

**T.R.**  
**GEBZE TECHNICAL UNIVERSITY**  
**GRADUATE SCHOOL**

**FUNCTORIAL CONSRUCTIONS FOR GRAPH ALGEBRAS**



**HATİCE NUR KOLCU**

**A THESIS OF MASTER OF SCIENCE**  
**DEPARTMENT OF MATHEMATICS**

**ADVISOR: ASSOC. PROF. DR. AYTEN KOÇ**

**MARCH 2024**

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*To my  
Family and  
My Love...*

## ABSTRACT

The celebrated Gelfand-Naimark Theorem states that the category of compact Hausdorff topological spaces is equivalent to the opposite category of commutative unital  $C^*$ -algebras. Topological constructions such as obtaining the sphere by gluing two discs (corresponding to the northern and southern hemispheres) along their boundaries (the equator) or obtaining the real projective space as the quotient of a sphere by antipodal identification, have their counterparts in contravariant  $C^*$ -algebra constructions.

The Gelfand-Naimark-Segal Theorem characterises all  $C^*$ -algebras as closed  $*$ -subalgebras of bounded linear transformations on a Hilbert space  $H$ . Unlike the Gelfand-Naimark Theorem there is no uniqueness statements in the Gelfand-Naimark-Segal Theorem specifying the Hilbert space  $H$  or the closed  $*$ -subalgebra. There is probably no good analog of the Gelfand-Naimark Theorem for all noncommutative  $C^*$ -algebras. However, this may be possible for the important subclass of graph  $C^*$ -algebras. Some of these graph  $C^*$ -algebras have been identified as quantum spaces, that is, deformations of the commutative  $C^*$ -algebras of complex valued continuous functions defined on compact Hausdorff topological spaces.

The Stone-Weierstrass Theorem provides finitely generated dense subalgebras of algebras of continuous functions on closed and bounded subsets of Euclidean space. These are usually polynomial algebras or their quotients.

We present noncommutative (algebraic) analogs of these constructions as dense finitely generated subalgebras, (co)functorially. The cofunctor  $L_{\mathbb{F}}$  assigns to each di(irected)graph  $\Gamma$  its Leavitt path algebra  $L_{\mathbb{F}}(\Gamma)$  with coefficients in the field  $\mathbb{F}$  and a graded  $*$ -algebra homomorphism to each admissible digraph morphism.

**Keywords:** Leavitt Path Algebra,  $C^*$ -algebra, Category, Functor.

## ÖZET

Gelfand-Naimark Teoremi, tıkız Hausdorff topolojik uzayları kategorisinin, deęişmeli birimli  $C^*$ -cebirlerinin karřıt kategorisine denk olduęunu ifade eder. Kuzey ve güney yarımkürelere karřılık gelen iki diski sınırları (ekvator) boyunca yapıřtırarak küreyi elde etmek veya antipodal tanımlamayla bir kürenin bölümü olarak real projektif uzayı elde etmek gibi topolojik inřaların  $C^*$ -cebirleri kategorisinde karřılıkları vardır.

Gelfand-Naimark-Segal Teoremi, tüm  $C^*$ -cebirlerini, bir  $H$  Hilbert uzayı üzerindeki sınırlı lineer dönüşümlerin kapalı  $*$ -alt cebirleri olarak karakterize eder. Gelfand-Naimark Teoreminin aksine, Gelfand-Naimark-Segal Teoreminde  $H$  Hilbert uzayını veya kapalı  $*$ -alt cebirini belirleyen teklik ifadesi yoktur. Deęişmeli olmayan tüm  $C^*$ -cebirleri için Gelfand-Naimark Teoreminin muhtemelen iyi bir karřılığı yoktur. Ancak  $C^*$ -cebirlerinin önemli bir alt sınıfı olan çizge  $C^*$ -cebirleri için bu mümkün olabilir. Çizge  $C^*$ -cebirlerinin bazıları kuantum uzaylar, yani tıkız Hausdorff topolojik uzayların üzerinde tanımlı kompleks deęerli sürekli fonksiyon  $C^*$ -cebirlerinin (deęişmeli olmayan) deformasyonları olarak verilmiřtir.

Stone-Weierstrass Teoremi, Öklid uzayının kapalı ve sınırlı alt kümeleri üzerindeki sürekli fonksiyon cebirlerinin sonlu üretilmiř yoğun alt cebirlerini saęlar. Bunlar genellikle polinom cebirleri veya bölümleridir.

Bu yapıların deęişmeli olmayan (cebirsel) analoglarını, yani quantum uzayların yoğun, sonlu üretilmiř alt cebirlerini kofunktoryel olarak veriyoruz.  $L_{\mathbb{F}}$  kofunktoru her yönlü  $\Gamma$  çizgesine  $\Gamma$  nın katsayıları  $\mathbb{F}$  cisminde gelen  $L_{\mathbb{F}}(\Gamma)$  Leavitt yol cebirini ve her yasal yönlü çizge morfizmasına bir dereceli  $*$ -cebiri homomorfizması atar.

**Anahtar Kelimeler:** Leavitt Yol Cebiri,  $C^*$ -cebiri, Kategori, Funktor.

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## LIST OF ABBREVIATIONS AND ACRONYMS

|   |  |
|---|--|
| $\mathbb{N}$                                | : Natural numbers $\{0, 1, 2, \dots\}$   |
| $\mathbb{Z}$                                | : Integers   |
| $\mathbb{R}$                                | : Real numbers   |
| $\mathbb{C}$                                | : Complex numbers  |
| $\mathbb{F}[x]$                             | : Polynomial algebra with coefficients from the field $\mathbb{F}$   |
| $\mathbb{F}[x_1, \dots, x_n]$               | : Polynomial algebra with coefficients from the field $\mathbb{F}$ in $n$ variables                        |
| $\mathbb{F}[x, x^{-1}]$                     | : Laurent polynomial algebra   |
| $M_n(\mathbb{F})$                           | : Matrix algebra of $n \times n$ -matrices over $\mathbb{F}$ with coefficients from the field $\mathbb{F}$ |
| $\mathbb{F}X$                               | : $\mathbb{F}$ -vector space with basis the set $X$  |
| $\mathbb{F}\langle x_1, \dots, x_n \rangle$ | : Noncommutative polynomial algebra over $\mathbb{F}$ in $n$ variables                                     |
| $\mathbb{F}\langle X \rangle$               | : Noncommutative polynomial algebra over $\mathbb{F}$ with variable set $X$                                |
| $L_{\mathbb{F}}(1, n)$                      | : Leavitt algebra of type $(1, n)$ over $\mathbb{F}$   |
| $\mathcal{A}_{\gamma}$                      | : $\gamma$ -component of a $G$ -graded algebra $\mathcal{A}$ where $\gamma \in G$                          |
| $(X, \ \cdot\ )$                            | : Normed vector space  |
| $\langle \cdot   \cdot \rangle$             | : Inner product  |
| $X \perp Y$                                 | : $X$ is orthogonal to $Y$   |
| $B^{\perp}$                                 | : Orthogonal complement of $B$   |
| $d(x, y)$                                   | : Distance between $x$ and $y$   |
| $(X, d)$                                    | : Metric space   |
| $\mathbf{C}(X)$                             | : Complex valued continuous functions on the topological space $X$   |
| $\mathcal{T}$                               | : Topology $\mathcal{T}$ on a set $X$  |
| $(X, \mathcal{T})$                          | : Topological space  |
| $\overline{A}$                              | : Closure of $A$   |
| $\mathcal{T}_A$                             | : Subspace topology on $A$   |
| $X \cong Y$                                 | : Homeomorphism from $X$ to $Y$  |
| $\mathcal{T}_{\sim}$                        | : Quotient topology on $X/\sim$  |
| $X/G$                                       | : Space of orbits of group $G$ acting on topological space $X$   |
| $S^1$                                       | : Circle: $\{z \in \mathbb{C} \mid  z  = 1\}$  |
| $D^2$                                       | : 2-Disk: $\{z \in \mathbb{C} \mid  z  \leq 1\}$   |
| $S^2$                                       | : 2-Sphere: $\{(z, t) \in \mathbb{C} \times \mathbb{R} \mid  z ^2 + t^2 = 1\}$                             |
| $\mathbb{R}P^2$                             | : Real projective plane: $(\mathbb{R}^3 \setminus \{(0, 0, 0)\})/(\mathbb{R} \setminus \{0\})$             |
| $Ob(\mathcal{C})$                           | : Class of objects of the category $\mathcal{C}$   |
| $\mathcal{C}(X, Y)$                         | : Set of morphisms from $X$ to $Y$ in the category $\mathcal{C}$   |
| $id_X$                                      | : Identity map/morphism on $X$   |
| $\mathcal{C}^{op}$                          | : Opposite category of $\mathcal{C}$   |
| <b>Set</b>                                  | : Category of sets and functions   |
| <b>set</b>                                  | : Category of finite sets and functions  |
| <b>Grp</b>                                  | : Category of groups and group homomorphisms   |
| <b>Vec<math>_{\mathbb{F}}</math></b>        | : Category of vector spaces and linear transformations   |
| <b>vec<math>_{\mathbb{F}}</math></b>        | : Category of finite dimensional vector spaces and linear transformations                                  |

|  |   |  |
|--|---|--|
| <b>Mod<sub>R</sub></b>                       | : | Category of right $R$ -modules and $R$ -module homomorphisms               |
| <b>mod<sub>R</sub></b>                       | : | Category of finitely generated modules and $R$ -module homomorphisms       |
| <b>Alg<sub><math>\mathbb{F}</math></sub></b> | : | Category of $\mathbb{F}$ -algebras and $\mathbb{F}$ -algebra homomorphisms |
| <b>C*alg</b>                                 | : | Category of $C^*$ -algebras and bounded $*$ -algebra homomorphisms         |
| <b>Top</b>                                   | : | Category of topological spaces and continuous functions                    |
| <b>Dgrf</b>                                  | : | Category of digraphs and digraph morphisms                                 |
| $\Gamma$                                     | : | Di(irected )graph  |
| $s$  | : | Source function  |
| $t$  | : | Target function  |
| $E$  | : | Set of arrows  |
| $V$  | : | Set of vertices  |
| $Path(\Gamma)$                               | : | Set of paths in $\Gamma$   |
| $\rightsquigarrow$                           | : | Preorder leadsto on $V$  |
| $\mathbf{\Gamma}$                            | : | Path category of $\Gamma$ (with objects $V$ and morphisms $Path(\Gamma)$ ) |
| $\mathbb{F}\Gamma$                           | : | Path algebra of $\Gamma$ over $\mathbb{F}$                                 |
| $\hat{\Gamma}$                               | : | Extended digraph of $\Gamma$   |
| $L_{\mathbb{F}}(\Gamma)$                     | : | Leavitt path algebra of $\Gamma$   |
| $CL_{\mathbb{F}}(\Gamma)$                    | : | Cohn path algebra of $\Gamma$  |
| $ x $  | : | Grade of the homogenous element $x$  |
| $ev_x$                                       | : | Evaluation at $x$  |
| $X \amalg_A Y$                               | : | Pushout of the diagram $X \longleftarrow A \longrightarrow Y$              |
| $X \times_A Y$                               | : | Pullback of the diagram $X \longrightarrow A \longleftarrow Y$             |
| <b>aDgrf</b>                                 | : | Category of digraphs and admissible morphisms                              |
| $L_{\mathbb{F}}$                             | : | Leavitt cofunctor  |

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# 1. INTRODUCTION

The celebrated Gelfand-Naimark Theorem essentially states that the category of compact Hausdorff topological spaces is equivalent to the opposite category of commutative unital  $C^*$ -algebras (given as the Banach algebras of complex valued continuous functions defined on the spaces). When these spaces are closed and bounded subsets of Euclidean space the Stone-Weierstrass Theorem provides finitely generated dense subalgebras (polynomial algebras or their quotients).

Topological constructions such as obtaining the sphere by gluing two disks (corresponding to the north and south hemispheres) along their boundary (the equator) or by collapsing the boundary sphere of a disc, obtaining the real projective space as the quotient of a sphere by the antipodal identification have their counterparts in contravariant  $C^*$ -algebra constructions. These are explained in 4.1 and 4.2 below.

Noncommutative deformation quantizations of these  $C^*$ -algebras are referred to as quantum topological spaces. Graph  $C^*$ -algebras form an important class which include quantum discs, quantum spheres and quantum real projective spaces. Recently noncommutative analogs of these commutative  $C^*$ -algebra constructions have been studied, for instance in [1], [2], [3]. A discussion of deformation quantizations is beyond the scope of this thesis. We will just accept that the directed graphs given in these references correspond to the relevant quantum spaces.

The analog of dense polynomial algebras of the Stone-Weierstrass Theorem in graph  $C^*$ -algebras turn out to be Leavitt path algebras. These are discussed in Chapters 3 and 4. In particular, the third section of Chapter 4 has the main results of this thesis.

The methods used will be well-developed tools of category theory, topology, functional analysis and mostly associative algebras (specifically Leavitt path algebras). Our viewpoint is categorical and the main objective is to extend the Leavitt path algebra construction to a cofunctor from a suitable subcategory of directed graphs and their morphisms to category of  $\mathbb{Z}$ -graded  $*$ -algebras.

## 2. PRELIMINARIES

### 2.1. Algebras

**Definition 1.** [4] A set  $G$  is called a monoid such that  $G$  is a set and  $\star$  is a binary operation on  $G$  satisfying the following:

- (i)  $(a \star b) \star c = a \star (b \star c)$  for all  $a, b, c \in G$ ,
- (ii) there is  $e \in G$  such that  $a \star e = a = e \star a$  for all  $a \in G$ .

The element  $e$  is called the identity of the operation  $\star$ .

**Definition 2.** [4] A group is an ordered pair  $(G, \star)$  where  $G$  is a set and  $\star$  is a binary operation on  $G$  satisfying the following:

- (i)  $(G, \star)$  is monoid,
- (ii) For each  $a \in G$  there is an element  $a^{-1}$  of  $G$ , such that  $a \star a^{-1} = a^{-1} \star a = e$ .

The element  $a^{-1}$  is called the inverse of the element  $a$  with respect to  $\star$ .

The group  $(G, \star)$  is called abelian if  $a \star b = b \star a$  for all  $a, b \in G$ .

**Definition 3.** [4] A ring is an ordered triple  $(R, +, \cdot)$  such that  $R$  is a nonempty set and  $+$  and  $\cdot$  are two binary operations called addition and multiplication respectively, on  $R$  satisfying the following:

- (i)  $(R, +)$  is an abelian group,
- (ii)  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$  for all  $a, b, c \in R$ ,
- (iii)  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$  for all  $a, b, c \in R$ ,
- (iv)  $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$  for all  $a, b, c \in R$ .

If  $R$  has a multiplicative identity, denoted by  $1_R$ , or  $1$  when the ring  $R$  is clear from the context, then  $R$  is called unital.

A ring  $R$  is called commutative if  $ab = ba$  for all  $a, b \in R$ .

A commutative unital ring is called a field if the set of nonzero elements forms a multiplicative group.

We usually write  $ab$  instead of  $a \cdot b$  to denote multiplication.

**Definition 4.** [4] Let  $R$  and  $S$  be rings. A ring homomorphism is a map  $\theta : R \longrightarrow S$  satisfying  $\theta(a + b) = \theta(a) + \theta(b)$  and  $\theta(ab) = \theta(a)\theta(b)$  for all  $a, b \in R$ .

**Definition 5.** [4] Let  $R$  be a ring and  $I$  be a nonempty subset of  $R$ . If

- (i)  $a + b \in I$  for all  $a, b \in I$
- (ii)  $ra \in I$  and  $ar \in I$  for all  $r \in R$  and all  $a \in I$  then  $I$  is said to be a (two-sided) ideal of  $R$ .

**Definition 6.** [4] Let  $R$  be a ring. An abelian group  $(M, +)$  is called a right  $R$ -module or right module over  $R$  with respect to a mapping  $\cdot : M \times R \longrightarrow M$  if for all  $r, s \in R$  and  $m, n \in M$ ,

- (i)  $(m + n) \cdot r = m \cdot r + n \cdot r$ ,
- (ii)  $(m \cdot r) \cdot s = m \cdot (rs)$
- (iii)  $m \cdot (s + r) = m \cdot s + m \cdot r$

If  $R$  has 1 and if  $m \cdot 1 = m$  for all  $m \in M$ , then  $M$  is called unital or a unital right  $R$ -module. Again we usually write  $mr$  instead of  $m \cdot r$ .

A vector space is a unital (left)  $\mathbb{F}$ -module where  $\mathbb{F}$  is a field.

**Definition 7.** [4] Let  $R$  be a ring and let  $M$  and  $N$  be  $R$ -module. An  $R$ -module homomorphism is a map  $\theta : M \longrightarrow N$  satisfying  $\theta(m + n) = \theta(m) + \theta(n)$  and  $\theta(mr) = \theta(m)r$  for all  $m, n \in M$  and  $r \in R$ .

A vector space homomorphism is called a linear transformation.

**Definition 8.** [4] An  $\mathbb{F}$ -algebra  $\mathcal{A}$ , where  $\mathbb{F}$  is a field, is a ring that is also an  $\mathbb{F}$ -vector space with the same additive operation  $+$  such that multiplication by scalars satisfies  $\lambda(ab) = (\lambda a)b = a(\lambda b)$  for all  $\lambda \in \mathbb{F}$  and for all  $a, b \in \mathcal{A}$ . An algebra with 1 is called unital.

**Definition 9.** [4] Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $\mathbb{F}$ -algebras. An algebra homomorphism is a map  $\theta : \mathcal{A} \longrightarrow \mathcal{B}$  which is both a ring homomorphism and a linear transformation.

A ring/module/algebra homomorphism  $\theta$  is called

- (i) a monomorphism if  $\theta$  is one-to-one,
- (ii) an epimorphism if  $\theta$  is onto,

(iii) an isomorphism if there is a homomorphism  $\phi$  such that  $\theta \circ \phi$  and  $\theta \circ \phi$  are identity maps.

**Fact 1.** [4] A ring/module/algebra homomorphism  $\theta$  is an isomorphism if and only if  $\theta$  is one-to-one and onto.

**Example 1.** The set of all polynomials in  $X$  with coefficients from a field  $\mathbb{F}$  is denoted by  $\mathbb{F}[x]$  where  $\mathbb{F}[x] = \{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in \mathbb{F} \text{ with finitely many } a_i \text{ nonzero} \}$ . The algebra  $(\mathbb{F}[x], +, \cdot)$  is called the polynomial algebra with coefficients from  $\mathbb{F}$  when addition and multiplication of the polynomials

$$p(x) = \sum_{i=0}^{\infty} a_i x^i \text{ and } q(x) = \sum_{i=0}^{\infty} b_i x^i$$

are defined by

$$p(x) + q(x) = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

and

$$p(x) \cdot q(x) = \sum_{k=0}^{\infty} c_k x^k \text{ where } c_k = \sum_{i=0}^k a_i b_{k-i}.$$

The polynomial algebra  $\mathbb{F}[x]$  satisfies  $\lambda(p(x)q(x)) = (\lambda p(x))q(x) = p(x)(\lambda q(x))$  for all  $\lambda \in \mathbb{F}$  and for all  $p(x), q(x) \in \mathbb{F}[x]$ . Hence  $\mathbb{F}[x]$  is an  $\mathbb{F}$ -algebra.

