ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE WAVE EQUATIONS WITH NONLINEAR DISSIPATIVE TERMS IN UNBOUNDED DOMAINS

by

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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ABSTRACT

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In this thesis we survey asymptotic behavior of solutions to the Cauchy problem defined in whole space for a wave equation with nonlinear dissipative term and it is shown that under certain initial conditions the solution of this problem decays to zero at a polynomial rate when t goes to infinity.

The thesis consists of three chapters. In the first chapter previous studies related to the given problem are discussed and necessary definitions and theorems are given. In the second chapter it is shown that the solution of the problem under consideration decays to zero polynomially as t tends to infinity. In the last chapter the result of the thesis and that obtained previously are compared and some open problems are discussed.

Keywords: Nonlinear Wave Equation, Dissipative Term, Asymptotic Behavior, Decay Rate of Energy.

ÖZET

SINIRLI OLMAYAN BÖLGELERDE DOĞRUSAL OLMAYAN SÖNÜM TERİMLİ DALGA DENKLEMLERİNİN ÇÖZÜMLERİNİN ASİMTOTİK DAVRANIŞI

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Bu tezde, doğrusal olmayan sönüm terimli bir dalga denklemi için tüm uzayda tanımlanmış başlangıç değer probleminin çözümünün asimtotik davranışı ele alınmakta ve belirli başlangıç koşulları altında problemin çözümünün t sonsuza giderken polinom oranıyla sıfıra yaklaştığı gösterilmektedir.

Tez üç bölümden oluşmaktadır. Birinci bölümde verilen probleme ilişkin önceki çalışmalardan bahsedilmekte ve tez için gerekli olan tanımlar ve teoremler verilmektedir. İkinci bölümde ise ele alınan problemin çözümünün polinom oranıyla sıfıra yaklaştığının ispatı verilmektedir. Son bölümde ise tezde verilen sonuçlar ile önceki çalışmalarda elde edilen sonuçlar karşılaştırılmakta ve bazı açık problemlere yer verilmektedir.

Anahtar Kelimeler: Dalga Denklemi, Sönüm Terimi, Asimtotik Davranış, Enerji Azalım Oranı.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

This chapter contains the introduction of the subject and some preliminary materials which will be employed in the subsequent parts of the thesis.

1.1 Introduction

The energy decay and asymptotic behavior of solutions for nonlinear wave equations with dissipative terms are important questions and a key starting point for many open problems. Here we discuss the energy decay and asymptotic behavior of the Cauchy problem for a nonlinear dissipative wave equations of the form:

$$u_{tt} - \Delta u + g(t, x, u_t) = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{1.1}$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where u is the function of $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ with the space dimension $n \geq 3$, $\mathbb{R}_+$ is the set of nonnegative real numbers, Δ is the n dimensional Laplacian, and g is a (nonlinear) monotone differentiable function.

From the physical point of view equations of type (1.1) represents a classical vibrating membrane with the resistance proportional to the velocity when the function g depends only on u_t (see [1]).

The following definition will be frequently used in the sequel.

Definition 1.1 *For a solution u of problem (1.1)-(1.2) we define the (entire) energy* *function (or functional) $E(t)$ (or $E(u,t)$) by*

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} (u_t^2(t,x) + |\nabla u(t,x)|^2)dx. \tag{1.3}$$

Hereafter for simplicity we will use the symbol $\int$ for the integration over $\mathbb{R}^n$ and suppress the variables t and x in the function u and its partial derivatives.

Global existence and uniqueness of problem (1.1)-(1.2) has been established by Lions and Strauss [1]: If

$$u_0 \in H^{s+1}(\mathbb{R}^n), \;\; u_1 \in H^s(\mathbb{R}^n) \quad \text{for} \quad 0 \le s \le 1,$$

then there exists a unique global solution in the class

$$u \in C(\mathbb{R}_+, H^{s+1}(\mathbb{R}^n)), \;\; u_t \in C(\mathbb{R}_+, H^s(\mathbb{R}^n)).$$

Asymptotic behavior and related energy decay problem for (1.1)-(1.2) have attracted considerable attention in the recent years.

In [2] Matsumura, by using the Fourier integral transform method, has obtained the exact polynomial decay

$$E(t) \le Ct^{-\frac{n}{2}-1},$$

if the dissipation is linear, that is, if $g(t,x,u_t) = u_t$. Here and hereafter C denotes a positive generic constant.

For nonlinear dissipations $g(t,x,u_t) = |u_t|^{m-1}u_t$ $(m > 1)$, a polynomial decay rate has been derived in the presence of a mass term in equation (1.1) by Nakao [3] (for compactly supported initial data) and Mochizuki and Motai [4]. These authors have considered the nonlinear dissipative Klein-Gordon equation

$$u_{tt} - \Delta u + u + |u_t|^{m-1}u_t = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.4}$$

and shown that if $1 < m < 1 + \frac{2}{n}$, then

$$E(t) \le C(1+t)^{-\gamma} \quad \text{as} \quad t \to \infty,$$

where $\gamma = \frac{2}{m-1} - n$. They also show that if $m > 1 + \frac{2}{n}$, then there exists a dense set of initial data in $H^1 \times L^2$ such that $E(t)$ does not decay.

The best known decay estimate for equation (1.1) with power dissipations is due to [4]. They show a logarithmic decay rate of energy for the equation

$$u_{tt} - \Delta u + |u_t|^{m-1} u_t = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{1.5}$$

with exponents satisfying $1 < m < 1 + \frac{2}{n}$:

$$E(t) \le C[\,\ln(2+t)]^{-\gamma},$$

where $\gamma > 0$ depends on the parameter m. Although the case of equation (1.4) suggests that $m > 1 + \frac{2}{n}$ could be sufficient, the corresponding non-decay result in [4] requires that $m > 1 + \frac{2}{n-1}$.

It is expected, however, that the energy of (1.5) decays at a polynomial rate. The main difficulty in establishing such results seems to be the lack of control of the L^2 norm of the solution. This is an essential difference with the equation in a bounded domain or the Klein-Gordon equation. There are works improving the logarithmic decay rate for the sake of making the dissipation linear for large $|x|$. Nakao and Jung [5] consider a dissipation, which is allowed to be nonlinear only in a ball, but outside that ball the dissipation must be linear. The linearity of the dissipation for large $|x|$ makes it possible to control the L^2 norm of the solution.

Let us note that the initial-boundary value problem for (1.1) over Ω, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, is quite different from the Cauchy problem for (1.1). Nakao [6] and Haraux [7] have found polynomial decay rates of E(t) under the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. The current state of this problem and its generalization for localized damping and source is presented in Lasiecka and Toundykov [8] (see also the references therein).

The aim of this thesis is to show that for all solutions of problem (1.2)-(1.5) with compactly supported data $(u_0, u_1) \in H^2 \times H^1$ under the condition $1 < m \leq \frac{n+2}{n+1}$, $E(t)$ decays polynomially. The main idea used here is the 'parabolic' effect coming from the presence of the damping term.

The thesis is organized as follows. Section 1.2 includes essential definitions and general facts. Especially we give a lemma, related with L^p estimates for certain convolution operators in weighted spaces satisfying the A_p condition (see Definition 1.24), which is used in Section 2.4.

Chapter 2 contains the main part of the thesis. Section 2.2 deals with the decay rate of the external energy defined as

$$E_{ext}(t) = \frac{1}{2}\int_{|x|>t^{(1+\delta)/2}} (u_t^2 + |\nabla u|^2)dx, \tag{1.6}$$

where u is the solution of (1.2)-(1.5). Here we show that if the energy is localized in the exterior region $|x| > t^{(1+\delta)/2}$ with $\delta > 0$ decays fast. Namely, we get the estimate

$$E_{ext}(t) \leq Ct^{\frac{n}{2}+1-\left(\frac{m+1}{m-1}-\frac{n}{2}\right)\delta} \ln t, \quad t \to \infty$$

for $\delta \in (0, 1)$. We see from this estimate the decay rate of $E_{ext}(t)$ is fast when $m \approx 1$. On the other hand, the decay of $E_{ext}(t)$ is slow when $m \approx 1 + \frac{2}{n}$ and no decay is expected if $m \geq 1 + \frac{2}{n}$. This observation is consistent with the non-decay result on the Klein-Gordon equation (1.4) in [4], since the wave equation (1.5) is expected to have slower energy decay. The exterior energy can be studied by weighted estimates. The strongest parabolic effects are manifested in the case of linear damping, $m = 1$, which allows a weight of the form

$$w(t, x) = e^{|x|^2/(2t)}.$$

In this case, the exterior energy of equation (1.5) decays exponentially(see [9]):

$$E_{ext}(t) \leq Ce^{-t^{2\delta}/2}, \quad t \to \infty.$$

A suitable weight for nonlinear dissipations is

$$w(t,x) \ = (1 + |x|^2/t)^{b/2}, \tag{1.7}$$

with exponent b depending on m and n. The idea to use such a weight comes from the asymptotic behavior of fast diffusion equations

$$\partial_t v^m - \Delta v = 0.$$

In fact, every positive solution v has the asymptotic profile

$$v(t,x) \sim t^{-n\gamma}(\alpha + \beta|x|^2/t^{2m\gamma})^{-1/(m-1)}, \tag{1.8}$$

with positive constants α, β, and $\gamma = (n - m(n-2))^{-1}$ (see [10] and the references therein). The wave equation with a nonlinear damping (1.5) is formally transformed into the fast diffusion equation if $\partial_t^2 u$ is neglected, the remaining terms are differentiated with respect to t, and $\partial_t u$ is replaced by v. In general these manipulations can not be justified, although they are valid when the damping is linear; $\partial_t^2 u$ is much smaller than the other two terms in equation (1.5) as $t \to \infty$, see [2]. Thus we expect the phenomenon to persist when the damping is close to linear. The part $|x|^2/t^{2m\gamma}$ of the weight in (1.8) asymptotically approaches $|x|^2/t$ when $m \to 1$, since $\gamma \to 1/2$ as $m \to 1$. This explains the weight in (1.7). The decay rate of the interior energy E_{int}(t), defined as

$$E_{int}(t) = \frac{1}{2}\int_{|x|\leq t^{(1+\delta)/2}} (u_t^2 + |\nabla u|^2)dx,$$

is much slower than the decay rate of the exterior energy $E_{ext}(t)$. This further restricts the decay rate of the total energy $E(t)$.

Section 2.3 contains the decay estimates of $||u_t||_{L^2}$. The result is based on the scaling invariance of equation (1.5). In fact, we have weighted estimate of second order involving the scaling operator $S = t\partial_t + x.\nabla_x$:

$$\frac{1}{2}\int ((Su)_t^2 + |\nabla Su|^2)dx \leq C.$$

As a consequence, we can derive

$$||u_t||_{L^{m+1}} \leq Ct^{-\frac{1}{m+1}}.$$

Using the fast decay of $E_{ext}(t)$ and the last inequality we get the following estimate

$$||u_t||_{L^2} \leq Ct^{\frac{(1+\delta)n}{4}\frac{m-1}{m+1}-\frac{1}{n+1}} \ln^{\frac{1}{2}} t, \quad t \to \infty,$$

where $\delta < 1$ if $m < 1 + \frac{2}{n}$. We should note that this is the only place where the higher regularity of initial data is essential. The other results hold for data in the energy space.

Section 2.4 is devoted to the weighted space-time L^p estimates for ∇u. First we establish space-time L^p estimates of ∇u in terms of $\partial_t u$. To do so we rewrite equation (1.5) in the parabolic form

$$(\partial_t + \sqrt{-\Delta})^2 u = -|\partial_t u|^{m-1}\partial_t u + 2\sqrt{-\Delta}\partial_t u$$

and then we obtain weighted L^p estimates for this equation which helps bound ∇u in terms of u_t. The final decay estimate is

$$\int_0^t \int (s + |x|^2)^{\frac{a}{2}-\frac{m+1}{2m}} |\nabla u|^{\frac{m+1}{m}} dxds \leq F_m \int_0^t \int (s + |x|^2)^{\frac{a}{2}} |\partial_s u|^{m+1} dxds + G_{m,a},$$

where m and a satisfy

$$n + \frac{a}{2} - \frac{m+1}{2(m-1)} < -1 \ \text{ and } \ \ 0 < a < 1.$$

A key feature of this result is that F_m depends on m but not on the parameter a. The other constant $G_{m,a}$ depends also on the initial data u_0 and u_1. It follows that the decay rates of ∇u and $\partial_t u$ are closely related, with the latter being slightly faster. The proof is here based on classical estimates for convolution operators in weighted L^p spaces. Similar results hold for all L^p norms of ∇u and can be used to study the regularizing effect of nonlinear dissipation.

