THE PAULI PRINCIPLE, REPRESENTATION THEORY, AND GEOMETRY OF FLAG VARIETIES

A DISSERTATION SUBMITTED TO
THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

By
Murat Altunbulak
July, 2008

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Prof. Dr. Alexander A. Klyachko (Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Prof. Dr. Metin Gürses

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assoc. Prof. Dr. Tuğrul Hakioğlu

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assoc. Prof. Dr. A. Sinan Sertöz

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Asst. Prof. Dr. Kostyantyn Zheltukhin

Approved for the Institute of Engineering and Science:

Prof. Dr. Mehmet B. Baray
Director of the Institute

ABSTRACT

THE PAULI PRINCIPLE, REPRESENTATION THEORY, AND GEOMETRY OF FLAG VARIETIES

Murat Altunbulak
P.h.D. in Mathematics
Supervisor: Prof. Dr. Alexander A. Klyachko
July, 2008

According to the Pauli exclusion principle, discovered in 1925, no two identical electrons may occupy the same quantum state. In terms of electron density matrix this amounts to an upper bound for its eigenvalues by 1. In 1926, it has been replaced by skew-symmetry of a multi-electron wave function. In this thesis we give two different solutions to a problem about the impact of this replacement on the electron density matrix, which goes far beyond the original Pauli principle.

Keywords: The Pauli principle, N-representability, Density matrix, Representation theory, Flag varieties.

ÖZET

PAULİ İLKESİ, TEMSİL KURAMI VE BAYRAK ÇEŞİTLEMLERİNİN GEOMETRİSİ

Murat Altunbulak
Matematik, Doktora
Tez Yöneticisi: Prof. Dr. Alexander A. Klyachko
Temmuz, 2008

1925 yılında keşfedilen Pauli dışarlama ilkesine göre aynı kuvantum durumunu aynı iki elektron işgal edemez. Elektron yoğunluk dizeyi cinsinden bunun anlamı özdeğerlerin 1 üst sınırı ile sınırlandırılmasıdır. 1926 yılında bu ilke çoklu-elektron dalga işlevinin eksi bakışıklılığı ile değiştirilmiştir. Bu savda yukarıda adı geçen değişikliğin elektron yoğunluk dizeyi üzerindeki Pauli ilkesini aşan etkisi hakkındaki bir problemin iki değişik çözümünü sunacağız.

Anahtar sözcükler: Pauli ilkesi, N-temsiledilebilirlik, Yoğunluk matrisi, Temsil kuramı, Bayrak çeşitlemleri.

Acknowledgement

Writing a doctoral thesis is a long and complicated process. It might not have been possible for me without the support and encouragement of the people whose names I want to mention here.

My supervisor Alexander Klyachko helped me to a great extent, encouraged me to the very end and made this thesis possible. I can not thank him enough.

I also want to thank the Professors Gürses, Klyachko, Hakioğlu, Sertöz and Zheltukhin, in my examining committee for their time and useful comments.

My roommates Hacı Osman and Cemal did everything to make things easier at home. Besides, I could always knock on their doors, and share a few drinks and words to dispel my boredom. They shared my “burden” and thanks to them I never felt alone.

My sister, Fatma Altunbulak Aksu, never gave up trying to motivate me when I felt down. She is a great sister.

My friend İnan Utku Türkmen (and his wife Berivan) did not hesitate to offer help, especially in the last difficult phases of thesis-writing. Many friends offered their time and knowledge generously; among those are Olcay Coşkun, Uğur Madran, Aslı Pekcan, Muhammed İ. Yıldız. I am sincerely thankful to all of them.

Last but not least, I want to express my gratitude to my parents, Ömer and Şengül Altunbulak, who always unconditionally supported me. I always knew that their prayers were with me. I am so happy that I did not let their efforts be in vain. Thank you...

Contents

Chapter 1

Introduction

1.1 Basics of quantum mechanics

1.1.1 Quantum states

A quantum system A is described by a complex Hilbert space $\mathcal{H}_A$ which is called *state space* of the system A. Throughout this study we only consider *finite systems*, for which $\dim \mathcal{H}_A < \infty$. An actual state of a quantum system is described by either a unit vector $|\psi\rangle \in \mathcal{H}_A$ or by a non-negative Hermitian operator $\rho : \mathcal{H}_A \rightarrow \mathcal{H}_A$ with $\operatorname{Tr} \rho = 1$, depending on whether the state is pure or mixed. Mixed states are ensembles of pure states, where each pure state $|\psi_i\rangle$ appears with some probability p_i, that is, the operator ρ is represented as $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ which is called *Density matrix.* Particularly, a pure state $|\psi\rangle$ is represented by the density matrix $|\psi\rangle\langle\psi|$, a projection operator onto $|\psi\rangle$. Following Dirac, we use the "bra-ket" notation for describing quantum state vectors: $|\psi\rangle$ denotes a column vector, while $\langle\psi|$ denotes its adjoint, or conjugate transpose, which is a row vector.

1.1.2 Observables

The information from a quantum system is obtained from a measurement which is defined by a Hermitian operator $X_A : \mathcal{H}_A \to \mathcal{H}_A$ called *an observable.* By measuring the system in state ρ with X_A, we get a random quantity $x_A \in \mathrm{Spec} X_A$ implicitly determined by expectations

$$\langle f(x_A), \rho\rangle = \mathrm{Tr}(\rho f(X_A)) = \langle\psi|f(X_A)|\psi\rangle$$

of arbitrary function $f(x)$ on $\mathrm{Spec}X_A$. The second equality is valid only for pure state $|\psi\rangle$.

1.1.3 Superposition principle

The superposition principle of quantum mechanics states that the linear combination $a|\psi\rangle + b|\varphi\rangle$ of two realizable physical states $|\psi\rangle$ and $|\varphi\rangle$ is also a realizable state. Despite being one of the most striking revelation in physics, it is not related to common sense. For example, it implies that the famous Schrödinger cat may occupy the state

$$|\psi\rangle = |\text{dead}\rangle + |\text{alive}\rangle,$$

which is an intermediate state between death and life.

It follows from the superposition principle of quantum mechanics that the state space of composite system AB splits into tensor product

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$

of state spaces of the components $\mathcal{H}_A$ and $\mathcal{H}_B$.

1.1.4 Reduced states

The reduced density matrix ρ_A of a density matrix ρ_{AB} of the composite system $\mathcal{H}_{AB}$ is defined by the relation

$$\langle X_A, \rho_{AB}\rangle = \mathrm{Tr}(\rho_{AB}X_A) = \mathrm{Tr}(\rho_A X_A) = \langle X_A, \rho_A\rangle, \quad \forall\ X_A : \mathcal{H}_A \to \mathcal{H}_A. \tag{1.1}$$

Note that, the trace form $\mathrm{Tr}(\rho_{AB}X_A)$ gives a linear functional in X_A, and it is a well-known fact that in an inner product space V every linear functional $f : V \to \mathbb{C}$ is given by the scalar product $f(x) = (x, y)$ for unique $y \in V$. In our case V is the space of all Hermitian operators $X_A : \mathcal{H}_A \to \mathcal{H}_A$ with trace form. Hence, there is a unique Hermitian operator ρ_A which satisfies the second equality in (1.1).

The relation (1.1) tells that *if we observe only the subsystem A of the composite system AB then we get the same results as if A would be in reduced state* ρ_A. That is, ρ_A represents a visible state of the subsystem A. This clarifies the terminology.

Remark 1.1.1 The above reduction $\rho_{AB} \mapsto \rho_A$ is known as *contraction* in differential geometry. For instance, Ricci curvature $\mathrm{Ric} : \mathcal{T} \to \mathcal{T}$ is defined as the contraction of Riemann curvature $R : \mathcal{T} \otimes \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}$, where $\mathcal{T}$ stands for tangent bundle. In tensor notation (using the Einstein summation convention) this means that

$$\mathrm{Ric}_i^j = R_{ik}^{jk}.$$

1.1.5 Schmidt decomposition

Identifying the pure state of the two component system

$$|\psi\rangle = \sum_{ij} \psi_{ij} |\alpha_i\rangle \otimes |\beta_j\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$$

with its matrix $\psi = [\psi_{ij}]$ in orthonormal bases $|\alpha_i\rangle, |\beta_j\rangle$ of $\mathcal{H}_A, \mathcal{H}_B$, we see that reduced matrices of $|\psi\rangle$ in respective bases are given by matrices

$$\rho_A = \psi^\dagger \psi, \quad \rho_B = \psi \psi^\dagger, \tag{1.2}$$

which are isospectral, that is, they have the same non-negative spectra

$$\mathrm{Spec}\rho_A = \mathrm{Spec}\rho_B = \lambda \tag{1.3}$$

except extra zeros if $\dim \mathcal{H}_A \neq \dim \mathcal{H}_B$. The isospectrality implies so called *Schmidt decomposition*

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}\, |\psi_i\rangle^A \otimes |\psi_i\rangle^B, \tag{1.4}$$

where $|\psi_i\rangle^A, |\psi_i\rangle^B$ are eigenvectors of ρ_A, ρ_B with the same eigenvalue λ_i.

1.2 The Pauli principle

1.2.1 Initial form of the Pauli principle

The Pauli exclusion principle, discovered in 1925, states that no two identical fermions (e.g. electrons) may occupy the same quantum state. It was discovered before the Quantum mechanics. Actually, Pauli stated his result only for the system of electrons, without mentioning quantum states.[1] In the language of density matrices it can be stated in the following way.

By the superposition principle, a state of N-electron system is given by state vector $|\psi\rangle^N \in \mathcal{H}^{\otimes N}$, where $\mathcal{H}$ is the state space of one electron. Let ρ_i be the reduced density matrix of i^{th} electron. The expectation value of the i^{th} electron in state $|\psi\rangle$ is given by the number $\langle\psi|\rho_i|\psi\rangle$; $|\psi\rangle \in \mathcal{H}$, which leads to the probability to find the i^{th} electron in state $|\psi\rangle$. The electron density matrix ρ of N electrons is defined as the sum of all its reduced matrices ρ_i: $\rho = \sum_i \rho_i$. Then the number $\langle\psi|\rho|\psi\rangle$ gives the expectation value of the number of electrons in state $|\psi\rangle$. In terms of the electron density matrix ρ the Pauli exclusion principle amounts to the inequality $\langle\psi|\rho|\psi\rangle \leq 1$, which bounds for its eigenvalues by 1, that is, $\mathrm{Spec}\rho \leq 1$.

1.2.2 Modern version

In 1926, the Pauli Exclusion Principle has been replaced by skew symmetry of a multi-electron wave function by Heisenberg and Dirac [12, Ch.4]. It can be explained in an easy way as follows. Assume that we have a state $|\psi\rangle$ of a system $\mathcal{H}^{\otimes N}$ of N identical particles. The indistinguishability of particles implies that if

---

[1] The original statement of the principle is as follows [30]: “There can never be two or more equivalent electrons in an atom.”

we permute the particles then we get the same state with a phase factor $e^{i\phi}$:

$$\pi|\psi\rangle = e^{i\phi}|\psi\rangle \ , \ \pi \in S_N.$$

This gives a one-dimensional representation of the symmetric group S_N of permutations of N letters. It's a well-known fact of representation theory that the group S_N has only two one-dimensional representations, namely identity and sign representations. Hence, we have either $e^{i\phi} \equiv 1$ or $e^{i\phi} \equiv \text{sign}\pi$. In the first case, which corresponds to bosonic particles (e.g. photons), we have symmetric tensors, while in the second case, which corresponds to fermionic particles (e.g. electrons), we have skew-symmetric tensors. As a result, the state space of N identical particles shrinks to symmetric tensors $S^N\mathcal{H} \subset \mathcal{H}^{\otimes N}$ for bosons and to skew-symmetric tensors $\wedge^N\mathcal{H} \subset \mathcal{H}^{\otimes N}$ for fermions. This implies the original Pauli principle, since $\psi \wedge \psi = 0$. In this case the density matrix of N-electron system becomes $\rho = N\rho_i$; $\text{Tr}\rho = N$.

1.2.3 Statement of the problem

In this thesis we study the impact of the above replacement on the electron density matrix. The problem which concerns the impact is the following:

> *What are the constraints on the electron density matrix ρ beyond the original Pauli principle,* $\text{Spec}\rho \leq 1$*?*

After A.J. Coleman [9], this problem became known as *N-representability* problem. Later, in mid 90's it was included by the National Research Council of USA in the list of ten most prominent research challenges in theoretical chemistry [36].

Actually, the above mentioned problem is known as *pure* N-representability, meaning that the operator ρ is the electron density matrix of a pure state. The general *mixed* N-representability problem concerns with the conditions on the operator ρ to be the particle density matrix of a mixed state ρ^N of a system of N identical particles. In [9], Coleman showed that the N-representability is unitary invariant. As a result, the constraints on the mixed state ρ^N and its particle density matrix ρ are expressible in terms of their spectra $\mu = \mathrm{Spec}\rho^N$ and $\lambda = \mathrm{Spec}\rho$. In this setting the above problem becomes:

> *What are the relations between the spectra μ and λ?*

1.3 Known results prior to 2006

There are a few cases where a complete solution of N-representability problem was known prior to 2006. Here, we give all known results. First, note that we always assume that all spectra are arranged in non-increasing order: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$, and unless otherwise stated, we assume that the density matrix of a mixed (or pure) state ρ^N of an N-particle system is normalized to $\mathrm{Tr}\rho^N = 1$, while for its particle density matrix ρ we have $\mathrm{Tr}\rho = N$.

1.3.1 Two-particle and two-hole systems

The simplest constraints on the electron density matrix beyond Pauli exclusion principle appear for two-electron system $\wedge^2\mathcal{H}_r$. The state vector $|\psi\rangle \in \wedge^2\mathcal{H}_r$ can be considered as a skew-symmetric bilinear form on $\mathcal{H}_r$. Its canonical form can be written as $|\psi\rangle = \sum_i a_i p_i \wedge q_i$, where p_i, q_j are orthonormal vectors in $\mathcal{H}_r$. This implies that the space $\mathcal{H}_r$ splits into direct sum of 2-dimensional spaces $\mathbb{C}^2$ (and extra 1-dimensional space if $r = \dim \mathcal{H}_r$ is odd). The reduced matrix ρ of $|\psi\rangle$ acts as a scalar on these 2-dimensional components, that is, $\rho = \sum_i a_i I_2$, where I_2 is identity operator on $\mathbb{C}^2$. Hence, the eigenvalues ($\lambda = \mathrm{Spec}\rho$) of the electron density matrix ρ are evenly degenerate; starting from the head $\lambda_{2i-1} = \lambda_{2i}$, except $\lambda_r = 0$ for odd $r = \dim \mathcal{H}_r$.

There is a similar result for two-hole system $\wedge^{r-2}\mathcal{H}_r$: constraints on ρ are given by double degeneracy of the spectrum, starting from tail $\lambda_{r-2i-1} = \lambda_{r-2i}$, except $\lambda_1 = 1$ for odd r.

1.3.2 The Borland-Dennis system

The simplest system beyond 2-electrons and 2-holes is $\wedge^3\mathcal{H}_6$ considered by Borland and Dennis in early 70's [5]. In this case, the N-representability conditions are given by the following (in)equalities:

$$\lambda_1 + \lambda_6 = \lambda_2 + \lambda_5 = \lambda_3 + \lambda_4 = 1, \quad \lambda_4 \le \lambda_5 + \lambda_6, \tag{1.5}$$

which are discovered by an extensive computer experiment. Borland and Dennis established the sufficiency of the relations (1.5) and refer to M.B. Ruskai and R.L. Kingsley for the complete proof. In 2007, Mary Beth Ruskai finally published the proof [35] derived from known constraints on the spectra of Hermitian matrices A, B, and $C = A + B$.

1.3.3 Peltzer-Brandstatter theorem

In 1971, Peltzer and Brandstatter claimed the following solution of N-representability problem:

False Theorem([31]): *For all systems $\wedge^N\mathcal{H}_r$ except two electrons $\wedge^2\mathcal{H}_r$, two holes $\wedge^{r-2}\mathcal{H}_r$, and Borland-Dennis system $\wedge^3\mathcal{H}_6$ the only restriction on one electron density matrix is given by Pauli constraint $\langle\psi|\rho|\psi\rangle \le 1$ ($\Leftrightarrow \mathrm{Spec}\rho \le 1$).*

It seems nobody has refuted the theorem before 2006, and the above results stood as the only known results for N-representability problem for more than 30 years. The first counter-example to Peltzer and Brandsttater's theorem appeared in [21].

1.4 New results

In this thesis we give a complete solution to this longstanding problem. The solution is given by *finite* set of linear inequalities. Before describing the general form of these inequalities let us introduce some notations and the general *mixed N-representability* problem.

Let a be a non-increasing sequence of numbers, $a : a_1 \geq a_2 \geq \ldots \geq a_r$. We call it as *test spectrum.* Now, define $a_I = \sum_{i \in I} a_i$ for $I = \{i_1, i_2, \ldots, i_N\} \subset \{1, 2, \ldots, r\}$. Denote by

$$\wedge^N a = \{\sum_{i \in I} a_i \ : \ I \subset \{1, 2, \ldots, r\}, |I| = N\}^{\downarrow},$$

the set of all possible sums a_I which are arranged in non-increasing order. For example, for $a = (5, 4, 3, 2, 1)$ and $N = 2$, $\wedge^2 a$ corresponds to the set

$$\wedge^2 a = \{a_{i_1} + a_{i_2} \ : \ 1 \leq i_1 < i_2 \leq 5\}^{\downarrow} = \{9, 8, 7, 7, 6, 6, 5, 5, 4, 3\}.$$

Let now, ρ^N be a mixed state of a system $\wedge^N \mathcal{H}_r$ of N fermions of rank r and ρ be its particle density matrix, and denote their spectra as $\mu = \mathrm{Spec}\rho^N$ and $\lambda = \mathrm{Spec}\rho$, respectively. The most general *mixed N-representability* problem concerns with the relations between these spectra.

1.4.1 General solution of mixed N-representability

The following theorem gives a solution of mixed N-representability problem. It is a special case of Theorem 3.2.1 which can be deduced from Berenstein and Sjamaar's results [3, Thm. 3.1.1].

Theorem 1.4.1 *For a mixed state ρ^N of a system $\wedge^N \mathcal{H}_r$ of N-fermions of rank r and its particle density matrix ρ all constraints on the spectra $\mu = \mathrm{Spec}\rho^N$ and $\lambda = \mathrm{Spec}\rho$, arranged in non-increasing order and normalized to $\mathrm{Tr}\rho^N = \sum_i \mu_i = 1$ and $\mathrm{Tr}\rho = \sum_i \lambda_i = N$ respectively, are given by the following inequalities*

$$\sum_i a_i \lambda_{v(i)} \leq \sum_k (\wedge^N a)_j \mu_{w(j)}, \qquad (a, v, w)$$

where $v \in S_r$ and $w \in S_{\binom{r}{N}}$ are permutations, subject to a topological condition $c_w^v(a) \neq 0$ explained below. □

To understand the topological nature of the coefficient $c_v^w(a)$ consider the flag variety $\mathcal{F}_a(\mathcal{H}_r)$ which can be understood as the set of Hermitian operators

$X : \mathcal{H}_r \to \mathcal{H}_r$ of given spectrum $a = \operatorname{Spec}(X)$, and morphism

$$\begin{array}{rcl}\varphi_a : \mathcal{F}_a(\mathcal{H}_r) & \to & \mathcal{F}_{\wedge^N a}(\wedge^N \mathcal{H}_r)\\ X & \mapsto & X^{(N)}\end{array}$$

where $X^{(N)} : |\psi_1\rangle \wedge |\psi_2\rangle \wedge \ldots \wedge |\psi_N\rangle \mapsto \sum_i |\psi_1\rangle \wedge |\psi_2\rangle \wedge \ldots \wedge X|\psi_i\rangle \wedge \ldots \wedge |\psi_N\rangle$. The coefficients $c_w^v(a)$ are defined via induced morphism of cohomologies

$$\begin{array}{rcl}\varphi_a^* : H^*(\mathcal{F}_{\wedge^N a}(\wedge^N \mathcal{H}_r)) & \to & H^*(\mathcal{F}_a(\mathcal{H}_r))\\ \sigma_w & \mapsto & \sum\limits_v c_w^v(a)\sigma_v\end{array}$$

written in the basis of Schubert cocycles σ_w. For details and the calculations of the coefficients $c_w^v(a)$ see $n°3.2.1$.

Remark 1.4.1 The coefficients $c_w^v(a)$ depend only on the order in which quantities $a_I = a_{i_1} + a_{i_2} + \cdots + a_{i_N}$, $I = \{i_1, i_2, \ldots, i_N\} \subset \{1, 2, \ldots, r\}$ appear in the spectrum $\wedge^N a$. The order changes when the test spectrum a crosses a hyperplane

$$H_{I|J} : a_I = a_J, \quad I \neq J.$$

The hyperplanes $H_{I|J}$ cut the set of all test spectra into a finite number of polyhedral cones called *cubicles*. For each cubicle one has to check inequality (a, v, w) only for its *extremal edges*. As a result N-representability amounts to a *finite system* of linear inequalities.

Remark 1.4.2 The solution of *pure N-representability* can be deduced from the above theorem by specialization $\mu_i = 0$ for $i \neq 1$. Recall that, a pure state $|\psi\rangle$ is represented by a projection operator $|\psi\rangle\langle\psi|$ onto $|\psi\rangle$. Hence,

$$\text{a state } \rho \text{ is pure} \Longleftrightarrow \operatorname{rk}\rho = 1 \Longleftrightarrow \operatorname{Spec}\rho = (1, 0, 0 \cdots, 0). \tag{1.6}$$

Example 1.4.1 Consider the system $\wedge^3\mathcal{H}_7$ of 3 electrons of rank 7. The constraints on the spectrum λ of the electron density matrix ρ of a pure state amounts to the following four inequalities

$$\begin{array}{ll}\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \leq 2, & \lambda_1 + \lambda_3 + \lambda_4 + \lambda_6 \leq 2,\\ \lambda_1 + \lambda_2 + \lambda_5 + \lambda_6 \leq 2, & \lambda_1 + \lambda_2 + \lambda_4 + \lambda_7 \leq 2.\end{array} \tag{1.7}$$

Now, let us see how these inequalities can be realized by Theorem 1.4.1. First, all the inequalities are obtained by using the same test spectra $a = (1, 1, 1, 1, 0, 0, 0)$. The shortest permutations which give the right hand sides of these inequalities are

$$\begin{aligned} v_1 &= (12345), \quad v_2 = (23465), \\ v_3 &= (35)(46), \quad v_4 = (34756), \end{aligned}$$

where v_i's are in S_7 and written in cycle decomposition. To interpret the right hand sides ($= 2$) of the inequalities in (1.7), we need:

$$\wedge^N a = (3, 3, 3, 3, 2, 2, \ldots, 2, 1, 1, \ldots, 1, 0).$$

Since for pure state $\mu = (1, 0, 0, \ldots, 0)$, then the shortest permutation w which produces 2 in the right hand side is the cyclic permutation $w = (12345) \in S_{35}$. The topological condition $c_w^{v_i}(a) = 1 \neq 0$ for all v_i. So the inequalities in (1.7) are all valid inequalities.

1.4.2 Grassmann inequalities

In this subsection, we give two type of inequalities which hold for a fixed N and arbitrary rank r. We call them as Grassmann inequalities first and second kind. Here, we only give their descriptions and examples for some N and rank r.

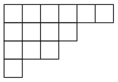
Grassmann inequalities of first kind

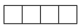
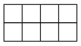
The spectrum λ of the particle density matrix ρ of N-fermion system $\wedge^N \mathcal{H}_r$ satisfies the following inequalities

$$\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_{N-1}} \leq N - 2, \tag{1.8}$$

with a few exceptions (Theorem 4.3.1), where the index set $I = \{i_1, i_2, \ldots, i_{N-1}\} \subset \{1, 2, \ldots, r\}$ is described by the Young diagram[2] σ_I of size

---

[2] A Young diagram $\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n)$ is an array of boxes, lined up at the left, with α_i boxes in the i^{th} row, with the rows arranged from top to bottom. For example,  is the Young diagram of $(6, 4, 3, 1)$. The size of a Young diagram α is defined as $|\alpha| = \sum_i \alpha_i$.

$r - N + 1$ in an $(N-1) \times (r-N+1)$ rectangular box, which is cut out by the polygonal line Γ_I connecting S–W and N–E corners of the rectangle, with i^{th} unit edge running to the North for $i \in I$ and to the East otherwise. For instance, the inequality $\lambda_1 + \lambda_6 \leq 1$, where $r = 6$ and $N = 3$, corresponds to the Young diagram  in a 2×4 rectangle .

In the simplest case $N = 3$, from (1.8) we get the inequalities

$$\lambda_{k+1} + \lambda_{r-k} \leq 1, \quad 0 \leq k < (r-1)/2$$

which hold for any even rank $r \geq 6$. This constraint forbids more than one electron to occupy *two* symmetric orbitals and supersedes the original Pauli principle. For $r = 6$, due to the normalization $\sum_i \lambda_i = 3$, the inequalities degenerate into Borland-Dennis *equalities* (1.5). For odd rank, the first inequality ($k = 0$) should be either skipped or replaced by weaker one $\lambda_1 + \lambda_r \leq 1 + \frac{2}{r-1}$.

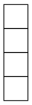
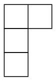
Grassmann inequalities of second kind

The following conditions

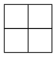
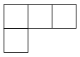
$$\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_{N+1}} \leq N - 1 \tag{1.9}$$

must be satisfied by the spectrum λ of the particle density matrix ρ of the system $\wedge^N \mathcal{H}_r$ for each Young diagram σ_I of size $N+1$ which fits in $(N+1) \times (r-N-1)$ rectangle, described as above, except for the row diagram, and for even N the column diagram.

For $N = 3$ the inequalities (1.9) amount to four inequalities listed below together with the corresponding diagrams

 $: \quad \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \leq 2,$  $: \quad \lambda_1 + \lambda_3 + \lambda_4 + \lambda_6 \leq 2,$

(1.10)

 $: \quad \lambda_1 + \lambda_2 + \lambda_5 + \lambda_6 \leq 2,$  $: \quad \lambda_1 + \lambda_2 + \lambda_4 + \lambda_7 \leq 2,$

which hold for arbitrary rank r and give all the constraints for $r \leq 7$. For $r = 6$ they turn into Borland-Dennis conditions (1.5).

1.5 Connection with representation theory

The Theorem 1.3 which gives a general solution of N-representability problem is not practical to find the explicit constraints on the density matrix of a given system $\wedge^N\mathcal{H}_r$ even for small ranks. A representation theoretical approach to the problem, discussed below, makes life a little bit easier. A combination of the two approaches leads to an algorithm for solution of the problem for any fixed rank (see Chapter 5). By the help of this algorithm, together with some other tools, we were able to find all constraints for the systems of rank $r \leq 10$ explicitly (see Chapter 6).

1.5.1 Irreducible representations of unitary group

Consider the m^{th} symmetric power $S^m(\wedge^N\mathcal{H}_r)$ of the irreducible representation $\wedge^N\mathcal{H}_r$ of the unitary group $\mathrm{U}(\mathcal{H}_r)$, which is no more irreducible. However, it can be decomposed into its irreducible components $\mathcal{H}^\lambda$ parameterized by Young diagrams

$$\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$$

of size $|\lambda| = \sum_i \lambda_i = N \cdot m$ which fit into $r \times m$ rectangle. In this setting, we have the following problem

> *Which irreducible representations $\mathcal{H}^\lambda$ of the unitary group $U(\mathcal{H}_r)$ can appear in the decomposition of $S^m(\wedge^N\mathcal{H}_r)$?*

A surprising result is that the solution of this problem coincides with the solution of N-representability problem.

To make the connection between these two different problems, let us treat the diagrams λ as *spectra*. We are interested in asymptotic behavior of these spectra as $m \to \infty$ and therefore normalize them to a fixed size $\widetilde{\lambda} = \lambda/m$, $\mathrm{Tr}\,\widetilde{\lambda} = N$. The following theorem gives an asymptotic solution for the pure N-representability problem.

Theorem 1.5.1 *Every $\widetilde{\lambda}$ obtained from irreducible component $\mathcal{H}^\lambda \subset S^m(\wedge^N \mathcal{H}_r)$ is a spectrum of the particle density matrix ρ of a pure state $\psi \in \wedge^N \mathcal{H}_r$. Moreover every one point reduced spectrum is a convex combination of such spectra $\widetilde{\lambda}$ with bounded $m \leq M$.* □

1.5.2 Practical algorithm

Note that, the set of all allowed spectra of the electron density matrix forms a convex polytope, called *Moment Polytope*. The above theorem gives an inner approximation to this polytope, while the Theorem 1.4.1 gives an outer approximation. Combining these two results leads to the following practical algorithm, which allows to find explicit constraints for the N-representability problem:

1. *Find all irreducible components $\mathcal{H}^\lambda \subset S^m(\wedge^N \mathcal{H}_r)$ for $m \leq M$.*
2. *Calculate the convex hull of the corresponding spectra $\widetilde{\lambda}$ which gives an inner approximation $\mathcal{P}_M^{\text{in}} \subset \mathcal{P}$ for the moment polytope $\mathcal{P}$.*
3. *Identify the facets of $\mathcal{P}_M^{\text{in}}$ that are given by the inequalities of Theorem 3.2.1. They cut out an outer approximation $\mathcal{P}_M^{\text{out}} \supset \mathcal{P}$.*
4. *Increase M and continue until $\mathcal{P}_M^{\text{in}} = \mathcal{P}_M^{\text{out}}$.*

1.6 Taking into account spin

Actually, the state space of a single particle with spin splits into the tensor product $\mathcal{H} = \mathcal{H}_r \otimes \mathcal{H}_s$ of the orbital component $\mathcal{H}_r$ and the spin component $\mathcal{H}_s$. The total N-fermion space decomposes into spin-orbital components as follows [38]

$$\wedge^N (\mathcal{H}_r \otimes \mathcal{H}_s) = \sum_{|\nu|=N} \mathcal{H}_r^\nu \otimes \mathcal{H}_s^{\nu^t}, \tag{1.11}$$

where ν^t stands for the transpose diagram, and $\mathcal{H}_r^\nu$ and $\mathcal{H}_s^{\nu^t}$ are irreducible representations of unitary groups $U(\mathcal{H}_r)$ and $U(\mathcal{H}_s)$ with Young diagrams ν and ν^t, respectively. In many physical systems, like electrons in an atom or a molecule, the total spin is a well defined quantity which singles out a specific component of this decomposition. We have to deal with pure states $\psi \in \mathcal{H}_r^{\nu} \otimes \mathcal{H}_s^{\nu^t}$. From n°1.1.5, the reduced states ρ_r^{ν} and $\rho_s^{\nu^t}$ of ψ are isospectral, that is, $\text{Spec}\rho_r^{\nu} = \text{Spec}\rho_s^{\nu^t}$. So we can identify the spectrum $\text{Spec}\rho_s^{\nu^t}$ with $\text{Spec}\rho_r^{\nu}$.

On the other hand, the *Schur-Weyl duality*

$$\mathcal{H}^{\otimes N} = \sum_{|\nu|=N} \mathcal{H}^{\nu} \otimes V^{\nu}, \tag{1.12}$$

between irreducible representations $\mathcal{H}^{\nu}$ and V^{ν} of the unitary $\mathrm{U}(\mathcal{H})$ and the symmetric S_N groups, respectively, allows to define the i^{th} reduced density matrix $\rho_i : \mathcal{H}$ for $\rho^{\nu} : \mathcal{H}^{\nu}$ as the reduced density matrix for $\rho^{\nu} \otimes 1$. The operator $\rho^{\nu} \otimes 1$ acting on the component $\mathcal{H}^{\nu} \otimes V^{\nu}$ commutes with S_N, and hence the reduced state ρ_i is independent of i.

The problem which we address here is the following:

> *What are the constraints on the spectra of $\rho^{\nu} : \mathcal{H}^{\nu}$ and its particle density matrix $\rho = N\rho_i : \mathcal{H}$?*

It is a variation of the N-representability problem. We call it as ν-representability.

As a result, by solving the above problem we may find all constraints on the spectra $\text{Spec}\rho_r^{\nu}$ and $\text{Spec}\rho_s^{\nu^t}$.

In Chapter 3 we give the formal solution of this problem which is the generalization of Theorem 1.4.1, and in Chapter 5 we give another solution which is the generalization of Theorem 1.5.1. Combining these two approaches gives an algorithm which is the modified version of the one given in previous section. In this new algorithm there is a small modification: instead of the symmetric power $S^m(\wedge^N \mathcal{H})$ we have to use the *plethysm* $[\mathcal{H}^{\nu}]^{\mu}$.

As an example let's consider the constraints on the mixed state ρ^{ν} and its reduced matrix ρ of a system of three electrons of the total spin $J = 1/2$. The problem is equivalent to ν-representability for $\nu = \begin{smallmatrix}\square & \square \\ \square & \end{smallmatrix}$ and $\text{Spec}\,\rho^{\nu} = (\mu_1, \mu_2)$. A calculation based on the above algorithm shows that the constraints amount to the following 5 inequalities

$$\lambda_1 - \lambda_2 \leq 1 + \mu_2, \quad \lambda_2 - \lambda_3 \leq 1 + \mu_2, \quad \lambda_1 - \lambda_3 \leq 2 - \mu_2,$$
$$\lambda_1 - \lambda_2 - \lambda_3 \leq 1, \quad 2\lambda_1 - \lambda_2 + \lambda_4 \leq 4 - \mu_2,$$

which are apparently independent of the rank.

The results stated in this thesis have appeared in [1]. Theoretical analyses have been provided by second author. Without his theoretical analyses, computer based calculations which yield ν- and N-representability constraints for some certain systems could not be achieved. His most valuable comment on this context should be emphasized here:

> "The theoretical results of the paper belong to the second author. They were often inspired by calculations, that at this stage couldn't be accomplished by a computer without intelligent human assistance and insight."

Chapter 2

Survey of Berenstein-Sjamaar's Results

In this chapter we rephrase the general result, given by Berenstein and Sjamaar in 2000 [3], in the form xsuitable for our purpose. Before describing their result we need some preliminary definitions and facts about representation theory, Lie algebra and geometry of flag varieties. The results are stated without proofs. We recommend books [15, 16, 17] for details.

2.1 Preliminaries

2.1.1 Representation theory

A *representation* of a group G in a finite dimensional complex vector space V is a homomorphism $\phi : G \to \mathrm{GL}_n(\mathbb{C})$ of G to the group $\mathrm{GL}_n(\mathbb{C})$ of automorphisms of V, where $n = \dim V$. For simplicity, we call V itself a representation of G and write gv for $\phi(g)(v)$. We use the notation $G : V$ for the representation V of G.

Example 2.1.1 Let X be a finite set, G be a finite group which acts on X by permutations, i.e., we have a homomorphism of groups $\phi : G \to S_X$, where S_X is the group of all permutations of X. The action of G on X can be extended linearly to an action on $\mathbb{C}X$, a vector space with basis X: $g\sum_x a_x x = \sum_x a_x gx$. With this action $\mathbb{C}X$ forms a representation of G called permutation representation. If we take $X = G$ then $\mathbb{C}G$ is called regular representation.

Operations on representations

Direct sum: Let $G : V$ and $G : W$ be two representations. Then the direct sum of these representations, $G : V \oplus W$ is defined by the action $g(v \oplus w) = gv \oplus gw$ for all $g \in G$, $v \in V$ and $w \in W$.

Direct product: Let $G : V$ and $G' : W$ be two representations of two different groups. Then the direct sum of $V \oplus W$ vector spaces is a representation of the group $G \times G'$ defined by the action $(g \times g')(v \oplus w) = gv \oplus g'w$ for all $g \in G$, $g' \in G'$, $v \in V$ and $w \in W$.

Dual representation: Let $G : V$ be a representation and V^* be its dual space, that is, the space of all linear functionals:

$$V^* = \{f : V \to \mathbb{C} \ : \ f(ax + by) = af(x) + bf(y)\,, a, b \in \mathbb{C}; x, y \in V\},$$

with an action of G:

$$gf(x) = f(g^{-1}x),\ g \in G,$$

then V^* is also a representation of G called *dual* representation.

Tensor product: The tensor product of two vector spaces V and W with bases $\{e_1, e_2, \ldots, e_n\}$ and $\{f_1.f_2, \ldots, f_m\}$, respectively, can be defined as the vector space $V \otimes W$ spanned by the pairs $e_i \otimes f_j$:

$$V \otimes W = \{\sum_{i,j} a_{ij} e_i \otimes f_j \ : \ a_{ij} \in \mathbb{C}\}.$$

If $G : V$ and $G : W$ are two representations then $G : V \otimes W$ is also a representation with the action of G given by $g(v \otimes w) = gv \otimes gw$ for $g \in G$, $v \in V$ and $w \in W$.

Irreducible representations

A subspace W of V is called a *subrepresentation* if it is invariant under the action of G, i.e., $gW \subset W$ for all $g \in G$. If V has no G-invariant subspace other than $\{0\}$ and V itself, then it is called *irreducible representation* of G. The main problem of representation theory is the classification of irreducible representations.

