WAVE ASYMPTOTICS USING MATCHED ASYMPTOTIC EXPANSIONS

by

KADİR KANAT

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Prof. Dr. Davut Köşker

Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Oğuz Yılmaz

Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality as a thesis for the degree of Master of Science.

Prof. Dr. Oğuz Yılmaz

Supervisor

Examining Committee Members

Prof. Dr. Mevlüt Teymur

Prof. Dr. Oğuz Yılmaz

Yrd. Doç. Dr. Davut Uğurlu

ABSTRACT

WAVE ASYMPTOTICS USING MATCHED ASYMPTOTIC EXPANSIONS

Kanat,Kadir

Master, Department of Mathematics

Supervisor: Prof. Dr. Oğuz Yılmaz

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First, basic definitions and theorems regarding asymptotic power series expansions are presented. Matched asymptotic expansions are presented next, together with some regular and singular simple problems. Finally, transmission past a partially immersed cylinder is analyzed using matched asymptotic expansions. Fluid domain is divided into three parts: Right and left inner regions, outer region and surface wave regions. First, right inner region approximation is carried out for the first two orders. Then by using matching technique, outer region is approximated. Again, by applying matching technique left inner region approximation is determined.

Keywords: Matched asymptotic expansions, Wave asymptotics, Transmission and diffraction of surface waves, Green's function, Wave-maker problem.

ÖZET

KARŞILAŞTIRMALI ASİMTOTİK AÇILIMLAR İLE

DALGA ASİMTOTİKLERİ

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İlk olarak, asimtotik kuvvet seri açılımları ile ilgili temel tanımlar ve teoremler sunulmuştur. Karşılaştırmalı asimtotik açılımlar konusu bazı düzgün ve tekil basit problemlerle beraber sunulmuştur. Son olarak, kısmen daldırılımış silindirdeki iletim karşılaştırmalı asimtotik açılımlar kullanılarak analiz edilmiştir. Akışkan alanı sağ ve sol iç bölgeler, dış bölge ve yüzey dalga bölgesi olmak üzere üç parçaya ayrıldı. İlk önce sağ iç bölge yaklaşımı ilk iki derece için yapıldı. Sonra karşılaştırma tekniği kullanılarak dış bölgeye yaklaşıldı. Tekrar karşılaştırma tekniği uygulanarak sol iç bölge yaklaşımı tanımlandı.

Anahtar Kelimeler: Karşılaştırmalı asimtotik açılımlar, Dalga asimtotikleri, Yüzey dalgalarının iletimi ve yansıması, Green fonksiyonu, Dalga-yapımı problemi.

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To My Family

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CHAPTER 1

FUNDAMENTAL CONCEPTS

1.1 Introduction

Simply stated, asymptotic analysis is that branch of mathematics devoted to the study of the behavior of functions at and near given points in their domains of definition [1]. Suppose then that $f(z)$ is a function of the complex variable z. Suppose further that we wish to study f near the point $z = z_0$. If f is analytic at $z = z_0$, then the desired behavior can be determined by studying its Taylor series expansion about $z = z_0$.

Now suppose that $z = z_0$ is a singularity of f. If it is either a pole or a branch point, then again the analysis can be reduced to the investigation of convergent series expansions. However, if $z = z_0$ is an essential singularity of f, then no such reduction is possible and the analysis is far more complicated. Partly for this reason, we shall find that most often our investigations will involve the study of functions near the points of essential singularity.

In this chapter we shall consider some of the more fundamental concepts of the subject. It is our aim to present enough introductory material so that the techniques to be presented in the future chapters can be well understood. Therefore, the present chapter shall consist of several definitions and theorems to place the subject on a firm mathematical foundation and also several heuristic discussions.

Let us begin by considering a particular example. We define

$$I(x) = xe^x \int_x^\infty \frac{e^{-t}}{t} dt. \tag{1.1}$$

Here x is real and nonnegative. The integral in (1.1) is often referred to as the *exponential integral* and is denoted by $E_1(x)$. The reason for the factor xe^x will be clear from the discussion below.

Suppose it is desired to approximate $I(x)$ at certain values of x. More precisely, suppose that an estimate of I correct to three significant figures is required. Our first inclination might be to seek a series representation of I and the use the appropriate partial sums to obtain the desired approximations.

By applying L'Hospital's rule in (1.1) we find that

$$\begin{aligned} \lim_{x\to 0^+} I &= 0 \\ \lim_{x\to\infty} I &= \lim_{x\to\infty}\left[\frac{\int_x^\infty t^{-1}e^{-t}dt}{x^{-1}e^{-x}}\right] = \lim_{x\to\infty}\left[1-\frac{1}{x+1}\right] = 1 \end{aligned} \tag{1.2}$$

To expand I in a series about $x=0$ is by no means a simple matter. The reason for this is due to the fact that $E_1(x)$ has a logarithmic singularity at the origin and hence $I(x)$ does not have a Taylor series expansion about $x=0$. Nevertheless an expansion can be obtained. Since

$$\begin{aligned} \frac{e^{-t}}{t} &= \sum_{n=0}^{\infty}\frac{(-1)^n t^{n-1}}{n!} = \frac{1}{t} - 1 + \frac{t}{2!} - \frac{t^2}{3!} + \cdots, \\ E_1(x) &= \int_x^\infty \frac{e^{-t}}{t}dt \\ &= \int_x^\infty (\frac{1}{t} - 1 + \frac{t}{2!} - \frac{t^2}{3!} + \cdots)dt \\ &= -\log x - \gamma + \Sigma_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n.n!}. \end{aligned} \tag{1.3}$$

Here γ is the so-called *Euler-Macheroni* constant and is defined by

$$\gamma = \lim_{m\to\infty}\left[\Sigma_{n=1}^{\infty}\frac{(-1)^{n+1}m^n}{n.n!} - \log m\right] = 0.5772157.... \tag{1.4}$$

Thus, upon combining (1.1) and (1.3) we obtain

$$I(x) = xe^x[-\log x - \gamma + \Sigma_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n.n!}]. \tag{1.5}$$

We note that the series in (1.5) converges for all x and that (1.5) is a series (although not a power series) representation of I about $x = 0$. When we examine (1.5) we discover that, for "moderate" values of x, the convergence is slow. Indeed, for $x = 10$, 40 terms must be retained to achieve an estimate of $I(10)$ accurate to three significant figures. Moreover, as x gets larger, the situation worsens. Upon reflection, we realize that (1.5) is an expansion of I about $x = 0$ and hence we should only expect to obtain accurate estimates using relatively few terms for x "small."

Thus, in trying to estimate I for $x \geq 10$ say, it is natural to seek an expansion about $x = \infty$. Such an expansion is readily obtained by repeatedly integrating by parts in (1.1). Indeed, after N integrations by parts, we obtain

$$
\begin{aligned}
I(x) &= xe^{x}\left[\frac{1}{x}e^{-x} - \int_x^\infty \frac{1}{t^2}e^{-t}dt\right] \\
&= xe^{x}\left[\frac{1}{x}e^{-x} - \frac{1}{x^2}e^{-x} + 2\int_x^\infty \frac{1}{t^3}e^{-t}dt\right] \\
&= xe^{x}\left[\frac{1}{x}e^{-x} - \frac{1}{x^2}e^{-x} + \frac{2}{x^3}e^{-x} - 6\int_x^\infty \frac{1}{t^4}e^{-t}dt\right] \\
&\;\;\cdot \\
&\;\;\cdot \\
I(x) &= \sum_{n=0}^{N-1} \frac{(-1)^n n!}{x^n} + (-1)^N N! x e^{x} \int_x^\infty \frac{e^{-t}}{t^{N+1}}dt \qquad (1.6)
\end{aligned}
$$

which is an exact expression.

At a first glance we are pleased with (1.6) because we have represented $I(x)$ by a series whose terms involve inverse powers of x. In fact, we let N go to infinity in (1.6) and set

$$I(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^n}. \qquad (1.7)$$

However, we realize that the series (1.7) diverges for all x. Indeed,

$$\left|\frac{(n+1)st\ term}{nth\ term}\right| = \frac{n}{x} \qquad (1.8)$$

which, for every fixed x, increases without bound as $n \to \infty$.

Although, generally convergent series are preferred, the divergent series (1.7) will be more useful than the convergent series (1.5) for us.

Let us set

$$S_N(x) = \sum_{n=0}^{N-1} \frac{(-1)^n n!}{x^n}, \tag{1.9}$$

$$\xi(x, N) = (-1)^N N! x e^x \int_x^\infty \frac{e^{-t}}{t^{N+1}} dt. \tag{1.10}$$

Thus $S_N(x)$ is the Nth partial sum of the divergent series (1.7) and $\xi(x, N)$ is the error made in approximating $I(x)$ by $S_N(x)$. We now make the observation that, because x is positive, $\xi(x, N)$ is positive and when N is even and negative when N is odd. This implies that

$$\begin{aligned} S_N(x) \leq I(x) \leq S_{N+1}(x), &\quad N \textit{ even} \\ S_{N+1}(x) \leq I(x) \leq S_N(x), &\quad N \textit{ odd}. \end{aligned} \tag{1.11}$$

Hence, for any x, the actual value of $I(x)$ must lie between two successive partial sums of the divergent series (1.7).

As (1.11) shows, a succession of upper and lower bounds for $I(x)$ can be obtained by evaluating $S_N(x)$, $N = 1, 2, \ldots$. We have not as yet, however, determined how good any of these bounds are. One point is clear; for fixed x, the best approximation of $I(x)$ by $S_N(x)$ is achieved for that integer N which minimizes $|\xi(x, N)|$. Furthermore, this optimum value of N, $\overline{N}$, must be finite because $\lim_{N\to\infty} |\xi(x, N)| = \infty$ due to the divergence of (1.7). Also, because $\xi(x, N)$ alternates in sign with N, $|\xi(x, N)|$ is less than $N!/x^N$, the absolute value of the first term omitted in (1.7) when approximated $I(x)$ by $S_N(x)$. Thus, for fixed x, we might expect that the estimate $S_N(x)$ improves with N so long as the absolute value of the ratio of successive terms remains less than or equal to 1. Hence, from

(1.8) we are led to predict that

$$\overline{N} = [x] = \text{Greatest integer less then or equal to x.} \tag{1.12}$$

Finally, we observe that for fixed N, $|\xi(x, N)|$ is a monotonically decreasing function of x and $\lim_{x\to\infty} |\xi(x, N)| = 0$. Thus, for any fixed N, $S_N(x)$ becomes a better approximation of I as x increases, but the error is zero only in the limit $x = \infty$.

We can conclude, therefore, that it would have been a mistake to discard the expansion about $x = \infty$ obtained via integration by parts , because a great deal of information about $I(x)$ has been obtained from (1.6). We might ask what the feature of the divergent series of (1.7) is that makes it so useful in approximating I for x, say, greater than 10 and why the utility of the convergent series (1.5) is rather limited in this region. The answer simply stated is that for $x \geq 10$, the convergent series "initially" diverges from the true value of $I(x)$ while the divergent series "initially" converges toward this value. Hence, we can obtain a reasonable approximation to I by taking relatively few terms of the divergent series whereas many more terms of the convergent series are needed to achieve the same degree of accuracy.

We might now conclude that, when x is "large," we should always use the partial sums $S_N(x)$ to approximate $I(x)$ rather than the partial sums of (1.5). In using the partial sums of (1.5) to approximate $I(x)$, we can make the error as small as we please, no matter how large x is, by simply taking sufficiently many terms. In using the partial sums $S_N(x)$ to approximate $I(x)$, the smallest error obtainable is dictated by the value of x. Moreover, this minimum error is never zero unless $x = \infty$. Thus, if we require $I(10)$ to four significant figures, then we cannot use any of the partial sums $S_N(10)$ because, at best, they afford three significant figures accuracy.

As a practical matter, in any problem where $I(x)$ is to be approximated, some a priori upper bound, say ξ_0, would be placed on the tolerable error. We know that for fixed N, $|\xi(x,N)|$ decreases monotonically to zero as x increases to infinity. Therefore, no matter how small ξ_0 is, we can find an x_0 such that

$$|\xi(x_0, \overline{N}(x_0))| \leq \xi_0. \tag{1.13}$$

Then, for all $x \geq x_0$, the partial sum $S_{\overline{N}(x_0)}(x)$ yields an estimate of $I(x)$ accurate to within the tolerable error. Actually as x increases from x_0, the number of terms required to achieve the desired accuracy decreases from $\overline{N}(x_0)$.

1.2 Order Relations

In this section x represents a complex variable. Let us suppose that $f(x)$ and $g(x)$ are two functions of x defined and continuous in a domain R.

Definition 1.1 ***LARGE "O" SYMBOL.*** *Suppose that, as $x \to x_0$ in R, there exists a constant k, that is, a quantity independent of x, and a neighborhood N_0 of x_0 such that*

$$|f(x)| \leq k|g(x)| \tag{1.14}$$

for all x in $N_0 \bigcap R$. Then we say that, as $x \to x_0$, $f(x)$ is large "O" of $g(x)$ and write symbolically

$$f(x) = O(g(x)), \qquad x \to x_0 \quad in \quad R. \tag{1.15}$$

Definition 1.2 ***SMALL "O" SYMBOL.*** *Suppose that, for any $\varepsilon > 0$, there exist a neighborhood N_ε of x_0 such that*

$$|f(x)| \leq \varepsilon|g(x)| \tag{1.16}$$

for all x in $N_\varepsilon \bigcap R$. Then we say that, as $x \to x_0$, $f(x)$ is small "o" of $g(x)$ and write symbolically

$$f(x) = o(g(x)), \quad x \to x_0 \quad in \quad R. \tag{1.17}$$

Thus, so long as $g(x)$ is not zero in R, $f = O(g(x))$ as $x \to x_0$ if the ratio $\frac{f}{g}$ remains bounded as $x \to x_0$ in R and $f = o(g(x))$ as $x \to x_0$ if the limit of the ratio is zero as $x \to x_0$ in R. We note that if $f = o(g(x))$ as $x \to x_0$, then necessarily $f = O(g(x))$ in this limit. The converse, however, need not be true.

Example 1.3 *Let x be a complex variable and let R be the sector $0 < |x| < \infty$, $|arg(x)| \leq \frac{\Pi}{2} - \delta$. If $0 < \delta \leq \frac{\pi}{2}$, then, as is easily verified,*

$$e^{-x} = o(x^m) \tag{1.18}$$

and as $|x| \to \infty$ in R for all complex numbers m. On the other hand, if $-\frac{\pi}{2} \leq \delta < 0$, then (1.18) does not hold for any complex number m as $|x| \to \infty$ in R due to the rapid growth of e^{-x} in the left half-plane $Re(x) < 0$.

There are many useful formulas involving combinations of order relations whose validity follows directly from the basic definitions. Below we list some important ones and give proofs for some of them. In each of these, the limit $x \to x_0$ in R is to be understood.

(1) $O(O(f)) = O(f)$

Suppose that $h_2(x) = O(h_1(x))$ and $h_1(x) = O(f(x))$, by using the definition of large "O", there exists constants k_1, k_2 such that

$$|h_2| \leq k_1|h_1(x)| \quad \textit{and} \quad |h_1| \leq k_2|f(x)|, \tag{1.19}$$

hence $|h_2| \leq k_3|f(x)|$ where $k_3 = k_1k_2$.

Therefore,

$$h_2 = O(h_1(x)) = O(O(f(x))) = O(f(x)). \square$$

(2) $O(fg) = O(f)O(g)$

Suppose that $h_1(x) = O(f(x))$ and $h_3(x) = O(g(x))$, by using the definition of large "O", there exists constants k_2, k_4 such that

$$|h_1| \leq k_2|f(x)| \quad and \quad |h_3| \leq k_4|g(x)|, \tag{1.20}$$

hence $|h_1h_3| = |h_1||h_3| \leq k_5|f(x)g(x)|$ where $k_5 = k_2k_4$.

Therefore,

$$h_1h_3 = O(f(x))O(g(x)) = O(f(x)g(x)). \square$$

(3) O(f)o(g)=o(fg).

(4) o(f)+O(f)=O(f).

(5) O(f)+O(f)=O(f)

(6) o(f)+o(f)=o(f).

(7) O(o(f))=o(O(f))=o(o(f))=o(f).

To conclude this section we wish to point out that there are several operations permissible with order relations. An important result is that an order relation can be integrated with respect to the independent variable. Indeed, suppose that R is an interval on the real line and $f = O(g)$ as $x \to x_0$ in R. Then

$$\int_x^{x_0} f(t)dt = O(\int_x^{x_0} |g(t)|dt), \quad x \to x_0 \ in \ R. \tag{1.21}$$

In general, order relations cannot be differentiated. That is, if $f = O(g)$ as $x \to x_0$, then it is not true in general that $f' = O(g')$ as $x \to x_0$.

1.3 Asymptotic Power Series Expansion

Let us briefly reconsider the function

$$I(x) = xe^x \int_x^{\infty} \frac{e^{-t}}{t} dt. \tag{1.22}$$

studied in Section 1.1 interpret the results obtained there in terms of the order relations of Section 1.2. We have from (1.6)

$$I(x) - \sum_{n=0}^{m-1} \frac{(-1)^n n!}{x^n} = \xi(x;m)$$
$$= (-1)^m m! x e^x \int_x^\infty \frac{e^{-t}}{t^{m+1}} dt, \quad m = 1, 2, ..., \quad (1.23)$$

which relates I, the mth partial sum of the divergent series (1.7) and the truncation error $\xi(x;m)$.

We now make the claim that $\xi(x;m) = O(x^{-m})$ as $x \to \infty$. Indeed, this follows by L'Hospital's rule which yields

$$\lim_{x\to\infty} \{x^m \xi(x;m)\} = (-1)^m m! \lim_{x\to\infty} \left\{ \frac{1}{x^{-(m+1)} e^{-x}} \int_x^\infty \frac{e^{-t}}{t^{m+1}} dt \right\} \quad (1.24)$$
$$= (-1)^m m!.$$

Thus as previously noted, the error made in approximating $I(x)$, as $x \to \infty$, by the first m terms of (1.6) is of the order of the first omitted term.

It is readily seen that the preceding result can be written in the following equivalent ways:

$$\lim_{x\to\infty} \left\{ x^m \left[I(x) - \sum_{n=0}^{m} \frac{(-1)^n n!}{x^n} \right] \right\} = 0 \quad , \quad m = 0, 1, 2, \ldots . \quad (1.25)$$

$$I(x) = \sum_{n=0}^{m} \frac{(-1)^n n!}{x^n} + O(x^{-m-1}) \quad , \quad x \to \infty, \; m = 0, 1, 2, \ldots . \quad (1.26)$$

It is the property expressed by (1.25) and (1.26) that makes the divergent series (1.7) useful in approximating $I(x)$ as $x \to \infty$. We now introduce the concept of asymptotic power series based on just this property.

For the present we assume that x is real variable whose domain is R and that x_0 is a finite point in $\overline{R}$, the closure of R. We then have the following.

Definition 1.4 *Let $f(x)$ be defined and continuous on R. The formal power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is said to be an asymptotic power series expansion of f,* *as $x \to x_0$ in R, if the conditions*

$$\lim_{x \to x_0} \left\{ (x - x_0)^{-m} \Big[f - \sum_{n=0}^{m} a_n (x - x_0)^n \Big] \right\} = 0, \quad m = 0, 1, 2, ... \tag{1.27}$$

are satisfied.

It is readily seen that conditions (1.27) are equivalent to

$$f(x) = \sum_{n=0}^{m} a_n (x - x_0)^n + O(x - x_0)^{m+1}, \quad x \to x_0 \text{ in } R, \quad m = 0, 1, 2, ... \tag{1.28}$$

We note that neither (1.27) and (1.28) implies the convergence of the formal power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$. They are simply statements concerning the behavior, as $x \to x_0$, of the error made in approximating f by the partial sums of this series. Thus, in general, we cannot set f equal to the series and hence we introduce the notation

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad x \to x_0 \tag{1.29}$$

to imply that conditions (1.27) hold and, in particular, to allow for the possible divergence of the right-hand side.