Finally, we combine the space-time estimates of ∇u with weighted energy estimates to derive a polynomial decay rate of the energy.

1.2 Definitions and basic facts

In this section we will give some definitions, notations and general facts. The main references of this section are [11], [12] and [13].

Definition 1.2 *Let Ω be an open, connected set (domain) in $\mathbb{R}^n$ and $1 \leq p$ be a real number. We denote by $L^p(\Omega)$ the class of all measurable functions u, defined on Ω for which*

$$\int_\Omega |u(x)|^p dx < \infty.$$

$L^p(\Omega)$ is a Banach space with the norm

$$||u||_{L^p} := \left(\int_\Omega |u(x)|^p dx \right)^{1/p} < \infty.$$

For $p = \infty$, $L^\infty(\Omega)$ is a Banach space with the norm

$$||u||_\infty = ess \sup_{x \in \Omega} |u(x)|.$$

For $p = 2$, $L^2(\Omega)$ *is a Hilbert space with the inner product*

$$(u, v) := \int_\Omega u(x)v(x)dx$$

for all u *and* $v \in L^2(\Omega)$.

Definition 1.3 *Let* $\Omega \subset \mathbb{R}^n$ *be an open, connected set and let* $\Phi : \Omega \to \mathbb{R}$ *be a function. Support of* Φ *can be defined as*

$$supp\{\Phi\} = \overline{\{x \in \Omega : \Phi(x) \neq 0\}}.$$

Definition 1.4 *Let* $\Omega \subset \mathbb{R}^n$ *be an open, connected set. A function* $u : \Omega \to \mathbb{R}$ *is said to be locally integrable if for every compact set* $K \subset \Omega$,

$$\int_K |u(x)|dx < \infty.$$

We denote by $L^1_{loc}(\Omega)$ *the space of locally integrable functions defined on* Ω.

Definition 1.5 $C_0^\infty(\Omega)$ *is the space of infinitely differentiable functions with compact support. This space is also known as the space of all test functions defined on* Ω.

Definition 1.6 *Let* $x \in \mathbb{R}^n$ *with coordinates* $x = (x_1, \cdots, x_n)$. *A multi-index is an n-tuple* $\alpha = (\alpha_1, \cdots, \alpha_n)$ $(\alpha_i \in \mathbb{N})$. *If we set*

$$|\alpha| = \sum_{i=1}^n \alpha_i,$$

then

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}}$$

represents the α*-th order partial differentiation operator.*

Definition 1.7 *Let* Ω *be a domain. Suppose* $u, v \in L^1_{loc}(\Omega)$*, and* α *is a multi-index. We say that v is the* α^{th}*-weak partial derivative of u, written*

$$D^\alpha u = v,$$

provided

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v \phi dx$$

for all test functions $\phi \in C_0^\infty(\Omega)$.

Definition 1.8 *A sequence of functions* $\{\phi_m\}$ *in* $C_0^\infty(\Omega)$ *is said to converge to* 0 *if there exists a fixed compact set* $K \subset \Omega$ *such that supp* $(\phi_m) \subset K$ *for all m and* $\{\phi_m\}$ *and all its derivatives converge uniformly to zero on K.*

Definition 1.9 *A linear functional T from* $C_0^\infty(\Omega)$ *to* $\mathbb{R}$ *is said to be a distribution, or generalized function if whenever* $\phi_m \to 0$ *in* C_0^∞*, we have* $T(\phi_m) \to 0$.

Definition 1.10 *The space* $C([0,T], X)$ *consists of all continuous functions* $u : [0,T] \to X$ *such that*

$$||u||_{C([0,T],X)} \equiv \max_{0 \le t \le T} ||u(.,t)||_X < \infty.$$

Definition 1.11 *Let k be a non-negative integer and let* $1 \le p \le \infty$*. Then we define* $W^{k,p}(\Omega)$ *to be set of all distributions* $u \in L^P(\Omega)$ *such that* $D^\alpha u \in L^P(\Omega)$ *for* $|\alpha| \le k$.

In $W^{k,p}(\Omega)$, we define a norm by

$$||u||_{k,p} := \Big(\sum_{|\alpha| \le k} ||D^\alpha u||_p^p \Big)^{1/p} \quad if \quad 1 \le p < \infty$$

and

$$||u||_{k,\infty} := \max_{0 \le |\alpha| \le k} ||D^\alpha u||_\infty \quad if \quad p = \infty.$$

For $p = 2$ we define an inner product by

$$(u, v)_k := \sum_{|\alpha| \le k} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx.$$

We also use the notation $H^k(\Omega)$ for $W^{k,2}(\Omega)$ and $L^2(\Omega)$ for $W^{0,2}(\Omega)$.

Definition 1.12 *By* $W_0^{k,p}(\Omega)$ *we denote the closure of* $C_0^\infty(\Omega)$ *in* $W^{k,p}(\Omega)$*.*

This means that $u \in W_0^{k,p}(\Omega)$ if and only if there exist functions $u_m \in C_0^\infty(\Omega)$ such that $u_m \to u \in W^{k,p}(\Omega)$.

Definition 1.13 *The gamma function,* $\Gamma(x)$*, is defined by*

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt.$$

for $x > 0$.

Theorem 1.14 *(The divergence theorem) Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *with smooth (or piecewise smooth) boundary* $\partial\Omega$*. Let* $\mathbf{F}$ *be a smooth vector field defined in* $\mathbb{R}^n$ *and let* $\mathbf{n}$ *be the unit outward-pointing normal of* $\partial\Omega$*. Then*

$$\int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} ds = \int_\Omega \nabla \cdot \mathbf{F} dx.$$

Lemma 1.15 *[14] Let*

$$P_t(x) = p_n \frac{t}{(t^2 + |x|^2)^{(n+1)/2}},$$

where $p_n = \Gamma[(n+1)/2]/\pi^{(n+1)/2}$, $x = (x_1, \cdots, x_n)$ *and* $t > 0$*. Then we have*

$$\int P_t(x)\, dx = 1, \quad \forall\, t > 0.$$

Theorem 1.16 *[12](Integration by parts formula). Let* Ω *be a bounded open set of* $\mathbb{R}^n$

with C^1 *boundary* $\partial\Omega$ *and let* u *and* $v \in C^1(\overline{\Omega})$. *Then*

$$\int_\Omega \frac{\partial u}{\partial x_i}\, v dx = -\int_\Omega u\, \frac{\partial v}{\partial x_i} dx + \int_{\partial\Omega} uv\, \nu^i dS \tag{1.9}$$

for each $i = 1, \cdots, n$, *where* ν^i *is the i-th component of the normal vector* ν *to* $\partial\Omega$.

Definition 1.17 *The Schwartz space, or the space of rapidly decreasing functions* f *on* $\mathbb{R}^n$, $\mathcal{S}(\mathbb{R}^n)$, *is given by*

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : \lim_{|x|\to\infty} |x^\beta D^\alpha f(x)| = 0 \right\}.$$

for all multi-indices α *and* β.

Definition 1.18 *A tempered distribution on* $\mathbb{R}^n$ *is a linear mapping* $\phi \to (f, \phi)$ *from* $\mathcal{S}(\mathbb{R}^n)$ *to* $\mathbb{R}$ *with the continuity property that* $(f, \phi_n) \to (f, \phi)$ *if* $\phi_n \to \phi \in \mathcal{S}(\mathbb{R}^n)$. *The set of all tempered distributions is denoted by* $\mathcal{S}'(\mathbb{R}^n)$. *We say that* $f_n \to f$ *in* $\mathcal{S}'(\mathbb{R}^n)$ *if* $(f, \phi_n) \to (f, \phi)$ *for very* $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Definition 1.19 *[15] Given a function* $u \in L^1(\mathbb{R}^n)$, *we define its Fourier transform by*

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) dx$$

and its inverse Fourier transform by

$$u(x) = \check{u}(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{u}(\xi) d\xi,$$

where $x = (x_1, \cdots, x_n)$, $\xi = (\xi_1, \cdots, \xi_n)$ *and* $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$.

Definition 1.20 *Let* $f, g \in L^1(\mathbb{R}^n)$. *The convolution* $f * g$ *of* f *and* g *defined on* $\mathbb{R}^n$ *is given by*

$$f * g(x) := \int_{\mathbb{R}^n} f(x-y) g(y)\, dy.$$

Theorem 1.21 *[14] If f and g belong to $L^1(\mathbb{R}^n)$, then*

$$(f * g)\hat{(}\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

Theorem 1.22 *[13] Let $1 < p$ and $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$. The $f * g$ is well defined, and further $f * g \in L^p(\mathbb{R}^n)$ with*

$$\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}.$$

Definition 1.23 *Let w be a non-negative locally integrable function on $\mathbb{R}^n$. The weighted space $L^p(w)$ consists of functions f whose p-th power is integrable with respect to the density function w, that is, the norm $\|f\|_{L^p(w)} < \infty$, where*

$$\|f\|_{L^p(w)} = \left(\int |f(x)|^p w(x) dx \right)^{1/p}.$$

Definition 1.24 *[11] Let $1 < p < \infty$. A weight w is said to satisfy the A_p condition in $\mathbb{R}^n$ if there exists a constant C such that*

$$\left(\frac{1}{|B|} \int_B w \right) \left(\frac{1}{|B|} \int_B w^{1-p'} \right)^{p-1} \leq C, \qquad p' = p/(p-1), \tag{1.10}$$

for all balls $B \subset \mathbb{R}^n$. The A_1 condition is

$$\frac{1}{|B|} \int_B w \leq Cwz, \qquad a.e. \qquad z \in B. \tag{1.11}$$

The following properties of A_p weights are consequences of the above definition.

Proposition 1.25

(1) $A_p \subset A_q \quad$ for $\quad 1 \leq p < q$

(2) $w \in A_p \Longleftrightarrow w^{1-p'} \in A_{p'}$

(3) If w_0 *and* $w_1 \in A_1$*, then* $w_0 w_1^{1-p} \in A_p$*.*

The following lemma is useful in showing the L^p and weighted L^p boundedness of operators defined in terms of convolutions.

Lemma 1.26 *Let* K *be a tempered distribution in* $\mathbb{R}^n$ *which coincides with a locally integrable function in* $\mathbb{R}^n \backslash \{0\}$*. Assume that*

$$|\hat{K}(\xi)| \leq C, \quad \xi \in \mathbb{R}^n$$

and

$$|\nabla K(x)| \leq \frac{C}{|x|^{m+1}}, \quad x \in \mathbb{R}^m \backslash \{0\}.$$

Then for $1 < p < \infty$*, the convolution* $K * f$ *satisfies*

$$||K * f||_{L^p} \leq C_p ||f||_{L^p}.$$

Moreover, for w *satisfying the* A_p *condition, we have*

$$||K * f||_{L^p(w)} \leq C_p(w) ||f||_{L^p(w)}.$$

The proof of the above lemma can be found in [11].

The next lemma verifies the A_p condition for two weights used in Section 2.4. This result is essentially known, but we give a short proof.

