ZONGULDAK BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

AN ALTERNATIVE DESCRIPTION OF THE TORSION SUBGROUP OF THE FREE CENTRE-BY-METABELIAN GROUP OF RANK AT LEAST FOUR IN TERMS OF GENERATORS

DEPARTMENT OF MATHEMATICS

MASTER OF SCIENCE THESIS

ŞULE AKGÜL

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“With this thesis it is declared that all the information in this thesis is obtained and presented according to academic rules and ethical principles. Also as required by the academic rules and ethical principles all works that are not result of this study are cited properly.”

Şule AKGÜL

ABSTRACT

Master of Science Thesis

AN ALTERNATIVE DESCRIPTION OF THE TORSION SUBGROUP OF THE FREE CENTRE-BY-METABELIAN GROUP OF RANK AT LEAST FOUR IN TERMS OF GENERATORS

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Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Prof. Dr. Seyhun KESİM
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Let F be a non-cyclic free group of rank *n*. Consider the quotient $F/[F'',F]$, the free centre-by-metabelian group of rank *n*. In 1973, C. K. Gupta proved by purely group theoretic means that it contains an elementary abelian 2-group of rank $\binom{n}{4}$ in its centre for $n \geq 4$, and exhibited an explicit generating set for this torsion subgroup. In this thesis, using homological methods, we provide an alternative explicit generating set for it, and identify this torsion subgroup as the isolator of an explicitly given subgroup in the quotient $F''/[F'',F']$.

Keywords: Free central extensions of groups, Homology of groups, Connecting homomorphisms.

Science Code: 403.01.01

ÖZET

Yüksek Lisans Tezi

RANKI EN AZ DÖRT OLAN SERBEST MERKEZİ METABELYEN GRUBUN TORSİON ALTGRUBUNUN ÜRETEÇLER CİNSİNDEN BİR ALTERNATİF TANIMI

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F rankı n olan ve devirli olmayan bir serbest grup olsun. Rankı n olan serbest merkezi metabelyen grup $F/[F'',F]$ i ele alalım. 1973 de C. K. Gupta grup teori araçları kullanarak $n \geq 4$ için bu grubun merkezinde rankı $\binom{n}{4}$ olan bir elemanter abelyen 2-grup içerdiğini ispatladı, ve bu torsiyon altgrubu üreten bir kümeyi belirgin bir biçimde sergiledi. Bu tezde, homolojik metotlar kullanarak, bu torsiyon altgrup için alternatif bir üreten kümeyi belirgin bir biçimde temin ediyoruz ve bu torsiyon altgrubu $F''/[F'',F']$ bölüm grubunda belirgin bir biçimde verilen bir altgrubun izolatörü olarak tanımlıyoruz.

Anahtar Kelimeler: Grupların serbest merkezi genişlemeleri, Grupların homolojisi, Bağlantı homomorfileri.

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TABLE OF CONTENTS

TABLE OF CONTENTS (continued)

LIST OF SYMBOLS AND ABBREVIATIONS

SYMBOLS

$H_k\left(G,B\right)$: The *k*-th homology group of *G* with coefficients in *B*

$\cong$: Isomorphic to

$\otimes$: Tensor product

S_n : The symmetric group of degree *n*

$\operatorname{sgn}(\sigma)$: The sign of the permutation σ

$t(G)$: The torsion subgroup of *G*, where *G* is any abelian group

$\ker\ \alpha$: The kernel of the homomorphism α

$\mathbb{Z}G$: The integral group ring of the group *G*

IG : The augmentation ideal of $\mathbb{Z}G$

G' : The commutator subgroup of *G*

$\mathbb{Z}$: The ring of integers

$\mathbb{Z}_n$: $\mathbb{Z}/n\mathbb{Z}$

$A \circ A$: The symmetric square of the module *A*

$A \wedge A$: The exterior square of the module *A*

CHAPTER 1

INTRODUCTION

In this thesis, we study elements of order 2 in the free centre-by-metabelian group which is a free central extension of a torsion-free group. In order to introduce this free central extension we make use of standard group theoretic notation for higher commutator subgroups which is explained in Chapter 2.

Let F be a free group of rank $n \geq 2$ with free generating set $X = \{x_1, ..., x_n\}$. Consider the quotient

$$F/[F'', F]. \tag{1.1}$$

It fits into the short exact sequence

$$1 \to F''/[F'', F] \to F/[F'', F] \to F/F'' \to 1 \tag{1.2}$$

where the left hand term is clearly central in the middle term, and hence (1.1) is a central extension of the right hand term F/F'' which is the free metabelian group of rank n. In fact, $F/[F'', F]$ is the free centre-by-metabelian group of rank n.

It is well-known that the right hand term in (1.2) is torsion-free, and that its centre is trivial (see [8]). Thus (1.1) is a free central extension of a torsion-free group, and its centre is precisely the left hand term in (1.2). A peculiar feature of the free central extension (1.1) is the presence of elements of order 2 in its centre $F''/[F'', F]$ for $n \geq 4$. Since the right hand term of (1.2) is, as we have just pointed out, torsion-free, any element of finite order in (1.1) must be contained in the centre, and it follows immediately that these elements form a subgroup.

This phenomenon was studied by Chander Kanta Gupta in 1973. Gupta proved that $F/[F'', F]$ is torsion-free for $n = 2$ and $n = 3$, and she discovered that, if $n \geq 4$, then $F/[F'', F]$ contains an elementary abelian 2-group of rank $\binom{n}{4}$ in its centre, an unexpected result at that time. Furthermore, she proved that this torsion subgroup is freely generated by the elements

$$\begin{aligned}
&\left[\left[x_i^{-1}, x_j^{-1}\right], [x_k, x_l]\right] \left[\left[x_i^{-1}, x_l^{-1}\right], [x_j, x_k]\right] \\
&\left[\left[x_i^{-1}, x_k^{-1}\right], [x_l, x_j]\right] \left[\left[x_l^{-1}, x_j^{-1}\right], [x_i, x_k]\right] \\
&\left[\left[x_j^{-1}, x_k^{-1}\right], [x_i, x_l]\right] \left[\left[x_k^{-1}, x_l^{-1}\right], [x_i, x_j]\right]
\end{aligned}$$

with $1 \leq i < j < k < l \leq n$ (see [1]). Gupta's proof of this remarkable result was purely group theoretical and consists of several pages of intricate commutator calculations.

In his pioneering 1977 paper [4], Yu. V. Kuz'min introduced homological methods into the study of Gupta's torsion elements, and subsequently, was able to identify the torsion subgroup of $F/[F'', F]$ with $H_4(F/F') \otimes \mathbb{Z}_2$, the fourth integral homology group of the free abelian group F/F' reduced modulo 2, and to obtain the same torsion elements.

In his 1989 paper [10], by deriving a 6-term exact sequence, Ralph Stöhr showed that if R is a normal subgroup of F and $G = F/R$ has no elements of order 2 then

$$t\left(F/\left[R', F\right]\right) \cong H_4\left(G, \mathbb{Z}_2\right)$$

where $H_4\left(G, \mathbb{Z}_2\right)$ denotes the fourth homology group of G with coefficients in the trivial G-module $\mathbb{Z}_2$. In the special case $R = F'$, $G = F/F'$ is free abelian, and the isomorphism $t\left(F/\left[R', F\right]\right) \cong H_4\left(G, \mathbb{Z}_2\right)$ turns into

$$t\left(F/[F'', F]\right) \cong H_4\left(F/F', \mathbb{Z}_2\right). \tag{1.3}$$

Now, (1.3) provides a description of the torsion subgroup of (1.1) as an abstract group in homological terms, and is consistent with Kuz'min's result, since $H_4\left(F/F', \mathbb{Z}_2\right)$ is isomorphic to $H_4(F/F') \otimes \mathbb{Z}_2$. The homology group $H_4\left(F/F', \mathbb{Z}_2\right)$ is an elementary abelian 2-group of rank $\binom{n}{4}$ for $4 \leq n =$rank F, and trivial for $4 > n$. Hence (1.3) implies that $F/[F'', F]$ is torsion-free for $n = 2$ and $n = 3$, and, if $n \geq 4$, then $F/[F'', F]$ contains elements of order 2. In 1997, in his Ph.D. thesis [6], using homological methods and doing a computation which is different from Kuz'min's computation, A. M. Sager described this torsion subgroup in group theoretic terms by identifying generators for it. His motivation for this work came from [10]. Furthermore, he showed that his generating set coincides with that of Gupta's.

In this thesis, our aims are to obtain an alternative description of the torsion subgroup of $F/[F'', F]$ in terms of generators of F for $n \geq 4$, to show that our generating set coincides

with that of Gupta's, and to identify this torsion subgroup as the isolator of an explicitly given subgroup in the lower central quotient $\gamma_2(F')/\gamma_3(F')$. As Sager, our motivation to achive these aims comes from [10], however we will do a compuation which is considerably shorter and more practical than Sager's computation. In our computation using the fact that the symmetric group S_n acts naturally as a group of automorphisms on the free group F by permuting the free generators, we will apply some S_n-actions induced by the S_n-action on the free group F. We will also use definitions of certain permutations of the symmetric groups S_3 and S_4 to keep statements of our findings as short as possible.
The key idea that allows to apply homological methods to the quotient

$$F''/[F'',F] \tag{1.4}$$

is to write it as a tensor product. Let G be a free abelian group given by the free presentation

$$1 \to F' \to F \to G \to 1. \tag{1.5}$$

The relation module of G stemming from (1.5) is the free abelian group $M = F'/F''$ with right G-action induced by conjugation in F. The quotient $F'/[F'',F']$ is a free nilpotent group of class 2 and, consequently, there is an isomorphism of G-modules

$$F''/[F'',F'] \cong F'/F'' \wedge F'/F'' = M \wedge M$$

where $M \wedge M$ denotes the exterior square of M, the G action being induced by conjugation in F. In view of the canonical isomorphism $F''/[F'',F'] \otimes_G \mathbb{Z} \cong F''/[F'',F]$, we then get

$$F''/[F'',F] \cong M \wedge M \otimes_G \mathbb{Z}. \tag{1.6}$$

Thus we have transformed the purely group theoretic problem of studying the quotient (1.4) into a problem about a tensor product involving the exterior square of the relation module M. Yu. V. Kuz'min was the first to exploit the isomorphism (1.6) (in a slightly different form).
In this thesis, we take advantage of the isomorphism (1.6), and use homological methods to identify elements of order 2 in $M \wedge M \otimes_G \mathbb{Z}$ (and hence in $F''/[F'',F]$). Since $t(F''/[F'',F]) = t(F/[F'',F])$, (1.3) can be restated as

$$t(M \wedge M \otimes_G \mathbb{Z}) \cong H_4(G, \mathbb{Z}_2) \tag{1.7}$$

where $G = F/F'$ is a free abelian group of rank n. We exploit the 6-term exact sequence which was used in the proof of (1.7) in [10] to compute the image of the homology group $H_4(G, \mathbb{Z}_2)$ in $M \wedge M \otimes_G \mathbb{Z}$ (and hence in $F''/[F'', F]$). Proofs of our main results (Theorem 5.1, Theorem 5.2 and Theorem 5.3) in Chapter 5 are by computing this image.

Basic results on homology of groups will be used without special references being given. These, however, can easily be found in [2].

The rest of the thesis is organized as follows. In Chapter 2, we introduce notation and some preliminary material required in this research. In Chapter 3, we review the result (1.3) as given in [10], and record its main ingredients including a 6-term exact sequence which plays a crucial role in our calculations in Chapter 4. In Chapter 4, we identify the image of $H_4(F/F', \mathbb{Z}_2)$ in $M \wedge M \otimes_G \mathbb{Z}$. In Chapter 5, we state and prove our main results one of which provides an explicit alternative set of generators for the torsion subgroup of the free centre-by-metabelian group of rank $n \geq 4$. Finally, in Chapter 6, we show that our generating set for the torsion subgroup of the free centre-by-metabelian group of rank $n \geq 4$ coincides with that of Gupta's.

