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GEBZE TECHNICAL UNIVERSITY

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FINITE CODIMENSIONAL MAXIMAL IDEALS OF LEAVITT PATH ALGEBRAS

MELİKE KAMAR

A THESIS OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

ADVISOR: DOÇ. DR. AYTEN KOÇ

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LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

LEAVITT YOL CEBİRLERİNİN SONLU KOBOYUTLU MAKSİMAL İDEALLERİ

MELİKE KAMAR

YÜKSEK LİSANS TEZİ

MATEMATİK ANABİLİM DALI

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Gebze Technical University Graduate Administration Board
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To my children,
Baran Alp and Can.

ABSTRACT

In this thesis finite codimensional maximal ideals of Leavitt path algebras (LPAs) are considered. Standard tools of abstract algebra, specifically the theory of associative algebras, their structure and representations are used. Some basic definitions and properties of digraphs and Leavitt path algebras are given. Maximal and graded ideals of Leavitt path algebras are studied.

Çolak described two-sided ideals of a Leavitt path algebra of a row-finite directed graph in 2011. The main result in Çolak's paper is that any ideal I of a Leavitt path algebra is generated by the subset $I \cap V$ of V and elements of the form $s(C) + \sum_i \lambda_i C^i$ for some cycle C with $s(C) \notin I$. On the other hand all finite dimensional quotients of Leavitt path algebras of row-finite directed graphs were classified by Koç-Özaydın in 2020. However, the kernel ideals of these quotients were not provided with generators of the form given by Çolak. This thesis makes the relationship between finite dimensional quotients and these canonical generators explicit: We provide canonical generators for finite codimensional maximal ideals of Leavitt path algebras and show that every such ideal is principal and each such quotient is isomorphic to a Leavitt path algebra with coefficients in an extension field of $\mathbb{F}$.

In the Leavitt path algebra literature there has been a significant amount of research dedicated to constructing a dictionary between the combinatorial properties of a directed graph and the algebraic properties of the corresponding Leavitt path algebra. This "dictionary" is presented in a tabular format in the fourth chapter of this thesis.

Keywords: **Leavitt Path Algebra, Maximal Ideal, Finite Codimensional Ideal.**

ÖZET

Bu tez de Leavitt yol cebirlerinin bölümü sonlu boyutlu olan maksimal idealleri ele alındı. Soyut cebirin standart teknikleri, özellikle birleşmeli cebirlerin yapı ve temsil teorileri kullanıldı. Bu bağlamda bazı temel tanımlar, yönlü çizgelerin ve Leavitt yol cebirlerinin önemli özellikleri ile dereceli ve maksimal idealleri incelendi.

Satır sonlu yönlü çizgelerin Leavitt yol cebirlerinin çift taraflı idealleri 2011 de Çolak tarafından belirlendi. Çolak'ın makalesindeki ana teorem, bir Leavitt yol cebirinin herhangi bir çift taraflı ideali, içerdiği köşeler ve C bir döngü olmak üzere $s(C) + \sum_i \lambda_i C^i$ formundaki elemanlar tarafından üretildiğini ifade eder. Diğer yandan satır sonlu, yönlü çizgelerin Leavitt yol cebirlerinin sonlu boyutlu bölümleri, Koç-Özaydın tarafından 2020 de sınıflandırıldı. Ancak bu bölümlerin çekirdek ideallerinin Çolak tarafından verilen formdaki üreteçleri belirlenmemişti. Bu tezin bir amacı bu çekirdeklerin üreteçlerini açıkca belirlemektir: sonlu (yönlü) çizgelerin Leavitt yol cebirlerinin sonlu boyutlu bölümlerinin çekirdeklerinin kanonik üreteçleri verilerek bu çekirdek ideallerin esas ideal olduğu ve bu bölümlerin (girdileri katsayı cisminin bir genişlemesinden gelen) matris cebirlerine izomorf oldukları gösterildi.

Leavitt yol cebiri literatüründe önemli araştırma temalarından biri de bir yönlü çizgenin geometrik/çizgesel özellikleri ile bu yönlü çizgenin tanımladığı Leavitt yol cebirının cebirsel özellikleri arasındaki ilişkileri saptamaktır. Bu tezin Sözlük başlıklı 4. bölümünde bu ilişkilerin bir kısmı bir tablo şeklinde verildi.

Anahtar Kelimeler: Leavitt Yol Cebiri, Maksimal İdeal, Sonlu Koboyutlu İdeal.

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LIST OF ABBREVIATIONS AND ACRONYMS

$\mathbb{Z}$	:	Integers
$\mathbb{Q}$	:	Rational numbers
$\mathbb{R}$	:	Real numbers
$\mathbb{C}$	:	Complex numbers
$\mathbb{N}$	:	Natural numbers $\{0, 1, 2, \dots\}$
1-1	:	one-to-one
$Ker\varphi$	:	Kernel of the homomorphism φ
$2\mathbb{Z}$	:	Ring of even integers
PID	:	Principal Ideal Domain
$\mathbb{F}[x]$	:	Polynomial algebra with coefficients from the field $\mathbb{F}$
$\mathbb{F}[x, x^{-1}]$	:	Laurent polynomial algebra
$\subset$	:	Proper subset
R/I	:	Quotient ring of R by I
$\subseteq$	:	Subset
$\langle X \rangle$	:	Module generated by a set X
$\mathbb{F}[x_1, x_2, \dots, x_n]$	:	Polynomial algebra with coefficients from the field $\mathbb{F}$ in n variables
$\delta_{i,j}$	:	Kronecker delta
$M_n(\mathbb{F})$	:	Matrix algebra of $n \times n$-matrices over $\mathbb{F}$
$\mathbb{F}X$	:	$\mathbb{F}$-vector space with basis the set X
$\mathbb{F}\langle x_1, x_2, \dots, x_n\rangle$	:	Noncommutative polynomial algebra over $\mathbb{F}$ in n variables
$\mathbb{F}\langle X\rangle$	:	Noncommutative polynomial algebra over $\mathbb{F}$ with variable set X
$L_{\mathbb{F}}(1, n)$	:	Leavitt $\mathbb{F}$-algebra of type $(1, n)$
$\mathcal{A}_\gamma$	:	γ-component of a G-graded algebra $\mathcal{A}$ where $\gamma \in G$
$Ob(\mathcal{C})$	:	Class of objects of the category $\mathcal{C}$
$\mathcal{C}(X, Y)$	:	Set of morphisms from X to Y in the category $\mathcal{C}$
id_X	:	Identity map/morphism on X
Set	:	Category of sets and functions
set	:	Category of finite sets and functions
Grp	:	Category of groups and group homomorphisms
$\mathbf{Vec}_{\mathbb{F}}$	:	Category of vector spaces and linear transformations
$\mathbf{vec}_{\mathbb{F}}$	:	Category of finite dimensional vector spaces and linear transformations
$\mathbf{Mod_R}$	:	Category of right R-modules and R-module homomorphisms
$\mathbf{mod}_{\mathbb{R}}$	:	Category of finitely generated right R-modules and R-module homomorphisms
$\mathbb{F}\Gamma$	:	Path algebra of Γ over $\mathbb{F}$
2^X	:	Power set of the set X
Γ	:	Di(rected)graph
V	:	Set of vertices
E	:	Set of arrows
s	:	Source function
t	:	Target function

$Path(\Gamma)$	:	Set of paths in Γ
$\rightsquigarrow$	:	Preorder leadsto on V
$V_{v\rightsquigarrow}$	:	Set of successors of a vertex v
$V_{\rightsquigarrow w}$	:	Set of predecessors of a vertex w
$\overline{S}$	:	Hereditary saturated closure of $S \subset V$
$\mathbf{\Gamma}$	:	Path category of Γ (with objects V and morphisms $Path(\Gamma)$)
$\hat{\Gamma}$	:	Extended digraph of Γ
LPA	:	Leavitt Path Algebra
$L_{\mathbb{F}}(\Gamma)$	:	Leavitt path algebra of Γ
$\|a\|$	:	Grade of the homogenous element a
$\mathbb{Z}_+$	:	Set of positive integers
E_{ij}	:	Elementary matrix
$\mathscr{L}_{gr}(L_{\mathbb{F}}(\Gamma))$	:	The lattice of graded ideals of $L_{\mathbb{F}}(\Gamma)$
$\mathscr{H}$	:	Set of hereditary and saturated subsets of V
Γ_M	:	Support subgraph of the module M
ACC	:	Ascending Chain Condition
DCC	:	Descending Chain Condition

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1. INTRODUCTION

Leavitt path algebras were introduced in 2004 by Ara, Moreno, Pardo and Abrams, Aranda Pino, independently [1], [2]. The concept of Leavitt path algebras is associated to directed graphs. A di(rected)graph is a mathematical structure consisting of vertices and arrows connecting these vertices.

There were initial studies on the ideal structures of Leavitt path algebras by Abrams and Aranda Pino on the hereditary saturated subsets of vertices of digraphs and also by Tomforde. Generators for two-sided ideals of a Leavitt path algebra of a (row-finite) digraph were given by Çolak [3] and Rangaswamy [4]. On the other hand Koç-Özaydın classified all finite dimensional quotients of Leavitt path algebras of (row-finite) digraphs [5]. In particular, they show that the Leavitt path algebra of a row-finite digraph Γ has a nonzero finite dimensional quotient if and only if Γ has a maximal sink or a maximal cycle.

This thesis consists of five chapters. The first chapter is this introduction containing information about the content of the thesis.

The second chapter consists of four subsections. The first of these includes fundamental topics of abstract algebra such as rings, modules, ideals, fields, vector spaces, etc., with theorems and examples, which are used in the thesis. In the second subsection, the definitions of algebra and some specific examples used in the thesis are provided. Additionally, the conditions for an algebra and ideal to be graded are given. In the other subsections the definition of a category and basic examples are discussed and partially ordered sets are considered.

In the third chapter, digraphs are defined, and their fundamental properties are discussed. Then the definition of a Leavitt path algebra is provided and examples are given showing that some well-known algebras are obtained as Leavitt path algebras of specific digraphs. The definition and theorems related to graded ideals of Leavitt path algebras are provided. The relationship between hereditary and saturated subsets of V in Γ and graded ideals of $L_{\mathbb{F}}(\Gamma)$ is explained.

In the third section of the third chapter we provide canonical generators for finite codimensional maximal ideals of LPAs and show that every such ideal is principal and each such quotient is isomorphic to a Leavitt path algebra with coefficients in an extension field of $\mathbb{F}$. This realizes one of the objectives of this thesis, namely to make the relationship between finite dimensional quotients and these canonical generators explicit.

In the Leavitt path algebra literature there has been a significant amount of research dedicated to constructing a dictionary between the combinatorial properties of a directed graph and the algebraic properties of the corresponding Leavitt path algebra. This dictionary presented in a tabular format is given in the fourth chapter.

2. PRELIMINARIES

2.1. Rings and Ideals

Definition 2.1. [6] A binary operation $*$ on a set S is a mapping that assigns to each ordered pair $(s_1, s_2) \in S \times S$ an element $s_1 * s_2 \in S$. In this case S is said to be closed under the binary operation.

If $s_1 * s_2 = s_2 * s_1$ for all $s_1,\ s_2 \in S$ then the binary operation $*$ is commutative. If $s_1 * (s_2 * s_3) = (s_1 * s_2) * s_3$ for all $s_1,\ s_2,\ s_3 \in S$ then $*$ is associative. If there exists an element e in S such that $s_1 * e = s_1 = e * s_1$ for all s_1 in S then e is called an identity.

Definition 2.2. [7] An ordered pair $(G, *)$ with $*$ being a binary operation defined on the set G is a group if the following are satisfied:

(i) $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$. (associativity)

(ii) In G, there is an element e, the identity of G, satisfying $a * e = e * a = a$ for all $a \in G$.

(iii) For every $a \in G$ there exists $a^{-1} \in G$, the inverse of a, satisfying $a * a^{-1} = a^{-1} * a = e$.

The group $(G, *)$ is abelian if $a * b = b * a$ for all $a, b \in G$.

Example 1. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are additive abelian groups where $e = 0$ and $a^{-1} = -a$ for all a.

A monoid is an ordered pair $(S, *)$ where S is a set and $*$ is a binary operation satisfying (i) and (ii).

Example 2. The sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are additive (commutative) monoids as well as commutative multiplicative monoids. Here 0 is the identity of addition and the multiplicative identity is 1.

Definition 2.3. [7] A ring is an ordered triple $(R, +, \cdot)$ where R is a set, and the two binary operations $+$ and $\cdot$ called addition and multiplication, satisfy the following:

(i) R under addition is an abelian group with 0_R being the identity,

(ii) R under multiplication is a monoid with 1_R being the identity,

(iii) multiplication is both left and right distributive over addition, i.e., for all $x, y, z \in R$:

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

and

$$(y + z) \cdot x = y \cdot x + z \cdot x$$

A ring R is commutative if $(R, \cdot)$ is a commutative monoid, i.e., for all $x, y \in R$

$$x \cdot y = y \cdot x$$

We will abbreviate the product $x \cdot y$ as xy when there is no risk of confusion. The identity elements of addition and multiplication in R are denoted by 0_R and 1_R. The subscripts on 0_R and 1_R may also be omitted when the context makes it clear.

Example 3. $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$ are commutative rings where $+$ and $\cdot$ have their standard meanings.

We will follow the usual abuse of notation where R stands in for the triple $(R, +, .)$.

Example 4. For any ring R the ring $M_n(R)$ is $n \times n$-matrices whose entries are from R, with the usual matrix addition and multiplication where 0 is the matrix with all entries zero and 1 is the identity matrix $I := I_n$ whose entries are 1 on the main diagonal and 0 elsewhere. We display $a \in M_n(R)$ as an $n \times n$ rectangular array

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

where $a_{ij} \in R$, or as (a_{ij}) where $1 \leq i, j \leq n$. Hence $(M_n(R), +, \cdot)$ is a ring with the operations:

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}) \text{ and } (a_{ij}) \cdot (b_{ij}) = (c_{ij})$$

where $c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$. The ring $(M_n(R), +, \cdot)$ is not commutative for $n > 1$.

Definition 2.4. [8] A field is a commutative ring in which every nonzero element is invertible.

The ring of integers $\mathbb{Z}$ is a commutative ring but it is not a field. The ring of rational numbers $\mathbb{Q}$ is a field.

Definition 2.5. [8] An element a of a commutative ring R is called a zero divisor if there is a nonzero element $b \in R$ such that $ab = 0$.

Definition 2.6. [8] A commutative ring R is an integral domain if there is no zero divisor in R.

Every field is an integral domain. The converse is not true, that is, every integral domain is not a field. For instance, $\mathbb{Z}$ is not a field but it is an integral domain. However, every finite integral domain is a field. [8]

Definition 2.7. [7] A subring S of a ring R is a subset that is both an additive and a multiplicative monoid.

Definition 2.8. [7] Let R and S be rings. A ring homomorphism is a map φ from R to S satisfying for all $a, b \in R$

$$\varphi(a+b) = \varphi(a) + \varphi(b) \quad \text{and} \quad \varphi(ab) = \varphi(a)\varphi(b).$$

Definition 2.9. [9] A homomorphism $\varphi : R \to S$ is

(i) a monomorphism if φ is 1-1,

(ii) a epimorphism if φ is onto S,

(iii) an isomorphism if φ is 1-1 and onto.

Definition 2.10. [8] Let $\varphi : R \longrightarrow S$ be a ring homomorphism. The kernel of φ, denoted by $Ker\varphi$, consists of those elements in R which are mapped by φ to the zero element of the ring S:

$$Ker\varphi = \{a \in R \mid \varphi(a) = 0_S\}.$$

That is, $Ker\varphi$ is the kernel of φ considered as a homomorphism of additive groups.

Definition 2.11. [8] A left ideal I of a ring R is an additive subgroup of R satisfying $r.x \in I$ for all $r \in R$ and for all $x \in I$. Similarly, a right ideal I of a ring R is an additive subgroup of R satisfying $x.r \in I$ for all $r \in R$ and for all $x \in I$. A (two-sided) ideal is both a left and a right ideal.

For all r in R we have $0r = 0$ because $0r = (0+0)r = 0r + 0r$ and similarly, $r0 = 0$. Hence $\{0\}$ and R are ideals of R.