Lemma 1.27 *[11] Let* (t, x) *be the standard coordinates in* $\mathbb{R}_+ \times \mathbb{R}^n$*. For* $1 < p < \infty$*, the following hold:*

(i) $\quad |x|^a \in A_p(\mathbb{R}_+ \times \mathbb{R}^n) \quad$ *if* $\quad -n < a < n(p-1)$*,*

(ii) $\quad t^b \in A_p(\mathbb{R}_+ \times \mathbb{R}^n) \quad$ *if* $\quad -1 < b < p - 1$*.*

Proof. we can write

$$a = a_1 + (1-p)a_2 \;\; with \;\; -n < a_1 \le 0, \qquad i = 1, 2.$$

By property (*ii*) in Lemma 1.27 with $w_i = |x|^{a_i}$, claim *(i)* follows from

$$|x|^{a_i} \in A_1(\mathbb{R}_+ \times \mathbb{R}^n) \quad \text{for} \quad -n < a_1 \le 0, \quad i = 1, 2.$$

Clearly it is sufficient to show that $|x|^{a_1}$ is an A_1 weight. Consider a ball B of radius r_0 centered at (t_0, x_0). The inequality to verify is

$$\frac{1}{|B|}\int_B |x|^{a_1} dxdt \le C|y|^{a_1}, \qquad a.e. \qquad (s, y) \in B,$$

where C is independent of B and (s, y). Since $(t_0 - s)^2 + (x_0 - y)^2 \le r_0^2$ and $a_1 \le 0$, the strongest inequality corresponds to $s = t_0$ and $|y| = |x_0| + r_0$:

$$\frac{1}{r_0^{n+1}}\int_{(t_0-t)^2+(x_0-x)^2\le r_0^2} |x|^{a_1} dxdt \le C(|x_0| + r_0)^{a_1}.$$

Notice that the range of t in included in the interval $[t_0 - r_0, t_0 + r_0]$. Thus

$$\frac{1}{r_0^n}\int_{|x_0-x|\le r_0} |x|^{a_1} dx \le C(|x_0| + r_0)^{a_1}$$

for all x_0 and r_0, will yield statement *(i)*. We rewrite this inequality as

$$\frac{1}{r_0^n(|x_0| + r_0)^{a_1}}\int_{|x_0-x|\le r_0} |x|^{a_1} dx \le C$$

and consider two cases for x_0 and r_0. If $|x_0| \ge 2r_0$, then $|x_0 - x| \le r_0$ implies $|x| \ge \frac{1}{2}|x_0|$. Hence

$$\frac{1}{r_0^n(|x_0| + r_0)^{a_1}}\int_{|x_0-x|\le r_0} |x|^{a_1} dx \le \frac{(\frac{1}{2}|x_0|)^{a_1} vol(\{x : |x_0 - x| \le r_0\})}{r_0^n(|x_0| + r_0)^{a_1}}$$

which is bounded by $\mathrm{vol}(\mathrm{B}^n)/2^{a_1}$, i.e., a constant depending only on a_1 and n.

If $|x_0| \leq 2r_0$, then $|x_0 - x| \leq r_0$ yields $|x| \leq 3r_0$. Thus

$$\frac{1}{r_0^n(|x_0| + r_0)^{a_1}} \int_{|x_0-x|\leq r_0} |x|^{a_1} dx \leq \frac{1}{r_0^n(|x_0| + r_0)^{a_1}} \int_{|x_0|\leq 3r_0} |x|^{a_1} dx.$$

The right-hand side is dominated by area $(S^{n-1})3^{a_1+n}/(a_1+n)$. This completes the proof of claim (i). Similarly, we can verify claim (ii). □

1.3 Inequalities

In this section we give some basic inequalities which will be employed frequently throughout the thesis.

Cauchy's inequality.

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}, \quad \forall a,\, b \in \mathbb{R}.$$

Young's inequality. Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \ \ (a, b > 0).$$

Cauchy-Schwarz's Inequality. Let H be a Hilbert space associated with the inner product $(.,.)$ and norm $||u|| = (u, u)^{1/2}$. Then

$$|(u, v)| \leq ||u||.||v||, \ \ \forall u, v \in H.$$

Hölder's Inequality. Assume $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then if $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, we have

$$\int_\Omega |uv| dx \leq ||u||_{L^p} ||v||_{L^q}.$$

Gagliardo-Nirenberg's inequality.

$$||D^j u||_p \leq C||D^\mu u||_r^\theta ||u||_q^{1-\theta} \quad \forall u \in \mathbb{C}_0^\mu(\mathbb{R}^n),$$

where $0 \leq \frac{j}{\mu} \leq \theta < 1$, $C = C(n, \mu, j, q, r, \theta) > 0$ are constants, and

$$\frac{1}{p} = \frac{j}{n} + \theta(\frac{1}{r} - \frac{\mu}{n}) + (1 - \theta)\frac{1}{q}.$$

Hardy's inequality. Let f be a nonnegative integrable function on $\mathbb{R}^n$ $(n \geq 1)$ and let d be a number such that $-n < d \leq 0$. Then the inequality

$$\int |x|^d |f(x)|^p dx \leq \left(\frac{p}{n+d}\right)^p \int |x|^{d+p} |\nabla f(x)|^p dx$$

holds (see [16]).

CHAPTER 2

DECAY ESTIMATES AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE WAVE EQUATION

2.1 Introduction

In this chapter we investigate asymptotic behavior and energy decay rates of the Cauchy problem given for equation (1.5) with the compactly supported initial data $u_0 \in H^2(\mathbb{R}^n)$ and $u_1 \in H^1(\mathbb{R}^n)$.

The main reference of this chapter is [16]. In [17], [18] and [19], it is shown that equation (1.5) with

$$u_0 \in H^2(\mathbb{R}^n), \quad u_1 \in H^1(\mathbb{R}^n) \cap L^{2m}(\mathbb{R}^n)$$

has a unique global solution u satisfying the following properties:

(a) $u \in C(\mathbb{R}_+, H^2(\mathbb{R}^n))$, $u_t \in C(\mathbb{R}_+, H^1(\mathbb{R}^n))$, $u_{tt} \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^n))$.

(b) The entire energy function $E(t)$ defined by (1.3) satisfies $E(t) \leq E(0)$ for all $t > 0$, where

$$E(0) = \frac{1}{2}\int (u_1^2 + |\nabla u_0|^2)dx.$$

(c) The solution u has a finite speed of propagation which means that if $u_0(x)$ and $u_1(x) = 0$ for $|x| > R$, then $u(t,x) = 0$ for $|x| > t + R$.

It follows from the energy identity that $E(t)$ is a decreasing function. A naturally arising question is whether the energy $E(t)$ decays to zero or not as $t \to \infty$.

2.2 Decay rate of the external energy

In this section we will show a weighted energy estimate which implies that $E_{ext}(t)$ decays fast as $t \to \infty$ if the damping is close to linear ($m \approx 1$).

Proposition 2.1 *Let u be a solution of equation (1.5) with compactly supported initial data $(u_0, u_1) \in H^1 \times L^2$. Then*

$$E_{ext}(t) \leq C t^{\frac{n}{2}+1-(\frac{m+1}{m-1}-\frac{n}{2})\delta} \ln t, \quad t \geq 2,$$

for any $n \geq 1$ and $\delta \in (0,1)$, $E_{ext}(t)$ is the external energy in (1.6).

Proof. We begin with deriving a weighted energy identity. For this reason, let w be a positive continuously differentiable function whose useful forms are introduced later. First let us multiply equation (1.5) by wu_t and rearrange the terms to get

$$(\frac{w}{2}(u_t^2 + |\nabla u|^2))_t - \mathrm{div}(wu_t \nabla u) = -w|u_t|^{m+1} + \frac{w_t}{2}(u_t^2 + |\nabla u|^2) + \nabla w \cdot u_t \nabla u.$$

Since the solution u is compactly supported, integrating the last expression over $\mathbb{R}^n$ and using the divergence theorem gives

$$\begin{aligned} \frac{d}{dt}\int \frac{w}{2}(u_t^2 + |\nabla u|^2)dx &= -\int w|u_t|^{m+1}dx + \int \frac{w_t}{2}(u_t^2 + |\nabla u|^2)dx \\ &\quad - \int \nabla w \cdot u_t \nabla u dx. \end{aligned} \tag{2.1}$$

Let

$$w(t,x) = (1 + \frac{|x|^2}{t})^{\frac{b}{2}},$$

where the optimal value of the parameter $b > 0$ will be chosen. We can assume that $t \geq 2$ and $|x| > 0$. To find estimates for the integrals on the right-hand side of identity

(2.1), we compute the derivatives

$$w_t = -\frac{b}{2}\frac{|x|^2}{t^2}(1+\frac{|x|^2}{t})^{\frac{b}{2}-1} \quad \text{and} \quad \nabla w = b\frac{x}{t}(1+\frac{|x|^2}{t})^{\frac{b}{2}-1}.$$

Observe that

$$\frac{|\nabla w|^2}{-w_t w} = 2b(1+|x|^2/t)^{-1}.$$

Let us find a bound for $\int \nabla w \cdot u_t \nabla u \, dx$. An application of the Cauchy-Schwarz and Cauchy's inequality yields

$$\begin{aligned} |\nabla w \cdot u_t \nabla u| &\leq |\sqrt{-w_t}\nabla u|.|\frac{u_t \nabla w}{\sqrt{-w_t}}| \\ &\leq \frac{1}{2}(-w_t)|\nabla u|^2 + \frac{1}{2}\frac{|\nabla w|^2}{-w_t}u_t^2. \end{aligned} \tag{2.2}$$

By using Young's inequality with $p = \frac{m+1}{2}$ and $q = \frac{m+1}{m-1}$ we have

$$\frac{1}{2}\frac{|\nabla w|^2}{-w_t}u_t^2 \leq \frac{1}{m+1}w|u_t|^{m+1} + \frac{m-1}{2(m+1)}(\frac{|\nabla w|^2}{-w_t w})^{\frac{m+1}{m-1}} w\chi, \tag{2.3}$$

where χ is the characteristic function

$$\chi(x) = \begin{cases} 1 \ , & |x| \leq t+R \\ 0 \ , & \text{otherwise} \end{cases}$$

of the ball $\{x : |x| \leq t+R\}$. By (2.2) and (2.3) we have

$$|\nabla w \cdot u_t \nabla u| \leq \frac{1}{2}(-w_t)|\nabla u|^2 + \frac{1}{m+1}w|u_t|^{m+1} + \frac{m-1}{2(m+1)}(\frac{|\nabla w|^2}{-w_t w})^{\frac{m+1}{m-1}} w\chi. \tag{2.4}$$

Thus it follows from (2.1) and (2.4) that

$$\begin{aligned}\frac{d}{dt}\int\frac{w}{2}(u_t^2+|\nabla u|^2)dx &\le -\frac{m}{m+1}\int w|u_t|^{m+1}dx+\int\frac{w_t}{2}|\nabla u|^2dx\\ &+\frac{m-1}{2(m+1)}\int(\frac{|\nabla w|^2}{-w_tw})^{\frac{m+1}{m-1}}w\chi dx. \end{aligned}\tag{2.5}$$

Neglecting the negative terms on the right-hand side of (2.5), we have

$$\begin{aligned}\frac{d}{dt}\int\frac{w}{2}(u_t^2+|\nabla u|^2)dx &\le \frac{m-1}{2(m+1)}\int(\frac{|\nabla w|^2}{-w_tw})^{\frac{m+1}{m-1}}w\chi dx\\ &= C\int_{|x|\le t+R}(1+|x|^2/t)^{\frac{b}{2}-\frac{m+1}{m-1}}dx, \end{aligned}\tag{2.6}$$

where C is a positive constant depending on m and b. Using the substitution $x=\sqrt{t}y$ in (2.6) we get

$$\frac{d}{dt}\int\frac{w}{2}(u_t^2+|\nabla u|^2)dx\le Ct^{n/2}\int_{|y|\le\sqrt{t}+R}(1+|y|^2)^{\frac{b}{2}-\frac{m+1}{m-1}}dy. \tag{2.7}$$

If $b/2-(m+1)/(m-1)\le -n/2$, then from (2.7) we have

$$\frac{d}{dt}\int\frac{w}{2}(u_t^2+|\nabla u|^2)dx\le Ct^{n/2}\int_{|y|\le\sqrt{t}+R}(1+|y|^2)^{-n/2}dy. \tag{2.8}$$

Since

$$Ct^{n/2}\int_{|y|\le\sqrt{t}+R}(1+|y|^2)^{-n/2}dy\le Ct^{n/2}\ln t,$$

from (2.8) we obtain

$$\frac{d}{dt}\int(1+|x|^2/t)^{b/2}(u_t^2+|\nabla u|^2)dx\le Ct^{n/2}\ln t.$$

Integrating this last inequality over the interval $[2, t]$, we have the weighted estimate

$$\int (1 + |x|^2/t)^{b/2}(u_t^2 + |\nabla u|^2)dx \leq Ct^{n/2+1} \ln t, \quad t \geq 2,$$

for all $b \leq 2(m + 1)/(m - 1) - n$. The result implies fast decay of the local energy in region where the quotient $|x|^2/t$ is large. In particular, since

$$(1 + |x|^2/t)^{b/2} \geq t^{b\delta/2} \quad \text{for} \quad |x| \geq t^{(1+\delta)/2},$$

we have

$$\begin{aligned} t^{b\delta/2} \int_{|x|\geq t^{(1+\delta)/2}} (u_t^2 + |\nabla u|^2)dx &\leq \int_{|x|\geq t^{(1+\delta)/2}} (1 + |x|^2/t)^{b/2}(u_t^2 + |\nabla u|^2)dx \\ &\leq Ct^{n/2+1} \ln t \end{aligned}$$

for $t \geq 2$. Therefore we have

$$E_{ext}(t) = \int_{|x|\geq t^{(1+\delta)/2}} (u_t^2 + |\nabla u|^2)dx \leq Ct^{n/2+1-b\delta/2} \ln t$$

for $t \geq 2$. The result follows from $b \leq 2(m + 1)/(m - 1) - n$. □

We should note that we have expressed the integrals in the above proof in polar coordinates.