Theorem 2.1.1 *For abelian group G, every irreducible representation is one dimensional.* $\square$

Let $G : V$ be a representation, and $(,)$ be Hermitian metric on V, that is, a map $(,) : V \times V \to \mathbb{C}$ satisfying the following properties for all $x, y, z \in V$ and $a, b \in \mathbb{C}$:

- Conjugate symmetry: $(x, y) = \overline{(y, x)}$,
- Positive definiteness: $(x, x) \geq 0$ and $(x, x) = 0$ if and only if $x = 0$,
- Linearity in the first variable: $(ax + by, z) = a(x, z) + b(y, z)$. Together with conjugate symmetry this implies semi-linearity in the second variable: $(x, ay + bz) = \overline{a}(x, z) + \overline{b}(x, y)$.

A metric (x, y) on V is said to be *G-invariant* if for all $g \in G$, $(gx, gy) = (x, y)$, i.e., g acts on V by unitary transformations.

Theorem 2.1.2 *Every finite dimensional representation V of G carries an G-invariant metric.* $\square$

Theorem 2.1.3 (Maschke) *Let $U \subset V$ be a subrepresentation of $G : V$. Then there exists a G-invariant subspace $W \subset V$ such that $V = U \oplus V$.* $\square$

Corollary 2.1.1 *Every finite dimensional representation V of G is the direct sum of irreducible representations.* $\square$

Let $G : V$ and $G : W$ be two representations. A *G-morphism* $\varphi : V \to W$ is a linear transformation commuting with the action of G, that is, $\varphi(gv) = g\varphi(v)$ for $g \in G$ and $v \in V$. Denote the set of all G-morphisms between V and W by $\mathrm{Hom}_G(V, W)$.

Theorem 2.1.4 (Schur's Lemma) *Let V and W be two irreducible representations of $G : V$. Then*

$$\mathrm{Hom}_G(V, W) = \begin{cases} 0, & \textit{if } V \ncong W \\ \mathbb{C}, & \textit{if } V \cong W \end{cases} .$$

□

Characters

A *character* of a representation $G : V$ is a complex valued function $\chi_V : G \to \mathbb{C}$ defined by $\chi_V(g) = \mathrm{Tr}_V(g)$, the trace of g on V. It is a significant notion in representation theory, because it characterizes the representation $G : V$.

Theorem 2.1.5 *Isomorphic representations have the same characters.* □

Here are some elementary properties of characters.

Properties of characters:

- $\chi_V(1) = \dim V$,
- $\chi_V(g^{-1}) = \overline{\chi_V(g)}$,
- $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$,
- $\chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g)$,
- χ_V is a central function on G, i.e., $\chi_V(g) = \chi_V(h^{-1}gh)$ for all $h \in G$,
- $\chi_{V^*}(g) = \overline{\chi_V(g)}$.

Orthogonality relations between characters: Let χ_i be characters of non-isomorphic irreducible representations V_i's of the group G. Then

$$(\chi_i, \chi_j) := \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} = \begin{cases} 1\ , & \text{if } i = j \\ 0\ , & \text{if } i \neq j \end{cases} ,$$

and

$$\sum_i \chi_i(g) \overline{\chi_i(h)} = \begin{cases} |C_G(g)|\ , & \text{if } h \in C_g \\ 0\ , & \text{otherwise} \end{cases} ,$$

where $C_G(g) = \{f \in G \ : \ fg = gf\}$ is the centralizer of g, and $C_g = \{f^{-1}gf \ : \ f \in G\}$ is the conjugacy class of g. The above relations between characters are known as 1st and 2nd orthogonality relations, respectively.

2.1.2 Irreducible representations of symmetric group S_n

Induced representations

Let H be a subgroup of G and U be a representation of H. Then the representation

$$U_H^G = \bigoplus_{x \in G/H} xU$$

of G is said to be induced by representation U of subgroup H. Here, xU is an isomorphic copy of U, and the action of G on U_H^G is given by

$$g \sum_{x \in G/H} xv_x = \sum_{x \in G/H} gxv_x,$$

where $v_x \in V$ for each x.

Example 2.1.2 Let $H \subset G$ and $H : U$ be trivial (identity) representation, i.e., each element $h \in H$ acts on U as an identity operator. Then the induced representation U_H^G is the permutation representation in set $X = G/H$.

Young tableaux

A *Young diagram* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a finite collection of boxes, or cells, arranged in left-justified rows such that the i^{th} row has length λ_i and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$. For example, the Young diagram $\lambda = (5, 3, 2)$ looks like

$$\lambda = \begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline & & \\ \cline{1-3} & \\ \cline{1-2} \end{array}$$

A way of putting a positive integer in each box of a Young diagram λ is called a *numbering* (when the entries are distinct) or *filling* of the diagram. A *semistandard tableau* (or simply tableau) is a filling which is weakly increasing in rows and strictly increasing in columns. A *standard tableau* is a tableau in which entries are the numbers from 1 to n, each occurring once, where $n = |\lambda|$ is the number of boxes of the diagram λ. Here are examples of semistandard and standard tableaux of shape $\lambda = (5, 3, 2)$

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 4 & 5 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 7 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 9 & 10 \\ \cline{1-2} \end{array}$$

Semistandard tableau Standard tableau

When we flip a diagram λ over its main diagonal (from upper left to lower right) we get a new diagram λ^t called the transpose (or conjugate) diagram of λ. For example, the transpose of the above diagram $\lambda = (5, 3, 2)$ is $\lambda = (3, 3, 2, 1, 1)$:

$$\lambda^t = \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1} \\ \cline{1-1} \end{array}$$

From Young diagrams to irreducible representations of S_n

Let $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_k$ be a Young diagram of size $n = |\lambda| = \sum_i \lambda_i$, and T be any tableau with shape λ. Let R_T be the group of permutations of numbers in rows of tableau T, and C_T be the group of permutations of numbers in columns of tableau T. These are subgroups of S_n and conjugate the following groups

$R_T \backsimeq R_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots S_{\lambda_k} \subset S_n$ and $C_T \backsimeq C_\lambda = S_{\lambda_1^t} \times S_{\lambda_2^t} \times \cdots S_{\lambda_l^t} \subset S_n$, where $l = \lambda_1$. Note that, $R_\lambda = C_{\lambda^t}$ and $C_\lambda = R_{\lambda^t}$.

Now, define two representations of S_n as

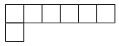
$$M_\lambda = (\mathrm{id})_{R_\lambda}^{S_n} \qquad \text{and} \qquad N_\lambda = (\mathrm{sgn})_{C_\lambda}^{S_n},$$

where id and sgn are the trivial and sign representations of S_n, respectively. They are both one-dimensional. The action of latter one is given by multiplication by the scalars $sgn(\pi) = \pm 1$, the sign of permutation π.

Theorem 2.1.6 *There exists unique irreducible representation* V^λ *of* S_n *such that* $V^\lambda \subset M_\lambda$ *and* $V^\lambda \subset N_\lambda$. $\Box$

The irreducible representation V^λ is called *Specht representation.*

Theorem 2.1.7 *Two Specht representations* V^λ *and* V^μ *are isomorphic if and only if* $\lambda = \mu$*, and every irreducible representation* V *of* S_n *is isomorphic to* V^λ *for some diagram* λ. $\Box$

Example 2.1.3 For row diagram $\lambda = (n)$, $R_\lambda = S_n$, and hence, $V^\lambda = M_\lambda = (\mathrm{id})_{S_n}^{S_n} = \mathrm{id}$. And for column diagram $\lambda = (1, 1, 1, \ldots, 1)$, $C_\lambda = S_n$. Therefore, $V^\lambda = N_\lambda = (\mathrm{sgn})_{S_n}^{S_n} = \mathrm{sgn}$. As an another example, consider the standard representation $\mathbb{C}^n$ of S_n. The action of S_n is given by permutation of coordinates. The representation $\mathbb{C}^n$ splits into its irreducible components as: $\mathbb{C}^n = (\mathrm{id}) \oplus V$, where the component V is the space spanned by $e_1, e_2, \ldots e_n$, the standard basis of $\mathbb{C}^n$, subject to condition $e_1 + e_2 + \cdots e_n = 0$. The irreducible representation V corresponds the Specht representation V^λ, where $\lambda = (n-1, 1) =$ .

2.1.3 Irreducible representations of unitary group

In this subsection we describe the irreducible representations of the unitary group $\mathrm{U}(\mathcal{H})$ which is one of the main objects for our study.

Lie groups, Lie algebras and their representations

A *Lie group* G is a smooth manifold equipped with a compatible group structure. Here, compatible means that the group operations (product and inverse) are smooth maps.

A *Lie algebra* over $\mathbb{R}$ (or $\mathbb{C}$) is a vector space $\mathfrak{g}$ with a skew-symmetric bilinear form $[,]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called Lie bracket, which satisfies the *Jacobi identity*:

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0, \text{ for all } A, B, C \in \mathfrak{g}.$$

In particular, for a Lie group G its Lie algebra $\mathfrak{g} = Lie(G)$ is defined as the tangent space $T_e(G)$ of G at its identity element e.

Example 2.1.4 In this thesis, we mostly deal with the Lie group $\mathrm{U}(\mathcal{H})$ which consists of all unitary operators on the complex Hilbert space $\mathcal{H}$. An operator $X : \mathcal{H} \to \mathcal{H}$ is said to be unitary if it satisfies the condition $XX^\dagger = I$, where $X^\dagger$ stands for the adjoint operator of X. The Lie algebra $\mathfrak{u}(\mathcal{H})$ of the unitary group $\mathrm{U}(\mathcal{H})$ consists of all skew-Hermitian operators $A = -A^\dagger$. However, we usually treat it as the algebra of Hermitian operators at the expense of a modified Lie bracket $[A, B] = i(AB - BA)$. Another example of Lie group we are interested in is the general linear group $\mathrm{GL}(\mathcal{H})$ consisting of all invertible linear operators on $\mathcal{H}$. It is the *complexification* of $\mathrm{U}(\mathcal{H})$, i.e., $\mathrm{GL}(\mathcal{H}) = \mathrm{U}(\mathcal{H}) \otimes \mathbb{C}$. The Lie algebra of $\mathrm{GL}(\mathcal{H})$ denoted by $\mathfrak{gl}(\mathcal{H})$ consists of all linear operators on $\mathcal{H}$.

Since Lie groups are also topological spaces, they may have some topological properties like compactness, connectedness, simply connectedness, etc.

Compactness: A topological space X is said to be *compact* if for every family of open sets which cover X, there is a finite sub-family which also covers X.

Connectedness: X is said to be *connected* if it can not be expressed as a union of two non-empty disjoint closed sets.

Simply connectedness: X is said to be *simply connected* if it is path connected and every closed path can be contracted continuously to a point. Here, path connected means that for every two point $x, y \in X$ there is a smooth path $\gamma : [0,1] \to X$ with initial point $\gamma(0) = x$ and terminal point $\gamma(1) = y$, and a path γ is closed if $\gamma(0) = \gamma(1)$.

A representation of a Lie group G is defined as for usual groups. A *representation of a Lie algebra* $\mathfrak{g}$ in a vector space V is a homomorphism $\phi : \mathfrak{g} :\longrightarrow \mathfrak{gl}(V)$ which preserves the Lie brackets of $\mathfrak{g}$ and $\mathfrak{gl}(V)$, i.e.,

$$\phi([X,Y]) = [\phi(X), \phi(Y)] = \phi(X)\phi(Y) - \phi(Y)\phi(X)\ , \quad \text{for all } \ X, Y \in \mathfrak{g}.$$

Example 2.1.5 (Adjoint representation) Let G be a connected Lie group, and $\mathfrak{g}$ be its Lie algebra. The group G acts on itself by inner automorphisms:

$$\begin{array}{rccccc} \mathrm{A}(g): & G & \longrightarrow & G & ; & g \in G. \\ & x & \mapsto & gxg^{-1} & & \end{array}$$

The differential of the above action defines a representation $\mathrm{Ad} = d\mathrm{A} : G \to \mathrm{GL}(\mathfrak{g})$ of Lie group G, called adjoint representation of G. It induces a representation of the Lie algebra $\mathfrak{g}$:

$$\mathrm{ad} = d\mathrm{Ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),$$

which is also called adjoint representation (of Lie algebra $\mathfrak{g}$).

In general, for any representation $\phi : G \to \mathrm{GL}(V)$ of connected Lie group G, its differential $d\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ defines a representation of Lie algebra $\mathfrak{g} = Lie(G)$. Moreover, the representation ϕ is irreducible if and only if $d\phi$ is irreducible.

Irreducible representations of $\mathrm{GL}(\mathcal{H})$

Now we will construct irreducible representations of the general linear group $\mathrm{GL}(\mathcal{H})$ which is the complexification of the unitary group $\mathrm{U}(\mathcal{H})$. Since both group are connected, there is a one-to-one correspondence between irreducible representations of $\mathrm{U}(\mathcal{H})$ and its complexification $\mathrm{GL}(\mathcal{H})$. As a result, the construction of irreducible representations of $\mathrm{GL}(\mathcal{H})$ yields the irreducible representations of $\mathrm{U}(\mathcal{H})$.

Let $\mathcal{H}$ be a finite dimensional Hilbert space, and consider its n^{th} tensor power $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ on which S_n acts by permutation of the components

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n \overset{\pi}{\mapsto} x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n} \ ; \qquad \pi \in S_n \ , \ \pi(k) = i_k \ ,$$

and $G = \mathrm{GL}(\mathcal{H})$ acts by diagonal transformation

$$g^{\otimes n} : x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto gx_1 \otimes gx_2 \otimes \cdots \otimes gx_n \ ; g \in G.$$

Clearly, these actions commute, i.e., $g\pi = \pi g$ for all $g \in G$ and $\pi \in S_n$. Being S_n-representation, $\mathcal{H}^{\otimes n}$ splits into its irreducible components as follows:

$$\mathcal{H}^{\otimes n} = \bigoplus_{|\lambda|=n} m_\lambda V^\lambda \text{ , where the multiplicity } m_\lambda = \dim(\mathrm{Hom}_{S_n}(V^\lambda, \mathcal{H}^{\otimes n})).$$

Here, V^λ's are Specht representations of S_n corresponding to Young diagram λ. Now, define

$$\mathcal{H}^\lambda := \mathrm{Hom}_{S_n}(V^\lambda, \mathcal{H}^{\otimes n}).$$

Since actions of G and S_n commute, $\mathcal{H}^\lambda$ defines a representation of $\mathrm{GL}(\mathcal{H})$ called *natural representation* of $\mathrm{GL}(\mathcal{H})$.

Example 2.1.6 For row diagram $\lambda = (n)$ the Specht representation V^λ corresponds to the trivial representation id of S_n, and hence $\mathcal{H}^\lambda = \mathrm{Hom}_{S_n}(\mathrm{id}, \mathcal{H}^{\otimes n}) = S^n\mathcal{H}$. For column diagram $\lambda = (1, 1, 1, \ldots, 1)$ we have $V^\lambda = \mathrm{sgn}$, which implies $\mathcal{H}^\lambda = \mathrm{Hom}_{S_n}(\mathrm{sgn}, \mathcal{H}^{\otimes n}) = \wedge^n\mathcal{H}$.

Theorem 2.1.8 *1.* $\mathcal{H}^\lambda \neq 0 \Leftrightarrow \#(\textit{rows of } \lambda) \leq \dim \mathcal{H} = d$.

2. $\mathcal{H}^\lambda$ *is irreducible representation of* $\mathrm{GL}(\mathcal{H})$. $\square$

Schur-Weyl duality

For each Young diagram λ there exists a natural map

$$\begin{array}{ccc} \mathcal{H}^{\lambda} \otimes V^{\lambda} & \longrightarrow & \mathcal{H}^{\otimes n} \\ \varphi \otimes x & \mapsto & \varphi(x) \end{array},$$

which is compatible with the actions of $\mathrm{GL}(\mathcal{H})$ and S_n. It induces an isomorphism known as *Schur-Weyl Duality*.

Theorem 2.1.9 (Schur-Weyl Duality) *The representation* $\mathcal{H}^{\otimes n}$ *splits into irreducible representations of* $\mathrm{GL}(\mathcal{H}) \times S_n$ *as*

$$\mathcal{H}^{\otimes n} \cong \bigoplus_{|\lambda|=n} \mathcal{H}^{\lambda} \otimes V^{\lambda}. \quad \square \tag{2.1}$$

Maximal tori and Cartan subalgebras

A *Cartan subgroup* T of a compact connected Lie group G is a maximal connected abelian subgroup (also called a *maximal torus*). Its Lie algebra is called *Cartan subalgebra* $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$.

For a Cartan subgroup T of a Lie algebra G, consider the normalizer of T

$$N_G(T) := \{g \in G \mid g^{-1}tg \in T, \ \forall\, t \in T\},$$

which is the maximal normal subgroup containing T. The quotient group $W_G = N_G(T)/T$ is finite group and called the Weyl group of G.

Let V be a representation of a Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$ be its fixed Cartan subalgebra. A *weight space* $V_\alpha \subset V$ of *weight* $\alpha \in \mathfrak{h}^*$ is defined by

$$V_\alpha := \{v \in V; \forall h \in \mathfrak{h} \quad h \cdot v = \alpha(h)v\}.$$

Similarly, we can define a weight space V_α for representation of a Lie group (resp. an associative algebra) as the subspace of eigenvectors of some maximal commutative subgroup (resp. subalgebra) of the eigenvalue α. Elements of the weight spaces are called *weight vectors*.

As an example, consider the group $G = \mathrm{Gl}_n(\mathbb{C})$ of invertible $n \times n$ complex matrices with a representation $G : V$. The diagonal subgroup of G

$$T = \{\mathrm{diag}(z_1, z_2, \ldots, z_n) \mid z_i \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}\}$$

is the Cartan subgroup of G. Clearly, $T \simeq \mathbb{C}^* \times \mathbb{C}^* \times \ldots \times \mathbb{C}^*$. Since T is abelian, the reduced representation $T : V$ splits into 1-dimensional components

$$V = \bigoplus_i V_i \ ; \ \dim V_i = 1$$

and since $\dim V_i = 1$, T acts on V_i (hence on V) as multiplication by scalars, i.e., $\forall t \in T$, $t : v \mapsto \chi(t)v$, $\forall v \in V$, where $\chi(t) \in \mathbb{C}^*$ and

$$\chi(t_1 t_2) = \chi(t_1).\chi(t_2), \qquad \forall t_1, t_2 \in T.$$

The homomorphism, just defined, $\chi : T \to \mathbb{C}^*$ is the character of T, which is explicitly defined by the formulae $\chi(t) = z_1^{a_1} z_2^{a_2} \ldots z_n^{a_n}$; $a_i \in \mathbb{Z}$, $t = \mathrm{diag}(z_1, z_2, \ldots, z_n) \in T$.

Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\chi_\alpha : T \to \mathbb{C}^*$ be the corresponding character of T. Then the n-tuple α is a weight of $G : V$ with the weight space $V_\alpha = \{x \in V \mid t \cdot x = \chi_\alpha(t) \cdot x\}$.

Note that, the set of weights of any representation of G is ordered lexicographically, i.e., $\alpha > \beta$ if the first nonzero $a_i - b_i$ is positive, where $\beta = (b_1, b_2, \ldots, b_n)$. The *highest weight* of $G : V$ is the maximal weight in lexicographical order. The corresponding weight vector is called *highest weight vector.* It follows from definition that, if α is a highest weight then $a_1 \geq a_2 \geq \ldots \geq a_n$.

The weights which occurs in the adjoint representation $G : \mathfrak{g}$ are called the *roots* of the Lie algebra and the corresponding subspaces $\mathfrak{g}_\alpha \subset \mathfrak{g}$ *root spaces.*

Positive roots and Weyl chambers

Let R be the set of roots of a Lie algebra $\mathfrak{g}$, and R^+ be the subset of R with the properties:

1. for each $\alpha \in R$, either $\alpha \in R^+$ or $-\alpha \in R^+$,
2. for any $\alpha, \beta \in R^+$ so that $\alpha + \beta$ is a root, then $\alpha + \beta \in R^+$.

Then the roots in R^+ is said to be *positive roots*. An element $\alpha \in R^+$ is called *simple* if it cannot be written as the sum of two positive roots. The set Δ of simple roots form a basis of an Euclidean space E [1] with the property that for any $\alpha \in R$ is a linear combination of elements of Δ with coefficients either all non-negative or all non-positive.

Now, let $\alpha^\perp$ be the hyperplane in E orthogonal to root α. Then all hyperplanes $\alpha^\perp$ defined by $\alpha \in R$ cuts the Euclidean space E into a finite number of open regions, called *Weyl chambers*. It is a fact that the Weyl group W_G acts transitively on the set of Weyl chambers. In particular, the number of Weyl chambers equals to the order of Weyl group W_G. Among the Weyl chambers one is special called *positive Weyl chamber*, and it is defined by the closed set

$$\mathfrak{h}^*_+ = \{u \in E \ : \ \langle u, \alpha \rangle \geq 0 \ \text{ for all } \ \alpha \in R^+\}.$$

A weight of a representation $G : V$ of the Lie group G which lies inside the positive Weyl chamber is called *dominant weight*.

2.1.4 Flag varieties and Schubert cocycles

We begin by introducing the notion of Borel subalgebras and Borel subgroups. First, note that a choice of Cartan subalgebra $\mathfrak{h}$ in a semisimple Lie algebra $\mathfrak{g}$ determines a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is a root space corresponding to root α. For each choice of positive roots R^+, we can associate a subalgebra

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha,$$

which is called a *Borel subalgebra*.

---

[1] In fact E is the space $i\mathfrak{h}$, where $\mathfrak{h}$ is Cartan subalgebra of $\mathfrak{g}$, with the inner product $\langle , \rangle$ defined by so called Killing form: $\langle x, y \rangle = \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}x \cdot \mathrm{ad}y)$.

If G is a Lie group with a semisimple Lie algebra $\mathfrak{g}$, the connected subgroup B of G with Lie algebra $\mathfrak{b}$ is called *Borel subgroup*. For example, in the group $GL_n(\mathbb{C})$, the subgroup of upper triangular matrices is a Borel subgroup.

Let now G be a Lie group with Borel subgroup G. The subgroup P of G satisfying $B \subset P \subset G$ is called *parabolic subgroup*, and its Lie algebra $\mathfrak{p}$ is called *parabolic subalgebra*.

For a Lie group G with a parabolic subgroup P, the homogenous space G/P form a variety which is called *generalized flag variety*. It is so called, because for the group $G = \mathrm{SL}_n(\mathbb{C}) = \{g \in \mathrm{GL}_n(\mathbb{C}) \ : \ \det g = 1\}$, and $P = B$ which is the group of all upper-triangular matrices in G, the quotient G/B corresponds to usual complete flag variety, i.e., the variety of all flags

$$G/B = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n\},$$

of subspaces with $\dim V_i = i$. Note that, the group $\mathrm{SL}_n(\mathbb{C})$ acts transitively on the complete flags $\{0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n \ : \ \dim V_i = i\}$ in $\mathbb{C}^n$ with stabilizer B. Hence, the homogeneous space G/B consists of all complete flags.

Given a semisimple Lie group G with a Borel subgroup B and a parabolic subgroup P, it is known that the homogeneous space G/P consists of finitely many B-orbits that may be parameterized by certain elements of the Weyl group W. The closure of the B-orbit associated to an element w of the Weyl group is called a *Schubert variety* in G/P, and its cohomology class σ_w is called *Schubert cocycle*. All Schubert cocycles form a basis for the cohomology ring $H^*(G/P)$.

2.2 Berenstein-Sjamaar theorem

Let M be a compact connected Lie group with Lie algebra $\mathfrak{m}$ and its dual coadjoint representation $\mathfrak{m}^*$. Let $\mathfrak{t} \subset \mathfrak{m}$ be a Cartan subalgebra and $\mathcal{O} \subset \mathfrak{m}^*$ be a coadjoint orbit of group M. The composition $\Delta : \mathcal{O} \hookrightarrow \mathfrak{m}^* \to \mathfrak{t}^*$ is called *moment map*. Here, $\mathfrak{t}^*$ stands for the dual representation of $\mathfrak{t}$. Kostant's theorem implies that the image of Δ is a convex polytope. It is spanned by W-orbit of some weight $\mu \in \mathfrak{t}^*$

which can be taken from a fixed positive Weyl chamber $\mathfrak{t}_+^*$, where $W = N(\mathfrak{t})/Z(\mathfrak{t})$ is the Weyl group of M. This gives a parameterization of coadjoint orbits $\mathcal{O}_\mu$ by dominant weights $\mu \in \mathfrak{t}_+^*$.

Example 2.2.1 Consider the unitary group $\mathrm{U}(n)$ and its Lie algebra $\mathfrak{u}(n)$ which consists of all Hermitian $n \times n$ matrices. Let's identify $\mathfrak{u}(n)$ with its dual using the invariant trace form $(A, B) = \mathrm{Tr}(AB)$. Then the (co)adjoint orbit $\mathcal{O}_\mu$ consists of all Hermitian matrices A of spectrum $\mu : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and the moment map $\Delta : \mathcal{O}_\mu \to \mathfrak{t}$ is given by orthogonal projection into Cartan subalgebra of diagonal matrices $\mathfrak{t}$. In this case, Kostant's theorem amounts to Horn's observation that the diagonal entries of Hermitian matrices of spectrum μ form a convex polytope with vertices $w\mu$ obtained from μ by permutations of coordinates μ_i. This is equivalent to the *majorization inequalities*

$$\begin{array}{rcl} d_1 & \leq & \mu_1 \\ d_1 + d_2 & \leq & \mu_1 + \mu_2 \\ d_1 + d_2 + d_3 & \leq & \mu_1 + \mu_2 + \mu_3 \\ \cdots & \cdots & \cdots \\ d_1 + d_2 + \cdots + d_n & = & \mu_1 + \mu_2 + \cdots + \mu_n \end{array} \tag{2.2}$$

for diagonal entries $d : d_1 \geq d_2 \geq \cdots \geq d_n$ of matrix A. We will use the notation $d \preceq \mu$ for these inequalities.

Let now, L be compact connected Lie subgroup of M with the inclusion $f : L \hookrightarrow M$. The inclusion f induces two morphisms $f_* : \mathfrak{l} \hookrightarrow \mathfrak{m}$ and $f^* : \mathfrak{m}^* \to \mathfrak{l}^*$ of Lie algebras and their duals. In [3] Berenstein and Sjamaar gave a decomposition of the projection $f^*(\mathcal{O}_\mu) \subset \mathfrak{l}^*$ of M-orbit $\mathcal{O}_\mu \subset \mathfrak{m}^*$ into L-orbits $\mathcal{O}_\lambda \subset \mathfrak{l}^*$. Below, we give their results in the form which is more suitable for our study.

Now, let us fix the Cartan subalgebras $\mathfrak{t}_L \hookrightarrow \mathfrak{t}_M$ of groups L, M, and for every *test spectrum* $a \in \mathfrak{t}_L$ consider the inclusion of the coadjoint orbits of groups L and M

$$\varphi_a : \mathcal{O}_a \hookrightarrow \mathcal{O}_{f_*(a)} \tag{2.3}$$

through a and $f_*(a)$ respectively. Topologically, the orbits correspond to the flag varieties:

$$\mathcal{O}_a = L^{\mathbb{C}}/P_a \tag{2.4}$$

where $P_a \subset L^{\mathbb{C}}$ is parabolic subgroup of complexified group $L^{\mathbb{C}}$ whose Lie algebra $\mathfrak{p}_a = \mathfrak{t}_L \oplus \bigoplus_{\alpha \in R'} \mathfrak{g}_\alpha$, where R' is the set of roots α such that $\langle \alpha, a \rangle \geq 0$.

The inclusion φ_a in (2.3) induces the morphism of cohomologies

$$\varphi_a^* : H^*(\mathcal{O}_{f_*(a)}) \to H^*(\mathcal{O}_a), \tag{2.5}$$

given in the bases of Schubert cocyles σ_w by coefficients $c_w^v(a)$ of the decomposition

$$\varphi_a^* : \sigma_w \mapsto \sum_v c_w^v(a)\sigma_v. \tag{2.6}$$

The coefficients $c_w^v(a)$ are of great significance to the next theorem which gives the main results of the paper [3] by Berenstein and Sjamaar written in the form which is more suitable for the intended applications. For its proof see [1].

Theorem 2.2.1 *In the above notations, the inclusion $\mathcal{O}_\lambda \subset f^*(\mathcal{O}_\mu)$ is equivalent to the following system of linear inequalities*

$$\langle \lambda, va \rangle \leq \langle \mu, w f_*(a) \rangle \tag{a, v, w}$$

for all $a \in \mathfrak{t}_L, v \in W_L, w \in W_M$ such that $c_w^v(a) \neq 0$. $\square$

Chapter 3

ν-Representability Problem

In this chapter we apply Theorem 2.2.1 to the morphism $f : \mathrm{U}(\mathcal{H}) \to \mathrm{U}(\mathcal{H}^\nu)$, where $\mathcal{H}^\nu$ is the irreducible representation of the unitary group $\mathrm{U}(\mathcal{H})$ with a Young diagram ν of order $N = |\nu|$. Recall that, if we take ν as a row diagram, then the corresponding irreducible representation $\mathcal{H}^\nu$ becomes $S^N\mathcal{H}$ which is the state space of N-boson system. On the other hand, if ν is a column diagram, then we get the irreducible representation $\wedge^N\mathcal{H}$ which describes the state space of N-fermion system. However, for the system of fermions with a spin, we need more general *para-statistics* representations $\mathcal{H}^\nu$. Note that, the state space of a single particle with spin splits into the tensor product $\mathcal{H} = \mathcal{H}_r \otimes \mathcal{H}_s$ of the orbital $\mathcal{H}_r$ and the spin $\mathcal{H}_s$ components. The state space of N-fermion system decomposes into spin-orbital components as follows [38]

$$\wedge^N \left(\mathcal{H}_r \otimes \mathcal{H}_s\right) = \sum_{|\nu|=N} \mathcal{H}_r^\nu \otimes \mathcal{H}_s^{\nu^t}, \tag{3.1}$$

where ν^t stands for the transpose diagram whose rows are the columns of ν. For the most physical systems, like electrons in an atom or a molecule, the total spin is a well defined quantity which singles out a specific component of this decomposition. In this setting, we concern the constraints on the spectra of the reduced states ρ_r^ν and $\rho_s^{\nu^t}$ of a pure state $|\psi\rangle \in \mathcal{H}_r^\nu \otimes \mathcal{H}_s^{\nu^t}$. From n°1.1.5, we have the isospectrality of ρ_r^ν and $\rho_s^{\nu^t}$: $\mathrm{Spec}\rho_r^\nu = \mathrm{Spec}\rho_s^{\nu^t}$. Hence, it is enough to find the constraints on ρ_r^ν.

3.1 Physical interpretation

Let's now consider the operator $X \in \mathfrak{u}(\mathcal{H})$ as an observable and treat a typical element of the dual space $\rho \in \mathfrak{u}(\mathcal{H})^*$ as a mixed state (For a while we ignore the positivity $\rho \geq 0$ and normalization condition $\mathrm{Tr}\rho = 1$). Then the expectation of X in state ρ gives a duality pairing

$$\langle X, \rho \rangle = \mathrm{Tr}_{\mathcal{H}}\, X\rho. \tag{3.2}$$

Now, we will explain the physical meaning of the projection $f^* : \mathfrak{u}(\mathcal{H}^\nu)^* \to \mathfrak{u}(\mathcal{H})^*$ which is uniquely determined by the equation

$$\langle f_*(X), \rho^\nu \rangle = \langle X, f^*(\rho^\nu) \rangle, \qquad X \in \mathfrak{u}(\mathcal{H}), \qquad \rho^\nu \in \mathfrak{u}(\mathcal{H}^\nu)^*,$$

where $f_* : \mathfrak{u}(\mathcal{H}) \to \mathfrak{u}(\mathcal{H}^\nu)$ is inclusion induced by f. In the above setting (3.2) this means that

$$\mathrm{Tr}_{\mathcal{H}^\nu}(X\rho^\nu) = \mathrm{Tr}_{\mathcal{H}}(Xf^*(\rho^\nu)), \qquad \forall X \in \mathfrak{u}(\mathcal{H}). \tag{3.3}$$

From the Schur-Weyl duality (Thm.2.1.9) we have

$$\mathcal{H}^{\otimes N} = \sum_{|\nu|=N} \mathcal{H}^\nu \otimes V^\nu, \tag{3.4}$$

where $\mathcal{H}^\nu$ and V^ν are irreducible representations, described in previous chapter, of $\mathrm{U}(\mathcal{H})$ and S_N respectively. One can treat $\mathcal{H}^{\otimes N}$ as a state space of N-particles. For identical particles all physical quantities should commute with S_N. Looking into the right hand side of (3.4) we see that such quantities are linear combinations of operators $\rho^\nu \otimes 1$ acting in the component $\mathcal{H}^\nu \otimes V^\nu$ and equal to zero elsewhere. In the case of a genuine mixed state ρ^ν, i.e. a nonnegative operator of trace 1, one can think the operator $(\rho^\nu \otimes 1)/\dim V^\nu$ as a mixed state of N identical particles obeying some para-statistics of type ν. Let $\rho_i : \mathcal{H}$ be its i-th *reduced state*. Since $\rho^\nu \otimes 1$ commutes with S_N, the reduced state $\rho = \rho_i$ is actually independent of i. However, sometimes we keep the index i just to indicate the tensor component where it operates.

Proposition 3.1.1 *In the above notations*

$$f^*(\rho^\nu) = N\rho. \tag{3.5}$$

Proof: We have to check that (3.5) fits the equation (3.3):

$$\mathrm{Tr}_{\mathcal{H}^\nu}(X\rho^\nu) = \mathrm{Tr}_{\mathcal{H}^\nu\otimes V^\nu} X\frac{\rho^\nu\otimes 1}{\dim V^\nu} = \mathrm{Tr}_{\mathcal{H}^{\otimes N}} X\frac{\rho^\nu\otimes 1}{\dim V^\nu} = \sum_i \mathrm{Tr}_{\mathcal{H}} X_i\rho_i = N\,\mathrm{Tr}_{\mathcal{H}} X\rho,$$

where X_i is a copy of X acting in the i-th component of $\mathcal{H}^{\otimes N}$, so that

$$\mathrm{Tr}_{\mathcal{H}^{\otimes N}} X_i\frac{\rho^\nu\otimes 1}{\dim V^\nu} = \mathrm{Tr}_{\mathcal{H}} X_i\rho_i$$

by definition (1.1) of reduced state. □

A general *ν-representability problem* deals with the relations between the spectrum μ of a mixed state ρ^ν and spectrum λ of its *particle density matrix* $N\rho$. The latter spectrum is known as the *occupation numbers* of the system in state ρ^ν. More precisely, the occupation numbers of *natural orbitals*. The natural orbitals are defined as eigenvectors of the particle density matrix.

3.2 General solution of the ν-representability problem

From now on, the lower index r denotes the dimension of the Hilbert space $\mathcal{H}_r$ known as the rank of the system. Note that the character of the representation $\mathcal{H}_r^\nu$, i.e. the trace of a diagonal operator

$$z = \mathrm{diag}(z_1, z_2, \ldots, z_r) \in \mathrm{U}(\mathcal{H}_r), \tag{3.6}$$

in some orthonormal basis e of $\mathcal{H}_r$, is given by *Schur's function* $S_\nu(z_1, z_2, \ldots, z_r)$. It has a purely combinatorial description in terms of the semistandard tableaux T of shape ν. Then the Schur function can be written as a sum of monomials $z^T = \prod_{i\in T} z_i$

$$S_\nu(z) = \sum_T z^T$$

corresponding to all semistandard tableaux T of shape ν. The monomials are actually the weights of representation $\mathcal{H}_r^\nu$, that is

$$z \cdot e_T = z^T e_T \tag{3.7}$$

for some basis e_T of $\mathcal{H}_r^\nu$ parameterized by the semistandard tableaux. Denote by $\mathfrak{t} \subset \mathfrak{u}(\mathcal{H}_r)$ and $\mathfrak{t}_\nu \subset \mathfrak{u}(\mathcal{H}_r^\nu)$ the Cartan subalgebras of real diagonal operators in the bases e and e_T respectively, so that the *differential* of the above group action $z : e_T \mapsto z^T e_T$ gives the morphism

$$f_* : \mathfrak{t} \to \mathfrak{t}_\nu, \qquad f_*(a) : e_T \mapsto a_T e_T, \tag{3.8}$$

where $a_T := \sum_{i \in T} a_i$. The orbits $\mathcal{O}_a$ and $\mathcal{O}_{f_*(a)}$ can be treated as flag varieties $\mathcal{F}_a(\mathcal{H}_r)$ and $\mathcal{F}_{a^\nu}(\mathcal{H}_r^\nu)$ consisting of Hermitian operators of spectra $a : a_1 \geq a_2 \geq \cdots \geq a_r$ and a^ν respectively. Here a^ν consists of the quantities a_T arranged in the non-increasing order

$$a^\nu := \{a_T \mid T = \text{semistandard tableau of shape } \nu\}^\downarrow. \tag{3.9}$$

Finally, we need the morphism

$$\varphi_a : \mathcal{F}_a(\mathcal{H}_r) \to \mathcal{F}_{a^\nu}(\mathcal{H}_r^\nu), \qquad X \mapsto f_*(X), \tag{3.10}$$

together with its cohomological version

$$\varphi_a^* : H^*(\mathcal{F}_{a^\nu}(\mathcal{H}_r^\nu)) \to H^*(\mathcal{F}_a(\mathcal{H}_r)), \tag{3.11}$$

given in the canonical bases by coefficients $c_w^v(a)$:

$$\varphi_a^* : \sigma_w \mapsto \sum_v c_w^v(a) \sigma_v. \tag{3.12}$$

Theorem 3.2.1 *In the above notations all constraints on the occupation numbers λ of the system $\mathcal{H}_r^\nu$ in a state ρ^ν of spectrum μ are given by the inequalities*

$$\sum_i a_i \lambda_{v(i)} \leq \sum_k a_k^\nu \mu_{w(k)} \tag{3.13}$$

for all test spectra a and permutations v, w such that $c_w^v(a) \neq 0$.