Example 5. Let's consider $(a) = \{ka \mid k \in \mathbb{Z}\}$ where $a \in \mathbb{Z}$.
(a) is an ideal of $\mathbb{Z}$ since

$$ka - la = (k-l)a,$$

$$k(la) = (kl)a, \qquad k,\ l \in \mathbb{Z}.$$

In particular, the ring of even integers is an ideal of $\mathbb{Z}$ since $(2) = 2\mathbb{Z}$. Note that $\mathbb{Z} = (1)$ and $\{0\} = (0)$.

Definition 2.12. [8] An ideal (a) generated by one element of a ring R is a principal ideal and given by $(a) = \{ \sum\limits_{finite} s_i a r_i \mid\ s_i, r_i \in R\}$.

Definition 2.13. [8] An integral domain R is a principal ideal domain, abbreviated PID, if every ideal is principal.

Example 6.
(i) The ring $\mathbb{Z}$ of integers is a principal ideal domain; in fact, if I is an ideal of the ring $\mathbb{Z}$ then $I = (k)$ for a unique nonnegative integer k.
(ii) The polynomial ring $\mathbb{F}[x]$ with coefficients in the field $\mathbb{F}$ is a principal ideal domain. Every ideal I of $\mathbb{F}[x]$ is of the form $(p(x))$ where $p(x)$ is a unique monic polynomial.
(iii) The Laurent polynomial ring $\mathbb{F}[x, x^{-1}]$ with coefficients in the field $\mathbb{F}$ is a principal ideal domain. Every ideal I of $\mathbb{F}[x, x^{-1}]$ is of the form $(p(x))$ where $p(x)$ is a unique monic polynomial with $p(0) \neq 0$.

Theorem 2.14. [8] If φ is a ring homomorphism from a ring R into a ring S then $Ker\varphi$ is an ideal of R.

Proof. Let $\varphi : R \to S$ be a ring homomorphism. $Ker\varphi \neq \emptyset$ since $0_R \in Ker\varphi$. Suppose that $a, b \in Ker\varphi$. Then $\varphi(a) = 0_S$ and $\varphi(b) = 0_S$. Since $Ker\varphi$ is a ring homomorphism $\varphi(a-b) = \varphi(a) - \varphi(b) = 0_S$. Thus $a - b \in Ker\varphi$, that is, $Ker\varphi$ is a subgroup of R. Let $r \in R$ and $a \in Ker\varphi$. Then $\varphi(a) = 0_S$. Hence we get $\varphi(ar) = \varphi(a)\varphi(r) = 0_S\varphi(r) = 0_S$ and $\varphi(ra) = \varphi(r)\varphi(a) = \varphi(r)0_S = 0_S$. So $ar \in Ker\varphi$ and $ra \in Ker\varphi$. Thus $Ker\varphi$ is an ideal of R. $\square$

Let J be an ideal of a ring R and R/J be the collection of all cosets of J in R, that is,

$$R/J = \{a + J \mid a \in R\}.$$

The set R/J forms a ring under the following operations:

$$(a+J)+(b+J)=(a+b)+J,$$

$$(a+J)(b+J)=ab+J.$$

These operations are well-defined: suppose that $a+J=a'+J$ and $b+J=b'+J$. Then $a-a'=j_1$ and $b-b'=j_2$ for some $j_1, j_2 \in J$. Hence we get

$$(a+b)-(a'+b')=(a-a')+(b-b')=j_1+j_2 \in J$$

and so $(a+b)+J=(a'+b')+J$. Therefore $(a+J)+(b+J)=(a'+J)+(b'+J)$. Also, we observe that

$$ab-a'b'=a(b-b')+(a-a')b'=aj_2+j_1b' \in J$$

since $aj_2 \in J$ and $j_1b' \in J$. Hence $ab+J=a'b'+J$ thus the multiplication in R/J is well-defined. If J is an ideal of the ring R then R/J is also a ring [8], called the quotient ring of R by J.

If $\varphi : R \longrightarrow R/J$ is given by $r \mapsto r+J$ then φ is a ring homomorphism: For all $r, s \in R$,

$$\varphi(r+s)=r+s+J=(r+J)+(s+J)=\varphi(r)+\varphi(s) \qquad \text{and}$$

$$\varphi(rs)=rs+J=(r+J)(s+J)=\varphi(r)\varphi(s).$$

And $Ker\varphi=\{a \in R \,|\, \varphi(a)=0+J\}=\{a \in R \,|\, a+J=J\}=\{a \in R \,|\, a \in J\}=J$.

Hence every ideal is realized as the kernel of some homomorphism.

Definition 2.15. [8] An ideal I of the ring R is called a maximal ideal provided that $I \neq R$ and whenever J is an ideal of R with $I \subset J \subseteq R$ then $J=R$.

Example 7. The maximal ideals of $\mathbb{Z}$ are (p) where p is a unique prime number: Assume that (p) is a maximal ideal. If the integer p is not prime then $p=ab$ with $1<a<b<p$. Hence the ideals (a) and (b) are such that $(p) \subset (a) \subset \mathbb{Z}$, $(p) \subset (b) \subset \mathbb{Z}$ contrary to the maximality of (p).

Definition 2.16. [8] A ring R is called a skew-field if every nonzero element r of R is invertible, that is, there is $s \in R$ such that $rs=1=sr$.

Definition 2.17. [10] A nonzero ring R is called simple if the only ideals of R are (0) and R. Equivalently, R is simple if and only if (0) is a maximal ideal.

If R is a simple ring then for any nonzero element $a \in R$, the ideal (a) is R. If R is commutative then R is simple if and only if R is a field.

Fields and skew-fields are simple rings: If $\mathbb{F}$ is a field or a skew-field and $I \neq (0)$ is an ideal of $\mathbb{F}$ then there is a nonzero element a in I. Every nonzero element $a \in I$ has an inverse a^{-1} in $\mathbb{F}$ since $\mathbb{F}$ is a field. Hence $aa^{-1} = 1 \in I$ and so $I = \mathbb{F}$, that is, $\mathbb{F}$ is simple.

Example 8. The ring $M_n(\mathbb{F})$ is simple where $\mathbb{F}$ is a field: Let J be a nonzero ideal of $M_n(\mathbb{F})$ where $n \in \mathbb{N}$ and let $0 \neq A \in J$. Let E_{ij} be the matrix unit such that (i,j)-position is 1 and zeros elsewhere. There is a nonzero entry $\alpha := \alpha_{ij}$ of A. Then $\alpha E_{11} = E_{i1} A E_{1j} \in \langle A \rangle$. Since $\mathbb{F}$ is field, we get $(\alpha^{-1} E_{11})(\alpha E_{11}) = E_{11} \in \langle A \rangle$. Similarly we have $E_{22} \ldots E_{nn} \in \langle A \rangle$. Hence the identity element $I = \sum\limits_{i=1}^{n} E_{ii} \in \langle A \rangle$. Therefore $\langle A \rangle = M_n(\mathbb{F})$. This implies that $J = M_n(\mathbb{F})$.

When $\mathbb{F}$ is a field and $n > 1$, $M_n(\mathbb{F})$ is neither a field nor a skew-field. Also, $M_n(R)$ is not simple if the ring R is not simple since $M_n(I)$ is an ideal of $M_n(R)$ when I is an ideal of R.

Definition 2.18. [10] An ideal I is called prime when I satisfies the following: If J_1 and J_2 are ideals of R such that $J_1 J_2 \subseteq I$ then $J_1 \subseteq I$ or $J_2 \subseteq I$.

When R is a commutative ring the usual definition of a prime ideal I is: if $ab \in I$ then $a \in I$ or $b \in I$ for all $a, b \in R$. For commutative rings the two definitions are equivalent.
If I is an ideal of a commutative ring R then I is prime if and only if R/I is an integral domain [8]. Hence $\{0\}$ is a prime ideal when R is an integral domain.

Theorem 2.19. [8] I is a maximal ideal of a commutative ring R if and only if R/I is a field.

Proof. $(\Rightarrow)$ Suppose that I is maximal ideal and $a + I \in R/I$ with $a + I \neq I$. Then $a \notin I$. So we have $I \subset I + (a)$. Since I is maximal then $I + (a) = R$. In addition since R has identity element there exists $u \in I$ and $r \in R$ such that $u + ra = 1$. Then $1 - ra = u \in I$ and also $1 - ra \in I$. This implies that $1 + I = ra + I = (r + I)(a + I)$, that is, $r + I$ is the reverse of $a + I$. Hence R/I is field.

$(\Leftarrow)$ Let R/I be a field and J be any ideal of R such that $I \subset J \subseteq R$. Since $I \neq J$ there exists an element $a \in J$ with $a \notin I$. Since R/I is field then $a + I \in R/I$ have an inverse such that $(a + I)(b + I) = ab + I = 1 + I$ for some $(b + I) \in R/I$. So $1 - ab \in I \subset J$. Moreover $ab \in J$ since $a \in J$. The identity $1 = (1 - ab) + ab \in J$ and also $J = R$, that is, I is a maximal ideal. $\square$

Theorem 2.20. [8] In a commutative ring every maximal ideal is a prime ideal.

However, every prime ideal is not a maximal ideal: for example, the ideal $\{0\}$ is not a maximal ideal of $\mathbb{Z}$ since $\{0\} \subset (2) \subset \mathbb{Z}$ while it is a prime ideal.

Definition 2.21. [8] Let $\mathbb{F}$ be a field. A nonconstant polynomial $f(x) \in \mathbb{F}[x]$ is said to be an irreducible polynomial in $\mathbb{F}[x]$ if $f(x)$ can not be expressed as the product of two polynomials of positive degree.

Theorem 2.22. [8] Let $\mathbb{F}$ be a field. Then the following are equivalent:

(i) $p(x)$ is irreducible in $\mathbb{F}[x]$.

(ii) The ideal $(\, p(x)\,)$ is a maximal ideal of $\mathbb{F}[x]$.

(iii) The ideal $(\, p(x)\,)$ is a prime ideal of $\mathbb{F}[x]$.

(iv) $F[x]/(\, p(x)\,)$ is a field.

Definition 2.23. [8] A ring R satisfies the ascending chain condition for (left/right/two-sided) ideals if given any sequence of ideals $J_1, J_2, \cdots$ of R with

$$J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq \cdots$$

there exists an integer n (depending on the sequence) such that $J_n = J_{n+1} = \cdots$.

Definition 2.24. [8] A ring R satisfies the descending chain condition for (left/right/two-sided) ideals if given any descending chain of ideals of R,

$$J_1 \supseteq J_2 \supseteq \cdots \supseteq J_m \supseteq \cdots$$

there exists an integer m such that $J_m = J_{m+1} = J_{m+2} = \cdots$.

Theorem 2.25. [8] A ring R satisfies the ascending chain condition for ideals if and only if every ideal of R is finitely generated.

A ring satisfying the ascending chain condition for left (respectively right) ideals is called a left (respectively right) Noetherian ring. A ring satisfying the descending chain condition for left (respectively right) ideals is said to be a left (respectively right) Artinian.

Definition 2.26. [9] Let R be a ring. An abelian group $(M, +)$ is called a right R-module over R with respect to a mapping $M \times R \to M$ if for all $m_1,\ m_2 \in M$ and $r, s \in R$

(i) $(m_1 + m_2)s = m_1 s + m_2 s$

(ii) $m_1(s + r) = m_1 s + m_1 r$

(iii) $m_1(sr) = (m_1 s)r$

(iv) $m_1 1_R = m_1$.

A left R-module is defined analogously.

R is a right R-module, denoted by R_R.

From now on all modules will be right modules unless explicitly stated otherwise.

Example 9. Let R be a ring and J be an ideal in R. The ideal J is an R-module from the definition of an ideal.

Example 10. The quotient ring R/J is a right R-module:

(i) We know that R/J has an abelian group structure.

(ii) Let $s,\ s' \in R$ such that $s + J = s' + J \in R/J$ and let $r \in R$. Thus $s - s' \in J$ and hence $sr - s'r = (s - s')r \in J$. Therefore $sr + J = s'r + J$. Let's define a mapping

$$R/J \times R \to R/J$$
$$(s + J, r) \mapsto sr + J.$$

Therefore R/J is a right R-module by this scalar multiplication.

Definition 2.27. [9] Let M be an R-module and $\emptyset \neq N \subseteq M$. If N is a subgroup of M and $ar \in N$ for all $a \in N,\ r \in R$ then N is called a submodule of M.

Definition 2.28. [7] An R-module M is called simple if M is nonzero and if the only submodules of M are 0 and M.

Definition 2.29. [7] If an R-module M is expressed as $M = \bigoplus_{i=1}^{n} M_i$ where each M_i is simple submodule of M then M is called semisimple.

Definition 2.30. [7] A map φ from an R-module M to an R-module N is called an R-module homomorphism if φ satisfies the following:

(i) $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in M$.

(ii) $\varphi(xr) = \varphi(x)r$ for all $r \in R, x \in M$.

Definition 2.31. [9] Let X be a subset of an R-module M. The submodule $\langle\ X\ \rangle$ of M generated by X is defined as the intersection of all submodules of M which contain X. If $M = \langle\ X\ \rangle$ and X is a finite set then M is finitely generated.

Definition 2.32. [11] If X is a subset of an R-module M such that every element m of M has a unique expression $m = \sum x_i r_i$ with $x_i \in X$ and $r_i \in R$ for $i = 1, 2, \cdots, k$ then X is a basis for M.

Definition 2.33. [11] An R-module M is free when M has a basis.

Definition 2.34. [9] An $\mathbb{F}$-module M is a vector space over a field $\mathbb{F}$. The elements of $\mathbb{F}$ are called scalars and the elements of M are called vectors. A submodule of M is a subspace of M. A subset S of M spans M if $M = \langle\ S\ \rangle$. The submodule N generated by a subset Y of M is called the span of Y and denoted by $\langle\ Y\ \rangle$.

Definition 2.35. [9] Let W be a vector space over the field $\mathbb{F}$. A subset S of W is said to be linearly independent over $\mathbb{F}$ if for every finite number of distinct elements $s_1, \cdots, s_n \in S$ and for any finite set of scalars $\{\alpha_1, \cdots, \alpha_n\}$, $\quad \alpha_1 s_1 + \cdots + \alpha_n s_n = 0$ implies that $\alpha_1 = \cdots = \alpha_n = 0$.

Definition 2.36. [9] Let W be a vector space over the field $\mathbb{F}$. A subset B of W is called a basis for W over $\mathbb{F}$ if B spans W, that is, $W = \langle\ B\ \rangle$, and B is linearly independent over $\mathbb{F}$.

Definition 2.37. [9] Let W be a vector space over the field $\mathbb{F}$. If W is spanned by a finite set of vectors, then W is said to be finite dimensional over $\mathbb{F}$ and the dimension of W is the number of elements in a basis for W.

Since a finite dimensional vector space W has a finite basis, W is free, i.e., every finite dimensional vector space is a finitely generated free module over the scalar field $\mathbb{F}$.

2.2. Algebras

Definition 2.38. [7] An $\mathbb{F}-$ algebra A, where $\mathbb{F}$ is a field, is a ring that is also an $\mathbb{F}-$vector space with the same additive operation $+$ such that multiplication by scalars satisfies $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for all $\lambda \in \mathbb{F}$ and for all $a, b \in A$.

Example 11. The set of all polynomials in x with coefficients from a field $\mathbb{F}$ is denoted by $\mathbb{F}[x]$ where $\mathbb{F}[x] = \{\sum_{i=0}^{n} a_i x^i \mid a_i \in \mathbb{F}\,,\ n \in \mathbb{N}\}$. The algebra $(\mathbb{F}[x], +, \cdot)$ is called the polynomial algebra with coefficients from $\mathbb{F}$ when addition and multiplication of the polynomials

$$p(x) = \sum_{i=0}^{n} a_i x^i \text{ and } q(x) = \sum_{i=0}^{m} b_i x^i$$

are defined by

$$p(x) + q(x) = \sum_{i=0}^{max\{m,n\}} (a_i + b_i) x^i$$

and

$$p(x) \cdot q(x) = \sum_{k=0}^{m+n} c_k x^k \quad \text{where} \quad c_k = \sum_{i+j=k} a_i b_j.$$

Here the convention is all coefficients a_i that are not explicitly given in the sum $\sum_{i=0}^{n} a_i x^i$ are defined to be zero.