It is interesting to consider the two limit cases. If the exponent is $m \approx 1$, we have that $E_{ext}(t)$ decays faster than any power of t. This is consistent with our knowledge of the linear case $m = 1$, in which the external energy decays exponentially.

The upper bound on admissible exponents m is determined from the condition $\delta < 1$. In fact, $\delta = 1$ means that the support is no longer suppressed inside a small subset of the ball $|x| \leq t + R$. Proposition 2.1 shows that $E_{ext}(t)$ (with $\delta = 1$) will decay as long as

$$\frac{n}{2} + 1 - \frac{m+1}{m-1} + \frac{n}{2} > 0.$$

Solving the last inequality for m, we obtain the condition $m < 1 + 2/n$.

2.3 Decay rates of u_t

In this section we obtain the decay rates of $||u_t||_{L^2}$ and $||u_t||_{L^{m+1}}$. First we observe that the invariance of equation (1.5) under scaling transforms allows weighted estimates of second order using the scaling operator

$$S = t\partial_t + x \cdot \nabla_x.$$

As a consequence we get the decay estimate of $||u_t||_{L^{m+1}}$. Then we can find the decay estimate for $||u_t||_{L^2}$ by the finite propagation speed and Hölder inequality.

We need the following lemmas.

Lemma 2.2 *Let u be a solution of equation (1.5) with compactly supported initial data $(u_0, u_1) \in H^2 \times H^1$ and consider the energy functional*

$$\varepsilon(u,t) = \frac{1}{2}\int (u_t^2 + |\nabla u|^2)dx.$$

Then

$$\varepsilon(Su,t) \leq 2\varepsilon(Su + \frac{2-m}{m-1}u, 0) + 2\Big(\frac{2-m}{m-1}\Big)^2 \varepsilon(u,0),$$

where $Su = tu_t + x \cdot \nabla u$ and the exponent $m > 1$. Hence $\varepsilon(Su,t) \leq C$ for all $t \geq 0$.

Proof. Consider the family of scaled functions

$$u_\lambda(t,x) = e^{\frac{2-m}{m-1}\lambda}u(e^\lambda t, e^\lambda x), \quad \lambda \in \mathbb{R},$$

and notice that

$$\frac{d}{d\lambda}u_\lambda(t,x)|_{\lambda=0} = \frac{2-m}{m-1}u + Su.$$

It can be easily seen that u_λ satisfies

$$(u_\lambda)_{tt} - \Delta u_\lambda + |(u_\lambda)_t|^{m-1}(u_\lambda)_t = 0 \tag{2.9}$$

Let $w = u_\lambda - u$, in which u is the solution of (1.5). Thus subtracting equation (1.5) from equation (2.9) gives

$$w_{tt} - \Delta w + |(u_\lambda)_t|^{m-1}(u_\lambda)_t - |u_t|^{m-1}u_t = 0. \tag{2.10}$$

Multiplying Eq.(2.10) by w_t, integrating over $\mathbb{R}^n$, using the divergence theorem and finite speed of propagation we obtain

$$\frac{d}{dt}\varepsilon(w,t) + \int \Big(|(u_\lambda)_t|^{m-1}(u_\lambda)_t - |u_t|^{m-1}u_t\Big)w_t\, dx \leq 0, \tag{2.11}$$

where

$$\varepsilon(w,t) = \int (w_t^2 + |\nabla w|^2)dx.$$

Since the function $f(u) = |u|^{m-1}u$ is monotone for $m \geq 1$ and for all $u \in \mathbb{R}$ and hence from (2.11) we have

$$\frac{d}{dt}\varepsilon(w,t) \leq 0.$$

Thus integrating this last inequality over $(0,t)$ we get

$$\varepsilon(w,t) \leq \varepsilon(w,0). \tag{2.12}$$

Dividing both sides of equation (2.12) by λ^2 leads to

$$\varepsilon(\frac{w}{\lambda}, t) \leq \varepsilon(\frac{w}{\lambda}, 0) \tag{2.13}$$

and then taking the limit as $\lambda \to 0$ we have

$$\varepsilon(\frac{dw}{d\lambda}|_{\lambda=0}, t) \leq \varepsilon(\frac{dw}{d\lambda}|_{\lambda=0}, 0). \tag{2.14}$$

Substituting

$$\frac{dw}{d\lambda}|_{\lambda=0} = Su + \frac{2-m}{m-1}u,$$

into (2.14) we find

$$\varepsilon(Su + \frac{2-m}{m-1}u, t) \leq \varepsilon(Su + \frac{2-m}{m-1}u, 0).$$

Using the inequalities

$$(u_t + v_t)^2 \leq 2(u_t^2 + v_t^2), \quad |\nabla u + \nabla v|^2 \leq 2(|\nabla u|^2 + |\nabla v|^2),$$

and by the definition of $\varepsilon(u, t)$ it is easy to see that

$$\varepsilon(u + v, t) \leq 2\varepsilon(u, t) + 2\varepsilon(v, t).$$

Thus employing this last inequality and the equality

$$\varepsilon(cu, t) = c^2\varepsilon(u, t),$$

where c is any constant, we complete the proof.

In the next lemma we express u_t in terms of $S u$ from the identities

$$tu_t = S u - x \cdot \nabla u, \tag{2.15}$$

$$tu_{tt} = (S u - x \cdot \nabla u)_t - u_t. \tag{2.16}$$

Lemma 2.3 *The following identities hold for* $t > 0$:

(i) $\int u_{tt} u_t dx = \frac{n-2}{2t} \int u_t^2 dx + \frac{1}{t} \int (S u)_t u_t dx,$

(ii) $- \int u_t \, \Delta u \, dx = \frac{n-2}{2t} \int |\nabla u|^2 dx + \frac{1}{t} \int \nabla (S u) \cdot \nabla u dx.$

Proof. To show (i), first we multiply (2.16) by u_t and obtain the following identity.

$$\begin{aligned}
tu_{tt}u_t &= (S u - x \cdot \nabla u)_t u_t - u_t^2 \\
&= (S u)_t u_t - (x \cdot \nabla u_t) u_t - u_t^2 \\
&= (S u)_t u_t - \frac{x}{2} \cdot \nabla u_t^2 - u_t^2 \\
&= (S u)_t u_t - \frac{1}{2} \mathrm{div}(x u_t^2) + (\frac{n}{2} - 1) u_t^2.
\end{aligned}$$

Afterwards integrating this identity over $\mathbb{R}^n$ we get

$$\int t u_{tt} u_t dx = \int (S u)_t u_t dx - \frac{1}{2} \int \mathrm{div}(x u_t^2) dx + \frac{n-2}{2} \int u_t^2 dx. \tag{2.17}$$

So the result follows by using the divergence theorem and compact support of u.

Identity (ii) is obtained as follows. Multiplying $-\Delta u$ by u_t, integrating over $\mathbb{R}^n$, using the divergence theorem and (2.15) we get

$$\begin{aligned}
-\int u_t \Delta u dx &= \int \nabla u \nabla u_t dx - \int \nabla \cdot (u_t \nabla u) dx \\
&= -\frac{1}{t} \int \nabla u \cdot \nabla (S u - x \cdot \nabla u) dx \\
&= -\frac{1}{t} \int \nabla u \cdot \nabla (S u) dx + \frac{1}{t} \int \nabla u \cdot \nabla (x \cdot \nabla u) dx.
\end{aligned} \tag{2.18}$$

Also integration by parts formula and finite propagation speed of u it is not difficult to see that

$$\int \nabla u \cdot \nabla(x \cdot \nabla u)dx = -\int (x \cdot \nabla u)\Delta u dx = (1 - \frac{n}{2}) \int |\nabla u|^2 dx.$$

Thus the result follows by substituting the last identity in (2.18). □

From this Lemma and Lemma 2.2 we deduce

Proposition 2.4 *Let u be a solution of (1.5) with compactly supported initial data $(u_0, u_1) \in H^2 \times H^1$. Then*

$$||u_t||_{L^{m+1}} \leq Ct^{-\frac{1}{m+1}}, \qquad t \geq 2,$$

for all dimensions $n \geq 1$ and exponents $m > 1$.

Proof. We multiply (1.5) by u_t and integrate on $\mathbb{R}^n$:

$$\int |u_t|^{m+1} dx = -\int u_{tt} u_t \, dx + \int \Delta u u_t \, dx.$$

It follows from Lemma 2.3 that

$$\int |u_t|^{m+1} dx = \frac{2-n}{2t} \int \left(u_t^2 + |\nabla u|^2\right) dx - \frac{1}{t} \int \left((Su)_t u_t + \nabla(Su) \cdot \nabla u\right) dx. \quad (2.19)$$

Also by the Cauchy-Schwarz and Hölder inequalities we can show that

$$\int (Su)_t u_t \, dx \leq \varepsilon^{1/2}(u,t)\varepsilon^{1/2}(Su,t),$$

$$\int \nabla(Su) \cdot \nabla u \, dx \leq \varepsilon^{1/2}(u,t)\varepsilon^{1/2}(Su,t).$$

Using these estimates in equation (2.19) we get

$$\int |u_t|^{m+1} dx = \frac{2-n}{2t}\varepsilon(u,t) + \frac{C}{t}\varepsilon^{1/2}(u,t)\varepsilon^{1/2}(Su,t). \tag{2.20}$$

Lemma 2.2 and (2.20) imply

$$\int |u_t|^{m+1} dx \le Ct^{-1}$$

for $t \ge 2$. Thus the proof is complete. □

Corollary 2.5 *Under the assumptions of Proposition 2.4,*

$$\|u_t\|_{L^2} \le Ct^{\frac{n}{2}\frac{m-1}{m+1}-\frac{1}{m+1}}, \qquad t \ge 2.$$

In particular, $\|u_t\|_{L^2} \to 0$ *as* $t \to \infty$ *if* $1 < m < 1 + \frac{2}{n}$ *holds.*

Proof. First by Hölder's inequality with $p = \frac{m+1}{2}$ and $q = \frac{m+1}{m-1}$ and the finite propagation speed of equation (1.5) we have

$$\|u_t\|_{L^2}^2 \le \left(\int_{|x|\le t+R} dx\right)^{\frac{m-1}{m+1}} \left(\int |u_t|^{m+1} dx\right)^{\frac{2}{m+1}},$$

which implies

$$\|u_t\|_{L^2} \le [vol(\mathbf{B}^n)(t+R)^n]^{\frac{1}{2}\frac{m-1}{m+1}} \|u_t\|_{L^{m+1}}.$$

Then by Proposition 2.4 we get

$$\begin{aligned} \|u_t\|_{L^2} &\le C\,[vol(\mathbf{B}^n)(t+R)^n]^{\frac{1}{2}\frac{m-1}{m+1}}\, t^{-\frac{1}{m+1}} \\ &\le Ct^{\frac{n}{2}\frac{m-1}{m+1}-\frac{1}{m+1}}. \end{aligned}$$

Also, the last inequality implies that if $1 < m < 1 + \frac{2}{n}$, then $||u_t||_{L^2} \to 0$ as $t \to \infty$. □

Now by combining Propositions 2.1 and 2.4 we are ready to prove Proposition 2.6 which gives stronger decay estimate for $||u_t||_{L^2}$.

