ABANT IZZET BAYSAL UNIVERSITY

THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

NONLINEAR BERNSTEIN TYPE OPERATORS AND ITS APPROXIMATION PROPERTIES

DOCTOR OF PHILOSOPHY

HÜSEYİN ERHAN ALTIN

BOLU, MAY 2016

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APPROVAL OF THE THESIS

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To my family, my wife and my supervisor...

DECLARATION

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

HÜSEYİN ERHAN ALTIN

ABSTRACT

NONLINEAR BERNSTEIN TYPE OPERATORS AND ITS APPROXIMATION PROPERTIES
PH.D. THESIS
HÜSEYİN ERHAN ALTIN
ABANT IZZET BAYSAL UNIVERSITY GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS
(SUPERVISOR: PROF. DR. HARUN KARSLI)

BOLU, MAY 2016

This thesis deals with approximation properties of certain nonlinear Bernstein type operators in the approximation theory. This thesis consists of four chapters. The first chapter is devoted to the introduction. The second chapter contains concepts, definitions and also some important inequalities which are needed in the further chapters. In the third chapter, after giving some information about the nonlinear singular integral operators, existence theorem for these operators are investigated. Moreover, Fatou type convergence properties of the derivatives of these operators are studied. Then, convergence of the nonlinear Bernstein type operators are examined and using the modulus of continuity, the rate of convergence of these operators are investigated. Furthermore, by means of this operators, the pointwise convergence on some specified points are studied. Then, their pointwise convergence to some functions which are derivatives of bounded variation are observed. Finally, a Voronovskaya type formula is provided for nonlinear Bernstein type operators. In the last chapter, conclusion of the thesis and some recommendations for further studies can be found.

KEYWORDS: Singular Integral, Fatou Type Convergence, Modulus of Continuity, Rate of Convergence, Nonlinear Bernstein Type Operators, Bounded Variation, Voronovskaya Type Formula

ÖZET

LİNEER OLMAYAN BERNSTEIN TİPİ OPERATÖRLER VE ONLARIN YAKLAŞIM ÖZELLİKLERİ

DOKTORA TEZİ

HÜSEYİN ERHAN ALTIN
ABANT İZZET BAYSAL ÜNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ
MATEMATİK ANABİLİM DALI
(TEZ DANIŞMANI: PROF. DR. HARUN KARSLI)

BOLU, MAYIS 2016

Bu tez, yaklaşım teorisindeki nonlinear Bernstein tipli operatörlerin yaklaşım özellikleri ile ilgili bir çalışmadır. Bu tez dört bölümden oluşmaktadır. Birinci bölüm giriş kısmına ayrılmıştır. İkinci bölümde ileri bölümlerde gerekli olan kavramlar, tanımlar ve bazı önemli eşitsizlikler verilmiştir. Üçüncü bölümde, lineer olmayan singüler integral operatörler ile ilgili bazı bilgiler verildikten sonra bu operatörler için varlık teoremi elde edilmiştir. Ayrıca bu operatörlerin türevlerinin Fatou tipli yakınsaklık özellikleri araştırılmıştır. Daha sonra lineer olmayan Bernstein tipli operatörlerin yakınsaklıkları incelenmiş ve süreklilik modülü kullanılarak, bu operatörler için yakınsaklık hızları araştırılmıştır. Ayrıca bu operatörler yardımıyla bazı özel noktalardaki noktasal yakınsaklık çalışılmıştır. Bunlara ek olarak, türevleri sınırlı salınımlı olan bazı fonksiyonlara bu operatörler yardımıyla noktasal yakınsaklık özellikleri incelenmiştir. Son olarak lineer olmayan Bernstein tipli operatörler için Voronovskaya tipli teorem elde edilmiştir. Son bölümde ise bu tezle ilgili sonuç ve daha sonraki çalışmalar için öneriler bulunabilir.

ANAHTAR KELİMELER: Singüler İntegral, Fatou Tipli Yakınsaklık, Süreklilik Modülü, Yakınsaklık Hızı, Lineer Olmayan Bernstein Operatörleri, Sınırlı Salınım, Voronovskaya Tipi Formüller

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LIST OF ABBREVIATIONS AND SYMBOLS

$(\mathbf{B_n f})$	: Classical Bernstein operators.
$(\mathbf{NB_n f})$	: Nonlinear Bernstein operators.
$\mathbf{p_{n,k}}(\mathbf{x})$	: Bernstein basis.
$\mathbf{\Lambda}$	: Set of indices.
$[\mathbf{a}]$	: The greatest integer less than or equal to a.
$\mathbf{C}(\mathbf{I})$	: Functions which are defined on I and continuous.
$\mathbf{L_1}(\mathbf{I})$	: Functions which are defined on I and integrable.
$\mathbf{L_p}(\mathbf{I})$	: Functions which are defined on I and integrable to the $p-$th power.
$\mathbf{BV}(\mathbf{I})$	: Set of all functions which are of bounded variation over I.
$\mathbf{DBV}(\mathbf{I})$	: Set of all differentiable functions which are of derivatives of bounded variation over I.
$\bigvee_{\mathbf{a}}^{\mathbf{b}}(\mathbf{f})$	: Total variation of the function f(x) on the interval [a,b].
$\mathbf{d_t f}(\mathbf{x},\mathbf{t})$	: Differential of the function f(x,t) with respect to the variable t.
$(\mathbf{f} * \mathbf{g})$	: The convolution of f and g.
$\boldsymbol{\omega_\psi}(\mathbf{f},\boldsymbol{\delta})$	: $\psi-$modulus of continuity of the function f(x).

ACKNOWLEDGEMENTS

I would like to express my gratitude to my supervisor Prof. Dr. Harun Karslı for valuable guidance, useful suggestions, penetrative advices, his kindness and patience. I would have not been able to succeed in completing my PhD studies without his golden support.

I also thanks to Prof. Dr. Ali Aral and Assis. Prof. Dr. İsmail Uğur Tiryaki for their valuable guidance and suggestions in the preparation of this thesis.

Finally, I wish to express my gratitude to my family, especially to my wife Kübra Altın, for the emotional support and encouragement.

1. INTRODUCTION

Approximation theory is the most popular area which is interested with the fundamental problem of analysis, approximating a given function f in some sensation or other by functions having particular features, and usually, by functions which have better features than f.

As many other fields of analysis, approximation theory has its major base in the mathematics of the 19th century. Indeed, functions were basically viewed by means of concrete formula, series, or as solutions of equations at the starting of the 19th century. The modern conception of a function differentiated by its imperative properties was established and approved with a result of the claims of Fourier and the consequences of Dirichlet.

We find some of the first results of approximation theory in the theory of Fourier series. These results involve conditions on a function that provide the pointwise or uniform convergence of its Fourier series.

The possibility of approximation is the first question we need to ask in approximation theory. Is the given family of functions from which we plan to approximate dense in the set of functions we desire to approximate? That means, can we approximate any function in our set, as well as we might desire, using arbitrary functions from our given family?

In 1885, Weierstrass proved the density of algebraic polynomials in the class of continuous real-valued functions on a compact interval, and also the density of trigonometric polynomials in the class of $2\pi-$periodic continuous real-valued functions. These results were the first important density theorems in approximation theory.

In 1912 S. Bernstein were introduced the Bernstein polynomials (Bernstein, 1912) to give especially a simple proof of the approximation theorem of Weierstrass.

Now, let's talk briefly about the polynomial of Bernstein. For a function $f\left(x\right)$ defined on the closed interval $[0,1]$ the expression

$$\left(B_n f\right)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k \left(1-x\right)^{n-k} \tag{1.1}$$

is called the Bernstein polynomial of order n of the function $f\left(x\right)$. $\left(B_n f\right)(x)$ is a polynomial in x of degree $\leq n$. For if $f(x)$ is continuous on $[0,1]$, then as it seen in (Lorentz, 1953),

$$\lim_{n\to\infty}(B_nf)(x)=f(x)$$

uniformly in $[0,1]$.

There are many other statements called singular integrals, which have, in common with the $(B_nf)(x)$, the peculiarity of approximating the generating function $f(x)$ and of reproducing some of its properties. The best-known singular integral is the Dirichlet's integral

$$s_n(x)=\frac{1}{\pi}\int_{-\pi}^{+\pi}f(t)\frac{\sin\left(n+\frac{1}{2}\right)(t-x)}{2\sin\frac{1}{2}(t-x)}dt,$$

representing the partial sums $s_n(x)$ of the Fourier series of the function $f(x)$ integrable on $[-\pi,+\pi]$. The Fejer integral is another example which represents the arithmetic means

$$\sigma_n=\frac{(s_0+s_1+...+s_n)}{(n+1)}$$

of the above $s_n(x)$. In general, a singular integral may be written in the form

$$\Phi_n(x)=\int_a^b f(t)\,K_n(x,t)\,dt, \tag{1.2}$$

where $K_n(x,t)$ is the kernel, defined for $a\leq x\leq b$, $a\leq t\leq b$, which has the property that for functions $f(x)$ of a certain class and in a certain sense, $\Phi_n(x)$ converges to $f(x)$ as $n\to\infty$.

The Bernstein polynomial (1.1) is a finite sum of a type corresponding to the integral (1.2). Both (1.1) and (1.2) are special cases of singular Stieltjes integrals. (1.1) may be written in the form of a Stieltjes integral in the variable t,

$$(B_nf)(x)=\int_0^1 f(t)\,d_tK_n(x,t)\,,$$

with the kernel

$$\begin{aligned}K_n(x,t)&=\sum_{k\leq nt}\binom{n}{k}x^k(1-x)^{n-k}\quad,\quad 0<t\leq 1\\ K_n(x,0)&=0\end{aligned}$$

which is constant in any interval

$$\frac{k}{n}\leq t<\frac{(k+1)}{n}\ ,\ k=0,1,...,n-1,$$

and has the jump

$$\binom{n}{k} x^k (1-x)^{n-k}$$

at the basic point of interpolation $t = k/n$. In this sense, the theory of the Bernstein polynomials, as well as the theory of Fourier series, is a chapter of the theory of singular integrals.

Now, a brief and technical explanation of the relation between approximation by linear and nonlinear operators will be stated. In approximation theory, applications were limited to linear operators because the concept of singularity of an integral operator was closely connected with its linearity. Then, about thirty years ago, the notion of singularity was extended to cover the case of nonlinear integral operators (Musielak, 1983). In (Musielak, 1983), the assumption of linearity of the singular integral operators was replaced by an assumption of a Lipschitz condition for the kernel function $K_\lambda(t,u)$ with respect to the second variable.

In this thesis, the Fatou type convergence properties of the $r-$th and $(r+1)-$th derivatives of nonlinear singular integral operators of type

$$(I_\lambda f)(x) = \int_a^b K_\lambda(t-x, f(t))\, dt, \quad x \in (a,b), \tag{1.3}$$

acting on functions defined on an arbitrary interval (a,b), where the kernel K_λ satisfies some suitable assumptions will be concerned. Also, as a main problem of this thesis, approximation properties of nonlinear Bernstein operators $(NB_n f)$ of the form

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) \;,\; 0 \le x \le 1 \;,\quad n \in \mathbb{N},$$

acting on bounded functions f on an interval $[0,1]$ will be studied, where $P_{n,k}$ satisfy some suitable assumptions.

2. AIM AND SCOPE OF THE STUDY

The main objective of this thesis is to consider the problem of approximating a given real-valued function f, defined on $[0,1]$, by means of a sequence of nonlinear Bernstein operators $(NB_n f)$.

In accordance with this purpose, a number of fundamental definitions and inequalities about approximation theory are listed that will be connected throughout this thesis.

2.1 Some Important Definitions

Definition 2.1.1. *(Butzer and Nessel, 1971) C denotes the set of all functions which are uniformly continuous and bounded on $\mathbb{R}$, endowed with the norm*

$$\|f\|_C = \sup_{x\in\mathbb{R}} |f(x)| .$$

Definition 2.1.2. *(Butzer and Nessel, 1971) L_p is the set of functions which are Lebesgue integrable to the pth power over $\mathbb{R}$ if $1 \leq p < \infty$, and essentially bounded (bounded almost everywhere) on $\mathbb{R}$ if $p = \infty$. For $f \in L_p$*

$$\|f\|_p = \left\{ \int_{-\infty}^{\infty} |f(x)|^p \, dx \right\}^{\frac{1}{p}}$$

if p is such that $1 \leq p < \infty$, and in case $p = \infty$

$$\|f\|_\infty = ess \sup_{x\in\mathbb{R}} |f(x)| .$$

Definition 2.1.3. *(Butzer and Nessel, 1971) Let f and g be two functions defined and measurable on $\mathbb{R}$. The expression*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-u)\, g(u)\, du$$

is called the convolution of f and g.

Definition 2.1.4. *(Butzer and Nessel, 1971) A point $x \in \mathbb{R}$ is called a D-point of the function $f \in L_1(I)$ if*

$$\lim_{h\to 0} \frac{1}{h} \int_0^h [f(x+u) - f(x)]\, du = 0$$

where I is an arbitrary given interval on $\mathbb{R}$.

Definition 2.1.5. *(Butzer and Nessel, 1971) A point $x \in \mathbb{R}$ is called a Lebesgue point or L-point of the function $f \in L_1(I)$ if*

$$\lim_{h\to 0} \frac{1}{h} \int_0^h |f(x+u) - f(x)|\, du = 0 \tag{2.1}$$

where I is an arbitrary given interval on $\mathbb{R}$.

Definition 2.1.6. *(Natanson, 1955) The point x_0 is called a limit point or point of accumulation of a point set E if every open interval containing this point contains at least one point of E distinct from the point x_0.*

Definition 2.1.7. *(Bardaro et al., 2003) Let Λ be an infinite set of indices. For the sake of simplicity, the case will be limited, when Λ is an infinite subset of the interval $[a, \lambda_0[$, where $a \in \mathbb{R}$ and $\lambda_0 \in \mathbb{R}$, $\lambda_0 > a$, or $\lambda_0 = +\infty$, and λ_0 is a point of accumulation of the set Λ. Convergence $a_\lambda \overset{\lambda}{\to} a_0$, where a_λ, $a_0 \in \mathbb{R}$, $\lambda \in \Lambda$, means that for every $\epsilon > 0$ there exists a left neighbourhood U_ϵ of λ_0, equal to an interval $]\lambda_0 - \delta, \lambda_0[$ when $\lambda_0 < +\infty$ and equal to a halfline $[\lambda_1, +\infty[$ in the case when $\lambda_0 = +\infty$, such that $|a_\lambda - a_0| < \epsilon$ for all $\lambda \in U_\epsilon \cap \Lambda$.*

Definition 2.1.8. *(Butzer and Nessel, 1971) Let λ be a parameter ranging over some set Λ which is either an interval (a, b) with $0 \le a < b \le +\infty$ or the set $\mathbb{N}$, and let λ_0 be one of the points a, b or $+\infty$. A set of functions $\{K_\lambda(x)\}$ will be called a kernel on the real line if $K_\lambda \in L_1$ for each $\lambda \in \Lambda$ and*

$$\int_{-\infty}^{\infty} K_\lambda(x)\, dx = 1\ ,\ \ \lambda \in \Lambda.$$

A kernel $\{K_\lambda(x)\}$ will be said to be real, bounded or continuous if $K_\lambda(x)$ is a real, bounded or continuous function of x for each $\lambda \in \Lambda$. A real kernel $\{K_\lambda(x)\}$ is even or positive if $K_\lambda(x) = K_\lambda(-x)$ or $K_\lambda(x) \ge 0$ a.e. for each $\lambda \in \Lambda$.

Definition 2.1.9. *(Butzer and Nessel, 1971) For $f \in X(\mathbb{R})$ the convolution*

$$I_\lambda(f; x) = (f * K_\lambda)(x) = \int_{-\infty}^{\infty} f(x-u)\, K_\lambda(u)\, du$$

defines a singular integral generated by the kernel $\{K_\lambda(x)\}$ where $X(\mathbb{R})$ is one of $C(\mathbb{R})$ or $L_p(\mathbb{R})$, $1 \le p < \infty$. The singular integral is said to be positive or continuous if the kernel is positive or continuous.

Definition 2.1.10. *(Butzer and Nessel, 1971) A kernel $\{K_\lambda(x)\}$ is called an approximate identity if there is some constant $M > 0$ with*

$$\|K_\lambda(x)\|_1 \leq M \ , \ \lambda \in \Lambda$$

and

$$\lim_{\lambda \to \lambda_0} \int\limits_{\delta \leq |x|} |K_\lambda(x)| \, dx = 0 \ , \ \delta > 0. \tag{2.2}$$

Apart from (2.2) we sometimes use

$$\lim_{\lambda \to \lambda_0} \left[\sup_{\delta \leq |x|} |K_\lambda(x)| \right] = 0 \ , \ \delta > 0.$$

Definition 2.1.11. *(Natanson, 1955) Let a function $f(x)$ be defined and bounded on the interval $[a, b]$. Subdivide $[a, b]$ into parts by means of the points*

$$x_0 = a < x_1 < ... < x_n = b$$

and form the sum

$$V = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \, .$$

The least upper bound of the set of all possible sums V is called the total variation of the function $f(x)$ on $[a, b]$ and is designated by $\bigvee\limits_a^b (f)$ or $\underset{a \leq x \leq b}{var} f(x)$. If

$$\bigvee_a^b (f) < +\infty,$$

then $f(x)$ is said to be a function of bounded (finite) variation on $[a, b]$. The set of all bounded variation function on $[a, b]$ is denoted by $BV[a, b]$.