Proof: Follows from Proposition 3.1.1 and Theorem 2.2.1. Remember that the left action of a permutation on "places" is inverse to its right action on indices. As a result, the permutations v and w, acting on a and $f_*(a) = a^\nu$ in Theorem 2.2.1, move to the indices of λ and μ in the inequality (3.13). $\square$

Remark 3.2.1 The coefficient $c_w^v(a)$ depends only on the order in which quantities a_T appear in the spectrum a^ν. The order changes when the test spectrum a crosses a hyperplane

$$H_{T|T'} : \sum_{i\in T} a_i = \sum_{j\in T'} a_j.$$

The hyperplanes cut the set of all test spectra into a finite number of polyhedral cones called *cubicles*. For each cubicle one has to check the inequality (3.13) only for its *extremal edges*. As a result, the ν-representability amounts to a *finite system* of linear inequalities.

3.2.1 Topological nature of the coefficients $c_w^v(a)$

Note that the inequalities (3.13) are subject to the topological condition $c_w^v(a) \neq 0$. So one has to calculate the coefficients $c_w^v(a)$ in order to give Theorem 3.2.1 full strength. We borrow from [1] the following calculation of these coefficients.

Canonical generators

To proceed we first need an alternative description of the cohomology of flag variety $\mathcal{F}_a(\mathcal{H}_r)$ [4]. Recall that the latter is understood here as the set of Hermitian operators in $\mathcal{H}_r$ of given spectrum a. To avoid technicalities, we assume the spectrum to be simple, i.e., $a_1 > a_2 > \cdots > a_r$. Let $\mathcal{E}_i$ be the *eigenbundle* on $\mathcal{F}_a(\mathcal{H}_r)$ whose fiber at $X \in \mathcal{F}_a(\mathcal{H}_r)$ is the eigenspace of operator X with eigenvalue a_i. Their Chern classes $x_i = c_1(\mathcal{E}_i)$ generate the cohomology ring $H^*(\mathcal{F}_a(\mathcal{H}_r))$ and we refer to them as the *canonical generators*. The elementary symmetric functions $\sigma_i(x)$ of the canonical generators are the characteristic classes of the trivial bundle

$\mathcal{H}_r$ and thus vanish. This identifies the cohomology with the *ring of coinvariants*

$$H^*(\mathcal{F}_a(\mathcal{H}_r)) = \mathbb{Z}[x_1, x_2, \dots, x_r]/(\sigma_1, \sigma_2, \dots, \sigma_r). \tag{3.14}$$

This approach to the cohomology is more functorial and by that reason leads to an easy calculation of the morphism (3.11)

$$\varphi_a^* : H^*(\mathcal{F}_{a^\nu}(\mathcal{H}^\nu)) \to H^*(\mathcal{F}_a(\mathcal{H})).$$

Recall that the spectrum a^ν consists of the quantities $a_T = \sum_{i\in T} a_i$ arranged in decreasing order, where T runs over all semistandard tableaux of shape ν. We define $x_T = \sum_{i\in T} x_i$ in a similar way.

Proposition 3.2.1 *Let x_i and x_k^ν be the canonical generators of $H^*(\mathcal{F}_a(\mathcal{H}))$ and $H^*(\mathcal{F}_{a^\nu}(\mathcal{H}^\nu))$ respectively. Then*

$$\varphi_a^*(x_k^\nu) = x_T, \qquad \textit{when} \qquad a_k^\nu = a_T. \tag{3.15}$$

In other words, $\varphi_a^*(x_k^\nu)$ is obtained from a_k^ν by the substitution $a_i \mapsto x_i$.

Proof: The eigenbundle $\mathcal{E}_i$ is equivariant with respect to the adjoint action $X \mapsto uXu^*$ of the unitary group $\mathrm{U}(\mathcal{H})$. Therefore it is uniquely determined by the linear representation of the centralizer $D = Z(X)$ in a fixed fiber $\mathcal{E}_i(X)$ or by its character $\varepsilon_i : D \to \mathbb{S}^1 = \{z \in \mathbb{C}^* \mid |z| = 1\}$. In the eigenbasis e of the operator X the centralizer becomes a diagonal torus with typical element $z = \mathrm{diag}(z_1, z_2, \dots, z_r)$ and the character $\varepsilon_i : z \mapsto z_i$.

Let now $X^\nu = \varphi_a(X)$, $D^\nu = Z(X^\nu)$, and e_T be the weight basis of $\mathcal{H}^\nu$, introduced in the beginning of this section, parameterized by semistandard tableaux T of shape ν and arranged in the order of eigenvalues a^ν. Then the character of the pull back $\varphi_a^{-1}(\mathcal{E}_k^\nu)$ is just the weight $\prod_{i\in T}\varepsilon_i$ of the k-th vector e_T, where the tableau T is determined from the equation $a_k^\nu = a_T$, cf. (3.7). Thus $\varphi_a^{-1}(\mathcal{E}_k^\nu) = \bigotimes_{i\in T}\mathcal{E}_i$ and we finally get

$$\varphi_a^*(x_k^\nu) = \varphi_a^*(c_1(\mathcal{E}_k^\nu)) = c_1(\varphi_a^{-1}(\mathcal{E}_k^\nu)) = c_1(\bigotimes_{i\in T}\mathcal{E}_i) = \sum_{i\in T} x_i = x_T. \quad \square$$

Remark 3.2.2 Formula (3.15) may look ambiguous for a degenerate spectrum a, while in fact it is perfectly self-consistent. Indeed, consider a small perturbation $\tilde{a}$, resolving multiple components of a, and the natural projection

$$\pi : \mathcal{F}_{\tilde{a}}(\mathcal{H}) \to \mathcal{F}_a(\mathcal{H})$$

that maps $\widetilde{X} = \sum_i \tilde{a}_i|e_i\rangle\langle e_i|$ into $X = \sum_i a_i|e_i\rangle\langle e_i|$, where e_i is an orthonormal eigenbasis of $\widetilde{X}$. It is known [4] that π induces isomorphism

$$\pi^* : H^*(\mathcal{F}_a(\mathcal{H})) \simeq H^*(\mathcal{F}_{\tilde{a}}(\mathcal{H}))^{W(D)}, \tag{3.16}$$

where on the right hand side stands algebra of invariants with respect to permutations of the canonical generators $\tilde{x}_i$ with the same unperturbed eigenvalue $a_i = \alpha$. Such permutations form Weyl group $W(D)$ of the maximal torus $\widetilde{D} = Z(\widetilde{X})$ in $D = Z(X)$. For example, characteristic classes of the eigenbundle $\mathcal{E}_\alpha$ with multiple eigenvalue $\alpha = a_i$ correspond to elementary symmetric functions of the respective variables $\tilde{x}_i$.

Equation (3.15), as it stands, depends on a specific ordering of the unresolved spectral values a_i and a_k^ν. However, when φ_a^* applied to invariant elements with respect to the above Weyl group, the ambiguity vanishes.

Note also, that Schubert cocycle $\sigma_w \in H^*(\mathcal{F}_{\tilde{a}}(\mathcal{H}))$ is invariant with respect to $W(D)$ if and only if w is the shortest representative in its left coset modulo $W(D)$. Such cocycles form the canonical basis of cohomology $H^*(\mathcal{F}_a(\mathcal{H}))$.

Schubert polynomials

To calculate the coefficients $c_w^v(a)$ we have to return back to the Schubert cocycles σ_w and express them via the canonical generators x_i. This can be accomplished by the *divided difference operators*

$$\partial_i : f(x_1, x_2, \ldots, x_n) \mapsto \frac{f(\ldots, x_i, x_{i+1}, \ldots) - f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}} \tag{3.17}$$

as follows. Write a permutation $w \in S_n$ as a product of the minimal number of transpositions $s_i = (i, i+1)$

$$w = s_{i_1} s_{i_2} \cdots s_{i_\ell}. \tag{3.18}$$

The number of factors $\ell(w) = \#\{i < j \mid w(i) > w(j)\}$ is called the *length* of the permutation w. The product

$$\partial_w := \partial_{i_1}\partial_{i_2}\cdots\partial_{i_\ell}$$

is independent of the reduced decomposition and in terms of these operators the Schubert cocycle σ_w is given by the equation

$$\sigma_w = \partial_{w^{-1}w_0}(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}), \tag{3.19}$$

where $w_0 = (n, n-1, \ldots, 2, 1)$ is the unique permutation of the maximal length.

The right hand side of equation (3.19) makes sense for independent variables x_i and in this setting it is called *Schubert polynomial* $S_w(x_1, x_2, \ldots, x_n)$, $\deg S_w = \ell(w)$. They were first introduced by Lascoux and Schützenberger [23, 25] who studied them in a long series of papers. See [26] for further references and a concise exposition of the theory. We borrow from [23] the following table, in which x, y, z stand for x_1, x_2, x_3.

w	S_w	w	S_w	w	S_w	w	S_w
3210	x^3y^2z	2301	x^2y^2	2031	$x^2y + x^2z$	1203	xy
2310	x^2y^2z	3021	$x^3y + x^3z$	2103	x^2y	2013	x^2
3120	x^3yz	3102	x^3y	3012	x^3	0132	$x + y + z$
3201	x^3y^2	1230	xyz	0231	$xy + yz + zx$	0213	$x + y$
1320	$x^2yz + xy^2z$	0321	$x^2y + x^2z + xy^2$	0312	$x^2 + xy + y^2$	1023	x
2130	x^2yz	1302	$x^2y + xy^2$	1032	$x^2 + xy + xz$	0123	1

Extra variables $x_{n+1}, x_{n+2}, \ldots$ being added to (3.19) leave Schubert polynomials unaltered. By that reason they are usually treated as polynomials in an infinite ordered alphabet $x = (x_1, x_2, \ldots)$. With this understanding every homogeneous polynomial can be decomposed into Schubert components as follows

$$f(x) = \sum_{\ell(w)=\deg(f)} \partial_w f \cdot S_w(x).$$

Applying this to the polynomial

$$\varphi_a^*(S_w(x^\nu)) = S_w(\varphi_a^*(x^\nu)) = \sum_{\ell(v)=\ell(w)} c_w^v(a) \cdot S_v(x),$$

and using Proposition 3.2.1 we finally arrive at the following result.

Theorem 3.2.2 *For the ν-representability problem the coefficients of the decomposition $\varphi_a^*(\sigma_w) = \sum_v c_w^v(a)\sigma_v$ are given by the formula*

$$c_w^v(a) = \partial_v S_w(x^\nu)\mid_{x_k^\nu \mapsto x_T}, \tag{3.20}$$

where the tableau T is derived from equation $a_k^\nu = a_T$, and the operator ∂_v acts on the variables x_i, replacing x_k^ν via specialization $x_k^\nu \mapsto x_T = \sum_{i\in T} x_i$. $\square$

Take notice that this equation is independent of an ordering of the unresolved spectral values a_k^ν. Indeed, Schubert polynomial $S_w(x^\nu)$ is symmetric in the respective variables x_k^ν, *provided* that w is the minimal representative in its left coset modulo centralizer of the spectrum a^ν in the symmetric group. Only such permutations correspond to Schubert cocycles $\sigma_w \in H^*(\mathcal{F}_{a^\nu}(\mathcal{H}^\nu))$, cf. Remark 3.2.2.

3.3 Constraints on spin and orbital occupation numbers

Let's now consider a system of N fermions of smallest possible spin $s = 1/2$, $\dim \mathcal{H}_s = 2$. For such systems spin-orbital decomposition (3.1) contains only the terms

$$\mathcal{H}_r^\nu \otimes \mathcal{H}_s^{\nu^t} \tag{3.21}$$

with at most two-column diagram ν. Denote the lengths of the columns of ν by α and β such that $\alpha \geq \beta$. They are determined by equations

$$\alpha + \beta = N, \qquad \alpha - \beta = 2J, \tag{3.22}$$

where J is the total spin of the system, so that $\mathcal{H}_s^{\nu^t} = \mathcal{H}_J$ is just the spin J representation of the group $\mathrm{SU}(\mathcal{H}_s) = \mathrm{SU}(2)$.

A pure N-fermion state of the system with total spin J is given by a vector

$$|\psi\rangle \in \mathcal{H}_r^\nu \otimes \mathcal{H}_J,$$

where the diagram ν is determined by equations (3.22). Let $\rho^\nu : \mathcal{H}_r^\nu$ and $\rho^J : \mathcal{H}_J$ be the reduced states of $|\psi\rangle$ in the orbital and spin components, respectively.

It is given in $n^\circ 1.1.5$ that the reduced states are isospectral $\mathrm{Spec}\rho^\nu = \mathrm{Spec}\rho^J$. Hence $\mathrm{Spec}\,\rho^\nu$ can be identified with the *spin occupation numbers*, $\mathrm{Spec}\rho^J$. On the other hand, in Theorem 3.2.1 we give the relations between $\mathrm{Spec}\rho^\nu$ and the orbital occupation numbers which is given by the spectrum of the particle density matrix $N\rho$. As a result, if we know the solution of the ν-representability problem for two-column diagrams, then we can produce all constraints on spin and orbital occupation numbers.

Corollary 3.3.1 *All constraints on spin and orbital occupation numbers of N-electron system in a pure state of total spin J are given by the inequalities* (3.13), *applied to two column diagram ν determined by equations* (3.22), *and bounded to mixed states ρ^ν of rank not exceeding dimensionality $2J+1$ of the spin space.*$\Box$

3.4 Basic inequalities

Since φ_a^* is a ring homomorphism, it maps unit into unit $\varphi_a^*(1) = 1$, that is $c_w^v(a) = 1$ for identical permutations v, w. As a result we have the following *basic inequality* for free

$$\sum_i a_i\lambda_i \le \sum_k a_k^\nu \mu_k,$$

which holds for all test spectra a. Let's analyse it in details for a *pure state* $\rho^\nu = |\psi\rangle\langle\psi|$. For pure states the right hand side of the above inequality is maximal, and since the spectrum $\mu = (1, 0, 0, \ldots, 0)$ for pure states, the above inequality becomes

$$\sum_i a_i\lambda_i \le a_1^\nu = \max_T \sum_{i\in T} a_i = \sum_i a_i\nu_i, \tag{3.23}$$

where $\nu_1 \ge \nu_2 \ge \cdots \ge 0$ are rows of ν. The maximum in the right hand side is attained for the tableau T of shape ν whose i^{th}-row is filled by i.

The normalization $\sum_i \lambda_i = N = \sum_j \nu_j$ allows to shift the test spectra into the positive domain $a_1 \ge a_2 \ge \cdots \ge 0$, so that they became nonnegative linear combinations of the fundamental weights

$$\omega_k = (\underbrace{1,1,\ldots,1}_{k},0,0,\ldots,0). \qquad (3.24)$$

Hence it is enough to check (3.23) for $a = \omega_k$, which gives the *majorization inequality* $\lambda \preceq \nu$, cf. Example 2.2.1. The equality holds only for coherent states, that is, for highest weight vectors of the representation.

Proposition 3.4.1 *The occupation numbers of a coherent state* $|\psi\rangle \in \mathcal{H}^\nu$ *are equal to* ν.

Proof: Consider a decomposition of the complexified Lie algebra

$$\mathfrak{u}(\mathcal{H}) \otimes \mathbb{C} = \mathfrak{gl}(\mathcal{H}) = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+,$$

into a diagonal Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \otimes \mathbb{C}$ accompanied with lower- and upper-triangular nilpotent subalgebras $\mathfrak{n}_\mp$. By definition $\mathfrak{n}_+$ annihilates the highest vector $|\psi\rangle \in \mathcal{H}^\nu$ of weight ν. Hence $\langle\psi|X^\pm|\psi\rangle = \langle X^\mp\psi|\psi\rangle = 0$ for all $X^\pm \in \mathfrak{n}_\pm$. Then by equation (3.3)

$$\langle\psi|X^\pm|\psi\rangle = \mathrm{Tr}_{\mathcal{H}^\nu}(X^\pm|\psi\rangle\langle\psi|) = \mathrm{Tr}_{\mathcal{H}}(X^\pm f^*(|\psi\rangle\langle\psi|)) = 0, \qquad \forall X^\pm \in \mathfrak{n}_\pm.$$

This means that $\rho = f^*(|\psi\rangle\langle\psi|)$ is a diagonal matrix. On the other hand $t|\psi\rangle = \langle t,\nu\rangle|\psi\rangle$ for $t \in \mathfrak{t}$, hence as above

$$\langle t,\nu\rangle = \langle\psi|t|\psi\rangle = \mathrm{Tr}_{\mathcal{H}^\nu}(t|\psi\rangle\langle\psi|) = \mathrm{Tr}_{\mathcal{H}}(tf^*(|\psi\rangle\langle\psi|)) = \mathrm{Tr}_{\mathcal{H}}(t\rho) = \langle t,\rho\rangle,$$

that is $\mathrm{Spec}\,\rho = \nu$. □

Another important property of the occupation numbers is the following convexity property.

Proposition 3.4.2 *The set of allowed occupation numbers, written in any order, form a convex set.*

Proof: Let ρ_1^ν, ρ_2^ν be mixed states, with the particle densities ρ_1, ρ_2, and the occupation numbers λ_1, λ_2. Apply to ρ_1, ρ_1^ν a unitary rotation $\rho_1 \mapsto U\rho_1U^*$,

$\rho_1^\nu \mapsto U\rho_1^\nu U^*$ that transforms orthonormal eigenvectors of ρ_1 into that of ρ_2 in a prescribed order. The resulting new operators ρ_1, ρ_2 commute and have the original spectra λ_1, λ_2. Then the particle density matrix $\rho = p_1\rho_1 + p_2\rho_2$ of the convex combination $\rho^\nu = p_1\rho_1^\nu + p_2\rho_2^\nu$ has spectrum $\lambda = p_1\lambda_1 + p_2\lambda_2$. Here, the convex combination means that the coefficients p_1 and p_2 are positive numbers such that $p_1 + p_2 = 1$. □

From the above two propositions we have the following result which characterizes occupation numbers of the system $\mathcal{H}^\nu$ in an unspecified mixed state.

Theorem 3.4.1 *The occupation numbers of the system $\mathcal{H}^\nu$ in an arbitrary mixed state satisfy the majorization inequality*

$$\lambda \preceq \nu, \tag{3.25}$$

and any such λ can be realized as the occupation numbers of some mixed state of $\mathcal{H}^\nu$.

Proof: Indeed, the polytope given by the majorization inequality (3.25) is just a convex hull of vectors obtained from ν by permutations of coordinates, cf. Example 2.2.1. Hence by Prop. 3.4.1 and Prop. 3.4.2 it consists of legitimate occupation numbers. □

Remark 3.4.1 For a column diagram ν the majorization inequality $\lambda \preceq \nu$ amounts to the *Pauli exclusion principle* $\lambda_i \leq 1$. In general, we refer to it as the *Pauli constraint*. The second part of Theorem 3.4.1 extends Coleman's result [9] for $\wedge^N\mathcal{H}$.

Recall, that the above theorem solves the ν-representability problem for unspecified mixed states. We will see later that for pure states the answer in general is much more complicated. Nevertheless, there are surprisingly many systems for which the majorization inequality alone is sufficient for pure ν-representability. We will consider them in the next section.

3.5 Pure moment polytope

One of the most striking features of Theorem 3.2.1 is the linearity of the constraints (3.13). As a result, the set of all allowed spectra (λ, μ) form a convex polytope, called (noncommutative) *moment polytope*. The convexity still holds for any fixed $\mu = \mathrm{Spec}\rho^\nu$, and in particular for the occupation numbers λ of all *pure* states. We refer to the latter case as the *pure moment polytope*. It sits inside the positive Weyl chamber, and its multiple kaleidoscopic reflections in the walls of the chamber generally form a *nonconvex* rosette, consisting of all legitimate occupation numbers written in an arbitrary order. It can be convex only if all constraints on the occupation numbers are given by the majorization inequality $\lambda \preceq \nu$ alone. Here we describe a class of representations $\mathcal{H}^\nu$ with this property.

As an example consider a system of $N \geq 2$ bosons in which case ν is a row diagram and the majorization inequality imposes no constraints on occupation numbers λ. By Theorem 3.4.1 this means that every nonnegative spectrum λ of trace N represents occupation numbers of some *mixed* state. It turns out that for bosons one can easily find a *pure* state which does the job:

$$|\psi\rangle = \sum_i \sqrt{\lambda_i} e_i^{\otimes N} \in S^N\mathcal{H},$$

where e_i is an orthonormal basis of $\mathcal{H}$. This makes the bosonic N-representability problem meaningless.

A more interesting physical example constitutes the so-called *closed shell*, meaning a system of electrons of total spin zero. The corresponding diagram ν consists of two columns of equal length. We will see shortly that in this case the Pauli constraint $\lambda \leq 2$ shapes the pure moment polytope.

Observe that it is enough to construct pure states whose occupation numbers are generators of the cone cut out of the Weyl chamber by the majorization inequality $\lambda \preceq \nu$. Then the convexity does the rest.

Recall, that in the proof of Theorem 3.4.1 we have already identified ν with the occupation numbers of a *coherent state*. Due to the majorization inequality $\lambda \preceq \nu$, the entropy of its reduced state is minimal possible. By that reason coherent states are generally considered as closest to classical ones [32]. At the other extreme one finds the so called *completely entangled* states $\psi \in \mathcal{H}^\nu$ whose particle density matrix $\rho = f^*(|\psi\rangle\langle\psi|)$ is scalar and the reduced entropy is maximal [22]. By definition (3.3) we have $\mathrm{Tr}_{\mathcal{H}}(X\rho) = \mathrm{Tr}_{\mathcal{H}^\nu}(X|\psi\rangle\langle\psi|) = \langle\psi|X|\psi\rangle$, so that the completely entangled states can be described by equation

$$\langle\psi|X|\psi\rangle = 0, \qquad \forall\, X \in \mathfrak{su}(\mathcal{H}). \tag{3.26}$$

Let's call a system $\mathcal{H}_r^\nu$ *exceptional* if the $\mathrm{SU}(\mathcal{H}_r)$-representation $\mathcal{H}_r^\nu$ is equivalent to one of the following: $\mathcal{H}_r$, its dual $\mathcal{H}_r^*$, and, for odd rank r, $\wedge^2\mathcal{H}_r$, $\wedge^2\mathcal{H}_r^*$. The Young diagram ν of an exceptional system can be obtained from $r \times m$ rectangle by adding an extra column of length $1, r-1, 2, r-2$ respectively.

One readily realizes that the exceptional systems contain no completely entangled states, because reduced matrix of $|\psi\rangle \in \wedge^2\mathcal{H}_r$ has even rank.

Proposition 3.5.1 *In every non-exceptional system $\mathcal{H}^\nu$ there exists a completely entangled state.*

Proof: The result is actually well known, but in a different context. The entanglement equation (3.26) is nothing but the stationarity condition for the length of vector $\langle\psi|\psi\rangle$ with respect to action of the *complexified* group $\mathrm{SL}(\mathcal{H})$. It is known [37] that every stationary point is actually a minimum, and an $\mathrm{SL}(\mathcal{H})$-orbit contains a minimal vector if and only if the orbit is closed. As a result, we end up with the problem of existence of a nonzero closed orbit, or, what is the same, the existence of a nonconstant polynomial invariant. The proposition just reproduces a known answer to the latter question [37]. □

By admitting other simple Lie groups we find only two more exceptional representations: the standard representation of the symplectic group $\mathrm{Sp}(n)$ and a halfspinor representation of $\mathrm{Spin}(10)$.

Now we can solve the pure ν-representability problem for a wide class of systems, including the above mentioned closed shell.

Theorem 3.5.1 *Suppose that all columns of Young diagram ν are multiple, which means that every number in the sequence of columns lengths $\nu_1^t \geq \nu_2^t \geq \nu_3^t \geq \cdots$ appears at least twice. Then all constraints on the occupation numbers of the system $\mathcal{H}^\nu$ in a pure state are given by the majorization inequality $\lambda \preceq \nu$ alone.*

Proof: We'll proceed by induction on the height of the diagram ν. The triviality of the bosonic N-representability problem provides a starting point for the induction.

Let now λ be a vertex of the polytope cut out of the positive Weyl chamber by the majorization inequality $\lambda \preceq \nu$. Take notice that the latter includes equation $\operatorname{Tr}\lambda = \operatorname{Tr}\nu$. Then the following alternative holds:

1. *Either all nonzero components of λ are equal,*
2. *Or one can split λ and ν into two parts $\lambda = \lambda'|\lambda''$, $\nu = \nu'|\nu''$ containing the first p components and the remaining ones, both satisfying the inequalities $\lambda' \preceq \nu'$, $\lambda'' \preceq \nu''$.*

Indeed, the second claim just tells that the p^{th} majorization inequality in (2.2) turns into equality. On the other hand, if all the majorization inequalities are strict, and λ contains different nonzero entries, then one can linearly vary these entries preserving the non-increasing order of λ and the majorization $\lambda \preceq \nu$. As a result we get a line segment in the polytope containing λ, which is impossible for a vertex.

We've to prove that every vertex λ represents occupation numbers of some pure state. Consider the above two cases separately.

Case 1 Let λ contains r equal nonzero entries and $\mathcal{H}_r \subset \mathcal{H}$ be a subspace of dimension r. The conditions of the theorem ensure that the system $\mathcal{H}_r^\nu$ is non-exceptional, hence by Proposition 3.5.1 it contains a state $\psi \in \mathcal{H}_r^\nu$ with occupation numbers equal to nonzero part of λ. In bigger system $\mathcal{H}^\nu \supset \mathcal{H}_r^\nu$ its occupation numbers will be extended by zeros.

Case 2 Let the system has rank $r = p+q$. Choose a decomposition $\mathcal{H}_r = \mathcal{H}_p \oplus \mathcal{H}_q$ and consider a restriction of the representation $\mathcal{H}_r^\nu$ onto subgroup $\mathrm{U}(\mathcal{H}_p) \times \mathrm{U}(\mathcal{H}_q)$

$$\mathcal{H}_r^\nu = \sum_{\mu,\pi} c_{\mu\pi}^\nu \mathcal{H}_p^\mu \otimes \mathcal{H}_q^\pi, \tag{3.27}$$

where $c_{\mu\pi}^\nu$ are the omnipresent Littlewood-Richardson coefficients. Observe that $c_{\nu'\nu''}^\nu = 1$, and therefore $\mathcal{H}_p^{\nu'} \otimes \mathcal{H}_q^{\nu''} \subset \mathcal{H}_r^\nu$. By induction hypothesis there exist states $|\psi'\rangle \in \mathcal{H}_p^{\nu'}$ and $|\psi''\rangle \in \mathcal{H}_q^{\nu''}$ with occupation numbers λ', λ'' and particle densities ρ', ρ'' respectively. Then decomposable state $|\psi\rangle = |\psi'\rangle \otimes |\psi''\rangle$ has particle density $\rho' \oplus \rho''$, and its occupation numbers are equal to $\lambda = \lambda' | \lambda''$. $\square$

Let's extract for a reference a useful corollary from the last part of the proof.

Corollary 3.5.1 *Suppose that the Littlewood-Richardson coefficient* $c_{\mu\pi}^\nu$ *is nonzero. Then merging of the occupation numbers* λ', λ'' *of the systems* $\mathcal{H}_p^\mu$, $\mathcal{H}_q^\pi$ *form legitimate occupation numbers of the system* $\mathcal{H}_{p+q}^\nu$. $\square$

Remark 3.5.1 The restriction on the column's multiplicities of the diagram ν is needed only to ensure that the components of any splitting $\nu = \nu'|\nu''|\nu'''|\ldots$ are non-exceptional. The latter condition holds for any two-row diagram $[\alpha,\beta], \beta \neq 1$ for $\dim \mathcal{H} \geq 3$. This gives examples of systems beyond Theorem 3.5.1, say for $\nu = [3,2]$, whose pure moment polytope is given by the majorization inequality alone. More such diagrams can be produced as follows: take ν as in Theorem 3.5.1 and remove one cell from its last row. This works when the last row contains at least three cells and rank of the system is bigger than the height of ν. A complete classification of all such systems is still missing.

3.6 Dadok-Kac construction

In the last two theorems we encounter the problem of construction a pure state with given occupation numbers. The problem lies at the very heart of the ν-representability and one shouldn't expect an easy solution. Nevertheless, there is a combinatorial construction that produces a state with *diagonal* density matrix, whose spectrum can be easily controlled. It has been used first by Borland and Dennis [5] to forecast the structure of the moment polytope for small fermionic systems. Later on Müller [29] formalized and advanced their approach to the limit. It fits into a general Dadok-Kac construction [11] that works for any representation.

Below we follow the notations introduced at the beginning of n° 3.2. Let $x = \mathrm{diag}(x_1, x_2, \ldots, x_r)$ be a typical element from Cartan subalgebra $\mathfrak{t} \subset \mathfrak{u}(\mathcal{H}_r)$. For a given semistandard tableau T call the linear form $\omega_T : x \mapsto x_T = \sum_{i \in T} x_i$ the *weight* of the basic vector $e_T \in \mathcal{H}_r^\nu$. We also need nonzero weights of the adjoint representation $\alpha_{ij} : x \mapsto x_i - x_j$, $i \neq j$ called *roots*. Let's turn the set of semistandard tableaux of shape ν into a graph by connecting T and T' each time $\omega_T - \omega_{T'}$ is a root, i.e. the contents of T and T', considered as multi-sets, differ by exactly one element.

Proposition 3.6.1 *Let* $\mathbf{T}$ *be a set of semistandard tableaux of shape* ν *containing no connected pairs. Then every state* $|\psi\rangle = \sum_{T \in \mathbf{T}} c_T e_T \in \mathcal{H}^\nu$ *with support* $\mathbf{T}$ *has a diagonal particle density matrix with entries*

$$\lambda_i = \sum_{T \ni i} |c_T|^2, \tag{3.28}$$

where every tableau T *is counted as many times as the index* i *appears in it.*

Proof: The proof refines the arguments used in Prop. 3.4.1, from which we borrow the notations. As in the Prop. 3.4.1 we have to prove $\langle\psi|X|\psi\rangle = 0$ for every $X \in \mathfrak{n}_+ + \mathfrak{n}_-$. It is enough to consider root vectors X_α that form a basis of $\mathfrak{n}_+ + \mathfrak{n}_-$. Then

$$\langle\psi|X_\alpha|\psi\rangle = \sum_{T, T' \in \mathbf{T}} \bar{c}_{T'} c_T \langle e_{T'}|X_\alpha|e_T\rangle.$$

Since $X_\alpha e_T$ has weight $\alpha + \omega_T$, it is orthogonal to $e_{T'}$, except $\omega_{T'} = \omega_T + \alpha$. The latter is impossible for $T, T' \in \mathbf{T}$, and therefore the reduced state of $|\psi\rangle$ is diagonal. A straightforward calculation gives the diagonal entries (3.28). $\square$

Take notice that for a fixed support $\mathbf{T}$ the set of unordered spectra (3.28) form a convex polytope. It is not known when this approach gives the whole moment polytope. The smallest fermionic system where it fails is $\wedge^3\mathcal{H}_8$, see Chapter 6.

Chapter 4

Beyond The Basic Constraints

Here we use the results from previous chapter to derive some general inequalities for the pure ν-representability problem beyond the Pauli constraint $\lambda \preceq \nu$. We begin with a complete solution of the ν-representability problem for two-row diagrams. Then we return back to the original N-representability problem which seems to be the most difficult one.

4.1 Two-row diagrams

For two-row diagram $\nu = [\alpha, \beta]$ the majorization inequality $\lambda \preceq \nu$ just tells that $\lambda_1 \leq \alpha$. As we know, for $\beta \neq 1$ it shapes the whole moment polytope, see Remark 3.5.1 to Theorem 3.5.1. Here we will explain the remaining case $\nu = [N-1, 1]$, and thus solve the pure ν-representability problem for all two-row diagrams. The result cannot be extended to three-row diagrams, nor even to three fermion systems, where the number of independent inequalities *increases* with the rank, see Corollary 4.3.1 below. For convenience and a future reference we collect in the next theorem all known facts.

Theorem 4.1.1 *For a system $\mathcal{H}_r^\nu$ of rank $r \geq 3$ with two-row diagram $\nu = [\alpha, \beta]$, $\alpha + \beta = N$ all constraints on the occupation numbers of a pure state are given by* *the following conditions*

1. *Basic inequality* $\lambda_1 \leq \alpha$ *for* $\beta \neq 1$.
2. *Inequality* $\lambda_1 - \lambda_2 \leq N - 2$ *for* $\nu = [N-1, 1]$, $N > 3$.
3. *Inequalities* $\lambda_1 - \lambda_2 \leq 1$, $\lambda_2 - \lambda_3 \leq 1$ *for* $\nu = [2, 1]$.
4. *Even degeneracy* $\lambda_{2i-1} = \lambda_{2i}$ *for* $\nu = [1, 1]$.

Proof: We have already addressed the cases 1 and 4 in Remark 3.5.1 and Introduction respectively.

Case 2: Necessity. To prove the inequality $\lambda_1 - \lambda_2 \leq N - 2$ we have to put it into the form of Theorem 3.2.1

$$\sum_i a_i \lambda_{v(i)} \leq \sum_k a_k^\nu \mu_{w(k)}. \tag{4.1}$$

This suggests the test spectrum $a = (1, 0, 0, \ldots, 0, -1)$ and the shortest permutation v that transforms it into $(1, -1, 0, 0, \ldots, 0)$, which is the cyclic one $v = (2, 3, 4, \ldots, r)$. Thus we get the left hand side of the inequality. To interpret its right hand side $N - 2$, notice that the spectrum a^ν starts with the terms

$$a^\nu = (\underbrace{N-1, N-1, \ldots, N-1}_{r-2}, N-2, \ldots),$$

corresponding to semi-standard tableaux T with first row of ones and the indices $2, 3, \ldots, r$ filling the unique place in the second row. Since for pure state $\mu = (1, 0, 0, \ldots, 0)$, then the shortest permutation w that produces $N - 2$ in the right hand side of (4.1) is also cyclic $w = (1, 2, 3, \ldots, r-1)$. The corresponding Schubert polynomial is just the monomial

$$S_w(x^\nu) = x_1^\nu x_2^\nu \cdots x_{r-2}^\nu.$$

This is a special case of Grassmann permutations discussed in the next n° 4.2. Specialization $x_k^\nu \mapsto x_T$ of Theorem 3.2.2 transforms it into the product

$$P(x) = \prod_{i=2}^{r-1} [(N-1)x_1 + x_i].$$

Taking the reduced decomposition $v = s_2 s_3 \cdots s_{r-1}$ we infer

$$c_w^v(a) = \partial_v P(x) = \partial_2 \partial_3 \cdots \partial_{r-1} P(x).$$

The right hand side is a constant, and the operators ∂_i do not touch x_1. Hence we can put $x_1 = 0$, that gives

$$c_w^v(a) = \partial_2 \partial_3 \cdots \partial_{r-1}(x_2 x_3 \cdots x_{r-1}) = 1.$$

Since $c_w^v(a) \neq 0$, the inequality follows from Theorem 3.2.1.

Case 2: Sufficiency. By the convexity it is enough to construct *extremal states* whose occupation numbers are vertices of the polytope cut out from the Weyl chamber by the inequality $\lambda_1 - \lambda_2 \leq N - 2$ and the normalization $\operatorname{Tr} \lambda = N$. The vertices are given first of all by the fundamental weights normalized to trace N

$$\omega_k = (\underbrace{N/k, N/k, \ldots, N/k}_{k}, 0, 0, \ldots, 0)$$

that generate the edges of the Weyl chamber, except for ω_1 forbidden by the constraint $\lambda_1 - \lambda_2 \leq N - 2$. The latter is replaced by the intersections τ_k of segments $[\omega_1, \omega_k]$ with the hyperplane $\lambda_1 - \lambda_2 = N - 2$

$$\tau_k = (\underbrace{N - 2 + 2/k, 2/k, \ldots, 2/k}_{k}, 0, 0, \ldots, 0).$$

Here we tacitly assume that $N > 3$, since otherwise ω_2 would be also forbidden. The same condition ensures that the system $\mathcal{H}_k^\nu$ is non-exceptional for $k \geq 2$, hence ω_k are occupation numbers of some pure states by Proposition 3.5.1.