Proposition 2.6 *Let u be a solution of equation (1.5) with compactly supported initial data* $(u_0, u_1) \in H^2 \times H^1$. *For any* $n \geq 1$,

$$||u_t||_{L^2} \leq Ct^{\frac{(1+\delta)n}{4}\frac{m-1}{m+1}-\frac{1}{m+1}} \ln^{\frac{1}{2}} t, \qquad t \geq 2,$$

where $\delta = \frac{(m-1)(m+n+3)}{(m+1)^2-(m-1)n}$. *Notice that* $\quad \delta < 1 \quad$ *if* $\quad m < 1 + \frac{2}{n}$.

Proof. We start the proof by splitting the norm $||u_t||_{L^2}$ over interior and exterior regions:

$$||u_t||_{L^2} \leq ||u_t||_{L^2(|x|\leq t^{(1+\delta)/2})} + ||u_t||_{L^2(|x|>t^{(1+\delta)/2})}. \tag{2.21}$$

The first term on the right-hand side of (2.21) is bounded by Hölder's inequality and Proposition 2.4, while the second term is bounded by Proposition 2.1:

$$\begin{aligned} ||u_t||_{L^2} &\leq \left[vol(\mathbf{B}^n)(t+R)^{\frac{1+\delta}{2}n}\right]^{\frac{1}{2}\frac{m-1}{m+1}} ||u_t||_{L^{m+1}(|x|<t^{\frac{1+\delta}{2}})} + 2^{\frac{1}{2}} E_{ext}^{\frac{1}{2}}(u,t) \\ &\leq Ct^{\frac{(1+\delta)}{4}\frac{m-1}{m+1}-\frac{1}{m+1}} + Ct^{\frac{1}{2}\left[\frac{n}{2}+1-\delta\left(\frac{m+1}{m-1}-\frac{n}{2}\right)\right]}(\ln t)^{\frac{1}{2}}. \end{aligned}$$

The optimal δ is such that the two powers of t are equal. Thus,

$$\delta = \frac{(m-1)(m+n+3)}{(m+1)^2-(m-1)n}.$$

It is easy to check that $\delta < 1$ when $1 < m < 1 + \frac{2}{n}$, so the estimate here is stronger than the estimate in Corollary 2.5. □

2.4 Weighted L^p estimates for ∇u

This section is devoted to establish an L^p estimate of ∇u in terms of $\partial_t u$ for solutions of the wave equation with nonlinear damping

$$\partial_t^2 u - \Delta u = -|\partial_t u|^{m-1}\partial_t u. \tag{2.22}$$

For our goal the exponents $p \leq 2$ are entirely sufficient, although the argument can be modified to cover all exponents $2 < p \leq m+1$. To obtain such an estimate we will require two weighted L^p estimates for linear equations of the form

$$\partial_t^2 u - \Delta u = f \qquad \text{in} \qquad \mathbb{R}_+ \times \mathbb{R}^n \tag{2.23}$$

which is rewritten as

$$\left(\partial_t + \sqrt{-\Delta}\right)^2 u = f + 2\sqrt{-\Delta}\partial_t u \qquad \text{in} \qquad \mathbb{R}_+ \times \mathbb{R}^n. \tag{2.24}$$

This is a parabolic equation for u with a new source depending on f and $\partial_t u$. We have an estimate of ∇u by the two terms in the source.

The following lemmas are needed to obtain of the main result of this section.

Lemma 2.7 *Assume that $n \geq 3$. Let u be a solution of equation (2.23) with zero initial data $u|_{t=0} = \partial_t u|_{t=0} = 0$. There exist constants C_p and $C_{p,d}$ such that u satisfies*

$$\begin{aligned}\left(\int_0^t \int (s+|x|^2)^{d/2}|\nabla u|^p dxds\right)^{1/p} &\leq C_p\left(\int_0^t \int |x|^d(\sqrt{-\Delta})^{-1} f|^p dxds\right)^{1/p} \\ &+ C_{p,d}\left(\int_0^t \int s^{d/2}|\partial_s u|^p dxds\right)^{1/p},\end{aligned}$$

whenever p and d satisfy $1 < p < \infty$ and $-2 < d \leq 0$.

Proof. The first step is to derive a suitable representation for ∇u. To obtain such a representation we need to solve equation (2.24) for u with zero initial data. Let us rewrite equation (2.24) as

$$(\partial_t + \sqrt{a})^2 u = g,$$

where $a = -\Delta$ and $g = f + 2\sqrt{-\Delta}\partial_t u$. Then letting $v = (\partial_t + \sqrt{a})u$, we have

$$\partial_t v + \sqrt{a}v = g.$$

The solution of this equation with $v|_{t=0}$ is

$$v = \int_0^t e^{-(t-s)\sqrt{a}} g(s)\, ds.$$

By back substitution we have

$$(\partial_t + \sqrt{a})u = \int_0^t e^{-(t-s)\sqrt{a}} g(s)\, ds.$$

Solving this last equation for u together with changing the order of integration we obtain

$$u(t) = \int_0^t (t-s)e^{-(t-s)\sqrt{-\Delta}} f(s)\, ds + 2\int_0^t (t-s)e^{-(t-s)\sqrt{-\Delta}}\sqrt{-\Delta}\partial_s u\, ds.$$

Applying ∇ to the both sides of this equation and using the facts that

$$e^{-t\sqrt{-\Delta}}\sqrt{-\Delta} = -\partial_t e^{-t\sqrt{-\Delta}}, \quad e^{-t\sqrt{-\Delta}} = -\partial_t e^{-t\sqrt{-\Delta}}(\sqrt{-\Delta})^{-1},$$

we get

$$\nabla u(t) = -\int_0^t (t-s)\partial_t \nabla e^{-(t-s)\sqrt{-\Delta}}(\sqrt{-\Delta})^{-1} f(s)\, ds - 2\int_0^t (t-s)\partial_t \nabla e^{-(t-s)\sqrt{-\Delta}}\partial_s u\, ds.$$

Now by introducing the operator

$$Tf(t,x) = -\int_0^t (t-s)\partial_t \nabla e^{-(t-s)\sqrt{-\Delta}} f(s,x)ds, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

we can rewrite ∇u in the form

$$\nabla u = T(\sqrt{-\Delta})^{-1} f + 2T\partial_t u. \tag{2.25}$$

The next step is to deduce weighted L^p estimates for T. It is easy to see that $T$is a convolution operator:

$$Tf = K_T * f \qquad \text{with} \qquad K_T(t,x) = -H(t)t\partial_t \nabla P_t(x),$$

where H is the Heaviside function

$$H(t) = \begin{cases} 1 \ , & t < 0 \\ 0 \ , & t \geq 0 \end{cases}$$

and $P_t(x)$ is the Poisson kernel

$$P_t(x) = (2\pi)^{-n} \int e^{-t|\xi| + i\xi \cdot x} d\xi.$$

We can compute the Fourier transform $\hat{K}_T(\tau, \xi)$ using the partial Fourier transform $\hat{P}_t(\xi) = e^{-t|\xi|}$ and identity $K_T(t,x) = -H(t)t\partial_t \nabla P_t(x)$. First note that

$$\begin{aligned} \nabla \hat{P}_t(\xi) &= (2\pi)^{-n} \int \nabla P_t(x) e^{-ix\xi} dx \\ &= (2\pi)^{-n} i\xi \int P_t(x) e^{-ix\xi} dx \\ &= i\xi \hat{P}_t(\xi). \end{aligned}$$

Then the Fourier transform of K_T with respect to x is

$$\begin{aligned}
\hat{K_T}(t,\xi) &= -H(t)t\partial_t\nabla\hat{P}_t(\xi)\\
&= -i\xi H(t)t\partial_t\hat{P}_t(\xi)\\
&= i\xi|\xi|H(t)t\hat{P}_t(\xi),
\end{aligned}$$

where we have used $\partial_t\hat{P}_t(\xi) = -|\xi|\hat{P}_t(\xi)$, and the Fourier transform of $\hat{K_T}(t,\xi)$ with respect to t is

$$\begin{aligned}
\hat{K_T}(\tau,\xi) &= (2\pi)^{-1}\int_{-\infty}^{\infty}\hat{K_T}(t,\xi)e^{-i\tau t}dt\\
&= \frac{i\xi|\xi|}{2\pi}\int_{-\infty}^{\infty}H(t)te^{-t|\xi|}e^{-it\tau}dt\\
&= \frac{i\xi|\xi|}{2\pi}\int_{0}^{\infty}te^{-t|\xi|}e^{-it\tau}dt\\
&= \frac{i\xi|\xi|}{2\pi}\int_{0}^{\infty}te^{-t(i\tau+|\xi|)}dt.
\end{aligned}$$

So after integration we get

$$\hat{K_T}(\tau,\xi) = \frac{i\xi|\xi|}{2\pi(i\tau+|\xi|)^2}$$

and hence we have

$$|\hat{K}(\tau,\xi)| \leq C, \qquad (\tau,\xi)\in\mathbb{R}\times\mathbb{R}^n\backslash\{0\}. \tag{2.26}$$

To apply Lemma 1.26 we need to find a bound for the derivatives of $K_T(t,x)$. Let us first compute

$$\partial_t K_T(t,x) = -\partial_t\nabla P_t(x) - t(\partial_t\nabla P_t(x))_t\partial_t K_T(t,x), \tag{2.27}$$

where the explicit formula of $P_t(x)$ is given in Lemma 1.15. After substituting the derivatives of $P_t(x)$

$$\begin{aligned}
\nabla P_t(x) &= (\frac{n+1}{2})p_n \frac{-2tx}{(t^2+|x|^2)^{\frac{n+3}{3}}}\,,\\
\partial_t \nabla P_t(x) &= -(n+1)p_n\Big[\frac{x}{(t^2+|x|^2)^{\frac{n+3}{2}}} - \frac{(n+3)t^2x}{(t^2+|x|^2)^{\frac{n+5}{2}}}\Big],\\
(\partial_t \nabla P_t(x))_t &= p_n\Big[\frac{3(n+1)(n+3)xt}{(t^2+|x|^2)^{\frac{n+5}{2}}} - \frac{(n+1)(n+3)^2t^3x}{(t^2+|x|^2)^{\frac{n+7}{2}}}\Big]
\end{aligned}$$

into (2.27) we get

$$\partial_t K_T(t,x) = p_n\Big[\frac{(n+1)x}{(t^2+|x|^2)^{\frac{n+3}{2}}} - \frac{4(n+1)(n+3)t^2x}{(t^2+|x|^2)^{\frac{n+5}{2}}} + \frac{(n+1)(n+3)t^4x}{(t^2+|x|^2)^{\frac{n+7}{2}}}\Big].$$

Using the basic inequality $|x| \le \sqrt{t^2+|x|^2}$, we see that

$$\begin{aligned}
|\partial_t K_T(t,x)| &\le \frac{|x|}{(t^2+|x|^2)^{\frac{n+3}{2}}} + \frac{t^2|x|}{(t^2+|x|^2)^{\frac{n+5}{2}}} + \frac{t^4|x|}{(t^2+|x|^2)^{\frac{n+7}{2}}}\\
&\le \frac{C}{(t^2+|x|^2)^{\frac{n+2}{2}}}.
\end{aligned} \tag{2.28}$$

In a similar manner, we can show that

$$|\nabla K_T(t,x)| \le \frac{C}{(t^2+x^2)^{(n+2)/2}}, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \{0\}. \tag{2.29}$$

Thus combining (2.28) and (2.29) we have

$$|(\partial_t, \nabla)K_T(t,x)| \le \frac{C}{(t^2+x^2)^{(n+2)/2}}, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \{0\}. \tag{2.30}$$

