ZONGULDAK BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

APPLICATIONS OF GENERALIZED WOODALL NUMBERS

DEPARTMENT OF MATHEMATICS

MASTER OF SCIENCE THESIS

ORHAN EREN

JANUARY 2024

ZONGULDAK BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

APPLICATIONS OF GENERALIZED WOODALL NUMBERS

DEPARTMENT OF MATHEMATICS

MASTER OF SCIENCE THESIS

Orhan EREN

ADVISOR : Prof. Dr. Yüksel SOYKAN

ZONGULDAK
JANUARY 2024

APPROVAL OF THE THESIS:

The thesis entitled “Applications of Generalized Woodall Numbers” and submitted by Orhan EREN has been examined and accepted by the jury as a Master of Science thesis in Department of Mathematics, Graduate School of Natural and Applied Sciences, Zonguldak Bülent Ecevit University. 23/01/2024

Advisor: Prof. Dr. Yüksel SOYKAN ....
Zonguldak Bülent Ecevit University, Faculty of Science, Department of Mathematics

Member: Assoc. Prof. Dr. Nazmiye GÖNÜL BİLGİN ....
Zonguldak Bülent Ecevit University, Faculty of Science, Department of Mathematics

Member: Assist. Prof. Dr. Erkan TAŞDEMİR ....
Kırklareli University, Pınarhisar Vocational School, Department of Electronic and Automation

---

Approved by the Graduate School of Natural and Applied Sciences. ..../..../2024

Prof. Dr. Fikret GÖLGELEYEN
Director

“With this thesis it is declared that all the information in this thesis is obtained and presented according to academic rules and ethical principles. Also as required by academic rules and ethical principles all works that are not result of this study are cited properly.”

Orhan EREN

ABSTRACT

Master of Science Thesis

APPLICATIONS OF GENERALIZED WOODALL NUMBERS

Orhan EREN

Zonguldak Bülent Ecevit University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Prof. Dr. Yüksel SOYKAN
January 2024, 113 pages

In this thesis, we consider application of generalized Woodall numbers to some special numbers. We summarize the chapters as follows.

In Chapter 1, we present the definitions and properties of some special numbers used throughout the thesis. We also review the literature on these numbers by scanning several research papers.

In Chapter 2, we define Gaussian generalized Woodall numbers and then we give generating functions and Binet's formulas of these numbers. Next, we obtain some identities, Simpson's formula, sum formulas and matrix formula for Gaussian generalized Woodall numbers. This chapter contains our original work.

In Chapter 3, we define dual generalized Woodall numbers, then we found generating functions and Binet's formulas of these numbers. Furthermore, we investigate some identities, the summation formulas and matrices related for these numbers. This chapter contains our original work.

ABSTRACT (continued)

In Chapter 4, we investigate hyperbolic generalized Woodall numbers. First, the hyperbolic generalized Woodall numbers are defined, then the generating functions and Binet's formula of these numbers are found. Furthermore, we obtain some identities, the summation formulas of these numbers. In addition, we obtain matrices related with these numbers. The work presented in this chapter is original.

In Chapter 5, we define dual hyperbolic generalized Woodall numbers and then we found generating functions and Binet's formulas of these numbers. Furthermore, we obtain some identities for these numbers. In addition, we obtain the summation formulas of these numbers with positive and negative subscripts. Finally, we investigate matrices related with dual hyperbolic generalized Woodall numbers. This chapter contains our original work.

Keywords: Generalized Woodall numbers, Gaussian generalized Woodall numbers, dual generalized Woodall numbers, hyperbolic generalized Woodall numbers and dual hyperbolic generalized Woodall numbers.

Science Code: 403.01.01

ÖZET

Yüksek Lisans Tezi

GENELLEŞTİRİLMİŞ WOODALL SAYILARININ UYGULAMALARI

Orhan EREN

Zonguldak Bülent Ecevit Üniversitesi
Fen Bilimleri Enstitüsü
Matematik Anabilim Dalı

Tez Danışmanı: Prof. Dr. Yüksel SOYKAN
Ocak 2024, 113 sayfa

Bu tezde, genelleştirilmiş Woodall sayılarının bazı özel sayılara uygulanmasını ele aldık. Bölümleri şu şekilde özetliyoruz.

Bölüm 1'de, tez boyunca kullanılan bazı özel sayıların tanımlarını ve özelliklerini sunuyoruz. Ayrıca çeşitli araştırma makalelerini tarayarak bu sayılara ilişkin literatürü de gözden geçiriyoruz.

Bölüm 2'de, Gaussian genelleştirilmiş Woodall sayılarını tanımladık ve ardından bu sayıların üreteç fonksiyonlarını ve Binet formüllerini verdik. Daha sonra, Gauss genelleştirilmiş Woodall sayıları için bazı özdeşlikler, Simpson formülü, toplam formülleri ve matris formülü elde ediyoruz. Bu bölümde orijinal çalışmamız yer almaktadır.

Bölüm 3'te, dual genelleştirilmiş Woodall sayılarını tanımladık, ardından bu sayıların üreteç fonksiyonlarını ve Binet formüllerini bulduk. Ayrıca, bazı özdeşlikleri, bu sayılara ilişkin toplama formüllerini ve matrisleri araştırıyoruz. Bu bölümde orijinal çalışmamız yer almaktadır.

ÖZET (devam ediyor)

Bölüm 4'te, hiperbolik genelleştirilmiş Woodall sayılarını inceliyoruz. Öncelikle, hiperbolik genelleştirilmiş Woodall sayıları tanımlanır, daha sonra bu sayıların üreteç fonksiyonları ve Binet formülü bulunur. Ayrıca, bu sayıların bazı özdeşliklerini, toplam formüllerini de elde ediyoruz. Ayrıca bu sayılarla ilgili matrisleri de elde ediyoruz. Bu bölümde sunulan çalışma orijinaldir.

Bölüm 5’'te, dual hiperbolik genelleştirilmiş Woodall sayılarını tanımladık ve ardından bu sayıların üreteç fonksiyonlarını ve Binet formüllerini bulduk. Ayrıca, bu sayılar için bazı özdeşlikler elde ediyoruz. Ayrıca bu sayıların pozitif ve negatif indisli toplam formüllerini de elde ediyoruz. Son olarak dual hiperbolik genelleştirilmiş Woodall sayılarına ilişkin matrisleri araştırıyoruz. Bu bölümde orijinal çalışmamız yer almaktadır.

Anahtar Kelimeler: Genelleştirilmiş Woodall sayıları, Gaussian genelleştirilmiş Woodall sayıları, dual genelleştirilmiş Woodall sayıları, hiperbolik genelleştirilmiş Woodall sayıları ve dual hiperbolik genelleştirilmiş Woodall sayıları.

Bilim Kodu: 403.01.01

ACKNOWLEDGEMENTS

In the completion of this work, first, I extend my infinite gratitude to my esteemed advisor, Prof. Dr. Yüksel SOYKAN which I will never forget, from whom I have benefited from immense knowledge and experience, over two years.

I also express my heartfelt thanks to my invaluable family, who have been with me throughout my entire educational journey, and to my dear spouse and beloved son, who supported me through all the challenges during this research period and in every phase of my life.

I dedicate this thesis to my wife Mutiye ARSLAN EREN and my son Yiğit EREN.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

TABLE OF CONTENTS (Continued)

LIST OF TABLES

LIST OF SYMBOLS AND ABBREVIATIONS

SYMBOLS

W_n : n-th generalized Woodall numbers
R_n : n-th Woodall numbers (sometimes called Riesel numbers)
C_n : n-th Cullen numbers
G_n : n-th modified Woodall numbers
H_n : n-th modified Cullen numbers
GW_n : n-th Gaussian generalized Woodall numbers
$\mathcal{D}\mathrm{W}_\mathrm{n}$: n-th dual generalized Woodall numbers
$\mathcal{H}\mathrm{W}_\mathrm{n}$: n-th hyperbolic generalized Woodall numbers
$\widehat{W}_\mathrm{n}$: n-th dual hyperbolic generalized Woodall numbers

CHAPTER 1

INTRODUCTION

In this chapter, we present the definitions and some properties of some special numbers used throughout the thesis. We also review the literature on these numbers by scanning several research papers.

1.1 GAUSSIAN NUMBERS

In this section, we give information about Gaussian numbers.

1.1.1 Definitions And Properties

A Gaussian integer z is a complex number whose real and imaginary parts are both integers, i.e., $z = a + ib$, $a, b \in Z$. These numbers are denoted by $Z[i]$. The norm of a Gaussian integer $a + ib$, $a, b \in Z$ is its Euclidean norm, that is,
$N(a + ib) = \sqrt{a^2 + b^2} = \sqrt{(a + ib)(a - ib)}$.
For more information about this kind of integers, see the work of Fraleigh [26]. If we use together sequences of integers defined recursively and Gaussian type integers, we obtain a new sequences of complex numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell Lucas and Gaussian Jacobsthal numbers; Gaussian Padovan and Gaussian Pell-Padovan numbers; Gaussian Tribonacci numbers.
In literature, there have been so many studies of these sequences of Gaussian numbers.

1.1.2 Literature Review For Gaussian Numbers

Now, we give some information about the Gaussian sequence from the literature.

- First we give Gaussian numbers with second order recurrence.

- In 1963, Horadam [36] introduced the concept of complex Fibonacci numbers called the Gaussian Fibonacci number numbers as

  $GF_n = F_n + iF_{n-1}$

  where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$. (in fact, he defined these numbers as $GF_n = F_n + iF_{n-1}$ and he called these numbers as complex Fibonacci numbers.)

- Pethe and Horadam [47] introduced generalized Gaussian Fibonacci numbers

  $GF_n = F_n + iF_{n-1}$

  where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

- Halıcı and Öz [32] studied Gaussian Pell and Pell Lucas numbers by written, respectively,

  $$\begin{aligned} GP_n &= P_n + iP_{n-1}, \\ GQ_n &= Q_n + iQ_{n-1}, \end{aligned}$$

  where $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, $Q_0 = 2$, $Q_1 = 2$.

- Aşçı and Gürel [2] presented Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers given by, respectively,

  $$\begin{aligned} GJ_n &= J_n + iJ_{n-1}, \\ Gj_n &= j_n + ij_{n-1}, \end{aligned}$$

  where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$ and $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 2$, $j_1 = 1$.

- Taşcı [68] introduced and studied Gaussian Mersenne numbers and define by

  $GM_n = M_n + iM_{n-1}$

  where $M_n = 3M_{n-1} - 2M_{n-2}$, $M_0 = 0$, $M_1 = 1$.

- Taşçı [66] introduced and studied Gaussian balancing and Lucas Balancing numbers and given by, respectively,

  $$\begin{aligned} GB_n &= B_n + iB_{n-1}, \\ GC_n &= C_n + iC_{n-1}, \end{aligned}$$

  where $B_n = 6B_{n-1} - B_{n-2}$, $B_0 = 0$, $B_1 = 1$ and $C_n = 6C_{n-1} - C_{n-2}$, $C_0 = 1$, $C_1 = 3$.

- Ertaş and Yılmaz [24] studied Gaussian Oresme numbers and given by

  $GT_n = T_n + iT_{n-1}$

  where $T_n = T_{n-1} - \frac{1}{4}T_{n-2}$, $T_0 = 0$, $T_1 = \frac{1}{2}$.

- Now, we present Gaussian numbers with third order recurrence relations.

  - Soykan, Taşdemir, Okumuş and Göcen [54] presented Gaussian generalized Tribonacci numbers given by

    $GW_n = W_n + iW_{n-1}$

    where $W_n = W_{n-1} + W_{n-2} + W_{n-3}$, with the initial condition W_0, W_1, W_2.

  - Taşcı [67] studied Gaussian Padovan and Gaussian Pell- Padovan numbers by written, respectively,

    $$\begin{aligned} GP_n &= P_n + iP_{n-1}, \\ GR_n &= R_n + iR_{n-1}, \end{aligned}$$

    where $P_n = P_{n-2} + P_{n-3}$, $P_0 = 1$, $P_1 = 1$, $P_2 = 1$, and $R_n = 2R_{n-2} + R_{n-3}$, $R_0 = 1$, $R_1 = 1$, $R_2 = 1$.

  - Cerda-Morales [13] defined Gaussian third-order Jacobsthal numbers by given

    $GJ_n = J_n + iJ_{n-1}$

    where $J_n = J_{n-1} + J_{n-2} + 2J_{n-3}$, $J_1 = 0$, $J_2 = 1$, $J_2 = 1$.

1.2 DUAL NUMBERS

In this section, we present Dual numbers.

1.2.1 Definitions And Properties

First of all, we need to state about hypercomplex number systems. In 1989, I. Kantor is worked the hypercomplex numbers systems, [38]. These numbers systems are extensions of real numbers. Some commutative examples of hypercomplex number systems; complex numbers, hyperbolic (double, split-complex) numbers [53] and dual numbers [25] are given below in order.

$$\begin{aligned}
\mathbb{C} &= \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\}, \\
\mathbb{H} &= \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}, \\
\mathbb{D} &= \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.
\end{aligned}$$

Some non-commutative examples of hypercomplex number systems are quaternions, [34],

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

octonions [6] and sedenions [56]. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras obtained from the real numbers $\mathbb{R}$ by a doubling procedure called the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions to form what is known as the 2^n-ions (see for example [9], [37], [45]).

Quaternions were invented by Irish mathematician W. R. Hamilton (1805-1865) [34] as an extension to the complex numbers. Hyperbolic numbers with complex coefficients are introduced by J. Cockle in 1848, [16]. H. H. Cheng and S. Thompson [14] introduced dual numbers with complex coefficients and called complex dual numbers. Akar, Yüce and Şahin [1] introduced dual hyperbolic numbers.

A dual number is a hyper-complex number and is defined by

$$q = a_0 + \varepsilon a_1$$

where a_0 and a_1 are real numbers.

The set of all dual numbers is denoted by

$$\mathbb{D} = \{a_0 + \varepsilon a_1 : a_0, a_1 \in \mathbb{R},\ \varepsilon^2 = 0,\ \varepsilon \neq 0\}.$$

The base elements $\{1,\, \varepsilon\}$ of dual numbers satisfy the following properties (commutative multiplications):

$$1.\varepsilon = \varepsilon,\ \varepsilon^2 = \varepsilon.\varepsilon = 0$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$).

Let m and n be two dual numbers as $m = a_0 + \varepsilon a_1$ and $n = b_0 + \varepsilon b_1$; the addition and subtraction of two dual numbers as m and n is

$m \mp n = a_0 \mp b_0 + \varepsilon(a_1 \mp b_1)$,

then, the multiplication of two dual numbers m and n is

$mn = a_0 b_0 + \varepsilon(a_1 b_0 + a_0 b_1)$.

1.2.2 Literature Review For Dual Numbers

Now, we provide some information related to dual numbers from the literature.

- Cheng and Thompson [14] introduced dual numbers with complex coefficients.
- Halici [33] studied Dual Fibonacci Octonions as

  $$p = \sum_{s=0}^{7} F_{n+s} e_s$$

where Fibonacci given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

- Gürses, Şentürk and Yüce [31] studied dual-generalized complex Fibonacci and Lucas numbers, respectively, as

$$\begin{aligned} \widetilde{\mathcal{F}}_n &= F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3}, \\ \widetilde{\mathcal{L}}_n &= L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3}, \end{aligned}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Aydın [5] studied Dual Jacobsthal Quaternions as

  $$QJ_{k;n} = J_{k;n} + i_1 J_{k;n+1} + i_2 J_{k;n+2} + i_3 J_{k;n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$.

- Nurkan, Guven, [46] studied Dual Fibonacci Quaternions as

  $$\widetilde{Q}n = (F_n + F_{n+1}) + i(F_{n+1} + F_{n+2}) + j(F_{n+2} + F_{n+3}) + k(F_{n+3} + F_{n+4})$$

where Fibonacci given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

- Yüce, Aydın, [71] studied Generalized Dual Fibonacci Quaternions as

$$Q_{\mathbb{D}} = \{\mathbb{D}_n = H_n + iH_{n+1} + jH_{n+2} + kH_{n+3} : H_n \text{ is } n\text{-th generalized Fibonacci number}\}$$

where

$i^2 = j^2 = k^2 = ijk = 0,\ ij = -ji = jk = -kj = ki = -ik = 0.$

- Aydın, Köklü and Yüce, [3] presented Generalized dual Pell quaternions as

$$Q_{\mathbb{D}} = \left\{\mathbb{D}_n^P = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3} : P_n \text{ is } n\text{-th generalized Pell number}\right\}$$

where

$i^2 = j^2 = k^2 = ijk = 0,\ ij = -ji = jk = -kj = ki = -ik = 0.$

Then, we give Hyperbolic numbers, which are another commutative example of hypercomplex number systems.

1.3 HYPERBOLIC NUMBERS

In this section, we present some information related to hyperbolic numbers.

1.3.1 Definitions And Properties

Hyperbolic numbers are defined by

$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R},\ j^2 = 1,\ j \neq \pm 1\}.$

The base elements $\{1, j\}$ of hyperbolic numbers satisfy the following properties (commutative multiplications):

$1.j = j,\ j^2 = j.j = 1$

where j symbolizes the hyperbolic unit ($j^2 = 1$).

Let m and n be two hyperbolic numbers as $m = a_0 + ja_1$ and $n = b_0 + jb_1$. The addition, subtraction and multiplication of two hyperbolic numbers m and n are defined as

$m + n = a_0 + b_0 + j(a_1 + b_1),$

$m - n = a_0 - b_0 + j(a_1 - b_1),$

$mn = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0).$

1.3.2 Literature Review For Hyperbolic Numbers

Now, we give some information about hyperbolic numbers from the literature.

- Richter, [49] worked On Hyperbolic Complex Numbers.

- Gürses, Şentürk and Yüce, [29] studied A Study on Dual-Generalized Complex and Hyperbolic-Generalized Complex numbers.

- Cockle, [16] worked the Hyperbolic numbers with complex coefficients.

- Aydın, [4] worked hyperbolic Fibonacci numbers given by

  $\widetilde{F}_n = F_n + hF_{n+1},\ (h^2 = 1)$

  where Fibonacci numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$ with the initial condition $F_1 = F_2 = 1$, $(n \geq 3)$.

- Dikmen, [19] worked hyperbolic Jacobsthal numbers given by

  $\widehat{J}_n = J_n + hJ_{n+1}\ ,\ (h^2 = 1)$

  where Jacobsthal numbers, respectively, given by $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$.

- Taş, [65] worked on hyperbolic Jacobsthal-Lucas sequence given by

  $HJ_n = J_n + hJ_{n+1}, (h^2 = 1)$

  where Jacobsthal-Lucas numbers, respectively, given by $J_{n+2} = J_{n+1} + 2J_n$, with the initial condition $J_0 = 2,\ J_1 = 1$.

- Soykan and Taşdemir, [63] worked on hyperbolic generalized Jacobsthal numbers given by

  $\widetilde{V}_n = V_n + hV_{n+1}\ ,\ (h^2 = 1)$

  where generalized Jacobsthal numbers are given by $V_n = V_{n-1} + 2V_{n-2}$, $V_0 = a$, $V_1 = b$ $(n \geq 2)$ with the initial values V_0, V_1 not all being zero.

- Dişkaya, Menken, Catarino, [21] worked on the hyperbolic Leonardo and hyperbolic Francois quaternions given by

  $$\begin{aligned} H\mathcal{L}_n &= \mathcal{L}_n e_0 + \mathcal{L}_{n+1} e_1 + \mathcal{L}_{n+2} e_2 + \mathcal{L}_{n+3} e_3, \\ H\mathcal{F}_n &= \mathcal{F}_n e_0 + \mathcal{F}_{n+1} e_1 + \mathcal{F}_{n+2} e_2 + \mathcal{F}_{n+3} e_3 \end{aligned}$$

  where Francois and Leonardo numbers, respectively, given by $\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2} + 1$, with the initial condition $\mathcal{F}_0 = 2$, $\mathcal{F}_1 = 1$ and $\mathcal{L}_{n+2} = \mathcal{L}_{n+1} + \mathcal{L}_n$, with the initial condition $\mathcal{L}_0 = 1$, $\mathcal{L}_1 = 1$.

- Dikmen and Altınsoy, [20] worked on third order hyperbolic Jacobsthal numbers given by

  $$\begin{aligned} \widehat{J}_n^{(3)} &= J_n^{(3)} + hJ_{n+1}^{(3)}, \\ \widehat{j}_n^{(3)} &= j_n^{(3)} + hj_{n+1}^{(3)} \end{aligned}$$

  where Jacobsthal numbers, respectively, given by $J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)}$, $J_0^{(3)} = 0$, $J_1^{(3)} = 1$, $J_2^{(3)} = 1$, $j_n^{(3)} = j_{n-1}^{(3)} + j_{n-2}^{(3)} + 2j_{n-3}^{(3)}$, $j_0^{(3)} = 2$, $j_1^{(3)} = 1$, $j_2^{(3)} = 5$.

1.4 DUAL HYPERBOLIC NUMBERS

In this section, we obtain about dual hyperbolic numbers.

1.4.1 Definitions And Properties

A dual hyperbolic number is a hyper-complex number and is defined by

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3$$

where a_0, a_1, a_2 and a_3 are real numbers.
The set of all dual hyperbolic numbers is denoted by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers satisfy the following properties (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

Let m and n be two dual numbers as $m = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $n = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$. The addition, subtraction and multiplication of two dual numbers m and n are defined as

$$m \mp n = a_0 \mp b_0 + j(a_1 \mp b_1) + \varepsilon(a_2 \mp b_2) + j\varepsilon(a_3 \mp b_3),$$

$$mn = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3).$$

The dual hyperbolic numbers form a commutative ring, real vector space and an algebra. But $\mathbb{H}_{\mathbb{D}}$ is not a field because every dual hyperbolic number doesn't have an inverse. For more information on the dual hyperbolic numbers, see [1].

Next, we present details about dual hyperbolic and some information related to dual hyperbolic sequences from the literature.

1.4.2 Literature Review For Dual Hyperbolic Numbers

- Akar, Yüce and Şahin, [1] presented the dual hyperbolic numbers.

- Bród, Liana, Włoch [10] studied dual hyperbolic generalized balancing numbers as

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $B_n = 6B_{n-1} - B_{n-2}$, $B_0 = 0$, $B_1 = 1$.

- Cihan, Azak, Güngör, Tosun, [18] studied dual hyperbolic Fibonacci and Lucas numbers given by

$$\begin{aligned} DHF_n &= F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3}, \\ DHL_n &= L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3} \end{aligned}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Soykan, Taşdemir, Okumuş, [62] studied on dual hyperbolic numbers with generalized Jacobsthal numbers components given by

  $$\begin{aligned}\widehat{J}_n &= Jn + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3},\\ \widehat{K}_n &= K_n + jK_{n+1} + \varepsilon K_{n+2} + j\varepsilon K_{n+3}\end{aligned}$$

  where Jacobsthal and Jacobsthal-Lucas numbers, respectively, given by $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$, $K_n = K_{n-1} + 2K_{n-2}$, $K_0 = 2$, $K_1 = 1$.

- Soykan, Gümüş, Göcen, [58] presented dual hyperbolic generalized Pell numbers given by

  $\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$

  where generalized Pell numbers are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a$, $V_1 = b$ $(n \geq 2)$ with the initial values V_0, V_1 not all being zero.

1.5 GENERALIZED WOODALL NUMBERS

Finally, in this section, we state the definition and some important properties of the generalized Woodall numbers used throughout the thesis.

1.5.1 Definitions And Properties

The generalized Woodall sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, 5, -8, 4)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 5W_{n-1} - 8W_{n-2} + 4W_{n-3} \tag{1.1}$$

with the initial values W_0, W_1, W_2 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-1)} - \frac{5}{4}W_{-(n-2)} + \frac{1}{4}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n. For more information on generalized Woodall numbers, see [59].

The first few generalized Woodall numbers with positive subscript and negative subscript are given in the following Table 1.1.

Table 1.1 A few generalized Woodall numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{1}{4}(8W_0 - 5W_1 + W_2)$
2	W_2	$\frac{1}{4}(11W_0 - 9W_1 + 2W_2)$
3	$4W_0 - 8W_1 + 5W_2$	$\frac{1}{16}(52W_0 - 47W_1 + 11W_2)$
4	$20W_0 - 36W_1 + 17W_2$	$\frac{1}{16}(57W_0 - 54W_1 + 13W_2)$
5	$68W_0 - 116W_1 + 49W_2$	$\frac{1}{64}(240W_0 - 233W_1 + 57W_2)$

Next, we can list some important properties of generalized Woodall numbers that are needed.

Now, we give Binet's formula of generalized Woodall numbers.

Theorem 1.1 *[59, Theorem 1.1] Binet's formula of generalized Woodall numbers can be given as*

$$W_n = (A_1 + A_2 n) \times 2^n + A_3$$

where

$$\begin{aligned} A_1 &= -W_2 + 4W_1 - 3W_0, \\ A_2 &= \frac{W_2 - 3W_1 + 2W_0}{2}, \\ A_3 &= W_2 - 4W_1 + 4W_0, \end{aligned}$$

that is,

$$W_n = ((-W_2 + 4W_1 - 3W_0) + \frac{W_2 - 3W_1 + 2W_0}{2} n) \times 2^n + (W_2 - 4W_1 + 4W_0). \tag{1.2}$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 5x^2 + 8x - 4 = (x-2)^2 (x-1) = 0,$$

where $\alpha = \beta = 2$, $\gamma = 1$.

Now, we define four specific cases of the sequence $\{W_n\}$.

1. The Woodall numbers $\{R_n\}$, sometimes called Riesel numbers, and also called Cullen numbers of the second kind, are numbers of the form

   $$R_n = n \times 2^n - 1.$$

The first few Woodall numbers are:

$1, 7, 23, 63, 159, 383, 895, 2047, 4607, 10239, 22527, 49151, 106495, \ldots$

(sequence A003261 in the OEIS [52]). Woodall numbers were first studied by Allan J. C. Cunningham and H. J. Woodall in [17] in 1917, inspired by James Cullen's earlier study of the similarly-defined Cullen numbers.

2. The Cullen numbers $\{C_n\}$ are numbers of the form

   $C_n = n \times 2^n + 1.$

   The first few Cullen numbers are:

   $1, 3, 9, 25, 65, 161, 385, 897, 2049, 4609, 10241, 22529, 49153, 106497, ...$

   (sequence A002064 in the OEIS). Woodall and Cullen sequences have been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [7,8,17,28,30,35,39,41,42,43,44] and references therein. Note that $\{R_n\}$ and $\{C_n\}$ hold the following relations:

   $$\begin{aligned} R_n &= 4R_{n-1} - 4R_{n-2} - 1, \\ C_n &= 4C_{n-1} - 4C_{n-2} + 1. \end{aligned}$$

   Note also that the sequences $\{R_n\}$ and $\{C_n\}$ satisfy the following third order linear recurrences:

   $$R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3}, \qquad R_0 = -1, R_1 = 1, R_2 = 7, \tag{1.3}$$

   $$C_n = 5C_{n-1} - 8C_{n-2} + 4C_{n-3}, \qquad C_0 = 1, C_1 = 3, C_2 = 9. \tag{1.4}$$

3. The modified Woodall numbers $\{G_n\}$ are numbers of the form

   $G_n = (n-1)\,2^n + 1$ (using initial conditions in (1.2)).

   The modified Woodall sequence $\{G_n\}_{n\geq 0}$ is defined, respectively, by the third order recurrence relation:

   $$G_n = 5G_{n-1} - 8G_{n-2} + 4G_{n-3}, \qquad G_0 = 0, G_1 = 1, G_2 = 5, \tag{1.5}$$

4. The modified Cullen numbers $\{H_n\}$ are numbers of the form

   $H_n = 2^{n+1} + 1$ (using initial conditions in (1.2)).

   The modified Cullen sequence $\{H_n\}_{n\geq 0}$ is defined, respectively, by the third order recurrence relation:

   $$H_n = 5H_{n-1} - 8H_{n-2} + 4H_{n-3}, \qquad H_0 = 3, H_1 = 5, H_2 = 9, \tag{1.6}$$

Then, the sequences $\{G_n\}_{n\geq 0}, \{H_n\}_{n\geq 0}, \{R_n\}$ and $\{C_n\}$ can be extended to negative subscripts by defining,

$$\begin{aligned}
G_{-n} &= 2G_{-(n-1)} - \frac{5}{4}G_{-(n-2)} + \frac{1}{4}G_{-(n-3)}, \\
H_{-n} &= 2H_{-(n-1)} - \frac{5}{4}H_{-(n-2)} + \frac{1}{4}H_{-(n-3)}, \\
R_{-n} &= 2R_{-(n-1)} - \frac{5}{4}R_{-(n-2)} + \frac{1}{4}R_{-(n-3)}, \\
C_{-n} &= 2C_{-(n-1)} - \frac{5}{4}C_{-(n-2)} + \frac{1}{4}C_{-(n-3)},
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.3), (1.4), (1.5) and (1.6) hold for all integer n.For more information on generalized Woodall numbers, see [59].
Next, we give some of the sum formulas of generalized Woodall numbers in the following propositions.

Proposition 1.2 *For the generalized Woodall numbers, we have the following formulas:*

- $\sum_{k=0}^{n} W_k = \frac{1}{2}W_2(2n - 2^{n+1}(n-1) + 2^{n+2}(n-2) + 6) - \frac{1}{2}W_1(8n - 2^{n+1}(3n-5) + 2^{n+2}(3n-8) + 22) + W_0(4n - 2^{n+1}(n-2) + 2^{n+2}(n-3) + 9)$.

- $\sum_{k=0}^{n} W_{k+1} = \frac{1}{2}W_2(2n + 2^{n+3}(n-1) - 2^{n+2}n + 8) - \frac{1}{2}W_1(8n - 2^{n+2}(3n-2) + 2^{n+3}(3n-5) + 30) + W_0(4n - 2^{n+2}(n-1) + 2^{n+3}(n-2) + 12)$.

- $\sum_{k=0}^{n} W_{k+2} = \frac{1}{2}W_2(2n - 2^{n+3}(n+1) + 2^{n+4}n + 10) + W_0(4n + 2^{n+4}(n-1) - 2^{n+3}n + 16) - \frac{1}{2}W_1(8n - 2^{n+3}(3n+1) + 2^{n+4}(3n-2) + 40)$.

- $\sum_{k=0}^{n} W_{k+3} = W_0(4n - 2^{n+4}(n+1) + 2^{n+5}n + 20) - \frac{1}{2}W_1(8n + 2^{n+5}(3n+1) - 2^{n+4}(3n+4) + 48) + \frac{1}{2}W_2(2n - 2^{n+4}(n+2) + 2^{n+5}(n+1) + 10)$.

Proof. For the proof, see Soykan [57]. □

Proposition 1.3 *For the generalized Woodall numbers, we have the following formulas:*

- $\sum_{k=0}^{n} W_{2k} = \frac{1}{9}W_0(36n-2^{2n+2}(2n-1)+2^{2n+4}(2n-3)+53)-\frac{1}{18}W_1(72n-2^{2n+2}(6n-2)+2^{2n+4}(6n-8)+120)+\frac{1}{18}W_2(18n+2^{2n+4}(2n-2)-2\times 2^{2n+2}n+32)$.

- $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{18}W_2(18n-2^{2n+3}(2n+1)+2^{2n+5}(2n-1)+40)-\frac{1}{18}W_1(72n-2^{2n+3}(6n+1)+2^{2n+5}(6n-5)+150)+\frac{1}{9}W_0(36n+2^{2n+5}(2n-2)-2\times 2^{2n+3}n+64)$.

- $\sum_{k=0}^{n} W_{2k+2} = \frac{1}{9}W_0(36n-2^{2n+4}(2n+1)+2^{2n+6}(2n-1)+80)-\frac{1}{18}W_1(72n-2^{2n+4}(6n+4)+2^{2n+6}(6n-2)+192)+\frac{1}{18}W_2(18n-2^{2n+4}(2n+2)+2\times 2^{2n+6}n+50)$.

- $\sum_{k=0}^{n} W_{2k+3} = \frac{1}{18}W_2(18n-2^{2n+5}(2n+3)+2^{2n+7}(2n+1)+58)-\frac{1}{18}W_1(72n+2^{2n+7}(6n+1)-2^{2n+5}(6n+7)+240)+\frac{1}{9}W_0(36n-2^{2n+5}(2n+2)+2\times 2^{2n+7}n+100)$.

- $\sum_{k=0}^{n} W_{2k+4} = \frac{1}{18}W_2(18n-2^{2n+6}(2n+4)+2^{2n+8}(2n+2)+50)+\frac{1}{9}W_0(36n-2^{2n+6}(2n+3)+2^{2n+8}(2n+1)+116)-\frac{1}{18}W_1(72n+2^{2n+8}(6n+4)-2^{2n+6}(6n+10)+264)$.

Proof. For the proof, see Soykan [57]. □

Proposition 1.4 *For the generalized Woodall numbers, we have the following formulas:*

- $\sum_{k=0}^{n} W_{-k} = 4W_0(n+\frac{1}{2^{n+1}}(n+4)-\frac{1}{2^{n+2}}(n+3)-1)+2W_1(\frac{1}{2^{n+2}}(3n+8)-2n-\frac{1}{2^{n+1}}(3n+11)+\frac{7}{2})+2W_2(\frac{1}{2}n+\frac{1}{2^{n+1}}(n+3)-\frac{1}{2^{n+2}}(n+2)-1)$.

- $\sum_{k=0}^{n} W_{-k+1} = 2W_2(\frac{1}{2}n+\frac{1}{2^{n}}(n+2)-\frac{1}{2^{n+1}}(n+1)-\frac{3}{2})+4W_0(n+\frac{1}{2^{n}}(n+3)-\frac{1}{2^{n+1}}(n+2)-2)+2W_1(\frac{1}{2^{n+1}}(3n+5)-2n-\frac{1}{2^{n}}(3n+8)+6)$.

- $\sum_{k=0}^{n} W_{-k+2} = 2W_2(\frac{1}{2}n+2^{1-n}(n+1)-\frac{1}{2^{n}}n-\frac{3}{2})+4W_0(n-\frac{1}{2^{n}}(n+1)+2^{1-n}(n+2)-3)-2W_1(2n+2^{1-n}(3n+5)-\frac{1}{2^{n}}(3n+2)-8)$.

- $\sum_{k=0}^{n} W_{-k+3} = 2W_2(\frac{1}{2}n+2^{2-n}n-2^{1-n}(n-1)+\frac{1}{2})+2W_1(2^{1-n}(3n-1)-2n-2^{2-n}(3n+2)+6)+4W_0(n-2^{1-n}n+2^{2-n}(n+1)-3)$.

Proof. For the proof, see Soykan [57]. □

Proposition 1.5 *For the generalized Woodall numbers, we have the following formulas:*

- $\sum_{k=0}^{n} W_{-2k} = \frac{8}{9}W_1(\frac{1}{2^{2n+4}}(6n+8) - \frac{9}{2}n - \frac{1}{2^{2n+2}}(6n+14)+3) + \frac{16}{9}W_0(\frac{9}{4}n + \frac{1}{2^{2n+2}}(2n+5) - \frac{1}{2^{2n+4}}(2n+3) - \frac{1}{2}) + \frac{8}{9}W_2(\frac{9}{8}n + \frac{1}{2^{2n+2}}(2n+4) - \frac{1}{2^{2n+4}}(2n+2) - \frac{7}{8})$.

- $\sum_{k=0}^{n} W_{-2k+1} = \frac{8}{9}W_1(\frac{1}{2^{2n+3}}(6n+5) - \frac{9}{2}n - \frac{1}{2^{2n+1}}(6n+11)+6) + \frac{16}{9}W_0(\frac{9}{4}n + \frac{1}{2^{2n+1}}(2n+4) - \frac{1}{2^{2n+3}}(2n+2) - \frac{7}{4}) + \frac{8}{9}W_2(\frac{9}{8}n + \frac{1}{2^{2n+1}}(2n+3) - \frac{1}{2^{2n+3}}(2n+1) - \frac{11}{8})$.

- $\sum_{k=0}^{n} W_{-2k+2} = \frac{8}{9}W_2(\frac{9}{8}n - \frac{2}{2^{2n+2}}n + \frac{1}{2^{2n}}(2n+2) - \frac{7}{8}) - \frac{16}{9}W_0(\frac{1}{2^{2n+2}}(2n+1) - \frac{9}{4}n - \frac{1}{2^{2n}}(2n+3) + \frac{11}{4}) + \frac{8}{9}W_1(\frac{1}{2^{2n+2}}(6n+2) - \frac{9}{2}n - \frac{1}{2^{2n}}(6n+8) + \frac{15}{2})$.

- $\sum_{k=0}^{n} W_{-2k+3} = \frac{8}{9}W_1(\frac{1}{2^{2n+1}}(6n-1) - \frac{9}{2}n - 2^{1-2n}(6n+5) + \frac{3}{2}) + \frac{8}{9}W_2(\frac{9}{8}n - \frac{1}{2^{2n+1}}(2n-1) + 2^{1-2n}(2n+1) + \frac{25}{8}) + \frac{16}{9}W_0(\frac{9}{4}n + 2^{1-2n}(2n+2) - \frac{2}{2^{2n+1}}n - \frac{7}{4})$.

- $\sum_{k=0}^{n} W_{-2k+4} = \frac{8}{9}W_2(\frac{9}{8}n + 2 \times 2^{2-2n}n - \frac{1}{2^{2n}}(2n-2) + \frac{137}{8}) + \frac{16}{9}W_0(\frac{9}{4}n + 2^{2-2n}(2n+1) - \frac{1}{2^{2n}}(2n-1) + \frac{25}{4}) - \frac{8}{9}W_1(\frac{9}{2}n + 2^{2-2n}(6n+2) - \frac{1}{2^{2n}}(6n-4) + \frac{57}{2})$.

Proof. For the proof, see Soykan [57]. □

Now, we recall the generating function and the Cassini identity for generalized Woodall numbers.

The generating function for generalized Woodall numbers is:

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 5W_0)x + (W_2 - 5W_1 + 8W_0)x^2}{1 - 5x + 8x^2 - 4x^3}. \tag{1.7}$$

The Cassini identity for generalized Woodall numbers is:

$$\begin{aligned} W_{n+1}W_{n-1} - W_n^2 &= \frac{1}{4}2^n(A + B2^n + Cn). \\ A &= 4W_1^2 + W_2^2 - 4W_0W_1 + 4W_0W_2 - 5W_1W_2. \\ B &= -4W_0^2 - 9W_1^2 - W_2^2 + 12W_0W_1 - 4W_0W_2 + 6W_1W_2. \\ C &= 8W_0^2 + 12W_1^2 + W_2^2 - 20W_0W_1 + 6W_0W_2 - 7W_1W_2. \end{aligned}$$

For further information about generalized Woodall numbers, see [59].

CHAPTER 2

GAUSSIAN GENERALIZED WOODALL NUMBERS

In this chapter, we define Gaussian generalized Woodall numbers and then we give generating functions and Binet's formulas of these numbers. Next, we obtain some identities, Simpson's formula, sum formulas and matrix formula for Gaussian generalized Woodall numbers. Our work which is given in this chapter is original. This chapter was published as the journal, see [61].

2.1 DEFINITIONS AND PROPERTIES

Gaussian generalized Woodall numbers $\{GW_n\}_{n\geq 0} = \{GW_n(GW_0, GW_1, GW_2)\}_{n\geq 0}$ are defined by

$$GW_n = 5GW_{n-1} - 8GW_{n-2} + 4GW_{n-3}, \tag{2.1}$$

with the initial conditions

$$GW_0 = W_0 + i(\frac{1}{4}(8W_0 - 5W_1 + W_2)),\ GW_1 = W_1 + iW_0,\ GW_2 = W_2 + iW_1,$$

not all being zero. The sequences $\{GW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GW_{-n} = 2GW_{-(n-1)} - \frac{5}{4}GW_{-(n-2)} + \frac{1}{4}GW_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integer n. Note that for $n \geq 0$, we get

$$GW_n = W_n + iW_{n-1} \tag{2.2}$$

and

$$GW_{-n} = W_{-n} + iW_{-n-1}. \tag{2.3}$$

The first few generalized Gaussian Woodall numbers with positive subscript and negative subscript are given in the following table.

Table 2.1. The first few generalized Gaussian Woodall numbers.

n	GW_n	GW_{-n}
0	$W_0+i\frac{1}{4}(8W_0-5W_1+W_2)$	$W_0+i\frac{1}{4}(8W_0-5W_1+W_2)$
1	W_1+iW_0	$\frac{1}{4}(8W_0-5W_1+W_2)+i\frac{1}{4}(11W_0-9W_1+2W_2)$
2	W_2+iW_1	$\frac{1}{4}(11W_0-9W_1+2W_2)+i\frac{1}{16}(52W_0-47W_1+11W_2)$

We consider four special cases of GW_n :

$GW_n(0,1,5+i)=GG_n$ is the sequence of Gaussian Modified Woodall numbers,

$GW_n(3+2i,5+3i,9+5i)=GH_n$ is the sequence of Gaussian Modified Cullen numbers,

$GW_n(-1-\frac{3}{2}i,1-i,7+i)=GR_n$ is the sequence of Gaussian Woodall numbers and

$GW_n(1+\frac{1}{2}i,3+i,9+3i)=GC_n$ is the sequence of Gaussian Cullen numbers.