The polynomial algebra $\mathbb{F}[x]$ satisfies $\lambda(p(x)q(x)) = (\lambda p(x))q(x) = p(x)(\lambda q(x))$ for all $\lambda \in \mathbb{F}$ and for all $p(x), q(x) \in \mathbb{F}[x]$. Hence $\mathbb{F}[x]$ is an $\mathbb{F}$-algebra.

Similarly, $\mathbb{F}[x_1, x_2, \cdots x_n]$ is the polynomial algebra in n variables with coefficients from $\mathbb{F}$.

Example 12. The set of all polynomials in x, x^{-1} with coefficients from a field $\mathbb{F}$ is denoted by $\mathbb{F}[x, x^{-1}]$ where $\mathbb{F}[x, x^{-1}] = \{\sum_{i=-m}^{n} a_i x^i \mid a_i \in \mathbb{F},\ m, n \in \mathbb{N}\}$. This algebra is called the Laurent polynomial algebra with coefficients from $\mathbb{F}$.

Example 13. The set of square matrices of a fixed size over a field $\mathbb{F}$, with the operations of matrix addition, matrix multiplication and multiplication by scalars is an $\mathbb{F}$-algebra.

The matrix algebra of $n \times n$-matrices over $\mathbb{F}$ is denoted by $M_n(\mathbb{F})$.

Definition 2.39. [9] For a set X the vector space $\mathbb{F}X$ is the set of formal $\mathbb{F}$-linear combinations of elements of X. The monoid algebra of a (multiplicative) monoid S over $\mathbb{F}$ is $\mathbb{F}S$ as a vector space with the multiplication of S extended linearly to $\mathbb{F}S$.

Example 14. If the monoid is $(\mathbb{N}, +)$ then the algebra $\mathbb{F}\mathbb{N}$ is isomorphic to the polynomial algebra $\mathbb{F}X$ where $n \leftrightarrow x^n$.

Example 15. The $\mathbb{F}$-algebra $\mathbb{F}\langle x_1, x_2, \ldots, x_n\rangle$ is called the noncommutative polynomial algebra over $\mathbb{F}$ in n noncommuting variables, that is, the monoid algebra of the multiplicative monoid of noncommuting monomials $x_{i_1}^{m_1} x_{i_2}^{m_2} \ldots x_{i_k}^{m_k}$ with $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$ and $m_i \in \mathbb{N}$ for $i \in \{1, 2, \ldots k\}$. Note that if some $m_j = 0$ then this monomial is identified with the monomial obtained by dropping x_{i_j} . In particular if all $m_j = 0$ then we have the empty product which is 1, the multiplicative identity of the monoid and the algebra. The subalgebra of polynomials with constant term 0, that is, the span of monomials $x_{i_1}^{m_1} x_{i_2}^{m_2} \ldots x_{i_k}^{m_k}$ with some $m_i > 0$ will be denoted $\mathbb{F}\langle x_1, x_2, \ldots x_n\rangle_+$ which is a nonunital algebra. For any set X the polynomial algebra $\mathbb{F}\langle X\rangle$ is the polynomial algebra in the noncommuting variables $x \in X$.

Algebras given by generators and relations: Let X be the set of generators. Then the relations are of the form $f_i = g_i$.

(i) If $f_i,\ g_i \in \mathbb{F}_+\langle X\rangle$ for all $i \in I$ then $\mathcal{A} := \mathbb{F}_+\langle X\rangle/(\{f_i - g_i\}_{i\in I})$ is the algebra generated by X subject to the relations $\{f_i = g_i\}_{i\in I}$ where $(\{f_i - g_i\}_{i\in I})$ is the ideal generated by $\{f_i - g_i\}_{i\in I}$.

(ii) If some f_i or g_i is not in $\mathbb{F}_+\langle X\rangle$, that is, f_i or g_i has a nonzero constant term, then the algebra generated by X subject to the relations $\{f_i = g_i\}_{i\in I}$ is $\mathcal{A} := \mathbb{F}\langle X\rangle/(\{f_i - g_i\}_{i\in I})$.

The definition of Leavitt path algebras is of the first kind, that is, a quotient of $\mathbb{F}_+\langle X\rangle$, such as the definition of most algebras below. However, the definition of the Jacobson algebra, defined below, is of the second kind.

To define an algebra homomorphism φ from the $\mathbb{F}$-algebra $\mathcal{A}$ generated by X subject to $\{f_i = g_i\}_{i\in I}$ to the $\mathbb{F}$-algebra $\mathcal{B}$ we specify $\varphi(X) \subseteq \mathcal{B}$ and check that the relations are satisfied, that is, $\varphi(f_i(X)) := f_i(\varphi(X)) = g_i(\varphi(X))$ for all $i \in I$.

Example 16. [12] Let $\mathbb{F}$ be a field and n a positive integer. Then the Leavitt $\mathbb{F}$-algebra of type $(1,n)$, denoted by $L_{\mathbb{F}}(1,n)$, is the $\mathbb{F}$-algebra generated by elements $\{x_i, y_i : i = 1, \cdots, n\}$ and subject to the following relations:

(i) $\sum_{i=1}^{n} y_i x_i = 1$

(ii) $x_i y_j = \delta_{i,j} 1$

where $\delta_{i,j}$ is the Kronecker delta. Note that $L_{\mathbb{F}}(1,1) \cong \mathbb{F}[x, x^{-1}]$.

Definition 2.40. [7] Let A and B be $\mathbb{F}$-algebras. If $\varphi : A \to B$ is a ring homomorphism and a linear transformation then φ is called an $\mathbb{F}$-algebra homomorphism.

Definition 2.41. [13] Let G be a group. An $\mathbb{F}$-algebra $\mathcal{A}$ is called a G-graded $\mathbb{F}$-algebra if

(i) $\mathcal{A} = \bigoplus_{\gamma \in G} \mathcal{A}_\gamma$ where $\mathcal{A}_\gamma$ are subspaces of $\mathcal{A}$,

(ii) $\mathcal{A}_\gamma A_\delta \subseteq \mathcal{A}_{\gamma\delta}$ for every $\gamma, \delta \in G$.

The subspace $\mathcal{A}_\gamma$ is called the γ-*component* of $\mathcal{A}$. The elements of $\bigcup_{\gamma \in G} \mathcal{A}_\gamma$ in a graded algebra $\mathcal{A}$ are called *homogeneous elements*. The non-zero elements of $\mathcal{A}_\gamma$ are called γ-*homogeneous*, and set to have grade γ . The element 0 is considered to be homogeneous of every grade.

Example 17. The polynomial algebra $\mathbb{F}[x]$ is a $\mathbb{Z}$-graded $\mathbb{F}$-algebra since $\mathbb{F}[x] = \bigoplus_{i \in \mathbb{N}} \mathbb{F}x^i$ with $\mathbb{F}[x]_i := \mathbb{F}x^i$ when $i \geq 0$ in particular $\mathbb{F}[x]_0 = \mathbb{F}$ and $\mathbb{F}[x]_1 = \mathbb{F}x$. Also $\mathbb{F}[x]_i := 0$ when $i < 0$. Note that $\mathbb{F}[x]_i \mathbb{F}[x]_j = (\mathbb{F}x^i)(\mathbb{F}x^j) = \mathbb{F}x^{i+j} = \mathbb{F}[x]_{i+j}$.

Example 18. The polynomial algebra $\mathbb{F}[x, y]$ is a $\mathbb{Z}$-graded $\mathbb{F}$-algebra by total degree and also a $\mathbb{Z} \times \mathbb{Z}$-graded $\mathbb{F}$-algebra where $\mathbb{F}[x, y]_i := \mathbb{F}x^i + \mathbb{F}x^{i-1}y + \cdots + \mathbb{F}y^i$ in the $\mathbb{Z}$-grading and $\mathbb{F}[x, y]_{(i,j)} := \mathbb{F}x^i y^j$ in the $\mathbb{Z} \times \mathbb{Z}$-grading for all $i,\ j \geq 0$ with the remaning homogeneous components being zero. Similarly $\mathbb{F}[x_1, x_2, \ldots, x_m]$ has a $\mathbb{Z}$-grading by total degree where homogeneous elements of degree m are exactly the homogeneous polynomials of degree m and also a $\mathbb{Z}^m$-grading where homogeneous components have dimension zero or 1.

Example 19. Let $\mathbb{F}$ be an arbitrary field and $\mathbb{F}[x, x^{-1}]$ be the Laurent polynomial algebra. An element of $\mathbb{F}[x, x^{-1}]$ is of the form $\sum\limits_{i \geq m} a_i x^i$ with $m \in \mathbb{Z}$ and finitely many non-zero a_i's. Then $\mathbb{F}[x, x^{-1}]$ has the standard $\mathbb{Z}$-grading $\mathbb{F}[x, x^{-1}]_n = \mathbb{F}x^n$ with $n \in \mathbb{Z}$.

Example 20. Any $\mathbb{F}$-algebra $\mathcal{A} = \mathcal{A}_0 \bigoplus \mathcal{A}_1$ where $\mathcal{A}_0$, $\mathcal{A}_1$ are subspaces of $\mathcal{A}$, is called a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{F}$-algebra if $\mathcal{A}_\gamma \mathcal{A}_\delta \subseteq \mathcal{A}_{\gamma+\delta}$ for every $\gamma, \delta \in \mathbb{Z}/2\mathbb{Z}$. Elements of the 0-component $\mathcal{A}_0$ are said to be even and elements of the 1-component $\mathcal{A}_1$ said to be odd. $\mathbb{Z}/2\mathbb{Z}$-algebras are also called superalgebras. Any $\mathbb{F}$-algebra has the trivial $\mathbb{Z}/2\mathbb{Z}$-grading where $\mathcal{A}_0 := \mathcal{A}$ and $\mathcal{A}_1 := 0$. Similarly, any $\mathbb{F}$-algebra $\mathcal{A}$ has the trivial $\mathbb{Z}$-grading where $\mathcal{A}_0 := \mathcal{A}$ and $\mathcal{A}_n := 0$ when $n \neq 0$.

Definition 2.42. [13] A graded algebra $\mathcal{A}$ is locally finite dimensional if each homogeneous summand is finite dimensional.

Definition 2.43. [13] Let G be a group. An ideal I of a G-graded $\mathbb{F}$-algebra $\mathcal{A}$ is said to be a graded ideal if $I \subseteq \sum_{\gamma \in G} (I \cap \mathcal{A}_\gamma)$, or , equivalently, if $x = \sum_{\gamma \in G} x_\gamma \in I$ implies $x_\gamma \in I$ for every $\gamma \in G$.

Definition 2.44. [13] Let $\mathbb{F}$ be a field and let G be a group. Given two G-graded $\mathbb{F}$-algebras $\mathcal{A} = \bigoplus_{\gamma \in G} \mathcal{A}_\gamma$ and $\mathcal{B} = \bigoplus_{\gamma \in G} \mathcal{B}_\gamma$, an $\mathbb{F}$-algebra homomorphism f from $\mathcal{A}$ to $\mathcal{B}$ is said to be a graded homomorphism if $f(\mathcal{A}_\gamma) \subseteq \mathcal{B}_\gamma$ for every $\gamma \in G$.

2.3. Categories

Definition 2.45. [14] A (locally small) category $\mathcal{C}$ consists of a class $Ob(\mathcal{C})$ of objects, and a set of morphisms $\mathcal{C}(X,Y)$ for each pair X, Y of objects such that:

(i) For each $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(Y,Z)$ there is a morphism $g \circ f \in \mathcal{C}(X,Z)$ which is called the composition of f with g . Composition is associative when defined.

(ii) For each object X , there is an identity morphism $\boldsymbol{id_X} \in \mathcal{C}(X,X)$ such that $id_X \circ f = f$ for all $f \in \mathcal{C}(Y,X)$ and $g \circ id_X = g$ for all $g \in \mathcal{C}(X,Y)$.

We will use the notation $X \xrightarrow{f} Y$ to mean $f \in \mathcal{C}(X,Y)$.

A subcategory of $\mathcal{C}$ has some of the objects of $\mathcal{C}$ and some of the morphisms of $\mathcal{C}$ with the same identity morphisms and the same compositions. A subcategory $\mathcal{B}$ of $\mathcal{C}$ is full if $\mathcal{B}(X,Y) = \mathcal{C}(X,Y)$ for all objects X, Y in $\mathcal{B}$. If $Ob(\mathcal{C})$ is a set then $\mathcal{C}$ is a small category.

Example 21. (i) The category **Set**: The objects of this category are sets and the morphisms of this category are functions $X \xrightarrow{f} Y$. The category **set** is the full subcategory whose objects are finite sets.

(ii) The category **Grp**: Objects are groups and morphisms are group homomorphisms. The subcategory of abelian groups is a full subcategory.

(iii) The category $\mathbf{Vec}_{\mathbb{F}}$: Objects are vector spaces over the field $\mathbb{F}$ and morphisms are linear transformations. The category $\mathbf{vec}_{\mathbb{F}}$ is the full subcategory whose objects are finite dimensional vector spaces.

(iv) The category $\mathbf{Mod}_R$: Objects are right R-modules where R is a ring and morphisms are R-module homomorphisms. The full subcategory $\mathbf{mod}_R$ has finitely generated modules as its objects. The previous example is a special case with the ring R being a field $\mathbb{F}$.

Definition 2.46. [14] A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}, \mathcal{D}$ are categories, assigns

(i) an object $\mathcal{F}(X)$ of $\mathcal{D}$ to each object X of $\mathcal{C}$,

(ii) a morphism $\mathcal{F}(f) \in \mathcal{D}(\mathcal{F}(X), \mathcal{F}(Y))$ to each $f \in \mathcal{C}(X,Y)$ such that

(a) $\mathcal{F}(id_X) = id_{\mathcal{F}(X)}$ for each object X of $\mathcal{C}$ and

(b) $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$ when $f \circ g$ is defined.

Definition 2.47. [14] Consider functors $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and $\mathcal{G} : \mathcal{C} \longrightarrow \mathcal{D}$. A natural transformation $\theta : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism such that

(i) θ give a morphism $\theta_X : \mathcal{F} \longrightarrow \mathcal{G}$ for each object X in $\mathcal{C}$.

(ii) For each pair of objects X and Y and each morphism $\mathcal{C}(X,Y)$ for all $f \in \mathcal{C}(X,Y)$ the diagram

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\theta_X"] & \mathcal{F}(Y) \arrow[d, "\theta_Y"] \\ \mathcal{G}(X) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(Y) \end{tikzcd}$$

commutes, that is, $\theta_Y \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \theta_X$.

We call θ_X the components of the natural transformation θ at the object X. A natural transformation θ with every component θ_X invertible in $\mathcal{D}$ is called a natural equivalence, we say that the functors $\mathcal{F}$ and $\mathcal{G}$ are naturally isomorphic denoted by $\mathcal{F} \cong \mathcal{G}$. The inverses θ_X^{-1} in $\mathcal{D}$ are the components of $\theta^{-1} : \mathcal{G} \longrightarrow \mathcal{F}$, that is, $(\theta^{-1})_X = \theta_X^{-1}$.

Example 22. A quiver representation is a functor ρ from the small category Γ with objects V and morphisms given by paths in Γ to the category $\mathbf{Vec}_{\mathbb{F}}$. The linear transformation assigned to the path $p = e_1 e_2 \dots e_n$ is the composition $s(p) = s(e_1) \xrightarrow{\rho(e_1)} t(e_1) = s(e_2) \xrightarrow{\rho(e_2)} \cdots \xrightarrow{\rho(e_n)} t(e_n) = t(p)$ from $\rho(s(p))$ to $\rho(t(p))$. A morphism of quiver representations is a natural transformation between two such functors. The objects of the category of quiver representations of Γ are quiver representations and its morphisms are natural transformations. This category is equivalent to the category $\mathbf{Mod}_{\mathbb{F}\Gamma}$ [15].

2.4. Posets

Definition 2.48. [7] A relation $\sim$ on a set S is a partial order if $\sim$ satisfies the following:

(i) $a \sim a$ for all $a \in S$ (the reflexive property)

(ii) if $a \sim b$ and $b \sim a$ then $a = b$ for all $a, b \in S$ (the antisymmetric property)

(iii) if $a \sim b$ and $b \sim c$ then $a \sim c$ for all $a, b, c \in S$ (the transitive property)

The pair $(S, \sim)$ is a partially ordered set (or, for short, poset).