It follows from (2.26) and (2.30) that K_T meets the conditions of Lemma 1.26. Hence T is a bounded operator in L^p for $1 < p < \infty$. Such operators are also bounded in $L^p(w)$ for any weight w satisfying the A_p condition in $\mathbb{R}_+ \times \mathbb{R}^n$. Lemma 1.27 shows that

$$w_1^d(t,x) = |x|^d, \qquad -n < d < n(p-1),$$

$$w_2^d(t,x) = t^{d/2}, \qquad -2 < d < 2(p-1),$$

are A_p weights, so T is a bounded operator in $L^p(w_k^d)$ for $k = 1, 2$. Thus

$$||Tf||_{L^p(w_1^d)} \leq C_p ||f||_{L^p(w_1^d)}, \qquad -2 \leq d \leq 0.$$

Moreover, the weighted estimates in $L^p(w_1^d)$ and $L^p(w_2^d)$ admit restrictions to each finite interval $[0, t] \subset R+$:

$$\left(\int_0^t \int w_k^d(s,x)|Tf(s,x)|^p dxds\right)^{1/p} \leq C_{p,d}^{(k)} \left(\int_0^t \int w_k^d(s,x)|f(s,x)|^p dxds\right)^{1/p} \quad (2.31)$$

for $k = 1, 2$. Here $C_{p,d}^{(1)} = C_p$ and $C_{p,d}^{(2)} = C_p(w_2^d)$. We will write $C_{p,d}$ for the latter constant. Both estimates are valid if $-2 < d \leq 0$. To complete the proof first we notice that

$$(s + |x|^2)^{d/2} \leq |x|^d \qquad \text{and} \qquad (s + |x|^2)^{d/2} \leq s^{d/2} \quad (2.32)$$

and so by (2.25) and the triangle inequality we have

$$\begin{aligned} & \left(\int_0^t \int (s + |x|^2)^{d/2} |\nabla u|^p dxds\right)^{1/p} \\ = & \left(\int_0^t \int (s + |x|^2)^{d/2} |T(\sqrt{-\Delta})^{-1} f + 2T\partial_t u|^p dxds\right)^{1/p} \\ \leq & \left(\int_0^t \int (s + |x|^2)^{d/2} |T(\sqrt{-\Delta})^{-1} f|^p dxds\right)^{1/p} \\ + & 2\left(\int_0^t \int (s + |x|^2)^{d/2} |T\partial_t u|^p dxds\right)^{1/p}. \end{aligned} \quad (2.33)$$

By (2.31) and (2.32) we have

$$\left(\int_0^t \int (s + |x|^2)^{d/2} |T(\sqrt{-\Delta})^{-1} f|^p dxds\right)^{1/p}$$

$$\leq C_p\Big(\int_0^t \int (s+|x|^2)^{d/2}|(\sqrt{-\Delta})^{-1}f|^p dxds\Big)^{1/p}$$
$$\leq C_p\Big(\int_0^t \int |x|^d|(\sqrt{-\Delta})^{-1}f|^p dxds\Big)^{1/p}.$$

and

$$\Big(\int_0^t \int (s+|x|^2)^{d/2}|T\partial_t u|^p dxds\Big)^{1/p}$$
$$\leq C_{p,d}\Big(\int_0^t \int (s+|x|^2)^{d/2}|\partial_t u|^p dxds\Big)^{1/p}$$
$$\leq C_{p,d}\Big(\int_0^t \int s^{d/2}|\partial_t u|^p dxds\Big)^{1/p}.$$

Therefore substituting the last inequalities in (2.33) completes the proof. □

The above result will be applied to the wave equation (2.22) with the damping treated as a source. To insure zero initial data in (2.7), we subtract the solution u_l of the linear parabolic equation

$$(\partial_t + \sqrt{-\Delta})^2 u_l = 0 \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^n \tag{2.34}$$

Formula (2.22) explains why u_l is more convenient than a solution of the wave equation with identical initial data.

Lemma 2.8 *Assume that $n \geq 3$. Let u_ℓ be a solution of equation (2.34) with the initial data $u_\ell|_{t=0} = u_0$ and $\partial_t u_\ell|_{t=0} = u_1$. There exists a constant $D_{p,d}$, such that u_ℓ satisfies*

$$\Big(\int_0^t \int (s+|x|^2)^{d/2}|\nabla u_l|^p dxds\Big)^{1/p}$$

$$\leq D_{p,d}\left(\sum_{k=0,1}(||(\sqrt{-\Delta})^k u_0||_{L^1} + ||(\sqrt{-\Delta})^k u_0||_{L^p}) + ||u_1||_{L^1} + ||u_1||_{L^p}\right)$$

whenever p and d satisfy $n/(n-1) \leq p \leq 2$ and $-2 \leq d \leq 0$.

Proof. First we solve equation (2.34) with the initial data u_0 and u_1. Using the Fourier transform we can show that

$$\begin{aligned}\hat{u_\ell}(t,\xi) &= te^{-t|\xi|}\Big(\hat{u_1}(\xi) + |\xi|\hat{u_0}(\xi)\Big) + e^{-t|\xi|}\hat{u_0}(\xi)\\ &= \Big(tP_t * (u_1 + \sqrt{-\Delta}u_0)\Big)\widehat{\ }(\xi) + (P_t * u_0)\widehat{\ }(\xi)\end{aligned}$$

and hence by the inverse Fourier transform we have

$$u_\ell(t,x) = P_t(x) * u_0(x) + tP_t(x) * (u_1(x) + \sqrt{-\Delta}u_0(x)),$$

where we have used the facts that $(\sqrt{-\Delta}u)\widehat{\ } = |\xi|\hat{u}(\xi)$ and $(f * g)\widehat{\ }(\xi) = \hat{f}(\xi)\hat{g}(\xi)$. To estimate the two convolutions in the right-hand side of the last equality, we apply three inequalities for $P_t(x)$ which are verified directly Lemma 1.15. First we note that by Lemma 1.15 and Theorem 1.22 the following inequalities hold:
If $0 < t \leq 1$,

$$||\nabla P_t * f||_{L^p} \leq C||\nabla f||_{L^p} \qquad \text{and} \qquad ||\nabla P_t * f||_{L^p} \leq \frac{C}{t}||f||_{L^p}. \tag{2.35}$$

If $t > 1$,

$$||\nabla P_t * f||_{L^p} \leq \frac{C}{t^{n(p-1)/p+1}}||f||_{L^1}. \tag{2.36}$$

We show only the estimate of $\nabla tP_t(x) * (u_1(x) + \sqrt{-\Delta}u_0(x))$, since the other estimate is similar. Let $f = u_1(x) + \sqrt{-\Delta}u_0(x)$. By the second inequality in (2.35)

$$\int_0^1 ||t\nabla P_t * f||^p_{L^p} t^{d/2} dt \leq C||f||^p_{L^p} \int_0^1 t^{d/2} dt. \tag{2.37}$$

Recall that $d > -2$, so the integral converges. Using an inequality of the form

$$(a+b)^p \leq 2^{p-1}(a^p + b^p),$$

where a and b are nonnegative real numbers and $p > 1$, (2.37) implies

$$\int_0^1 ||t\nabla P_t * f||_{L^p}^p t^{d/2} dt \leq C(||u_1||_{L^p}^p + ||\sqrt{-\Delta} u_0||_{L^p}^p).$$

We apply (2.36) to bound

$$\int_1^\infty ||t\nabla P_t * f||_{L^p}^p t^{d/2} dt \leq C||f||_{L^1}^p \int_1^\infty t^{d/2-n(p-1)} dt.$$

Notice that $d/2 - n(p-1) \leq -n/(n-1)$ for $p \geq n/(n-1)$. Hence the integral converges

$$\int_1^\infty ||t\nabla P_t * f||_{L^p}^p t^{d/2} dt \leq C(||u_1||_{L^1}^p + ||\sqrt{-\Delta} u_0||_{L^1}^p).$$

Adding the estimates for $t \in (0, 1]$ and $[1, \infty)$, we obtain

$$\int_0^\infty ||t\nabla P_t * f||_{L^p}^p t^{d/2} dt \leq C(||u_1||_{L^p}^p + ||\sqrt{-\Delta} u_0||_{L^p}^p + ||u_1||_{L^1}^p + ||\sqrt{-\Delta} u_0||_{L^1}^p).$$

for $p \geq n/(n-1)$. The proof is complete for this term. There is a similar estimate for the other term $\nabla P_t(x) * u_0(x)$. The main difference is that we apply the first inequality in (2.35) to the integral on $(0, 1]$. □

The following proposition, which gives the weighted L^p estimate of ∇u is readily followed by the lemmas given above.

Proposition 2.9 *Assume that $n \geq 3$ and let u be a solution of (2.22) with the initial data*

$$u|_{t=0} = u_0, \qquad \partial_t u|_{t=0} = u_1.$$

Then there exist constants B_p, $C_{p,d}$, and $D_{p,d}$, such that

$$\left(\int_0^t \int (s + |x|^2)^{d/2} |\nabla u|^p dx ds\right)^{1/p}$$

$$
\begin{aligned}
&\leq B_p\left(\int_0^t\int |x|^{d+p}|\partial_s u|^{mp}dxds\right)^{1/p}+C_{p,d}\left(\int_0^t\int s^{d/2}|\partial_s u|^p dxds\right)^{1/p}\\
&+D_{p,d}\left(\sum_{k=0,1}(\|(\sqrt{-\Delta})^k u_0\|_{L^1}+\|(\sqrt{-\Delta})^k u_0\|_{L^p})+\|u_1\|_{L^1}+\|u_1\|_{L^p}\right)
\end{aligned}
$$

for any p and d satisfying

$$
\frac{n}{n-1}\leq p\leq 2 \qquad \textit{and} \qquad -2<d\leq 0.
$$

Here every constant depends only on its subscripts p, or p and d.

Proof. We first rewrite equation (2.22) as

$$
(\partial_t+\sqrt{-\Delta})^2 u=-|\partial_t u|^{m-1}\partial_t u+2\sqrt{-\Delta}\partial_t u.
$$

Then applying Lemma 2.7 and Lemma 2.8, we have

$$
\begin{aligned}
&\left(\int_0^t\int (s+|x|^2)^{d/2}|\nabla u|^p dxds\right)^{1/p}\\
&\leq C_p\left(\int_0^t\int |x|^d|(\sqrt{-\Delta})^{-1}|\partial_s u|^{m-1}\partial_s u|^p dxds\right)^{1/p}\\
&+C_{p,d}\left(\int_0^t\int s^{d/2}|\partial_s u|^p dxds\right)^{1/p}\\
&+D_{p,d}\left(\sum_{k=0,1}(\|(\sqrt{-\Delta})^k u_0\|_{L^1}+\|(\sqrt{-\Delta})^k u_0\|_{L^p})+\|u_1\|_{L^1}+\|u_1\|_{L^p}\right)
\end{aligned}
\tag{2.38}
$$

with p and d satisfying the conditions there. To complete the proof we simplify the integral

$$
C_p\int_0^t\int |x|^d|(\sqrt{-\Delta})^{-1}|\partial_s u|^{m-1}\partial_s u|^p dxds,
$$

so that the constant remains independent of d. First we use Hardy's inequality with

$$
f=(\sqrt{-\Delta})^{-1}|\partial_s u|^{m-1}\partial_s u,\quad 0\leq s\leq t\ \text{ and }\ d>-n:
$$

$$\left(\int_0^t \int |x|^d |(\sqrt{-\Delta})^{-1} |\partial_s u|^{m-1} \partial_s u|^p dx ds\right)^{1/p}$$
$$\le \frac{p}{n+d} \left(\int_0^t \int |x|^{d+p} |\nabla(\sqrt{-\Delta})^{-1} |\partial_s u|^{m-1} \partial_s u|^p dx ds\right)^{1/p}. \tag{2.39}$$

Next we need weighted L^p estimates for the Riesz transform

$$Rf = \nabla(\sqrt{-\Delta})^{-1} f \qquad \text{or} \qquad Rf(x) = (2\pi)^{-n} \int \frac{i\xi}{|\xi|} e^{i\xi\cdot x} \hat{f}(\xi) d\xi, \qquad x \in \mathbb{R}^n.$$

It is well known that R is a convolution transform:

$$Rf = K_R * f \quad \text{with} \quad K_R(x) = r_n \frac{x}{|x|^{n+1}}, \qquad r_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}.$$