Now, we formally define them.

Gaussian modified Woodall numbers, Gaussian modified Cullen numbers, Gaussian Woodall numbers and Gaussian Cullen numbers with the initial conditions are defined by

$$\begin{aligned}
GG_n &= 5GG_{n-1}-8GG_{n-2}+4GG_{n-3}, \quad GG_0=0, GG_1=1, GG_2=5+i, \\
GH_n &= 5GH_{n-1}-8GH_{n-2}+4GH_{n-3}, \quad GH_0=3+2i, GH_1=5+3i, GH_2=9+5i, \\
GR_n &= 5GR_{n-1}-8GR_{n-2}+4GR_{n-3}, \quad GR_0=-1-\frac{3}{2}i, GR_1=1-i, GR_2=7+i,
\end{aligned}$$

and

$$GC_n=5GC_{n-1}-8GC_{n-2}+4GC_{n-3}, \quad GC_0=1+\frac{1}{2}i, GC_1=3+i, GC_2=9+3i.$$

Note that for all integers n, we obtain

$$\begin{aligned}
GG_n &= G_n+iG_{n-1}, \\
GH_n &= H_n+iH_{n-1}, \\
GR_n &= R_n+iR_{n-1},
\end{aligned}$$

and

$$GC_n=C_n+iC_{n-1}.$$

Some values of Gaussian modified Woodall numbers, Gaussian modified Cullen numbers, Gaussian Woodall numbers and Gaussian Cullen numbers with positive and negative subscripts are given in the following table.

Table 2.2. Some values of special cases of generalized Gaussian Woodall numbers.

n	0	1	2	3	4	5	6
GG_n	0	1	$5+i$	$17+5i$	$49+17i$	$129+49i$	$321+129i$
GG_{-n}	0	$\frac{1}{4}i$	$\frac{1}{4}+\frac{1}{2}i$	$\frac{1}{2}+\frac{11}{16}i$	$\frac{11}{16}+\frac{13}{16}i$	$\frac{13}{16}+\frac{57}{64}i$	$\frac{57}{64}+\frac{15}{16}i$
GH_n	$3+2i$	$5+3i$	$9+5i$	$17+9i$	$33+17i$	$65+33i$	$129+65i$
GH_{-n}	$3+2i$	$2+\frac{3}{2}i$	$\frac{3}{2}+\frac{5}{4}i$	$\frac{5}{4}+\frac{9}{8}i$	$\frac{9}{8}+\frac{17}{16}i$	$\frac{17}{16}+\frac{33}{32}i$	$\frac{33}{32}+\frac{65}{64}i$
GR_n	$-1-\frac{3}{2}i$	$1-i$	$7+i$	$23+7i$	$63+23i$	$159+63i$	$383+159i$
GR_{-n}	$-1-\frac{3}{2}i$	$-\frac{3}{2}-\frac{3}{2}i$	$-\frac{3}{2}-\frac{11}{8}i$	$\frac{11}{8}-\frac{5}{4}i$	$-\frac{5}{4}-\frac{37}{32}i$	$-\frac{37}{32}-\frac{35}{32}i$	$-\frac{35}{32}-\frac{135}{128}i$
GC_n	$1+\frac{1}{2}i$	$3+i$	$9+3i$	$25+9i$	$65+25i$	$161+65i$	$385+161i$
GC_{-n}	$1+\frac{1}{2}i$	$\frac{1}{2}+\frac{1}{2}i$	$\frac{1}{2}+\frac{5}{8}i$	$\frac{5}{8}+\frac{3}{4}i$	$\frac{3}{4}+\frac{27}{32}i$	$\frac{27}{32}+\frac{29}{32}i$	$\frac{29}{32}+\frac{121}{128}i$

We now present Binet's formula for the Gaussian generalized Woodall numbers.

2.1.1 Binet's Formula For Gaussian Generalized Woodall Numbers

Theorem 2.1 *The Binet's formula for the Gaussian generalized Woodall numbers is* $GW_n = (((-W_2+4W_1-3W_0)+\frac{W_2-3W_1+2W_0}{2}n)2^n+(W_2-4W_1+4W_0))+i(((-W_2+4W_1-3W_0)+\frac{W_2-3W_1+2W_0}{2}(n-1))2^{n-1}+(W_2-4W_1+4W_0))$.

Proof. The proof follows from (1.2) and (2.2). □

The previous Theorem gives the following results, as special cases.

Corollary 2.1 *For all n we have the following Binet's Formulas:*

(a) $GG_n = i2^{n-1}(n-2)+2^n(n-1)+1+i$.

(b) $GH_n = 2i2^{n-1}+2\times 2^n+1+i$.

(c) $GR_n = i2^{n-1}(n-1)+2^n n-1-i$.

(d) $GC_n = i2^{n-1}(n-1)+2^n n+1+i$.

The next theorem presents the generating function of Gaussian generalized Woodall numbers.

2.1.2 Generating Function For Gaussian Generalized Woodall Numbers

Theorem 2.2 *The generating function of Gaussian generalized Woodall numbers is given as*

$$f_{GW_n}(x)=\sum_{n=0}^{\infty}GW_nx^n=\frac{GW_0+(GW_1-5GW_0)x+(GW_2-5GW_1+8GW_0)x^2}{1-5x+8x^2-4x^3}. \tag{2.4}$$

Proof. Let

$$f_{GW_n}(x)=\sum_{n=0}^{\infty}GW_nx^n$$

be generating function of Gaussian generalized Woodall numbers. Then using the definition of Gaussian Woodall numbers, and subtracting $xf(x)$, $x^2f(x)$ and $x^3f(x)$ from $f(x)$ we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
&(1-5x+8x^2-4x^3)f_{GW_n}(x)\\
=\;&\sum_{n=0}^{\infty}GW_nx^n-5x\sum_{n=0}^{\infty}GW_nx^n+8x^2\sum_{n=0}^{\infty}GW_nx^n-4x^3\sum_{n=0}^{\infty}GW_nx^n,\\
=\;&\sum_{n=0}^{\infty}GW_nx^n-5\sum_{n=0}^{\infty}GW_nx^{n+1}+8\sum_{n=0}^{\infty}GW_nx^{n+2}-4\sum_{n=0}^{\infty}GW_nx^{n+3},\\
=\;&\sum_{n=0}^{\infty}GW_nx^n-5\sum_{n=1}^{\infty}GW_{n-1}x^n+8\sum_{n=2}^{\infty}GW_{n-2}x^n-4\sum_{n=3}^{\infty}GW_{n-3}x^n,\\
=\;&(GW_0+GW_1x+GW_2x^2)-5(GW_0x+GW_1x^2)+8GW_0x^2\\
&+\sum_{n=3}^{\infty}(GW_n-5GW_{n-1}+8GW_{n-2}-4GW_{n-3})x^n,\\
=\;&GW_0+GW_1x+GW_2x^2-5GW_0x-5GW_1x^2+8GW_0x^2,\\
=\;&GW_0+(GW_1-5GW_0)x+(GW_2-5GW_1+8GW_0)x^2.
\end{aligned}$$

Now, it follows that

$$f_{GW_n}(x)=\frac{GW_0+(GW_1-5GW_0)x+(GW_2-5GW_1+8GW_0)x^2}{1-5x+8x^2-4x^3}.$$

This completes the proof. □

The previous theorem gives the following results as particular examples:

$$f_{GG_n}(x)=\frac{x+ix^2}{1-5x+8x^2-4x^3}, \tag{2.5}$$

$$f_{GH_n}(x)=\frac{(8+6i)\,x^2-(10+7i)\,x+3+2i}{1-5x+8x^2-4x^3}, \tag{2.6}$$

$$f_{GR_n}(x) = \frac{-(6+6i)\,x^2 + (6+\frac{13}{2}i)x - 1 - \frac{3}{2}i}{1-5x+8x^2-4x^3}, \tag{2.7}$$

$$f_{GC_n}(x) = \frac{(2+2i)\,x^2 - (2+\frac{3}{2}i)x + 1 + \frac{1}{2}i}{1-5x+8x^2-4x^3}. \tag{2.8}$$

2.2 SOME IDENTITIES RELATED TO GAUSSIAN WOODALL NUMBERS

In this section, we obtain some identities of Gaussian modified Woodall, Gaussian modified Cullen, Gaussian Woodall and Gaussian Cullen numbers.

Theorem 2.3 *The following equations hold for all integer n:*

$$GH_n = 2GG_{n+2} - 7GG_{n+1} + 6GG_n, \tag{2.9}$$

$$GH_n = 3GG_{n+1} - 10GG_n + 8GG_{n-1}, \tag{2.10}$$

$$GR_n = -2GC_{n+2} + 8GC_{n+1} - 7GC_n,$$

$$GG_n = -\frac{1}{2}GC_{n+2} + \frac{3}{2}GC_{n+1}, \tag{2.11}$$

$$GC_n = -\frac{7}{4}GR_{n+3} + \frac{27}{4}GR_{n+2} - 6GR_{n+1}, \tag{2.12}$$

$$GH_n = -\frac{1}{2}GR_{n+3} + \frac{5}{2}GR_{n+2} - 3GR_{n+1}, \tag{2.13}$$

$$GH_n = 5GG_n - 16GG_{n-1} + 12GG_{n-2}. \tag{2.14}$$

Proof. To proof identity (2.9), we can write

$$GH_n = aGG_{n+2} + bGG_{n+1} + cGG_n$$

and solving the system of equations

$$\begin{aligned} GH_0 &= aGG_2 + bGG_1 + cGG_0, \\ GH_1 &= aGG_3 + bGG_2 + cGG_1, \\ GH_2 &= aGG_4 + bGG_3 + cGG_2, \end{aligned}$$

we find that $a = 2$, $b = -7$, $c = 6$. Or using the relations $GH_n = H_n + iH_{n-1}$, $GG_n = G_n + iG_{n-1}$ and identity $H_n = 2G_{n+2} - 7G_{n+1} + 6G_n$, we obtain the identity (2.9). The others can be found similarly. □

Lemma 2.4 ([frontczakconvo2018]) *Suppose that* $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ *is the generating function of the sequence* $\{a_n\}_{n\geq 0}$. *Then the generating functions of the sequences* $\{a_{2n}\}_{n\geq 0}$ *and* $\{a_{2n+1}\}_{n\geq 0}$ *are given as*

$$f_{a_{2n}}(x) = \sum_{n=0}^{\infty} a_{2n} x^n = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2}$$

and

$$f_{a_{2n+1}}(x) = \sum_{n=0}^{\infty} a_{2n+1} x^n = \frac{f(\sqrt{x}) - f(-\sqrt{x})}{2\sqrt{x}}$$

respectively.

The next Theorem presents the generating functions of even and odd-indexed Gaussian generalized Woodall sequences.

Theorem 2.5 *The generating functions of the sequences* GW_{2n} *and* GW_{2n+1} *are given by*

$$f_{GW_{2n}}(x) = \frac{GW_0 - (9GW_0 - GW_2)x + (44GW_0 - 36GW_1 + 8GW_2)x^2}{1 - 9x + 24x^2 - 16x^3}$$

and

$$f_{GW_{2n+1}}(x) = \frac{GW_1 + (4GW_0 - 17GW_1 + 5GW_2)x + (32GW_0 - 20GW_1 + 4GW_2)x^2}{1 - 9x + 24x^2 - 16x^3}$$

respectively.

Proof. Both statements are consequences of Lemma (2.4) applied to (2.4) and some lengthy algebraic calculations. □
The previous theorem gives the following corollaries as particular examples.

Corollary 2.2 *We have the followings:*

(a) $f_{GR_{2n}}(x) = -\frac{(24+22i)x^2-(16+\frac{29}{2}i)x+1+\frac{3}{2}i}{1-9x+24x^2-16x^3}$ *and* $f_{GR_{2n+1}} = (x)\frac{-(24+24i)x^2+(14+16i)x+1-i}{1-9x+24x^2-16x^3}$.

(b) $f_{GC_{2n}}(x) = \frac{(8+10i)x^2-\frac{3}{2}ix+1+\frac{1}{2}i}{1-9x+24x^2-16x^3}$ *and* $f_{GC_{2n+1}}(x) = \frac{(8+8i)x^2-2x+3+i}{1-9x+24x^2-16x^3}$.

(c) $f_{GG_{2n}}(x) = \frac{(4+8i)x^2+(5+i)x}{1-9x+24x^2-16x^3}$ *and* $f_{GG_{2n+1}}(x) = \frac{4ix^2+(8+5i)x+1}{1-9x+24x^2-16x^3}$.

(d) $f_{GH_{2n}}(x) = \frac{(24+20i)x^2-(18+13i)x+3+2i}{1-9x+24x^2-16x^3}$ *and* $f_{GH_{2n+1}}(x) = \frac{(32+24i)x^2-(28+18i)x+5+3i}{1-9x+24x^2-16x^3}$.

From Corollary (2.2) we can obtain the following corollary which presents the identities of Gaussian Woodall sequences.

Corollary 2.3 *We have the following identities:*

(a) $(4+8i)\,GH_{2n-4}+(5+i)\,GH_{2n-2}=(24+20i)GG_{2n-4}-(18+13i)\,GG_{2n-2}+(3+2i)GG_{2n}$.

(b) $(4+8i)\,GH_{2n-3}+(5+i)\,GH_{2n-1}=(32+24i)\,GG_{2n-4}-(28+18i)\,GG_{2n-2}+(5+3i)GG_{2n}$.

(c) $-(24+24i)GG_{2n-4}+(14+16i)GG_{2n-2}+(1-i)GG_{2n}=(4+8i)GR_{2n-3}+(5+i)GR_{2n-1}$.

(d) $-(24+24i)GG_{2n-3}+(14+16i)GG_{2n-1}+(1-i)GG_{2n+1}=4iGR_{2n-3}+(8+5i)GR_{2n-1}+GR_{2n+1}$.

(e) $(8+10i)\,GG_{2n-4}-\frac{3}{2}iGG_{2n-2}+(1+\frac{1}{2}i)GG_{2n}=(4+8i)GC_{2n-4}+(5+i)\,GC_{2n-2}$.

(f) $(8+10i)GG_{2n-3}-\frac{3}{2}iGG_{2n-1}+(1+\frac{1}{2}i)GG_{2n+1}=4iGC_{2n-4}+(8+5i)GC_{2n-2}+GC_{2n}$.

(g) $(8+8i)\,GG_{2n-4}-2GG_{2n-2}+(3+i)GG_{2n}=(4+8i)GC_{2n-3}+(5+i)\,GC_{2n-1}$.

(h) $(8+8i)GG_{2n-3}-2GG_{2n-1}+(3+i)GG_{2n+1}=4iGC_{2n-3}+(8+5i)\,GC_{2n-1}+GC_{2n+1}$.

(i) $-(24+22i)GG_{2n-4}+(16+\frac{29}{2}i)GG_{2n-2}-(1+\frac{3}{2}i)GG_{2n}=(4+8iGR_{2n-4}+(5+i)\,GR_{2n-2}$.

(j) $-(24+22i)GG_{2n-3}+(16+\frac{29}{2}i)GG_{2n-1}-(1+\frac{3}{2}i)GG_{2n+1}=4iGR_{2n-4}+(8+5i)\,GR_{2n-2}+GR_{2n}$.

(k) $4iGH_{2n-4}+(8+5i)\,GH_{2n-2}+GH_{2n}=(24+20i)\,GG_{2n-3}-(18+13i)\,GG_{2n-1}+(3+2i)GG_{2n+1}$.

(l) $4iGH_{2n-3}+(8+5i)\,GH_{2n-1}+GH_{2n+1}=(32+24i)\,GG_{2n-3}-(28+18i)\,GG_{2n-1}+(5+3i)GG_{2n+1}$.

(m) $-(24+22i)C(2n-3)+(16+\frac{29}{2}i)C(2n-1)-(1+\frac{3}{2}i)C(2n+1)=(8+8i)\,R(2n-4)-2R(2n-2)+(3+i)R(2n)$.

(n) $-(24+24i)\,C(2n-4)+(14+16i)\,C(2n-2)+(1-i)C(2n)=(8+10i)\,R(2n-3)-\frac{3}{2}iR(2n-1)+(1+\frac{1}{2}i)R(2n+1).$

Proof. From (2.2) we obtain

$$((4+8i)\,x^2+(5+i)\,x)f_{GH_{2n}}=((24+20i)\,x^2-(18+13i)\,x+3+2i)f_{GG_{2n}}.$$

The LHS (left hand side) is equal to

$$\begin{aligned}
LHS &= ((5+i)\,x+(4+8i)\,x^2)\sum_{n=0}^{\infty}GH_{2n}x^n\\
&= (5+i)\,x\sum_{n=0}^{\infty}GH_{2n}x^n+(4+8i)\,x^2\sum_{n=0}^{\infty}GH_{2n}x^n\\
&= (5+i)\sum_{n=0}^{\infty}GH_{2n}x^{n+1}+(4+8i)\sum_{n=0}^{\infty}GH_{2n}x^{n+2}\\
&= (5+i)\sum_{n=1}^{\infty}GH_{2n-2}x^n+(4+8i)\sum_{n=2}^{\infty}GH_{2n-4}x^n\\
&= (5+i)\,GH_0x\sum_{n=2}^{\infty}GH_{2n-2}x^n+(4+8i)\sum_{n=2}^{\infty}GH_{2n-4}x^n\\
&= (5+i)\,(3+2i)x+\sum_{n=2}^{\infty}((4+8i)\,GH_{2n-4}+(5+i)\,GH_{2n-2})\,x^n
\end{aligned}$$

whereas the RHS is

RHS

$$\begin{aligned}
&= (3+2i-(18+13i)\,x+(24+20i)\,x^2)\sum_{n=0}^{\infty}GG_{2n}x^n\\
&= (3+2i)\sum_{n=0}^{\infty}GG_{2n}x^n-(18+13i)\,x\sum_{n=0}^{\infty}GG_{2n}x^n+(24+20i)\,x^2\sum_{n=0}^{\infty}GG_{2n}x^n\\
&= (3+2i)\sum_{n=0}^{\infty}GG_{2n}x^n-(18+13i)\sum_{n=0}^{\infty}GG_{2n}x^{n+1}+(24+20i)\sum_{n=0}^{\infty}GG_{2n}x^{n+2}\\
&= (3+2i)\sum_{n=0}^{\infty}GG_{2n}x^n-(18+13i)\sum_{n=1}^{\infty}GG_{2n-2}x^n+(24+20i)\sum_{n=2}^{\infty}GG_{2n-4}x^n\\
&= (3+2i)(GG_0+GG_2x)\sum_{n=2}^{\infty}GG_{2n}x^n-(18+13i)\,(GG_0x)\sum_{n=2}^{\infty}GG_{2n-2}x^n\\
&\quad +(24+20i)\sum_{n=2}^{\infty}GG_{2n-4}x^n\\
&= (3+2i)\,(5+i)\,x+\sum_{n=2}^{\infty}((24+20i)GG_{2n-4}-(18+13i)\,GG_{2n-2}+(3+2i)GG_{2n})\,x^n.
\end{aligned}$$

Compare the coefficients and the proof of the first identity (a) is done. The other identities can be proved similarly. □

We present an identity related to Gaussian generalized Woodall numbers and Woodall numbers.

Theorem 2.6 *For all $n, m \in \mathbb{Z}$, the following identity holds:*

$$GW_{m+n} = G_{m+1}GW_n + (-8G_m + 4G_{m-1})GW_{n-1} + 4G_mGW_{n-2}. \tag{2.15}$$

Proof. First, we assume that $m \geq 0$. We prove the identity (2.15) by induction on m. If $m = 0$ then

$$GW_n = G_1GW_n + (-8G_0 + 4G_{-1})GW_{n-1} + 4G_0GW_{n-2}$$

which is true because $G_{-1} = 0$, $G_0 = 0$, $G_1 = 1$. Assume that the equality holds for $m \leq k$. For $m = k + 1$, we have

$$\begin{aligned}
GW_{(k+1)+n} &= 5GW_{n+k} - 8GW_{n+k-1} + 4GW_{n+k-2} \\
&= 5(G_{k+1}GW_n + (-8G_k + 4G_{k-1})GW_{n-1} + 4G_kGW_{n-2}) \\
&\quad -8(G_kGW_n + (-8G_{k-1} + 4G_{k-2})GW_{n-1} + 4G_{k-1}GW_{n-2}) \\
&\quad +4(G_{k-1}GW_n + (-8G_{k-2} + 4G_{k-3})GW_{n-1} + 4G_{k-2}GW_{n-2}) \\
&= (5G_{k+1} - 8G_k + 4G_{k-1})GW_n + (-8(G_k + G_{k-1} + G_{k-2}) \\
&\quad +4(G_{k-1} + G_{k-2} + G_{k-3}))GW_{n-1} + 4(G_k + G_{k-1} + G_{k-2})GW_{n-2} \\
&= G_{k+2}GW_n + (-8G_{k+1} + 4G_k)GW_{n-1} + 4G_{k+1}GW_{n-2} \\
&= G_{(k+1)+1}GW_n + (-8G_{k+1} + 4G_{(k+1)-1})GW_{n-1} + 4G_{k+1}GW_{n-2}.
\end{aligned}$$

By mathematical induction on m, this proves (2.6). The other cases can be shown similarly. □

The previous theorem gives the following results as particular examples:

For all $n, m \in \mathbb{Z}$, we have (taking $GW_n = GG_n$ or $GW_n = GH_n$ or $GW_n = GR_n$ or $GW_n = GC_n$)

$$\begin{aligned}
GG_{m+n} &= G_{m+1}GG_n + (-8G_m + 4G_{m-1})GG_{n-1} + 4G_mGG_{n-2}, \\
GH_{m+n} &= G_{m+1}GH_n + (-8G_m + 4G_{m-1})GH_{n-1} + 4G_mGH_{n-2}, \\
GR_{m+n} &= G_{m+1}GR_n + (-8G_m + 4G_{m-1})GR_{n-1} + 4G_mGR_{n-2}, \\
GC_{m+n} &= G_{m+1}GC_n + (-8G_m + 4G_{m-1})GC_{n-1} + 4G_mGC_{n-2}.
\end{aligned}$$

2.3 SIMPSON'S FORMULA

In this section, we present Simpson's formula of generalized Gaussian Woodall numbers.

Theorem 2.7 *(Simpson's formula of generalized Gaussian Woodall numbers). For all integers n, we have*

$$\begin{aligned}\begin{vmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{vmatrix} &= 4^n \begin{vmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{vmatrix} \\ &= 4^n(\frac{1}{800}+\frac{7}{800}i)(W_0-W_1+\frac{1}{4}W_2)((2-14i)W_0 \\ &\quad -(3-21i)W_1+(1-7i)W_2)^2. \end{aligned}$$

Proof. Use [55, Theorem 3.1]. □

From the Theorem (2.7) we get the following corollary.

Corollary 2.4 *For all integers n, we get the following identities.*

(a) $\begin{vmatrix} GG_{n+2} & GG_{n+1} & GG_n \\ GG_{n+1} & GG_n & GG_{n-1} \\ GG_n & GG_{n-1} & GG_{n-2} \end{vmatrix} = (1-7i)\,2^{2n-4}.$

(b) $\begin{vmatrix} GH_{n+2} & GH_{n+1} & GH_n \\ GH_{n+1} & GH_n & GH_{n-1} \\ GH_n & GH_{n-1} & GH_{n-2} \end{vmatrix} = 0.$

(c) $\begin{vmatrix} GR_{n+2} & GR_{n+1} & GR_n \\ GR_{n+1} & GR_n & GR_{n-1} \\ GR_n & GR_{n-1} & GR_{n-2} \end{vmatrix} = -\,(1-7i)\,2^{2n-4}.$

(d) $\begin{vmatrix} GC_{n+2} & GC_{n+1} & GC_n \\ GC_{n+1} & GC_n & GC_{n-1} \\ GC_n & GC_{n-1} & GC_{n-2} \end{vmatrix} = (1-7i)\,2^{2n-4}.$

2.4 SUM FORMULAS

In this section, we give some sum formulas of generalized Gaussian Woodall numbers.

Theorem 2.8 *For all integers $n \geq 0$, we have the following formulas:*

(a) $\sum_{k=0}^{n} GW_k = \frac{1}{2}W_2(2n - 2^{n+1}(n-1) + 2^{n+2}\,(n-2) + 6) - \frac{1}{2}W_1(8n - 2^{n+1}(3n-5) + 2^{n+2}(3n-8) + 22) + W_0(4n - 2^{n+1}(n-2) + 2^{n+2}(n-3) + 9) + i(\frac{1}{4}(28 + 16n - 5 \times 2^{n+2} + 2^{n+2}n)W_0 + (-33 - 16n + 7 \times 2^{n+2} - 3 \times 2^{n+1}n)W_1 + (9 + 4n - 2^{n+3} + 2^{n+1}n)W_2)$.

(b) $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{18}W_2(18n - 2^{2n+3}(2n+1) + 2^{2n+5}(2n-1) + 40) - \frac{1}{18}W_1(72n - 2^{2n+3}(6n+1) + 2^{2n+5}(6n-5) + 150) + \frac{1}{9}W_0(36n + 2^{2n+5}(2n-2) - 2 \times 2^{2n+3}n + 64) + i(\frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) - \frac{1}{18}W_1(72n - 2^{2n+2}(6n-2) + 2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32))$.

(c) $\sum_{k=0}^{n} GW_{2k} = \frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) - \frac{1}{18}W_1(72n - 2^{2n+2}(6n-2) + 2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32) + i((\frac{1}{9}W_0(36n - 2^{2n+1}(2n-2) + 2^{2n+3}(2n-4) + 46) + \frac{1}{18}W_2((18n - 2^{2n+1}(2n-1) + 2^{2n+3}(2n-3) + \frac{53}{2}) - \frac{1}{18}W_1(72n - 2^{2n+1}(6n-5) + 2^{2n+3}(6n-11) + \frac{201}{2}))$.

Proof.

(a) When we use (2.2),

$$\sum_{k=0}^{n} GW_k = \sum_{k=0}^{n} W_k + i\sum_{k=0}^{n} W_{k-1}.$$

So, then we obtain

$$\begin{aligned}\sum_{k=0}^{n} W_k &= \frac{1}{2}W_2(2n - 2^{n+1}(n-1) + 2^{n+2}(n-2) + 6) - \frac{1}{2}W_1(8n - 2^{n+1}(3n-5) \\ &\quad + 2^{n+2}(3n-8) + 22) + W_0(4n - 2^{n+1}(n-2) + 2^{n+2}(n-3) + 9))\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{n} W_{k-1} &= (\frac{1}{4}((28 + 16n - 5 \times 2^{n+2} + 2^{n+2}n)W_0 + (-33 - 16n + 7 \times 2^{n+2} \\ &\quad -3 \times 2^{n+1}n)W_1 + (9 + 4n - 2^{n+3} + 2^{n+1}n)W_2))\end{aligned}$$

from (59, Theorem 6.1). We get

$$\begin{aligned}\sum_{k=0}^{n} GW_k &= \frac{1}{2}W_2(2n-2^{n+1}(n-1)+2^{n+2}(n-2)+6)-\frac{1}{2}W_1(8n-2^{n+1}(3n-5)\\ &\quad +2^{n+2}(3n-8)+22)+W_0(4n-2^{n+1}(n-2)+2^{n+2}(n-3)+9))\\ &\quad +i(\frac{1}{4}((28+16n-5\times 2^{n+2}+2^{n+2}n)W_0+(-33-16n+7\times 2^{n+2}\\ &\quad -3\times 2^{n+1}n)W_1+(9+4n-2^{n+3}+2^{n+1}n)W_2)).\end{aligned}$$

(b) When we use (2.2), we get

$$\sum_{k=0}^{n} GW_{2k+1}=\sum_{k=0}^{n} W_{2k+1}+i\sum_{k=0}^{n} W_{2k}$$

and so, from (1.3), we know that:

$$\begin{aligned}\sum_{k=0}^{n} W_{2k+1} &= \frac{1}{18}W_2(18n-2^{2n+3}(2n+1)+2^{2n+5}(2n-1)+40)\\ &\quad -\frac{1}{18}W_1(72n-2^{2n+3}(6n+1)+2^{2n+5}(6n-5)+150)\\ &\quad +\frac{1}{9}W_0(36n+2^{2n+5}(2n-2)-2\times 2^{2n+3}n+64)\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{n} W_{2k} &= ((\frac{1}{9}W_0(36n-2^{2n+2}(2n-1)+2^{2n+4}(2n-3)+53)\\ &\quad -\frac{1}{18}W_1(72n-2^{2n+2}(6n-2)+2^{2n+4}(6n-8)+120)\\ &\quad +\frac{1}{18}W_2(18n+2^{2n+4}(2n-2)-2\times 2^{2n+2}n+32)).\end{aligned}$$

We get

$$\begin{aligned}\sum_{k=0}^{n} GW_{2k+1} &= \frac{1}{18}W_2(18n-2^{2n+3}(2n+1)+2^{2n+5}(2n-1)+40)\\ &\quad -\frac{1}{18}W_1(72n-2^{2n+3}(6n+1)+2^{2n+5}(6n-5)+150)\\ &\quad +\frac{1}{9}W_0(36n+2^{2n+5}(2n-2)-2\times 2^{2n+3}n+64)\\ &\quad +i((\frac{1}{9}W_0(36n-2^{2n+2}(2n-1)+2^{2n+4}(2n-3)+53)\\ &\quad -\frac{1}{18}W_1(72n-2^{2n+2}(6n-2)+2^{2n+4}(6n-8)+120)\\ &\quad +\frac{1}{18}W_2(18n+2^{2n+4}(2n-2)-2\times 2^{2n+2}n+32)).\end{aligned}$$

(c) From (1.3) and (59, Theorem 6.1), we get

$$\begin{aligned}\sum_{k=0}^{n} W_{2k} &= ((\frac{1}{9}W_0(36n - 2^{2n+2}((2n-1) + 2^{2n+4}((2n-3)+53) \\ &\quad -\frac{1}{18}W_1(72n - 2^{2n+2}(6n-2) + 2^{2n+4}(6n-8) + 120) \\ &\quad +\frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2\times 2^{2n+2}n + 32))\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{n} W_{2k-1} &= (\frac{1}{9}W_0(36n - 2^{2n+1}(2n-2) + 2^{2n+3}(2n-4) + 46) \\ &\quad +\frac{1}{18}W_2(18n - 2^{2n+1}(2n-1) + 2^{2n+3}(2n-3) + \frac{53}{2}) \\ &\quad -\frac{1}{18}W_1(72n - 2^{2n+1}(6n-5) + 2^{2n+3}(6n-11) + \frac{201}{2})).\end{aligned}$$

So, we use

$$\sum_{k=0}^{n} GW_{2k} = \sum_{k=0}^{n} W_{2k} + i\sum_{k=0}^{n} W_{2k-1},$$

we get

$$\begin{aligned}\sum_{k=0}^{n} GW_{2k} &= \frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) \\ &\quad -\frac{1}{18}W_1(72n - 2^{2n+2}(6n-2) + 2^{2n+4}(6n-8) + 120) \\ &\quad +\frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2\times 2^{2n+2}n + 32) \\ &\quad +i(\frac{1}{9}W_0(36n - 2^{2n+1}(2n-2) + 2^{2n+3}(2n-4) + 46) \\ &\quad +\frac{1}{18}W_2((18n - 2^{2n+1}(2n-1) + 2^{2n+3}(2n-3) + \frac{53}{2}) \\ &\quad -\frac{1}{18}W_1(72n - 2^{2n+1}(6n-5) + 2^{2n+3}(6n-11) + \frac{201}{2})).\end{aligned}$$

This completes the proof. □

As special cases of the above Theorem, we have the following four Corollary. First, taking $GW_n = GG_n$ with $GG_0 = 0, GG_1 = 1, GG_2 = 5 + i$, we get:

Corollary 2.5 *(Sum of the Gaussian modified Woodall numbers). For $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=0}^{n} GG_k = (1+i)n + (2+i)2^n n - (4+3i)2^n + (4+3i)$.

(b) $\sum_{k=0}^{n} GG_{2k+1} = \frac{4}{9}((\frac{9}{4}+\frac{9}{4}i)n - (4+5i)2^{2n} + (12+6i)2^{2n}n + (\frac{25}{4}+5i))$.

(c) $\sum_{k=0}^{n} GG_{2k} = \frac{4}{9}((\frac{9}{4}+\frac{9}{4}i)n - (5+4i)2^{2n} + (6+3i)2^{2n}n + (5+4i))$.

Second, taking $GW_n = GH_n$ with $GH_0 = 3+2i, GH_1 = 5+3i, GH_2 = 9+5i$, we have the following corollary:

Corollary 2.6 *(Sum of the Gaussian modified Cullen numbers). For $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=0}^{n} GH_k = 2^{n+2} + n - 1 + i(n + 2^{n+1})$.

(b) $\sum_{k=0}^{n} GH_{2k+1} = \frac{1}{3}(2^{2n+4} + 3n - 1) + i(\frac{1}{3}(2^{2n+3} + 3n + 1))$.

(c) $\sum_{k=0}^{n} GH_{2k} = \frac{1}{3}(2^{2n+3} + 3n + 1) + i(n + \frac{1}{3}2^{2n+2} + \frac{2}{3})$.

Third, taking $GW_n = GR_n$ with $GR_0 = -1 - \frac{3}{2}i, GR_1 = 1 - i, GR_2 = 7 + i$, we get the following corollary:

Corollary 2.7 *(Sum of the Gaussian Woodall numbers). For $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=0}^{n} GR_k = (n-1)(2^{n+1} - 1) + i(2^{n+1}(n-1) - n - 2^n n + \frac{1}{2})$.

(b) $\sum_{k=0}^{n} GR_{2k+1} = \frac{1}{9}((6n+1)2^{2n+3} - 9n + 1) + i(\frac{1}{9}((3n-1)2^{2n+3} - 9n - 1))$.

(c) $\sum_{k=0}^{n} GR_{2k} = \frac{1}{9}((3n-1)2^{2n+3} - 9n - 1) + i(\frac{1}{9}2^{2n+3}(2n-1) - \frac{1}{9}2^{2n+1}(2n+1) - n - \frac{7}{18})$.

Fourth, taking $GW_n = GC_n$ with $GC_0 = 1 + \frac{1}{2}i, GC_1 = 3 + i, GC_2 = 9 + 3i$, we have the following corollary:

Corollary 2.8 *(Sum of the Gaussian Cullen numbers). For $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=0}^{n} GC_k = (n-1)2^{n+1} + n + 3 + i(n + 2^{n+1}(n-1) - 2^n n + \frac{5}{2})$.

(b) $\sum_{k=0}^{n} GC_{2k+1} = \frac{1}{9}((6n+1)2^{2n+3} + 9n + 19) + i(\frac{1}{9}((3n-1)2^{2n+3} + 9n + 17))$.

(c) $\sum_{k=0}^{n} GC_{2k} = \frac{2}{9}((\frac{9}{2}+\frac{9}{2}i)n - (4+5i)\,2^{2n} + (12+6i)\,2^{2n}n + (\frac{17}{2}+\frac{29}{4}i))$.

2.5 MATRIX FORMULA OF GW_n

Consider the sequence $\{G_n\}$ which is defined by the third-order recurrence relation

$$G_n = 5G_{n-1} - 8G_{n-2} + 4G_{n-3}$$

with the initial conditions

$$G_0 = 0, G_1 = 1, G_2 = 5.$$

We define the square matrix A of order 3 as

$$A = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 4$. We give the following Lemma.

Note that

$$A^n = \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & G_{n-2} \end{pmatrix}.$$

For the proof see [60].

Lemma 2.9 *For all integers, the following identity is true*

$$\begin{pmatrix} GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

Proof. We suppose that $n \geq 0$. We prove the required equality by induction on n. If $n = 0$ we obtain

$$\begin{pmatrix} GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. So, the following identity is true.

$$\begin{pmatrix} GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

For $n = k+1$, we get

$$\begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix}$$

$$= \begin{pmatrix} 5GW_{k+2} - 8GW_{k+1} + 4GW_k \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix}.$$

Consequently, by induction on n, the proof is finished. The case $n < 0$ can be proved similarly. □

Theorem 2.10 *We assume that the matrices N_{GW} and E_{GW} are defined as follows*

$$N_{GW} = \begin{pmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{pmatrix},$$

$$E_{GW} = \begin{pmatrix} GG_{n+2} & GG_{n+1} & GG_n \\ GG_{n+1} & GG_n & GG_{n-1} \\ GG_n & GG_{n-1} & GG_{n-2} \end{pmatrix}.$$

Then, the following identity is true between A^n, N_{GW} and E_{GW} :

$$A^n N_{GW} = E_{GW}.$$

Proof. Note that one gets

$$\begin{aligned}
A^n N_{GW} &= \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & G_{n-2} \end{pmatrix} \begin{pmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}
\end{aligned}$$

such that

$$\begin{aligned}
b_{11} &= GW_2 G_{n+1} + GW_1 \left(4G_{n-1} - 8G_n\right) + GW_0 G_n, \\
b_{12} &= GW_1 G_{n+1} + GW_0 \left(4G_{n-1} - 8G_n\right) + GW_{-1} G_n, \\
b_{13} &= GW_0 G_{n+1} + GW_{-1} \left(4G_{n-1} - 8G_n\right) + GW_{-2} G_n, \\
b_{21} &= GW_2 G_n + GW_1 \left(4G_{n-2} - 8G_{n-1}\right) + GW_0 G_{n-1}, \\
b_{22} &= GW_1 G_n + GW_0 \left(4G_{n-2} - 8G_{n-1}\right) + GW_{-1} G_{n-1}, \\
b_{23} &= GW_0 G_n + GW_{-1} \left(4G_{n-2} - 8G_{n-1}\right) + GW_{-2} G_{n-1}, \\
b_{31} &= GW_2 G_{n-1} + GW_1 \left(4G_{n-3} - 8G_{n-2}\right) + GW_0 G_{n-2}, \\
b_{32} &= GW_1 G_{n-1} + GW_0 \left(4G_{n-3} - 8G_{n-2}\right) + GW_{-1} G_{n-2}, \\
b_{33} &= GW_0 G_{n-1} + GW_{-1} \left(4G_{n-3} - 8G_{n-2}\right) + GW_{-2} G_{n-2}.
\end{aligned}$$

Using the Theorem (2.6) the proof is completed. □

We have the following identities for N_{GW}, E_{GW} :

$$N_{GG} = \begin{pmatrix} 5+i & 1 & 0 \\ 1 & 0 & \frac{1}{4}i \\ 0 & \frac{1}{4}i & \frac{1}{4}+\frac{1}{2}i \end{pmatrix}, \quad N_{GH} = \begin{pmatrix} 9+5i & 5+3i & 3+2i \\ 5+3i & 3+2i & 2+\frac{3}{2}i \\ 3+2i & 2+\frac{3}{2}i & \frac{3}{2}+\frac{5}{4}i \end{pmatrix},$$

$$N_{GR} = \begin{pmatrix} 7+i & 1-i & -1-\frac{3}{2}i \\ 1-i & -1-\frac{3}{2}i & -\frac{3}{2}-\frac{3}{2}i \\ -1-\frac{3}{2}i & -\frac{3}{2}-\frac{3}{2}i & -\frac{3}{2}-\frac{11}{8}i \end{pmatrix}, \quad N_{GC} = \begin{pmatrix} 9+3i & 3+i & 1+\frac{1}{2}i \\ 3+i & 1+\frac{1}{2}i & \frac{1}{2}+\frac{1}{2}i \\ 1+\frac{1}{2}i & \frac{1}{2}+\frac{1}{2}i & \frac{1}{2}+\frac{5}{8}i \end{pmatrix}.$$

and

$$E_{GG} = \begin{pmatrix} GG_{n+2} & GG_{n+1} & GG_n \\ GG_{n+1} & GG_n & GG_{n-1} \\ GG_n & GG_{n-1} & GG_{n-2} \end{pmatrix} \quad , \quad E_{GH} = \begin{pmatrix} GH_{n+2} & GH_{n+1} & GH_n \\ GH_{n+1} & GH_n & GH_{n-1} \\ GH_n & GH_{n-1} & GH_{n-2} \end{pmatrix},$$

$$E_{GR} = \begin{pmatrix} GR_{n+2} & GR_{n+1} & GR_n \\ GR_{n+1} & GR_n & GR_{n-1} \\ GR_n & GR_{n-1} & GR_{n-2} \end{pmatrix} \quad , \quad E_{GC} = \begin{pmatrix} GC_{n+2} & GC_{n+1} & GC_n \\ GC_{n+1} & GC_n & GC_{n-1} \\ GC_n & GC_{n-1} & GC_{n-2} \end{pmatrix}.$$

From the previous theorem presented, we have the following corollary.

Corollary 2.9 *The following identities are true:*

(a) $A^n N_{GG} = E_{GG}$.