Similarly,  $\mathbb{F}[x_1, x_2, \dots, x_n]$  is the polynomial algebra in  $n$  variables with coefficients from  $\mathbb{F}$ .

**Example 2.** The set of all polynomials in  $x, x^{-1}$  with coefficients from a field  $\mathbb{F}$  is denoted by  $\mathbb{F}[x, x^{-1}]$  where  $\mathbb{F}[x, x^{-1}] = \{ \sum_{i=-m}^n a_i x^i \mid a_i \in \mathbb{F}, m, n \in \mathbb{N} \}$ . This algebra is called the Laurent polynomial algebra with coefficients from  $\mathbb{F}$ .

**Example 3.** The set of square matrices of a fixed size over a field  $\mathbb{F}$ , with the operations of matrix addition, matrix multiplication and multiplication by scalars is an  $\mathbb{F}$ -algebra. The matrix algebra of  $n \times n$ -matrices over  $\mathbb{F}$  is denoted by  $M_n(\mathbb{F})$ .

**Definition 10.** [5] For a set  $X$  the vector space  $\mathbb{F}X$  is the set of formal  $\mathbb{F}$ -linear combinations of elements of  $X$ . The monoid algebra of a (multiplicative) monoid  $S$  over  $\mathbb{F}$  is  $\mathbb{F}S$  as a vector space with the multiplication of  $S$  extended linearly to  $\mathbb{F}S$ .

**Example 4.** If the monoid is  $(\mathbb{N}, +)$  then the algebra  $\mathbb{F}\mathbb{N}$  is isomorphic to the polynomial algebra  $\mathbb{F}X$  where  $n \leftrightarrow x^n$ .

**Example 5.** The  $\mathbb{F}$ -algebra  $\mathbb{F}\langle x_1, x_2, \dots, x_n \rangle$  is called the noncommutative polynomial algebra over  $\mathbb{F}$  in  $n$  noncommuting variables, that is, the monoid algebra of the multiplicative monoid of noncommuting monomials  $x_{i_1}^{m_1} x_{i_2}^{m_2} \dots x_{i_k}^{m_k}$  with  $i_1, i_2, \dots, i_k \in \{1, 2, \dots, n\}$  and  $m_i \in \mathbb{N}$  for  $i \in \{1, 2, \dots, k\}$ . Note that if some  $m_j = 0$  then this monomial is identified with the monomial obtained by dropping  $x_{i_j}$ . In particular if all  $m_j = 0$  then we have the empty product which is 1, the multiplicative identity of the monoid and the algebra. The subalgebra of polynomials with constant term 0, that is, the span of monomials  $x_{i_1}^{m_1} x_{i_2}^{m_2} \dots x_{i_k}^{m_k}$  with some  $m_i > 0$  will be denoted  $\mathbb{F}\langle x_1, x_2, \dots, x_n \rangle_+$  which is a nonunital algebra. For any set  $X$  the polynomial algebra  $\mathbb{F}\langle X \rangle$  is the polynomial algebra in the noncommuting variables  $x \in X$  and  $\mathbb{F}\langle X \rangle_+$  is as above, that is,  $\mathbb{F}\langle X \rangle_+$  is the (nonunital) subalgebra of  $\mathbb{F}\langle X \rangle$  consisting of polynomials with constant term zero.

Algebras defined by generators and relations: If  $X$  is the set of generators then the relations are the form  $\{f_i = g_i\}_{i \in I}$

- (i) If  $f_i, g_i \in \mathbb{F}\langle X \rangle_+$  for all  $i \in I$  then  $\mathcal{A} := \mathbb{F}\langle X \rangle_+ / (\{f_i - g_i\}_{i \in I})$  is the algebra generated by  $X$  subject to the relations  $\{f_i = g_i\}_{i \in I}$  where  $(\{f_i - g_i\}_{i \in I})$  is the ideal generated by  $\{f_i - g_i\}_{i \in I}$ .
- (ii) If some  $f_i$  or  $g_i$  is not in  $\mathbb{F}\langle X \rangle_+$ , that is, some  $f_i$  or  $g_i$  has a nonzero constant term, then the algebra generated by  $X$  subject to the relations  $\{f_i = g_i\}_{i \in I}$  is  $\mathcal{A} := \mathbb{F}\langle X \rangle / (\{f_i - g_i\}_{i \in I})$ .

The definition of most algebras below, such as Leavitt path algebras, is of the first kind, that is, as a quotient of  $\mathbb{F}\langle X \rangle_+$ . But the definition of the Jacobson algebra (also defined below) is of the second kind.

To define an algebra homomorphism  $\varphi$  from the  $\mathbb{F}$ -algebra  $\mathcal{A}$  generated by  $X$  subject to the relations  $\{f_i = g_i\}_{i \in I}$  to the  $\mathbb{F}$ -algebra  $\mathcal{B}$  we specify  $\varphi(X) = \{\varphi(x)\}_{x \in X} \subseteq \mathcal{B}$  for all  $x \in X$  and check that the relations are satisfied, that is,  $f_i(\varphi(X)) = g_i(\varphi(X))$  for all  $i \in I$ .

**Example 6.** The matrix algebra  $M_n(\mathbb{F})$  where  $\mathbb{F}$  is a field and  $n$  is a positive integer, may be realized as the algebra generated by the  $n^2$  generators  $\{x_{ij}\}$  with  $1 \leq i, j \leq n$

subject to the  $n^4$  relations  $x_{ij}x_{kl} = \delta_{j,k}x_{il}$  for all  $1 \leq i, j, k, l \leq n$  where  $\delta_{j,k}$  is the Kronecker delta. In the quotient  $\mathbb{F}\langle\{x_{ij}\}\rangle_+ / (\{x_{ij}x_{kl} - \delta_{j,k}x_{il}\}) \cong M_n(\mathbb{F})$  the image of  $x_{ij} \in \mathbb{F}\langle\{x_{ij}\}\rangle_+$  is the elementary matrix  $E_{ij} \in M_n(\mathbb{F})$  whose  $(i, j)$  entry is 1 and all other entries are 0.

**Example 7.** [6] Let  $\mathbb{F}$  be a field and  $n$  a positive integer. Then the Leavitt  $\mathbb{F}$ -algebra of type  $(1, n)$ , denoted by  $L_{\mathbb{F}}(1, n)$ , is the  $\mathbb{F}$ -algebra generated by elements  $\{x_i, y_i : i = 1, \dots, n\}$  and subject to the following relations:

$$(i) \sum_{i=1}^n y_i x_i = 1$$

$$(ii) x_i y_j = \delta_{i,j} .$$

Note that  $L_{\mathbb{F}}(1, 1) \cong \mathbb{F}[x, x^{-1}]$ .

**Definition 11.** [7] Let  $G$  be a group. An  $\mathbb{F}$ -algebra  $\mathcal{A}$  is called a  $G$ -graded  $\mathbb{F}$ -algebra if

$$(i) \mathcal{A} = \bigoplus_{\gamma \in G} \mathcal{A}_{\gamma} \text{ where } \mathcal{A}_{\gamma} \text{ are subspaces of } \mathcal{A},$$

$$(ii) \mathcal{A}_{\gamma} \mathcal{A}_{\delta} \subseteq \mathcal{A}_{\gamma\delta} \text{ for every } \gamma, \delta \in G.$$

The subspace  $\mathcal{A}_{\gamma}$  is called the  $\gamma$ -component of  $\mathcal{A}$ . The elements of  $\bigcup_{\gamma \in G} \mathcal{A}_{\gamma}$  in a graded algebra  $\mathcal{A}$  are called homogeneous elements. The non-zero elements of  $\mathcal{A}_{\gamma}$  are called  $\gamma$ -homogeneous, and set to have grade  $\gamma$ . The element 0 is considered to be homogeneous of every grade.

**Example 8.** The polynomial algebra  $\mathbb{F}[x]$  is a  $\mathbb{Z}$ -graded  $\mathbb{F}$ -algebra since  $\mathbb{F}[x] = \bigoplus_{i \in \mathbb{N}} \mathbb{F}x^i$  with  $\mathbb{F}[x]_i := \mathbb{F}x^i$  when  $i \geq 0$ , in particular  $\mathbb{F}[x]_0 = \mathbb{F}$  and  $\mathbb{F}[x]_1 = \mathbb{F}x$ . Also  $\mathbb{F}[x]_i := 0$  when  $i < 0$ . Note that  $\mathbb{F}[x]_i \mathbb{F}[x]_j = (\mathbb{F}x^i)(\mathbb{F}x^j) = \mathbb{F}x^{i+j} = \mathbb{F}[x]_{i+j}$ . The polynomial algebra  $\mathbb{F}[x, y]$  has a  $\mathbb{Z}$ -grading (by total degree) and a  $\mathbb{Z} \times \mathbb{Z}$ -grading where  $\mathbb{F}[x, y]_i := \mathbb{F}x^i + \mathbb{F}x^{i-1}y + \dots + \mathbb{F}y^i$  in the  $\mathbb{Z}$ -grading and  $\mathbb{F}[x, y]_{(i,j)} := \mathbb{F}x^i y^j$  in the  $\mathbb{Z} \times \mathbb{Z}$ -grading for all  $i, j \geq 0$ , with the remaining homogeneous components being zero. Similarly  $\mathbb{F}[x_1, x_2, \dots, x_n]$  has a  $\mathbb{Z}$ -grading by total degree (where homogeneous elements of degree  $n$  are exactly the homogeneous polynomials of degree  $n$ ) and a  $\mathbb{Z}^n$ -grading where homogeneous components have dimension 0 or 1.

**Example 9.** If  $\mathbb{F}$  be a be an arbitrary field and  $\mathbb{F}[x, x^{-1}]$  be the Laurent polynomial algebra. An element of  $\mathbb{F}[x, x^{-1}]$  is of the form  $\sum_{i \geq m} a_i x^i$  with  $m \in \mathbb{Z}$  and finitely many non-zero  $a_i$ 's. Then  $\mathbb{F}[x, x^{-1}]$  has the  $\mathbb{Z}$ -grading where  $\mathbb{F}[x, x^{-1}]_n = \mathbb{F}x^n$  for

all  $n \in \mathbb{Z}$ .

**Example 10.** A super algebra  $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$  is a  $\mathbb{Z}/2\mathbb{Z}$ -graded algebra. Elements of the 0-component  $\mathcal{A}_0$  are said to be even and elements of the 1-component  $\mathcal{A}_1$  are said to be odd. Any  $\mathbb{F}$ -algebra has the trivial  $\mathbb{Z}/2\mathbb{Z}$ -grading where  $\mathcal{A}_0 := \mathcal{A}$  and  $\mathcal{A}_1 := 0$ . Similarly, any  $\mathbb{F}$ -algebra  $\mathcal{A}$  has the trivial  $\mathbb{Z}$ -grading where  $\mathcal{A}_0 := \mathcal{A}$  and  $\mathcal{A}_n := 0$  when  $n \neq 0$ .

**Definition 12.** [8] Let  $\mathbb{F}$  be a field with an automorphism  $\mathbb{F} \xrightarrow{\bar{\cdot}} \mathbb{F}$  satisfying  $\overline{\bar{\lambda}} = \lambda$  for all  $\lambda$  in  $\mathbb{F}$ . The  $\mathbb{F}$ -algebra  $\mathcal{A}$  is a  $*$ -algebra if there is a map  $*$  :  $\mathcal{A} \rightarrow \mathcal{A}$  satisfying the following for all  $x, y \in \mathcal{A}$  and for all  $\lambda \in \mathbb{F}$  :

(i)  $(x + y)^* = x^* + y^*$

(ii)  $(xy)^* = y^*x^*$

(iii)  $(x^*)^* = x$

(iv)  $(\lambda x)^* = \bar{\lambda}x^*$ .

If  $\mathcal{A}$  is unital then  $1^* = 1$ : For all  $x \in \mathcal{A}$ , we have  $1^*x = ((1^*x)^*)^* = (x^*1)^* = (x^*)^* = x$ , similarly  $x1^* = x$ . Hence  $1^* = 1^*1 = 1$ .

When  $\mathcal{A}$  is also a  $G$ -graded algebra we say that  $\mathcal{A}$  is a graded  $*$ -algebra if  $\mathcal{A}_\gamma^* = \mathcal{A}_{\gamma^{-1}}$  for all  $\gamma \in G$ .

## 2.2. $C^*$ -algebras

**Definition 13.** [9] Let  $X$  be a vector space over  $\mathbb{F}$  where  $\mathbb{F}$  is either  $\mathbb{R}$  or  $\mathbb{C}$ . A norm on  $X$  is a map  $\|\cdot\| : X \rightarrow \mathbb{R}$  that satisfies the following properties:

(i)  $\|x\| \geq 0$  for all  $x \in X$

(ii)  $\|x\| = 0$  if and only if  $x = 0$  for all  $x \in X$

(iii)  $\|\lambda x\| = |\lambda|\|x\|$  for all  $x \in X$  and for all  $\lambda \in \mathbb{F}$

(iv)  $\|x + y\| \leq \|x\| + \|y\|$  for all  $x, y \in X$ .

$(X, \|\cdot\|)$  is called a normed vector space or a normed space for short.

A linear transformation  $f : X \rightarrow Y$  where  $X$  and  $Y$  are normed spaces is called bounded if  $\sup_{\|x\| \leq 1} \|f(x)\|$  is finite.  $\|f\| := \sup_{\|x\| \leq 1} \|f(x)\|$  is the operator norm of a bounded linear transformation  $f$ .

If  $X$  and  $Y$  are normed spaces then  $X$  and  $Y$  are isometrically isomorphic if there is an onto linear transformation  $\varphi : X \longrightarrow Y$  and  $\|\varphi(x)\| = \|x\|$  for all  $x \in X$ . Note that  $\varphi$  is necessarily one-to-one since if  $\varphi(x) = 0$  then  $\|x\| = \|\varphi(x)\| = \|0\| = 0$ , hence  $x = 0$ .

**Definition 14.** [10] Let  $X$  be a vector space over  $\mathbb{C}$  (respectively  $\mathbb{R}$ ). An inner product on  $X$  is a map  $X \times X \longrightarrow \mathbb{C}$ , (respectively  $\mathbb{R}$ ) denoted by  $(u, v) \mapsto \langle u|v \rangle$ , satisfying the following properties:

- (i)  $\langle u|u \rangle \in \mathbb{R}$  and  $\langle u|u \rangle \geq 0$  for all  $u \in X$ ,
- (ii)  $\langle u|u \rangle = 0$  if and only if  $u = 0$  for all  $u \in X$ ,
- (iii)  $\langle u|v \rangle = \overline{\langle v|u \rangle}$  for all  $u, v \in X$ ,
- (iv)  $\langle u + \alpha v|w \rangle = \langle u|w \rangle + \alpha \langle v|w \rangle$  for all  $u, v, w \in X$  and  $\alpha \in \mathbb{C}$  (respectively  $\mathbb{R}$ ).

An inner product on  $X$  defines a norm on  $X$  given by  $\|u\|^2 = \langle u|u \rangle$ .

**Definition 15.** [10] Elements  $x$  and  $y$  of an inner product space  $X$  are said to be orthogonal, denoted by  $x \perp y$ , if  $\langle x|y \rangle = 0$ . Subsets  $A, B$  of  $X$  are orthogonal, denoted  $A \perp B$ , if  $a \perp b$  for all  $a \in A$  and all  $b \in B$ .

An orthogonal set  $A$  in an inner product space  $X$  is a subset of  $X$  whose elements are pairwise orthogonal. An orthonormal set  $A \subset X$  is an orthogonal set in  $X$  whose elements have norm 1, that is,  $\langle x|y \rangle = \delta_{x,y}$ , for all  $x, y \in A$  where  $\delta$  is the Kronecker delta. The orthogonal complement of  $B \subset X$  is the subspace of all vectors in  $X$  that are orthogonal to  $B$ , denoted  $B^\perp := \{x \in X \mid x \perp b \text{ for all } b \in B\}$ .

**Definition 16.** [11] Suppose  $d : X \times X \longrightarrow \mathbb{R}$  and that for all  $x, y, z \in X$ :

- (i)  $d(x, y) \geq 0$
- (ii)  $d(x, y) = 0$  if and only if  $x = y$ ;
- (iii)  $d(x, y) = d(y, x)$ ;
- (iv)  $d(x, y) \leq d(x, z) + d(z, y)$ ;

$d$  is called a metric on  $X$  and  $(X, d)$  is called a metric space.

**Examples.**

- (i) The standard metric on  $\mathbb{R}$  or  $\mathbb{C}$  is defined as  $d(x, y) := |x - y|$ .

(ii) The standard metric on  $\mathbb{R}^n$  or  $\mathbb{C}^n$  is defined as  $d(x, y) := \sqrt{\sum_{i=1}^n \|(x_i - y_i)\|^2}$  when  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_n)$  are in  $\mathbb{R}^n$  or  $\mathbb{C}^n$ .

A normed space  $X$  is a metric space where  $d(x, y) := \|x - y\|$  for all  $x, y \in X$ .

**Definition 17.** [11] A sequence  $\{x_n\}_{n \in \mathbb{N}}$  in a metric space  $(X, d)$  converges to  $x \in X$  if for all  $\epsilon > 0$  there is an  $m \in \mathbb{N}$  such that  $d(x_n, x) < \epsilon$  for all  $n \geq m$ . If  $\{x_n\}$  converges some  $x \in X$  then we say that  $\{x_n\}$  is convergent sequence. A subset  $D$  of a metric space  $X$  is dense if for all  $x \in X$  there is a sequence  $\{x_n\} \subseteq D$  converging to  $x$ .

**Definition 18.** [11] A sequence  $\{x_n\}_{n \in \mathbb{N}}$  in a metric space is called a Cauchy sequence if for any given  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that if  $n, m \geq N$  then  $d(x_n, x_m) < \epsilon$ . A metric space  $X$  is complete if all Cauchy sequences in  $X$  are convergent.

**Definition 19.** [10] A complete normed space is called a Banach space. A Hilbert space  $H$  is a complete inner product space. Hence every Hilbert spaces is a Banach space. A Banach space is said to be separable if it has a countable dense subset, equivalently if it contains a countable subset whose span is dense. A linearly independent subset whose span is dense is called a topological basis.

**Example 11.** The set  $l_2$  of all infinite sequences  $x = (x_1, x_2, \dots)$  of complex (or real) numbers satisfying  $\sum_{k=1}^{\infty} |x_k|^2 < \infty$ , equipped with pointwise addition  $(x_1, x_2, \dots) + (y_1, y_2, \dots) := (x_1 + y_1, x_2 + y_2, \dots)$  and pointwise scalar multiplication  $\lambda(x_1, x_2, \dots) := (\lambda x_1, \lambda x_2, \dots)$  is an inner product space where  $\langle (x_1, x_2, \dots) | (y_1, y_2, \dots) \rangle := \sum_{k=1}^{\infty} x_k \overline{y_k}$ .