Definition 2.1.12. *(Natanson, 1955) Let $f(x)$ and $g(x)$ be finite functions defined on the closed interval $[a, b]$. Subdivide $[a, b]$ into parts by means of the points*

$$x_0 = a < x_1 < ... < x_n = b,$$

choose a point ξ_k in $[x_k, x_{k+1}]$ for $k = 0, ..., n-1$, and form the sum

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k) \left[g(x_{k+1}) - g(x_k) \right] .$$

If the sum σ tends to a finite limit I as

$$\lambda = \max(x_{k+1} - x_k) \to 0,$$

independently of both the method of subdivision and the choice of the points ξ_k, this limit is called the Stieltjes integral of the function $f(x)$ with respect to the function $g(x)$ and is designated by

$$\int_a^b f(x)\,dg(x).$$

The existence of one of the integrals $\int_a^b f(x)\,dg(x)$ or $\int_a^b g(x)\,df(x)$ implies the existence of the other. In this case, the equality

$$\int_a^b f(x)\,dg(x) + \int_a^b g(x)\,df(x) = [f(x)\,g(x)]_a^b \tag{2.3}$$

holds, where

$$[f(x)\,g(x)]_a^b = f(b)\,g(b) - f(a)\,g(a).$$

The relation (2.3) is called the formula for integration by parts.

2.2 Some Important Inequalities

Definition 2.2.1. *(Zygmund, 2002) A function $\phi(x)$ defined in an open or closed interval (a,b) is said to be convex if for every pair of points P_1, P_2 on the curve $y = \phi(x)$ the points of the arc P_1P_2 are below, or on, the chord P_1P_2. For example, x^r, with $r \geq 1$, is convex in $(0, +\infty)$.*

Jensen's inequality states that for any system of positive numbers $p_1, p_2, ..., p_n$, and for any system of points $x_1, x_2, ..., x_n$ in (a,b),

$$\phi\left(\frac{p_1x_1 + p_2x_2 + ... + p_nx_n}{p_1 + p_2 + ... + p_n}\right) \leq \frac{p_1\phi(x_1) + ... + p_n\phi(x_n)}{p_1 + ... + p_n}.$$

Definition 2.2.2. *(Butzer and Nessel, 1971) Let $f(x,y)$ be defined and measurable on $\mathbb{R}^2$. If $\|f(\circ,y)\|_{X(\mathbb{R})} \in L_1$, then*

$$\left\|\int_{-\infty}^{+\infty} f(\circ,y)\,dy\right\|_{X(\mathbb{R})} \leq \int_{-\infty}^{+\infty} \|f(\circ,y)\|_{X(\mathbb{R})}\,dy.$$

This inequality known as the Hölder-Minkowski Inequality or the generalized Minkowski inequality.

Lemma 2.2.3. *(Natanson, 1960) Let a summable function $f(x)$ possessing the property*

$$M = \sup_{0<h\leq b-a} \left\{ \frac{1}{h} \left| \int_{a}^{a+h} f(x)\, dx \right| \right\} < +\infty \tag{2.4}$$

defined on the closed interval $[a, b]$.

For any nonnegative decreasing function $g(x)$, defined and summable on $[a, b]$, the integral

$$\int_{a}^{b} f(x)\, g(x)\, dx \tag{2.5}$$

exists (it might be improper for $x = a$) and the inequality

$$\left| \int_{a}^{b} f(x)\, g(x)\, dx \right| \leq M \int_{a}^{b} g(x)\, dx \tag{2.6}$$

is valid.

3. METHODS AND RESULTS

In this chapter, by using the method of Pierre Joseph Louis Fatou, the convergence of certain family of nonlinear singular integral operators are investigated. Then, by using the methods applied in linear positive operator theory, singular integral operator theory and bounded variation theory, the approximation properties of certain nonlinear Bernstein type operators are examined.

3.1 Fatou Type Convergence of Nonlinear Singular Integral Operators

In this section the Fatou type pointwise convergence of $r-th$ and $(r+1)-th$ derivatives of certain family of nonlinear singular integral operators $(I_\lambda f)$ of the form

$$(I_\lambda f)(x) = \int_a^b K_\lambda (t-x, f(t))\, dt \ , \ x \in (a,b)$$

acting on functions defined on an arbitrary interval (a,b) is concerned, where the kernel function K_λ satisfies some suitable assumptions. Actually, convergence of the derivatives of convolution type nonlinear singular integral operators are studied in (Karsli, 2007) and the results presented in this section are extension of those and can be found in (Karsli and Altin, 2013).

In the convergence theorem of this section, the Fatou type convergence is discussed, that is, the convergence is restricted to some subsets of the plane. In other words, whenever the first parameter tends to an accumulation point x_0, at which the function f has finite $r-th$ and $(r+1)-th$ derivatives, the second parameter tends to an accumulation point λ_0 of a given index set Λ.

3.1.1 Nonlinear Singular Integral Operators

Definition 3.1.1. *Let Λ be a nonempty set of indices with a topology and λ_0 be an accumulation point of Λ in this topology. $\mathcal{U}(\theta)$ denote the family of all neighborhoods of the neutral element θ of $\mathbb{R}$, and x_0 is a fixed accumulation point of $\mathbb{R}$.*

Now some convenient assumptions about the kernel function K_λ are given, which are used in the further theorems. Throughout this section, it assumed that the function $K_\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions;

a$'$) Let $L_\lambda(t)$ be an integrable function such that for any fixed $r \in \mathbb{N}$

$$\left[\frac{\partial^r}{\partial x^r} K_\lambda(t-x,u) - \frac{\partial^r}{\partial x^r} K_\lambda(t-x,v)\right] = \frac{\partial^r}{\partial x^r} L_\lambda(t-x)\,[u-v] \qquad (3.1)$$

holds for every $t, u, v \in \mathbb{R}$ and for any $\lambda \in \Lambda$.

b$'$) $\lim\limits_{\lambda \to \lambda_0} \int\limits_{\mathbb{R}\setminus U} L_\lambda(t)\,dt = 0$, for every $U \in \mathcal{U}(0)$.

c$'$) $\lim\limits_{\lambda \to \lambda_0} \left[\sup\limits_{|t|\geq\delta} L_\lambda(t)\right] = 0$, for every $\delta > 0$.

d$'$) $\lim\limits_{\lambda \to \lambda_0} \int\limits_{\mathbb{R}} L_\lambda(t)\,dt = 1$.

Definition 3.1.2. *A function $\widetilde{f} \in L_1(\mathbb{R})$ is defined as*

$$\widetilde{f}(t) := \begin{cases} f(t) & , \quad t \in (a,b), \\ 0 & , \quad t \notin (a,b) \end{cases}$$

where $f \in L_1(a,b)$.

Definition 3.1.3. *Let the kernel function $K_\lambda(t,u)$ be continuous in $\mathbb{R}$ for every $t \in \mathbb{R}$, then the kernel function is called Carathéodory kernel function.*

Theorem 3.1.4. *Let $1 \leq p < \infty$ and assume that $K_\lambda(t,u)$ is a Caratheodory kernel function. If $f \in L_p(a,b)$, then $(I_\lambda f) \in L_p(a,b)$ and*

$$\|I_\lambda f\|_{L_p(a,b)} \leq H(\lambda)\,\|f\|_{L_p(a,b)}$$

for every $\lambda \in \Lambda$, where

$$H^p(\lambda) = \int_a^b |L_\lambda(t-x)|^p\,dx.$$

Proof. First of $\|I_\lambda f\|_{L_p(a,b)}$ is written as

$$\|I_\lambda f\|_{L_p(a,b)} = \left\| \int_a^b K_\lambda(t-x, f(t))\,dt \right\|_p$$

and using generalized Minkowsky inequality

$$\begin{aligned}\left\|I_\lambda f\right\|_{L_p(a,b)} &\leq \int_a^b \left\|K_\lambda(t-x,f(t))\right\|_p dt \\ &= \int_a^b \left(\int_a^b \left|K_\lambda(t-x,f(t))\right|^p dx\right)^{\frac{1}{p}} dt\end{aligned}$$

and taking into account (a′)

$$\begin{aligned}\left\|I_\lambda f\right\|_{L_p(a,b)} &\leq \int_a^b \left(\int_a^b \left(\left|L_\lambda(t-x)\right|\left|f(t)\right|\right)^p dx\right)^{\frac{1}{p}} dt \\ &= \int_a^b \left(\left|f(t)\right|^p \int_a^b \left|L_\lambda(t-x)\right|^p dx\right)^{\frac{1}{p}} dt \\ &\leq H(\lambda)\left\|f\right\|_{L_p(a,b)}\end{aligned}$$

where $H^p(\lambda) = \int_a^b \left|L_\lambda(t-x)\right|^p dx$. This kind of existence theorem is also valid in general function spaces (Bardaro et al., 2003). □

3.1.2 Convergence of the Derivatives

Before starting the approximation properties of the operators $(I_\lambda f)$, the following definition is introduced:

$$D_r := \left\{(x,\lambda) \in I \times \Lambda : \left|x-x_0\right|^v \int_{x_0-x-\delta}^{x_0-x+\delta} |t|^{r-v} \left|\frac{\partial^r}{\partial t^r} L_\lambda(t)\right| dt < C_v\right\}, \tag{3.2}$$

where $I=(a,b)$ is an arbitrary interval in $\mathbb{R}$ and $C_v > 0$, $(v = 1,2,...,r)$ are any constants.

Now, the approximation for finite $r-th$ order derivatives of the operator $(I_\lambda f)$ in $L_1(I)$ can be investigated.

Theorem 3.1.5. *Let the function $L_\lambda(t)$ and its derivatives $\frac{\partial^v}{\partial t^v} L_\lambda(t)$, $(v = 1,2,...,r)$ be continuous with respect to t on $(-\infty,\infty)$ and $L_\lambda(t)$ be integrable with respect to t for each fixed $\lambda \in \Lambda$. Suppose that the conditions (c′) and (d′) together with*

$$\sup_{\lambda\in\Lambda} \int_{x_0-x-\delta}^{x_0-x+\delta} |t|^r \left|\frac{\partial^r}{\partial t^r} L_\lambda(t)\right| dt < \infty \tag{3.3}$$

and

$$\lim_{\lambda\to\lambda_0} \sup_{0<\delta\leq|t|} \left| \frac{\partial^r}{\partial t^r} L_\lambda(t) \right| = 0 \tag{3.4}$$

hold for every $\delta > 0$, are satisfied. Suppose also that the function $f \in L_1(I)$ has at x_0 a finite $r - th$ derivative.

Then

$$\lim \frac{\partial^r}{\partial x^r} (I_\lambda f)(x) = f^{(r)}(x_0)$$

as $(x, \lambda) \to (x_0, \lambda_0)$ and $(x, \lambda) \in D_r$.

Proof. Suppose that

$$a < x_0 < b$$

and

$$0 < |x_0 - x| < \delta \ (\delta > 0).$$

Set up a function

$$g(t) = f(x_0) + (t - x_0) f'(x_0) + ... + \frac{(t - x_0)^r}{r!} f^{(r)}(x_0). \tag{3.5}$$

In this way

$$g^{(r)}(t) = f^{(r)}(x_0)$$

holds. First of all this theorem could be proven for the function $g(t)$. For this sense a function $\widetilde{g} \in L_1(\mathbb{R})$ is introduced as;

$$\widetilde{g}(t) := \begin{cases} g(t) & , \quad t \in (a, b), \\ 0 & , \quad t \notin (a, b). \end{cases} \tag{3.6}$$

Applying the operator I_λ to the function $g(t)$, one has

$$(I_\lambda g)(x) = \int_a^b K_\lambda(t - x, g(t))\, dt.$$

Now using (3.6), the last equality can be rewritten as follows;

$$(I_\lambda g)(x) = \int_{\mathbb{R}} K_\lambda \left(t - x, \widetilde{g}(t)\right) dt = \left(I_\lambda \widetilde{g}\right)(x). \tag{3.7}$$

Differentiating r times (3.7) with respect to x, one has

$$\begin{aligned}\frac{\partial^r}{\partial x^r}\left(I_\lambda g\right)(x) &= \frac{\partial^r}{\partial x^r}\int\limits_{\mathbb{R}} K_\lambda\left(t-x,\widetilde{g}(t)\right)dt \\ &= \int\limits_{\mathbb{R}} \widetilde{g}(t)\,\frac{\partial^r}{\partial x^r}L_\lambda(t-x)\,dt \\ &= (-1)^r\int\limits_{\mathbb{R}} \widetilde{g}(t)\,\frac{\partial^r}{\partial t^r}L_\lambda(t-x)\,dt \\ &= \int\limits_{\mathbb{R}} \widetilde{g}^{(r)}(t)\,L_\lambda(t-x)\,dt.\end{aligned}$$

Using (3.5) in the last equality,

$$\begin{aligned}\left|\frac{\partial^r}{\partial x^r}\left(I_\lambda g\right)(x)-f^{(r)}(x_0)\right| &= \left|\int\limits_a^b f^{(r)}(x_0)\,L_\lambda(t-x)\,dt-f^{(r)}(x_0)\right| \\ &= \left|f^{(r)}(x_0)\right|\left|\int\limits_a^b L_\lambda(t-x)\,dt-1\right|.\end{aligned}$$

Taking into account the condition (d′) one has

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)}\frac{\partial^r}{\partial x^r}\left(I_\lambda g\right)(x)=f^{(r)}(x_0). \qquad (3.8)$$

Now denote

$$I_\lambda(x):=\frac{\partial^r}{\partial x^r}\left(I_\lambda g\right)(x)-\frac{\partial^r}{\partial x^r}\left(I_\lambda f\right)(x).$$

Under favour of (3.8), it is sufficient to show that

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)}\left|I_\lambda(x)\right|=0,$$

for completing the proof of the theorem.

$I_\lambda(x)$ can be regulated as follows;

$$\begin{aligned}
|I_\lambda(x)| &= \left|\int_{\mathbb{R}} \widetilde{f}(t)\frac{\partial^r}{\partial t^r}L_\lambda(t-x)\,dt - \int_{\mathbb{R}} \widetilde{g}(t)\frac{\partial^r}{\partial t^r}L_\lambda(t-x)\,dt\right| \\
&= \left|\int_a^b [f(t)-g(t)]\frac{\partial^r}{\partial t^r}L_\lambda(t-x)\,dt\right| \\
&\leq \int_a^{x_0-\delta} |f(t)-g(t)|\left|\frac{\partial^r}{\partial t^r}L_\lambda(t-x)\right|dt \\
&\quad + \int_{x_0-\delta}^{x_0+\delta} |f(t)-g(t)|\left|\frac{\partial^r}{\partial t^r}L_\lambda(t-x)\right|dt \\
&\quad + \int_{x_0+\delta}^{b} |f(t)-g(t)|\left|\frac{\partial^r}{\partial t^r}L_\lambda(t-x)\right|dt \\
&=: I_1(x,\lambda)+I_2(x,\lambda)+I_3(x,\lambda).
\end{aligned}$$

Now $I_1(x,\lambda)$, $I_2(x,\lambda)$ and $I_3(x,\lambda)$ will be estimated separately. The following method is used to estimate $I_1(x,\lambda)$ and $I_3(x,\lambda)$.

$$\begin{aligned}
I_1(x,\lambda) &= \int_a^{x_0-\delta} |f(t)-g(t)|\left|\frac{\partial^r}{\partial t^r}L_\lambda(t-x)\right|dt \\
&\leq \sup_{0<\delta\leq|t|}\left|\frac{\partial^r}{\partial t^r}L_\lambda(t-x)\right|\int_a^b |f(t)-g(t)|\,dt
\end{aligned}$$

and with the same idea

$$\begin{aligned}
I_3(x,\lambda) &= \int_{x_0+\delta}^{b} |f(t)-g(t)|\left|\frac{\partial^r}{\partial t^r}L_\lambda(t-x)\right|dt \\
&\leq \sup_{0<\delta\leq|t|}\left|\frac{\partial^r}{\partial t^r}L_\lambda(t-x)\right|\int_a^b |f(t)-g(t)|\,dt.
\end{aligned}$$

On the other hand, $I_2(x,\lambda)$ can be written as follows;

$$I_2(x,\lambda) = \int_{x_0-\delta}^{x_0+\delta}\left|\frac{f(t)-g(t)}{(t-x_0)^r}\right|\left|(t-x_0)^r\right|\left|\frac{\partial^r}{\partial t^r}L_\lambda(t-x)\right|dt.$$

For every $\varepsilon>0$ there exists a $\delta>0$ such that;

$$I_2(x,\lambda) \leq \varepsilon\int_{x_0-\delta}^{x_0+\delta}\left|(t-x_0)^r\right|\left|\frac{\partial^r}{\partial t^r}L_\lambda(t-x)\right|dt.$$

Let

$$I_{2,1}(x,\lambda) := \int\limits_{x_0-\delta}^{x_0+\delta} |(t-x_0)^r| \left|\frac{\partial^r}{\partial t^r} L_\lambda(t-x)\right| dt$$
$$= \int\limits_{x_0-x-\delta}^{x_0-x+\delta} |(t+x-x_0)^r| \left|\frac{\partial^r}{\partial t^r} L_\lambda(t)\right| dt.$$

Now $I_{2,1}(x,\lambda)$ can be separated as follows:

$$\begin{aligned} I_{2,1}(x,\lambda) &= \int\limits_{x_0-x-\delta}^{x_0-x+\delta} |(t+x-x_0)^r - t^r + t^r| \left|\frac{\partial^r}{\partial t^r} L_\lambda(t)\right| dt \\ &\leq \int\limits_{x_0-x-\delta}^{x_0-x+\delta} |(t+x-x_0)^r - t^r| \left|\frac{\partial^r}{\partial t^r} L_\lambda(t)\right| dt \\ &\quad + \int\limits_{x_0-x-\delta}^{x_0-x+\delta} |t|^r \left|\frac{\partial^r}{\partial t^r} L_\lambda(t)\right| dt \\ &=: I_{2,1,1}(x,\lambda) + I_{2,1,2}(x,\lambda). \end{aligned}$$

$I_{2,1,2}(x,\lambda)$ is finite with (3.3). So it is sufficient to show that $I_{2,1,1}(x,\lambda)$ is finite.