(b) $A^n N_{GH} = E_{GH}$.

(c) $A^n N_{GR} = E_{GR}$.

(d) $A^n N_{GC} = E_{GC}$.

CHAPTER 3

DUAL GENERALIZED WOODALL NUMBERS

In this chapter, we define dual generalized Woodall numbers and present generating functions and Binet's formulas for them. Furthermore, we obtain some identities for dual generalized Woodall numbers. In addition, we obtain the summation formulas of the dual generalized Woodall numbers with positive and negative subscripts. Finally, we investigate matrices related to dual generalized Woodall numbers.

3.1 DEFINITIONS AND PROPERTIES

We now define dual generalized Woodall numbers over $\mathbb{D}$. The nth dual generalized Woodall number is

$$\mathcal{D}W_n = W_n + \varepsilon W_{n+1}. \tag{3.1}$$

with the initial values $\mathcal{D}W_0$, $\mathcal{D}W_1$, $\mathcal{D}W_2$. (3.1) can be written to negative subscripts by defining,

$$\mathcal{D}W_{-n} = W_{-n} + \varepsilon W_{-n+1}. \tag{3.2}$$

So, identity (3.1) holds for all integers n.
The special cases, the nth dual modified Woodall, the nth dual modified Cullen, the nth dual Woodall and the nth dual Cullen numbers are given as

$$\begin{aligned}
\mathcal{D}G_n &= G_n + \varepsilon G_{n+1}, \\
\mathcal{D}H_n &= H_n + \varepsilon H_{n+1}, \\
\mathcal{D}R_n &= R_n + \varepsilon R_{n+1}, \\
\mathcal{D}C_n &= C_n + \varepsilon C_{n+1}.
\end{aligned}$$

Then, using (3.1), the following identity can be expressed true for every integers $n \geq 0$,

$$\mathcal{D}W_n = 5\mathcal{D}W_{n-1} - 8\mathcal{D}W_{n-2} + 4\mathcal{D}W_{n-3}. \tag{3.3}$$

Using (3.2), the sequence $\{\mathcal{D}W_n\}_{n\geq 0}$ can be written as

$$\mathcal{D}W_{-n} = -2\mathcal{D}W_{-(n-1)} - \frac{5}{4}\mathcal{D}W_{-(n-2)} + \frac{1}{4}\mathcal{D}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (3.3) holds for all integers n.
The initial several dual generalized Woodall numbers with positive subscript and negative subscript are given in the following Table 3.1.

Table 3.1. A few dual generalized Woodall numbers.

n	$\mathcal{D}W_n$	$\mathcal{D}W_{-n}$
0	$\mathcal{D}W_0$	$\mathcal{D}W_0$
1	$\mathcal{D}W_1$	$\frac{1}{4}(8\mathcal{D}W_0 - 5\mathcal{D}W_1 + \mathcal{D}W_2)$
2	$\mathcal{D}W_2$	$\frac{1}{4}(11\mathcal{D}W_0 - 9\mathcal{D}W_1 + 2\mathcal{D}W_2)$
3	$4\mathcal{D}W_0 - 8\mathcal{D}W_1 + 5\mathcal{D}W_2$	$\frac{1}{16}(52\mathcal{D}W_0 - 47\mathcal{D}W_1 + 11\mathcal{D}W_2)$
4	$20\mathcal{D}W_0 - 36\mathcal{D}W_1 + 17\mathcal{D}W_2$	$\frac{1}{16}(57\mathcal{D}W_0 - 54\mathcal{D}W_1 + 13\mathcal{D}W_2)$
5	$68\mathcal{D}W_0 - 116\mathcal{D}W_1 + 49\mathcal{D}W_2$	$\frac{1}{64}(240\mathcal{D}W_0 - 233\mathcal{D}W_1 + 57\mathcal{D}W_2)$

Note that

$$\begin{aligned}
\mathcal{D}W_0 &= W_0 + \varepsilon W_1, \\
\mathcal{D}W_1 &= W_1 + \varepsilon W_2, \\
\mathcal{D}W_2 &= W_2 + \varepsilon W_3 = W_2 + \varepsilon(4W_0 - 8W_1 + 5W_2).
\end{aligned}$$

For dual modified Woodall numbers (taking $W_n = G_n$, $G_0 = 0$, $G_1 = 1$, $G_2 = 5$), we get

$$\begin{aligned}
\mathcal{D}G_0 &= G_0 + \varepsilon G_1 = \varepsilon, \\
\mathcal{D}G_1 &= G_1 + \varepsilon G_2 = 1 + 5\varepsilon, \\
\mathcal{D}G_2 &= G_2 + \varepsilon G_3 = 5 + 17\varepsilon,
\end{aligned}$$

and for dual modified Cullen numbers (taking $W_n = H_n$, $H_0 = 3$, $H_1 = 5$, $H_2 = 9$), we get

$$\begin{aligned}
\mathcal{D}H_0 &= H_0 + \varepsilon H_1 = 3 + 5\varepsilon, \\
\mathcal{D}H_1 &= H_1 + \varepsilon H_2 = 5 + 9\varepsilon, \\
\mathcal{D}H_2 &= H_2 + \varepsilon H_3 = 9 + 17\varepsilon,
\end{aligned}$$

and for dual Woodall numbers (taking $W_n = R_n$, $R_0 = -1$, $R_1 = 1$, $R_2 = 7$), we get

$$\begin{aligned}
\mathcal{D}R_0 &= R_0 + \varepsilon R_1 = -1 + \varepsilon, \\
\mathcal{D}R_1 &= R_1 + \varepsilon R_2 = 1 + 7\varepsilon, \\
\mathcal{D}R_2 &= R_2 + \varepsilon R_3 = 7 + 23\varepsilon,
\end{aligned}$$

and for dual Cullen numbers (taking $W_n = C_n$, $C_0 = 1$, $C_1 = 3$, $C_2 = 9$), we get

$$\begin{aligned}
\mathcal{D}C_0 &= C_0 + \varepsilon C_1 = 1 + 3\varepsilon, \\
\mathcal{D}C_1 &= C_1 + \varepsilon C_2 = 3 + 9\varepsilon, \\
\mathcal{D}C_2 &= C_2 + \varepsilon C_3 = 9 + 25\varepsilon.
\end{aligned}$$

A few dual modified Woodall numbers, dual modified Cullen numbers, dual Woodall numbers and dual Cullen numbers with positive subscript and negative subscript are given in the following Table 3.2, Table 3.3, Table 3.4 and Table 3.5.

Table 3.2. Dual Modified Woodall Numbers

n	$\mathcal{D}G_n$	$\mathcal{D}G_{-n}$
0	ε	ε
1	$5\varepsilon + 1$	0
2	$17\varepsilon + 5$	$\frac{1}{4}$
3	$49\varepsilon + 17$	$\frac{1}{4}\varepsilon + \frac{1}{2}$
4	$129\varepsilon + 49$	$\frac{1}{2}\varepsilon + \frac{11}{16}$
5	$321\varepsilon + 129$	$\frac{11}{16}\varepsilon + \frac{13}{16}$

and

Table 3.3. Dual Modified Cullen Numbers

n	$\mathcal{D}H_n$	$\mathcal{D}H_{-n}$
0	$5\varepsilon + 3$	$5\varepsilon + 3$
1	$9\varepsilon + 5$	$3\varepsilon + 2$
2	$17\varepsilon + 9$	$2\varepsilon + \frac{3}{2}$
3	$33\varepsilon + 17$	$\frac{3}{2}\varepsilon + \frac{5}{4}$
4	$65\varepsilon + 33$	$\frac{5}{4}\varepsilon + \frac{9}{8}$
5	$129\varepsilon + 65$	$\frac{9}{8}\varepsilon + \frac{17}{16}$

and

Table 3.4. Dual Woodall Numbers

n	$\mathcal{D}R_n$	$\mathcal{D}R_{-n}$
0	$\varepsilon-1$	$\varepsilon-1$
1	$7\varepsilon+1$	$-\varepsilon-\frac{3}{2}$
2	$23\varepsilon+7$	$-\frac{3}{2}\varepsilon-\frac{3}{2}$
3	$63\varepsilon+23$	$-\frac{3}{2}\varepsilon-\frac{11}{8}$
4	$159\varepsilon+63$	$-\frac{11}{8}\varepsilon-\frac{5}{4}$
5	$383\varepsilon+159$	$-\frac{5}{4}\varepsilon-\frac{37}{32}$

and

Table 3.5. Dual Cullen Numbers

n	$\mathcal{D}C_n$	$\mathcal{D}C_{-n}$
0	$3\varepsilon+1$	$3\varepsilon+1$
1	$9\varepsilon+3$	$\varepsilon+\frac{1}{2}$
2	$25\varepsilon+9$	$\frac{1}{2}\varepsilon+\frac{1}{2}$
3	$65\varepsilon+25$	$\frac{1}{2}\varepsilon+\frac{5}{8}$
4	$161\varepsilon+65$	$\frac{5}{8}\varepsilon+\frac{3}{4}$
5	$385\varepsilon+161$	$\frac{3}{4}\varepsilon+\frac{27}{32}$

Now, we state Binet's formula for the dual generalized Woodall numbers and in the rest of the paper, we fix the following notations:

$$\begin{aligned}
\widehat{\alpha} &= 1+2\varepsilon,\\
\widehat{\beta} &= 2\varepsilon,\\
\widehat{\gamma} &= 1+\varepsilon.
\end{aligned}$$

Note that we have the following identities:

$$\begin{aligned}
\widehat{\alpha}^2 &= 1+4\varepsilon,\\
\widehat{\beta}^2 &= 0,\\
\widehat{\gamma}^2 &= 1+2\varepsilon,\\
\widehat{\alpha\beta} &= 2\varepsilon,\\
\widehat{\alpha\gamma} &= 1+3\varepsilon,\\
\widehat{\beta\gamma} &= 2\varepsilon,\\
\widehat{\alpha\beta\gamma} &= 2\varepsilon.
\end{aligned}$$

Next, we present Binet's formula.

3.1.1 Binet's Formula For Dual Generalized Woodall Numbers

Theorem 3.1 *(Binet's Formula) For any integer n, the nth dual generalized Woodall number is*

$$\mathcal{D}W_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2n\widehat{\alpha})2^n + A_3\widehat{\gamma}. \tag{3.4}$$

Proof. Using Binet's formula

$$W_n = (A_1 + A_2n)2^n + A_3$$

of the generalized Woodall numbers, we obtain

$$\begin{aligned}
\mathcal{D}W_n &= W_n + \varepsilon W_{n+1} \\
&= (A_1 + A_2n)2^n + A_3 + \varepsilon((A_1 + A_2(n+1))2^{n+1} + A_3) \\
&= A_1 2^n + A_2 n 2^n + A_3 \\
&\quad + \varepsilon A_1 2^{n+1} + \varepsilon A_2 n 2^{n+1} + \varepsilon A_2 2^{n+1} + \varepsilon A_3 \\
&= A_1 2^n(1+2\varepsilon) + A_2 n 2^n(1+2\varepsilon) + A_2 2^n(2\varepsilon) + A_3(1+\varepsilon) \\
&= A_1 2^n\widehat{\alpha} + A_2 n 2^n\widehat{\alpha} + A_2 2^n\widehat{\beta} + A_3\widehat{\gamma} \\
&= (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2n\widehat{\alpha})2^n + A_3\widehat{\gamma}.
\end{aligned}$$

This proves (3.4). □

As special cases, for any integer n, Binet's Formula of nth dual modified Woodall number, dual modified Cullen number, dual Woodall number and dual Cullen number are

- $\mathcal{D}G_n = (-\widehat{\alpha} + \widehat{\beta} + n\widehat{\alpha})2^n + \widehat{\gamma}$,

  $\mathcal{D}G_n = 1 + (n-1)2^n + \varepsilon(1 + n2^{n+1})$.

- $\mathcal{D}H_n = (2\widehat{\alpha})2^n + \widehat{\gamma}$,

  $\mathcal{D}H_n = 1 + 2^{n+1} + \varepsilon(1 + 2^{n+2})$.

- $\mathcal{D}R_n = (\widehat{\beta} + n\widehat{\alpha})2^n - \widehat{\gamma}$,

  $\mathcal{D}R_n = -1 + n2^n + \varepsilon(-1 + 2^{n+1} + n2^{n+1})$.

- $\mathcal{D}C_n = (\widehat{\beta} + n\widehat{\alpha})2^n + \widehat{\gamma}$,

  $\mathcal{D}C_n = 1 + n2^n + \varepsilon(1 + 2^{n+1} + n2^{n+1})$.

Next, we give a generating function for the dual generalized Woodall numbers.

3.1.2 Generating Function For Dual Generalized Woodall Numbers

Theorem 3.2 *The generating function for the dual generalized Woodall numbers is*

$$\sum_{n=0}^{\infty}\mathcal{D}W_nx^n=\frac{\mathcal{D}W_0+(\mathcal{D}W_1-5\mathcal{D}W_0)x+(\mathcal{D}W_2-5\mathcal{D}W_1+8\mathcal{D}W_0)x^2}{1-5x+8x^2-4x^3}. \tag{3.5}$$

Proof. Let

$$g(x)=\sum_{n=0}^{\infty}\mathcal{D}W_nx^n$$

be generating function of the dual generalized Woodall numbers. Then, using the definition of the dual generalized Woodall numbers, and subtracting $xg(x)$, $x^2g(x)$ and $x^3g(x)$ from $g(x)$, we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
&(1-5x+8x^2-4x^3)g(x)\\
=\;&\sum_{n=0}^{\infty}\mathcal{D}W_nx^n-5x\sum_{n=0}^{\infty}\mathcal{D}W_nx^n+8x^2\sum_{n=0}^{\infty}\mathcal{D}W_nx^n-4x^3\sum_{n=0}^{\infty}\mathcal{D}W_nx^n\\
=\;&\sum_{n=0}^{\infty}\mathcal{D}W_nx^n-5\sum_{n=0}^{\infty}\mathcal{D}W_nx^{n+1}+8\sum_{n=0}^{\infty}\mathcal{D}W_nx^{n+2}-4\sum_{n=0}^{\infty}\mathcal{D}W_nx^{n+3}\\
=\;&\sum_{n=0}^{\infty}\mathcal{D}W_nx^n-5\sum_{n=1}^{\infty}\mathcal{D}W_{n-1}x^n+8\sum_{n=2}^{\infty}\mathcal{D}W_{n-2}x^n-4\sum_{n=3}^{\infty}\mathcal{D}W_{n-3}x^n\\
=\;&(\mathcal{D}W_0+\mathcal{D}W_1x+\mathcal{D}W_2x^2)-5(\mathcal{D}W_0x+\mathcal{D}W_1x^2)+8\mathcal{D}W_0x^2\\
&+\sum_{n=3}^{\infty}(\mathcal{D}W_n-5\mathcal{D}W_{n-1}+8\mathcal{D}W_{n-2}-4\mathcal{D}W_{n-3})x^n\\
=\;&\mathcal{D}W_0+(\mathcal{D}W_1-5\mathcal{D}W_0)x+(\mathcal{D}W_2-5\mathcal{D}W_1+8\mathcal{D}W_0)x^2.
\end{aligned}$$

Note that we used the recurrence relation $\mathcal{D}W_n=5\mathcal{D}W_{n-1}-8\mathcal{D}W_{n-2}+4\mathcal{D}W_{n-3}$. Rearranging the above equation, we get

$$g(x)=\frac{\mathcal{D}W_0+(\mathcal{D}W_1-5\mathcal{D}W_0)x+(\mathcal{D}W_2-5\mathcal{D}W_1+8\mathcal{D}W_0)x^2}{1-5x+8x^2-4x^3}.$$

The proof is finished. □

As special cases, the generating functions for the dual modified Woodall, dual modified Cullen, dual Woodall and dual Cullen numbers are

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{D}G_nx^n &= \frac{\varepsilon+x}{1-5x+8x^2-4x^3},\\
\sum_{n=0}^{\infty}\mathcal{D}H_nx^n &= \frac{5\varepsilon+3+(-16\varepsilon-10)\,x+(12\varepsilon+8)\,x^2}{1-5x+8x^2-4x^3},\\
\sum_{n=0}^{\infty}\mathcal{D}R_nx^n &= \frac{-1+\varepsilon+(2\varepsilon+6)x+(-4\varepsilon-6)x^2}{1-5x+8x^2-4x^3}
\end{aligned}$$

and

$$\sum_{n=0}^{\infty}\mathcal{D}C_n x^n = \frac{3\varepsilon+1+(-6\varepsilon-2)\,x+(4\varepsilon+2)\,x^2}{1-5x+8x^2-4x^3}$$

respectively.

Now, we obtain Binet's formula from the generating function.

We obtain Binet's formula of dual generalized Woodall number $\{\mathcal{D}W_n\}$ by the use of generating function for $\mathcal{D}W_n$.

Theorem 3.3 *(Binet's formula of dual generalized Woodall numbers)*

$$\mathcal{D}W_n = (A_1\widehat{\alpha}+A_2\widehat{\beta}+A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}. \tag{3.6}$$

Proof. Let

$$\sum_{n=0}^{\infty}\mathcal{D}W_n x^n = \frac{\mathcal{D}W_0+(\mathcal{D}W_1-5\mathcal{D}W_0)x+(\mathcal{D}W_2-5\mathcal{D}W_1+8\mathcal{D}W_0)x^2}{1-5x+8x^2-4x^3}.$$

Then, we write

$$\frac{\mathcal{D}W_0+(\mathcal{D}W_1-5\mathcal{D}W_0)x+(\mathcal{D}W_2-5\mathcal{D}W_1+8\mathcal{D}W_0)x^2}{(1-x)\,(1-2x)^2} = \frac{d_1}{(1-x)}+\frac{d_2}{(1-2x)}+\frac{d_3}{(1-2x)^2}. \tag{3.7}$$

So,

$\mathcal{D}W_0+(\mathcal{D}W_1-5\mathcal{D}W_0)x+(\mathcal{D}W_2-5\mathcal{D}W_1+8\mathcal{D}W_0)x^2 = (\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3)+(-4d_1-3d_2-d_3)x+(4d_1+2d_2)x^2$.

We get

$$\begin{aligned}
\mathcal{D}W_0 &= d_1+d_2+d_3,\\
\mathcal{D}W_1-5\mathcal{D}W_0 &= -4d_1-3d_2-d_3,\\
\mathcal{D}W_2-5\mathcal{D}W_1+8\mathcal{D}W_0 &= 4d_1+2d_2.
\end{aligned}$$

If we solve these simultaneous equations, we get

$$\begin{aligned}
d_1 &= 4\mathcal{D}W_0-4\mathcal{D}W_1+\mathcal{D}W_2,\\
d_2 &= -4\mathcal{D}W_0+\frac{11}{2}\mathcal{D}W_1-\frac{3}{2}\mathcal{D}W_2,\\
d_3 &= \mathcal{D}W_0-\frac{3}{2}\mathcal{D}W_1+\frac{1}{2}\mathcal{D}W_2.
\end{aligned}$$

Thus, (3.7) can be written as

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{D}W_n x^n &= d_1 \frac{1}{(1-x)} + d_2 \frac{1}{(1-2x)} + d_3 \frac{1}{(2x-1)^2}, \\
&= d_1 \sum_{n=0}^{\infty} x^n + d_2 \sum_{n=0}^{\infty} 2^n x^n + d_3 \sum_{n=0}^{\infty} 2^n (n+1) x^n, \\
&= \sum_{n=0}^{\infty} (d_1 + d_2 2^n + d_3 2^n (n+1)) x^n, \\
&= \sum_{n=0}^{\infty} (4\mathcal{D}W_0 - 4\mathcal{D}W_1 + \mathcal{D}W_2 + (-4\mathcal{D}W_0 + \frac{11}{2}\mathcal{D}W_1 - \frac{3}{2}\mathcal{D}W_2)2^n \\
&\quad +(\mathcal{D}W_0 - \frac{3}{2}\mathcal{D}W_1 + \frac{1}{2}\mathcal{D}W_2)2^n(n+1))x^n, \\
&= \sum_{n=0}^{\infty} (4\mathcal{D}W_0 - 4\mathcal{D}W_1 + \mathcal{D}W_2 + (-4\mathcal{D}W_0 + \frac{11}{2}\mathcal{D}W_1 - \frac{3}{2}\mathcal{D}W_2)2^n \\
&\quad +(\mathcal{D}W_0 - \frac{3}{2}\mathcal{D}W_1 + \frac{1}{2}\mathcal{D}W_2)2^n + (\mathcal{D}W_0 - \frac{3}{2}\mathcal{D}W_1 + \frac{1}{2}\mathcal{D}W_2)2^n n)x^n, \\
&= \sum_{n=0}^{\infty} (4\mathcal{D}W_0 - 4\mathcal{D}W_1 + \mathcal{D}W_2 + (\mathcal{D}W_0 - \frac{3}{2}\mathcal{D}W_1 + \frac{1}{2}\mathcal{D}W_2)n2^n \\
&\quad +(-3\mathcal{D}W_0 + 4\mathcal{D}W_1 - \mathcal{D}W_2)2^n)x^n, \\
&= \sum_{n=0}^{\infty} ((-3\mathcal{D}W_0 + 4\mathcal{D}W_1 - \mathcal{D}W_2) + (\mathcal{D}W_0 - \frac{3}{2}\mathcal{D}W_1 + \frac{1}{2}\mathcal{D}W_2)n)2^n \\
&\quad +4\mathcal{D}W_0 - 4\mathcal{D}W_1 + \mathcal{D}W_2)x^n.
\end{aligned}$$

This gives

$$\mathcal{D}W_n = (\mathcal{D}A_1 + \mathcal{D}A_2 n)2^n + \mathcal{D}A_3$$

where

$$\begin{aligned}
\mathcal{D}A_1 &= -3\mathcal{D}W_0 + 4\mathcal{D}W_1 - \mathcal{D}W_2 = -3W_0 + 4W_1 - W_2 + \varepsilon(-4W_0 + 5W_1 - W_2), \\
\mathcal{D}A_2 &= \mathcal{D}W_0 - \frac{3}{2}\mathcal{D}W_1 + \frac{1}{2}\mathcal{D}W_2 = W_0 - \frac{3}{2}W_1 + \frac{1}{2}W_2 + \varepsilon(2W_0 - 3W_1 + W_2), \\
\mathcal{D}A_3 &= 4\mathcal{D}W_0 - 4\mathcal{D}W_1 + \mathcal{D}W_2 = W_2 - 4W_1 + 4W_0 + \varepsilon(W_2 - 4W_1 + 4W_0).
\end{aligned}$$

Note that the following equalities are true:

$$\begin{aligned}
A_1\widehat{\alpha} + A_2\widehat{\beta} &= (-W_2 + 4W_1 - 3W_0)(1+2\varepsilon) + (\frac{W_2 - 3W_1 + 2W_0}{2})(2\varepsilon) \\
&= -3W_0 + 4W_1 - W_2 + \varepsilon(-4W_0 + 5W_1 - W_2).
\end{aligned}$$

$$\begin{aligned}
A_2\widehat{\alpha} &= \frac{W_2 - 3W_1 + 2W_0}{2}(1+2\varepsilon) \\
&= W_0 - \frac{3}{2}W_1 + \frac{1}{2}W_2 + \varepsilon(2W_0 - 3W_1 + W_2).
\end{aligned}$$

$A_3\widehat{\gamma} = W_2 - 4W_1 + 4W_0 + \varepsilon(W_2 - 4W_1 + 4W_0).$

Therefore, we can write the following equality:

$$\mathcal{D}W_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2n\widehat{\alpha})2^n + A_3\widehat{\gamma}.$$

The proof is finished. $\square$

Next, using Theorem 3.3, we present Binet's formulas of dual modified Woodall, dual modified Cullen, dual Woodall and dual Cullen numbers.

3.2 SOME IDENTITIES FOR DUAL GENERALIZED WOODALL NUMBERS

We now present a few special identities for the dual generalized Woodall sequence $\{\mathcal{D}W_n\}$. The following theorem presents Simpson's identity for the dual generalized Woodall numbers.

Theorem 3.4 *(Simpson's formula for dual generalized Woodall sequence) For all integers n, we have*

$$\begin{vmatrix} \mathcal{D}W_{n+2} & \mathcal{D}W_{n+1} & \mathcal{D}W_n \\ \mathcal{D}W_{n+1} & \mathcal{D}W_n & \mathcal{D}W_{n-1} \\ \mathcal{D}W_n & \mathcal{D}W_{n-1} & \mathcal{D}W_{n-2} \end{vmatrix} = 4^n \begin{vmatrix} \mathcal{D}W_2 & \mathcal{D}W_1 & \mathcal{D}W_0 \\ \mathcal{D}W_1 & \mathcal{D}W_0 & \mathcal{D}W_{-1} \\ \mathcal{D}W_0 & \mathcal{D}W_{-1} & \mathcal{D}W_{-2} \end{vmatrix}. \tag{3.8}$$

Proof. First, we assume that $n \geq 0$. For the proof, we use mathematical induction on n. For $n = 0$ identity (3.8) is true. Now, we obtain is true for $n = k$. Hence we write the following identity

$$\begin{vmatrix} \mathcal{D}W_{k+2} & \mathcal{D}W_{k+1} & \mathcal{D}W_k \\ \mathcal{D}W_{k+1} & \mathcal{D}W_k & \mathcal{D}W_{k-1} \\ \mathcal{D}W_k & \mathcal{D}W_{k-1} & \mathcal{D}W_{k-2} \end{vmatrix} = 4^k \begin{vmatrix} \mathcal{D}W_2 & \mathcal{D}W_1 & \mathcal{D}W_0 \\ \mathcal{D}W_1 & \mathcal{D}W_0 & \mathcal{D}W_{-1} \\ \mathcal{D}W_0 & \mathcal{D}W_{-1} & \mathcal{D}W_{-2} \end{vmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned}
&\begin{vmatrix} \mathcal{D}W_{k+3} & \mathcal{D}W_{k+2} & \mathcal{D}W_{k+1} \\ \mathcal{D}W_{k+2} & \mathcal{D}W_{k+1} & \mathcal{D}W_{k} \\ \mathcal{D}W_{k+1} & \mathcal{D}W_{k} & \mathcal{D}W_{k-1} \end{vmatrix} \\
= &\begin{vmatrix} 5\mathcal{D}W_{k+2} - 8\mathcal{D}W_{k+1} + 4\mathcal{D}W_{k} & \mathcal{D}W_{k+2} & \mathcal{D}W_{k+1} \\ 5\mathcal{D}W_{k+1} - 8\mathcal{D}W_{k} + 4\mathcal{D}W_{k-1} & \mathcal{D}W_{k+1} & \mathcal{D}W_{k} \\ 5\mathcal{D}W_{k} - 8\mathcal{D}W_{k-1} + 4\mathcal{D}W_{k-2} & \mathcal{D}W_{k} & \mathcal{D}W_{k-1} \end{vmatrix} \\
= & \; 5\begin{vmatrix} \mathcal{D}W_{k+2} & \mathcal{D}W_{k+2} & \mathcal{D}W_{k+1} \\ \mathcal{D}W_{k+1} & \mathcal{D}W_{k+1} & \mathcal{D}W_{k} \\ \mathcal{D}W_{k} & \mathcal{D}W_{k} & \mathcal{D}W_{k-1} \end{vmatrix} - 8\begin{vmatrix} \mathcal{D}W_{k+1} & \mathcal{D}W_{k+2} & \mathcal{D}W_{k+1} \\ \mathcal{D}W_{k} & \mathcal{D}W_{k+1} & \mathcal{D}W_{k} \\ \mathcal{D}W_{k-1} & \mathcal{D}W_{k} & \mathcal{D}W_{k-1} \end{vmatrix} \\
& \; +4\begin{vmatrix} \mathcal{D}W_{k} & \mathcal{D}W_{k+2} & \mathcal{D}W_{k+1} \\ \mathcal{D}W_{k-1} & \mathcal{D}W_{k+1} & \mathcal{D}W_{k} \\ \mathcal{D}W_{k-2} & \mathcal{D}W_{k} & \mathcal{D}W_{k-1} \end{vmatrix} \\
= & \; 4\begin{vmatrix} \mathcal{D}W_{k+2} & \mathcal{D}W_{k+1} & \mathcal{D}W_{k} \\ \mathcal{D}W_{k+1} & \mathcal{D}W_{k} & \mathcal{D}W_{k-1} \\ \mathcal{D}W_{k} & \mathcal{D}W_{k-1} & \mathcal{D}W_{k-2} \end{vmatrix} = 4^{k+1}\begin{vmatrix} \mathcal{D}W_{2} & \mathcal{D}W_{1} & \mathcal{D}W_{0} \\ \mathcal{D}W_{1} & \mathcal{D}W_{0} & \mathcal{D}W_{-1} \\ \mathcal{D}W_{0} & \mathcal{D}W_{-1} & \mathcal{D}W_{-2} \end{vmatrix}.
\end{aligned}$$

In the other case, if we take $n < 0$, the proof can be done similarly. Thus, the proof is concluded. □

From theorem (3.4), we get the following corollary.

Corollary 3.1 *(Simpson's formula for dual generalized Woodall sequence's special cases)*

(a) $$\begin{vmatrix} \mathcal{D}G_{k+2} & \mathcal{D}G_{k+1} & \mathcal{D}G_{k} \\ \mathcal{D}G_{k+1} & \mathcal{D}G_{k} & \mathcal{D}G_{k-1} \\ \mathcal{D}G_{k} & \mathcal{D}G_{k-1} & \mathcal{D}G_{k-2} \end{vmatrix} = -4^{n-1}(9 + 9\varepsilon).$$

(b) $$\begin{vmatrix} \mathcal{D}H_{k+2} & \mathcal{D}H_{k+1} & \mathcal{D}H_{k} \\ \mathcal{D}H_{k+1} & \mathcal{D}H_{k} & \mathcal{D}H_{k-1} \\ \mathcal{D}H_{k} & \mathcal{D}H_{k-1} & \mathcal{D}H_{k-2} \end{vmatrix} = 0.$$

(c) $$\begin{vmatrix} \mathcal{D}R_{k+2} & \mathcal{D}R_{k+1} & \mathcal{D}R_{k} \\ \mathcal{D}R_{k+1} & \mathcal{D}R_{k} & \mathcal{D}R_{k-1} \\ \mathcal{D}R_{k} & \mathcal{D}R_{k-1} & \mathcal{D}R_{k-2} \end{vmatrix} = 4^{n-1}(9 + 9\varepsilon).$$

(d) $\begin{vmatrix} \mathcal{D}C_{k+2} & \mathcal{D}C_{k+1} & \mathcal{D}C_{k} \\ \mathcal{D}C_{k+1} & \mathcal{D}C_{k} & \mathcal{D}C_{k-1} \\ \mathcal{D}C_{k} & \mathcal{D}C_{k-1} & \mathcal{D}C_{k-2} \end{vmatrix} = -4^{n-1}(9+9\varepsilon).$

Theorem 3.5 *(Catalan's identity) For all integers n and m, the following identity holds* $\mathcal{D}W_{n+m}\mathcal{D}W_{n-m} - \mathcal{D}W_n^2 = 2^{n-m}(-2^{m+n}m^2\widehat{\alpha}^2A_2^2 + A_2A_3(-2^{m+1}\widehat{\beta}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} + 2^{2m}\widehat{\beta}\widehat{\gamma} - m\widehat{\alpha}\widehat{\gamma} + n\widehat{\alpha}\widehat{\gamma} - 2^{m+1}n\widehat{\alpha}\widehat{\gamma} + 2^{2m}m\widehat{\alpha}\widehat{\gamma} + 2^{2m}n\widehat{\alpha}\widehat{\gamma}) + A_1A_3(\widehat{\alpha}\widehat{\gamma} - 2^{m+1}\widehat{\alpha}\widehat{\gamma} + 2^{2m}\widehat{\alpha}\widehat{\gamma}))$.

Proof. Using Binet's formula $\mathcal{D}W_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2n\widehat{\alpha})2^n + A_3\widehat{\gamma}$, we get the required identity. □

As special cases of the above theorem, we give Catalan's identity of dual modified Woodall, dual modified Cullen, dual Woodall and dual Cullen numbers. Firstly, we present Catalan's identity of dual Woodall numbers.

Corollary 3.2 *(Catalan's identity for the dual modified Woodall numbers) For all integers n and m, the following identity holds* $\mathcal{D}G_{n+m}\mathcal{D}G_{n-m} - \mathcal{D}G_n^2 = -2^{n-m}(\widehat{\alpha}\widehat{\gamma} - \widehat{\beta}\widehat{\gamma} + 2^{2m}\widehat{\alpha}\widehat{\gamma} - 2^{2m}\widehat{\beta}\widehat{\gamma} - 2^{m+1}\widehat{\alpha}\widehat{\gamma} + 2^{m+1}\widehat{\beta}\widehat{\gamma} + m\widehat{\alpha}\widehat{\gamma} - n\widehat{\alpha}\widehat{\gamma} + 2^{m+n}m^2\widehat{\alpha}^2 - 2^{2m}m\widehat{\alpha}\widehat{\gamma} - 2^{2m}n\widehat{\alpha}\widehat{\gamma} + 2^{m+1}n\widehat{\alpha}\widehat{\gamma})$.

Proof. Take $W_n = G_n$ in Theorem 3.5. □

Secondly, we give Catalan's identity of dual modified Cullen numbers.

Corollary 3.3 *(Catalan's identity for the dual modified Cullen numbers) For all integers n and m, the following identity holds*

$$\mathcal{D}H_{n+m}\mathcal{D}H_{n-m} - \mathcal{D}H_n^2 = 2^{n-m}(2\widehat{\alpha}\widehat{\gamma} + 2 \times 2^{2m}\widehat{\alpha}\widehat{\gamma} - 2 \times 2^{m+1}\widehat{\alpha}\widehat{\gamma}).$$

Proof. Take $W_n = H_n$ in Theorem 3.5. □

Thirdly, we give Catalan's identity of dual Woodall numbers.

Corollary 3.4 *(Catalan's identity for the dual Woodall numbers) For all integers n and m, the following identity holds* $\mathcal{D}R_{n+m}\mathcal{D}R_{n-m} - \mathcal{D}R_n^2 = -2^{n-m}(\widehat{\beta}\widehat{\gamma} + 2^{2m}\widehat{\beta}\widehat{\gamma} - 2^{m+1}\widehat{\beta}\widehat{\gamma} - m\widehat{\alpha}\widehat{\gamma} + n\widehat{\alpha}\widehat{\gamma} + 2^{m+n}m^2\widehat{\alpha}^2 + 2^{2m}m\widehat{\alpha}\widehat{\gamma} + 2^{2m}n\widehat{\alpha}\widehat{\gamma} - 2^{m+1}n\widehat{\alpha}\widehat{\gamma})$.

Proof. Take $W_n = R_n$ in Theorem 3.5. □

Fourthly, we give Catalan's identity of dual Cullen numbers.

Corollary 3.5 *(Catalan's identity for the dual Cullen numbers) For all integers n and m, the following identity holds*

$$\mathcal{D}C_{n+m}\mathcal{D}C_{n-m}-\mathcal{D}C_n^2 = 2^{n-m}(\widehat{\beta}\widehat{\gamma}+2^{2m}\widehat{\beta}\widehat{\gamma}-2^{m+1}\widehat{\beta}\widehat{\gamma}-m\widehat{\alpha}\widehat{\gamma}+n\widehat{\alpha}\widehat{\gamma}-2^{m+n}m^2\widehat{\alpha}^2+2^{2m}m\widehat{\alpha}\widehat{\gamma}+2^{2m}n\widehat{\alpha}\widehat{\gamma}-2^{m+1}n\widehat{\alpha}\widehat{\gamma}).$$

Proof. Take $W_n = C_n$ in Theorem 3.5. □

Note that for $m = 1$ in Catalan's identity, we get Cassini's identity for the dual generalized Woodall sequence.

Corollary 3.6 *(Cassini's identity) For all integers n, the following identity holds*

$$\mathcal{D}W_{n+1}\mathcal{D}W_{n-1}-\mathcal{D}W_n^2 = 2^{n-1}(A_2A_3(3\widehat{\alpha}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}+n\widehat{\alpha}\widehat{\gamma})-2^{n+1}A_2^2\widehat{\alpha}^2+A_1A_3\widehat{\alpha}\widehat{\gamma}).$$

As special cases of Cassini's identity, we give Cassini's identity of dual modified Woodall, dual modified Cullen, dual Woodall and dual Cullen numbers. Firstly, we present Cassini's identity of dual modified Woodall numbers.

Corollary 3.7 *(Cassini's identity of dual modified Woodall numbers) For all integers n, the following identity holds*

$$\mathcal{D}G_{n+1}\mathcal{D}G_{n-1}-\mathcal{D}G_n^2 = 2^{n-1}(2\widehat{\alpha}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}-2^{n+1}\widehat{\alpha}^2+n\widehat{\alpha}\widehat{\gamma}).$$

Secondly, we give Cassini's identity of dual modified Cullen numbers.

Corollary 3.8 *(Cassini's identity of dual modified Cullen numbers) For all integers n, the following identity holds*

$$\mathcal{D}H_{n+1}\mathcal{D}H_{n-1}-\mathcal{D}H_n^2 = 2^n\widehat{\alpha}\widehat{\gamma}.$$

Fourthly, we give Cassini's identity of dual Woodall numbers.

Corollary 3.9 *(Cassini's identity of dual Woodall numbers) For all integers n, the following identity holds*

$$\mathcal{D}R_{n+1}\mathcal{D}R_{n-1}-\mathcal{D}R_n^2 = -2^{n-1}(3\widehat{\alpha}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}+2^{n+1}\widehat{\alpha}^2+n\widehat{\alpha}\widehat{\gamma}).$$

Thirdly, we give Cassini's identity of dual Cullen numbers.

Corollary 3.10 *(Cassini's identity of dual Cullen numbers) For all integers n, the following identity holds*

$$\mathcal{D}C_{n+1}\mathcal{D}C_{n-1}-\mathcal{D}C_n^2=2^{n-1}(3\widehat{\alpha}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}-2^{n+1}\widehat{\alpha}^2+n\widehat{\alpha}\widehat{\gamma}).$$

Theorem 3.6 *For all integers m, n, G_n is woodall numbers, the following identity is true:*

$$\mathcal{D}W_{n+m}=\mathcal{D}W_nG_{m+1}+\mathcal{D}W_{n-1}(-8G_m+4G_{m-1})+4\mathcal{D}W_{n-2}G_m. \tag{3.9}$$

Proof. The identity (3.9) can be proved by mathematical induction on m. First of all, we assume that $m\geq 0$ and $n\geq 0$. If $m=0$ we get

$$\mathcal{D}W_n=\mathcal{D}W_nG_1+\mathcal{D}W_{n-1}(-8G_0+4G_{-1})+4\mathcal{D}W_{n-2}G_0$$

which is true by seeing that $G_{-1}=0$, $G_{-2}=\frac{1}{4}$, $G_{-3}=\frac{1}{2}$. We assume that the identity given holds for $m=k$. For $m=k+1$, we get

$$\begin{aligned}
\mathcal{D}W_{n+(k+1)} &= 5\mathcal{D}W_{n+k}-8\mathcal{D}W_{n+k-1}+4\mathcal{D}W_{n+k-2}\\
&= 5(\mathcal{D}W_nG_{k+1}+\mathcal{D}W_{n-1}(-8G_k+4G_{k-1})+4\mathcal{D}W_{n-2}G_k)\\
&\quad -8(\mathcal{D}W_nG_k+\mathcal{D}W_{n-1}(-8G_{k-1}+4G_{k-2})+4\mathcal{D}W_{n-2}G_{k-1})\\
&\quad +4(\mathcal{D}W_nG_{k-1}+\mathcal{D}W_{n-1}(-8G_{k-2}+4G_{k-3})+4\mathcal{D}W_{n-2}G_{k-2})\\
&= \mathcal{D}W_n(5G_{k+1}-8G_k+4G_{k-1})+\mathcal{D}W_{n-1}(-8(5G_k-8G_{k-1}+4G_{k-2})\\
&\quad +4(5G_{k-1}-8G_{k-2}+4G_{k-3}))+4\mathcal{D}W_{n-2}(5G_k-8G_{k-1}+4G_{k-2})\\
&= \mathcal{D}W_nG_{k+2}+\mathcal{D}W_{n-1}(-8G_{k+1}+4G_k)+4\mathcal{D}W_{n-2}G_{k+1}\\
&= \mathcal{D}W_nG_{(k+1)+1}+\mathcal{D}W_{n-1}(-8G_{(k+1)}+4G_{(k+1)-1})+4\mathcal{D}W_{n-2}G_{(k+1)}.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves (3.6). The other cases of m, n can be proved similarly. □

3.3 LINEAR SUMS FOR DUAL GENERALIZED WOODALL NUMBERS

In this section, we present the summation formulas of the dual generalized Woodall numbers with positive and negative subscripts.

Now, we give the formulas that give the summation of the dual generalized Woodall numbers in the following theorem.