To deal with the remaining vertices τ_k we invoke the Dadok-Kac construction n° 3.6 and observe that the state

$$\psi_k = \begin{ytableau} 1 & k & k & \cdot & \cdot & \cdot & k \\ k \end{ytableau} + \frac{1}{\sqrt{2}} \sum_{2 \leq i < k} \begin{ytableau} i & i & k & \cdot & \cdot & \cdot & k \\ k \end{ytableau}$$

has a disconnected support and the occupation numbers τ_k, $k \geq 2$. Here for clarity we write tableau T instead of the weight vector e_T and skip an overall normalization factor.

Case 3. Here we only briefly sketch the proof that follows a similar scheme. The second inequality in the form $\lambda_2 - \lambda_3 \leq N - 2$ holds for all N, but it becomes redundant for $N > 3$. It can be deduced from Theorem 3.2.1 by calculation of the coefficient $c^v_w(a)$ for the same a and w as above, but with another permutation $v = (1,2)(3,4,\ldots,r)$. Then, keeping the notations of Case 2, we get

$$\begin{aligned} c^v_w(a) &= \partial_3\partial_4\cdots\partial_{r-1}\partial_1 P(x_1,x_2,\ldots,x_{r-1}) \\ &= \partial_3\partial_4\cdots\partial_{r-1}\frac{P(x_1,x_2,\ldots,x_{r-1}) - P(x_2,x_1,\ldots,x_{r-1})}{x_1 - x_2}. \end{aligned}$$

The operators ∂_k, $k \geq 3$ do not affect variables x_1, x_2. Therefore we can pass in the fraction to the limit $x_1, x_2 \to 0$ equal to $(N-2)x_3x_4\cdots x_{r-1}$, that gives $c^v_w(a) = N - 2 \neq 0$.

To prove sufficiency of the above inequalities we again have to look at the vertices of a polytope cut out of the Weyl chamber by the constraints $\lambda_1 - \lambda_2 \leq 1$, $\lambda_2 - \lambda_3 \leq 1$, $\operatorname{Tr}\lambda = 3$. This time, along with $\omega_k, k \geq 3$ and $\tau_k, k \geq 2$, there are vertices of another type

$$\eta_k = (\underbrace{1+1/k, 1+1/k, 1/k, 1/k, \ldots, 1/k}_{k}, 0, 0, \ldots, 0)$$

for $k \geq 3$. They represent occupation numbers of the following states with disconnected support

$$\psi_k = \sqrt{k+1}\,\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} + \sqrt{2}\,\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \sum_{3<i\leq k}\begin{array}{|c|c|}\hline 2 & i \\ \hline i \\ \cline{1-1}\end{array}. \qquad \square$$

Remark 4.1.1 Two-row diagrams naturally appear in description of bosonic systems, like photons where polarization plays rôle of spin. Representation with diagram $\begin{array}{|c|c|}\hline \, & \, \\ \hline \, \\ \cline{1-1}\end{array}$ can be applied both for bosons and fermions. In this case we calculated all constraints on the spin and orbital occupation numbers for small ranks, see n° 6.1. It appears that the constraints are stable and independent of the rank.

4.2 Grassmann inequalities

Here, we return back to the initial pure N-representability problem. Now, let us consider a special type of constraint on the occupation numbers of the system

$\wedge^N\mathcal{H}_r$ with 0/1 coefficients

$$\lambda_{i_1}+\lambda_{i_2}+\cdots+\lambda_{i_p}\le b. \tag{4.2}$$

We call them as *Grassmann inequality* (this terminology will be clear soon). For example, all constraints (1.7) for the system $\wedge^3\mathcal{H}_7$ are Grassmannian. We assume that the Grassmann inequality is *essential*, which means that it defines a facet of the moment polytope. Then it should fit into the form of Theorem 3.2.1 with

$$a=(\underbrace{1,1,\ldots,1}_{p},0,0,\ldots,0)$$

and the *Grassmann permutation* or *shuffle*

$$v=[i_1,i_2,\ldots,i_p,j_1,j_2,\ldots,j_q]:=[I,J], \tag{4.3}$$

where I and J are increasing sequences of lengths p and q so that $p+q=r$. This is the shortest permutation which produces the left hand side of inequality (4.2). Note that, our terminology comes from the observation that for the above test spectrum a the flag variety $\mathcal{F}_a(\mathcal{H})$ reduces to the *Grassmannian* $\mathrm{Gr}_p^q(\mathcal{H})$ which consists of all subspaces in $\mathcal{H}$ of dimension p and codimension q.

We can associate the Grassmann permutation $v=[I,J]$ with a Young diagram γ_I^J. Let Γ be a polygonal line connecting SW and NE corners of $p\times q$ rectangle, with k-th unit step running to the North for $k\in I$ and to the East for $k\in J$. Then the line Γ cuts out from the rectangle a Young diagram $\gamma=\gamma_I^J\subset p\times q$ located at its NW corner. We'll refer to I and J as the *vertical* and *horizontal* sequences of the diagram γ and denote the corresponding shuffle by $v_\gamma=[I,J]$. The length of the shuffle v_γ is equal to the size $|\gamma|$ of the diagram γ and its Schubert polynomial reduces to the much better understood Schur function

$$S_{v_\gamma}(x)=S_\gamma(x_1,x_2,\ldots,x_p).$$

Observe that $\gamma_{p-k+1}=i_k-k$, and the size of the Young diagram γ related to its vertical sequence by the equation

$$|\gamma|=\sum_{1\le k\le p}(i_k-k). \tag{4.4}$$

To get the strongest inequality (4.2) we chose w to be cyclic permutation

$$w = (1, 2, \dots, \ell + 1) = [2, 3, \dots, \ell + 1, 1, \ell + 2, \ell + 3 \dots, r]$$

of length $\ell = \ell(v) = |\gamma|$ for which the right hand side $b = (\wedge^N a)_{\ell+1}$ of (4.1) is minimal and equal to $(\ell + 1)^{\text{th}}$ term of the non-increasing sequence

$$\wedge^N a = \{a_K := a_{k_1} + a_{k_2} + \cdots + a_{k_N} \mid 1 \leq k_1 < k_2 < \cdots < k_N \leq r\}^{\downarrow}.$$

The sequence consists of nonnegative numbers m each taken with multiplicity

$$\binom{p}{m}\binom{q}{N-m}.$$

Recall that w also should be the minimal representative in its left coset modulo stabilizer of $\wedge^N a$. For the cyclic permutation this amounts to the inequality $(\wedge^N a)_\ell > (\wedge^N a)_{\ell+1} = b$, which tells that the first ℓ terms of $\wedge^N a$ contain all the components bigger than b. The number of such terms is bounded by the inequality

$$\sum_{m>b} \binom{p}{m}\binom{q}{N-m} = \ell = |\gamma| \leq pq. \tag{4.5}$$

To avoid sporadic constraints, assume that the inequality we are looking for is *stable*, i.e. remains valid for arbitrary big rank r. Then the left hand side should be linear in $q = r - p$ and the sum contains at most two terms: $m = N$ and $m = N - 1$. Thus we end up with two possibilities

1. $b = N - 2$, $p = N - 1$, $\ell = r - p$, which gives the inequality

   $$\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_{N-1}} \leq N - 2, \tag{4.6}$$

   with $\sum_k (i_k - k) = r - p$.

2. $b = N - 1$, $p \geq N$, $\ell = \binom{p}{N}$, which gives the inequality

   $$\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_p} \leq N - 1, \tag{4.7}$$

   with $\sum_k (i_k - k) = \binom{p}{N}$.

We will refer to them as the Grassmann inequalities of the first and second kind respectively. For the inequalities of the first kind the sum $\sum_k (i_k - k) = r - p$ increases with the rank, and therefore some of the involved occupation numbers should move away from the head of the spectrum. In contrast, the constraints of the second kind deal only with a few leading occupation numbers that are independent of the rank. We analyze them below for $p = N+1$ and postpone the first kind to the next section. The final result is that these inequalities actually hold true with very few exceptions.

The cyclic permutation w is a special type of shuffle with column Young diagram of height ℓ. The corresponding Schur function is just the monomial

$$S_w(y) = y_1 y_2 \dots y_\ell.$$

Applying to S_w the specialization of Theorem 3.2.2 we arrive at the product

$$P(x) = \prod_{1\le k_1<k_2<\cdots<k_N\le p} (x_{k_1} + x_{k_2} + \cdots + x_{k_N}) = \sum_\gamma c_\gamma S_\gamma(x_1, x_2, \dots, x_p). \quad (4.8)$$

Being symmetric, it can be expressed via Schur functions and, by Theorem 3.2.1, each time $S_\gamma(x)$ enters into the decomposition with nonzero coefficient $c_\gamma \neq 0$ we get inequality

$$\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_p} \le N - 1, \quad (4.9)$$

where $i_1 < i_2 < \cdots < i_p$ is the vertical sequence of Young diagram $\gamma \subset p \times q$, $|\gamma| = \binom{p}{N}$.

The product $P(x)$ represents the top Chern class of the exterior power $\wedge^N \mathcal{E}_p$ of the tautological bundle $\mathcal{E}_p$ on Grassmannian Gr_p^q and the decomposition (4.8) has been discussed in this context [24]. However, known results are very limited.

Example 4.2.1 For $N = 2$ and any $p \ge N$ the product

$$P(x) = \prod_{1\le i<j\le p} (x_i + x_j) = S_\delta(x_1, x_2, \dots, x_p)$$

is just Schur function with triangular Young diagram $\delta = [p-1, p-2, \dots, 0]$, see [27]. This gives for two fermion system $\wedge^2\mathcal{H}$ the inequality

$$\lambda_1 + \lambda_3 + \lambda_5 + \lambda_7 \cdots \le 1, \quad (4.10)$$

that, due to the normalization $\sum_i \lambda_i = 2$, degenerates into equality and implies even degeneracy $\lambda_{2i-1} = \lambda_{2i}$ of the occupation numbers.

On the other hand, for arbitrary N and minimal value $p = N$ we get

$$P(x) = x_1 + x_2 + \cdots + x_N = S_{\square}(x).$$

The vertical sequence of the one-box diagram $\square$ gives a nontrivial inequality

$$\lambda_1 + \lambda_2 + \cdots + \lambda_{N-1} + \lambda_{N+1} \leq N - 1 \tag{4.11}$$

that forces N-th electron into N-th orbital, when the preceding orbitals are fully occupied. We improve it below.

To the rest of this section we focus upon the next case $p = N+1$ that provides an infinite series of inequalities. Observe that in this setting a row diagram γ of length $N + 1 = \binom{p}{N}$ produces a *false* inequality

$$\lambda_1 + \lambda_2 + \cdots + \lambda_N + \lambda_{2N+2} \leq N - 1, \quad (?) \tag{4.12}$$

that fails for a coherent state given by one Slater determinant $e_1 \wedge e_2 \wedge \ldots \wedge e_N$. Similarly, the column inequality

$$\lambda_2 + \lambda_3 + \ldots + \lambda_{N+2} \leq N - 1 \qquad\qquad (?) \tag{4.13}$$

fails for even N. Indeed, in this case the system $\wedge^N \mathcal{H}_{N+2} \subset \wedge^N \mathcal{H}_r$ is non-exceptional and hence, by Proposition 3.5.1, the spectrum

$$\lambda = \frac{1}{N+2}(\underbrace{N, N, \ldots, N}_{N+2}, 0, 0 \ldots, 0)$$

represents legitimate occupation numbers violating the inequality.

Quite unexpectedly, all the other diagrams produce a valid constraint. In plain language the result can be stated as follows.

Theorem 4.2.1 *The occupation numbers of N-fermion system $\wedge^N \mathcal{H}$ in a pure state satisfy the following constraint*

$$\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_{N+1}} \leq N - 1$$

each time $\sum_k (i_k - k) = N+1$, *except for inequality* (4.12) *and, for even* N, *inequality* (4.13)*.*

Proof: For $p = N+1$ the decomposition (4.8) takes the form

$$\begin{aligned} P(x) &= \prod_{1\le i\le N+1} (x_1 + x_2 + \cdots + \widehat{x_i} + \cdots + x_{N+1}) = \prod_{1\le i\le N+1} (\sigma_1 - x_i) \\ &= \sum_{0\le k\le N+1} (-1)^k \sigma_1^{N+1-k} \sigma_k = \sum_\gamma c_\gamma S_\gamma(x_1, x_2, \ldots, x_{N+1}), \end{aligned}$$

where $\sigma_k(x) = S_{[1^k]}(x)$ are elementary symmetric functions, or what is the same Schur functions for the column diagram $[1^k]$.

For Young diagrams $\tau \subset \gamma$ denote by $t(\gamma/\tau)$ the number of standard tableaux of skew shape γ/τ. Then

$$c_\gamma = \sum_{k\ge 0} (-1)^k t(\gamma/[1^k]). \tag{4.14}$$

Indeed, the coefficient at S_γ in $\sigma_1^{N+1-k}\sigma_k = S_{[1]}^{N+1-k} S_{[1^k]}$ is equal to the number of ways to build γ from the column diagram $[1^k]$ by adding cells one at a time. Numbering the cells in the order of their appearance gives a standard tableaux of shape $\gamma/[1^k]$ that encodes the whole building process. Thus the coefficient is $t(\gamma/[1^k])$ and the equation (4.14) follows.

For a column diagram γ we infer from the last equation

$$c_\gamma = \sum_{k=0}^{N+1} (-1)^k = \begin{cases} 0, & N \equiv 0 \mod 2, \\ 1, & N \equiv 1 \mod 2. \end{cases}$$

Henceforth we assume that γ is not a column. Let's combine successive even and odd terms of the sum (4.14)

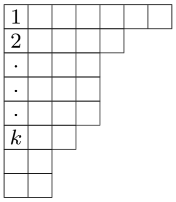
$$c_\gamma = \sum_{i\ge 0} [t(\gamma/[1^{2i}]) - t(\gamma/[1^{2i+1}])]. \tag{4.15}$$

We claim that

$$t(\gamma/[1^k]) - t(\gamma/[1^{k+1}]) = t(\gamma/[2, 1^{k-1}]), \tag{4.16}$$

where meaningless terms understood as zeros, e.g. the right hand side for $k = 0$.

Indeed, the building process can be described as an extension of the partially filled tableau



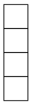
to a full standard tableau of shape γ. One can put the number $k+1$ either just below k or next to 1. For the first choice the number of ways to complete the tableau is $t(\gamma/[1^{k+1}])$, while for another one the number is $t(\gamma/[2,1^{k-1}])$. Hence $t(\gamma/[1^k]) = t(\gamma/[1^{k+1}]) + t(\gamma/[2,1^{k-1}])$.

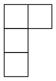
Combining the last two equations we arrive at the following representation of the coefficient c_γ as a sum of nonnegative terms

$$c_\gamma = \sum_{i>0} t(\gamma/[2,1^{2i-1}]). \tag{4.17}$$

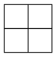
For a row diagram all terms vanish, while otherwise $t(\gamma/[2,1]) \neq 0$. Hence $c_\gamma > 0$ if the diagram is neither a row nor a column. The result now follows from Theorem 3.2.1. □

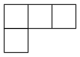
Example 4.2.2 For $N = 3$ the theorem gives four inequalities listed below together with the corresponding diagrams


: $\lambda_2+\lambda_3+\lambda_4+\lambda_5 \le 2,$


: $\lambda_1+\lambda_3+\lambda_4+\lambda_6 \le 2,$

(4.18)


: $\lambda_1+\lambda_2+\lambda_5+\lambda_6 \le 2,$


: $\lambda_1+\lambda_2+\lambda_4+\lambda_7 \le 2.$

They are valid for arbitrary rank r and give all constraints on the occupation numbers for $r \le 7$.

Observe also an improved version of the inequality (4.11)

$$\lambda_1+\lambda_2+\cdots+\lambda_{N-1}+\lambda_{N+1}+\lambda_{2N+1} \le N-1, \tag{4.19}$$

coming from the diagram $[N,1]$, and another inequality

$$\lambda_2+\lambda_3+\cdots+\lambda_{N+2}\le N-1,$$

originated from a column diagram and valid only for *odd* N.

Remark 4.2.1 We have considered above only Grassmann inequalities of the lowest levels $p=N,N+1$. The higher levels provide further improvements. For example, the inequalities (4.11) and (4.19) are just the first terms of an infinite series corresponding to increasing values of p

$$\lambda_{i_1}+\lambda_{i_2}+\lambda_{i_3}+\cdots+\lambda_{i_p}\le N-1, \tag{4.20}$$

where $i_k=k+\binom{k-1}{N-1}$. For $N=2$ this gives the inequality (4.10) and the double degeneracy of the occupation numbers, while for $N=3$ we get the inequality

$$\lambda_1+\lambda_2+\lambda_4+\lambda_7+\lambda_{11}+\lambda_{16}+\cdots\le 2,$$

where the differences between the successive indices are natural numbers $1,2,3,4,\ldots$. The details will not be discussed here.

4.3 Grassmann inequalities of the first kind

Formally we have such an inequality

$$\lambda_{i_1}+\lambda_{i_2}+\cdots+\lambda_{i_{N-1}}\le N-2 \tag{4.21}$$

each time the Schur function $S_\gamma=S_{v_\gamma}$ enters into the decomposition

$$P(x)=\prod_{N\le j\le r}(x_1+x_2+\cdots+x_{N-1}+x_j)=\sum_{\ell(v)=\ell}c_vS_v(x). \tag{4.22}$$

Here γ is a Young diagram of size $\ell=r-N+1$ with the vertical sequence formed by the indices in the above inequality, and v_γ is the corresponding shuffle. In contrast to the previous case, the product is *not* a symmetric function and its decomposition into Schubert polynomials is a challenge.

Let's try a simple case of a row diagram that produces the inequality

$$\lambda_1+\lambda_2+\cdots+\lambda_{N-2}+\lambda_r\leq N-2. \tag{4.23}$$

A close look shows that it *fails* for odd $\ell=r-N+1=2m-1$ for the spectrum

$$\lambda=(\underbrace{1,1,\ldots,1}_{N-2},\underbrace{1/m,1/m,\ldots,1/m}_{2m})$$

obtained by merging of the occupation numbers of the systems $\wedge^{N-2}\mathcal{H}_{N-2}$ and $\wedge^2\mathcal{H}_{2m}$, see Corollary 3.5.1 of Theorem 3.5.1. Nevertheless

Proposition 4.3.1 *The inequality* (4.23) *holds for even* $\ell=r-N+1$. *In this case the Schur function with a row diagram enters into the decomposition* (4.22) *with unit coefficient.*

Proof: The row diagram γ corresponds to the cyclic permutation

$$v=v_\gamma=(r,r-1,\ldots,N,N-1)=s_{r-1}s_{r-2}\cdots s_{N-1},$$

where $s_i=(i,i+1)$ are transpositions. We have to calculate the coefficient c_v of the decomposition (4.22) given by the equation

$$c_v=\partial_vP(x)=\partial_{r-1}\partial_{r-2}\cdots\partial_{N-1}P(x).$$

The operator ∂_v does not affect the variables $x_i, i<N-1$, so we can set them to zero and deal with the polynomial

$$P_0(x)=\prod_{N\leq i\leq r}(x_{N-1}+x_i)=\sum_{N\leq i_1<i_2<\cdots<i_k\leq r}x_{N-1}^{\ell-k}x_{i_1}x_{i_2}\cdots x_{i_k}.$$

We claim that

$$\partial_vx_{N-1}^{\ell-k}x_{i_1}x_{i_2}\cdots x_{i_k}=\begin{cases}(-1)^k & \text{for } i_s=r-k+s,\\ 0 & \text{otherwise.}\end{cases} \tag{4.24}$$

Let start with the second case $i_1\leq r-k=\ell+N-k-1$. In the following calculation we set to zero all variables that are not affected by the subsequent operators ∂_j. With this convention we get

$$\partial_{i_1-2}\partial_{i_1-3}\cdots\partial_{N-1}x_{N-1}^{\ell-k}x_{i_1}x_{i_2}\cdots x_{i_k}=x_{i_1-1}^{\ell+N-k-i_1}x_{i_1}x_{i_2}\cdots x_{i_k}. \tag{4.25}$$

The resulting monomial is divisible by s_{i_1-1}-invariant factor $x_{i_1-1}x_{i_1}$ that commutes with operator ∂_{i_1-1}. Hence everything vanishes in the next step as a result of the action ∂_{i_1-1} and setting $x_{i_1-1}=0$.

In the case $i_1 = r-k+1 = \ell+N-k$ the right hand side of (4.25) is just the product of the last k variables $x_{r-k+1}x_{r-k+2}\cdots x_r$ and application of the remaining operators ∂_j, $r-k \le j \le r-1$ gives $(-1)^k$.

Finally, from the equation (4.24) we infer

$$c_v = \sum_{0\le k\le \ell} (-1)^k = \begin{cases} 1, & \ell \text{ is even}, \\ 0, & \ell \text{ is odd}, \end{cases} \tag{4.26}$$

and the result follows from Theorem 3.2.1. □

Remark 4.3.1 The inequality (4.23) is most appealing for $N=3$

$$\lambda_1 + \lambda_r \le 1, \tag{4.27}$$

where it supersedes the Pauli principle $\lambda_1 \le 1$ for even r. M.B. Ruskai also conjectured inequality (4.27) in her analysis of three-fermion and three-hole systems [35].

Observe the following result, anticipated by many experts. It may appear not so trivial if compared with Theorems 3.5.1 and 4.1.1.

Corollary 4.3.1 *No finite set of inequalities gives all constraints on occupation numbers of N-fermion system $\wedge^N\mathcal{H}$, $N>1$ of arbitrary big rank.*

Proof: Indeed, a finite set Q of linear inequalities $L_\alpha(\lambda) \le b_\alpha$ includes only finitely many occupation numbers λ_i, $i<M$. Every inequality that follows from Q is a nonnegative combination of the inequalities from Q, the ordering conditions $\lambda_i - \lambda_{i-1} \le 0$, and a multiple of the normalization equation $\sum_{i=1}^r \lambda_i = N$.

Suppose now that the inequality of Proposition 4.3.1

$$\lambda_1 + \lambda_2 + \cdots + \lambda_{N-2} + \lambda_r \le N-2 \tag{4.28}$$

can be deduced from the system Q for some $r \gg M$ and even $\ell = r - N + 1$. The coefficients at λ_i in the left side for $i \geq M$ should come from the following linear combination with non-negative coefficients a_i

$$\begin{gathered} a_1(\lambda_2 - \lambda_1) + a_2(\lambda_3 - \lambda_2) + \cdots + a_{r-1}(\lambda_r - \lambda_{r-1}) - a_r\lambda_r = \\ -\lambda_1 a_1 + \lambda_2(a_1 - a_2) + \cdots + \lambda_{r-1}(a_{r-2} - a_{r-1}) + \lambda_r(a_{r-1} - a_r) \end{gathered}$$

amended with a multiple of the normalization equation. The Abel transformation shown in the second line implies that the coefficients a_i should form an arithmetical progression $a_i = ai + b$ for $M \leq i < r$, while $a_r = ar + b - 1 \geq 0$.

Suppose now that $a \geq 0$. Then the same combination of inequalities from Q that produces (4.28) and the same coefficients a_i for $i < r$ together with $a_r = ar + b \geq 0, \quad a_{r+1} = a(r+1) + b - 1 \geq 0$ would give a *false* inequality of rank $r+1$ obtained from (4.28) by replacing $r \mapsto r+1$. Recall that the inequality (4.28) *fails* for odd $\ell = r - N + 1$. For $a \leq 0$ a similar consideration gives a false inequality of rank $r - 1$.□

Proposition 4.3.1 can be extended to two-row diagrams $\gamma = [\ell - k, k]$. For three fermions this leads to the constraints

$$\lambda_{k+1} + \lambda_{r-k} \leq 1, \quad \text{for} \quad k + 1 < r - k, \tag{4.29}$$

that prohibit more than one electron to occupy *two* complementary orbitals. It holds both for even and odd r for $k > 0$. The corresponding coefficients $c_\gamma = c(\ell, k)$ of the decomposition (4.22) satisfy the recurrence relation $c(\ell, k) = c(\ell - 1, k) + c(\ell - 1, k - 1)$ and form the left half of the Pascal triangle

$$\begin{array}{ccccccccccccccccccc}
&&&&&&&&& 0 &&&&&&&&& \\
&&&&&&&& 1 && -1 &&&&&&&& \\
&&&&&&& 0 && 0 && 0 &&&&&&& \\
&&&&&& 1 && 0 && 0 && -1 &&&&&& \\
&&&&& 0 && 1 && 0 && -1 && 0 &&&&& \\
&&&& 1 && 1 && 1 && -1 && -1 && -1 &&&& \\
&&& 0 && 2 && 2 && 0 && -2 && -2 && 0 &&& \\
&& 1 && 2 && 4 && 2 && -2 && -4 && -2 && -1 && \\
& 0 && 3 && 6 && 6 && 0 && -6 && -6 && -3 && 0 & \\
1 && 3 && 9 && 12 && 6 && -6 && -12 && -9 && -3 && -1
\end{array}$$

with apex at $\ell = -1$, and 0/1 boundary condition for $k = 0$ set by equation (4.26). We return to the Pascal recurrence relation in a more general framework below, see equation (4.35).

Observe a zero in the forth line of the Pascal triangle, corresponding to diagram $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$. In general, a column diagram should have zero coefficient, because it produces inequality

$$\lambda_1 + \lambda_2 + \cdots + \widehat{\lambda_{N-\ell}} + \cdots + \lambda_N \leq N - 2 \qquad (?) \tag{4.30}$$

that *fails* for a coherent state given by one Slater determinant.

It turns out that the Grassmann inequality of the first kind (4.21) holds for all diagrams, except for a column and an odd row.

Theorem 4.3.1 *The occupation numbers of N-fermion system $\wedge^N \mathcal{H}_r$ in a pure state satisfy the following constraint*

$$\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_{N-1}} \leq N - 2 \tag{4.31}$$

each time $\sum_k (i_k - k) = r - N + 1$, except for inequality (4.30) *and, for odd $\ell = r - N + 1$, inequality* (4.23).

Proof: We've to show that Schur function $S_\gamma(x) = S_{v_\gamma}(x)$ enters into the decomposition

$$P_r(x) = \prod_{N \le j \le r} (x_1 + x_2 + \cdots + x_{N-1} + x_j) = \sum_{\ell(v)=\ell} c_v S_v(x), \tag{4.32}$$

provided that $\gamma \subset p \times q$ is neither a column nor an odd row. Here $p = N - 1$, $q = \ell = |\gamma| = r - p$.

Note first of all, that the coefficients of this decomposition are nonnegative for $v \in S_r$ and can be positive only for shuffles $v = v_\gamma$. The first claim holds in general for the coefficients $c_v^w(a)$ of Theorem 3.2.1

$$\varphi_a^*(\sigma_w) = \sum_v c_w^v(a)\sigma_v$$

since the cycle $\varphi_a^{-1}(\sigma_w) \subset \mathcal{F}_a(\mathcal{H}_r)$ is *effective*. Here v runs over representatives of minimal length in left coset modulo stabilizer of a. To include all permutations $v \in S_r$ one has to deal with a small perturbation $\tilde{a}$ that resolves multiple entries of a. However, since $\varphi_{\tilde{a}}^{-1}(\sigma_w) \subset \mathcal{F}_{\tilde{a}}(\mathcal{H}_r)$ is pull back of $\varphi_a^{-1}(\sigma_w) \subset \mathcal{F}_a(\mathcal{H}_r)$ via natural projection $\pi : \mathcal{F}_{\tilde{a}}(\mathcal{H}_r) \to \mathcal{F}_a(\mathcal{H}_r)$ defined in Remark 3.2.2, then decomposition of $\varphi_{\tilde{a}}^{-1}(\sigma_w)$ and $\varphi_a^{-1}(\sigma_w)$ involve the same Schubert cycles σ_v. This prove the second claim. Let's add as a warning, that the decomposition (4.32) actually *contains* Schubert polynomials S_v with permutations $v \notin S_r$.

The rest of the proof is purely algebraic. We'll proceed by induction on r keeping N fixed. For the first meaningful case $r = N + 1$, $\ell = 2$, as we know, only row diagram $\square\square$ appears in the decomposition.

Suppose now the induction hypothesis holds for $P_r(x)$, and consider the next polynomial

$$\begin{aligned} P_{r+1}(x) &= (x_1 + x_2 + \cdots + x_{N-1} + x_{r+1})P_r(x) \\ &= (x_1 + x_2 + \cdots + x_{N-1} + x_{r+1}) \sum_{\ell(v)=\ell} c_v S_v(x). \end{aligned} \tag{4.33}$$

We can find its Schubert components using a version of Monk's formula

$$(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \cdots)S_v(x) = \sum_{\ell(vt_{ij})=\ell(v)+1} (\alpha_i - \alpha_j) S_{vt_{ij}},$$

where $t_{ij} = (i, j)$, $i < j < \infty$ is a transposition, see [26, p. 86]. For a typical term of (4.33) this gives

$$\begin{aligned}(x_1 + x_2 + \quad \cdots \quad &+ x_{N-1} + x_{r+1})S_v \\ &= \sum_{1\le i<N\le j\ne r+1} S_{vt_{ij}} - \sum_{N\le j\ne r+1} \operatorname{sgn}(r+1-j) S_{vt_{j,r+1}}, \qquad (4.34)\end{aligned}$$

where the sums include only those transpositions t for which $\ell(vt) = \ell(v) + 1$. We are interested in the terms $u_\gamma = vt \in S_{r+1}$ that are shuffles coming from a Young diagram $\gamma \subset p \times (\ell + 1)$ of size $\ell + 1$. Let's single out the row diagram for which Proposition 4.3.1 gives the coefficient c_γ. The remaining shuffles u_γ do not move the last index $r + 1$, and therefore permutation $v = u_\gamma t_{i,j}$ has a bigger length than u_γ for $j \ge r + 1$. Hence a non-row Schur component S_γ in (4.34) comes from the sum

$$\sum_{1\le i<N\le j\le r} S_{vt_{ij}}$$

for $v = u_\gamma t_{ij}$, $\ell(v) = \ell(u_\gamma) - 1 = |\gamma| - 1$. Then $v \in S_r$, and $S_v(x)$ enters into decomposition (4.32) only for a shuffle $v = v_\tau$. In this case the relation $v_\tau = u_\gamma t_{ij}$ just means that τ is obtained from γ by removing a cell. As a result, we arrive at the recurrence relation

$$c_\gamma = \sum_{\gamma/\tau=\text{cell}} c_\tau, \qquad (4.35)$$

that holds for all *non-row* diagrams γ. This implies that $c_\gamma > 0$ if one can obtain an even row from γ by removing cells one at a time from a non-row diagram. This can be done for any diagram different from a column or an odd row. The inequality (4.31) now follows from Theorem 3.2.1. □

Example 4.3.1 *For four fermion system $\wedge^4\mathcal{H}_r$ the theorem gives inequality*

$$\lambda_i + \lambda_j + \lambda_k \le 2,$$

that holds for odd rank $r \ge 7$ and pairwise distinct indices satisfying equation $i+j+k = r+3$. For even r one has to exclude the row inequality $\lambda_1+\lambda_2+\lambda_r \le 2$.

For two-row diagrams equation (4.35) amounts to the Pascal recurrence relation discussed in Remark 4.3.1. In general, it allows to get an explicit formula for the coefficient c_γ that is surprisingly similar to the one given in the proof of Theorem 4.3.1, where we borrow the notations.

Corollary 4.3.2

$$c_\gamma = \sum_{k\geq 0}(-1)^k t(\gamma/[k]) = \sum_{i>0} t(\gamma/[2i,1]), \tag{4.36}$$

where the second equality holds for diagrams γ different from rows and columns.

Proof: Applying the equation (4.35) recurrently in conjunction with Proposition 4.3.1 we find out that c_γ is equal to the number of ways to obtain an even row from γ by removing cells one at a time from a non-row diagram. If γ is not a row or a column, then the last step in the process will be $[2i, 1] \mapsto [2i]$. Encoding the process by the standard tableaux, we arrived at the second formula. The first one follows from the identity $t(\gamma/[2i, 1]) = t(\gamma/[2i]) - t(\gamma/[2i + 1])$, cf. the proof of Theorem 4.3.1, and holds for all diagrams. □

Chapter 5

Connection With Representation Theory

The solution of ν-representability problem suggested by Theorem 3.2.1 is not feasible, except for very small systems. For example, for four fermions $\wedge^4\mathcal{H}_8$ we confront with an immense symmetric group of degree $\binom{8}{4} = 70$. Besides, listing of the extremal edges for systems of this size is all but impossible. A representation theoretical interpretation of the ν-representability discussed below allows to circumvent these difficulties.

Let's consider a composition of the Schur functors $\mathcal{H} \mapsto \mathcal{H}^\nu$ called a *plethysm*

$$[\mathcal{H}^\nu]^\mu = \sum_{|\lambda|=|\nu|\cdot|\mu|} m_\lambda^\mu \mathcal{H}^\lambda. \tag{5.1}$$

It splits into $\mathrm{U}(\mathcal{H})$ irreducible components $\mathcal{H}^\lambda$ of multiplicity m_λ^μ. It is instructive to treat the diagrams λ and μ as *spectra*. We are interested in their asymptotic behavior for $m_\lambda^\mu \neq 0$ and $|\mu| \to \infty$. Therefore we normalize them to a fixed size $\widetilde{\mu} = \mu/|\mu|$, $\widetilde{\lambda} = \lambda/|\mu|$, so that $\operatorname{Tr}\widetilde{\mu} = 1$ and $\operatorname{Tr}\widetilde{\lambda} = N = |\nu|$.

Theorem 5.0.2 *Every time $m_\lambda^\mu \neq 0$ the couple $(\widetilde{\lambda}, \widetilde{\mu})$ belongs to the moment polytope of the system $\mathcal{H}^\nu$, i.e. there exists its mixed state ρ^ν of spectrum $\widetilde{\mu}$, with occupation numbers $\widetilde{\lambda}$. Moreover every point of the moment polytope is a convex combination of such spectra $(\widetilde{\lambda}, \widetilde{\mu})$ of a bounded size $|\mu| \leq M < \infty$.*$\Box$

The theorem is a special case of Mumford's description of the moment polytope, see his appendix in [28]. It also holds in more general Berenstein-Sjamaar settings [3]. A similar result holds for general quantum marginal problem [13, 6, 20, 7].

5.1 Practical algorithm

For a fixed M the convex hull of the spectra $(\widetilde{\lambda}, \widetilde{\mu})$ from Theorem 5.0.2 gives an inner approximation to the moment polytope, while any set of inequalities of Theorem 3.2.1 amounts to its outer approximation. This suggests the following approach to the mixed ν-representability problem, which combines both theorems.

1. Find all irreducible components $\mathcal{H}^\lambda \subset [\mathcal{H}^\nu]^\mu$ for $|\mu| \leq M$.
2. Calculate the convex hull of the corresponding spectra $(\widetilde{\lambda}, \widetilde{\mu})$ that gives an inner approximation $\mathcal{P}_M^{\text{in}} \subset \mathcal{P}$ for the moment polytope $\mathcal{P}$.
3. Identify the facets of $\mathcal{P}_M^{\text{in}}$ that are given by the inequalities of Theorem 3.2.1. They cut out an outer approximation $\mathcal{P}_M^{\text{out}} \supset \mathcal{P}$.
4. Increase M and continue until $\mathcal{P}_M^{\text{in}} = \mathcal{P}_M^{\text{out}}$.

The algorithm became practical by generosity of the authors of `LiE` package [8], who made it publicly available. It allows to handle plethysms efficiently. We also benefit from `Convex` package by Franz [14], who apply a similar approach to the quantum marginal problem for three qutrits [13, 20].

One can incorporate in the algorithm additional constraints on spectrum of the mixed state ρ^ν. In many problems this is just a restriction on the rank $\operatorname{rk} \rho^\nu \leq p$, that bounds the number of rows of μ. For example, a pure state $\rho^\nu = |\psi\rangle\langle\psi|$ has rank one, the corresponding diagram $\mu = [m]$ reduces to a row, and the plethysm amounts to the symmetric power $S^m(\mathcal{H}^\nu)$. More generally, for spin-orbital occupation numbers of a system of electrons of total spin J, we have to deal with mixed states of rank $2J+1$, see Corollary 3.3.1 to Theorem 3.2.1, and respectively with the diagrams μ of at most that height.