This infinite series converges because  $\sum_{k=1}^{\infty} |x_k|^2$  and  $\sum_{k=1}^{\infty} |y_k|^2$  are both convergent [10]. Also  $l_2$  is a separable Hilbert space since  $\{e_1, e_2, e_3, \dots\}$  is a countable orthonormal topological basis where  $e_i := (0, \dots, 0, 1, 0, \dots)$  is the sequence with only the  $i^{\text{th}}$  coordinate nonzero. In fact every separable Hilbert space is isometrically isomorphic to  $l_2$  via Gramm-Schmidt orthogonalization [10].

**Definition 20.** [10] A normed algebra  $\mathcal{A}$  is a normed space which is an algebra such that  $\|xy\| \leq \|x\| \|y\|$  for all  $x, y \in \mathcal{A}$  and if  $\mathcal{A}$  has an identity  $\mathbf{1}$  then  $\|\mathbf{1}\| = 1$ .

**Definition 21.** [10] A Banach algebra is a complete normed algebra  $\mathcal{A}$ .

**Definition 22.** [10] A  $C^*$ -algebra  $\mathcal{A}$  is a Banach  $*$ -algebra satisfying the  $C^*$  identity such that  $\|ff^*\| = \|f\|^2$  for all  $f \in \mathcal{A}$ .

**Example 12.** Let  $X$  be a compact Hausdorff topological space and  $\mathbf{C}(X)$  be the algebra of complex valued continuous functions.  $\mathbf{C}(X)$  is a commutative unital  $C^*$ -algebra where  $f^* = \bar{f}$  and  $\|f\| = \|f\|_\infty = \sup_{x \in X} \|f(x)\|$ .

**Theorem 2.1. (Gelfand-Naimark, 1943 [12] )** If  $\mathcal{A}$  is a unital commutative  $C^*$ -algebra then  $\mathcal{A}$  is isometrically isomorphic to  $\mathbf{C}(X)$ , the  $C^*$ -algebra of complex valued continuous functions for a unique (up to homeomorphism) compact Hausdorff topological space  $X$ .

**Theorem 2.2. (Gelfand-Naimark-Segal, 1947 [12] )** Every  $C^*$ -algebra is isometrically isomorphic to a closed  $*$ -subalgebra of  $B(H)$ , the bounded linear operators on some Hilbert space  $H$ , with the operator norm.

If  $\mathcal{A}$  is a closed  $*$ -subalgebra of  $B(H)$ , then  $\mathcal{A}$  is a  $C^*$ -algebra with the operator norm:  $\mathcal{A}$  is complete since it is closed, hence it is a Banach  $*$ -algebra. The  $C^*$  identity holds for  $\mathcal{A}$  since it holds for  $B(H)$  in generally.

**Definition/Theorem 1.** [13] If  $K$  is a closed subspace of the Hilbert space  $H$  then  $H \cong K \oplus K^\perp$ . This decomposition defines  $\alpha_K \in B(H)$ , the (orthogonal) projection onto  $K$ . If  $K'$  is another closed subspace of  $H$  then  $\alpha_K \alpha_{K'} = 0$  if and only if  $K' \perp K$ . An operator  $\alpha \in B(H)$  for some Hilbert space  $H$  is a projection if and only if  $\alpha$  is self-adjoint and idempotent, that is,  $\alpha^* = \alpha$  and  $\alpha^2 = \alpha$ . Hence if  $\alpha^* = \alpha = \alpha^2$  then  $\alpha = \alpha_K$  for a (unique) closed subspace  $K (= Im\alpha)$  of  $H$ . We define a projection in a  $C^*$ -algebra to be a self-adjoint idempotent element. We say that projections  $\alpha$  and  $\beta$  are orthogonal if  $\alpha\beta = 0$ .

If  $\varphi$  is a bounded linear transformation from the Hilbert space  $H$  to the Hilbert space  $H'$  then  $\varphi$  is a partial isometry if  $\varphi|_{\text{ker}\varphi^\perp}$  is an isometry. If  $\varphi \in B(H)$  then  $\varphi$  is a partial isometry if and only if  $\varphi\varphi^*$  and  $\varphi^*\varphi$  are projections. We define an element  $\varphi$  of a  $C^*$ -algebra to be a partial isometry if  $\varphi\varphi^*$  and  $\varphi^*\varphi$  are projections. If  $\varphi$  is partial isometry so is  $\varphi^*$ . [14]

**Definition 23.** [13] Let  $\Gamma = (V, E, s, t)$  be a finite digraph. A Cuntz–Krieger  $\Gamma$ -family in a  $C^*$ -algebra consists of a set of projections  $\{\alpha_v \mid v \in V\}$  and a set of partial isometries  $\{\varphi_e \mid e \in E\}$  satisfying:

$$(V) \quad \alpha_v \alpha_w = \delta_{v,w} \alpha_v$$

$$(E) \quad \alpha_{s(e)} \varphi_e = \varphi_e$$

$$(CK1) \quad \varphi_e^* \varphi_e = \alpha_{t(e)}$$

$$(CK2) \quad \alpha_v = \sum_{s(e)=v} \varphi_e \varphi_e^* \text{ whenever } v \text{ is not a sink.}$$

The graph  $C^*$ -algebra  $C^*(\Gamma)$  of  $\Gamma$  is a  $C^*$ -algebra with a Cuntz–Krieger  $\Gamma$ -family  $\{\alpha_v | v \in V\}$  and  $\{\varphi_e | e \in E\}$  satisfying the universal property: For every Cuntz–Krieger  $\Gamma$ -family  $\{\beta_v | v \in V\}$  and  $\{\psi_e | e \in E\}$  in a  $C^*$ -algebra  $\mathcal{B}$ , there is a unique homomorphism  $\theta : C^*(\Gamma) \longrightarrow \mathcal{B}$  by  $\theta(\alpha_v) = \beta_v$  and  $\theta(\varphi_e) = \psi_e$  for all  $v \in V$  and for all  $e \in E$ . The graph  $C^*$ -algebra  $C^*(\Gamma)$  is unique up to unique isomorphism.

## 2.3. Topology

**Definition 24.** [11] A topology  $\mathcal{T}$  on a set  $X$  is a collection of subsets of  $X$  such that

- (i)  $\emptyset, X \in \mathcal{T}$ ,
- (ii) If  $U_i \in \mathcal{T}$  for each  $i \in I$  then  $\bigcup_{i \in I} U_i \in \mathcal{T}$ ,
- (iii) If  $U_1, \dots, U_n \in \mathcal{T}$  then  $\bigcap_{i=1}^n U_i \in \mathcal{T}$ .

The pair  $(X, \mathcal{T})$  is called a topological space.

A set  $U \subseteq X$  is called open if  $U \in \mathcal{T}$ .

A subset  $F$  in  $X$  is called closed if its complement is open, that is,  $X \setminus F \in \mathcal{T}$ .

### Examples.

- (i) Suppose  $X$  is a set and  $\mathcal{T} = \{\emptyset, X\}$  is called the trivial topology on  $X$  and it is the smallest possible topology on  $X$ . The only open (or closed) sets are  $\emptyset$  and  $X$ .
- (ii) Suppose  $\mathcal{T} = \mathcal{P}(X)$  where  $\mathcal{P}(X)$  is the set of all subsets of  $X$ . Then  $\mathcal{T}$  is called the discrete topology on  $X$  and it is the largest possible topology on  $X$ .  $(X, \mathcal{T})$  is called a discrete topological space.
- (iii) For a set  $X$ , let  $\mathcal{T} = \{U \subseteq X : U = \emptyset \text{ or } X - U \text{ is finite}\}$ . Then  $\mathcal{T}$  is called the cofinite topology on  $X$ .

**Remark 1.** The set  $A$  is called countable if there is a one-to-one map  $f : A \longrightarrow \mathbb{N}$ .

**Definition 25.** [11] Suppose  $A$  is a subset of the topological space  $X$ . The closure of  $A$  in  $X$ , denoted by  $\overline{A}$  is the intersection of all closed subsets of  $X$  containing  $A$ . A

subset  $D$  of  $X$  is called dense if  $\overline{D} = X$ . The space  $(X, \mathcal{T})$  is called separable if there is a countable dense set  $D$  in  $X$ .

**Definition 26.** [11] Suppose  $(X, \mathcal{T})$  is a topological space and  $A \subseteq X$ . The subspace topology on  $A$  is defined as  $\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\}$  and  $(A, \mathcal{T}_A)$  is called a subspace of  $(X, \mathcal{T})$ .

**Definition 27.** [11] Suppose  $(X, \mathcal{T})$  and  $(Y, \mathcal{T}')$  are topological spaces. A function  $f : X \rightarrow Y$  is continuous if  $f^{-1}(U) \in \mathcal{T}$  for all  $U \in \mathcal{T}'$ .

**Definition 28.** [11] A function  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')$  is called a homeomorphism if  $f$  is a bijection and  $f$  and  $f^{-1}$  are both continuous. If there is a homeomorphism from  $X$  to  $Y$ , then  $X$  and  $Y$  are called homeomorphic, denoted  $X \cong Y$ .

**Definition 29.** [11] Suppose  $(X_i, \mathcal{T}_i)$  are topological spaces for  $i \in I$ . The open sets of the product topology on  $\prod_{i \in I} X_i$  are arbitrary unions of sets of the form  $\prod_{i \in I} U_i$  where each  $U_i$  is open in  $X_i$  and all but finitely many  $U_i = X_i$ . Projection function  $pr_j : \prod_{i \in I} X_i \rightarrow X_j$  are continuous when  $\prod_{i \in I} X_i$  has the product topology and the product topology is the smallest topology making all  $pr_j$  with  $j \in I$  continuous.

**Definition 30.** [11] If  $f : X \rightarrow Y$  with  $(Y, \mathcal{T}')$  is a function from  $X$  to the topological space  $Y$  then  $\mathcal{T} := \{f^{-1}(U) \mid U \in \mathcal{T}'\}$  is the weak topology on  $X$  induced by  $f$ . This is the smallest topology on  $X$  making  $f$  continuous.

**Definition 31.** [11] If  $\sim$  on equivalence relations on the topological space  $(X, \mathcal{T})$  then the quotient topology on  $X/\sim$  is  $\mathcal{T}_\sim := \{U \subset X/\sim \mid q^{-1}(U) \in \mathcal{T}\}$  where  $q : X \rightarrow X/\sim$  is the quotient map.

**Definition 32.** [11]  $(X, d)$  is a metric space then the topology on  $\mathbf{X}$  induced by the  $\mathbf{d}$  consists of arbitrary unions of sets of the form  $B(x; r) := \{y \in X \mid d(x, y) < r\}$  where  $x \in X$  and  $r > 0$  in  $\mathbb{R}$ .

### Examples.

- (i) The 2-disk  $D^2 := \{z \in \mathbb{C} \mid |z| \leq 1\}$  has the subspace topology where the topology on  $\mathbb{C}$  is induced by the metric  $d(x, y) := |x - y|$ . Similarly, the circle  $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$  has the subspace topology via the inclusion  $S^1 \hookrightarrow D^2$ .
- (ii) The 2-sphere  $S^2 := \{(z, t) \in \mathbb{C} \times \mathbb{R} \mid |z|^2 + t^2 = 1\}$  has the subspace topology with  $\mathbb{C} \times \mathbb{R}$  having the product of the standard topologies. The topology on

$S^2$  is also induced by a metric, for instance, the restriction of the usual distance function on  $\mathbb{C} \times \mathbb{R}$ , that is,  $d((z, t), (w, s)) := \sqrt{|z - w|^2 + (t - s)^2}$ , for all  $(z, t), (w, s) \in S^2$ .

- (iii) The real projective plane  $\mathbb{R}P^2$  is the set of all lines through the origin in  $\mathbb{R}^3$  with the quotient topology on  $(\mathbb{R}^3 \setminus \{(0, 0, 0)\})/(\mathbb{R} \setminus \{0\})$  where the multiplicative group  $\mathbb{R} \setminus \{0\}$  acts via scalar multiplication. Equivalently,  $\mathbb{R}P^2$  may be realized as  $S^2/(\{1, -1\})$  where the multiplicative group  $\{1, -1\}$  acts on the 2-sphere  $S^2$  via scalar multiplication (this is called the antipodal action). Real projective space  $\mathbb{R}P^n$  is defined similarly by replacing  $\mathbb{R}^3$  with  $\mathbb{R}^{n+1}$ . The topology on  $\mathbb{R}P^n$  is also induced by a metric where the distance between two lines is defined as  $\sin(\theta)$  where  $\theta$  is the angle between the lines.

## 2.4. Categories and Functors

**Definition 33.** [12] A (locally small) category  $\mathcal{C}$  consists of a class  $Ob(\mathcal{C})$  of objects, and a set of morphisms  $\mathcal{C}(X, Y)$  for each pair  $X, Y$  of objects such that:

- (i) For each  $f \in \mathcal{C}(X, Y)$  and  $g \in \mathcal{C}(Y, Z)$  there is a morphism  $g \circ f \in \mathcal{C}(X, Z)$  which is called the composition of  $f$  with  $g$ . Composition is associative when defined.
- (ii) For each object  $X$ , there is an identity morphism  $id_X \in \mathcal{C}(X, X)$  such that  $id_X \circ f = f$  for all  $f \in \mathcal{C}(Y, X)$  and  $g \circ id_X = g$  for all  $g \in \mathcal{C}(X, Y)$ .

We will use the notation  $X \xrightarrow{f} Y$  to mean  $f \in \mathcal{C}(X, Y)$ .

The opposite of a category  $\mathcal{C}$ , denoted  $\mathcal{C}^{op}$ , has the same objects as  $\mathcal{C}$  but the morphisms are reversed, that is,  $\mathcal{C}^{op}(X, Y) := \mathcal{C}(Y, X)$ . A subcategory of  $\mathcal{C}$  has some of the objects of  $\mathcal{C}$  and some of the morphisms of  $\mathcal{C}$  with the same identity morphisms and the same compositions. A subcategory  $\mathcal{B}$  of  $\mathcal{C}$  is full if  $\mathcal{B}(X, Y) = \mathcal{C}(X, Y)$  for all objects  $X, Y$  in  $\mathcal{B}$ . If  $Ob(\mathcal{C})$  is a set then  $\mathcal{C}$  is a small category.

**Example 13.**

- (i) The category **Set**: Objects are sets and morphisms are functions  $X \xrightarrow{f} Y$ . The category **Set** is the full subcategory whose objects are finite sets.
- (ii) The category **Grp**: Objects are groups and morphisms are group homomorphisms.

The subcategory of abelian groups is a full subcategory.

- (iii) The category  $\mathbf{Vec}_{\mathbb{F}}$ : Objects are vector spaces over the field  $\mathbb{F}$  and morphisms are linear transformations. The category  $\mathbf{vec}_{\mathbb{F}}$  is the full subcategory whose objects are finite dimensional vector spaces.
- (iv) The category  $\mathbf{Mod}_R$ : Objects are right  $R$ -modules where  $R$  is a ring and morphisms are  $R$ -module homomorphisms. The full subcategory  $\mathbf{mod}_R$  has finitely generated modules as its objects. The previous example is a special case with the ring  $R$  being a field  $\mathbb{F}$ .
- (v) The category  $\mathbf{Alg}_{\mathbb{F}}$ : Objects are  $\mathbb{F}$ -algebras and morphisms are algebra homomorphisms, that is,  $\mathbb{F}$ -linear ring homomorphisms. The subcategory of graded  $\mathbb{F}$ -algebras and graded algebra homomorphisms is not a full subcategory.
- (vi) The category  $\mathbf{C}^*\mathbf{Alg}$ : Objects are  $C^*$ -algebras and morphisms are  $C^*$ -algebra morphisms, that is, bounded linear ring homomorphisms.
- (vii) The category  $\mathbf{Top}$ : Objects are topological spaces and morphisms are continuous functions  $X \xrightarrow{f} Y$ .

**Definition 34.** [12] A **functor**  $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$  where  $\mathcal{C}, \mathcal{D}$  are categories, assigns

- (i) an object  $\mathcal{F}(X)$  of  $\mathcal{D}$  to each object  $X$  of  $\mathcal{C}$ ,
- (ii) a morphism  $\mathcal{F}(f) \in \mathcal{D}(\mathcal{F}(X), \mathcal{F}(Y))$  to each  $f \in \mathcal{C}(X, Y)$  such that
  - (a)  $\mathcal{F}(id_X) = id_{\mathcal{F}(X)}$  for each object  $X$  of  $\mathcal{C}$  and
  - (b)  $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$  when  $f \circ g$  is defined.

A cofunctor  $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$  is a functor from  $\mathcal{C}$  to  $\mathcal{D}^{op}$  or equivalently, a from  $\mathcal{C}^{op}$  to  $\mathcal{D}$ .

**Definition 35.** [12] Consider functors  $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$  and  $\mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$ . A natural transformation  $\theta : \mathcal{F} \rightarrow \mathcal{G}$  is a morphism such that

- (i)  $\theta$  give a morphism  $\theta_X : \mathcal{F} \rightarrow \mathcal{G}$  for each object  $X$  in  $\mathcal{C}$ .
- (ii) For each pair of objects  $X$  and  $Y$  and each morphism  $f \in \mathcal{C}(X, Y)$  the diagram

$$\begin{array}{ccc}
 \mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y) \\
 \downarrow \theta_X & & \downarrow \theta_Y \\
 \mathcal{G}(X) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(Y)
 \end{array}$$

commutes, that is,  $\theta_Y \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \theta_X$ .

We call  $\theta_X$  the components of the natural transformation  $\theta$  at the object  $X$ . A natural transformation  $\theta$  with every component  $\theta_X$  invertible in  $\mathcal{D}$  is called a natural equivalence, we say that the functors  $\mathcal{F}$  and  $\mathcal{G}$  are naturally isomorphic denoted by  $\mathcal{F} \cong \mathcal{G}$ . The inverses  $\theta_X^{-1}$  in  $\mathcal{D}$  are the components of  $\theta^{-1} : \mathcal{G} \longrightarrow \mathcal{F}$ , that is,  $(\theta^{-1})_X = \theta_X^{-1}$ .

**Example 14.** [15] A quiver representation is a functor  $\rho$  from the small category  $\Gamma$  with objects  $V$  and morphisms given by paths in  $\Gamma$  to the category  $\mathbf{Vec}_{\mathbb{F}}$ . The linear transformation assigned to the path  $p = e_1 e_2 \dots e_n$  is the composition  $s(p) = s(e_1) \xrightarrow{\rho(e_1)} t(e_1) = s(e_2) \xrightarrow{\rho(e_2)} \dots \xrightarrow{\rho(e_n)} t(e_n) = t(p)$  from  $\rho(s(p))$  to  $\rho(t(p))$ . A morphism of quiver representations is a natural transformation between two such functors. The objects of the category of quiver representations of  $\Gamma$  are quiver representations and its morphisms are natural transformations. This category is equivalent to the category  $\mathbf{Mod}_{\mathbb{F}\Gamma}$  [15].

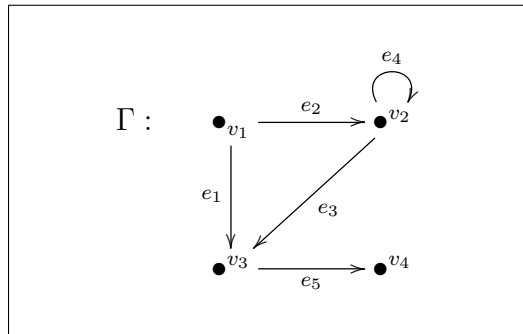
### 3. LEAVITT PATH ALGEBRAS

**Definition 36.** [7] A di(irected )graph  $\Gamma$  is a 4-tuple  $(V, E, s, t)$  where  $V$  is the set of vertices,  $E$  is the set of arrows (directed edges),  $s$  and  $t : E \rightarrow V$  are the source and the target functions.