Theorem 3.7 *For $n \geq 0$, dual generalized Woodall numbers have the following formulas:*

(a) $\sum_{k=0}^{n} \mathcal{D}W_k = (3 + n - 3 \times 2^n + 2^n n + 4\varepsilon + \varepsilon n - 2^{n+2}\varepsilon + 2^{n+1}\varepsilon n)W_2 + (-11 - 4n + 11 \times 2^n - 3 \times 2^n n - 15\varepsilon - 4\varepsilon n + 2^{n+4}\varepsilon - 3 \times 2^{n+1}\varepsilon n)W_1 + (9 + 4n - 2^{n+3} + 2^{n+1}n + 12\varepsilon + 4\varepsilon n - 3 \times 2^{n+2}\varepsilon + 2^{n+2}\varepsilon n)W_0.$

(b) $\sum_{k=0}^{n} \mathcal{D}W_{2k} = (\frac{16}{9} + n - \frac{1}{9}2^{2n+4} + \frac{1}{3}2^{2n+2}n + \frac{20}{9}\varepsilon + \varepsilon n - \frac{5}{9}2^{2n+2}\varepsilon + \frac{1}{3}2^{2n+3}\varepsilon n)W_2 + (-\frac{20}{3} - 4n + \frac{5}{3}2^{2n+2} - 2^{2n+2}n - \frac{25}{3}\varepsilon + \frac{7}{3}2^{2n+2}\varepsilon - 4\varepsilon n - 2^{2n+3}\varepsilon n)W_1 + (\frac{53}{9} - \frac{11}{9}2^{2n+2} + 4n + \frac{1}{3}2^{2n+3}n + \frac{64}{9}\varepsilon - \frac{1}{9}2^{2n+6}\varepsilon + 4\varepsilon n + \frac{1}{3}2^{2n+4}\varepsilon n)W_0.$

(c) $\sum_{k=0}^{n} \mathcal{D}W_{2k+1} = (\frac{20}{9} - \frac{5}{9}2^{2n+2} + n + \frac{1}{3}2^{2n+3}n + \frac{25}{9}\varepsilon - \frac{1}{9}2^{2n+4}\varepsilon + \varepsilon n + \frac{1}{3}2^{2n+4}\varepsilon n)W_2 + (-\frac{25}{3} + \frac{7}{3}2^{2n+2} - 4n - 2^{2n+3}n + \frac{1}{3}2^{2n+5}\varepsilon - \frac{32}{3}\varepsilon - 4\varepsilon n - 2^{2n+4}\varepsilon n)W_1 + (\frac{64}{9} - \frac{1}{9}2^{2n+6} + 4n + \frac{1}{3}2^{2n+4}n + \frac{80}{9}\varepsilon - \frac{5}{9}2^{2n+4}\varepsilon + 4\varepsilon n + \frac{1}{3}2^{2n+5}\varepsilon n)W_0.$

Proof. It can be obtained by using Proposition 1.2 and Proposition 1.3.

(a) We can derive the following using the formulas in Proposition 1.2.

$$\sum_{k=0}^{n} \mathcal{D}W_k = \sum_{k=0}^{n} W_k + \varepsilon \sum_{k=0}^{n} W_{k+1}.$$

$$\begin{aligned}
\sum_{k=0}^{n} \mathcal{D}W_k \;=\; & \frac{1}{2}W_2(2n - 2^{n+1}(n-1) + 2^{n+2}(n-2) + 6) \\
& -\frac{1}{2}W_1(8n - 2^{n+1}(3n-5) + 2^{n+2}(3n-8) + 22) \\
& +W_0(4n - 2^{n+1}(n-2) + 2^{n+2}(n-3) + 9) + \varepsilon(\frac{1}{2}W_2(2n \\
& +2^{n+3}(n-1) - 2^{n+2}n + 8) - \frac{1}{2}W_1(8n - 2^{n+2}(3n-2) \\
& +2^{n+3}(3n-5) + 30) + W_0(4n - 2^{n+2}(n-1) \\
& +2^{n+3}(n-2) + 12)).
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{n} \mathcal{D}W_k \;=\; & (3 + n - 3 \times 2^n + 2^n n + 4\varepsilon + \varepsilon n - 2^{n+2}\varepsilon + 2^{n+1}\varepsilon n)W_2 \\
& +(-11 - 4n + 11 \times 2^n - 3 \times 2^n n - 15\varepsilon - 4\varepsilon n + 2^{n+4}\varepsilon \\
& -3 \times 2^{n+1}\varepsilon n)W_1 + (9 + 4n - 2^{n+3} + 2^{n+1}n + 12\varepsilon + 4\varepsilon n \\
& -3 \times 2^{n+2}\varepsilon + 2^{n+2}\varepsilon n)W_0.
\end{aligned}$$

The proof is finished. □

(b) We can derive the following using the formulas in Proposition 1.3.

$$\sum_{k=0}^{n} \mathcal{D}W_{2k} = \sum_{k=0}^{n} W_{2k} + \varepsilon \sum_{k=0}^{n} W_{2k+1}.$$

$$\begin{aligned}
\sum_{k=0}^{n} \mathcal{D}W_{2k} &= \frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) \\
&\quad -\frac{1}{18}W_1(72n - 2^{2n+2}(6n-2) + 2^{2n+4}(6n-8) + 120) \\
&\quad +\frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32) \\
&\quad +\varepsilon(\frac{1}{18}W_2(18n - 2^{2n+3}(2n+1) + 2^{2n+5}(2n-1) + 40) \\
&\quad -\frac{1}{18}W_1(72n - 2^{2n+3}(6n+1) + 2^{2n+5}(6n-5) + 150) \\
&\quad +\frac{1}{9}W_0(36n + 2^{2n+5}(2n-2) - 2 \times 2^{2n+3}n + 64)).
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{n} \mathcal{D}W_{2k} &= (\frac{16}{9} + n - \frac{1}{9}2^{2n+4} + \frac{1}{3}2^{2n+2}n + \frac{20}{9}\varepsilon + \varepsilon n - \frac{5}{9}2^{2n+2}\varepsilon \\
&\quad +\frac{1}{3}2^{2n+3}\varepsilon n)W_2 + (-\frac{20}{3} - 4n + \frac{5}{3}2^{2n+2} - 2^{2n+2}n - \frac{25}{3}\varepsilon \\
&\quad +\frac{7}{3}2^{2n+2}\varepsilon - 4\varepsilon n - 2^{2n+3}\varepsilon n)W_1 + (\frac{53}{9} - \frac{11}{9}2^{2n+2} + 4n \\
&\quad +\frac{1}{3}2^{2n+3}n + \frac{64}{9}\varepsilon - \frac{1}{9}2^{2n+6}\varepsilon + 4\varepsilon n + \frac{1}{3}2^{2n+4}\varepsilon n)W_0.
\end{aligned}$$

The proof is completed. □

(c) We can derive the following using the formulas in Proposition 1.3.

$$\sum_{k=0}^{n} \mathcal{D}W_{2k+1} = \sum_{k=0}^{n} W_{2k+1} + \varepsilon \sum_{k=0}^{n} W_{2k+2}.$$

$$\begin{aligned}
\sum_{k=0}^{n} \mathcal{D}W_{2k+1} &= \frac{1}{18}W_2(18n - 2^{2n+3}(2n+1) + 2^{2n+5}(2n-1) + 40) \\
&\quad -\frac{1}{18}W_1(72n - 2^{2n+3}(6n+1) + 2^{2n+5}(6n-5) + 150) \\
&\quad +\frac{1}{9}W_0(36n + 2^{2n+5}(2n-2) - 2 \times 2^{2n+3}n + 64) \\
&\quad +\varepsilon(\frac{1}{9}W_0(36n - 2^{2n+4}(2n+1) + 2^{2n+6}(2n-1) + 80) \\
&\quad -\frac{1}{18}W_1(72n - 2^{2n+4}(6n+4) + 2^{2n+6}(6n-2) + 192) \\
&\quad +\frac{1}{18}W_2(18n - 2^{2n+4}(2n+2) + 2 \times 2^{2n+6}n + 50)).
\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n}\mathcal{D}W_{2k+1} &= (\frac{20}{9}-\frac{5}{9}2^{2n+2}+n+\frac{1}{3}2^{2n+3}n+\frac{25}{9}\varepsilon-\frac{1}{9}2^{2n+4}\varepsilon+\varepsilon n \\ &\quad +\frac{1}{3}2^{2n+4}\varepsilon n)W_2+(-\frac{25}{3}+\frac{7}{3}2^{2n+2}-4n-2^{2n+3}n \\ &\quad +\frac{1}{3}2^{2n+5}\varepsilon-\frac{32}{3}\varepsilon-4\varepsilon n-2^{2n+4}\varepsilon n)W_1+(\frac{64}{9}-\frac{1}{9}2^{2n+6} \\ &\quad +4n+\frac{1}{3}2^{2n+4}n+\frac{80}{9}\varepsilon-\frac{5}{9}2^{2n+4}\varepsilon+4\varepsilon n+\frac{1}{3}2^{2n+5}\varepsilon n)W_0.\end{aligned}$$

The proof is finished. □

As a first special case of the above theorem, we have the following summation formulas for dual Woodall numbers:

Corollary 3.11 *For* $n \geq 0$, *dual modified Woodall numbers have the following properties:*

(a) $\sum_{k=0}^{n}\mathcal{D}G_k = 4+n+2^{n+1}n-2^{n+2}+\varepsilon(5-5\times 2^{n+2}+n+2^{n+4}+2^{n+2}n)$.

(b) $\sum_{k=0}^{n}\mathcal{D}G_{2k} = \frac{20}{9}+n+\frac{2}{3}2^{2n+2}n+\frac{5}{3}2^{2n+2}-\frac{5}{9}2^{2n+4}+\varepsilon(\frac{25}{9}-\frac{4}{9}2^{2n+2}+n+\frac{2}{3}2^{2n+3}n)$.

(c) $\sum_{k=0}^{n}\mathcal{D}G_{2k+1} = \frac{25}{9}+n+\frac{2}{3}2^{2n+3}n-\frac{4}{9}2^{2n+2}+\varepsilon(\frac{29}{9}-\frac{5}{9}2^{2n+4}+\frac{1}{3}2^{2n+5}+n+\frac{2}{3}2^{2n+4}n)$.

As a second special case of the above theorem, we have the following summation formulas for dual modified Cullen numbers:

Corollary 3.12 *For* $n \geq 0$, *dual modified Cullen numbers have the following properties:*

(a) $\sum_{k=0}^{n}\mathcal{D}H_k = -1+n-6\times 2^n n-3\times 2^{n+3}+3\times 2^{n+1}n+28\times 2^n+\varepsilon(-3-18\times 2^{n+2}+$
$5\times 2^{n+4}+n-6\times 2^{n+1}n+3\times 2^{n+2}n)$.

(b) $\sum_{k=0}^{n}\mathcal{D}H_{2k} = \frac{1}{3}+n-2^{2n+3}n+2^{2n+3}n+\frac{14}{3}2^{2n+2}-2^{2n+4}+\varepsilon(-\frac{1}{3}+\frac{20}{3}2^{2n+2}-\frac{1}{3}2^{2n+6}$
$+n-2^{2n+4}n+2^{2n+4}n)$.

(c) $\sum_{k=0}^{n}\mathcal{D}H_{2k+1} = -\frac{1}{3}+n-2\times 2^{2n+3}n+2^{2n+4}n+\frac{20}{3}2^{2n+2}-\frac{1}{3}2^{2n+6}+\varepsilon(-\frac{5}{3}-\frac{8}{3}2^{2n+4}+\frac{5}{3}2^{2n+5}$
$+n-2^{2n+5}n+2^{2n+5}n)$.

As a third special case of the above theorem, we have the following summation formulas for dual Woodall numbers:

Corollary 3.13 *For $n \geq 0$, dual Woodall numbers have the following properties:*

(a) $\sum_{k=0}^{n} \mathcal{D}R_k = 1 - n + 4 \times 2^n n + 2^{n+3} - 2^{n+1} n - 10 \times 2^n + \varepsilon(1 - 2^{n+4} + 2^{n+4} - n + 2^{n+3} n - 2^{n+2} n)$.

(b) $\sum_{k=0}^{n} \mathcal{D}R_{2k} = -\frac{1}{9} - n + \frac{4}{3} 2^{2n+2} n - \frac{1}{3} 2^{2n+3} n + \frac{26}{9} 2^{2n+2} - \frac{7}{9} 2^{2n+4} + \varepsilon(\frac{1}{9} - n - \frac{14}{9} 2^{2n+2} + \frac{1}{9} 2^{2n+6} + \frac{4}{3} 2^{2n+3} n - \frac{1}{3} 2^{2n+4} n)$.

(c) $\sum_{k=0}^{n} \mathcal{D}R_{2k+1} = \frac{1}{9} - n + \frac{4}{3} 2^{2n+3} n - \frac{1}{3} 2^{2n+4} n - \frac{14}{9} 2^{2n+2} + \frac{1}{9} 2^{2n+6} + \varepsilon(-\frac{1}{9} + \frac{1}{3} 2^{2n+5} - \frac{2}{9} 2^{2n+4} - n + \frac{4}{3} 2^{2n+4} n - \frac{1}{3} 2^{2n+5} n)$.

As a fourth special case of the above theorem, we have the following summation formulas for dual Cullen numbers:

Corollary 3.14 *For $n \geq 0$, dual Cullen numbers have the following properties.*

(a) $\sum_{k=0}^{n} \mathcal{D}C_k = 3 + n - 2^{n+3} + 2^{n+1} n + 6 \times 2^n + \varepsilon(3 + n + 2^{n+2} n)$.

(b) $\sum_{k=0}^{n} \mathcal{D}C_{2k} = \frac{17}{9} + n + \frac{1}{3} 2^{2n+3} n - \frac{2}{9} 2^{2n+2} + \varepsilon(\frac{19}{9} + n + \frac{1}{9} 2^{2n+3} + \frac{1}{3} 2^{2n+4} n)$.

(c) $\sum_{k=0}^{n} \mathcal{D}C_{2k+1} = \frac{19}{9} + n + \frac{1}{3} 2^{2n+4} n + \frac{1}{9} 2^{2n+3} + \varepsilon(\frac{17}{9} + \frac{4}{9} 2^{2n+4} + n + \frac{1}{3} 2^{2n+5} n)$.

Next, we introduce the formulas that give the summation of the dual generalized Woodall numbers with negative subscripts in the following theorem.

Theorem 3.8 *For $n \geq 0$, dual generalized Woodall numbers have the following formulas:*

(a) $\sum_{k=0}^{n} \mathcal{D}W_{-k} = (-2 + \frac{2}{2^n} - 3\varepsilon + n + \frac{3}{2^n}\varepsilon + \frac{1}{2 \times 2^n} n + \varepsilon n + \frac{1}{2^n}\varepsilon n) W_2 + (7 - \frac{7}{2^n} + 12\varepsilon - 4n - \frac{11}{2^n}\varepsilon - \frac{3}{2 \times 2^n} n - 4\varepsilon n - \frac{3}{2^n}\varepsilon n) W_1 + (-4 + \frac{5}{2^n} - 8\varepsilon + 4n + \frac{8}{2^n}\varepsilon + \frac{1}{2^n} n + 4\varepsilon n + \frac{2}{2^n}\varepsilon n) W_0$.

(b) $\sum_{k=0}^{n} \mathcal{D}W_{-2k} = (-\frac{7}{9} + \frac{7}{9 \times 2^{2n}} - \frac{11}{9}\varepsilon + n + \frac{11}{9 \times 2^{2n}}\varepsilon + \frac{1}{3 \times 2^{2n}} n + \varepsilon n + \frac{2}{3 \times 2^{2n}}\varepsilon n) W_2 + (\frac{8}{3} - \frac{8}{3 \times 2^{2n}} + \frac{16}{3}\varepsilon - 4n - \frac{13}{3 \times 2^{2n}}\varepsilon - \frac{1}{2^{2n}} n - 4\varepsilon n - \frac{2}{2^{2n}}\varepsilon n) W_1 + (-\frac{8}{9} + \frac{17}{9 \times 2^{2n}} - \frac{28}{9}\varepsilon + 4n + \frac{28}{9 \times 2^{2n}}\varepsilon + \frac{2}{3 \times 2^{2n}} n + 4\varepsilon n + \frac{4}{3 \times 2^{2n}}\varepsilon n) W_0$.

(c) $\sum_{k=0}^{n} \mathcal{D}W_{-2k+1} = (-\frac{11}{9} + \frac{11}{9 \times 2^{2n}} - \frac{7}{9}\varepsilon + n + \frac{16}{9 \times 2^{2n}}\varepsilon + \frac{2}{3 \times 2^{2n}} n + \varepsilon n + \frac{4}{3 \times 2^{2n}}\varepsilon n) W_2 + (\frac{16}{3} - \frac{13}{3 \times 2^{2n}} + \frac{20}{3}\varepsilon - 4n - \frac{20}{3 \times 2^{2n}}\varepsilon - \frac{2}{2^{2n}} n - 4\varepsilon n - \frac{4}{2^{2n}}\varepsilon n) W_1 + (-\frac{28}{9} + \frac{28}{9 \times 2^{2n}} - \frac{44}{9}\varepsilon + 4n + \frac{44}{9 \times 2^{2n}}\varepsilon + \frac{4}{3 \times 2^{2n}} n + 4\varepsilon n + \frac{8}{3 \times 2^{2n}}\varepsilon n) W_0$.

Proof. It can be obtained by using Proposition 1.4 and Proposition 1.5.

(a) We can derive the following using the formulas in Proposition 1.4.

$$\sum_{k=0}^{n}\mathcal{D}W_{-k}=\sum_{k=0}^{n}W_{-k}+\varepsilon\sum_{k=0}^{n}W_{-k+1}.$$

$$\begin{aligned}\sum_{k=0}^{n}\mathcal{D}W_{-k} &= 4W_0(n+\frac{1}{2^{n+1}}(n+4)-\frac{1}{2^{n+2}}(n+3)-1)\\ &\quad +2W_1(\frac{1}{2^{n+2}}(3n+8)-2n-\frac{1}{2^{n+1}}(3n+11)+\frac{7}{2})\\ &\quad +2W_2(\frac{1}{2}n+\frac{1}{2^{n+1}}(n+3)-\frac{1}{2^{n+2}}(n+2)-1)\\ &\quad +\varepsilon(2W_2(\frac{1}{2}n+\frac{1}{2^{n}}(n+2)-\frac{1}{2^{n+1}}(n+1)-\frac{3}{2})+4W_0(n+\frac{1}{2^{n}}(n+3)\\ &\quad -\frac{1}{2^{n+1}}(n+2)-2)+2W_1(\frac{1}{2^{n+1}}(3n+5)-2n-\frac{1}{2^{n}}(3n+8)+6)).\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n}\mathcal{D}W_{-k} &= (-2+\frac{2}{2^{n}}-3\varepsilon+n+\frac{3}{2^{n}}\varepsilon+\frac{1}{2\times 2^{n}}n+\varepsilon n+\frac{1}{2^{n}}\varepsilon n)W_2\\ &\quad +(7-\frac{7}{2^{n}}+12\varepsilon-4n-\frac{11}{2^{n}}\varepsilon-\frac{3}{2\times 2^{n}}n-4\varepsilon n-\frac{3}{2^{n}}\varepsilon n)W_1\\ &\quad +(-4+\frac{5}{2^{n}}-8\varepsilon+4n+\frac{8}{2^{n}}\varepsilon+\frac{1}{2^{n}}n+4\varepsilon n+\frac{2}{2^{n}}\varepsilon n)W_0.\end{aligned}$$

This proves (a). We can prove (b) and (c) similar way. □

As a first special case of the above theorem, we have the following summation formulas for dual modified Woodall numbers:

Corollary 3.15 *For $n\geq 0$, dual modified Woodall numbers have the following properties:*

(a) $\sum_{k=0}^{n}\mathcal{D}G_{-k}=-3+n+\frac{n+3}{2^{n}}+\varepsilon(-3+n+\frac{2n+4}{2^{n}})$.

(b) $\sum_{k=0}^{n}\mathcal{D}G_{-2k}=-\frac{11}{9}+n+\frac{11+6n}{9\times 2^{2n}}+\varepsilon(-\frac{7}{9}+n+\frac{16+12n}{9\times 2^{2n}})$.

(c) $\sum_{k=0}^{n}\mathcal{D}G_{-2k+1}=-\frac{7}{9}+n+\frac{16+12n}{9\times 2^{2n}}+\varepsilon(\frac{25}{9}+n+\frac{20+24n}{9\times 2^{2n}})$.

As a second special case of the above theorem, we have the following summation formulas for dual modified Cullen numbers:

Corollary 3.16 *For $n \geq 0$, dual modified Cullen numbers have the following properties:*

(a) $\sum_{k=0}^{n} \mathcal{D}H_{-k} = 5 + n - \frac{2}{2^n} + \varepsilon(9 - \frac{4}{2^n} + n)$.

(b) $\sum_{k=0}^{n} \mathcal{D}H_{-2k} = \frac{11}{3} + n - \frac{2}{3\times 2^{2n}} + \varepsilon(\frac{19}{3} - \frac{4}{3\times 2^{2n}} + n)$.

(c) $\sum_{k=0}^{n} \mathcal{D}H_{-2k+1} = \frac{19}{3} + n - \frac{4}{3\times 2^{2n}} + \varepsilon(\frac{35}{3} - \frac{8}{3\times 2^{2n}} + n)$.

As a third special case of the above theorem, we have the following summation formulas for dual Woodall numbers:

Corollary 3.17 *For $n \geq 0$, dual Woodall numbers have the following properties:*

(a) $\sum_{k=0}^{n} \mathcal{D}R_{-k} = -3 - n + \frac{2+n}{2^n} + \varepsilon(-1 - n + \frac{2+2n}{2^n})$.

(b) $\sum_{k=0}^{n} \mathcal{D}R_{-2k} = -\frac{17}{9} - n + \frac{8+6n}{9\times 2^{2n}} + \varepsilon(-\frac{1}{9} - n + \frac{10+12n}{9\times 2^{2n}})$.

(c) $\sum_{k=0}^{n} \mathcal{D}R_{-2k+1} = -\frac{1}{9} - n + \frac{10+12n}{9\times 2^{2n}} + \varepsilon(\frac{55}{9} - n + \frac{8+24n}{9\times 2^{2n}})$.

As a fourth special case of the above theorem, we have the following summation formulas for dual Cullen numbers:

Corollary 3.18 *For $n \geq 0$, dual Cullen numbers have the following properties:*

(a) $\sum_{k=0}^{n} \mathcal{D}C_{-k} = -1 + n + \frac{2+n}{2^n} + \varepsilon(1 + \frac{2+2n}{2^n} + n)$.

(b) $\sum_{k=0}^{n} \mathcal{D}C_{-2k} = \frac{1}{9} + n + \frac{8+6n}{9\times 2^{2n}} + \varepsilon(\frac{17}{9} + \frac{10+12n}{9\times 2^{2n}} + n)$.

(c) $\sum_{k=0}^{n} \mathcal{D}C_{-2k+1} = \frac{17}{9} + n + \frac{10+12n}{9\times 2^{2n}} + \varepsilon(\frac{73}{9} + \frac{8+24n}{9\times 2^{2n}} + n)$.

3.4 MATRICES RELATED TO DUAL GENERALIZED WOODALL NUMBERS

In this section, we investigate matrices related to dual generalized Woodall numbers. Now, $\{G_n\}$ defined by the third-order recurrence relation as follows

$G_n = 5G_{n-1} - 8G_{n-2} + 4G_{n-3}$ with the initial conditions $G_0 = 0,\ G_1 = 1,\ G_2 = 5$.

We present the square matrix A of order 3 as

$$A = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 4$. Then, we give the following Lemma.

Note that

$$A^n = \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & 4G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & 4G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & 4G_{n-2} \end{pmatrix}.$$

For the proof see [60].

Lemma 3.9 *For all integers n the following identity is true.*

$$\begin{pmatrix} \mathcal{D}W_{n+2} \\ \mathcal{D}W_{n+1} \\ \mathcal{D}W_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \mathcal{D}W_2 \\ \mathcal{D}W_1 \\ \mathcal{D}W_0 \end{pmatrix}.$$

Proof. First, we suppose that $n \geq 0$. Lemma (3.9) can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} \mathcal{D}W_2 \\ \mathcal{D}W_1 \\ \mathcal{D}W_0 \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} \mathcal{D}W_2 \\ \mathcal{D}W_1 \\ \mathcal{D}W_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} \mathcal{D}W_{k+2} \\ \mathcal{D}W_{k+1} \\ \mathcal{D}W_k \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \mathcal{D}W_2 \\ \mathcal{D}W_1 \\ \mathcal{D}W_0 \end{pmatrix}$$

For $n = k + 1$, we get

$$\begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \mathcal{D}W_2 \\ \mathcal{D}W_1 \\ \mathcal{D}W_0 \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} \mathcal{D}W_2 \\ \mathcal{D}W_1 \\ \mathcal{D}W_0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{D}W_{k+2} \\ \mathcal{D}W_{k+1} \\ \mathcal{D}W_k \end{pmatrix}$$

$$= \begin{pmatrix} 5\mathcal{D}W_{k+2} - 8\mathcal{D}W_{k+1} + 4\mathcal{D}W_k \\ \mathcal{D}W_{k+2} \\ \mathcal{D}W_{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{D}W_{k+3} \\ \mathcal{D}W_{k+2} \\ \mathcal{D}W_{k+1} \end{pmatrix}.$$

If we suppose that $n < 0$ the proof can be done similarly. Consequently, by mathematical induction on n, the proof is completed. □

Theorem 3.10 *If we define the matrices $N_{\mathcal{D}W}$ and $E_{\mathcal{D}W}$ as follow.*

$$N_{\mathcal{D}W} = \begin{pmatrix} \mathcal{D}W_2 & \mathcal{D}W_1 & \mathcal{D}W_0 \\ \mathcal{D}W_1 & \mathcal{D}W_0 & \mathcal{D}W_{-1} \\ \mathcal{D}W_0 & \mathcal{D}W_{-1} & \mathcal{D}W_{-2} \end{pmatrix}, \; E_{\mathcal{D}W} = \begin{pmatrix} \mathcal{D}W_{n+2} & \mathcal{D}W_{n+1} & \mathcal{D}W_n \\ \mathcal{D}W_{n+1} & \mathcal{D}W_n & \mathcal{D}W_{n-1} \\ \mathcal{D}W_n & \mathcal{D}W_{n-1} & \mathcal{D}W_{n-2} \end{pmatrix}.$$

then the following identity is true:

$$A^n N_{\mathcal{D}W} = E_{\mathcal{D}W}.$$

Proof. For the proof, we can use the following identities.

$$A^n N_{\mathcal{D}W} = \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & 4G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & 4G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & 4G_{n-2} \end{pmatrix} \begin{pmatrix} \mathcal{D}W_2 & \mathcal{D}W_1 & \mathcal{D}W_0 \\ \mathcal{D}W_1 & \mathcal{D}W_0 & \mathcal{D}W_{-1} \\ \mathcal{D}W_0 & \mathcal{D}W_{-1} & \mathcal{D}W_{-2} \end{pmatrix},$$

$$= \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

where

$$\begin{aligned}
b_{11} &= \mathcal{D}W_2G_{n+1} + \mathcal{D}W_1\left(-8G_n + 4G_{n-1}\right) + \mathcal{D}W_0 4G_n,\\
b_{12} &= \mathcal{D}W_1G_{n+1} + \mathcal{D}W_0\left(-8G_n + 4G_{n-1}\right) + \mathcal{D}W_{-1} 4G_n,\\
b_{13} &= \mathcal{D}W_0G_{n+1} + \mathcal{D}W_{-1}\left(-8G_n + 4G_{n-1}\right) + \mathcal{D}W_{-2} 4G_n,\\
b_{21} &= \mathcal{D}W_2G_n + \mathcal{D}W_1\left(-8G_n + 4G_{n-1}\right) + \mathcal{D}W_0 4G_{n-1},\\
b_{22} &= \mathcal{D}W_1G_n + \mathcal{D}W_0\left(-8G_n + 4G_{n-1}\right) + \mathcal{D}W_{-1} 4G_{n-1},\\
b_{23} &= \mathcal{D}W_0G_n + \mathcal{D}W_{-1}\left(-8G_n + 4G_{n-1}\right) + \mathcal{D}W_{-2} 4G_{n-1},\\
b_{31} &= \mathcal{D}W_2G_{n-1} + \mathcal{D}W_1\left(-8G_n + 4G_{n-1}\right) + \mathcal{D}W_0 4G_{n-2},\\
b_{32} &= \mathcal{D}W_1G_{n-1} + \mathcal{D}W_0\left(-8G_n + 4G_{n-1}\right) + \mathcal{D}W_{-1} 4G_{n-2},\\
b_{33} &= \mathcal{D}W_0G_{n-1} + \mathcal{D}W_{-1}\left(-8G_n + 4G_{n-1}\right) + \mathcal{D}W_{-2} 4G_{n-2},
\end{aligned}$$

Using the Theorem (3.6) the proof is done. □

From Theorem (3.10), we can write the following corollary.

Corollary 3.19 *We have the following identity.*

(a) *If we define $N_{\mathcal{D}G}$ and $E_{\mathcal{D}G}$ as follows.*

$$N_{\mathcal{D}G} = \begin{pmatrix} \mathcal{D}G_2 & \mathcal{D}G_1 & \mathcal{D}G_0 \\ \mathcal{D}G_1 & \mathcal{D}G_0 & \mathcal{D}G_{-1} \\ \mathcal{D}G_0 & \mathcal{D}G_{-1} & \mathcal{D}G_{-2} \end{pmatrix}, \quad E_{\mathcal{D}G} = \begin{pmatrix} \mathcal{D}G_{n+2} & \mathcal{D}G_{n+1} & \mathcal{D}G_n \\ \mathcal{D}G_{n+1} & \mathcal{D}G_n & \mathcal{D}G_{n-1} \\ \mathcal{D}G_n & \mathcal{D}G_{n-1} & \mathcal{D}G_{n-2} \end{pmatrix},$$

then we get

$$A^nN_{\mathcal{D}G} = E_{\mathcal{D}G}.$$

(b) *If we define $N_{\mathcal{D}H}$ and $E_{\mathcal{D}H}$ as follows.*

$$N_{\mathcal{D}H} = \begin{pmatrix} \mathcal{D}H_2 & \mathcal{D}H_1 & \mathcal{D}H_0 \\ \mathcal{D}H_1 & \mathcal{D}H_0 & \mathcal{D}H_{-1} \\ \mathcal{D}H_0 & \mathcal{D}H_{-1} & \mathcal{D}H_{-2} \end{pmatrix}, \quad E_{\mathcal{D}H} = \begin{pmatrix} \mathcal{D}H_{n+2} & \mathcal{D}H_{n+1} & \mathcal{D}H_n \\ \mathcal{D}H_{n+1} & \mathcal{D}H_n & \mathcal{D}H_{n-1} \\ \mathcal{D}H_n & \mathcal{D}H_{n-1} & \mathcal{D}H_{n-2} \end{pmatrix},$$

then we get

$$A^nN_{\mathcal{D}H} = E_{\mathcal{D}H}.$$

(c) *If we define* $N_{\mathcal{D}R}$ *and* $E_{\mathcal{D}R}$ *as follows.*

$$N_{\mathcal{D}R} = \begin{pmatrix} \mathcal{D}R_2 & \mathcal{D}R_1 & \mathcal{D}R_0 \\ \mathcal{D}R_1 & \mathcal{D}R_0 & \mathcal{D}R_{-1} \\ \mathcal{D}R_0 & \mathcal{D}R_{-1} & \mathcal{D}R_{-2} \end{pmatrix}, \ E_{\mathcal{D}R} = \begin{pmatrix} \mathcal{D}R_{n+2} & \mathcal{D}R_{n+1} & \mathcal{D}R_n \\ \mathcal{D}R_{n+1} & \mathcal{D}R_n & \mathcal{D}R_{n-1} \\ \mathcal{D}R_n & \mathcal{D}R_{n-1} & \mathcal{D}R_{n-2} \end{pmatrix}.$$

then we get

$$A^n N_{\mathcal{D}R} = E_{\mathcal{D}R}.$$

(d) *If we define* $N_{\mathcal{D}C}$ *and* $E_{\mathcal{D}C}$ *as follows.*

$$N_{\mathcal{D}C} = \begin{pmatrix} \mathcal{D}C_2 & \mathcal{D}C_1 & \mathcal{D}C_0 \\ \mathcal{D}C_1 & \mathcal{D}C_0 & \mathcal{D}C_{-1} \\ \mathcal{D}C_0 & \mathcal{D}C_{-1} & \mathcal{D}C_{-2} \end{pmatrix}, \ E_{\mathcal{D}C} = \begin{pmatrix} \mathcal{D}C_{n+2} & \mathcal{D}C_{n+1} & \mathcal{D}C_n \\ \mathcal{D}C_{n+1} & \mathcal{D}C_n & \mathcal{D}C_{n-1} \\ \mathcal{D}C_n & \mathcal{D}C_{n-1} & \mathcal{D}C_{n-2} \end{pmatrix},$$

then we get

$$A^n N_{\mathcal{D}C} = E_{\mathcal{D}C}.$$

CHAPTER 4

HYPERBOLIC GENERALIZED WOODALL NUMBERS

In this chapter, we investigate hyperbolic generalized Woodall numbers. First, we define these numbers, then we include some properties about these numbers. Our work which is given in this chapter is original. This chapter was published as the journal, see [23].

4.1 DEFINITIONS AND PROPERTIES

We now define hyperbolic generalized Woodall numbers over $\mathbb{H}$. The nth hyperbolic generalized Woodall number is

$$\mathcal{H}W_n = W_n + jW_{n+1}. \tag{4.1}$$

with the initial values $\mathcal{H}W_0$, $\mathcal{H}W_1$, $\mathcal{H}W_2$. (4.1) can be written to negative subscripts by defining,

$$\mathcal{H}W_{-n} = W_{-n} + jW_{-n+1}. \tag{4.2}$$

So, identity (4.1) holds for all integers n.
The special cases of the nth hyperbolic generalized Woodall numbers are given as

$$\begin{aligned}
\mathcal{H}G_n &= G_n + jG_{n+1}, \\
\mathcal{H}H_n &= H_n + jH_{n+1}, \\
\mathcal{H}R_n &= R_n + jR_{n+1}, \\
\mathcal{H}C_n &= C_n + jC_{n+1}.
\end{aligned}$$

Hence, using (4.1), for $n \geq 0$, the following identity is true.

$$\mathcal{H}W_n = 5\mathcal{H}W_{n-1} - 8\mathcal{H}W_{n-2} + 4\mathcal{H}W_{n-3}. \tag{4.3}$$

Using (4.2), the sequence $\{\mathcal{H}W_n\}_{n\geq 0}$ can be written as

$$\mathcal{H}W_{-n} = -2\mathcal{H}W_{-(n-1)} - \frac{5}{4}\mathcal{H}W_{-(n-2)} + \frac{1}{4}\mathcal{H}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (4.3) holds for all integers n.

The few hyperbolic generalized Woodall numbers with positive subscript and negative subscript are given in the following Table 4.1.

Table 4.1. A few hyperbolic generalized Woodall numbers.

n	$\mathcal{H}W_n$	$\mathcal{H}W_{-n}$
0	$\mathcal{H}W_0$	$\mathcal{H}W_0$
1	$\mathcal{H}W_1$	$\frac{1}{4}(8\mathcal{H}W_0 - 5\mathcal{H}W_1 + \mathcal{H}W_2)$
2	$\mathcal{H}W_2$	$\frac{1}{4}(11\mathcal{H}W_0 - 9\mathcal{H}W_1 + 2\mathcal{H}W_2)$
3	$4\mathcal{H}W_0 - 8\mathcal{H}W_1 + 5\mathcal{H}W_2$	$\frac{1}{16}(52\mathcal{H}W_0 - 47\mathcal{H}W_1 + 11\mathcal{H}W_2)$
4	$20\mathcal{H}W_0 - 36\mathcal{H}W_1 + 17\mathcal{H}W_2$	$\frac{1}{16}(57\mathcal{H}W_0 - 54\mathcal{H}W_1 + 13\mathcal{H}W_2)$
5	$68\mathcal{H}W_0 - 116\mathcal{H}W_1 + 49\mathcal{H}W_2$	$\frac{1}{64}(240\mathcal{H}W_0 - 233\mathcal{H}W_1 + 57\mathcal{H}W_2)$

Note that

$$\begin{aligned} \mathcal{H}W_0 &= W_0 + jW_1, \\ \mathcal{H}W_1 &= W_1 + jW_2, \\ \mathcal{H}W_2 &= W_2 + jW_3 = W_2 + j(4W_0 - 8W_1 + 5W_2). \end{aligned}$$

For hyperbolic modified Woodall numbers (taking $W_n = G_n$, $G_0 = 0$, $G_1 = 1$, $G_2 = 5$), we get

$$\begin{aligned} \mathcal{H}G_0 &= G_0 + jG_1 = j, \\ \mathcal{H}G_1 &= G_1 + jG_2 = 1 + 5j, \\ \mathcal{H}G_2 &= G_2 + jG_3 = 5 + 17j \end{aligned}$$

and for hyperbolic modified Cullen numbers (taking $W_n = H_n$, $H_0 = 3$, $H_1 = 5$, $H_2 = 9$), we get

$$\begin{aligned} \mathcal{H}H_0 &= H_0 + jH_1 = 3 + 5j, \\ \mathcal{H}H_1 &= H_1 + jH_2 = 5 + 9j, \\ \mathcal{H}H_2 &= H_2 + jH_3 = 9 + 17j \end{aligned}$$

and for hyperbolic Woodall numbers (taking $W_n = R_n$, $R_0 = -1$, $R_1 = 1$, $R_2 = 7$), we get

$$\begin{aligned} \mathcal{H}R_0 &= R_0 + jR_1 = -1 + j, \\ \mathcal{H}R_1 &= R_1 + jR_2 = 1 + 7j, \\ \mathcal{H}R_2 &= R_2 + jR_3 = 7 + 23j \end{aligned}$$

and for hyperbolic Cullen numbers (taking $W_n = C_n$, $C_0 = 1$, $C_1 = 3$, $C_2 = 9$), we get

$$\begin{aligned}
\mathcal{H}C_0 &= C_0 + jC_1 = 1 + 3j, \\
\mathcal{H}C_1 &= C_1 + jC_2 = 3 + 9j, \\
\mathcal{H}C_2 &= C_2 + jC_3 = 9 + 25j.
\end{aligned}$$

A few hyperbolic modified Woodall numbers, hyperbolic modified Cullen numbers, hyperbolic Woodall numbers and hyperbolic Cullen numbers with positive subscript and negative subscript are given in Table 4.2, Table 4.3, Table 4.4 and Table 4.5.

Table 4.2. Hyperbolic Modified Woodall Numbers

n	$\mathcal{H}G_n$	$\mathcal{H}G_{-n}$
0	j	j
1	$1+5j$	0
2	$5+17j$	$\frac{1}{4}$
3	$17+49j$	$\frac{1}{2}+\frac{1}{4}j$
4	$49+129j$	$\frac{11}{16}+\frac{1}{2}j$
5	$129+321j$	$\frac{13}{16}+\frac{11}{16}j$

Table 4.3. Hyperbolic Modified Cullen Numbers

n	$\mathcal{H}H_n$	$\mathcal{H}H_{-n}$
0	$3+5j$	$3+5j$
1	$5+9j$	$2+3j$
2	$9+17j$	$\frac{3}{2}+3j$
3	$17+33j$	$\frac{5}{4}+\frac{3}{2}j$
4	$33+65j$	$\frac{9}{8}+\frac{3}{2}j$
5	$65+129j$	$\frac{17}{16}+\frac{9}{8}j$

Table 4.4. Hyperbolic Woodall Numbers

n	$\mathcal{H}R_n$	$\mathcal{H}R_{-n}$
0	$-1+j$	$-1+j$
1	$1+7j$	$-\frac{3}{2}-j$
2	$7+23j$	$-\frac{3}{2}-\frac{3}{2}j$
3	$23+63j$	$-\frac{11}{8}-\frac{3}{2}j$
4	$63+159j$	$-\frac{5}{4}-\frac{11}{8}j$
5	$159+383j$	$-\frac{37}{32}-\frac{5}{4}j$

Table 4.5. Hyperbolic Cullen Numbers

n	$\mathcal{H}C_n$	$\mathcal{H}C_{-n}$
0	$1+3j$	$1+3j$
1	$3+9j$	$\frac{1}{2}+j$
2	$9+25j$	$\frac{1}{2}+j$
3	$25+65j$	$\frac{5}{8}+\frac{1}{2}j$
4	$65+161j$	$\frac{3}{4}+\frac{1}{2}j$
5	$161+385j$	$\frac{27}{32}+\frac{3}{4}j$

Now, we give Binet's formula for the hyperbolic generalized Woodall numbers and in the rest of the paper, we fix the following notations:

$$\begin{aligned}
\widehat{\alpha} &= 1+2j, \\
\widehat{\beta} &= 2j, \\
\widehat{\gamma} &= 1+j.
\end{aligned}$$

Note that we have the following identities:

$$\begin{aligned}
\widehat{\alpha}^2 &= 5+4j, \\
\widehat{\beta}^2 &= 4, \\
\widehat{\gamma}^2 &= 2+2j, \\
\widehat{\alpha}\widehat{\beta} &= 4+2j, \\
\widehat{\alpha}\widehat{\gamma} &= 3+3j, \\
\widehat{\beta}\widehat{\gamma} &= 2+2j, \\
\widehat{\alpha}\widehat{\beta}\widehat{\gamma} &= 6+6j.
\end{aligned}$$

4.1.1 Binet's Formula For Hyperbolic Generalized Woodall Numbers

Now, we give Binet's formula in the following theorem.