Example 23. $(\mathbb{R}, \leq)$ is a poset since the usual order $\leq$ in $\mathbb{R}$ is reflexive, antisymmetric and transitive.

Example 24. The power set 2^X of a given set X, the set of all subsets of X, with the set inclusion relation: $(2^X, \subseteq)$ is a poset.

3. LPAs AND THEIR FINITE CODIMENSIONAL MAXIMAL IDEALS

3.1. Digraphs

Definition 3.1. [13] A di(rected)graph Γ is a four-tuple (V, E, s, t) where V is the set of vertices, E is the set of arrows (directed edges), s and $t : E \longrightarrow V$ are the source and the target functions.

If $s(e) = u$ and $t(e) = w$ then we say that u *emits* e and that w *receives* e. If a vertex does not receive any arrow then it is called a *source*. The set of all arrows emitted by $u \in V$ is denoted by $s^{-1}(u)$ while $t^{-1}(u)$ denotes the set of all arrows received by u. If $s^{-1}(w) = \emptyset$ then w is called a *sink*. If u is not a sink and $0 < |s^{-1}(u)| < \infty$ then the vertex u is called a *regular* vertex.

A *path* p of length $n > 0$ is a sequence of arrows $e_1e_2 \cdots e_n$ such that $t(e_i) = s(e_{i+1})$ for $i = 1, 2, 3, \ldots, n-1$. The *length* of a path p, denoted by $l(p)$, is the number of arrows in p. The *source* of a path p, denoted by $s(p)$, is the source of its initial arrow $s(e_1)$ while the *target* of a path p, denoted by $t(p)$, is the target of its final arrow $t(e_n)$. Every vertex v is a path of length 0 with $s(v) = v = t(v)$. The set of all paths in Γ is denoted by $Path(\Gamma)$. If $p = e_1 \cdots e_n$ and $q = e_1 \cdots e_m$ for some $m \leq n$ then q is an *initial segment* of p.

Example 25. Let us consider the following digraph $\Gamma = (V, E, s, t)$: The set of vertices and the set of arrows are $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_1, e_2, e_3, e_4\}$ with $s(e_1) = v_1$, $s(e_2) = t(e_1) = v_2$, $s(e_3) = t(e_3) = s(e_4) = v_3$, $t(e_4) = v_4$. The vertex v_1 is source and the vertex v_4 is sink in Γ. The source of the path $p = e_2e_3$ in Γ is $s(p) = s(e_2) = v_2$ and the target of p is $t(p) = t(e_3) = v_3$.

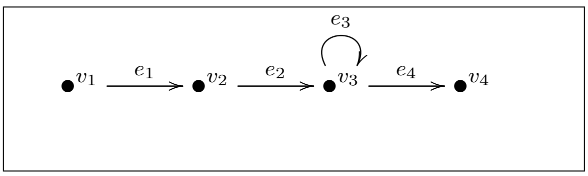

Figure 3.1: Digraph Γ

A path p with $l(p) > 0$ is called a *cycle* if $s(p) = t(p)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$. An arrow e with $s(e) = t(e)$, equivalently, a cycle of length 1, is called a *loop*. If there is no cycle in Γ then Γ is said to be *acyclic*. An arrow e is called an *exit* of the path $p = f_1 \cdots f_n$ if there is an $i \in \{1, \ldots, n\}$ such that $s(e) = s(f_i)$ but $e \neq f_i$.

Example 26. Consider the following digraph Γ:

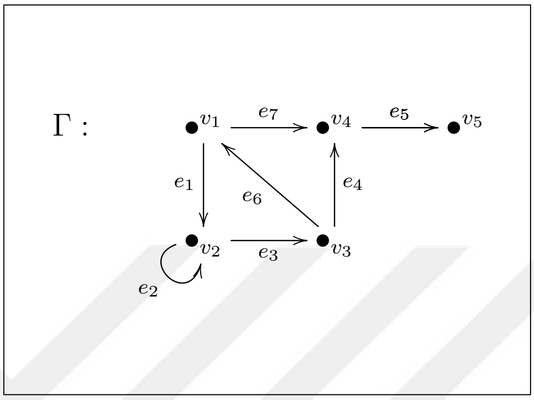

Figure 3.2: Digraph Γ

Let's choose a path $C = e_1e_3e_6$ in Γ. The path C is a cycle of length 3 and $s(C) = v_1$. The arrows e_2, e_4 and e_7 are exits of C and also e_2 is a cycle with length 1, that is, a loop.

If E and V are both finite then the digraph Γ is said to be *finite*. The digraph Γ is called *row-finite* if $s^{-1}(u)$ is a finite set for every $u \in V$.

Definition 3.2. [16] The preorder leadsto, denoted by $\rightsquigarrow$, on V is defined as $u \rightsquigarrow w$ if there is a path $p \in Path(\Gamma)$ from u to w. The set of successors of a vertex u is

$$V_{u\rightsquigarrow} = \{w \in V \mid u \rightsquigarrow w\}.$$

If $X \subseteq V$ then we define $V_{X\rightsquigarrow} = \bigcup_{u \in X} V_{u\rightsquigarrow}$. The set of predecessors of a vertex u is

$$V_{\rightsquigarrow u} = \{w \in V \mid w \rightsquigarrow u\}.$$

If $X \subseteq V$ then we define $V_{\rightsquigarrow X} = \bigcup_{u \in X} V_{\rightsquigarrow u}$.

For instance, $V_{v_4\rightsquigarrow} = \{v_4, v_5\}$ and $V_{\rightsquigarrow\{v_1,v_2\}} = \{v_1,\ v_2,\ v_3\}$ in the Figure 3.2 above.

If Γ is a finite digraph whose cycles are pairwise disjoint then there is a partial order $\rightsquigarrow$ on the set of sinks and cycles in Γ. Let U be the set of sinks and cycles in Γ. Then $(U, \rightsquigarrow)$ is a poset since it satisfies the following relations: For all $x, y, z \in U$

(i) $x \rightsquigarrow x$ since there is a path of length zero from x to x.

(ii) If $x \rightsquigarrow y$ and $y \rightsquigarrow x$ then x and y are either sinks or cycles since there is no path of nonzero length from any sink. If both of them are sinks then $x = y$. If both of them are cycles then there is a path p from the cycle x to the cycle y and there is a path q from y to x since $x \rightsquigarrow y$ and $y \rightsquigarrow x$. Then $x = y$ otherwise the cycle x and the cycle pq are not pairwise disjoint in Γ.

(iii) If $x \rightsquigarrow y$ and $y \rightsquigarrow z$ then there exists a path p from x to y and also there exists a path q from y to z. Hence we have the path pq with $t(p) = s(y) = s(q)$ from x to z, that is, $x \rightsquigarrow z$.

Definition 3.3. [13] Let Γ be a digraph and $H \subseteq V$.

(i) H is hereditary if $u \in H$ and $u \rightsquigarrow w$ then $w \in H$.

(ii) H is saturated if $0 < |s^{-1}(u)| < \infty$ and $\{t(e) \mid e \in s^{-1}(u)\} \subseteq H$ then $u \in H$.

Example 27. Let's give some hereditary saturated subsets for the following digraph:

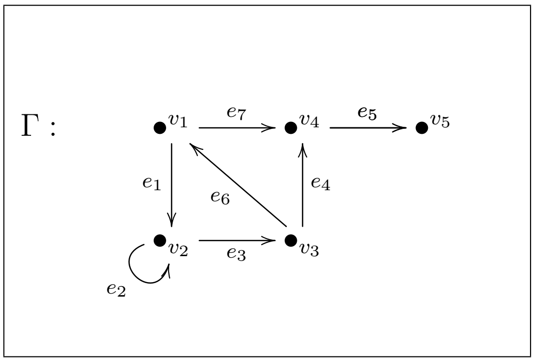

Figure 3.3: Digraph Γ

Let $H_1 = \{v_4, v_5\}, H_2 = \{v_4\}$ and $H_3 = \{v_1, v_4\}$ be the subsets of V. The set H_1 is hereditary and saturated. H_2 is saturated but not hereditary since $v_4 \in H_2$ and $v_4 \rightsquigarrow v_5$ but $v_5 \notin H_2$. The set H_3 is not hereditary since $v_1 \in H_3$ and $v_1 \rightsquigarrow v_2$ but $v_2 \notin H_3$. Also H_3 is not saturated since $\{t(e)|e \in s^{-1}(v_3)\} = \{v_1, v_4\} \subseteq H_3$ but $v_3 \notin H_3$.

If H_1 and H_2 are hereditary saturated subsets of V then $H_1 \cap H_2$ is hereditary saturated: Let $u \in H_1 \cap H_2$ then $u \in H_1$ and $u \in H_2$. If there is a path from u to w then we have $w \in H_1$ and $w \in H_2$ since H_1 and H_2 are hereditary. So $w \in H_1 \cap H_2$, that is, $H_1 \cap H_2$ is hereditary. Assume that v is not a sink and $\{t(e) \mid e \in s^{-1}(v)\} \subseteq H_1 \cap H_2$. Then $\{t(e) \mid e \in s^{-1}(v)\}$ is the subset of both H_1 and H_2 and so $v \in H_1$ and $v \in H_2$ since H_1 and H_2 are saturated. Hence $v \in H_1 \cap H_2$. Therefore $H_1 \cap H_2$ is saturated.

Example 28. However, the union of saturated subsets is not saturated: For example,

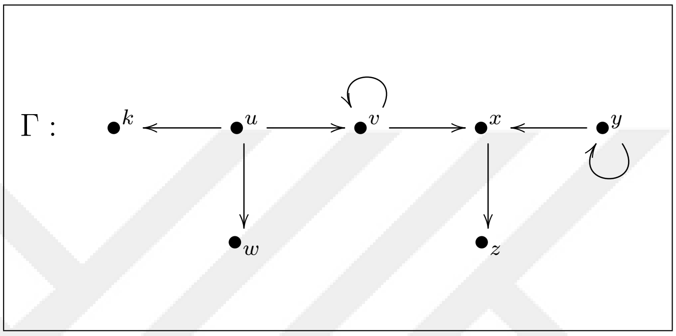

Figure 3.4: Digraph Γ

$H_1 = \{x, z, w, v\}$ and $H_2 = \{x, y, z, k\}$ are saturated but $H_1 \cup H_2 = \{x, y, z, v, w, k\}$ is not saturated since $u \notin H_1 \cup H_2$ while $\{t(e) \mid e \in s^{-1}(u)\} = \{v, w, k\} \subseteq H_1 \cup H_2$.

Definition 3.4. [13] The hereditary saturated *closure* $\overline{S}$ of $S \subseteq V$ is the smallest hereditary and saturated subset of V containing S.

Since the intersection of hereditary saturated sets are also hereditary and saturated, the hereditary saturated closure of S is the intersection of all hereditary and saturated subsets of V containing S. When Γ is row-finite, the hereditary saturated closure of $S \subseteq V$ consists of all vertices $u \in V$ such that every path starting at u and ending at a sink and every infinite path starting at u meets a successor of some $s \in S$.

Definition 3.5. [17] A digraph $\Lambda = (V', E', s', t')$ is a subgraph of a digraph $\Gamma = (V, E, s, t)$ if $V' \subseteq V$ and $E' \subseteq E$.

Definition 3.6. [17] A subgraph Λ of a digraph Γ is called a full subgraph of Γ if every arrow e in Γ with $s(e)$ and $t(e)$ in Λ is an arrow of Λ.

Definition 3.7. [17] A digraph morphism from a digraph $\Gamma = (V, E, s, t)$ to a digraph

$\Lambda = (V', E', s', t')$ is a pair $\varphi = (\varphi_0, \varphi_1)$ of functions $\varphi_0 : V \to V'$ and $\varphi_1 : E \to E'$ satisfying

$$s'(\varphi_1(e)) = \varphi_0(s(e)) \quad \text{and} \quad t'(\varphi_1(e)) = \varphi_0(t(e))$$

for all $e \in E$. A digraph morphism $\varphi : \Gamma \longrightarrow \Lambda$ is an isomorphism if there exists a digraph morphism $\Psi : \Lambda \longrightarrow \Gamma$ such that $\varphi \circ \Psi = id_\Lambda$ and $\Psi \circ \varphi = id_\Gamma$.

In the remainder of this thesis all digraphs will be finite unless explicitly stated otherwise.

A digraph Γ can also be regarded as a category whose objects are the vertices in Γ, with morphisms being the paths in Γ. Identity morphisms are also identified with the vertices and composition is given by concatenation of paths. Unlike the previous examples of categories above $\mathbf{\Gamma}$ is a small category. Note that the order of composition (of paths) in $\mathbf{\Gamma}$ is written from left to right, the opposite of the order we compose functions.

3.2. Leavitt Path Algebras (LPAs)

In this section our main objective is to define Leavitt path algebras, abbreviated LPAs. First we start with the definition of a path algebra. The input for both algebras is a digraph Γ and a coefficient field $\mathbb{F}$.

Definition 3.8. [13] The path algebra $\mathbb{F}\Gamma$ of a digraph Γ over a field $\mathbb{F}$ is defined as the $\mathbb{F}$-algebra generated by $V \sqcup E$ with the relations

(V) $\quad uv = \delta_{u,v} u$ for all $u, v \in V$,

(E) $\quad s(e)e = e = et(e)$ for all $e \in E$

where $\delta_{i,j}$ is the Kronecker delta.

Note that by the relation (V), each $v \in V$ is an idempotent in $\mathbb{F}\Gamma$ and the elements of V are mutually orthogonal in $\mathbb{F}\Gamma$. The path algebra $\mathbb{F}\Gamma$ has all paths in Γ as an $\mathbb{F}$-basis with multiplication given by concatenation pq if $t(p) = s(q)$ and $pq := 0$ if $t(p) \neq s(q)$.

Definition 3.9. [13] The extended digraph $\hat{\Gamma} = (V, E \sqcup E^*, s, t)$ of Γ is obtained by adding a dual arrow e^* for each e in E where $E^* = \{e^* \mid e \in E\}$, $s(e^*) = t(e)$ and $t(e^*) = s(e)$ for all $e \in E$.

Example 29. The extended digraph of Γ in Figure 3.3:

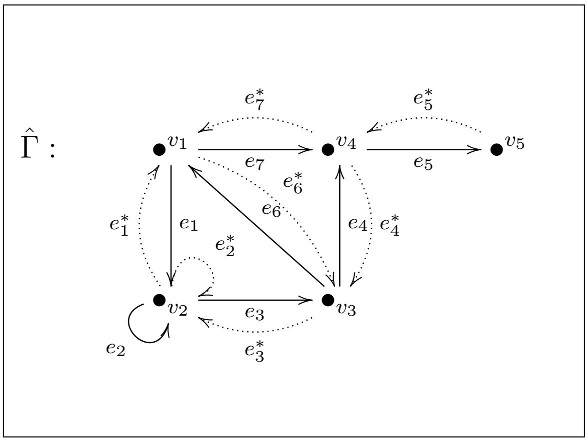

Figure 3.5: Extended digraph $\hat{\Gamma}$

Definition 3.10. [13] The Leavitt Path Algebra (LPA) $L_{\mathbb{F}}(\Gamma)$ of a finite digraph Γ with coefficients in the field $\mathbb{F}$ is defined as the $\mathbb{F}$-algebra generated by $V \sqcup E \sqcup E^*$ with the relations:

(V) $\quad uv = \delta_{u,v} u$ for all $u, v \in V$,

(E) $\quad s(e)e = e = et(e)$ and $t(e)e^* = e^* = e^*s(e)$ for all $e \in E$,

$(CK1)$ $\quad e^*f = \delta_{e,f}\, t(e)$ for all $e, f \in E$,

$(CK2)$ $\quad u = \sum\limits_{s(e)=u} ee^*$ for every vertex u with $0 < |s^{-1}(u)|$.

The relations $(CK1)$ and $(CK2)$ are called the *Cuntz-Krieger* relations. The relations (V) state that the vertices are mutually orthogonal idempotents. In the algebras $\mathbb{F}\Gamma$ and $L_{\mathbb{F}}(\Gamma)$ the multiplicative unit 1 is $\sum\limits_{u \in V} u$.