5.2 Particle-hole duality

Here is another application of Theorem 5.0.2. Recall, that we arrived at the ν-representability problem from the spin-orbital decompositions (3.1) of n° 3. In this setting the Young diagram ν comes together with a rectangular frame $r\times s\supset\nu$, where r and s are dimensions of the orbital and spin spaces respectively. Let ν^* be the *complementary diagram* to ν in the frame $r\times s$, that is $\nu_i^*=s-\nu_{r+1-i}$. One can think about the representation $\mathcal{H}_r^{\nu^*}$ as describing the *holes* of the system $\mathcal{H}_r^\nu$. These are dual systems with a natural pairing $\mathcal{H}_r^\nu\otimes\mathcal{H}_r^{\nu^*}\to\mathcal{H}_r^{r\times s}=\det(\mathcal{H}_r)^{\otimes s}$, that can be extended to a pairing of the plethysms $[\mathcal{H}_r^\nu]^\mu\otimes[\mathcal{H}_r^{\nu^*}]^\mu\to\det(\mathcal{H}_r)^{\otimes sm}$, where $m=|\mu|$. The latter duality means that if $\mathcal{H}_r^\lambda$ is a component of $[\mathcal{H}_r^\nu]^\mu$, then $\mathcal{H}_r^{\lambda^*}$ is a component of $[\mathcal{H}_r^{\nu^*}]^\mu$ of the same multiplicity. Here λ^* is the complementary diagram to $\lambda\subset r\times sm$. In view of Theorem 5.0.2 this implies

Corollary 5.2.1 *The moment polytope of the hole system $\mathcal{H}_r^{\nu^*}$ is obtained from the moment polytope of $\mathcal{H}_r^\nu$ by the transformation $(\lambda,\mu)\mapsto(\lambda^*,\mu)$, where $\lambda_i^*=s-\lambda_{r+1-i}$.* $\square$

Chapter 6

Explicit Constraints For Some Small Systems

Here we give full constraints on the occupation numbers of some small systems. We start with a simple example of constraints on spin and orbital occupation numbers for a system of three electrons of total spin $J = 1/2$. It is equivalent to mixed ν-representability problem with $\nu = \yng(2,1)$. Then we pass on to an example of mixed N-representability conditions for the system $\wedge^2\mathcal{H}_4$. Finally, we analyse the pure N-representability problem for systems of rank $r \leq 10$. Because of particle-hole duality (n° 5.2) and known solutions for 2-fermion systems, we restrict ourselves to the range $3 \leq N \leq r/2$.

In order to find full constraints on occupation numbers we first try the algorithm n° 5.1. Unfortunately, due to computer limitation, it can be accomplished only for very small systems. For the pure N-representability problem these are the systems($\wedge^3\mathcal{H}_6,\ \wedge^3\mathcal{H}_7\ \wedge^4\mathcal{H}_8$) for which Borland and Dennis foresaw 35 yeas ago [5]. To move further we use any tool available, from a clever guess to a numerical optimization. As a result we managed to get a complete list of the constraints for the systems of rank not exceeding 10. For $r \leq 8$ we provide a rigorous proof below. We also have a proof for system $\wedge^3\mathcal{H}_9$ based on other ideas, but we prefer to give them in our successive works. For the remaining cases the constraints are complete only *beyond a reasonable doubt.* To resolve the doubt one has to verify independently that the vertices of the constructed polytope are genuine occupation numbers. We did this using a variety of methods for most of the vertices, but some still evaded all the efforts. For the latter we resort to the numerical optimization to check that they indeed can be approached very closely within the moment polytope. The biggest system we treated $\wedge^5\mathcal{H}_{10}$ is bounded by 161 inequalities.

6.1 Spin and orbital occupation numbers

Here we consider a simple example of constraints on spin μ and orbital λ occupation numbers for a system of three electrons of the total spin $J = 1/2$. By Corollary 3.3.1 to Theorem 3.2.1 the problem is equivalent to mixed ν-representability for $\nu = (2,1) = \ydiagram{2,1}$ and $\mathrm{Spec}\rho^\nu = (\mu_1, \mu_2)$. A calculation based on the algorithm n° 5.1 shows that the constraints amount to 5 inequalities

$$\lambda_1 - \lambda_2 \leq 1 + \mu_2, \quad \lambda_2 - \lambda_3 \leq 1 + \mu_2, \quad \lambda_1 - \lambda_3 \leq 2 - \mu_2$$
$$\lambda_1 - \lambda_2 - \lambda_3 \leq 1, \quad 2\lambda_1 - \lambda_2 + \lambda_4 \leq 4 - \mu_2,$$

which apparently are independent of the rank. We test them for $r = 4, 5$. In Tables 6.1 and 6.2 one can find the corresponding permutations v, w (written in cycle decomposition) and the coefficients $c_v^w(a)$ in the setting of Theorem 3.2.1. Recall that λ and μ are arranged in the non-increasing order and are normalized to the traces 3 and 1 respectively.

For pure ν-representability, in which case the state ρ^ν is pure, we manage to find all constraints on occupation numbers for systems of rank not exceeding 6. In appendix we give all pure ν-representability conditions for systems of rank 5 and 6. For rank 4 systems the only interesting case corresponds to the diagram $\ydiagram{2,1}$, and the constraints for this case can be obtained from the above inequality by using the specialization $\mu_1 = 1$ and $\mu_2 = 0$.

Inequalities	$v \in S_4$	$w \in S_{20}$	$c_w^v(a)$
$\lambda_1 - \lambda_3 \leq 2 - \mu_2$	(3 4)	(2 3)	1
$\lambda_2 - \lambda_3 \leq 1 + \mu_2$	(1 2)(3 4)	(1 2 3)	1
$\lambda_1 - \lambda_2 \leq 1 + \mu_2$	(2 3 4)	(1 2 3)	1
$\lambda_1 - \lambda_2 - \lambda_3 \leq 1$	(2 4 3)	(1 3 2)	1
$2\lambda_1 - \lambda_2 + \lambda_4 \leq 4 - \mu_2$	(2 4)	(1 3 4 2)	1

Table 6.1: Mixed ν-representability conditions for system $\mathcal{H}_4^\nu$ with $\nu = (2, 1)$.

Inequalities	$v \in S_5$	$w \in S_{40}$	$c_w^v(a)$
$\lambda_1 - \lambda_3 \leq 2 - \mu_2$	(3 4 5)	(2 3 4)	1
$\lambda_2 - \lambda_3 \leq 1 + \mu_2$	(1 2)(3 4 5)	(1 2 3 4)	1
$\lambda_1 - \lambda_2 \leq 1 + \mu_2$	(2 3 4 5)	(1 2 3 4)	1
$\lambda_1 - \lambda_2 - \lambda_3 \leq 1$	(2 4)(3 5)	(1 3)(2 4)	1
$2\lambda_1 - \lambda_2 + \lambda_4 \leq 4 - \mu_2$	(2 4 5)	(1 3 4 5 2)	1

Table 6.2: Mixed ν-representability conditions for system $\mathcal{H}_5^\nu$ with $\nu = (2, 1)$.

6.2 Mixed N-representability

Using formula (3.20) we deduce from Theorem 3.2.1 the following inequalities between the spectrum $\mu = \mathrm{Spec}\rho'$ of mixed state ρ' and the spectrum $\lambda = \mathrm{Spec}\rho$ of its reduced state ρ for the system $\wedge^2\mathcal{H}_4$

$$\begin{aligned}
2\lambda_1 &\leq \mu_1 + \mu_2 + \mu_3 \\
2\lambda_4 &\geq \mu_4 + \mu_5 + \mu_6 \\
2(\lambda_1 - \lambda_4) &\leq \mu_1 + \mu_2 - \mu_5 - \mu_6 \\
\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 &\leq \mu_1 - \mu_6 \\
\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 &\leq \min(\mu_1 - \mu_5, \mu_2 - \mu_6) \\
|\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4| &\leq \min(\mu_1 - \mu_4, \mu_2 - \mu_5, \mu_3 - \mu_6) \\
2\max(\lambda_1 - \lambda_3, \lambda_2 - \lambda_4) &\leq \min(\mu_1 + \mu_3 - \mu_5 - \mu_6, \mu_1 + \mu_2 - \mu_4 - \mu_6) \\
2\max(\lambda_1 - \lambda_2, \lambda_3 - \lambda_4) &\leq \min(\mu_1 + \mu_3 - \mu_4 - \mu_6, \mu_2 + \mu_3 - \mu_5 - \mu_6, \\
&\quad\ \mu_1 + \mu_2 - \mu_4 - \mu_5).
\end{aligned} \tag{6.1}$$

Inequalities	$v \in S_6$	$w \in S_{20}$	$c_w^v(a)$
$\lambda_1 + \lambda_6 \leq 1$	(2 6 5 4 3)		1
$\lambda_2 + \lambda_5 \leq 1$	(1 2 5 4 3)	(1 2 3 4 5)	1
$\lambda_3 + \lambda_4 \leq 1$	(1 3)(2 4)		1
$\lambda_4 \leq \lambda_5 + \lambda_6$	(1 4 3 2)	(1 2 3 4)	1

Table 6.3: N-representability inequalities for system $\wedge^3\mathcal{H}_6$. By normalization condition, the first group of inequalities amounts to equations in (1.5).

As we've mentioned in the Introduction these inequalities impose constraints on spectra of Riemann $\mathrm{R} = \rho'$ and Ricci $\mathrm{Ric} = \rho$ curvatures of four-manifold. Note that, the traces of the operators ρ' and ρ are related by the equation $\mathrm{Tr}\rho' = \mathrm{Tr}\rho$.

Criterion for mixed $N-$representability for system $\wedge^2\mathcal{H}_5$ is given by 460 independent linear inequalities which cannot be reproduced here.

6.3 Pure N-representability

The known solution for two fermions, together with the particle-hole duality n° 5.2, bound the pure $N-$representability problem to the range $3 \leq N \leq r/2$. For rank $r \leq 8$ this leaves us with systems $\wedge^3\mathcal{H}_6$, $\wedge^3\mathcal{H}_7$, $\wedge^3\mathcal{H}_8$, and $\wedge^4\mathcal{H}_8$.

For three of them $\wedge^3\mathcal{H}_6$, $\wedge^3\mathcal{H}_7$ and $\wedge^4\mathcal{H}_8$ the algorithm n° 5.1 runs flawlessly and terminates at $M = 4, 8, 10$, respectively. The independent constraints grouped by the test spectra a, together with the coefficients $c_w^v(a)$, and cycle decomposition of the permutations v, w are given in Tables 6.3–6.5.

6.3.1 System $\wedge^3\mathcal{H}_8$

The remaining system $\wedge^3\mathcal{H}_8$ is much harder to resolve. We spent almost six months to decompose plethysm $S^m(\wedge^3\mathcal{H}_8)$ up to degree $m = 24$, but still have had a inconsistency between the inner and the outer approximations to the moment

Inequalities	$v \in S_7$	$w \in S_{35}$	$c_w^v(a)$
$\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \leq 2$	(1 2 3 4 5)	(1 2 3 4 5)	1
$\lambda_1 + \lambda_3 + \lambda_4 + \lambda_6 \leq 2$	(2 3 4 6 5)		1
$\lambda_1 + \lambda_2 + \lambda_4 + \lambda_7 \leq 2$	(3 4 7 6 5)		1
$\lambda_1 + \lambda_2 + \lambda_5 + \lambda_6 \leq 2$	(3 5)(4 6)		1

Table 6.4: N-representability inequalities for system $\wedge^3\mathcal{H}_7$.

Inequalities	$v \in S_8$	$w \in S_{70}$	$c_w^v(a)$
$\lambda_1 \leq 1$	(1)	(1)	1
$\lambda_5 - \lambda_6 - \lambda_7 - \lambda_8 \leq 0$	(1 5 4 3 2)	(1 2 3 4 5)	1
$\lambda_1 - \lambda_2 - \lambda_7 - \lambda_8 \leq 0$	(2 3 4 5 6)		1
$\lambda_1 - \lambda_3 - \lambda_6 - \lambda_8 \leq 0$	(3 4 5 7 6)		1
$\lambda_1 - \lambda_4 - \lambda_6 - \lambda_7 \leq 0$	(4 5 8 7 6)		1
$\lambda_1 - \lambda_4 - \lambda_5 - \lambda_8 \leq 0$	(4 6)(5 7)		1
$\lambda_3 - \lambda_4 - \lambda_7 - \lambda_8 \leq 0$	(1 3 2)(4 5 6)		1
$\lambda_2 - \lambda_4 - \lambda_6 - \lambda_8 \leq 0$	(1 2)(4 5 7 6)		1
$\lambda_2 + \lambda_3 + \lambda_5 - \lambda_8 \leq 2$	(1 2 3 5 4)	(1 2 3 4 5)	1
$\lambda_1 + \lambda_3 + \lambda_6 - \lambda_8 \leq 2$	(2 3 6 5 4)		1
$\lambda_1 + \lambda_2 + \lambda_7 - \lambda_8 \leq 2$	(3 7 6 5 4)		1
$\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 \leq 2$	(4 5 6 7 8)		1
$\lambda_1 + \lambda_4 + \lambda_5 - \lambda_8 \leq 2$	(2 4)(3 5)		1
$\lambda_1 + \lambda_2 + \lambda_5 - \lambda_6 \leq 2$	(3 5 4)(6 7 8)		1
$\lambda_1 + \lambda_3 + \lambda_5 - \lambda_7 \leq 2$	(2 3 5 4)(7 8)		1

Table 6.5: N-representability inequalities for system $\wedge^4\mathcal{H}_8$.

polytope. Actually all facets of $\mathcal{P}_{24}^{\text{in}}$, except the following one

$$\lambda_1 + \lambda_5 + \lambda_6 \geq 1, \quad (?)$$

fit into the form (3.13) of Theorem 3.2.1. Because of computer limitations we could not continue to decompose further symmetric powers. Therefore at this stage we resort to use a numerical minimization of the linear form $L(\lambda) = \lambda_1 + \lambda_5 + \lambda_6$, obtained from the *bad facet* (?), over all particle density matrices. It turns out that the form attains its minimum, equal to $\frac{27}{28}$, at the vertex

$$\frac{1}{28}(15, 15, 15, 15, 6, 6, 6, 6, 6). \tag{6.2}$$

Adding this vertex gives a polytope $\mathcal{P}$ whose all facets are covered by Theorem 3.2.1. Thus $\mathcal{P}$ is the genuine moment polytope for $\wedge^3\mathcal{H}_8$ given by 31 independent inequalities listed in Table 6.6.

We are actually unhappy with employment of the numerical optimization, which can produce no rigorous result. Nevertheless, it provides a helpful hint. In the next subsection we will see that the vertex (6.2) indeed belongs to the moment polytope. This gives an independent check of the above pure N-representability conditions for $\wedge^3\mathcal{H}_8$.

6.3.2 Extremal States

In this part we produce an extremal state for every *extremal spectrum* (occupation numbers) of each systems $\wedge^3\mathcal{H}_6$, $\wedge^3\mathcal{H}_7$, $\wedge^3\mathcal{H}_8$, and $\wedge^4\mathcal{H}_8$ analyzed above, using a special case of Dadok-Kac construction (n° 3.6). This gives a computer independent check of N-representability conditions for fermion systems of rank $r \leq 8$.

Independent sets and systems $\wedge^3\mathcal{H}_6$, $\wedge^3\mathcal{H}_7$, and $\wedge^4\mathcal{H}_8$

Recall that N-fermion system $\wedge^N\mathcal{H}_r$ corresponds to one-column diagram $\nu = [1, 1, \ldots, 1]$ of size N. For simplicity we will write $I = [i, j, \ldots, k]$ instead of

Inequalities	$v \in S_8$	$w \in S_{56}$	$c^v_w(a)$
$\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \leq 2$	(1 2 3 4 5)	(1 2 3 4 5)	1
$\lambda_1 + \lambda_2 + \lambda_4 + \lambda_7 \leq 2$	(3 4 7 6 5)		1
$\lambda_1 + \lambda_3 + \lambda_4 + \lambda_6 \leq 2$	(2 3 4 6 5)		1
$\lambda_1 + \lambda_2 + \lambda_5 + \lambda_6 \leq 2$	(3 5)(4 6)		1
$\lambda_1 + \lambda_2 - \lambda_3 \leq 1$	(3 4 5 6 7 8)	(1 2 3 4 5 6)	1
$\lambda_2 + \lambda_5 - \lambda_7 \leq 1$	(1 2 5 4 3)(7 8)		1
$\lambda_1 + \lambda_6 - \lambda_7 \leq 1$	(2 6 5 4 3)(7 8)		1
$\lambda_2 + \lambda_4 - \lambda_6 \leq 1$	(1 2 4 3)(6 7 8)		1
$\lambda_1 + \lambda_4 - \lambda_5 \leq 1$	(2 4 3)(5 6 7 8)		1
$\lambda_3 + \lambda_4 - \lambda_7 \leq 1$	(1 3)(2 4)(7 8)		1
$\lambda_1 + \lambda_8 \leq 1$	(2 8 7 6 5 4 3)	(1 2 3 4 5 6 7)	1
$\lambda_2 - \lambda_3 - \lambda_6 - \lambda_7 \leq 0$	(1 2)(3 4 5 8 7 6)	(1 2 3 4 5 6 7)	1
$\lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 \leq 0$	(1 4 3 2)(5 8 7 6)		1
$\lambda_1 - \lambda_3 - \lambda_5 - \lambda_7 \leq 0$	(3 4 6)(5 8 7)		1
$\lambda_2 + \lambda_3 + 2\lambda_4 - \lambda_5 - \lambda_7 + \lambda_8 \leq 2$	(1 4 8 7 5)	(1 2 3 . . . 10 11)	1
$\lambda_1 + \lambda_3 + 2\lambda_4 - \lambda_5 - \lambda_6 + \lambda_8 \leq 2$	(1 4 8 6 7 5 2)		1
$\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_4 - \lambda_5 + \lambda_8 \leq 2$	(1 2)(3 4 8 5 6 7)		1
$\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_5 - \lambda_6 + \lambda_8 \leq 2$	(1 2)(3 5 4 8 6 7)		1
$\lambda_1 + \lambda_2 - 2\lambda_3 - \lambda_4 - \lambda_5 \leq 0$	(3 6 4 7 5 8)	(1 2 3 . . . 11 12)	1
$\lambda_1 - \lambda_2 - \lambda_3 + \lambda_6 - 2\lambda_7 \leq 0$	(2 6)(3 4 5 8 7)		1
$\lambda_1 - \lambda_3 - \lambda_4 - \lambda_5 + \lambda_8 \leq 0$	(2 8 5 7 4 6 3)	(1 2 3 . . . 12 13)	1
$\lambda_1 - \lambda_2 - \lambda_3 - \lambda_7 + \lambda_8 \leq 0$	(2 8 7 3 4 5 6)		1
$2\lambda_1 - \lambda_2 + \lambda_4 - 2\lambda_5 - \lambda_6 + \lambda_8 \leq 1$	(2 4 3 8 5 7 6)	(1 2 3 . . . 12 13)	1
$\lambda_3 + 2\lambda_4 - 2\lambda_5 - \lambda_6 - \lambda_7 + \lambda_8 \leq 1$	(1 4)(2 3 8 5)		1
$2\lambda_1 - \lambda_2 - \lambda_4 + \lambda_6 - 2\lambda_7 + \lambda_8 \leq 1$	(2 6)(3 8 7 4)		1
$2\lambda_1 + \lambda_2 - 2\lambda_3 - \lambda_4 - \lambda_6 + \lambda_8 \leq 1$	(3 8)(4 5 7 6)		1
$\lambda_1 + 2\lambda_2 - 2\lambda_3 - \lambda_5 - \lambda_6 + \lambda_8 \leq 1$	(1 2)(3 8)(5 7 6)		1
$2\lambda_1 - 2\lambda_2 - \lambda_3 - \lambda_4 + \lambda_6 - 3\lambda_7 + \lambda_8 \leq 0$	(2 6 4 5 3 8 7)	(1 2 3 . . . 14 15)	1
$-\lambda_1 + \lambda_3 + 2\lambda_4 - 3\lambda_5 - 2\lambda_6 - \lambda_7 + \lambda_8 \leq 0$	(1 4 2 3 8 5)(6 7)		1
$2\lambda_1 + \lambda_2 - 3\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6 + \lambda_8 \leq 0$	(3 8)(4 7)		1
$\lambda_1 + 2\lambda_2 - 3\lambda_3 - \lambda_4 - 2\lambda_5 - \lambda_6 + \lambda_8 \leq 0$	(1 2)(3 8)(4 7 5)		1

Table 6.6: N-representability inequalities for system $\wedge^3\mathcal{H}_8$.

semistandard tableau $T = \begin{array}{|c|}\hline i\\\hline j\\\hline \cdot\\\hline \cdot\\\hline \cdot\\\hline k\\\hline\end{array}$, and $\psi = \sum_I c_I \cdot I$ instead of state $|\psi\rangle = \sum_T c_T \cdot e_T \in \wedge^N \mathcal{H}_r$. In this notation disconnected support $\mathbf{T}$ corresponds to the set Δ of sequences I with the property that any $I, J \in \Delta$ are either equal or differ by at least two elements. Following Müller [29] we call such support Δ an *independent set* of indices I. In Proposition 3.6.1 we proved that a state with independent support has a diagonal reduced density matrix. In general reduced matrix $\rho = [\rho_{ij}]$ of a state $\psi = \sum_I c_I \cdot I$ has entries

$$\rho_{ij} = \sum_{\substack{i \in I, j \in J, \\ I \backslash i = J \backslash j}} (-1)^{\pi_I(i)+\pi_J(j)} c_I c_J^*,$$

where $\pi_I(i)$ is the number of indices in I which are less then i.

Borland and Dennis [5] observed that every state $|\psi\rangle \in \wedge^3 \mathcal{H}_6$ in eigenbasis of its particle density matrix has independent support

$$\Delta = \{[123], [145], [246], [356]\}.$$

This implies the above mentioned constraints (1.5) on occupation numbers for this system.

In general, spectra of particle density matrices for states with fixed independent support Δ form a convex polytope $\mathcal{P}_\Delta$, contained in the moment polytope $\mathcal{P}$. For the systems $\wedge^3 \mathcal{H}_7$ and $\wedge^4 \mathcal{H}_8$ the polytope $\mathcal{P}_\Delta$ coincides with the momentum polytope for the following independent supports

$$\Delta_1 = \{\ [123],\ [145],\ [167],\ [246],\ [257],\ [347],\ [356]\ \};$$

$$\begin{aligned}\Delta_2 = \{\ & [1234],\ [1256],\ [1278],\ [1357],\ [1368],\ [1458],\ [1467],\\ & [2358],\ [2367],\ [2457],\ [2468],\ [3456],\ [3478],\ [5678]\ \},\end{aligned}$$

respectively. This gives us unique extremal state $|\psi\rangle$ with these supports for each extremal spectra. We list all extremal states for every vertex of the moment polytope for these systems in Tables 6.7-6.8. The first 4 lines of table 6.7 give the extremal states for the system $\wedge^3 \mathcal{H}_6$.

Extremal states	Vertices
$[123]$	$(1:1:1:0:0:0:0)$
$[123]+[145]$	$(2:1:1:1:1:0:0)$
$[123]+[145]+[246]+[356]$	$(1:1:1:1:1:1:0)$
$\sqrt{2}[123]+[145]+[246]$	$(3:3:2:2:1:1:0)$
$[123]+[145]+[167]+[246]+[257]+[347]+[356]$	$(1:1:1:1:1:1:1)$
$\sqrt{2}[123]+[167]+[246]+[257]+[145]$	$(2:2:1:1:1:1:1)$
$\sqrt{2}[123]+\sqrt{2}[145]+[246]+[257]+[347]+[356]$	$(2:2:2:2:2:1:1)$
$[123]+[145]+[167]$	$(3:1:1:1:1:1:1)$
$\sqrt{2}[123]+[145]+[246]+[347]$	$(3:3:3:3:1:1:1)$
$\sqrt{3}[123]+\sqrt{2}[145]+[246]+[257]$	$(5:5:3:3:3:1:1)$

Table 6.7: Vertices of the moment polytope of $\wedge^3\mathcal{H}_7$ and the corresponding extremal states.

Extremal states	Vertices
$[1234]$	$(1:1:1:1:0:0:0:0)$
$[1234]+[1256]+[3456]$	$(1:1:1:1:1:1:0:0)$
$[1234]+[1256]$	$(2:2:1:1:1:1:0:0)$
$[1234]+[1256]+[1357]+[1467]+[2367]+[2457]+[3456]$	$(1:1:1:1:1:1:1:0)$
$[1234]+[1256]+[1357]+[1467]$	$(2:1:1:1:1:1:1:0)$
$\sqrt{2}[1234]+[1256]+[1357]+[2367]$	$(2:2:2:1:1:1:1:0)$
$\sqrt{2}[1234]+[1256]+[1357]+[2457]+[3456]$	$(2:2:2:2:2:1:1:0)$
$\sqrt{3}[1234]+\sqrt{2}[1256]+[1357]+[2457]$	$(3:3:2:2:2:1:1:0)$
$\sqrt{2}[1234]+\sqrt{2}[1256]+[1357]+[1467]+[2367]+[2457]$	$(3:3:2:2:2:2:2:0)$
$\sqrt{2}[1234]+[1256]+[1357]$	$(4:3:3:2:2:1:1:0)$
$[1234]+[5678]$	$(1:1:1:1:1:1:1:1)$
$\sqrt{2}[1234]+[1256]+[1278]+[1357]+[1368]$	$(3:2:2:1:1:1:1:1)$
$[1234]+[1256]+[1278]$	$(3:3:1:1:1:1:1:1)$
$\sqrt{3}[1234]+[1256]+[1357]+[1458]+[2358]+[2457]+[3456]$	$(3:3:3:3:3:1:1:1)$
$\sqrt{2}[1234]+\sqrt{2}[1256]+[1357]+[1368]+[1458]+[1467]$	$(4:2:2:2:2:2:1:1)$
$2[1234]+\sqrt{2}[1256]+[1357]+[1458]+[2358]+[2457]$	$(4:4:3:3:3:1:1:1)$
$2[1234]+\sqrt{2}[1256]+[1357]+[1368]+[2358]+[2367]$	$(4:4:4:2:2:2:1:1)$
$\sqrt{2}[1234]+[1256]+[1357]+[1458]$	$(5:3:3:3:3:1:1:1)$
$\sqrt{3}[1234]+[1256]+[1357]+[2358]$	$(5:5:5:3:3:1:1:1)$
$[1234]+[1256]+[1278]+[1357]+[1368]+[1458]+[1467]$	$(7:3:3:3:3:3:3:3)$
$\sqrt{3}[1234]+\sqrt{2}[1256]+[1357]+[1368]$	$(7:5:5:3:3:3:1:1)$
$\sqrt{3}[1234]+[1256]+[1278]+[1357]+[1368]+[2358]+[2367]$	$(7:7:7:3:3:3:3:3)$

Table 6.8: Vertices of the moment polytope of $\wedge^4\mathcal{H}_8$ and the corresponding extremal states.

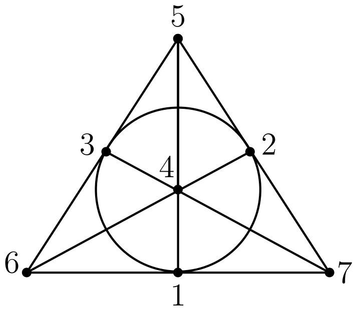


Figure 6.1: Lines in Cayley projective plane over $\mathbb{F}_2$.

The above independent sets have a nice geometric interpretation: Δ_1 is the set of lines in Cayley projective plane over binary field $\mathbb{F}_2$ (see figure 6.1), and Δ_2 is the set of affine planes in 3-space $\mathbb{F}_2^3$ over this field. This explains why any two triplets in Δ_1 have exactly one index in common, and any two quadruples in Δ_2 intersect in at most two indices.

Extremal states for the system $\wedge^3\mathcal{H}_8$

In this case there are 4 maximal independent sets [29]. None of them covers all extremal spectra, but all together they produce extremal states for most vertices, with 11 exceptions. The latter include the problematic vertex (6.2), for which there is no extremal state with independent support.

After some guesses and trials we managed to construct all missed extremal states using the following supports

$$\Delta_3 = \{[123], [145], [178], [246], [258], [347], [368], [567], [\mathbf{356}]\},$$
$$\Delta_4 = \{[123], [145], [178], [246], [258], [347], [368], [567], [\mathbf{124}]\},$$

which are obtained by adding an extra triplet, typeset in boldface, to a independent set. In particular, for the problematic vertex (6.2) we found the following state

$$|\psi\rangle = 2[123] + \sqrt{10}[145] + \sqrt{5}[347] + \sqrt{2}[\mathbf{356}] + \sqrt{2}[258] + 2[368] + [178].$$

The particle density matrices of these states are not diagonal, but the diagonal blocks are small. This allow us to compile Table 6.9 of extremal states for $\wedge^3\mathcal{H}_8$.

This gives a computer independent proof of N-representability condition for all systems of rank $r \leq 8$, provided one takes for granted the coefficients $c_w^w(a)$ in Tables 6.3-6.6.

Extremal states	Vertices
$[123]$	(1:1:1:0:0:0:0:0)
$[123]+[145]$	(2:1:1:1:1:0:0:0)
$[123]+[145]+[246]+[356]$	(1:1:1:1:1:1:0:0)
$\sqrt{2}[123]+[145]+[246]$	(3:3:2:2:1:1:0:0)
$[123]+[145]+[167]+[246]+[257]+[347]+[356]$	(1:1:1:1:1:1:1:0)
$\sqrt{2}[123]+[167]+[246]+[257]+[145]$	(2:2:1:1:1:1:1:0)
$\sqrt{2}[123]+\sqrt{2}[145]+[246]+[257]+[347]+[356]$	(2:2:2:2:2:1:1:0)
$[123]+[145]+[167]$	(3:1:1:1:1:1:1:0)
$\sqrt{2}[123]+[145]+[246]+[347]$	(3:3:3:3:1:1:1:0)
$\sqrt{3}[123]+\sqrt{2}[145]+[246]+[257]$	(5:5:3:3:3:1:1:0)
$[178]+[368]+[258]+[567]+[347]+[246]+[145]+[123]$	(1:1:1:1:1:1:1:1)
$\sqrt{2}[178]+[368]+[567]+[246]+\sqrt{2}[145]+\sqrt{2}[123]$	(2:1:1:1:1:1:1:1)
$\sqrt{2}[178]+[258]+[567]+\sqrt{2}[246]+[145]+\sqrt{3}[123]$	(2:2:1:1:1:1:1:1)
$\sqrt{3}[123]+\sqrt{3}[145]+[246]+\sqrt{2}[347]+[356]+\sqrt{2}[258]$	(3:3:3:3:3:1:1:1)
$\sqrt{3}[178]+\sqrt{2}[567]+[347]+[246]+2[145]+\sqrt{5}[123]$	(4:2:2:2:2:2:1:1)
$[178]+[246]+[145]+\sqrt{2}[123]$	(4:3:2:2:1:1:1:1)
$[178]+[258]+[246]+[145]+\sqrt{2}[123]$	(4:4:2:2:2:2:1:1)
$[258]+[567]+[145]+\sqrt{3}[123]$	(4:4:3:3:1:1:1:1)
$\sqrt{2}[145]+[246]+[347]+[\mathbf{356}]+\sqrt{2}[368]$	(4:4:4:4:2:1:1:1]
$\sqrt{2}[178]+[246]+[145]+\sqrt{2}[123]$	(5:3:2:2:2:2:1:1)
$[368]+[347]+\sqrt{2}[145]+\sqrt{3}[123]$	(5:5:3:3:2:1:1:1)
$2[123]+\sqrt{10}[145]+\sqrt{5}[347]+\sqrt{2}[\mathbf{356}]+\sqrt{2}[258]+2[368]+[178]$	(5:5:5:5:2:2:2:2)
$[178]+[567]+\sqrt{2}[145]+\sqrt{3}[123]$	(6:3:3:3:2:2:1:1)
$2[123]+\sqrt{2}[246]+\sqrt{3}[\mathbf{356}]+\sqrt{5}[567]+2[258]$	(6:5:5:5:2:2:1:1)
$\sqrt{2}[178]+[258]+\sqrt{2}[246]+[145]+\sqrt{3}[123]$	(6:6:3:3:3:2:2:2)
$2\sqrt{2}[145]+\sqrt{2}[246]+\sqrt{2}[347]+\sqrt{3}[\mathbf{356}]+\sqrt{3}[368]$	(6:6:4:4:4:1:1:1)
$2\sqrt{3}[123]+\sqrt{6}[145]+\sqrt{2}[\mathbf{356}]+2[567]+\sqrt{3}[258]+\sqrt{3}[178]$	(7:5:5:5:2:2:2:2)
$\sqrt{2}[145]+2[246]+[347]+[\mathbf{356}]+\sqrt{2}[368]$	(7:7:4:4:4:2:1:1)
$\sqrt{3}[246]+\sqrt{2}[347]+\sqrt{6}[258]+2[368]+2\sqrt{2}[178]+[\mathbf{124}]$	(9:5:5:5:3:3:3:3)
$\sqrt{3}[258]+[567]+\sqrt{2}[347]+\sqrt{2}[246]+2[123]$	(9:6:4:4:4:3:3:3)
$3[145]+\sqrt{6}[246]+3[347]+2[\mathbf{356}]+\sqrt{3}[258]+\sqrt{14}[368]$	(9:8:8:8:3:3:3:3)
$\sqrt{2}[178]+[258]+\sqrt{3}[246]+\sqrt{2}[145]+\sqrt{5}[123]$	(9:9:5:5:3:3:3:2)
$2[123]+\sqrt{2}[246]+\sqrt{2}[\mathbf{356}]+\sqrt{3}[567]+\sqrt{3}[258]+\sqrt{2}[368]$	(9:9:9:9:4:4:2:2)
$2\sqrt{2}[145]+\sqrt{6}[246]+\sqrt{6}[347]+\sqrt{5}[\mathbf{356}]+\sqrt{2}[258]+3[368]$	(10:10:10:10:4:4:3:3)
$\sqrt{5}[178]+[347]+\sqrt{2}[246]+\sqrt{2}[145]+2[123]$	(11:6:6:5:5:5:2:2)
$\sqrt{3}[178]+[258]+2[246]+\sqrt{2}[145]+\sqrt{6}[123]$	(11:11:6:6:4:4:3:3)
$\sqrt{3}[178]+\sqrt{2}[567]+[246]+2[145]+\sqrt{5}[123]$	(12:6:6:5:5:5:3:3)
$[123]+\sqrt{3}[145]+2[347]+2[\mathbf{356}]+\sqrt{3}[258]+\sqrt{3}[368]$	(12:12:7:7:4:4:4:4)

Table 6.9: Vertices of the moment polytope of $\wedge^3\mathcal{H}_8$ and the corresponding extremal states. The first ten lines give the same data for $\wedge^3\mathcal{H}_7$.

6.3.3 Systems of rank 9 and 10

For these systems the results are less definite. As in the previous case ($\wedge^3\mathcal{H}_8$) we failed in our attempt to get full constraints on occupation numbers by using the algorithm n° 5.1. In this cases there are many inconsistent inequalities which we call them as bad facets, meaning that the facets don't fit into the form (3.13) of Theorem 3.2.1. To resolve the inconsistency between inner and outer approximations to the moment polytope we again resort to use a numerical minimization of the linear forms, obtained from those bad facets, over all particle density matrices, recursively. After several successive minimization processes we managed to find all constraints on occupation numbers for systems of rank 9 and 10. However, only for the smallest system $\wedge^3\mathcal{H}_9$ we have a rigorous justification of completeness for the system of 52 independent inequalities. For the next system $\wedge^4\mathcal{H}_9$ we found 60 constraints, which give a polytope with 103 vertices. For all of them, except for the following two vertices

$$[16, 16, 16, 6, 6, 6, 6, 6, 6]/21, \qquad [20, 14, 14, 14, 14, 4, 4, 4, 4]/23,$$

we have proved rigorously that they belong to the moment polytope. The remaining two vertices were checked only numerically. It turns out that the same two vertices would provide the completeness of 125 constraints for $\wedge^4\mathcal{H}_{10}$. The occupation numbers of the remaining systems $\wedge^3\mathcal{H}_{10}$ and $\wedge^5\mathcal{H}_{10}$ are bounded by 93 and 161 inequalities, but many vertices are still waiting a confirmation by non-numerical methods.

The facets of the moment polytopes for all systems of ranks 9 and 10 are given in appendix, together with the permutations v, w and the coefficients $c_v^w(a)$ in the setting of Theorem 3.2.1.

Chapter 7

Appendix A: Further Constraints

A.1 N-representability Constraints for Systems of rank 9 and 10

Here, we give all constraints on the electron density matrices of the systems of rank $r = 9$ and 10. Because of particle-hole duality we only give constraints for the systems $\wedge^N \mathcal{H}_r$ with $N \leq [r/2]$. The spectra $\lambda = \text{Spec}\rho$ are arranged in non-increasing order and, normalized to $\text{Tr}\rho = \sum_i \lambda_i = N$. We group the inequalities by the extremal edges. Also, we list the permutations $v \in S_r$ and $w \in S_{\binom{r}{N}}$ which give the non-zero coefficients $c_v^w(a)$. In fact, the coefficients $c_v^w(a)$ are all equal to 1 for all permutations v and w.