Theorem 4.1 *(Binet's Formula) For any integer n, Binet's formula of nth hyperbolic generalized Woodall number is*

$$\mathcal{H}W_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}. \tag{4.4}$$

Proof. Using Binet's formula given below

$W_n = (A_1 + A_2 n)2^n + A_3,$

we obtain

$$\begin{aligned}
\mathcal{H}W_n &= W_n + jW_{n+1} \\
&= (A_1 + A_2 n)2^n + A_3 + j((A_1 + A_2(n+1))2^{n+1} + A_3) \\
&= A_1 2^n + A_2 n 2^n + A_3 \\
&\quad + jA_1 2^{n+1} + jA_2 n 2^{n+1} + jA_2 2^{n+1} + jA_3 \\
&= A_1 2^n(1+2j) + A_2 n 2^n (1+2j) + A_2 2^n(2j) + A_3(1+j) \\
&= A_1 2^n \widehat{\alpha} + A_2 n 2^n \widehat{\alpha} + A_2 2^n \widehat{\beta} + A_3 \widehat{\gamma} \\
&= (A_1 \widehat{\alpha} + A_2 \widehat{\beta} + A_2 n \widehat{\alpha})2^n + A_3 \widehat{\gamma}.
\end{aligned}$$

This proves (4.4). □

As special cases, for any integer n, Binet's Formula of nth hyperbolic modified Woodall number, hyperbolic modified Cullen number, hyperbolic Woodall number and hyperbolic Cullen number are

- $\mathcal{H}G_n = (-\widehat{\alpha} + \widehat{\beta} + n\widehat{\alpha})2^n + \widehat{\gamma},$

  $\mathcal{H}G_n = 1 + (n-1)2^n + j(1 + n2^{n+1}).$

- $\mathcal{H}H_n = (2\widehat{\alpha})2^n + \widehat{\gamma},$

  $\mathcal{H}H_n = 1 + 2^{n+1} + j(1 + 2^{n+2}).$

- $\mathcal{H}R_n = (\widehat{\beta} + n\widehat{\alpha})2^n - \widehat{\gamma},$

  $\mathcal{H}R_n = -1 + n2^n + j(-1 + 2^{n+1} + n2^{n+1}).$

- $\mathcal{H}C_n = (\widehat{\beta} + n\widehat{\alpha})2^n + \widehat{\gamma},$

  $\mathcal{H}C_n = 1 + n2^n + j(1 + 2^{n+1} + n2^{n+1}).$

Next, we present the generating function of the hyperbolic generalized Woodall numbers in the following theorem.

4.1.2 Generating Function For Hyperbolic Generalized Woodall Numbers

Theorem 4.2 *The generating function of the hyperbolic generalized Woodall numbers is*

$$\sum_{n=0}^{\infty} \mathcal{H}W_n x^n = \frac{\mathcal{H}W_0 + (\mathcal{H}W_1 - 5\mathcal{H}W_0)x + (\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0)x^2}{1 - 5x + 8x^2 - 4x^3}. \tag{4.5}$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} \mathcal{H}W_n x^n$$

be generating function of the hyperbolic generalized Woodall numbers. Then, using the definition of the hyperbolic generalized Woodall numbers, and subtracting $xg(x)$, $x^2g(x)$ and $x^3g(x)$ from $g(x)$, we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
&(1 - 5x + 8x^2 - 4x^3)g(x) \\
= \; & \sum_{n=0}^{\infty} \mathcal{H}W_n x^n - 5x\sum_{n=0}^{\infty} \mathcal{H}W_n x^n + 8x^2\sum_{n=0}^{\infty} \mathcal{H}W_n x^n - 4x^3\sum_{n=0}^{\infty} \mathcal{H}W_n x^n \\
= \; & \sum_{n=0}^{\infty} \mathcal{H}W_n x^n - 5\sum_{n=0}^{\infty} \mathcal{H}W_n x^{n+1} + 8\sum_{n=0}^{\infty} \mathcal{H}W_n x^{n+2} - 4\sum_{n=0}^{\infty} \mathcal{H}W_n x^{n+3} \\
= \; & \sum_{n=0}^{\infty} \mathcal{H}W_n x^n - 5\sum_{n=1}^{\infty} \mathcal{H}W_{n-1} x^n + 8\sum_{n=2}^{\infty} \mathcal{H}W_{n-2} x^n - 4\sum_{n=3}^{\infty} \mathcal{H}W_{n-3} x^n \\
= \; & (\mathcal{H}W_0 + \mathcal{H}W_1 x + \mathcal{H}W_2 x^2) - 5(\mathcal{H}W_0 x + \mathcal{H}W_1 x^2) + 8\mathcal{H}W_0 x^2 \\
& + \sum_{n=3}^{\infty} (\mathcal{H}W_n - 5\mathcal{H}W_{n-1} + 8\mathcal{H}W_{n-2} - 4\mathcal{H}W_{n-3}) x^n \\
= \; & \mathcal{H}W_0 + (\mathcal{H}W_1 - 5\mathcal{H}W_0)x + (\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0)x^2.
\end{aligned}$$

Note that, we have the recurrence relation $\mathcal{H}W_n = 5\mathcal{H}W_{n-1} - 8\mathcal{H}W_{n-2} + 4\mathcal{H}W_{n-3}$. Rearranging the above equation, we get

$$g(x) = \frac{\mathcal{H}W_0 + (\mathcal{H}W_1 - 5\mathcal{H}W_0)x + (\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0)x^2}{1 - 5x + 8x^2 - 4x^3}.$$

The proof is finished. □

As special cases, the generating functions for the hyperbolic modified Woodall, hyperbolic modified Cullen, hyperbolic Woodall and hyperbolic Cullen numbers are

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{H}G_n x^n &= \frac{j + x}{1 - 5x + 8x^2 - 4x^3}, \\
\sum_{n=0}^{\infty} \mathcal{H}H_n x^n &= \frac{5j + 3 + (-16j - 10)\,x + (12j + 8)\,x^2}{1 - 5x + 8x^2 - 4x^3}, \\
\sum_{n=0}^{\infty} \mathcal{H}R_n x^n &= \frac{-1 + j + (2j + 6)x + (-4j - 6)x^2}{1 - 5x + 8x^2 - 4x^3}
\end{aligned}$$

and

$$\sum_{n=0}^{\infty}\mathcal{H}C_n x^n = \frac{3j+1+(-6j-2)\,x+(4j+2)\,x^2}{1-5x+8x^2-4x^3}$$

respectively.

Now, we obtain Binet's formula of hyperbolic generalized Woodall number $\{\mathcal{H}W_n\}$ by the use of generating function for $\mathcal{H}W_n$.

Theorem 4.3 *(The Binet's formula of hyperbolic generalized Woodall numbers)*

$$\mathcal{H}W_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}. \tag{4.6}$$

Proof. Let

$$\sum_{n=0}^{\infty}\mathcal{H}W_n x^n = \frac{\mathcal{H}W_0 + (\mathcal{H}W_1 - 5\mathcal{H}W_0)x + (\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0)x^2}{1-5x+8x^2-4x^3}.$$

Then, we write

$$\frac{\mathcal{H}W_0 + (\mathcal{H}W_1 - 5\mathcal{H}W_0)x + (\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0)x^2}{(1-x)\,(1-2x)^2} = \frac{d_1}{(1-x)} + \frac{d_2}{(1-2x)} + \frac{d_3}{(1-2x)^2}. \tag{4.7}$$

So,

$\mathcal{H}W_0 + (\mathcal{H}W_1 - 5\mathcal{H}W_0)x + (\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0)x^2 = (d_1 + d_2 + d_3) + (-4d_1 - 3d_2 - d_3)x + (4d_1 + 2d_2)x^2$.

We get

$$\begin{aligned}
\mathcal{H}W_0 &= d_1 + d_2 + d_3, \\
\mathcal{H}W_1 - 5\mathcal{H}W_0 &= -4d_1 - 3d_2 - d_3, \\
\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0 &= 4d_1 + 2d_2.
\end{aligned}$$

If we solve this simultaneous equation, we obtain

$$\begin{aligned}
d_1 &= 4\mathcal{H}W_0 - 4\mathcal{H}W_1 + \mathcal{H}W_2, \\
d_2 &= -4\mathcal{H}W_0 + \frac{11}{2}\mathcal{H}W_1 - \frac{3}{2}\mathcal{H}W_2, \\
d_3 &= \mathcal{H}W_0 - \frac{3}{2}\mathcal{H}W_1 + \frac{1}{2}\mathcal{H}W_2.
\end{aligned}$$

Thus, (4.7) can be written as

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{H}W_nx^n &= d_1\frac{1}{(1-x)}+d_2\frac{1}{(1-2x)}+d_3\frac{1}{(2x-1)^2}, \\
&= d_1\sum_{n=0}^{\infty}x^n+d_2\sum_{n=0}^{\infty}2^nx^n+d_3\sum_{n=0}^{\infty}2^n(n+1)x^n, \\
&= \sum_{n=0}^{\infty}(d_1+d_22^n+d_32^n(n+1))x^n, \\
&= \sum_{n=0}^{\infty}(4\mathcal{H}W_0-4\mathcal{H}W_1+\mathcal{H}W_2+(-4\mathcal{H}W_0+\frac{11}{2}\mathcal{H}W_1-\frac{3}{2}\mathcal{H}W_2)2^n \\
&\quad +(\mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2)2^n(n+1))x^n, \\
&= \sum_{n=0}^{\infty}(4\mathcal{H}W_0-4\mathcal{H}W_1+\mathcal{H}W_2+(-4\mathcal{H}W_0+\frac{11}{2}\mathcal{H}W_1-\frac{3}{2}\mathcal{H}W_2)2^n \\
&\quad +(\mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2)2^n+(\mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2)2^nn)x^n, \\
&= \sum_{n=0}^{\infty}(4\mathcal{H}W_0-4\mathcal{H}W_1+\mathcal{H}W_2+(\mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2)n2^n \\
&\quad +(-3\mathcal{H}W_0+4\mathcal{H}W_1-\mathcal{H}W_2)2^n)x^n, \\
&= \sum_{n=0}^{\infty}((-3\mathcal{H}W_0+4\mathcal{H}W_1-\mathcal{H}W_2)+(\mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2)n)2^n \\
&\quad +4\mathcal{H}W_0-4\mathcal{H}W_1+\mathcal{H}W_2)x^n.
\end{aligned}$$

This gives

$$\mathcal{H}W_n=(\mathcal{H}A_1+\mathcal{H}A_2n)2^n+\mathcal{H}A_3$$

where

$$\begin{aligned}
\mathcal{H}A_1 &= -3\mathcal{H}W_0+4\mathcal{H}W_1-\mathcal{H}W_2, \\
\mathcal{H}A_2 &= \mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2, \\
\mathcal{H}A_3 &= 4\mathcal{H}W_0-4\mathcal{H}W_1+\mathcal{H}W_2.
\end{aligned}$$

Note that the following equalities are true:

$$\begin{aligned}
A_1\widehat{\alpha}+A_2\widehat{\beta} &= (-W_2+4W_1-3W_0)(1+2j)+(\frac{W_2-3W_1+2W_0}{2})(2j) \\
&= -3W_0+4W_1-W_2+j(-4W_0+5W_1-W_2).
\end{aligned}$$

$$\begin{aligned}
A_2\widehat{\alpha} &= \frac{W_2-3W_1+2W_0}{2}(1+2j) \\
&= W_0-\frac{3}{2}W_1+\frac{1}{2}W_2+j(2W_0-3W_1+W_2).
\end{aligned}$$

$A_3\widehat{\gamma} = W_2 - 4W_1 + 4W_0 + j(W_2 - 4W_1 + 4W_0).$

Therefore, we can write the following equality:

$\mathcal{H}W_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}.$

The proof is finished. □

Next, using Theorem 4.3, we present Binet's formulas of hyperbolic modified Woodall, hyperbolic modified Cullen, hyperbolic Woodall and hyperbolic Cullen numbers.

4.2 SOME IDENTITIES FOR HYPERBOLIC GENERALIZED WOODALL NUMBERS

We now investigate a few special identities for the hyperbolic generalized Woodall sequence $\{\mathcal{H}W_n\}$. The following theorem presents Simpson's identity for the hyperbolic generalized Woodall numbers.

Theorem 4.4 *(The Simpson's formula of hyperbolic generalized Woodall sequence)For all integers n we have*

$$\begin{vmatrix} \mathcal{H}W_{n+2} & \mathcal{H}W_{n+1} & \mathcal{H}W_n \\ \mathcal{H}W_{n+1} & \mathcal{H}W_n & \mathcal{H}W_{n-1} \\ \mathcal{H}W_n & \mathcal{H}W_{n-1} & \mathcal{H}W_{n-2} \end{vmatrix} = 4^n \begin{vmatrix} \mathcal{H}W_2 & \mathcal{H}W_1 & \mathcal{H}W_0 \\ \mathcal{H}W_1 & \mathcal{H}W_0 & \mathcal{H}W_{-1} \\ \mathcal{H}W_0 & \mathcal{H}W_{-1} & \mathcal{H}W_{-2} \end{vmatrix}. \tag{4.8}$$

Proof. First, we assume that $n \geq 0$. For the proof, we use mathematical induction on n. For $n = 0$ identity (4.8) is true. Now, we take identity (4.8) is true for $n = k$. Hence, we write the following identity

$$\begin{vmatrix} \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} & \mathcal{H}W_k \\ \mathcal{H}W_{k+1} & \mathcal{H}W_k & \mathcal{H}W_{k-1} \\ \mathcal{H}W_k & \mathcal{H}W_{k-1} & \mathcal{H}W_{k-2} \end{vmatrix} = 4^k \begin{vmatrix} \mathcal{H}W_2 & \mathcal{H}W_1 & \mathcal{H}W_0 \\ \mathcal{H}W_1 & \mathcal{H}W_0 & \mathcal{H}W_{-1} \\ \mathcal{H}W_0 & \mathcal{H}W_{-1} & \mathcal{H}W_{-2} \end{vmatrix}.$$

For $n = k+1$, we get

$$\begin{aligned}
&\begin{vmatrix} \mathcal{H}W_{k+3} & \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} \\ \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} & \mathcal{H}W_{k} \\ \mathcal{H}W_{k+1} & \mathcal{H}W_{k} & \mathcal{H}W_{k-1} \end{vmatrix} \\
= &\begin{vmatrix} 5\mathcal{H}W_{k+2} - 8\mathcal{H}W_{k+1} + 4\mathcal{H}W_{k} & \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} \\ 5\mathcal{H}W_{k+1} - 8\mathcal{H}W_{k} + 4\mathcal{H}W_{k-1} & \mathcal{H}W_{k+1} & \mathcal{H}W_{k} \\ 5\mathcal{H}W_{k} - 8\mathcal{H}W_{k-1} + 4\mathcal{H}W_{k-2} & \mathcal{H}W_{k} & \mathcal{H}W_{k-1} \end{vmatrix} \\
= &\ 5\begin{vmatrix} \mathcal{H}W_{k+2} & \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} \\ \mathcal{H}W_{k+1} & \mathcal{H}W_{k+1} & \mathcal{H}W_{k} \\ \mathcal{H}W_{k} & \mathcal{H}W_{k} & \mathcal{H}W_{k-1} \end{vmatrix} - 8\begin{vmatrix} \mathcal{H}W_{k+1} & \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} \\ \mathcal{H}W_{k} & \mathcal{H}W_{k+1} & \mathcal{H}W_{k} \\ \mathcal{H}W_{k-1} & \mathcal{H}W_{k} & \mathcal{H}W_{k-1} \end{vmatrix} \\
&+4\begin{vmatrix} \mathcal{H}W_{k} & \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} \\ \mathcal{H}W_{k-1} & \mathcal{H}W_{k+1} & \mathcal{H}W_{k} \\ \mathcal{H}W_{k-2} & \mathcal{H}W_{k} & \mathcal{H}W_{k-1} \end{vmatrix} \\
= &\ 4\begin{vmatrix} \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} & \mathcal{H}W_{k} \\ \mathcal{H}W_{k+1} & \mathcal{H}W_{k} & \mathcal{H}W_{k-1} \\ \mathcal{H}W_{k} & \mathcal{H}W_{k-1} & \mathcal{H}W_{k-2} \end{vmatrix} = 4^{k+1}\begin{vmatrix} \mathcal{H}W_{2} & \mathcal{H}W_{1} & \mathcal{H}W_{0} \\ \mathcal{H}W_{1} & \mathcal{H}W_{0} & \mathcal{H}W_{-1} \\ \mathcal{H}W_{0} & \mathcal{H}W_{-1} & \mathcal{H}W_{-2} \end{vmatrix}.
\end{aligned}$$

The other case (for $n < 0$) can be done similarly. Thus, the proof is finished. □

From theorem (4.4), we get the following corollary.

Corollary 4.1 *(Simpson's formula of hyperbolic generalized Woodall sequence's special cases)*

(a) $$\begin{vmatrix} \mathcal{H}G_{k+2} & \mathcal{H}G_{k+1} & \mathcal{H}G_{k} \\ \mathcal{H}G_{k+1} & \mathcal{H}G_{k} & \mathcal{H}G_{k-1} \\ \mathcal{H}G_{k} & \mathcal{H}G_{k-1} & \mathcal{H}G_{k-2} \end{vmatrix} = -4^{n-1}(9+9j).$$

(b) $$\begin{vmatrix} \mathcal{H}H_{k+2} & \mathcal{H}H_{k+1} & \mathcal{H}H_{k} \\ \mathcal{H}H_{k+1} & \mathcal{H}H_{k} & \mathcal{H}H_{k-1} \\ \mathcal{H}H_{k} & \mathcal{H}H_{k-1} & \mathcal{H}H_{k-2} \end{vmatrix} = 0.$$

(c) $$\begin{vmatrix} \mathcal{H}R_{k+2} & \mathcal{H}R_{k+1} & \mathcal{H}R_{k} \\ \mathcal{H}R_{k+1} & \mathcal{H}R_{k} & \mathcal{H}R_{k-1} \\ \mathcal{H}R_{k} & \mathcal{H}R_{k-1} & \mathcal{H}R_{k-2} \end{vmatrix} = 4^{n-1}(9+9j).$$

(d) $\begin{vmatrix} \mathcal{H}C_{k+2} & \mathcal{H}C_{k+1} & \mathcal{H}C_{k} \\ \mathcal{H}C_{k+1} & \mathcal{H}C_{k} & \mathcal{H}C_{k-1} \\ \mathcal{H}C_{k} & \mathcal{H}C_{k-1} & \mathcal{H}C_{k-2} \end{vmatrix} = -4^{n-1}(9+9j).$

Theorem 4.5 *(Catalan's identity) For all integers n and m, the following identity holds* $\mathcal{H}W_{n+m}\mathcal{H}W_{n-m} - \mathcal{H}W_n^2 = 2^{n-m}(-2^{m+n}m^2\widehat{\alpha}^2A_2^2 + A_2A_3(-2^{m+1}\widehat{\beta}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} + 2^{2m}\widehat{\beta}\widehat{\gamma} - m\widehat{\alpha}\widehat{\gamma} + n\widehat{\alpha}\widehat{\gamma} - 2^{m+1}n\widehat{\alpha}\widehat{\gamma} + 2^{2m}m\widehat{\alpha}\widehat{\gamma} + 2^{2m}n\widehat{\alpha}\widehat{\gamma}) + A_1A_3(\widehat{\alpha}\widehat{\gamma} - 2^{m+1}\widehat{\alpha}\widehat{\gamma} + 2^{2m}\widehat{\alpha}\widehat{\gamma}))$.

Proof. Using Binet's formula $\mathcal{H}W_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2n\widehat{\alpha})2^n + A_3\widehat{\gamma}$, we get the required identity. □

As special cases of the above theorem, we give Catalan's identity of hyperbolic modified Woodall, hyperbolic modified Cullen, hyperbolic Woodall and hyperbolic Cullen numbers. Firstly, we present Catalan's identity of hyperbolic Woodall numbers.

Corollary 4.2 *(Catalan's identity for the hyperbolic modified Woodall numbers) For all integers n and m, the following identity holds*

$$\begin{aligned} \mathcal{H}G_{n+m}\mathcal{H}G_{n-m} - \mathcal{H}G_n^2 &= -2^{n-m}(\widehat{\alpha}\widehat{\gamma} - \widehat{\beta}\widehat{\gamma} + 2^{2m}\widehat{\alpha}\widehat{\gamma} - 2^{2m}\widehat{\beta}\widehat{\gamma} - 2^{m+1}\widehat{\alpha}\widehat{\gamma} + 2^{m+1}\widehat{\beta}\widehat{\gamma} \\ &\quad +m\widehat{\alpha}\widehat{\gamma} - n\widehat{\alpha}\widehat{\gamma} + 2^{m+n}m^2\widehat{\alpha}^2 - 2^{2m}m\widehat{\alpha}\widehat{\gamma} - 2^{2m}n\widehat{\alpha}\widehat{\gamma} \\ &\quad +2^{m+1}n\widehat{\alpha}\widehat{\gamma}). \end{aligned}$$

Proof. Take $W_n = G_n$ in Theorem 4.5. □

Secondly, we give Catalan's identity of hyperbolic modified Cullen numbers.

Corollary 4.3 *(Catalan's identity for the hyperbolic modified Cullen numbers) For all integers n and m, the following identity holds*

$$\mathcal{H}H_{n+m}\mathcal{H}H_{n-m} - \mathcal{H}H_n^2 = 2^{n-m}(2\widehat{\alpha}\widehat{\gamma} + 2 \times 2^{2m}\widehat{\alpha}\widehat{\gamma} - 2 \times 2^{m+1}\widehat{\alpha}\widehat{\gamma}).$$

Proof. Take $W_n = H_n$ in Theorem 4.5. □

Thirdly, we give Catalan's identity of hyperbolic Woodall numbers.

Corollary 4.4 *(Catalan's identity for the hyperbolic Woodall numbers) For all integers n and m, the following identity holds*

$$\begin{aligned} \mathcal{H}R_{n+m}\mathcal{H}R_{n-m} - \mathcal{H}R_n^2 &= -2^{n-m}(\widehat{\beta}\widehat{\gamma} + 2^{2m}\widehat{\beta}\widehat{\gamma} - 2^{m+1}\widehat{\beta}\widehat{\gamma} - m\widehat{\alpha}\widehat{\gamma} + n\widehat{\alpha}\widehat{\gamma} + 2^{m+n}m^2\widehat{\alpha}^2 \\ &\quad +2^{2m}m\widehat{\alpha}\widehat{\gamma} + 2^{2m}n\widehat{\alpha}\widehat{\gamma} - 2^{m+1}n\widehat{\alpha}\widehat{\gamma}). \end{aligned}$$

Proof. Take $W_n = R_n$ in Theorem 4.5. $\square$

Fourthly, we give Catalan's identity of hyperbolic Cullen numbers.

Corollary 4.5 *(Catalan's identity for the hyperbolic Cullen numbers) For all integers n and m, the following identity holds*

$$\begin{aligned}\mathcal{H}C_{n+m}\mathcal{H}C_{n-m} - \mathcal{H}C_n^2 &= 2^{n-m}(\widehat{\beta}\widehat{\gamma} + 2^{2m}\widehat{\beta}\widehat{\gamma} - 2^{m+1}\widehat{\beta}\widehat{\gamma} - m\widehat{\alpha}\widehat{\gamma} + n\widehat{\alpha}\widehat{\gamma} - 2^{m+n}m^2\widehat{\alpha}^2 \\ &\quad +2^{2m}m\widehat{\alpha}\widehat{\gamma} + 2^{2m}n\widehat{\alpha}\widehat{\gamma} - 2^{m+1}n\widehat{\alpha}\widehat{\gamma}).\end{aligned}$$

Proof. Take $W_n = C_n$ in Theorem 4.5. $\square$

Note that for $m = 1$ in Catalan's identity, we get Cassini's identity for the hyperbolic generalized Woodall sequence.

Corollary 4.6 *(Cassini's identity) For all integers n, the following identity holds*

$$\mathcal{H}W_{n+1}\mathcal{H}W_{n-1} - \mathcal{H}W_n^2 = 2^{n-1}(A_2A_3(3\widehat{\alpha}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} + n\widehat{\alpha}\widehat{\gamma}) - 2^{n+1}A_2^2\widehat{\alpha}^2 + A_1A_3\widehat{\alpha}\widehat{\gamma}).$$

As special cases of Cassini's identity, we give Cassini's identity of hyperbolic modified Woodall, hyperbolic modified Cullen, hyperbolic Woodall and hyperbolic Cullen numbers. Firstly, we present Cassini's identity of hyperbolic modified Woodall numbers.

Corollary 4.7 *(Cassini's identity of hyperbolic modified Woodall numbers) For all integers n, the following identity holds*

$$\mathcal{H}G_{n+1}\mathcal{H}G_{n-1} - \mathcal{H}G_n^2 = 2^{n-1}(2\widehat{\alpha}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} - 2^{n+1}\widehat{\alpha}^2 + n\widehat{\alpha}\widehat{\gamma}).$$

Secondly, we give Cassini's identity of hyperbolic modified Cullen numbers.

Corollary 4.8 *(Cassini's identity of hyperbolic modified Cullen numbers) For all integers n, the following identity holds*

$$\mathcal{H}H_{n+1}\mathcal{H}H_{n-1} - \mathcal{H}H_n^2 = 2^n\widehat{\alpha}\widehat{\gamma}.$$

Fourth, we give Cassini's identity of hyperbolic Woodall numbers.

Corollary 4.9 *(Cassini's identity of hyperbolic Woodall numbers) For all integers n, the following identity holds*

$$\mathcal{H}R_{n+1}\mathcal{H}R_{n-1} - \mathcal{H}R_n^2 = -2^{n-1}(3\widehat{\alpha}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} + 2^{n+1}\widehat{\alpha}^2 + n\widehat{\alpha}\widehat{\gamma}).$$

Thirdly, we give Cassini's identity of hyperbolic Cullen numbers.

Corollary 4.10 *(Cassini's identity of hyperbolic Cullen numbers) For all integers n, the following identity holds*

$$\mathcal{H}C_{n+1}\mathcal{H}C_{n-1}-\mathcal{H}C_n^2=2^{n-1}(3\widehat{\alpha}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}-2^{n+1}\widehat{\alpha}^2+n\widehat{\alpha}\widehat{\gamma}).$$

Theorem 4.6 *For all integers m, n, G_n is woodall numbers, the following identity is true:*

$$\mathcal{H}W_{n+m}=\mathcal{H}W_nG_{m+1}+\mathcal{H}W_{n-1}(-8G_m+4G_{m-1})+4\mathcal{H}W_{n-2}G_m. \tag{4.9}$$

Proof. The identity (4.9) can be proved by mathematical induction on m. First, we assume that $m\geq 0$ and $n\geq 0$. If $m=0$ we get

$$\mathcal{H}W_n=\mathcal{H}W_nG_1+\mathcal{H}W_{n-1}(-8G_0+4G_{-1})+4\mathcal{H}W_{n-2}G_0$$

which is true by seeing that $G_{-1}=0$, $G_{-2}=\frac{1}{4}$, $G_{-3}=\frac{1}{2}$. We assume that the identity given holds for $m=k$. For $m=k+1$, we get

$$\begin{aligned}
\mathcal{H}W_{n+(k+1)} &= 5\mathcal{H}W_{n+k}-8\mathcal{H}W_{n+k-1}+4\mathcal{H}W_{n+k-2}\\
&= 5(\mathcal{H}W_nG_{k+1}+\mathcal{H}W_{n-1}(-8G_k+4G_{k-1})+4\mathcal{H}W_{n-2}G_k)\\
&\quad -8(\mathcal{H}W_nG_k+\mathcal{H}W_{n-1}(-8G_{k-1}+4G_{k-2})+4\mathcal{H}W_{n-2}G_{k-1})\\
&\quad +4(\mathcal{H}W_nG_{k-1}+\mathcal{H}W_{n-1}(-8G_{k-2}+4G_{k-3})+4\mathcal{H}W_{n-2}G_{k-2})\\
&= \mathcal{H}W_n(5G_{k+1}-8G_k+4G_{k-1})+\mathcal{H}W_{n-1}(-8(5G_k-8G_{k-1}+4G_{k-2})\\
&\quad +4(5G_{k-1}-8G_{k-2}+4G_{k-3}))+4\mathcal{H}W_{n-2}(5G_k-8G_{k-1}+4G_{k-2})\\
&= \mathcal{H}W_nG_{k+2}+\mathcal{H}W_{n-1}(-8G_{k+1}+4G_k)+4\mathcal{H}W_{n-2}G_{k+1}\\
&= \mathcal{H}W_nG_{(k+1)+1}+\mathcal{H}W_{n-1}(-8G_{(k+1)}+4G_{(k+1)-1})+4\mathcal{H}W_{n-2}G_{(k+1)}.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves (4.6). Similarly, we can show for the other cases. □

4.3 LINEAR SUMS FOR HYPERBOLIC GENERALIZED WOODALL NUMBERS

In this section, we investigate the summation formulas of the hyperbolic generalized Woodall numbers with positive and negative subscripts.

Now, we present the formulas that give the summation of the hyperbolic generalized Woodall numbers in the following theorem.

Theorem 4.7 *For $n \geq 0$, hyperbolic generalized Woodall numbers have the following formulas:*

(a) $\sum_{k=0}^{n} \mathcal{H}W_k = (3 + n - 3 \times 2^n + 2^n n + 4j + jn - 2^{n+2}j + 2^{n+1}jn)W_2 + (-11 - 4n + 11 \times 2^n - 3 \times 2^n n - 15j - 4jn + 2^{n+4}j - 3 \times 2^{n+1}jn)W_1 + (9 + 4n - 2^{n+3} + 2^{n+1}n + 12j + 4jn - 3 \times 2^{n+2}j + 2^{n+2}jn)W_0.$

(b) $\sum_{k=0}^{n} \mathcal{H}W_{2k} = (\frac{16}{9} + n - \frac{1}{9}2^{2n+4} + \frac{1}{3}2^{2n+2}n + \frac{20}{9}j + jn - \frac{5}{9}2^{2n+2}j + \frac{1}{3}2^{2n+3}jn)W_2 + (-\frac{20}{3} - 4n + \frac{5}{3}2^{2n+2} - 2^{2n+2}n - \frac{25}{3}j + \frac{7}{3}2^{2n+2}j - 4jn - 2^{2n+3}jn)W_1 + (\frac{53}{9} - \frac{11}{9}2^{2n+2} + 4n + \frac{1}{3}2^{2n+3}n + \frac{64}{9}j - \frac{1}{9}2^{2n+6}j + 4jn + \frac{1}{3}2^{2n+4}jn)W_0.$

(c) $\sum_{k=0}^{n} \mathcal{H}W_{2k+1} = (\frac{20}{9} - \frac{5}{9}2^{2n+2} + n + \frac{1}{3}2^{2n+3}n + \frac{25}{9}j - \frac{1}{9}2^{2n+4}j + jn + \frac{1}{3}2^{2n+4}jn)W_2 + (-\frac{25}{3} + \frac{7}{3}2^{2n+2} - 4n - 2^{2n+3}n + \frac{1}{3}2^{2n+5}j - \frac{32}{3}j - 4jn - 2^{2n+4}jn)W_1 + (\frac{64}{9} - \frac{1}{9}2^{2n+6} + 4n + \frac{1}{3}2^{2n+4}n + \frac{80}{9}j - \frac{5}{9}2^{2n+4}j + 4jn + \frac{1}{3}2^{2n+5}jn)W_0.$

Proof. It can be obtained by using Proposition 1.2 and Proposition 1.3.

(a) We can derive the following using the formulas in Proposition 1.2.

$$\sum_{k=0}^{n} \mathcal{H}W_k = \sum_{k=0}^{n} W_k + j\sum_{k=0}^{n} W_{k+1}.$$

$$\begin{aligned}
\sum_{k=0}^{n} \mathcal{H}W_k \quad = \quad & \frac{1}{2}W_2(2n - 2^{n+1}(n-1) + 2^{n+2}(n-2) + 6) - \frac{1}{2}W_1(8n \\
& -2^{n+1}(3n-5) + 2^{n+2}(3n-8) + 22) + W_0(4n - 2^{n+1}(n-2) \\
& +2^{n+2}(n-3) + 9) + j(\frac{1}{2}W_2(2n + 2^{n+3}(n-1) - 2^{n+2}n + 8) \\
& -\frac{1}{2}W_1(8n - 2^{n+2}(3n-2) + 2^{n+3}(3n-5) + 30) + W_0(4n \\
& -2^{n+2}(n-1) + 2^{n+3}(n-2) + 12)).
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{n} \mathcal{H}W_k \quad = \quad & (3 + n - 3 \times 2^n + 2^n n + 4j + jn - 2^{n+2}j + 2^{n+1}jn)W_2 \\
& +(-11 - 4n + 11 \times 2^n - 3 \times 2^n n - 15j - 4jn + 2^{n+4}j \\
& -3 \times 2^{n+1}jn)W_1 + (9 + 4n - 2^{n+3} + 2^{n+1}n + 12j + 4jn \\
& -3 \times 2^{n+2}j + 2^{n+2}jn)W_0.
\end{aligned}$$

The proof is finished. □

(b) We can derive the following using the formulas in Proposition 1.3.

$$\sum_{k=0}^{n} \mathcal{H}W_{2k} = \sum_{k=0}^{n} W_{2k} + j\sum_{k=0}^{n} W_{2k+1}.$$

$$\begin{aligned}
\sum_{k=0}^{n} \mathcal{H}W_{2k} &= \frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) - \frac{1}{18}W_1(72n \\
&\quad -2^{2n+2}(6n-2) + 2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n \\
&\quad +2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32) + j(\frac{1}{18}W_2(18n - 2^{2n+3}(2n+1) \\
&\quad +2^{2n+5}(2n-1) + 40) - \frac{1}{18}W_1(72n - 2^{2n+3}(6n+1) + 2^{2n+5}(6n-5) \\
&\quad +150) + \frac{1}{9}W_0(36n + 2^{2n+5}(2n-2) - 2 \times 2^{2n+3}n + 64)).
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{n} \mathcal{H}W_{2k} &= (\frac{16}{9} + n - \frac{1}{9}2^{2n+4} + \frac{1}{3}2^{2n+2}n + \frac{20}{9}j + jn - \frac{5}{9}2^{2n+2}j + \frac{1}{3}2^{2n+3}jn)W_2 \\
&\quad +(-\frac{20}{3} - 4n + \frac{5}{3}2^{2n+2} - 2^{2n+2}n - \frac{25}{3}j + \frac{7}{3}2^{2n+2}j - 4jn \\
&\quad -2^{2n+3}jn)W_1 + (\frac{53}{9} - \frac{11}{9}2^{2n+2} + 4n + \frac{1}{3}2^{2n+3}n + \frac{64}{9}j - \frac{1}{9}2^{2n+6}j \\
&\quad +4jn + \frac{1}{3}2^{2n+4}jn)W_0.
\end{aligned}$$

The proof is completed. □

(c) We can derive the following using the formulas in Proposition 1.3.

$$\sum_{k=0}^{n} \mathcal{H}W_{2k+1} = \sum_{k=0}^{n} W_{2k+1} + j\sum_{k=0}^{n} W_{2k+2}.$$

$$\begin{aligned}
\sum_{k=0}^{n} \mathcal{H}W_{2k+1} &= \frac{1}{18}W_2(18n - 2^{2n+3}(2n+1) + 2^{2n+5}(2n-1) + 40) \\
&\quad -\frac{1}{18}W_1(72n - 2^{2n+3}(6n+1) + 2^{2n+5}(6n-5) + 150) \\
&\quad +\frac{1}{9}W_0(36n + 2^{2n+5}(2n-2) - 2 \times 2^{2n+3}n + 64) \\
&\quad +j(\frac{1}{9}W_0(36n - 2^{2n+4}(2n+1) + 2^{2n+6}(2n-1) + 80) \\
&\quad -\frac{1}{18}W_1(72n - 2^{2n+4}(6n+4) + 2^{2n+6}(6n-2) + 192) \\
&\quad +\frac{1}{18}W_2(18n - 2^{2n+4}(2n+2) + 2 \times 2^{2n+6}n + 50)).
\end{aligned}$$

$$
\begin{aligned}
\sum_{k=0}^{n} \mathcal{H}W_{2k+1} &= (\frac{20}{9} - \frac{5}{9}2^{2n+2} + n + \frac{1}{3}2^{2n+3}n + \frac{25}{9}j - \frac{1}{9}2^{2n+4}j + jn \\
&\quad + \frac{1}{3}2^{2n+4}jn)W_2 + (-\frac{25}{3} + \frac{7}{3}2^{2n+2} - 4n - 2^{2n+3}n + \frac{1}{3}2^{2n+5}j \\
&\quad - \frac{32}{3}j - 4jn - 2^{2n+4}jn)W_1 + (\frac{64}{9} - \frac{1}{9}2^{2n+6} + 4n + \frac{1}{3}2^{2n+4}n \\
&\quad + \frac{80}{9}j - \frac{5}{9}2^{2n+4}j + 4jn + \frac{1}{3}2^{2n+5}jn)W_0.
\end{aligned}
$$

The proof is finished. □

As a first special case of the above theorem, we have the following summation formulas for hyperbolic Woodall numbers:

Corollary 4.11 *For $n \geq 0$, hyperbolic modified Woodall numbers have the following properties:*

(a) $\sum_{k=0}^{n} \mathcal{H}G_k = 4 + n + 2^{n+1}n - 2^{n+2} + j(5 - 5 \times 2^{n+2} + n + 2^{n+4} + 2^{n+2}n).$

(b) $\sum_{k=0}^{n} \mathcal{H}G_{2k} = \frac{20}{9} + n + \frac{2}{3}2^{2n+2}n + \frac{5}{3}2^{2n+2} - \frac{5}{9}2^{2n+4} + j(\frac{25}{9} - \frac{4}{9}2^{2n+2} + n + \frac{2}{3}2^{2n+3}n).$

(c) $\sum_{k=0}^{n} \mathcal{H}G_{2k+1} = \frac{25}{9} + n + \frac{2}{3}2^{2n+3}n - \frac{4}{9}2^{2n+2} + j(\frac{29}{9} - \frac{5}{9}2^{2n+4} + \frac{1}{3}2^{2n+5} + n + \frac{2}{3}2^{2n+4}n).$

As a second special case of the above theorem, we have the following summation formulas for hyperbolic modified Cullen numbers:

Corollary 4.12 *For $n \geq 0$, hyperbolic modified Cullen numbers have the following properties:*

(a) $\sum_{k=0}^{n} \mathcal{H}H_k = -1 + n - 6 \times 2^{n}n - 3 \times 2^{n+3} + 3 \times 2^{n+1}n + 28 \times 2^{n} + j(-3 - 18 \times 2^{n+2} + 5 \times 2^{n+4} + n - 6 \times 2^{n+1}n + 3 \times 2^{n+2}n).$

(b) $\sum_{k=0}^{n} \mathcal{H}H_{2k} = \frac{1}{3} + n - 2^{2n+3}n + 2^{2n+3}n + \frac{14}{3}2^{2n+2} - 2^{2n+4} + j(-\frac{1}{3} + \frac{20}{3}2^{2n+2} - \frac{1}{3}2^{2n+6} + n - 2^{2n+4}n + 2^{2n+4}n).$

(c) $\sum_{k=0}^{n} \mathcal{H}H_{2k+1} = -\frac{1}{3} + n - 2 \times 2^{2n+3}n + 2^{2n+4}n + \frac{20}{3}2^{2n+2} - \frac{1}{3}2^{2n+6} + j(-\frac{5}{3} - \frac{8}{3}2^{2n+4} + \frac{5}{3}2^{2n+5} + n - 2^{2n+5}n + 2^{2n+5}n).$

As a third special case of the above theorem, we have the following summation formulas for hyperbolic Woodall numbers:

Corollary 4.13 *For $n \geq 0$, hyperbolic Woodall numbers have the following properties:*

(a) $\sum_{k=0}^{n} \mathcal{H}R_k = 1 - n + 4 \times 2^n n + 2^{n+3} - 2^{n+1} n - 10 \times 2^n + j(1 - 2^{n+4} + 2^{n+4} - n + 2^{n+3} n - 2^{n+2} n)$.