Definition 1.5 *Assume that (1.27) is satisfied for $m = 0, ..., N - 1$, but not for $m = N$. Then we say that $\sum_{n=0}^{N-1} a_n (x - x_0)^n$ is an asymptotic power series of f as $x \to x_0$ to N terms and write*

$$f(x) \sim \sum_{n=0}^{N-1} a_n (x - x_0)^n, \quad x \to x_0. \tag{1.30}$$

Under the assumptions made we can now only conclude that

$$f(x) = \sum_{n=0}^{N-1} a_n (x - x_0)^n + o(x - x_0)^{N-1}, \quad x \to x_0. \tag{1.31}$$

It is a simple matter to adapt our definition of asymptotic power series to the important case where $x_0 = \infty$. Indeed, we have the following.

Definition 1.6 *The formal series $\sum_{n=0}^{\infty} a_n x^{-n}$ is said to be an asymptotic power* *series expansion of f as $x \to \infty$ if the following equivalent sets of conditions are satisfied:*

$$\lim_{x\to\infty} \left\{x^m\Big[f - \sum_{n=0}^{m} a_n x^{-n}\Big]\right\} = 0, \quad m = 0, 1, 2, ..., \tag{1.32}$$

$$f(x) = \sum_{n=0}^{m} a_n x^{-n} + O(x^{-m-1}), \quad x \to \infty, \quad m = 0, 1, 2, ... \,. \tag{1.33}$$

If (1.32) and (1.33) hold, then we write

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}, \quad x \to \infty. \tag{1.34}$$

Thus, the analysis of Section 1.1 coupled with (1.24) shows that

$$xe^x \int_x^\infty \frac{e^{-t}}{t} dt \sim \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^n}, \quad x \to \infty. \tag{1.35}$$

As we have noted, the asymptotic conditions (1.27) imply neither the convergence nor the divergence of the formal series $\sum_{n=0}^{\infty} a_n x^{-n}$. If the series converges to $f(x)$ throughout some neighborhood of $x = x_0$, then f is a real analytic function in this neighborhood and can be studied by using the powerful theorems associated with such functions. The more interesting case arises when the formal series actually diverges in R, except of course at $x = x_0$. In that event $f(x)$ must have some sort of singularity at $x = x_0$ in the sense that $x = x_0$ must be a point of nonanalyticity for f.

If the asymptotic power series diverges, then for any value of $x \neq x_0$, the optimum number of terms of the series to be used in approximating f must be finite and the corresponding optimum error is not zero.

Taylor's theorem with remainder states, $(0 < \theta_m < 1)$,

$$\begin{aligned} f(x) &= \sum_{n=0}^{m} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{1}{(m+1)!}(x-x_0)^{m+1} f^{(m+1)}(x_0 + \theta_m[x - x_0]), \\ &= \sum_{n=0}^{m} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + O(x-x_0)^{m+1}, \quad x \to x_0,\ m = 0, 1, ..., N-1 \end{aligned} \tag{1.36}$$

Then it immediately follows that

$$f(x) \sim \sum_{n=0}^{N-1} \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}, \quad x \to x_0 \tag{1.37}$$

is an asymptotic power series expansion of f to N terms as $x \to x_0$.

Let us now suppose that f is infinitely differentiable at $x = x_0$. We then say that f belongs to the class $C^\infty(x_0)$. We can then let N go to infinity in (1.37) to obtain the infinite asymptotic power series expansion

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}, \quad x \to x_0. \tag{1.38}$$

Note that we have not set $f(x)$ equal to its Taylor series, for nothing we have assumed implies the convergence of this series. In other words, f need not be analytic at $x = x_0$.

To pursue this matter a bit further, let us define the function space $\tilde{C}^\infty(x_0)$ which includes those functions infinitely differentiable at $x = x_0$ but not analytic there. Such functions arise often in asymptotic analysis. Unfortunately, they are easier to define then to construct. If $f \in \tilde{C}^\infty(x_0)$, then $f^{(n)}(x_0)$ is finite for each n but the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

does not converge to f throughout any neighborhood of $x = x_0$. This can happen in two ways. The Taylor series may actually converge throughout some neighborhood of $x = x_0$ but not to $f(x)$. Alternately, the series might diverge for all $x \neq x_0$.

Example 1.7 *Consider*

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0, \\ 0, & x \leq 0. \end{cases} \tag{1.39}$$

Here $f(x)$ and all of its derivatives are continuous and equal to zero at $x = 0$. Although f is infinitely differentiable at $x = 0$, it is not analytic there because its Taylor series sum to zero for all x, a result which disagrees with f for x positive. However, because f is infinitely differentiable at $x = 0$ we do have the asymptotic power series expansion

$$f(x) \sim 0, \quad x \to 0. \tag{1.40}$$

Example 1.8 *Let us now consider*

$$f(x) = \sum_{m=0}^{\infty} e^{-m} \cos m^2 x; \quad -1 \le x \le 1. \tag{1.41}$$

We first note that the infinite series is uniformly convergent for $x \in [-1, 1]$ as are all of the series obtained by successively differentiating (1.41) term-by-term. Therefore, we can conclude that $f(x)$ and all of its derivatives are continuous in $[-1, 1]$ and furthermore the derivatives of f can be obtained by successive term-by-term differentiation of (1.41).

At the origin only the derivatives of even order are nonzero and we have

$$f^{(2n)}(0) = (-1)^n \sum_{m=0}^{\infty} e^{-m} (m^2)^{2n}, \quad n = 0, 1, 2, \ldots . \tag{1.42}$$

Thus, because f is infinitely differentiable at the origin, we can immediately write

$$f(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Big(\sum_{m=0}^{\infty} e^{-m} m^{4n} \Big), \quad x \to 0. \tag{1.43}$$

Consider now the absolute value of the term of degree $2n$, $n > 0$, in (1.43).

We have

$$\frac{|x^{2n}|}{(2n)!} \Big(\sum_{m=0}^{\infty} e^{-m} m^{4n} \Big) \ge \Big(\frac{|x|}{2n} \Big)^{2n} \Big(\sum_{m=0}^{\infty} e^{-m} m^{4n} \Big) \ge \Big(\frac{|x| m^2}{2n} \Big)^{2n} e^{-m} \tag{1.44}$$

which holds for $m = 0, 1, 2, \ldots$, because one term of a sum of positive number is certainly less than the sum. Furthermore, for $x \ne 0$, strict inequality holds in

(1.44). If we set $m = 2n$*, then (1.44) becomes*

$$\frac{|x|^{2n}}{(2n)!}\Big(\sum_{m=0}^{\infty} e^{-m} m^{4n}\Big) > \Big(\frac{2n|x|}{e}\Big)^{2n}, \qquad x \neq 0. \tag{1.45}$$

Because the right-hand side of (1.45) exceeds 1 for all n *greater than* $e/2|x|$*, the series in (1.43) diverges for all nonzero* x*. Therefore,* f *is infinitely differentiable at* $x = 0$ *but is not analytic there.*

We wish to comment here, however, that the terms of any asymptotic expansion can depend on one or more parameters. If the asymptotic conditions are satisfied as these parameters range over some domain in parameter space, then we say that the asymptotic expansion holds uniformly in the parameter(s) on this domain.

CHAPTER 2

MATCHED ASYMPTOTIC EXPANSIONS

One possible way of defining a function $f(x,\varepsilon)$ is as the solution of a differential equation in which x is the independent variable and ε occurs as a parameter. If one cannot solve this differential equation for arbitrary ε (as, for example, if the differential equation is nonlinear with $\varepsilon \neq 0$) can one calculate the asymptotic expansion of the solution by considering a sequence of simpler differential equations governing each term of this expansion? Here, we consider a simple example to introduce some ideas.

The first-order equation [2]

$$\varepsilon\frac{dy}{dx} + y = \frac{\varepsilon[x(\varepsilon-1)+\varepsilon^2]e^{-x}}{(x+\varepsilon)^2}, \quad 0 \leq x \leq \infty,\ 0 < \varepsilon \ll 1 \tag{2.1}$$

with the initial condition

$$y(0) = 0, \tag{2.2}$$

has the exact solution

$$y = f(x,\varepsilon) \equiv e^{-x/\varepsilon} - \varepsilon\frac{e^{-x}}{x+\varepsilon}. \tag{2.3}$$

Ignoring temporarily the origin of Eqn.(2.3), we see that $f(x,\varepsilon)$ defines a well behaved function, and it is interesting to consider the asymptotic expansion of this function as $\varepsilon \to 0$. In each case, when we fix x to be some positive value and apply the limit process, we obtain all a_k's, $k = 0, 1, 2, \cdots$, such that

$$a_0(x) = \lim_{\varepsilon\to 0}\frac{f(x,\varepsilon)}{\phi_0(\varepsilon)} = \lim_{\varepsilon\to 0}\left(e^{-x/\varepsilon} - \varepsilon\frac{e^{-x}}{x+\varepsilon}\right) = 0 \tag{2.4}$$

$$a_1(x) = \lim_{\varepsilon\to 0}\frac{f(x,\varepsilon)}{\phi_1(\varepsilon)} = \lim_{\varepsilon\to 0}\left(\frac{e^{-x/\varepsilon}}{\varepsilon} - \frac{e^{-x}}{x+\varepsilon}\right) = -\frac{e^{-x}}{x} \tag{2.5}$$

$$
\begin{aligned}
a_2(x) &= \lim_{\varepsilon\to 0}\frac{f(x,\varepsilon)-a_1(x)\phi_1(\varepsilon)}{\phi_2(\varepsilon)} = \lim_{\varepsilon\to 0}\left(\frac{e^{-x/\varepsilon}-\varepsilon\frac{e^{-x}}{x+\varepsilon}+\frac{e^{-x}}{x}\varepsilon}{\varepsilon^2}\right) = \frac{e^{-x}}{x^2} \quad (2.6)\\
&\vdots\\
a_k(x) &= \lim_{\varepsilon\to 0}\frac{f(x,\varepsilon)-\sum_{n=1}^{k-1}a_n(x)\phi_n(\varepsilon)}{\phi_k(\varepsilon)} = \frac{(-1)^k e^{-x}}{x^k}, \quad k=1,2,\cdots, \quad (2.7)
\end{aligned}
$$

with $\phi_n(\varepsilon)=\varepsilon^n$. Now, if we reorganize the equation

$$
f=\sum_{n=1}^{N}a_n(x)\phi_n(\varepsilon)+O(\phi_{N+1}),
$$

we find the following expansion for f, called an "outer" expansion

$$
\begin{aligned}
f &= -\varepsilon\frac{e^{-x}}{x}+\varepsilon^2\frac{e^{-x}}{x^2}-\varepsilon^3\frac{e^{-x}}{x^3}+O(\varepsilon^4)\\
&\equiv \sum_{n=0}^{N}\varepsilon^n h_n(x)+O(\varepsilon^{n+1}) \qquad (2.8)
\end{aligned}
$$

and the contribution of the $e^{-x/\varepsilon}$ term is smaller than any term in the series in (2.8). We shall refer to such a term as a "transcendentally small" term (abbreviated as T.S.T) in this limit.

Clearly (2.8) is not uniformly valid near $x=0$. In fact, it is singular there, and this expansion is not a good approximation of the function defined by (2.3) no matter how small ε is if we allow x also to become small.

It is therefore natural to seek another expansion of (2.3) which adequately approximates this function near $x=0$. Since the combination x/ε occurs in the first term one is led to the change of variables $x^*=x/\varepsilon$

$$
y=g(x^*,\varepsilon)\equiv e^{-x^*}-\frac{e^{-\varepsilon x^*}}{x^*+1}. \qquad (2.9)
$$

With $x^*=x/\varepsilon$ (2.9) defines the same function as f. However, the asymptotic expansion of g with x^* fixed as $\varepsilon\to 0$ is quite different. To obtain the asymptotic expansion of g, we find all b_k's, $k = 0, 1, \cdots$, such that

$$
\begin{aligned}
b_0(x^*) &= \lim_{\varepsilon \to 0} \frac{g(x^*, \varepsilon)}{\phi_0(\varepsilon)} = e^{-x^*} - \frac{1}{x^* + 1} \\
b_1(x^*) &= \lim_{\varepsilon \to 0} \frac{g(x^*, \varepsilon) - b_0(x^*)\phi_0(\varepsilon)}{\phi_1(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{e^{-x^*} - \frac{e^{-\varepsilon x^*}}{x^*+1} - e^{-x^*} + \frac{1}{x^*+1}}{\varepsilon} = \frac{x^*}{x^* + 1} \\
b_2(x^*) &= \lim_{\varepsilon \to 0} \frac{g(x^*, \varepsilon) - b_0(x^*)\phi_0(\varepsilon) - b_1(x^*)\phi_1(\varepsilon)}{\phi_2(\varepsilon)} = -\frac{x^{*2}}{2(x^* + 1)} \\
&\vdots \\
b_k(x^*) &= \lim_{\varepsilon \to 0} \frac{g(x^*, \varepsilon) - \sum_{n=0}^{k-1} b_n(x^*)\phi_n(\varepsilon)}{\phi_k(\varepsilon)} = \frac{(-1)^{k+1} x^{*k}}{k!(x^* + 1)}, \quad k = 1, 2, \cdots.
\end{aligned}
$$

Now, if we replace all b_k's, $k = 0, 1, \cdots$, in the below equation

$$
g = \sum_{n=1}^{N} b_n(x^*)\phi_n(\varepsilon) + O(\phi_{N+1}),
$$

then we can see that y has the expansion which will be referred to as the "inner" expansion

$$
\begin{aligned}
g &= e^{-x^*} - \frac{1}{x^* + 1} + \frac{\varepsilon x^*}{x^* + 1} - \frac{\varepsilon^2 x^{*2}}{2(x^* + 1)} + \frac{\varepsilon^3 x^{*3}}{6(x^* + 1)} + O(\varepsilon^4) \\
&\equiv \sum_{n=0}^{N} \varepsilon^n g_n(x^*) + O(x^{*N+1}).
\end{aligned} \tag{2.10}
$$

Now, this expansion is accurate for small x. In particular, the condition $y = 0$ at $x = x^* = 0$ is satisfied. However, the result fails to be uniformly valid for x^* large. Thus, the two expansions (2.8) and (2.10) have mutually exclusive domains of validity. Hence, depending on the magnitude of x compared to ε one expansion or the other should be used.

We can find a transition region between two mutually exclusive domains, which is contained in both (2.8) and (2.10). Let us consider the expansion which would result from (2.8) by letting $\varepsilon \to 0$ with $x_\eta = x/\eta(\varepsilon)$ fixed for some $\eta(\varepsilon)$ such that $\varepsilon \ll \eta \ll 1$. Thus, in such an expansion, we consider $x \to 0$ in the limit but at a slower rate than in the case leading to (2.10). In a sense, the above defines an "intermediate" limit process .

Setting $x = \eta(\varepsilon)x_\eta$, we write (2.3) in the form

$$y = l(x_\eta, \eta, \varepsilon) \equiv e^{-\eta x_\eta/\varepsilon} - \frac{\varepsilon e^{-\eta x_\eta}}{\eta x_\eta + \varepsilon}. \tag{2.11}$$

Since we need the asymptotic expansion of l, we find Taylor series for c_k's, $k = 0, 1, \cdots$, such that

$$c_0(x_\eta) = \lim_{\varepsilon\to 0}\left(e^{-\eta x_\eta/\varepsilon} - \frac{\varepsilon e^{-\eta x_\eta}}{\eta x_\eta + \varepsilon}\right) = 0$$

$$\begin{aligned}
c_1(x_\eta) &= \lim_{\varepsilon\to 0}\frac{l(x_\eta,\varepsilon) - c_0(x_\eta)\phi_0(\varepsilon)}{\phi_1(\varepsilon)} = \lim_{\varepsilon\to 0}\left(\frac{e^{-\eta x_\eta/\varepsilon}}{\varepsilon} - \frac{e^{-\eta x_\eta}}{\eta x_\eta + \varepsilon}\right)\\
&= -\frac{e^{-\eta x_\eta}}{\eta x_\eta} = -\frac{\left(1 - \eta x_\eta + \frac{\eta^2 x_\eta^2}{2!} - \frac{\eta^3 x_\eta^3}{3!} + \cdots\right)}{\eta x_\eta}\\
&= -\frac{1}{\eta x_\eta} + 1 - \frac{\eta x_\eta}{2} + \frac{\eta^2 x_\eta^2}{6} - \cdots
\end{aligned}$$

$$\begin{aligned}
c_2(x_\eta) &= \lim_{\varepsilon\to 0}\frac{l(x_\eta,\varepsilon) - c_0(x_\eta)\phi_0(\varepsilon) - c_1(x_\eta)\phi_1(\varepsilon)}{\phi_2(\varepsilon)} = \lim_{\varepsilon\to 0}\frac{e^{-\eta x_\eta/\varepsilon} - \frac{\varepsilon e^{-\eta x_\eta}}{\eta x_\eta+\varepsilon} + \frac{\varepsilon e^{-\eta x_\eta}}{\eta x_\eta}}{\varepsilon^2}\\
&= \lim_{\varepsilon\to 0}\left(\frac{e^{-\eta x_\eta/\varepsilon}}{\varepsilon^2} + \frac{e^{-\eta x_\eta}}{\eta x_\eta(\eta x_\eta + \varepsilon)}\right)\\
&= \frac{e^{-\eta x_\eta}}{\eta^2 x_\eta^2} = \frac{\left(1 - \eta x_\eta + \frac{\eta^2 x_\eta^2}{2!} + \cdots\right)}{\eta^2 x_\eta^2}\\
&= \frac{1}{\eta^2 x_\eta^2} - \frac{1}{\eta x_\eta} + \frac{1}{2} - \cdots
\end{aligned}$$

$$\begin{aligned}
c_3(x_\eta) &= \lim_{\varepsilon\to 0}\frac{l(x_\eta,\varepsilon) - c_0(x_\eta)\phi_0(\varepsilon) - c_1(x_\eta)\phi_1(\varepsilon) - c_2(x_\eta)\phi_2(\varepsilon)}{\phi_3(\varepsilon)}\\
&= \lim_{\varepsilon\to 0}\left(\frac{e^{-\eta x_\eta/\varepsilon}}{\varepsilon^3} - \frac{1}{\varepsilon^2}\left[\frac{e^{-\eta x_\eta}}{\eta x_\eta + \varepsilon} - \frac{e^{-\eta x_\eta}}{\eta x_\eta} + \frac{\varepsilon e^{-\eta x_\eta}}{\eta^2 x_\eta^2}\right]\right)\\
&= \lim_{\varepsilon\to 0}\left(\frac{e^{-\eta x_\eta/\varepsilon}}{\varepsilon^3} - \frac{e^{-\eta x_\eta}}{\eta^2 x_\eta^2(\eta x_\eta + \varepsilon)}\right)\\
&= -\frac{e^{-\eta x_\eta}}{\eta^3 x_\eta^3} = -\frac{\left(1 - \eta x_\eta + \frac{\eta^2 x_\eta^2}{2!} + \cdots\right)}{\eta^3 x_\eta^3}\\
&= -\frac{1}{\eta^3 x_\eta^3} + \frac{1}{\eta^2 x_\eta^2} - \frac{1}{2\eta x_\eta} + \cdots
\end{aligned}$$

$$
\begin{aligned}
c_4(x_\eta) &= \lim_{\varepsilon\to 0} \frac{l(x_\eta,\varepsilon) - c_0(x_\eta)\phi_0(\varepsilon) - c_1(x_\eta)\phi_1(\varepsilon) - c_2(x_\eta)\phi_2(\varepsilon) - c_3(x_\eta)\phi_3(\varepsilon)}{\phi_4(\varepsilon)} \\
&= \lim_{\varepsilon\to 0} \Big(\frac{e^{-\eta x_\eta/\varepsilon}}{\varepsilon^4} - \frac{1}{\varepsilon^3}\Big[\frac{e^{-\eta x_\eta}}{\eta x_\eta + \varepsilon} - \frac{e^{-\eta x_\eta}}{\eta x_\eta} + \frac{\varepsilon e^{-\eta x_\eta}}{\eta^2 x_\eta^2} - \frac{\varepsilon^2 e^{-\eta x_\eta}}{\eta^3 x_\eta^3} \Big] \Big) \\
&= \frac{e^{-\eta x_\eta}}{\eta^4 x_\eta^4} = \frac{\left(1 - \eta x_\eta + \frac{\eta^2 x_\eta^2}{2!} + \cdots\right)}{\eta^4 x_\eta^4} \\
&= \frac{1}{\eta^4 x_\eta^4} - \frac{1}{\eta^3 x_\eta^3} + \frac{1}{2\eta^2 x_\eta^2} - \cdots \\
&\ \vdots \\
c_k(x_\eta) &= \lim_{\varepsilon\to 0} \frac{l(x_\eta,\varepsilon) - \sum_{n=0}^{k-1} c_n(x_\eta)\phi_n(\varepsilon)}{\phi_k(\varepsilon)} = \frac{(-1)^k e^{-\eta x_\eta}}{\eta^k x_\eta^k}, \quad k = 1, 2, \cdots,
\end{aligned}
$$

and replace all c_k's, $k = 0, 1, \cdots$, in the below equation

$$
l = c_0(x_\eta)\phi_0(\varepsilon) + c_1(x_\eta)\phi_1(\varepsilon) + c_2(x_\eta)\phi_2(\varepsilon) + \cdots .
$$

Hence, with x_η fixed (2.11) has the following expansion, called the "intermediate" expansion

$$
\begin{aligned}
l = & -\frac{\varepsilon}{\eta x_\eta} + \frac{\varepsilon^2}{\eta^2 x_\eta^2} + \varepsilon - \frac{\varepsilon^3}{\eta^3 x_\eta^3} - \frac{\varepsilon^2}{\eta x_\eta} - \frac{\varepsilon \eta x_\eta}{2} \\
& + O(\frac{\varepsilon^4}{\eta^4}) + O(\frac{\varepsilon^3}{\eta^2}) + O(\varepsilon^2) + O(\varepsilon\eta^2). \qquad (2.12)
\end{aligned}
$$

In the above the term $e^{-\eta x_\eta/\varepsilon}$ is transcendentally small does not appear as long as $\varepsilon|\log\varepsilon| \ll \eta$. Henceforth, we shall always ignore such a term and automatically require that $\varepsilon|\log\varepsilon| \ll \eta$ in our calculations. This restricts somewhat the range of η to $\varepsilon|\log\varepsilon| \ll \eta \ll 1$.