CHAPTER 2

PRELIMINARIES AND NOTATION

In this chapter, we introduce notation and some preliminary material required in this research.

2.1 GROUPS

Let G be a group and $a, b \in G$. The conjugate of a by b is $a^b = b^{-1}ab$ and the commutator of a and b is defined as $[a, b] = a^{-1}b^{-1}ab$. The subgroup of G generated by the set $\{[a, b] \mid a, b \in G\}$ is called the commutator subgroup of G and denoted by G'. If H and K are subgroups of a group G, then $[H, K]$ is the subgroup of G generated by the elements $[h, k]$ where $h \in H$, $k \in K$. In particular, the commutator subgroup of G can be written as $G' = [G, G]$. The left-normed commutator $[a_1, ..., a_n]$ is defined for $n > 2$ by setting $[a_1, ..., a_n] = [[a_1, ..., a_{n-1}], a_n]$. The lower central series of G is the chain of its fully invariant subgroups

$$G = \gamma_1(G) \geq \gamma_2(G) \geq .... \geq \gamma_i(G) \geq \gamma_{i+1}(G) \geq ...$$

where, for $i \geq 1$, $\gamma_{i+1}(G) = [\gamma_i(G), G]$. Let F be a free group of rank $n \geq 2$. The quotient group $F'/\gamma_{c+1}(F')$ is the free nilpotent (of class c) group of rank n. Let G be a group. The subgroups $G^{(j)}$ for $j = 0, 1, 2, ...$ are defined by induction. If we set $G^{(0)} = G$ and $G^{(j+1)} = \left[G^{(j)}, G^{(j)}\right]$ we get a descending chain of fully invariant subgroups in G of the form

$$G = G^{(0)} \supseteq G^{(1)} \supseteq G^{(2)} \supseteq ... \supseteq G^{(j)} \supseteq ...$$

where $G^{(1)} = G' = [G, G]$, $G^{(2)} = G'' = [G', G'] = [[G, G], [G, G]]$. It is called the derived series of G. The second member $G^{(2)}$ is called the second derived subgroup of G. Let F be a free group of rank $n \geq 2$. The free metabelian group of rank n is defined as F/F'' i.e. the quotient of F by its second derived subgroup. For an arbitrary abelian group

G, we let $t(G)$ denote the torsion subgroup of G. For a permutation σ in the symmetric group S_n, the sign of σ will be denoted by $\operatorname{sgn}(\sigma)$. Let A be a free abelian group and B a subgroup of A. The isolator of B in A is defined as

$$\mathrm{Is}_A B = \{x \in A \mid x^m \in B \text{ for some } m\}.$$

2.2 MODULES FOR GROUPS

2.2.1 G-Modules

Let G be a group (written multiplicatively), and let $\mathbb{Z}G$ be the integral group ring of G. A module over $\mathbb{Z}G$ will be referred to as a G-module. We will usually work with right G-modules. A G-module is $\mathbb{Z}$-free, if its underlying abelian group is a free $\mathbb{Z}$-module. The tensor product over $\mathbb{Z}$ will be denoted by $\otimes$ instead of $\otimes_{\mathbb{Z}}$. If B and C are right G-modules, the tensor product $B \otimes C$ can be endowed with a G-module structure by defining $(b \otimes c)g = bg \otimes cg$ for $b \in B$, $c \in C$ and $g \in G$. This type of action is called diagonal action. By forgetting the G-module structure of B, $B \otimes C$ becomes a G-module, using only the structure of C, by defining $(b \otimes c)g = b \otimes cg$ for $b \in B$, $c \in C$ and $g \in G$. This type of action is called single action. A G-module A is called trivial if $ag = a$ for all $a \in A$, $g \in G$. The ring of integers $\mathbb{Z}$ will always be regarded as a trivial G-module. By $\mathbb{Z}_n$ we denote the quotient $\mathbb{Z}/n\mathbb{Z}$, which is also regarded as a trivial G-module. In general, we may regard every abelian group as a trivial G-module. Now let A be a $\mathbb{Z}$-free G-module. It is well-known that if B is a free G-module then the tensor product $A \otimes B$ is G-free (see, eg., Chapter VI of [2]). This will be used without specific references being given.

2.2.2 Tensor, Exterior and Symmetric Powers

For a $\mathbb{Z}$-free right G-module A, we denote the n-th tensor power of A by

$$T^n A = \underbrace{A \otimes ... \otimes A}_{n},$$

$T^0 A = \mathbb{Z}$. The tensor power $T^n A$ will be regarded as a right G-module with diagonal action. Let $\mathfrak{a}_n$ be the submodule of $T^n A$ generated by all elements of type $a_1 \otimes ... \otimes a_n$ where $a_i = a_j$ for some $i \neq j$ and $a_1, ..., a_n \in A$. The n-th exterior power of A is defined by $\Lambda^n A = T^n A/\mathfrak{a}_n$. In $T^n A$, we let $\mathfrak{b}_n$ be the submodule generated by all elements of type $a_1 \otimes ... \otimes a_n - a_{\rho(1)} \otimes ... \otimes a_{\rho(n)}$ where ρ ranges over all permutations of $\{1, 2, ..., n\}$ and $a_1, ..., a_n \in A$. Then, the n-th symmetric power of A is defined by $A^n = T^n A/\mathfrak{b}_n$. The canonical images of $a_1 \otimes ... \otimes a_n$ under the natural projections $T^n A \to \Lambda^n A$ and $T^n A \to A^n$ are denoted by $a_1 \wedge ... \wedge a_n$ and $a_1 \circ ... \circ a_n$ respectively. Both $\Lambda^n A$ and A^n will be regarded as G-modules with diagonal action. For $n = 2$ we also use the notation $\Lambda^2 A = A \wedge A$ for the exterior square of A and $A^2 = A \circ A$ for the symmetric square of A. The exterior and symmetric squares of A are defined by

$$A \wedge A = A \otimes A/\left\langle \{a \otimes a \mid a \in A\} \right\rangle$$

and

$$A \circ A = A \otimes A/\left\langle \{a_1 \otimes a_2 - a_2 \otimes a_1 \mid a_1, a_2 \in A\} \right\rangle,$$

respectively. Thus, we have the relations $a \wedge a = 0$ and $a_1 \wedge a_2 = -a_2 \wedge a_1$ in $A \wedge A$, and the relation $a_1 \circ a_2 = a_2 \circ a_1$ in $A \circ A$ for all $a, a_1, a_2 \in A$.

2.2.3 The Relation and Augmentation Sequences

For the following material we refer to [2].

Let G be a (multiplicative) group. As usual $\mathbb{Z}G$ denotes the integral group ring of the group G. The map $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ defined by

$$\varepsilon\left(\sum_{g \in G} n_g g\right) = \sum_{g \in G} n_g$$

is called the augmentation map. This map is a ring homomorphism. The kernel of ε, denoted by IG, is called the augmentation ideal of $\mathbb{Z}G$. As an abelian group the augmentation ideal IG is free on the set $W = \{g - 1 \mid g \in G,\ g \neq 1\}$. The short exact sequence

$$0 \to IG \xrightarrow{\imath} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $\imath$ is the inclusion map, will be referred to as the augmentation sequence. If F is a free group with basis X, then the augmentation ideal IF is a free F-module on the set $\{x - 1 \mid x \in X\}$ (see, e.g., Theorem 5.5 of [2]).

Let G be a group given by a free presentation

$$1 \to R \to F \xrightarrow{\pi} G \to 1 \tag{2.1}$$

where F is a free group (with basis X, say), R is a normal subgroup of F and $G \cong F/R$. The free abelian group $R_{ab} = R/R'$ carries, by conjugation, the structure of an F-module. Since R operates trivially R_{ab} may be regarded as a right G-module by defining

$$rR'g = \left(x^{-1}rx\right) R'$$

where $r \in R$, $x \in F$, $g \in G$ with $g = \pi(x)$. The G-module R_{ab} is called the relation module of G associated with the free presentation (2.1). The free presentation (2.1) yields a short exact sequence of G-modules

$$0 \to R_{ab} \xrightarrow{\mu} P \xrightarrow{\sigma} IG \to 0 \tag{2.2}$$

where $P = IF \otimes_F \mathbb{Z}G$ is a free G-module on the set $\{(x-1) \otimes 1 \mid x \in X\}$ with respect to single action, the monomorphism μ is given by, for $r \in R$, $\mu\left(rR'\right) = (r-1) \otimes 1$ and the epimorphism σ is, on the free generators of P, given by $\sigma\left((x-1) \otimes 1\right) = \pi(x) - 1$ (see, e.g., Theorem 6.3 of [2]). The exact sequence (2.2) is usually called the relation sequence stemming from (2.1), and the injection μ is usually called the Magnus embedding.

For our applications we consider the following special case. We put $G = F/F'$. This means that G is a free abelian group of rank n with free generators $b_1, ..., b_n$ where $b_i = x_i F'$ $(1 \leq i \leq n)$. In this case the relation module is $F'_{ab} = F'/F''$ which we will denote by M. The relation module $M = F'/F''$ is (as a G-module) generated by the elements $[x_i, x_j] F''$ $(1 \leq i < j \leq n)$. In what follows we will abuse notation by suppressing the F'' when working in the relation module (unless otherwise stated). For example $[x_i, x_j] F''$ will be written as $[x_i, x_j]$. P is a free G-module of rank n with free generators $e_i = (x_i - 1) \otimes 1$ $(1 \leq i \leq n)$. Let us now write the Magnus embedding and the epimorphism σ in a form convenient for our calculations in Chapter 4. One easily calculates that

$$\mu\left([x_i, x_j]\right) = e_i\left(b_j - 1\right) - e_j\left(b_i - 1\right).$$

The epimorphism σ is, on the free generators of P, given by $\sigma\left(e_i\right) = b_i - 1$. As an immediate application of the Magnus embedding $\mu : M \to P$, one has

$$\mu\left([x_1, x_2]\left(b_3 - 1\right) + [x_2, x_3]\left(b_1 - 1\right) + [x_3, x_1]\left(b_2 - 1\right)\right) = 0.$$

Hence, in M, we have

$$[x_1, x_2]\,(b_3 - 1) + [x_2, x_3]\,(b_1 - 1) + [x_3, x_1]\,(b_2 - 1) = 0,$$

or

$$[x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2] = 0$$

as μ is injective. We will refer to this result as the Jacobi identity.