(b) $\sum_{k=0}^{n} \mathcal{H}R_{2k} = -\frac{1}{9} - n + \frac{4}{3} 2^{2n+2} n - \frac{1}{3} 2^{2n+3} n + \frac{26}{9} 2^{2n+2} - \frac{7}{9} 2^{2n+4} + j(\frac{1}{9} - n - \frac{14}{9} 2^{2n+2} + \frac{1}{9} 2^{2n+6} + \frac{4}{3} 2^{2n+3} n - \frac{1}{3} 2^{2n+4} n)$.

(c) $\sum_{k=0}^{n} \mathcal{H}R_{2k+1} = \frac{1}{9} - n + \frac{4}{3} 2^{2n+3} n - \frac{1}{3} 2^{2n+4} n - \frac{14}{9} 2^{2n+2} + \frac{1}{9} 2^{2n+6} + j(-\frac{1}{9} + \frac{1}{3} 2^{2n+5} - \frac{2}{9} 2^{2n+4} - n + \frac{4}{3} 2^{2n+4} n - \frac{1}{3} 2^{2n+5} n)$.

As a fourth special case of the above theorem, we have the following summation formulas for hyperbolic Cullen numbers:

Corollary 4.14 *For $n \geq 0$, hyperbolic Cullen numbers have the following properties.*

(a) $\sum_{k=0}^{n} \mathcal{H}C_k = 3 + n - 2^{n+3} + 2^{n+1} n + 6 \times 2^n + j(3 + n + 2^{n+2} n)$.

(b) $\sum_{k=0}^{n} \mathcal{H}C_{2k} = \frac{17}{9} + n + \frac{1}{3} 2^{2n+3} n - \frac{2}{9} 2^{2n+2} + j(\frac{19}{9} + n + \frac{1}{9} 2^{2n+3} + \frac{1}{3} 2^{2n+4} n)$.

(c) $\sum_{k=0}^{n} \mathcal{H}C_{2k+1} = \frac{19}{9} + n + \frac{1}{3} 2^{2n+4} n + \frac{1}{9} 2^{2n+3} + j(\frac{17}{9} + \frac{4}{9} 2^{2n+4} + n + \frac{1}{3} 2^{2n+5} n)$.

We next introduce the formulas that allow us to find the sum of hyperbolic generalized Woodall numbers with negative subscripts in the following theorem.

Theorem 4.8 *For $n \geq 0$, hyperbolic generalized Woodall numbers have the following formulas:*

(a) $\sum_{k=0}^{n} \mathcal{H}W_{-k} = (-2 + \frac{2}{2^n} - 3j + n + \frac{3}{2^n} j + \frac{1}{2 \times 2^n} n + jn + \frac{1}{2^n} jn) W_2 + (7 - \frac{7}{2^n} + 12j - 4n - \frac{11}{2^n} j - \frac{3}{2 \times 2^n} n - 4jn - \frac{3}{2^n} jn) W_1 + (-4 + \frac{5}{2^n} - 8j + 4n + \frac{8}{2^n} j + \frac{1}{2^n} n + 4jn + \frac{2}{2^n} jn) W_0$.

(b) $\sum_{k=0}^{n} \mathcal{H}W_{-2k} = (-\frac{7}{9} + \frac{7}{9 \times 2^{2n}} - \frac{11}{9} j + n + \frac{11}{9 \times 2^{2n}} j + \frac{1}{3 \times 2^{2n}} n + jn + \frac{2}{3 \times 2^{2n}} jn) W_2 + (\frac{8}{3} - \frac{8}{3 \times 2^{2n}} + \frac{16}{3} j - 4n - \frac{13}{3 \times 2^{2n}} j - \frac{1}{2^{2n}} n - 4jn - \frac{2}{2^{2n}} jn) W_1 + (-\frac{8}{9} + \frac{17}{9 \times 2^{2n}} - \frac{28}{9} j + 4n + \frac{28}{9 \times 2^{2n}} j + \frac{2}{3 \times 2^{2n}} n + 4jn + \frac{4}{3 \times 2^{2n}} jn) W_0$.

(c) $\sum_{k=0}^{n}\mathcal{H}W_{-2k+1} = (-\frac{11}{9}+\frac{11}{9\times 2^{2n}}-\frac{7}{9}j+n+\frac{16}{9\times 2^{2n}}j+\frac{2}{3\times 2^{2n}}n+jn+\frac{4}{3\times 2^{2n}}jn)W_2+(\frac{16}{3}-\frac{13}{3\times 2^{2n}}+\frac{20}{3}j-4n-\frac{20}{3\times 2^{2n}}j-\frac{2}{2^{2n}}n-4jn-\frac{4}{2^{2n}}jn)W_1+(-\frac{28}{9}+\frac{28}{9\times 2^{2n}}-\frac{44}{9}j+4n+\frac{44}{9\times 2^{2n}}j+\frac{4}{3\times 2^{2n}}n+4jn+\frac{8}{3\times 2^{2n}}jn)W_0.$

Proof. It can be obtained by using Proposition 1.4 and Proposition 1.5.

(a) We can derive the following using the formulas in Proposition 1.4.

$$\sum_{k=0}^{n}\mathcal{H}W_{-k}=\sum_{k=0}^{n}W_{-k}+j\sum_{k=0}^{n}W_{-k+1}.$$

$$\begin{aligned}\sum_{k=0}^{n}\mathcal{H}W_{-k} &= 4W_0(n+\frac{1}{2^{n+1}}(n+4)-\frac{1}{2^{n+2}}(n+3)-1)\\ &\quad +2W_1(\frac{1}{2^{n+2}}(3n+8)-2n-\frac{1}{2^{n+1}}(3n+11)+\frac{7}{2})\\ &\quad +2W_2(\frac{1}{2}n+\frac{1}{2^{n+1}}(n+3)-\frac{1}{2^{n+2}}(n+2)-1)\\ &\quad +j(2W_2(\frac{1}{2}n+\frac{1}{2^{n}}(n+2)-\frac{1}{2^{n+1}}(n+1)-\frac{3}{2})\\ &\quad +4W_0(n+\frac{1}{2^{n}}(n+3)-\frac{1}{2^{n+1}}(n+2)-2)\\ &\quad +2W_1(\frac{1}{2^{n+1}}(3n+5)-2n-\frac{1}{2^{n}}(3n+8)+6)).\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n}\mathcal{H}W_{-k} &= (-2+\frac{2}{2^{n}}-3j+n+\frac{3}{2^{n}}j+\frac{1}{2\times 2^{n}}n+jn\\ &\quad +\frac{1}{2^{n}}jn)W_2+(7-\frac{7}{2^{n}}+12j-4n-\frac{11}{2^{n}}j-\frac{3}{2\times 2^{n}}n\\ &\quad -4jn-\frac{3}{2^{n}}jn)W_1+(-4+\frac{5}{2^{n}}-8j+4n+\frac{8}{2^{n}}j\\ &\quad +\frac{1}{2^{n}}n+4jn+\frac{2}{2^{n}}jn)W_0.\end{aligned}$$

This proves (a). We can be proved (b) and (c) similarly way using Proposition 1.5.
□

As a first special case of the above theorem, we have the following summation formulas for hyperbolic modified Woodall numbers:

Corollary 4.15 *For $n \geq 0$, hyperbolic modified Woodall numbers have the following properties:*

(a) $\sum_{k=0}^{n}\mathcal{H}G_{-k}=-3+n+\frac{n+3}{2^n}+j(-3+n+\frac{2n+4}{2^n})$.

(b) $\sum_{k=0}^{n}\mathcal{H}G_{-2k}=-\frac{11}{9}+n+\frac{11+6n}{9\times 2^{2n}}+j(-\frac{7}{9}+n+\frac{16+12n}{9\times 2^{2n}})$.

(c) $\sum_{k=0}^{n}\mathcal{H}G_{-2k+1}=-\frac{7}{9}+n+\frac{16+12n}{9\times 2^{2n}}+j(\frac{25}{9}+n+\frac{20+24n}{9\times 2^{2n}})$.

As a second special case of the above theorem, we have the following summation formulas for hyperbolic modified Cullen numbers:

Corollary 4.16 *For $n\geq 0$, hyperbolic modified Cullen numbers have the following properties:*

(a) $\sum_{k=0}^{n}\mathcal{H}H_{-k}=5+n-\frac{2}{2^n}+j(9-\frac{4}{2^n}+n)$.

(b) $\sum_{k=0}^{n}\mathcal{H}H_{-2k}=\frac{11}{3}+n-\frac{2}{3\times 2^{2n}}+j(\frac{19}{3}-\frac{4}{3\times 2^{2n}}+n)$.

(c) $\sum_{k=0}^{n}\mathcal{H}H_{-2k+1}=\frac{19}{3}+n-\frac{4}{3\times 2^{2n}}+j(\frac{35}{3}-\frac{8}{3\times 2^{2n}}+n)$.

As a third special case of the above theorem, we have the following summation formulas for hyperbolic Woodall numbers:

Corollary 4.17 *For $n\geq 0$, hyperbolic Woodall numbers have the following properties:*

(a) $\sum_{k=0}^{n}\mathcal{H}R_{-k}=-3-n+\frac{2+n}{2^n}+j(-1-n+\frac{2+2n}{2^n})$.

(b) $\sum_{k=0}^{n}\mathcal{H}R_{-2k}=-\frac{17}{9}-n+\frac{8+6n}{9\times 2^{2n}}+j(-\frac{1}{9}-n+\frac{10+12n}{9\times 2^{2n}})$.

(c) $\sum_{k=0}^{n}\mathcal{H}R_{-2k+1}=-\frac{1}{9}-n+\frac{10+12n}{9\times 2^{2n}}+j(\frac{55}{9}-n+\frac{8+24n}{9\times 2^{2n}})$.

As a fourth special case of the above theorem, we have the following summation formulas for hyperbolic Cullen numbers:

Corollary 4.18 *For $n\geq 0$, hyperbolic Cullen numbers have the following properties:*

(a) $\sum_{k=0}^{n}\mathcal{H}C_{-k}=-1+n+\frac{2+n}{2^n}+j(1+\frac{2+2n}{2^n}+n)$.

(b) $\sum_{k=0}^{n}\mathcal{H}C_{-2k}=\frac{1}{9}+n+\frac{8+6n}{9\times 2^{2n}}+j(\frac{17}{9}+\frac{10+12n}{9\times 2^{2n}}+n)$.

(c) $\sum_{k=0}^{n}\mathcal{H}C_{-2k+1}=\frac{17}{9}+n+\frac{10+12n}{9\times 2^{2n}}+j(\frac{73}{9}+\frac{8+24n}{9\times 2^{2n}}+n)$.

4.4 MATRICES RELATED TO HYPERBOLIC GENERALIZED WOODALL NUMBERS

In this section, we present matrices related to hyperbolic generalized Woodall numbers. Now, $\{G_n\}$ defined by the third-order recurrence relation as follows

$G_n = 5G_{n-1} - 8G_{n-2} + 4G_{n-3}$ with the initial conditions $G_0 = 0,\ G_1 = 1,\ G_2 = 5.$

We present the square matrix A of order 3 as

$$A = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 4$. Then, we give the following Lemma.
Note that

$$A^n = \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & 4G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & 4G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & 4G_{n-2} \end{pmatrix}.$$

For the proof see [60].

Lemma 4.9 *For all integers n the following identity is true.*

$$\begin{pmatrix} \mathcal{H}W_{n+2} \\ \mathcal{H}W_{n+1} \\ \mathcal{H}W_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix}.$$

Proof. First, we suppose that $n \geq 0$. Lemma (4.9) can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} \mathcal{H}W_{k+2} \\ \mathcal{H}W_{k+1} \\ \mathcal{H}W_k \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix}$$

For $n = k + 1$, we get

$$\begin{aligned}
\begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix} &= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix} \\
&= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{H}W_{k+2} \\ \mathcal{H}W_{k+1} \\ \mathcal{H}W_k \end{pmatrix} \\
&= \begin{pmatrix} 5\mathcal{H}W_{k+2} - 8\mathcal{H}W_{k+1} + 4\mathcal{H}W_k \\ \mathcal{H}W_{k+2} \\ \mathcal{H}W_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{H}W_{k+3} \\ \mathcal{H}W_{k+2} \\ \mathcal{H}W_{k+1} \end{pmatrix}.
\end{aligned}$$

If we suppose that $n < 0$ the proof can be done similarly. Consequently, by mathematical induction on n, the proof is completed. □

Theorem 4.10 *If we define the matrices $N_{\mathcal{H}W}$ and $E_{\mathcal{H}W}$ as follow,*

$$N_{\mathcal{H}W} = \begin{pmatrix} \mathcal{H}W_2 & \mathcal{H}W_1 & \mathcal{H}W_0 \\ \mathcal{H}W_1 & \mathcal{H}W_0 & \mathcal{H}W_{-1} \\ \mathcal{H}W_0 & \mathcal{H}W_{-1} & \mathcal{H}W_{-2} \end{pmatrix}, \quad E_{\mathcal{H}W} = \begin{pmatrix} \mathcal{H}W_{n+2} & \mathcal{H}W_{n+1} & \mathcal{H}W_n \\ \mathcal{H}W_{n+1} & \mathcal{H}W_n & \mathcal{H}W_{n-1} \\ \mathcal{H}W_n & \mathcal{H}W_{n-1} & \mathcal{H}W_{n-2} \end{pmatrix}.$$

then the following identity is true:

$$A^n N_{\mathcal{H}W} = E_{\mathcal{H}W}.$$

Proof. For the proof, we can use the following identities:

$$\begin{aligned}
A^n N_{\mathcal{H}W} &= \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & 4G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & 4G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & 4G_{n-2} \end{pmatrix} \begin{pmatrix} \mathcal{H}W_2 & \mathcal{H}W_1 & \mathcal{H}W_0 \\ \mathcal{H}W_1 & \mathcal{H}W_0 & \mathcal{H}W_{-1} \\ \mathcal{H}W_0 & \mathcal{H}W_{-1} & \mathcal{H}W_{-2} \end{pmatrix}, \\
&= \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
b_{11} &= \mathcal{H}W_2G_{n+1} + \mathcal{H}W_1\left(-8G_n + 4G_{n-1}\right) + \mathcal{H}W_0 4G_n, \\
b_{12} &= \mathcal{H}W_1G_{n+1} + \mathcal{H}W_0\left(-8G_n + 4G_{n-1}\right) + \mathcal{H}W_{-1} 4G_n, \\
b_{13} &= \mathcal{H}W_0G_{n+1} + \mathcal{H}W_{-1}\left(-8G_n + 4G_{n-1}\right) + \mathcal{H}W_{-2} 4G_n, \\
b_{21} &= \mathcal{H}W_2G_n + \mathcal{H}W_1\left(-8G_n + 4G_{n-1}\right) + \mathcal{H}W_0 4G_{n-1}, \\
b_{22} &= \mathcal{H}W_1G_n + \mathcal{H}W_0\left(-8G_n + 4G_{n-1}\right) + \mathcal{H}W_{-1} 4G_{n-1}, \\
b_{23} &= \mathcal{H}W_0G_n + \mathcal{H}W_{-1}\left(-8G_n + 4G_{n-1}\right) + \mathcal{H}W_{-2} 4G_{n-1}, \\
b_{31} &= \mathcal{H}W_2G_{n-1} + \mathcal{H}W_1\left(-8G_n + 4G_{n-1}\right) + \mathcal{H}W_0 4G_{n-2}, \\
b_{32} &= \mathcal{H}W_1G_{n-1} + \mathcal{H}W_0\left(-8G_n + 4G_{n-1}\right) + \mathcal{H}W_{-1} 4G_{n-2}, \\
b_{33} &= \mathcal{H}W_0G_{n-1} + \mathcal{H}W_{-1}\left(-8G_n + 4G_{n-1}\right) + \mathcal{H}W_{-2} 4G_{n-2},
\end{aligned}$$

Using the Theorem (4.6) the proof is done. □

From Theorem (4.10), we can write the following corollary.

Corollary 4.19 *We have the following identity.*

(a) *If we define $N_{\mathcal{H}G}$ and $E_{\mathcal{H}G}$ as follows,*

$$N_{\mathcal{H}G} = \begin{pmatrix} \mathcal{H}G_2 & \mathcal{H}G_1 & \mathcal{H}G_0 \\ \mathcal{H}G_1 & \mathcal{H}G_0 & \mathcal{H}G_{-1} \\ \mathcal{H}G_0 & \mathcal{H}G_{-1} & \mathcal{H}G_{-2} \end{pmatrix}, \; E_{\mathcal{H}G} = \begin{pmatrix} \mathcal{H}G_{n+2} & \mathcal{H}G_{n+1} & \mathcal{H}G_n \\ \mathcal{H}G_{n+1} & \mathcal{H}G_n & \mathcal{H}G_{n-1} \\ \mathcal{H}G_n & \mathcal{H}G_{n-1} & \mathcal{H}G_{n-2} \end{pmatrix},$$

then we get

$$A^nN_{\mathcal{H}G} = E_{\mathcal{H}G}.$$

(b) *If we define $N_{\mathcal{H}H}$ and $E_{\mathcal{H}H}$ as follows,*

$$N_{\mathcal{H}H} = \begin{pmatrix} \mathcal{H}H_2 & \mathcal{H}H_1 & \mathcal{H}H_0 \\ \mathcal{H}H_1 & \mathcal{H}H_0 & \mathcal{H}H_{-1} \\ \mathcal{H}H_0 & \mathcal{H}H_{-1} & \mathcal{H}H_{-2} \end{pmatrix}, \; E_{\mathcal{H}H} = \begin{pmatrix} \mathcal{H}H_{n+2} & \mathcal{H}H_{n+1} & \mathcal{H}H_n \\ \mathcal{H}H_{n+1} & \mathcal{H}H_n & \mathcal{H}H_{n-1} \\ \mathcal{H}H_n & \mathcal{H}H_{n-1} & \mathcal{H}H_{n-2} \end{pmatrix},$$

then we get

$$A^nN_{\mathcal{H}H} = E_{\mathcal{H}H}.$$

(c) *If we define* $N_{\mathcal{H}R}$ *and* $E_{\mathcal{H}R}$ *as follows,*

$$N_{\mathcal{H}R}=\begin{pmatrix} \mathcal{H}R_2 & \mathcal{H}R_1 & \mathcal{H}R_0 \\ \mathcal{H}R_1 & \mathcal{H}R_0 & \mathcal{H}R_{-1} \\ \mathcal{H}R_0 & \mathcal{H}R_{-1} & \mathcal{H}R_{-2} \end{pmatrix},\ E_{\mathcal{H}R}=\begin{pmatrix} \mathcal{H}R_{n+2} & \mathcal{H}R_{n+1} & \mathcal{H}R_n \\ \mathcal{H}R_{n+1} & \mathcal{H}R_n & \mathcal{H}R_{n-1} \\ \mathcal{H}R_n & \mathcal{H}R_{n-1} & \mathcal{H}R_{n-2} \end{pmatrix},$$

then we get

$$A^n N_{\mathcal{H}R} = E_{\mathcal{H}R}.$$

(d) *If we define* $N_{\mathcal{H}C}$ *and* $E_{\mathcal{H}C}$ *as follows,*

$$N_{\mathcal{H}C}=\begin{pmatrix} \mathcal{H}C_2 & \mathcal{H}C_1 & \mathcal{H}C_0 \\ \mathcal{H}C_1 & \mathcal{H}C_0 & \mathcal{H}C_{-1} \\ \mathcal{H}C_0 & \mathcal{H}C_{-1} & \mathcal{H}C_{-2} \end{pmatrix},\ E_{\mathcal{H}C}=\begin{pmatrix} \mathcal{H}C_{n+2} & \mathcal{H}C_{n+1} & \mathcal{H}C_n \\ \mathcal{H}C_{n+1} & \mathcal{H}C_n & \mathcal{H}C_{n-1} \\ \mathcal{H}C_n & \mathcal{H}C_{n-1} & \mathcal{H}C_{n-2} \end{pmatrix},$$

then we get

$$A^n N_{\mathcal{H}C} = E_{\mathcal{H}C}.$$

CHAPTER 5

DUAL HYPERBOLIC GENERALIZED WOODALL NUMBERS

In this chapter, we define dual hyperbolic generalized Woodall numbers and present generating functions and Binet's formulas for them.

5.1 DEFINITIONS AND PROPERTIES

We now define dual hyperbolic generalized Woodall numbers over $\mathbb{H}_{\mathbb{D}}$. The nth dual hyperbolic generalized Woodall number is

$$\widehat{W}_n = W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3}. \tag{5.1}$$

with the initial values $\widehat{W}_0$, $\widehat{W}_1$, $\widehat{W}_2$. (5.1) can be written to negative subscripts by defining,

$$\widehat{W}_{-n} = W_{-n} + jW_{-n+1} + \varepsilon W_{-n+2} + j\varepsilon W_{-n+3}. \tag{5.2}$$

So, identity (5.1) holds for all integers n.
The special cases of the nth dual hyperbolic generalized Woodall numbers are given as

$$\begin{aligned}
\widehat{G}_n &= G_n + jG_{n+1} + \varepsilon G_{n+2} + j\varepsilon G_{n+3}, \\
\widehat{H}_n &= H_n + jH_{n+1} + \varepsilon H_{n+2} + j\varepsilon H_{n+3}, \\
\widehat{R}_n &= R_n + jR_{n+1} + \varepsilon R_{n+2} + j\varepsilon R_{n+3}, \\
\widehat{C}_n &= C_n + jC_{n+1} + \varepsilon C_{n+2} + j\varepsilon C_{n+3}.
\end{aligned}$$

Then, using (5.1), the following identity is true for every integers $n \geq 0$.

$$\widehat{W}_n = 5\widehat{W}_{n-1} - 8\widehat{W}_{n-2} + 4\widehat{W}_{n-3}. \tag{5.3}$$

Using (5.2), the sequence $\{\widehat{W}_n\}_{n\geq 0}$ can be extended to negative subscripts as

$$\widehat{W}_{-n} = -2\widehat{W}_{-(n-1)} - \frac{5}{4}\widehat{W}_{-(n-2)} + \frac{1}{4}\widehat{W}_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrence (5.3) holds for all integers n.
The few dual hyperbolic generalized Woodall numbers with positive subscript and negative subscript are given in the following Table 5.1.

Table 5.1. A few dual hyperbolic generalized Woodall numbers.

n	$\widehat{W}_n$	$\widehat{W}_{-n}$
0	$\widehat{W}_0$	$\widehat{W}_0$
1	$\widehat{W}_1$	$\frac{1}{4}(8\widehat{W}_0-5\widehat{W}_1+\widehat{W}_2)$
2	$\widehat{W}_2$	$\frac{1}{4}(11\widehat{W}_0-9\widehat{W}_1+2\widehat{W}_2)$
3	$4\widehat{W}_0-8\widehat{W}_1+5\widehat{W}_2$	$\frac{1}{16}(52\widehat{W}_0-47\widehat{W}_1+11\widehat{W}_2)$
4	$20\widehat{W}_0-36\widehat{W}_1+17\widehat{W}_2$	$\frac{1}{16}(57\widehat{W}_0-54\widehat{W}_1+13\widehat{W}_2)$
5	$68\widehat{W}_0-116\widehat{W}_1+49\widehat{W}_2$	$\frac{1}{64}(240\widehat{W}_0-233\widehat{W}_1+57\widehat{W}_2)$

Note that

$$\begin{aligned}
\widehat{W}_0 &= W_0+jW_1+\varepsilon W_2+j\varepsilon W_3 = W_0+jW_1+\varepsilon W_2+j\varepsilon(4W_0-8W_1+5W_2), \\
\widehat{W}_1 &= W_1+jW_2+\varepsilon W_3+j\varepsilon W_4 = W_1+jW_2+\varepsilon(4W_0-8W_1+5W_2) \\
&\quad +j\varepsilon(20W_0-36W_1+17W_2), \\
\widehat{W}_2 &= W_2+jW_3+\varepsilon W_4+j\varepsilon W_5 = W_2+j(4W_0-8W_1+5W_2) \\
&\quad +\varepsilon(20W_0-36W_1+17W_2)+j\varepsilon(68W_0-116W_1+49W_2).
\end{aligned}$$

The special cases of dual hyperbolic generalized Woodall numbers, we obtain the following initial conditions.

$$\begin{aligned}
\widehat{G}_0 &= G_0+jG_1+\varepsilon G_2+j\varepsilon G_3 = j+5\varepsilon+17j\varepsilon, \\
\widehat{G}_1 &= G_1+jG_2+\varepsilon G_3+j\varepsilon G_4 = 1+5j+17\varepsilon+49j\varepsilon, \\
\widehat{G}_2 &= G_2+jG_3+\varepsilon G_4+j\varepsilon G_5 = 5+17j+49\varepsilon+129j\varepsilon.
\end{aligned}$$

$$\begin{aligned}
\widehat{H}_0 &= H_0+jH_1+\varepsilon H_2+j\varepsilon H_3 = 3+5j+9\varepsilon+17j\varepsilon, \\
\widehat{H}_1 &= H_1+jH_2+\varepsilon H_3+j\varepsilon H_4 = 5+9j+17\varepsilon+33j\varepsilon, \\
\widehat{H}_2 &= H_2+jH_3+\varepsilon H_4+j\varepsilon H_5 = 9+17j+33\varepsilon+65j\varepsilon.
\end{aligned}$$

$$\begin{aligned}
\widehat{R}_0 &= R_0+jR_1+\varepsilon R_2+j\varepsilon R_3 = -1+j+7\varepsilon+23j\varepsilon, \\
\widehat{R}_1 &= R_1+jR_2+\varepsilon R_3+j\varepsilon R_4 = 1+7j+23\varepsilon+63j\varepsilon, \\
\widehat{R}_2 &= R_2+jR_3+\varepsilon R_4+j\varepsilon R_5 = 7+23j+63\varepsilon+159j\varepsilon.
\end{aligned}$$

$$\begin{aligned}
\widehat{C}_0 &= C_0+jC_1+\varepsilon C_2+j\varepsilon C_3 = 1+3j+9\varepsilon+25j\varepsilon, \\
\widehat{C}_1 &= C_1+jC_2+\varepsilon C_3+j\varepsilon C_4 = 3+9j+25\varepsilon+65j\varepsilon, \\
\widehat{C}_2 &= C_2+jC_3+\varepsilon C_4+j\varepsilon C_5 = 9+25j+65\varepsilon+161j\varepsilon.
\end{aligned}$$

Some dual hyperbolic modified Woodall numbers, dual hyperbolic modified Cullen numbers, dual hyperbolic Woodall numbers and dual hyperbolic Cullen numbers with positive subscript and negative subscript are presented in tables which are given below.

Table 5.2. Dual hyperbolic modified Woodall numbers

n	$\widehat{G}_n$	$\widehat{G}_{-n}$
0	$j+5\varepsilon+17j\varepsilon$	$j+5\varepsilon+17j\varepsilon$
1	$1+5j+17\varepsilon+49j\varepsilon$	$\varepsilon+5j\varepsilon$
2	$5+17j+49\varepsilon+129j\varepsilon$	$\frac{1}{4}+j\varepsilon$
3	$17+49j+129\varepsilon+321j\varepsilon$	$\frac{1}{2}+\frac{1}{4}j$
4	$49+129j+321\varepsilon+769j\varepsilon$	$\frac{11}{16}+\frac{1}{2}j+\frac{1}{4}\varepsilon$
5	$129+321j+769\varepsilon+1793j\varepsilon$	$\frac{13}{16}+\frac{11}{16}j+\frac{1}{2}\varepsilon+\frac{1}{4}j\varepsilon$

The other table is as follows:

Table 5.3. Dual hyperbolic modified Cullen numbers

n	$\widehat{H}_n$	$\widehat{H}_{-n}$
0	$3+5j+9\varepsilon+17j\varepsilon$	$3+5j+9\varepsilon+17j\varepsilon$
1	$5+9j+17\varepsilon+33j\varepsilon$	$2+3j+5\varepsilon+9j\varepsilon$
2	$9+17j+33\varepsilon+65j\varepsilon$	$\frac{3}{2}+2\varepsilon+3j+5j\varepsilon$
3	$17+33j+65\varepsilon+129j\varepsilon$	$\frac{5}{4}+\frac{3}{2}j+2\varepsilon+3j\varepsilon$
4	$33+65j+129\varepsilon+257j\varepsilon$	$\frac{9}{8}+\frac{5}{4}\varepsilon+\frac{3}{2}j+2j\varepsilon$
5	$65+129j+257\varepsilon+513j\varepsilon$	$\frac{17}{16}+\frac{9}{8}j+\frac{5}{4}\varepsilon+\frac{3}{2}j\varepsilon$

The other table is as follows:

Table 5.4. Dual hyperbolic Woodall numbers

n	$\widehat{R}_n$	$\widehat{R}_{-n}$
0	$-1+j+7\varepsilon+23j\varepsilon$	$-1+j+7\varepsilon+23j\varepsilon$
1	$1+7j+23\varepsilon+63j\varepsilon$	$-\frac{3}{2}-j+\varepsilon+7j\varepsilon$
2	$7+23j+63\varepsilon+159j\varepsilon$	$-\frac{3}{2}-\frac{3}{2}j-\varepsilon+j\varepsilon$
3	$23+63j+159\varepsilon+383j\varepsilon$	$-\frac{11}{8}-\frac{3}{2}j-\frac{3}{2}\varepsilon-j\varepsilon$
4	$63+159j+383\varepsilon+895j\varepsilon$	$-\frac{5}{4}-\frac{11}{8}j-\frac{3}{2}\varepsilon-\frac{3}{2}j\varepsilon$
5	$159+383j+895\varepsilon+2047j\varepsilon$	$-\frac{37}{32}-\frac{5}{4}j-\frac{11}{8}\varepsilon-\frac{3}{2}j\varepsilon$

and the other table is as follows:

Table 5.5. Dual hyperbolic Cullen numbers

n	$\widehat{C}_n$	$\widehat{C}_{-n}$
0	$1+3j+9\varepsilon+25j\varepsilon$	$1+3j+9\varepsilon+25j\varepsilon$
1	$3+9j+25\varepsilon+65j\varepsilon$	$\frac{1}{2}+j+3\varepsilon+9j\varepsilon$
2	$9+25j+65\varepsilon+161j\varepsilon$	$\frac{1}{2}+\frac{1}{2}\varepsilon+j+3j\varepsilon$
3	$25+65j+161\varepsilon+385j\varepsilon$	$\frac{5}{8}+\frac{1}{2}j+\frac{1}{2}\varepsilon+j\varepsilon$
4	$65+161j+385\varepsilon+897j\varepsilon$	$\frac{3}{4}+\frac{5}{8}\varepsilon+\frac{1}{2}j+\frac{1}{2}j\varepsilon$
5	$161+385j+897\varepsilon+2049j\varepsilon$	$\frac{27}{32}+\frac{3}{4}j+\frac{5}{8}\varepsilon+\frac{1}{2}j\varepsilon$

Now, we give Binet's formula for the dual hyperbolic generalized Woodall numbers and in the rest of the chapter, we fix the following notations:

$$\widehat{\alpha}=1+2j+4\varepsilon+8j\varepsilon, \widehat{\beta}=2j+8\varepsilon+24j\varepsilon, \widehat{\gamma}=1+j+\varepsilon+j\varepsilon.$$

Note that we have the following identities:

$$\begin{aligned}
\widehat{\alpha}^2 &= 5+4j+40\varepsilon++32j\varepsilon,\\
\widehat{\beta}^2 &= 4+96\varepsilon+32j\varepsilon,\\
\widehat{\gamma}^2 &= 2+2j+4\varepsilon+4j\varepsilon,\\
\widehat{\alpha}\widehat{\beta} &= (1+2j+4\varepsilon+8j\varepsilon)(2j+8\varepsilon+24j\varepsilon),\\
\widehat{\alpha}\widehat{\gamma} &= 3+3j+15\varepsilon+15j\varepsilon,\\
\widehat{\beta}\widehat{\gamma} &= 2+2j+34\varepsilon+34j\varepsilon,\\
\widehat{\alpha}\widehat{\beta}\widehat{\gamma} &= 6+6j+126\varepsilon+126j\varepsilon.
\end{aligned}$$

Now, we present Binet's formula in the following theorem.

5.1.1 Binet's Formula For Dual Hyperbolic Generalized Woodall Numbers

Theorem 5.1 *(Binet's Formula) For any integer n, the nth dual hyperbolic generalized Woodall number is*

$$\widehat{W}_n=(A_1\widehat{\alpha}+A_2\widehat{\beta}+A_2n\widehat{\alpha})2^n+A_3\widehat{\gamma}. \tag{5.4}$$

Proof. Using Binet's formula

$$W_n=(A_1+A_2n)2^n+A_3$$

of the generalized Woodall numbers, we obtain

$$\begin{aligned}
\widehat{W}_n &= W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3} \\
&= (A_1 + A_2 n)2^n + A_3 + j((A_1 + A_2(n+1))2^{n+1} + A_3) \\
&\quad +\varepsilon((A_1 + A_2(n+2))2^{n+2} + A_3) + j\varepsilon((A_1 + A_2(n+3))2^{n+3} + A_3) \\
&= A_1 2^n + A_2 n 2^n + A_3 \\
&\quad +jA_1 2^{n+1} + jA_2 n 2^{n+1} + jA_2 2^{n+1} + jA_3 \\
&\quad +\varepsilon A_1 2^{n+2} + \varepsilon A_2 n 2^{n+2} + 2\varepsilon A_2 2^{n+2} + \varepsilon A_3 \\
&\quad +j\varepsilon A_1 2^{n+3} + j\varepsilon A_2 n 2^{n+3} + 3j\varepsilon A_2 2^{n+3} + j\varepsilon A_3 \\
&= A_1 2^n (1 + 2j + 4\varepsilon + 8j\varepsilon) + A_2 n 2^n (1 + 2j + 4\varepsilon + 8j\varepsilon) + A_2 2^n (2j + 8\varepsilon + 24j\varepsilon) \\
&\quad +A_3(1 + j + \varepsilon + j\varepsilon) \\
&= A_1 2^n \widehat{\alpha} + A_2 n 2^n \widehat{\alpha} + A_2 2^n \widehat{\beta} + A_3 \widehat{\gamma} \\
&= (A_1 \widehat{\alpha} + A_2 \widehat{\beta} + A_2 n \widehat{\alpha}) 2^n + A_3 \widehat{\gamma}.
\end{aligned}$$

This proves (5.4). □

As special cases, for any integer n, Binet's Formula of nth dual hyperbolic modified Woodall number, dual hyperbolic modified Cullen number, dual hyperbolic Woodall number and dual hyperbolic Cullen number are

- $\widehat{G}_n = (-\widehat{\alpha} + \widehat{\beta} + n\widehat{\alpha})2^n + \widehat{\gamma}$,

  $\widehat{G}_n = 1 + (n-1)2^n + j(1 + n2^{n+1}) + \varepsilon(1 + 2^{n+2} + n2^{n+2}) + j\varepsilon(1 + 2^{n+4} + n2^{n+3})$.

- $\widehat{H}_n = (2\widehat{\alpha})2^n + \widehat{\gamma}$,

  $\widehat{H}_n = 1 + 2^{n+1} + j(1 + 2^{n+2}) + \varepsilon(1 + 2^{n+3}) + j\varepsilon(1 + 2^{n+4})$.

- $\widehat{R}_n = (\widehat{\beta} + n\widehat{\alpha})2^n - \widehat{\gamma}$,

  $\widehat{R}_n = -1 + n2^n + j(-1 + 2^{n+1} + n2^{n+1}) + \varepsilon(-1 + 2^{n+3} + n2^{n+2}) + j\varepsilon(-1 + 3 \times 2^{n+3} + n2^{n+3})$.

- $\widehat{C}_n = (\widehat{\beta} + n\widehat{\alpha})2^n + \widehat{\gamma}$,

  $\widehat{C}_n = 1 + n2^n + j(1 + 2^{n+1} + n2^{n+1}) + \varepsilon(1 + 2^{n+3} + n2^{n+2}) + j\varepsilon(1 + 3 \times 2^{n+3} + n2^{n+3})$.

Next, we give a generating function for dual hyperbolic generalized Woodall numbers.

5.1.2 Generating Function For Dual Hyperbolic Generalized Woodall Numbers

Theorem 5.2 *The generating function for the dual hyperbolic generalized Woodall numbers is*

$$\sum_{n=0}^{\infty} \widehat{W}_n x^n = \frac{\widehat{W}_0 + (\widehat{W}_1 - 5\widehat{W}_0)x + (\widehat{W}_2 - 5\widehat{W}_1 + 8\widehat{W}_0)x^2}{1 - 5x + 8x^2 - 4x^3}. \tag{5.5}$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} \widehat{W}_n x^n$$

be generating function of the dual hyperbolic generalized Woodall numbers. Then, using the definition of the dual hyperbolic generalized Woodall numbers, and subtracting $xg(x)$, $x^2 g(x)$ and $x^3 g(x)$ from $g(x)$, we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
(1 - 5x + 8x^2 - 4x^3)g(x) &= \sum_{n=0}^{\infty} \widehat{W}_n x^n - 5x \sum_{n=0}^{\infty} \widehat{W}_n x^n + 8x^2 \sum_{n=0}^{\infty} \widehat{W}_n x^n - 4x^3 \sum_{n=0}^{\infty} \widehat{W}_n x^n \\
&= \sum_{n=0}^{\infty} \widehat{W}_n x^n - 5 \sum_{n=0}^{\infty} \widehat{W}_n x^{n+1} + 8 \sum_{n=0}^{\infty} \widehat{W}_n x^{n+2} - 4 \sum_{n=0}^{\infty} \widehat{W}_n x^{n+3} \\
&= \sum_{n=0}^{\infty} \widehat{W}_n x^n - 5 \sum_{n=1}^{\infty} \widehat{W}_{n-1} x^n + 8 \sum_{n=2}^{\infty} \widehat{W}_{n-2} x^n - 4 \sum_{n=3}^{\infty} \widehat{W}_{n-3} x^n \\
&= (\widehat{W}_0 + \widehat{W}_1 x + \widehat{W}_2 x^2) - 5(\widehat{W}_0 x + \widehat{W}_1 x^2) + 8\widehat{W}_0 x^2 \\
&\quad + \sum_{n=3}^{\infty} (\widehat{W}_n - 5\widehat{W}_{n-1} + 8\widehat{W}_{n-2} - 4\widehat{W}_{n-3}) x^n \\
&= \widehat{W}_0 + (\widehat{W}_1 - 5\widehat{W}_0)x + (\widehat{W}_2 - 5\widehat{W}_1 + 8\widehat{W}_0)x^2.
\end{aligned}$$

Note that we used the recurrence relation $\widehat{W}_n = 5\widehat{W}_{n-1} - 8\widehat{W}_{n-2} + 4\widehat{W}_{n-3}$. Rearranging the above equation, we get

$$g(x) = \frac{\widehat{W}_0 + (\widehat{W}_1 - 5\widehat{W}_0)x + (\widehat{W}_2 - 5\widehat{W}_1 + 8\widehat{W}_0)x^2}{1 - 5x + 8x^2 - 4x^3}.$$

The proof is finished. □

As special cases, the generating functions for the dual hyperbolic modified Woodall, dual hyperbolic modified Cullen, dual hyperbolic Woodall and dual hyperbolic Cullen numbers are

$\sum_{n=0}^{\infty} \widehat{G}_n x^n = \frac{j+5\varepsilon+17j\varepsilon+(1-36j\varepsilon-8\varepsilon)x+(4\varepsilon+20j\varepsilon)x^2}{1-5x+8x^2-4x^3}$,

$\sum_{n=0}^{\infty} \widehat{H}_n x^n = \frac{5j+9\varepsilon+17j\varepsilon+3+(-16j-28\varepsilon-52j\varepsilon-10)x+(12j+20\varepsilon+36j\varepsilon+8)x^2}{1-5x+8x^2-4x^3}$,

$\sum_{n=0}^{\infty} \widehat{R}_n x^n = \frac{-1+j+7\varepsilon+23j\varepsilon+(2j-12\varepsilon-52j\varepsilon+6)x+(4\varepsilon-4j+28j\varepsilon-6)x^2}{1-5x+8x^2-4x^3}$,

and

$\sum_{n=0}^{\infty} \widehat{C}_n x^n = \frac{3j+9\varepsilon+25j\varepsilon+1+(-6j-20\varepsilon-60j\varepsilon-2)x+(4j+12\varepsilon+36j\varepsilon+2)x^2}{1-5x+8x^2-4x^3}$,

respectively.

Now, we obtain Binet's formula of dual hyperbolic generalized Woodall number $\{\widehat{W_n}\}$ by the use of generating function for $\widehat{W_n}$.