If we now reexpand the outer and inner expansions using the above intermediate limit process, we find that if a sufficient number of terms are included both eventually give (2.12). This means that the outer expansion, which was constructed under the assumption $\varepsilon \to 0$ with fixed $x \neq 0$ is actually valid in the extended sense $\varepsilon \to 0$, $x_\eta = x/\eta(\varepsilon)$ fixed for some class of functions $\eta(\varepsilon) \ll 1$. Similarly, the inner expansion which was constructed under the assumption $\varepsilon \to 0$, $x^* = x/\varepsilon$ fixed $\neq \infty$ is actually valid in the extended sense $\varepsilon \to 0$, $x_\eta = x/\eta(\varepsilon)$ fixed, for some class of functions $\eta(\varepsilon)$ such that $\varepsilon|\log\varepsilon| \ll \eta$.

We will demonstrate next that for this example, the extended domains of validity of the inner and outer expansions overlap in the following sense. For each $R = 0, 1, 2, \cdots$ there exist integers P, and Q, and functions $\eta_1(\varepsilon)$ and $\eta_2(\varepsilon)$ with $\eta_1 \ll \eta_2$ such that

$$\lim_{\varepsilon\to 0} \frac{\left[\sum_{n=0}^{P} h_n(\eta x_\eta)\varepsilon^n - \sum_{n=0}^{Q} g_n(\eta x_\eta/\varepsilon)\varepsilon^n\right]}{\varepsilon^R} = 0 \quad , x_\eta \text{ fixed,} \tag{2.13}$$

for all η satisfying $\eta_1 \ll \eta \ll \eta_2$.

Equation (2.13) is a matching condition for the inner and outer expansions in their common overlap domain of validity which is defined as the class of functions $\eta(\varepsilon)$ satisfying the condition $\eta_1 \ll \eta \ll \eta_2$.

To demonstrate the result let us first take $R = 0$. Assuming that $P = Q = 0$, the question now is whether

$$\lim_{\varepsilon\to 0}[h_0(\eta x_\eta) - g_0(\eta x_\eta/\varepsilon)] = 0 \quad , x_\eta \text{ fixed.} \tag{2.14}$$

We know that from the Eqn.(2.10)

$$g_0 = e^{-x^*} - \frac{1}{1+x^*} = e^{\frac{-\eta x_\eta}{\varepsilon}} - \frac{\varepsilon}{\varepsilon + \eta x_\eta} \tag{2.15}$$

in terms of x_η, and $h_0 = 0$, but we need to find g_0 as the following form

$$g_0 = \sum_{n=1}^{N} d_n(x_\eta)\phi_n(\varepsilon) + O(\phi_{N+1}) \tag{2.16}$$

with $\phi_n = \varepsilon^n$, hence we obtain all d_k's, $k = 0, 1, 2, \cdots$, such that

$$d_0(x_\eta) \;=\; \lim_{\varepsilon\to 0}\left(e^{\frac{-\eta x_\eta}{\varepsilon}} - \frac{\varepsilon}{\varepsilon + \eta x_\eta}\right) = 0$$

$$
\begin{aligned}
d_1(x_\eta) &= \lim_{\varepsilon\to 0}\Big(\frac{e^{\frac{-\eta x_\eta}{\varepsilon}} - \frac{\varepsilon}{\varepsilon+\eta x_\eta}}{\varepsilon}\Big) = -\frac{1}{\eta x_\eta} \\
d_2(x_\eta) &= \lim_{\varepsilon\to 0}\Big(\frac{e^{\frac{-\eta x_\eta}{\varepsilon}} - \frac{\varepsilon}{\varepsilon+\eta x_\eta} + \frac{\varepsilon}{\eta x_\eta}}{\varepsilon^2}\Big) = \frac{1}{\eta^2 x_\eta^2} \\
d_3(x_\eta) &= \lim_{\varepsilon\to 0}\Big(\frac{e^{\frac{-\eta x_\eta}{\varepsilon}} - \frac{\varepsilon}{\varepsilon+\eta x_\eta} + \frac{\varepsilon}{\eta x_\eta} - \frac{\varepsilon^2}{\eta^2 x_\eta^2}}{\varepsilon^3}\Big) = -\frac{1}{\eta^3 x_\eta^3} \\
&\vdots \\
d_k(x_\eta) &= \lim_{\varepsilon\to 0}\frac{g_0(x_\eta,\varepsilon) - \sum_{n=0}^{k-1} d_n(x_\eta)\phi_n(\varepsilon)}{\phi_k(\varepsilon)} = \frac{(-1)^k}{\eta^k x_\eta^k}, \quad k = 1, 2, \cdots.
\end{aligned}
$$

Finally, we replace all d_k's in the Eqn.(2.16) and we get

$$
g_0 = -\frac{\varepsilon}{\eta x_\eta} + \frac{\varepsilon^2}{\eta^2 x_\eta^2} - \frac{\varepsilon^3}{\eta^3 x_\eta^3} + O(\frac{\varepsilon^4}{\eta^4}) + T.S.T. \tag{2.17}
$$

Clearly (2.14) is satisfied by any η such that $\varepsilon|\log\varepsilon| \ll \eta$. Thus, the overlap domain for the matching to $O(1)$ is determined by

$$
\eta_1 = \varepsilon|\log\varepsilon|, \quad \eta_2 = 1.
$$

To order ε the matching condition (with $P = Q = 1$ and $R = 1$) becomes

$$
\lim_{\varepsilon\to 0}\Big[\frac{h_0(\eta x_\eta) + \varepsilon h_1(\eta x_\eta) - g_0(\eta x_\eta/\varepsilon) - \varepsilon g_1(\eta x_\eta/\varepsilon)}{\varepsilon}\Big] = 0, \quad x_\eta \text{ fixed.} \tag{2.18}
$$

Similarly, we obtain h_1 from the Eqn.(2.8) and g_1 from the Eqn.(2.10) such that

$$
\begin{aligned}
h_1 &= -\frac{e^{-x}}{x} = -\frac{e^{-\eta x_\eta}}{\eta x_\eta} \\
g_1 &= \frac{x^*}{x^*+1} = \frac{\eta x_\eta}{\eta x_\eta + \varepsilon}.
\end{aligned}
$$

Expanding h_1 in terms of x_η gives

$$
\begin{aligned}
h_1 &= -\frac{e^{-\eta x_\eta}}{\eta x_\eta} = -\frac{1 - \eta x_\eta + \frac{\eta^2 x_\eta^2}{2} - \frac{\eta^3 x_\eta^3}{6} + \frac{\eta^4 x_\eta^4}{24} - \cdots}{\eta x_\eta} \\
&= -\frac{1}{\eta x_\eta} + 1 - \frac{\eta x_\eta}{2} + \frac{\eta^2 x_\eta^2}{6} - \frac{\eta^3 x_\eta^3}{24} + \cdots.
\end{aligned}
$$

To expand g_1 in the terms of x_η, we consider that g_1 can be written as the following form

$$g_1 = \sum_{n=1}^{N} e_n(x_\eta)\phi_n(\varepsilon) + O(\phi_{N+1}) \tag{2.19}$$

where

$$\begin{aligned}
e_0(x_\eta) &= \lim_{\varepsilon\to 0} \frac{\eta x_\eta}{\eta x_\eta + \varepsilon} = 1 \\
e_1(x_\eta) &= \lim_{\varepsilon\to 0} \frac{\frac{\eta x_\eta}{\eta x_\eta+\varepsilon} - 1}{\varepsilon} = -\frac{1}{\eta x_\eta} \\
e_2(x_\eta) &= \lim_{\varepsilon\to 0} \frac{\frac{\eta x_\eta}{\eta x_\eta+\varepsilon} - 1 + \frac{\varepsilon}{\eta x_\eta}}{\varepsilon^2} = \frac{1}{\eta^2 x_\eta^2} \\
&\vdots \\
e_k(x_\eta) &= \lim_{\varepsilon\to 0} \frac{g_1(x_\eta,\varepsilon) - \sum_{n=0}^{k-1} e_n(x_\eta)\phi_n(\varepsilon)}{\phi_k(\varepsilon)} = \frac{(-1)^k}{\eta^k x_\eta^k}, \quad k = 1, 2, \cdots.
\end{aligned}$$

Hence, by replacing all e_k's in the Eqn.(2.19), we get

$$g_1 = 1 - \frac{\varepsilon}{\eta x_\eta} + \frac{\varepsilon^2}{\eta^2 x_\eta^2} - \frac{\varepsilon^3}{\eta^3 x_\eta^3} + \cdots$$

We can use the previous results for h_1 and g_1 to calculate

$$\varepsilon h_1 = -\frac{\varepsilon}{\eta x_\eta} + \varepsilon - \frac{\varepsilon\eta x_\eta}{2} + \frac{\varepsilon\eta^2 x_\eta^2}{6} + O(\varepsilon\eta^3) \tag{2.20}$$

$$\varepsilon g_1 = \varepsilon - \frac{\varepsilon^2}{\eta x_\eta} + O(\frac{\varepsilon^3}{\eta^2}). \tag{2.21}$$

Substituting (2.17), (2.20) and (2.21) into (2.18) and cancelling identical terms gives to lowest order

$$\lim_{\varepsilon\to 0}\left[-\frac{\varepsilon}{\eta^2 x_\eta^2} - \frac{\eta x_\eta}{2} + \frac{\varepsilon}{\eta x_\eta} + \frac{\varepsilon^2}{\eta^3 x_\eta^3} + O(\eta^2) + O(\frac{\varepsilon^2}{\eta^2}) + O(\frac{\varepsilon^3}{\eta^4})\right] \quad , x_\eta \textit{ fixed.} \tag{2.22}$$

All terms automatically vanish as long as $\varepsilon|\log\varepsilon| \ll \eta \ll 1$ except the first term. This will vanish if $\varepsilon/\eta^2 \to 0$, i.e.,$\varepsilon^{1/2} \ll \eta$. Since $\varepsilon|\log\varepsilon| \ll \varepsilon^{1/2}$, the overlap domain to $O(\varepsilon)$ is $\varepsilon^{1/2} \ll \eta \ll 1$, i.e., $\eta_1 = \varepsilon^{1/2}, \ \eta_2 = 1$.

To carry out the matching to $O(\varepsilon^2)$, we first attempt to show that

$$\lim_{\varepsilon \to 0} \frac{h_0 + \varepsilon h_1 + \varepsilon^2 h_2 - g_0 - \varepsilon g_1 - \varepsilon^2 g_2}{\varepsilon^2} = 0 \quad , x_\eta \ fixed, \tag{2.23}$$

i.e., we assume that $P = Q = 2$.

We find h_2 from the Eqn.(2.8) and expand it in the terms of x_η such that

$$\begin{aligned} h_2 &= \frac{e^{-\eta x_\eta}}{\eta^2 x_\eta^2} = \frac{1 - \eta x_\eta + \frac{\eta^2 x_\eta^2}{2} - \frac{\eta^3 x_\eta^3}{6} + \frac{\eta^4 x_\eta^4}{24} - \cdots}{\eta^2 x_\eta^2} \\ &= \frac{1}{\eta^2 x_\eta^2} - \frac{1}{\eta x_\eta} + \frac{1}{2} - \frac{\eta x_\eta}{6} + \frac{\eta^2 x_\eta^2}{24} - \cdots. \end{aligned}$$

Now we have to expand g_2 in the terms of x_η, so we now consider the case that

$$g_2 = -\frac{x^{*2}}{2(x^* + 1)} = -\frac{\eta^2 x_\eta^2}{2\varepsilon(\varepsilon + \eta x_\eta)}.$$

In this case, suppose that

$$g_2 = \sum_{n=1}^{N} p_n(x_\eta)\phi_n(\varepsilon) + O(\phi_{N+1}) \tag{2.24}$$

where

$$\begin{aligned} p_{-1}(x_\eta) &= \lim_{\varepsilon \to 0} -\frac{\varepsilon \eta^2 x_\eta^2}{2\varepsilon(\varepsilon + \eta x_\eta)} = -\frac{\eta x_\eta}{2} \\ p_0(x_\eta) &= \lim_{\varepsilon \to 0} \left(-\frac{\eta^2 x_\eta^2}{2\varepsilon(\varepsilon + \eta x_\eta)} + \frac{\eta x_\eta}{2\varepsilon} \right) = \frac{1}{2} \\ p_1(x_\eta) &= \lim_{\varepsilon \to 0} \frac{-\frac{\eta^2 x_\eta^2}{2\varepsilon(\varepsilon + \eta x_\eta)} + \frac{\eta x_\eta}{2\varepsilon} - \frac{1}{2}}{\varepsilon} = -\frac{1}{2\eta x_\eta} \\ p_2(x_\eta) &= \lim_{\varepsilon \to 0} \frac{-\frac{\eta^2 x_\eta^2}{2\varepsilon(\varepsilon + \eta x_\eta)} + \frac{\eta x_\eta}{2\varepsilon} - \frac{1}{2} + \frac{\varepsilon}{2\eta x_\eta}}{\varepsilon^2} = \frac{1}{2\eta^2 x_\eta^2} \\ &\vdots \\ p_k(x_\eta) &= \lim_{\varepsilon \to 0} \frac{g_2(x_\eta, \varepsilon) - \sum_{n=0}^{k-1} p_n(x_\eta)\phi_n(\varepsilon)}{\phi_k(\varepsilon)} = \frac{(-1)^k}{2\eta^k x_\eta^k}, \quad k = 1, 2, \cdots. \end{aligned}$$

Hence, by replacing all p_k's in the Eqn.(2.24), we get

$$g_2 = -\frac{\eta x_\eta}{2\varepsilon} + \frac{1}{2} - \frac{\varepsilon}{2\eta x_\eta} + \frac{\varepsilon^2}{2\eta^2 x_\eta^2} + O(\frac{\varepsilon^3}{\eta^3})$$

We now calculate

$$\varepsilon^2 h_2 = \frac{\varepsilon^2}{\eta^2 x_\eta^2} - \frac{\varepsilon^2}{\eta x_\eta} + \frac{\varepsilon^2}{2} + O(\varepsilon^2\eta) \tag{2.25}$$

$$\varepsilon^2 g_2 = -\frac{\varepsilon\eta x_\eta}{2} + \frac{\varepsilon^2}{2} + O(\frac{\varepsilon^3}{\eta}). \tag{2.26}$$

After cancelling out all terms which match identically we are left with the requirement

$$\lim_{\varepsilon\to 0}\Big[\frac{\varepsilon}{\eta^3 x_\eta^3} + \frac{\eta^2 x_\eta^2}{6\varepsilon} + O(\frac{\varepsilon}{\eta^2}) + O(\frac{\varepsilon^2}{\eta^4}) + O(\frac{\eta^3}{\varepsilon}) + O(\eta) + O(\frac{\varepsilon}{\eta})\Big] = 0 \quad , x_\eta \ fixed \tag{2.27}$$

The first term in the above comes from g_0 while the second comes from εh_1. Since the remainders contain the reciprocals of η^2/ε and ε/η^3, it is clear that these terms are singular. Their counter parts must be found in the higher order terms. In fact, it is easy to see that the leading term in $\varepsilon^3 g_3$ will match with $-\varepsilon^3/\eta^3 x_\eta^3$.

We know from the Eqn.(2.8)

$$\begin{aligned} h_3 &= -\frac{e^{-\eta x_\eta}}{\eta^3 x_\eta^3} = -\frac{1 - \eta x_\eta + \frac{\eta^2 x_\eta^2}{2} - \frac{\eta^3 x_\eta^3}{6} + \frac{\eta^4 x_\eta^4}{24} - \cdots}{\eta^3 x_\eta^3} \\ &= -\frac{1}{\eta^3 x_\eta^3} + \frac{1}{\eta^2 x_\eta^2} - \frac{1}{2\eta x_\eta} + \frac{1}{6} - \frac{\eta x_\eta}{24} + \cdots \end{aligned}$$

in the terms of x_η.

In the Eqn.(2.10), we observe that

$$g_3 = \frac{x^{*3}}{6(x^*+1)} = \frac{\eta^3 x_\eta^3}{6\varepsilon^2(\varepsilon + \eta x_\eta)}.$$

Similarly, we consider that

$$g_3 = \sum_{n=1}^{N} r_n(x_\eta)\phi_n(\varepsilon) + O(\phi_{N+1}) \tag{2.28}$$

where

$$\begin{aligned}
r_{-2} &= \lim_{\varepsilon\to 0} \frac{\varepsilon^2\eta^3 x_\eta^3}{6\varepsilon^2(\varepsilon+\eta x_\eta)} = \frac{\eta^2 x_\eta^2}{6} \\
r_{-1} &= \lim_{\varepsilon\to 0} \frac{\frac{\eta^3 x_\eta^3}{6\varepsilon^2(\varepsilon+\eta x_\eta)} - \frac{\eta^2 x_\eta^2}{6\varepsilon^2}}{\varepsilon^{-1}} = -\frac{\eta x_\eta}{6} \\
r_0 &= \lim_{\varepsilon\to 0} \left(\frac{\eta^3 x_\eta^3}{6\varepsilon^2(\varepsilon+\eta x_\eta)} - \frac{\eta^2 x_\eta^2}{6\varepsilon^2} + \frac{\eta x_\eta}{6\varepsilon}\right) = \frac{1}{6} \\
r_1 &= \lim_{\varepsilon\to 0} \frac{\frac{\eta^3 x_\eta^3}{6\varepsilon^2(\varepsilon+\eta x_\eta)} - \frac{\eta^2 x_\eta^2}{6\varepsilon^2} + \frac{\eta x_\eta}{6\varepsilon} - \frac{1}{6}}{\varepsilon} = -\frac{1}{6\eta x_\eta} \\
&\vdots \\
r_k(x_\eta) &= \lim_{\varepsilon\to 0} \frac{g_3(x_\eta,\varepsilon) - \sum_{n=0}^{k-1} r_n(x_\eta)\phi_n(\varepsilon)}{\phi_k(\varepsilon)} = \frac{(-1)^k}{6\eta^k x_\eta^k}, \quad k = 1, 2, \cdots.
\end{aligned}$$

Hence, by replacing all r_k's in the Eqn.(2.28), we get

$$g_3 = \frac{\eta^2 x_\eta^2}{6\varepsilon^2} - \frac{\eta x_\eta}{6\varepsilon} + \frac{1}{6} - \frac{\varepsilon}{6\eta x_\eta} + O(\frac{\varepsilon^2}{\eta^2})$$

More precisely,

$$\varepsilon^3 g_3 = \varepsilon\eta^2 \frac{x_\eta^2}{6} + O(\varepsilon^2\eta) \tag{2.29}$$

$$\varepsilon^3 h_3 = -\frac{\varepsilon^3}{\eta^3 x_\eta^3} + O(\frac{\varepsilon^3}{\eta^2}) \tag{2.30}$$

Now, the matching condition to $O(\varepsilon^2)$ with $P = Q = 3$ is satisfied, as long as $\eta^3/\varepsilon \to 0$ to eliminate the remainder term in εh_1. We must also require that $(\varepsilon/\eta^2 \to 0)$ to eliminate the remainder terms in εg_1 and $\varepsilon^3 h^3$. The above two conditions imply that η must be constrained as follows

$$\varepsilon^{1/2} \ll \eta \ll \varepsilon^{1/3}, \tag{2.31}$$

i.e., $\eta_1 = \varepsilon^{1/2}$ $\eta_2 = \varepsilon^{1/3}$.

