdot\|_{\ell_p(\hat{E}(r,t))}$ and $\|\cdot\|_{\ell_\infty(\hat{E}(r,t))}$.

Proof. The proof is considered only for $\ell_p(\hat{E}(r,t))$. Verifying the completeness of $\ell_p(\hat{E}(r,t))$ is sufficient to obtain the proof. Let $\zeta^n = (\zeta_i^n)$ be a Cauchy sequence such that $(\zeta_i^n) = (\zeta_1^n, \zeta_2^n, \dots) \in \ell_p(\hat{E}(r,t))$ for each $n \in \mathbb{N}$. Now given $\varepsilon > 0$ there has a natural value N such that for every $n, m \ge N$, we shall have

$$\|\zeta^n - \zeta^m\|_{\ell_p(\hat{E}(r,t))} = \left(\sum_i \left|\hat{E}_i(r,t)(\zeta^n - \zeta^m)\right|^p\right)^{\frac{1}{p}} < \varepsilon$$

and then

$$\sum_i \left|\hat{E}_i(r,t)(\zeta^n - \zeta^m)\right|^p < \varepsilon^p. \quad (7.1)$$

Then for every $n, m \ge N$ and for all $i \in \mathbb{N}$, we see that

$$\left|\hat{E}_i(r,t)(\zeta^n - \zeta^m)\right|^p < \varepsilon.$$

Hence for all $i \in \mathbb{N}$, we get

$$\left|\hat{E}_i(r,t)(\zeta^n - \zeta^m)\right| \to 0$$

as $n, m \to \infty$. This means that $\hat{E}_i(r,t)(\zeta^n) = (\hat{E}_i(r,t)(\zeta_i^1), \hat{E}_i(r,t)(\zeta_i^2), \dots)$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, then it becomes convergent, so there is $\zeta = (\zeta_i)$ such that

$(\hat{E}_i(r,t)(\zeta^n)) \to (\hat{E}_i(r,t)(\zeta))$ in $\mathbb{C}$ as $n \to \infty$ and for each $i \in \mathbb{N}$. We are required to verify that $(\hat{E}_i(r,t)(\zeta)) \in \ell_p(\hat{E}(r,t))$. Now letting $m \to \infty$ in (7.1), we obtain for $n > N$

$$\sum_i \left|\hat{E}_i(r,t)(\zeta^n - \zeta)\right|^p < \varepsilon^p. \tag{7.2}$$

This intends that $(\zeta^n - \zeta) \in \ell_p(\hat{E}(r,t))$, and since $\zeta^n \in \ell_p(\hat{E}(r,t))$, then we have

$$\begin{aligned}
\sum_i \left|\hat{E}_i(r,t)(\zeta)\right|^p &= \sum_i \left|\hat{E}_i(r,t)(\zeta^n - \zeta) + \hat{E}_i(r,t)(\zeta^n)\right|^p \\
&\le \sum_i 2^p \left(\left|\hat{E}_i(r,t)(\zeta^n - \zeta)\right|^p + \left|\hat{E}_i(r,t)(\zeta^n)\right|^p\right) \\
&= 2^p \sum_i \left|\hat{E}_i(r,t)(\zeta^n - \zeta)\right|^p + 2^p \sum_i \left|\hat{E}_i(r,t)(\zeta^n)\right|^p \\
&< \infty.
\end{aligned}$$

Furthermore, by adding limits in (7.2) and letting $n \to \infty$, we have

$$\lim_{n\to\infty} \|\zeta^n - \zeta\|_{\ell_p,\hat{E}(r,t))} = 0.$$

So indeed, the sequence ζ^n converges to ζ and they are both in $\ell_p(\hat{E}(r,t))$ for $1 \le p < \infty$. So, we conclude the completeness of $\ell_p(\hat{E}(r,t))$. Therefore, it is a Banach space. □

Theorem 7.1.6 The sequence space $\ell_p(\hat{E}(r,t))$ is linearly isomorphic to ℓ_p for $1 \le p < \infty$.

Proof. Suppose that $M: \ell_p(\hat{E}(r,t)) \to \ell_p$ is a transformation and considered as $M\zeta = \gamma$ with (5.3), then for $\zeta \in \ell_p(\hat{E}(r,t))$, it gives us $\gamma = \hat{E}_i(r,t)\zeta \in \ell_p$. The linearity of M is clearly holding. It's obvious that $\zeta = 0$ whenever $M\zeta = 0$. It follows that M is injective. Let $\gamma = (\gamma_i) \in \ell_p$ and define $\zeta = (\zeta_i)$ as follows:

$$\zeta_i = \frac{1}{r} \sum_{m=1}^{i} \left(-\frac{t}{r}\right)^{i-m} \frac{L_i^2}{L_{m-1}L_m} \gamma_m.$$

By using (5.3), one can write

$$\|\zeta\|_{\ell_p(\hat{E}(r,t))} = \left(\sum_i \left| r\frac{L_{i-1}}{L_i}\zeta_i + t\frac{L_i}{L_{i-1}}\zeta_{i-1}\right|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_i \left| \frac{L_{i-1}}{L_i} \sum_{m=1}^{i} \left(-\frac{t}{r}\right)^{i-m} \frac{L_i^2}{L_{m-1}L_m}\gamma_m + \frac{t}{r}\frac{L_i}{L_{i-1}} \sum_{m=1}^{i-1} \left(-\frac{t}{r}\right)^{i-m-1} \frac{L_{i-1}^2}{L_{m-1}L_m}\gamma_m \right|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_i |\gamma_i|^p\right)^{\frac{1}{p}} = \|\gamma\|_{\ell_p}.$$

Which yields

$$\|\zeta\|_{\ell_p(\hat{E}(r,t))} < \infty.$$

Therefore, the function is surjective and so it is linear bijective. Also, the norm is preserved. Thus, for $1 \leq p < \infty$ we obtain that $\ell_p(\hat{E}(r,t))$ and ℓ_p are linearly isomorphic. □

Theorem 7.1.7 For $1 \leq p < q$, the following inclusion relationship is satisfied.