Theorem 5.3 *(Binet's formula of dual hyperbolic generalized Woodall numbers)*

$$\widehat{W_n} = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}. \tag{5.6}$$

Proof. Let

$$\sum_{n=0}^{\infty} \widehat{W_n} x^n = \frac{\widehat{W_0} + (\widehat{W_1} - 5\widehat{W_0})x + (\widehat{W_2} - 5\widehat{W_1} + 8\widehat{W_0})x^2}{1 - 5x + 8x^2 - 4x^3}.$$

Then, we write

$$\frac{\widehat{W_0} + (\widehat{W_1} - 5\widehat{W_0})x + (\widehat{W_2} - 5\widehat{W_1} + 8\widehat{W_0})x^2}{(1-x)\,(1-2x)^2} = \frac{d_1}{(1-x)} + \frac{d_2}{(1-2x)} + \frac{d_3}{(1-2x)^2}. \tag{5.7}$$

So,

$$\widehat{W_0} + (\widehat{W_1} - 5\widehat{W_0})x + (\widehat{W_2} - 5\widehat{W_1} + 8\widehat{W_0})x^2 = (d_1+d_2+d_3) + (-4d_1-3d_2-d_3)x + (4d_1+2d_2)x^2.$$

We get

$$\begin{aligned}
\widehat{W_0} &= d_1 + d_2 + d_3, \\
\widehat{W_1} - 5\widehat{W_0} &= -4d_1 - 3d_2 - d_3, \\
\widehat{W_2} - 5\widehat{W_1} + 8\widehat{W_0} &= 4d_1 + 2d_2.
\end{aligned}$$

If we solve this simultaneous equation, we get

$$\begin{aligned}
d_1 &= 4\widehat{W_0} - 4\widehat{W_1} + \widehat{W_2}, \\
d_2 &= -4\widehat{W_0} + \frac{11}{2}\widehat{W_1} - \frac{3}{2}\widehat{W_2}, \\
d_3 &= \widehat{W_0} - \frac{3}{2}\widehat{W_1} + \frac{1}{2}\widehat{W_2}.
\end{aligned}$$

Thus, (5.7) can be written as

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{W}_n x^n &= d_1\frac{1}{(1-x)}+d_2\frac{1}{(1-2x)}+d_3\frac{1}{(2x-1)^2}, \\
&= d_1\sum_{n=0}^{\infty}x^n+d_2\sum_{n=0}^{\infty}2^n x^n+d_3\sum_{n=0}^{\infty}2^n(n+1)x^n, \\
&= \sum_{n=0}^{\infty}(d_1+d_2 2^n+d_3 2^n(n+1))x^n, \\
&= \sum_{n=0}^{\infty}(4\widehat{W}_0-4\widehat{W}_1+\widehat{W}_2+(-4\widehat{W}_0+\frac{11}{2}\widehat{W}_1-\frac{3}{2}\widehat{W}_2)2^n+(\widehat{W}_0-\frac{3}{2}\widehat{W}_1 \\
&\quad +\frac{1}{2}\widehat{W}_2)2^n(n+1))x^n, \\
&= \sum_{n=0}^{\infty}(4\widehat{W}_0-4\widehat{W}_1+\widehat{W}_2+(-4\widehat{W}_0+\frac{11}{2}\widehat{W}_1-\frac{3}{2}\widehat{W}_2)2^n+(\widehat{W}_0-\frac{3}{2}\widehat{W}_1 \\
&\quad +\frac{1}{2}\widehat{W}_2)2^n+(\widehat{W}_0-\frac{3}{2}\widehat{W}_1+\frac{1}{2}\widehat{W}_2)2^n n)x^n, \\
&= \sum_{n=0}^{\infty}(4\widehat{W}_0-4\widehat{W}_1+\widehat{W}_2+(\widehat{W}_0-\frac{3}{2}\widehat{W}_1+\frac{1}{2}\widehat{W}_2)n2^n+(-3\widehat{W}_0+4\widehat{W}_1 \\
&\quad -\widehat{W}_2)2^n)x^n, \\
&= \sum_{n=0}^{\infty}((-3\widehat{W}_0+4\widehat{W}_1-\widehat{W}_2)+(\widehat{W}_0-\frac{3}{2}\widehat{W}_1+\frac{1}{2}\widehat{W}_2)n)2^n+4\widehat{W}_0-4\widehat{W}_1 \\
&\quad +\widehat{W}_2)x^n.
\end{aligned}$$

This gives

$$\widehat{W}_n=(\widehat{A_1}+\widehat{A_2}n)2^n+\widehat{A_3}$$

where

$$\begin{aligned}
\widehat{A_1} &= -3\widehat{W}_0+4\widehat{W}_1-\widehat{W}_2, \\
\widehat{A_2} &= \widehat{W}_0-\frac{3}{2}\widehat{W}_1+\frac{1}{2}\widehat{W}_2, \\
\widehat{A_3} &= 4\widehat{W}_0-4\widehat{W}_1+\widehat{W}_2.
\end{aligned}$$

Note that the following equalities are true:

$A_1\widehat{\alpha}+A_2\widehat{\beta}$

$$\begin{aligned}
&= (-W_2+4W_1-3W_0)(1+2j+4\varepsilon+8j\varepsilon)+(\frac{W_2-3W_1+2W_0}{2})(2j+8\varepsilon+24j\varepsilon) \\
&= -3W_0+4W_1-W_2+j(-4W_0+5W_1-W_2)+\varepsilon(-4W_0+4W_1)+j\varepsilon(-4W_1+4W_2).
\end{aligned}$$

$A_2\widehat{\alpha}$

$$\begin{aligned}
&= \frac{W_2-3W_1+2W_0}{2}(1+2j+4\varepsilon+8j\varepsilon) \\
&= W_0-\frac{3}{2}W_1+\frac{1}{2}W_2+j(2W_0-3W_1+W_2)+\varepsilon(4W_0-6W_1+2W_2) \\
&\quad +j\varepsilon(8W_0-12W_1+4W_2).
\end{aligned}$$

$A_3\widehat{\gamma}$

$$= W_2-4W_1+4W_0+j(W_2-4W_1+4W_0)+\varepsilon(W_2-4W_1+4W_0)+j\varepsilon(W_2-4W_1+4W_0).$$

Therefore, we can write the following equality:

$$\widehat{W}_n = (A_1\widehat{\alpha}+A_2\widehat{\beta}+A_2n\widehat{\alpha})2^n+A_3\widehat{\gamma}.$$

The proof is finished. □

Next, using Theorem 5.3, we present Binet's formulas of dual hyperbolic modified Woodall, dual hyperbolic modified Cullen, dual hyperbolic Woodall and dual hyperbolic Cullen numbers.

5.2 SOME IDENTITIES FOR DUAL HYPERBOLIC GENERALIZED WOODALL NUMBERS

We now present a few special identities for the dual hyperbolic generalized Woodall sequence $\{\widehat{W}_n\}$. The following theorem presents Simpson's identity for the dual hyperbolic generalized Woodall numbers.

Theorem 5.4 *(Simpson's formula for dual hyperbolic generalized Woodall sequence)For all integers n we have*

$$\begin{vmatrix} \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \end{vmatrix} = 4^n \begin{vmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{vmatrix}. \tag{5.8}$$

Proof. First, we assume that $n \geq 0$. For the proof, we use mathematical induction on n. For $n = 0$ identity (5.8) is true. Now, we take (5.8) is true for $n = k$. Therefore, we write the following identity

$$\begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k+1} & \widehat{W}_k & \widehat{W}_{k-1} \\ \widehat{W}_k & \widehat{W}_{k-1} & \widehat{W}_{k-2} \end{vmatrix} = 4^k \begin{vmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{vmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned}
\begin{vmatrix} \widehat{W}_{k+3} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_{k} \\ \widehat{W}_{k+1} & \widehat{W}_{k} & \widehat{W}_{k-1} \end{vmatrix}
&= \begin{vmatrix} 5\widehat{W}_{k+2} - 8\widehat{W}_{k+1} + 4\widehat{W}_{k} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ 5\widehat{W}_{k+1} - 8\widehat{W}_{k} + 4\widehat{W}_{k-1} & \widehat{W}_{k+1} & \widehat{W}_{k} \\ 5\widehat{W}_{k} - 8\widehat{W}_{k-1} + 4\widehat{W}_{k-2} & \widehat{W}_{k} & \widehat{W}_{k-1} \end{vmatrix} \\
&= 5\begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k+1} & \widehat{W}_{k+1} & \widehat{W}_{k} \\ \widehat{W}_{k} & \widehat{W}_{k} & \widehat{W}_{k-1} \end{vmatrix} - 8\begin{vmatrix} \widehat{W}_{k+1} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k} & \widehat{W}_{k+1} & \widehat{W}_{k} \\ \widehat{W}_{k-1} & \widehat{W}_{k} & \widehat{W}_{k-1} \end{vmatrix} \\
&\quad +4\begin{vmatrix} \widehat{W}_{k} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k-1} & \widehat{W}_{k+1} & \widehat{W}_{k} \\ \widehat{W}_{k-2} & \widehat{W}_{k} & \widehat{W}_{k-1} \end{vmatrix} \\
&= 4\begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_{k} \\ \widehat{W}_{k+1} & \widehat{W}_{k} & \widehat{W}_{k-1} \\ \widehat{W}_{k} & \widehat{W}_{k-1} & \widehat{W}_{k-2} \end{vmatrix} = 4^{k+1}\begin{vmatrix} \widehat{W}_{2} & \widehat{W}_{1} & \widehat{W}_{0} \\ \widehat{W}_{1} & \widehat{W}_{0} & \widehat{W}_{-1} \\ \widehat{W}_{0} & \widehat{W}_{-1} & \widehat{W}_{-2} \end{vmatrix}.
\end{aligned}$$

$n < 0$ can be proved similarly. Thus, the proof is finished. □

From theorem (5.4), we get the following corollary.

Corollary 5.1 *(Simpson's formula for dual hyperbolic generalized Woodall sequence's special cases)*

(a) $$\begin{vmatrix} \widehat{G}_{k+2} & \widehat{G}_{k+1} & \widehat{G}_{k} \\ \widehat{G}_{k+1} & \widehat{G}_{k} & \widehat{G}_{k-1} \\ \widehat{G}_{k} & \widehat{G}_{k-1} & \widehat{G}_{k-2} \end{vmatrix} = -4^{n-1}(9 + 9j + 9\varepsilon + 153j\varepsilon).$$

(b) $$\begin{vmatrix} \widehat{H}_{k+2} & \widehat{H}_{k+1} & \widehat{H}_{k} \\ \widehat{H}_{k+1} & \widehat{H}_{k} & \widehat{H}_{k-1} \\ \widehat{H}_{k} & \widehat{H}_{k-1} & \widehat{H}_{k-2} \end{vmatrix} = 0.$$

(c) $$\begin{vmatrix} \widehat{R}_{k+2} & \widehat{R}_{k+1} & \widehat{R}_{k} \\ \widehat{R}_{k+1} & \widehat{R}_{k} & \widehat{R}_{k-1} \\ \widehat{R}_{k} & \widehat{R}_{k-1} & \widehat{R}_{k-2} \end{vmatrix} = 4^{n-1}(9 + 9j + 9\varepsilon + 153j\varepsilon).$$

(d) $$\begin{vmatrix} \widehat{C}_{k+2} & \widehat{C}_{k+1} & \widehat{C}_{k} \\ \widehat{C}_{k+1} & \widehat{C}_{k} & \widehat{C}_{k-1} \\ \widehat{C}_{k} & \widehat{C}_{k-1} & \widehat{C}_{k-2} \end{vmatrix} = -4^{n-1}(9 + 9j + 9\varepsilon + 153j\varepsilon).$$

Theorem 5.5 *(Catalan's identity) For all integers n and m, the following identity holds:* $\widehat{W}_{n+m}\widehat{W}_{n-m}-\widehat{W}_n^2 = 2^{n-m}(-2^{m+n}m^2\widehat{\alpha}^2A_2^2+A_2A_3(-2^{m+1}\widehat{\beta}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}+2^{2m}\widehat{\beta}\widehat{\gamma}-m\widehat{\alpha}\widehat{\gamma}+n\widehat{\alpha}\widehat{\gamma}-2^{m+1}n\widehat{\alpha}\widehat{\gamma}+2^{2m}m\widehat{\alpha}\widehat{\gamma}+2^{2m}n\widehat{\alpha}\widehat{\gamma})+A_1A_3(\widehat{\alpha}\widehat{\gamma}-2^{m+1}\widehat{\alpha}\widehat{\gamma}+2^{2m}\widehat{\alpha}\widehat{\gamma}))$.

Proof. Using Binet's formula $\widehat{W}_n=(A_1\widehat{\alpha}+A_2\widehat{\beta}+A_2n\widehat{\alpha})2^n+A_3\widehat{\gamma}$, we get the required identity. □

As special cases of the above theorem, we give Catalan's identity of dual hyperbolic modified Woodall, dual hyperbolic modified Cullen, dual hyperbolic Woodall and dual hyperbolic Cullen numbers. Firstly, we present Catalan's identity of dual hyperbolic Woodall numbers.

Corollary 5.2 *(Catalan's identity for the dual hyperbolic modified Woodall numbers) For all integers n and m, the following identity holds:*

$$\begin{aligned}\widehat{G}_{n+m}\widehat{G}_{n-m}-\widehat{G}_n^2 &= -2^{n-m}(\widehat{\alpha}\widehat{\gamma}-\widehat{\beta}\widehat{\gamma}+2^{2m}\widehat{\alpha}\widehat{\gamma}-2^{2m}\widehat{\beta}\widehat{\gamma}-2^{m+1}\widehat{\alpha}\widehat{\gamma}+2^{m+1}\widehat{\beta}\widehat{\gamma}+m\widehat{\alpha}\widehat{\gamma}\\ &\quad -n\widehat{\alpha}\widehat{\gamma}+2^{m+n}m^2\widehat{\alpha}^2-2^{2m}m\widehat{\alpha}\widehat{\gamma}-2^{2m}n\widehat{\alpha}\widehat{\gamma}+2^{m+1}n\widehat{\alpha}\widehat{\gamma}).\end{aligned}$$

Proof. Take $W_n=G_n$ in Theorem 3.5. □

Secondly, we give Catalan's identity of dual hyperbolic modified Cullen numbers.

Corollary 5.3 *(Catalan's identity for the dual hyperbolic modified Cullen numbers) For all integers n and m, the following identity holds:*

$$\widehat{H}_{n+m}\widehat{H}_{n-m}-\widehat{H}_n^2=2^{n-m}(2\widehat{\alpha}\widehat{\gamma}+2\times 2^{2m}\widehat{\alpha}\widehat{\gamma}-2\times 2^{m+1}\widehat{\alpha}\widehat{\gamma}).$$

Proof. Take $W_n=H_n$ in Theorem 3.5. □

Thirdly, we give Catalan's identity of dual hyperbolic Woodall numbers.

Corollary 5.4 *(Catalan's identity for the dual hyperbolic Woodall numbers) For all integers n and m, the following identity holds:*
$\widehat{R}_{n+m}\widehat{R}_{n-m}-\widehat{R}_n^2=-2^{n-m}(\widehat{\beta}\widehat{\gamma}+2^{2m}\widehat{\beta}\widehat{\gamma}-2^{m+1}\widehat{\beta}\widehat{\gamma}-m\widehat{\alpha}\widehat{\gamma}+n\widehat{\alpha}\widehat{\gamma}+2^{m+n}m^2\widehat{\alpha}^2+2^{2m}m\widehat{\alpha}\widehat{\gamma}+2^{2m}n\widehat{\alpha}\widehat{\gamma}-2^{m+1}n\widehat{\alpha}\widehat{\gamma})$.

Proof. Take $W_n=R_n$ in Theorem 3.5. □

Fourthly, we give Catalan's identity of dual hyperbolic Cullen numbers.

Corollary 5.5 *(Catalan's identity for the dual hyperbolic Cullen numbers) For all integers n and m, the following identity holds:*
$\widehat{C}_{n+m}\widehat{C}_{n-m} - \widehat{C}_n^2 = 2^{n-m}(\widehat{\beta}\widehat{\gamma} + 2^{2m}\widehat{\beta}\widehat{\gamma} - 2^{m+1}\widehat{\beta}\widehat{\gamma} - m\widehat{\alpha}\widehat{\gamma} + n\widehat{\alpha}\widehat{\gamma} - 2^{m+n}m^2\widehat{\alpha}^2 + 2^{2m}m\widehat{\alpha}\widehat{\gamma} + 2^{2m}n\widehat{\alpha}\widehat{\gamma} - 2^{m+1}n\widehat{\alpha}\widehat{\gamma})$.

Proof. Take $W_n = C_n$ in Theorem 3.5. □
Note that for $m = 1$ in Catalan's identity, we get Cassini's identity for the dual hyperbolic generalized Woodall sequence.

Corollary 5.6 *(Cassini's identity) For all integers n, the following identity holds:*

$$\widehat{W}_{n+1}\widehat{W}_{n-1} - \widehat{W}_n^2 = 2^{n-1}(A_2A_3(3\widehat{\alpha}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} + n\widehat{\alpha}\widehat{\gamma}) - 2^{n+1}A_2^2\widehat{\alpha}^2 + A_1A_3\widehat{\alpha}\widehat{\gamma}).$$

As special cases of Cassini's identity, we give Cassini's identity of dual hyperbolic modified Woodall, dual hyperbolic modified Cullen, dual hyperbolic Woodall and dual hyperbolic Cullen numbers. Firstly, we present Cassini's identity of dual hyperbolic modified Woodall numbers.

Corollary 5.7 *(Cassini's identity of dual hyperbolic modified Woodall numbers) For all integers n, the following identity holds:*

$$\widehat{G}_{n+1}\widehat{G}_{n-1} - \widehat{G}_n^2 = 2^{n-1}(2\widehat{\alpha}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} - 2^{n+1}\widehat{\alpha}^2 + n\widehat{\alpha}\widehat{\gamma}).$$

Secondly, we give Cassini's identity of dual hyperbolic modified Cullen numbers.

Corollary 5.8 *(Cassini's identity of dual hyperbolic modified Cullen numbers) For all integers n, the following identity holds:*

$$\widehat{H}_{n+1}\widehat{H}_{n-1} - \widehat{H}_n^2 = 2^n\widehat{\alpha}\widehat{\gamma}.$$

Fourthly, we give Cassini's identity of dual hyperbolic Woodall numbers.

Corollary 5.9 *(Cassini's identity of dual hyperbolic Woodall numbers) For all integers n, the following identity holds:*

$$\widehat{R}_{n+1}\widehat{R}_{n-1} - \widehat{R}_n^2 = -2^{n-1}(3\widehat{\alpha}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} + 2^{n+1}\widehat{\alpha}^2 + n\widehat{\alpha}\widehat{\gamma}).$$

Thirdly, we give Cassini's identity of dual hyperbolic Cullen numbers.

Corollary 5.10 *(Cassini's identity of dual hyperbolic Cullen numbers) For all integers* n*, the following identity holds:*

$$\widehat{C}_{n+1}\widehat{C}_{n-1}-\widehat{C}_n^2=2^{n-1}(3\widehat{\alpha}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}-2^{n+1}\widehat{\alpha}^2+n\widehat{\alpha}\widehat{\gamma}).$$

Theorem 5.6 *For all integers* m*,* n*,* G_n *is woodall numbers, the following identity is true:*

$$\widehat{W}_{n+m}=\widehat{W}_nG_{m+1}+\widehat{W}_{n-1}(-8G_m+4G_{m-1})+4\widehat{W}_{n-2}G_m. \tag{5.9}$$

Proof. The identity (5.9) can be proved by mathematical induction on m. Firstly, we assume that $m\geq 0$ and $n\geq 0$. If $m=0$ we get

$$\widehat{W}_n=\widehat{W}_nG_1+\widehat{W}_{n-1}(-8G_0+4G_{-1})+4\widehat{W}_{n-2}G_0$$

which is true by seeing that $G_{-1}=0$, $G_{-2}=\frac{1}{4}$, $G_{-3}=\frac{1}{2}$. We assume that the identity given holds for $m=k$. For $m=k+1$, we get

$$\begin{aligned}
\widehat{W}_{n+(k+1)} &= 5\widehat{W}_{n+k}-8\widehat{W}_{n+k-1}+4\widehat{W}_{n+k-2}\\
&= 5(\widehat{W}_nG_{k+1}+\widehat{W}_{n-1}(-8G_k+4G_{k-1})+4\widehat{W}_{n-2}G_k)\\
&\quad -8(\widehat{W}_nG_k+\widehat{W}_{n-1}(-8G_{k-1}+4G_{k-2})+4\widehat{W}_{n-2}G_{k-1})\\
&\quad +4(\widehat{W}_nG_{k-1}+\widehat{W}_{n-1}(-8G_{k-2}+4G_{k-3})+4\widehat{W}_{n-2}G_{k-2})\\
&= \widehat{W}_n(5G_{k+1}-8G_k+4G_{k-1})+\widehat{W}_{n-1}(-8(5G_k-8G_{k-1}+4G_{k-2})\\
&\quad +4(5G_{k-1}-8G_{k-2}+4G_{k-3}))+4\widehat{W}_{n-2}(5G_k-8G_{k-1}+4G_{k-2})\\
&= \widehat{W}_nG_{k+2}+\widehat{W}_{n-1}(-8G_{k+1}+4G_k)+4\widehat{W}_{n-2}G_{k+1}\\
&= \widehat{W}_nG_{(k+1)+1}+\widehat{W}_{n-1}(-8G_{(k+1)}+4G_{(k+1)-1})+4\widehat{W}_{n-2}G_{(k+1)}.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves (5.6). Similarly, we can show for the other cases. □

5.3 LINEAR SUMS FOR DUAL HYPERBOLIC GENERALIZED WOODALL NUMBERS

In this section, we give the summation formulas of the dual hyperbolic generalized Woodall numbers with positive and negative subscripts.
Now, we present the formulas that give the summation of the dual hyperbolic generalized Woodall numbers.

Theorem 5.7 *For $n \geq 0$, dual hyperbolic generalized Woodall numbers have the following formulas:*

(a) $\sum_{k=0}^{n} \widehat{W}_k = (3 + n - 3 \times 2^n + 2^n n + 4j + jn - 2^{n+2} j + 2^{n+1} jn + 5\varepsilon + n\varepsilon - 2^{n+2}\varepsilon + 2^{n+2} n\varepsilon + 5j\varepsilon + jn\varepsilon + 2^{n+3} jn\varepsilon) W_2 + (-11 - 4n + 11 \times 2^n - 3 \times 2^n n - 15j - 4jn + 2^{n+4} j - 3 \times 2^{n+1} jn - 20\varepsilon - 4n\varepsilon + 5 \times 2^{n+2}\varepsilon - 3 \times 2^{n+2} n\varepsilon - 24j\varepsilon - 4jn\varepsilon + 2^{n+4} j\varepsilon - 3 \times 2^{n+3} jn\varepsilon) W_1 + (9 + 4n - 2^{n+3} + 2^{n+1} n + 12j + 4jn - 3 \times 2^{n+2} j + 2^{n+2} jn + 16\varepsilon + 4n\varepsilon - 2^{n+4}\varepsilon + 2^{n+3} n\varepsilon + 20j\varepsilon + 4jn\varepsilon - 2^{n+4} j\varepsilon + 2^{n+4} jn\varepsilon) W_0.$

(b) $\sum_{k=0}^{n} \widehat{W}_{2k} = (\frac{16}{9} + n - \frac{1}{9} 2^{2n+4} + \frac{1}{3} 2^{2n+2} n + \frac{20}{9} j + jn - \frac{5}{9} 2^{2n+2} j + \frac{1}{3} 2^{2n+3} jn + \frac{25}{9}\varepsilon + n \varepsilon - \frac{1}{9} 2^{2n+4}\varepsilon + \frac{1}{3} 2^{2n+4} n\varepsilon + \frac{29}{9} j\varepsilon + \frac{1}{9} 2^{2n+4} j\varepsilon + jn\varepsilon + \frac{32}{3} 2^{2n} jn\varepsilon) W_2 + (-\frac{20}{3} - 4n + \frac{5}{3} 2^{2n+2} - 2^{2n+2} n - \frac{25}{3} j + \frac{7}{3} 2^{2n+2} j - 4jn - 2^{2n+3} jn - \frac{32}{3}\varepsilon + \frac{1}{3} 2^{2n+5}\varepsilon - 4n\varepsilon - 2^{2n+4} n\varepsilon - \frac{40}{3} j\varepsilon + \frac{1}{3} 2^{2n+4} j\varepsilon - 4jn\varepsilon - 2^{2n+5} jn\varepsilon) W_1 + (\frac{53}{9} - \frac{11}{9} 2^{2n+2} + 4n + \frac{1}{3} 2^{2n+3} n + \frac{64}{9} j - \frac{1}{9} 2^{2n+6} j + 4jn + \frac{1}{3} 2^{2n+4} jn + \frac{80}{9}\varepsilon - \frac{5}{9} 2^{2n+4}\varepsilon + 4n\varepsilon + \frac{1}{3} 2^{2n+5} n\varepsilon + \frac{100}{9} j\varepsilon - \frac{1}{9} 2^{2n+6} j\varepsilon + 4jn \varepsilon + \frac{1}{3} 2^{2n+6} jn\varepsilon) W_0.$

(c) $\sum_{k=0}^{n} \widehat{W}_{2k+1} = (\frac{20}{9} - \frac{5}{9} 2^{2n+2} + n + \frac{1}{3} 2^{2n+3} n + \frac{25}{9} j - \frac{1}{9} 2^{2n+4} j + jn + \frac{1}{3} 2^{2n+4} jn + \frac{29}{9}\varepsilon + n \varepsilon + \frac{1}{9} 2^{2n+4}\varepsilon + \frac{1}{3} 2^{2n+5} n\varepsilon + \frac{25}{9} j\varepsilon + \frac{1}{9} 2^{2n+7} j\varepsilon + jn\varepsilon + \frac{1}{3} 2^{2n+6} jn\varepsilon) W_2 + (-\frac{25}{3} + \frac{7}{3} 2^{2n+2} - 4n - 2^{2n+3} n + \frac{1}{3} 2^{2n+5} j - \frac{32}{3} j - 4jn - 2^{2n+4} jn - \frac{40}{3}\varepsilon + \frac{1}{3} 2^{2n+4}\varepsilon - 4n\varepsilon - 2^{2n+5} n\varepsilon - 4jn\varepsilon - \frac{44}{3} j\varepsilon - \frac{1}{3} 2^{2n+6} j\varepsilon - 2^{2n+6} jn\varepsilon) W_1 + (\frac{64}{9} - \frac{1}{9} 2^{2n+6} + 4n + \frac{1}{3} 2^{2n+4} n + \frac{80}{9} j - \frac{5}{9} 2^{2n+4} j + 4jn + \frac{1}{3} 2^{2n+5} jn + \frac{100}{9}\varepsilon - \frac{1}{9} 2^{2n+6}\varepsilon + 4n\varepsilon + \frac{1}{3} 2^{2n+6} n\varepsilon + \frac{116}{9} j\varepsilon + \frac{1}{9} 2^{2n+6} j\varepsilon + 4jn\varepsilon + \frac{1}{3} 2^{2n+7} jn\varepsilon) W_0.$

Proof. It can be obtained by using Proposition 1.2 and Proposition 1.3.

(a) We can derive the following using the formulas in Proposition 1.2.

$$\sum_{k=0}^{n} \widehat{W}_k = \sum_{k=0}^{n} W_k + j \sum_{k=0}^{n} W_{k+1} + \varepsilon \sum_{k=0}^{n} W_{k+2} + j\varepsilon \sum_{k=0}^{n} W_{k+3}.$$

$$\begin{aligned}\sum_{k=0}^{n}\widehat{W}_k &= \frac{1}{2}W_2(2n-2^{n+1}(n-1)+2^{n+2}(n-2)+6)-\frac{1}{2}W_1(8n-2^{n+1}(3n-5)\\ &+2^{n+2}(3n-8)+22)+W_0(4n-2^{n+1}(n-2)+2^{n+2}(n-3)+9)\\ &+j(\frac{1}{2}W_2(2n+2^{n+3}(n-1)-2^{n+2}n+8)-\frac{1}{2}W_1(8n-2^{n+2}(3n-2)\\ &+2^{n+3}(3n-5)+30)+W_0(4n-2^{n+2}(n-1)+2^{n+3}(n-2)+12))\\ &+\varepsilon(\frac{1}{2}W_2(2n-2^{n+3}(n+1)+2^{n+4}n+10)+W_0(4n+2^{n+4}(n-1)\\ &-2^{n+3}n+16)-\frac{1}{2}W_1(8n-2^{n+3}(3n+1)+2^{n+4}(3n-2)+40))\\ &+j\varepsilon(W_0(4n-2^{n+4}(n+1)+2^{n+5}n+20)-\frac{1}{2}W_1(8n+2^{n+5}(3n+1)\\ &-2^{n+4}(3n+4)+48)+\frac{1}{2}W_2(2n-2^{n+4}(n+2)+2^{n+5}(n+1)+10)).\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n}\widehat{W}_k &= (3+n-3\times 2^n+2^n n+4j+jn-2^{n+2}j+2^{n+1}jn+5\varepsilon+n\varepsilon\\ &-2^{n+2}\varepsilon+2^{n+2}n\varepsilon+5j\varepsilon+jn\varepsilon+2^{n+3}jn\varepsilon)W_2+(-11-4n\\ &+11\times 2^n-3\times 2^n n-15j-4jn+2^{n+4}j-3\times 2^{n+1}jn-20\varepsilon\\ &-4n\varepsilon+5\times 2^{n+2}\varepsilon-3\times 2^{n+2}n\varepsilon-24j\varepsilon-4jn\varepsilon+2^{n+4}j\varepsilon\\ &-3\times 2^{n+3}jn\varepsilon)W_1+(9+4n-2^{n+3}+2^{n+1}n+12j+4jn\\ &-3\times 2^{n+2}j+2^{n+2}jn+16\varepsilon+4n\varepsilon-2^{n+4}\varepsilon+2^{n+3}n\varepsilon+20j\varepsilon\\ &+4jn\varepsilon-2^{n+4}j\varepsilon+2^{n+4}jn\varepsilon)W_0.\end{aligned}$$

The proof is finished. □

(b) We can derive the following using the formulas in Proposition 1.3.

$$\sum_{k=0}^{n}\widehat{W}_{2k}=\sum_{k=0}^{n}W_{2k}+j\sum_{k=0}^{n}W_{2k+1}+\varepsilon\sum_{k=0}^{n}W_{2k+2}+j\varepsilon\sum_{k=0}^{n}W_{2k+3}.$$

$$
\begin{aligned}
\sum_{k=0}^{n}\widehat{W}_{2k} = {} & \frac{1}{9}W_0(36n-2^{2n+2}(2n-1)+2^{2n+4}(2n-3)+53)-\frac{1}{18}W_1(72n \\
& -2^{2n+2}(6n-2)+2^{2n+4}(6n-8)+120)+\frac{1}{18}W_2(18n+2^{2n+4}(2n-2) \\
& -2\times 2^{2n+2}n+32)+j(\frac{1}{18}W_2(18n-2^{2n+3}(2n+1)+2^{2n+5}(2n-1) \\
& +40)-\frac{1}{18}W_1(72n-2^{2n+3}(6n+1)+2^{2n+5}(6n-5)+150) \\
& +\frac{1}{9}W_0(36n+2^{2n+5}(2n-2)-2\times 2^{2n+3}n+64))+\varepsilon(\frac{1}{9}W_0(36n \\
& -2^{2n+4}(2n+1)+2^{2n+6}(2n-1)+80)-\frac{1}{18}W_1(72n-2^{2n+4}(6n+4) \\
& +2^{2n+6}(6n-2)+192)+\frac{1}{18}W_2(18n-2^{2n+4}(2n+2)+2\times 2^{2n+6}n \\
& +50))+j\varepsilon(\frac{1}{18}W_2((18n-2^{2n+5}(2n+3)+2^{2n+7}(2n+1)+58) \\
& -\frac{1}{18}W_1(72n+2^{2n+7}(6n+1)-2^{2n+5}(6n+7)+240)+\frac{1}{9}W_0(36n \\
& -2^{2n+5}(2n+2)+2\times 2^{2n+7}n+100)).
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{n}\widehat{W}_{2k} = {} & (\frac{16}{9}+n-\frac{1}{9}2^{2n+4}+\frac{1}{3}2^{2n+2}n+\frac{20}{9}j+jn-\frac{5}{9}2^{2n+2}j+\frac{1}{3}2^{2n+3}jn+\frac{25}{9}\varepsilon \\
& +n\varepsilon-\frac{1}{9}2^{2n+4}\varepsilon+\frac{1}{3}2^{2n+4}n\varepsilon+\frac{29}{9}j\varepsilon+\frac{1}{9}2^{2n+4}j\varepsilon+jn\varepsilon+\frac{32}{3}2^{2n}jn\varepsilon)W_2 \\
& +(-\frac{20}{3}-4n+\frac{5}{3}2^{2n+2}-2^{2n+2}n-\frac{25}{3}j+\frac{7}{3}2^{2n+2}j-4jn-2^{2n+3}jn \\
& -\frac{32}{3}\varepsilon+\frac{1}{3}2^{2n+5}\varepsilon-4n\varepsilon-2^{2n+4}n\varepsilon-\frac{40}{3}j\varepsilon+\frac{1}{3}2^{2n+4}j\varepsilon-4jn\varepsilon \\
& -2^{2n+5}jn\varepsilon)W_1+(\frac{53}{9}-\frac{11}{9}2^{2n+2}+4n+\frac{1}{3}2^{2n+3}n+\frac{64}{9}j-\frac{1}{9}2^{2n+6}j \\
& +4jn+\frac{1}{3}2^{2n+4}jn+\frac{80}{9}\varepsilon-\frac{5}{9}2^{2n+4}\varepsilon+4n\varepsilon+\frac{1}{3}2^{2n+5}n\varepsilon+\frac{100}{9}j\varepsilon \\
& -\frac{1}{9}2^{2n+6}j\varepsilon+4jn\varepsilon+\frac{1}{3}2^{2n+6}jn\varepsilon)W_0.
\end{aligned}
$$

The proof is completed. □

(c) We can derive the following using the formulas in Proposition 1.3.

$$\sum_{k=0}^{n}\widehat{W}_{2k+1}=\sum_{k=0}^{n}W_{2k+1}+j\sum_{k=0}^{n}W_{2k+2}+\varepsilon\sum_{k=0}^{n}W_{2k+3}+j\varepsilon\sum_{k=0}^{n}W_{2k+4}.$$

$$\begin{aligned}\sum_{k=0}^{n}\widehat{W}_{2k+1} &= \frac{1}{18}W_2(18n-2^{2n+3}(2n+1)+2^{2n+5}(2n-1)+40)\\
&-\frac{1}{18}W_1(72n-2^{2n+3}(6n+1)+2^{2n+5}(6n-5)+150)\\
&+\frac{1}{9}W_0(36n+2^{2n+5}(2n-2)-2\times 2^{2n+3}n+64)\\
&+j(\frac{1}{9}W_0(36n-2^{2n+4}(2n+1)+2^{2n+6}(2n-1)+80)\\
&-\frac{1}{18}W_1(72n-2^{2n+4}(6n+4)+2^{2n+6}(6n-2)+192)\\
&+\frac{1}{18}W_2(18n-2^{2n+4}(2n+2)+2\times 2^{2n+6}n+50))\\
&+\varepsilon(\frac{1}{18}W_2((18n-2^{2n+5}(2n+3)+2^{2n+7}(2n+1)+58)\\
&-\frac{1}{18}W_1(72n+2^{2n+7}(6n+1)-2^{2n+5}(6n+7)+240)\\
&+\frac{1}{9}W_0(36n-2^{2n+5}(2n+2)+2\times 2^{2n+7}n+100))\\
&+j\varepsilon(\frac{1}{18}W_2(18n-2^{2n+6}(2n+4)+2^{2n+8}(2n+2)+50)\\
&+\frac{1}{9}W_0(36n-2^{2n+6}(2n+3)+2^{2n+8}(2n+1)+116)\\
&-\frac{1}{18}W_1(72n+2^{2n+8}(6n+4)-2^{2n+6}(6n+10)+264)).\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n}\widehat{W}_{2k+1} &= (\frac{20}{9}-\frac{5}{9}2^{2n+2}+n+\frac{1}{3}2^{2n+3}n+\frac{25}{9}j-\frac{1}{9}2^{2n+4}j+jn+\frac{1}{3}2^{2n+4}jn\\
&+\frac{29}{9}\varepsilon+n\varepsilon+\frac{1}{9}2^{2n+4}\varepsilon+\frac{1}{3}2^{2n+5}n\varepsilon+\frac{25}{9}j\varepsilon+\frac{1}{9}2^{2n+7}j\varepsilon+jn\varepsilon\\
&+\frac{1}{3}2^{2n+6}jn\varepsilon)W_2+(-\frac{25}{3}+\frac{7}{3}2^{2n+2}-4n-2^{2n+3}n+\frac{1}{3}2^{2n+5}j\\
&-\frac{32}{3}j-4jn-2^{2n+4}jn-\frac{40}{3}\varepsilon+\frac{1}{3}2^{2n+4}\varepsilon-4n\varepsilon-2^{2n+5}n\varepsilon-4jn\varepsilon\\
&-\frac{44}{3}j\varepsilon-\frac{1}{3}2^{2n+6}j\varepsilon-2^{2n+6}jn\varepsilon)W_1+(\frac{64}{9}-\frac{1}{9}2^{2n+6}+4n\\
&+\frac{1}{3}2^{2n+4}n+\frac{80}{9}j-\frac{5}{9}2^{2n+4}j+4jn+\frac{1}{3}2^{2n+5}jn+\frac{100}{9}\varepsilon-\frac{1}{9}2^{2n+6}\varepsilon\\
&+4n\varepsilon+\frac{1}{3}2^{2n+6}n\varepsilon+\frac{116}{9}j\varepsilon+\frac{1}{9}2^{2n+6}j\varepsilon+4jn\varepsilon+\frac{1}{3}2^{2n+7}jn\varepsilon)W_0.\end{aligned}$$

The proof is finished. □

As a first special case of the above theorem, we have the following summation formulas for dual hyperbolic Woodall numbers:

Corollary 5.11 *For $n \geq 0$, dual hyperbolic modified Woodall numbers have the following properties:*

(a) $\sum_{k=0}^{n} \widehat{G}_k = 4 + n + 2^{n+1}n - 2^{n+2} + j(5 - 5 \times 2^{n+2} + n + 2^{n+4} + 2^{n+2}n) + \varepsilon(5 + n + 2^{n+3}n) + j\varepsilon(1 + 2^{n+4} + n + 2^{n+4}n)$.

(b) $\sum_{k=0}^{n} \widehat{G}_{2k} = \frac{20}{9} + n + \frac{2}{3}2^{2n+2}n + \frac{5}{3}2^{2n+2} - \frac{5}{9}2^{2n+4} + j(\frac{25}{9} - \frac{4}{9}2^{2n+2} + n + \frac{2}{3}2^{2n+3}n) + \varepsilon(\frac{29}{9} + n - \frac{5}{9}2^{2n+4} + \frac{1}{3}2^{2n+5} + \frac{2}{3}2^{2n+4}n) + j\varepsilon(\frac{25}{9} + n + \frac{8}{9}2^{2n+4} + \frac{160}{3}2^{2n}n - 2^{2n+5}n)$.

(c) $\sum_{k=0}^{n} \widehat{G}_{2k+1} = \frac{25}{9} + n + \frac{2}{3}2^{2n+3}n - \frac{4}{9}2^{2n+2} + j(\frac{29}{9} - \frac{5}{9}2^{2n+4} + \frac{1}{3}2^{2n+5} + n + \frac{2}{3}2^{2n+4}n) + \varepsilon(\frac{25}{9} + n + \frac{8}{9}2^{2n+4} + \frac{2}{3}2^{2n+5}n) + j\varepsilon(-\frac{7}{9} + n - \frac{1}{3}2^{2n+6} + \frac{5}{9}2^{2n+7} + \frac{2}{3}2^{2n+6}n)$.

As a second special case of the above theorem, we have the following summation formulas for dual hyperbolic modified Cullen numbers:

Corollary 5.12 *For $n \geq 0$, dual hyperbolic modified Cullen numbers have the following properties:*

(a) $\sum_{k=0}^{n} \widehat{H}_k = -1 + n - 6 \times 2^n n - 3 \times 2^{n+3} + 3 \times 2^{n+1}n + 28 \times 2^n + j(-3 - 18 \times 2^{n+2} + 5 \times 2^{n+4} + n - 6 \times 2^{n+1}n + 3 \times 2^{n+2}n) + \varepsilon(-7 + 16 \times 2^{n+2} - 3 \times 2^{n+4} + n - 6 \times 2^{n+2}n + 3 \times 2^{n+3} n) + j\varepsilon(-15 + 2 \times 2^{n+4} + n - 6 \times 2^{n+3}n + 3 \times 2^{n+4}n)$.

(b) $\sum_{k=0}^{n} \widehat{H}_{2k} = \frac{1}{3} + n - 2^{2n+3}n + 2^{2n+3}n + \frac{14}{3}2^{2n+2} - 2^{2n+4} + j(-\frac{1}{3} + \frac{20}{3}2^{2n+2} - \frac{1}{3}2^{2n+6} + n - 2^{2n+4}n + 2^{2n+4}n) + \varepsilon(-\frac{5}{3} + n - \frac{8}{3}2^{2n+4} + \frac{5}{3}2^{2n+5} - 2^{2n+5}n + 2^{2n+5}n) + j\varepsilon(-\frac{13}{3} + n + \frac{8}{3}2^{2n+4} - \frac{1}{3}2^{2n+6} + 96 \times 2^{2n}n - 5 \times 2^{2n+5}n + 2^{2n+6}n)$.