Let G be the free abelian group given by the free presentation (1.5). The commutator subgroup F' of F is free by Schreier's theorem. We consider the lower central quotient $\gamma_n(F')/\gamma_{n+1}(F')$ of the free group F' which will be written additively. This quotient (a free abelian group) is generated by the left-normed commutators of weight i

$$[a_1, a_2, ..., a_i]\,\gamma_{i+1}(F')$$

where $a_1, a_2, ..., a_i \in F'$. It carries, by conjugation, the structure of an F-module. Since F' operates trivially, it may be regarded as a right G-module by defining

$$[a_1, a_2, ..., a_i]\,\gamma_{i+1}(F')b_i = [a_1, a_2, ..., a_i]^{x_i}\,\gamma_{i+1}(F')$$

where $a_1, a_2, ..., a_i \in F'$, $x_i \in F$, $b_i = x_iF' \in G$. For future reference, we record that this action yields

$$[a_1, a_2, ..., a_i]\,\gamma_{i+1}(F')\,(b_i - 1) = [a_1, a_2, ..., a_i, x_i]\,\gamma_{i+1}(F'). \tag{2.3}$$

Indeed, bearing in mind that the quotient $\gamma_n(F')/\gamma_{n+1}(F')$ is written additively, we have

$$\begin{aligned}
[a_1, a_2, ..., a_i]\,\gamma_{i+1}(F')\,(b_i - 1) &= [a_1, a_2, ..., a_i]^{x_i}\,\gamma_{i+1}(F') - [a_1, a_2, ..., a_i]\,\gamma_{i+1}(F') \\
&= [a_1, a_2, ..., a_i]^{x_i}\,[a_1, a_2, ..., a_i]^{-1}\,\gamma_{i+1}(F') \\
&= x_i^{-1}\,[a_1, a_2, ..., a_i]\,x_i\,[a_1, a_2, ..., a_i]^{-1}\,\gamma_{i+1}(F') \\
&= [a_1, a_2, ..., a_i]^{-1}\,x_i^{-1}\,[a_1, a_2, ..., a_i]\,x_i\gamma_{i+1}(F') \\
&= [a_1, a_2, ..., a_i, x_i]\,\gamma_{i+1}(F').
\end{aligned}$$

The quotient $F'/\,[F'', F']$ is a free nilpotent group of class 2 and, consequently, there is an isomorphism of G-modules

$$F''/\,[F'', F'] \cong F'/F'' \wedge F'/F'' = M \wedge M, \tag{2.4}$$

the G action being induced by conjugation in F. Explicitly, the isomorphism (2.4) acts as follows. For a_1, $a_2 \in F'$,

$$[a_1, a_2]\,[F'', F'] \leftrightarrow a_1 F'' \wedge a_2 F''.$$

In Chapter 4 we will examine $M \wedge M$, and in Section 5 we will use the isomorphism (2.4) to restate our findings in the language of group theory. In view of the canonical isomorphism $F''/\,[F'', F'] \otimes_G \mathbb{Z} \cong F''/\,[F'', F]$, trivializing the G-action (i.e. tensoring with the trivial G-module $\mathbb{Z}$) on both sides in (2.4) yields an isomorphism

$$F''/\,[F'', F] \cong M \wedge M \otimes_G \mathbb{Z}. \tag{2.5}$$

Explicitly, the isomorphism (2.5) acts as follows. For a_1, $a_2 \in F'$,

$$[a_1, a_2]\,[F'', F] \leftrightarrow a_1 F'' \wedge a_2 F'' \otimes 1.$$

In Chapter 4 we will obtain a generating set for the torsion subgroup of $M \wedge M \otimes_G \mathbb{Z}$, and in Section 5 we will use the isomorphism (2.5) to prove Theorem 5.1.

We make some conventions for our applications in Chapter 4. Let F be a free group of rank $n \geq 2$ on a free generating set $X = \{x_1, ..., x_n\}$. The symmetric group S_n acts naturally as a group of automorphisms on the free group F by permuting the free generators. We will write this action on the right. The symmetric group S_n acts also on the free abelian group $G = F/F'$, the regular module $\mathbb{Z}G$, the augmentation ideal IG, the relation module $M = F'/F''$, the free G-module $P = IF \otimes_F \mathbb{Z}G$, and the tensor products involving G, $\mathbb{Z}G$, IG, M, P and their tensor, exterior and symmetric powers as a group of automorphisms. The S_n-actions are all induced by the S_n-action on the free group F.

2.3 HOMOLOGY OF FREE ABELIAN GROUPS

The homology of free abelian groups is well understood (see, e.g., Chapter VI of [5]). We recall the construction of a well-known free resolution of the trivial module $\mathbb{Z}$ for a free abelian group G, the so-called Koszul complex, and show how it is used to determine the k-th homology group of G with coefficients in a trivial G-module B denoted by $H_k(G, B)$. This free resolution will play a crucial part in our calculations in Chapter 4.

Let G be a free abelian group of rank n with free generators $b_1, ..., b_n$. Let P be a free G-module with free generators $e_1, ..., e_n$, and let $\Lambda^k P$ denote the k-th exterior power over

$\mathbb{Z}G$ of the free G-module P. For $k > 1$, $\Lambda^k P$ is a free G-module of rank $\binom{n}{k}$ with free generators $e_{i_1} \wedge ... \wedge e_{i_k}$ $(1 \leq i_1 < ... < i_k \leq n)$. In particular, $\Lambda^n P$ is a free G-module of rank 1 with basis $\{e_1 \wedge ... \wedge e_n\}$. We extend the definition of $\Lambda^k P$ to the cases $k = 0$ and $k = 1$ by setting $\Lambda^0 P = \mathbb{Z}G$ and $\Lambda^1 P = P$. Define differentials $d_k : \Lambda^k P \to \Lambda^{k-1} P$ by setting

$$d_k \left(e_{i_1} \wedge ... \wedge e_{i_k}\right) = \sum_{j=1}^{k} (-1)^{j-1} \left(b_{i_j} - 1\right) \left(e_{i_1} \wedge ... \wedge \widehat{e}_{i_j} \wedge ... \wedge e_{i_k}\right)$$

and $d_1(e_j) = b_j - 1$, (here $\widehat{e}_{i_j}$ indicates the omission of e_{i_j}). To simplify notation we put $\Lambda^k P = P_k$ for $k = 1, 2, ..., m$. Then

$$\mathcal{P} : 0 \to P_m \xrightarrow{d_m} P_{m-1} \xrightarrow{d_{m-1}} ... \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ is the augmentation map, is a free resolution (the Koszul complex) of the trivial G-module $\mathbb{Z}$. For a proof of this fact we refer to [5]. Let B be a trivial G-module. Then, by definition $H_k(G, B) = H_k(G, B \otimes_G \mathcal{P})$. By the definition of the differentials d_k one has $d_k(P_k) \subseteq IG \cdot P_{k-1}$. This implies that the induced differentials $1 \otimes d_k$ on $B \otimes_G \mathcal{P}$ are all zero maps. Hence $H_k(G, B) = B \otimes_G P_k$. Since P_k is a free G-module of rank $\binom{n}{k}$, $H_k(G, B)$ is a direct sum of $\binom{n}{k}$ isomorphic copies of the trivial G-module B (we follow the convention that if $n < k$ then $\binom{n}{k} = 0$ and hence $H_k(G, B) = 0$). In particular, for $B = \mathbb{Z}$, the integral homology group $H_k(G, \mathbb{Z}) = H_k(G)$ is a free abelian group of rank $\binom{n}{k}$, for $B = \mathbb{Z}_n$, the homology group $H_k(G, \mathbb{Z}_n)$ is a direct sum of $\binom{n}{k}$ isomorphic copies of $\mathbb{Z}_n$, and for $B = \mathbb{Z}_p$ where p is a prime, $H_k(G, \mathbb{Z}_p)$ is an elementary abelian p-group of rank $\binom{n}{k}$.

CHAPTER 3

DESCRIPTION OF THE TORSION SUBGROUP OF $F/[F'', F]$ IN HOMOLOGICAL TERMS

In this chapter, we record one of the main ingredients of the proof of the isomorphism

$$t\left(M \wedge M \otimes_G \mathbb{Z}\right) \cong H_4(G, \mathbb{Z}_2), \tag{3.1}$$

as given in [10], in a form that will be convenient for computing the image of $H_4(G, \mathbb{Z}_2)$ in $M \wedge M \otimes_G \mathbb{Z}$ in Chapter 4.

This main ingredient is the 6-term sequence of G-modules

$$\mathcal{J} : 0 \longrightarrow M \wedge M \xrightarrow{\alpha_4} M \otimes P \xrightarrow{\alpha_3} P \circ P \xrightarrow{\alpha_2} \mathbb{Z}G \circ \mathbb{Z}G \xrightarrow{\alpha_1} \mathbb{Z}G \xrightarrow{\widetilde{\varepsilon}} \mathbb{Z}_2 \longrightarrow 0.$$

The differentials in $\mathcal{J}$ are as follows.

$\alpha_1 : \mathbb{Z}G \circ \mathbb{Z}G \to \mathbb{Z}G$ is, for $a_1, a_2 \in \mathbb{Z}G$, defined by

$$\alpha_1\left(a_1 \circ a_2\right) = \varepsilon\left(a_1\right) a_2 + \varepsilon\left(a_2\right) a_1$$

where $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ is the augmentation map from the augmentation sequence.

$\alpha_2 : P \circ P \to \mathbb{Z}G \circ \mathbb{Z}G$ is, for $p_1, p_2 \in P$, defined by

$$\alpha_2\left(p_1 \circ p_2\right) = \sigma(p_1) \circ \sigma(p_2)$$

where σ is the map $P \to IG$ from the relation sequence.

$\alpha_3 : M \otimes P \to P \circ P$ is, for $m \in M$, $p \in P$, defined by

$$\alpha_3\left(m \otimes p\right) = \mu(m) \circ p$$

where $\mu : M \to P$ is the Magnus embedding from the relation sequence.

$\alpha_4 : M \wedge M \to M \otimes P$ is, for $m_1, m_2 \in M$, defined by

$$\alpha_4\left(m_1 \wedge m_2\right) = m_1 \otimes \mu(m_2) - m_2 \otimes \mu(m_1)$$

where, again, μ is is the Magnus embedding from the relation sequence.

$\widetilde{\varepsilon} : \mathbb{Z}G \to \mathbb{Z}_2$ is, for $a \in \mathbb{Z}G$, defined by

$\widetilde{\varepsilon}\,(a) = \varepsilon\,(a) + 2\mathbb{Z}$

where, again, ε is the augmentation map from the augmentation sequence.
By Lemma 3.1 of [10], $\mathcal{J}$ is exact. Now $M \otimes P$ is G-free, since M is $\mathbb{Z}$-free and P is G-free. Moreover, by Corollary 3.13 of [9], $P \circ P$ and $\mathbb{Z}G \circ \mathbb{Z}G$ are also G-free. Obviously $\mathbb{Z}G$ is G-free. Hence, the four middle terms $M \otimes P$, $P \circ P$, $\mathbb{Z}G \circ \mathbb{Z}G$ and $\mathbb{Z}G$ in $\mathcal{J}$ are homologically trivial. Consequently, the exact sequence $\mathcal{J}$ yields (by dimension shifting) the isomorphism

$H_4(G, \mathbb{Z}_2) \cong \ker\left(M \wedge M \otimes_G \mathbb{Z} \to M \otimes P \otimes_G \mathbb{Z}\right).$

Since $M \otimes P \otimes_G \mathbb{Z}$ is a free abelian group, the torsion subgroup $t\,(M \wedge M \otimes_G \mathbb{Z})$ must be contained in that kernel and this proves (3.1).

CHAPTER 4

THE IMAGE OF $H_4(G, \mathbb{Z}_2)$ IN $M \wedge M \otimes_G \mathbb{Z}$

In this chapter, in order to identify the image of $H_4(G, \mathbb{Z}_2)$ in $M \wedge M \otimes_G \mathbb{Z}$, we work out the connecting homomorphism $H_4(G, \mathbb{Z}_2) \to H_0(G, M \wedge M)$ stemming from the exact sequence $\mathcal{J}$ which is introduced in Chapter 3. We use the double complex $\mathcal{J} \otimes_G \mathcal{P}$, where $\mathcal{P}$ is the Koszul complex which is introduced in Chapter 2. In our calculations in this chapter, we will make use of the following.