$$\ell_p\left(\hat{E}(r,t)\right) \subseteq \ell_q\left(\hat{E}(r,t)\right)$$

Proof. By using Theorem 7.1.6, we possess a transformation $M: \ell_p(\hat{E}(r,t)) \to \ell_p$ that is considered as $M\zeta = \hat{E}_i(r,t)\zeta$ $(i \in \mathbb{N})$. Now if $\zeta \in \ell_p(\hat{E}(r,t))$ then it gives us $M\zeta \in \ell_p$. Since $\ell_p \subset \ell_q$ for $1 \leq p < q$, so we have $M\zeta \in \ell_q$. Thus $\zeta \in \ell_q(\hat{E}(r,t))$. So that for $1 \leq p < q$ the inclusion $\ell_p(\hat{E}(r,t)) \subset \ell_q(\hat{E}(r,t))$ is held. Hence the proof. □

7.2. On Some Sequence Sets Based on the Lucas Band Matrix and a Sequence of Modulus Functions

In this subpart of our study, we generalized some sequence spaces depended on the Lucas band matrix $\hat{E}(r,t)$ and a sequence consisting of modulus functions $G = (g_i)$ in $\mathbb{G}$, and study some interesting results through newly introduced sequence spaces.

Definition 7.2.1 Assume that $G = (g_i)$ is given as a sequence of modulus functions in $\mathbb{G}$ and $v = (v_i)$ is considered a sequence consisting of strictly positive real numbers. Let $1 \leq p < \infty$, then with the help of (5.3), we generalized the following sequence spaces

$$\ell_p\left(G^k, v, \hat{E}(r,t)\right) = \left\{\zeta = (\zeta_i) \in \omega: \sum_i \left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^p < \infty\right\}$$

and

$$\ell_\infty\left(G^k, v, \hat{E}(r,t)\right) = \left\{\zeta = (\zeta_i) \in \omega: \sup_i\left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right] < \infty\right\}.$$

Remark 7.2.2 The above sequence spaces can be redefined with the help of (1.1) as follows

$$\ell_p\left(G^k, v, \hat{E}(r,t)\right) = \left(\ell_p\right)_{v_i g_i^k \hat{E}(r,t)} \text{ and } \ell_\infty\left(G^k, v, \hat{E}(r,t)\right) = (\ell_\infty)_{v_i g_i^k \hat{E}(r,t)} \quad (7.3)$$

Remark 7.2.3 If $k = 1$ then the spaces $\ell_p(G^k, v, \hat{E}(r,t))$ and $\ell_\infty(G^k, v, \hat{E}(r,t))$ will reduce to $\ell_p(G, v, \hat{E}(r,t))$ and $\ell_\infty(G, v, \hat{E}(r,t))$ of Mohiuddine at el. [47], respectively, as well as if we put $v_i = 1$ and $g_i(u) = u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then the sequence spaces $\ell_p(G^k, v, \hat{E}(r,t))$ and $\ell_\infty(G^k, v, \hat{E}(r,t))$ will become the same as $\ell_p(\hat{E}(r,t))$ and $\ell_\infty(\hat{E}(r,t))$ of Karakas [57], respectively.

Theorem 7.2.4 Consider that $G = (g_i)$ is a sequence consisting of modulus functions in $\mathbb{G}$. Then the sequence sets $\ell_p(G^k, v, \hat{E}(r,t))$ and $\ell_\infty(G^k, v, \hat{E}(r,t))$ are linear spaces over $\mathbb{C}$.

Proof. Let $\zeta = (\zeta_i)$, $\xi = (\xi_i) \in \ell_p(G^k, v, \hat{E}(r,t))$ and $\alpha, \sigma \in \mathbb{C}$. Then there are positive integers M_α and N_σ such that $|\alpha| \le M_\alpha$ and $|\sigma| \le N_\sigma$. Since g_i is increasing for each $i \in \mathbb{N}$, then we shall have

$$\sum_i \left[v_i g_i^k\left(\left|\alpha \hat{E}_i(r,t)(\zeta) + \sigma \hat{E}_i(r,t)(\xi)\right|\right)\right]^p$$
$$\le \sum_i \left[v_i g_i^k\left(|\alpha|\left|\hat{E}_i(r,t)(\zeta)\right|\right) + v_i g_i^k\left(|\sigma|\left|\hat{E}_i(r,t)(\xi)\right|\right)\right]^p$$
$$\le \sum_i 2^p \left[\left(v_i g_i^k\left(|\alpha|\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right)^p + \left(v_i g_i^k\left(|\sigma|\left|\hat{E}_i(r,t)(\xi)\right|\right)\right)^p\right]$$
$$\le (2M_\alpha)^p \sum_i \left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^p + (2N_\sigma)^p \sum_i \left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\xi)\right|\right)\right]^p$$
$$< \infty.$$

This verifies that $\ell_p(G^k, v, \hat{E}(r,t))$ is a linear space. The proof for $\ell_\infty(G^k, v, \hat{E}(r,t))$ is similarly done. Hence the proof. □

Theorem 7.2.5 Assume that $G = (g_i)$ is a sequence consisting of modulus functions in $\mathbb{G}$. The sequence sets $\ell_p(G^k, v, \hat{E}(r,t))$ and $\ell_\infty(G^k, v, \hat{E}(r,t))$ are normed linear spaces for $1 \le p < \infty$, respectively, normed by

$$\|\zeta\|_{\ell_p(G^k, v, \hat{E}(r,t))} = \left(\sum_{i=1}^{\infty} \left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^p\right)^{\frac{1}{p}} \tag{7.4}$$

and

$$\|\zeta\|_{\ell_\infty(G^k, v, \hat{E}(r,t))} = \sup_i \left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]. \tag{7.5}$$