Using the obvious identity

$$a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + ... + ab^{n-2} + b^{n-1}\right) \tag{3.9}$$

$I_{2,1,1}(x,\lambda)$ can be rewritten as,

$$\begin{aligned} I_{2,1,1}(x,\lambda) &= |x-x_0| \int\limits_{x_0-x-\delta}^{x_0-x+\delta} \left|(t+x-x_0)^{r-1}+...+t^{r-1}\right| \left|\frac{\partial^r}{\partial t^r} L_\lambda(t)\right| dt \\ &\leq |x-x_0| \int\limits_{x_0-x-\delta}^{x_0-x+\delta} \left|(t+x-x_0)^{r-1}\right| \left|\frac{\partial^r}{\partial t^r} L_\lambda(t)\right| dt \\ &\quad +|x-x_0| \int\limits_{x_0-x-\delta}^{x_0-x+\delta} \left|(t+x-x_0)^{r-2}\, t\right| \left|\frac{\partial^r}{\partial t^r} L_\lambda(t)\right| dt \\ &\quad . \\ &\quad . \\ &\quad . \\ &\quad +|x-x_0| \int\limits_{x_0-x-\delta}^{x_0-x+\delta} \left|(t+x-x_0)\, t^{r-2}\right| \left|\frac{\partial^r}{\partial t^r} L_\lambda(t)\right| dt \\ &\quad +|x-x_0| \int\limits_{x_0-x-\delta}^{x_0-x+\delta} |t|^{r-1} \left|\frac{\partial^r}{\partial t^r} L_\lambda(t)\right| dt. \end{aligned} \tag{3.10}$$

Applying the formula (3.9) successively to the right-hand side of (3.10), it is seen that $I_{2,1,1}(x,\lambda)$ is less than or equal to the linear combinations of

$$|x-x_0|^v \int\limits_{x_0-x-\delta}^{x_0-x+\delta} |t|^{r-v} \left|\frac{\partial^r}{\partial t^r} L_\lambda(t)\right| dt \ , \ \ (v=1,2,...,r).$$

Taking into account of (3.2) and (3.3), it is seen that $I_{2,1}(x,\lambda)$ is finite on any planar set D_r.

Hence

$$\begin{aligned} |I_\lambda(x)| &\leq I_1(x,\lambda)+I_2(x,\lambda)+I_3(x,\lambda) \\ &\leq \varepsilon I_2(x,\lambda) + 2\sup_{0<\delta\leq|t|} \left|\frac{\partial^r}{\partial t^r} L_\lambda(t-x)\right| \int\limits_a^b |f(t)-g(t)|\, dt. \end{aligned}$$

Since $f(t)-g(t)$ belongs to $L_1(a,b)$, in view of (3.2), (3.3), (3.4) and the condition (c′),

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} |I_\lambda(x)| = 0,$$

that is,

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \frac{\partial^r}{\partial x^r}(I_\lambda f)(x) = \lim_{(x,\lambda)\to(x_0,\lambda_0)} \frac{\partial^r}{\partial x^r}(I_\lambda g)(x).$$

This completes the proof of the theorem. □

Secondly, the approximation for finite $(r+1)-th$ derivatives of the operator $(I_\lambda f)$ in $L_1(I)$ will be investigated.

Theorem 3.1.6. *Let the function $L_\lambda(t)$ and its derivatives $\frac{\partial^v}{\partial t^v}L_\lambda(t)$, $(v=1,2,...,r,r+1)$ be continuous with respect to t on $(-\infty,\infty)$ and $L_\lambda(t)$ be integrable with respect to t for each fixed $\lambda\in\Lambda$. Suppose that the conditions (c′) and (d′) are satisfied. Also, it is assumed that the equations*

$$\sup_{\lambda\in\Lambda}\int\limits_{x_0-x-\delta}^{x_0-x+\delta}|t|^{r+1}\left|\frac{\partial^{r+1}}{\partial t^{r+1}}L_\lambda(t)\right|dt<\infty$$

and

$$\lim_{\lambda\to\lambda_0}\sup_{0<\delta\le|t|}\left|\frac{\partial^{r+1}}{\partial t^{r+1}}L_\lambda(t)\right|=0$$

hold for every $\delta>0$. Suppose also that the function $f\in L_1(I)$ has at x_0 finite derivatives at $f_+^{(r+1)}(x_0)$ and $f_-^{(r+1)}(x_0)$.

Then

$$\lim\frac{\partial^{r+1}}{\partial x^{r+1}}(I_\lambda f)(x)=Bf_+^{(r+1)}(x_0)+(1-B)f_-^{(r+1)}(x_0)$$

as $(x,\lambda)\to(x_0,\lambda_0)$ and $(x,\lambda)\in D_{r+1}$, where

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)}\int\limits_{x_0}^{\infty}L_\lambda(t-x)\,dt=B\ ,\ 0\le B\le 1. \tag{3.11}$$

Proof. Assume that

$$a<x_0<b$$

and

$$0<x_0-x<\frac{\delta}{2}\,(\delta>0).$$

Set up a function

$$g(t):=\begin{cases} f(x_0)+...+\frac{(t-x_0)^r}{r!}f^{(r)}(x_0)+\frac{(t-x_0)^{r+1}}{(r+1)!}f_-^{(r+1)}(x_0) & ,\quad a<t<x_0,\\ f(x_0)+...+\frac{(t-x_0)^r}{r!}f^{(r)}(x_0)+\frac{(t-x_0)^{r+1}}{(r+1)!}f_+^{(r+1)}(x_0) & ,\quad x_0\le t<b.\end{cases} \tag{3.12}$$

Here, it is noted that

$$(I_\lambda g)(x)=\int\limits_a^b K_\lambda(t-x,g(t))\,dt=\int\limits_{\mathbb{R}}K_\lambda\left(t-x,\widetilde{g}(t)\right)dt.$$

Differentiating both sides of the last equality $(r+1)$ times with respect to x, one has

$$\frac{\partial^{r+1}}{\partial x^{r+1}}(I_\lambda g)(x)=\int\limits_a^b g^{(r+1)}(t)\,L_\lambda(t-x)\,dt.$$

The last equality can be rewritten with (3.12) in the form

$$\frac{\partial^{r+1}}{\partial x^{r+1}}(I_\lambda g)(x)=f_-^{(r+1)}(x_0)\int_a^{x_0}L_\lambda(t-x)\,dt+f_+^{(r+1)}(x_0)\int_{x_0}^{b}L_\lambda(t-x)\,dt.$$

The hypothesis (3.11) yields

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)}\frac{\partial^{r+1}}{\partial x^{r+1}}(I_\lambda g)(x)=Bf_+^{(r+1)}(x_0)+(1-B)\,f_-^{(r+1)}(x_0).$$

Define $I_\lambda(x)$ as

$$|I_\lambda(x)|:=\left|\frac{\partial^{r+1}}{\partial x^{r+1}}(I_\lambda g)(x)-\frac{\partial^{r+1}}{\partial x^{r+1}}(I_\lambda f)(x)\right|.$$

In order to complete the proof of the theorem, it is sufficient to show that

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)}|I_\lambda(x)|=0.$$

$I_\lambda(x)$ can be regulated as follows;

$$\begin{aligned}|I_\lambda(x)| &= \left|\int_a^b[f(t)-g(t)]\frac{\partial^{r+1}}{\partial t^{r+1}}L_\lambda(t-x)\,dt\right|\\ &\le \int_a^{x_0-\delta}|f(t)-g(t)|\left|\frac{\partial^{r+1}}{\partial t^{r+1}}L_\lambda(t-x)\right|dt\\ &\quad+\int_{x_0-\delta}^{x_0}\left|\frac{f(t)-g(t)}{(t-x_0)^{r+1}}\right|\left|(t-x_0)^{r+1}\right|\left|\frac{\partial^{r+1}}{\partial t^{r+1}}L_\lambda(t-x)\right|dt\\ &\quad+\int_{x_0+\delta}^{b}|f(t)-g(t)|\left|\frac{\partial^{r+1}}{\partial t^{r+1}}L_\lambda(t-x)\right|dt\\ &\quad+\int_{x_0}^{x_0+\delta}\left|\frac{f(t)-g(t)}{(t-x_0)^{r+1}}\right|\left|(t-x_0)^{r+1}\right|\left|\frac{\partial^{r+1}}{\partial t^{r+1}}L_\lambda(t-x)\right|dt\\ &=: I_1(x,\lambda)+I_2(x,\lambda)+I_3(x,\lambda)+I_4(x,\lambda).\end{aligned}\tag{3.13}$$

Since f has at x_0 a finite $(r+1)-th$ right and left derivatives, then for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$\begin{aligned}I_2(x,\lambda) &= \int_{x_0-\delta}^{x_0}\left|\frac{f(t)-g(t)}{(t-x_0)^{r+1}}\right|\left|(t-x_0)^{r+1}\right|\left|\frac{\partial^{r+1}}{\partial t^{r+1}}L_\lambda(t-x)\right|dt\\ &\le \varepsilon\int_{x_0-\delta}^{x_0}\left|(t-x_0)^{r+1}\right|\left|\frac{\partial^{r+1}}{\partial t^{r+1}}L_\lambda(t-x)\right|dt\end{aligned}\tag{3.14}$$

and

$$\begin{aligned} I_4(x,\lambda) &= \int_{x_0}^{x_0+\delta} \left| \frac{f(t)-g(t)}{(t-x_0)^{r+1}} \right| \left| (t-x_0)^{r+1} \right| \left| \frac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x) \right| dt \\ &\leq \varepsilon \int_{x_0}^{x_0+\delta} \left| (t-x_0)^{r+1} \right| \left| \frac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x) \right| dt. \end{aligned} \tag{3.15}$$

Thus one has

$$I_2(x,\lambda) + I_4(x,\lambda) \leq \varepsilon \int_{x_0-\delta}^{x_0+\delta} \left| (t-x_0)^{r+1} \right| \left| \frac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x) \right| dt.$$

Setting

$$I_{2,1}(x,\lambda) + I_{4,1}(x,\lambda) := \int_{x_0-\delta}^{x_0+\delta} \left| (t-x_0)^{r+1} \right| \left| \frac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x) \right| dt$$

and using the same method as in the proof of Theorem 3.1.5., $I_{2,1}(x,\lambda) + I_{4,1}(x,\lambda)$ is deduced less than or equal to a linear combination of

$$|x-x_0|^v \int_{x_0-x-\delta}^{x_0-x+\delta} |t|^{r+1-v} \left| \frac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t) \right| dt \ , \ \ (v = 1, 2, ..., r+1).$$

By virtue of (3.2), the term $I_2(x,\lambda) + I_4(x,\lambda)$ is bounded.

Now, the integrals $I_1(x,\lambda)$ and $I_3(x,\lambda)$ are considered, respectively.

$$\begin{aligned} I_1(x,\lambda) &= \int_a^{x_0-\delta} |f(t)-g(t)| \left| \frac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x) \right| dt \\ &\leq M \sup_{\frac{\delta}{2}<|u|} \left| \frac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(u) \right| \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} I_3(x,\lambda) &= \int_{x_0+\delta}^{b} |f(t)-g(t)| \left| \frac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x) \right| dt \\ &\leq M \sup_{\frac{\delta}{2}<|u|} \left| \frac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(u) \right|. \end{aligned} \tag{3.17}$$

Using (3.14), (3.15), (3.16) and (3.17) in (3.13) one has

$$\begin{aligned} |I_\lambda(x)| &\leq I_1(x,\lambda) + I_2(x,\lambda) + I_3(x,\lambda) + I_4(x,\lambda) \\ &\leq \varepsilon \left[I_{2,1}(x,\lambda) + I_{4,1}(x,\lambda) \right] \\ &\quad +2M \sup_{\frac{\delta}{2}<|u|} \left| \frac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(u) \right|. \end{aligned} \tag{3.18}$$

Under the hypothesis of the theorem, (3.18) yields

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} |I_\lambda(x)| = \lim_{(x,\lambda)\to(x_0,\lambda_0)} \left| \frac{\partial^{r+1}}{\partial x^{r+1}} (I_\lambda g)(x) - \frac{\partial^{r+1}}{\partial x^{r+1}} (I_\lambda f)(x) \right| = 0.$$

This completes the proof. □

Example 3.1.7. *Define a function $K_\lambda(t,u)$ as*

$$K_\lambda(t,u) = \begin{cases} (r+1)\lambda^{r+1} t^r u + u & , \quad t \in \left[0, \frac{1}{\lambda}\right], \\ 0 & , \quad t \notin \left[0, \frac{1}{\lambda}\right], \end{cases}$$

where $\Lambda = [1,\infty)$ is a set of indices with natural topology and $\lambda_0 = \infty$ is an accumulation point of Λ in this topology.

It is seen that $K_\lambda(t,u)$ is a kernel, that is, $K_\lambda(t,0) = 0$.

For every $u \in \mathbb{R}$, it is seen that

$$\frac{\partial^r}{\partial x^r} K_\lambda(t-x,u) = \begin{cases} (-1)^r (r+1)! \lambda^{r+1} u & , \quad t-x \in \left[0, \frac{1}{\lambda}\right], \\ 0 & , \quad t-x \notin \left[0, \frac{1}{\lambda}\right]. \end{cases}$$

By using (3.1), it is obtained that

$$\frac{\partial^r}{\partial x^r} L_\lambda(t-x) = \begin{cases} (-1)^r (r+1)! \lambda^{r+1} & , \quad t-x \in \left[0, \frac{1}{\lambda}\right], \\ 0 & , \quad t-x \notin \left[0, \frac{1}{\lambda}\right]. \end{cases}$$

And this implies that

$$L_\lambda(t-x) = \begin{cases} (r+1)\lambda^{r+1}(t-x)^r & , \quad t-x \in \left[0, \frac{1}{\lambda}\right], \\ 0 & , \quad t-x \notin \left[0, \frac{1}{\lambda}\right]. \end{cases}$$

Moreover it is seen that

$$\int_{\mathbb{R}} L_\lambda(x)\, dt = \int_{\left[0,\frac{1}{\lambda}\right]} (r+1)\lambda^{r+1} t^r dt = 1 < \infty.$$

It is also easy to see that

$$\lim_{\lambda\to\infty} \int_{\mathbb{R}\setminus U} L_\lambda(t)\, dt = 0,$$

for every $U \in \mathcal{U}(0)$ and

$$\lim_{\lambda\to\infty} \left[\sup_{|t|\geq\delta} L_\lambda(t) \right] = 0,$$

for every $\delta > 0$.

3.2 Nonlinear Bernstein Type Operators and its Approximation Properties

In this section, approximation properties of certain families of nonlinear Bernstein operators NB_nf of the form

$$(NB_nf)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) \quad , \quad 0 \leq x \leq 1 \quad , \quad n \in \mathbb{N}, \tag{3.19}$$

acting on bounded functions f on an interval $[0,1]$ are concerned, where $P_{n,k}$ satisfy some suitable assumptions.

Before introducing convergence theorems for nonlinear Bernstein type operators, some preliminary definitions are pointed out, that will be useful in this section.

Definition 3.2.1. *A function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is defined as continuous, concave and nondecreasing such that*

$$\varphi(0) = 0,$$

and

$$\varphi(u) > 0 \text{ for } u > 0.$$

Such function is called a φ-function and the class of all the φ-functions will be denoted by Ψ.

Definition 3.2.2. *It is defined that $\mu : \mathbb{N} \to \mathbb{R}^+$ is an increasing and continuous function such that*

$$\lim_{n\to\infty} \mu(n) = \infty.$$

Definition 3.2.3. *Let $\{P_{n,k}\}_{n\in\mathbb{N}}$ be a sequence functions $P_{n,k} : [0,1]\, x\, \mathbb{R} \to \mathbb{R}$ defined by*

$$P_{n,k}(x, u) = p_{n,k}(x) H_n(u) \tag{3.20}$$

for every $x \in [0,1], u \in \mathbb{R}$, where $H_n : \mathbb{R} \to \mathbb{R}$ is such that $H_n(0) = 0$ and $p_{n,k}(x)$ is the Bernstein basis defined as

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Now some convenient assumptions are given about the kernel function, which are used in the further theorems. Assume that the following conditions hold:

a) $H_n : \mathbb{R} \to \mathbb{R}$ *is such that*

$$|H_n(u) - H_n(v)| \leq \psi(|u - v|), \quad \psi \in \Psi,$$

holds for every $u, v \in \mathbb{R}$, for every $n \in \mathbb{N}$. That is, H_n satisfies a $(L-\psi)$ Lipschitz condition.

b) *Now set*

$$K_n(x,u) := \begin{cases} \sum\limits_{k \le nu} p_{n,k}(x) & , \quad 0 < u \le 1, \\ \\ 0 & , \quad u = 0. \end{cases}$$

and

$$B_n(x) := \int\limits_{x - x/n^{\gamma/\beta}}^{x + (1-x)/n^{\gamma/\beta}} d_t\left(K_n(x,t)\right) \quad \text{for any fixed } x \in (0,1), \tag{3.21}$$

where $\beta > 0$, $\gamma \ge 1$ and

$$\lambda_n(x,t) := \int\limits_0^t d_u\left(K_n(x,u)\right). \tag{3.22}$$

c) Denote $r_n(u)$ as

$$r_n(u) := H_n(u) - u,$$

where $u \in \mathbb{R}$ *and* $n \in \mathbb{N}$. Assume that *for* n sufficiently large

$$\sup_u |r_n(u)| \le \frac{1}{\mu(n)},$$

holds.