Moreover K_R satisfies the conditions of Lemma 1.26. Since the weight

$$w_1^{d+p}(x) = |x|^{d+p}, \qquad -n < d + p < n(p-1),$$

satisfies the $A_p(\mathbb{R}^n)$ condition, according to Lemma 1.27, R is a bounded operator in the space $L^p(w_1^{d+p})$:

$$\|\nabla(\sqrt{-\Delta})^{-1} f\|_{L^p(w_1^{d+p})} \le K_p(w_1^{d+p}) \|f\|_{L^p(w_1^{d+p})}.$$

The range for d is $-n-p < d < n(p-1)-p$, so it includes the range $-2 \le d \le 0$ under the condition $0 \le n(p-1) - p$, or

$$p \ge \frac{n}{n-1}. \tag{2.40}$$

We use a standard convexity argument to find a uniform constant K_p depending on the constants $K_p(w_1^{-2})$ and $K_p(w_1^0)$:

$$\|\nabla(\sqrt{-\Delta})^{-1}f\|_{L^p(w_1^{d+p})} \leq K_p\|f\|_{L^p(w_1^{d+p})}, \quad -2 \leq d \leq 0.$$

Combining this inequality with Hardy's inequality ,we have

$$\left(\int_0^t \int |x|^d |(\sqrt{-\Delta})^{-1}|\partial_s u|^{m-1}\partial_s u|^p dxds\right)^{1/p} \quad \leq \quad \frac{pK_p}{n+p}\left(\int_0^t \int |x|^{d+p}|\partial_s u|^{mp}dxds\right)^{1/p}$$

whenever the conditions $-2 \leq d \leq 0$ and (2.40) hold. A substitution into inequality (2.38) yields the final estimate

$$\begin{aligned}
&\left(\int_0^t \int (s+|x|^2)^{d/2}|\nabla u|^p dxds\right)^{1/p} \\
&\leq B_p\left(\int_0^t \int |x|^{d+p}|\partial_s u|^{mp}dxds\right)^{1/p} \\
&+C_{p,d}\left(\int_0^t \int s^{d/2}|\partial_s u|^p dxds\right)^{1/p} \\
&+D_{p,d}\left(\sum_{k=0,1}(\|(\sqrt{-\Delta})^k u_0\|_{L^1} + \|(\sqrt{-\Delta})^k u_0\|_{L^p}) + \|u_1\|_{L^1} + \|u_1\|_{L^p}\right)
\end{aligned}$$

with $B_p = pC_pK_p/(n-2)$. The proof is complete. □

We conclude this section with a corollary of Proposition 2.9 relating the L^{m+1} norm of $\partial_t u$ and the $L^{(m+1)/m}$ norm of ∇u.

Corollary 2.10 *Assume that $n \geq 3$. Let u be a solution of (2.22) with initial data*

$$u|_{t=0} = u_0, \qquad \partial_t u|_{t=0} = u_1.$$

There exist constant F_m and $G_{m,a}$, such that

$$\int_0^t \int (s+|x|^2)^{\frac{a}{2}-\frac{m+1}{2m}}|\nabla u|^{\frac{m+1}{m}}dxds \leq F_m \int_0^t \int (s+|x|^2)^{\frac{a}{2}}|\partial_s u|^{m+1}dxds + G_{m,a}, \quad (2.41)$$

for m and a satisfying

$$n + \tfrac{a}{2} - \tfrac{m+1}{2(m-1)} < -1 \qquad \textit{and} \qquad 0 < a < 1$$

The constant $G_{m,a}$ depends also on u_0 and u_1

Proof. Recall that $||u_t||_{L^{m+1}}$ can be bounded by the damping in equation (2.22). We choose $p = (m+1)/m$ and $d = a-(m+1)/m$ in (2.9). Here $a \in (0, 1)$. The corresponding estimate is

$$\begin{aligned}
&\left(\int_0^t \int (s+|x|^2)^{\frac{a}{2}-\frac{m+1}{2m}} |\nabla u|^{\frac{m+1}{m}} dxds\right)^{\frac{m}{m+1}} \\
&\quad \leq B_{(m+1)/m} \left(\int_0^t \int |x|^a |\partial_s u|^{m+1} dxds\right)^{\frac{m}{m+1}} \\
&\quad +C_{(m+1)/m, a-(m+1)/m} \left(\int_0^t \int s^{\frac{a}{2}-\frac{m+1}{2m}} |\partial_s u|^{\frac{m+1}{m}} dxds\right)^{\frac{m}{m+1}} + C_0,
\end{aligned} \tag{2.42}$$

where C_0 is a constant depending on m, d, and the initial data. We need an upper bound on the $L^{(m+1)/m}$ norm of $\partial_t u$, i.e., the integral

$$I_2 = \int_0^t \int s^{\frac{a}{2}-\frac{m+1}{2m}} |\partial_s u|^{\frac{m+1}{m}} dxds.$$

We will consider separately the cases of small and large t. If $t \leq 1$, we have

$$I_2 \leq \sup_{0 \leq s \leq 1} ||\partial_s u||^{(m+1)/m}_{L^{(m+1)/m}} \int_0^t s^{\frac{a}{2}-\frac{m+1}{2m}} ds.$$

Since $(m + 1)/m \leq 2$ and $u(t,\cdot)$ is compactly supported, we obtain

$$I_2 \leq C(1 + E(0)), \qquad t \leq 1.$$

If $t > 1$, the finite propagation speed and Hölder's inequality yield

$$\int_1^t \int s^{\frac{a}{2}-\frac{m+1}{2m}} |\partial_s u|^{\frac{m+1}{m}} dxds$$

$$\leq \left(\int_1^t \int_{|x| \leq s+R} s^{\frac{a}{2} - \frac{m+1}{2(m-1)}} dx ds \right)^{\frac{m-1}{m}} \left(\int_1^t \int s^{\frac{a}{2}} |\partial_s u|^{m+1} dx ds \right)^{\frac{1}{m}}. \qquad (2.43)$$

From vol $B(s+R) \leq C(s+R)^n$, the first factor is bounded by

$$\int_1^t s^{n + \frac{a}{2} - \frac{m+1}{2(m-1)}} ds \leq K$$

if $n + a/2 - (m+1)/(2(m-1)) < -1$. Applying Young's inequality,

$$\begin{aligned} \int_1^t \int s^{\frac{a}{2} - \frac{m+1}{2m}} |\partial_s u|^{\frac{m+1}{m}} dx ds &\leq K^{\frac{m-1}{m}} \left(\int_1^t \int s^{\frac{a}{2}} |\partial_s u|^{m+1} dx ds \right)^{\frac{1}{m}} \\ &\leq C_\epsilon + \epsilon \int_0^t \int s^{\frac{a}{2}} |\partial_s u|^{m+1} dx ds, \end{aligned}$$

for any $\epsilon > 0$. Thus

$$I_2 \leq C_\epsilon (1 + E(0)) + \epsilon \int_0^t \int s^{\frac{a}{2}} |\partial_s u|^{m+1} dx ds, \qquad t > 1.$$

The estimates for I_2 together with estimate (2.42) yield the result. □

It is sufficient to choose $F_m = (2B_{(m+1)/m})^{(m+1)/m}$, which is still a constant independent of a. The optimal a will be determined in Section 2.5. The above condition on m is equivalent to

$$m < 1 + \frac{1}{n + 1/2 + a/2}.$$

Thus $m \leq 1 + 1/(n+1)$ will be a stronger condition independent of a. However, such a restriction does not seem optimal. We can do better if we use the suppressed support instead of finite propagation speed in (2.43). The resulting sharper estimate of the first integral will imply a weaker restriction on m. We do not pursue this estimate since it is unlikely to give the optimal condition $m < 1 + 2/n$.

2.5 Polynomial decay of the entire energy

In this section we prove that the energy function given in (1.5) decays to zero at a polynomial rate as $t \to \infty$.

Theorem 2.11 *Let u be a solution of Eq.(1.5) with compactly supported initial data $(u_0, u_1) \in H^2 \times H^1$. For $n \geq 3$ and $(n+2)/(n+1) \geq m > 1$, the entire energy defined by (1.3) decays polynomially:*

$$E(t) \leq Ct^{-a/2}, t \to \infty,$$

where the positive exponent a depends only on m and n.

Proof. We can assume that $t \geq 2$. The weighted energy identity is

$$\frac{d}{dt}\int \frac{w}{2}(u_t^2 + |\nabla u|^2)dx = -\int w|u_t|^{m+1}dx + R_1 + R_2, \tag{2.44}$$

where

$$R_1 = \int \frac{w}{2}(u_t^2 + |\nabla u|^2)dx,$$

$$R_2 = \int \nabla w \cdot u_t \nabla u dx.$$

Here the weight is $w(t,x) = (t + |x|^2)^{a/2}$ with a small constant $a \in (0,1)$. The exact conditions on a will be given later. An important property of w is that

$$w_t = \frac{a}{2}(t + |x|^2)^{a/2-1}, \qquad \nabla w = ax(t + |x|^2)^{a/2-1},$$

so w_t is much smaller than $|\nabla w|$ when $|x|$ is large. More precisely, $|\nabla w| = 2|x|w_t$. Our goal is to show that the right-hand side of identity (2.44) belongs to $L^1(\mathbb{R})$ for sufficiently small a. The computations are elementary, based on Young's and Hölder's inequalities, with the exception of an $L^{(m+1)/m}$ estimate for ∇u established in Corollary 2.10. For convenience we restate this result. Assume that m and a satisfy

$$n + \frac{a}{2} - \frac{m+1}{2(m-1)} < -1 \quad \text{and} \quad 0 < a < 1.$$

There exist constants F_m and $G_{m,a}$ such that the solution of equation (1.5) satisfies

$$\int_0^t \int (s + |x|^2)^{\frac{a}{2} - \frac{m+1}{2m}} |\nabla u|^{\frac{m+1}{m}} dxds \leq F_m \int_0^t \int (s + |x|^2)^{\frac{a}{2}} |u_s|^{m+1} dxds + G_{m,a}. \tag{2.45}$$

The constant $G_{m,a}$ depends also on the initial data u_0 and u_1. We begin with an upper bound of R_2. Applying Young's inequality, we obtain

$$\begin{aligned} |\nabla w \cdot u_t \nabla u| &\leq a|x|(t + |x|^2)^{\frac{a}{2}-1} |u_t||\nabla u| \\ &\leq \frac{a}{m+1}(t + |x|^2)^{\frac{a}{2}} |u_t|^{m+1} \\ &+ \frac{am}{m+1}|x|^{\frac{m+1}{m}} (t + |x|^2)^{\frac{a}{2} - \frac{m+1}{m}} |\nabla u|^{\frac{m+1}{m}}. \end{aligned}$$

Since $|x| \leq (t + |x|^2)^{1/2}$, we have the following estimate after integration:

$$R_2 \leq \frac{a}{m+1} \int (t + |x|^2)^{\frac{a}{2}} |u_t|^{m+1} dx \; + \frac{am}{m+1} \int (t + |x|^2)^{\frac{\pi}{2} - \frac{m+1}{2m}} |\nabla u|^{\frac{m+1}{m}} dx. \tag{2.46}$$

The two terms will be bounded separately. We proceed to derive a similar estimate for R_1. One part of R_1 is readily bounded by Young's inequality:

$$\begin{aligned} \frac{w_t}{2} u_t^2 &= \frac{a}{4}(t + |x|^2)^{\frac{a}{2}-1} |u_t|^2 \\ &\leq \frac{a}{4(m+1)}(t + |x|^2)^{\frac{a}{2}} |u_t|^{m+1} \\ &+ \frac{a(m-1)}{4(m+1)}(t + |x|^2)^{\frac{a}{2} - \frac{m+1}{m-1}}. \end{aligned}$$

Hence,

$$\begin{aligned}\int \frac{w_t}{2} u_t^2 dx &\leq \frac{a}{4(m+1)} \int (t+|x|^2)^{\frac{a}{2}} |u_t|^{m+1} dx \\ &+ \frac{a(m-1)}{4(m+1)} \int (t+|x|^2)^{\frac{a}{2}-\frac{m+1}{m-1}} dx. \qquad (2.47)\end{aligned}$$