Proof. Assume that $\zeta = (\zeta_i), \xi = (\xi_i) \in \ell_p(G^k, v, \hat{E}(r,t))$. The first norm axiom N_1 is clearly held. Then we concentrate on the other axioms. (N_2) Let α be a scalar and put $v = (v_i)$ as a sequence such that $(v_i) = (u_i h_i)$, where (u_i) is a sequence consisting of strictly positive real numbers and (h_i) is stated as follows

$$h_i = \frac{|\alpha| g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)}{g_i^k\left(\left|\hat{E}_i(r,t)(\alpha\zeta)\right|\right)}$$

with $g_i^k\left(\left|\hat{E}_i(r,t)(\alpha\zeta)\right|\right) \neq 0$ for all $i \in \mathbb{N}$. Then we shall have

$$\begin{aligned}\|\alpha\zeta\|_{\ell_p(G^k,v,\hat{E}(r,t))} &= \left(\sum_i \left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\alpha\zeta)\right|\right)\right]^p\right)^{\frac{1}{p}} \\ &= \left(\sum_i \left[u_i \frac{|\alpha| g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)}{g_i^k\left(\left|\hat{E}_i(r,t)(\alpha\zeta)\right|\right)} g_i^k\left(\left|\hat{E}_i(r,t)(\alpha\zeta)\right|\right)\right]^p\right)^{\frac{1}{p}} \\ &= |\alpha|\left(\sum_i \left[u_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^p\right)^{\frac{1}{p}} = |\alpha|\|\zeta\|_{\ell_p(G^k,v,\hat{E}(r,t))}.\end{aligned}$$

Using Minkowski's inequality (2.2), the axiom N_3 is easily held. Hence the proof □

Theorem 7.2.6 Assume that $G = (g_i)$ is a sequence consisting of modulus functions in $\mathbb{G}$. The sequence spaces $\ell_p(G^k, v, \hat{E}(r,t))$ and $\ell_\infty(G^k, v, \hat{E}(r,t))$ are Banach spaces, respectively, with $\|\cdot\|_{\ell_p(G^k,v,\hat{E}(r,t))}$ and $\|\cdot\|_{\ell_\infty(G^k,v,\hat{E}(r,t))}$, for $1 \leq p < \infty$.

The proof of this theorem is followed from Theorem 7.1.5, so we leave it here.

Theorem 7.2.7 Assume that $G = (g_i)$ is a sequence consisting of modulus functions in $\mathbb{G}$. Then the given sequence sets $\ell_p(G^k, v, \hat{E}(r,t))$ and $\ell_\infty(G^k, v, \hat{E}(r,t))$ are BK-spaces for $1 \leq p < \infty$, respectively, with the norms defined in (7.4) and (7.5).

Proof. The verification is simply obtained. Since the conditions of (7.3) hold, $\hat{E}(r,t)$ is a triangle matrix and both of ℓ_p and ℓ_∞ are BK-spaces considering their typical norms. Then by Theorem 4.3.12 of Wilansky [58], the proof can be done straightforwardly. Therefore, these two sequence sets are BK spaces. Hence the proof. □

Remark 7.2.8 It is clear that $\|\zeta\|_{\ell_p(G^k,v,\hat{E}(r,t))} \neq \||\zeta|\|_{\ell_p(G^k,v,\hat{E}(r,t))}$ and $\|\zeta\|_{\ell_\infty(G^k,v,\hat{E}(r,t))} \neq \||\zeta|\|_{\ell_\infty(G^k,v,\hat{E}(r,t))}$. This means that the sequence spaces $\ell_p(G^k, v, \hat{E}(r,t))$ and $\ell_\infty(G^k, v, \hat{E}(r,t))$ are of non-absolute type. From the above nonequalities, it has come to notice that the absolute property may not hold for $\ell_p(G^k, v, \hat{E}(r,t))$ and $\ell_\infty(G^k, v, \hat{E}(r,t))$ from at least one sequence in which $|\zeta| = \left(|\zeta_i|\right)$ and $1 \leq p < \infty$.

Theorem 7.2.9 Assume that $G = (g_i)$ is a sequence consisting of modulus functions in $\mathbb{G}$. Then for $1 \leq p < q$, the following inclusion relationship is satisfied.

$$\ell_p\left(G^k, v, \hat{E}(r,t)\right) \subseteq \ell_q\left(G^k, v, \hat{E}(r,t)\right)$$

Proof. Let $\zeta = (\zeta_i) \in \ell_p(G^k, v, \widehat{E}(r,t))$. Then for $p \geq 1$, it implies that

$$\left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^p \leq 1$$

for some fixed $i_0 \in \mathbb{N}$ and for all $i \geq i_0$. Since every modulus in G is increasing and $p < q$, then

$$\begin{aligned}\sum_{i=1}^{\infty}\left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^q &= \sum_{i=1}^{i_0}\left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^q + \sum_{i=i_0+1}^{\infty}\left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^q \\ &\leq \sum_{i=1}^{i_0}\left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^q + \sum_{i=i_0+1}^{\infty}\left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^p < \infty.\end{aligned}$$

Therefore $\zeta \in \ell_q(G^k, v, \widehat{E}(r,t))$. Hence the proof. □

Theorem 7.2.10 Assume that $G = (g_i)$ is a sequence consisting of modulus functions in $\mathbb{G}$. Then the indicated inclusion relationship below is valid.

$$\ell_p \subset \ell_p\left(G^k, v, \hat{E}(r,t)\right) \text{ for } 1 \leq p < \infty.$$

Proof. To verify the validity of the inclusion we need to find a number $D > 0$ such that $\|\zeta\|_{\ell_p(G^k, v, \hat{E}(r,t))} \leq D\|\zeta\|_{\ell_p}$ for $\zeta \in \ell_p$. From the Lucas sequence, we write $\frac{L_{i-1}}{L_i} \leq 2$ and $\frac{L_i}{L_{i-1}} \leq 3$ $(i \in \mathbb{N})$. Also, for $r \neq 0$ and $t \neq 0$, there are positive integers T_r and T_t such that $|r| \leq T_r$ and $|t| \leq T_t$. Now we assume that $\zeta \in \ell_p$ $(1 \leq p < \infty)$, then with the use of (5.3), we may write

$$\begin{aligned}\sum_i\left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^p &= \sum_i\left[v_i g_i^k\left(\left|r\frac{L_{i-1}}{L_i}\zeta_i + s\frac{L_i}{L_{i-1}}\zeta_{i-1}\right|\right)\right]^p \\ &\leq \sum_i 6^{p-1}\left[v_i g_i^k\left(|2r\zeta_i| + |3s\zeta_{i-1}|\right)\right]^p \\ &\leq 6^{2p-1}\, max\{T_r, T_t\}\left(\sum_i\left[v_i g_i^k\left(|\zeta_i|\right)\right]^p + \sum_i\left[v_i g_i^k\left(|\zeta_{i-1}|\right)\right]^p\right).\end{aligned}$$