Example 3.2.4. *For every $n \in \mathbb{N}$ and $u \in \mathbb{R}$, let*

$$H_n(u) = \begin{cases} n \log(1 + u/n) & , \quad 0 \le u < 1 \\ nu \log(1 + 1/n) & , \quad u \ge 1, \end{cases}$$

where the definition of H_n can be extended in odd-way for $u < 0$. Then $H_n : \mathbb{R} \to \mathbb{R}$ is such that $H_n(0) = 0$; moreover it is easy to show that

$$|H_n(u) - H_n(v)| \le |u - v|$$

for every $u, v \in \mathbb{R}$ and $n \in \mathbb{N}$. This is a simple example of H_n which can be found in (Bardaro et al., 2003).

Definition 3.2.5. *For any function f for which the one-sided limits $f(x+)$, $f(x-)$ exist at every point $x \in (0,1)$, the function $f_x(t)$ is defined as*

$$f_x(t) = \begin{cases} f(t) - f(x+) & , \quad x < t \le 1, \\ 0 & , \quad t = x, \\ f(t) - f(x-) & , \quad 0 \le t < x. \end{cases} \tag{3.23}$$

Lemma 3.2.6. *For all $x \in (0,1)$ and for each $n \in \mathbb{N}$, let*

$$NB_n((t-x)^\beta; x) \leq \frac{B_n(x)}{n^{\gamma/\beta}},\ (\beta > 0) \tag{3.24}$$

holds, where $B_n(x)$ is defined as (3.21). Then one has

$$\lambda_n(x,t) = \int_0^t d_u\left(K_n(x,u)\right) \leq \frac{B_n(x)}{(x-t)^\beta n^{\gamma/\beta}}\ ,\quad 0 \leq t < x, \tag{3.25}$$

and

$$1-\lambda_n(x,t) = \int_t^1 d_u\left(K_n(x,u)\right) \leq \frac{B_n(x)}{(t-x)^\beta n^{\gamma/\beta}}\ ,\quad x < t < 1. \tag{3.26}$$

Proof. (3.25) can be rewritten as

$$\begin{aligned}\lambda_n(x,t) &= \int_0^t d_u\left(K_n(x,u)\right) \leq \int_0^t \left(\frac{x-u}{x-t}\right)^\beta d_u\left(K_n(x,u)\right)\\ &\leq \frac{1}{(x-t)^\beta}\int_0^1 |u-x|^\beta\, d_u\left(K_n(x,u)\right).\end{aligned}$$

According to (3.24), one has

$$\lambda_n(x,t) \leq \frac{B_n(x)}{(x-t)^\beta n^{\gamma/\beta}}.$$

In the same way (3.26) can be rewritten as

$$\begin{aligned}1-\lambda_n(x,t) &= \int_t^1 d_u\left(K_n(x,u)\right) \leq \int_t^1 \left(\frac{u-x}{t-x}\right)^\beta d_u\left(K_n(x,u)\right)\\ &\leq \frac{1}{(t-x)^\beta}\int_0^1 |u-x|^\beta\, d_u\left(K_n(x,u)\right).\end{aligned}$$

In view of (3.24), one has

$$1-\lambda_n(x,t) \leq \frac{B_n(x)}{(t-x)^\beta n^{\gamma/\beta}}.$$

This completes the proof. □

3.2.1 Convergence and Rate of Convergence by Modulus of Continuity

In this section it will be considered that the problem of approximating a function f, belonging to a continuous function space, by means of a nonlinear Bernstein type operators. First of all the following convergence theorem for these operators can be given.

Theorem 3.2.7. *Let $f : I \to \mathbb{R}$, $f \in C(I)$ and suppose that a kernel satisfies (a), (b) and (c). Then*

$$\|NB_n f - f\|_C \to 0 \text{ as } n \to \infty$$

where $I = [0,1]$ and $\psi \in \Psi$.

Proof. The difference $|NB_n f(x) - f(x)|$ can be rewritten as;

$$\begin{aligned} |NB_n f(x) - f(x)| &= \left| \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) - f(x) \right| \\ &= \left| \sum_{k=0}^{n} \left\{ H_n\left(f\left(\frac{k}{n}\right)\right) - f(x) \right\} p_{n,k}(x) \right| \end{aligned}$$

and using triangle inequality

$$\begin{aligned} |NB_n f(x) - f(x)| &\leq \sum_{k=0}^{n} \left| H_n\left(f\left(\frac{k}{n}\right)\right) - H_n(f(x)) \right| p_{n,k}(x) \\ &\quad + \sum_{k=0}^{n} |H_n(f(x)) - f(x)| \, p_{n,k}(x) \\ &= : I_{n,1}(x) + I_{n,2}(x). \end{aligned}$$

Considering $I_{n,2}(x)$ and taking into account (c) one has

$$\begin{aligned} I_{n,2}(x) &= \sum_{k=0}^{n} |H_n(f(x)) - f(x)| \, p_{n,k}(x) \\ &\leq \frac{1}{\mu(n)}. \end{aligned}$$

Now $I_{n,1}(x)$ can be rewritten by using (a) as follows;

$$\begin{aligned} I_{n,1}(x) &= \sum_{k=0}^{n} \left| H_n\left(f\left(\frac{k}{n}\right)\right) - H_n(f(x)) \right| p_{n,k}(x) \\ &\leq \sum_{k=0}^{n} \psi\left(\left| f\left(\frac{k}{n}\right) - f(x) \right|\right) p_{n,k}(x). \end{aligned}$$

Using Stieltjes integral form of the Bernstein operators

$$\begin{aligned} I_{n,1}(x) &= \int_0^1 \psi(|f(t)-f(x)|)\, d_t(K_n(x,t)) \\ &= \int_{|t-x|<\delta} \psi(|f(t)-f(x)|)\, d_t(K_n(x,t)) \\ &\quad + \int_{|t-x|\geq\delta} \psi(|f(t)-f(x)|)\, d_t(K_n(x,t)) \\ &\leq \psi(\epsilon)+\psi(2\|f\|_C)\ \epsilon \end{aligned}$$

holds true, since ψ is non-decreasing and concave function. Finally, it is obtained that

$$|NB_n f(x)-f(x)| \leq \psi(\epsilon)+\psi(2\|f\|_C)\ \epsilon+\frac{1}{\mu(n)}$$

and so, since $\frac{1}{\mu(n)} \to 0$ when $n\to\infty$, it is obtained that

$$\lim_{n\to\infty}\sup_x |NB_n f(x)-f(x)| \leq \psi(\epsilon)+\psi(2\|f\|_C)\ \epsilon.$$

Hence the assertion follows, $\epsilon>0$ being arbitrary. □

A problem of interest in Approximation Theory is studying the degree of approximation. The degree of approximation of a function $f(x)$, $a\leq x\leq b$ by polynomials may be simply described in terms of its modulus of continuity $\omega(\delta)=\omega(f;\delta)$. For each $\delta>0$, $\omega(f;\delta)$ is the maximum of $|f(x)-f(y)|$ for all $a\leq x\leq b,\ a\leq y\leq b$, $|x-y|<\delta$; $\omega(f;\delta)$ clearly decreases to zero with δ if $f(x)$ is continuous. Now, the certain modulus of continuity will be defined.

Definition 3.2.8. *Let $f\in C[a,b]$ and $\delta>0$ be given. Then the $\psi-$modulus of continuity is given by;*

$$\omega_\psi(f;\delta)=\sup_{|t-x|\leq\delta\ ,\ t,x\in[a,b]} \psi(|f(t)-f(x)|)\,. \tag{3.27}$$

In the next lemma some properties for the $\psi-$modulus of continuity will be indicated, which will be often used in the sequel.

Lemma 3.2.9. $\omega_\psi(f;\delta)$ *has the following properties;*

(i) $\omega_\psi(f;\delta)\geq 0$,

(ii) If $\delta_1\leq\delta_2$, *then* $\omega_\psi(f;\delta_1)\leq\omega_\psi(f;\delta_2)$,

(iii) Let $m \in \mathbb{N}$*, then* $\omega_\psi(f; m\delta) \leq m.\omega_\psi(f;\delta)$,

(iv) Let $\lambda \in \mathbb{R}^+$*, then* $\omega_\psi(f;\lambda\delta) \leq (\lambda+1)\,\omega_\psi(f;\delta)$,

(v) $\lim_{\delta\to 0^+} \omega_\psi(f;\delta) = 0$,

(vi) $\psi(|f(t) - f(x)|) \leq \omega_\psi(f;|t-x|)$,

(vii) $\psi(|f(t) - f(x)|) \leq \left(\frac{|t-x|}{\delta} + 1\right)\omega_\psi(f;\delta)$.

Proof. The proofs of properties which are obtained in the similar ways with the classical ones are given.

(i) Since $|f(t) - f(x)| \geq 0$ and $\psi(u) > 0$ for $u > 0$ from the definition of ψ function, then $\omega_\psi(f;\delta) = \sup_{|x-y|\leq\delta} \psi(|f(x) - f(y)|) \geq 0$ for every $x, y \in [a,b]$.

(ii) If $\delta_1 \leq \delta_2$ then the zone $|t-x| \leq \delta_2$ is bigger than the zone $|t-x| \leq \delta_1$. Since ψ is non-decreasing, it is obvious from the definition that, if the zone gets bigger its supremum value gets bigger.

(iii) From the definition it can be written;

$$\omega_\psi(f;m\delta) = \sup_{|t-x|\leq m\delta\ ,\ t,x\in[a,b]} \psi(|f(t) - f(x)|)\ ,\ m \in \mathbb{N}.$$

If $|t-x| \leq m\delta$ then

$$x - m\delta \leq t \leq x + m\delta.$$

Let $t = x + mh$, then $|h| \leq \delta$ and one has

$$\begin{aligned}
\omega_\psi(f;m\delta) &= \max_{|h|\leq\delta\ ,\ t,x\in[a,b]} \psi(|f(x+mh) - f(x)|) \\
&= \max_{|h|\leq\delta\ ,\ t,x\in[a,b]} \psi\left(\left|\sum_{k=0}^{m-1}[f(x+(k+1)h) - f(x+kh)]\right|\right)
\end{aligned}$$

then by using triangle inequality and since ψ is non-decreasing and concave, it can be attained that

$$\begin{aligned}
\omega_\psi(f;m\delta) &\leq \max_{|h|\leq\delta\ ,\ t,x\in[a,b]} \psi\left(\sum_{k=0}^{m-1}|f(x+(k+1)h) - f(x+kh)|\right) \\
&\leq \max_{|h|\leq\delta\ ,\ t,x\in[a,b]} \sum_{k=0}^{m-1}\psi(|f(x+(k+1)h) - f(x+kh)|) \\
&\leq \sum_{k=0}^{m-1}\max_{|h|\leq\delta\ ,\ t,x\in[a,b]} \psi(|f(x+(k+1)h) - f(x+kh)|) \\
&= \omega_\psi(f;\delta) + ... + \omega_\psi(f;\delta).
\end{aligned}$$

Therefore, it is obtained that

$$\omega_{\psi}(f;m\delta)\leq m.\omega_{\psi}(f;\delta).$$

(iv) Let $\lambda\in\mathbb{R}^{+}$ then $[\lambda]\leq\lambda\leq[\lambda]+1$. Then by using (ii) it can be written;

$$\omega_{\psi}(f;\lambda\delta)\leq\omega_{\psi}(f;([\lambda]+1)\delta).$$

Due to $[\lambda]$ is positive, (iii) can be used to the right side of inequality to get

$$\omega_{\psi}(f;([\lambda]+1)\delta)\leq(\lambda+1)\omega_{\psi}(f;\delta).$$

In conclusion, it is obtained that

$$\omega_{\psi}(f;\lambda\delta)\leq(\lambda+1)\omega_{\psi}(f;\delta).$$

(v) Let's consider the inequality $|t-x|\leq\delta$. $\delta\to0^{+}$ means that $t\to x$. Due to function f is continuous, if $t\to x$ then $|f(t)-f(x)|\to0$. And using the continuity of the function ψ and $\psi(0)=0$, assertion follows.

(vi) Let $\delta=|t-x|$ in the $\omega_{\psi}(f;\delta)$ then it is attained that

$$\omega_{\psi}(f;|t-x|)=\sup_{t,x\in[a,b]}\psi(|f(t)-f(x)|).$$

Due to supremum of $\psi(|f(t)-f(x)|)$ equal to $\omega_{\psi}(f;|t-x|)$, then

$$\psi(|f(t)-f(x)|)\leq\omega_{\psi}(f;|t-x|).$$

(vii) From (vi),

$$\psi(|f(t)-f(x)|)\leq\omega_{\psi}\left(f;\frac{|t-x|}{\delta}\delta\right).$$

Then using (iv), it is obtained that

$$\psi(|f(t)-f(x)|)\leq\left(\frac{|t-x|}{\delta}+1\right)\omega_{\psi}(f;\delta).$$

This completes the proof. □

Theorem 3.2.10. *If $f(x)$ is continuous and $\omega_{\psi}(f;\delta)$ the $\psi-$modulus of continuity of $f(x)$ given in (3.27), then*

$$|NB_{n}f(x)-f(x)|\leq\psi(\epsilon)+\frac{5}{4}\omega_{\psi}(f;\delta)+\frac{1}{\mu(n)}$$

where $\delta=n^{-\frac{1}{2}}$.

Proof. The difference can be written as in the proof of the Theorem 3.2.7.,

$$\begin{aligned} |NB_n f(x) - f(x)| &= \left| \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) - f(x) \right| \\ &\leq I_{n,1}(x) + I_{n,2}(x) \end{aligned}$$

where

$$\begin{aligned} I_{n,2}(x) &= \sum_{k=0}^{n} |H_n(f(x)) - f(x)| \, p_{n,k}(x) \\ &\leq \frac{1}{\mu(n)}. \end{aligned}$$

Now, $I_{n,1}(x)$ can be considered and if $I_{n,1}(x)$ is thought as two sum;

$$\begin{aligned} I_{n,1}(x) &\leq \sum_{\left|\frac{k}{n}-x\right| \leq \delta} \psi\left(\left| f\left(\frac{k}{n}\right) - f(x) \right|\right) p_{n,k}(x) \\ &\quad + \sum_{\left|\frac{k}{n}-x\right| > \delta} \psi\left(\left| f\left(\frac{k}{n}\right) - f(x) \right|\right) p_{n,k}(x) \\ &\leq \psi(\epsilon) + I_{n,1,2}(x). \end{aligned}$$

Taking into account that $\omega_\psi(f;\delta)$ the $\psi-$modulus of continuity, $I_{n,1,2}(x)$ can be written as

$$\begin{aligned} I_{n,1,2}(x) &= \sum_{\left|\frac{k}{n}-x\right| > \delta} \psi\left(\left| f\left(\frac{k}{n}\right) - f(x) \right|\right) p_{n,k}(x) \\ &\leq \omega_\psi(f;\delta) \sum_{\left|\frac{k}{n}-x\right| > \delta} \left(\frac{\left|\frac{k}{n} - x\right|}{\delta} + 1 \right) p_{n,k}(x) \\ &\leq \omega_\psi(f;\delta) \left\{ 1 + \delta^{-1} \sum_{\left|\frac{k}{n}-x\right| > \delta} \left| \frac{k}{n} - x \right| p_{n,k}(x) \right\} \\ &\leq \omega_\psi(f;\delta) \left\{ 1 + \delta^{-2} \sum_{k=0}^{n} \left(\frac{k}{n} - x \right)^2 p_{n,k}(x) \right\} \\ &\leq \omega_\psi(f;\delta) \left\{ 1 + \left(4n\delta^2\right)^{-1} \right\}. \end{aligned}$$

In conclusion;

$$\begin{aligned} |NB_n f(x) - f(x)| &\leq \psi(\epsilon) + \omega_\psi(f;\delta) \left\{ 1 + \left(4n\delta^2\right)^{-1} \right\} + \frac{1}{\mu(n)} \\ &\leq \psi(\epsilon) + \frac{5}{4}\omega_\psi(f;\delta) + \frac{1}{\mu(n)} \end{aligned}$$

where $\delta = n^{-\frac{1}{2}}$. □

3.2.2 Some Convergence Results for Nonlinear Bernstein Type Operators on Some Specified Points

In this section, some approximation results which are dealing with the rate of pointwise convergence of the nonlinear Bernstein operators $NB_n f$ to the limit $\psi o\left|f\right|$ and f of functions of bounded variation on the interval $[0,1]$ will be obtained. At the point x, which is a discontinuity of the first kind of f and of its derivative, it will be proved that $\left(NB_n f\right)(x)$ converge to the limit $f\left(x\right)$ which can be found in (Karsli et al., 2014).

The counterpart of such kind of results for positive linear operators of functions of bounded variation in the Jordan sense were first obtained in (Bojanic, 1979) and (Cheng, 1983).

In order to prove the main approximation result, the following preliminary lemma is needed.

Lemma 3.2.11. *(Zeng, 1998) For all $x \in (0,1)$ and for all $n > \frac{256}{25x(1-x)}$, the following inequality holds*

$$p_{n,k}\left(x\right) \leq \frac{1}{\sqrt{2enx(1-x)}}, \tag{3.28}$$

where $e = 2.71...$ is the Napierian constant.