If  $s(e) = v$  and  $t(e) = w$  then we say that  $v$  emits  $e$  and that  $w$  receives  $e$ . A vertex which does not receive any arrow is called a source. A vertex which emits no arrow is called a sink. For  $v \in V$ ,  $s^{-1}(v) := \{e \in E \mid s(e) = v\}$  and  $t^{-1}(v) := \{e \in E \mid t(e) = v\}$ . If  $0 < |s^{-1}(v)| < \infty$  then  $v$  is called a regular vertex.

A path  $p$  of length  $n > 0$  is a sequence of arrows  $e_1 \dots e_n$  such that  $t(e_i) = s(e_{i+1})$  for  $i = 1, \dots, n-1$ . The length of a path  $p$ , denoted by  $l(p)$ , is the number of arrows in  $p$ . The source of a path  $p$ , denoted by  $s(p)$ , is the source of its initial arrow, i.e.,  $s(e_1)$  while the target of a path  $p$ , denoted by  $t(p)$ , is the target of its last arrow, i.e.,  $t(e_n)$ . Every vertex  $v$  is a path of length 0 with  $s(v) = v = t(v)$ . The set of paths in  $\Gamma$  is denoted by  $Path(\Gamma)$ . For a path  $p = e_1 \dots e_n \in Path(\Gamma)$ , if  $q = e_1 \dots e_m$  for some  $m \leq n$  then  $q$  is an initial segment of  $p$ .

**Example 15.** Let us consider the digraph  $\Gamma$  where  $E = \{e_1, e_2, e_3, e_4, e_5\}$  and  $V = \{v_1, v_2, v_3, v_4\}$ .  $s(e_1) = s(e_2) = v_1$ ,  $s(e_3) = t(e_2) = s(e_4) = t(e_4) = v_2$ ,  $t(e_1) = t(e_3) = s(e_5) = v_3$ ,  $t(e_5) = v_4$ . The source of the path  $p = e_2 e_4 e_3 e_5$  is  $s(p) = s(e_2) = v_1$  and the target of  $p$  is  $t(p) = t(e_5) = v_4$ .

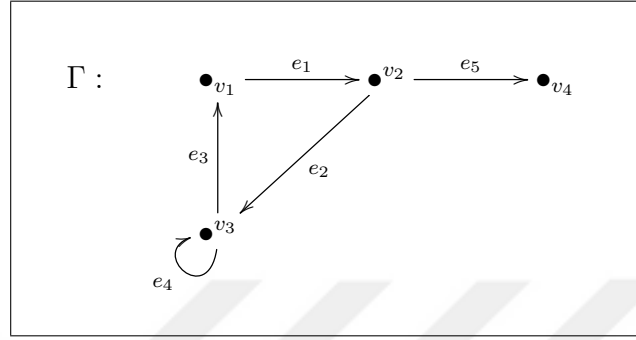


**Figure 3.1:** Digraph  $\Gamma$

A path  $p$  with  $l(p) > 0$  is called a cycle if  $s(p) = t(p)$  and  $s(e_i) \neq s(e_j)$  for every

$i \neq j$ . An arrow  $e$  with  $s(e) = t(e)$ , equivalently, a cycle of length 1, is called a loop. If there is no cycle in a digraph  $\Gamma$  then it is said to be acyclic. An arrow  $e$  is an exit of the path  $p = e_1 \dots e_n$  if there exists an  $i \in \{1, \dots, n\}$  such that  $s(e) = s(e_i)$  but  $e \neq e_i$ .

**Example 16.** Consider the following digraph:



**Figure 3.2:** Digraph  $\Gamma$

Let's choose a path  $C = e_1 e_2 e_3$ . The path  $C$  is a cycle with  $s(C) = v_1 = t(C)$  and  $l(C) = 3$ . The arrows  $e_4$  and  $e_5$  are the exits of  $C$ . The arrow  $e_4$  is a loop (= cycle with length 1).

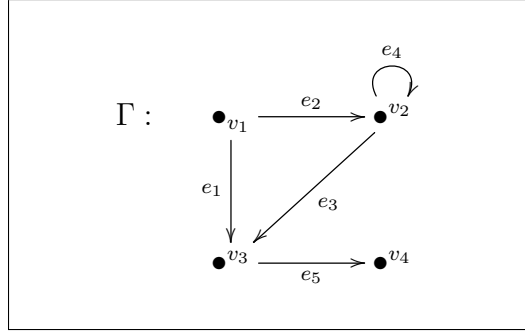
The digraph  $\Gamma$  is finite when  $E$  and  $V$  are finite. An infinite path has infinitely many arrows. In this thesis we will only work with finite digraphs.

**Definition 37.** [16] The preorder leadsto, denoted by  $\rightsquigarrow$ , on  $V$  is defined as  $v \rightsquigarrow w$  if there is a path  $p \in Path(\Gamma)$  from  $v$  to  $w$ . The set of successors of a vertex  $v$  is  $V_{v \rightsquigarrow} = \{w \in V \mid v \rightsquigarrow w\}$ . If  $X \subseteq V$  then we define  $V_{X \rightsquigarrow} = \bigcup_{v \in X} V_{v \rightsquigarrow}$ .

**Definition 38.** [7] Let  $\Gamma$  be a digraph and  $H \subseteq V$ .

- (i)  $H$  is hereditary if  $v \in H$  and  $v \rightsquigarrow w$  then  $w \in H$ .
- (ii)  $H$  is saturated if  $0 < |s^{-1}(v)| < \infty$  and  $\{t(e) \mid e \in s^{-1}(v)\} \subseteq H$  then  $v \in H$ .

**Example 17.** In the following digraph we see that  $H = \{v_2, v_3, v_4\}$  is a hereditary subset of  $V$  but is not saturated since  $\{t(e) \mid s(e) = v_1\} = \{v_2, v_3\} \subset H$  but  $v_1 \notin H$ . The subset  $S = \{v_1, v_2\}$  is saturated but not hereditary. The subset  $\{v_3, v_4\}$  is hereditary and saturated.



**Figure 3.3:** Digraph  $\Gamma$

**Definition 39.** [12] The category **Dgrf**: Objects are digraphs and morphisms are functions  $\varphi$  from  $V_\Gamma \sqcup E_\Gamma$  to  $V_\Lambda \sqcup E_\Lambda$  such that  $\varphi(V_\Gamma) \subseteq V_\Lambda$  and  $\varphi(E_\Gamma) \subseteq E_\Lambda$  from  $\Gamma = (V_\Gamma, E_\Gamma)$  to  $\Lambda = (V_\Lambda, E_\Lambda)$  satisfying  $\varphi(s(e)) = s(\varphi(e))$  and  $\varphi(t(e)) = t(\varphi(e))$  for all  $e \in E_\Gamma$ .

A digraph  $\Gamma$  can also be regarded as a category whose objects are the vertices in  $\Gamma$ , with morphisms being the paths in  $\Gamma$ . Identity morphisms correspond to the vertices, that is, paths of length 0. Composition is given by concatenation of paths. Unlike the previous examples of categories above  $\Gamma$  is a small category. Note that the order of composition (of paths) in  $\Gamma$  is written from left to right, the opposite of the order we compose functions.

**Definition 40.** [7] Let  $\mathbb{F}$  be a field and  $\Gamma$  be a digraph. The path algebra of  $\Gamma$  over  $\mathbb{F}$ , denoted by  $\mathbb{F}\Gamma$ , is defined as the  $\mathbb{F}$ -algebra generated by  $V \sqcup E$  with the relations

(V)  $vw = \delta_{v,w}v$  for all  $v, w \in V$

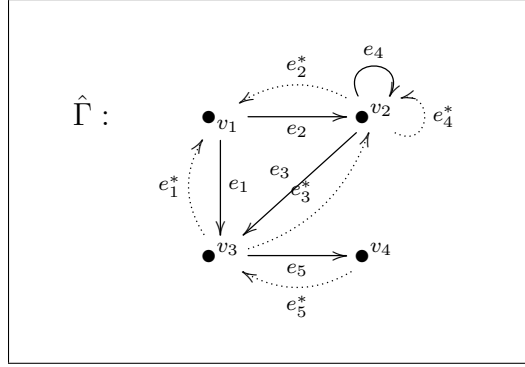
(E)  $s(e)e = e = et(e)$  for all  $e \in E$

where  $\delta_{i,j}$  is the Kronecker delta.

The paths in  $\Gamma$  form an  $\mathbb{F}$ -basis of the path algebra.

**Definition 41.** [7] The extended digraph  $\hat{\Gamma} = (V, E \sqcup E^*, s, t)$  of  $\Gamma$  is obtained by adding a dual arrow  $e^*$  for each  $e$  in  $E$  where  $E^* = \{e^* \mid e \in E\}$ ,  $s(e^*) = t(e)$  and  $t(e^*) = s(e)$  for all  $e \in E$ .

**Example 18.** The extended digraph  $\hat{\Gamma}$  of the digraph  $\Gamma$  in Example 15 is as follows:



**Figure 3.4:** Digraph  $\hat{\Gamma}$

**Definition 42.** [7] The Leavitt Path Algebra (LPA)  $L_{\mathbb{F}}(\Gamma)$  of a digraph  $\Gamma$  with coefficients in the field  $\mathbb{F}$  is the quotient of the path algebra  $\mathbb{F}\hat{\Gamma}$  satisfying the relations below, that is,  $L_{\mathbb{F}}(\Gamma) \cong \mathbb{F}\hat{\Gamma}/I$  where  $I$  is the ideal generated by the elements in  $\mathbb{F}\hat{\Gamma}$  corresponding to:

$$(CK1) \quad e^*f = \delta_{e,f} t(e) \text{ for all } e, f \in E$$

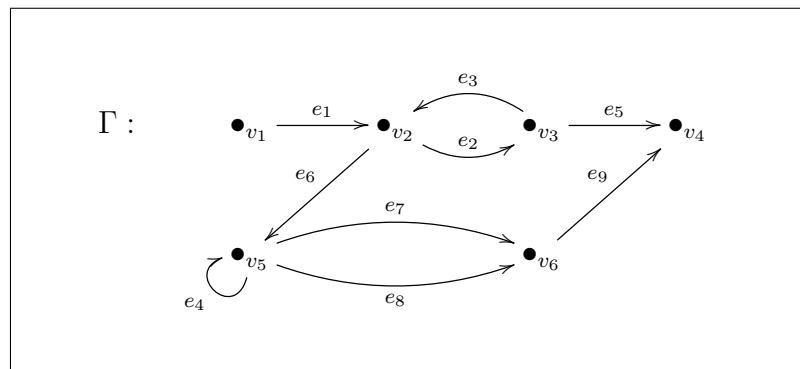
$$(CK2) \quad v = \sum_{s(e)=v} ee^* \text{ for every vertex } v \text{ with } 0 < |s^{-1}(v)| < \infty$$

When  $V$  is finite, the algebras  $\mathbb{F}\hat{\Gamma}$  and  $L_{\mathbb{F}}(\Gamma)$  are unital:  $1 = \sum_{v \in V} v$ . If  $V$  is infinite then  $\mathbb{F}\hat{\Gamma}$  and  $L_{\mathbb{F}}(\Gamma)$  are non-unital.

The conditions (CK1) and (CK2) are called the Cuntz-Krieger relations.

We get the Cohn path algebra  $CL_{\mathbb{F}}(\Gamma)$  when we impose the relations (CK1) in addition to (V) and (E). Hence  $L_{\mathbb{F}}(\Gamma)$  is a quotient of  $CL_{\mathbb{F}}(\Gamma)$ , which is a quotient of  $\mathbb{F}\hat{\Gamma}$ .

**Example 19.** Consider the following digraph:



**Figure 3.5:** Digraph  $\Gamma$

Here are some computations in  $L_{\mathbb{F}}(\Gamma)$ :

$v_6e_9 = e_9 = e_9v_4$  by (E) while  $v_4e_9^* = e_9^* = e_9^*v_6$  by (E).

$e_2^*e_2 = v_3$  while  $e_2^*e_6 = 0$  by (CK1).

$v_2 = e_2e_2^* + e_6e_6^*$  by (CK2).

$v_5 = e_4e_4^* + e_7e_7^* + e_8e_8^*$  and  $v_3 = e_5e_5^* + e_3e_3^*$  by (CK2).

We see that  $v_1e_1 = e_1$  while  $e_1v_1 = 0$ . Thus  $L_{\mathbb{F}}(\Gamma)$  is not commutative.

**Lemma 3.1.** [7] Let  $p$  and  $q$  be paths in  $\Gamma$ .

(i)  $p^*q = 0$  unless  $q$  is an initial segment of  $p$ , that is,  $p = qp'$  for some path  $p'$  or  $p$  is an initial segment of  $q$ , that is,  $q = pq'$  for some path  $q'$ .

(ii)  $L_{\mathbb{F}}(\Gamma)$  is spanned by  $\{pq^* \mid p, q \in Path(\Gamma), t(p) = t(q)\}$ .

$L_{\mathbb{F}}(\Gamma)$  has a  $\mathbb{Z}$ -grading given by  $|v| = 0$  for all  $v \in V$ ,  $|e| = 1$  and  $|e^*| = -1$  and for all  $e \in E$ . This defines a grading on  $L_{\mathbb{F}}(\Gamma)$  (since all the relations are homogeneous):

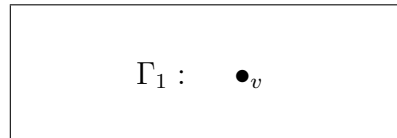
**Lemma 3.2.** [7]  $L_{\mathbb{F}}(\Gamma)$  is a  $\mathbb{Z}$ -graded algebra with  $|pq^*| := l(p) - l(q)$ . That is,

$$L_{\mathbb{F}}(\Gamma) = \bigoplus_{n \in \mathbb{Z}} L_{\mathbb{F}}(\Gamma)_n$$

where  $L_{\mathbb{F}}(\Gamma)_n = \left\{ \sum_{i=1}^m \lambda_i p_i q_i^* : l(p_i) - l(q_i) = n \right\}$  for each  $n \in \mathbb{Z}$ , with each  $\lambda_i \in \mathbb{F}$  and each  $p_i, q_i \in Path(\Gamma)$ .

We define an operator  $*$  on  $V \sqcup E \sqcup E^*$  by  $v^* = v$  for all  $v \in V$  and  $(f^*)^* = f$  for all  $f \in E$ . And also we extend the operator  $*$  on paths of  $\hat{\Gamma}$  by  $p^* = (f_1 f_2 \dots f_k)^* = f_k^* f_{k-1}^* \dots f_1^*$  and so to  $L_{\mathbb{F}}(\Gamma)$  linearly. The operator  $*$  induces a grade-reversing involutive anti-automorphism and  $|a^*| = -|a|$  for all homogeneous  $a$  in  $L_{\mathbb{F}}(\Gamma)$ .

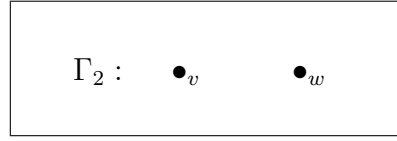
**Examples.** (i) Let  $\Gamma_1$  consist of a single vertex  $v$  and no arrow:



**Figure 3.6:** Digraph  $\Gamma_1$

We see that  $\mathbb{F}\Gamma_1 \cong \mathbb{F}$  and  $L_{\mathbb{F}}(\Gamma_1) \cong \mathbb{F}$  where  $v \leftrightarrow 1$ .

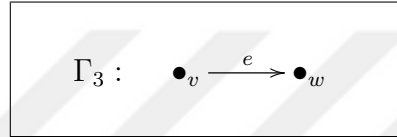
(ii) Adding an extra vertex  $w$  we have  $\Gamma_2$  as follows:



**Figure 3.7:** Digraph  $\Gamma_2$

Then  $\mathbb{F}\Gamma_2 \cong \mathbb{F}^2$  and  $L_{\mathbb{F}}(\Gamma_2) \cong \mathbb{F}^2$ .

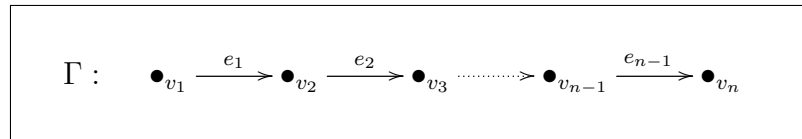
(iii) If we add an arrow from  $v$  to  $w$  then we have:



**Figure 3.8:** Digraph  $\Gamma_3$

The path algebra  $\mathbb{F}\Gamma_3$  is isomorphic to the algebra of upper triangular  $2 \times 2$ -matrices. The Leavitt path algebra  $L_{\mathbb{F}}(\Gamma_3)$  is generated by the elements  $v$ ,  $w$ ,  $e$  and  $e^*$ . We have  $L_{\mathbb{F}}(\Gamma_3) \cong M_2(\mathbb{F})$  where  $v \leftrightarrow E_{11}$ ,  $w \leftrightarrow E_{22}$ ,  $e \leftrightarrow E_{12}$  and  $e^* \leftrightarrow E_{21}$ .

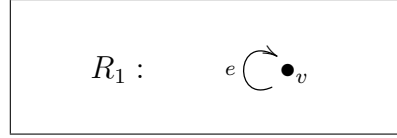
(iv) Now we generalize the above digraph  $\Gamma_3$  by a digraph with  $n$  vertices:



**Figure 3.9:** Digraph  $\Gamma$

The path algebra  $\mathbb{F}\Gamma$  is isomorphic to the algebra of upper triangular  $n \times n$ -matrices. We have  $L_{\mathbb{F}}(\Gamma) \cong M_n(\mathbb{F})$ , the matrix algebra with coefficients in  $\mathbb{F}$ , where  $v_i \leftrightarrow E_{ii}$ ,  $e_i \leftrightarrow E_{i(i+1)}$  and  $e_i^* \leftrightarrow E_{(i+1)i}$ .

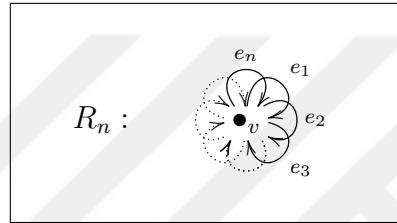
(v) Adding a loop to  $\Gamma_1$  in Figure 3.6 we get:



**Figure 3.10:** Digraph  $R_1$

Then  $\mathbb{F}R_1 \cong \mathbb{F}[x]$  where  $v \leftrightarrow 1$  and  $e \leftrightarrow x$ . Also  $L_{\mathbb{F}}(R_1) \cong \mathbb{F}[x, x^{-1}]$ , the algebra of Laurent polynomials with coefficient in  $\mathbb{F}$ , where  $v \leftrightarrow 1$ ,  $e \leftrightarrow x$  and  $e^* \leftrightarrow x^{-1}$ .