Example 30. Let Γ be the digraph described by

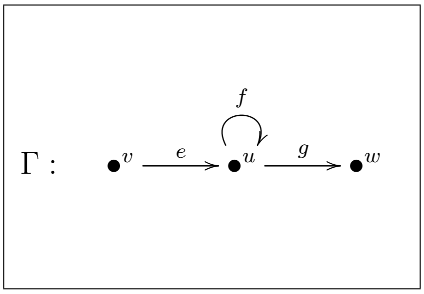

Figure 3.6: Digraph Γ

Some computations in $L_{\mathbb{F}}(\Gamma)$ are following :

$$vv = v \quad \text{and} \quad vu = 0 \quad \textit{by}\ (V)$$

$$ve = e = eu \quad \text{and} \quad e^*v = e^* = ue^* \quad \textit{by}\ (E)$$

$$g^*g = w \quad \text{but} \quad g^*f = 0 \quad \text{by}\ \ (CK1)$$

$$v = ee^* \quad \text{and} \quad u = ff^* + gg^* \quad \text{by}\ (CK2)$$

Lemma 3.11. [13] Let p and q be paths.

(i) $p^*q = 0$ unless q is an initial segment of p, that is, $p = qp'$ for some path p' or p is an initial segment of q, that is, $q = pq'$ for some path q'.

(ii) $L_{\mathbb{F}}(\Gamma)$ is spanned by $\{pq^* | p, q \in Path(\Gamma), t(p) = t(q)\}$.

(iii) If C is a cycle with no exit in Γ then $CC^* = s(C) = C^*C$.

Proof. [18]

(i) Let $p = e_1, ...e_n$ and $q = f_1, ...f_m$. Then $p^*q = (e_1, ...e_n)^*(f_1, ...f_m) = e_n^*...(e_1^*f_1)...f_m$ is 0 or $e_n^*...e_2^*t(e_1)f_2...f_m$ since $e_1^*f_1 = \delta_{e_1,f_1}t(e_1)$ by (CK1). If $p^*q \neq 0$ then $p^*q = e_n^*...e_2^*t(e_1)f_2...f_m = e_n^*...e_2^*f_2...f_m$ since $e_2^*t(e_1) = e_2^*$ by (E). Similarly, we get $p^*q = 0$ or $p^*q = e_n^*...e_3^*f_3...f_m$ by using (CK1) and (E). Continuing this process, we get $p^*q = 0$ if $e_i \neq f_i$ for some $1 \leq i \leq min\{n, m\}$. Thus if $p^*q \neq 0$ then $e_i = f_i$ for all $1 \leq i \leq min\{n, m\}$, that is, q is an initial segment of p or p is an initial segment of q.

If p is a path of length 0, that is, $p = v \in V$ then $p^*q = v^*q = vs(q)q = \delta_{v,s(q)}vq$ by (E) and (V). Thus $p^*q = 0$ unless $v = s(q)$, i.e., $q = s(q)q = pq'$ where $q' = q$. Similarly, if $q = v \in V$ then $p^*q = p^*v = p^*(s(p))v = p^*\delta_{s(p),v}v$. Hence

$p^*q = 0$ unless $s(p) = v$, that is, $p = (s(p))p = vp = qp'$ where $p' = p$. Thus $p^*q = 0$ unless q is an initial segment of p or p is an initial segment of q.

(ii) If $p = qr$ then p^*q simplifies to r^* since $p^*q = (qr)^*q = r^*q^*q = r^*t(q) = r^*$. Similarly, $p^*q = p^*pr = t(p)r = r$ if $q = pr$. Hence (i) implies that multiplying terms of the form pq^* we get 0 or another term of this form. Also, if $t(p) \neq t(q)$ then $pq^* = 0$ by (V) and (E). Thus $L_{\mathbb{F}}(\Gamma)$ is generated as an $\mathbb{F}$-module by pq^* with $p, q \in Path(\Gamma)$ and $t(p) = t(q)$.

(iii) As in (ii), $C^*C = t(C) = s(C)$ after applying (CK1), (V) and (E) repeatedly. Since C has no exit, $s(e) = e^*e$ for each arrow e on C by (CK2). Therefore $CC^* = s(C) = C^*C$. $\square$

$L_{\mathbb{F}}(\Gamma)$ has a $\mathbb{Z}$-grading given by $|u| = 0$ for $u \in V$, $|e| = 1$ and $|e^*| = -1$ and for $e \in E$. This defines a grading on $L_{\mathbb{F}}(\Gamma)$ since all the relations are homogeneous. Hence $|pq^*| = l(p) - l(q)$ and

$$L_{\mathbb{F}}(\Gamma) = \bigoplus_{n \in \mathbb{Z}} L_{\mathbb{F}}(\Gamma)_n$$

where $L_{\mathbb{F}}(\Gamma)_n = \left\{ \sum_{i=1}^{m} k_i p_i q_i^* : l(p_i) - l(q_i) = n \right\}$ for each $n \in \mathbb{Z}$, with each $k_i \in \mathbb{F}$ and each $p_i, q_i \in Path(\Gamma)$.

We have $*$ defined on $V \sqcup E \sqcup E^*$ where $u^* = u$ for all $u \in V$ and $(e^*)^* = e$ for all $e \in E$. We extend the operator $*$ to $Path(\hat{\Gamma})$ by $(e_1 e_2 \dots e_m)^* = e_m^* \dots e_2^* e_1^*$ and then to $L_{\mathbb{F}}(\Gamma)$ linearly. The operator $*$ induces a grade-reversing involutive anti-automorphism $L_{\mathbb{F}}(\Gamma)$, that is, $(a + b)^* = a^* + b^*$, $(\lambda a)^* = \lambda a^*$, $(ab)^* = b^*a^*$, $(a^*)^* = a$ for all $a,\ b \in L_{\mathbb{F}}(\Gamma)$ and $\lambda \in \mathbb{F}$, moreover $|a^*| = -|a|$ for all homogeneous a in $L_{\mathbb{F}}(\Gamma)$.

Some well-known algebras can be described as Leavitt path algebras:

Example 31. If Γ is a single vertex with no arrow then $L_{\mathbb{F}}(\Gamma) \cong \mathbb{F}v \cong \mathbb{F}$.

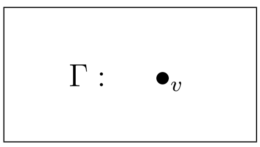

Figure 3.7: Digraph Γ with single vertex

Adding an extra vertex w we have the following digraph Λ and so $L_{\mathbb{F}}(\Lambda) \cong \mathbb{F}^2$.

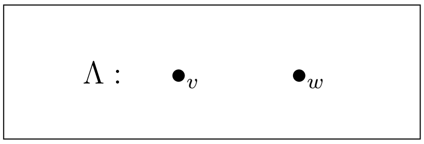

Figure 3.8: Digraph Λ

Lemma 3.12. [13] Let $\mathbb{F}$ be any field. Then the Leavitt path algebra of the following digraph R_1 is isomorphic to $\mathbb{F}[x, x^{-1}]$, the algebra of Laurent polynomials with coefficients in $\mathbb{F}$.

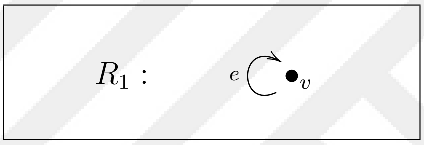

Figure 3.9: Digraph R_1

Proof. In the algebra $L_{\mathbb{F}}(R_1)$ the multiplicative unit 1 is v. The digraph R_1 consist of one vertex and one arrow. By the Cuntz-Kriger relations (CK1) and (CK2) we get $e^*e = v = 1$ and $ee^* = v = 1$ in $L_{\mathbb{F}}(R_1)$.

Let's define $\varphi : L_{\mathbb{F}}(R_1) \to \mathbb{F}[x, x^{-1}]$ by $\varphi(v) = 1, \varphi(e) = x$ and $\varphi(e^*) = x^{-1}$. Then φ is a homomorphism since φ preserves the Leavitt path algebra relations: For $v \in V$ and $e \in E$

(V) $\varphi(vv) := \varphi(v)\varphi(v) = 1.1 = 1 = \varphi(v)$.

(E) $\varphi(s(e)e) := \varphi(s(e))\varphi(e) = 1.x = x = \varphi(e)$ and

$\varphi(et(e)) := \varphi(e)\varphi(t(e)) = x.1 = x = \varphi(e)$,

$\varphi(e^*s(e)) := \varphi(e^*)\varphi(s(e)) = x^{-1}.1 = x^{-1} = \varphi(e^*)$ and

$\varphi(t(e)e^*) := \varphi(t(e))\varphi(e^*) = 1.x^{-1} = x^{-1} = \varphi(e^*)$.

(CK1) $\varphi(e^*e) := \varphi(e^*)\varphi(e) = x^{-1}.x = 1 = \varphi(t(e))$.

(CK2) $\varphi(ee^*) := \varphi(e)\varphi(e^*) = x.x^{-1} = 1 = \varphi(v)$.

Since $e^*e = v = 1$ and $ee^* = v = 1$ in $L_{\mathbb{F}}(R_1)$ any element of the form pq^* equals e^n or

$(e^*)^m$ or v where $n, m > 0$. Hence the set $\{e^n \mid n > 0\} \cup \{v\} \cup \{(e^*)^m \mid m > 0\}$ spans $L_{\mathbb{F}}(R_1)$ as a vector space. The image of this set is $\{x^n \,|\, n > 0\} \cup \{1\} \cup \{(x^{-1})^m \,|\, m > 0\}$ which is a basis for $\mathbb{F}[x, x^{-1}]$. Hence $\{e^n \,|n > 0\} \cup \{v\} \cup \{(e^*)^m \mid m > 0\}$ is linearly independent and φ is an isomorphism. $\square$

Lemma 3.13. [13] If $n \geq 2$ is any positive integer and R_n is the rose with n petals then $L_{\mathbb{F}}(R_n)$ is isomorphic to $L_{\mathbb{F}}(1, n)$, the Leavitt $\mathbb{F}$-algebra of type $(1, n)$.

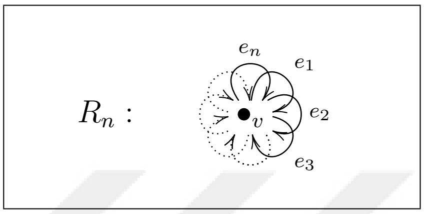

Figure 3.10: A rose with n petals

Proof. The rose with n petals digraph has one vertex and n loops. We know that $L_{\mathbb{F}}(1, n)$ is the $\mathbb{F}$-algebra on the $2n$ variables $x_1, \cdots, x_n, y_1, \cdots, y_n$ and subject to the relations $\sum_{i=1}^{n} x_i y_i = 1$ and $y_i x_j = \delta_{i,j} 1$, $(1 \leq i, j \leq n)$. We define the map $\varphi : L_{\mathbb{F}}(R_n) \to L_{\mathbb{F}}(1, n)$ by $\varphi(v) = 1_R$, $\varphi(e_i) = x_i$ and $\varphi(e_i^*) = y_i$. The relations given above are precisely the relations provided by the (CK1) and (CK2) on $L_{\mathbb{F}}(R_n)$ while the relations (V) and (E) are automatically satisfied in this case. $\square$

Lemma 3.14. [13] If $\mathbb{F}$ is a field and $n \in \mathbb{Z}_+$ with $n \geq 1$ then $L_{\mathbb{F}}(A_n) \cong M_n(\mathbb{F})$ where A_n is the following digraph with n vertices and $n - 1$ arrows:

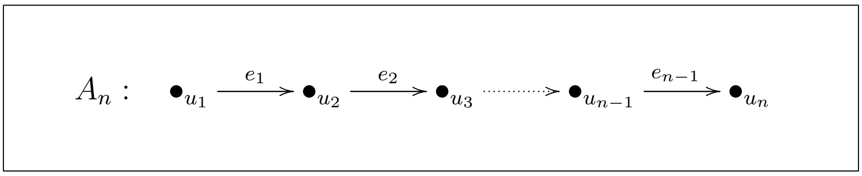

Figure 3.11: Digraph A_n

Proof. Let $\{E_{ij} | 1 \leq i, j \leq n\}$ denote the elementary matrices in $M_n(\mathbb{F})$. We define the map $\varphi : L_{\mathbb{F}}(A_n) \to M_n(\mathbb{F})$ by setting $\varphi(u_i) = E_{ii}$, $\varphi(e_i) = E_{i(i+1)}$, and $\varphi(e_i^*) = E_{(i+1)i}$. Then

(V) $\varphi(u_i)\varphi(u_j) = E_{ii}E_{jj} = \delta_{i,j}E_{ii} = \delta_{i,j}\varphi(u_i)$ for all $u_i,\ u_j \in V$.

(E) $\varphi(s(e_i))\varphi(e_i) = \varphi(u_i)\varphi(e_i) = E_{ii}E_{i(i+1)} = E_{i(i+1)} = \varphi(e_i)$,
$\varphi(e_i)\varphi(t(e_i)) = \varphi(e_i)\varphi(u_{i+1}) = E_{i(i+1)}E_{(i+1)(i+1)} = E_{i(i+1)} = \varphi(e_i)$,
$\varphi(e_i^*)\varphi(s(e_i)) = \varphi(e_i^*)\varphi(u_i) = E_{(i+1)i}E_{ii} = E_{(i+1)i} = \varphi(e_i^*)$ and
$\varphi(t(e_i))\varphi(e_i^*) = \varphi(u_{i+1})\varphi(e_i^*) = E_{(i+1)(i+1)}E_{(i+1)i} = E_{(i+1)i} = \varphi(e_i^*)$ for all $e_i \in E$.

(CK1) $\varphi(e_i^*)\varphi(e_j) = E_{(i+1)i}E_{j(j+1)} = \delta_{i,j}E_{(i+1)(i+1)} = \delta_{i,j}\varphi(u_{i+1}) = \delta_{i,j}\varphi(t(e_i))$.
(CK2) $\varphi(e_ie_i^*) := \varphi(e_i)\varphi(e_i^*) = E_{i(i+1)}E_{(i+1)i} = E_{ii} = \varphi(u_i)$,
that is, φ preserves the Leavitt path algebra relations. Hence φ is a homomorphism. Note that $\{pq^* \mid t(p) = t(q) = u_n\}$ spans $L_{\mathbb{F}}(A_n)$ using Lemma 3.11 and (CK2) : $u_i = e_ie_i^*$ for all $1 \leq i \leq n$. Let $p_i := e_ie_{i+1}\dots e_{n-1}$ so $\varphi(p_ip_j^*) = E_{ij}$. Since the image of $\{pq^* \mid t(p) = t(q) = u_n\}$ is linearly independent $\{pq^* \mid t(p) = t(q) = u_n\}$ also is linearly independent. Therefore φ maps a basis of $L_{\mathbb{F}}(A_n)$ to a basis of $M_n(\mathbb{F})$ and thus φ is an isomorphism. $\square$

Example 32. Let's consider the Toeplitz digraph:

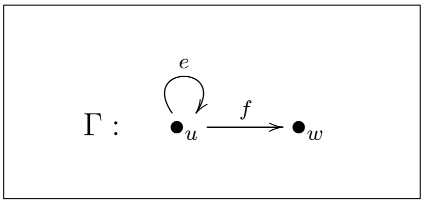

Figure 3.12: Toeplitz digraph

Then $L_{\mathbb{F}}(\Gamma) \cong \mathbb{F}\langle x, y\rangle/(1-xy)$, the Jacobson algebra, where $1 \leftrightarrow u+w$, $\quad y \leftrightarrow e+f$, $x \leftrightarrow e^* + f^*$, $\quad yx \leftrightarrow u$, $\quad 1 - yx \leftrightarrow w$, $\quad y^2x \leftrightarrow e \quad$ and $\quad y(1-yx) \leftrightarrow f$.

In the following theorem "cycle" means a geometric cycle, that is, $C = e_1e_2\dots e_n$ is considered equivalent to $e_2e_3\dots e_1$.