Taking both sides of the above inequality to the power of $1/p$ then using Minkowski's inequality (2.2), for $1 < p \leq \infty$, we have

$$\|\zeta\|_{\ell_p(G^k,v,\hat{E}(r,t))} \leq 36\, max\{T_r, T_t\}\, \|\zeta\|_{\ell_p} \tag{7.6}$$

For $p = 1$, the inequality (7.6) is easily obtained. Hence the proof. □

Example 7.2.11 The sequence $\zeta = (\zeta_i) = \left(\frac{1}{r}\left(-\frac{s}{r}\right)^i L_i^2\right)$ assures the strictness of the above inclusion relationship since $\zeta \in \ell_p(G^k, v, \widehat{E}(r,t)) - \ell_p$.

Theorem 7.2.12 Assume that $G = (g_i)$ is a sequence consisting of modulus functions in $\mathbb{G}$, and let $\beta_m = v_i g_i^k(\hat{E}_m(r,t)(\zeta))$. Then for $1 \leq p < \infty$, the sequence $\left(h^{(m)}\right)_{i=1}^{\infty}$ provides a basis for $\ell_p(G^k, v, \hat{E}(r,t))$ which is formed as

$$\left(h^{(m)}\right)_i = \begin{cases} \frac{1}{r}\left(-\frac{s}{r}\right)^{m-i} \dfrac{L_i{}^2}{L_{m-1}L_m}, & i \geq m \\ 0, & m > i. \end{cases}$$

For that, every $\zeta \in \ell_p\left(G^k, v, \widehat{E}(r,t)\right)$ is uniquely represented in the following form

$$\zeta = \sum_m \beta_m h^{(m)}. \tag{7.7}$$

Proof. By using the sequence $\left(h^{(m)}\right)_i$ we get $v_i g_i^k\left(\hat{E}(r,t)\left(h^{(m)}\right)\right) = e^{(m)} \in \ell_p$ where $e^{(m)} = (0, 0, \dots, 0, 1, 0, \dots)$ (i.e. 1 at the *m*th place and zero elsewhere) for each $m \in \mathbb{N}$. Hence $h^{(m)} \in \ell_p(G^k, v, \hat{E}(r,t))$. In addition, let $\zeta \in \ell_p(G^k, v, \widehat{E}(r,t))$ and for every $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, take

$$\zeta^{(n)} = \sum_{m=1}^{n} \beta_m h^{(m)}.$$

Thus

$$v_i g_i^k\left(\hat{E}\,(r,t)\left(\zeta^{(n)}\right)\right) = \sum_{m=1}^{n} v_i g_i^k\left(\hat{E}_m(r,t)(\zeta)\right) v_i g_i^k\left(\hat{E}\,(r,t)\left(h^{(m)}\right)\right)$$

$$= \sum_{m=1}^{n} \beta_m\, e^{(m)}.$$

Also

$$v_i g_i^k\left(\hat{E}_i\,(r,t)\left(\zeta - \zeta^{(n)}\right)\right) = \begin{cases} v_i g_i^k\left(\hat{E}_i\,(r,t)(\zeta)\right), & i > n \\ 0, & 0 \leq i \leq n. \end{cases}$$

Then, there has $n_0 \in \mathbb{N}_0$ such that

$$\sum_{i=n_0+1}^{\infty}\left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^p \leq \left(\frac{\varepsilon}{2}\right)^p$$

for any $\varepsilon > 0$. Therefore, for every $n > n_0$, we shall have

$$\begin{aligned}\left\|\zeta-\zeta^{(n)}\right\|_{\ell_p(G^k,v,\hat{E}(r,t))} &= \left(\sum_{i=n+1}^{\infty}\left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{i=n_0+1}^{\infty}\left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^p\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

This concludes that

$$\lim_{n\to\infty}\left\|\zeta-\zeta^{(n)}\right\|_{\ell_p(G^k,v,\hat{E}(r,t))} = 0.$$

Moreover, to show that (7.7) is unique, let us consider

$$\zeta = \sum_m \xi_m h^{(m)}$$

for $\zeta \in \ell_p(G^k, v, \widehat{E}(r,t))$. Then, we have

$$\begin{aligned}v_i g_i^k\left(\hat{E}_i(r,t)(\zeta)\right) &= \sum_m \xi_m\left(v_i g_i^k\left(\hat{E}_i(r,t)\left(h^{(m)}\right)\right)\right) \\ &= \sum_m \xi_m e_i^{(m)} = \xi_i.\end{aligned}$$

Hence the proof. □

Theorem 7.2.13 Assume that $G = (g_i)$ is a sequence consisting of modulus functions in $\mathbb{G}$. Then the Gurarii's modulus of convexity for $\ell_p(G^k, v, \hat{E}(r,t))$, $1 \leq p < \infty$ is expressed as follows

$$\gamma_{\ell_p(G^k,v,\hat{E}(r,t))}(\varepsilon) \leq 1-\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}, \quad \varepsilon \in [0,2].$$

Proof. Take $\zeta \in \ell_p(G^k, v, \widehat{E}(r,t))$. Then,

$$\|\zeta\|_{\ell_p(G^k,v,\hat{E}(r,t))} = \left\|v_i g_i^k\left(\hat{E}_i(r,t)(\zeta)\right)\right\|_{\ell_p} = \left(\sum_i\left[v_i g_i^k\left(\left|\hat{E}_n(r,t)(\zeta)\right|\right)\right]^p\right)^{\frac{1}{p}}$$