Theorem 3.2.12. *Let X be the set of all bounded Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$ and $\psi \in \Psi$ be such that $\psi o\left|f\right| \in BV\left([0,1]\right)$. Suppose that $P_{n,k}(x,u)$ satisfies conditions (a), (b) and (c). Then for every $x \in (0,1)$ and for all $n > \frac{256}{25x(1-x)}$, the following inequality holds*

$$\begin{aligned} & \left(NB_n f\right)(x) - \left[\psi\left(\left|\frac{f(x+)+f(x-)}{2}\right|\right) + \psi\left(\left|\frac{f(x+)-f(x-)}{2}\right|\right)\right] \\ \leq\ & B_n^*(x) n^{-\gamma/\beta}\left[\bigvee_0^1 \psi\left(\left|f_x\right|\right) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \psi\left(\left|f_x\right|\right)\right] \\ & + B_n(x) \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi\left(\left|f_x\right|\right) \\ & + \psi\left(\left|f(x) - \frac{f(x+)+f(x-)}{2}\right| \frac{e_n(x)}{\sqrt{2enx(1-x)}}\right) \end{aligned}$$

where

$$B_n^*(x) = B_n(x) \max\{x^{-\beta}, (1-x)^{-\beta}\},\ \ (\beta > 0)\,,$$

and

$$e_n(x) = \begin{cases} 1 \ , \ x = \frac{k'}{n} \ \text{for some} \ k' \in IN \\ 0 \ , \ x \neq \frac{k'}{n} \ \text{for all} \ k' \in IN \end{cases}$$

and $\bigvee_0^1 \psi(|f_x|)$ *is the total variation of* $\psi(|f_x|)$ *on* $[0,1]$.

Proof. For any $f \in X$, Bojanic-Cheng decomposition yields

$$\begin{aligned} f(t) &= \frac{f(x+)+f(x-)}{2} + f_x(t) + \frac{f(x+)-f(x-)}{2} sgn(t-x) \\ &+\delta_x(t)\left[f(x) - \frac{f(x+)+f(x-)}{2}\right], \end{aligned} \tag{3.29}$$

where $f_x(t)$ is defined as in (3.23) and

$$\delta_x(t) = \begin{cases} 1 \ , \ x = t, \\ 0 \ , \ x \neq t. \end{cases}$$

Now applying the operator (3.19) to (3.29), and using (3.20) with a condition (a),

$$\begin{aligned} (NB_n f)(x) &= \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) \\ &= \sum_{k=0}^{n} p_{n,k}(x) H_n\left(f\left(\frac{k}{n}\right)\right) \\ &\leq \sum_{k=0}^{n} p_{n,k}(x)\psi\left(\left|f\left(\frac{k}{n}\right)\right|\right) \\ &\leq \psi\left(\left|\frac{f(x+)+f(x-)}{2}\right|\right)\sum_{k=0}^{n} p_{n,k}(x) \\ &\quad +\sum_{k=0}^{n} p_{n,k}(x)\psi\left(\left|f_x\left(\frac{k}{n}\right)\right|\right) \\ &\quad +\sum_{k=0}^{n} p_{n,k}(x)\psi\left(\left|\frac{f(x+)-f(x-)}{2}\right|\left|sgn\left(\frac{k}{n}-x\right)\right|\right) \\ &\quad +\sum_{k=0}^{n} p_{n,k}(x)\psi\left(\left|f(x) - \frac{f(x+)+f(x-)}{2}\right|\delta_x\left(\frac{k}{n}\right)\right). \end{aligned}$$

Hence from (3.29),

$$\begin{aligned} (NB_n f)(x) &\leq \left[\psi\left(\left|\frac{f(x+)+f(x-)}{2}\right|\right)\right. \\ &\quad \left.+\psi\left(\left|\frac{f(x+)-f(x-)}{2}\right|\right)\right]\sum_{k=0}^{n} p_{n,k}(x) \\ &\quad +\sum_{k=0}^{n} p_{n,k}(x)\psi\left(\left|f_x\left(\frac{k}{n}\right)\right|\right) \\ &\quad +\psi\left(\sum_{k=0}^{n} p_{n,k}(x)\left|f(x) - \frac{f(x+)+f(x-)}{2}\right|\delta_x\left(\frac{k}{n}\right)\right). \end{aligned}$$

Here it is noted that

$$\sum_{k=0}^{n} p_{n,k}(x)\delta_x\left(\frac{k}{n}\right) = e_n(x)p_{n,k'}(x)$$

where

$$e_n(x) = 1$$

if there exists a k' such that $x = k'/n$ and

$$e_n(x) = 0$$

if $x \neq k'/n$ for all $k \in \{0, 1, ..., n\}$. Thus,

$$\begin{aligned}
(NB_n f)(x) &\leq \left[\psi\left(\left|\frac{f(x+)+f(x-)}{2}\right|\right)\right. \\
&\quad \left.+\psi\left(\left|\frac{f(x+)-f(x-)}{2}\right|\right)\right]\sum_{k=0}^{n} p_{n,k}(x) \\
&\quad +\sum_{k=0}^{n} p_{n,k}(x)\psi\left(\left|f_x\left(\frac{k}{n}\right)\right|\right) \\
&\quad \psi\left(\left|f(x)-\frac{f(x+)+f(x-)}{2}\right| e_n(x)p_{n,k'}(x)\right).
\end{aligned}$$

In view of (3.28), it is obtained that

$$\begin{aligned}
&(NB_n f)(x) - \left[\psi\left(\left|\frac{f(x+)+f(x-)}{2}\right|\right)+\psi\left(\left|\frac{f(x+)-f(x-)}{2}\right|\right)\right] \\
\leq\ &\sum_{k=0}^{n} p_{n,k}(x)\psi\left(\left|f_x\left(\frac{k}{n}\right)\right|\right) \\
&+\psi\left(\left|f(x)-\frac{f(x+)+f(x-)}{2}\right|\frac{e_n(x)}{\sqrt{2enx(1-x)}}\right). \qquad (3.30)
\end{aligned}$$

In order to complete the proof of theorem, an estimation is needed for the term

$$\sum_{k=0}^{n} p_{n,k}(x)\psi\left(\left|f_x\left(\frac{k}{n}\right)\right|\right)$$

in (3.30). In view of (b) last integral is split into three parts as follows;

$$\begin{aligned}
\sum_{k=0}^{n} p_{n,k}(x)\psi\left(\left|f_x\left(\frac{k}{n}\right)\right|\right) &\leq \left(\int_{0}^{x-x/n^{\gamma/\beta}} + \int_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} + \int_{x+(1-x)/n^{\gamma/\beta}}^{1}\right) \\
&\quad \psi\left(|f_x(t)|\right) d_t\left(K_n(x,t)\right) \\
&= : |I_1(n,x)| + |I_2(n,x)| + |I_3(n,x)|. \qquad (3.31)
\end{aligned}$$

First, $I_2(n,x)$ is estimated. Since

$$f_x(x) = 0$$

and

$$\psi(0) = 0,$$

$I_2(n,x)$ can be written for $t \in \left[x - x/n^{\gamma/\beta}, x + (1-x)/n^{\gamma/\beta}\right]$,

$$|I_2(n,x)| = \int\limits_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \left[\psi(|f_x(t)|) - \psi(|f_x(x)|)\right] d_t\left(K_n(x,t)\right),$$

and also by the condition (b)

$$\begin{aligned} |I_2(n,x)| &\leq \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi(|f_x|) \int\limits_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} d_t\left(K_n(x,t)\right) \\ &\leq B_n(x) \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi(|f_x|). \end{aligned} \tag{3.32}$$

Next, $I_1(n,x)$ is estimated. Using partial Lebesgue-Stieltjes integration, it is obtained that

$$\begin{aligned} |I_1(n,x)| &= \int\limits_0^{x-x/n^{\gamma/\beta}} \psi(|f_x(t)|)\, d_t\left(K_n(x,t)\right) \\ &= \psi\left(\left|f_x(x - x/n^{\gamma/\beta})\right|\right) K_n(x, x - x/n^{\gamma/\beta}) \\ &\quad - \int\limits_0^{x-x/n^{\gamma/\beta}} K_n(x,t) d_t\left(\psi(|f_x(t)|)\right). \end{aligned}$$

Let $y = x - x/n^{\gamma/\beta}$. By (3.25), it is clear that

$$K_n(x,y) \leq B_n(x)(x-y)^{-\beta} n^{\gamma(\beta-1)/\beta}. \tag{3.33}$$

Here, it should be noted that

$$\begin{aligned} \psi\left(\left|f_x(x - x/n^{\gamma/\beta})\right|\right) &= \psi\left(\left|f_x(x - x/n^{\gamma/\beta})\right|\right) - \psi(|f_x(x)|) \\ &\leq \bigvee_{x-x/n^{\gamma/\beta}}^{x} \psi(|f_x|). \end{aligned}$$

Using partial integration and applying (3.33), it is obtained that

$$\begin{aligned}
|I_1(n,x)| &\leq \bigvee_{x-x/n^{\gamma/\beta}}^{x} \psi\left(|f_x|\right)\left|K_n(x,x-x/n^{\gamma/\beta})\right| \\
&\quad + \int_0^{x-x/n^{\gamma/\beta}} K_n(x,t)d_t\left(-\bigvee_t^x \psi\left(|f_x|\right)\right) \\
&\leq \bigvee_{x-x/n^{\gamma/\beta}}^{x} \psi\left(|f_x|\right)B_n(x)x^{-\beta}n^{\gamma(\beta-1)/\beta} \\
&\quad + B_n(x)n^{-\gamma/\beta}\int_0^{x-x/n^{\gamma/\beta}} (x-t)^{-\beta}d_t\left(-\bigvee_t^x \psi\left(|f_x|\right)\right) \\
&= \bigvee_{x-x/n^{\gamma/\beta}}^{x} \psi\left(|f_x|\right)B_n(x)x^{-\beta}n^{\gamma(\beta-1)/\beta} \\
&\quad + B_n(x)n^{-\gamma/\beta}\left[-x^{-\beta}/n^{-\gamma}\bigvee_{x-x/n^{\gamma/\beta}}^{x} \psi\left(|f_x|\right)\right. \\
&\quad \left. + x^{-\beta}\bigvee_0^x \psi\left(|f_x|\right) + \int_0^{x-x/n^{\gamma/\beta}} \bigvee_t^x \psi\left(|f_x|\right)\frac{\beta}{(x-t)^{\beta+1}}dt\right] \\
&= B_n(x)n^{-\gamma/\beta}\left[x^{-\beta}\bigvee_0^x \psi\left(|f_x|\right)\right. \\
&\quad \left. + \int_0^{x-x/n^{\gamma/\beta}} \bigvee_t^x \psi\left(|f_x|\right)\frac{\beta}{(x-t)^{\beta+1}}dt\right].
\end{aligned}$$

By changing the variable t by $x-x/u^{1/\beta}$ in the last integral,

$$\begin{aligned}
\int_0^{x-x/n^{\gamma/\beta}} \bigvee_t^x \psi\left(|f_x|\right)\frac{\beta}{(x-t)^{\beta+1}}dt &= \frac{1}{x^\beta}\int_1^{n^\gamma} \bigvee_{x-x/u^{1/\beta}}^{x} \psi\left(|f_x|\right)\,du \\
&\leq \frac{1}{x^\beta}\sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x} \psi\left(|f_x|\right).
\end{aligned}$$

Consequently, it obtained that

$$|I_1(n,x)| \leq B_n(x)n^{-\gamma/\beta}x^{-\beta}\left[\bigvee_0^x \psi\left(|f_x|\right) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x} \psi\left(|f_x|\right)\right]. \tag{3.34}$$

Using the similar method for $I_3(n,x)$, it is found that

$$|I_3(n,x)| \leq B_n(x)n^{-\gamma/\beta}(1-x)^{-\beta}\left[\bigvee_x^1 \psi\left(|f_x|\right) + \sum_{k=1}^{[n^\gamma]} \bigvee_x^{x+(1-x)/k^{1/\beta}} \psi\left(|f_x|\right)\right]. \tag{3.35}$$

Combining (3.32), (3.34) and (3.35) in (3.31), it is obtained that

$$\begin{aligned}
&\sum_{k=0}^{n} p_{n,k}(x)\psi\left(\left|f_x\left(\frac{k}{n}\right)\right|\right) \\
\leq\ & B_n(x)n^{-\gamma/\beta}x^{-\beta}\left[\bigvee_0^x \psi(|f_x|) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x} \psi(|f_x|)\right] \\
&+B_n(x)n^{-\gamma/\beta}(1-x)^{-\beta}\left[\bigvee_x^1 \psi(|f_x|)\right. \\
&\left.+\sum_{k=1}^{[n^\gamma]} \bigvee_{x}^{x+(1-x)/k^{1/\beta}} \psi(|f_x|)\right] \\
&+B_n(x)\bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi(|f_x|) \\
\leq\ & B_n^*(x)n^{-\gamma/\beta}\left[\bigvee_0^1 \psi(|f_x|) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \psi(|f_x|)\right] \\
&+B_n(x)\bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi(|f_x|). \qquad (3.36)
\end{aligned}$$

Collecting (3.30) and (3.36), desired result is attained. This completes the proof of the theorem. □

A corollary follow from this theorem.

Corollary 3.2.13. *If $f \in C[0,1]$ is chosen in the Theorem 3.2.12., then*

$$\begin{aligned}
&(NB_nf)(x) - \psi(|f(x)|) \\
\leq\ & B_n^*(x)n^{-\gamma/\beta}\left[\bigvee_0^1 \psi(|f_x|) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \psi(|f_x|)\right] \\
&+B_n(x)\bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi(|f_x|)
\end{aligned}$$

where $B_n^*(x) = B_n(x)\max\{x^{-\beta},(1-x)^{-\beta}\}$, $(\beta > 0)$.

Theorem 3.2.14. *Let $\psi \in \Psi$ and $f \in X$ be such that $\psi o |f| \in BV([0,1])$. Suppose that $P_{n,k}(x,u)$ satisfies conditions (a), (b) and (c). Then for every $x \in (0,1)$ and for all*

$n > \frac{256}{25x(1-x)}$, *the following inequality holds*

$$\begin{aligned}
&\left|\left(NB_nf\right)(x) - \frac{f(x+)+f(x-)}{2}\right| \\
\leq\ & B_n^*(x)n^{-\gamma/\beta}\left[\bigvee_0^1 \psi\left(|f_x|\right) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \psi\left(|f_x|\right)\right] \\
&+B_n(x) \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi\left(|f_x|\right) \\
&+\psi\left(\left|f(x) - \frac{f(x+)+f(x-)}{2}\right| \frac{e_n(x)}{\sqrt{2enx(1-x)}}\right) \\
&+\psi\left(\frac{|f(x+)-f(x-)|}{2}\right) + \frac{1}{\mu(n)}
\end{aligned}$$

where $B_n^*(x) = B_n(x)\max\{x^{-\beta}, (1-x)^{-\beta}\}$, $(\beta > 0)$.

Proof. For any $f \in X$,

$$\begin{aligned}
&\left|\left(NB_nf\right)(x) - \frac{f(x+)+f(x-)}{2}\right| \\
=\ & \left|\sum_{k=0}^{n} p_{n,k}(x) H_n\left(f\left(\frac{k}{n}\right)\right) - \frac{f(x+)+f(x-)}{2}\right| \\
\leq\ & \sum_{k=0}^{n} p_{n,k}(x)\psi\left(\left|f\left(\frac{k}{n}\right) - \frac{f(x+)+f(x-)}{2}\right|\right) \\
&+\sum_{k=0}^{n} p_{n,k}(x)\left|H_n\left(\frac{f(x+)+f(x-)}{2}\right) - \frac{f(x+)+f(x-)}{2}\right| \\
\leq\ & \sum_{k=0}^{n} p_{n,k}(x)\psi\left(\left|f_x\left(\frac{k}{n}\right)\right|\right) \\
&+\sum_{k=0}^{n} p_{n,k}(x)\psi\left(\left|f(x) - \frac{f(x+)+f(x-)}{2}\right|\delta_x\left(\frac{k}{n}\right)\right) \\
&+\psi\left(\left|\frac{f(x+)-f(x-)}{2}\right|\right) \\
&+\sum_{k=0}^{n} p_{n,k}(x)\left|H_n\left(\frac{f(x+)+f(x-)}{2}\right) - \frac{f(x+)+f(x-)}{2}\right|.
\end{aligned}$$

As in the proof of the Theorem 3.2.12., one has

$$\begin{aligned}
&\left|(NB_nf)(x) - \frac{f(x+)+f(x-)}{2}\right| \\
\leq\ & B_n^*(x)n^{-\gamma/\beta}\left[\bigvee_0^1 \psi(|f_x|) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \psi(|f_x|)\right] \\
&+B_n(x) \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi(|f_x|) \\
&+\psi\left(\left|f(x) - \frac{f(x+)+f(x-)}{2}\right| \frac{e_n(x)}{\sqrt{2enx(1-x)}}\right) \\
&+\psi\left(\frac{|f(x+)-f(x-)|}{2}\right) \\
&+\sum_{k=0}^{n} p_{k,n}(x)\left|H_n\left(\frac{f(x+)+f(x-)}{2}\right) - \frac{f(x+)+f(x-)}{2}\right|.
\end{aligned}$$

In view of (c), $\sup_u |H_n(u) - u| \leq \frac{1}{\mu(n)}$ is had. This completes the proof of the theorem. □

Several simple corollaries follow from this theorem.