Theorem 3.15. [19] Let Γ be a finite digraph whose cycles have no exit. Then

$$L_{\mathbb{F}}(\Gamma) \cong \left(\bigoplus_{j=1}^{k} M_{m_j}(\mathbb{F})\right) \oplus \left(\bigoplus_{i=1}^{l} M_{n_i}(\mathbb{F}[x, x^{-1}])\right)$$

where k is the number of sinks in Γ, say $w_1, \cdots, w_k$, and m_j is the number of paths ending at the sink w_j for each $1 \leq j \leq k$;
l is the number of cycles in Γ, say $D_1, \ldots, D_l$, and n_i is the number of paths ending at $s(D_i)$ of the ith cycle D_i not containing D_i for each $1 \leq i \leq l$.

Example 33. Consider the following digraph:

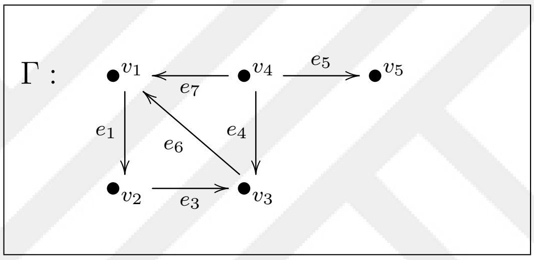

Figure 3.13: Digraph Γ

Then $L_{\mathbb{F}}(\Gamma) \cong M_5(\mathbb{F}[x, x^{-1}]) \oplus M_2(\mathbb{F})$.

Example 34. Consider the following digraph:

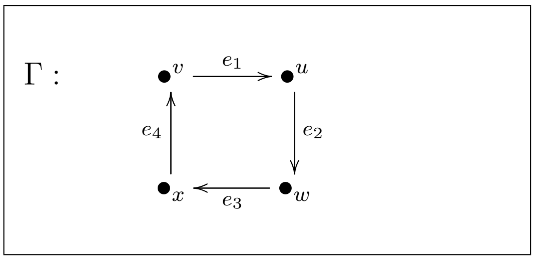

Figure 3.14: Digraph Γ

Then $L_{\mathbb{F}}(\Gamma) \cong M_4(\mathbb{F}[x, x^{-1}])$.

Example 35. Consider the digraphs Γ and Λ:

$$\Gamma: \quad \bullet^{u} \longrightarrow \bullet^{v} \longrightarrow \bullet^{w} \qquad\qquad \Lambda: \quad \bullet^{x} \longrightarrow \bullet^{y} \longleftarrow \bullet^{z}$$

Figure 3.15: Digraphs Γ and Λ

Then $L_{\mathbb{F}}(\Gamma) \cong M_3(\mathbb{F})$ and $L_{\mathbb{F}}(\Lambda) \cong M_3(\mathbb{F})$. Hence $L_{\mathbb{F}}(\Gamma) \cong L_{\mathbb{F}}(\Lambda)$ while $\Gamma \ncong \Lambda$.

Lemma 3.16. [13] Let Γ be a finite digraph. Then

$$L_{\mathbb{F}}(\Gamma) = \bigoplus_{v \in V} L_{\mathbb{F}}(\Gamma)v.$$

Lemma 3.17. [13] Let Γ be a finite digraph and $\mathbb{F}$ any field. If I is an ideal of $L_{\mathbb{F}}(\Gamma)$ then $I \cap V$ is hereditary and saturated.

Proof. If $v \in I \cap V$ and $e \in E$ with $s(e) = v$ then by (CK1) and (E) we have $t(e) = e^*e = e^*s(e)e = e^*ve \in I$ since $v \in I$ and I is an ideal of $L_{\mathbb{F}}(\Gamma)$. Hence $I \cap V$ is hereditary.

To show that $I \cap V$ is saturated, suppose that $v \in V$ is not a sink and $t(e) \in I$ for all $e \in s^{-1}(v)$. By (CK2) and (E), we get $v = \sum\limits_{s(e)=v} ee^* = \sum\limits_{s(e)=v} et(e)e^* \in I$. Hence $I \cap V$ is saturated. □

Fact: If φ is a graded homomorphism from $L_{\mathbb{F}}(\Gamma)$ then $Ker\varphi$ is graded.

Proof. If $x \in Ker\varphi$ then $\varphi(x) = 0$ and also $x = x_1 + ... + x_m$ where each x_i is a homogenous component of x because $L_{\mathbb{F}}(\Gamma)$ is graded. Since φ is a graded homomorphism $\varphi(x) = \sum\limits_{i=1}^{m} \varphi(x_i)$ where each $\varphi(x_i)$ is a homogeneous component of $\varphi(x)$. As $\varphi(x) = 0$, each homogeneous component $\varphi(x_i) = 0$. Hence each $x_i \in Ker\varphi$, that is, $Ker\varphi$ is a graded ideal. □

Lemma 3.18. [13] (i) If I is a nonzero graded ideal of $L_{\mathbb{F}}(\Gamma)$ then $I \cap V \neq \emptyset$.

(ii) A graded homomorphism φ from $L_{\mathbb{F}}(\Gamma)$ is 1-1 if and only if $V \cap Ker\varphi = \emptyset$.

Proof. (i) If I is a nonzero graded ideal of $L_{\mathbb{F}}(\Gamma)$ then there is $0 \neq x \in I$ and $x = \sum_{i=1}^{n} \lambda_i p_i q_i^*$ where $t(p_i) = t(q_i)$ and $\lambda_i \neq 0$ for all i. We may assume that all the monomials $p_i q_i^*$ have the same grade because I is a graded ideal. There is $v \in V$ such that $xv \neq 0$ since $0 \neq x = x \sum u$ where the sum is over $u \in \{s(q_i)\}_{i=1}^{n}$ and xv is also homogeneous because v is homogeneous. Now $0 \neq \sum_{j=1}^{m} \lambda_j p_j q_j^* \in I$ with $s(q_j) = v$ for $j = 1, 2, \cdots, m$ after deleting the terms with $q_i^* v = 0$.

Case 1: If v is sink then $q_j = v$, hence $0 \neq \sum_{j=1}^{m} \lambda_j p_j \in I$. Since $\sum_{j=1}^{m} \lambda_j p_j$ is homogeneous, the paths p_j are distinct and have the same length. Therefore $\lambda_1^{-1} p_1^* \sum_{j=1}^{m} \lambda_j p_j = v \in I \cap V$ by Lemma 3.11.

Case 2: If v is a nonsink then $v = \sum_{s(f)=v} ff^*$, hence $0 \neq (\sum_{j=1}^{m} \lambda_j p_j q_j^*)(\sum_{s(f)=v} ff^*)$. So there is $e \in E$ with $0 \neq (\sum_{j=1}^{m} \lambda_j p_j q_j^*)e = \sum \lambda_j p_j q_j^* e \in I$ and $\sum \lambda_j p_j q_j^* e$ is homogeneous. If $q_j^* e \neq 0$ and $q_j \notin V$ then $l(e^* q_j) < l(q_j)$. Repeating this process we end up with a nonzero homogeneous element $\sum \mu_k p_k \in I$. Hence Case 2 is reduced to Case 1 and we are done.

(ii) If φ a graded homomorphism from $L_{\mathbb{F}}(\Gamma)$ then by Lemma 3.2, $Ker\varphi$ must be graded. $Ker\varphi = \{0\}$ if and only if $V \cap Ker\varphi = \emptyset$ by (i). Thus φ is one-to-one if and only if $V \cap Ker\varphi = \emptyset$. □

Definition 3.19. [13] A digraph Γ satisfies Condition (L) if every cycle in Γ has an exit.

Theorem 3.20. [3] Let Γ be a finite digraph. Let I be a (two-sided) ideal of $L_{\mathbb{F}}(\Gamma)$. Then I is generated by the subset $I \cap V$ of V and elements of the form $s(C) + \sum_{k=1}^{m} \lambda_k C^k$ for some cycle C with $s(C) \notin I$ and $0 \neq \lambda_1, \lambda_2, \ldots, \lambda_m$ in $\mathbb{F}$ with $m > 0$.

Corollary 3.21. If I is a graded ideal of $L_{\mathbb{F}}(\Gamma)$ then $I = (I \cap V)$.

Proof. Let I be a graded ideal of $L_{\mathbb{F}}(\Gamma)$. By Theorem 3.20, the ideal I is generated by $I \cap V$ and elements of the form $s(C) + \sum_{k=1}^{m} \lambda_k C^k$ for some cycle C with $s(C) \notin I$ and $0 \neq \lambda_m, ..., \lambda_2, \lambda_1$ in $\mathbb{F}$ with $m > 0$. But I does not contain any element of the form $s(C) + \sum_{k=1}^{m} \lambda_k C^k$ for some cycle C with $s(C) \notin I$: If $s(C) + \sum_{k=1}^{m} \lambda_k C^k \in I$ then $s(C) \in I$ since I is a graded ideal, $|s(C)| = 0$ and $|\lambda_k C^k| > 0$ for $1 \leq k \leq m$. Hence $I = (I \cap V)$. □

Corollary 3.22. Let Γ be a finite digraph and $\mathbb{F}$ any field. If I is an ideal of $L_{\mathbb{F}}(\Gamma)$ then the largest graded ideal of $L_{\mathbb{F}}(\Gamma)$ contained in I is $(I \cap V)$.

Proof. Let I be an ideal of $L_{\mathbb{F}}(\Gamma)$. Suppose that J is a graded ideal contained in I. Then $J \cap V \subseteq I \cap V$ since $J \subseteq I$. Hence $(J \cap V) \subseteq (I \cap V) \subseteq I$. Also by Corollary 3.21 we get $J = (J \cap V)$ since J is graded ideal. Thus $J \subseteq (I \cap V)$ and $(I \cap V)$ is a graded ideal contained in I. Therefore every graded ideal contained in I is contained in $(I \cap V)$. Hence $(I \cap V)$ is the largest graded ideal contained in I. $\square$

Definition 3.23. [13] The quotient digraph Γ/H is the full subgraph on a hereditary saturated subset H of V obtained from Γ by deleting all vertices in H and all arrows touching vertices in H.

Example 36. When Γ is the Toeplitz digraph:

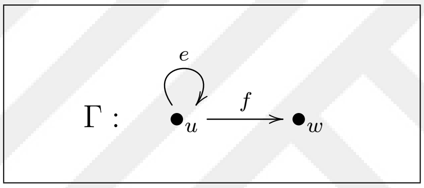

Figure 3.16: The Toeplitz digraph

$H = \{w\}$ is a hereditary and saturated subset of V. Then the quotient digraph Γ/H is :

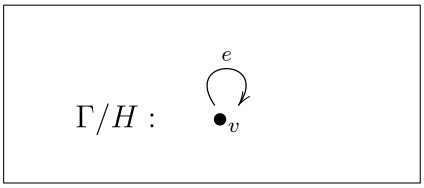

Figure 3.17: Quotient digraph Γ/H

Theorem 3.24 below, the classification of graded ideals, gives an order preserving one-to-one correspondence between the poset of graded ideals of a Leavitt path algebra and the poset of hereditary saturated subsets of vertices under inclusion.

Let Γ be a finite digraph and let $\mathbb{F}$ be any field. Denote by $\mathscr{L}_{gr}(L_{\mathbb{F}}(\Gamma))$ the poset of graded ideals of $L_{\mathbb{F}}(\Gamma)$ with order given by inclusion which becomes a lattice with join and meet being ideal sum and ideal intersection, respectively.

The set of hereditary saturated subsets of V will be denoted by $\mathscr{H}_{\Gamma}$ (or by $\mathscr{H}$ when Γ is clear from the context.)

$\mathscr{H}$ is a partially ordered set under inclusion, that is, $H \leq H'$ if $H \subseteq H'$. In fact, $\mathscr{H}$ becomes a lattice, with join $\vee$ defined as $H \vee H' := \overline{H \bigcup H'}$ and meet $\wedge$ being $H \wedge H' := H \bigcap H'$.

Theorem 3.24. [13] If Γ is a finite digraph and $\mathbb{F}$ is any field then the following maps φ and its inverse ψ define an order preserving one-to-one correspondence:

$$\varphi : \mathscr{L}_{gr}(L_{\mathbb{F}}(\Gamma)) \to \mathscr{H} \quad \text{via} \quad \varphi(I) = I \cap V$$

and

$$\psi : \mathscr{H} \to \mathscr{L}_{gr}(L_{\mathbb{F}}(\Gamma)) \quad \text{via} \quad \psi(H) = (H).$$

Theorem 3.25. [13] If Γ is a finite digraph and $\mathbb{F}$ is a field then

$$L_{\mathbb{F}}(\Gamma)/(H) \cong L_{\mathbb{F}}(\Gamma/H)$$

where (H) is the ideal generated by a hereditary saturated subset H of vertices and Γ/H is as above.

Proof. [18] We define a $*$-homomorphism from $L_{\mathbb{F}}(\Gamma/H)$ to $L_{\mathbb{F}}(\Gamma)/(H)$ by sending all vertices v to $v + (H)$ and all arrows e to $e + (H)$. The relations (V), (E) and (CK1) are satistied. If $v \notin H$ is not a sink in Γ then v is not a sink in Γ/H since H is saturated. In $L_{\mathbb{F}}(\Gamma)$

$$v = \sum_{s(e)=v,\, t(e)\notin H} ee^* + \sum_{s(f)=v,\, t(f)\in H} ff^*$$

by (CK2). Since the second sum is in (H) we see that (CK2) in $L_{\mathbb{F}}(\Gamma/H)$ is satisfied and thus we have a homomorphism.

We define a $*$-homomorphism from $L_{\mathbb{F}}(\Gamma)$ to $L_{\mathbb{F}}(\Gamma/H)$ by sending all vertices v to v if $v \notin H$ and to 0 otherwise, similarly e to e if e is an arrow in Γ/H and to 0 otherwise. The relations (V) and (E) are immediate to verify. Most cases of (CK1) are also easy to check. For the case of $e^*e = t(e)$, if $t(e) \notin H$ then $s(e) \notin H$ since H is hereditary, hence e is in Γ/H and the relation holds. When v is not a sink in Γ then v is not a sink in Γ/H, breaking up the right-hand side of (CK2) as above shows that the relations is satisfied. The ideal (H) is in kernel of this homomorphism, so we have the induced homomorphism from $L_{\mathbb{F}}(\Gamma)/(H)$ to $L_{\mathbb{F}}(\Gamma/H)$. The $*$-homomorphisms above are both $\mathbb{Z}$-graded and are inverses of each other. $\square$

For a cycle D with $s(D) = u$, we will use these notations:

$$D^0 := u \text{ and } D^{-l} := (D^*)^l \quad \text{for all} \quad l \in \mathbb{N}.$$

Definition 3.26. [13] Let Γ be a finite digraph and let $\mathbb{F}$ be any field. For every cycle D with $s(D) = u$ and every polynomial $g(x) = \sum_{i=m}^{n} \lambda_i x^i \in \mathbb{F}[x, x^{-1}]$ $(m \leq n;\, m, n \in \mathbb{Z})$, the element $\sum_{i=m}^{n} \lambda_i D^i$ of $L_{\mathbb{F}}(\Gamma)$ will be denoted by $g(D)$.

Lemma 3.27. [13] Let Γ be a finite digraph and $\mathbb{F}$ any field. If D is a cycle with no exit then

$$uL_{\mathbb{F}}(\Gamma)u = \left\{ \sum_{i=m}^{n} \lambda_i D^i \;\middle|\; \lambda_i \in \mathbb{F},\;\; m \leq n,\;\; m,\, n \in \mathbb{Z} \right\} \cong \mathbb{F}[x, x^{-1}]$$

where $s(D) := u \leftrightarrow 1,\;\; D \leftrightarrow x \;$ and $\; D^* \leftrightarrow x^{-1}$.