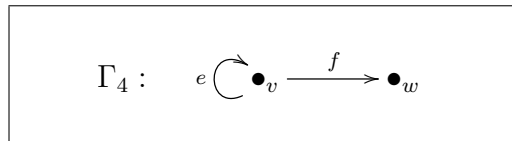
(vi) We generalize the example above to  $R_n$ , a rose with  $n$  petals.



**Figure 3.11:** A rose with  $n$  petals

$\mathbb{F}R_n \cong \mathbb{F}\langle x_1, x_2, \dots, x_n \rangle$  where  $x_i \leftrightarrow e_i$  and  $v \leftrightarrow 1$ . The Leavitt path algebra  $L_{\mathbb{F}}(R_n)$  is isomorphic to  $L(1, n)$ , the Leavitt algebra of type  $(1, n)$  in Example 7, where  $v \leftrightarrow 1$ ,  $e_i \leftrightarrow y_i$  and  $e_i^* \leftrightarrow x_i$ .

(vii) Let's consider the Toeplitz digraph:



**Figure 3.12:** Toeplitz digraph

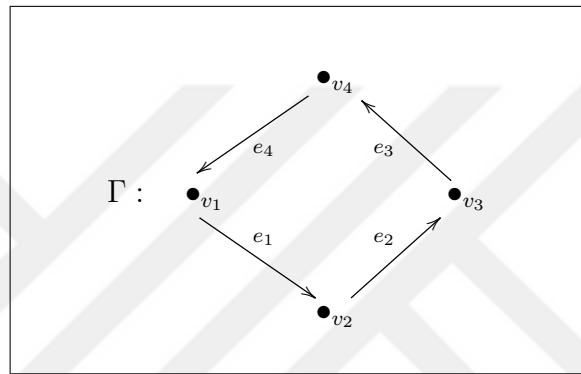
The Leavitt path algebra of the Toeplitz digraph is isomorphic to the Jacobson algebra  $\mathbb{F}\langle x, y \rangle / (1 - xy)$  where  $v + w \leftrightarrow 1$ ,  $e + f \leftrightarrow y$ ,  $e^* + f^* \leftrightarrow x$ ,  $v \leftrightarrow yx$ ,  $w \leftrightarrow 1 - yx$ ,  $e \leftrightarrow y^2x$  and  $f \leftrightarrow y(1 - yx)$ .

**Lemma 3.3.** [17] Let  $\Gamma$  be a finite digraph whose cycles have no exit then

$$L_{\mathbb{F}}(\Gamma) \cong \left( \bigoplus_{i=1}^k M_{m_i}(\mathbb{F}[x, x^{-1}]) \right) \oplus \left( \bigoplus_{j=1}^l M_{n_j}(\mathbb{F}) \right)$$

where  $k$  is the number of cycles (up to circular shifts) in  $\Gamma$ , say  $C_1, \dots, C_k$ , and  $m_i$  is the number of paths ending at  $s(C_i)$  which do not contain the cycle itself for  $i = 1, \dots, k$ ; similarly  $l$  is the number of sinks in  $\Gamma$ , say  $w_1, \dots, w_l$ , and  $n_j$  is the number of paths ending at the sink  $w_j$  for every  $j = 1, \dots, l$ .

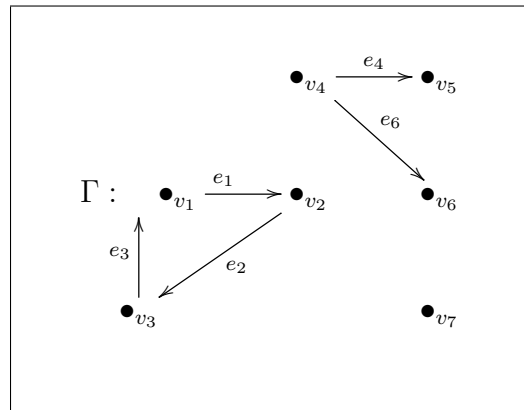
**Example 20.** Let's consider the digraph  $\Gamma$  :



**Figure 3.13:** Cycle of length 4

$\Gamma$  is a cycle,  $C = e_1e_2e_3e_4$ . The number of paths ending at  $s(C) = v_1$  is 4. Then the Leavitt path algebra of  $\Gamma$  is isomorphic to  $M_4(\mathbb{F}[x, x^{-1}])$ .

**Example 21.** Consider the following digraph  $\Gamma$  :



**Figure 3.14:** Cycle of length 3

$\Gamma$  has the cycle  $C = e_1 e_2 e_3$  and the number of paths ending at  $s(C)$  is 3. Also  $v_5, v_6$  and  $v_7$  are sinks in  $\Gamma$ . The numbers of paths ending at each sink are 2, 2 and 1, respectively. Hence  $L_{\mathbb{F}}(\Gamma)$  is isomorphic to  $M_3(\mathbb{F}[x, x^{-1}]) \oplus M_2(\mathbb{F}) \oplus M_2(\mathbb{F}) \oplus M_1(\mathbb{F})$ .

**Definition 43.** [15] A quiver representation  $\rho$  assigns a (possibly infinite dimensional) vector space  $\rho(v)$  to each vertex  $v$  and a linear transformation  $\rho(e) : \rho(s(e)) \rightarrow \rho(t(e))$  to each arrow  $e$ . A morphism of quiver representations  $\varphi : \rho \rightarrow \sigma$  is a family of linear transformations  $\{\varphi_v : \rho(v) \rightarrow \sigma(v)_{v \in V}\}$  such that for all  $e \in E$  the diagram

$$\begin{array}{ccc} \rho(s(e)) & \xrightarrow{\rho(e)} & \rho(t(e)) \\ \downarrow \varphi_{s(e)} & & \downarrow \varphi_{t(e)} \\ \sigma(s(e)) & \xrightarrow{\sigma(e)} & \sigma(t(e)) \end{array}$$

commutes.

**Theorem 3.4.** [16] If  $\Gamma = (V, E, s, t)$  is a finite digraph then  $\mathbf{Mod}_{L_{\mathbb{F}}(\Gamma)}$ , the category of unital right  $L_{\mathbb{F}}(\Gamma)$ -modules, is equivalent to the full subcategory of quiver representations  $\rho$  of  $\Gamma$  satisfying the following condition (I):

For every nonsink  $v \in V$ ,

$$\bigoplus_{s(e)=v} \rho(e) : \rho(v) \longrightarrow \bigoplus_{s(e)=v} \rho(t(e))$$

is an isomorphism. The right  $L_{\mathbb{F}}(\Gamma)$ -module corresponding to the quiver representation  $\rho$  is  $M = \bigoplus_{v \in V} \rho(v)$  as an  $\mathbb{F}$ -module and the actions of generators of  $L_{\mathbb{F}}(\Gamma)$  are given by the compositions

$$v : M = \bigoplus_{u \in V} \rho(u) \xrightarrow{pr_{\rho(v)}} \rho(v) \hookrightarrow \bigoplus_{u \in V} \rho(u) = M$$

$$e : M = \bigoplus_{v \in V} \rho(v) \xrightarrow{pr_{s(e)}} \rho(s(e)) \xrightarrow{\rho(e)} \rho(t(e)) \hookrightarrow \bigoplus_{v \in V} \rho(v) = M$$

$$e^* : M = \bigoplus_{v \in V} \rho(v) \xrightarrow{pr_{t(e)}} \rho(t(e)) \hookrightarrow \bigoplus_{s(f)=s(e)} \rho(t(f)) \xrightarrow{(\bigoplus \rho(f))^{-1}} \rho(s(e)) \hookrightarrow \bigoplus_{v \in V} \rho(v) = M$$

**Lemma 3.5.** [18] If  $\Gamma$  is a finite digraph then the standard homomorphism (sending  $V \sqcup E$  to themselves) from the path algebra  $\mathbb{F}\Gamma$  to the Leavitt path algebra  $L_{\mathbb{F}}(\Gamma)$  is injective. Hence we may think of  $\mathbb{F}\Gamma$  as a subalgebra of  $L_{\mathbb{F}}(\Gamma)$ .

**Lemma 3.6.** [19] If  $p$  and  $q$  are in  $Path(\Gamma)$  with  $t(p) = t(q)$  then  $pq^* \neq 0$  in  $L_{\mathbb{F}}(\Gamma)$ .

**Theorem 3.7.** [20] (The Graded Uniqueness Theorem) Let  $\Gamma$  be a finite digraph and  $\mathbb{F}$  any field. If  $A$  is a  $\mathbb{Z}$ -graded ring, and  $\varphi : L_{\mathbb{F}}(\Gamma) \longrightarrow A$  is a graded ring homomorphism with  $\varphi(v) \neq 0$  for every vertex  $v \in V$ , then  $\varphi$  is injective.

A basis for  $L_{\mathbb{F}}(\Gamma)$ : We know that  $\{pq^* \mid p, q \in Path(\Gamma), t(p) = t(q)\}$  spans  $L_{\mathbb{F}}(\Gamma)$  by Lemma 3.1 (ii). However, in general  $\{pq^* \mid p, q \in Path(\Gamma), t(p) = t(q)\}$  is not linearly independent. For instance, if the vertex  $v$  is not a sink then (CK2) shows that  $\{pq^* \mid p, q \in Path(\Gamma), t(p) = t(q)\}$  is linearly dependent. A basis which is a subset of  $\{pq^* \mid p, q \in Path(\Gamma), t(p) = t(q)\}$  is given in [21]: For each vertex  $v \in V$  that is not a sink we (arbitrarily) choose a distinguished arrow  $e_v$  with  $s(e_v) = v$ .

**Theorem 3.8.** [21] If  $D$  is the set of distinguished arrows in  $\Gamma$  then

$$\{pq^* \mid p, q \in Path(\Gamma), t(p) = t(q), pq^* \text{ does not contain } ee^* \text{ with } e \in D\}$$

is a basis of  $L_{\mathbb{F}}(\Gamma)$ .

**Theorem 3.9.** [7]  $L_{\mathbb{C}}(\Gamma)$  is isomorphic to a dense subalgebra of  $C^*(\Gamma)$ .

## 4. FUNCTORIAL CONSTRUCTIONS

### 4.1. The Gelfand-Naimark Theorem

**Theorem 4.1.** [12] (Gelfand-Naimark, categorical version) Let  $\mathbf{C}(X)$  be the  $C^*$ -algebra of all continuous functions from the compact Hausdorff topological space  $X$  to  $\mathbb{C}$ , the complex plane as a topological space (where  $\|f\|_\infty = \sup_{x \in X} |f(x)|$  and  $f^* = \bar{f}$ ). We can extend  $\mathbf{C}$  to a cofunctor from the full subcategory of compact Hausdorff topological spaces of  $\mathbf{Top}$  to the category of  $C^*$ -algebras by  $\mathbf{C}(\varphi)(f) := f \circ \varphi$  for all  $f \in \mathbf{C}(Y)$  where  $\varphi : X \rightarrow Y$  is a continuous function. This defines a cofunctor  $\mathbf{C}$  giving an equivalence with the opposite category of the full subcategory of unital commutative  $C^*$ -algebras in  $\mathbf{C}^* \mathbf{Alg}$ .

Given a unital commutative  $C^*$ -algebra  $\mathcal{A}$  we recover the compact Hausdorff topological space (up to homeomorphism) as  $\mathbf{X}(\mathcal{A})$ : the set of unital  $C^*$ -algebra homomorphisms from  $\mathcal{A}$  to  $\mathbb{C}$  with the weak topology induced by  $\mathbf{X}(\mathcal{A}) \rightarrow \prod_{\|a\|=1} \mathbb{C}$  where  $\alpha \mapsto (\alpha(a)) \in \prod_{\|a\|=1} \mathbb{C}$  and the target has the product topology. The  $C^*$ -algebra homomorphism corresponding to  $x \in X$  from  $\mathbf{C}(X)$  to  $\mathbb{C}$  is the evaluation map  $ev_x$  defined as  $ev_x(f) = f(x)$ .

**Lemma 4.2.** If  $x \in X$  then  $ev_x : \mathbf{C}(X) \rightarrow \mathbb{C}$  is a  $C^*$ -algebra homomorphism.

*Proof.* Suppose  $f$  and  $g$  are in  $\mathbf{C}(X)$ . Then

$$ev_x(f + g) = (f + g)(x) = f(x) + g(x) = ev_x(f) + ev_x(g) \quad (4.1)$$

$$ev_x(fg) = (fg)(x) = f(x)g(x) = ev_x(f)ev_x(g). \quad (4.2)$$

Also for all  $\lambda \in \mathbb{C}$  the scalar product  $\lambda f$  is also the product of the constant function  $\lambda$  with  $f$ . hence  $ev_x$  is linear. Thus  $ev_x$  is an algebra homomorphism. Now we need to show  $ev_x(f^*) = ev_x(f)^*$ .

$$ev_x(f^*) = f^*(x) = \overline{f(x)} = \overline{ev_x(f)} = (ev_x(f))^*. \quad (4.3)$$

Finally,

$$\|ev_x(f)\| = |ev_x(f)| = |f(x)| \leq \sup_{y \in X} |f(y)| = \|f\| \quad (4.4)$$

Hence,  $\|ev_x\| \leq 1$ , that is  $ev_x$  is bounded.  $\square$

We can extend  $\mathbf{X}$  to a cofunctor by defining  $\mathbf{X}(\Phi)(\alpha) := \alpha \circ \Phi$  for all  $\alpha \in \mathbf{X}(\mathcal{B})$  where  $\Phi : \mathcal{A} \rightarrow \mathcal{B}$  is a  $C^*$ -algebra homomorphism between unital commutative  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ . The compositions  $\mathbf{C} \circ \mathbf{X}$  and  $\mathbf{X} \circ \mathbf{C}$  are naturally isomorphic to the identity functors on the subcategory of unital commutative  $C^*$ -algebras and compact Hausdorff topological spaces respectively.

Under the Gelfand-Naimark equivalence the inclusion of a closed subset  $Y$  into a compact Hausdorff topological space  $X$  corresponds to the epimorphism  $\mathbf{C}(X) \rightarrow \mathbf{C}(Y)$  given by restriction, that is,  $f \mapsto f|_Y$  for all  $f \in \mathbf{C}(X)$ . Here  $Y$  is compact under the subspace topology since  $Y$  is closed in  $X$  [11] and any subspace of a Hausdorff topological space is also Hausdorff. This restriction map is onto by the Tietze Extension Theorem [11]. That is, the cofunctor  $\mathbf{C}$  sends inclusion of a closed subspace  $Y$  to the restriction of functions to  $Y$ :

$$Y \hookrightarrow X \quad \xrightarrow{\mathbf{C}} \quad \mathbf{C}(X) \xrightarrow{|_Y} \mathbf{C}(Y)$$

### Examples.

- (i)  $S^1 \hookrightarrow D^2$  corresponds to the restriction  $\mathbf{C}(D^2) \xrightarrow{|_{S^1}} \mathbf{C}(S^1)$ .
- (ii)  $D^2 \xrightarrow{f_+} S^2$  where  $f_+(z) := (z, \sqrt{1 - |z|^2})$  is essentially the inclusion of the upper hemisphere into the 2-sphere. The corresponding  $C^*$ -algebra homomorphism  $\mathbf{C}(S^2) \rightarrow \mathbf{C}(D^2)$  sends  $g \in \mathbf{C}(S^2)$  to  $g \circ f_+$  which maps  $z \in D^2$  to  $g(z, \sqrt{1 - |z|^2})$ .
- (iii)  $D^2 \xrightarrow{f_-} S^2$  where  $f_-(z) := (z, -\sqrt{1 - |z|^2})$  is essentially the inclusion of the lower hemisphere into the 2-sphere. The corresponding  $C^*$ -algebra homomorphism  $\mathbf{C}(S^2) \rightarrow \mathbf{C}(D^2)$  sends  $g \in \mathbf{C}(S^2)$  to  $g \circ f_-$  which maps  $z \in D^2$  to  $g(z, -\sqrt{1 - |z|^2})$ .

If  $G$  is finite group acting (on the left) continuously on the compact Hausdorff topological space  $X$  then there is also a right action of  $G$  on  $\mathbf{C}(X)$  provided by the cofunctor  $\mathbf{C}$ :

For all  $f \in \mathbf{C}(X)$  and for all  $g \in G$  the function  $fg := \mathbf{C}(g)(f) \in \mathbf{C}(X)$  is defined as  $(fg)(x) := f(gx)$  for all  $x \in X$ . Under the Gelfand-Naimark correspondence the quotient topological space  $X/G$ , which is also compact and Hausdorff in the quotient topology because  $G$  is finite, corresponds to the subalgebra of functions invariant under  $G$ :

$$\mathbf{C}(X/G) \cong \mathbf{C}(X)^G := \{f \in \mathbf{C}(X) \mid f(gx) = f(x) \text{ for all } x \in X \text{ and for all } g \in G\}$$

That is, the Gelfand-Naimark cofunctor  $\mathbf{C}$  sends the quotient map from  $X$  to the orbit space  $X/G$  to the inclusion of  $G$ -invariant (complex valued) continuous functions into all continuous functions defined on  $X$ :

$$X \longrightarrow X/G \quad \xrightarrow{\mathbf{C}} \quad \mathbf{C}(X/G) \cong \mathbf{C}(X)^G \hookrightarrow \mathbf{C}(X)$$

**Example 22.** If  $G = \{1, -1\}$  is acting antipodally on  $S^2$  then the quotient  $S^2/G$  is homeomorphic to  $\mathbb{R}P^2$ , the real projective plane and

$$\mathbf{C}(\mathbb{R}P^2) \cong \{f \in \mathbf{C}(S^2) \mid f \text{ is even, that is } f(-x) = f(x) \text{ for all } x \in S^2\}.$$

In the category **Top** we have the quotient map in (4.5) on the left which corresponds to the inclusion of the subalgebra of even functions into all continuous functions in the category of  $C^*$ -algebras on the right, via the Gelfand-Naimark Theorem.

$$S^2 \longrightarrow \mathbb{R}P^2 \quad \xrightarrow{\mathbf{C}} \quad \mathbf{C}(S^2) \longleftarrow \mathbf{C}(\mathbb{R}P^2) \quad (4.5)$$

**Example 23.** If  $G = \{1, -1\}$  is acting on  $S^2$  with the non-identity element sending  $(z, t)$  to  $(z, -t)$  for all  $(z, t) \in S^2$  then  $S^2/G \cong D^2$  and

$$\mathbf{C}(D^2) \cong \{f \in \mathbf{C}(S^2) \mid f(z, -t) = f(z, t) \text{ for all } (z, t) \in S^2\}.$$

Note that  $f(z, -t) = f(z, t)$  for all  $(z, t) \in S^2$  implies that  $f$  depends only on  $z$  since there are at most two points, namely  $(z, t)$  and  $(z, -t)$  for the same  $z$ .