For the system $\wedge^3 \mathcal{H}_9$:

Inequalities	$v \in S_9$	$w \in S_{84}$
$\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 \leq 1$	(3 4 7)(5 8)(6 9)	(1 2 3 … 10 11)

$2\lambda_1 + \lambda_2 + \lambda_4 + \lambda_6 + \lambda_8 \leq 3$	(3 4 6)(5 8 7)	
$\lambda_1 + 2\lambda_2 + \lambda_4 + \lambda_5 + \lambda_8 \leq 3$	(1 2)(3 4 5 8 7 6)	(1 2 3 4 5 6 7)
$\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + \lambda_8 \leq 3$	(1 4 3 2)(5 8 7 6)	

$2\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_4 + \lambda_8 + \lambda_9 \leq 3$	(3 4 8 7 6 5 9)	
$2\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_6 + \lambda_7 + \lambda_8 \leq 3$	(3 6 4 7 5 8 9)	
$2\lambda_1 + \lambda_2 + 2\lambda_4 - \lambda_5 + \lambda_8 + \lambda_9 \leq 3$	(2 4 8 7 6 3)(5 9)	
$2\lambda_1 + \lambda_2 + \lambda_6 + \lambda_7 + 2\lambda_8 - \lambda_9 \leq 3$	(2 8 5 7 4 6 3)	
$2\lambda_1 + \lambda_4 + \lambda_5 + \lambda_6 + 2\lambda_8 - \lambda_9 \leq 3$	(2 8 7 3 4 5 6)	(1 2 3 ... 12 13)
$2\lambda_1 + \lambda_4 + \lambda_5 + 2\lambda_6 - \lambda_7 + \lambda_8 \leq 3$	(2 6)(3 4 5 8 9 7)	
$\lambda_1 + 2\lambda_2 + 2\lambda_4 - \lambda_6 + \lambda_8 + \lambda_9 \leq 3$	(1 2 4 8 7 5 9 6 3)	
$\lambda_1 + 2\lambda_2 + 2\lambda_4 - \lambda_5 + \lambda_7 + \lambda_9 \leq 3$	(1 2 4 7 6 3)(5 9)	
$2\lambda_2 + \lambda_3 + \lambda_4 + 2\lambda_5 - \lambda_6 + \lambda_9 \leq 3$	(1 2 5 9 6)	
$\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 - \lambda_6 + \lambda_9 \leq 3$	(1 3 2 4 5 9 6)	

$\lambda_1 + \lambda_2 + \lambda_4 + \lambda_7 \leq 2$	(3 4 7 6 5)	
$\lambda_1 + \lambda_2 + \lambda_5 + \lambda_6 \leq 2$	(3 5)(4 6)	
$\lambda_1 + \lambda_3 + \lambda_4 + \lambda_6 \leq 2$	(2 3 4 6 5)	(1 2 3 4 5)
$\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \leq 2$	(1 2 3 4 5)	

$3\lambda_1 + 2\lambda_2 - 2\lambda_3 + 2\lambda_4 - 2\lambda_5 - \lambda_6 + \lambda_9 \leq 3$	(3 4 9 5 7 6 8)	
$3\lambda_1 + 2\lambda_2 - 2\lambda_3 - \lambda_4 + \lambda_7 + 2\lambda_8 - 2\lambda_9 \leq 3$	(3 8)(4 7)	
$3\lambda_1 - \lambda_2 + \lambda_5 + 2\lambda_6 - 2\lambda_7 + 2\lambda_8 - 2\lambda_9 \leq 3$	(2 6 4 5 3 8 7)	
$2\lambda_1 + 3\lambda_2 - 2\lambda_3 + 2\lambda_4 - 2\lambda_5 - \lambda_6 + \lambda_8 \leq 3$	(1 2)(3 4 8)(5 7 6 9)	
$2\lambda_1 + 3\lambda_2 - 2\lambda_3 + 2\lambda_4 - \lambda_5 - 2\lambda_6 + \lambda_9 \leq 3$	(1 2)(3 4 9 6 8)(5 7)	(1 2 3 ... 14 15)
$2\lambda_1 + 3\lambda_2 - 2\lambda_3 - \lambda_5 + \lambda_7 + 2\lambda_8 - 2\lambda_9 \leq 3$	(1 2)(3 8)(4 7 5)	
$2\lambda_1 + 2\lambda_2 - \lambda_3 + 3\lambda_4 - 2\lambda_5 - 2\lambda_6 + \lambda_9 \leq 3$	(1 4 9 6 8 5 7 3 2)	
$\lambda_2 + 2\lambda_3 + 3\lambda_4 - 2\lambda_5 - \lambda_6 + 2\lambda_8 - 2\lambda_9 \leq 3$	(1 4 2 3 8 5)(6 7)	

$\lambda_1 + \lambda_2 - \lambda_3 \leq 1$	(3 4 5 6 7 8 9)	
$\lambda_1 + \lambda_4 - \lambda_5 \leq 1$	(2 4 3)(5 6 7 8 9)	
$\lambda_1 + \lambda_6 - \lambda_7 \leq 1$	(2 6 5 4 3)(7 8 9)	
$\lambda_1 + \lambda_8 - \lambda_9 \leq 1$	(2 8 7 6 5 4 3)	(1 2 3 4 5 6 7)
$\lambda_2 + \lambda_4 - \lambda_6 \leq 1$	(1 2 4 3)(6 7 8 9)	
$\lambda_2 + \lambda_5 - \lambda_7 \leq 1$	(1 2 5 4 3)(7 8 9)	
$\lambda_3 + \lambda_4 - \lambda_7 \leq 1$	(1 3)(2 4)(7 8 9)	

$2\lambda_1 + 3\lambda_2 + 2\lambda_4 + \lambda_6 + \lambda_7 + 2\lambda_8 \leq 5$	(1 2)(3 4 8 5 6 7)	
$2\lambda_1 + 3\lambda_2 + \lambda_4 + 2\lambda_5 + \lambda_7 + 2\lambda_8 \leq 5$	(1 2)(3 5 4 8 6 7)	
$2\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_4 + \lambda_7 + 2\lambda_8 \leq 5$	(1 4 8 6 7 5 2)	(1 2 3 ... 10 11)
$\lambda_1 + 2\lambda_2 + 2\lambda_3 + 3\lambda_4 + \lambda_6 + 2\lambda_8 \leq 5$	(1 4 8 7 5)	

$3\lambda_1 + 2\lambda_2 - \lambda_3 + 2\lambda_4 - \lambda_5 + \lambda_7 + \lambda_9 \leq 4$	(3 4 7 8)(5 9)	
$3\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_5 + 2\lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(3 6 4 5 9 7 8)	
$3\lambda_1 + \lambda_3 + 2\lambda_4 - \lambda_5 + \lambda_7 + 2\lambda_8 - \lambda_9 \leq 4$	(2 4 3 8 5 7 6)	
$3\lambda_1 + \lambda_3 + 2\lambda_4 - \lambda_5 + 2\lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(2 4 3 6)(5 9 7 8)	
$3\lambda_1 + \lambda_3 + \lambda_5 + 2\lambda_6 - \lambda_7 + 2\lambda_8 - \lambda_9 \leq 4$	(2 6)(3 8 7 4)	
$2\lambda_1 + 3\lambda_2 - \lambda_3 + 2\lambda_4 - \lambda_6 + \lambda_7 + \lambda_9 \leq 4$	(1 2)(3 4 7 8)(5 9 6)	
$2\lambda_1 + 3\lambda_2 - \lambda_3 + 2\lambda_4 - \lambda_5 + \lambda_7 + \lambda_8 \leq 4$	(1 2)(3 4 7 9 5 8)	
$2\lambda_1 + 3\lambda_2 - \lambda_3 + \lambda_4 + \lambda_7 + 2\lambda_8 - \lambda_9 \leq 4$	(1 2)(3 8)(5 7 6)	(1 2 3 ... 12 13)
$2\lambda_1 + 3\lambda_2 - \lambda_3 + 2\lambda_5 + \lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(1 2)(3 5 9 7 8)(4 6)	
$2\lambda_1 + 2\lambda_3 + 3\lambda_4 - \lambda_5 + \lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(1 4 6 2)(5 9 7 8)	
$2\lambda_1 + 2\lambda_2 + 3\lambda_4 - \lambda_5 - \lambda_6 + \lambda_7 + \lambda_9 \leq 4$	(1 4 7 8 5 9 6 3 2)	
$\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_4 - \lambda_5 + 2\lambda_8 - \lambda_9 \leq 4$	(1 4)(2 3 8 5)	
$3\lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 - \lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(1 2 4 3 5 9 7 8 6)	
$2\lambda_2 + 2\lambda_3 + 3\lambda_4 + \lambda_5 - \lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(1 4 5 9 7 8 6)	

For the system $\wedge^4\mathcal{H}_9$:

Inequalities	$v \in S_9$	$w \in S_{126}$

$\lambda_1 \leq 1$	(1)	(1)
$2\lambda_1 + \lambda_2 + \lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_6 \leq 2$	(4 7 5 8 6 9)	(1 2 3 … 11 12)
$2\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 + \lambda_7 - 2\lambda_8 \leq 2$	(3 7)(4 5 6 9 8)	
$\lambda_1 + \lambda_4 + 2\lambda_5 - 2\lambda_6 - \lambda_7 - \lambda_8 \leq 2$	(1 5 3 4 2)(6 9)	
$\lambda_2 + \lambda_3 + 2\lambda_5 - 2\lambda_6 - \lambda_7 - \lambda_8 \leq 2$	(1 5 4)(6 9)	
$\lambda_1 + \lambda_2 + \lambda_9 \leq 2$	(3 9 8 7 6 5 4)	(1 2 3 4 5 6 7)
$2\lambda_1 + \lambda_2 + \lambda_3 + \lambda_5 + \lambda_8 \leq 4$	(4 5 8 7 6)	(1 2 3 4 5 1)
$2\lambda_1 + \lambda_2 + \lambda_3 + \lambda_6 + \lambda_7 \leq 4$	(4 6)(5 7)	
$2\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 \leq 4$	(3 4 5 7 6)	
$2\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 \leq 4$	(2 3 4 5 6)	
$\lambda_1 + 2\lambda_2 + \lambda_3 + \lambda_5 + \lambda_7 \leq 4$	(1 2)(4 5 7 6)	
$\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_5 + \lambda_6 \leq 4$	(1 3 2)(4 5 6)	
$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\lambda_5 \leq 4$	(1 5 4 3 2)	
$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_5 + \lambda_9 \leq 3$	(4 5 9 8 7 6)	(1 2 3 4 5 6)
$3\lambda_1 + \lambda_2 + 2\lambda_3 - \lambda_4 + \lambda_5 - \lambda_6 + \lambda_9 \leq 5$	(2 3)(4 5 9 6 7 8)	(1 2 3 … 10 11)
$3\lambda_1 + \lambda_2 + 2\lambda_3 - \lambda_4 + \lambda_6 - \lambda_7 + \lambda_9 \leq 5$	(2 3)(4 6 5 9 7 8)	
$3\lambda_1 + \lambda_2 + \lambda_4 + 2\lambda_5 - \lambda_6 - \lambda_7 + \lambda_9 \leq 5$	(2 5 9 7 8 6 3)	
$3\lambda_1 + \lambda_3 + \lambda_4 + 2\lambda_5 - \lambda_6 - \lambda_8 + \lambda_9 \leq 5$	(2 5 9 8 6)	
$2\lambda_1 + 3\lambda_2 + \lambda_3 - \lambda_4 + \lambda_5 - \lambda_6 + \lambda_9 \leq 5$	(1 2)(4 5 9 6 7 8)	
$2\lambda_1 + 3\lambda_2 + \lambda_3 - \lambda_4 + \lambda_7 - \lambda_8 + \lambda_9 \leq 5$	(1 2)(4 7 6 5 9 8)	
$2\lambda_1 + 3\lambda_2 + \lambda_5 - \lambda_6 + \lambda_7 - \lambda_8 + \lambda_9 \leq 5$	(1 2)(3 5 9 8 6)(4 7)	

$3\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_5 - 2\lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(3 5 4 9 6 8 7)	(1 2 3 … 12 13)
$3\lambda_1 + 2\lambda_2 - \lambda_3 - \lambda_5 + \lambda_7 - 2\lambda_8 + \lambda_9 \leq 4$	(3 7)(4 9 8 5)	
$3\lambda_1 + 2\lambda_2 + \lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_7 + \lambda_9 \leq 4$	(4 9)(5 6 8 7)	
$3\lambda_1 + \lambda_2 + 2\lambda_3 - 2\lambda_4 - \lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(2 3)(4 9)(6 8 7)	
$3\lambda_1 + \lambda_4 + 2\lambda_5 - 2\lambda_6 - \lambda_7 - \lambda_8 + \lambda_9 \leq 4$	(2 5)(3 4 9 6)	
$2\lambda_1 + 3\lambda_2 + \lambda_3 - 2\lambda_4 - \lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(1 2)(4 9)(6 8 7)	
$2\lambda_1 + 3\lambda_2 + \lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_8 + \lambda_9 \leq 4$	(1 2)(4 9)(5 6 7)	
$2\lambda_1 + 3\lambda_2 - \lambda_4 + \lambda_5 - 2\lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(1 2)(3 5)(4 9 6 8 7)	
$2\lambda_1 + 3\lambda_2 - \lambda_4 - \lambda_5 + \lambda_7 - 2\lambda_8 + \lambda_9 \leq 4$	(1 2)(3 7 4 9 8 5)	
$2\lambda_1 + 3\lambda_2 - \lambda_3 + \lambda_5 - 2\lambda_6 - \lambda_8 + \lambda_9 \leq 4$	(1 2)(3 5 4 9 6 7)	
$2\lambda_1 + 3\lambda_2 - \lambda_3 - \lambda_6 + \lambda_7 - 2\lambda_8 + \lambda_9 \leq 4$	(1 2)(3 7)(4 9 8 6 5)	

$\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 \leq 2$	(4 5 6 7 8 9)	(1 2 3 4 5 6)
$\lambda_1 + \lambda_2 + \lambda_5 - \lambda_6 \leq 2$	(3 5 4)(6 7 8 9)	
$\lambda_1 + \lambda_2 + \lambda_7 - \lambda_8 \leq 2$	(3 7 6 5 4)(8 9)	
$\lambda_1 + \lambda_3 + \lambda_5 - \lambda_7 \leq 2$	(2 3 5 4)(7 8 9)	
$\lambda_1 + \lambda_3 + \lambda_6 - \lambda_8 \leq 2$	(2 3 6 5 4)(8 9)	
$\lambda_1 + \lambda_4 + \lambda_5 - \lambda_8 \leq 2$	(2 4)(3 5)(8 9)	
$\lambda_2 + \lambda_3 + \lambda_5 - \lambda_8 \leq 2$	(1 2 3 5 4)(8 9)	

$2\lambda_1 + 2\lambda_2 + \lambda_3 + \lambda_5 + \lambda_7 + \lambda_9 \leq 5$	(4 5 7)(6 9 8)	(1 2 3 4 5 6 7)
$2\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_5 + \lambda_6 + \lambda_9 \leq 5$	(2 3)(4 5 6 9 8 7)	
$2\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\lambda_5 + \lambda_9 \leq 5$	(2 5 4 3)(6 9 8 7)	

$2\lambda_1 + \lambda_2 - \lambda_4 - \lambda_5 - \lambda_6 + \lambda_9 \leq 2$	(3 9 6 8 5 7 4)	(1 2 3… 12 13)
$2\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 - \lambda_8 + \lambda_9 \leq 2$	(3 9 8 4 5 6 7)	

$2\lambda_1 + \lambda_3 + \lambda_4 + 2\lambda_5 - \lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(2 5 9 7 8 6)	(1 2 3 … 11 12)

$3\lambda_1+\lambda_2+2\lambda_3-2\lambda_4+\lambda_5-2\lambda_6-\lambda_7\leq 4$	(2 3)(4 5 8)(6 9)	
$2\lambda_1+3\lambda_2+\lambda_3-2\lambda_4+\lambda_5-2\lambda_6-\lambda_7\leq 4$	(1 2)(4 5 8)(6 9)	
$2\lambda_1+3\lambda_2+\lambda_3-2\lambda_4-\lambda_5+\lambda_7-2\lambda_8\leq 4$	(1 2)(4 7 5 6 9 8)	
$2\lambda_1+3\lambda_2-\lambda_3+\lambda_5-2\lambda_6+\lambda_7-2\lambda_8\leq 4$	(1 2)(3 5 4 7)(6 9 8)	
$2\lambda_1+\lambda_2+3\lambda_3-2\lambda_4+\lambda_6-2\lambda_7-\lambda_8\leq 4$	(1 3 2)(4 6 9 7 8)	
$2\lambda_1+\lambda_2+\lambda_4+3\lambda_5-2\lambda_6-2\lambda_7-\lambda_8\leq 4$	(1 5 3 2)(6 9 7 8)	(1 2 3 … 10 11)
$2\lambda_1+\lambda_3+\lambda_4+3\lambda_5-2\lambda_6-\lambda_7-2\lambda_8\leq 4$	(1 5 2)(6 9 8)	
$\lambda_1+2\lambda_2+3\lambda_3-2\lambda_4+\lambda_5-2\lambda_7-\lambda_8\leq 4$	(1 3)(4 5 6 9 7 8)	
$\lambda_1+2\lambda_2+3\lambda_3-2\lambda_4+\lambda_5-\lambda_6-2\lambda_8\leq 4$	(1 3)(4 5 7 6 9 8)	
$\lambda_1+2\lambda_2+\lambda_3+3\lambda_5-2\lambda_6-2\lambda_7-\lambda_8\leq 4$	(1 5 4 3)(6 9 7 8)	
$\lambda_1+\lambda_2+2\lambda_3+3\lambda_5-2\lambda_6-\lambda_7-2\lambda_8\leq 4$	(1 5 4 2 3)(6 9 8)	

$5\lambda_1+3\lambda_2+2\lambda_3-2\lambda_4-\lambda_5+\lambda_8+2\lambda_9\leq 8$	(4 9)(5 8)	
$5\lambda_1+3\lambda_2-\lambda_3+\lambda_6+2\lambda_7-2\lambda_8+2\lambda_9\leq 8$	(3 7 5 6 4 9 8)	(1 2 3 … 14, 15)
$5\lambda_1+2\lambda_2+3\lambda_3-2\lambda_4-\lambda_6+\lambda_8+2\lambda_9\leq 8$	(2 3)(4 9)(5 8 6)	
$5\lambda_1+\lambda_3+2\lambda_4+3\lambda_5-2\lambda_6-\lambda_7+2\lambda_9\leq 8$	(2 5 3 4 9 6)(7 8)	

For the system $\wedge^3\mathcal{H}_{10}$:

Inequalities	$v\in S_{10}$	$w\in S_{120}$
$\lambda_1+\lambda_2-\lambda_3+\lambda_4-\lambda_5-\lambda_6\leq 1$	(3 4 7 10 6 9 5 8)	(1 2 3 … 13 14)
$\lambda_1+\lambda_{10}\leq 1$	(2 10 9 8 7 6 5 4 3)	(1 2 3 … 8 9)
$2\lambda_1+\lambda_5+\lambda_6+\lambda_7+\lambda_8+2\lambda_{10}\leq 3$	(2 10 9 4 6 8 3 5 7)	(1 2 3 … 20 21)
$2\lambda_1+\lambda_2+\lambda_7+\lambda_8+\lambda_9+2\lambda_{10}\leq 3$	(2 10 6 9 5 8 4 7 3)	
$3\lambda_1+2\lambda_2-2\lambda_3-\lambda_4+\lambda_9+2\lambda_{10}\leq 3$	(3 10)(4 9)	(1 2 3 … 22 23)
$3\lambda_1-\lambda_2+\lambda_7+2\lambda_8-2\lambda_9+2\lambda_{10}\leq 3$	(2 8 6 4 7 5 3 10 9)	

$\lambda_1 + \lambda_3 + \lambda_4 + \lambda_6 \leq 2$	(2 3 4 6 5)	
$\lambda_1 + \lambda_2 + \lambda_4 + \lambda_7 \leq 2$	(3 4 7 6 5)	(1 2 3 4 5)
$\lambda_1 + \lambda_2 + \lambda_5 + \lambda_6 \leq 2$	(3 5)(4 6)	
$\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \leq 2$	(1 2 3 4 5)	

$2\lambda_1 + \lambda_3 + 2\lambda_4 - \lambda_5 + \lambda_6 + \lambda_{10} \leq 3$	(2 4 6)(5 10)	
$2\lambda_1 + \lambda_3 + \lambda_4 + 2\lambda_6 - \lambda_7 + \lambda_{10} \leq 3$	(2 6)(5 10 7)	
$\lambda_1 + 2\lambda_3 + 2\lambda_4 + \lambda_6 - \lambda_7 + \lambda_{10} \leq 3$	(1 3)(2 4 6)(5 10 7)	
$2\lambda_1 + \lambda_2 + \lambda_6 + \lambda_7 + 2\lambda_8 - \lambda_9 \leq 3$	(2 8 5 7 4 6 3)(9 10)	
$2\lambda_1 + \lambda_4 + \lambda_5 + 2\lambda_6 - \lambda_7 + \lambda_8 \leq 3$	(2 6)(3 4 5 8 9 10 7)	
$2\lambda_1 + \lambda_4 + \lambda_5 + \lambda_6 + 2\lambda_8 - \lambda_9 \leq 3$	(2 8 7 3 4 5 6)(9 10)	
$2\lambda_1 + \lambda_2 + \lambda_5 + 2\lambda_6 - \lambda_7 + \lambda_{10} \leq 3$	(2 6 3)(4 5 10 7)	
$\lambda_1 + 2\lambda_2 + 2\lambda_5 + \lambda_6 - \lambda_7 + \lambda_{10} \leq 3$	(1 2 5 10 7 4 6 3)	
$2\lambda_2 + \lambda_3 + \lambda_4 + 2\lambda_5 - \lambda_7 + \lambda_{10} \leq 3$	(1 2 5 10 7 6)	
$\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 - \lambda_7 + \lambda_{10} \leq 3$	(1 3 2 4 5 10 7 6)	
$2\lambda_2 + \lambda_3 + 2\lambda_4 + \lambda_5 - \lambda_6 + \lambda_{10} \leq 3$	(1 2 4 5 10 6)	
$2\lambda_1 + \lambda_2 + 2\lambda_4 - \lambda_5 + \lambda_7 + \lambda_{10} \leq 3$	(2 4 7 6 3)(5 10)	(1 2 3 … 13 14)
$\lambda_1 + 2\lambda_2 + 2\lambda_4 - \lambda_6 + \lambda_7 + \lambda_{10} \leq 3$	(1 2 4 7 5 10 6 3)	
$2\lambda_1 + \lambda_2 + 2\lambda_4 - \lambda_5 + \lambda_8 + \lambda_9 \leq 3$	(2 4 8 7 6 3)(5 9 10)	
$\lambda_1 + 2\lambda_2 + 2\lambda_4 - \lambda_6 + \lambda_8 + \lambda_9 \leq 3$	(1 2 4 8 7 5 9 10 6 3)	
$\lambda_1 + 2\lambda_2 + 2\lambda_4 - \lambda_5 + \lambda_7 + \lambda_9 \leq 3$	(1 2 4 7 6 3)(5 9 10)	
$2\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_4 + \lambda_7 + \lambda_{10} \leq 3$	(3 4 7 6 5 10)	
$2\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_4 + \lambda_8 + \lambda_9 \leq 3$	(3 4 8 7 6 5 9 10)	
$2\lambda_2 + \lambda_3 + \lambda_4 + 2\lambda_5 - \lambda_6 + \lambda_9 \leq 3$	(1 2 5 9 10 6)	
$\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 - \lambda_6 + \lambda_9 \leq 3$	(1 3 2 4 5 9 10 6)	
$2\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_5 + \lambda_6 + \lambda_{10} \leq 3$	(3 5 10)(4 6)	
$2\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_6 + \lambda_7 + \lambda_8 \leq 3$	(3 6 4 7 5 8 9 10)	

$\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_5 - \lambda_6 + \lambda_8 - \lambda_9 \leq 2$	(1 2)(3 5 4 8)(6 7 10 9)	
$\lambda_1 + \lambda_3 + 2\lambda_4 - \lambda_5 - \lambda_6 + \lambda_8 - \lambda_9 \leq 2$	(1 4 8 5 2)(6 7 10 9)	
$\lambda_2 + \lambda_3 + 2\lambda_4 - \lambda_5 - \lambda_7 + \lambda_8 - \lambda_9 \leq 2$	(1 4 8 5)(7 10 9)	(1 2 3 … 13 14)
$\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_4 - \lambda_5 + \lambda_8 - \lambda_9 \leq 2$	(1 2)(3 4 8)(5 6 7 10 9)	

$\lambda_2 + \lambda_4 - \lambda_6 \leq 1$	(1 2 4 3)(6 7 8 9 10)	
$\lambda_2 + \lambda_5 - \lambda_7 \leq 1$	(1 2 5 4 3)(7 8 9 10)	
$\lambda_1 + \lambda_4 - \lambda_5 \leq 1$	(2 4 3)(5 6 7 8 9 10)	
$\lambda_1 + \lambda_6 - \lambda_7 \leq 1$	(2 6 5 4 3)(7 8 9 10)	(1 2 3 4 5 6 7 8)
$\lambda_3 + \lambda_4 - \lambda_7 \leq 1$	(1 3)(2 4)(7 8 9 10)	
$\lambda_1 + \lambda_2 - \lambda_3 \leq 1$	(3 4 5 6 7 8 9 10)	
$\lambda_1 + \lambda_8 - \lambda_9 \leq 1$	(2 8 7 6 5 4 3)(9 10)	

$2\lambda_1 + \lambda_5 + \lambda_6 + \lambda_7 + 2\lambda_8 - \lambda_9 + \lambda_{10} \leq 3$	(2 8 3 5 7)(4 6 10 9)	
$2\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_7 + \lambda_8 + \lambda_9 + \lambda_{10} \leq 3$	(3 7 4 8 5 9 6 10)	(1 2 3 … 19 20)

$2\lambda_1 + \lambda_2 + \lambda_4 + \lambda_6 + \lambda_8 + \lambda_{10} \leq 3$	(3 4 6 10 9 7)(5 8)	
$\lambda_1 + 2\lambda_2 + \lambda_4 + \lambda_5 + \lambda_8 + \lambda_{10} \leq 3$	(1 2)(3 4 5 8 6 10 9 7)	(1 2 3 … 10 11)
$\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + \lambda_8 + \lambda_{10} \leq 3$	(1 4 3 2)(5 8 6 10 9 7)	

$\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_5 - \lambda_6 + \lambda_{10} \leq 2$	(1 2)(3 5 4 10 6 7 8 9)	
$\lambda_1 + \lambda_3 + 2\lambda_4 - \lambda_5 - \lambda_6 + \lambda_{10} \leq 2$	(1 4 10 6 7 8 9 5 2)	(1 2 3 … 16 17)
$\lambda_2 + \lambda_3 + 2\lambda_4 - \lambda_5 - \lambda_7 + \lambda_{10} \leq 2$	(1 4 10 7 8 9 5)	

$2\lambda_1 + 3\lambda_2 - 2\lambda_3 - \lambda_5 + \lambda_7 + 2\lambda_8 - 2\lambda_9 - \lambda_{10} \leq 3$	(1 2)(3 8 10 9)(4 7 5)	
$\lambda_2 + 2\lambda_3 + 3\lambda_4 - 2\lambda_5 - \lambda_6 + 2\lambda_8 - 2\lambda_9 - \lambda_{10} \leq 3$	(1 4 2 3 8 10 9 5)(6 7)	
$3\lambda_1 - \lambda_2 + \lambda_3 + 2\lambda_4 - 2\lambda_5 + 2\lambda_6 - 2\lambda_7 - \lambda_8 \leq 3$	(2 4 3 6 10 7)(5 9)	
$2\lambda_1 - \lambda_2 + 2\lambda_3 + 3\lambda_4 - 2\lambda_5 + \lambda_6 - 2\lambda_7 - \lambda_8 \leq 3$	(1 4 6 10 7 2)(5 9)	
$3\lambda_1 + 2\lambda_2 - 2\lambda_3 - \lambda_4 + \lambda_7 + 2\lambda_8 - 2\lambda_9 - \lambda_{10} \leq 3$	(3 8 10 9)(4 7)	
$3\lambda_1 - \lambda_2 + \lambda_5 + 2\lambda_6 - 2\lambda_7 + 2\lambda_8 - 2\lambda_9 - \lambda_{10} \leq 3$	(2 6 4 5 3 8 10 9 7)	
$3\lambda_1 + 2\lambda_2 - 2\lambda_3 - \lambda_4 + \lambda_5 + 2\lambda_6 - 2\lambda_7 - \lambda_8 \leq 3$	(3 6 10 7 4 5 9)	
$2\lambda_1 + 3\lambda_2 - 2\lambda_3 - \lambda_4 + 2\lambda_5 + \lambda_6 - 2\lambda_7 - \lambda_8 \leq 3$	(1 2)(3 5 9)(4 6 10 7)	
$-\lambda_1 + 3\lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 - 2\lambda_6 - 2\lambda_7 - \lambda_8 \leq 3$	(1 2 4 3 5 9 6 10 7)	(1 2 3 … 16 17)
$-\lambda_1 + 2\lambda_2 + 2\lambda_3 + 3\lambda_4 + \lambda_5 - 2\lambda_6 - 2\lambda_7 - \lambda_8 \leq 3$	(1 4 5 9 6 10 7)	
$2\lambda_1 + 3\lambda_2 - 2\lambda_3 + 2\lambda_4 - 2\lambda_5 - \lambda_6 + \lambda_8 - \lambda_{10} \leq 3$	(1 2)(3 4 8 10 5 7 6 9)	
$2\lambda_1 + 3\lambda_2 - 2\lambda_3 + 2\lambda_4 - 2\lambda_5 - \lambda_6 + \lambda_7 - \lambda_9 \leq 3$	(1 2)(3 4 7 6 10 5 8 9)	
$3\lambda_1 + 2\lambda_2 - 2\lambda_3 + 2\lambda_4 - 2\lambda_5 - \lambda_6 + \lambda_7 - \lambda_8 \leq 3$	(3 4 7 6 10 5 9)	
$2\lambda_1 + 2\lambda_2 - \lambda_3 + 3\lambda_4 - 2\lambda_5 - 2\lambda_6 + \lambda_7 - \lambda_8 \leq 3$	(1 4 7 3 2)(5 9)(6 10)	
$2\lambda_1 + 3\lambda_2 - 2\lambda_3 + 2\lambda_4 - \lambda_5 - 2\lambda_6 + \lambda_7 - \lambda_8 \leq 3$	(1 2)(3 4 7 5 9)(6 10)	
$3\lambda_1 + 2\lambda_2 - 2\lambda_3 + 2\lambda_4 - 2\lambda_5 - \lambda_6 + \lambda_9 - \lambda_{10} \leq 3$	(3 4 9)(5 7 6 8 10)	
$2\lambda_1 + 2\lambda_2 - \lambda_3 + 3\lambda_4 - 2\lambda_5 - 2\lambda_6 + \lambda_9 - \lambda_{10} \leq 3$	(1 4 9 5 7 3 2)(6 8 10)	
$2\lambda_1 + 3\lambda_2 - 2\lambda_3 + 2\lambda_4 - \lambda_5 - 2\lambda_6 + \lambda_9 - \lambda_{10} \leq 3$	(1 2)(3 4 9)(5 7)(6 8 10)	

$4\lambda_1 + 3\lambda_2 - 4\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_7 + \lambda_9 + 2\lambda_{10} \leq 3$	(3 10)(4 9)(5 6 8 7)	
$4\lambda_1 - 2\lambda_2 - \lambda_3 - \lambda_5 + \lambda_7 + 3\lambda_8 - 4\lambda_9 + 2\lambda_{10} \leq 3$	(2 8 5 4 7 3 10 9)	(1 2 3 … 25 26)

$3\lambda_1 + \lambda_3 + \lambda_5 + 2\lambda_6 - \lambda_7 + \lambda_9 + 2\lambda_{10} \leq 4$	(2 6 9 8 4 3 10 7)	
$3\lambda_1 + \lambda_3 + 2\lambda_4 - \lambda_5 + \lambda_7 + \lambda_9 + 2\lambda_{10} \leq 4$	(2 4 3 10 5 7)(6 9 8)	
$3\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_5 + \lambda_7 + \lambda_9 + 2\lambda_{10} \leq 4$	(3 10)(4 5 7)(6 9 8)	(1 2 3 … 19 20)
$\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_4 - \lambda_5 + \lambda_9 + 2\lambda_{10} \leq 4$	(1 4)(2 3 10 5)(6 9 8 7)	
$3\lambda_1 + \lambda_3 + \lambda_5 + \lambda_7 + 2\lambda_8 - \lambda_9 + 2\lambda_{10} \leq 4$	(2 8 4 3 10 9 6 7)	

$2\lambda_1 + 3\lambda_2 - \lambda_3 + \lambda_4 + \lambda_7 + 2\lambda_8 - \lambda_9 \leq 4$	(1 2)(3 8 10 9)(5 7 6)	(1 2 3 … 14 15)
$\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_4 - \lambda_5 + 2\lambda_8 - \lambda_9 \leq 4$	(1 4)(2 3 8 10 9 5)	
$3\lambda_1 + \lambda_3 + 2\lambda_4 - \lambda_5 + 2\lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(2 4 3 6)(5 9)(7 8 10)	
$3\lambda_1 + \lambda_3 + 2\lambda_4 - \lambda_5 + \lambda_7 + 2\lambda_8 - \lambda_9 \leq 4$	(2 4 3 8 10 9 5 7 6)	
$2\lambda_1 + 2\lambda_3 + 3\lambda_4 - \lambda_5 + \lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(1 4 6 2)(5 9)(7 8 10)	
$3\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_5 + \lambda_7 + 2\lambda_8 - \lambda_9 \leq 4$	(3 8 10 9)(4 5 7 6)	
$3\lambda_1 + \lambda_3 + \lambda_5 + 2\lambda_6 - \lambda_7 + 2\lambda_8 - \lambda_9 \leq 4$	(2 6)(3 8 10 9 7 4)	
$3\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_5 + 2\lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(3 6 4 5 9)(7 8 10)	
$2\lambda_1 + 3\lambda_2 - \lambda_3 + 2\lambda_5 + \lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(1 2)(3 5 9)(4 6)(7 8 10)	
$3\lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 - \lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(1 2 4 3 5 9 6)(7 8 10)	
$2\lambda_2 + 2\lambda_3 + 3\lambda_4 + \lambda_5 - \lambda_6 - \lambda_7 + \lambda_9 \leq 4$	(1 4 5 9 6)(7 8 10)	
$2\lambda_1 + 3\lambda_2 - \lambda_3 + 2\lambda_4 - \lambda_5 + \lambda_7 + \lambda_8 \leq 4$	(1 2)(3 4 7 9)(5 8 10)	
$3\lambda_1 + 2\lambda_2 - \lambda_3 + 2\lambda_4 - \lambda_5 + \lambda_7 + \lambda_9 \leq 4$	(3 4 7 8 10 5 9)	
$2\lambda_1 + 2\lambda_2 + 3\lambda_4 - \lambda_5 - \lambda_6 + \lambda_7 + \lambda_9 \leq 4$	(1 4 7 8 10 6 3 2)(5 9)	
$2\lambda_1 + 3\lambda_2 - \lambda_3 + 2\lambda_4 - \lambda_6 + \lambda_7 + \lambda_9 \leq 4$	(1 2)(3 4 7 8 10 6 5 9)	
$4\lambda_1 - 2\lambda_2 - \lambda_3 - \lambda_5 + \lambda_7 + 2\lambda_8 - 3\lambda_9 + 3\lambda_{10} \leq 3$	(2 10 9)(3 8 5 4 7)	(1 2 3 … 26 27)
$4\lambda_1 + 2\lambda_2 - 3\lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_7 + \lambda_9 + 3\lambda_{10} \leq 3$	(2 10 3)(4 9)(5 6 8 7)	

For the system $\wedge^4\mathcal{H}_{10}$:

Inequalities	$v \in S_{10}$	$w \in S_{210}$
$\lambda_1 \leq 1$	(1)	(1)
$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_5 + \lambda_9 \leq 3$	(4 5 9 8 7 6)	(1 2 3 4 5 6)
$\lambda_1 + \lambda_2 + 2\lambda_3 - \lambda_4 + 2\lambda_5 - 2\lambda_6 - \lambda_7 - 2\lambda_8 \leq 3$	(1 3)(2 5 9 6 10 8 7 4)	(1 2 3 … 14 15)
$\lambda_1 + 2\lambda_2 + \lambda_3 - \lambda_4 + 2\lambda_5 - 2\lambda_6 - 2\lambda_7 - \lambda_8 \leq 3$	(1 2 5 9 6 10 7 4 3)	
$2\lambda_1 + \lambda_2 + 2\lambda_3 - 2\lambda_4 + \lambda_5 - 2\lambda_6 - \lambda_7 - \lambda_8 \leq 3$	(2 3)(4 5 9)(6 10)	
$\lambda_1 + 2\lambda_2 + 2\lambda_3 - 2\lambda_4 + \lambda_5 - \lambda_6 - 2\lambda_7 - \lambda_8 \leq 3$	(1 2 3)(4 5 9)(6 10 7)	