Proof. Let $D = e_1 \cdots e_n$. First of all we show that each $p \in Path(\Gamma)$ with $s(p) = u$ is of the form $D^m r_j$ where $m \in \mathbb{Z}_+$, $r_j = e_1 \cdots e_j$ for $1 \leq j < n$, $r_0 = u$ and $|p| = mn + j$. Let's use induction on $|p|$: If $|p| = 1$ and $s(p) = s(e_1)$ then $p = e_1$ since the cycle D has no exit. Now suppose that the result holds for any $q \in Path(\Gamma)$ with $s(q) = u$, $|q| \leq sn + t$ and consider any $p \in Path(\Gamma)$ with $s(p) = u$ and $|p| = sn + t + 1$. We can write $p = p'f$ with $p' \in Path(\Gamma)$, $s(p') = u$, $f \in E$ and $|p'| = sn + t$, so by the induction hypothesis $p' = D^s e_1 \cdots e_t$. Since D has no exits, $s(f) = t(e_t) = s(e_{t+1})$ implies $f = e_{t+1}$. Thus $p = p'e_{t+1} = D^s e_1 \cdots e_{t+1}$.

Now let $p,\, q \in Path(\Gamma)$ with $s(p) = s(q) = u$.

(i) If $|p| = |q|$ and $pq^* \neq 0$ then we have $pq^* = D^l e_1 \cdots e_k e_k^* \cdots e_1^* D^{-l} = u$ since D has no exit.

(ii) If $|p| > |q|$ and $pq^* \neq 0$ then $pq^* = D^{d+j} e_1 \cdots e_k e_k^* \cdots e_1^* D^{-j} = D^d$ with $d \in \mathbb{N}$.

(iii) If $|p| < |q|$ and $pq^* \neq 0$ then $pq^* = D^j e_1 \cdots e_k e_k^* \cdots e_1^* D^{-j-d} = D^{-d}$ with $d \in \mathbb{N}$.

Let $\alpha \in uL_{\mathbb{F}}(\Gamma)u$. Then we write $\alpha = \lambda u + \sum_{i=1}^{l} \lambda_i p_i q_i^*$ where $\lambda_i,\ \lambda \in \mathbb{F}$ and $p_i, q_i\ Path(\Gamma)$ such that $s(p_i) = s(q_i) = u$ for all $1 \leq i \leq l$. Thus we get $\alpha = \sum_{i=0}^{l} \lambda_i D^{m_i}$ where $|p_i q_i^*| = m_i n$ for some $m_i \in \mathbb{Z}$.

Now we define $\varphi : \mathbb{F}[x, x^{-1}] \to L_{\mathbb{F}}(\Gamma)$ by setting $\varphi(1) = u,\ \varphi(x) = D$ and $\varphi(x^{-1}) = D^*$. Hence φ is an $\mathbb{F}$-algebra monomorphism with image $uL_{\mathbb{F}}(\Gamma)u$. Thus $uL_{\mathbb{F}}(\Gamma)u$ is isomorphic to the Laurent polynomial algebra $\mathbb{F}[x, x^{-1}]$. □

Definition 3.28. [5] A cycle C is maximal if no other cycle leads to C (in particular, a maximal cycle is disjoint from all other cycles). A sink w is maximal if there is no cycle C which leads to w.

Example 37. Consider the cycles $C_1 = e_9 e_{10} e_{11}$, $C_2 = e_5 e_3 e_6$, $C_3 = e_4$ and the sinks v_1 and v_6 in the following digraph:

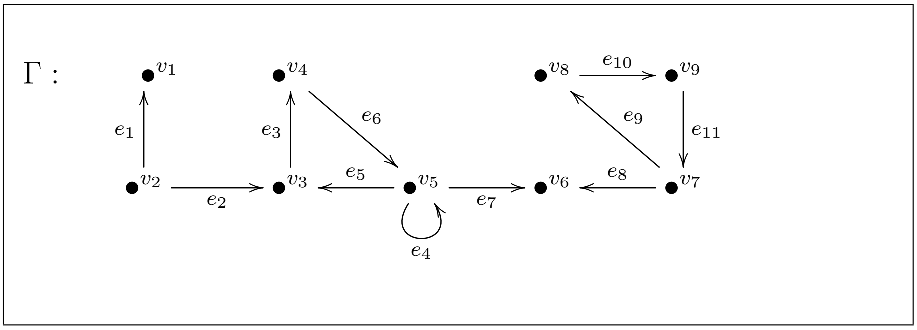

Figure 3.18: Digraph Γ

The cycle C_1 is maximal in Γ but C_2 and C_3 are not maximal since C_2 leads to C_3 and C_3 leads to C_2. The sink v_1 is maximal but v_6 is not maximal since C_1 leads to v_7.

Definition 3.29. [5] If M is a right $L_{\mathbb{F}}(\Gamma)$-module then the support subgraph of M, denoted by Γ_M, is the full subgraph of Γ on $V_M := \{v \in V \mid Mv \neq 0\}$, the support of M.

Example 38. If I is an ideal of $L_{\mathbb{F}}(\Gamma)$ then $V \setminus V_{L_{\mathbb{F}}(\Gamma)/I} = I \cap V$.

Lemma 3.30. [5] If M is a finite dimensional $L_{\mathbb{F}}(\Gamma)$-module then the cycles of Γ_M, the support subgraph of M, have no exit. Hence V_M is contained in $\bigcup V_{\rightsquigarrow v}$ where v is either a maximal sink or the source of a maximal cycle.

Theorem 3.31. [5] Let Γ be a finite digraph. Γ has a maximal sink or cycle if and only if $L_{\mathbb{F}}(\Gamma)$ has a nonzero finite dimensional quotient.

Example 39. Let's consider the following digraphs R_n and Γ:

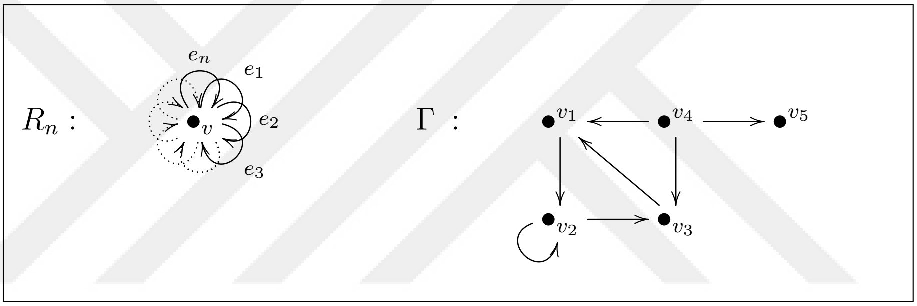

Figure 3.19: Digraphs R_n and Γ

Then $L_{\mathbb{F}}(R_n)$ has no nonzero finite dimensional quotient since R_n has no maximal sink or cycle. $L_{\mathbb{F}}(\Gamma)$ has a nonzero finite dimensional quotient since the vertex v_5 is a maximal sink in Γ.

A quiver representation ρ assigns a (possibly infinite dimensional) vector space $\rho(v)$ to each vertex v and a linear transformation $\rho(e) : \rho(s(e)) \to \rho(t(e))$ to each arrow e. A morphism of quiver representatitons $\varphi : \rho \to \sigma$ is a family of linear transformations $\{\varphi_v : \rho(v) \to \sigma(v)\}_{v \in V}$ such that $\forall e \in E$ the diagram

$$\begin{tikzcd} \rho(s(e)) \arrow[r, "\rho(e)"] \arrow[d, "\varphi_{s(e)}"] & \rho(t(e)) \arrow[d, "\varphi_{t(e)}"] \\ \sigma(s(e)) \arrow[r, "\sigma(e)"] & \sigma(t(e)) \end{tikzcd}$$

commutes. The category of quiver representations of Γ is the category of functors from the path category of the digraph Γ (whose objects are the vertices V and the morphisms are the paths in Γ) to the category of $\mathbb{F}$-vector spaces. [15]

We may view the category $Mod_{L_{\mathbb{F}}(\Gamma)}$ as a subcategory of $Mod_{\mathbb{F}\Gamma}$, the category of modules over the path algebra $\mathbb{F}\Gamma$, or equivalently, the category of quiver representations of Γ by using the following theorem:

Theorem 3.32. [5] If $\Gamma = (V, E, s, t)$ is a finite digraph then $Mod_{L_{\mathbb{F}}(\Gamma)}$, the category of right $L_{\mathbb{F}}(\Gamma)$-modules, is equivalent to the full subcategory of quiver representations ρ of Γ satisfying the following condition (I):
For every nonsink $v \in V$,

$$\bigoplus_{s(e)=v} \rho(e) : \rho(v) \longrightarrow \bigoplus_{s(e)=v} \rho(t(e))$$

is an isomorphism.

The right $L_{\mathbb{F}}(\Gamma)$-module corresponding to the quiver representation ρ is $M = \bigoplus_{v\in V} \rho(v)$ as an $\mathbb{F}$-module and the actions of generators of $L_{\mathbb{F}}(\Gamma)$ are given by the compositions

$$u : M = \bigoplus_{v\in V} \rho(v) \xrightarrow{pr_{\rho(u)}} \rho(u) \hookrightarrow \bigoplus_{v\in V} \rho(v) = M$$

$$e : M = \bigoplus_{v\in V} \rho(v) \xrightarrow{pr_{s(e)}} \rho(s(e)) \xrightarrow{\rho(e)} \rho(t(e)) \hookrightarrow \bigoplus \rho(v) = M$$

$$e^* : M = \bigoplus_{v\in V} \rho(v) \xrightarrow{pr_{t(e)}} \rho(t(e)) \hookrightarrow \bigoplus_{s(f)=s(e)} \rho(t(f)) \xrightarrow{(\bigoplus \rho(f))^{-1}} \rho(s(e)) \hookrightarrow \bigoplus_{v\in V} \rho(v) = M$$

We denote by $\mathbb{F}X$, the free $\mathbb{F}$-module with basis X.

Example 40. Let $\rho(v) = \mathbb{F}\mathbb{Z}$ for all $v \in V$. Pick an isomorphism

$$\varphi_v : \rho(v) \longrightarrow \bigoplus_{s(e)=v} \rho(t(e))$$

for each non-sink $v \in V$. Let $\rho(e)$ be the composition

$$\rho(s(e)) \xrightarrow{\varphi_{s(e)}} \bigoplus_{s(f)=s(e)} \rho(t(f)) \xrightarrow{pr_e} \rho(t(e))$$

and $\rho(e^*)$ be the composition

$$\rho(t(e)) \overset{\varphi_{s(e)}}{\hookrightarrow} \bigoplus_{s(f)=s(e)} \rho(t(f)) \xrightarrow{\varphi_e^{-1}} \rho(s(e))$$

Since condition (I) is satisfied by construction, we have an $L_{\mathbb{F}}(\Gamma)$-module.

We may think of the path algebra $\mathbb{F}\Gamma$ as a subalgebra of the Leavitt path algebra $L_{\mathbb{F}}(\Gamma)$ since the standard homomorphism, mapping $V \sqcup E$ to themselves, from $\mathbb{F}\Gamma$ to $L_{\mathbb{F}}(\Gamma)$ is injective [20].

Lemma 3.33. [18] If p and q are in $Path(\Gamma)$ with $t(p) = t(q)$ then $pq^* \neq 0$ in $L_{\mathbb{F}}(\Gamma)$.

It follows from Lemma 3.33 that every vertex v in $L_{\mathbb{F}}(\Gamma)$ is nonzero. More generally all paths and all dual paths are nonzero in $L_{\mathbb{F}}(\Gamma)$.

Remark 3.34. The paragraph above and the last lemma are consequences of the representation in an Example 40 with $Mv = \mathbb{F}\mathbb{Z}$ for all $v \in V$. In [5] this is given as an application of Theorem 3.32 realizing $L_{\mathbb{F}}(\Gamma)$ representations as quiver representations, but it can also be proved independently of that theorem.

3.3. Finite Codimensional Maximal Ideals of LPAs

This section contains the main results of this thesis classifying all finite codimensional maximal ideals of Leavitt Path algebras (of finite digraphs).

Definition 3.35. [21] An ideal I of an $\mathbb{F}$-algebra A is finite codimensional if A/I is finite dimensional as an $\mathbb{F}$-vector space.

Theorem 3.36. [22] If Γ is a finite digraph then every ideal of $L_{\mathbb{F}}(\Gamma)$ is principal.

Proof. By Theorem 3.20, canonical generators for I are $H := I \cap V$ and $\{f_i(C_i)\}_{i=1}^n$ with C_i distinct cycles in Γ and $f_i(x) \in \mathbb{F}[x]$ with $f_i(0) = 1 \neq f_i(x)$ for $i = 1, 2, \ldots, n$. Both H and $\{f_i(x)\}_{i=1}^n$ are finite since Γ is finite. We claim that

$$I = \big(\sum_{v \in H} v + \sum_{i=1}^{n} f_i(C_i)\big).$$

We have

$$s(C_j)\big[\sum_{v \in H} v + \sum_{i=1}^{n} f_i(C_i)\big] = f_j(C_j) \quad \text{for } j = 1, 2, \ldots, n$$

and

$$u\big[\sum_{v \in H} v + \sum_{i=1}^{n} f_i(C_i)\big] = u \quad \text{for all } u \in H$$

showing that all the generators of I are in the ideal generated by $\sum\limits_{v\in H} v + \sum\limits_{i=1}^{n} f_i(C_i)$. Clearly $\sum\limits_{v\in H} v + \sum\limits_{i=1}^{n} f_i(C_i)$ is contained in I, therefore $I = \big(\sum\limits_{v\in H} v + \sum\limits_{i=1}^{n} f_i(C_i)\big)$. $\square$

Theorem 3.37. If R and S are rings then maximal ideals of $R \times S$ are either of the form $I \times S$ or $R \times I$ where I is a maximal ideal of R or S.

Proof. If J is an ideal of $R\times S$ then $R\times S \longrightarrow R/(R\cap J)\times S/(S\cap J)$ is an epimorphism where $(r, s) \mapsto (r + R \cap J,\ s + S \cap J)$. Here R is identified with $\{(r, 0)\}_{r\in R} \subseteq R \times S$ and similarly S is identified with $\{(0, s)\}_{s\in S} \subseteq R \times S$, hence we may identify $R\cap J$ and $S \cap J$ with ideals of R and S, respectively. The kernel of this epimorphism is J. Hence the induced homomorphism $(R\times S)/J \longrightarrow R/(R\cap J)\times S/(S\cap J)$ is an isomorphism. We have a one-to-one correspondence between the ideals of $R \times S$ containing J and the ideals of $R/(R \cap J) \times S/(S \cap J)$ via this isomorphism. If J is a maximal ideal of $R \times S$ and $R \cap J \neq R$ then the projection $R/(R \cap J) \times S/(S \cap J) \longrightarrow R/(R \cap J)$ must be an isomorphism (otherwise its kernel would be a proper non-zero ideal). Hence $S \cap J = S$ and $I := R \cap J$ is a maximal ideal. Similarly, if $S \cap J \neq S$ we conclude that the maximal ideal J is of the form $(R \cap J) \times S$. $\square$

Corollary 3.38. If $R_1, R_2, \ldots, R_n$ are rings then maximal ideals of $R_1 \times R_2 \times \cdots \times R_n$ are of the form $I_1 \times R_2 \times \cdots \times R_n$ or $R_1 \times I_2 \times R_3 \times \cdots \times R_n$ or $\ldots$ or $R_1 \times R_2 \times \cdots \times R_{n-1} \times I_n$ where I_i is a maximal ideal of R_i for $i = 1, 2, \ldots, n$.

Proof. This is a standard proof by mathematical induction. The case $n = 1$ is clear. For $n > 1$ we have $R_1 \times R_2 \times \cdots \times R_n = R_1 \times S$ where $S = R_2 \times \cdots \times R_n$. By Theorem 3.37 a maximal ideal of $R \times S$ is of the form $I_1 \times S = I_1 \times R_2 \times \cdots \times R_n$ where I_1 is a maximal ideal of R_1 or $R_1 \times I$ where I is a maximal ideal of $R_2 \times \cdots \times R_n$. Now the induction hypothesis yields the desired conclusion. $\square$

Theorem 3.39. [22] If Γ is finite digraph and I is finite codimensional maximal ideal of $L_{\mathbb{F}}(\Gamma)$ then $L_{\mathbb{F}}(\Gamma)/I \cong M_n(\mathbb{E})$ where $\mathbb{E}$ is a finite field extension of $\mathbb{F}$.

1. If I is graded then $\mathbb{E} = \mathbb{F}$ and n is the number of paths in Γ ending at w for some maximal sink w. Conversely, every maximal sink w defines a finite codimensional maximal graded ideal $(V \setminus V_{\rightsquigarrow w})$. Hence there is a bijection between maximal sinks of Γ and finite codimensional maximal graded ideals of $L_{\mathbb{F}}(\Gamma)$.