We also take the sequences $a = (a_m)$ and $b = (b_m)$, where

$$a_m = \left(\left(v_i g_i^k\right)^{-1} \hat{E}^{-1}(r,t) \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}, \left(v_i g_i^k\right)^{-1} \hat{E}^{-1}(r,t) \left(\frac{\varepsilon}{2}\right), 0, 0, \dots \right)$$

and

$$b_m = \left(\left(v_i g_i^k\right)^{-1} \hat{E}^{-1}(r,t) \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}, \left(v_i g_i^k\right)^{-1} \hat{E}^{-1}(r,t) \left(-\frac{\varepsilon}{2}\right), 0, 0, \dots \right).$$

Where $\hat{E}^{-1}(r,t)$ represents the inverse of the matrix $\hat{E}(r,t)$ and $\varepsilon \in [0,2]$. The $\hat{E}$-transforms of the sequences a and b are given by

$$v_i g_i^k \hat{E}(r,t)(a) = \left(\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}, \left(\frac{\varepsilon}{2}\right), 0, 0, \dots \right)$$

and

$$v_i g_i^k \hat{E}(r,t)(b) = \left(\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}, \left(-\frac{\varepsilon}{2}\right), 0, 0, \dots \right).$$

Then, we have

$$\left\| v_i g_i^k \hat{E}(r,t)(a) \right\|_{\ell_p} = \|a\|_{\ell_p(G^k, v, \hat{E}(r,t)).} = 1,$$

and

$$\left\| v_i g_i^k \hat{E}(r,t)(b) \right\|_{\ell_p} = \|b\|_{\ell_p(G^k, v, \hat{E}(r,t)).} = 1.$$

Hence, $a, b \in S_{\ell_p(G^k, v, \hat{E}(r,t))}$, and

$$\left\| v_i g_i^k \hat{E}(r,t)(a) - v_i g_i^k \hat{E}(r,t)(b) \right\|_{\ell_p} = \|a - b\|_{\ell_p(G^k, v, \hat{E}(r,t))} = \varepsilon.$$

Now, for $\alpha \in [0,1]$,

$$\|\alpha a + (1-\alpha) b\|_{\ell_p(G^k, v, \hat{E}(r,t))}^p = \left\| \alpha v_i g_i^k \hat{E}(r,t)(a) + (1-\alpha) v_i g_i^k \hat{E}(r,t)(b) \right\|_{\ell_p}^p$$

$$= 1 - \left(\frac{\varepsilon}{2}\right)^p + |2\alpha - 1|^p \left(\frac{\varepsilon}{2}\right)^p.$$

From here,

$$\inf_{\alpha \in [0,1]} \|\alpha a + (1-\alpha) b\|_{\ell_p(G^k, v, \hat{E}(r,t))} = \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}.$$

Therefore, for $1 \le p < \infty$,

$$\gamma_{\ell_p(G^k,v,\hat{E}(r,t))}(\varepsilon) \leq 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}.$$

Hence the proof. □

Corollary 7.2.14

(i) If $\varepsilon = 2$, then $\gamma_{\ell_p(G^k,v,\hat{E}(r,s))}(\varepsilon) \leq 1$ and so that $\ell_p\left(G^k, v, \hat{E}(r,s)\right)$ is strictly convex.

(ii) If $0 < \varepsilon < 2$, then $0 < \gamma_{\ell_p(G^k,v,\hat{E}(r,s))}(\varepsilon) < 1$ and so that $\ell_p\left(G^k, v, \hat{E}(r,s)\right)$ is uniformly convex.

Theorem 7.2.15 Consider that $G = (g_i)$ is a sequence consisting of modulus functions in $\mathbb{G}$. If $g_i(u) \leq u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then $\ell_p(v, \hat{E}(r,t)) \subset \ell_p(G^k, v, \hat{E}(r,t))$.

Proof. Let $\zeta \in \ell_p(v, \hat{E}(r,t))$. Since (G_i) is increasing for each $i \in \mathbb{N}$, and $g_i(u) \leq u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then according to (6.1), we may write

$$\sum_i \left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^p \leq \sum_i \left[v_i g_i^{k-1}\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^p \leq \cdots \leq \sum_i \left[v_i \left|\hat{E}_i(r,t)(\zeta)\right|\right]^p.$$

So that

$$\sum_i \left[v_i g_i^k\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^p < \infty.$$

Therefore, we obtain that $\zeta \in \ell_p(G^k, v, \hat{E}(r,t))$. Hence the proof. □

Corollary 7.2.16 Assume that $G = (g_i)$ is a sequence consisting of modulus functions in $\mathbb{G}$. If $g_i(u) \leq u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then

$$\ell_p\left(v, \hat{E}(r,t)\right) \subset \ell_p\left(G, v, \hat{E}(r,t)\right) \subset \cdots \subset \ell_p\left(G^k, v, \hat{E}(r,t)\right).$$

Theorem 7.2.17 Consider that $G = (g_i)$ is a sequence consisting of modulus functions in $\mathbb{G}$. If $g_i(u) \geq u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then $\ell_p(G^k, v, \hat{E}(r,t)) \subset \ell_p(v, \hat{E}(r,t))$.

Proof. By using (6.2), the proof can be clearly done similar to the above proof. So, we leave it.

Corollary 7.2.18 Assume that $G = (g_i)$ is a sequence consisting of modulus functions in $\mathbb{F}$. If $g_i(u) \geq u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then

$$\ell_p\left(G^k, v, \hat{E}(r,t)\right) \subset \ell_p\left(G^{k-1}, v, \hat{E}(r,t)\right) \subset \cdots \subset \ell_p\left(v, \hat{E}(r,t)\right).$$

Corollary 7.2.19 Assume that $G = (g_i)$ is a sequence consisting of modulus functions in $\mathbb{G}$. Let $k_1 < k_2$ and $u \in \mathbb{R}^+ \cup \{0\}$, then we have the cases below:

(i) If $g_i(u) \leq u$ for all $i \in \mathbb{N}$, then $\ell_p(G^{k_1}, v, \hat{E}(r,t)) \subset \ell_p(G^{k_2}, v, \hat{E}(r,t))$.