If  $A$  is a closed subset of compact Hausdorff topological space  $X$  then  $X/A$ , the quotient topological space obtained by collapsing the subset  $A$  to a single point is also compact

Hausdorff. The quotient map  $q : X \rightarrow X/A$  corresponds to the monomorphism  $\mathbf{C}(q) : \mathbf{C}(X/A) \rightarrow \mathbf{C}(X)$  with  $\mathbf{C}(X/A)$  isomorphic to the preimage of the subalgebra of constant functions on  $A$  under  $\mathbf{C}(X) \xrightarrow{\mathbf{C}(\iota)=|^A} \mathbf{C}(A)$  where  $\iota : A \hookrightarrow X$ . We can also realize  $\mathbf{C}(X/A)$  as the pullback of the diagram  $\mathbf{C}(\{\text{pt}\}) \cong \mathbb{C} \hookrightarrow \mathbf{C}(A) \xleftarrow{|^A} \mathbf{C}(X)$  corresponding to the pushout diagram  $\{\text{pt}\} \longleftarrow A \hookrightarrow X$  in the topological category where  $\{\text{pt}\}$  is the topological space consisting of a single point. These functorial constructions will be explained in the next section.

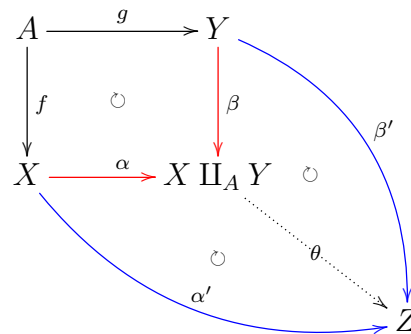
**Example 24.** Collapsing  $A = \{0, 1\}$  in  $X = [0, 1]$  we obtain  $[0, 1]/\{0, 1\} \cong S^1$ . In the  $C^*$ -algebra category  $\mathbf{C}(S^1) \cong \{f \in \mathbf{C}([0, 1]) \mid f(0) = f(1)\}$ .

**Example 25.** Collapsing  $A = S^1$  in  $X = D^2$  we obtain  $D^2/S^1 \cong S^2$ . In the  $C^*$ -algebra category  $\mathbf{C}(S^2) \cong \{f \in \mathbf{C}(D^2) \mid f|_{S^1} \text{ is constant}\}$ .

**Example 26.** We can also obtain the sphere by gluing two disks (the northern and southern hemispheres) along their boundaries (the equator). Here  $X$  is the disjoint union  $D^2 \sqcup D^2$  and  $\sim$  identifies each point on the boundary circle  $S^1$  of one disk with the same point on the other disk:  $(D^2 \sqcup D^2)/\sim \cong S^2$ . In the  $C^*$ -algebra category we get  $\mathbf{C}(S^2)$  as the pullback of the diagram  $\mathbf{C}(D^2) \xrightarrow{|_{S^1}} \mathbf{C}(S^1) \xleftarrow{|_{S^1}} \mathbf{C}(D^2)$  corresponding to the pushout of  $D^2 \hookrightarrow S^1 \hookrightarrow D^2$  in the topological category, to be explained below.

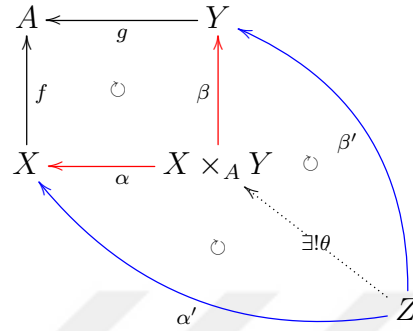
## 4.2. Pushouts and Pullbacks

**Definition 44.** [12] Given two morphisms  $f \in \mathcal{C}(A, X)$  and  $g \in \mathcal{C}(A, Y)$ , that is a diagram  $X \xleftarrow{f} A \xrightarrow{g} Y$  in a category  $\mathcal{C}$ , a pushout is a triple  $(X \amalg_A Y, \alpha, \beta)$  with  $\alpha \circ f = \beta \circ g$ , satisfying the universal property: for every triple  $(Z, \alpha', \beta')$  where  $\alpha' : X \rightarrow Z$  and  $\beta' : Y \rightarrow Z$  with  $\alpha' \circ f = \beta' \circ g$ , there exists a unique morphism  $\theta : X \amalg_A Y \rightarrow Z$  making the diagram



commutative. When a pushout exists, it is unique up to (unique) isomorphism.

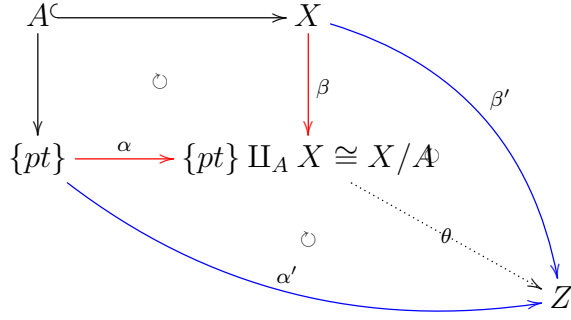
A pullback of the diagram  $X \xrightarrow{f} A \xleftarrow{g} Y$  is the dual notion of a pushout obtained by reversing the directions of all the morphisms. When a pullback exists, it is unique up to (unique) isomorphism:



In the category **Set** the pushout of  $f : A \longrightarrow X$  and  $g : A \longrightarrow Y$  always exists.  $X \amalg_A Y$  is the quotient of the disjoint union  $X \amalg Y$  where the elements  $f(a)$  and  $g(a)$  identified for each  $a \in A$ . The morphisms  $\alpha : X \longrightarrow X \amalg_A Y$  and  $\beta : Y \longrightarrow X \amalg_A Y$  are the composition of the inclusions of  $X$  and  $Y$  into  $X \amalg Y$  followed by the projection onto  $X \amalg_A Y$ . Given  $\alpha' : X \longrightarrow Z$  and  $\beta' : Y \longrightarrow Z$  with  $\alpha' \circ f = \beta' \circ g$ , the function  $\theta : X \amalg_A Y \longrightarrow Z$  is defined as  $\theta([x]) := \alpha'(x)$  for all  $x \in X$  and  $\theta([y]) := \beta'(y)$  for all  $y \in Y$ . The function  $\theta$  is well-defined because  $\alpha' \circ f = \beta' \circ g$ .

In the category **Top** also the pushout of  $f : A \longrightarrow X$  and  $g : A \longrightarrow Y$  always exists. As a set  $X \amalg_A Y$  is defined just like in **Set** with the quotient topology coming from the quotient map  $X \amalg Y \longrightarrow X \amalg_A Y$ . (Open sets in  $X \amalg Y$  are unions of an open set in  $X$  and an open set in  $Y$ , a subset of  $X \amalg_A Y$  is open if and only if its preimage in  $X \amalg Y$  is open.) The continuous functions  $\alpha, \beta$  and  $\theta$  are also defined exactly as in **Set**.

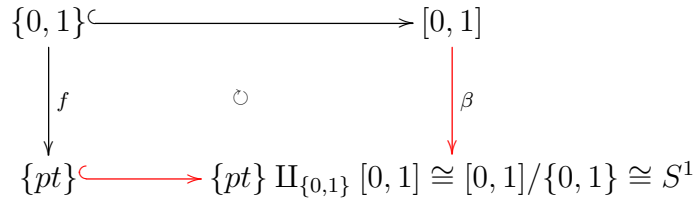
Collapsing a subspace  $A$  of the topological space  $X$  to obtain the quotient topological space  $X/A$  is a special case of the pushout construction coming from the diagram  $\{pt\} \longleftarrow A \hookrightarrow X$  where  $\{pt\}$  is the topological space consisting of single point:



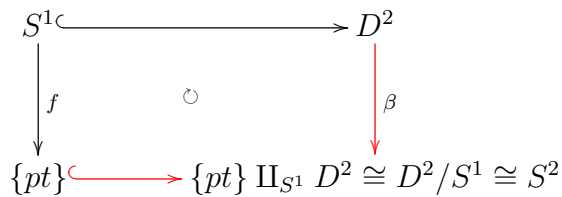
Here  $\beta$  is essentially the quotient map  $X \rightarrow X/A$ . For the diagram to be commutative the composition  $A \rightarrow \{pt\} \xrightarrow{\alpha'} Z$  must be the same as the composition  $A \rightarrow X \xrightarrow{\beta'} Z$ , that is,  $\beta'(a) = \alpha'(pt)$  for all  $a \in A \subseteq X$ . Hence  $\theta$  must send the element in  $X/A$  corresponding to the subset  $A$  to  $\alpha'(pt)$ . For all  $x \in X \setminus A$ , we have  $\theta(x) = \beta'(x)$ .

A comprehensive reference for pullback and pushout constructions in the category of  $C^*$ -algebras is [22].

**Example 27.** From the previous section we have: Collapsing  $A = \{0, 1\}$  in  $X = [0, 1]$  we obtain  $[0, 1]/\{0, 1\} \cong S^1$ . In the  $C^*$ -algebra category  $\mathbf{C}(S^1) \cong \{f \in \mathbf{C}([0, 1]) \mid f(0) = f(1)\}$ . This corresponds to the pushout diagram of



**Example 28.** From the previous section we have: Collapsing  $A = S^1$  in  $X = D^2$  we obtain  $D^2/S^1 \cong S^2$ . In the  $C^*$ -algebra category  $\mathbf{C}(S^2) \cong \{f \in \mathbf{C}(D^2) \mid f|_{S^1} \text{ is constant}\}$ . This corresponds to the pushout diagram of



**Example 29.** From the previous section: We obtain the sphere by gluing two disks (the northern and southern hemispheres) along their boundaries (the equator). This construction is realized by the pushout of  $D^2 \leftarrow S^1 \hookrightarrow D^2$  as in the diagram

$$\begin{array}{ccc}
 S^1 \hookrightarrow & \longrightarrow & D_+^2 \\
 \downarrow & \circlearrowleft & \downarrow \\
 D_-^2 \hookrightarrow & \longrightarrow & S^2 \cong (D_-^2 \sqcup D_+^2)/\sim
 \end{array}$$

The pushout of a diagram  $D^n \leftarrow S^{n-1} \hookrightarrow X$  where  $X$  is a compact Hausdorff topological space is very important in algebraic topology because all compact regular CW-complexes are constructed by repeated applications of this procedure, starting with  $X$  being a finite discrete set of points [23]. Below we will give discrete/algebraic non-commutative analogs of this construction where the compact regular CW-complex will correspond to a finite digraph, which will be an algebraic noncommutative version of a quantum topological space. The  $C^*$ -algebra of complex valued continuous functions on the compact regular CW-complex will correspond to a graph  $C^*$ -algebra and the Leavitt path algebra (with complex coefficients) of the same graph will replace the algebra of polynomial functions (analogous to the Stone-Weierstrass theorem).

In the category **Dgrf** the pushout of  $f : \Delta \longrightarrow \Gamma$  and  $g : \Delta \longrightarrow \Lambda$  always exists. The vertex set of the pushout digraph  $\Gamma \amalg_{\Delta} \Lambda$  is the pushout  $V_{\Gamma} \amalg_{V_{\Delta}} V_{\Lambda}$  of the vertex sets and similarly the arrow set of  $\Gamma \amalg_{\Delta} \Lambda$  is the pushout  $E_{\Gamma} \amalg_{E_{\Delta}} E_{\Lambda}$  of the arrow sets. The source function for the pushout digraph is the unique morphism  $\theta_s$  from  $E_{\Gamma} \amalg_{E_{\Delta}} E_{\Lambda}$  to  $V_{\Gamma} \amalg_{V_{\Delta}} V_{\Lambda}$  coming from the pushout diagram:

$$\begin{array}{ccccc}
 E_{\Delta} & \xrightarrow{g} & E_{\Lambda} & & \\
 \downarrow f & \circlearrowleft & \downarrow \beta_E & \searrow s_{\Lambda} & \\
 E_{\Gamma} & \xrightarrow{\alpha_E} & E_{\Gamma} \amalg_{E_{\Delta}} E_{\Lambda} & \circlearrowleft & V_{\Lambda} \\
 \downarrow s_{\Gamma} & & \downarrow \theta_s & & \downarrow \beta_V \\
 V_{\Gamma} & \xrightarrow{\alpha_V} & V_{\Gamma} \amalg_{V_{\Delta}} V_{\Lambda} & & 
 \end{array}$$

Similarly the target function for the pushout digraph is the unique morphism  $\theta_t$  from  $E_\Gamma \amalg_{E_\Delta} E_\Lambda$  to  $V_\Gamma \amalg_{V_\Delta} V_\Lambda$  coming from the pushout diagram:

$$\begin{array}{ccccc}
E_\Delta & \xrightarrow{g} & E_\Lambda & & \\
\downarrow f & \circlearrowleft & \downarrow \beta_E & \searrow t_\Lambda & \\
E_\Gamma & \xrightarrow{\alpha_E} & E_\Gamma \amalg_{E_\Delta} E_\Lambda & \circlearrowleft & V_\Lambda \\
& \searrow t_\Gamma & \circlearrowleft & \circlearrowleft & \downarrow \beta_V \\
& & V_\Gamma & \xrightarrow{\alpha_V} & V_\Gamma \amalg_{V_\Delta} V_\Lambda \\
& & & \circlearrowleft & 
\end{array}$$

$\theta_t$  (dotted arrow from  $E_\Gamma \amalg_{E_\Delta} E_\Lambda$  to  $V_\Gamma \amalg_{V_\Delta} V_\Lambda$ )

Hence  $s([e]) := [s_\Gamma(e)]$ ,  $t([e]) := [t_\Gamma(e)]$  for  $e \in E_\Gamma$  and  $s([e]) := [s_\Lambda(e)]$ ,  $t([e]) := [t_\Lambda(e)]$  for  $e \in E_\Lambda$ . The digraph morphisms  $\alpha$  and  $\beta$  are given by  $\alpha(x) := [x]$  for all  $x \in V_\Gamma \sqcup E_\Gamma$  and  $\beta(y) := [y]$  for all  $y \in V_\Lambda \sqcup E_\Lambda$ . Also  $\theta_s([x]) = \alpha_V s_\Gamma(x) = [s_\Gamma(x)]$  and  $\theta_t([x]) = \alpha_V t_\Gamma(x) = [t_\Gamma(x)]$  for  $x \in E_\Gamma$ . Similarly  $\theta_s([y]) = \beta_V s_\Lambda(y) = [t_\Lambda(y)]$  and  $\theta_t([y]) = \beta_V t_\Lambda(y) = [t_\Lambda(y)]$  for  $y \in E_\Lambda$ .

We will be considering pushout diagrams in **Dgrf** with  $f$  and  $g$  being one-to-one, identifying  $\Delta$  with a subgraph of both  $\Gamma$  and  $\Lambda$ . Then we can think of the pushout  $\Gamma \amalg_\Delta \Lambda$  as  $\Gamma \cup \Lambda$  where  $E_{\Gamma \amalg_\Delta \Lambda} = E_\Gamma \amalg_{E_\Delta} E_\Lambda = E_\Gamma \cup E_\Lambda$  and  $V_{\Gamma \amalg_\Delta \Lambda} = V_\Gamma \amalg_{V_\Delta} V_\Lambda = V_\Gamma \cup V_\Lambda$ . Also, the morphisms  $\alpha$  and  $\beta$  in the pushout diagram become the inclusions of the digraphs  $\Gamma$  and  $\Lambda$  into  $\Gamma \cup \Lambda$ .

In the categories **Vec $_{\mathbb{F}}$** , **Mod $_{\mathbb{R}}$** , **Alg $_{\mathbb{F}}$**  and **C\*-Alg**, pullbacks exist. Given  $f : X \longrightarrow A$  and  $g : Y \longrightarrow A$  the pullback object is  $X \times_A Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}$  and  $\alpha, \beta$  are restrictions to  $X \times_A Y$  of the projections from  $X \times Y$  to  $X$  and  $Y$  respectively. For  $\alpha' : Z \longrightarrow X$  and  $\beta' : Z \longrightarrow Y$  with  $f \circ \alpha' = g \circ \beta'$  the unique morphism  $\theta$  is defined as  $\theta(z) := (\alpha'(z), \beta'(z))$  for all  $z \in Z$ . In fact all this remains correct in the categories **Set** and **Top** also, but we will not need that in this thesis.

Pullbacks also exists in the category of  $\mathbb{Z}$ -graded algebras and the category of  $*$ -algebras; There is a natural  $\mathbb{Z}$ -grading on the product  $A \times B$  of two  $\mathbb{Z}$ -graded algebras  $A$  and  $B$  given as  $(A \times B)_n := A_n \times B_n$ . The inclusion and the projection maps between  $A$  or  $B$  and  $A \times B$  are graded homomorphisms, hence the pullback object is a graded subalgebra of  $A \times B$  and the morphisms  $\alpha, \beta$  of the pullback diagram are graded homomorphisms. Similarly, when  $A$  and  $B$  are  $*$ -algebras so is  $A \times B$  where  $(a, b)^* := (a^*, b^*)$ . The

inclusion and the projection maps are  $*$ -algebra homomorphisms, the pullback object is a  $*$ -subalgebra of  $A \times B$ , the homomorphisms  $\alpha$  and  $\beta$  are  $*$ -homomorphisms.

The Gelfand-Naimark cofunctor  $\mathbf{C}$  from the category compact Hausdorff topological spaces (and continuous functions) to the category of  $C^*$ -algebras (and bounded  $*$ -algebra homomorphisms) sends pushout diagrams to pullback diagrams since  $\mathbf{C}$  gives an equivalence. For instance applying  $\mathbf{C}$  to the pullback diagram in Example 29 we get:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 S^1 & \hookrightarrow & D_+^2 \\
 \downarrow & \circlearrowleft & \downarrow \\
 D_-^2 & \hookrightarrow & S^2
 \end{array} & \xrightarrow{\mathbf{C}} & \begin{array}{ccc}
 \mathbf{C}(S^1) & \longleftarrow & \mathbf{C}(D_+^2) \\
 \uparrow & \circlearrowleft & \uparrow \\
 \mathbf{C}(D_-^2) & \longleftarrow & \mathbf{C}(S^2)
 \end{array}
 \end{array}$$

Hence in the  $C^*$ -algebra category we recover  $\mathbf{C}(S^2)$  as the pullback of the diagram  $\mathbf{C}(D^2) \xrightarrow{|_{S^1}} \mathbf{C}(S^1) \xleftarrow{|_{S^1}} \mathbf{C}(D^2)$  corresponding to the pushout of  $D^2 \hookrightarrow S^1 \hookrightarrow D^2$  in the topological category.

**Example 30.** In the category **Top** of topological spaces we have the pushout diagram on the left which corresponds to the pullback diagram in the category of  $C^*$ -algebras on the right via the Gelfand-Naimark Theorem. On the left all morphisms are embeddings, on the right all morphisms are given by restrictions:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 S^1 & \hookrightarrow & D_+^2 \\
 \downarrow & \circlearrowleft & \downarrow \\
 D_-^2 & \hookrightarrow & S^2
 \end{array} & \xrightarrow{\mathbf{C}} & \begin{array}{ccc}
 \mathbf{C}(S^1) & \longleftarrow & \mathbf{C}(D_+^2) \\
 \uparrow & \circlearrowleft & \uparrow \\
 \mathbf{C}(D_-^2) & \longleftarrow & \mathbf{C}(S^2)
 \end{array}
 \end{array}$$

### 4.3. The Cofunctor $L_{\mathbb{F}}$

The  $C^*$ -algebras  $C(X)$  for  $X = S^1, D^2, S^2$  are not finitely generated as algebras. As  $\mathbb{C}$ -vector spaces they have no explicit (Hamel) bases (they do have Schauder bases). However, polynomial functions are finitely generated and dense in the space of continuous functions by the Stone-Weierstrass Theorem. Also polynomial functions have convenient Gröbner bases. We give noncommutative analogs of these dense, finitely

generated subalgebras with explicit well-behaved bases [19] for quantum discs, spheres, etc. This is achieved via a cofunctor from a subcategory of digraphs to the category of graded  $*$ -algebras, which assigns  $L_{\mathbb{F}}(\Gamma)$  to  $\Gamma$  at the level of objects. This cofunctor sends quotients to "inclusions" and pushouts to pullbacks. In particular when the coefficient field is  $\mathbb{C}$ , we can recover quantum discs, spheres and projective spaces [3] as graph  $C^*$ -algebras [R05] by taking completions. Moreover, the cofunctor  $L_{\mathbb{F}}$  is defined for all fields  $\mathbb{F}$ , not just  $\mathbb{C}$  or  $\mathbb{R}$ .