$\lambda_1+2\lambda_2+2\lambda_3-\lambda_4+3\lambda_5-3\lambda_6-2\lambda_7-2\lambda_8\leq 4$	(1 5 9 8 7 4)(6 10)	(1 2 3 . . . 16 17)
$2\lambda_1+2\lambda_2+3\lambda_3-3\lambda_4+\lambda_5-2\lambda_6-2\lambda_7-\lambda_8\leq 4$	(1 3 2)(4 5 9 7 8 6 10)	

$3\lambda_1+\lambda_2+2\lambda_3-\lambda_4+\lambda_6-\lambda_7+\lambda_9-\lambda_{10}\leq 5$	(2 3)(4 6 5 9 7 8)	(1 2 3 . . . 10 11)
$2\lambda_1+\lambda_2+\lambda_3+\lambda_4+3\lambda_5-\lambda_6-\lambda_7-\lambda_8\leq 5$	(1 5 4 3 2)(6 9 7 10 8)	
$3\lambda_1+\lambda_2+\lambda_4+2\lambda_5-\lambda_6-\lambda_7+\lambda_9-\lambda_{10}\leq 5$	(2 5 9 7 8 6 3)	
$3\lambda_1+\lambda_2+2\lambda_3-\lambda_4+\lambda_5-\lambda_6+\lambda_9-\lambda_{10}\leq 5$	(2 3)(4 5 9 6 7 8)	
$2\lambda_1+3\lambda_2+\lambda_3-\lambda_4+\lambda_5-\lambda_6+\lambda_9-\lambda_{10}\leq 5$	(1 2)(4 5 9 6 7 8)	
$2\lambda_1+3\lambda_2+\lambda_5-\lambda_6+\lambda_7-\lambda_8+\lambda_9-\lambda_{10}\leq 5$	(1 2)(3 5 9 8 6)(4 7)	
$2\lambda_1+3\lambda_2+\lambda_3-\lambda_4+\lambda_7-\lambda_8+\lambda_9-\lambda_{10}\leq 5$	(1 2)(4 7 6 5 9 8)	
$3\lambda_1+\lambda_3+\lambda_4+2\lambda_5-\lambda_6-\lambda_8+\lambda_9-\lambda_{10}\leq 5$	(2 5 9 8 6)	
$2\lambda_1+\lambda_2+3\lambda_3-\lambda_4+\lambda_5+\lambda_6-\lambda_7-\lambda_8\leq 5$	(1 3 2)(4 5 6 9 7 10 8)	
$2\lambda_1+3\lambda_2+\lambda_3-\lambda_4+\lambda_5-\lambda_6+\lambda_7-\lambda_8\leq 5$	(1 2)(4 5 7 10 8)(6 9)	

$\lambda_1+\lambda_3+\lambda_5-\lambda_7\leq 2$	(2 3 5 4)(7 8 9 10)	(1 2 3 4 5 6 7)
$\lambda_1+\lambda_4+\lambda_5-\lambda_8\leq 2$	(2 4)(3 5)(8 9 10)	
$\lambda_1+\lambda_2+\lambda_9-\lambda_{10}\leq 2$	(3 9 8 7 6 5 4)	
$\lambda_1+\lambda_2+\lambda_7-\lambda_8\leq 2$	(3 7 6 5 4)(8 9 10)	
$\lambda_2+\lambda_3+\lambda_5-\lambda_8\leq 2$	(1 2 3 5 4)(8 9 10)	
$\lambda_1+\lambda_3+\lambda_6-\lambda_8\leq 2$	(2 3 6 5 4)(8 9 10)	
$\lambda_1+\lambda_2+\lambda_5-\lambda_6\leq 2$	(3 5 4)(6 7 8 9 10)	
$\lambda_1+\lambda_2+\lambda_3-\lambda_4\leq 2$	(4 5 6 7 8 9 10)	

$2\lambda_1+\lambda_2+\lambda_3+\lambda_5+\lambda_8\leq 4$	(4 5 8 7 6)	(1 2 3 4 5)
$2\lambda_1+\lambda_3+\lambda_4+\lambda_5+\lambda_6\leq 4$	(2 3 4 5 6)	
$2\lambda_1+\lambda_2+\lambda_3+\lambda_6+\lambda_7\leq 4$	(4 6)(5 7)	
$2\lambda_1+\lambda_2+\lambda_4+\lambda_5+\lambda_7\leq 4$	(3 4 5 7 6)	
$\lambda_1+\lambda_2+\lambda_3+\lambda_4+2\lambda_5\leq 4$	(1 5 4 3 2)	
$\lambda_1+\lambda_2+2\lambda_3+\lambda_5+\lambda_6\leq 4$	(1 3 2)(4 5 6)	
$\lambda_1+2\lambda_2+\lambda_3+\lambda_5+\lambda_7\leq 4$	(1 2)(4 5 7 6)	

$2\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\lambda_5 + \lambda_9 \leq 5$	(2 5 4 3)(6 9 8 7)	
$2\lambda_1 + 2\lambda_2 + \lambda_3 + \lambda_5 + \lambda_7 + \lambda_9 \leq 5$	(4 5 7)(6 9 8)	(1 2 3 4 5 6 7)
$2\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_5 + \lambda_6 + \lambda_9 \leq 5$	(2 3)(4 5 6 9 8 7)	

$2\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 + \lambda_5 - \lambda_6 - \lambda_7 \leq 3$	(4 5 8)(6 9)(7 10)	
$\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_5 - \lambda_6 - \lambda_7 - \lambda_8 \leq 3$	(1 5 4 3 2)(6 9 7 10 8)	
$\lambda_1 + 2\lambda_2 + \lambda_3 - \lambda_4 + \lambda_5 - \lambda_6 - \lambda_8 \leq 3$	(1 2)(4 5 7 10 8)(6 9)	(1 2 3 ... 10 11)
$\lambda_1 + \lambda_2 + 2\lambda_3 - \lambda_4 + \lambda_5 - \lambda_7 - \lambda_8 \leq 3$	(1 3 2)(4 5 6 9 7 10 8)	

$2\lambda_1 + 3\lambda_2 + 4\lambda_3 - \lambda_4 + 5\lambda_5 - 4\lambda_6 - 2\lambda_7 - 3\lambda_8 + \lambda_{10} \leq 8$	(1 5 10 6 9 8 7 4)(2 3)	
$2\lambda_1 + 3\lambda_2 + 4\lambda_3 - \lambda_4 + 5\lambda_5 - 4\lambda_6 - 3\lambda_7 - 2\lambda_8 + \lambda_9 \leq 8$	(1 5 9 7 4)(2 3)(6 10)	
$3\lambda_1 + 4\lambda_2 + 5\lambda_3 - 4\lambda_4 + 2\lambda_5 - 2\lambda_6 - 3\lambda_7 - \lambda_8 + \lambda_{10} \leq 8$	(1 3)(4 5 10)(6 9 7 8)	
$4\lambda_1 + 3\lambda_2 + 5\lambda_3 - 4\lambda_4 + 2\lambda_5 - 3\lambda_6 - 2\lambda_7 - \lambda_8 + \lambda_{10} \leq 8$	(1 3 2)(4 5 10)(6 9)(7 8)	(1 2 3 ... 18 19)
$3\lambda_1 + 4\lambda_2 + 5\lambda_3 - 4\lambda_4 + 2\lambda_5 - 3\lambda_6 - 2\lambda_7 - \lambda_8 + \lambda_9 \leq 8$	(1 3)(4 5 9 6 10)(7 8)	
$2\lambda_1 + 4\lambda_2 + 3\lambda_3 - \lambda_4 + 5\lambda_5 - 4\lambda_6 - 3\lambda_7 - 2\lambda_8 + \lambda_{10} \leq 8$	(1 5 10 6 9 7 4)	

$3\lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4 + 3\lambda_5 + \lambda_8 + 2\lambda_9 \leq 8$	(2 5 9 7 8 6)	(1 2 3 ... 11 12)

$2\lambda_1 + \lambda_2 + 3\lambda_3 + \lambda_5 + 2\lambda_6 - \lambda_7 + \lambda_{10} \leq 6$	(1 3 6 10 7 4 2)	
$\lambda_1 + 2\lambda_2 + 3\lambda_3 + 2\lambda_5 + \lambda_6 - \lambda_7 + \lambda_{10} \leq 6$	(1 3 5 6 10 7 4)	
$3\lambda_1 + 2\lambda_2 + \lambda_5 + \lambda_6 + 2\lambda_7 - \lambda_8 + \lambda_9 \leq 6$	(3 7)(4 5 6 9 10 8)	
$3\lambda_1 + 2\lambda_2 + \lambda_5 + \lambda_6 + \lambda_7 + 2\lambda_9 - \lambda_{10} \leq 6$	(3 9 8 4 5 6 7)	
$3\lambda_1 + 2\lambda_2 + \lambda_3 + \lambda_7 + \lambda_8 + 2\lambda_9 - \lambda_{10} \leq 6$	(3 9 6 8 5 7 4)	
$3\lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_5 - \lambda_7 + \lambda_9 + \lambda_{10} \leq 6$	(2 3 5 9 8 6 10 7 4)	
$3\lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_5 - \lambda_6 + \lambda_8 + \lambda_{10} \leq 6$	(2 3 5 8 7 4)(6 10)	
$3\lambda_1 + \lambda_3 + 2\lambda_4 + 2\lambda_5 + \lambda_6 - \lambda_7 + \lambda_{10} \leq 6$	(2 4 3 5 6 10 7)	
$3\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5 + 2\lambda_6 - \lambda_7 + \lambda_{10} \leq 6$	(2 3 6 10 7)	(1 2 3 ... 12 13)
$3\lambda_1 + 2\lambda_2 + \lambda_3 + 2\lambda_5 - \lambda_6 + \lambda_9 + \lambda_{10} \leq 6$	(3 5 9 8 7 4)(6 10)	
$3\lambda_1 + 2\lambda_2 + 2\lambda_3 - \lambda_4 + \lambda_7 + \lambda_8 + \lambda_9 \leq 6$	(4 7 5 8 6 9 10)	
$2\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + 3\lambda_5 - \lambda_7 + \lambda_{10} \leq 6$	(1 5 3 4 2)(6 10 7)	
$2\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + 3\lambda_5 - \lambda_6 + \lambda_9 \leq 6$	(1 5 3 4 2)(6 9 10)	
$3\lambda_1 + 2\lambda_2 + 2\lambda_3 - \lambda_4 + \lambda_5 + \lambda_9 + \lambda_{10} \leq 6$	(4 5 9 8 7 6 10)	
$\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4 + 3\lambda_5 - \lambda_7 + \lambda_{10} \leq 6$	(1 5 4)(6 10 7)	
$2\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4 + 3\lambda_5 - \lambda_6 + \lambda_{10} \leq 6$	(1 5 4 2)(6 10)	
$\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4 + 3\lambda_5 - \lambda_6 + \lambda_9 \leq 6$	(1 5 4)(6 9 10)	

$2\lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4 + 5\lambda_5 - 2\lambda_6 - 2\lambda_7 - \lambda_8 \leq 8$	(1 5 4 2 3)(6 9)(7 10)	
$3\lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4 + 5\lambda_5 - 2\lambda_6 - 2\lambda_7 - \lambda_8 \leq 8$	(1 5 2)(6 9)(7 10)	
$5\lambda_1 + \lambda_3 + 2\lambda_4 + 3\lambda_5 - 2\lambda_6 - \lambda_7 + 2\lambda_9 - 2\lambda_{10} \leq 8$	(2 5 3 4 9 6)(7 8)	
$5\lambda_1 + 2\lambda_2 + 3\lambda_3 - 2\lambda_4 - \lambda_6 + \lambda_8 + 2\lambda_9 - 2\lambda_{10} \leq 8$	(2 3)(4 9)(5 8 6)	
$5\lambda_1 + 3\lambda_2 - \lambda_3 + \lambda_6 + 2\lambda_7 - 2\lambda_8 + 2\lambda_9 - 2\lambda_{10} \leq 8$	(3 7 5 6 4 9 8)	
$5\lambda_1 + 3\lambda_2 + 2\lambda_3 - 2\lambda_4 - \lambda_5 + \lambda_8 + 2\lambda_9 - 2\lambda_{10} \leq 8$	(4 9)(5 8)	(1 2 3 ... 14 15)
$5\lambda_1 + 2\lambda_2 + 3\lambda_3 - 2\lambda_4 + 2\lambda_5 - 2\lambda_6 - \lambda_7 + \lambda_9 \leq 8$	(2 3)(4 5 9)(6 8 7 10)	
$5\lambda_1 + 2\lambda_2 + 2\lambda_3 - \lambda_4 + 3\lambda_5 - 2\lambda_6 - 2\lambda_7 + \lambda_{10} \leq 8$	(2 5 10 7 9 6 8 4 3)	
$5\lambda_1 + 2\lambda_2 + 3\lambda_3 - 2\lambda_4 + 2\lambda_5 - \lambda_6 - 2\lambda_7 + \lambda_{10} \leq 8$	(2 3)(4 5 10 7 9)(6 8)	
$5\lambda_1 + 3\lambda_2 + 2\lambda_3 - 2\lambda_4 + 2\lambda_5 - 2\lambda_6 - \lambda_7 + \lambda_{10} \leq 8$	(4 5 10 6 8 7 9)	

$2\lambda_1 + 3\lambda_2 + \lambda_3 - 2\lambda_4 + \lambda_5 - 2\lambda_6 - \lambda_8 - \lambda_9 \leq 4$	(1 2)(4 5 7 8 9)(6 10)	
$2\lambda_1 + 3\lambda_2 - \lambda_4 + \lambda_5 - 2\lambda_6 + \lambda_7 - 2\lambda_8 - \lambda_9 \leq 4$	(1 2)(3 5)(4 7)(6 10 8 9)	
$\lambda_1 + 3\lambda_2 + \lambda_3 - \lambda_4 + 2\lambda_5 - 2\lambda_6 - 2\lambda_8 - \lambda_9 \leq 4$	(1 2 5 7 4 3)(6 10 8 9)	
$2\lambda_1 + 3\lambda_3 - \lambda_4 + \lambda_5 + \lambda_6 - 2\lambda_7 - 2\lambda_8 - \lambda_9 \leq 4$	(1 3 5 2)(4 6 10 8 9 7)	
$2\lambda_1 + \lambda_2 + 3\lambda_3 - 2\lambda_4 + \lambda_6 - \lambda_7 - 2\lambda_8 - \lambda_9 \leq 4$	(1 3 2)(4 6 10 8 9)	
$\lambda_1 + \lambda_2 + 3\lambda_3 - \lambda_4 + 2\lambda_5 - 2\lambda_7 - 2\lambda_8 - \lambda_9 \leq 4$	(1 3)(2 5 6 10 8 9 7 4)	
$\lambda_1 + 2\lambda_2 + 3\lambda_3 - 2\lambda_4 + \lambda_5 - \lambda_7 - 2\lambda_8 - \lambda_9 \leq 4$	(1 3)(4 5 6 10 8 9)	
$2\lambda_1 + 3\lambda_2 + \lambda_3 - 2\lambda_4 - \lambda_6 + \lambda_7 - 2\lambda_8 - \lambda_9 \leq 4$	(1 2)(4 7 6 10 8 9)	
$\lambda_1 + 3\lambda_2 + 2\lambda_3 - 2\lambda_4 + \lambda_5 - \lambda_6 - 2\lambda_8 - \lambda_9 \leq 4$	(1 2 3)(4 5 7 6 10 8 9)	
$3\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 + 2\lambda_5 - 2\lambda_6 - 2\lambda_8 - \lambda_9 \leq 4$	(2 5 7 3)(6 10 8 9)	
$3\lambda_1 + \lambda_2 + 2\lambda_3 - 2\lambda_4 - \lambda_5 + \lambda_6 - 2\lambda_8 - \lambda_9 \leq 4$	(2 3)(4 6 10 8 9)(5 7)	
$3\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_5 - 2\lambda_6 + \lambda_7 - 2\lambda_8 - \lambda_9 \leq 4$	(3 5 4 7)(6 10 8 9)	
$3\lambda_1 + 2\lambda_2 + \lambda_3 - 2\lambda_4 - \lambda_5 + \lambda_7 - 2\lambda_8 - \lambda_9 \leq 4$	(4 7 5 6 10 8 9)	(1 2 3 ... 12 13)
$3\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4 + 2\lambda_5 - 2\lambda_7 - 2\lambda_8 - \lambda_9 \leq 4$	(2 5 6 10 8 9 7)	
$3\lambda_1 - \lambda_2 + 2\lambda_3 + \lambda_5 + \lambda_6 - 2\lambda_7 - 2\lambda_8 - \lambda_9 \leq 4$	(2 3 5 4 6 10 8 9 7)	
$2\lambda_1 + \lambda_2 + 3\lambda_3 - 2\lambda_4 + \lambda_5 - 2\lambda_7 - \lambda_8 - \lambda_9 \leq 4$	(1 3 2)(4 5 6 10 7 8 9)	
$3\lambda_1 + \lambda_2 + 2\lambda_3 - 2\lambda_4 + \lambda_5 - \lambda_6 - 2\lambda_7 - \lambda_9 \leq 4$	(2 3)(4 5 8 9)(6 10 7)	
$3\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 + 2\lambda_5 - 2\lambda_6 - 2\lambda_7 - \lambda_9 \leq 4$	(2 5 8 9 6 10 7 4 3)	
$3\lambda_1 + 2\lambda_2 + \lambda_3 - 2\lambda_4 + \lambda_5 - 2\lambda_6 - \lambda_7 - \lambda_9 \leq 4$	(4 5 8 9)(6 10)	
$2\lambda_1 + \lambda_2 + \lambda_4 + 3\lambda_5 - 2\lambda_6 - \lambda_7 - 2\lambda_8 - \lambda_9 \leq 4$	(1 5 3 2)(6 10 8 9)	
$\lambda_1 + 2\lambda_2 + \lambda_3 + 3\lambda_5 - 2\lambda_6 - \lambda_7 - 2\lambda_8 - \lambda_9 \leq 4$	(1 5 4 3)(6 10 8 9)	
$2\lambda_1 + \lambda_3 + \lambda_4 + 3\lambda_5 - \lambda_6 - 2\lambda_7 - 2\lambda_8 - \lambda_9 \leq 4$	(1 5 2)(6 10 8 9 7)	
$\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_5 - \lambda_6 - 2\lambda_7 - 2\lambda_8 - \lambda_9 \leq 4$	(1 5 4 2 3)(6 10 8 9 7)	
$2\lambda_1 + \lambda_2 + \lambda_3 + 3\lambda_5 - 2\lambda_6 - 2\lambda_7 - \lambda_8 - \lambda_9 \leq 4$	(1 5 4 3 2)(6 10 7 8 9)	

$3\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_5 - 2\lambda_6 - \lambda_7 + \lambda_9 - 2\lambda_{10} \leq 4$	(3 5 4 9 6 8 7)	
$\lambda_1 + 2\lambda_2 + 3\lambda_3 - 2\lambda_4 + \lambda_5 - \lambda_6 - 2\lambda_8 - \lambda_{10} \leq 4$	(1 3)(4 5 7 6 9)(8 10)	
$2\lambda_1 + 3\lambda_2 + \lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_8 + \lambda_9 - 2\lambda_{10} \leq 4$	(1 2)(4 9)(5 6 7)	
$2\lambda_1 + 3\lambda_2 + \lambda_3 - 2\lambda_4 - \lambda_5 + \lambda_7 - 2\lambda_8 - \lambda_{10} \leq 4$	(1 2)(4 7 5 6 9)(8 10)	
$2\lambda_1 + 3\lambda_2 - \lambda_3 + \lambda_5 - 2\lambda_6 + \lambda_7 - 2\lambda_8 - \lambda_{10} \leq 4$	(1 2)(3 5 4 7)(6 9)(8 10)	
$2\lambda_1 + 3\lambda_2 - \lambda_3 - \lambda_6 + \lambda_7 - 2\lambda_8 + \lambda_9 - 2\lambda_{10} \leq 4$	(1 2)(3 7)(4 9 8 6 5)	
$2\lambda_1 + 3\lambda_2 - \lambda_3 + \lambda_5 - 2\lambda_6 - \lambda_8 + \lambda_9 - 2\lambda_{10} \leq 4$	(1 2)(3 5 4 9 6 7)	
$2\lambda_1 + 3\lambda_2 + \lambda_3 - 2\lambda_4 - \lambda_6 - \lambda_7 + \lambda_9 - 2\lambda_{10} \leq 4$	(1 2)(4 9)(6 8 7)	
$2\lambda_1 + 3\lambda_2 - \lambda_4 + \lambda_5 - 2\lambda_6 - \lambda_7 + \lambda_9 - 2\lambda_{10} \leq 4$	(1 2)(3 5)(4 9 6 8 7)	
$2\lambda_1 + 3\lambda_2 - \lambda_4 - \lambda_5 + \lambda_7 - 2\lambda_8 + \lambda_9 - 2\lambda_{10} \leq 4$	(1 2)(3 7 4 9 8 5)	
$3\lambda_1 + 2\lambda_2 - \lambda_3 - \lambda_5 + \lambda_7 - 2\lambda_8 + \lambda_9 - 2\lambda_{10} \leq 4$	(3 7)(4 9 8 5)	
$3\lambda_1 + 2\lambda_2 + \lambda_3 - 2\lambda_4 - \lambda_5 - \lambda_7 + \lambda_9 - 2\lambda_{10} \leq 4$	(4 9)(5 6 8 7)	(1 2 3 … 12 13)
$2\lambda_1 + 3\lambda_2 + \lambda_3 - 2\lambda_4 + \lambda_5 - 2\lambda_6 - \lambda_7 - \lambda_{10} \leq 4$	(1 2)(4 5 8 10 6 9)	
$2\lambda_1 + \lambda_2 + \lambda_4 + 3\lambda_5 - 2\lambda_6 - 2\lambda_7 - \lambda_8 - \lambda_{10} \leq 4$	(1 5 3 2)(6 9)(7 8 10)	
$3\lambda_1 + \lambda_2 + 2\lambda_3 - 2\lambda_4 + \lambda_5 - 2\lambda_6 - \lambda_7 - \lambda_{10} \leq 4$	(2 3)(4 5 8 10 6 9)	
$2\lambda_1 + \lambda_3 + \lambda_4 + 3\lambda_5 - 2\lambda_6 - \lambda_7 - 2\lambda_8 - \lambda_{10} \leq 4$	(1 5 2)(6 9)(8 10)	
$2\lambda_1 + \lambda_2 + 3\lambda_3 - 2\lambda_4 + \lambda_6 - 2\lambda_7 - \lambda_8 - \lambda_{10} \leq 4$	(1 3 2)(4 6 9)(7 8 10)	
$3\lambda_1 + \lambda_4 + 2\lambda_5 - 2\lambda_6 - \lambda_7 - \lambda_8 + \lambda_9 - 2\lambda_{10} \leq 4$	(2 5)(3 4 9 6)	
$3\lambda_1 + \lambda_2 + 2\lambda_3 - 2\lambda_4 - \lambda_6 - \lambda_7 + \lambda_9 - 2\lambda_{10} \leq 4$	(2 3)(4 9)(6 8 7)	
$\lambda_1 + 2\lambda_2 + \lambda_3 + 3\lambda_5 - 2\lambda_6 - 2\lambda_7 - \lambda_8 - \lambda_{10} \leq 4$	(1 5 4 3)(6 9)(7 8 10)	
$\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_5 - 2\lambda_6 - \lambda_7 - 2\lambda_8 - \lambda_{10} \leq 4$	(1 5 4 2 3)(6 9)(8 10)	
$\lambda_1 + 2\lambda_2 + 3\lambda_3 - 2\lambda_4 + \lambda_5 - 2\lambda_7 - \lambda_8 - \lambda_{10} \leq 4$	(1 3)(4 5 6 9)(7 8 10)	

$2\lambda_1 + 3\lambda_2 + 4\lambda_3 - 2\lambda_4 + 2\lambda_5 - 2\lambda_7 - \lambda_8 + \lambda_{10} \leq 7$	(1 3)(4 5 10 7 9)	
$2\lambda_1 + 2\lambda_2 + 3\lambda_3 + 4\lambda_5 - 2\lambda_6 - \lambda_7 - 2\lambda_8 + \lambda_{10} \leq 7$	(1 5 10 8 7 9 6 4 2 3)	
$2\lambda_1 + 2\lambda_2 + 3\lambda_3 + 4\lambda_5 - 2\lambda_6 - 2\lambda_7 - \lambda_8 + \lambda_9 \leq 7$	(1 5 9 6 4 2 3)(7 10)	(1 2 3 … 15 16)
$4\lambda_1 + 2\lambda_2 + 3\lambda_3 - 2\lambda_4 + 2\lambda_5 - 2\lambda_6 - \lambda_7 + \lambda_{10} \leq 7$	(2 3)(4 5 10 6 8 7 9)	
$2\lambda_1 + 3\lambda_2 + 2\lambda_3 + 4\lambda_5 - 2\lambda_6 - 2\lambda_7 - \lambda_8 + \lambda_{10} \leq 7$	(1 5 10 7 9 6 4 3)	

For the system $\wedge^5\mathcal{H}_{10}$:

Inequalities	$v \in S_{10}$	$w \in S_{252}$
$\lambda_1 \leq 1$	(1)	(1)
$\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6+\lambda_7 \leq 4$	(1 2 3 4 5 6 7)	(1 2 3 4 5 6 7)
$\lambda_1+\lambda_2+\lambda_3+\lambda_{10} \leq 3$	(4 10 9 8 7 6 5)	(1 2 3 4 5 6 7)
$\lambda_2+\lambda_3+\lambda_4-\lambda_5-\lambda_9-\lambda_{10} \leq 2$	(1 2 3 4)(5 6 7 8)	(1 2 3 4 5 6 7)
$\lambda_1+\lambda_2+\lambda_6-\lambda_7-\lambda_8-\lambda_9 \leq 2$	(3 6 5 4)(7 10 9 8)	
$\lambda_2+\lambda_3+\lambda_6-\lambda_7-\lambda_9-\lambda_{10} \leq 2$	(1 2 3 6 5 4)(7 8)	
$\lambda_1+\lambda_2+\lambda_3-\lambda_5-\lambda_7-\lambda_9 \leq 2$	(5 6 8)(7 10 9)	
$\lambda_1+\lambda_2+\lambda_4-\lambda_5-\lambda_8-\lambda_9 \leq 2$	(3 4)(5 6 7 10 9 8)	
$\lambda_2+\lambda_4+\lambda_6-\lambda_8-\lambda_9-\lambda_{10} \leq 2$	(1 2 4)(3 6 5)	
$2\lambda_1+2\lambda_2+3\lambda_3-\lambda_4-\lambda_6+\lambda_8-2\lambda_9+\lambda_{10} \leq 6$	(1 3 2)(4 8)(5 10 9 6)	(1 2 3 … 14 15)
$2\lambda_1+2\lambda_2+3\lambda_3+\lambda_4-2\lambda_5-\lambda_6-\lambda_8+\lambda_{10} \leq 6$	(1 3 2)(5 10)(6 7 9 8)	
$\lambda_2+2\lambda_3+3\lambda_4+\lambda_6+2\lambda_7-2\lambda_8-\lambda_9-\lambda_{10} \leq 6$	(1 4 2 3 7 5 6)(8 9 10)	
$2\lambda_1+2\lambda_2+3\lambda_3-\lambda_4+\lambda_6-2\lambda_7-\lambda_8+\lambda_{10} \leq 6$	(1 3 2)(4 6 5 10 7 9 8)	
$3\lambda_2+\lambda_4+2\lambda_5+\lambda_6+2\lambda_7-2\lambda_8-\lambda_9-\lambda_{10} \leq 6$	(1 2 5 6)(3 7)(8 9 10)	
$3\lambda_2+\lambda_4+\lambda_5+2\lambda_6+2\lambda_7-\lambda_8-2\lambda_9-\lambda_{10} \leq 6$	(1 2 6)(3 7)(9 10)	
$\lambda_2+2\lambda_3+\lambda_4+2\lambda_5+3\lambda_6-2\lambda_8-\lambda_9-\lambda_{10} \leq 6$	(1 6)(2 3 5 4)(8 9 10)	
$2\lambda_1+3\lambda_2+2\lambda_3+\lambda_4-2\lambda_5-\lambda_6-\lambda_7+\lambda_{10} \leq 6$	(1 2)(5 10)(6 8)(7 9)	
$\lambda_2+\lambda_3+2\lambda_4+2\lambda_5+3\lambda_6-\lambda_8-2\lambda_9-\lambda_{10} \leq 6$	(1 6)(2 4)(3 5)(9 10)	
$2\lambda_1+3\lambda_2+2\lambda_3-\lambda_4-\lambda_5+\lambda_8-2\lambda_9+\lambda_{10} \leq 6$	(1 2)(4 8)(5 10 9)	

$\lambda_1 + 2\lambda_2 + \lambda_3 + 3\lambda_4 + 3\lambda_6 - 2\lambda_8 - 2\lambda_9 - \lambda_{10} \leq 7$	(1 4)(2 6 5 3)(8 10 9)	
$\lambda_1 + 3\lambda_2 + 3\lambda_4 + 2\lambda_6 + \lambda_7 - 2\lambda_8 - 2\lambda_9 - \lambda_{10} \leq 7$	(1 2 4)(3 6)(5 7)(8 10 9)	
$2\lambda_1 + 3\lambda_2 + 3\lambda_4 - \lambda_5 + \lambda_6 + \lambda_7 - 2\lambda_8 - 2\lambda_9 \leq 7$	(1 2 4 6 3)(5 7 10 9 8)	
$2\lambda_1 + 3\lambda_2 + \lambda_3 + 3\lambda_4 - 2\lambda_5 + \lambda_6 - 2\lambda_8 - \lambda_9 \leq 7$	(1 2 4 3)(5 6 7 10 8 9)	
$\lambda_1 + 3\lambda_2 + 2\lambda_4 + \lambda_5 + 3\lambda_6 - 2\lambda_8 - 2\lambda_9 - \lambda_{10} \leq 7$	(1 2 6 3 4)(8 10 9)	
$2\lambda_1 + 3\lambda_2 + 3\lambda_3 + \lambda_4 - 2\lambda_5 - \lambda_7 + \lambda_8 - 2\lambda_9 \leq 7$	(1 2 3)(5 8 7 10 9)	
$2\lambda_1 + 3\lambda_2 + 3\lambda_3 + \lambda_4 - 2\lambda_5 + \lambda_6 - 2\lambda_7 - \lambda_9 \leq 7$	(1 2 3)(5 6 8 9)(7 10)	
$2\lambda_1 + 3\lambda_2 + \lambda_3 + 3\lambda_4 - 2\lambda_5 + \lambda_7 - \lambda_8 - 2\lambda_9 \leq 7$	(1 2 4 3)(5 7 10 9)	
$\lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_5 + 3\lambda_6 - 2\lambda_7 - \lambda_9 - 2\lambda_{10} \leq 7$	(1 3 2 6 4)(7 8 9)	
$\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_5 + 3\lambda_6 - 2\lambda_7 - 2\lambda_9 - \lambda_{10} \leq 7$	(1 2 6 4)(7 8 10 9)	
$\lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4 + 3\lambda_6 - 2\lambda_7 - 2\lambda_9 - \lambda_{10} \leq 7$	(1 3 2 6 5 4)(7 8 10 9)	(1 2 3 … 10 11)
$2\lambda_1 + 3\lambda_2 + \lambda_4 + \lambda_5 + 3\lambda_6 - \lambda_7 - 2\lambda_8 - 2\lambda_9 \leq 7$	(1 2 6 3)(7 10 9 8)	
$2\lambda_1 + 3\lambda_2 + \lambda_3 + \lambda_4 + 3\lambda_6 - 2\lambda_7 - 2\lambda_8 - \lambda_9 \leq 7$	(1 2 6 5 4 3)(7 10 8 9)	
$2\lambda_1 + 3\lambda_2 + 3\lambda_3 - \lambda_5 + \lambda_6 - 2\lambda_7 + \lambda_8 - 2\lambda_9 \leq 7$	(1 2 3)(4 6)(5 8)(7 10 9)	
$\lambda_1 + 3\lambda_2 + 3\lambda_3 + 2\lambda_6 - 2\lambda_7 + \lambda_8 - 2\lambda_9 - \lambda_{10} \leq 7$	(1 2 3 6 4)(5 8 10 9 7)	
$3\lambda_1 + 2\lambda_2 + \lambda_3 + 3\lambda_4 - 2\lambda_5 + \lambda_7 - 2\lambda_8 - \lambda_9 \leq 7$	(2 4 3)(5 7 10 8 9)	
$\lambda_1 + 3\lambda_2 + 3\lambda_3 + 2\lambda_4 - 2\lambda_5 + \lambda_8 - 2\lambda_9 - \lambda_{10} \leq 7$	(1 2 3 4)(5 8 10 9)	
$\lambda_1 + 2\lambda_2 + 3\lambda_3 + 3\lambda_4 - 2\lambda_5 + \lambda_6 - 2\lambda_9 - \lambda_{10} \leq 7$	(1 3 2 4)(5 6 7 8 10 9)	
$2\lambda_1 + 3\lambda_2 + \lambda_3 + \lambda_5 + 3\lambda_6 - 2\lambda_7 - \lambda_8 - 2\lambda_9 \leq 7$	(1 2 6 4 3)(7 10 9)	
$\lambda_1 + 3\lambda_2 + 2\lambda_3 + 3\lambda_4 - 2\lambda_5 + \lambda_7 - 2\lambda_9 - \lambda_{10} \leq 7$	(1 2 4)(5 7 8 10 9)	
$3\lambda_1 + 2\lambda_2 + 3\lambda_3 - \lambda_4 + \lambda_6 - 2\lambda_7 + \lambda_8 - 2\lambda_9 \leq 7$	(2 3)(4 6 5 8)(7 10 9)	
$\lambda_1 + 3\lambda_2 + 3\lambda_4 + \lambda_6 + 2\lambda_7 - 2\lambda_8 - \lambda_9 - 2\lambda_{10} \leq 7$	(1 2 4)(3 7 5 6)(8 9)	
$3\lambda_1 + 2\lambda_2 + \lambda_4 + \lambda_5 + 3\lambda_6 - 2\lambda_7 - \lambda_8 - 2\lambda_9 \leq 7$	(2 6 3)(7 10 9)	

$\lambda_1 + 3\lambda_2 + 2\lambda_3 + 3\lambda_4 - 2\lambda_5 + \lambda_8 - \lambda_9 - 2\lambda_{10} \leq 7$	(1 2 4)(5 8 9)	
$\lambda_1 + 2\lambda_2 + 3\lambda_3 + 3\lambda_4 - 2\lambda_5 + \lambda_7 - \lambda_9 - 2\lambda_{10} \leq 7$	(1 3 2 4)(5 7 8 9)	
$3\lambda_1 + 2\lambda_2 + \lambda_3 + \lambda_5 + 3\lambda_6 - 2\lambda_7 - 2\lambda_8 - \lambda_9 \leq 7$	(2 6 4 3)(7 10 8 9)	
$3\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_4 - 2\lambda_5 + \lambda_6 - \lambda_7 - 2\lambda_9 \leq 7$	(2 4)(5 6 8 7 10 9)	
$\lambda_1 + 3\lambda_2 + \lambda_3 + 2\lambda_4 + 3\lambda_6 - 2\lambda_7 - \lambda_8 - 2\lambda_{10} \leq 7$	(1 2 6 5 3 4)(7 9)	
$3\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_4 - 2\lambda_5 + \lambda_6 - 2\lambda_8 - \lambda_9 \leq 7$	(2 4)(5 6 7 10 8 9)	
$\lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4 + 3\lambda_6 - 2\lambda_7 - \lambda_8 - 2\lambda_{10} \leq 7$	(1 3 2 6 5 4)(7 9)	
$\lambda_1 + 3\lambda_2 + 2\lambda_3 + 3\lambda_4 - 2\lambda_5 + \lambda_6 - \lambda_7 - 2\lambda_{10} \leq 7$	(1 2 4)(5 6 8 7 9)	
$3\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + 3\lambda_6 - 2\lambda_7 - \lambda_8 - 2\lambda_9 \leq 7$	(2 6 5 3 4)(7 10 9)	(1 2 3 ... 10 11)
$3\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4 + 3\lambda_6 - 2\lambda_7 - 2\lambda_8 - \lambda_9 \leq 7$	(2 6 5 4)(7 10 8 9)	
$\lambda_1 + 2\lambda_2 + 3\lambda_3 + 3\lambda_4 - 2\lambda_5 + \lambda_6 - \lambda_8 - 2\lambda_{10} \leq 7$	(1 3 2 4)(5 6 7 9)	
$3\lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4 - 2\lambda_5 - \lambda_6 + \lambda_8 - 2\lambda_9 \leq 7$	(2 3)(5 8 6 7 10 9)	
$\lambda_1 + 3\lambda_2 + \lambda_4 + 2\lambda_5 + 3\lambda_6 - 2\lambda_8 - \lambda_9 - 2\lambda_{10} \leq 7$	(1 2 6 3 5 4)(8 9)	
$\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_4 + 3\lambda_6 - 2\lambda_8 - \lambda_9 - 2\lambda_{10} \leq 7$	(1 4)(2 6 5)(8 9)	
$3\lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4 - 2\lambda_5 + \lambda_6 - 2\lambda_7 - \lambda_8 \leq 7$	(2 3)(5 6 9)(7 10)	
$\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_4 + 3\lambda_6 - \lambda_7 - 2\lambda_9 - 2\lambda_{10} \leq 7$	(1 4)(2 6 5)(7 8)	
$3\lambda_1 + 3\lambda_2 + \lambda_3 + 2\lambda_4 - 2\lambda_5 + \lambda_6 - 2\lambda_7 - \lambda_8 \leq 7$	(3 4)(5 6 9)(7 10)	