This process can be continued indefinitely. We note that the overlap domain shrinks as the order of the matching increases, and this is typical. We have carried out the matching to $O(\varepsilon^2)$ to point out the fact that terms of order ε^3 were needed. In general, the choice of P and Q is not a priori obvious and must be deduced from the matching.

We have adopted the somewhat cumbersome formulation of writing both inner and outer variables in terms of x_η to insure that singularities are taken into account, and perhaps more importantly, to exhibit directly the overlap domain by requiring all remainder terms to vanish to a given order.

We now turn to the question of finding one asymptotic expansion which is uniformly valid for all x. Proceeding heuristically, we assume (since the ingredients of both expansions are needed in their respective regions) that this uniformly valid "composite" expansion consists of the sum of the inner and outer expansions, less those terms which are common to both. The common terms are those that are matched and should not appear twice in the composite expansion. Of course, the final result must be checked to verify that it leads to the inner or outer expansions under the appropriate limit processes.

To $O(1) h_0 = 0$ therefore $g_0(x^*)$ is uniformly valid. This can be verified in two ways, the first directly, by noting that

$$\lim_{\varepsilon \to 0} g(x^*, \varepsilon) - g_0(x^*) = 0 \quad for\ all\ x^* \tag{2.32}$$

or indirectly by noting that the outer limit process, when applied to g_0, gives zero. To order ε, we note by comparing Equations (2.17), (2.20) and (2.21) that in the sum $g_0 + \varepsilon(h_1 + g_1)$ the term $\varepsilon - 1/x^*$ is common both in expansions thus, the composite expansion which is uniformly valid to $O(\varepsilon)$ is

$$\begin{aligned} y &= k(x, \varepsilon) = g_0 + \varepsilon\Big(h_1 + g_1 - 1 + \frac{1}{\varepsilon x^*}\Big) + O(\varepsilon^2) \\ &= e^{-x^*} - \frac{1}{1+x^*} + \varepsilon\Big[-\frac{e^{-x}}{x} - \frac{1}{1+x^*} + \frac{1}{\varepsilon x^*}\Big] + O(\varepsilon^2). \end{aligned} \tag{2.33}$$

To prove that (2.33) is indeed uniformly valid we must show that in the limit as $\varepsilon \to 0$, x fixed $\neq 0$ (2.33) reduces to εh_1, and that in the limit $\varepsilon \to 0$, x^* fixed it reduces to $g_0 + \varepsilon g_1$.

To show that (2.33) leads to the outer expansion in the limit $\varepsilon \to 0$, x fixed $\neq 0$, we rewrite it in the form

$$k = e^{-x/\varepsilon} - \frac{\varepsilon}{\varepsilon + x} + \varepsilon\Big[-\frac{e^{-x}}{x} - \frac{\varepsilon}{\varepsilon + x} + \frac{1}{x}\Big] + O(\varepsilon^2). \tag{2.34}$$

The term $e^{-x/\varepsilon}$ is transcendentally small. Developing $\varepsilon/(\varepsilon + x)$ gives

$$\frac{\varepsilon}{\varepsilon + x} = \frac{\varepsilon}{x(1+\varepsilon/x)} = \frac{\varepsilon}{x}\Big(1 - \frac{\varepsilon}{x} + \cdots\Big) = \frac{\varepsilon}{x} - \frac{\varepsilon^2}{x^2} + \cdots \tag{2.35}$$

and the leading term in the above cancels the term $1/x$ in the bracketed expression of (2.34). Thus, to $O(\varepsilon)$ we have

$$\lim_{\varepsilon \to 0} \frac{k}{\varepsilon} = -\frac{e^{-x}}{x} = h_1(x) \quad x^* \text{ fixed.} \tag{2.36}$$

To show that (2.33) gives the inner expansion, we rewrite it in terms of x^* as follows

$$k = e^{-x^*} - \frac{1}{1+x^*} + \varepsilon\Big[\frac{-e^{-\varepsilon x^*} + 1}{\varepsilon x^*} - \frac{1}{1+x^*}\Big] + O(\varepsilon^2). \tag{2.37}$$

Developing $1 - e^{-\varepsilon x^*}$ gives $\varepsilon x^* + \cdots$. Therefore

$$\lim_{\varepsilon \to 0} \frac{k - g_0(x^*)}{\varepsilon} = \frac{x^*}{1+x^*} = g_1(x^*) \quad x \text{ fixed.} \tag{2.38}$$

This proves that (2.33) is the uniformly valid asymptotic expansion of (2.3) to $O(\varepsilon)$. Similar, more tedious arguments can be used to calculate the next term in k.

The important feature to note here is that the uniformly valid expansion is *not* in the form

$$f(x,\varepsilon) - \sum_{n=1}^{M} a_n(x)\phi_n(\varepsilon) = 0(\phi_M) \; as \; \varepsilon \to \varepsilon_0 \tag{2.39}$$

for each $M = 1, 2, \cdots, N$, because the dependence on x and ε *cannot* be factored.

The above ideas lead to the following definitions:

We say that $\sum_{n=1}^{N} f_n(x^*)\mu_n(\varepsilon)$ is a "limit process" expansion as $\varepsilon \to 0$ of $f(x,\varepsilon)$ with respect to the sequence μ_n if it can be derived by making the change of variable $x = s(\varepsilon)x^*$ for a specific $s(\varepsilon)$ in f, and then proceeding with usual calculation for the f_n with x^* fixed in the limit $\varepsilon \to 0$.

Thus, the outer and inner expansions discussed in the above example are limit process expansions with $s = 1$ and $s = \varepsilon$, respectively. The composite expansion is not a limit process expansion. In fact, it is a general asymptotic expansion in the sense of the following definition.

We say that $\sum_{n=1}^{N} k_n(x,\varepsilon)$ is a "general asymptotic expansion" as $\varepsilon \to 0$ of $f(x,\varepsilon)$ to N terms with respect to the sequence μ_n if for each $M = 1, 2, \cdots, N$

$$\lim_{\varepsilon \to 0} \frac{f(x,\varepsilon) - \sum_{n=1}^{M} k_n(x,\varepsilon)}{\mu_n(\varepsilon)} = 0. \tag{2.40}$$

Thus for the example considered $\mu_1 = 1, \mu_2 = \varepsilon, \cdots$ and

$$k_1 = e^{-x/\varepsilon} - \frac{\varepsilon}{\varepsilon + x} \tag{2.41}$$

$$k_2 = \frac{\varepsilon(1 - e^{-x})}{x} - \frac{\varepsilon^2}{\varepsilon + x} \tag{2.42}$$

etc.

We now turn to the very important question of whether the outer and inner expansions can be derived from the differential equation (2.1) without knowledge of the exact solution (2.3). This is crucial, if the above ideas are to be exploited for problems where an exact solution is not available. Assume an outer expansion of the form

$$y = \mu_1(\varepsilon)h_1(x) + \mu_2(\varepsilon)h_2(x) + \mu_3(\varepsilon)h_3(x) + \cdots \tag{2.43}$$

with μ_1, μ_2,etc. unknown. Substituting (2.43) into (2.1) and developing the outer expansion of the right hand side of (2.1) gives ($' = d/dx$)

$$\begin{aligned} &\varepsilon\mu_1 h_1' + \varepsilon\mu_2 h_2' + \varepsilon\mu_3 h_3' + \mu_1 h_1 + \mu_2 h_2 + \mu_3 h_3 + O(\mu_4) \\ &\quad = -\varepsilon\frac{e^{-x}}{x} + \varepsilon^2\frac{e^{-x}}{x^2}(2+x) - \varepsilon^3\frac{e^{-x}(3-x)}{x^3} + O(\varepsilon^4). \end{aligned} \tag{2.44}$$

The lowest order term on the left hand side is $\mu_1 h_1$, while the leading term on the right hand side is $-\varepsilon e^{-x}/x$. Therefore, we must choose $\mu_1 = \varepsilon$ and find $h_1 = -e^{-x}/x$.

In the next approximation (with $\mu_1 = \varepsilon$) the equation

$$\varepsilon^2 h_1' + \mu_2 h_2 = \varepsilon^2\frac{e^{-x}}{x^2}(2+x) \tag{2.45}$$

contains the lowest order terms.

Thus, unless we set $\mu_2 = \varepsilon^2$ we have a contradiction $[h_1' \neq (e^{-x}/x^2)(2+x)]$. This fixes $\mu_2 = \varepsilon^2$ and (2.45) defines h_2 to be

$$h_2 = \frac{e^{-x}}{x^2}(2+x) - h_1' = \frac{e^{-x}}{x^2}. \tag{2.46}$$

Similarly, consistency of the terms of order ε^3 requires that $\mu_3 = \varepsilon^3$ and one then calculates

$$h_3 = -\frac{e^{-x}}{x^3} \tag{2.47}$$

and the process can be continued indefinitely. We see for this example that applying the outer limit to the differential equation does indeed reproduce the outer expansion. Since we are dealing with a first-order equation the calculation of the outer expansion does not involve the solution of differential equations.

Consider next the inner limit applied to the differential equation (2.1). Writing

(2.1) and $y(0) = 0$ in terms of x^* and expanding the right hand side in the limit process $\varepsilon \to 0$, x^* fixed gives

$$\frac{dy}{dx^*} + y = -\frac{x^*}{(1+x^*)^2} + \varepsilon\frac{(x^{*2}+x^*+1)}{(1+x^*)^2} - \varepsilon^2\frac{(x^{*3}+2x^{*2}+2x^*)}{2(1+x^*)^2} + O(\varepsilon^3) \quad (2.48)$$

$$y(0) = 0 \quad (2.49)$$

We assume an inner expansion of the form

$$y = v_0(\varepsilon)g_0(x^*) + v_1(\varepsilon)g_1(x^*) + v_2(\varepsilon)g_2(x^*) + \cdots \quad (2.50)$$

The boundary condition (2.49) then implies that $g_n(0) = 0,\ n = 0, 1, \cdots$. Substituting (2.50) into (2.48) gives the first approximation

$$\frac{dg_0}{dx^*} + g_0 = \begin{cases} 0 & \text{if } v_0 \gg 1, \\ \frac{-x^*}{(1+x^*)^2} & \text{if } v_0 = 1. \end{cases} \quad (2.51)$$

Since the solution of (2.51) subject to $g_0(0) = 0$ identically zero, the first non-vanishing term in (2.50) will correspond to the choice $v_0 = 1$ in which case the solution is

$$g_0(x^*) = e^{-x^*} - \frac{1}{1+x^*}. \quad (2.52)$$

Repeating this argument to higher orders , we see that unless we choose $v_1 = \varepsilon$,$v_2 = \varepsilon^2$ etc. The corresponding $g_i(x^*)$ will be identically zero. Setting $v_i = \varepsilon^i$ then provides a set of nonhomogeneous equations which can be solved and which reproduce the inner expansion (2.10).

In the next chapter we will build on the above ideas and consider less trivial examples where considering all limit process expansions and matching will provide the uniformly valid general asymptotic expansion for cases where the differential equation is hard (or impossible) to solve exactly if $\varepsilon \neq 0$.

It can be noted here that the ideas introduced in order to construct and match asymptotic expansions depended in no way on the linearity of the particular

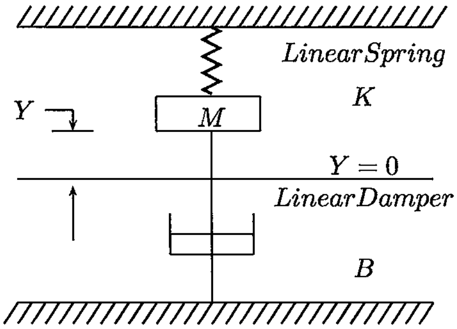

Figure 2.1: Spring-Mass System with Damping

example. Hence one can expect that these methods can be applied to nonlinear cases also.

2.1 Limit Process Expansions Applied to Ordinary Differential Equations

2.1.1 Linear Oscillator: Regular Perturbation

As a first example [3], consider a case for which the exact solution is easily found: the respond of a linear spring-mass-damping system initially at rest to an impulse I_0 (see Figure (2.1)). The equation and initial conditions are

$$\begin{aligned} M\frac{d^2Y}{dT^2} &+ B\frac{dY}{dT} + KY = I_0\delta(T) \\ Y(0^-) &= \frac{dY(0^-)}{dT} = 0, \end{aligned} \tag{2.53}$$

where δ is the Dirac delta function.

Problem (2.53) can be replaced by an equivalent one, (2.54) by considering an impulse-momentum balance across $T = 0$ or by integrating Equation (2.53) from $T = 0^-$ to $T = 0^+$:

$$\begin{aligned} M\frac{d^2Y}{dT^2} + B\frac{dY}{dT} + KY &= 0, \quad T > 0 \\ Y(0^+) &= 0 \end{aligned}$$

$$\frac{dY(0^+)}{dT} = \frac{I_0}{M} \tag{2.54}$$

The solution defined by this problem is the fundamental solution of this linear equation.

The "regular" perturbation is concerned with an approximation for small damping coefficient B. For small B, we expect the motion to be a slightly damped oscillation close to the free simple harmonic oscillation of the system - the solution of Equation (2.54) with $B = 0$.

Non-Dimensionalization

For the introduction of dimensionless coordinates, a suitable time scale is $\sqrt{M/K}$, the reciprocal of the natural frequency of free undamped motion, since this scale remains in the limit $B \to 0$. The length scale A, a measure of the amplitude, can be chosen arbitrarily and this choice will not affect the resulting dimensionless differential equation since it is linear. Actually, we will choose A in a form convenient for normalizing the initial velocity.

Setting

$$t^* = \frac{T}{(M/K)^{1/2}}, \quad y = \frac{Y}{A} \tag{2.55}$$

we have

$$M\frac{d^2(Ay)}{d(t^*\sqrt{M/K})^2} + B\frac{d(Ay)}{d(t^*\sqrt{M/K})} + Ay = 0$$

$$\frac{d^2y}{dt^{*2}} + 2\varepsilon^*\frac{dy}{dt^*} + y = 0, \tag{2.56}$$

where

$$\varepsilon^* = \frac{B}{2(MK)^{1/2}}.$$

In these variables

$$Y(0^+) = Ay(0^+) = 0 \quad \Rightarrow \quad y(0^+) = 0$$

and

$$\frac{dY}{dT} = \frac{d(Ay)}{d(t^*\sqrt{M/K})} \quad = \quad \frac{A}{\sqrt{M/K}}\frac{dy}{dt^*} = \frac{I_0}{M} \Rightarrow \frac{dy}{dt^*} = \frac{I_0}{\sqrt{MKA}}.$$

We find that $dy(0^+)/dt^* = 1$ if we set $A = I_0/(MK)^{1/2}$. It is seen that the solution of the problem only involves the one parameter ε^*. Small damping corresponds to ε^* small so that an expansion with the limit process $\varepsilon^* \to 0$ is considered.

For this example it is evident that nontrivial correction terms can be obtained only if y is expanded in powers of ε^* as follows

$$y(t^*, \varepsilon^*) = g_0(t^*) + \varepsilon^* g_1(t^*) + \cdots. \tag{2.57}$$

By substituting the expansion (2.57) into Equation (2.56) and the initial conditions, we obtain the following sequence of linear problems to solve in order

$$\begin{aligned}
\frac{d^2 g_0}{dt^{*2}} + g_0 = 0 \quad &; \qquad g_0(0) = 0, \quad \frac{dg_0(0)}{dt^*} = 0 \\
\frac{d^2 g_1}{dt^{*2}} + g_1 = -2\frac{dg_0}{dt^*} \quad &; \qquad g_1(0) = \frac{dg_1(0)}{dt^*} = 0 \\
\frac{d^2 g_i}{dt^{*2}} + g_i = -2\frac{dg_{i-1}}{dt^*} \quad &; \qquad g_i(0) = \frac{dg_i(0)}{dt^*} = 0 \quad i = 1, 2, \cdots.
\end{aligned} \tag{2.58}$$

The equation for g_0 is that of the free undamped motion; correction g_1 is computed by the damping acting on the velocity of the free motion g_0, and so on. The solutions are

$$\begin{aligned}
g_0 &= \sin t^*, \\
g_1 &= -t^* \sin t^*,
\end{aligned} \tag{2.59}$$

etc. Thus, as far as the solution has been carried the result is

$$y = \sin t^* - \varepsilon^* t^* \sin t^* + O(\varepsilon^2 t^{*2}). \tag{2.60}$$

In this approximation, the amplitude decays linearly with time. If we consider the exact solution

$$y(t^*, \varepsilon^*) = \frac{e^{-\varepsilon^* t^*}}{\sqrt{1-\varepsilon^{*2}}} \sin(\sqrt{1-\varepsilon^{*2}}t^*) \tag{2.61}$$

we see that the amplitude $1-\varepsilon^* t^*$ is the expansion of $e^{-\varepsilon^* t^*}$ to this order. Clearly, this expansion is only valid for t^* fixed and finite. Thus, this regular perturbation is identical to an inner limit process expansion as discussed in Chapter 2. For any finite time interval $0 < t^* < t_0^*$, ε^* can be chosen sufficiently small so that Equation (2.60) is a good approximation uniformly valid to $O(\varepsilon)$ of the exact result of (2.61).

Because in this limit $e^{-\varepsilon^* t^*}$ and $\sin\sqrt{1-\varepsilon^{*2}}t^*$ are approximated by power series in $\varepsilon^* t^*$ and $\varepsilon^{*2} t^*$, the results are not uniformly valid over the entire time interval. Now we consider a perturbation problem for Equation (2.53) which is singular at $T = 0$.

2.1.2 Linear Oscillator: Singular Problem

The singular problem is connected with approximations of Equation (2.53) for small values of the mass M. The difficulty near $T = 0$ arises from the fact that the limit equation with $M = 0$ is first order, so that both initial conditions of (2.54) cannot be satisfied. The loss of an initial or boundary condition in a problem leads, in general, to the occurrence of a boundary layer.

First, a discussion of this problem is given based on physical reasoning. After a short time interval, it can be expected that the motion of a system is described by the limit form of Equation (2.53) with $M = 0$. Thus, we have

$$B\frac{dY}{dT} + KY = I_0\delta(T). \tag{2.62}$$

The initial condition in velocity is lost. By integrating both sides of Eqn.(2.62) from 0^- to 0^+, we get

$$B(Y(0^+) + Y(0^-)) = I_0,$$

hence the effect of the impulse is to jump the initial displacement from $Y(0^-) = 0$ to

$$Y(0^+) = \frac{I_0}{B}. \tag{2.63}$$

Hence, the solution of the following homogeneous equation

$$B\frac{dY}{dT} + KY = 0$$

is

$$Y = \frac{I_0}{B}e^{-KT/B}. \tag{2.64}$$

The solution demonstrates the exponential decay after the short initial interval in which the displacement increased infinitely rapidly from 0 to I_0/B. In order to describe the motion during the initial instants, we remark that inertia is certainly dominant at $T = 0$ (impulse-momentum balance) and that due to the large initial velocity, damping is important immediately but the spring is not, since deflection must be achieved before the spring is effective. Thus, in the initial instants, Equation (2.53) can be approximated by

$$M\frac{d^2Y}{dT^2} + B\frac{dY}{dT} = I_0\delta(T) \tag{2.65}$$

with the solution

$$Y(T) = \frac{I_0}{B}(1 - e^{-BT/M}). \tag{2.66}$$

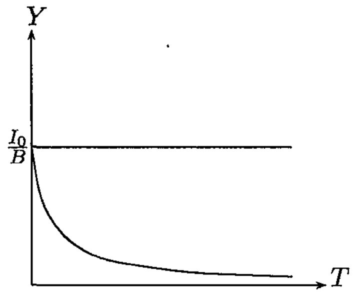

Figure 2.2: Solution of Spring-Mass System after a short time

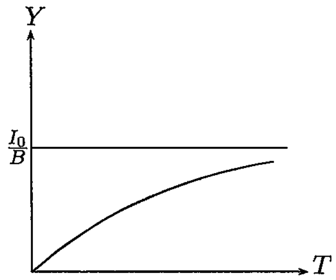

Figure 2.3: Solution of Spring-Mass System at initial instants

This solution shows the approach of the deflection in a very short time ($M \to 0$) to the starting value for the decay solution of Equation (2.64). The curves shown in Figures (2.2) and (2.3) give an overall picture of the motion.