(c) $\sum_{k=0}^{n} \widehat{H}_{2k+1} = -\frac{1}{3} + n - 2 \times 2^{2n+3}n + 2^{2n+4}n + \frac{20}{3}2^{2n+2} - \frac{1}{3}2^{2n+6} + j(-\frac{5}{3} - \frac{8}{3}2^{2n+4} + \frac{5}{3}2^{2n+5} + n - 2^{2n+5}n + 2^{2n+5}n) + \varepsilon(-\frac{13}{3} + n + \frac{8}{3}2^{2n+4} - \frac{1}{3}2^{2n+6} - 2^{2n+6}n + 2^{2n+6}n) + j\varepsilon(-\frac{29}{3} + n - \frac{4}{3}2^{2n+6} + 2^{2n+7} - 2^{2n+7}n + 2^{2n+7}n)$.

As a third special case of the above theorem, we have the following summation formulas for dual hyperbolic Woodall numbers:

Corollary 5.13 *For $n \geq 0$, dual hyperbolic Woodall numbers have the following properties:*

(a) $\sum_{k=0}^{n}\widehat{R}_k = 1-n+4\times 2^n n+2^{n+3}-2^{n+1}n-10\times 2^n+j(1-2^{n+4}+2^{n+4}-n+2^{n+3}n-2^{n+2}n)+\varepsilon(-1-2^{n+3}+2^{n+4}-n+2^{n+4}n-2^{n+3}n)+j\varepsilon(-9+2^{n+5}-n+2^{n+5}n-2^{n+4}n)$.

(b) $\sum_{k=0}^{n}\widehat{R}_{2k} = -\frac{1}{9}-n+\frac{4}{3}2^{2n+2}n-\frac{1}{3}2^{2n+3}n+\frac{26}{9}2^{2n+2}-\frac{7}{9}2^{2n+4}+j(\frac{1}{9}-n-\frac{14}{9}2^{2n+2}+\frac{1}{9}2^{2n+6}+\frac{4}{3}2^{2n+3}n-\frac{1}{3}2^{2n+4}n)+\varepsilon(-\frac{1}{9}-n-\frac{2}{9}2^{2n+4}+\frac{1}{3}2^{2n+5}+\frac{4}{3}2^{2n+4}n-\frac{1}{3}2^{2n+5}n)+j\varepsilon(-\frac{17}{9}-n+\frac{10}{9}2^{2n+4}+\frac{1}{9}2^{2n+6}+\frac{224}{3}2^{2n}n-2^{2n+5}n-\frac{1}{3}2^{2n+6}n)$.

(c) $\sum_{k=0}^{n}\widehat{R}_{2k+1} = \frac{1}{9}-n+\frac{4}{3}2^{2n+3}n-\frac{1}{3}2^{2n+4}n-\frac{14}{9}2^{2n+2}+\frac{1}{9}2^{2n+6}+j(-\frac{1}{9}+\frac{1}{3}2^{2n+5}-\frac{2}{9}2^{2n+4}-n+\frac{4}{3}2^{2n+4}n-\frac{1}{3}2^{2n+5}n)+\varepsilon(-\frac{17}{9}-n+\frac{10}{9}2^{2n+4}+\frac{1}{9}2^{2n+6}+\frac{4}{3}2^{2n+5}n-\frac{1}{3}2^{2n+6}n)+j\varepsilon(-\frac{73}{9}-n-\frac{4}{9}2^{2n+6}+\frac{7}{9}2^{2n+7}+\frac{4}{3}2^{2n+6}n-\frac{1}{3}2^{2n+7}n)$.

As a fourth special case of the above theorem, we have the following summation formulas for dual hyperbolic Cullen numbers:

Corollary 5.14 *For $n\geq 0$, dual hyperbolic Cullen numbers have the following properties.*

(a) $\sum_{k=0}^{n}\widehat{C}_k = 3+n-2^{n+3}+2^{n+1}n+6\times 2^n+j(3+n+2^{n+2}n)+\varepsilon(1+2^{n+3}+n+2^{n+3}n)+j\varepsilon(-7+2^{n+5}+n+2^{n+4}n)$.

(b) $\sum_{k=0}^{n}\widehat{C}_{2k} = \frac{17}{9}+n+\frac{1}{3}2^{2n+3}n-\frac{2}{9}2^{2n+2}+j(\frac{19}{9}+n+\frac{1}{9}2^{2n+3}+\frac{1}{3}2^{2n+4}n)+\varepsilon(\frac{17}{9}+n+\frac{4}{9}2^{2n+4}+\frac{1}{3}2^{2n+5}n)+j\varepsilon(\frac{1}{9}+n+\frac{7}{9}2^{2n+5}+\frac{1}{3}2^{2n+6}n)$.

(c) $\sum_{k=0}^{n}\widehat{C}_{2k+1} = \frac{19}{9}+n+\frac{1}{3}2^{2n+4}n+\frac{1}{9}2^{2n+3}+j(\frac{17}{9}+\frac{4}{9}2^{2n+4}+n+\frac{1}{3}2^{2n+5}n)+\varepsilon(\frac{1}{9}+n+\frac{7}{9}2^{2n+5}+\frac{1}{3}2^{2n+6}n)+j\varepsilon(-\frac{55}{9}+n+\frac{5}{9}2^{2n+7}+\frac{1}{3}2^{2n+7}n)$.

Now, we introduce the formulas that give the summation of the dual hyperbolic generalized Woodall numbers with negative subscripts.

Theorem 5.8 *For $n\geq 0$, dual hyperbolic generalized Woodall numbers have the following formulas:*

(a) $\sum_{k=0}^{n}\widehat{W}_{-k} = (-2+\frac{2}{2^n}-3j+n-3\varepsilon+\frac{3}{2^n}j+\frac{1}{2\times 2^n}n+jn+\frac{4}{2^n}\varepsilon+j\varepsilon+n\varepsilon+\frac{1}{2^n}jn+\frac{4}{2^n}j\varepsilon+\frac{2}{2^n}n\varepsilon+jn\varepsilon+\frac{4}{2^n}jn\varepsilon)W_2+(7-\frac{7}{2^n}+12j-4n+16\varepsilon-\frac{11}{2^n}j-\frac{3}{2\times 2^n}n-4jn-\frac{16}{2^n}\varepsilon+12j\varepsilon-4n\varepsilon-\frac{3}{2^n}jn-\frac{20}{2^n}j\varepsilon-\frac{6}{2^n}n\varepsilon-4jn\varepsilon-\frac{12}{2^n}jn\varepsilon)W_1+(-4+\frac{5}{2^n}-8j+4n-12\varepsilon+\frac{8}{2^n}j+\frac{1}{2^n}n+4jn+\frac{12}{2^n}\varepsilon-12j\varepsilon+4n\varepsilon+\frac{2}{2^n}jn+\frac{16}{2^n}j\varepsilon+\frac{4}{2^n}n\varepsilon+4jn\varepsilon+\frac{8}{2^n}jn\varepsilon)W_0$.

(b) $\sum_{k=0}^{n}\widehat{W}_{-2k}=(-\frac{7}{9}+\frac{7}{9\times 2^{2n}}-\frac{11}{9}j+n-\frac{7}{9}\varepsilon+\frac{11}{9\times 2^{2n}}j+\frac{1}{3\times 2^{2n}}n+jn+\frac{16}{9\times 2^{2n}}\varepsilon+\frac{25}{9}j\varepsilon+ n\varepsilon+\frac{2}{3\times 2^{2n}}jn+\frac{20}{9\times 2^{2n}}j\varepsilon+\frac{4}{3\times 2^{2n}}n\varepsilon+jn\varepsilon+\frac{8}{3\times 2^{2n}}jn\varepsilon)W_2+(\frac{8}{3}-\frac{8}{3\times 2^{2n}}+\frac{16}{3}j-4n+ \frac{20}{3}\varepsilon-\frac{13}{3\times 2^{2n}}j-\frac{1}{2^{2n}}n-4jn-\frac{20}{3\times 2^{2n}}\varepsilon+\frac{4}{3}j\varepsilon-4n\varepsilon-\frac{2}{2^{2n}}jn-\frac{28}{3\times 2^{2n}}j\varepsilon-\frac{4}{2^{2n}}n\varepsilon-4jn\varepsilon-\frac{8}{2^{2n}} jn\varepsilon)W_1+(-\frac{8}{9}+\frac{17}{9\times 2^{2n}}-\frac{28}{9}j+4n-\frac{44}{9}\varepsilon+\frac{28}{9\times 2^{2n}}j+\frac{2}{3\times 2^{2n}}n+4jn+\frac{44}{9\times 2^{2n}}\varepsilon-\frac{28}{9}j\varepsilon+ 4n\varepsilon+\frac{4}{3\times 2^{2n}}jn+\frac{64}{9\times 2^{2n}}j\varepsilon+\frac{8}{3\times 2^{2n}}n\varepsilon+4jn\varepsilon+\frac{16}{3\times 2^{2n}}jn\varepsilon)W_0.$

(c) $\sum_{k=0}^{n}\widehat{W}_{-2k+1}=(-\frac{11}{9}+\frac{11}{9\times 2^{2n}}-\frac{7}{9}j+n+\frac{25}{9}\varepsilon+\frac{16}{9\times 2^{2n}}j+\frac{2}{3\times 2^{2n}}n+jn+\frac{20}{9\times 2^{2n}}\varepsilon+\frac{137}{9}j \varepsilon+n\varepsilon+\frac{4}{3\times 2^{2n}}jn+\frac{16}{9\times 2^{2n}}j\varepsilon+\frac{8}{3\times 2^{2n}}n\varepsilon+jn\varepsilon+\frac{16}{3\times 2^{2n}}jn\varepsilon)W_2+(\frac{16}{3}-\frac{13}{3\times 2^{2n}}+\frac{20}{3}j- 4n+\frac{4}{3}\varepsilon-\frac{20}{3\times 2^{2n}}j-\frac{2}{2^{2n}}n-4jn-\frac{28}{3\times 2^{2n}}\varepsilon-\frac{76}{3}j\varepsilon-4n\varepsilon-\frac{4}{2^{2n}}jn-\frac{32}{3\times 2^{2n}}j\varepsilon-\frac{8}{2^{2n}}n\varepsilon- 4jn\varepsilon-\frac{16}{2^{2n}}jn\varepsilon)W_1+(-\frac{28}{9}+\frac{28}{9\times 2^{2n}}-\frac{44}{9}j+4n-\frac{28}{9}\varepsilon+\frac{44}{9\times 2^{2n}}j+\frac{4}{3\times 2^{2n}}n+4jn+\frac{64}{9\times 2^{2n}}\varepsilon+ \frac{100}{9}j\varepsilon+4n\varepsilon+\frac{8}{3\times 2^{2n}}jn+\frac{80}{9\times 2^{2n}}j\varepsilon+\frac{16}{3\times 2^{2n}}n\varepsilon+4jn\varepsilon+\frac{32}{3\times 2^{2n}}jn\varepsilon)W_0.$

Proof. It can be obtained by using Proposition 1.4 and Proposition 1.5.

(a) We can derive the following using the formulas in Proposition 1.4.

$$\sum_{k=0}^{n}\widehat{W}_{-k}=\sum_{k=0}^{n}W_{-k}+j\sum_{k=0}^{n}W_{-k+1}+\varepsilon\sum_{k=0}^{n}W_{-k+2}+j\varepsilon\sum_{k=0}^{n}W_{-k+3}.\sum_{k=0}^{n}\widehat{W}_{-k}.$$

$$\begin{aligned}\sum_{k=0}^{n}\widehat{W}_{-k} &= 4W_0(n+\frac{1}{2^{n+1}}(n+4)-\frac{1}{2^{n+2}}(n+3)-1)+2W_1(\frac{1}{2^{n+2}}(3n+8)-2n\\ &\quad -\frac{1}{2^{n+1}}(3n+11)+\frac{7}{2})+2W_2(\frac{1}{2}n+\frac{1}{2^{n+1}}(n+3)-\frac{1}{2^{n+2}}(n+2)-1)\\ &\quad +j(2W_2(\frac{1}{2}n+\frac{1}{2^{n}}(n+2)-\frac{1}{2^{n+1}}(n+1)-\frac{3}{2})+4W_0(n+\frac{1}{2^{n}}(n+3)\\ &\quad -\frac{1}{2^{n+1}}(n+2)-2)+2W_1(\frac{1}{2^{n+1}}(3n+5)-2n-\frac{1}{2^{n}}(3n+8)+6))\\ &\quad +\varepsilon(2W_2(\frac{1}{2}n+2^{1-n}(n+1)-\frac{1}{2^{n}}n-\frac{3}{2})+4W_0(n-\frac{1}{2^{n}}(n+1)\\ &\quad +2^{1-n}(n+2)-3)-2W_1(2n+2^{1-n}(3n+5)-\frac{1}{2^{n}}(3n+2)-8))\\ &\quad +j\varepsilon(2W_2(\frac{1}{2}n+2^{2-n}n-2^{1-n}(n-1)+\frac{1}{2})+2W_1(2^{1-n}(3n-1)-2n\\ &\quad -2^{2-n}(3n+2)+6)+4W_0(n-2^{1-n}n+2^{2-n}(n+1)-3)).\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{n}\widehat{W}_{-k} &= (-2+\frac{2}{2^n}-3j+n-3\varepsilon+\frac{3}{2^n}j+\frac{1}{2\times 2^n}n+jn+\frac{4}{2^n}\varepsilon+j\varepsilon+n\varepsilon \\
&+\frac{1}{2^n}jn+\frac{4}{2^n}j\varepsilon+\frac{2}{2^n}n\varepsilon+jn\varepsilon+\frac{4}{2^n}jn\varepsilon)W_2+(7-\frac{7}{2^n}+12j-4n \\
&+16\varepsilon-\frac{11}{2^n}j-\frac{3}{2\times 2^n}n-4jn-\frac{16}{2^n}\varepsilon+12j\varepsilon-4n\varepsilon-\frac{3}{2^n}jn-\frac{20}{2^n}j\varepsilon \\
&-\frac{6}{2^n}n\varepsilon-4jn\varepsilon-\frac{12}{2^n}jn\varepsilon)W_1+(-4+\frac{5}{2^n}-8j+4n-12\varepsilon+\frac{8}{2^n}j \\
&+\frac{1}{2^n}n+4jn+\frac{12}{2^n}\varepsilon-12j\varepsilon+4n\varepsilon+\frac{2}{2^n}jn+\frac{16}{2^n}j\varepsilon+\frac{4}{2^n}n\varepsilon+4jn\varepsilon \\
&+\frac{8}{2^n}jn\varepsilon)W_0.
\end{aligned}$$

This proves (a). We can be proved (b) and (c) similarly way using Proposition 1.5.
□

As a first special case of the above theorem, we have the following summation formulas for dual hyperbolic modified Woodall numbers:

Corollary 5.15 *For* $n \geq 0$, *dual hyperbolic modified Woodall numbers have the following properties:*

(a) $\sum_{k=0}^{n}\widehat{G}_{-k} = -3+n+\frac{n+3}{2^n}+j(-3+n+\frac{2n+4}{2^n})+\varepsilon(1+n+\frac{4+4n}{2^n})+j\varepsilon(17+n+\frac{8}{2^n}n)$.

(b) $\sum_{k=0}^{n}\widehat{G}_{-2k} = -\frac{11}{9}+n+\frac{11+6n}{9\times 2^{2n}}+j(-\frac{7}{9}+n+\frac{16+12n}{9\times 2^{2n}})+\varepsilon(\frac{25}{9}+n+\frac{20+24n}{9\times 2^{2n}})+j\varepsilon(\frac{137}{9}+ n+\frac{16+48n}{9\times 2^{2n}})$.

(c) $\sum_{k=0}^{n}\widehat{G}_{-2k+1} = -\frac{7}{9}+n+\frac{16+12n}{9\times 2^{2n}}+j(\frac{25}{9}+n+\frac{20+24n}{9\times 2^{2n}})+\varepsilon(\frac{137}{9}+\frac{16+48n}{9\times 2^{2n}}+n)+j\varepsilon(\frac{457}{9}+n + \frac{-16+96n}{9\times 2^{2n}})$.

As a second special case of the above theorem, we have the following summation formulas for dual hyperbolic modified Cullen numbers:

Corollary 5.16 *For* $n \geq 0$, *dual hyperbolic modified Cullen numbers have the following properties:*

(a) $\sum_{k=0}^{n}\widehat{H}_{-k} = 5+n-\frac{2}{2^n}+j(9-\frac{4}{2^n}+n)+\varepsilon(17-\frac{8}{2^n}+n)+j\varepsilon(33-\frac{16}{2^n}+n)$.

(b) $\sum_{k=0}^{n}\widehat{H}_{-2k} = \frac{11}{3}+n-\frac{2}{3\times 2^{2n}}+j(\frac{19}{3}-\frac{4}{3\times 2^{2n}}+n)+\varepsilon(\frac{35}{3}-\frac{8}{3\times 2^{2n}}+n)+j\varepsilon(\frac{67}{3}-\frac{16}{3\times 2^{2n}}+n)$.

(c) $\sum_{k=0}^{n}\widehat{H}_{-2k+1}=\frac{19}{3}+n-\frac{4}{3\times 2^{2n}}+j(\frac{35}{3}-\frac{8}{3\times 2^{2n}}+n)+\varepsilon(\frac{67}{3}-\frac{16}{3\times 2^{2n}}+n)+j\varepsilon(\frac{131}{3}-\frac{32}{3\times 2^{2n}}+n)$.

As a third special case of the above theorem, we have the following summation formulas for dual hyperbolic Woodall numbers:

Corollary 5.17 *For $n\geq 0$, dual hyperbolic Woodall numbers have the following properties:*

(a) $\sum_{k=0}^{n}\widehat{R}_{-k}=-3-n+\frac{2+n}{2^{n}}+j(-1-n+\frac{2+2n}{2^{n}})+\varepsilon(7-n+\frac{4}{2^{n}}n)+j\varepsilon(31-\frac{8}{2^{n}}-n+\frac{8}{2^{n}}n)$.

(b) $\sum_{k=0}^{n}\widehat{R}_{-2k}=-\frac{17}{9}-n+\frac{8+6n}{9\times 2^{2n}}+j(-\frac{1}{9}-n+\frac{10+12n}{9\times 2^{2n}})+\varepsilon(\frac{55}{9}-n+\frac{8+24n}{9\times 2^{2n}})+j\varepsilon(\frac{215}{9}-n+\frac{-8+48n}{9\times 2^{2n}})$.

(c) $\sum_{k=0}^{n}\widehat{R}_{-2k+1}=-\frac{1}{9}-n+\frac{10+12n}{9\times 2^{2n}}+j(\frac{55}{9}-n+\frac{8+24n}{9\times 2^{2n}})+\varepsilon(\frac{215}{9}-n+\frac{-8+48n}{9\times 2^{2n}})+j\varepsilon(\frac{631}{9}-n+\frac{-64+96n}{9\times 2^{2n}})$.

As a fourth special case of the above theorem, we have the following summation formulas for dual hyperbolic Cullen numbers:

Corollary 5.18 *For $n\geq 0$, dual hyperbolic Cullen numbers have the following properties:*

(a) $\sum_{k=0}^{n}\widehat{C}_{-k}=-1+n+\frac{2+n}{2^{n}}+j(1+\frac{2+2n}{2^{n}}+n)+\varepsilon(9+n+\frac{4}{2^{n}}n)+j\varepsilon(33+n+\frac{-8+8n}{2^{n}})$.

(b) $\sum_{k=0}^{n}\widehat{C}_{-2k}=\frac{1}{9}+n+\frac{8+6n}{9\times 2^{2n}}+j(\frac{17}{9}+\frac{10+12n}{9\times 2^{2n}}+n)+\varepsilon(\frac{73}{9}+\frac{8+24n}{9\times 2^{2n}}+n)+j\varepsilon(\frac{233}{9}-\frac{8-48n}{9\times 2^{2n}}+n)$.

(c) $\sum_{k=0}^{n}\widehat{C}_{-2k+1}=\frac{17}{9}+n+\frac{10+12n}{9\times 2^{2n}}+j(\frac{73}{9}+\frac{8+24n}{9\times 2^{2n}}+n)+\varepsilon(\frac{233}{9}-\frac{8-48n}{9\times 2^{2n}}+n)+j\varepsilon(\frac{649}{9}-\frac{64-96n}{9\times 2^{2n}}+n)$.

5.4 MATRICES RELATED TO DUAL HYPERBOLIC GENERALIZED WOODALL NUMBERS

In this section, we investigate matrices related to dual hyperbolic generalized Woodall numbers.

Now, we recall $\{G_n\}$ defined by the third-order recurrence relation as follows

$G_n=5G_{n-1}-8G_{n-2}+4G_{n-3}$ with the initial conditions $G_0=0,\ G_1=1,\ G_2=5$.

We present the square matrix A of order 3 as

$$A=\begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A=4$. Then, we give the following Lemma.

Note that

$$A^n=\begin{pmatrix} G_{n+1} & -8G_n+4G_{n-1} & 4G_n \\ G_n & -8G_{n-1}+4G_{n-2} & 4G_{n-1} \\ G_{n-1} & -8G_{n-2}+4G_{n-3} & 4G_{n-2} \end{pmatrix}.$$

For the proof see [60].

Lemma 5.9 *For all integers n the following identity is true.*

$$\begin{pmatrix} \widehat{W}_{n+2} \\ \widehat{W}_{n+1} \\ \widehat{W}_n \end{pmatrix}=\begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n\begin{pmatrix} \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}.$$

Proof. First, we suppose that $n\geq 0$. Lemma (5.9) can be given by mathematical induction on n. If $n=0$ we get

$$\begin{pmatrix} \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}=\begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0\begin{pmatrix} \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n=k$. Thus the following identity is true.

$$\begin{pmatrix} \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_k \end{pmatrix}=\begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k\begin{pmatrix} \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}$$

For $n = k + 1$, we get

$$\begin{aligned}
\begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} &= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} \\
&= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_k \end{pmatrix} \\
&= \begin{pmatrix} 5\widehat{W}_{k+2} - 8\widehat{W}_{k+1} + 4\widehat{W}_k \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix}.
\end{aligned}$$

If we suppose that $n < 0$ the proof can be done similarly. Consequently, by mathematical induction on n, the proof is completed. □

Theorem 5.10 *If we define the matrices* $N_{\widehat{W}}$ *and* $E_{\widehat{W}}$ *as follow.*

$$N_{\widehat{W}} = \begin{pmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{pmatrix}, \ E_{\widehat{W}} = \begin{pmatrix} \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \end{pmatrix}.$$

then the following identity is true:

$$A^n N_{\widehat{W}} = E_{\widehat{W}}.$$

Proof. For the proof, we can use the following identities.

$$\begin{aligned}
A^n N_{\widehat{W}} &= \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & 4G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & 4G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & 4G_{n-2} \end{pmatrix} \begin{pmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{pmatrix}, \\
&= \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}
\end{aligned}$$

where

$$
\begin{aligned}
b_{11} &= \widehat{W}_2 G_{n+1} + \widehat{W}_1 \left(-8G_n + 4G_{n-1}\right) + \widehat{W}_0 4G_n, \\
b_{12} &= \widehat{W}_1 G_{n+1} + \widehat{W}_0 \left(-8G_n + 4G_{n-1}\right) + \widehat{W}_{-1} 4G_n, \\
b_{13} &= \widehat{W}_0 G_{n+1} + \widehat{W}_{-1} \left(-8G_n + 4G_{n-1}\right) + \widehat{W}_{-2} 4G_n, \\
b_{21} &= \widehat{W}_2 G_n + \widehat{W}_1 \left(-8G_n + 4G_{n-1}\right) + \widehat{W}_0 4G_{n-1}, \\
b_{22} &= \widehat{W}_1 G_n + \widehat{W}_0 \left(-8G_n + 4G_{n-1}\right) + \widehat{W}_{-1} 4G_{n-1}, \\
b_{23} &= \widehat{W}_0 G_n + \widehat{W}_{-1} \left(-8G_n + 4G_{n-1}\right) + \widehat{W}_{-2} 4G_{n-1}, \\
b_{31} &= \widehat{W}_2 G_{n-1} + \widehat{W}_1 \left(-8G_n + 4G_{n-1}\right) + \widehat{W}_0 4G_{n-2}, \\
b_{32} &= \widehat{W}_1 G_{n-1} + \widehat{W}_0 \left(-8G_n + 4G_{n-1}\right) + \widehat{W}_{-1} 4G_{n-2}, \\
b_{33} &= \widehat{W}_0 G_{n-1} + \widehat{W}_{-1} \left(-8G_n + 4G_{n-1}\right) + \widehat{W}_{-2} 4G_{n-2},
\end{aligned}
$$

Using the Theorem (5.6) the proof is done. □

From Theorem (5.10), we can write the following corollary.

Corollary 5.19 *We have the following identity.*

(a) *If we define $N_{\widehat{G}}$ and $E_{\widehat{G}}$ as follows.*

$$
N_{\widehat{G}} = \begin{pmatrix} \widehat{G}_2 & \widehat{G}_1 & \widehat{G}_0 \\ \widehat{G}_1 & \widehat{G}_0 & \widehat{G}_{-1} \\ \widehat{G}_0 & \widehat{G}_{-1} & \widehat{G}_{-2} \end{pmatrix}, \; E_{\widehat{G}} = \begin{pmatrix} \widehat{G}_{n+2} & \widehat{G}_{n+1} & \widehat{G}_n \\ \widehat{G}_{n+1} & \widehat{G}_n & \widehat{G}_{n-1} \\ \widehat{G}_n & \widehat{G}_{n-1} & \widehat{G}_{n-2} \end{pmatrix},
$$

then we get

$$A^n N_{\widehat{G}} = E_{\widehat{G}}.$$

(b) *If we define $N_{\widehat{H}}$ and $E_{\widehat{H}}$ as follows.*

$$
N_{\widehat{H}} = \begin{pmatrix} \widehat{H}_2 & \widehat{H}_1 & \widehat{H}_0 \\ \widehat{H}_1 & \widehat{H}_0 & \widehat{H}_{-1} \\ \widehat{H}_0 & \widehat{H}_{-1} & \widehat{H}_{-2} \end{pmatrix}, \; E_{\widehat{H}} = \begin{pmatrix} \widehat{H}_{n+2} & \widehat{H}_{n+1} & \widehat{H}_n \\ \widehat{H}_{n+1} & \widehat{H}_n & \widehat{H}_{n-1} \\ \widehat{H}_n & \widehat{H}_{n-1} & \widehat{H}_{n-2} \end{pmatrix},
$$

then we get

$$A^n N_{\widehat{H}} = E_{\widehat{H}}.$$

(c) *If we define* $N_{\widehat{R}}$ *and* $E_{\widehat{R}}$ *as follows.*

$$N_{\widehat{R}} = \begin{pmatrix} \widehat{R}_2 & \widehat{R}_1 & \widehat{R}_0 \\ \widehat{R}_1 & \widehat{R}_0 & \widehat{R}_{-1} \\ \widehat{R}_0 & \widehat{R}_{-1} & \widehat{R}_{-2} \end{pmatrix},\ E_{\widehat{R}} = \begin{pmatrix} \widehat{R}_{n+2} & \widehat{R}_{n+1} & \widehat{R}_n \\ \widehat{R}_{n+1} & \widehat{R}_n & \widehat{R}_{n-1} \\ \widehat{R}_n & \widehat{R}_{n-1} & \widehat{R}_{n-2} \end{pmatrix}.$$

then we get

$$A^n N_{\widehat{R}} = E_{\widehat{R}}.$$

(d) *If we define* $N_{\widehat{C}}$ *and* $E_{\widehat{C}}$ *as follows.*

$$N_{\widehat{C}} = \begin{pmatrix} \widehat{C}_2 & \widehat{C}_1 & \widehat{C}_0 \\ \widehat{C}_1 & \widehat{C}_0 & \widehat{C}_{-1} \\ \widehat{C}_0 & \widehat{C}_{-1} & \widehat{C}_{-2} \end{pmatrix},\ E_{\widehat{C}} = \begin{pmatrix} \widehat{C}_{n+2} & \widehat{C}_{n+1} & \widehat{C}_n \\ \widehat{C}_{n+1} & \widehat{C}_n & \widehat{C}_{n-1} \\ \widehat{C}_n & \widehat{C}_{n-1} & \widehat{C}_{n-2} \end{pmatrix},$$

then we get

$$A^n N_{\widehat{C}} = E_{\widehat{C}}.$$

REFERENCES

[1] **Akar M, Yüce S and Şahin Ş** (2008) On the Dual Hyperbolic Numbers and the Complex Hyperbolic Numbers. *Journal of Computer Science & Computational Mathematics*, 8(1), 1-6.

[2] **Aşcı M and Gürel E** (2013) Gaussian Jacobsthal and Gaussian Jacobsthal Lucas Numbers. *Ars Combinatoria. Soc.*, 111.

[3] **Aydın F T, Köklü K and Yüce S** (2017) Generalized Dual Pell Quaternions. *Notes on Number Theory and Discrete Mathematics*, 23(4), 66-84.

[4] **Aydın F T** (2019) Hyperbolic Fibonacci Sequence. *Universal Journal of Mathematics and Applications*, 2 (2), 59-64.

[5] **Aydın F T** (2020) Dual Jacobsthal Quaternions. *Communications in Advanced Mathematical Sciences*, Vol. III, No. 3, 130-142.

[6] **Baez J** (2002) The octonions. *Bull. Amer. Math. Soc.*, 39(2), 145-205.

[7] **Berrizbeitia P, Fernandes J G, González M J, Luca FV and Huguet J M** (2012) On Cullen numbers which are Both Riesel and Sierpiński numbers. *Journal of Number Theory*, 132, 2836-2841.

[8] **Bilu Y, Marques D and Togbe A** (2019) Generalized Cullen numbers in linear recurrence sequences. *Journal of Number Theory*, 202, 412-425.

[9] **Biss D K, Dugger D and Isaksen D C** (2008) Large annihilators in Cayley-Dickson algebras. *Communication in Algebra*, 36 (2), 632-664.

[10] **Bród D, Liana A and Włoch I** (2020) Two Generalizations of Dual-Hyperbolic Balancing Numbers. *Symmetry*, 12(11), 1866.

[11] **Bruce I** (1984) A modified Tribonacci sequence. *The Fibonacci Quarterly*, 22:3, pp. 244-246.

[12] **Catalani M** (2002) Identities for Tribonacci-related sequences. - *arXiv preprint*, https://arxiv.org/pdf/math/0209179.pdf math/0209179.

[13] **Cerda-Morales G** (2022) On Gauss third-order Jacobsthal numbers and their applications. *Annals of the Alexandru Ioan Cuza University-Mathematics*, 67(2), 231-241.

[14] **Cheng H H, and Thompson S** (1996) Dual Polynomials and Complex Dual Numbers for Analysis of Spatial Mechanisms. *Proc. of ASME 24th Biennial Mechanisms Conference*, Irvine, CA, August, 19-22.

REFERENCES (continued)

[15] **Choi E** (2013) Modular tribonacci Numbers by Matrix Method. *J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.*, Volume 20, Number 3 (August 2013), 207-221.

[16] **Cockle J** (1849) On a New Imaginary in Algebra. *Philosophical magazine*, London-Dublin-Edinburgh, 3(34), 37-47.

[17] **Cunningham A J C** (1917) Woodall, H. J., Factorisation of Q=(2^{q}∓q) and (q.2^{q}∓1). *Messenger of Mathematics*, 47, 1-38.

[18] **Cihan A, Azak A Z, Güngör M A and Tosun M** (2019) A Study on Dual Hyperbolic Fibonacci and Lucas Numbers. *An. Şt. Univ. Ovidius Constanta*, 27(1), 35-48.

[19] **Dikmen C M** (2019) Hyperbolic Jacobsthal Numbers. *Asian Research Journal of Mathematics*, 15(4), 1-9.

[20] **Dikmen C M and Altınsoy M** (2022) On Third Order Hyperbolic Jacobsthal Numbers, *Konuralp Journal of Mathematics*, 10 (1), 118-126.

[21] **Dişkaya O, Menken H and Catarino P M M C** (2023) On the Hyperbolic Leonardo and Hyperbolic Francois Quaternions. *Journal of New Theory*, 42, 74-85.

[22] **Elia M** (2001) Derived Sequences, The Tribonacci Recurrence and Cubic Forms. *The Fibonacci Quarterly*, 39:2, 107-115.

[23] **Eren O and Soykan Y** (2024) On Hyperbolic Generalized Woodall Numbers, *Asian Journal of Advanced Research and Reports*, 18(2), 43-69.

[24] **Ertaş A and Yılmaz F** (2023) On Quaternions with Gaussian Oresme Coefficients. *Turk. J. Math. Comput. Sci*., 15(1), 192-202.

[25] **Fjelstad P and Gal S G** (1998) n-dimensional Hyperbolic Complex Numbers. *Advances in Applied Clifford Algebras*, 8(1), 47-68.

[26] **Fraleigh J B** (1976) A First Course In Abstract Algebra, (2nd ed.). *Addison-Wesley, Reading*, ISBN0-201-01984-1.

[27] **Frontczak R** (2018) Convolutions for Generalized Tribonacci Numbers and Related Results, *International Journal of Mathematical Analysis*, 12(7), 307 – 324.

[28] **Grantham J and Graves H** (2021) The abc conjecture implies that only finitely many s-Cullen numbers are repunits. *http://arxiv.org/abs/2009.04052v3*, MathNT.

[29] **Gurses N, Senturk G Y and Yuce S** (2021) A Study on Dual-Generalized Complex and Hyperbolic-Generalized Complex Numbers. *Journal of Science*, 34(1), 180-194.

[30] **Guy R** (1994) Unsolved Problems in Number Theory (2nd ed.). *Springer-Verlag*, New York.

[31] **Gürses N, Şentürk G Y and Yüce S** (2022) A comprehensive survey of dual-generalized complex Fibonacci and Lucas numbers. *Sigma J Eng Nat Sci*, Vol. 40, No. 1, 179-187.

[32] **Halıcı S and Öz S** (2016) On Some Gusian Pell and Pell-Lucas Numbers. *Ordu Univ. J. Sci. Tech.*, 6(1),8-18.

[33] **Halıcı S** (2015) On Dual Fibonacci Octonions. *Advances in Applied Clifford Algebras*, 25, 905-914.

[34] **Hamilton W R** (1969) Elements of Quaternions. *Chelsea Publishing Company*, New York.

[35] **Hooley C** (1976) Applications of the sieve methods to the theory of numbers, *Cambridge University Press*, Cambridge.

[36] **Horadam A F** (1963) Complex Fibonacci Numbers and Fibonacci quaternions, *Amer. Math. Monthly*, 70, 289-291.

[37] **Imaeda K and Imaeda M** (2000) Sedenions: algebra and analysis. *Applied Mathematics and Computation*, 115, 77-88.

[38] **Kantor I and Solodovnikov A** (1989) Hypercomplex Numbers. *Springer-Verlag*, New York.

[39] **Keller W** (1995) New Cullen primes. *Math. Comput*. 64, 1733-1741.

[40] **Lin P Y** (1988) De Moivre-Type Identities For The Tribonacci Numbers. *The Fibonacci Quarterly*, 26, pp. 131-134.

[41] **Luca F and Stanica P** (2004) Cullen numbers in binary recurrent sequences, FT. Howard (ed.), Applications of Fibonacci Numbers : Proceedings of the Tenth International Research Conference on Fibonacci Numbers and their Applications. *Kluwer Academic Publishers*, 9, 167-175.

[42] **Marques D** (2014) On Generalized Cullen and Woodall numbers that are also Fibonacci numbers. *Journal of Integer Sequences*, 17, Article 14.9.4.

[43] **Marques D** (2015) Fibonacci s-Cullen and s-Woodall numbers. *Journal of Integer Sequences*, 18, Article 15.1.4.

[44] **Meher N K and Rout S S** (2020) Cullen numbers in sums of terms of recurrence sequence. *http://arxiv.org/abs/2010.10014v1*, MathNT.

[45] **Moreno G** (1998) The zero divisors of the Cayley-Dickson algebras over the real numbers. *Bol. Soc. Mat. Mexicana*, 3(4), 13-28.

[46] **Nurkan S K and Guven İ A** (2015) Dual Fibonacci Quaternions. *Advances in Applied Clifford Algebras*, 25, 403-414.

[47] **Pethe S and Horadam A F** (1986) Generalised Gaussian Fibonacci numbers. *Bull. Austral. Math. Soc.*, Vol.33, pp.37-48.

[48] **Pethe S** (1988) Some Identities for Tribonacci Sequences. *The Fibonacci Quarterly*, 26, 144-246.

[49] **Richter W-D** (2022) On Hyperbolic Complex Numbers. *Appl. Sci.*, 12, 5844.

[50] **Scott A, Delaney T and Hoggatt Jr V** (1977) The Tribonacci sequence. *The Fibonacci Quarterly,* 15(3), 193-200.

[51] **Shannon A** (1977) Tribonacci numbers and Pascal's pyramid. *The Fibonacci Quarterly,* 15:3, 268-275.

[52] **Sloane N J A** The on-line encyclopedia of integer sequences. *http://oeis.org/.*

[53] **Sobczyk G** (1995) The Hyperbolic Number Plane. *The College Mathematics Journal*, 26(4), 268-280.

[54] **Soykan Y, Taşdemir E, Okumuş İ and Göcen M** (2018) Gaussian Generalized Tribonacci Numbers. *Journal of Progressive Research in Mathematics*, 14(2).

[55] **Soykan Y** (2019) Simson Identity of Generalized m-step Fibonacci Numbers. *Int. J. Adv. Appl. Math. and Mech.,* 7(2), 45-56.

[56] **Soykan Y** (2019) Tribonacci and Tribonacci-Lucas Sedenions. *Mathematics*, 7(1), 74.

[57] **Soykan Y** (2019) On Summing Formulas For Generalized Fibonacci and Gaussian Generalized Fibonacci Numbers. *Advances in Research*, 20(2), 1-15.

[58] **Soykan Y, Gümüş M and Göcen M** (2021) A study on dual hyperbolic generalized Pell numbers. *Malaya Journal Of Matematik*, 09(03), 99-116.

[59] **Soykan Y and İrge V** (2022) Generalized Woodall Numbers: An Investigation of Properties of Woodall and Cullen Numbers via Their Third Order Linear Recurrence Relations. *Universal Journal of Mathematics and Applications*, 5(2), 69-81.

[60] **Soykan Y** (2020) A Study On Generalized (r,s,t)-Numbers. *MathLAB Journal*, 7, 101-129.

[61] **Soykan Y and Eren O** (2023) Gaussian Generalized Woodall Numbers, *Archives of Current Research International*, 23(8), 48-68.

[62] **Soykan Y, Taşdemir E and Okumuş İ** (2022) On dual hyperbolic numbers with generalized Jacobsthal numbers components. *Indian J Pure Appl Math*, 54, 824-840.

[63] **Soykan Y and Taşdemir E** (2022) A study on hyperbolic numbers with generalized Jacobsthal numbers components. *Int. J. Nonlinear Anal. Appl.,* 2, 1965-1981.

[64] **Spickerman W** (1981) Binet's formula for the Tribonacci sequence. *The Fibonacci Quarterly*, 20, pp.118-120.

[65] **Taş S** (2022) On Hyperbolic Jacobsthal-Lucas Sequence. *Fundamental Journal of Mathematics and Applications*, 5 (1), 16-20.

[66] **Taşcı D** (2018) Gaussiam Balancing And Gaussian Lucas Balancing Numbers. *Journal of Science and Arts*, 3(44),661-666.

[67] **Taşcı D** (2018) Gaussian Padovan and Gaussian Pell- Padovan numbers. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 67(2), 82-88.

[68] **Taşcı D** (2021) On Gaussian Mersenne Numbers. *Journal of Science and Arts,* 4(57),1012-1028.

[69] **Yalavigi C C** (1972) Properties of Tribonacci numbers. *The Fibonacci Quarterly*, 10 : 3, pp. 231-246.

[70] **Yilmaz N and Taskara N** (2014) Tribonacci and Tribonacci-Lucas Numbers via the Determinants of Special Matrices. *Applied Mathematical Sciences*, 8, no. 39, 1947-1955.

[71] **Yüce S and Aydın F T** (2016) Generalized Dual Fibonacci Quaternions, *Applied Mathematics E-notes*, 16, 276-289.

CURRICULUM VITAE

Orhan EREN completed her primary, secondary and high school education in ZONGULDAK. He completed his undergraduate degree in mathematics at Bolu Abant İzzet Baysal University in 2011. In 2017, he started working as a mathematics teacher at the Ministry of National Education in Van. He has been working at Kdz Ereğli since 2021. I was accepted to Zonguldak Bülent Ecevit University for a master's degree in 2022 and graduated in 2024. He is married and has 1 child.