Let  $\varphi$  be a digraph morphism. If  $\varphi(f) = e$  for  $f \in E_{\Gamma}$  then  $\varphi(t(f)) = t(\varphi(f)) = t(e)$ . Hence  $t$  maps  $\varphi^{-1}(e)$  to  $\varphi^{-1}(t(e))$  for all  $e \in E_{\Lambda}$ .

**Definition 45.** [24] A digraph morphism  $\varphi : \Gamma \longrightarrow \Lambda$  is admissible if

- (i)  $t|_{\varphi^{-1}(e)}$  is a bijection from  $\varphi^{-1}(e)$  to  $\varphi^{-1}(t(e))$  for all  $e \in E_{\Lambda}$ .
- (ii) If  $w$  is a sink in  $\Gamma$  then  $\varphi(w)$  is sink in  $\Lambda$ .

If  $\varphi$  is one-to-one, that is,  $\varphi$  is essentially the inclusion of a subgraph into a digraph, then (i) is equivalent to: If  $t(e)$  is in the image of  $\varphi$  then  $e$  is also in the image of  $\varphi$  for all  $e \in E_{\Lambda}$ .

Clearly, the identity morphism on a digraph  $\Gamma$  is admissible.

**Lemma 4.3.** [24] Composition of admissible digraph morphisms are admissible. Hence we have a subcategory **aDgrf** of **Dgrf** whose objects are all digraphs and whose morphisms are only the admissible digraph morphisms.

*Proof.* Let  $f \in \mathcal{C}(\Gamma, \Lambda)$  and  $g \in \mathcal{C}(\Lambda, \Delta)$  be admissible digraph morphisms. All we need to show  $g \circ f \in \mathcal{C}(\Gamma, \Delta)$  is an admissible digraph morphism. If  $w$  is a sink in  $\Gamma$  then  $f(w)$  is a sink in  $\Lambda$  by (ii), and similarly  $g(f(w))$  is a sink in  $\Delta$ . Hence  $g \circ f$  satisfies (ii). Now we need to show  $t|_{(g \circ f)^{-1}(e)} = t|_{f^{-1} \circ g^{-1}(e)}$  is a bijection from  $(g \circ f)^{-1}(e) = (f^{-1} \circ g^{-1})(e)$  to  $(g \circ f)^{-1}(t(e)) = (f^{-1} \circ g^{-1})(t(e))$  for all  $e \in E_{\Delta}$ . We know that  $t|_{f^{-1}(d)}$  and  $t|_{g^{-1}(e)}$  are bijections from  $f^{-1}(d)$  to  $f^{-1}(t(d))$  and from  $g^{-1}(e)$  to  $g^{-1}(t(e))$ , respectively. For  $e \in E_{\Delta}$  we have  $(f^{-1} \circ g^{-1})(e) = \bigsqcup_{g(d)=e} f^{-1}(d)$  where  $\bigsqcup$  denotes disjoint union. For each  $d$ ,  $t|_{f^{-1}(d)}$  is a bijection from  $f^{-1}(d)$  to  $f^{-1}(t(d))$ . Since  $(f^{-1} \circ g^{-1})(t(e)) = \bigsqcup_{u \in g^{-1}(t(e))} f^{-1}(u) = \bigsqcup_{g(t(d))=t(e)} f^{-1}(t(d)) = \bigsqcup_{t(g(d))=t(e)} f^{-1}(t(d))$  by admissible property (i) applied to  $g$ . Since  $t|_{(g \circ f)^{-1}(d)}$  maps each subset  $f^{-1}(d)$  bijectively to  $f^{-1}(t(d))$ ,

we see that  $t|_{(g \circ f)^{-1}(e)}$  is also a bijection from  $(g \circ f)^{-1}(e) = \bigsqcup_{g(d)=e} f^{-1}(d)$  to  $(g \circ f)^{-1}(t(e)) = \bigsqcup_{t(g(d))=t(e)} f^{-1}(t(d))$ . Hence a composition of admissible digraph morphisms satisfies (i) and (ii), thus it is admissible.  $\square$

The cofunctor  $L_{\mathbb{F}}$  is defined from the category  $\mathbf{aDgrf}$  to  $\mathbf{Alg}_{\mathbb{F}}$ , its image is in the subcategory of graded  $*$ -algebras and graded  $*$ -algebra homomorphisms.

Let  $\varphi : \Gamma \longrightarrow \Lambda$  is a digraph morphism.

We define  $L_{\mathbb{F}}(\varphi) : V_{\Lambda} \sqcup E_{\Lambda} \longrightarrow L_{\mathbb{F}}(\Gamma)$  as

$$L_{\mathbb{F}}(\varphi)(v) := \sum_{\varphi(u)=v} u \quad \text{for all } v \in V_{\Lambda}$$

$$\text{and } L_{\mathbb{F}}(\varphi)(e) := \sum_{\varphi(f)=e} f \quad \text{for all } e \in E_{\Lambda}.$$

**Theorem 4.4.** [24] Let  $\Gamma$  and  $\Lambda$  be finite digraphs. If  $\varphi : \Gamma \longrightarrow \Lambda$  is a digraph morphism then  $L_{\mathbb{F}}(\varphi)$  defines a (graded)  $*$ -algebra homomorphism if and only if  $\varphi$  is admissible.

*Proof.* If  $\varphi$  is admissible then we need to check that the relations (V), (E), (CK1) and (CK2) are satisfied when we apply  $L_{\mathbb{F}}(\varphi)$  to  $L_{\mathbb{F}}(\Lambda)$ .

(V): If  $v \in V_{\Lambda}$  then  $L_{\mathbb{F}}(\varphi)(v) = \sum_{\varphi(u)=v} u$ , so

$$L_{\mathbb{F}}(\varphi)(v)L_{\mathbb{F}}(\varphi)(v) = \left( \sum_{\varphi(u)=v} u \right)^2 = \sum_{\varphi(u)=v} u^2 = \sum_{\varphi(u)=v} u = L_{\mathbb{F}}(\varphi)(v)$$

since vertices are orthogonal idempotents. If  $v' \neq v$  then  $L_{\mathbb{F}}(\varphi)(v)L_{\mathbb{F}}(\varphi)(v') = \left( \sum_{\varphi(u)=v} u \right) \left( \sum_{\varphi(u')=v'} u' \right) = 0 = L_{\mathbb{F}}(\varphi)(0) = L_{\mathbb{F}}(\varphi)(vv')$  since preimages of distinct vertices are distinct.

(E): If  $e \in E_{\Lambda}$  is not in the image of  $\varphi$  then  $L_{\mathbb{F}}(\varphi)(e) = 0$ . Hence  $L_{\mathbb{F}}(\varphi)(s(e))L_{\mathbb{F}}(\varphi)(e) = 0 = L_{\mathbb{F}}(\varphi)(e) = L_{\mathbb{F}}(\varphi)(e)L_{\mathbb{F}}(\varphi)(t(e))$  and similarly for  $t(e)e^* = e^* = e^*s(e)$ . When  $e \in E_{\Lambda}$  is in the image of  $\varphi$ , let  $\{e_i\} = \varphi^{-1}(e)$ . Now  $L_{\mathbb{F}}(\varphi)(s(e))e = \left( \sum u \right) \left( \sum e_i \right)$  where the first sum is over  $\varphi^{-1}(s(e))$ . Since  $\varphi(s(e_i)) = s(\varphi(e_i)) = s(e)$

we have  $s(e_i) \in \varphi^{-1}(s(e))$ . Hence  $(\sum u)e_i = s(e_i)e_i = e_i$  and  $L_{\mathbb{F}}(\varphi)(s(e)e) = (\sum u)(\sum e_i) = \sum e_i = L_{\mathbb{F}}(\varphi)(e)$ . Similarly for  $e = et(e)$ , also for  $t(e)e^* = e^*$  and  $e^*s(e) = e^*$ .

**(CK1)**: If  $e \neq f$  in  $E_{\Lambda}$ . Then  $\varphi^{-1}(e)$  and  $\varphi^{-1}(f)$  are disjoint subsets of  $E_{\Gamma}$ , hence

$$L_{\mathbb{F}}(\varphi)(e^*f) = L_{\mathbb{F}}(\varphi)(e^*)L_{\mathbb{F}}(\varphi)(f) = \left( \sum_{\varphi(g)=e} g^* \right) \left( \sum_{\varphi(h)=f} h \right) = 0$$

by (CK1) in  $L_{\mathbb{F}}(\Gamma)$ . So  $L_{\mathbb{F}}(\varphi)(e^*f) = L_{\mathbb{F}}(\varphi)(0)$  when  $e \neq f$ .

$$L_{\mathbb{F}}(\varphi)(e^*e) = \left( \sum_{\varphi(g)=e} g^* \right) \left( \sum_{\varphi(h)=e} h \right) = \sum_{\varphi(g)=e} g^*g$$

since  $g^*h = 0$  by (CK1) when  $g \neq h$ . Also  $\sum_{\varphi(g)=e} g^*g = \sum_{\varphi(g)=e} t(g)$  by (CK1) in  $L_{\mathbb{F}}(\Gamma)$ . So  $L_{\mathbb{F}}(\varphi)(e^*e) = \sum_{\varphi(g)=e} t(g) = \sum_{\varphi(v)=t(e)} v = L_{\mathbb{F}}(\varphi)(t(e))$  for all  $e \in E_{\Lambda}$  where the middle equality follows from condition (i) of admissibility.

Condition (i) is necessary for (CK1): otherwise for some  $e \in E_{\Lambda}$  we would have

$$L_{\mathbb{F}}(\varphi)(e^*e) = \sum_{\varphi(g)=e} t(g) \neq \sum_{\varphi(v)=t(e)} v = L_{\mathbb{F}}(\varphi)(t(e))$$

because vertices are linearly independent in  $L_{\mathbb{F}}(\Gamma)$ , so  $\sum_{\varphi(g)=e} t(g) = \sum_{\varphi(v)=t(e)} v$  if and only if  $\{t(g)\}_{\varphi(g)=e} = \{v\}_{\varphi(v)=t(e)}$ .

**(CK2)**: If  $v \in V_{\Lambda}$  is not a sink then

$$L_{\mathbb{F}}(\varphi)\left(\sum_{s(e)=v} ee^*\right) = \sum_{s(e)=v} L_{\mathbb{F}}(\varphi)(e)L_{\mathbb{F}}(\varphi)(e^*) = \sum_{s(e)=v} \left( \sum_{\varphi(f)=e} f \right) \left( \sum_{\varphi(g)=e} g^* \right).$$

Now  $f = ft(f)$  and  $g^* = t(g)g^*$  by (E). If  $fg^* = ft(f)t(g)g^* \neq 0$  then  $t(f) = t(g)$  by (V), hence  $f = g$  by (i). Therefore

$$\sum_{s(e)=v} \left( \sum_{\varphi(f)=e} f \right) \left( \sum_{\varphi(g)=e} g^* \right) = \sum_{s(e)=v} \sum_{\varphi(f)=e} ff^* = \sum_{\varphi(u)=v} \sum_{s(f)=u} ff^*$$

changing the order of summation using  $\varphi(s(f)) = s(\varphi(f)) = s(e)$  because this sum is over all arrows  $f$  in  $\Gamma$  with source being a preimage of  $v$  in  $\Lambda$ . Since  $\sum_{s(f)=e} ff^* = u$  by (CK2) and (ii) we have

$$L_{\mathbb{F}}(\varphi)\left(\sum_{s(e)=v} ee^*\right) = \sum_{\varphi(u)=v} u = L_{\mathbb{F}}(\varphi)(v).$$

Condition (ii) is necessary: If there is a sink in  $\Gamma$  whose image  $v$  under  $\varphi$  is not a sink in  $\Lambda$  then the computation of  $L_{\mathbb{F}}(\varphi)\left(\sum_{s(e)=v} ee^*\right)$  above would yield  $\sum w = 0$  where the sum is over all sinks  $w$  with  $\varphi(w) = v$ . As above, this is not possible since vertices are linearly independent.

We have show that  $\varphi$  being an admissible digraph morphism is necessary and sufficient for  $L_{\mathbb{F}}(\varphi)$  to define an algebra homomorphism (We will denote the algebra homomorphism also by  $L_{\mathbb{F}}(\varphi)$ ). From the definition of  $L_{\mathbb{F}}(\varphi)$  it is clear that this is a graded  $*$ -homomorphism.  $\square$

**Remark 2.** To realize  $\mathbb{F}\Gamma$  as a cofunctor, using the same definition for the images of vertices and arrows as in  $L_{\mathbb{F}}(\varphi)$  we don't need either of the two conditions in Definition 45. To realize the Cohn algebra construction as a cofunctor we only need condition (i).

**Proposition 4.5.** [24] If  $\varphi : \Gamma \longrightarrow \Lambda$  is an onto admissible digraph morphism then  $L_{\mathbb{F}}(\varphi) : L_{\mathbb{F}}(\Lambda) \longrightarrow L_{\mathbb{F}}(\Gamma)$  is a (graded  $*$ -algebra) monomorphism. If  $\varphi$  is one-to-one then  $L_{\mathbb{F}}(\varphi)$  is an epimorphism.

*Proof.* Suppose  $\varphi : \Gamma \longrightarrow \Lambda$  is an onto admissible digraph morphism. For any  $v \in \Lambda$  we know  $L_{\mathbb{F}}(\varphi)(v) = \sum_{\varphi(u)=v} u$ . From Lemma 3.6 we know that  $w \neq 0$  for all vertices  $w$  in  $\Gamma$ . If  $\varphi(w) = v$  then  $w\left(\sum_{\varphi(u)=v} u\right) = w \neq 0$  hence  $\sum_{\varphi(u)=v} u \neq 0$ . Since  $\varphi$  is onto  $L_{\mathbb{F}}(\varphi)(v) \neq 0$  for every vertex  $v \in \Lambda$ . From the Graded Uniqueness Theorem (Theorem 3.7) we see that  $L_{\mathbb{F}}(\varphi)$  is a monomorphism.

Suppose  $\varphi : \Gamma \longrightarrow \Lambda$  is a one-to-one admissible digraph morphism. For any vertex  $v$  in  $\Gamma$  we have  $\varphi^{-1}(\varphi(v)) = \{v\}$  since  $\varphi$  is one-to-one. Therefore  $L_{\mathbb{F}}(\varphi)(\varphi(v)) = v$  for every  $v$  in  $\Gamma$ , showing that  $v$  is in the image of  $L_{\mathbb{F}}(\varphi)$ . Similarly, for any arrow  $e$  in  $\Gamma$  we have  $\varphi^{-1}(\varphi(e)) = \{e\}$  since  $\varphi$  is one-to-one. Therefore  $L_{\mathbb{F}}(\varphi)(\varphi(e)) = e$  for every  $e$  in  $\Gamma$ , showing that  $e$  is in the image of  $L_{\mathbb{F}}(\varphi)$ . Since  $L_{\mathbb{F}}(\varphi)$  is a  $*$ -homomorphism

$e^* = L_{\mathbb{F}}(\varphi)(\varphi(e)^*)$  is in the image of  $L_{\mathbb{F}}(\varphi)$  for every arrow  $e$  in  $\Gamma$ . The algebra  $L_{\mathbb{F}}(\Gamma)$  is generated by  $V \sqcup E \sqcup \hat{E}$  and all of these generators are in the image of  $L_{\mathbb{F}}(\varphi)$ . Thus  $L_{\mathbb{F}}(\varphi)$  is an epimorphism.  $\square$

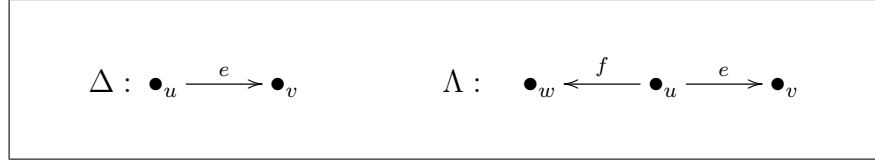
We are mostly interested in one-to-one admissible digraph morphisms  $\varphi : \Delta \longrightarrow \Lambda$  where we identify  $\Delta$  with the subgraph of  $\Lambda$  which is the image of the digraph morphism, so we may think of  $\varphi$  as an inclusion. If  $\varphi : \Delta \hookrightarrow \Lambda$  is admissible and  $p, q \in Path(\Lambda)$  with  $t(p) = t(q)$  then either

- (i)  $L_{\mathbb{F}}(\varphi)(pq^*) = 0$  when  $p$  or  $q$  involves a vertex or an arrow not in  $\Delta$ ; or
- (ii)  $L_{\mathbb{F}}(\varphi)(pq^*) = pq^* \in L_{\mathbb{F}}(\Delta)$  when  $p, q \in Path(\Delta)$ .

Here it is important to realize that  $pq^* \in L_{\mathbb{F}}(\Lambda)$  is the input of  $L_{\mathbb{F}}(\varphi)$  while  $L_{\mathbb{F}}(\varphi)(pq^*) \in L_{\mathbb{F}}(\Delta)$ . We want to define a linear transformation  $\sigma : L_{\mathbb{F}}(\Delta) \longrightarrow L_{\mathbb{F}}(\Lambda)$  which is a right inverse of  $L_{\mathbb{F}}(\varphi)$  when  $\Delta$  is an admissible subgraph of  $\Lambda$ . (Note that  $\sigma$  will not be an algebra homomorphism in general.) First we need to recall the basis of  $L_{\mathbb{F}}(\Lambda)$  given at the end of Section 3.

If  $v$  is a nonsink in  $\Lambda$  and  $v$  is in the subgraph  $\Delta$  then  $v$  is a nonsink in  $\Delta$  by admissibility. In this case we pick  $e_v$ , the distinguished arrow corresponding to  $v$ , in  $\Delta$ . If  $v$  is not in  $\Delta$  then we pick  $e_v$  in  $\Lambda$  arbitrarily. So we have picked the distinguished arrows for both  $\Lambda$  and  $\Delta$  compatibly. The bases for  $L_{\mathbb{F}}(\Delta)$  and  $L_{\mathbb{F}}(\Lambda)$  consist of elements of the form  $pq^*$  not containing an  $ee^*$  for any distinguished arrow  $e$ . Hence any basis vector for  $L_{\mathbb{F}}(\Delta)$  is also a basis vector for  $L_{\mathbb{F}}(\Lambda)$ . Now we define  $\sigma : L_{\mathbb{F}}(\Delta) \longrightarrow L_{\mathbb{F}}(\Lambda)$  by sending a basis vector  $pq^*$  of  $L_{\mathbb{F}}(\Delta)$  to "itself" considered as an element of  $L_{\mathbb{F}}(\Lambda)$ . By construction  $L_{\mathbb{F}}(\varphi) \circ \sigma = id_{L_{\mathbb{F}}(\Delta)}$ , that is,  $\sigma$  is a right inverse of  $L_{\mathbb{F}}(\varphi)$  where  $\varphi$  is the inclusion of  $\Delta$  into  $\Lambda$ . The linear transformation  $\sigma$  is not an algebra homomorphism in general. Also,  $\sigma$  depends on the choice of the distinguished arrows, so  $\sigma$  is not unique. Hence  $\sigma$  is just a right inverse of  $L_{\mathbb{F}}(\varphi)$  regarded as a linear transformation.