Following our physical considerations, we aim to construct suitable asymptotic expansions for expressing these physical ideas and to show how to join these expansions. The method uses expansions valid after a short time (away from the initial point) and expansions valid near the initial point.

Outer Expansion

For the expansion valid away from the initial point, we find that natural variables are those based on a time scale for decay (B/K) and on an amplitude linear in I_0. Let

$$t = \frac{K}{B}T, \quad y = B\frac{Y}{I_0}$$

so that Equation (2.54) reads

$$M\frac{d^2(\frac{I_0}{B}y)}{d(\frac{B}{K}t)^2} + B\frac{d(\frac{I_0}{B}y)}{d(\frac{B}{K}t)} + K\frac{I_0}{B}y = \varepsilon\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0 \tag{2.67}$$

where $\varepsilon = MK/B^2$, with initial conditions

$$y(0) = 0, \quad \frac{dy}{dt}(0) = \frac{1}{\varepsilon}. \tag{2.68}$$

The expansion valid away from the initial point (outer expansion) is associated with the limit

$$\varepsilon \to 0, \quad t \text{ fixed (outer limit)}. \tag{2.69}$$

An asymptotic expansion in the form

$$y(t,\varepsilon) = v_1(\varepsilon)h_1(t) + v_2(\varepsilon)h_2(t) + \cdots, \tag{2.70}$$

ia assumed.

The equations resulting from repeated application of the outer limit to Equation (2.67) or equating terms of the same order of magnitude when Equation (2.70) is substituted in Equation (2.67) are

$$\frac{dh_1}{dt} + h_1 = 0, \tag{2.71}$$

$$\frac{dh_2}{dt} + h_2 = \begin{cases} -\frac{d^2h_1}{dt^2} & \text{if } \frac{v_1\varepsilon}{v_2} = 1, \\ \\ 0 & \text{if } \frac{v_1\varepsilon}{v_2} \to 0. \end{cases} \tag{2.72}$$

The initial conditions for this set of equations, as well as the orders of the various $v_i(\varepsilon)$, are unknown and have to be found by matching with an expansion valid near $t = 0$ (inner expansion), which takes account of the initial conditions.

The solution for the outer expansion are

$$h_1 = A_1 e^{-t} \tag{2.73}$$

$$h_2 = A_2 e^{-t} - A_1 t e^{-t}. \tag{2.74}$$

The term $-A_1 t e^{-t}$ above would be missing if it turned out that $v_1 \varepsilon / v_2 \to 0$.

Inner Expansion

Various limits can be considered in which a representative time t approaches the origin in the $\varepsilon - t$ space at varying rates. That is, limits and associated asymptotic expansions can be considered for which t_η is fixed, where

$$t_\eta = \frac{t}{\eta(\varepsilon)}, \quad \eta(\varepsilon) \to 0 \; as \; \varepsilon \to 0.$$

In terms of this variable, Equation (2.67) is

$$\frac{\varepsilon}{\eta^2} \frac{d^2 y}{dt_\eta^2} + \frac{1}{\eta} \frac{dy}{dt_\eta} + y = 0 \tag{2.75}$$

Three cases evidently arise, yielding different limit equations, which would be satisfied by the dominant term of corresponding asymptotic expansions.

Case I

Inner-inner limit $\eta \ll \varepsilon$ or $\varepsilon / \eta \to \infty$,

$$\frac{d^2 y}{dt_\eta^2} = 0. \tag{2.76}$$

This is not a distinguished limit as Equation (2.76) holds for a class of functions different orders. In this limit the governing equation is of second order so that the initial conditions can be accounted for. However, the use of an inner-inner limit is unnecessary for this problem since the expansion associated with it is contained in Case II. The expansion associated with this limit is valid only in a very small time interval, $t \leq k\eta(\varepsilon)$, around $t = 0$, the inertia-dominated regime.

Case II

Initial-layer (boundary layer) limit, $\eta = \varepsilon$

$$\frac{d^2y}{dt_\eta^2} + \frac{dy}{dt_\eta} = 0 \tag{2.77}$$

This is called a *distinguished* limit since $\eta(\varepsilon)$ cannot belong to a class of functions with different orders but is definitely $O(\varepsilon)$ as $t \to 0$. The corresponding expansion will be a limit-process expansion. The initial conditions can be satisfied. This limit yields the boundary-layer equation derived previously by physical reasoning. As t_η in Equation (2.77) approaches zero, the solution of Equation (2.76) are obtained and, in that sense, Case I is contained in Case II.

Case III

Intermediate limit, $\varepsilon \ll \eta(\varepsilon) \ll 1$, or $\varepsilon/\eta(\varepsilon) \to 0$

$$\frac{dy}{dt_\eta} = 0. \tag{2.78}$$

This is also is not a distinguished limit but consists of a class of limits (and expansions) intermediate to the boundary-layer limit and outer limit. Equation (2.78) can handle neither the initial conditions nor the expected behavior at infinity. In this example, limits of Case III are also superfluous since they are contained in both those of Case II and the outer limit.

Thus, consider the asymptotic expansion associated with the initial layer limit

$$y(t;\varepsilon) = \mu_1(\varepsilon)g_1(t^*) + \mu_2(\varepsilon)g_2(t^*) + \cdots, \tag{2.79}$$

where $t^* = t/\varepsilon$, and with the sequence of approximate equations which result,

$$\frac{d^2g_1}{dt^{*2}} + \frac{dg_1}{dt^*} = 0 \tag{2.80}$$

$$\frac{d^2 g_2}{dt^{*2}} + \frac{dg_2}{dt^*} = \begin{cases} -g_1 & \text{if } \frac{\varepsilon\mu_1}{\mu_2} \to 1, \\ \\ 0 & \text{if } \frac{\varepsilon\mu_1}{\mu_2} \to 0. \end{cases} \tag{2.81}$$

The initial conditions fix $\mu_1(\varepsilon)$ since

$$\frac{dy}{dt} = \frac{\mu_1}{\varepsilon}\frac{dg_1}{dt^*} + \frac{\mu_2}{\varepsilon}\frac{dg_2}{dt^*} + \cdots = \frac{1}{\varepsilon} \quad as\ t^* \to 0$$

Thus, $\mu_1 = 1$, and the initial conditions associated with (2.80) and (2.81) are

$$g_1(0) = 0, \quad \frac{dg_1(0)}{dt^*} = 1, \tag{2.82}$$

$$g_2(0) = 0, \quad \frac{dg_2(0)}{dt^*} = 0. \tag{2.83}$$

The solution of the initial-layer Equation (2.80) is thus

$$g_1 = 1 - e^{-t^*}. \tag{2.84}$$

A nonzero term is needed for g_2 if corrections to the first term are to be found. Thus,μ_2 must be chosen equal to ε, and Equation (2.81) reads

$$\frac{d^2 g_2}{dt^{*2}} + \frac{dg_2}{dt^*} = -(1 - e^{-t^*}).$$

The solution of this is

$$g_2 = (2 - t^*) - (2 + t^*)e^{-t^*}. \tag{2.85}$$

The inner expansion can be carried out in this way to any order.

Matching

The matching of inner and outer expansions serves to determine both the orders of magnitude of the v_i and the constants of integration $A_1, A_2, \cdots$ in the outer expansion. Crudely speaking, the idea of matching is that the behavior of the outer expansion as $t \to 0$ and the inner expansion as $t^* \to \infty$ is in agreement. More formally, as discussed in Chapter 2, there is a domain in which both expansions are valid, an overlap domain, and in which these expansions agree. At first, the outer expansion is assumed valid, that is, truly asymptotic to the exact solution for $t \geq k_1 \geq 0$ (and for as many terms as are considered). Correspondingly, the inner expansion is valid for $0 \leq t^* \leq k_2$ or $0 \leq t \leq k_2\varepsilon$. If the regions of validity can be extended so that the outer expansion is valid in $t \leq \eta_1(\varepsilon)$ and the inner expansion extended to lap domain. If we assume that overlap exists in this case, as already indicated for the first time, an intermediate limit (corresponding to Case III) must hold in the overlap domain.

Accordingly, we define the intermediate variable t_η

$$t_\eta = \frac{t}{\eta(\varepsilon)}$$

for some, yet to be specified, class of functions $\eta(\varepsilon)$ contained in the class $\varepsilon \ll \eta(\varepsilon) \ll 1$. We will investigate whether the inner and outer expansions when expressed in terms of t_η agree to some order.

More precisely, matching to $O(1)$ requires that we be able to find a class of functions $\eta(\varepsilon)$ such that $(R = 0, P = Q = 1)$

$$\lim_{\varepsilon \to 0} \left\{ v_1(\varepsilon) h_1(\eta t_\eta) - g_1\left(\frac{\eta t_\eta}{\varepsilon}\right) \right\} = 0, \quad t_\eta \text{ fixed} \tag{2.86}$$

or, using the expressions (2.73) and (2.84) in terms of t_η that we have calculated for h_1 and g_1, this means that

$$\lim_{\varepsilon \to 0} \{ v_1(\varepsilon) A_1 e^{-\eta t_\eta} - [1 - e^{-(\eta/\varepsilon) t_\eta}] \} = 0, \quad t_\eta \text{ fixed.} \tag{2.87}$$

We can find the overlap domain for the matching to order unity by using the fact that $e^{-(\eta/\varepsilon)t_\eta}$ is transcendentally small, i.e.,

$$e^{-(\eta/\varepsilon)t_\eta} \ll \varepsilon \Rightarrow -\frac{\eta}{\varepsilon} \ll \log\varepsilon \Rightarrow \varepsilon|\log\varepsilon| \ll \eta(\varepsilon) \ll 1$$

hence Equation (2.87) can only hold if we choose $v_1(\varepsilon) = 1$ and $A_1 = 1$. Assuming that $v_2 = \varepsilon v_1 = \varepsilon$, matching to order ε will provide the value of A_2. The matching condition, $(R = 1, P = Q = 2)$, now assuming that the first two terms in each expansion are sufficient, is

$$Z = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big\{v_1(\varepsilon)h_1(\eta t_\eta) + v_2(\varepsilon)h_2(\eta t_\eta) - g_1\Big(\frac{\eta t_\eta}{\varepsilon}\Big) - \varepsilon g_2\Big(\frac{\eta t_\eta}{\varepsilon}\Big)\Big\} = 0, \quad t_\eta \text{ fixed} \tag{2.88}$$

If we take account of terms which have already been matched, expand the exponential $e^{-\eta t_\eta}$, and neglect transcendentally small terms, Eqn.(2.88) reduces to

$$\begin{aligned} Z &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{e^{-\eta t_\eta} + v_2(\varepsilon)(A_2 e^{-\eta t_\eta} - t_\eta e^{-\eta t_\eta}) - (1 - e^{-\eta t_\eta/\varepsilon}) \\ &- \varepsilon[(2 - (\eta t_\eta/\varepsilon)) - (2 + (\eta t_\eta/\varepsilon))e^{-\eta t_\eta/\varepsilon}]\} \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{(1 - \eta t_\eta + \cdots) + v_2(\varepsilon)[A_2(1 - \eta t_\eta + \cdots) - t_\eta(1 - \eta t_\eta + \cdots)] \\ &- (1 - e^{-\eta t_\eta/\varepsilon}) - \varepsilon[(2 - (\eta t_\eta/\varepsilon)) - (2 + (\eta t_\eta/\varepsilon))e^{-\eta t_\eta/\varepsilon}]\} \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{-\eta t_\eta + O(\eta^2) + \varepsilon A_2 + \eta t_\eta - 2\varepsilon + O(\varepsilon^2)\} = 0, \quad t_\eta \text{ fixed.} \end{aligned} \tag{2.89}$$

The $O(\eta^2)$ remainder results from the next term in the expansion of $e^{-\eta t_\eta}$ while the $O(\varepsilon^2)$ remainder accounts for the third term, $\varepsilon^2 g_2$, in the inner expansion.

The terms of order η cancel identically, as they should (for otherwise $\eta/\varepsilon \to \infty$ matching would fail to order ε). The terms of order ε in the numerator will cancel if we set $A_2 = 2$, and the most important remainder term is $O(\eta^2/\varepsilon)$ and will vanish if $\eta(\varepsilon) \leq \varepsilon^{1/2}$. Thus, the overlap domain for the matching to order ε is $\varepsilon|\log\varepsilon| \ll \eta \ll \varepsilon^{1/2}$ and is smaller than that associated with the $O(1)$ matching.

We note that all the assumptions that were made prior to the matching justified a posteriori: It was necessary to retain the term $-A_1 t e^{-t}$ in h_2, hence the choice $v_2 = \varepsilon$ was appropriate. The number of terms needed in each expansion to order 1 and ε was one and two respectively. Failure of the matching would have required these assumptions be reexamined and modified or abandoned. For example, in some problems a homogeneous solution (corresponding to the choice $v_1\varepsilon \ll v_2$)might be needed to carry out the matching, or the number of terms required in one of the expansions might exceed those in the other, etc.

The result achieved so far are summarized as follows:

$$y(t:\varepsilon) = e^{-t} + \varepsilon[2-t]e^{-t} + \cdots (outer), \tag{2.90}$$

$$y(t:\varepsilon) = [1-e^{-t^*}] + \varepsilon[(2-t^*) - (2+t^*)e^{-t^*}] + \cdots,\ t^* = \frac{t}{\varepsilon}\ (inner). \tag{2.91}$$

A uniformly valid asymptotic expansion, of the general form of Equation (2.40) can be constructed from Equations (2.90) and (2.91). The inner and outer expansions have some terms in common, namely those terms that are matched (cancel out in the matching). If the two expansions in Equations (2.90) and (2.91) are added together and the common part subtracted, then a uniformly valid $(0 \leq t \leq t_1)$ representation results. It is clear from the occurrence of the term εte^{-t} in the outer expansion that uniform validity does not extend to $t = \infty$. The remedy for this is discussed before. The common part (cp) is, for this case,

$$cp = 1 + \varepsilon(2-t^*) + \cdots.$$

In a boundary-layer problems such as this, it is possible and desirable to subtract the common part from the inner expansion, so that the part of the uniform expansion left in inner variables decays exponentially. The resulting expansion is

$$y(t:\varepsilon) = \{e^{-t} - e^{-t^*}\} + \varepsilon\{(2-t)e^{-t} - (2+t^*)e^{-t^*}\} + \cdots. \tag{2.92}$$

This expansion falls into the class of general asymptotic expansions of the form

$$y(t:\varepsilon) = F_0(t:\varepsilon) + \varepsilon F_1(t:\varepsilon) + \cdots.$$

defined by Equation (2.40). For this case, the general expansion takes the form of a composite expansion: each term is composed of an inner part which decays to zero and an outer part:

$$y(t:\varepsilon) = [h_1(t) + g_1^*(t^*)] + \varepsilon[h_2(t) + g_2^*(t^*)] + \cdots, \quad t^* = \frac{t}{\varepsilon} \tag{2.93}$$

where the g_i^* are defined in Equation (2.92).

Such a form as Equation (2.93) can be assumed *a priori* in many boundary-layer problems. The $g_i^*(t/\varepsilon)$ should have the property of correcting the incorrect (in general) boundary condition of the first outer solution h_1 and should decay exponentially from the boundary.

Finally, in this example the first term of the uniformly valid approximation gives a good description of the physical phenomenon for small M. In physical variables, we have,

$$y \cong \frac{I_0}{B}\{e^{-KT/B} - e^{-BT/M}\}. \tag{2.94}$$

The motion shows a rapid rise to peak at $T \cong (M/B)\log(B^2/KM)$ and an eventual decay.

If we are interested in extending the range of uniform validity of the outer expressions to $t = \infty$ it can be done by considering a more general outer limit process. There is a cumulative effect of inertia which slowly shifts the time scale for decay. For example a variable t^+

$$t^+ = t\{1 + a_1\varepsilon + a_2\varepsilon^2 + \cdots\} \tag{2.95}$$

can be considered and the associated limit process has $t^+(t,\varepsilon)$ fixed. The constant a_k are to be found so as to enforce uniform validity at infinity. The outer expansion is now of the form

$$y(t:\varepsilon) = h_1^+(t^+) + \varepsilon h_2^+(t^+) + \cdots. \tag{2.96}$$

Note that

$$\frac{dy}{dt} = \frac{dh_1^+}{dt^+} + \varepsilon\left\{\frac{dh_2^+}{dt^+} + a_1\frac{dh_1^+}{dt^+}\right\} + \cdots$$

$$\frac{d^2y}{dt^2} = \frac{d^2h_1^+}{dt^{+2}} + \varepsilon\left\{\frac{d^2h_2^+}{dt^{+2}} + 2a_1\frac{d^2h_1^+}{dt^{+2}}\right\} + \cdots.$$

Thus the sequence replacing (2.71), (2.72) is

$$\frac{dh_1^+}{dt^+} + h_1^+ = 0 \tag{2.97}$$

$$\frac{dh_2^+}{dt^+} + h_2^+ = -\frac{d^2h_1^+}{dt^{+2}} - a_1\frac{dh_1^+}{dt^+}. \tag{2.98}$$

The term εte^{-t} which caused trouble at infinity arose from the right hand side of (2.72) in particular from the presence of e^{-t}, a solution of the homogeneous equation. This term can be eliminated by the choice of a_1. In this example $a_1 = 1$ evidently makes the right hand side of (2.98) equal to zero.

The outer expansion is now

$$y(t:\varepsilon) = A_1e^{-t^*} + \varepsilon A_2e^{-t^*}\cdots, \qquad t^* = t(1+\varepsilon+\cdots). \tag{2.99}$$

This expansion is now uniform near infinity.

CHAPTER 3

AN EXAMPLE OF WAVE ASYMPTOTICS

3.1 Introduction

The method of matched asymptotic expansions, described in Van Dyke's book [4], has had considerable success in dealing with a wide variety of problems, particularly in fluid mechanics. The method does not have full rigour, except in a narrow class of special problems. But the technique does provide a rational procedure in the sense that it can often be used to predict asymptotic approximations to high order, at least in principle.

3.2 Transmission past a partially immersed cylinder

The example (treated by [5], [6], [7], [8], [9]) concerns the transmission of short surface waves by a partially immersed cylinder that intersects the free surface vertically. This problem has been dealt with rigorously by Ursell [9]. Ursell reduced to problem to that of an integral equation that is amenable to solution by iteration. Here we follow the heuristic method of Leppington [7].