(ii) If $g_i(u) \geq u$ for all $i \in \mathbb{N}$, then $\ell_p(G^{k_2}, v, \hat{E}(r,t)) \subset \ell_p(G^{k_1}, v, \hat{E}(r,t))$.

The proof of (i) and (ii) can be done by using (6.1) and (6.2), respectively.

Theorem 7.2.20 Consider that $G = (g_i)$ and $H = (h_i)$ are two sequences of modulus functions in $\mathbb{G}$. If $g_i(u) \leq h_i(u)$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then

(i) $\ell_p(G^k \circ H, v, \hat{E}(r,t)) \subset \ell_p(G^{k+1}, v, \hat{E}(r,t))$.

(ii) $\ell_p(H^{k+1}, v, \hat{E}(r,t)) \subset \ell_p(H^k \circ G, v, \hat{E}(r,t))$.

Proof. (i) From (6.3), we have

$$g_i^{k+1}(u) \leq g_i^k\big(h_i(u)\big).$$

Now let $\zeta \in \ell_p(G^k \circ H, v, \hat{E}(r,t))$. Then according to the above inequality, we may write

$$\sum_i \left[v_i g_i^{k+1}\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right]^p \leq \sum_i \left[v_i g_i^k\left(h_i\left(\left|\hat{E}_i(r,t)(\zeta)\right|\right)\right)\right]^p < \infty$$

for $1 \leq p < \infty$. This concludes that $\zeta \in \ell_p(G^{k+1}, v, \hat{E}(r,t))$. The proof of (ii) can be done similarly to the above steps by using (6.4). □

8. CONCLUSIONS

During this study, the modulus functions g and h, the lacunary sequences $\theta = (k_r)$ and $\vartheta = (s_r)$, the sequences of modulus functions $G = (g_i)$ and $H = (h_i)$, the Lucas transform $\hat{E}(r,t)$, and a sequence of strictly positive reals $v = (v_i)$ were generally used to obtain some new results, where $(r, i \in \mathbb{N})$.

Firstly, the idea of g^k-density of a natural numbers' subset was established, then by the use of θ, this definition has been extended to introduce the idea of lacunary statistical convergence according to g^k, where $g^k = g \circ g \circ \dots \circ g$ (k times). Also, the connections between the sets of $S_\theta(g^k)$ and S_θ, $S_\theta(g^{k+1})$ and $S_\theta(g^k \circ h)$, $S_\theta(g^k)$ and $S_\theta(g^{k,m})$, $S_\theta(g^{k,m})$ and $S_\theta(g^{m,k})$ have been talked about. After that, the idea of lacunary strong convergence according to g^k was defined, then the relations between the sets of $N_\theta(g^k)$ and N_θ, $N_\theta(g^{k_1})$ and $N_\theta(g^{k_2})$, $N_\theta(g^k)$ and $N_\vartheta(g^k)$, $N_\theta(g^k)$ and S_θ were determined, where $k_1 \leq k_2$.

In addition, the concept of G^k-density of a natural numbers' subset has been established, where $G^k = (g_n^k) = \{g_1^k, g_2^k, \dots\}$ $(k \in \mathbb{N})$. Later, with the help of a lacunary sequence θ and the Lucas transform $\hat{E}(r,t)$ this definition was extended to define the thought of lacunary statistical $\hat{E}(r,t)$-convergence according to G^k. Also some connections between the sets of $S_\theta(\hat{E}(r,t), G^k)$ and $S_\theta(\hat{E}(r,t))$, $S_\theta(\hat{E}(r,t), G^k)$ and $S_\vartheta(\hat{E}(r,t), G^k)$, $S_\theta(\hat{E}(r,t), G^{k+1})$ and $S_\theta(\hat{E}(r,t), G^k \circ H)$ have been examined. Furthermore, the concept of lacunary strong $\hat{E}(r,t)$-convergence according to G^k was introduced. Then the connections between the sets of $N_\theta(\hat{E}(r,t), G^k)$ and $N_\vartheta(\hat{E}(r,t), G^k)$, $N_\theta(\hat{E}(r,t), G^k \circ H)$ and $N_\theta(\hat{E}(r,t), G^{k+1})$, $N_\theta^0(G^k)$ and $N_\theta^0(\hat{E}(r,t), G^k)$ have been determined. Also, the relations of the set $N_\theta(\hat{E}(r,t), G^k)$-strongly convergent sequences to the sets of $S_\theta(\hat{E}(r,t), G^k)$-statistically convergent sequences and $w(\hat{E}(r,t), G^k)$-strongly Cesaro summable sequences were talked about.

Moreover, two sequence spaces depended on the Lucas band matrix and a sequence of modulus functions, such as $\ell_p(G^k, v, \hat{E}(r,t))$ and $\ell_\infty(G^k, v, \hat{E}(r,t))$ were generalized. Then these sequence spaces were established as BK spaces with their identified norms. The basis and a geometrical property such as the Gurarii's modulus of convexity for $\ell_p(G^k, v, \hat{E}(r,t))$ have been discussed. Also, the relation between ℓ_p and $\ell_p(G^k, v, \hat{E}(r,t))$ was examined.

REFERENCES

[1] Vembu, R. and Bowrna, A., (2016). A generalization of natural density, *Malaya Journal of Matematik*, 4(03), pp: 388-391. https://doi.org/10.26637/mjm403/006

[2] Zygmond, A, (1979). Trigonometric Series, *Cambridge University Press*, United Kingdom.