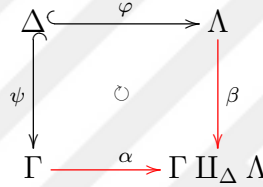
**Example 31.** For the admissible inclusion  $\Delta \hookrightarrow \Lambda$  in Figure 4.1 we have  $u = ee^*$  in  $L_{\mathbb{F}}(\Delta)$  while  $u = ee^* + ff^*$  in  $L_{\mathbb{F}}(\Lambda)$  by (CK2). In  $\Delta$  and  $\Lambda$  we need to take  $e$  as the distinguished arrow at  $u$  and  $\sigma(u) = u$ . If  $\sigma$  was an algebra homomorphism then we would have  $ee^* = ee^* + ff^*$  in  $L_{\mathbb{F}}(\Lambda)$ . This is not possible since  $ff^* \neq 0$  by Lemma 3.6.



**Figure 4.1:**  $\sigma$  is not an algebra homomorphism

**Lemma 4.6.** [24] In the category **aDgrf**, the pushout of  $\Gamma \xleftarrow{\psi} \Delta \xrightarrow{\varphi} \Lambda$  exists when  $\varphi$  and  $\psi$  are one-to-one admissible digraph morphisms. That is, the digraph morphisms  $\alpha$  and  $\beta$  of the pushout diagram in **Dgrf** are admissible.

*Proof.* Let the following be a pushout diagram



where  $\varphi$  and  $\psi$  are one-to-one admissible digraphs morphisms. Since  $\varphi$  and  $\psi$  are one-to-one we may regard  $\Delta$  as the common subgraph of  $\Gamma$  and  $\Lambda$  hence  $\Gamma \amalg_{\Delta} \Lambda = \Gamma \cup \Lambda$  where  $E_{\Gamma} \amalg_{E_{\Delta}} E_{\Lambda} = E_{\Gamma} \cup E_{\Lambda}$  and  $V_{\Gamma} \amalg_{V_{\Delta}} V_{\Lambda} = V_{\Gamma} \cup V_{\Lambda}$ . We need to show that  $\Gamma \xrightarrow{\alpha} \Gamma \cup \Lambda$  and  $\Lambda \xrightarrow{\beta} \Gamma \cup \Lambda$  are admissible. To check first condition of admissibility it is enough to show that if  $t(e)$  is in  $\Gamma$  (or  $\Lambda$ ) for  $e$  in  $\Gamma \cup \Lambda$  then  $e$  is in  $\Gamma$  (respectively  $\Lambda$ ) because  $\alpha$  and  $\beta$  are inclusions.

If  $e$  is in  $\Gamma \cup \Lambda$  and  $t(e)$  is in  $\Gamma$  then  $e$  is in  $\Gamma$  or  $\Lambda$ . If  $e$  is in  $\Gamma$  then we are done. If  $e$  is in  $\Lambda$  then  $t(e)$  is also in  $\Lambda$ . Then  $t(e)$  is in  $\Delta$  since  $t(e)$  is in both  $\Gamma$  and  $\Lambda$ . From the admissibility of the inclusion  $\Delta \hookrightarrow \Gamma$  we get that  $e$  is in  $\Delta$ . Since  $\Delta$  is a subgraph of  $\Gamma$  we see that  $e$  is also in  $\Gamma$ . Replacing  $\Gamma$  with  $\Lambda$  in the discussion above we get that  $\Lambda \hookrightarrow \Gamma \cup \Lambda$  is admissible also.

If  $v$  is in  $\Gamma$  and  $v$  is not a sink in  $\Gamma \cup \Lambda$  then there is an  $e$  in  $\Gamma \cup \Lambda$  with  $s(e) = v$ . If  $e$  is in  $\Gamma$  then  $v$  is not a sink in  $\Gamma$ . If  $e$  is in  $\Lambda$  then  $s(e) = v$  is in both  $\Gamma$  and  $\Lambda$ . Hence  $v$  is in  $\Delta$ . Since the inclusion  $\Delta \hookrightarrow \Lambda$  is admissible,  $v$  is not a sink in  $\Delta$  because  $v$  is not a sink in  $\Lambda$ . So there is an arrow  $f$  in  $\Delta$  with  $s(f) = v$ . But  $\Delta$  is

a subgraph of  $\Gamma$  so  $f$  is also in  $\Gamma$  hence  $v$  is not a sink in  $\Gamma$ . We have shown that for all  $v$  in  $\Gamma$  if  $v$  is not a sink  $\Gamma \cup \Lambda$  then  $v$  is not a sink in  $\Gamma$ . The contrapositive of this statement says that if  $v$  is a sink in  $\Gamma$  then  $v$  is a sink in  $\Gamma \cup \Lambda$ . Thus the second condition of admissibility is satisfied for  $\Gamma \xrightarrow{\alpha} \Gamma \cup \Lambda$ . The proof of the second condition of admissibility for  $\Lambda \xrightarrow{\beta} \Gamma \cup \Lambda$  is basically the same.  $\square$

Now we are ready to state and prove our main result.

**Theorem 4.7.** [24] If we apply the cofunctor  $L_{\mathbb{F}}$  to the pushout diagram of  $\Gamma \xleftarrow{\psi} \Delta \xrightarrow{\varphi} \Lambda$  in  $\mathbf{adgrf}$  with  $\varphi$  and  $\psi$  one-to-one then in the category  $\mathbf{Alg}_{\mathbb{F}}$ , we get the pullback diagram of  $L_{\mathbb{F}}(\Gamma) \xrightarrow{L_{\mathbb{F}}(\psi)} L_{\mathbb{F}}(\Delta) \xleftarrow{L_{\mathbb{F}}(\varphi)} L_{\mathbb{F}}(\Lambda)$ .

*Proof.* We may identify  $\Delta$  with a common subgraph of  $\Lambda$  and  $\Gamma$  since  $\varphi$  and  $\psi$  are one-to-one, so the pushout object  $\Gamma \amalg_{\Delta} \Lambda$  becomes the union of the  $\Gamma$  and  $\Lambda$  with  $\Delta$  in the intersection. Moreover the inclusions  $\alpha : \Gamma \rightarrow \Gamma \amalg_{\Delta} \Lambda$  and  $\beta : \Lambda \rightarrow \Gamma \amalg_{\Delta} \Lambda$  are admissible by Lemma 4.6. Hence we may apply the cofunctor  $L_{\mathbb{F}}$  to the admissible digraph inclusions  $\alpha$  and  $\beta$  as well as  $\varphi$  and  $\psi$ . From the commutative diagram of digraphs (and admissible digraph morphisms) we get the commutative diagram of algebras as shown below:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \Delta & \xrightarrow{\varphi} & \Lambda \\
 \psi \downarrow & \circlearrowleft & \downarrow \beta \\
 \Gamma & \xrightarrow{\alpha} & \Gamma \amalg_{\Delta} \Lambda
 \end{array} & \xrightarrow{L_{\mathbb{F}}} & \begin{array}{ccc}
 L_{\mathbb{F}}(\Delta) & \xleftarrow{L_{\mathbb{F}}(\varphi)} & L_{\mathbb{F}}(\Lambda) \\
 L_{\mathbb{F}}(\psi) \uparrow & \circlearrowleft & \uparrow L_{\mathbb{F}}(\beta) \\
 L_{\mathbb{F}}(\Gamma) & \xleftarrow{L_{\mathbb{F}}(\alpha)} & L_{\mathbb{F}}(\Gamma \amalg_{\Delta} \Lambda)
 \end{array}
 \end{array}$$

From the universal property of the pullback we get the diagram

$$\begin{array}{ccccc}
 L_{\mathbb{F}}(\Delta) & \xleftarrow{L_{\mathbb{F}}(\varphi)} & L_{\mathbb{F}}(\Lambda) & & \\
 \uparrow L_{\mathbb{F}}(\psi) & & \uparrow G & & \\
 L_{\mathbb{F}}(\Gamma) & \xleftarrow{F} & L_{\mathbb{F}}(\Gamma) \times_{L_{\mathbb{F}}(\Delta)} L_{\mathbb{F}}(\Lambda) & \xleftarrow{L_{\mathbb{F}}(\beta)} & L_{\mathbb{F}}(\Lambda) \\
 & & \downarrow \theta & & \\
 & & L_{\mathbb{F}}(\Gamma \amalg_{\Delta} \Lambda) & & 
 \end{array}$$

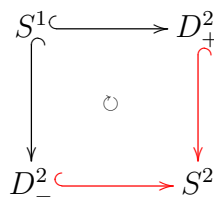
$L_{\mathbb{F}}(\alpha) \xleftarrow{\quad} L_{\mathbb{F}}(\Gamma) \quad L_{\mathbb{F}}(\beta) \xleftarrow{\quad} L_{\mathbb{F}}(\Lambda)$

Here  $L_{\mathbb{F}}(\Gamma) \times_{L_{\mathbb{F}}(\Delta)} L_{\mathbb{F}}(\Lambda) = \{(a, b) \in L_{\mathbb{F}}(\Gamma) \times L_{\mathbb{F}}(\Lambda) \mid L_{\mathbb{F}}(\psi)(a) = L_{\mathbb{F}}(\varphi)(b)\}$  and the morphism  $\theta$  maps  $c \in L_{\mathbb{F}}(\Gamma \amalg_{\Delta} \Lambda)$  to the ordered pair  $(L_{\mathbb{F}}(\alpha)(c), L_{\mathbb{F}}(\beta)(c))$ . The morphisms  $F$  and  $G$  are restrictions of the projections to  $L_{\mathbb{F}}(\Gamma)$  and  $L_{\mathbb{F}}(\Lambda)$  respectively. We will be done when we show that  $\theta$  is an isomorphism.

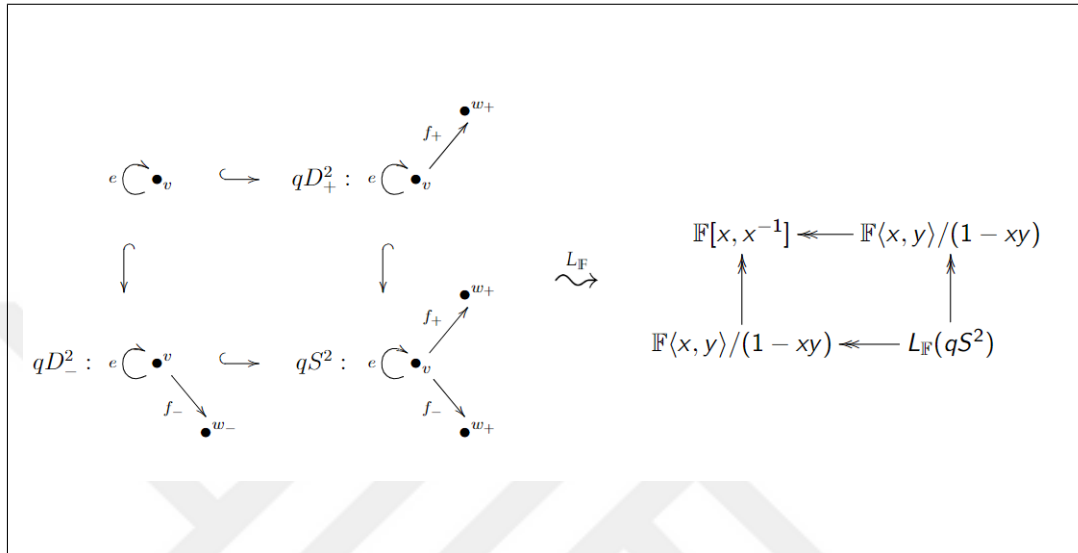
We know from the discussion on the pullback diagrams of  $\mathbb{Z}$ -graded algebras that  $\theta$  is a graded algebra homomorphism. By the Graded Uniqueness Theorem, i.e., Theorem 3.7 it is enough to show that the image of every vertex in  $\Gamma \amalg_{\Delta} \Lambda$  is nonzero. Each vertex  $v$  in  $\Gamma \amalg_{\Delta} \Lambda$  must be a vertex in  $\Gamma$  or a vertex in  $\Lambda$  since  $\Gamma \amalg_{\Delta} \Lambda$  is the union of the subgraphs  $\Gamma$  and  $\Lambda$ . Hence  $L_{\mathbb{F}}(\alpha)(v)$  or  $L_{\mathbb{F}}(\beta)(v)$  is nonzero by the definition of  $L_{\mathbb{F}}$  because  $\alpha$  and  $\beta$  are the inclusions of  $\Gamma$  and  $\Lambda$  into  $\Gamma \amalg_{\Delta} \Lambda$ . Thus  $\theta(v) = (L_{\mathbb{F}}(\alpha)(v), L_{\mathbb{F}}(\beta)(v)) \neq 0$  for all  $v \in \Gamma \amalg_{\Delta} \Lambda$  and  $\theta$  is one-to-one.

To see that  $\theta$  is onto we will find a basis of  $L_{\mathbb{F}}(\Gamma \amalg_{\Delta} \Lambda)$  which restrict to bases of  $L_{\mathbb{F}}(\Gamma)$ ,  $L_{\mathbb{F}}(\Lambda)$  and  $L_{\mathbb{F}}(\Delta)$  as in the discussion above Lemma 4.6. That is, if  $v$  is a nonsink in  $\Gamma \amalg_{\Delta} \Lambda$  and  $v$  is a vertex in  $\Delta$  then  $v$  is a nonsink in  $\Delta$  by admissibility. We pick a distinguished arrow  $e_v$  from  $\Delta$ . If  $v$  is a nonsink and not in  $\Delta$  then it must be a nonsink in either  $\Gamma$  or  $\Lambda$ , so we pick  $e_v$  accordingly. Now we may define the linear transformations  $\sigma_1 : L_{\mathbb{F}}(\Gamma) \rightarrow L_{\mathbb{F}}(\Gamma \amalg_{\Delta} \Lambda)$  and  $\sigma_2 : L_{\mathbb{F}}(\Lambda) \rightarrow L_{\mathbb{F}}(\Gamma \amalg_{\Delta} \Lambda)$  sending basis elements  $p_i q_i^*$  of  $L_{\mathbb{F}}(\Gamma)$  or  $L_{\mathbb{F}}(\Lambda)$  to "themselves" considered as elements of  $L_{\mathbb{F}}(\Gamma \amalg_{\Delta} \Lambda)$ . Given an element  $(a, b) \in L_{\mathbb{F}}(\Gamma) \times_{L_{\mathbb{F}}(\Delta)} L_{\mathbb{F}}(\Lambda)$  we have  $a = \sum_{i=1}^k \lambda_i p_i q_i^* + \sum_{i=k+1}^n \lambda_i p_i q_i^*$  where each  $p_i q_i^*$  is a basis vector of  $L_{\mathbb{F}}(\Gamma)$  and  $p_i q_i^*$  is a basis vector of  $L_{\mathbb{F}}(\Delta)$  if and only if  $i \leq k$ . Hence  $L_{\mathbb{F}}(\alpha)(a) = \sum_{i=1}^k \lambda_i p_i q_i^* = L_{\mathbb{F}}(\beta)(b)$  and  $b = \sum_{j=1}^m \mu_j p_j q_j^* + \sum_{i=1}^k \lambda_i p_i q_i^*$  with  $L_{\mathbb{F}}(\beta)(\sum_{j=1}^m \mu_j p_j q_j^*) = 0$ . Now  $\theta(\sum_{i=1}^n \lambda_i p_i q_i^* + \sum_{j=1}^m \mu_j p_j q_j^*) = (a, b)$ . Thus  $\theta$  is onto and we are done.  $\square$

In Figure 4.2 we give the noncommutative algebraic analog of obtaining the sphere by identifying two discs (the northern and southern hemispheres) along their boundaries (the equator) shown in the diagram



When  $\mathbb{F} = \mathbb{C}$  completing LPAs yields the graph  $C^*$ -algebras identified as the quantum versions of  $D^2$  and  $S^2$ . Each kernel of the epimorphisms of the LPAs on the right are ideals generated by a sink ( $w_+$  or  $w_-$ ).



**Figure 4.2:** Cofunctor  $L_{\mathbb{F}}$

The adjective quantum is used because while the mathematical model for classical mechanics is a commutative algebra, the algebra of continuous functions on the state space, the mathematical model for quantum mechanics is a noncommutative algebra of bounded linear operators on a Hilbert space.

## 5. CONCLUSION

The main objective of this thesis is to realize the construction of Leavitt path algebras of di(irected )graphs as the object part of a cofunctor from a subcategory of digraphs and digraph morphisms to the category of  $\mathbb{Z}$ -graded  $*$ -algebras and their morphisms. To this end we determine which digraph morphisms are "admissible", that is, enable us to extend the Leavitt path algebra construction (contravariantly) to digraph morphisms. Already at the object level this provides noncommutative analogs of polynomial algebras for some quantum topological spaces. For instance:

**Theorem 4.4.** Let  $\Gamma$  and  $\Lambda$  be finite digraphs. If  $\varphi : \Gamma \longrightarrow \Lambda$  is a digraph morphism then  $L_{\mathbb{F}}(\varphi)$  defines a graded  $*$ -algebra homomorphism if and only if  $\varphi$  is admissible. Moreover we give some realizations of pushout to pullback constructions in this context. Specifically:

**Theorem 4.7.** If we apply the cofunctor  $L_{\mathbb{F}}$  to the pushout diagram of  $\Gamma \xleftarrow{\psi} \Delta \xrightarrow{\varphi} \Lambda$  in  $\mathbf{aDgrf}$  with  $\varphi$  and  $\psi$  one-to-one then in the category  $\mathbf{Alg}_{\mathbb{F}}$ , we get the pullback diagram of  $L_{\mathbb{F}}(\Gamma) \xrightarrow{L_{\mathbb{F}}(\psi)} L_{\mathbb{F}}(\Delta) \xleftarrow{L_{\mathbb{F}}(\varphi)} L_{\mathbb{F}}(\Lambda)$ .

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## **BIOGRAPHY**

Hatice Nur KOLCU graduated from Gebze Technical University, Faculty of Science, Mathematics Department in 2022. She started her master's degree as a student of Gebze Technical University, Institute of Graduate School, Mathematics Department in 2022.



## **PUBLICATIONS AND PRESENTATIONS FROM THE THESIS**

[1] Koç A., Kolcu H.N., (2023) "Functorial constructions for graph algebras" 7. GTU Lisansüstü Araştırmalar Sempozyumu, Gebze Technical University, Türkiye, May 31-June 1.