Corollary 3.2.15. *If $f \in C[0,1]$ is chosen in the Theorem 3.2.14., then*

$$\begin{aligned}
&|(NB_nf)(x) - f(x)| \\
\leq\ & B_n^*(x)n^{-\gamma/\beta}\left[\bigvee_0^1 \psi(|f_x|) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \psi(|f_x|)\right] \\
&+B_n(x) \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} \psi(|f_x|)
\end{aligned}$$

where $B_n^*(x) = B_n(x)\max\{x^{-\beta}, (1-x)^{-\beta}\}$, $(\beta > 0)$.

Corollary 3.2.16. *If $f \in C[0,1]$ and $\psi(t) = t$ (that is, strongly Lipschitz condition) in the Theorem 3.2.14., then*

$$\begin{aligned}
&|(NB_nf)(x) - f(x)| \\
\leq\ & B_n^*(x)n^{-\gamma/\beta}\left[\bigvee_0^1 (|f_x|) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} (|f_x|)\right] \\
&+B_n(x) \bigvee_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} (|f_x|)
\end{aligned}$$

where $B_n^*(x) = B_n(x)\max\{x^{-\beta}, (1-x)^{-\beta}\}$, $(\beta > 0)$.

It should be noted that Corollary (3.2.16) is a very special case of Corollary (3.2.15).

The next task in this section is to estimate the rate of pointwise convergence for the nonlinear sequence of Bernstein operators (3.19) to the point x, at the Lebesgue points of f, as $n \to \infty$. In (Karsli, 2013), this type convergence results for nonlinear singular integral operators can be found and also the next consequence is suitable to see in (Karsli and Altin, 2015a). As a preliminary, the simple lemma about Lebesgue point is proven.

Lemma 3.2.17. *Let $x \in \mathbb{R}$ is a Lebesgue point of the function f, then*

$$\left| \int_0^h \psi\left(\left|f\left(x+t\right) - f\left(x\right)\right|\right) dt \right| = o\left(|h|\right) \ as\ h \to 0$$

where $\psi \in \Psi$.

Proof. In order to prove the lemma, the following two statements will be shown.

$$\left| \int_0^h \psi\left(\left|f\left(x+t\right) - f\left(x\right)\right|\right) dt \right| = o\left(h\right) \ as\ h \to 0^+,$$

and

$$\left| \int_h^0 \psi\left(\left|f\left(x+t\right) - f\left(x\right)\right|\right) dt \right| = o\left(-h\right) \ as\ h \to 0^-.$$

Since ψ is concave, one has for $h < 0$ and $h > 0$, respectively,

$$\frac{1}{-h} \int_h^0 \psi\left(\left|f\left(x+t\right) - f\left(x\right)\right|\right) dt \le \psi\left(\frac{1}{-h} \int_h^0 \left|f\left(x+t\right) - f\left(x\right)\right| dt \right)$$

and

$$\frac{1}{h} \int_0^h \psi\left(\left|f\left(x+t\right) - f\left(x\right)\right|\right) dt \le \psi\left(\frac{1}{h} \int_0^h \left|f\left(x+t\right) - f\left(x\right)\right| dt \right).$$

Hence, by continuity of ψ and $\psi\left(0\right) = 0$, the desired result is reached. □

Theorem 3.2.18. *Let $\psi \in \Psi$ and $f \in L_1\left([0,1]\right)$ be such that $\psi \circ |f| \in BV\left([0,1]\right)$. Suppose that $P_{n,k}\left(x,u\right)$ satisfies condition (a), (b) and (c). Then at each point $x \in (0,1)$ for which (2.1) holds we have for each $\epsilon > 0$ and for sufficiently large $n \in \mathbb{N}$,*

$$\begin{aligned} \left|\left(NB_n f\right)\left(x\right) - f\left(x\right)\right| \ \le\ & \epsilon B_n^*\left(x\right) \left(n^{\frac{\gamma}{\beta}}\right)^{\beta-1} \\ & + \frac{B_n^*\left(x\right)}{n^{\frac{\gamma}{\beta}}} \left[\bigvee_0^1 \psi\left(|f_x|\right) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \psi\left(|f_x|\right) \right] \\ & + \frac{1}{\mu\left(n\right)} \end{aligned}$$

where $B_n^*(x) = B_n(x) \max\left\{x^{-\beta}, (1-x)^{-\beta}\right\}$, $(\beta > 0)$.

Proof. Suppose that

$$x + \delta < 1,$$

and

$$x - \delta > 0$$

for any $0 < \delta$.

Let

$$\begin{aligned} |I_n(x)| := & \quad |(NB_n f)(x) - f(x)| \\ = & \left| \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) - f(x) \right|. \end{aligned}$$

From (3.19) and using triangle inequality, $|I_n(x)|$ can be rewritten as follows:

$$\begin{aligned} |I_n(x)| \leq & \left| \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) - \sum_{k=0}^{n} P_{n,k}(x, f(x)) \right| \\ & + \left| \sum_{k=0}^{n} P_{n,k}(x, f(x)) - f(x) \right| \\ = & : I_{n,1}(x) + I_{n,2}(x). \end{aligned}$$

From (c) it is easy to see that the second term of the right-hand-side of the above inequality is less than or equal to $\frac{1}{\mu(n)}$. Indeed;

$$\begin{aligned} I_{n,2}(x) &= \left| \sum_{k=0}^{n} P_{n,k}(x, f(x)) - f(x) \right| \\ &= \left| \sum_{k=0}^{n} p_{n,k}(x) H_n(f(x)) - \sum_{k=0}^{n} p_{n,k}(x) f(x) \right| \\ &= |H_n(f(x)) - f(x)| \sum_{k=0}^{n} p_{n,k}(x) \\ &\leq \frac{1}{\mu(n)} \end{aligned}$$

holds for n sufficiently large.

As to the first term, by (a) and using Lebesgue-Stieltjes integral representation of Bernstein polynomial, the following inequality is available.

$$\begin{aligned} I_{n,1}(x) &\leq \sum_{k=0}^{n} \psi\left(\left|f\left(\frac{k}{n}\right)-f(x)\right|\right) p_{n,k}(x) \\ &= \int_{0}^{1} \psi\left(\left|f(t)-f(x)\right|\right) d_t\left(K_n(x,t)\right). \end{aligned}$$

According to (b), the last integral can be split into three terms as follows:

$$\begin{aligned} I_{n,1}(x) &\leq \left(\int_{0}^{x-x/n^{\frac{\gamma}{\beta}}} + \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x+(1-x)/n^{\frac{\gamma}{\beta}}} + \int_{x+(1-x)/n^{\frac{\gamma}{\beta}}}^{1} \right) \\ &\quad \psi\left(\left|f(t)-f(x)\right|\right) d_t\left(K_n(x,t)\right) \\ &= : I_1(n,x) + I_2(n,x) + I_3(n,x). \end{aligned}$$

First, $I_2(n,x)$ is estimated. It is available for $t \in \left[x - x/n^{\frac{\gamma}{\beta}}, x + (1-x)/n^{\frac{\gamma}{\beta}}\right]$

$$\begin{aligned} |I_2(n,x)| &= \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x+(1-x)/n^{\frac{\gamma}{\beta}}} \psi\left(\left|f(t)-f(x)\right|\right) d_t\left(K_n(x,t)\right) \\ &\leq \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi\left(\left|f(t)-f(x)\right|\right) d_t\left(K_n(x,t)\right) \\ &\quad + \int_{x}^{x+(1-x)/n^{\frac{\gamma}{\beta}}} \psi\left(\left|f(t)-f(x)\right|\right) d_t\left(K_n(x,t)\right) \\ &= : I_{2,1}(n,x) + I_{2,2}(n,x). \end{aligned}$$

Setting

$$F(t) := \int_{t}^{x} \psi\left(\left|f(y)-f(x)\right|\right) dy,$$

then, according to the Lemma 3.2.17., for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$F(t) \leq \epsilon\,(x-t) \tag{3.37}$$

for all $0 < x - t \leq \delta$.

Now, this δ is fixed and $I_{2,1}(n,x)$ and $I_{2,2}(n,x)$ are estimated respectively.

First of all, the Lebesgue-Stieltjes integral representation is recalled and by using (3.22),

$I_{2,1}(n,x)$ can be written as

$$\begin{aligned} I_{2,1}(n,x) &= \int\limits_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi(|f(t)-f(x)|)\, d_t K_n(x,t) \\ &= \int\limits_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi(|f(t)-f(x)|)\, \frac{\partial}{\partial t}\lambda_n(x,t)\, dt \\ &= \int\limits_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \frac{\partial}{\partial t}\lambda_n(x,t)\, dF(t). \end{aligned} \tag{3.38}$$

By applying partial Lebesgue-Stieltjes integration (3.38) and using (3.37), it is obtained that

$$\begin{aligned} I_{2,1}(n,x) &= F\left(x-x/n^{\frac{\gamma}{\beta}}\right)\frac{\partial}{\partial t}\left(\lambda_n\left(x,x-x/n^{\frac{\gamma}{\beta}}\right)\right) \\ &\quad + \int\limits_{x-x/n^{\frac{\gamma}{\beta}}}^{x} F(t)\,\frac{\partial^2}{\partial t^2}(\lambda_n(x,t))\, dt \\ &\leq \epsilon\, x/n^{\frac{\gamma}{\beta}}\frac{\partial}{\partial t}\left(\lambda_n\left(x,x-x/n^{\frac{\gamma}{\beta}}\right)\right) \\ &\quad +\epsilon \int\limits_{x-x/n^{\frac{\gamma}{\beta}}}^{x} (x-t)\,\frac{\partial^2}{\partial t^2}(\lambda_n(x,t))\, dt. \end{aligned}$$

Integration by parts again gives

$$\begin{aligned} I_{2,1}(n,x) &= \epsilon\, x/n^{\frac{\gamma}{\beta}}\frac{\partial}{\partial t}\left(\lambda_n\left(x,x-x/n^{\frac{\gamma}{\beta}}\right)\right) \\ &\quad +\epsilon\left\{-x/n^{\frac{\gamma}{\beta}}\frac{\partial}{\partial t}\left(\lambda_n\left(x,x-x/n^{\frac{\gamma}{\beta}}\right)\right)\right. \\ &\quad \left. + \int\limits_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \frac{\partial}{\partial t}(\lambda_n(x,t))\, dt\right\} \\ &= \epsilon \int\limits_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \frac{\partial}{\partial t}(\lambda_n(x,t))\, dt \\ &= \epsilon \int\limits_{x-x/n^{\frac{\gamma}{\beta}}}^{x} d_t(K_n(x,t)) \\ &\leq \epsilon\, B_n(x)\, x^{-\beta}\left(n^{\frac{\gamma}{\beta}}\right)^{\beta-1}. \end{aligned}$$

A similar method can be used for $I_{2,2}(n,x)$. Then, the following inequality is found,

$$\begin{aligned} I_{2,2}(n,x) &\leq \epsilon \int\limits_{x}^{x+(1-x)/n^{\frac{\gamma}{\beta}}} d_t\left(K_n(x,t)\right) \\ &\leq \epsilon\, B_n(x)(1-x)^{-\beta}\left(n^{\frac{\gamma}{\beta}}\right)^{\beta-1}. \end{aligned}$$

Next, $I_1(n,x)$ is estimated. Using partial Lebesgue-Stieltjes integration, it is obtained that

$$\begin{aligned} |I_1(n,x)| &= \int\limits_{0}^{x-x/n^{\frac{\gamma}{\beta}}} \psi\left(|f(t)-f(x)|\right) d_t\left(K_n(x,t)\right) \\ &= \int\limits_{0}^{x-x/n^{\frac{\gamma}{\beta}}} \psi\left(|f_x(t)|\right) \frac{\partial}{\partial t}\left(\lambda_n(x,t)\right) dt \\ &= \psi\left(\left|f_x\left(x-\frac{x}{n^{\frac{\gamma}{\beta}}}\right)\right|\right) \lambda_n\left(x, x-\frac{x}{n^{\frac{\gamma}{\beta}}}\right) \\ &\quad - \int\limits_{0}^{x-x/n^{\frac{\gamma}{\beta}}} \lambda_n(x,t)\, d_t\left(\psi\left(|f_x(t)|\right)\right). \end{aligned}$$

Let $y = x - x/n^{\frac{\gamma}{\beta}}$. By (3.25), it is clear that

$$\lambda_n(x,y) \leq B_n(x)(x-y)^{-\beta}\left(n^{\frac{\gamma}{\beta}}\right)^{\beta-1}. \tag{3.39}$$

Here it should be noted that

$$\begin{aligned} \psi\left(\left|f_x\left(x-\frac{x}{n^{\frac{\gamma}{\beta}}}\right)\right|\right) &= \left|\psi\left(\left|f_x\left(x-\frac{x}{n^{\frac{\gamma}{\beta}}}\right)\right|\right) - \psi\left(|f_x(x)|\right)\right| \\ &\leq \bigvee_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi\left(|f_x|\right). \end{aligned}$$

Using partial integration and applying (3.39), it is obtained that

$$\begin{aligned}
|I_1(n,x)| &\leq \bigvee_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi(|f_x|) \left|\lambda_n\left(x, x-\frac{x}{n^{\frac{\gamma}{\beta}}}\right)\right| \\
&\quad + \int_0^{x-x/n^{\frac{\gamma}{\beta}}} \lambda_n(x,t)\, d_t\left(-\bigvee_t^x \psi(|f_x|)\right) \\
&\leq \bigvee_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi(|f_x|)\, B_n(x)\, x^{-\beta}\left(n^{\frac{\gamma}{\beta}}\right)^{\beta-1} \\
&\quad + \frac{B_n(x)}{n^{\frac{\gamma}{\beta}}} \int_0^{x-x/n^{\frac{\gamma}{\beta}}} (x-t)^{-\beta}\, d_t\left(-\bigvee_t^x \psi(|f_x|)\right) \\
&= \bigvee_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi(|f_x|)\, B_n(x)\, x^{-\beta}\left(n^{\frac{\gamma}{\beta}}\right)^{\beta-1} \\
&\quad + + \frac{B_n(x)}{n^{\frac{\gamma}{\beta}}}\left[-x^{-\beta}n^{\gamma} \bigvee_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi(|f_x|) + x^{-\beta}\bigvee_0^x \psi(|f_x|)\right. \\
&\quad \left. + \int_0^{x-x/n^{\frac{\gamma}{\beta}}} \bigvee_t^x \psi(|f_x|)\, \frac{\beta}{(x-t)^{\beta+1}} dt\right] \\
&= \frac{B_n(x)}{n^{\frac{\gamma}{\beta}}}\left[x^{-\beta}\bigvee_0^x \psi(|f_x|) + \int_0^{x-x/n^{\frac{\gamma}{\beta}}} \bigvee_t^x \psi(|f_x|)\, \frac{\beta}{(x-t)^{\beta+1}} dt\right].
\end{aligned}$$

Changing the variable t by $x - x/u^{1/\beta}$ in the last integral, it is available that

$$\begin{aligned}
\int_0^{x-x/n^{\frac{\gamma}{\beta}}} \bigvee_t^x \psi(|f_x|)\, \frac{\beta}{(x-t)^{\beta+1}} dt &= \frac{1}{x^{\beta}} \int_1^{n^{\gamma}} \bigvee_{x-x/u^{1/\beta}}^{x} \psi(|f_x|)\, du \\
&\leq \frac{1}{x^{\beta}} \sum_{k=1}^{[n^{\gamma}]} \bigvee_{x-x/k^{1/\beta}}^{x} \psi(|f_x|).
\end{aligned}$$

Consequently, it is obtained that

$$|I_1(n,x)| \leq \frac{B_n(x)}{n^{\frac{\gamma}{\beta}}} x^{-\beta}\left[\bigvee_0^x \psi(|f_x|) + \sum_{k=1}^{[n^{\gamma}]} \bigvee_{x-x/k^{1/\beta}}^{x} \psi(|f_x|)\right].$$

By using a similar method, it can be found that

$$|I_3(n,x)| \leq \frac{B_n(x)}{n^{\frac{\gamma}{\beta}}} (1-x)^{-\beta}\left[\bigvee_x^1 \psi(|f_x|) + \sum_{k=1}^{[n^{\gamma}]} \bigvee_x^{x+(1-x)/k^{1/\beta}} \psi(|f_x|)\right].$$

By collecting the above estimates, the required result is attained. $\square$

With the aid of the Theorem 3.2.18., it is easy to obtain another convergence theorem concerning the convergence result.

Theorem 3.2.19. *Let $\psi \in \Psi$ and $f \in L_1([0,1])$ be such that $\psi \circ |f| \in BV([0,1])$. Suppose that $P_{n,k}(x,u)$ satisfies condition (a), (b) and (c). Then at each point $x \in (0,1)$ for which (2.1) holds we have*

$$\lim_{n\to\infty} |(NB_n f)(x) - f(x)| = 0.$$

Proof. From the Theorem 3.2.18. and the definition of ψ function the result is reached, by the arbitrariness of $\epsilon > 0$. $\square$

Corollary 3.2.20. *Let $\psi \in \Psi$ and $f \in L_1([0,1])$ be such that $\psi \circ |f| \in BV([0,1])$. Suppose that $P_{n,k}(x,u)$ satisfies condition (a), (b) and (c). Then*

$$\lim_{n\to\infty} |(NB_n f)(x) - f(x)| = 0$$

holds almost everywhere in $(0,1)$.

Proof. Since almost all $x \in (0,1)$ are Lebesgue points of the function $f \in L_1([0,1])$, then the assertion follows by the Theorem 3.2.19. $\square$

3.2.3 Some Convergence Results in the Space DBV

In this section the rate of pointwise convergence of the nonlinear Bernstein type operators (3.19) to the limit f will be investigated, where f and $\psi \circ |f|$ are functions whose derivatives are of bounded variation on the interval $[0,1]$. At the point $x \in (0,1)$ convergence of $(NB_n f)(x)$ to the limit $f(x)$ will be proved, which can be found in (Karsli et al., 2016).