Lemma 4.1 *Let A and B be right G-modules. For the module $A \otimes B$ (diagonal action) we have the following relation. Let $a \in A$, $b \in B$ and $g \in G$, then*

$$(a \otimes b)(g-1) = a(g-1) \otimes b(g-1) + a(g-1) \otimes b + a \otimes b(g-1).$$

Proof. By the definition of the diagonal action, one has

$$(a \otimes b)(g-1) = ag \otimes bg - a \otimes b. \tag{4.1}$$

But $ag = a(g-1) + a$ and $bg = b(g-1) + b$. Hence

$$ag \otimes bg = a(g-1) \otimes b(g-1) + a(g-1) \otimes b + a \otimes b(g-1) + a \otimes b. \tag{4.2}$$

The lemma follows by substituting (4.2) into (4.1). ■

4.1 THE CONNECTING HOMOMORPHISM $H_4(G, \mathbb{Z}_2) \to H_0(G, M \wedge M)$

The group $H_4(G, \mathbb{Z}_2)$ is (as a $\mathbb{Z}_2$-module) freely generated by the cycles

$$1 \otimes e_i \wedge e_j \wedge e_k \wedge e_l \in \mathbb{Z}_2 \otimes_G P_4$$

with $1 \leq i < j < k < l \leq n$.
For simplicity, we consider the cycle

$$1 \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4. \tag{4.3}$$

We start with (4.3), and then we alternate with taking inverse images and applying differentials as indicated in the following staircase diagram.

$$\begin{tikzcd}
 & & & & & \mathbb{Z}_2 \otimes_G P_4 \\
 & & & & \mathbb{Z}G \otimes_G P_3 & \mathbb{Z}G \otimes_G P_4 \arrow[l] \arrow[u] \\
 & & & \mathbb{Z}G \circ \mathbb{Z}G \otimes_G P_2 & \mathbb{Z}G \circ \mathbb{Z}G \otimes_G P_3 \arrow[l] \arrow[u] & \\
 & & P \circ P \otimes_G P_1 & P \circ P \otimes_G P_2 \arrow[l] \arrow[u] & & \\
 & M \otimes P \otimes_G \mathbb{Z}G & M \otimes P \otimes_G P_1 \arrow[l] \arrow[u] & & & \\
M \wedge M \otimes_G \mathbb{Z} & M \wedge M \otimes_G \mathbb{Z}G \arrow[l] \arrow[u] & & & &
\end{tikzcd}$$

An inverse image of (4.3) in $\mathbb{Z}G \otimes_G P_4$ is

$$1 \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4. \tag{4.4}$$

By applying the differential $\mathbb{Z}G \otimes_G P_4 \to \mathbb{Z}G \otimes_G P_3$ to (4.4), we obtain

$$\sum_{m=1}^{4} (-1)^{m-1} (b_m - 1) \otimes e_1 \wedge ... \wedge \widehat{e}_m \wedge ... \wedge e_4. \tag{4.5}$$

We simplify notation by setting $B_m = b_m - 1$.

In order to get an inverse image of (4.5) in $\mathbb{Z}G \circ \mathbb{Z}G \otimes_G P_3$, we consider the following element in $\mathbb{Z}G \circ \mathbb{Z}G$

$$(1 \circ 1)(b_m - 1).$$

Since the module $\mathbb{Z}G \circ \mathbb{Z}G$ is a homomorphic image of the module $\mathbb{Z}G \otimes \mathbb{Z}G$, Lemma 4.1 with $A = B = \mathbb{Z}G$ yields the following

$$\begin{aligned} (1 \circ 1)(b_m - 1) &= B_m \circ B_m + B_m \circ 1 + 1 \circ B_m \\ &= B_m \circ B_m + 2(B_m \circ 1) \\ &\equiv B_m \circ B_m \pmod 2. \end{aligned}$$

Hence

$$(1 \circ 1)(b_m - 1) + B_m \circ B_m \in 2(\mathbb{Z}G \circ \mathbb{Z}G)$$

and

$$\frac{1}{2}\sum_{m=1}^{4} (-1)^{m-1} ((1 \circ 1)(b_m - 1) + B_m \circ B_m) \otimes e_1 \wedge ... \wedge \widehat{e}_m \wedge ... \wedge e_4. \tag{4.6}$$

is an inverse image of (4.5) in $\mathbb{Z}G \circ \mathbb{Z}G \otimes_G P_3$. This is because we have $\alpha_1(1 \circ 1) = 2$ and $B_m \circ B_m \in \ker \alpha_1$.

By applying the differential $\mathbb{Z}G \circ \mathbb{Z}G \otimes_G P_3 \to \mathbb{Z}G \circ \mathbb{Z}G \otimes_G P_2$ to (4.6), we obtain

$$\frac{1}{2}\sum_{\rho} \operatorname{sgn}(\rho)(((B_1 \circ B_1)(b_2 - 1) - (B_2 \circ B_2)(b_1 - 1)) \otimes e_3 \wedge e_4)\rho \tag{4.7}$$

where ρ ranges over all permutations of $\{1, 2, 3, 4\}$ with $\rho(1) < \rho(2)$, $\rho(3) < \rho(4)$. This is because the term

$$\sum_{m=1}^{4} (-1)^{m-1} (1 \circ 1)(b_m - 1) \otimes e_1 \wedge ... \wedge \widehat{e}_m \wedge ... \wedge e_4$$

from (4.6) is the image of

$1 \circ 1 \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in \mathbb{Z}G \circ \mathbb{Z}G \otimes_G P_4$

under $1 \otimes d_4$ and hence it is in the kernel of $1 \otimes d_3$.
In order to get an inverse image of (4.7) in $P \circ P \otimes_G P_2$, we consider the following element in $P \circ P$

$$(e_1 \circ e_1)(b_2 - 1) - (e_2 \circ e_2)(b_1 - 1). \tag{4.8}$$

Since the module $P \circ P$ is a homomorphic image of the module $P \otimes P$, Lemma 4.1 with $A = B = P$ yields the following.
We have

$$\begin{aligned}(e_1 \circ e_1)(b_2 - 1) &= e_1B_2 \circ e_1B_2 + e_1B_2 \circ e_1 + e_1 \circ e_1B_2 \\ &= e_1B_2 \circ e_1B_2 + 2(e_1B_2 \circ e_1) \\ &\equiv e_1B_2 \circ e_1B_2 \pmod 2,\end{aligned}$$

and similarly

$$(e_2 \circ e_2)(b_1 - 1) \equiv e_2B_1 \circ e_2B_1 \pmod 2.$$

Hence

$$(e_1 \circ e_1)(b_2 - 1) - (e_2 \circ e_2)(b_1 - 1) \equiv e_1B_2 \circ e_1B_2 - e_2B_1 \circ e_2B_1 \pmod 2.$$

Furthermore, we have

$$\begin{aligned}(e_1B_2 - e_2B_1) \circ (e_1B_2 - e_2B_1) &= e_1B_2 \circ e_1B_2 + e_2B_1 \circ e_2B_1 - 2e_1B_2 \circ e_2B_1 \\ &\equiv e_1B_2 \circ e_1B_2 - e_2B_1 \circ e_2B_1 \pmod 2.\end{aligned}$$

Recall that $\mu([x_1, x_2]) = e_1B_2 - e_2B_1$ where μ is the Magnus embedding and hence we can identify $e_1B_2 - e_2B_1$ with $[x_1, x_2]$. Then, we have

$$(e_1 \circ e_1)(b_2 - 1) - (e_2 \circ e_2)(b_1 - 1) \equiv [x_1, x_2] \circ [x_1, x_2] \pmod 2.$$

Hence

$$(e_1 \circ e_1)(b_2 - 1) - (e_2 \circ e_2)(b_1 - 1) + [x_1, x_2] \circ [x_1, x_2] \in 2(P \circ P)$$

and

$$\frac{1}{2}\sum_{\rho}\operatorname{sgn}(\rho)\left(\left((e_1\circ e_1)(b_2-1)-(e_2\circ e_2)(b_1-1)+[x_1,x_2]\circ[x_1,x_2]\right)\otimes e_3\wedge e_4\right)\rho. \tag{4.9}$$

is an inverse image of (4.7) in $P\circ P\otimes_G P_2$. This is because we have $\alpha_2(e_1\circ e_1)=B_1\circ B_1$, $\alpha_2(e_2\circ e_2)=B_2\circ B_2$ and $[x_1,x_2]\circ[x_1,x_2]\in\ker\alpha_2$.

By applying the differential $P\circ P\otimes_G P_2\to P\circ P\otimes_G P_1$ to (4.9), we obtain

$$\frac{1}{2}\sum_{\lambda}\operatorname{sgn}(\lambda)\left(\left([x_1,x_2]^2(b_3-1)-[x_1,x_3]^2(b_2-1)+[x_2,x_3]^2(b_1-1)\right)\otimes e_4\right)\lambda \tag{4.10}$$

where λ ranges over all permutations of $\{1,2,3,4\}$ with $\lambda(1)<\lambda(2)<\lambda(3)$ and

$$[x_1,x_2]^2=[x_1,x_2]\circ[x_1,x_2]\,,\ [x_1,x_3]^2=[x_1,x_3]\circ[x_1,x_3]\,,\ [x_2,x_3]^2=[x_2,x_3]\circ[x_2,x_3]\,.$$

This is because the term

$$\sum_{\rho}\operatorname{sgn}(\rho)\left(\left((e_1\circ e_1)(b_2-1)-(e_2\circ e_2)(b_1-1)\right)\otimes e_3\wedge e_4\right)\rho$$

from (4.9) is the image of

$$\sum_{m=1}^{4}(-1)^{m-1}(e_m\circ e_m)\otimes e_1\wedge\ldots\wedge\widehat{e}_m\wedge\ldots\wedge e_4\in P\circ P\otimes_G P_3$$

under $1\otimes d_3$ and hence it is in the kernel of $1\otimes d_2$.

In order to get an inverse image of (4.10) in $M\otimes P\otimes_G P_1$, we consider the following element in $M\otimes P$

$$([x_1,x_2]\otimes[x_1,x_2])(b_3-1)-([x_1,x_3]\otimes[x_1,x_3])(b_2-1)+([x_2,x_3]\otimes[x_2,x_3])(b_1-1)$$

i.e.

$$\sum_{\theta}\operatorname{sgn}(\theta)([x_1,x_2]\otimes[x_1,x_2])(b_3-1)\theta \tag{4.11}$$

where θ ranges over all permutations of $\{1,2,3\}$ with $\theta(1)<\theta(2)$.

Using (2.3) and Lemma 4.1 with $A=M$ and $B=P$, we can decompose (4.11) into a sum

$$S_1+S_2$$

where

$$S_1=\sum_{\theta}\operatorname{sgn}(\theta)([x_1,x_2,x_3]\otimes[x_1,x_2,x_3])\theta$$

and

$$S_2 = \sum_{\theta} \mathrm{sgn}\,(\theta)\,([x_1,x_2,x_3] \otimes [x_1,x_2] + [x_1,x_2] \otimes [x_1,x_2,x_3])\,\theta.$$

Lemma 4.2 *We have*

$$S_1 \equiv \alpha_4\,([x_1,x_2,x_3] \wedge [x_2,x_3,x_1]) \;\; \textit{(mod 2)}$$

and

$$S_2 \equiv \alpha_4 \left(\sum_{\theta} sgn\,(\theta)\,([x_1,x_2,x_3] \wedge [x_1,x_2])\,\theta \right) \;\; \textit{(mod 2)}$$

where θ ranges over all permutations of $\{1,2,3\}$ with $\theta\,(1) < \theta\,(2)$.