[3] Steinhaus, H, (1951). *Sur la convergence ordinaire et la convergence asymptotique*. *in Colloquium Mathematicae.*

[4] Fast, H, (1951). Sur la convergence statistique. *in Colloquium Mathematicae*. http://eudml.org/doc/209960

[5] Schoenberg, I.J., (1959). The integrability of certain functions and related summability methods, *The American Mathematical Monthly*, 66(5), pp: 361-775. https://doi.org/10.1080/00029890.1959.11989303

[6] Fridy, J.A., (1985). On statistical convergence, *Analysis*, 5(4), pp: 301-314. https://doi.org/10.1524/anly.1985.5.4.301

[7] Šalát, T., (1980). On statistically convergent sequences of real numbers, *Mathematica Slovaca*, 30(2), pp: 139-150. http://dml.cz/dmlcz/136236

[8] Bhardwaj, V.K. and Bala, I., (2007). On weak statistical convergence, *International Journal of Mathematics and Mathematical Sciences*, 2007. https://doi.org/10.1155/2007/38530

[9] Colak, R., (2011). On λ-statistical convergence, Conference on Summabiilty and Applications, *Commerce University*, *Istanbul*.

[10] Güngör, M., Et, M., and Altin, Y., (2004). Strongly (Vσ, λ, q)-summable sequences defined by Orlicz functions, *Applied Mathematics and Computation*, 157(2), pp: 561-571. https://doi.org/10.1016/j.amc.2003.08.051

[11] Çolak, R. and Bektaş, Ç., (2011). Errata to:"λ-statistical convergence of order α" Acta Mathematica Scientia 2011, 31B (3): 953–959, *Acta Mathematica Scientia*, 31(5), pp: 2099-2100. https://doi.org/10.1016/S0252-9602(11)60288-9

[12] Nakano, H., (1951). Modulared sequence spaces, *Proceedings of the Japan Academy*, 27(9), pp: 508-512. https://doi.org/10.3792/pja/1195571225

[13] Aizpuru, A., Listán-García, M., and Rambla-Barreno, F., (2014). Density by moduli and statistical convergence, *Quaestiones Mathematicae*, 37(4), pp: 525-530. https://doi.org/10.2989/16073606.2014.981683

[14] Bhardwaj, V.K., Dhawan, S., and Gupta, S., (2016). Density by moduli and statistical boundedness. in *Abstract and Applied Analysis*. Hindawi. https://doi.org/10.1155/2016/2143018

[15] Altin, Y., Altinok, H., and Colak, R., (2006). On some seminormed sequence spaces defined by a modulus function, *Kragujevac Journal of Mathematics*, 29(29), pp: 121-132.

[16] Hatim, M.I. and Bektaş, Ç.A., (2023). A study on lacunary strong convergence according to modulus functions, *Proceedings of the Bulgarian Academy of Sciences*, 76(10), pp: 1485-1485. https://doi.org/10.7546/CRABS.2023.10.01

[17] Freedman, A.R., Sember, J.J., and Raphael, M., (1978). Some Cesaro-type summability spaces, *Proceedings of the London Mathematical Society*, 3(3), pp: 508-520. https://doi.org/10.1112/plms/s337.3.508

[18] Bektas, Ç.A., (2011). Some New Type of Lacunary Generalized Difference Sequence Spaces Defined by a Sequence of Moduli, *Int. J. Open Problems Compt. Math.*, 4(2). pp: 3.

[19] Bhardwaj, V.K. and Dhawan, S., (2016). Density by moduli and lacunary statistical convergence. in *Abstract and Applied Analysis*. Hindawi. https://doi.org/10.1155/2016/9365037

[20] Ibrahim, I.S., Çolak, R., (2021). On strong lacunary summability of order α with respect to modulus functions, *Annals of the University of Craiova-Mathematics and Computer Science Series* 48(1), pp: 127-136. https://doi.org/10.52846/ami.v48i1.1399

[21] Kandemir, H.Ş., Et, M., and Çakalli, H., (2023). On $S_{\alpha}^{\beta}(\theta, A, F)$-convergence and strong $N_{\alpha}^{\beta}(\theta, A, F)$-convergence, *Boletim da Sociedade Paranaense de Matemática*, 41, pp: 1-8. https://doi.org/10.5269/bspm.51415

[22] Kandemir, H.S., Et, M., and Çakalli, H., (2023). Strongly lacunary convergence of order α in neutrosophic normed spaces, *Transactions of A. Razmadze Mathematical Institute*, 177(1). https://rmi.tsu.ge/transactions/TRMI-volumes/177-1/v177(1)-9.pdf