It should be noted that the rate of convergence for functions in $DBV(I)$ for positive linear operators were first obtained in (Bojanic and Cheng, 1989) and (Bojanic and Cheng, 1992).

Definition 3.2.21. *(Bojanic and Cheng, 1989) Let $DBV(I)$ denotes the class of differentiable functions defined on a set $I \subset \mathbb{R}$, whose derivatives are bounded variation on I and will be denoted as*

$$DBV(I) = \{f : f' \in BV(I)\}.$$

Theorem 3.2.22. *Let $\psi \in \Psi$ and f be a function with derivatives of bounded variation on $[0,1]$. Then for every $x \in (0,1)$, we have for sufficiently large n,*

$$\begin{aligned} |(NB_n f)(x) - f(x)| &\leq \left| \frac{f'(x+) - f'(x-)}{2} \right| \sqrt{\frac{x(1-x)}{n}} \\ &\quad + \frac{2}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-x/k}^{x+(1-x)/k} \left(f'_x\right) + \frac{1}{\mu(n)} \end{aligned} \tag{3.40}$$

where $\bigvee_a^b \left(f'_x\right)$ is the total variation of f'_x on $[a,b]$.

Proof. The difference between $(NB_n f)(x)$ and $f(x)$ can be written as a singular Stieltjes integral as follows;

$$\begin{aligned} (NB_n f)(x) - f(x) &= \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) - f(x) \\ &= \sum_{k=0}^{n} p_{n,k}(x) H_n\left(f\left(\frac{k}{n}\right)\right) - f(x) \\ &= \int_0^1 H_n(f(t))\, d_t K_n(x,t) - f(x) \\ &= \int_0^1 [H_n(f(t)) - f(t)]\, d_t K_n(x,t) \\ &\quad + \int_0^1 [f(t) - f(x)]\, d_t K_n(x,t) \\ &= : I_{n,1}(x) + I_{n,2}(x). \end{aligned}$$

Firstly, it is considered that

$$I_{n,2}(x) = \int_0^1 [f(t) - f(x)]\, d_t K_n(x,t). \tag{3.41}$$

Since $f(t) \in DBV[0,1]$ (3.41) can be rewritten as follows;

$$\begin{aligned}
I_{n,2}(x) &= \int_0^x [f(t) - f(x)]\, d_t K_n(x,t) \\
&\quad + \int_x^1 [f(t) - f(x)]\, d_t K_n(x,t) \\
&= -\int_0^x \left[\int_t^x f'(u)\, du \right] d_t K_n(x,t) \\
&\quad + \int_x^1 \left[\int_x^t f'(u)\, du \right] d_t K_n(x,t) \\
&= : -I_{n,2,1}(x) + I_{n,2,2}(x),
\end{aligned}$$

where

$$I_{n,2,1}(x) = \int_0^x \left[\int_t^x f'(u)\, du \right] d_t K_n(x,t) \tag{3.42}$$

and

$$I_{n,2,2}(x) = \int_x^1 \left[\int_x^t f'(u)\, du \right] d_t K_n(x,t). \tag{3.43}$$

For any $f(t) \in DBV[0,1]$, $f'(t)$ can be decomposed into four parts by using (3.23) as

$$\begin{aligned}
f'(t) &= \frac{f'(x+) + f'(x-)}{2} + f'_x(t) + \frac{f'(x+) - f'(x-)}{2} sgn(t-x) \\
&\quad + \delta_x(t) \left[f'(x) - \frac{f'(x+) + f'(x-)}{2} \right],
\end{aligned}$$

where

$$\delta_x(t) = \begin{cases} 1 \ , & x = t, \\ 0 \ , & x \neq t. \end{cases}$$

If this equality used in (3.42) and (3.43), the following expressions is had.

$$\begin{aligned}
I_{n,2,1}(x) &= \int_0^x \Bigg\{ \int_t^x \frac{1}{2} \left(f'(x+) + f'(x-) \right) \\
&\quad + f'_x(u) + \frac{f'(x+) - f'(x-)}{2} sgn(u-x) \\
&\quad + \delta_x(u) \left[f'(x) - \frac{1}{2} \left(f'(x+) + f'(x-) \right) \right] du \Bigg\} d_t K_n(x,t)
\end{aligned}$$

and

$$
\begin{aligned}
I_{n,2,2}(x) &= \int\limits_{x}^{1}\left\{\int\limits_{x}^{t}\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right. \\
&+f_x'(u)+\frac{f'(x+)-f'(x-)}{2}sgn(u-x) \\
&\left.+\delta_x(u)\left[f'(x)-\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right]du\right\}d_tK_n(x,t).
\end{aligned}
$$

Firstly, $I_{n,2,1}(x)$ is evaluated.

$$
\begin{aligned}
I_{n,2,1}(x) &= \frac{f'(x+)+f'(x-)}{2}\int\limits_{0}^{x}(x-t)\,d_tK_n(x,t) \\
&+\int\limits_{0}^{x}\left[\int\limits_{t}^{x}f_x'(u)\,du\right]d_tK_n(x,t) \\
&-\frac{f'(x+)-f'(x-)}{2}\int\limits_{0}^{x}(x-t)\,d_tK_n(x,t) \\
&+\left[f'(x)-\frac{f'(x+)+f'(x-)}{2}\right]\int\limits_{0}^{x}\left[\int\limits_{t}^{x}\delta_x(u)du\right]d_tK_n(x,t).
\end{aligned}
$$

It is obvious that

$$
\int\limits_{t}^{x}\delta_x(u)du=0.
$$

From this fact that, it will be obtained as

$$
\begin{aligned}
I_{n,2,1}(x) &= \frac{f'(x+)+f'(x-)}{2}\int\limits_{0}^{x}(x-t)\,d_tK_n(x,t) \\
&+\int\limits_{0}^{x}\left[\int\limits_{t}^{x}f_x'(u)\,du\right]d_tK_n(x,t) \\
&-\frac{f'(x+)-f'(x-)}{2}\int\limits_{0}^{x}(x-t)\,d_tK_n(x,t).
\end{aligned}
\tag{3.44}
$$

Using a similar method, for evaluating $I_{n,2,2}(x)$,

$$
\begin{aligned}
I_{n,2,2}(x) &= \frac{f'(x+)+f'(x-)}{2}\int_x^1 (t-x)\,d_tK_n(x,t) \\
&+\int_x^1\left[\int_x^t f_x'(u)\,du\right]d_tK_n(x,t) \\
&+\frac{f'(x+)-f'(x-)}{2}\int_x^1 (t-x)\,d_tK_n(x,t) \\
&+\left[f'(x)-\frac{f'(x+)+f'(x-)}{2}\right]\int_x^1\left[\int_x^t \delta_x(u)du\right]d_tK_n(x,t)
\end{aligned}
$$

and it is found that

$$
\begin{aligned}
I_{n,2,2}(x) &= \frac{f'(x+)+f'(x-)}{2}\int_x^1 (t-x)\,d_tK_n(x,t) \\
&+\int_x^1\left[\int_x^t f_x'(u)\,du\right]d_tK_n(x,t) \\
&+\frac{f'(x+)-f'(x-)}{2}\int_x^1 (t-x)\,d_tK_n(x,t)\,. \qquad (3.45)
\end{aligned}
$$

By combining (3.44) and (3.45), it will be attained that

$$
\begin{aligned}
-I_{n,2,1}(x)+I_{n,2,2}(x) &= \frac{f'(x+)+f'(x-)}{2}\int_0^1 (t-x)\,d_tK_n(x,t) \\
&+\frac{f'(x+)-f'(x-)}{2}\int_0^1 |t-x|\,d_tK_n(x,t) \\
&-\int_0^x\left[\int_t^x f_x'(u)\,du\right]d_tK_n(x,t)\,. \\
&+\int_x^1\left[\int_x^t f_x'(u)\,du\right]d_tK_n(x,t)
\end{aligned}
$$

From the last expression, (3.41) can be rewritten as follows

$$\begin{aligned} I_{n,2}(x) &= \frac{f'(x+)+f'(x-)}{2}\int_0^1 (t-x)\,d_tK_n(x,t) \\ &\quad +\frac{f'(x+)-f'(x-)}{2}\int_0^1 |t-x|\,d_tK_n(x,t) \\ &\quad -\int_0^x \left[\int_t^x f'_x(u)\,du\right] d_tK_n(x,t) \\ &\quad +\int_x^1 \left[\int_x^t f'_x(u)\,du\right] d_tK_n(x,t)\,. \end{aligned} \tag{3.46}$$

On the other hand, since

$$\int_0^1 |t-x|\,d_tK_n(x,t) = B_n(|t-x|\,;x)$$

and

$$\int_0^1 (t-x)\,d_tK_n(x,t) = B_n((t-x)\,;x)\,,$$

by using these equalities in (3.46) and taking absolute value, (3.46) can be re-expressed as follows;

$$\begin{aligned} |I_{n,2}(x)| &\leq \left|\frac{f'(x+)+f'(x-)}{2}\right| |B_n((t-x);x)| \\ &\quad +\left|\frac{f'(x+)-f'(x-)}{2}\right| |B_n(|t-x|\,;x)| \\ &\quad +\left|-\int_0^x \left[\int_t^x f'_x(u)\,du\right] d_tK_n(x,t)\right| \\ &\quad +\left|\int_x^1 \left[\int_x^t f'_x(u)\,du\right] d_tK_n(x,t)\right|. \end{aligned}$$

Using Lebesgue-Stieltjes integration, and according to (3.22), it is obtained that

$$\int_0^x \left[\int_t^x f'_x(u)\,du\right] d_tK_n(x,t) = \int_0^x f'_x(t)\,\lambda_n(x,t)\,dt.$$

Thus

$$\left|-\int_0^x \left[\int_t^x f'_x(u)\,du\right] d_tK_n(x,t)\right| \leq \int_0^x |f'_x(t)|\,\lambda_n(x,t)\,dt$$

and

$$\left|-\int_0^x\left[\int_t^x f_x'(u)\,du\right]d_tK_n(x,t)\right| \leq \int_0^{x-\frac{x}{\sqrt{n}}}\left|f_x'(t)\right|\lambda_n(x,t)\,dt + \int_{x-\frac{x}{\sqrt{n}}}^x \left|f_x'(t)\right|\lambda_n(x,t)\,dt.$$

Since

$$f_x'(x)=0$$

and

$$\lambda_n(x,t)\leq 1,$$

one has

$$\begin{aligned}\int_{x-\frac{x}{\sqrt{n}}}^x \left|f_x'(t)\right|\lambda_n(x,t)\,dt &= \int_{x-\frac{x}{\sqrt{n}}}^x \left|f_x'(t)-f_x'(x)\right|\lambda_n(x,t)\,dt\\ &\leq \int_{x-\frac{x}{\sqrt{n}}}^x \bigvee_t^x\left(f_x'\right)dt.\end{aligned}$$

By the change of variables $t=x-\frac{x}{\sqrt{n}}$, it is obtained that

$$\int_{x-\frac{x}{\sqrt{n}}}^x \bigvee_t^x\left(f_x'\right)dt \leq \bigvee_{x-\frac{x}{\sqrt{n}}}^x\left(f_x'\right)\int_{x-\frac{x}{\sqrt{n}}}^x dt.$$

Besides from (3.25), it can be written as

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}}\left|f_x'(t)\right|\lambda_n(x,t)\,dt &\leq \frac{x(1-x)}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\left|f_x'(t)-f_x'(x)\right|\frac{dt}{(x-t)^2}\\ &\leq \frac{x(1-x)}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\bigvee_t^x\left(f_x'\right)\frac{dt}{(x-t)^2}.\end{aligned}$$

By the change of variables $t=x-\frac{x}{u}$ again, it is available that

$$\begin{aligned}\frac{x(1-x)}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\bigvee_t^x\left(f_x'\right)\frac{dt}{(x-t)^2} &= \frac{x(1-x)}{n}\int_1^{\sqrt{n}}\bigvee_{x-\frac{x}{u}}^x\left(f_x'\right)\frac{\left(\frac{x}{u^2}\right)du}{\left(\frac{x}{u}\right)^2}\\ &\leq \frac{(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^x\left(f_x'\right)\end{aligned}$$

and hence it is obtained that

$$\left|-\int_{0}^{x}\left[\int_{t}^{x}f_{x}'(u)\,du\right]d_{t}K_{n}(x,t)\right| \leq \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f_{x}') + \frac{(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f_{x}').$$

Since

$$\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f_{x}') \leq \frac{2x}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f_{x}'),$$

it follows that

$$\begin{aligned}\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f_{x}') + \frac{(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f_{x}') &\leq \frac{2x}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f_{x}') \\ &\quad + \frac{2(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f_{x}') \\ &\leq \frac{2}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f_{x}').\end{aligned}$$

Therefore

$$\left|-\int_{0}^{x}\left[\int_{t}^{x}f_{x}'(u)\,du\right]d_{t}K_{n}(x,t)\right| \leq \frac{2}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f_{x}').$$

Using a similar method for estimating, then one has

$$\left|\int_{x}^{1}\left[\int_{x}^{t}f_{x}'(u)\,du\right]d_{t}K_{n}(x,t)\right| \leq \frac{(1-x)}{\sqrt{n}}\bigvee_{x}^{x+\frac{(1-x)}{\sqrt{n}}}(f_{x}') + \frac{x}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{(1-x)}{k}}(f_{x}').$$

Furthermore, since

$$\frac{1-x}{\sqrt{n}}\bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}}(f_{x}') \leq \frac{2(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_{x}'),$$

the following inequality can be written

$$\begin{aligned}\frac{1-x}{\sqrt{n}}\bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}}(f_x')+\frac{x}{n}\sum_{k=1}^{\left[\sqrt{n}\right]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_x') &\le \frac{2(1-x)}{n}\sum_{k=1}^{\left[\sqrt{n}\right]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_x') \\ &\quad +\frac{2x}{n}\sum_{k=1}^{\left[\sqrt{n}\right]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_x') \\ &\le \frac{2}{n}\sum_{k=1}^{\left[\sqrt{n}\right]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_x').\end{aligned}$$

Thus, it is obtained that

$$\left|\int_x^1\left[\int_x^t f_x'(u)\,du\right]d_tK_n(x,t)\right|\le\frac{2}{n}\sum_{k=1}^{\left[\sqrt{n}\right]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_x').$$

By collecting the estimates, (3.40) is attained, i.e.,

$$|I_{n,2}(x)|\le\left|\frac{f'(x+)-f'(x-)}{2}\right|\sqrt{\frac{x(1-x)}{n}}+\frac{2}{n}\sum_{k=1}^{\left[\sqrt{n}\right]}\bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}}(f_x'),$$

and

$$\begin{aligned}|I_{n,1}(x)| &= \left|\int_0^1[H_n(f(t))-f(t)]\,d_tK_n(x,t)\right| \\ &\le \int_0^1|H_n(f(t))-f(t)|\,d_tK_n(x,t) \\ &\le \frac{1}{\mu(n)}\end{aligned}$$

holds for sufficiently large n. This completes the proof of the theorem. □

Theorem 3.2.23. *Let $\psi\in\Psi$ and $f\in X$ be such that $\psi\circ|f|\in DBV([0,1])$. Then for every $x\in(0,1)$, we have for sufficiently large n,*

$$\begin{aligned}|(NB_nf)(x)-f(x)| &\le \frac{\left|(\psi\circ|f|)'(x-)-(\psi\circ|f|)'(x+)\right|}{2}\sqrt{\frac{x(1-x)}{n}} \\ &\quad +\frac{2}{n}\sum_{k=1}^{\left[\sqrt{n}\right]}\bigvee_{x-x/k}^{x+(1-x)/k}(\psi\circ|f|)_x'+\frac{1}{\mu(n)}\end{aligned}$$

where $\bigvee_a^b(\psi\circ|f|)_x'$ is the total variation of $(\psi\circ|f|)_x'$ on $[a,b]$.