Proof. By the definition of the permutation θ, we have

$$\begin{aligned} S_1 &= \sum_{\theta} \mathrm{sgn}\,(\theta)\,([x_1,x_2,x_3] \otimes [x_1,x_2,x_3])\,\theta \\ &= ([x_1,x_2,x_3] \otimes [x_1,x_2,x_3])\,((1) - (2\ 3) + (1\ 2\ 3)) \\ &= [x_1,x_2,x_3] \otimes [x_1,x_2,x_3] - [x_1,x_3,x_2] \otimes [x_1,x_3,x_2] + [x_2,x_3,x_1] \otimes [x_2,x_3,x_1]\,. \end{aligned}$$

Since $[x_1,x_3,x_2] = [x_1,x_2,x_3] + [x_2,x_3,x_1]$ by the Jacobi identity, we get

$$\begin{aligned} S_1 &= -\,[x_1,x_2,x_3] \otimes [x_2,x_3,x_1] - [x_2,x_3,x_1] \otimes [x_1,x_2,x_3] \\ &\equiv [x_1,x_2,x_3] \otimes [x_2,x_3,x_1] - [x_2,x_3,x_1] \otimes [x_1,x_2,x_3] \\ &\equiv \alpha_4\,([x_1,x_2,x_3] \wedge [x_2,x_3,x_1]) \;\; (\mathrm{mod}\ 2). \end{aligned}$$

We immediately see that

$$\begin{aligned} S_2 &= \sum_{\theta} \mathrm{sgn}\,(\theta)\,([x_1,x_2,x_3] \otimes [x_1,x_2] + [x_1,x_2] \otimes [x_1,x_2,x_3])\,\theta \\ &\equiv \sum_{\theta} \mathrm{sgn}\,(\theta)\,([x_1,x_2,x_3] \otimes [x_1,x_2] - [x_1,x_2] \otimes [x_1,x_2,x_3])\,\theta \\ &\equiv \alpha_4 \left(\sum_{\theta} \mathrm{sgn}\,(\theta)\,([x_1,x_2,x_3] \wedge [x_1,x_2])\,\theta \right) \;\; (\mathrm{mod}\ 2). \end{aligned}$$

■

For convenience, we write, for θ as in (4.11),

$$S\,(1,2,3) = [x_1,x_2,x_3] \wedge [x_2,x_3,x_1] + \sum_{\theta} \mathrm{sgn}\,(\theta)\,([x_1,x_2,x_3] \wedge [x_1,x_2])\,\theta.$$

By Lemma 4.2, we see that

$$\sum_{\theta}\operatorname{sgn}(\theta)\left(\left[x_1,x_2\right]\otimes\left[x_1,x_2\right]\right)\left(b_3-1\right)\theta\equiv\alpha_4\left(S\left(1,2,3\right)\right)\pmod 2$$

and hence

$$\sum_{\theta}\operatorname{sgn}(\theta)\left(\left[x_1,x_2\right]\otimes\left[x_1,x_2\right]\right)\left(b_3-1\right)\theta+\alpha_4\left(S\left(1,2,3\right)\right)\in 2\left(M\otimes P\right).$$

Since $\alpha_4\left(S\left(1,2,3\right)\right)\in\ker\alpha_3$, we see that, for λ as in (4.10) and for θ as in (4.11),

$$\frac{1}{2}\sum_{\lambda}\operatorname{sgn}(\lambda)\left(\left(\sum_{\theta}\operatorname{sgn}(\theta)\left(\left[x_1,x_2\right]\otimes\left[x_1,x_2\right]\right)\left(b_3-1\right)\theta+\alpha_4\left(S\left(1,2,3\right)\right)\right)\otimes e_4\right)\lambda\quad(4.12)$$

is an inverse image of (4.10) in $M\otimes P\otimes_G P_1$.

By applying the differential $M\otimes P\otimes_G P_1\to M\otimes P\otimes_G\mathbb{Z}G$ to (4.12), we obtain

$$\frac{1}{2}\sum_{\lambda}\operatorname{sgn}(\lambda)\,\alpha_4\left(S\left(1,2,3\right)\right)\left(b_4-1\right)\lambda\qquad(4.13)$$

where λ is as in (4.10). Here we identify $M\otimes P\otimes_G\mathbb{Z}G$ with $M\otimes P$. This is because the term

$$\sum_{\lambda}\operatorname{sgn}(\lambda)\left(\sum_{\theta}\operatorname{sgn}(\theta)\left(\left[x_1,x_2\right]\otimes\left[x_1,x_2\right]\right)\left(b_3-1\right)\theta\otimes e_4\right)\lambda$$

from (4.12) is the image of

$$\sum_{\rho}\operatorname{sgn}(\rho)\left(\left[x_1,x_2\right]\otimes\left[x_1,x_2\right]\otimes e_3\wedge e_4\right)\rho\in M\otimes P\otimes_G P_2$$

under $1\otimes d_2$ and hence it is in the kernel of $1\otimes d_1$.

Now, we can take the following as the inverse image of (4.13) in $M\wedge M$

$$\frac{1}{2}\sum_{\lambda}\operatorname{sgn}(\lambda)\,S\left(1,2,3\right)\left(b_4-1\right)\lambda$$

where λ is as in (4.10). Here we identify $M\wedge M\otimes_G\mathbb{Z}G$ with $M\wedge M$.

Let

$$y=y\left(x_1,x_2,x_3,x_4\right)=\sum_{\lambda}\operatorname{sgn}(\lambda)\,S\left(1,2,3\right)\left(b_4-1\right)\lambda.$$

By the definitions of the permutations λ and θ, one has

$$\begin{aligned}y\left(x_1,x_2,x_3,x_4\right) &= \sum_{\lambda}\operatorname{sgn}(\lambda)\left(\left[x_1,x_2,x_3\right]\wedge\left[x_2,x_3,x_1\right]\right)\left(b_4-1\right)\lambda\\ &\quad+\sum_{\tau}\operatorname{sgn}(\tau)\left(\left[x_1,x_2,x_3\right]\wedge\left[x_1,x_2\right]\right)\left(b_4-1\right)\tau\end{aligned}$$

where λ and τ range over all permutations of $\{1,2,3,4\}$ with $\lambda(1) < \lambda(2) < \lambda(3)$ and $\tau(1) < \tau(2)$.

Now our aim is to write y as a linear combination of terms that have a coefficient of 2. The module $M \wedge M$ is a homomorphic image of the module $M \otimes M$. Thus using (2.3) and Lemma 4.1 with $A = B = M$, we can decompose $y(x_1, x_2, x_3, x_4)$ into a sum

$$T_1 + T_2 + T_3 + T_4$$

where

$$\begin{aligned}
T_1 &= \sum_{\lambda} \operatorname{sgn}(\lambda)\left([x_1,x_2,x_3,x_4] \wedge [x_2,x_3,x_1,x_4]\right)\lambda, \\
T_2 &= \sum_{\tau} \operatorname{sgn}(\tau)\left([x_1,x_2,x_3,x_4] \wedge [x_1,x_2]\right)\tau, \\
T_3 &= \sum_{\lambda} \operatorname{sgn}(\lambda)\left([x_1,x_2,x_3,x_4] \wedge [x_2,x_3,x_1]\right)\lambda \\
&\quad + \sum_{\lambda} \operatorname{sgn}(\lambda)\left([x_1,x_2,x_3] \wedge [x_2,x_3,x_1,x_4]\right)\lambda \\
&\quad + \sum_{\tau} \operatorname{sgn}(\tau)\left([x_1,x_2,x_3,x_4] \wedge [x_1,x_2,x_4]\right)\tau, \\
T_4 &= \sum_{\tau} \operatorname{sgn}(\tau)\left([x_1,x_2,x_3] \wedge [x_1,x_2,x_4]\right)\tau.
\end{aligned}$$

Lemma 4.3 *We have* $T_1 = 0$, $T_2 = 0$,

$$T_3 = 2\sum_{\lambda} sgn(\lambda)\left([x_1,x_2,x_3,x_4] \wedge [x_2,x_3,x_1]\right)\lambda$$

and

$$T_4 = 2\sum_{\rho} sgn(\rho)\left([x_1,x_2,x_3] \wedge [x_1,x_2,x_4]\right)\rho$$

where λ *and* ρ *range over all permutations of* $\{1,2,3,4\}$ *with* $\lambda(1) < \lambda(2) < \lambda(3)$ *and* $\rho(1) < \rho(2)$, $\rho(3) < \rho(4)$.

Proof. By the definition of the permutation λ, one has

$$\begin{aligned}
T_1 &= \sum_{\lambda} \operatorname{sgn}(\lambda)\left([x_1,x_2,x_3,x_4] \wedge [x_2,x_3,x_1,x_4]\right)\lambda \\
&= \left([x_1,x_2,x_3,x_4] \wedge [x_2,x_3,x_1,x_4]\right)\left((1) - (3\ 4) + (2\ 3\ 4) - (1\ 2\ 3\ 4)\right) \\
&= [x_1,x_2,x_3,x_4] \wedge [x_2,x_3,x_1,x_4] - [x_1,x_2,x_4,x_3] \wedge [x_2,x_4,x_1,x_3] \\
&\quad + [x_1,x_3,x_4,x_2] \wedge [x_3,x_4,x_1,x_2] - [x_2,x_3,x_4,x_1] \wedge [x_3,x_4,x_2,x_1] \\
&= [x_1,x_2,x_3,x_4] \wedge [x_4,x_3,x_1,x_2] + [x_1,x_2,x_4,x_3] \wedge [x_3,x_4,x_1,x_2] \\
&= 0.
\end{aligned}$$

By the definition of the permutation τ, one has, for ρ ranging over all permutations of $\{1,2,3,4\}$ with $\rho(1)<\rho(2)$, $\rho(3)<\rho(4)$,

$$\begin{aligned} T_2 &= \sum_{\tau}\operatorname{sgn}(\tau)\left([x_1,x_2,x_3,x_4]\wedge[x_1,x_2]\right)\tau \\ &= \sum_{\rho}\operatorname{sgn}(\rho)\left([x_1,x_2,x_3,x_4]\wedge[x_1,x_2]-[x_1,x_2,x_4,x_3]\wedge[x_1,x_2]\right)\rho \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} T_4 &= \sum_{\tau}\operatorname{sgn}(\tau)\left([x_1,x_2,x_3]\wedge[x_1,x_2,x_4]\right)\tau \\ &= \sum_{\rho}\operatorname{sgn}(\rho)\left([x_1,x_2,x_3]\wedge[x_1,x_2,x_4]-[x_1,x_2,x_4]\wedge[x_1,x_2,x_3]\right)\rho \\ &= 2\sum_{\rho}\operatorname{sgn}(\rho)\left([x_1,x_2,x_3]\wedge[x_1,x_2,x_4]\right)\rho. \end{aligned}$$

Consider the following term in T_3

$$\sum_{\tau}\operatorname{sgn}(\tau)\left([x_1,x_2,x_3,x_4]\wedge[x_1,x_2,x_4]\right)\tau.$$

By the definition of the permutation τ, we have

$$\begin{aligned} \sum_{\tau}\operatorname{sgn}(\tau)\left([x_1,x_2,x_3,x_4]\wedge[x_1,x_2,x_4]\right)\tau &= -\sum_{\tau}\operatorname{sgn}(\tau)\left([x_1,x_2,x_3,x_4]\wedge[x_1,x_2,x_4]\right)(3\ 4)\,\tau \\ &= \sum_{\tau}\operatorname{sgn}(\tau)\left([x_1,x_2,x_3]\wedge[x_1,x_2,x_4,x_3]\right)\tau. \end{aligned}$$

On the other hand, by the definitions of the permutations λ and τ, it is not hard to see that the term

$$\sum_{\tau}\operatorname{sgn}(\tau)\left([x_1,x_2,x_3]\wedge[x_1,x_2,x_3,x_4]\right)\tau$$

can be written as

$$\sum_{\lambda}\operatorname{sgn}(\lambda)\left([x_1,x_2,x_3]\wedge[x_1,x_2,x_3,x_4]\right)\left((1)-(2\ 3)+(1\ 2\ 3)\right)\lambda.$$

Hence, we get

$$\begin{aligned} \sum_{\tau}\operatorname{sgn}(\tau)\left([x_1,x_2,x_3]\wedge[x_1,x_2,x_3,x_4]\right)\tau &= \sum_{\lambda}\operatorname{sgn}(\lambda)\left([x_1,x_2,x_3]\wedge[x_1,x_2,x_3,x_4]\right)\lambda \\ &\quad -\sum_{\lambda}\operatorname{sgn}(\lambda)\left([x_1,x_3,x_2]\wedge[x_1,x_3,x_2,x_4]\right)\lambda \\ &\quad +\sum_{\lambda}\operatorname{sgn}(\lambda)\left([x_2,x_3,x_1]\wedge[x_2,x_3,x_1,x_4]\right)\lambda. \end{aligned}$$

and consequently

$$\begin{aligned} T_3 &= \sum_\lambda \operatorname{sgn}(\lambda)\left([x_1,x_2,x_3,x_4]\wedge[x_2,x_3,x_1]+[x_1,x_2,x_3]\wedge[x_2,x_3,x_1,x_4]\right)\lambda \\ &\quad +\sum_\lambda \operatorname{sgn}(\lambda)\left([x_1,x_2,x_3]\wedge[x_1,x_2,x_3,x_4]-[x_1,x_3,x_2]\wedge[x_1,x_3,x_2,x_4]\right)\lambda \\ &\quad +\sum_\lambda \operatorname{sgn}(\lambda)\left([x_2,x_3,x_1]\wedge[x_2,x_3,x_1,x_4]\right)\lambda \\ &= \sum_\lambda \operatorname{sgn}(\lambda)\left([x_1,x_2,x_3,x_4]\wedge[x_2,x_3,x_1]+[x_1,x_2,x_3]\wedge[x_1,x_3,x_2,x_4]\right)\lambda \\ &\quad +\sum_\lambda \operatorname{sgn}(\lambda)\left([x_1,x_3,x_2,x_4]\wedge[x_1,x_3,x_2]+[x_2,x_3,x_1]\wedge[x_2,x_3,x_1,x_4]\right)\lambda \\ &= \sum_\lambda \operatorname{sgn}(\lambda)\left([x_1,x_2,x_3,x_4]\wedge[x_2,x_3,x_1]+[x_1,x_2,x_3,x_4]\wedge[x_2,x_3,x_1]\right)\lambda \\ &= 2\sum_\lambda \operatorname{sgn}(\lambda)\left([x_1,x_2,x_3,x_4]\wedge[x_2,x_3,x_1]\right)\lambda. \end{aligned}$$