The remaining part of R_1, which involves $\frac{w_t}{2}|\nabla u|^2$ needs a different application Young's inequality:

$$\begin{aligned}\frac{w_t}{2}|\nabla u|^2 &= \frac{a}{4}(t+|x|^2)^{\frac{a}{2}-1}|\nabla u|^2 \\ &\leq \frac{am}{4(m+1)}(t+|x|^2)^{\frac{a}{2}-\frac{m+1}{2m}}|\nabla u|^{\frac{m+1}{m}} + \frac{a}{4(m+1)}(t+|x|^2)^{\frac{a}{2}-\frac{m+1}{2}}|\nabla u|^{m+1}.\end{aligned}$$

Integrating the last estimate, we have

$$\begin{aligned}\int \frac{w_t}{2}|\nabla u|^2 dx &\leq \frac{am}{4(m+1)} \int (t+|x|^2)^{\frac{a}{2}-\frac{m+1}{m}} |\nabla u|^{\frac{m+1}{m}} dx \\ &+ \frac{a}{4(m+1)} \int (t+|x|^2)^{\frac{a}{2}-\frac{m+1}{2}} |\nabla u|^{m+1} dx. \qquad (2.48)\end{aligned}$$

We can now substitute estimates (2.46), (2.47) and (2.48) into identity (2.44) and integrate the resulting estimate on $[2, t]$:

$$\begin{aligned}&\frac{1}{2}\int (s+|x|^2)^{\frac{a}{2}}(|u_s|^2+|\nabla u|^2)dx\Big|_2^t \\ &\leq d_1(t)+d_2(t)+d_3\int_2^t \int (s+|x|^2)^{\frac{a}{2}-\frac{m+1}{2m}}|\nabla u|^{\frac{m+1}{m}} \\ &-d_4 \int_2^t \int (s+|x|^2)^{\frac{a}{2}}|u_s|^{m+1} dx ds,\end{aligned}$$

where the functions and constants are defined as follows:

$$\begin{aligned}d_1(t) &= \frac{a(m-1)}{4(m+1)}\int_2^t \int (s+|x|^2)^{\frac{a}{2}-\frac{m+1}{m-1}} dx ds, \\ d_2(t) &= \frac{a}{4(m+1)}\int_2^t \int (s+|x|^2)^{\frac{a}{2}-\frac{m+1}{2}}|\nabla u|^{m+1} dx ds,\end{aligned}$$

$$d_3 = \frac{5am}{4(m+1)}, \qquad d_4 = 1 - \frac{5a}{4(m+1)}.$$

Clearly $d_3 \to 0$, while the other constant $d_4 \to 1$ as $a \to 0$. Moreover, the functions $d_i(t)$, $i = 1, 2$, are bounded on $\mathbb{R}_+$. See Lemma 2.12 below. The main difficulty is to bound the third term by the fourth (damping) term. Using estimate (2.45) in the last inequality we obtain

$$\begin{aligned} \frac{1}{2}\int (s+|x|^2)^{\frac{a}{2}}(|u_s|^2 + |\nabla u|^2)dx\Big|_2^t &\leq d_1(t) + d_2(t) + G_{m,a} \\ &+ (d_3F_m - d_4)\int_2^t \int (s+|x|^2)^{\frac{a}{2}}|u_s|^{m+1}dxds. \end{aligned}$$

We can choose $a \in (0,1)$ sufficiently small to insure $d_3F_m - d_4 < 0$. Assuming that $d_1(t)$ and $d_2(t)$ are bounded on $\mathbb{R}_+$, we obtain

$$\frac{1}{2}\int (t+|x|^2)^{\frac{a}{2}}(|u_t|^2 + |\nabla u|^2)dx \leq C.$$

□

It remains to verify the claim $d_i(t) \leq C$ for $i = 1, 2$.

Lemma 2.12 *Assume that* $m \leq 1 + 1/(n+1)$. *Then*

$$\text{(i)} \quad \mathrm{d_1(t) \leq C}, \qquad \text{(ii)} \quad \mathrm{d_2(t) \leq C}, \qquad \mathrm{t \geq 2}.$$

Proof. Part (i) follows if the exponent in $d_1(t)$ satisfies

$$\frac{a}{2} - \frac{m+1}{m-1} < -n - 1.$$

This is equivalent to $m < 1 + 4/(2n+a)$, so it holds for $m \leq 1 + 1/(n+1)$.

To verify part (ii), we use the Gagliardo-Nirenberg inequality. To use this inequality we need uniform bounds for $||\nabla u||_{L^2}$ and $||\Delta u||_{L^2}$. Since $u \in C(\mathbb{R}_+, H^2(\mathbb{R}^n))$, we have

$$\sup_{t\geq 0} ||\Delta u||_{L^2} \leq C. \tag{2.49}$$

A uniform bound for ∇u can be obtained by multiplying equation (1.5) by u_t and integrating over $\mathbb{R}^n$. So we have

$$\sup_{t\geq 0} ||\nabla u||_{L^2} \leq C. \tag{2.50}$$

Now we can apply the Gagliadro-Nirenberg inequality with $j = 0$, $\mu = 1$, $p = m + 1$ and $r = q = 2$ to have

$$||\nabla u||_{L^{m+1}} \leq C||\Delta u||^{\theta}_{L^2}||u||^{1-\theta}_{L^2}.$$

Here $\theta = \frac{m-1}{2(m+1)}$ and it is easy to verify that $\theta < 1$ if $m \leq 1 + \frac{1}{n+1}$. Hence by estimates (2.49) and (2.50) we have

$$\sup_{t\geq 0} ||\nabla u||_{L^{m+1}} \leq C. \tag{2.51}$$

This estimate implies

$$\begin{aligned} d_2(t) &= \frac{a}{4(m+1)}\int_2^t \int (s + |x|^2)^{\frac{a}{2}-\frac{m+1}{2}}|\nabla u|^{m+1}dxds \\ &\leq \frac{a}{4(m+1)}\int_2^t \int s^{\frac{a}{2}-\frac{m+1}{2}}|\nabla u|^{m+1}dxds \\ &\leq \frac{a}{4(m+1)}\sup_{t\geq 0} ||\nabla u||^{m+1}_{L^{m+1}}\int_2^\infty s^{\frac{a}{2}-\frac{m+1}{2}}ds \\ &\leq C\int_2^\infty s^{\frac{a}{2}-\frac{m+1}{2}}ds, \end{aligned}$$

where we have used the fact that

$$(s+|x|^2)^{\frac{a}{2}-\frac{m+1}{2}} \leq s^{\frac{a}{2}-\frac{m+1}{2}} \text{ if } a/2-(m+1)/2 < -1.$$

It is obvious that the integral converges if $a/2-(m+1)/2 < -1$. This condition is met for sufficiently small a. We have completed the proof of the lemma. □

CHAPTER 3

CONCLUSION

In the thesis the energy decay problem of solutions to the Cauchy problem for the wave equation with nonlinear dissipative term have been considered:

$$u_{tt} - \Delta u + |u_t|^{m-1}u_t = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n.$$

It is shown that under the assumptions:

1. The initial conditions

   $$u_0 \in H^2(\mathbb{R}^n), \quad u_1 \in H^1(\mathbb{R}^n)$$

   are compactly supported functions,

2. m is a parameter satisfying $1 < m \leq 1 + \frac{1}{n+1}$ when space dimensions $n \geq 3$,

the solution $u = u(t, x)$ to the above problem decays to zero at a polynomial order as $t \to \infty$. Here the suitable weights, which come from the asymptotic behavior of fast diffusion equations are constructed. So far we have seen in the previous studies the best decay rate for this problem is logarithmic under the conditions

$$u_0 \in H^1(\mathbb{R}^n), \quad u_1 \in L^2(\mathbb{R}^n)$$

and

$$1 < m < 1 + \frac{2}{n}$$

for $n \geq 3$(see [4]). We note that in [4], the authors have essentially derived a polynomial decay rate in the presence of a mass term:

$$u_{tt} - \Delta u + u + |u_t|^{m-1} u_t = 0.$$

As a result, for the future studies some open problems about the problem under question may be listed (see [16]). There are two types of open problems, which are either relatively accessible or very difficult.

One can generalize the polynomial decay in Theorem 2.11 for less regular initial data $(u_0, u_1) \in H^2 \times L^2$. Basically this means to prove Proposition 2.6 without using the scaling operator S and estimate more carefully $d_2(t)$ in Lemma 2.12.

The problem can be considered by removing the requirement of compactly supported initial data.

One can show that the polynomial decay in Theorem 2.11 holds for all exponents $m < 1 + 2/n$.

It seems that $m = 1 + 2/n$ is the critical number for decay or non-decay of the energy. Namely, there exists a dense set of initial data for the nonlinear dissipative wave equation (1.5), such that $E(t)$ does not decay if $m > 1 + 2/n$.

The hard open question is to find the exact decay rate of the energy. Here the regularizing effect of the nonlinear damping on ∇u can be used. The scaling invariance of (1.5) with $m > 1$ may be crucial, as it helps transform local estimates into global ones.

REFERENCES

[1] J. L. Lions, W. A. Strauss, *Some non-linear evolution equations*, Bull. Soc. Math. France **93** (1965), 43–96.

[2] A. Matsumura, *On the asymptotic behavior of solutions of semi-linear wave equations*, Publ. Res. Inst. Math. Sci. **12** (1976/77), no. 1, 169–189.

[3] M. Nakao, *Energy decay of the wave equation with a nonlinear dissipative term*, Funkcial. Ekvac. **26** (1983), no. 3, 237–250.

[4] K. Mochizuki, T. Motai, *On energy decay-nondecay problems for wave equations with nonlinear dissipative term in* $\mathbb{R}^n$, J. Math. Soc. Japan, **47** (1995), no. 3, 405–421.

[5] M. Nakao, I. Jung, *Energy decay for the wave equation in exterior domains with some half-linear dissipation*, Differential Integral Equations, **16** (2003), no. 8, 927–948.

[6] M. Nakao, *Asymptotic stability of the bounded or almost periodic solution of the wave equation with nonlinear dissipative term*, J. Math. Anal. Appl. **58** (1977), no. 2, 336–343.

[7] A. Haraux, *Comportement l'infini pour une quation d'ondes non linaire dissipative,* C. R. Acad. Sci. Paris Sr. A-B **287** (1978), no. 7, A507–A509.

[8] I. Lasiecka, D. Toundykov, *Energy decay rates for the semilinear wave equation with nonlinear localized damping and a source terms-an intrinsic approach*, preprint (2005).

[9] G. Todorova, B. Yordanov, *Critical exponent for a nonlinear wave equation with damping*, J. Differential Equations **174** (2001), no. 2, 464–489.

[10] J. A. Carrillo, J. L. Vazquez, *Fine asymptotics for fast diffusion equations*, Comm. Partial Differential Equations, **28** (2003), no. 5-6, 1023–1056.

[11] J. Duoandikoetxea, Fourier Analysis, translated and revised by David Cruz-Uribe, Providence, RI: American Mathematical Society, (2000).

[12] L. C. Evans, Partial Differential Equations, Providence, RI: American Mathematical Society, (1998).

[13] S. Kesavan, Topics in Functional Analysis and Applications, John Wiley and Sons, New York, (1989).

[14] Stein E. M., Weiss, G., Introduction to Fourier Analysis on Euclidean Spaces, Princeton, New Jersey Princeton University Press, (1971).

[15] R. S. Strichartz, A Guide to Distribution Theory and Fourier Transforms, CRC Press, London, (1994).

[16] G. Todorova, B. Yordanov, *The energy decay problem for wave equation with nonlinear dissipative terms in* $\mathbb{R}^n$, Indiana Univ. Math. J., **56**(2007), no. 1, 389–416.

[17] J. Liang, *Nonlinear hyperbolic smoothing at a focal point*, preprint (2004).

[18] T. Motai, *Asymptotic behavior of solutions to the Klein-Gordon equation with a nonlinear dissipative term*, Tsukuba J. Math. **15** (1991), no. 1, 151–160.

[19] J. Serrin, G. Todorova and E. Vitillaro, *Existence for a nonlinear wave equation with damping and source terms*, Differential Integral Equations, **16** (2003), no. 1, 13–50.