Proof. The difference between $(NB_nf)(x)$ and $f(x)$ can be written as a singular Stieltjes integral as follows;

$$\begin{aligned}
|(NB_nf)(x)-f(x)| &= \left|\int_0^1 H_n(f(t))\,d_tK_n(x,t)-f(x)\right| \\
&= \left|\int_0^1 [H_n(f(t))-H_n(f(x))]\,d_tK_n(x,t)\right. \\
&\quad \left.+\int_0^1 [H_n(f(x))-f(x)]\,d_tK_n(x,t)\right| \\
&\leq \int_0^1 |H_n(f(t))-H_n(f(x))|\,d_tK_n(x,t) \\
&\quad +\int_0^1 |H_n(f(x))-f(x)|\,d_tK_n(x,t) \\
&\leq \int_0^1 |H_n(f(x))-f(x)|\,d_tK_n(x,t) \\
&\quad +\int_0^1 \psi(|f(t)-f(x)|)\,d_tK_n(x,t) \\
&=: I_{n,1}(x)+I_{n,2}(x).
\end{aligned}$$

Note that for a concave function ψ

$$-\psi(|f(t)-f(x)|)\leq \psi(|f(t)|)-\psi(|f(x)|)$$

holds. Firstly, it is considered that

$$I_{n,2}(x)=\int_0^1 \psi(|f(t)-f(x)|)\,d_tK_n(x,t). \tag{3.47}$$

Since $(\psi\circ|f|)(t)\in DBV[0,1]$, (3.47) can be rewritten as follows:

$$\begin{aligned}
-I_{n,2}(x) &\leq \int_x^0 [\psi(|f(t)|)-\psi(|f(x)|)]\,d_tK_n(x,t) \\
&\quad+\int_1^x [\psi(|f(t)|)-\psi(|f(x)|)]\,d_tK_n(x,t) \\
&= \int_0^x \left[\int_t^x (\psi\circ|f|)'(u)\,du\right] d_tK_n(x,t) \\
&\quad+\int_x^1 \left[\int_t^x (\psi\circ|f|)'(u)\,du\right] d_tK_n(x,t) \\
&=: I_{n,2,1}(x)-I_{n,2,2}(x)
\end{aligned}$$

where

$$I_{n,2,1}(x)=\int_0^x \left[\int_t^x (\psi\circ|f|)'(u)\,du\right] d_tK_n(x,t) \tag{3.48}$$

and

$$I_{n,2,2}(x)=\int_x^1 \left[\int_x^t (\psi\circ|f|)'(u)\,du\right] d_tK_n(x,t). \tag{3.49}$$

For any $(\psi\circ|f|)(t)\in DBV[0,1]$, $(\psi\circ|f|)(t)$ is decomposed into four parts by using (3.23) as

$$\begin{aligned}
(\psi\circ|f|)'(t) &= \frac{(\psi\circ|f|)'(x+)+(\psi\circ|f|)'(x-)}{2} \\
&\quad+(\psi\circ|f|)'_x(t) \\
&\quad+\frac{(\psi\circ|f|)'(x+)-(\psi\circ|f|)'(x-)}{2}sgn(t-x) \\
&\quad+\delta_x(t)\Big[(\psi\circ|f|)'(x) \\
&\quad-\frac{(\psi\circ|f|)'(x+)+(\psi\circ|f|)'(x-)}{2}\Big],
\end{aligned}$$

where

$$\delta_x(t)=\begin{cases}1 &, \quad x=t,\\ 0 &, \quad x\neq t.\end{cases}$$

If this equality used in (3.48) and (3.49), the following expressions become available.

$$\begin{aligned} I_{n,2,1}(x) &= \int_0^x \left\{ \int_t^x \frac{(\psi\circ|f|)'(x+)+(\psi\circ|f|)'(x-)}{2} \right. \\ &\quad + (\psi\circ|f|)'_x(u) \\ &\quad + \frac{(\psi\circ|f|)'(x+)-(\psi\circ|f|)'(x-)}{2} sgn(u-x) \\ &\quad + \delta_x(u)\left[(\psi\circ|f|)'(x)\right. \\ &\quad \left.\left. - \frac{(\psi\circ|f|)'(x+)+(\psi\circ|f|)'(x-)}{2}\right] du \right\} d_tK_n(x,t) \end{aligned}$$

and

$$\begin{aligned} I_{n,2,2}(x) &= \int_x^1 \left\{ \int_x^t \frac{(\psi\circ|f|)'(x+)+(\psi\circ|f|)'(x-)}{2} \right. \\ &\quad + (\psi\circ|f|)'_x(u) \\ &\quad + \frac{(\psi\circ|f|)'(x+)-(\psi\circ|f|)'(x-)}{2} sgn(u-x) \\ &\quad + \delta_x(u)\left[(\psi\circ|f|)'(x)\right. \\ &\quad \left.\left. - \frac{(\psi\circ|f|)'(x+)+(\psi\circ|f|)'(x-)}{2}\right] du \right\} d_tK_n(x,t). \end{aligned}$$

Firstly, $I_{n,2,1}(x)$ is evaluated.

$$\begin{aligned} I_{n,2,1}(x) &= \frac{(\psi\circ|f|)'(x+)+(\psi\circ|f|)'(x-)}{2} \int_0^x (x-t)\, d_tK_n(x,t) \\ &\quad + \int_0^x \left[\int_t^x (\psi\circ|f|)'_x(u)\, du\right] d_tK_n(x,t) \\ &\quad - \frac{(\psi\circ|f|)'(x+)-(\psi\circ|f|)'(x-)}{2} \int_0^x (x-t)\, d_tK_n(x,t) \\ &\quad + \left[(\psi\circ|f|)'(x) - \frac{(\psi\circ|f|)'(x+)+(\psi\circ|f|)'(x-)}{2}\right] \\ &\quad \int_0^x \left[\int_t^x \delta_x(u)\, du\right] d_tK_n(x,t). \end{aligned}$$

It is obvious that

$$\int_t^x \delta_x(u)\, du = 0.$$

From this fact that, it is obtained as

$$\begin{aligned} I_{n,2,1}(x) &= \frac{(\psi\circ|f|)'(x+)+(\psi\circ|f|)'(x-)}{2}\int_0^x (x-t)\,d_tK_n(x,t) \\ &+\int_0^x\left[\int_t^x (\psi\circ|f|)_x'(u)\,du\right]d_tK_n(x,t) \\ &-\frac{(\psi\circ|f|)'(x+)-(\psi\circ|f|)'(x-)}{2}\int_0^x (x-t)\,d_tK_n(x,t)\,. \end{aligned} \tag{3.50}$$

Using a similar method, for evaluating $I_{n,2,2}(x)$,

$$\begin{aligned} I_{n,2,2}(x) &= \frac{(\psi\circ|f|)'(x+)+(\psi\circ|f|)'(x-)}{2}\int_x^1 (t-x)\,d_tK_n(x,t) \\ &+\int_x^1\left[\int_x^t (\psi\circ|f|)_x'(u)\,du\right]d_tK_n(x,t) \\ &-\frac{(\psi\circ|f|)'(x+)-(\psi\circ|f|)'(x-)}{2}\int_x^1 (t-x)\,d_tK_n(x,t) \\ &+\left[(\psi\circ|f|)'(x)-\frac{(\psi\circ|f|)'(x+)+(\psi\circ|f|)'(x-)}{2}\right] \\ &\int_x^1\left[\int_x^t \delta_x(u)\,du\right]d_tK_n(x,t) \end{aligned}$$

and it is found that

$$\begin{aligned} I_{n,2,2}(x) &= \frac{(\psi\circ|f|)'(x+)+(\psi\circ|f|)'(x-)}{2}\int_x^1 (t-x)\,d_tK_n(x,t) \\ &+\int_x^1\left[\int_x^t (\psi\circ|f|)_x'(u)\,du\right]d_tK_n(x,t) \\ &-\frac{(\psi\circ|f|)'(x+)-(\psi\circ|f|)'(x-)}{2}\int_x^1 (t-x)\,d_tK_n(x,t)\,. \end{aligned} \tag{3.51}$$

Combining (3.50) and (3.51), it is obtained that

$$\begin{aligned} I_{n,2,1}(x) - I_{n,2,2}(x) &= \frac{(\psi \circ |f|)'(x+) + (\psi \circ |f|)'(x-)}{2} \int_0^1 (t-x)\, d_t K_n(x,t) \\ &\quad - \frac{(\psi \circ |f|)'(x+) - (\psi \circ |f|)'(x-)}{2} \int_0^1 |t-x|\, d_t K_n(x,t) \\ &\quad + \int_0^x \left[\int_t^x (\psi \circ |f|)'_x(u)\, du \right] d_t K_n(x,t) \\ &\quad - \int_x^1 \left[\int_x^t (\psi \circ |f|)'_x(u)\, du \right] d_t K_n(x,t). \end{aligned}$$

On the other hand, note that

$$\begin{aligned} \int_0^1 |t-x|\, d_t K_n(x,t) &= B_n(|t-x|; x) \\ &\leq \sqrt{\left(B_n (t-x)^2\right)(x)} \\ &= \sqrt{\frac{x(1-x)}{n}} \end{aligned}$$

and

$$\int_0^1 (t-x)\, d_t K_n(x,t) = B_n((t-x))(x) = 0.$$

Therefore $I_{n,2,1}(x) - I_{n,2,2}(x)$ can be estimated as follows;

$$\begin{aligned} I_{n,2,1}(x) - I_{n,2,2}(x) &\leq \frac{(\psi \circ |f|)'(x+) - (\psi \circ |f|)'(x-)}{2} \sqrt{\frac{x(1-x)}{n}} \\ &\quad + \int_0^x \left[\int_t^x (\psi \circ |f|)'_x(u)\, du \right] d_t K_n(x,t) \\ &\quad - \int_x^1 \left[\int_x^t (\psi \circ |f|)'_x(u)\, du \right] d_t K_n(x,t). \end{aligned}$$

By using Lebesgue-Stieltjes integration, and according to (3.22), it is obtained that

$$\int_0^x \left[\int_t^x (\psi \circ |f|)'_x(u)\, du \right] d_t K_n(x,t) = \int_0^x (\psi \circ |f|)'_x(t)\, \lambda_n(x,t)\, dt.$$

and

$$\int_0^x \left[\int_t^x (\psi\circ|f|)'_x(u)\,du\right] d_tK_n(x,t) \leq \int_0^{x-\frac{x}{\sqrt{n}}} \left|(\psi\circ|f|)'_x(t)\right| \lambda_n(x,t)\,dt + \int_{x-\frac{x}{\sqrt{n}}}^{x} \left|(\psi\circ|f|)'_x(t)\right| \lambda_n(x,t)\,dt.$$

By using the method in the proof of the Theorem 3.2.22., one has

$$\int_0^x \left[\int_t^x (\psi\circ|f|)'_x(u)\,du\right] d_tK_n(x,t) \leq \frac{2}{n}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x} (\psi\circ|f|)'_x.$$

and

$$-\int_x^1 \left[\int_x^t f'_x(u)\,du\right] d_tK_n(x,t) \leq \frac{2}{n}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{1-x}{k}} (\psi\circ|f|)'_x.$$

In conclusion it is obtained that

$$I_{n,2}(x) \leq \frac{\left|(\psi\circ|f|)'(x+) - (\psi\circ|f|)'(x-)\right|}{2}\sqrt{\frac{x(1-x)}{n}} + \frac{2}{n}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}} (\psi\circ|f|)'_x.$$

Since

$$I_{n,1}(x) = \int_0^1 |H_n(f(t)) - f(t)|\,d_tK_n(x,t) \leq \frac{1}{\mu(n)}$$

holds for sufficiently large n, the proof of the theorem is now complete.

□

3.2.4 A Voronovskaya-Type Theorem for Nonlinear Bernstein Type Operators

In the final section of this chapter a Voronovskaya-type theorem is mentioned. In 1932 Voronovskaya gave the first asymptotic formula for the pointwise approximation of continuous functions which have a second derivative at a certain point $x \in I = [0,1]$. From than on, not a few generalized versions of Voronovskaya's formula have been worked, for several discrete operators, which present generalized versions of the classical Bernstein operator.

Thereby, a formula of Voronovskaya-type was given for a general class of discrete operators in (Bardaro and Mantellini, 2009). In this section, an asymptotic formula is provided, which can be found in (Karsli and Altin, 2015b), for a certain nonlinear Bernstein type operators of the form (3.19).

In this instance the theory is quite different because only some estimates of the error of approximation can be obtained in terms of limsup and absolute moments. It is noted that the results are strict extension of the classical ones, nominately, the results deal with the linear Bernstein operators.

The following definition is helpful in connection with the main theorem.

Definition 3.2.24. *It will be expressed that the sequence $(P_{n,k})_{n\in\mathbb{N}}$ is $\psi-$singular if the following assumptions are satisfied;*

(V.1) For every $x \in I$ and $\delta > 0$ there holds

$$\psi\left(\sum_{\left|\frac{k}{n}-x\right|\geq\delta}\left|\frac{k}{n}-x\right|p_{n,k}(x)\right) = o\left(n^{-1}\right)\ ,\ (n\to\infty)\,.$$

(V.2) For every $u \in \mathbb{R}$ and for every $x \in I$,

$$\lim_{n\to\infty} n\left[\sum_{k=0}^{n} P_{n,k}(x,u) - u\right] = 0. \tag{3.52}$$

Before formulating the main theorem of this section, the following lemma is given in virtue of (Bojanic and Cheng, 1989).

Lemma 3.2.25. *The first order absolute moment for Bernstein operator is defined as*

$$M_1(p_{n,k},x)=\sum_{k=0}^{n}\left|\left(\frac{k}{n}-x\right)\right|p_{n,k}(x) \tag{3.53}$$

and the inequality

$$M_1(p_{n,k},x)\leq\left(\frac{2x(1-x)}{\pi}\right)^{\frac{1}{2}}\frac{1}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right)$$

is derived for this moment.

Theorem 3.2.26. *Let $f\in L_1[0,1]$ be a function such that $f'(x)$ exists at a point $x\in(0,1)$. Let's assume that the sequence $(P_{n,k})_{n\in\mathbb{N}}$ is $\psi-$singular and*

$$\limsup_{n\to\infty} n\,\psi(M_1(p_{n,k},x))=l_1(x)\in\mathbb{R}, \tag{3.54}$$

where $M_1(p_{n,k},x)$ is the first order absolute moment of Bernstein operators given in (3.53). Then,

$$\limsup_{n\to\infty} n\ |(NB_nf)(x)-f(x)|\leq M\,l_1(x),$$

where $M>0$ be a sufficiently large integer.

Proof. Since f is differentiable at the point x, then there exists a bounded function h such that

$$\lim_{y\to 0}h(y)=0.$$

By Taylor's formula, it is available that

$$f\left(\frac{k}{n}\right)=f(x)+\left(\frac{k}{n}-x\right)f'(x)+\left(\frac{k}{n}-x\right)h\left(\frac{k}{n}-x\right).$$

Now, it can be written that

$$\begin{aligned}
n\ |(NB_nf)(x)-f(x)| &= n\left|\sum_{k=0}^{n}\left\{H_n\left(f\left(\frac{k}{n}\right)\right)-f(x)\right\}p_{n,k}(x)\right|\\
&\leq n\sum_{k=0}^{n}\psi\left(\left|f\left(\frac{k}{n}\right)-f(x)\right|\right)p_{n,k}(x)\\
&\quad +n\left|\sum_{k=0}^{n}\{H_n(f(x))-f(x)\}p_{n,k}(x)\right|\\
&=: I_1(x)+I_2(x).
\end{aligned}$$

By assumption (3.52), term $I_2(x)$ tends to zero.The first term can be estimated in the following way: Let $M>0$ be an integer such that

$$|f'(x)|+\left|h\left(\frac{k}{n}-x\right)\right|\leq M.$$

Using sub-additivity of the function $\psi(x)$ and $x \geq 0$, it is available that

$$\begin{aligned} I_1(x) &= n \sum_{k=0}^{n} \psi\left(\left|\left(\frac{k}{n}-x\right) f'(x) + \left(\frac{k}{n}-x\right) h\left(\frac{k}{n}-x\right)\right|\right) p_{n,k}(x) \\ &\leq n \sum_{k=0}^{n} \psi\left(\left|\left(\frac{k}{n}-x\right)\right| \left[|f'(x)| + \left|h\left(\frac{k}{n}-x\right)\right|\right]\right) p_{n,k}(x) \\ &\leq n \left\{\sum_{k=0}^{n} \psi\left(M\left|\left(\frac{k}{n}-x\right)\right|\right) p_{n,k}(x)\right\} \\ &\leq n\, M \left\{\sum_{k=0}^{n} \psi\left(\left|\left(\frac{k}{n}-x\right)\right|\right) p_{n,k}(x)\right\}. \end{aligned}$$

In virtue of Jensen's Inequality, it can be written that

$$\begin{aligned} I_1(x) &\leq n\, M\, \psi\left(\sum_{k=0}^{n} \left|\left(\frac{k}{n}-x\right)\right| p_{n,k}(x)\right) \\ &= n\, M\, \psi\left(M_1(p_{n,k}, x)\right). \end{aligned}$$

In view of (3.54), one has

$$\limsup_{n\to\infty} n\ \left|(NB_n f)(x) - f(x)\right| \leq M\ l_1(x).$$

This completes the proof of the theorem. □

4. CONCLUSION AND RECOMMENDATION

After the approach of (Musielak, 1983), several mathematicians have undertaken the program of extending approximation by nonlinear operators in many ways, including pointwise and uniform convergence, Korovkin type theorems in abstract function spaces, sampling series and so on. We note that approximation with nonlinear integral operators of convolution type was introduced by J. Musielak in (Musielak, 1983) and widely developed in (Bardaro et al., 2003). Especially, nonlinear integral operators of type

$$(I_\lambda f)(x) = \int_a^b K_\lambda(t-x, f(t))\, dt, \quad x \in (a,b),$$

and its special cases were studied by several mathematican in some Lebesgue spaces (Bardaro et al., 2008, 2011), (Gadjiev, 1963; Karsli, 2006, 2007, 2008), (Karsli and Gupta, 2008; Karsli and Ibikli, 2007), (Swiderski and Wachnicki, 2000) and (Taberski, 1962). Such developments delineated a theory which is nowadays referred to the theory of approximation by nonlinear integral operators.

As a main problem of this thesis, approximation properties of nonlinear Bernstein operators $(NB_n f)$ of the form

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) \quad , \quad 0 \le x \le 1 \quad , \quad n \in \mathbb{N},$$

acting on bounded functions f on an interval $[0,1]$, where $P_{n,k}$ satisfy some suitable assumptions is studied.

Actually we consider the problem of approximating a given real-valued function f, defined on $[0,1]$, by means of a sequence of nonlinear Bernstein operators $(NB_n f)$. Operators like positive, linear, convolution, moment and sampling play an important role in several branches of Mathematics, for instance in reconstruction of signals and images, in Fourier analysis, operator theory, probability theory and approximation theory.

For further reading, other kinds of convergence results of linear and nonlinear operators in the Lebesgue spaces, Musielak-Orlicz spaces, $BV-$spaces and $BV_\varphi-$spaces can be studied. Also, the applications of these type operators on signal analysis and image prossessing can be studied.

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