■

By Lemma 4.3, we see that $y(x_1,x_2,x_3,x_4)=T_3+T_4\in 2(M\wedge M)$. Let

$$u(x_1,x_2,x_3,x_4)=\frac{1}{2}y(x_1,x_2,x_3,x_4).$$

Then, we have

$$\begin{aligned} u(x_1,x_2,x_3,x_4) &= \sum_\lambda \operatorname{sgn}(\lambda)\left([x_1,x_2,x_3,x_4]\wedge[x_2,x_3,x_1]\right)\lambda \\ &\quad +\sum_\rho \operatorname{sgn}(\rho)\left([x_1,x_2,x_3]\wedge[x_1,x_2,x_4]\right)\rho \end{aligned}$$

where λ and ρ range over all permutations of $\{1,2,3,4\}$ with $\lambda(1)<\lambda(2)<\lambda(3)$ and $\rho(1)<\rho(2)$, $\rho(3)<\rho(4)$.

Consequently, the image of $u(x_1,x_2,x_3,x_4)$ in $M\wedge M\otimes_G\mathbb{Z}$ is the image of $1\otimes e_1\wedge e_2\wedge e_3\wedge e_4$ under the connecting homomorphism $H_4(G,\mathbb{Z}_2)\to M\wedge M\otimes_G\mathbb{Z}$. Now we are able to state the following results which have been published recently in [3].

Proposition 4.4 *The elements $u(x_i,x_j,x_k,x_l)\otimes 1$ with $1\leq i<j<k<l\leq n$ freely generate $t(M\wedge M\otimes_G\mathbb{Z})$, and they have order 2.*

Proof. The elements $1\otimes e_i\wedge e_j\wedge e_k\wedge e_l$ with $1\leq i<j<k<l\leq n$ freely generate $H_4(G,\mathbb{Z}_2)$, and they have order 2. The images of these elements in $t(M\wedge M\otimes_G\mathbb{Z})$ under the connecting homomorphism $H_4(G,\mathbb{Z}_2)\to t(M\wedge M\otimes_G\mathbb{Z})$, which is an isomorphism, are the elements $u(x_i,x_j,x_k,x_l)\otimes 1$. The proposition follows. ■

Proposition 4.5 *The elements* $y\left(x_i, x_j, x_k, x_l\right)$ *with* $1 \leq i < j < k < l \leq n$ *freely generate a free abelian group* A *of rank* $\binom{n}{4}$ *in* $M \wedge M$ *such that the isolator of* A *in* $M \wedge M$ *generates modulo* $(M \wedge M) \cdot IG$ *the torsion subgroup of* $M \wedge M \otimes_G \mathbb{Z}$*.*

Proof. By Proposition 4.4 the elements $\frac{1}{2}y\left(x_i, x_j, x_k, x_l\right) \otimes 1$ have order 2. This means that the elements $y\left(x_i, x_j, x_k, x_l\right)$ belong to $\ker\left(M \wedge M \otimes_G \mathbb{Z}G \rightarrow M \wedge M \otimes_G \mathbb{Z}\right)$ which is $(M \wedge M) \cdot IG$. Now the isomorphism (2.4) yields

$$(M \wedge M) \cdot IG \cong F''/\left[F'', F'\right] \cdot IG.$$

Moreover one has, for $m \in F''$, $x \in F$ and $\pi(x) = g \in G$,

$$\begin{aligned} m\left[F'', F'\right](g-1) &= m\left[F'', F'\right] g\left(m\left[F'', F'\right]\right)^{-1} \\ &= x^{-1} m x\left[F'', F'\right] m^{-1}\left[F'', F'\right] \\ &= x^{-1} m x m^{-1}\left[F'', F'\right] \\ &= m^{-1} x^{-1} m x\left[F'', F'\right] \\ &= [m, x]\left[F'', F'\right]. \end{aligned}$$

This shows that $F''/\left[F'', F'\right] \cdot IG \cong \left[F'', F\right]/\left[F'', F'\right]$, hence

$$(M \wedge M) \cdot IG \cong \left[F'', F\right]/\left[F'', F'\right].$$

$\ker\left(M \wedge M \otimes_G \mathbb{Z} \rightarrow M \otimes P \otimes_G \mathbb{Z}\right)$ coincides with $t\left(M \wedge M \otimes_G \mathbb{Z}\right)$. Thus, $M \wedge M \otimes_G \mathbb{Z}$ is a direct sum of the free abelian group $M \otimes P \otimes_G \mathbb{Z}$ and $t\left(M \wedge M \otimes_G \mathbb{Z}\right)$ which is a 2-group. Consider the subgroup H of $M \wedge M$ generated by the elements $u\left(x_i, x_j, x_k, x_l\right)$ the number of which is $\binom{n}{4}$. Since $M \wedge M$ is free abelian, H is free abelian and we have rank $H \leq \binom{n}{4}$. We show that in fact rank $H = \binom{n}{4}$. Indeed, consider the epimorphism $H \rightarrow t\left(M \wedge M \otimes_G \mathbb{Z}\right)$. If we had rank $H < \binom{n}{4}$, then it would not be possible to map H onto $t\left(M \wedge M \otimes_G \mathbb{Z}\right)$ which is an elementary abelian 2-group of rank $\binom{n}{4}$. This implies that the elements $y\left(x_i, x_j, x_k, x_l\right)$ freely generate a free abelian group A in $M \wedge M$. It is easy to see that H is contained in the isalator of A in $M \wedge M$. Moreover, we show that H is the isalator of A in $M \wedge M$. Now, suppose there exists $x \in \mathrm{Is}_{M \wedge M} A$ such that x does not belong to H. Then we have $x \in M \wedge M$ with $mx \in A$ for some m (we can choose such m positive and smallest possible), hence the image of x has order m in $M \wedge M \otimes_G \mathbb{Z}$. Since we have elements of order 2 in $M \wedge M \otimes_G \mathbb{Z}$, we get $m = 2$ and $2x \in A$, hence $x \in H$ giving contradiction. The proposition follows. ■

CHAPTER 5

STATEMENTS AND PROOFS OF MAIN RESULTS

In this chapter, using the results of Chapter 4, we state and prove our main results which have also been published recently in [3].

Theorem 5.1 *Let* F *be a free group of rank* n *with free generating set* $X = \{x_1, x_2, ..., x_n\}$ *for* $n \geq 4$. *Let*

$$\begin{aligned} w\left(x_1, x_2, x_3, x_4\right) &= \prod_{\lambda}\left(\left[\left[x_1, x_2, x_3, x_4\right], \left[x_2, x_3, x_1\right]\right]\lambda\right)^{sgn(\lambda)} \\ &\quad \prod_{\rho}\left(\left[\left[x_1, x_2, x_3\right], \left[x_1, x_2, x_4\right]\right]\rho\right)^{sgn(\rho)} \end{aligned}$$

where λ *and* ρ *range over all permutations of* $\{1, 2, 3, 4\}$ *with* $\lambda(1) < \lambda(2) < \lambda(3)$ *and* $\rho(1) < \rho(2)$, $\rho(3) < \rho(4)$. *Then the torsion subgroup of* $F/\left[F'', F\right]$ *is freely generated by the elements* $w\left(x_i, x_j, x_k, x_l\right)$ *with* $1 \leq i < j < k < l \leq n$.

Proof. We use the isomorphism (2.5) to translate the result obtained for $t(M \wedge M \otimes_G \mathbb{Z})$ into the setting of $t\left(F''/\left[F'', F\right]\right)$ (hence into the setting of $t\left(F/\left[F'', F\right]\right)$). By Proposition 4.4 it is freely generated by the elements $w\left(x_i, x_j, x_k, x_l\right)$ with $1 \leq i < j < k < l \leq n$. ■

Theorem 5.2 *Let* F *be a free group of rank* n *with free generating set* $X = \{x_1, x_2, ..., x_n\}$ *for* $n \geq 4$. *Let*

$$\begin{aligned} v\left(x_1, x_2, x_3, x_4\right) &= \prod_{\lambda}\left(\left[\left[x_1, x_2, x_3\right], \left[x_2, x_{3,}x_1\right], x_4\right]\lambda\right)^{sgn(\lambda)} \\ &\quad \prod_{\tau}\left(\left[\left[x_1, x_2, x_3\right], \left[x_1, x_2\right], x_4\right]\tau\right)^{sgn(\tau)} \end{aligned}$$

where λ *and* τ *range over all permutations of* $\{1, 2, 3, 4\}$ *with* $\lambda(1) < \lambda(2) < \lambda(3)$ *and* $\tau(1) < \tau(2)$. *Then the elements* $v(x_i, x_j, x_k, x_l)$ *with* $1 \leq i < j < k < l \leq n$ *freely generate a free abelian group* A *of rank* $\binom{n}{4}$ *in the quotient* $F''/\left[F'', F'\right]$ *such that the isolator of* A *in* $F''/\left[F'', F'\right]$ *generates modulo* $\left[F'', F\right]$ *the torsion subgroup of* $F/\left[F'', F\right]$.

Proof. We use the isomorphism (2.4) to translate the result obtained for $M \wedge M$ into the setting of the quotient $F''/[F'', F']$. By Proposition 4.5 the elements $v(x_i, x_j, x_k, x_l)$ with $1 \leq i < j < k < l \leq n$ freely generate a free abelian group A of rank $\binom{n}{4}$ in it such that the isolator of A in it generates modulo $[F'', F]$ the torsion subgroup of $F/[F'', F]$. ■

Theorem 5.3 *Let* $w(x_1, x_2, x_3, x_4)$ *and* $v(x_1, x_2, x_3, x_4)$ *be as in Theorem 5.1 and Theorem 5.2. Then the following relation holds in* $F''/[F'', F']$

$$w^2(x_1, x_2, x_3, x_4) = v(x_1, x_2, x_3, x_4).$$

Proof. Recall the elements $y(x_1, x_2, x_3, x_4)$ and $u(x_1, x_2, x_3, x_4)$ in $M \wedge M$ from Chapter 4. The isomorphism (2.4) takes $y(x_1, x_2, x_3, x_4)$ to $v(x_1, x_2, x_3, x_4)$, and it takes $u(x_1, x_2, x_3, x_4)$ to $w(x_1, x_2, x_3, x_4)$. The relation $2u(x_1, x_2, x_3, x_4) = y(x_1, x_2, x_3, x_4)$ in $M \wedge M$ implies the relation $w^2(x_1, x_2, x_3, x_4) = v(x_1, x_2, x_3, x_4)$ in $F''/[F'', F']$. ■

CHAPTER 6

DOES REALLY OUR GENERATING SET COINCIDE WITH THAT OF CHANDER KANTA GUPTA'S?