Cartesian coordinates (x, y) are chosen so that the undisturbed free surface is at $y = 0$, the $y - axis$ points downwards and the cross section S meets the free surface (FS) at $x = \pm a$. The geometry shown in Fig.(3.1), with the intersection points denoted by $P(x = a, y = 0)$ and $Q(x = -a, y = 0)$. The fluid is forced by an incident wave of potential $\Re\{\phi_{inc} e^{-iwt}\}$ where $\Re$ denotes the real part of a complex quantity and

$$\phi_{inc} = \exp\{\frac{-i(x-a)-y}{\varepsilon}\} \tag{3.1}$$

where $\varepsilon = g/w^2$. Water depth is assumed to be infinite. Henceforth the time

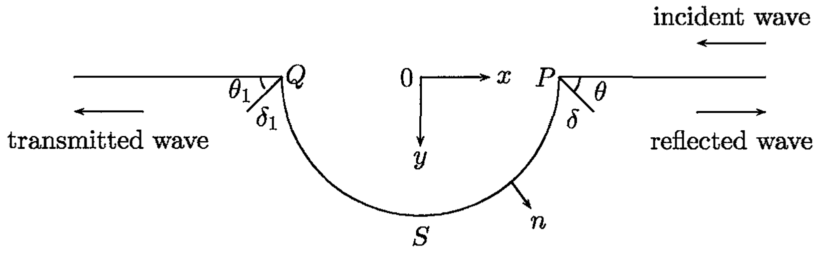

Figure 3.1: Transmission past a partially immersed cylinder.

factor e^{-iwt} and the symbol $\Re$ will usually be suppressed. The velocity potential $\Re\{\phi(x, y)e^{-iwt}\}$ is specified by the following equations

$$\nabla^2\phi = 0 \quad \text{in} \quad \text{the} \quad \text{fluid} \tag{3.2}$$

$$\phi + \varepsilon\phi_y = 0 \quad \text{on} \quad \text{FS} \tag{3.3}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on} \quad \text{S} \tag{3.4}$$

$$\phi - \phi_{inc} \sim \widetilde{R}\exp\{i(x+a)/\varepsilon - (y/\varepsilon)\} \quad as \quad x \to \infty \tag{3.5}$$

$$\phi \sim \widetilde{T}\exp\{-i(x-a)/\varepsilon - (y/\varepsilon)\} \quad as \quad x \to -\infty \tag{3.6}$$

where $\varepsilon = g/w^2$ is a measure of the wavelength, n denotes the outward normal from S and ϕ_y denotes the partial derivative of ϕ with respect to y. The complex reflection and transmission coefficients $\widetilde{R}$ and $\widetilde{T}$ are unknowns, to be determined. Of particular interest is the asymptotic form of the potential ϕ, and the constants $\widetilde{R}$ and $\widetilde{T}$ in the limit $\varepsilon/a \to +0$. For details, reader is referred to [10] and [11].

The solution depends critically on the shape of the scatterer S near the intersection points P and Q. Suppose that near P,S has the form

$$x - a = -f(y) \equiv \sum_{r=N}^{\infty} \frac{\alpha_r}{r!} y^r \tag{3.7}$$

for $0 \leq y < y_0$, where $N \geq 2$. If $N = 2$ then α_N is simply the curvature of S at point P. If $N > 2$, then α_N can be thought of as a generalized curvature there:

the larger the value of N, the flatter is the curve S at P. Similarly, near Q we take

$$x + a = f_1(y) = \sum_{r=M}^{\infty} \frac{\beta_r}{r!} y^r \quad M \geq 2. \tag{3.8}$$

Details are given here for the special case in which S is a semi-circle of radius a, whence $M = N = 2$ and $\alpha_2 = \beta_2 = 1/a$.

The asymptotic solution (*as* $\varepsilon \to +0$) is determined by the method of matched expansions . Different forms of approximation are postulated in various overlapping regions, namely: (*i*) the outer region (at distances large compared with ε from the free surface;) (*ii*) inner regions (within distances $\delta \ll a$ and $\delta_1 \ll a$ from the intersection points P and Q); and (*iii*) surface wave regions (near the free surface) where surface waves proportional to $\exp\{(\pm ix - y)/\varepsilon\}$ are significant.

3.2.1 Outer region

Now, consider that ϕ has the form $\phi \sim g_0(\varepsilon)\phi_0(x, y) + g_1(\varepsilon)\phi_1(x, y) + \cdots$. Letting $\varepsilon \to +0$ in the specifications (3.2) – (3.6), and with the aid of Equation (3.9)

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} &= (g_0\phi_{0xx} + g_1\phi_{1xx} + g_2\phi_{2xx} + \cdots) \\
&+ (g_0\phi_{0yy} + g_1\phi_{1yy} + g_2\phi_{2yy} + \cdots) = 0, \\
g_0(\phi_{0xx} + \phi_{0yy}) &= 0 \Rightarrow \phi_{0xx} + \phi_{0yy} = 0 \Rightarrow \phi_0 \quad \text{is} \quad \text{harmonic}, \\
g_1(\phi_{1xx} + \phi_{1yy}) &= 0 \Rightarrow \phi_{1xx} + \phi_{1yy} = 0 \Rightarrow \phi_1 \quad \text{is} \quad \text{harmonic}, \\
g_2(\phi_{2xx} + \phi_{2yy}) &= 0 \Rightarrow \phi_{2xx} + \phi_{2yy} = 0 \Rightarrow \phi_2 \quad \text{is} \quad \text{harmonic},
\end{aligned} \tag{3.9}$$

ϕ_0 is a harmonic function.

Since

$$\frac{\partial \phi}{\partial n} = \frac{\partial((g_0\phi_0) + (g_1\phi_1) + (g_2\phi_2) + \cdots)}{\partial n} = 0 \Rightarrow \frac{\partial \phi_0}{\partial n} = \frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n} = 0 \quad \text{on } S \tag{3.10}$$

and

$$\begin{aligned}
\phi+\varepsilon\phi_y &= (g_0\phi_0+g_1\phi_1+g_2\phi_2+\cdots)+\varepsilon(g_0\phi_{0y}+g_1\phi_{1y}+g_2\phi_{2y}+\cdots)=0\\
g_0\phi_0 &= 0\Rightarrow\phi_0=0 \quad on\ FS,\\
g_1\phi_1 &+ \varepsilon g_0\phi_{0y}=g_1\phi_1=0\Rightarrow\phi_1=0 \quad on\ FS,\\
g_2\phi_2 &+ \varepsilon g_1\phi_{1y}=g_2\phi_2=0\Rightarrow\phi_2=0 \quad on\ FS,
\end{aligned} \tag{3.11}$$

ϕ_0 is subject to the conditions

$$\left.\begin{array}{lll}
\partial\phi_0/\partial n=0 & on & S\\
\phi_0=0 & on & FS\\
\phi_0\to 0 & as & x^2+y^2\to\infty
\end{array}\right\}. \tag{3.12}$$

These homogeneous conditions admit a non-trivial solution only if ϕ_0 is singular somewhere. It transpires that ϕ_0 has a dipole singularity at the point P.

3.2.2 Right inner region

Near the intersection point $P(x=a, y=0)$ the dominant length scale is the (small) wavelength parameter ε. Thus, the appropriate scaled variables (X, Y) and Φ are introduced by the formulae

$$x=a+\varepsilon X, \quad y=\varepsilon Y; \quad \phi=\Phi(X,Y). \tag{3.13}$$

In particular the variables $\delta=((x-a)^2+y^2)^{\frac{1}{2}}$ and $R=(X^2+Y^2)^{\frac{1}{2}}$ are related by the identity $\delta=((\varepsilon X)^2+(\varepsilon Y)^2)^{1/2}=\varepsilon(X^2+Y^2)^{1/2}=\varepsilon R$. In terms of (X, Y) the circle $S(x^2+y^2=a^2)$ has the form

$$x^2+y^2=(\varepsilon X+a)^2+(\varepsilon Y)^2=a^2\Rightarrow\varepsilon X+a=(a^2+(-\varepsilon^2Y^2))^{1/2}. \tag{3.14}$$

By using the binomial expansion $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots$, we obtain that

$$\begin{aligned}
\varepsilon X + a &= (a^2)^{1/2} + (1/2)(a^2)^{-1/2}(-\varepsilon^2 Y^2) + \frac{(1/2)(-1/2)}{2}(a^2)^{-3/2}(-\varepsilon^2 Y^2)^2 + \cdots \\
\varepsilon X + a &= a - (\varepsilon^2 Y^2)/(2a) - (\varepsilon^4 Y^4)/(8a^3) + \cdots \\
X &= -(\varepsilon/2a)Y^2 - (\varepsilon^3/8a^3)Y^4)\cdots = -F(Y). \qquad (3.15)
\end{aligned}$$

The boundary condition (3.4) on S is

$$\Phi_X + F'(Y)\Phi_Y = 0 \quad on \quad X = -F(Y), \qquad (3.16)$$

since $\frac{\partial\phi}{\partial n} = 0 \Rightarrow \overrightarrow{grad}\phi.\overrightarrow{n} = (\phi_x \overrightarrow{i} + \phi_y \overrightarrow{j}).(1.\overrightarrow{i} + F_y \overrightarrow{j}) = \phi_x + \phi_y F_y = 0$. It is convenient to transform this condition on to the plane $X = 0$, using Taylor series,

$$\Phi_X(X,Y) = \Phi_X(0,Y) + X\Phi_{XX}(0,Y) + \cdots. \qquad (3.17)$$

Thus the right inner potential is the harmonic function Φ subject to the following body boundary condition, where $(X = 0)$ and $(Y > 0)$,

$$\Phi_X + (\frac{\varepsilon}{2a})(2Y\Phi_Y - Y^2\Phi_{XX}) + \frac{\varepsilon^3}{8a^3}(4Y^3\Phi_Y - Y^4\Phi_{XX}) + \cdots = 0. \qquad (3.18)$$

In order to derive this, we need the derivative of (3.15),

$$F'(Y) = (\varepsilon/a)Y + (\varepsilon^3/2a^3)Y^3 + \cdots. \qquad (3.19)$$

Since $\phi_y = \frac{dY}{dy}\Phi_Y = \frac{1}{\varepsilon}\Phi_Y$, equation (3.3) becomes the free surface boundary condition

$$\Phi + \Phi_Y = 0, \quad Y = 0, \quad X > 0. \qquad (3.20)$$

If we put $x = a + \varepsilon X$, and $y = \varepsilon Y$ in the equations (3.1) and (3.5) we obtain the radiation condition

$$\Phi \sim \exp(-iX - Y) + \widetilde{R}\exp(iX - Y + 2ia/\varepsilon) \tag{3.21}$$

plus wave-free terms, as $R = (X^2 + Y^2)^{1/2} \to \infty$.

3.2.3 Left inner region

A similar analysis is appropriate for the left inner region, near $Q(x = -a, y = 0)$. Coordinates (X_1, Y_1) are defined by

$$x = -a - \varepsilon X_1, \quad y = \varepsilon Y_1; \quad \phi = \Psi(X_1, Y_1) \tag{3.22}$$

whence $\delta_1 = \varepsilon R_1$, *with* $R_1 = (X_1^2 + Y_1^2)^{\frac{1}{2}}$. The harmonic function Ψ is subject to

$$\Psi_{X_1} \;+\; (\varepsilon/2a)(2Y_1\Psi_{Y_1} - Y_1^2\Psi_{X_1X_1}) + \frac{\varepsilon^3}{8a^3}(4Y_1^3\Psi_{Y_1} - Y_1^4\Psi_{X_1X_1}) + \cdots = 0 \tag{3.23}$$

where $(X_1 = 0,\ Y_1 > 0)$, and

$$\Psi + \Psi_{Y_1} = 0, \quad X_1 > 0, \quad Y_1 = 0 \tag{3.24}$$

and

$$\Psi \sim \widetilde{T}\exp(iX_1 - Y_1 + (2ia/\varepsilon)) \tag{3.25}$$

plus wave-free terms, as $R_1 = (X_1^2 + Y_1^2)^{\frac{1}{2}} \to \infty$.

3.2.4 Matching

The approximations are required to match together smoothly in the various regions of overlap. The wave terms (i.e those proportional to $\exp((\pm ix - y)/\varepsilon)$ generated in the inner regions are simply continued into the wave region near the free surface. The inner potentials Φ and Ψ will also include wave-free terms (such as $R\sin\theta$) that are required to match with the outer potential ϕ. Roughly, $\phi(\delta \to 0) \sim \Phi(R \to \infty)$ and $\phi(\delta_1 \to 0) \sim \Psi(R_1 \to \infty)$; that is, the inner limit of the outer potential must be asymptotically equivalent to the outer limit of the inner approximations. The precise conditions are those proposed by Van Dyke [4].

3.2.5 Right inner approximation

It is tentatively assumed that

$$\Phi \sim \Phi_0(X,Y) + \varepsilon\Phi_1(X,Y) + \varepsilon^2\Phi_2(X,Y) + \cdots \tag{3.26}$$

as $\varepsilon/a \to +0$; both Φ_0 and Φ_1 are harmonic. Note that it is not assumed that the asymptotic expansion includes only powers of ε, since intermediate terms (for example $\varepsilon^2 \log\varepsilon$), might well occur. A more systematic approach would allow for the possibility of an intermediate term, say $h(\varepsilon)\Phi_1^*$, with $h(\varepsilon)$ such that $\varepsilon \ll h(\varepsilon) \ll 1$, and to be determined by matching. It transpires that there is no such intermediate term in this case (so $\Phi_1^* = 0$) and this step is omitted for the sake of brevity.

Substitution of (3.26) into (3.18) gives us

$$\begin{aligned}(\Phi_{0X} &+ \varepsilon\Phi_{1X} + \cdots)\\ &+ (\varepsilon/2a)[2Y(\Phi_0 + \varepsilon\Phi_{1Y} + \cdots) - Y^2(\Phi_{0XX} + \varepsilon\Phi_{1XX} + \cdots)] + \cdots = 0\end{aligned}$$

hence, we obtain the following body boundary conditions

$$\begin{aligned} \Phi_{0X} &= 0 \quad as \quad X=0, \quad Y>0, \\ \Phi_{1X} &= -(1/2a)(2Y\Phi_{0Y} - Y^2\Phi_{0XX}) \quad as \quad X=0, \quad Y>0, \\ \Phi_{2X} &= -(1/2a)(2Y\Phi_{1Y} - Y^2\Phi_{1XX}) \quad as \quad X=0, \quad Y>0. \end{aligned} \tag{3.27}$$

Now again we replace (3.26) in (3.24) to obtain the free surface boundary conditions

$$\begin{aligned} \Phi_0 + \Phi_{0Y} &= 0 \quad as \quad X>0, \quad Y=0, \\ \Phi_1 + \Phi_{1Y} &= 0 \quad as \quad X>0, \quad Y=0, \\ \Phi_2 + \Phi_{2Y} &= 0 \quad as \quad X>0, \quad Y=0. \end{aligned} \tag{3.28}$$

Finally, we get

$$\Phi_0 = \exp(-iX - Y) + \exp(iX - Y), \tag{3.29}$$

which satisfies the boundary conditions (3.27),(3.28) and the radiation condition (3.21) as $R = (X^2 + Y^2)^{1/2} \to \infty$. For the right inner potential Φ, we find the first approximation

$$\Phi \sim \Phi_0 = \exp(-iX - Y) + \exp(iX - Y), \quad as \quad \varepsilon \to 0.$$

Hence, we find from Eqn.(3.21)

$$\widetilde{R} \sim \exp(-2ia/\varepsilon).$$

Now, we observe Φ_1 corresponds to a simple wave-maker problem (See Figure (3.2)), with solution

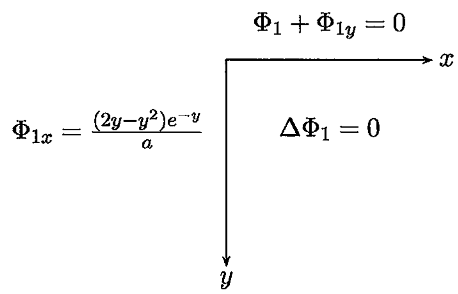

Figure 3.2: Φ_1 of the Right Inner Approximation

$$\Phi_1(X,Y) = -\frac{1}{a}\int_0^\infty G(X,Y;0,Y')(2Y' - Y'^2)e^{-Y'}dY', \tag{3.30}$$

where G is the fundamental source solution, or Green's function which satisfies the homogeneous boundary condition given in Fig.(3.3) [12]. We derive (3.30) from the Green's Formula [12],

$$\int\int(\Phi_1\nabla^2 G - G\nabla^2\Phi_1)dA = \oint(\Phi_1\nabla G - G\nabla\Phi_1)\cdot\mathbf{n}ds, \tag{3.31}$$

where the double integral is over the semicircular region shown in Fig.(3.4). The contour for the line integral is shown in Fig.(3.4) and $\mathbf{n}$ is the unit outward normal vector. Since Φ_1 is harmonic, we get

$$\begin{aligned}\Phi_1 &= \oint(\Phi_1\nabla G - G\nabla\Phi_1)\cdot\mathbf{n}ds \\ &= \int_{L_1}(\Phi_1\nabla G - G\nabla\Phi_1)\cdot\mathbf{n}ds + \int_{L_2}(\Phi_1\nabla G - G\nabla\Phi_1)\cdot\mathbf{n}ds \\ &+ \int_{C_R}(\Phi_1\nabla G - G\nabla\Phi_1)\cdot\mathbf{n}ds \end{aligned} \tag{3.32}$$

with

$$\begin{aligned}\int_{L_1}(\Phi_1\nabla G - G\nabla\Phi_1)\cdot\mathbf{n}ds &= \int_0^\infty(-\Phi_1\frac{\partial G}{\partial Y} + G\frac{\partial\Phi_1}{\partial Y})dX \\ &= \int_0^\infty(\frac{\partial\Phi_1}{\partial Y})(-G) + G\frac{\partial\Phi_1}{\partial Y})dX \\ &= 0,\end{aligned}$$

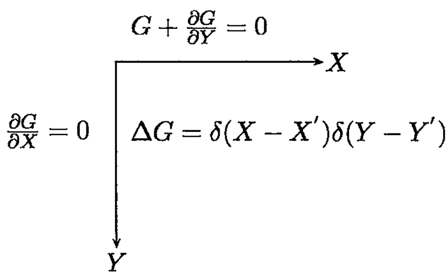

Figure 3.3: Boundary Value Problem for Green's Function

and

$$\begin{aligned}\int_{L_2} (\Phi_1 \nabla G - G \nabla \Phi_1) \cdot \mathbf{n} ds &= \int_{\infty}^{0} (-\Phi_1 \frac{\partial G}{\partial X} + G \frac{\partial \Phi_1}{\partial X}) dY \\ &= -\frac{1}{a} \int_{0}^{\infty} G(2Y - Y^2) e^{-Y} dY.\end{aligned}$$

We evaluate the integral over the circular arc as follows:

$$\lim_{R \to \infty} \int_{C_R} (\Phi_1 \nabla G - G \nabla \Phi_1) \cdot n ds = \lim_{R \to \infty} \int_{0}^{\frac{\pi}{2}} (\Phi_1 \frac{\partial G}{\partial R} - G \frac{\partial \Phi_1}{\partial R}) R d\theta = 0.$$

The last equation gives us the radiation condition

$$\lim_{R \to \infty} R(\Phi_1 \frac{\partial G}{\partial R} - G \frac{\partial \Phi_1}{\partial R}) = 0.$$

Using the Green's function, which will be found next, we determine that Φ_1 must behave like $(1/R)$ as $R \to \infty$.

Quarter domain in Fig.(3.3) can be reflected (See Fig.(3.5)) [13].