2. If I is not graded then $\mathbb{E} = \mathbb{F}[x]/(f(x))$ and n is the number of paths p with $t(p) = s(C)$ but $p \neq qC$ in Γ for some maximal cycle C. Conversely, every pair consisting of a maximal cycle C and an irreducible polynomial $f(x)$ with $f(0) = 1 \neq f(x)$ defines a finite codimensional maximal ideal $(\{V \setminus V_{\rightsquigarrow s(C)}\} \cup \{f(C)\})$. Hence there is a bijection between pairs $\{C, f(x)\}$ and finite codimensional maximal ideals of $L_{\mathbb{F}}(\Gamma)$ that are not graded.

Proof. Since $L_{\mathbb{F}}(\Gamma)/I$ is finite dimensional, its support is contained in $\bigcup_{v \in X} V_{\rightsquigarrow v}$ where X is the set of maximal sinks and sources of maximal cycles in Γ by Lemma 3.30. Hence $H := V \setminus \bigcup_{v \in X} V_{\rightsquigarrow v} \subseteq I$. Since H is hereditary and saturated, we have $L_{\mathbb{F}}(\Gamma)/(H) \cong L_{\mathbb{F}}(\Gamma/H)$ by Theorem 3.25. The cycles in Γ/H have no exit. Also,

$$L_{\mathbb{F}}(\Gamma)/(H) \cong L_{\mathbb{F}}(\Gamma/H) \cong \left(\bigoplus_{j=1}^{k} M_{m_j}(\mathbb{F})\right) \oplus \left(\bigoplus_{i=1}^{l} M_{n_i}(\mathbb{F}[x, x^{-1}])\right) \cong \bigoplus_{v \in X} L_{\mathbb{F}}(\Gamma_{\rightsquigarrow v})$$

by Theorem 3.15 where each summand $M_{m_j}(\mathbb{F})$ corresponds to a maximal sink and each summand $M_{n_i}(\mathbb{F}[x, x^{-1}])$ corresponds to a maximal cycle. The maximal ideal I of $L_{\mathbb{F}}(\Gamma)$ corresponds to a maximal ideal J of $\bigoplus_{v \in X} L_{\mathbb{F}}(\Gamma_{\rightsquigarrow v})$ since $(H) \subseteq I$. By Corollary 3.38 the ideal J contains all but one summand, say $L_{\mathbb{F}}(\Gamma_{\rightsquigarrow w})$, of $\bigoplus_{v \in X} L_{\mathbb{F}}(\Gamma_{\rightsquigarrow v})$ and the intersection of J with that summand is a maximal ideal K of that summand. We get an isomorphism between $L_{\mathbb{F}}(\Gamma)/I$ and $L_{\mathbb{F}}(\Gamma_{\rightsquigarrow w})/K$.

Now there are two cases:

(i) If $w = s(C)$ for some maximal cycle C in Γ then $L_{\mathbb{F}}(\Gamma_{\rightsquigarrow w}) \cong M_n(\mathbb{F}[x, x^{-1}])$ and K corresponds to a maximal ideal K' of $M_n(\mathbb{F}[x, x^{-1}])$. Then $M_n(\mathbb{F}[x, x^{-1}])/K' \cong M_n(\mathbb{E})$ where $f(x) \in \mathbb{F}[x]$ is an irreducible polynomial with $f(0) = 1 \neq f(x)$ and $\mathbb{E} := \mathbb{F}[x, x^{-1}]/(f(x)) \cong \mathbb{F}[x]/(f(x))$. Now $I = (\{V \setminus V_{\rightsquigarrow w}\} \cup \{f(C)\})$. The ideal I is not graded because $f(C) = w + \sum_{i=1}^{m} \lambda_i C^i \in I$ but $w \notin I$ since $f(x) \neq 1$ so $0 \neq \sum_{i=1}^{m} \lambda_i C^i \in \mathbb{F}\Gamma \subseteq L_{\mathbb{F}}(\Gamma)$.

Conversely, each maximal cycle C and each irreducible $f(x) \in \mathbb{F}[x]$ with $f(0) = 1 \neq f(x)$ determines the ideal $I := (\{V \setminus V_{\rightsquigarrow s(C)}\} \cup \{f(C)\})$ which is maximal because $L_{\mathbb{F}}(\Gamma)/I \cong M_n(\mathbb{E})$ is a simple ring. Again $\mathbb{E} \cong \mathbb{F}[x]/(f(x))$ and n is the number of paths p in Γ with $t(p) = s(C)$ and $p \neq qC$ for any $q \in Path(\Gamma)$ by Theorem 3.15.

(ii) If w is a maximal sink then $L_{\mathbb{F}}(\Gamma_{\rightsquigarrow w}) \cong M_n(\mathbb{F})$, a simple ring, hence $K = 0$.

Therefore $I = (V_{\rightsquigarrow w})$ is a graded ideal, n is the number of paths ending at w. Conversely, for any maximal sink w the subgraph $\Gamma_{\rightsquigarrow w}$ is acyclic and contains only one sink. Choosing $I = (V \setminus V_{\rightsquigarrow w})$ we have $L_{\mathbb{F}}(\Gamma)/I \cong M_n(\mathbb{F})$ where n is as above. That is, there is one-to-one correspondence between graded maximal ideals of $L_{\mathbb{F}}(\Gamma)$ and maximal sinks in Γ.

□

Corollary 3.40. [22] If I is a finite codimensional maximal ideal of $L_{\mathbb{F}}(\Gamma)$ then $L_{\mathbb{F}}(\Gamma)/I \cong L_{\mathbb{E}}(A_n)$ where $\mathbb{E}$ is a finite field extension of $\mathbb{F}$ and A_n is

$$A_n : \quad \bullet_{u_1} \xrightarrow{e_1} \bullet_{u_2} \xrightarrow{e_2} \bullet_{u_3} \cdots\cdots\rightarrow \bullet_{u_{n-1}} \xrightarrow{e_{n-1}} \bullet_{u_n}$$

Figure 3.20: Digraph A_n

We finish this section with an important example of a digraph Γ, namely the Toeplitz digraph, which contains a cycle with an exit (so it is not covered by Theorem 3.15). Nevertheless we can determine all ideals of $L_{\mathbb{F}}(\Gamma)$.

Example 41. Let's consider all ideals of $L_{\mathbb{F}}(\Gamma)$ when Γ is the Toeplitz digraph:

$$\Gamma : \quad \begin{tikzcd} \bullet_u \arrow[loop left, "e"] \arrow[r, "f"] & \bullet_w \end{tikzcd}$$

Figure 3.21: Toeplitz digraph

There are 3 hereditary saturated subsets: $\emptyset$, $\{w\}$ and $\{u, w\}$. If $\{u, w\} \subset I$ then the ideal $I = L_{\mathbb{F}}(\Gamma)$. Note that (w) is the unique minimal ideal of $L_{\mathbb{F}}(\Gamma)$: If $s(e) + \sum_{i=1}^{m} \lambda_i e^i \in I$

as in Çolak's theorem then

$$f^*[s(e)+\sum_{i=1}^{m}\lambda_i e^i]f \overset{(CK1)}{=} f^*s(e)f = f^*uf \overset{(E)}{=} f^*f \overset{(CK1)}{=} t(f) = w \in I.$$

Hence if $0 \neq I$ then $w \in I$. When $I = (w)$ then $L_{\mathbb{F}}(\Gamma)/I \cong L_{\mathbb{F}}(\Gamma/\{w\}) \cong \mathbb{F}[x, x^{-1}]$ by Lemma 3.12. Thus (w) is neither finite codimensional nor maximal. If $I = (w, s(e)+\sum_{i=1}^{m}\lambda_i e^i)$ with $\lambda_m \neq 0$ then $L_{\mathbb{F}}(\Gamma)/I \cong L_{\mathbb{F}}(\Gamma/\{w\})/(s(e)+\sum_{i=1}^{m}\lambda_i e^i) \cong \mathbb{F}[x, x^{-1}]/(1+\sum_{i=1}^{m}\lambda_i x^i) \cong \mathbb{F}[x]/(1+\sum_{i=1}^{m}\lambda_i x^i)$. Therefore I is finite codimensional since $dim^{\mathbb{F}}\big(\mathbb{F}[x]/(1+\sum_{i=1}^{m}\lambda_i x^i)\big) = m$. Also, I is maximal if and only if $1+\sum_{i=1}^{m}\lambda_i x^i$ is an irreducible polynomial. The only remaining ideal of $L_{\mathbb{F}}(\Gamma)$ is (0) generated by hereditary saturated subset $\emptyset$ which is neither maximal nor finite codimensional.

4. DICTIONARY

Constructing a dictionary between combinatorial/geometric properties of the (finite) digraph of Γ and algebraic properties of $L_{\mathbb{F}}(\Gamma)$ has been a major endeavour in the Leavitt path algebra literature. We summarize most of these results in the table below.

Table 4.1: Dictionary

Finite Digraph Γ	**Leavitt Path Algebra** $L_{\mathbb{F}}(\Gamma)$
Γ acyclic [23]	$L_{\mathbb{F}}(\Gamma)$ finite dimensional $\Updownarrow$ $L_{\mathbb{F}}(\Gamma) \cong \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{F})$ k: the number of sinks in Γ n_i: the number of paths ending at ith sink $\Updownarrow$ $L_{\mathbb{F}}(\Gamma)$ left Artinian $\Updownarrow$ $L_{\mathbb{F}}(\Gamma)$ right Artinian $\Updownarrow$ $L_{\mathbb{F}}(\Gamma)$ semisimple
Cycles in Γ have no exit [23] & [19]	$L_{\mathbb{F}}(\Gamma)$ locally finite dimensional $\Updownarrow$ $L_{\mathbb{F}}(\Gamma) \cong \left(\bigoplus_{i=1}^{l} M_{n_i}(\mathbb{F}[x, x^{-1}])\right) \oplus \left(\bigoplus_{j=1}^{k} M_{m_j}(\mathbb{F})\right)$ l: number of cycles in Γ n_i: number of paths ending at $s(C_i)$ of the ith cycle C_i not containing C_i k: number of sinks in Γ m_j: number of paths ending at the jth sink $\Updownarrow$ $L_{\mathbb{F}}(\Gamma)$ left Noetherian $\Updownarrow$ $L_{\mathbb{F}}(\Gamma)$ right Noetherian

Finite Digraph Γ	**Leavitt Path Algebra** $L_{\mathbb{F}}(\Gamma)$
Cycles in Γ have no exit continued [24]	$L_{\mathbb{F}}(\Gamma)$ a principal ideal ring (All left or right ideals are principal)
Cycles in Γ pairwise disjoint [25]	$L_{\mathbb{F}}(\Gamma)$ has polynomial growth
Γ has intersecting cycles [18]	$L_{\mathbb{F}}(\Gamma)$ has exponential growth
Γ has a maximal sink/cycle [5]	$L_{\mathbb{F}}(\Gamma)$ has a nonzero finite dimensional quotient
Each cycle in Γ has an exit and $\mathcal{H} = \{\emptyset, V\}$ [2]	$L_{\mathbb{F}}(\Gamma)$ simple

A finitely generated $\mathbb{F}$-algebra A is said to have polynomial growth if there is a finite dimensional subspace W of A containing 1 such that $A = \bigcup_{n=1}^{\infty} W^n$ and $dim^{\mathbb{F}}(W^n) \leq n^k$ for some positive integer k. The algebra A has exponential growth if $dim^{\mathbb{F}}(W^n) \geq b^n$ for some $b > 1$ with W as above. There are finitely generated algebras of intermediate growth, having neither polynomial nor exponential growth. [26]

However, the Leavitt path algebra $L_{\mathbb{F}}(\Gamma)$ of a finite digraph Γ can not have intermediate growth as implied by the table above.

There are several clear implications between some of the properties stated in the table above. For instance, if Γ is acyclic then it is vacuously true that cycles in Γ have no exit or if cycles in Γ have no exit then they are pairwise disjoint. An implication that is not so obvious is: *If* $L_{\mathbb{F}}(\Gamma)$ *has polynomial growth then* $L_{\mathbb{F}}(\Gamma)$ *has a nonzero finite dimensional quotient.* [5]

The digraph Γ below shows that the converse of this fact is not correct:

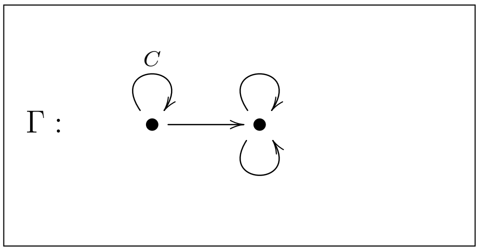

Figure 4.1: Digraph Γ

Γ has a maximal cycle, namely the loop C. Hence $L_{\mathbb{F}}(\Gamma)$ has a nonzero finite dimensional quotient. However Γ has intersecting cycles, so $L_{\mathbb{F}}(\Gamma)$ has exponential, not polynomial growth. [5]

5. CONCLUSIONS

The main objective of this thesis is to synthesize the theorems of Çolak [3] and Rangaswamy [4] on the structure of two sided ideals of Leavitt path algebras with the theorems of Koç-Özaydın [5] on finite dimensional quotients of Leavitt path algebras. In particular, we have:

Theorem 3.39 If Γ is finite digraph and I is finite codimensional maximal ideal of $L_{\mathbb{F}}(\Gamma)$ then $L_{\mathbb{F}}(\Gamma)/I \cong M_n(\mathbb{E})$ where $\mathbb{E}$ is a finite field extension of $\mathbb{F}$.

1. If I is graded then $\mathbb{E} = \mathbb{F}$ and n is the number of paths in Γ ending at w for some maximal sink w. Conversely, every maximal sink w defines a finite codimensional maximal graded ideal $(V \setminus V_{\rightsquigarrow w})$. Hence there is a bijection between maximal sinks of Γ and finite codimensional maximal graded ideals of $L_{\mathbb{F}}(\Gamma)$.
2. If I is not graded then $\mathbb{E} = \mathbb{F}[x]/(f(x))$ and n is the number of paths p with $t(p) = s(C)$ but $p \neq qC$ in Γ for some maximal cycle C. Conversely, every pair consisting of a maximal cycle C and an irreducible polynomial $f(x)$ with $f(0) = 1 \neq f(x)$ defines a finite codimensional maximal ideal $(\{V \setminus V_{\rightsquigarrow sC}\} \cup \{f(C)\})$. Hence there is a bijection between pairs $\{C, f(x)\}$ and finite codimensional maximal ideals of $L_{\mathbb{F}}(\Gamma)$ that are not graded.

Corollary 3.40 If I is a finite codimensional maximal ideal of $L_{\mathbb{F}}(\Gamma)$ then $L_{\mathbb{F}}(\Gamma)/I \cong L_{\mathbb{E}}(A_n)$ where $\mathbb{E}$ is a finite field extension of $\mathbb{F}$ and A_n is:

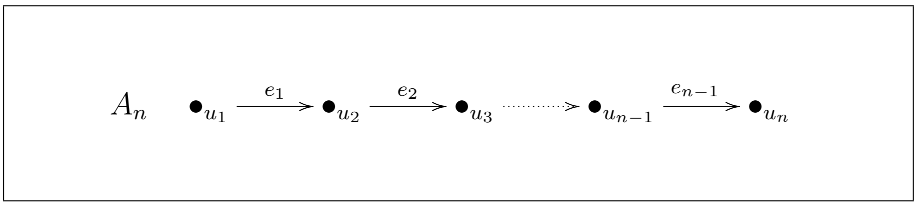

Figure 5.1: Digraph A_n

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Melike Kamar graduated from Çanakkale Milli Piyango Anatolian High School in 2007 and graduated from Marmara University, Faculty of Science, Mathematics Department in 2011. She started her master's degree as a student of Gebze Technical University, Mathematics Department. She has been working as a teacher at the Ministry of Education since 2013.

PUBLICATIONS AND PRESENTATIONS FROM THE THESIS

[1] Kamar M., Koç A., (2024) ”Finite Codimensional Maximal Ideals of Leavitt Path Algebras” Workshop on Nonlinear Systems IV, 02-04 January, Gebze Technical University, Türkiye.