In order to answer the question raised in the title of this final chapter, we start with Gupta's elements and try to convert them to our elements. For simplicity, we consider the element

$$\begin{aligned}
&\left[\left[x_1^{-1}, x_2^{-1}\right], [x_3, x_4]\right] \left[\left[x_1^{-1}, x_4^{-1}\right], [x_2, x_3]\right] \\
&\left[\left[x_1^{-1}, x_3^{-1}\right], [x_4, x_2]\right] \left[\left[x_4^{-1}, x_2^{-1}\right], [x_1, x_3]\right] \\
&\left[\left[x_2^{-1}, x_3^{-1}\right], [x_1, x_4]\right] \left[\left[x_3^{-1}, x_4^{-1}\right], [x_1, x_2]\right].
\end{aligned}$$

Since this element has order 2, it is equal to its inverse which is the following element

$$\begin{aligned}
&\left[[x_1, x_2], \left[x_3^{-1}, x_4^{-1}\right]\right] \left[[x_1, x_4], \left[x_2^{-1}, x_3^{-1}\right]\right] \\
&\left[[x_1, x_3], \left[x_4^{-1}, x_2^{-1}\right]\right] \left[[x_4, x_2], \left[x_1^{-1}, x_3^{-1}\right]\right] \\
&\left[[x_2, x_3], \left[x_1^{-1}, x_4^{-1}\right]\right] \left[[x_3, x_4], \left[x_1^{-1}, x_2^{-1}\right]\right].
\end{aligned} \tag{6.1}$$

Thus, for simplicity, we can consider (6.1). For $b_i, b_j \in G$, we have

$$[x_i, x_j]\,(b_i b_j)^{-1} = [x_i, x_j]^{(x_i x_j)^{-1}} = x_i x_j x_i^{-1} x_j^{-1} x_i x_j x_j^{-1} x_i^{-1} = x_i x_j x_i^{-1} x_j^{-1} = \left[x_i^{-1}, x_j^{-1}\right].$$

Thus (6.1) is equal to the following

$$\begin{aligned}
&\left[[x_1, x_2], [x_3, x_4]^{(x_3 x_4)^{-1}}\right] \left[[x_1, x_4], [x_2, x_3]^{(x_2 x_3)^{-1}}\right] \\
&\left[[x_1, x_3], [x_4, x_2]^{(x_4 x_2)^{-1}}\right] \left[[x_4, x_2], [x_1, x_3]^{(x_1 x_3)^{-1}}\right] \\
&\left[[x_2, x_3], [x_1, x_4]^{(x_1 x_4)^{-1}}\right] \left[[x_3, x_4], [x_1, x_2]^{(x_1 x_2)^{-1}}\right].
\end{aligned} \tag{6.2}$$

Using the fact that $\left[[,], [,]^{x^{-1}}\right] = [[,]^{x}, [,]]$ modulo $[F'', F]$, applying the isomorphism (2.5), and, for $a, b \in F'$, writing $a \wedge_\star b$ instead of $a \wedge b \otimes 1$, we see that (6.2) can be identified with the following

$$\begin{aligned}
&[x_1, x_2]\, b_3 b_4 \wedge_\star [x_3, x_4] + [x_1, x_4]\, b_2 b_3 \wedge_\star [x_2, x_3] \\
&+ [x_1, x_3]\, b_4 b_2 \wedge_\star [x_4, x_2] + [x_4, x_2]\, b_1 b_3 \wedge_\star [x_1, x_3] \\
&+ [x_2, x_3]\, b_1 b_4 \wedge_\star [x_1, x_4] + [x_3, x_4]\, b_1 b_2 \wedge_\star [x_1, x_2].
\end{aligned} \tag{6.3}$$

Now, it is easy to see that (6.3) can be written as

$$\sum_{\rho} \operatorname{sgn}(\rho)\left(\left[x_1, x_2\right] b_3 b_4 \wedge_{\star}\left[x_3, x_4\right]\right) \rho \tag{6.4}$$

where ρ ranges over all permutations of $\{1,2,3,4\}$ with $\rho(1)<\rho(2)$, $\rho(3)<\rho(4)$. By writing $b_i = B_i + 1$ for $i = 3, 4$, (6.4) can be rewritten as

$$\begin{aligned} & \sum_{\rho} \operatorname{sgn}(\rho)\left(\left[x_1, x_2\right] B_3 B_4 \wedge_{\star}\left[x_3, x_4\right]+\left[x_1, x_2\right] B_3 \wedge_{\star}\left[x_3, x_4\right]+\left[x_1, x_2\right] B_4 \wedge_{\star}\left[x_3, x_4\right]\right) \rho \\ & +\sum_{\rho} \operatorname{sgn}(\rho)\left(\left[x_1, x_2\right] \wedge_{\star}\left[x_3, x_4\right]\right) \rho . \end{aligned}$$

Now by the definition of the permutation ρ, we get

$$\begin{aligned} \sum_{\rho} \operatorname{sgn}(\rho)\left(\left[x_1, x_2\right] \wedge_{\star}\left[x_3, x_4\right]\right) \rho &= \left[x_1, x_2\right] \wedge_{\star}\left[x_3, x_4\right]-\left[x_1, x_3\right] \wedge_{\star}\left[x_2, x_4\right] \\ & +\left[x_2, x_3\right] \wedge_{\star}\left[x_1, x_4\right]+\left[x_1, x_4\right] \wedge_{\star}\left[x_2, x_3\right] \\ & -\left[x_2, x_4\right] \wedge_{\star}\left[x_1, x_3\right]+\left[x_3, x_4\right] \wedge_{\star}\left[x_1, x_2\right] \\ &= 0. \end{aligned}$$

Hence (6.4) can be rewritten as

$$\sum_{\rho} \operatorname{sgn}(\rho)\left(\left[x_1, x_2\right] B_3 B_4 \wedge_{\star}\left[x_3, x_4\right]+\left[x_1, x_2\right] B_3 \wedge_{\star}\left[x_3, x_4\right]+\left[x_1, x_2\right] B_4 \wedge_{\star}\left[x_3, x_4\right]\right) \rho \tag{6.5}$$

We simplify notation by setting

$$\left[x_{i_1}, x_{i_2}\right]=a_{i_1 i_2}, \ \left[x_{i_1}, x_{i_2}\right] B_{i_3} \ldots B_{i_n}=\left[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n}\right]=a_{i_1 i_2 i_3 \ldots i_n}$$

for $n \geq 3$. Then (6.5) becomes as follows

$$\sum_{\rho} \operatorname{sgn}(\rho)\left(a_{1234} \wedge_{\star} a_{34}+a_{123} \wedge_{\star} a_{34}+a_{124} \wedge_{\star} a_{34}\right) \rho . \tag{6.6}$$

Since $a_{1234} = a_{1324} + a_{3214}$ by the Jacobi identity, (6.6) is equal to the following

$$\begin{aligned} & \sum_{\rho} \operatorname{sgn}(\rho)\left(a_{1324} \wedge_{\star} a_{34}+a_{3214} \wedge_{\star} a_{34}\right) \rho \\ & +\sum_{\rho} \operatorname{sgn}(\rho)\left(a_{123} \wedge_{\star} a_{34}+a_{124} \wedge_{\star} a_{34}\right) \rho . \end{aligned} \tag{6.7}$$

Using the fact that the action is trivial, we can rewrite (6.7) as follows

$$\begin{aligned} & \sum_{\rho} \operatorname{sgn}(\rho)\left(a_{3124} \wedge_{\star} a_{342}+a_{314} \wedge_{\star} a_{342}+a_{2314} \wedge_{\star} a_{341}+a_{234} \wedge_{\star} a_{341}\right) \rho \\ & +\sum_{\rho} \operatorname{sgn}(\rho)\left(a_{123} \wedge_{\star} a_{34}+a_{124} \wedge_{\star} a_{34}\right) \rho \end{aligned} \tag{6.8}$$

Now consider the term $a_{314} \wedge_\star a_{342}$ in (6.8). By the definition of the permutation ρ, we have

$$\begin{aligned}
\sum_\rho \operatorname{sgn}(\rho)\,(a_{314} \wedge_\star a_{342})\,\rho &= \sum_\rho \operatorname{sgn}(\rho)\,(a_{314} \wedge_\star a_{342})\,(1\ 3)\,(2\ 4)\,\rho \\
&= \sum_\rho \operatorname{sgn}(\rho)\,(a_{132} \wedge_\star a_{124})\,\rho \\
&= \sum_\rho \operatorname{sgn}(\rho)\,(a_{123} \wedge_\star a_{124} + a_{231} \wedge_\star a_{124})\,\rho.
\end{aligned}$$

Thus (6.8) can be decomposed into a sum $A + B + C$ where

$$\begin{aligned}
A &= \sum_\rho \operatorname{sgn}(\rho)\,(a_{3124} \wedge_\star a_{342} + a_{2314} \wedge_\star a_{341})\,\rho, \\
B &= \sum_\rho \operatorname{sgn}(\rho)\,(a_{123} \wedge_\star a_{124})\,\rho, \\
C &= \sum_\rho \operatorname{sgn}(\rho)\,(a_{231} \wedge_\star a_{124} + a_{234} \wedge_\star a_{341})\,\rho \\
&\quad + \sum_\rho \operatorname{sgn}(\rho)\,(a_{123} \wedge_\star a_{34} + a_{124} \wedge_\star a_{34})\,\rho.
\end{aligned}$$

By the definition of the permutation ρ, we have

$$\begin{aligned}
C &= \sum_\rho \operatorname{sgn}(\rho)\,(a_{231} \wedge_\star a_{124} + a_{234} \wedge_\star a_{341}\,(1\ 3)\,(2\ 4))\,\rho \\
&\quad + \sum_\rho \operatorname{sgn}(\rho)\,(a_{123} \wedge_\star a_{34} + a_{124} \wedge_\star a_{34})\,\rho \\
&= \sum_\rho \operatorname{sgn}(\rho)\,(a_{231} \wedge_\star a_{124} + a_{412} \wedge_\star a_{123})\,\rho \\
&\quad + \sum_\rho \operatorname{sgn}(\rho)\,(a_{123} \wedge_\star a_{34} + a_{124} \wedge_\star a_{34})\,\rho.
\end{aligned}$$

Now, one calculates that

$$\begin{aligned}
a_{231} \wedge_\star a_{124} + a_{412} \wedge_\star a_{123} &= a_{214} \wedge_\star a_{231} + a_{412} \wedge_\star a_{123} \\
&= a_{241} \wedge_\star a_{231} + a_{412} \wedge_\star a_{231} + a_{412} \wedge_\star a_{123} \\
&= a_{241} \wedge_\star a_{231} + a_{412} \wedge_\star a_{132} \\
&= a_{421} \wedge_\star a_{321} + a_{412} \wedge_\star a_{132}.
\end{aligned}$$

Using the fact that the action is trivial, we get

$$a_{421} \wedge_\star a_{321} = a_{241} \wedge_\star a_{32} + a_{24} \wedge_\star a_{321} = a_{241} \wedge_\star a_{32} + a_{231} \wedge_\star a_{24}$$

and

$$a_{412} \wedge_\star a_{132} = a_{142} \wedge_\star a_{13} + a_{14} \wedge_\star a_{132} = a_{142} \wedge_\star a_{13} + a_{312} \wedge_\star a_{14}.$$

Thus

$$\begin{aligned} C &= \sum_{\rho} \operatorname{sgn}(\rho)\left(a_{241} \wedge_\star a_{32} + a_{231} \wedge_\star a_{24} + a_{142} \wedge_\star a_{13}\right)\rho \\ &\quad + \sum_{\rho} \operatorname{sgn}(\rho)\left(a_{312} \wedge_\star a_{14} + a_{123} \wedge_\star a_{34} + a_{124} \wedge_\star a_{34}\right)\rho. \end{aligned}$$