3.2.6 Green's Function for the Wave Maker Problem

Green's function can be written as

$$G = K_F + K_G$$

where K_F is the free space Green's function

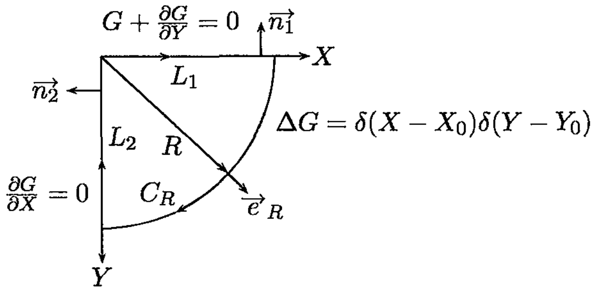

Figure 3.4: Contour used in Green's formula

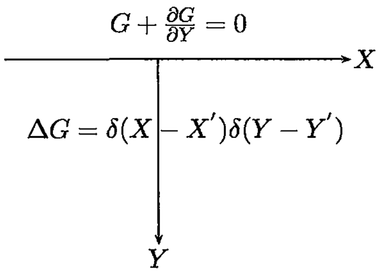

Figure 3.5: Reflection of Green's Function to the Region $y > 0$

$$K_F = \tfrac{1}{4\pi}\ln[(X - X')^2 + (Y - Y')^2]$$

and K_G

$$K_G = -\tfrac{1}{4\pi}\ln[(X - X')^2 + (Y + Y')^2] - v(X, X', Y', Y)$$

is required to satisfy the boundary condition at $Y = 0$, where $v(X, X', Y', Y)$ satisfies the Laplace's equation and nonhomogeneous boundary conditions

$$\Delta G = \Delta(K_F + K_G) = \delta(X - X')\delta(Y - Y').$$

Since $\Delta K_F = \delta(X - X')\delta(Y - Y')$, $\Delta K_G = -\frac{1}{4\pi}\Delta[\ln((X - X')^2 + (Y + Y')^2)] - \Delta v = 0$. Nevertheless $\ln((X - X')^2 + (Y + Y')^2)$ has no singular point in the first and second quadrant of the region shown in the Fig.(3.5), so $\Delta[\ln((X - X')^2 + (Y + Y')^2)] = 0$ and this implies

$$\Delta v = 0. \tag{3.33}$$

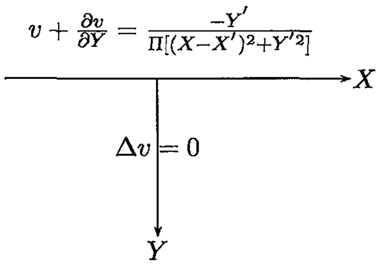

Figure 3.6: Boundary Value Problem for v

If we organize the following equations at $Y = 0$

$$\begin{aligned} G + G_Y &= K_G + \frac{\partial K_G}{\partial Y} + K_F + \frac{\partial K_F}{\partial Y} = 0 \\ K_G + \frac{\partial K_G}{\partial Y} &= -K_F - \frac{\partial K_F}{\partial Y} \end{aligned}$$

$$-\tfrac{1}{4\pi}\ln((X - X')^2 + Y'^2) - v - \tfrac{1}{4\pi}\tfrac{2Y'}{(X-X')^2+Y'^2} - \tfrac{\partial v}{\partial Y} =$$
$$-\tfrac{1}{4\pi}\ln((X - X')^2 + Y'^2) - \tfrac{1}{4\pi}\tfrac{-2Y'}{(X-X')^2+Y'^2}$$

we get

$$v + \frac{\partial v}{\partial Y} = -\frac{1}{\pi}\frac{Y'}{(X - X')^2 + Y'^2}. \tag{3.34}$$

Applying the Fourier Transformation in X

$$V(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} v(X)e^{iwX}dw \tag{3.35}$$

to the boundary value problem (3.33) and (3.40)(See Fig.(3.6)) we obtain

$$\frac{\partial^2 V}{\partial Y^2} - \lambda^2 V = 0 \tag{3.36}$$

$$V + \frac{\partial V}{\partial Y} = -\frac{Y'}{\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{i\lambda X}}{(X - X')^2 + Y'^2}dX. \tag{3.37}$$

Using the substitution $X - X' = t$ in (3.37) results in

$$V + \tfrac{\partial V}{\partial Y} = -\tfrac{1}{\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\tfrac{Y'e^{i\lambda(t+X')}}{t^2+Y'^2}dt = -\tfrac{e^{i\lambda X'}}{\pi}.\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tfrac{Y'e^{i\lambda t}}{t^2+Y'^2}dt = -\tfrac{e^{i\lambda X'}}{\sqrt{2\pi}}e^{-|\lambda|Y'}$$

$$V+\frac{\partial V}{\partial Y}=-\frac{1}{\sqrt{2\pi}}e^{i\lambda X'-|\lambda|Y'} \quad at \quad Y=0. \tag{3.38}$$

The solution of (3.36) is

$$V=ce^{|\lambda|Y}. \tag{3.39}$$

If we replace (3.39) in (3.38), we obtain c as follows,

$$c-|\lambda|c=\tfrac{-1}{\sqrt{2\pi}}e^{i\lambda X'-|\lambda|Y'} \quad \text{implies} \quad c=\tfrac{-1}{\sqrt{2\pi}}\tfrac{e^{i\lambda X'-|\lambda|Y'}}{1-|\lambda|}.$$

Hence,

$$V=\tfrac{-1}{\sqrt{2\pi}(1-|\lambda|)}e^{i\lambda X'-|\lambda|Y'-|\lambda|Y}.$$

Now we apply Inverse Fourier Transform

$$v(X)=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}V(w)e^{-iwX}dw$$

to V to obtain

$$\begin{aligned} v &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\lambda(X'-X)-|\lambda|(Y+Y')}}{1-|\lambda|}d\lambda \\ v &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\cos[\lambda(X-X')]e^{-|\lambda|(Y+Y')}}{1-|\lambda|}d\lambda-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{i\sin[\lambda(X-X')]e^{-|\lambda|(Y+Y')}}{1-|\lambda|}d\lambda \\ v &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\cos[\lambda(X-X')]e^{-|\lambda|(Y+Y')}}{1-|\lambda|}d\lambda. \end{aligned} \tag{3.40}$$

The Green's function $G(X,Y;X',Y';K)$ can be written as

$$G=\frac{1}{4\pi}\ln\frac{(X-X')^2+(Y-Y')^2}{(X-X')^2+(Y+Y')^2}+\frac{1}{\pi}PV\int_{0}^{\infty}\frac{\cos[\lambda(X-X')]e^{-\lambda(Y+Y')}}{K-\lambda}d\lambda \tag{3.41}$$

where PV denotes the principal value and $K=1$. For the PV integral in (3.40) we use the contour in Fig.(3.7). Let

$$\begin{aligned} I^* &= \int_{L}f(z)dz=2\pi i f(K)(z-K)=-2\pi i e^{iK(X-X')}e^{-K(Y+Y')} \\ &= \int_{L_1}f(z)dz+\int_{L_2}f(z)dz+\int_{L_3}f(z)dz+\lim_{\delta\to 0}\int_{C_\delta}f(z)dz+\lim_{R\to\infty}\int_{C_R}f(z)dz \end{aligned} \tag{3.42}$$

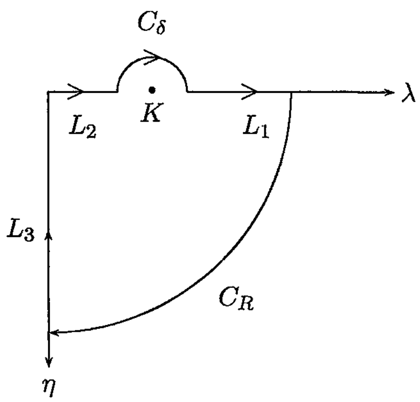

Figure 3.7: Contour for the Improper Integral

where

$$
\begin{aligned}
L &= L_1 \cup L_2 \cup L_3 \cup C_\delta(K) \cup C_R(K) \\
z = \lambda + i\eta \quad and \quad f(z) &= \frac{e^{iz(X-X')}e^{-z(Y+Y')}}{K-z} \\
C_\delta(K) \quad is \quad the \quad circle \quad z - K &= \delta e^{i\theta} \quad \pi < \theta < 2\pi \\
C_R(0) \quad is \quad the \quad circle \quad z &= Re^{i\theta} \quad 0 < \theta < \frac{\pi}{2}.
\end{aligned}
$$

We find that

$$
\begin{aligned}
\lim_{\delta\to 0}\int_{C_\delta} f(z)dz &= \lim_{\delta\to 0}\int_{\pi}^{2\pi} \frac{e^{i(K+\delta e^{i\theta})(X-X')}e^{-(K+\delta e^{i\theta})(Y+Y')}}{K-(K+\delta e^{i\theta})} i\delta e^{i\theta} d\theta \\
&= -\int_{\pi}^{2\pi} e^{iK(X-X')}e^{-K(Y+Y')} i d\theta = -i\pi e^{iK(X-X')}e^{-K(Y+Y')}.
\end{aligned} \tag{3.43}
$$

Nevertheless, along the line L_3, $z = i\eta$ so,

$$
\begin{aligned}
\int_{L_3} f(z)dz &= \int_{\infty}^{0} \frac{e^{-(X-X')\eta}e^{-i(Y+Y')\eta}}{K-i\eta} i d\eta \\
&= \int_{\infty}^{0} \frac{e^{-(X-X')\eta}e^{-i(Y+Y')\eta}(i\eta+K)}{(K-i\eta)(K+i\eta)} i d\eta \\
&= -\int_{\infty}^{0} \frac{\eta e^{-(X-X')\eta}[\cos[\eta(Y+Y')] - i\sin[\eta(Y+Y')]]}{\eta^2+K^2} d\eta
\end{aligned}
$$

$$+ \int_{\infty}^{0} \frac{Ke^{-(X-X')\eta}[i\cos[\eta(Y+Y')] + \sin[\eta(Y+Y')]]}{\eta^2+K^2} d\eta \quad (3.44)$$

and when we use the substitution $\eta = Kt$ and reorganize Eqn.(3.44), we get

$$\int_{L_3} f(z)dz = \int_0^{\infty} \frac{t\cos[K(Y+Y')t] - \sin[K(Y+Y')t]}{1+t^2} e^{-K(X-X')t} dt$$
$$-i\int_0^{\infty} \frac{\cos[K(Y+Y')t] + t\sin[K(Y+Y')t]}{1+t^2} e^{-K(X-X')t} dt. \quad (3.45)$$

We now show that the value of the integral on C_R approaches 0 as R tends to ∞. To do this, we have to know that

$$|\Re \int_{C_R} f(z)dz| \leq |\int_{C_R} f(z)dz| \leq \frac{\pi}{2} R \sup_{z\epsilon C_R} |f(z)| \quad (3.46)$$

where $\frac{\pi}{2}R$ is the length of C_R and $\Re$ denotes the real part. By using the Jordan's Inequality, the following is obtained,

$$\begin{aligned} e^{-R(X-X')\sin\theta} &\leq e^{\frac{-2R(X-X')\theta}{\pi}} \quad when \quad 0 \leq \theta \leq \frac{\pi}{2} \\ e^{-R(Y+Y')\cos\theta} &\leq e^{-R(1-\frac{2}{\pi}\theta)(Y+Y')} \quad when \quad 0 \leq \theta \leq \frac{\pi}{2}. \end{aligned} \quad (3.47)$$

By using Eqn.(3.46) and Eqn.(3.47),

$$\begin{aligned} \int_{C_R} f(z)dz &= \int_{C_R} \frac{e^{iRe^{i\theta}(X-X')} e^{-Re^{i\theta}(Y+Y')}}{K - Re^{i\theta}} iRe^{i\theta} d\theta \\ &= \int_{C_R} ie^{R(X-X')[i\cos\theta - \sin\theta]} e^{-R(Y+Y')[\cos\theta + i\sin\theta]} \frac{Re^{i\theta}}{K - Re^{i\theta}} d\theta \\ &= \int_{C_R} ie^{iR(X-X')\cos\theta} e^{-iR(Y+Y')\sin\theta} e^{-R[(X-X')\sin\theta + (Y+Y')\cos\theta]} \frac{Re^{i\theta}}{K - Re^{i\theta}} d\theta \\ &\leq \int_{C_R} |e^{-R(X-X')\sin\theta}||e^{-R(Y+Y')\cos\theta}|| \frac{Re^{i\theta}}{K - Re^{i\theta}} |d\theta \\ &\leq \int_{C_R} e^{-\frac{2}{\pi}R\theta(X-X')} e^{-R(1-\frac{2}{\pi}\theta)(Y+Y')} \frac{R}{K-R} d\theta \\ &\leq \int_{C_R} e^{-R(Y+Y')} e^{\frac{2\theta R}{\pi}((Y+Y')-(X-X'))} \frac{R}{K-R} d\theta \end{aligned} \quad (3.48)$$

Consequently,

$$|\int_{C_R} f(z)dz| \leq \frac{\pi[e^{-R(X-X')} - e^{-R(Y+Y')}]}{2(K-R)[(Y+Y')-(X+X')]} \to 0 \quad as \quad R\to\infty. \quad (3.49)$$

Now by the Eqn.(3.42),

$$\begin{aligned}\int_{L_1} \frac{e^{iz(X-X')}e^{-z(Y+Y')}}{K-z}dz &+ \int_{L_2} \frac{e^{iz(X-X')}e^{-z(Y+Y')}}{K-z}dz = \\ -i\pi e^{iK(X-X')}e^{-K(Y+Y')} &- \int_0^\infty \frac{t\cos[K(Y+Y')t] - \sin[K(Y+Y')t]}{1+t^2} e^{-K(X-X')t}dt \\ &+ i\int_0^\infty \frac{\cos[K(Y+Y')t] + t\sin[K(Y+Y')t]}{1+t^2} e^{-K(X-X')t}dt.\end{aligned} \quad (3.50)$$

It is obvious that,

$$\int_{L_1} \frac{\cos[\lambda(X-X')]e^{-\lambda(Y+Y')}}{K-\lambda}d\lambda + \int_{L_2} \frac{\cos[\lambda(X-X')]e^{-\lambda(Y+Y')}}{K-\lambda}d\lambda \quad (3.51)$$

is the real part of integral on the left side in Eqn.(3.50). Finally,

$$\begin{aligned} G &= \frac{1}{2\pi}\ln\frac{r}{r_1} + \sin[K(X-X')]e^{-K(Y+Y')} \\ &- \frac{1}{\pi}\int_0^\infty \frac{t\cos[K(Y+Y')t] - \sin[K(Y+Y')t]}{1+t^2} e^{-K(X-X')t}dt \end{aligned} \quad (3.52)$$

where $K = 1$, $r^2 = (X-X')^2 + (Y-Y')^2$ and $r_1^2 = (X-X')^2 + (Y+Y')^2$.

Therefore, we find that

$$\begin{aligned} \Phi_1(X,Y) &= -\frac{1}{a}\int_0^\infty \sin[X]e^{-Y}(2Y' - Y'^2)e^{-2Y'}dY' \\ &- \frac{1}{4\pi a}\int_0^\infty \log\{\frac{X^2+(Y-Y')^2}{X^2+(Y+Y')^2}\}(2Y' - Y'^2)e^{-Y'}dY' \\ &+ \frac{1}{\pi a}\int_0^\infty\int_0^\infty \frac{e^{-Xt}}{1+t^2} t\cos[(Y+Y')t](2Y' - Y'^2)e^{-Y'}dtdY' \\ &- \frac{1}{\pi a}\int_0^\infty\int_0^\infty \frac{e^{-Xt}}{1+t^2}\sin[(Y+Y')t](2Y' - Y'^2)e^{-Y'}dtdY'. \end{aligned} \quad (3.53)$$

The first term in the expression for $\Phi_1(X,Y)$ yields the surface wave contribution

$$\begin{aligned}\Phi_1 &\sim -\frac{1}{a}\sin[X]e^{-Y}\int_0^\infty (2Y'-Y'^2)e^{-2Y'}dY' \\ &\sim -\frac{1}{4a}\sin[X]e^{-Y} \quad as \quad X\to\infty. \end{aligned} \tag{3.54}$$

Since Φ_0 has no wave-free term at all, there is no contribution to match with an outer expansion. Now the form of the boundary condition (3.18) suggests that

$$\Phi \sim \Phi_0 + \varepsilon\Phi_1. \tag{3.55}$$

By using (3.21),(3.54) and (3.55) we find that the reflection constant $\widetilde{R}$ has the form

$$\widetilde{R} \sim [1+(i\varepsilon/4a)]\exp(-2ia/\varepsilon). \tag{3.56}$$

The double-integral term can be simplified by performing the Y' integral first. Let

$$\Phi_1(X,Y) \sim \frac{1}{\pi a}\int_0^\infty \frac{\widetilde{I}}{1+t^2}e^{-Xt}dt$$

where

$$\widetilde{I} = \int_0^\infty (t\cos[(Y+Y')t]-\sin[(Y+Y')t])(2Y'-Y'^2)e^{-Y'}dY'.$$

Since

$$\begin{aligned}\widetilde{I} &= \int_0^\infty (t\cos[(Y+Y')t]-\sin[(Y+Y')t])(2Y'-Y'^2)e^{-Y'}dY' \\ &= 2t\int_0^\infty \cos[(Y+Y')t]Y'e^{-Y'}dY' - t\int_0^\infty \cos[(Y+Y')t]Y'^2e^{-Y'}dY' \\ &- 2\int_0^\infty \sin[(Y+Y')t]Y'e^{-Y'}dY' + \int_0^\infty \sin[(Y+Y')t]Y'^2e^{-Y'}dY'\end{aligned}$$

$$
\begin{aligned}
&= \frac{2t}{1+t^2}[\frac{1-t^2}{1+t^2}\cos(Yt) - \frac{2t}{1+t^2}\sin(Yt)] \\
&- \frac{2t}{(1+t^2)^{3/2}}[\frac{1-3t^2}{(1+t^2)^{3/2}}\cos(Yt) - \frac{3t-t^3}{(1+t^2)^{3/2}}\sin(Yt)] \\
&- \frac{2}{1+t^2}[\frac{2t}{1+t^2}\cos(Yt) + \frac{1-t^2}{1+t^2}\sin(Yt)] \\
&+ \frac{2}{(1+t^2)^{3/2}}[\frac{3t-t^3}{(1+t^2)^{3/2}}\cos(Yt) + \frac{1-3t^2}{(1+t^2)^{3/2}}\sin(Yt)] \\
&= \frac{2t-2t^3}{(1+t^2)^2}\cos(Yt) - \frac{4t^2}{(1+t^2)^2}\sin(Yt), \qquad (3.57)
\end{aligned}
$$

we find that

$$
\begin{aligned}
\Phi_1(X,Y) &\sim \frac{1}{\pi a}\int_0^\infty \frac{\widetilde{I}}{1+t^2}e^{-Xt}dt \\
&\sim \frac{1}{\pi a}[\int_0^\infty \frac{2t-2t^3}{(1+t^2)^3}\cos(Yt)e^{-Xt}dt - \int_0^\infty \frac{4t^2}{(1+t^2)^3}\sin(Yt)e^{-Xt}dt] \\
&\sim -\frac{2(Y^2-X^2)}{\pi a(X^2+Y^2)^2} - \frac{8(Y^3-3X^2Y)}{\pi a(X^2+Y^2)^3} + \cdots \\
&\sim \frac{2\cos 2\theta}{\pi a R^2} + \frac{8\sin\theta}{\pi a R^3}[2\cos 2\theta + 1] + \cdots \qquad (3.58)
\end{aligned}
$$

where (R,θ) are given by $X = R\cos\theta$ and $Y = R\sin\theta$.

Setting $X = 0$ in (3.53), we have

$$
\begin{aligned}
\Phi_1(0,Y) &= -\frac{1}{2\pi a}\int_0^\infty \log|\frac{Y-Y'}{Y+Y'}|(2Y' - Y'^2)e^{-Y'}dY' \\
&+ \frac{1}{\pi a}\int_0^\infty \frac{1}{1+t^2}\int_0^\infty t\cos[(Y+Y')t](2Y' - Y'^2)e^{-Y'}dY'dt \\
&- \frac{1}{\pi a}\int_0^\infty \frac{1}{1+t^2}\int_0^\infty \sin[(Y+Y')t](2Y' - Y'^2)e^{-Y'}dY'dt \qquad (3.59)
\end{aligned}
$$

together with the exponentially decreasing wave term.