[23] Pehlivan, S. and Fisher, B., (1994). On some sequence spaces, *Indian Journal of Pure and Applied Mathematics*, 25, pp: 1067-1072.
[24] Başar, F., (2012). Summability Theory and its Applications, *Chapman and Hall/CRC*, Istanbul, Monographs. https://doi.org/10.1201/9781003294153
[25] Mursaleen, M. and Noman, A.K., (2010). On the spaces of λ-convergent and bounded sequences, *Thai Journal of Mathematics*, 8(2), pp: 311-329.
[26] Mursaleen, M. and Noman, A.K., (2011). On some new sequence spaces of non-absolute type related to the spaces l_p and l_∞ I, *Filomat*, **25**(2), pp: 33-51. https://www.jstor.org/stable/24895537
[27] Karakaş, M., (2015). A new regular matrix defined by Fibonacci numbers and its applications, *Bitlis Eren Üniversitesi Fen Bilimleri Dergisi*, 4(2). https://doi.org/10.17798/beufen.78452
[28] Candan, M. and Kara, E.E., (2015). A study of topological and geometrical characteristics of new Banach sequence spaces, *Gulf Journal of Mathematics*, 3(4). https://doi.org/10.56947/gjom.v3i4.50
[29] Başar, F. and Altay, B., (2003). On the space of sequences of p-bounded variation and related matrix mappings, *Ukrainian Mathematical Journal*, 55, pp: 136-147. https://doi.org/10.1023/A:1025080820961
[30] Savaş, E., Karakaya, V., and Şimşek, N., (2009). Some l(p) type new sequence spaces and their geometric properties. in *Abstract and Applied Analysis*. Hindawi. https://doi.org/10.1155/2009/696971
[31] Kizmaz, H., (1981). On certain sequence spaces, *Canadian Mathematical Bulletin*, 24(2), pp: 169-176.
[32] Et, M. and Çolak, R., (1995). On some generalized difference sequence spaces, *soochow journal of mathematics*, 21(4), pp: 377-386.
[33] Mursaleen, 1996. Generalized spaces of difference sequences, *Journal of Mathematical Analysis and Applications*, 203(3), 738-745. https://doi.org/10.1006/jmaa.1996.0409
[34] Bektas, C.A. and Colak, R., (2003). Generalized difference sequence spaces defined by a sequence of moduli, *Soochow Journal of Mathematics*, 29(2), pp: 215-220.
[35] Bektaş, Ç., Et, M., and Çolak, R., (2004). Generalized difference sequence spaces and their dual spaces, *Journal of Mathematical Analysis and Applications*, 292(2), pp: 423-432. https://doi.org/10.1016/j.jmaa.2003.12.006
[36] Polat, H., Karakaya, V., and Şimşek, N., (2011). Difference sequence spaces derived by using a generalized weighted mean, *Applied Mathematics Letters*, 24(5), pp: 608-614. https://doi.org/10.1016/j.aml.2010.11.020
[37] Colak, R. and Et, M., (2005). On some difference sequence sets and their topological properties, *Bulletin of the Malaysian Mathematical Sciences Society. Second Series*, 28(2), pp: 125-130.
[38] Altin, Y., (2009). Properties of some sets of sequences defined by a modulus function, *Acta Mathematica Scientia*, 29(2), pp: 427-434. https://doi.org/10.1016/S0252-9602(09)60042-4
[39] Kirişçi, M. and Başar, F., (2010). Some new sequence spaces derived by the domain of generalized difference matrix, *Computers & Mathematics with Applications*, 60(5), pp: 1299-1309. https://doi.org/10.1016/j.camwa.2010.06.010
[40] Choudhary, B., (1987). Functional Analysis with Applications, John Wiely, Sons, New Delhi, India.
[41] Rao, K.C., (2006). Functional Analysis and Applications, Alpha Science International Ltd., Oxford, U.K.
[42] Kreyszig, E., (1991). Introductory Functional Analysis with Applications, John Wiley & Sons.
[43] Trench, W.F., (2003). Introduction to Real Analysis, Library of Congress Cataloging-in-Publication.
[44] Maddox, I.J., (1970). Elements of Functional Analysis, Cambridge University Press.
[45] Clarkson, J.A., (1936). Uniformly convex spaces, *Transactions of the American Mathematical Society*, 40(3), pp: 396-414. DOI:10.1090/S0002-9947-1936-1501880-4
[46] Gurarii, V., (1967). On the differential properties of the modulus of convexity in a Banach space, *Mat. Issled*, 2, 8.
[47] Mohiuddine, S., Raj, K., and Choudhary, A., (2021). Difference sequence spaces based on Lucas band matrix and modulus function, *São Paulo Journal of Mathematical Sciences*, pp: 1-12. https://doi.org/10.1007/s40863-020-00203-2

[48] Obata, N., (1988). Density of natural numbers and the Lévy group, *Journal of Number Theory*, 30(3), pp: 288-297. https://doi.org/10.1016/0022-314X(88)90003-0

[49] Nakano, H., (1953). Concave modulars, *Journal of the Mathematical society of Japan*, 5(1), pp: 29-49. https://doi.org/10.2969/jmsj/00510029

[50] Çolak, R., (2020) Some Relations Between the Sets of f-Statistically Convergent Sequences. in *4th International Conference on Computational Mathematics and Engineering Sciences (CMES-2019) 4*. Springer. https://doi.org/10.1007/978-3-030-39112-6_25

[51] Ibrahim, I.S. and Çolak, R., (2021). On strong lacunary summability of order α with respect to modulus functions, *Annals of the University of Craiova-Mathematics and Computer Science Series*, 48(1), pp: 127-136. https://doi.org/10.52846/ami.v48i1.1399

[52] Bilalov, B. and Nazarova, T., (2015). On statistical convergence in metric spaces, *Journal of Mathematics Research*, 7(1), pp: 37. DOI:10.5539/jmr.v7n1p37

[53] Bhardwaj, V.K. and Dhawan, S., (2015). f-Statistical convergence of order α and strong Cesàro summability of order α with respect to a modulus, *Journal of Inequalities and Applications*, 2015(1), pp: 1-14. https://doi.org/10.1186/s13660-015-0850-x

[54] Maddox, I., (1987) Inclusions between FK spaces and Kuttner's theorem. *in Mathematical Proceedings of the Cambridge Philosophical Society*. Cambridge University Press. https://doi.org/10.1017/S0305004100066883

[55] Koshy, T., (2019). Fibonacci and Lucas Numbers with Applications, Volume 2, John Wiley & Sons.

[56] Vajda, S., (2008). Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications, Courier Corporation.

[57] Karakas, M. and Karakas, A., (2018). A study on lucas difference sequence spaces $l_p(\widehat{E}(r,t))$ and $l_\infty(\widehat{E}(r,t))$, *Maejo International Journal of Science and Technology*, 12(01), pp: 70-78. http://dspace.beu.edu.tr/handle/20.500.12643/13053

[58] Wilansky, A., (1984). Summability Through Functional Analysis, Elseiver Science Publishers B.V., Amsterdam.

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ACADEMIC ACTIVITIES

Conferences

- ICMME/ Firat University, Elazig, Turkey, 12-14 May 2016.
- CMES/ Ordu University, Ordu, Turkey, 20-22 May 2022.
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Papers

- M.I. Hatim, Ç.A. Bektaş, (2023). A study on lacunary strong convergence according to modulus functions, *Proceedings of the Bulgarian Academy of Sciences*, 76(10), pp. 1475-1485.
- M.I. Hatim, Ç.A. Bektaş, (2024). On some sequence sets based on the Lucas band matrix and modulus functions, *Filomat*, 38(11).