By the definition of the permutation ρ, we have

$$\begin{aligned} C &= a_{241} \wedge_\star a_{32}\left((1) + (1\ 2\ 3) + (2\ 4\ 3) + (1\ 3)(2\ 4) - (2\ 3) - (1\ 2\ 4\ 3)\right) \\ &\quad + a_{231} \wedge_\star a_{24}\left((1) + (1\ 2\ 3) + (2\ 4\ 3) + (1\ 3)(2\ 4) - (2\ 3) - (1\ 2\ 4\ 3)\right) \\ &\quad + a_{142} \wedge_\star a_{13}\left((1) + (1\ 2\ 3) + (2\ 4\ 3) + (1\ 3)(2\ 4) - (2\ 3) - (1\ 2\ 4\ 3)\right) \\ &\quad + a_{312} \wedge_\star a_{14}\left((1) + (1\ 2\ 3) + (2\ 4\ 3) + (1\ 3)(2\ 4) - (2\ 3) - (1\ 2\ 4\ 3)\right) \\ &\quad + a_{123} \wedge_\star a_{34}\left((1) + (1\ 2\ 3) + (2\ 4\ 3) + (1\ 3)(2\ 4) - (2\ 3) - (1\ 2\ 4\ 3)\right) \\ &\quad + a_{124} \wedge_\star a_{34}\left((1) + (1\ 2\ 3) + (2\ 4\ 3) + (1\ 3)(2\ 4) - (2\ 3) - (1\ 2\ 4\ 3)\right). \end{aligned}$$

Hence

$$\begin{aligned} C &= a_{241} \wedge_* a_{32} + a_{342} \wedge_* a_{13} + a_{431} \wedge_* a_{24} + a_{423} \wedge_* a_{14} - a_{341} \wedge_* a_{23} - a_{432} \wedge_* a_{14} \\ &\quad + a_{231} \wedge_* a_{24} + a_{312} \wedge_* a_{34} + a_{421} \wedge_* a_{43} + a_{413} \wedge_* a_{42} - a_{321} \wedge_* a_{34} - a_{412} \wedge_* a_{43} \\ &\quad + a_{142} \wedge_* a_{13} + a_{243} \wedge_* a_{21} + a_{134} \wedge_* a_{12} + a_{324} \wedge_* a_{31} - a_{143} \wedge_* a_{12} - a_{234} \wedge_* a_{21} \\ &\quad + a_{312} \wedge_* a_{14} + a_{123} \wedge_* a_{24} + a_{214} \wedge_* a_{13} + a_{134} \wedge_* a_{32} - a_{213} \wedge_* a_{14} - a_{124} \wedge_* a_{23} \\ &\quad + a_{123} \wedge_* a_{34} + a_{231} \wedge_* a_{14} + a_{142} \wedge_* a_{23} + a_{341} \wedge_* a_{12} - a_{132} \wedge_* a_{24} - a_{241} \wedge_* a_{13} \\ &\quad + a_{124} \wedge_* a_{34} + a_{234} \wedge_* a_{14} + a_{143} \wedge_* a_{23} + a_{342} \wedge_* a_{12} - a_{134} \wedge_* a_{24} - a_{243} \wedge_* a_{13}. \end{aligned}$$

C can be decomposed into a sum

$$C_{1,2} + C_{1,3} + C_{1,4} + C_{2,3} + C_{2,4} + C_{3,4}$$

where

$$\begin{aligned} C_{1,2} &= a_{243} \wedge_* a_{21} - a_{234} \wedge_* a_{21} + a_{342} \wedge_* a_{12} + a_{134} \wedge_* a_{12} - a_{143} \wedge_* a_{12} + a_{341} \wedge_* a_{12}, \\ C_{1,3} &= a_{142} \wedge_* a_{13} + a_{214} \wedge_* a_{13} - a_{241} \wedge_* a_{13} + a_{342} \wedge_* a_{13} + a_{324} \wedge_* a_{31} - a_{243} \wedge_* a_{13}, \\ C_{1,4} &= a_{312} \wedge_* a_{14} + a_{231} \wedge_* a_{14} - a_{213} \wedge_* a_{14} + a_{423} \wedge_* a_{14} + a_{234} \wedge_* a_{14} - a_{432} \wedge_* a_{14}, \\ C_{2,3} &= a_{241} \wedge_* a_{32} + a_{142} \wedge_* a_{23} - a_{124} \wedge_* a_{23} + a_{143} \wedge_* a_{23} + a_{134} \wedge_* a_{32} - a_{341} \wedge_* a_{23}, \\ C_{2,4} &= a_{431} \wedge_* a_{24} + a_{413} \wedge_* a_{42} - a_{134} \wedge_* a_{24} + a_{231} \wedge_* a_{24} + a_{123} \wedge_* a_{24} - a_{132} \wedge_* a_{24}, \\ C_{3,4} &= a_{312} \wedge_* a_{34} + a_{123} \wedge_* a_{34} - a_{321} \wedge_* a_{34} + a_{124} \wedge_* a_{34} + a_{421} \wedge_* a_{43} - a_{412} \wedge_* a_{43}. \end{aligned}$$

We see that

$$\begin{aligned}
C_{1,2} &= (a_{423}+a_{234}+a_{342})\wedge_{*} a_{12}+(a_{134}+a_{413}+a_{341})\wedge_{*} a_{12},\\
C_{1,3} &= (a_{142}+a_{214}+a_{421})\wedge_{*} a_{13}+(a_{342}+a_{234}+a_{423})\wedge_{*} a_{13},\\
C_{1,4} &= (a_{312}+a_{231}+a_{123})\wedge_{*} a_{14}+(a_{423}+a_{234}+a_{342})\wedge_{*} a_{14},\\
C_{2,3} &= (a_{421}+a_{142}+a_{214})\wedge_{*} a_{23}+(a_{143}+a_{314}+a_{431})\wedge_{*} a_{23},\\
C_{2,4} &= (a_{431}+a_{143}+a_{314})\wedge_{*} a_{24}+(a_{231}+a_{123}+a_{312})\wedge_{*} a_{24},\\
C_{3,4} &= (a_{312}+a_{123}+a_{231})\wedge_{*} a_{34}+(a_{124}+a_{241}+a_{412})\wedge_{*} a_{34}.
\end{aligned}$$

Using the Jacobi identity, we get $C_{1,2}=0$, $C_{1,3}=0$, $C_{1,4}=0$, $C_{2,3}=0$, $C_{2,4}=0$ and $C_{3,4}=0$ and hence $C=0$.

Now let us consider A. By the definition of the permutation ρ, one has

$$\begin{aligned}
A = {} & a_{3124}\wedge_{\star} a_{342}+a_{2314}\wedge_{\star} a_{341}-a_{2134}\wedge_{\star} a_{243}-a_{3214}\wedge_{\star} a_{241}\\
& -a_{1243}\wedge_{\star} a_{134}-a_{4123}\wedge_{\star} a_{132}+a_{1234}\wedge_{\star} a_{143}+a_{3124}\wedge_{\star} a_{142}\\
& +a_{1342}\wedge_{\star} a_{124}+a_{4132}\wedge_{\star} a_{123}+a_{2143}\wedge_{\star} a_{234}+a_{4213}\wedge_{\star} a_{231}.
\end{aligned}$$

A can be decomposed into a sum $A_1+A_2+A_3+A_4$ where

$$\begin{aligned}
A_1 &= a_{3124}\wedge_{\star} a_{342}-a_{2134}\wedge_{\star} a_{243}+a_{2143}\wedge_{\star} a_{234},\\
A_2 &= a_{2314}\wedge_{\star} a_{341}-a_{1243}\wedge_{\star} a_{134}+a_{1234}\wedge_{\star} a_{143},\\
A_3 &= -a_{3214}\wedge_{\star} a_{241}+a_{3124}\wedge_{\star} a_{142}+a_{1342}\wedge_{\star} a_{124},\\
A_4 &= -a_{4123}\wedge_{\star} a_{132}+a_{4132}\wedge_{\star} a_{123}+a_{4213}\wedge_{\star} a_{231}.
\end{aligned}$$

Using the Jacobi identity, we get

$$\begin{aligned}
A_1 &= a_{2341}\wedge_{\star} a_{432}=-a_{2341}\wedge_{\star} a_{342},\\
A_2 &= a_{1324}\wedge_{\star} a_{341}=a_{1342}\wedge_{\star} a_{341},\\
A_3 &= a_{1234}\wedge_{\star} a_{421}=-a_{1243}\wedge_{\star} a_{241},\\
A_4 &= a_{1234}\wedge_{\star} a_{231}
\end{aligned}$$

and hence

$$\begin{aligned}
A &= -a_{2341}\wedge_{\star} a_{342}+a_{1342}\wedge_{\star} a_{341}-a_{1243}\wedge_{\star} a_{241}+a_{1234}\wedge_{\star} a_{231}\\
&= \sum_{\lambda}\operatorname{sgn}(\lambda)\,(a_{1234}\wedge_{\star} a_{231})\,\lambda
\end{aligned}$$

where λ ranges over all permutations of $\{1,2,3,4\}$ with $\lambda(1) < \lambda(2) < \lambda(3)$. Consequently, we see that

$$\begin{aligned}\sum_{\rho} \operatorname{sgn}(\rho)\left([x_1,x_2]\, b_3 b_4 \wedge_{\star} [x_3,x_4]\right)\rho &= \sum_{\lambda} \operatorname{sgn}(\lambda)\left([x_1,x_2,x_3,x_4] \wedge_{\star} [x_2,x_3,x_1]\right)\lambda \\ &\quad + \sum_{\rho} \operatorname{sgn}(\rho)\left([x_1,x_2,x_3] \wedge_{\star} [x_1,x_2,x_4]\right)\rho.\end{aligned}$$

Now, by applying the isomorphism (2.5), we have

$$\begin{aligned} w(x_1,x_2,x_3,x_4) &= \left[[x_1,x_2],[x_3^{-1},x_4^{-1}]\right]\left[[x_1,x_4],[x_2^{-1},x_3^{-1}]\right] \\ &\quad \left[[x_1,x_3],[x_4^{-1},x_2^{-1}]\right]\left[[x_4,x_2],[x_1^{-1},x_3^{-1}]\right] \\ &\quad \left[[x_2,x_3],[x_1^{-1},x_4^{-1}]\right]\left[[x_3,x_4],[x_1^{-1},x_2^{-1}]\right] \\ &= \prod_{\lambda}\left(\left[[x_1,x_2,x_3,x_4],[x_2,x_3,x_1]\right]\lambda\right)^{\operatorname{sgn}(\lambda)} \\ &\quad \prod_{\rho}\left(\left[[x_1,x_2,x_3],[x_1,x_2,x_4]\right]\rho\right)^{\operatorname{sgn}(\rho)} \end{aligned}$$

where λ and ρ range over all permutations of $\{1,2,3,4\}$ with $\lambda(1) < \lambda(2) < \lambda(3)$ and $\rho(1) < \rho(2)$, $\rho(3) < \rho(4)$. Thus we have shown that, indeed, our generating set for the torsion subgroup of the free center-by-metabelian group of rank $n \geq 4$ coincides with that of Chander Kanta Gupta's.

REFERENCES

[1] **Gupta C K** (1973) The free centre-by-metabelian groups. *J. Aust. Math. Soc.,* 16(3):294-299.

[2] **Hilton P and Stammbach U** (1971) *A course in Homological Algebra.* Berlin: Springer.

[3] **Kesim S and Akgül Ş** (2023) An alternative description of the torsion subgroup of the free centre-by-metabelian group of rank at least four in terms of generators. *Commun. Algebra.,* DOI: 10.1080/00927872.2023.2263098.

[4] **Kuz'min Yu V** (1977) Free centre-by-metabelian groups, Lie algebras and $\mathfrak{D}$-groups. *Math. USSR Izv*., 11(1):1-30.

[5] **Maclane S** (1963) *Homology*, New York: Academic Press; Berlin: Springer-Verlag.

[6] **Sager A M** (1997) On elements of finite order in free central extensions of groups. *PhD Thesis*, UMIST, Department of Mathematics, Manchester, 152 pp.

[7] **Serre J P** (1992) *Lie algebras and Lie groups*. Berlin Heidelberg: Springer-Verlag.

[8] **Shmel'kin A L** (1965) Wreath products and varieties of groups. *Izv. Akad. Nauk SSSR Ser. Mat.,* 29:149-170. (in Russian).

[9] **Stöhr R** (1987) On torsion in free central extensions of some torsion-free groups. *J. Pure Appl. Algebra*, 46(2-3):249-289.

[10] **Stöhr R** (1989) On elements of order four in certain free central extensions of groups. *Math. Proc. Camb. Phil. Soc.*, 106(1):13-28.

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