Hence

$$
\begin{aligned}
\Phi_1(0,Y) &\sim \frac{1}{\pi a}\int_0^\infty \frac{\widetilde{I}}{1+t^2}dt \\
&\sim \frac{1}{\pi a}[\int_0^\infty \frac{2t-2t^3}{(1+t^2)^3}\cos(Yt)dt - \int_0^\infty \frac{4t^2}{(1+t^2)^3}\sin(Yt)dt] \\
&\sim \frac{1}{\pi a}[-\frac{2}{Y^2} + \frac{8}{Y^3} - \frac{48}{Y^4} + \cdots]. \qquad (3.60)
\end{aligned}
$$

It is readily seen that its contribution to Φ_1 is of order Y^{-2}. The main contribution, of order Y^{-1}, arises from the logarithmic term which can be obtained by partial integration as follows:

$$\begin{aligned}\Phi_1(0,Y';X,Y) &\sim -\frac{1}{4\pi a}\int_0^\infty \log\{\frac{X^2+(Y-Y')^2}{X^2+(Y+Y')^2}\}(2Y'-Y'^2)e^{-Y'}dY' \\ &\sim -\frac{1}{4\pi a}\int_0^\infty \frac{4Y^3+4YX^2-4YY'}{(X^2+(Y+Y')^2)((X^2+(Y-Y')^2)}Y'^2e^{-Y'}dY',\end{aligned} \tag{3.61}$$

setting $X=0$ and $Y'=Yt$, we have

$$\Phi_1(0,Y) \sim -\frac{Y^2}{\pi a}\int_0^\infty \frac{t^2}{1-t^2}e^{-Yt}dt. \tag{3.62}$$

Since $|t|<1$, and the use of Watson's lemma shows that

$$\begin{aligned}\Phi_1(0,Y) &\sim -\frac{Y^2}{\pi a}\int_0^\infty t^2(1-t^2+t^4-\cdots)e^{-Yt}dt \\ &\sim -\frac{Y^2}{\pi a}\int_0^\infty (t^2-t^4+t^6-\cdots)e^{-Yt}dt \\ &\sim -\frac{Y^2}{\pi a}[\frac{2}{Y^3}-\frac{24}{Y^5}+\frac{720}{Y^7}-\cdots] \quad as \quad Y\to\infty.\end{aligned} \tag{3.63}$$

To find the general far-field form for the harmonic function $\Phi_1(X,Y)$ we use the substitution $Y'=Yt$ in (3.61) then we obtain

$$\begin{aligned}\Phi_1(X,Y) &\sim -\frac{Y^4}{\pi a}\int_0^\infty \frac{-Y^2t^4+(X^2+Y^2)t^2}{Y^4t^4+(2Y^2X^2-2Y^4)t^2+(X^2+Y^2)^2}e^{-Yt}dt \\ &\sim -\frac{Y^4}{\pi a}\int_0^\infty (\frac{1}{X^2+Y^2}t^2+\frac{Y^4-3X^2Y^2}{(X^2+Y^2)^3}t^4+\cdots)e^{-Yt}dt \\ &\sim -\frac{Y^4}{\pi a}[\frac{2}{Y^3(X^2+Y^2)}+\frac{24(Y^2-3X^2)}{Y^5(X^2+Y^2)^3}+\cdots] \\ &\sim -\frac{2Y}{\pi a(X^2+Y^2)}-\frac{24(Y^2-3X^2)}{\pi aY(X^2+Y^2)^3}+\cdots \\ &\sim -\frac{2\sin\theta}{\pi aR}-\frac{24}{\pi aR^5}(-3\csc\theta+4\sin\theta)+\cdots \quad as\ R\to\infty,\end{aligned} \tag{3.64}$$

together with the surface wave (3.54), where (R, θ) are given by $X = R\cos\theta$ and $Y = R\sin\theta$. Hence the first order wave-free function in terms of outer coordinates is

$$\Phi_1 = (-2\varepsilon/a\pi)R^{-1}\sin\theta = (-2\varepsilon^2/a\pi)\delta^{-1}\sin\theta. \qquad (3.65)$$

3.2.7 Outer approximation

Expression (3.65) suggests the outer development

$$\phi \sim \varepsilon^2\phi_0 \qquad as \quad \varepsilon \to 0$$

with

$$\phi_0 \sim (-2/a\pi)\delta^{-1}\sin\theta \qquad as \quad \delta \to 0. \qquad (3.66)$$

Thus the harmonic function ϕ_0 is subject to the boundary condition (3.66), together with (3.12). The solution can be found by conformal transformation.

Let $H = \phi_0 + i\psi_0$, where ϕ_0(velocity potentials) and ψ_0(stream lines) satisfies the Cauchy-Riemann equations. So, we get

$$\psi_0 = -\frac{2\cos\theta}{a\pi\delta} + p_1, \qquad (3.67)$$

where p_1 is a real constant. Therefore,

$$\begin{aligned} \phi_0 &= -\frac{2}{a\pi}\frac{y}{(x-a)^2+y^2} \\ \psi_0 &= -\frac{2}{a\pi}\left(\frac{x-a}{(x-a)^2+y^2} + p_2\right) \end{aligned} \qquad (3.68)$$

where

$$\begin{aligned} \delta^2 &= (x-a)^2 + y^2, \qquad p_2 = (-a\pi p_1/2), \\ x &= a + \delta\cos\theta, \qquad y = \delta\sin\theta. \end{aligned}$$

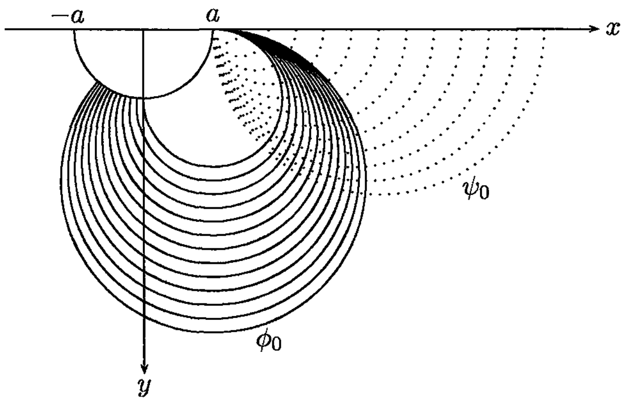

Figure 3.8: Graphs of ϕ_0 and ψ_0

Both ϕ_0 and ψ_0 must be zero at $x = -a$, (see Fig.(3.8)), so we find that $p_2 = (1/2a)$.

Hence

$$\begin{aligned} H &= -\frac{2}{a\pi}\frac{y}{(x-a)^2+y^2} - \frac{2i}{a\pi}(\frac{x-a}{(x-a)^2+y^2} + \frac{1}{2a}) \\ &= -\frac{i}{\pi a^2}(\frac{z+a}{z-a}), \end{aligned} \tag{3.69}$$

where

$$z = x + iy, \qquad x = a + \delta\cos\theta, \quad and \qquad y = \delta\sin\theta.$$

Now, if we consider that

$$\begin{aligned} \phi_0 &= c \\ \psi_0 &= c_1, \end{aligned} \tag{3.70}$$

we find the curves for ϕ_0 and ψ_0 as shown in Fig.(3.8). In this case, for the smallest circles in Fig.(3.8)

$$\begin{aligned}\phi_0 &= c = -\frac{1}{(1.1)\pi} = -0,289372... \\ \psi_0 &= c_1 = -\frac{1.2}{(0.2)\pi} = -1,909861... \quad ,\end{aligned} \tag{3.71}$$

and for the largest circles

$$\begin{aligned}\phi_0 &= c = -\frac{1}{(2.2)\pi} = -0,144686... \\ \psi_0 &= c_1 = -\frac{3.4}{(2.4)\pi} = -0,450939... \quad .\end{aligned} \tag{3.72}$$

We expand the rational function in (3.69) into Taylor series about $z = -a$, we get

$$\frac{z+a}{z-a} = -\frac{z+a}{2a(1-\frac{z+a}{2a})} = -(\frac{z+a}{2a} + \frac{(z+a)^2}{4a^2} + \frac{(z+a)^3}{8a^3} + \cdots), \quad |z+a| < 2a. \tag{3.73}$$

By taking the real part of (3.73),

$$\begin{aligned}\phi_0 &\sim -\frac{1}{2\pi a^3}[y + \frac{y(x+a)}{a}] - \frac{1}{8\pi a^5}[3(x+a)^2 y - y^3] + \cdots, \\ &\sim -\frac{1}{2\pi a^3}[\delta_1 \sin\theta_1 - (\delta_1^2/2a)\sin 2\theta_1] \\ &- \frac{\delta_1^3}{8\pi a^5}[3\sin\theta_1 - 4\sin^3\theta_1] + \cdots.\end{aligned} \tag{3.74}$$

Hence, using the variables of left inner approximation, (X_1, Y_1) or (R_1, θ_1) we get,

$$\begin{aligned}\phi &\sim \varepsilon^2\phi_0 = \varepsilon^2\{-\frac{1}{2\pi a^3}[\varepsilon Y_1 - \frac{\varepsilon^2}{a}X_1 Y_1] - \frac{1}{8\pi a^5}[3\varepsilon^3 X_1^2 Y_1 - \varepsilon^3 Y_1^3]\} + \cdots \\ &= -\frac{\varepsilon^3}{2\pi a^3}R_1\sin\theta_1 + \frac{\varepsilon^4}{4\pi a^4}R_1^2 \sin 2\theta_1 - \frac{\varepsilon^5 R_1^3}{8\pi a^5}[3\sin\theta_1 - 4\sin^3\theta_1] + \cdots,\end{aligned} \tag{3.75}$$

where

$$\begin{aligned} x &= -a - \varepsilon X_1, \qquad y = \varepsilon Y_1, \qquad \delta_1 = \varepsilon R_1, \\ X_1 &= R_1 \cos\theta_1, \qquad Y_1 = R_1 \sin\theta_1. \end{aligned}$$

3.2.8 Left inner approximation

Equation (3.75) suggests the left inner form:

$$\Psi \sim \varepsilon^3 \Psi_0 + g(\varepsilon)\Psi_1 + \varepsilon^4 \Psi_2 \tag{3.76}$$

where $g(\varepsilon)$ is to be determined. Now, if we replace the harmonic function Ψ given by Eqn.(3.76) in Eqn.(3.23) we obtain the following body boundary condition,

$$\begin{aligned} \varepsilon^3 \Psi_{0X_1} &+ g(\varepsilon)\Psi_{1X_1} + \varepsilon^4 \Psi_{2X_1} + (\varepsilon/2a)[2Y_1(\varepsilon^3 \Psi_{0Y_1} + g(\varepsilon)\Psi_{1Y_1} + \varepsilon^4 \Psi_{2Y_1}) \\ -Y_1^2(\varepsilon^3 \Psi_{0X_1X_1} &+ g(\varepsilon)\Psi_{1X_1X_1} + \varepsilon^4 \Psi_{2X_1X_1})] + \frac{\varepsilon^3}{8a^3}[4Y_1^3(\varepsilon^3 \Psi_{0Y_1} + g(\varepsilon)\Psi_{1Y_1} + \varepsilon^4 \Psi_{2Y_1}) \\ -Y_1^4(\varepsilon^3 \Psi_{0X_1X_1} &+ g(\varepsilon)\Psi_{1X_1X_1} + \varepsilon^4 \Psi_{2X_1X_1})] + \cdots = 0 \quad as \quad X_1 = 0, \quad Y_1 > 0. \end{aligned} \tag{3.77}$$

From the Eqn.(3.77), we find the following boundary conditions for Ψ_0, Ψ_1 and Ψ_2,

$$\begin{aligned} \Psi_{0X_1} &= 0 \quad as \quad X_1 = 0, \quad Y_1 > 0, \\ \Psi_{1X_1} &= 0 \quad as \quad X_1 = 0, \quad Y_1 > 0, \\ \Psi_{2X_1} &= -(1/2a)[Y_1\Psi_{0Y_1} - Y_1^2\Psi_{0X_1X_1}] = -(\pi a^4)^{-1}Y_1 \quad as \quad X_1 = 0, \quad Y_1 > 0. \end{aligned} \tag{3.78}$$

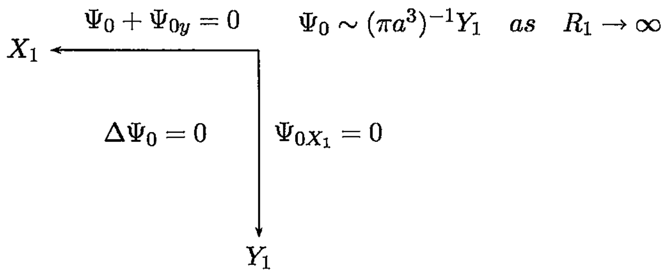

Figure 3.9: Boundary Value Problem for Ψ_0

Now again we replace the harmonic function Ψ given by Eqn.(3.76) in Eqn.(3.24) to obtain the free surface boundary condition,

$$(\varepsilon^3\Psi_0 + g(\varepsilon)\Psi_1 + \varepsilon^4\Psi_2) + (\varepsilon^3\Psi_{0Y_1} + g(\varepsilon)\Psi_{1Y_1} + \varepsilon^4\Psi_{2Y_1}) = 0, \quad X_1 > 0, \quad Y_1 = 0. \tag{3.79}$$

Then the following boundary conditions occur;

$$\begin{aligned} \Psi_0 + \Psi_{0Y_1} &= 0 \quad as \quad X_1 = 0, \quad Y_1 > 0, \\ \Psi_1 + \Psi_{1Y_1} &= 0 \quad as \quad X_1 = 0, \quad Y_1 > 0, \\ \Psi_2 + \Psi_{2Y_1} &= 0 \quad as \quad X_1 = 0, \quad Y_1 > 0. \end{aligned} \tag{3.80}$$

Matching with ϕ requires that

$$\begin{aligned} \Psi_0 &\sim -(2\pi a^3)^{-1} R_1 \sin\theta_1 \quad as \quad R_1 \to \infty \\ &\sim -(2\pi a^3)^{-1} Y_1 \quad as \quad R_1 \to \infty \end{aligned} \tag{3.81}$$

and

$$\begin{aligned} \Psi_2 &\sim (4\pi a^4)^{-1} R_1^2 \sin 2\theta_1 \quad as \quad R_1 \to \infty \\ &\sim (2\pi a^4)^{-1} X_1 Y_1 \quad as \quad R_1 \to \infty. \end{aligned} \tag{3.82}$$

Also, Ψ_1 can be of order R_1^2 at most as $R_1 \to \infty$.

To obtain Ψ_0, we need to use the method of separation of variables. Now, the harmonic function Ψ_0 can be written as,

$$\Psi_0 = W(X_1)P(Y_1). \tag{3.83}$$

Seperation of variables gives,

$$\frac{W_{X_1X_1}}{W} = -\frac{P_{Y_1Y_1}}{P} = -\lambda. \tag{3.84}$$

There are three possibilities for the constant λ;

Case I ($\lambda > 0$):

$$\begin{aligned} W(X_1) &= c_1 \cos\sqrt{\lambda}X_1 + c_2 \sin\sqrt{\lambda}X_1 \\ P(Y_1) &= c_3 e^{\sqrt{\lambda}Y_1} + c_4 e^{-\sqrt{\lambda}Y_1} \end{aligned}$$

Case II ($\lambda < 0$):

$$\begin{aligned} W(X_1) &= c_5 e^{\sqrt{-\lambda}X_1} + c_6 e^{-\sqrt{-\lambda}X_1} \\ P(Y_1) &= c_7 \cos\sqrt{-\lambda}Y_1 + c_8 \sin\sqrt{-\lambda}Y_1 \end{aligned}$$

Case III ($\lambda = 0$):

$$\begin{aligned} W(X_1) &= AX_1 + B \\ P(Y_1) &= CY_1 + D. \end{aligned}$$

Briefly, the solutions for $W(X_1)$ in Case I and $P(Y_1)$ in Case II oscillates. However, we do not expect oscillations for the boundary value problem shown in Fig.(3.9). Finally, the solutions in Case III are the only ones suitable for this problem. So, we find that

$$\Psi_0 = (AX_1 + B)(CY_1 + D).$$

If we impose the boundary conditions (3.78) and (3.80) to the above equation, we get $A = 0$, $C = -D$ and

$$\Psi_0 = BC(Y_1 - 1). \tag{3.85}$$

After we use the matching condition in (3.81), we obtain that $BC = -(1/2\pi a^3)$.

The boundary value problem for Ψ_2 is a simple wave maker problem, so the solution (3.30) can be used. In (3.30), if X, Y and Φ_1 are replaced by X_1, Y_1 and Ψ_2 we get,

$$\Psi_2(X_1, Y_1) = -\frac{1}{2\pi a^4}\int_0^\infty G(X_1, Y_1; 0, Y')Y'dY' \tag{3.86}$$

where

$$\begin{aligned} G(X_1, Y_1; X', Y') &= \sin[(X_1 - X')]e^{-(Y_1+Y')} + \frac{1}{4\pi}\log\{\frac{(X_1 - X')^2 + (Y_1 - Y')^2}{(X_1 - X')^2 + (Y_1 + Y')^2}\} \\ &- \frac{1}{\pi}\int_0^\infty \frac{t\cos(Y_1 + Y')t - \sin(Y_1 + Y')t}{1 + t^2} e^{-(X_1 - X')t}dt, \end{aligned} \tag{3.87}$$

thus

$$\begin{aligned} \Psi_2 &= -\frac{1}{2\pi a^4}\sin[X_1]e^{-Y_1}\int_0^\infty e^{-Y'}Y'dY' - \frac{1}{8\pi^2 a^4}\int_0^\infty \log\{\frac{X_1^2 + (Y_1 - Y')^2}{X_1^2 + (Y_1 + Y')^2}\}Y'dY' \\ &+ \frac{1}{2\pi^2 a^4}\int_0^\infty\int_0^\infty \frac{t\cos(Y_1 + Y')t - \sin(Y_1 + Y')t}{1 + t^2} e^{-X_1 t}Y'dtdY'. \end{aligned} \tag{3.88}$$

Hence

$$\Psi_2 \sim -\frac{1}{2\pi a^4}\sin[X_1]e^{-Y_1} \tag{3.89}$$

for large R_1, plus wave-free terms. Finally, we find that

$$\begin{aligned} \varepsilon^4\Psi_2 &\sim \widetilde{T}\exp(iX_1 - Y_1 + (2ia/\varepsilon)) \\ -\frac{\varepsilon^4}{2\pi a^4}\sin[X_1] &\sim [Re(\widetilde{T}) + iIm(\widetilde{T})][\cos(X_1 + (2a/\varepsilon)) + i\sin(X_1 + (2a/\varepsilon))] \end{aligned}$$

$$ \tag{3.90} $$

from (3.89), (3.76) and (3.25), hence

$$
\begin{aligned}
\frac{\varepsilon^4}{2\pi a^4} &= \cos(2a/\varepsilon)Im(\widetilde{T}) + \sin(2a/\varepsilon)Re(\widetilde{T}), \\
0 &= \cos(2a/\varepsilon)Re(\widetilde{T}) - \sin(2a/\varepsilon)Im(\widetilde{T}).
\end{aligned} \tag{3.91}
$$

Therefore, the transmission coefficient $\widetilde{T}$ is

$$ \widetilde{T} \sim (i/2\pi)(\varepsilon/a)^4 \exp(-2ia/\varepsilon). \tag{3.92} $$

This is the well-known result derived by Ursell [9].

REFERENCES

[1] N. Bleistein, and R.A. Handelsman, Asymptotic Expansions of Integrals. *Holt,Rinehart and Winston*, (1976)

[2] J. Kevorkian and J.D. Cole, Perturbation Methods in Applied Mathematics. *Springer-Verlag*, (1981)

[3] J.D. Cole, Perturbation Methods in Applied Mathematics. *Blaisdell Publishing Company*, (1968)

[4] M.V. Dyke, Perturbation Methods in Fluid Mechanics. *The Parabolic Press*, (1975)

[5] P.A. Martin, and G.R. Wickham, Wave Asymptotics. *Cambridge University Press*, (1992)

[6] F.G. Leppington, On the Radiation and Scattering of Short Surface Waves. Part 1. *J.Fluid Mech.*, **56**, (1972), 101-119

[7] F.G. Leppington, On the Radiation and Scattering of Short Surface Waves. Part 2. *J.Fluid Mech.*, **59**, (1973), 129-146

[8] F.G. Leppington, On the Radiation and Scattering of Short Surface Waves. Part 3. *J.Fluid Mech.*, **59**, (1973), 147-157

[9] F. Ursell, The Transmission of Surface Waves Under Surface Obstacles. *Proc. Camb. Phil. Soc.*, **57**, (1961), 638-668

[10] S.H. Lamb, Hydrodynamics. *Cambridge University Press*, (1932)

[11] C.C. Mei, The Applied Dynamics of Ocean Surface Waves. *World Scientific Publishing Co.,Pte.,Ltd.*, (1989)

[12] R. Haberman, Elementary Applied Partial Differential Equations with Fourier Series and Boundary Value Problems. *Prentice Hall, Inc.*, (1998)

[13] J.W. Brown, and R.V. Churchill, Complex Variables and Applications. *McGraw-Hill, Inc.*, (1996)

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