$3\lambda_1 + 4\lambda_2 + 5\lambda_3 + \lambda_4 - 4\lambda_5 - 3\lambda_6 - 2\lambda_7 - \lambda_8 + 2\lambda_{10} \leq 9$	(1 3)(4 10 5)(6 9)(7 8)	(1 2 3 ... 19 20)
$3\lambda_1 + 4\lambda_2 + 5\lambda_3 - 3\lambda_4 - 2\lambda_5 - \lambda_6 + \lambda_8 - 4\lambda_9 + 2\lambda_{10} \leq 9$	(1 3)(4 10 9)(5 8)(6 7)	

$-\lambda_1 + 5\lambda_2 + \lambda_4 + 2\lambda_5 + 3\lambda_6 + 4\lambda_7 - 4\lambda_8 - 3\lambda_9 - 2\lambda_{10} \leq 9$	(1 2 7)(3 6)(4 5)(8 10)	(1 2 3 ... 19 20)
$-\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_4 + 4\lambda_5 + 5\lambda_6 - 4\lambda_8 - 3\lambda_9 - 2\lambda_{10} \leq 9$	(1 6 7)(2 5)(3 4)(8 10)	

$\lambda_1 + 2\lambda_2 + 2\lambda_4 - \lambda_5 + \lambda_7 - 2\lambda_8 - \lambda_9 \leq 4$	(1 2 4 7 10 8 5 3)	
$\lambda_2 + 2\lambda_3 + \lambda_5 + 2\lambda_6 - \lambda_7 - 2\lambda_9 - \lambda_{10} \leq 4$	(1 3 2 6 4 5)(7 8)(9 10)	
$2\lambda_1 + 2\lambda_2 + \lambda_3 - \lambda_4 - \lambda_5 + \lambda_8 - 2\lambda_9 \leq 4$	(4 8)(5 6 7 10 9)	
$\lambda_2 + 2\lambda_3 + 2\lambda_4 - \lambda_5 + \lambda_8 - \lambda_9 - 2\lambda_{10} \leq 4$	(1 3 2 4 8 5)	
$2\lambda_1 + \lambda_3 + \lambda_4 + 2\lambda_6 - 2\lambda_7 - \lambda_8 - \lambda_9 \leq 4$	(2 6 5)(7 10)	
$\lambda_2 + 2\lambda_3 + 2\lambda_4 - \lambda_5 + \lambda_7 - 2\lambda_9 - \lambda_{10} \leq 4$	(1 3 2 4 7 8 5)(9 10)	
$2\lambda_2 + \lambda_3 + 2\lambda_4 - \lambda_5 + \lambda_8 - 2\lambda_9 - \lambda_{10} \leq 4$	(1 2 4 8 5)(9 10)	
$2\lambda_1 + \lambda_2 + \lambda_5 + 2\lambda_6 - 2\lambda_7 - \lambda_8 - \lambda_9 \leq 4$	(2 6 4 5 3)(7 10)	
$2\lambda_1 + 2\lambda_2 + \lambda_3 + \lambda_4 - 2\lambda_5 - \lambda_6 - \lambda_7 \leq 4$	(5 8 6 9 7 10)	(1 2 3 … 11 12)
$\lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_6 - \lambda_7 - 2\lambda_9 - \lambda_{10} \leq 4$	(1 4 3 2 6 5)(7 8)(9 10)	
$\lambda_1 + 2\lambda_2 + \lambda_4 + 2\lambda_6 - 2\lambda_7 - \lambda_8 - \lambda_9 \leq 4$	(1 2 6 5 3)(7 10)	
$\lambda_1 + 2\lambda_2 + \lambda_5 + 2\lambda_6 - \lambda_7 - 2\lambda_8 - \lambda_9 \leq 4$	(1 2 6 4 5 3)(7 10 8)	
$2\lambda_2 + \lambda_4 + \lambda_5 + 2\lambda_6 - 2\lambda_8 - \lambda_9 - \lambda_{10} \leq 4$	(1 2 6 3 4 5)(8 9 10)	
$\lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_6 - 2\lambda_8 - \lambda_9 - \lambda_{10} \leq 4$	(1 4 3 2 6 5)(8 9 10)	
$2\lambda_2 + 2\lambda_4 + \lambda_6 + \lambda_7 - 2\lambda_8 - \lambda_9 - \lambda_{10} \leq 4$	(1 2 4 7 5)(3 6)(8 9 10)	

$2\lambda_2 - \lambda_3 + \lambda_6 + \lambda_7 - \lambda_8 - 2\lambda_9 - 2\lambda_{10} \leq 2$	(1 2 6 5 4)(3 7)	
$\lambda_2 + 2\lambda_3 - \lambda_4 + \lambda_6 - 2\lambda_7 - 2\lambda_9 - \lambda_{10} \leq 2$	(1 3 6 8 10 9 7 4)	
$\lambda_1 + 2\lambda_2 + \lambda_4 - 2\lambda_5 - \lambda_6 - 2\lambda_8 - \lambda_9 \leq 2$	(1 2)(3 4)(5 7 6 10 8 9)	
$2\lambda_1 + \lambda_2 - \lambda_3 + \lambda_6 - 2\lambda_7 - 2\lambda_8 - \lambda_9 \leq 2$	(3 6 10 8 9 7)	
$\lambda_2 + \lambda_3 + 2\lambda_4 - 2\lambda_5 - \lambda_7 - \lambda_8 - 2\lambda_{10} \leq 2$	(1 4)(5 6 9)	
$\lambda_1 + 2\lambda_2 - \lambda_4 + \lambda_6 - 2\lambda_7 - 2\lambda_8 - \lambda_9 \leq 2$	(1 2)(3 6 10 8 9 7 4)	
$\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_6 - 2\lambda_7 - \lambda_8 - 2\lambda_9 \leq 2$	(1 2)(3 6 10 9 7)	
$\lambda_2 + \lambda_3 + 2\lambda_4 - 2\lambda_5 - \lambda_6 - \lambda_9 - 2\lambda_{10} \leq 2$	(1 4)(5 7 6 8 9)	(1 2 3 … 11 12)
$\lambda_1 + 2\lambda_2 + \lambda_4 - 2\lambda_5 - 2\lambda_7 - \lambda_8 - \lambda_9 \leq 2$	(1 2)(3 4)(5 6 10 7 8 9)	
$\lambda_2 + \lambda_3 + 2\lambda_4 - 2\lambda_5 - \lambda_7 - 2\lambda_9 - \lambda_{10} \leq 2$	(1 4)(5 6 8 10 9)	
$\lambda_2 + 2\lambda_3 + \lambda_4 - 2\lambda_5 - \lambda_6 - 2\lambda_9 - \lambda_{10} \leq 2$	(1 3 4)(5 7 6 8 10 9)	
$\lambda_1 + \lambda_2 + 2\lambda_3 - \lambda_4 - \lambda_5 - 2\lambda_7 - 2\lambda_9 \leq 2$	(1 3 2)(4 6 10 9 7)(5 8)	
$\lambda_4 + \lambda_5 + 2\lambda_6 - \lambda_7 - \lambda_8 - 2\lambda_9 - 2\lambda_{10} \leq 2$	(1 6 3 5 2 4)	
$\lambda_1 + \lambda_2 + 2\lambda_3 - 2\lambda_5 - 2\lambda_7 - \lambda_8 - \lambda_9 \leq 2$	(1 3 2)(5 6 10 7 8 9)	
$\lambda_1 + \lambda_2 + 2\lambda_3 - 2\lambda_5 - \lambda_6 - \lambda_7 - 2\lambda_9 \leq 2$	(1 3 2)(5 8 7 6 10 9)	

$\lambda_2+\lambda_3+\lambda_4+\lambda_6-\lambda_{10}\le 3$	(1 2 3 4 6 5)	
$\lambda_1+\lambda_3+\lambda_4+\lambda_6-\lambda_9\le 3$	(2 3 4 6 5)(9 10)	
$\lambda_1+\lambda_2+\lambda_5+\lambda_6-\lambda_9\le 3$	(3 5)(4 6)(9 10)	
$\lambda_1+\lambda_2+\lambda_3+\lambda_6-\lambda_7\le 3$	(4 6 5)(7 8 9 10)	
$\lambda_1+\lambda_2+\lambda_4+\lambda_7-\lambda_9\le 3$	(3 4 7 6 5)(9 10)	(1 2 3 4 5 6)
$\lambda_1+\lambda_2+\lambda_4+\lambda_6-\lambda_8\le 3$	(3 4 6 5)(8 9 10)	
$\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5\le 3$	(5 6 7 8 9 10)	
$\lambda_1+\lambda_2+\lambda_3+\lambda_8-\lambda_9\le 3$	(4 8 7 6 5)(9 10)	

$\lambda_1+2\lambda_2+\lambda_3+\lambda_4+\lambda_6+\lambda_9\le 5$	(1 2)(5 6 9 8 7)	
$2\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_6+\lambda_{10}\le 5$	(5 6 10 9 8 7)	
$\lambda_1+2\lambda_2+\lambda_3+\lambda_4+\lambda_7+\lambda_8\le 5$	(1 2)(5 7)(6 8)	
$\lambda_1+2\lambda_2+\lambda_3+\lambda_5+\lambda_6+\lambda_8\le 5$	(1 2)(4 5 6 8 7)	(1 2 3 4 5 6)
$\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+2\lambda_6\le 5$	(1 6 5 4 3 2)	
$\lambda_1+\lambda_2+2\lambda_3+\lambda_4+\lambda_6+\lambda_8\le 5$	(1 3 2)(5 6 8 7)	
$\lambda_1+2\lambda_2+\lambda_4+\lambda_5+\lambda_6+\lambda_7\le 5$	(1 2)(3 4 5 6 7)	
$\lambda_1+\lambda_2+\lambda_3+2\lambda_4+\lambda_6+\lambda_7\le 5$	(1 4 3 2)(5 6 7)	

$2\lambda_2+\lambda_3+2\lambda_4+\lambda_5+3\lambda_6-2\lambda_8-2\lambda_9-\lambda_{10}\le 6$	(1 6)(3 4)(8 10 9)	
$3\lambda_2+2\lambda_4+\lambda_5+2\lambda_6+\lambda_7-2\lambda_8-2\lambda_9-\lambda_{10}\le 6$	(1 2 4 5 7 3 6)(8 10 9)	
$2\lambda_2+\lambda_3+3\lambda_4+2\lambda_6+\lambda_7-2\lambda_8-2\lambda_9-\lambda_{10}\le 6$	(1 4 3 6)(5 7)(8 10 9)	
$3\lambda_2+\lambda_4+2\lambda_5+\lambda_6+2\lambda_7-\lambda_8-2\lambda_9-2\lambda_{10}\le 6$	(1 2 5 6)(3 7)	
$\lambda_2+2\lambda_3+2\lambda_4+\lambda_5+3\lambda_6-\lambda_7-2\lambda_9-2\lambda_{10}\le 6$	(1 6)(2 3 4)(7 8)	
$2\lambda_2+\lambda_3+\lambda_4+2\lambda_5+3\lambda_6-2\lambda_8-\lambda_9-2\lambda_{10}\le 6$	(1 6)(3 5 4)(8 9)	
$2\lambda_2+\lambda_3+3\lambda_4+\lambda_6+2\lambda_7-2\lambda_8-\lambda_9-2\lambda_{10}\le 6$	(1 4 3 7 5 6)(8 9)	
$\lambda_2+2\lambda_3+\lambda_4+2\lambda_5+3\lambda_6-\lambda_8-2\lambda_9-2\lambda_{10}\le 6$	(1 6)(2 3 5 4)	(1 2 3 … 12 13)
$\lambda_2+2\lambda_3+3\lambda_4+\lambda_6+2\lambda_7-\lambda_8-2\lambda_9-2\lambda_{10}\le 6$	(1 4 2 3 7 5 6)	
$\lambda_2+2\lambda_3+3\lambda_4+2\lambda_6+\lambda_7-2\lambda_8-\lambda_9-2\lambda_{10}\le 6$	(1 4 2 3 6)(5 7)(8 9)	
$3\lambda_2+2\lambda_4+\lambda_5+\lambda_6+2\lambda_7-2\lambda_8-\lambda_9-2\lambda_{10}\le 6$	(1 2 4 5 6)(3 7)(8 9)	
$3\lambda_2+\lambda_4+2\lambda_5+2\lambda_6+\lambda_7-2\lambda_8-\lambda_9-2\lambda_{10}\le 6$	(1 2 5 7 3 6)(8 9)	
$2\lambda_2+2\lambda_3+3\lambda_4-\lambda_5+\lambda_7+\lambda_8-2\lambda_9-2\lambda_{10}\le 6$	(1 4 7 6)(5 8)	
$\lambda_2+2\lambda_3+2\lambda_4+\lambda_5+3\lambda_6-2\lambda_8-\lambda_9-2\lambda_{10}\le 6$	(1 6)(2 3 4)(8 9)	
$2\lambda_1+2\lambda_2+\lambda_4+\lambda_5+3\lambda_6-2\lambda_7-2\lambda_8-\lambda_9\le 6$	(1 6 3 2)(7 10 8 9)	

$3\lambda_1 + 2\lambda_2 + 3\lambda_3 - \lambda_4 - \lambda_7 + \lambda_8 - 2\lambda_9 + \lambda_{10} \leq 7$	(2 3)(4 8)(5 10 9 7 6)	
$3\lambda_1 + 2\lambda_2 + 3\lambda_3 - \lambda_5 - \lambda_6 + \lambda_8 - 2\lambda_9 + \lambda_{10} \leq 7$	(2 3)(4 8 5 10 9 6)	
$3\lambda_1 + 3\lambda_2 + 2\lambda_3 - \lambda_4 - \lambda_6 + \lambda_8 - 2\lambda_9 + \lambda_{10} \leq 7$	(4 8)(5 10 9 6)	
$3\lambda_1 + 3\lambda_2 + \lambda_3 + 2\lambda_4 - 2\lambda_5 - \lambda_7 - \lambda_8 + \lambda_{10} \leq 7$	(3 4)(5 10)(7 9 8)	
$3\lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4 - 2\lambda_5 - \lambda_6 - \lambda_9 + \lambda_{10} \leq 7$	(2 3)(5 10)(6 7 8)	
$3\lambda_1 + 2\lambda_2 + 3\lambda_3 - \lambda_4 + \lambda_6 - 2\lambda_7 - \lambda_9 + \lambda_{10} \leq 7$	(2 3)(4 6 5 10 7 8)	
$3\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 - 2\lambda_5 - \lambda_6 - \lambda_8 + \lambda_{10} \leq 7$	(5 10)(6 7 9 8)	(1 2 3 … 12 13)
$3\lambda_1 + 3\lambda_2 + 2\lambda_3 - \lambda_4 + \lambda_6 - 2\lambda_7 - \lambda_8 + \lambda_{10} \leq 7$	(4 6 5 10 7 9 8)	
$3\lambda_1 + 2\lambda_2 + 3\lambda_3 - \lambda_5 + \lambda_6 - 2\lambda_7 - \lambda_8 + \lambda_{10} \leq 7$	(2 3)(4 6)(5 10 7 9 8)	
$3\lambda_1 + 3\lambda_2 + \lambda_5 + 2\lambda_6 - 2\lambda_7 - \lambda_8 - \lambda_9 + \lambda_{10} \leq 7$	(3 6)(4 5 10 7)	
$3\lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4 - 2\lambda_5 - \lambda_7 - \lambda_8 + \lambda_{10} \leq 7$	(2 3)(5 10)(7 9 8)	
$2\lambda_1 + 3\lambda_2 + 3\lambda_3 - \lambda_5 + \lambda_6 - 2\lambda_7 - \lambda_9 + \lambda_{10} \leq 7$	(1 2 3)(4 6)(5 10 7 8)	
$2\lambda_1 + 3\lambda_2 + 3\lambda_3 + \lambda_4 - 2\lambda_5 - \lambda_7 - \lambda_9 + \lambda_{10} \leq 7$	(1 2 3)(5 10)(7 8)	
$2\lambda_1 + 3\lambda_2 + 3\lambda_3 - \lambda_5 - \lambda_7 + \lambda_8 - 2\lambda_9 + \lambda_{10} \leq 7$	(1 2 3)(4 8 5 10 9 7 6)	
$\lambda_1 + 2\lambda_2 + 3\lambda_3 + 3\lambda_4 - 2\lambda_5 + \lambda_8 - \lambda_9 - \lambda_{10} \leq 7$	(1 3 2 4)(5 8 9 10)	

$2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 5\lambda_4 - 2\lambda_5 + \lambda_7 + 2\lambda_8 - \lambda_9 \leq 12$	(1 4)(2 3)(5 8 10)(6 7)	
$4\lambda_1 + 5\lambda_2 + \lambda_4 + 2\lambda_5 + 3\lambda_6 - 2\lambda_7 - \lambda_8 + 2\lambda_{10} \leq 12$	(1 2)(3 6 4 5 10 7)(8 9)	
$4\lambda_1 + 5\lambda_2 + 3\lambda_3 - \lambda_4 + \lambda_7 + 2\lambda_8 - 2\lambda_9 + 2\lambda_{10} \leq 12$	(1 2)(4 8 6 7 5 10 9)	(1 2 3 … 15 16)
$4\lambda_1 + 5\lambda_2 + 3\lambda_3 + 2\lambda_4 - 2\lambda_5 - \lambda_6 + \lambda_9 + 2\lambda_{10} \leq 12$	(1 2)(5 10)(6 9)	
$4\lambda_1 + 5\lambda_2 + 2\lambda_3 + 3\lambda_4 - 2\lambda_5 - \lambda_7 + \lambda_9 + 2\lambda_{10} \leq 12$	(1 2)(3 4)(5 10)(6 9 7)	

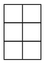
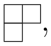
$2\lambda_1 + 3\lambda_2 + \lambda_4 + 2\lambda_5 + 4\lambda_6 - 3\lambda_7 - 2\lambda_8 - \lambda_9 \leq 8$	(1 6 3)(4 5)(7 10)(8 9)	
$2\lambda_2 + 3\lambda_3 + 4\lambda_4 - \lambda_5 + \lambda_7 + 2\lambda_8 - 3\lambda_9 - 2\lambda_{10} \leq 8$	(1 4 8 5 7 6)(2 3)(9 10)	
$\lambda_2 + 2\lambda_3 + 3\lambda_4 + 2\lambda_5 + 4\lambda_6 - \lambda_7 - 3\lambda_9 - 2\lambda_{10} \leq 8$	(1 6)(2 4 5)(7 8)(9 10)	(1 2 3 … 15 16)
$4\lambda_2 + \lambda_4 + 2\lambda_5 + 2\lambda_6 + 3\lambda_7 - \lambda_8 - 3\lambda_9 - 2\lambda_{10} \leq 8$	(1 2 7 3 5 4 6)(9 10)	
$\lambda_2 + 2\lambda_3 + 2\lambda_4 + 3\lambda_5 + 4\lambda_6 - \lambda_8 - 3\lambda_9 - 2\lambda_{10} \leq 8$	(1 6)(2 5)(9 10)	

$2\lambda_1 + 2\lambda_2 + \lambda_3 - \lambda_5 - \lambda_6 - \lambda_7 + \lambda_{10} \leq 4$	(4 10 7 9 6 8 5)	(1 2 3 … 12 13)
$2\lambda_1 + 2\lambda_2 + \lambda_3 - \lambda_4 - \lambda_5 - \lambda_9 + \lambda_{10} \leq 4$	(4 10 9 5 6 7 8)	

$2\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + 2\lambda_6 + 2\lambda_7 - \lambda_9 - \lambda_{10} \leq 6$	(1 2 6 5 4 3 7)	(1 2 3 … 12 13)
$\lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 + 2\lambda_6 + \lambda_7 - \lambda_9 - \lambda_{10} \leq 6$	(1 4 2 5 3 6 7)	

$\lambda_1 + 7\lambda_2 + 2\lambda_4 + 3\lambda_5 + 4\lambda_6 + 5\lambda_7 - 3\lambda_8 - 2\lambda_9 - \lambda_{10} \leq 15$	(1 2 7 3 6)(4 5)(8 10)	(1 2 3 … 18 19)
$\lambda_1 + 2\lambda_2 + 3\lambda_3 + 4\lambda_4 + 5\lambda_5 + 7\lambda_6 - 3\lambda_8 - 2\lambda_9 - \lambda_{10} \leq 15$	(1 6)(2 5)(3 4)(8 10)	
$3\lambda_1 + 4\lambda_2 + 5\lambda_3 - 3\lambda_4 - 2\lambda_5 - \lambda_6 + 2\lambda_8 - 5\lambda_9 + \lambda_{10} \leq 9$	(1 3)(4 8 5 10 9)(6 7)	(1 2 3 … 18 19)
$3\lambda_1 + 4\lambda_2 + 5\lambda_3 + 2\lambda_4 - 5\lambda_5 - 3\lambda_6 - 2\lambda_7 - \lambda_8 + \lambda_{10} \leq 9$	(1 3)(5 10)(6 9)(7 8)	
$\lambda_2 + 2\lambda_3 + 2\lambda_4 + 3\lambda_5 + 4\lambda_6 - 2\lambda_8 - 2\lambda_9 - \lambda_{10} \leq 8$	(1 6)(2 5)(8 10 9)	(1 2 3 … 16 17)
$4\lambda_2 + \lambda_4 + 2\lambda_5 + 2\lambda_6 + 3\lambda_7 - 2\lambda_8 - 2\lambda_9 - \lambda_{10} \leq 8$	(1 2 7 3 5 4 6)(8 10 9)	
$3\lambda_1 + 4\lambda_2 + 4\lambda_3 + 2\lambda_4 - 2\lambda_5 - \lambda_6 + \lambda_9 + 2\lambda_{10} \leq 11$	(1 2 3)(5 10)(6 9)	(1 2 3 … 16 17)
$3\lambda_1 + 4\lambda_2 + 4\lambda_3 - \lambda_4 + \lambda_7 + 2\lambda_8 - 2\lambda_9 + 2\lambda_{10} \leq 11$	(1 2 3)(4 8 6 7 5 10 9)	

A.2 ν-representability Constraints for the Systems of rank 5 and 6

In this part we give the list of constraints on orbital occupation numbers λ for a system $\mathcal{H}_r^\nu$ of $N = |\nu|$ electrons of total spin $J \geq 1/2$ and rank $r = 5, 6$. The problem is equivalent to pure ν-representability for two column diagram ν of size N. In the following list we skip the systems with the Young diagrams $\nu =$  and $\nu =$ , because in this cases the moment polytopes are given by the Pauli constraint $\lambda_1 \leq 2$ only. For the Young diagram $\nu =$ , the corresponding systems $\mathcal{H}_r^\nu$ have the following ν-representability conditions:

$$\lambda_1 - \lambda_2 \leq 1, \qquad \lambda_2 - \lambda_3 \leq 1.$$

That is why we also exclude these systems from the following list. The pure ν-representability conditions for the remaining systems, together with particle-hole duality, are listed below. Remind that the spectra λ are arranged in non-increasing order and normalized by $\sum_i \lambda_i = N = |\nu|$, and that the inequalities are grouped by extremal edges. We also provide the permutations v and w in the setting of Theorem 3.2.1 which give nonzero coefficients $c_v^w(a)$. In fact, $c_v^w(a) = 1$ for all v and w.

Systems of rank 5

- The system with the diagram $\nu = \ydiagram{2,1}$

Inequalities	$v \in S_5$	$w \in S_{45}$
$\lambda_2 + 2\lambda_3 - \lambda_4 - 2\lambda_5 \leq 3$	(13)	
$\lambda_1 + 2\lambda_2 - \lambda_3 - 2\lambda_4 \leq 3$	(12)(354)	
$2\lambda_2 + \lambda_3 - 2\lambda_4 - \lambda_5 \leq 3$	(123)(45)	(1234)
$2\lambda_1 - \lambda_2 + \lambda_4 - 2\lambda_5 \leq 3$	(24)	
$2\lambda_1 + \lambda_2 - 2\lambda_3 - \lambda_4 \leq 3$	(35)	
$\lambda_2 - \lambda_4 - \lambda_5 \leq 1$	(12)	
$\lambda_1 - \lambda_3 - \lambda_5 \leq 1$	(34)	(12)

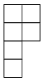
- The system with the diagram $\nu = \ydiagram{2,2,1}$

Inequalities	$v \in S_5$	$w \in S_{75}$
$\lambda_1 \leq 2$	(1)	(1)
$\lambda_2 - \lambda_3 - \lambda_5 \leq 1$	(12)(34)	
$\lambda_3 - \lambda_4 - \lambda_5 \leq 1$	(132)	(123)
$\lambda_1 + \lambda_3 - \lambda_4 \leq 3$	(23)(45)	
$\lambda_1 + \lambda_2 - \lambda_3 \leq 3$	(345)	(123)

- The system with the diagram $\nu = \ydiagram{2,1,1}$

Inequalities	$v \in S_5$	$w \in S_{24}$

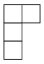
$2\lambda_1 - 2\lambda_2 - \lambda_3 + \lambda_5 \leq 1$	(25)(34)	
$-\lambda_1 + \lambda_3 + 2\lambda_4 - 2\lambda_5 \leq 1$	(14)(23)	(1234567)
$\lambda_2 - 2\lambda_3 - \lambda_4 \leq -1$	(12)(35)	
$-\lambda_1 + \lambda_3 - 2\lambda_5 \leq -1$	(134)	(12345)
$\lambda_2 + 2\lambda_3 - \lambda_4 \leq 3$	(13)(45)	
$2\lambda_1 - \lambda_3 + \lambda_5 \leq 3$	(253)	(12345)
$\lambda_3 - \lambda_4 - \lambda_5 \leq 0$	(132)	
$\lambda_2 - \lambda_3 - \lambda_5 \leq 0$	(12)(34)	(123)
$\lambda_1 - \lambda_3 - \lambda_4 \leq 0$	(354)	
$\lambda_1 + \lambda_3 - \lambda_4 \leq 2$	(23)(45)	
$\lambda_1 + \lambda_2 - \lambda_3 \leq 2$	(345)	(123)
$\lambda_2 + \lambda_3 - \lambda_5 \leq 2$	(123)	

- The system with the diagram $\nu =$ 

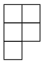
Inequalities	$v \in S_5$	$w \in S_{45}$
$\lambda_1 \leq 2$	(1)	(1)
$2\lambda_1 - \lambda_2 + \lambda_4 - 2\lambda_5 \leq 3$	(24)	
$2\lambda_1 + \lambda_2 - 2\lambda_3 - \lambda_4 \leq 3$	(35)	
$\lambda_1 + 2\lambda_2 - \lambda_3 - 2\lambda_4 \leq 3$	(12)(354)	(1234)
$2\lambda_2 + \lambda_3 - 2\lambda_4 - \lambda_5 \leq 3$	(123)(45)	
$\lambda_2 + 2\lambda_3 - \lambda_4 - 2\lambda_5 \leq 3$	(13)	

$\lambda_1 + \lambda_3 - \lambda_5 \leq 3$	(23)	
$\lambda_1 + \lambda_2 - \lambda_4 \leq 3$	(45)	(12)

Systems of rank 6

- The system with the diagram $\nu =$ 

Inequalities	$v \in S_6$	$w \in S_{105}$
$\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 \leq 2$	(35)(46)	
$\lambda_2 + \lambda_3 - \lambda_4 - \lambda_5 \leq 2$	(123)(465)	(12345)
$\lambda_2 - \lambda_4 - \lambda_5 \leq 1$	(12)(465)	
$\lambda_1 - \lambda_3 - \lambda_5 \leq 1$	(3465)	(1234)
$2\lambda_1 - \lambda_2 - \lambda_4 + \lambda_6 \leq 3$	(2645)	(12345678)
$\lambda_1 + 2\lambda_2 + 3\lambda_3 - \lambda_5 \leq 7$	(13)(56)	
$\lambda_1 + 3\lambda_2 + 2\lambda_3 - \lambda_4 \leq 7$	(123)(456)	
$2\lambda_1 + 3\lambda_2 - \lambda_4 + \lambda_5 \leq 7$	(12)(3564)	(12345)
$3\lambda_1 + \lambda_3 + 2\lambda_4 - \lambda_5 \leq 7$	(24)(56)	
$3\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_5 \leq 7$	(356)	

- The system with the diagram $\nu =$ 

Inequalities	$v \in S_6$	$w \in S_{210}$
$\lambda_1 \leq 2$	(1)	(1)

$\lambda_1 + 2\lambda_2 + \lambda_4 \leq 6$	(12)(34)	
$\lambda_1 + \lambda_2 + 2\lambda_3 \leq 6$	(132)(123)	
$\lambda_1 + \lambda_3 - \lambda_4 \leq 3$	(23)(456)	
$\lambda_1 + \lambda_2 - \lambda_3 \leq 3$	(3456)	(1234)

- The system with the diagram $\nu = \ydiagram{2,1,1,1}$

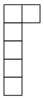
Inequalities	$v \in S_6$	$w \in S_{84}$
$\lambda_1 - \lambda_2 \leq 1$	(23456)	(12345)
$\lambda_2 + \lambda_3 - \lambda_5 - \lambda_6 \leq 2$	(123)	
$\lambda_1 + \lambda_3 - \lambda_4 - \lambda_6 \leq 2$	(23)(45)	
$\lambda_1 + \lambda_2 - \lambda_4 - \lambda_5 \leq 2$	(465)	(123)
$\lambda_1 + \lambda_2 - \lambda_3 - \lambda_6 \leq 2$	(345)	
$\lambda_2 + 2\lambda_3 - \lambda_4 - 2\lambda_6 \leq 3$	(13)(45)	
$2\lambda_2 + \lambda_4 - 2\lambda_5 - \lambda_6 \leq 3$	(1243)(56)	
$2\lambda_1 + \lambda_3 - 2\lambda_4 - \lambda_5 \leq 3$	(23)(46)	(12345)
$2\lambda_1 - \lambda_3 + \lambda_5 - 2\lambda_6 \leq 3$	(253)	
$\lambda_1 + \lambda_2 + 2\lambda_3 - \lambda_6 \leq 5$	(132)	
$\lambda_1 + 2\lambda_2 + \lambda_4 - \lambda_6 \leq 5$	(12)(34)	
$\lambda_1 + 2\lambda_2 + \lambda_3 - \lambda_5 \leq 5$	(12)(56)	(123)
$2\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 \leq 5$	(456)	
$2\lambda_1 + \lambda_2 + \lambda_5 - \lambda_6 \leq 5$	(354)	
$\lambda_1 - \lambda_2 - \lambda_4 - \lambda_6 \leq 0$	(2354)	(1234)

$\lambda_1 + 2\lambda_2 + 2\lambda_3 + 3\lambda_4 \leq 9$	(14)	(123456)
$-2\lambda_1 - \lambda_3 - 4\lambda_5 - 3\lambda_6 \leq -5$	(124)(56)	
$-2\lambda_1 - \lambda_2 - 3\lambda_5 - 4\lambda_6 \leq -5$	(1324)	(123456)
$\lambda_2 - \lambda_5 - \lambda_6 \leq 1$	(12)	(12)
$\lambda_1 - \lambda_4 - \lambda_6 \leq 1$	(45)	(12)
$3\lambda_1 - \lambda_2 + \lambda_4 + 2\lambda_5 - 3\lambda_6 \leq 6$	(25)(34)	
$\lambda_2 + 2\lambda_3 + 3\lambda_4 - \lambda_5 - 3\lambda_6 \leq 6$	(14)(23)	(1234567)
$\lambda_2 - 2\lambda_3 - \lambda_4 - 3\lambda_6 \leq -1$	(12)(35)	
$-\lambda_1 + \lambda_2 - 3\lambda_5 - 2\lambda_6 \leq -1$	(1234)(56)	(12345)
$-\lambda_1 + \lambda_3 - 2\lambda_5 - 3\lambda_6 \leq -1$	(134)	

- The system with the diagram $\nu = \ydiagram{2,2,1}$

Inequalities	$v \in S_6$	$w \in S_{189}$
$\lambda_1 \leq 2$	(1)	(1)
$\lambda_1 + \lambda_3 - \lambda_5 - \lambda_6 \leq 3$	(23)	
$\lambda_1 + \lambda_2 - \lambda_4 - \lambda_6 \leq 3$	(45)	(12)

$3\lambda_1 + \lambda_3 + 2\lambda_4 - \lambda_5 - 2\lambda_6 \leq 9$	(24)	
$3\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_5 - 2\lambda_6 \leq 9$	(35)	
$3\lambda_1 + 2\lambda_3 + \lambda_4 - 2\lambda_5 - \lambda_6 \leq 9$	(234)(56)	
$3\lambda_1 + \lambda_2 + 2\lambda_3 - \lambda_4 - 2\lambda_5 \leq 9$	(23)(465)	
$3\lambda_1 + 2\lambda_2 + \lambda_3 - 2\lambda_4 - \lambda_5 \leq 9$	(46)	
$2\lambda_1 + 3\lambda_2 - \lambda_4 + \lambda_5 - 2\lambda_6 \leq 9$	(12)(354)	(1234)
$\lambda_1 + 2\lambda_2 + 3\lambda_3 - \lambda_5 - 2\lambda_6 \leq 9$	(13)	
$\lambda_1 + 3\lambda_2 + 2\lambda_3 - \lambda_4 - 2\lambda_6 \leq 9$	(123)(45)	
$2\lambda_1 + \lambda_2 + 3\lambda_3 - 2\lambda_5 - \lambda_6 \leq 9$	(132)(56)	
$2\lambda_1 + 3\lambda_2 + \lambda_3 - 2\lambda_4 - \lambda_6 \leq 9$	(12)(456)	

- The system with the diagram $\nu =$ 

Inequalities	$v \in S_6$	$w \in S_{35}$
$\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 \leq 1$	(35)(46)	
$\lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 \leq 1$	(13)(24)	(12345)
$\lambda_1 + 2\lambda_2 - \lambda_4 + \lambda_6 \leq 4$	(12)(364)	
$\lambda_2 + 2\lambda_3 + \lambda_4 - \lambda_5 \leq 4$	(134)(56)	
$2\lambda_1 + \lambda_3 - \lambda_4 + \lambda_6 \leq 4$	(2364)	
$\lambda_2 + 2\lambda_3 + \lambda_5 - \lambda_6 \leq 4$	(1354)	(123456)
$\lambda_1 + 2\lambda_2 - \lambda_3 + \lambda_5 \leq 4$	(12)(356)	
$\lambda_1 + \lambda_3 + 2\lambda_4 - \lambda_5 \leq 4$	(142)(56)	
$\lambda_2 + \lambda_3 + 2\lambda_4 - \lambda_6 \leq 4$	(14)	
$2\lambda_1 + \lambda_2 - \lambda_3 + \lambda_6 \leq 4$	(36)	

$3\lambda_1 - 2\lambda_2 - \lambda_3 + \lambda_5 + 2\lambda_6 \leq 4$	(26)(35)	
$-\lambda_1 + \lambda_3 + 2\lambda_4 + 3\lambda_5 - 2\lambda_6 \leq 4$	(15)(24)	(123456789 10 11)
$\lambda_2 - \lambda_4 - \lambda_6 \leq 0$	(12)(45)	
$\lambda_3 - \lambda_5 - \lambda_6 \leq 0$	(132)	(123)
$\lambda_1 - \lambda_4 - \lambda_5 \leq 0$	(465)	
$\lambda_1 + \lambda_2 - \lambda_4 \leq 2$	(456)	
$\lambda_1 + \lambda_3 - \lambda_5 \leq 2$	(23)(56)	(123)
$\lambda_2 + \lambda_3 - \lambda_6 \leq 2$	(123)	
$-\lambda_1 + \lambda_3 + \lambda_4 - 2\lambda_6 \leq 0$	(13245)	(1234567)
$2\lambda_1 - \lambda_3 - \lambda_4 + \lambda_6 \leq 2$	(26453)	(1234567)
$3\lambda_1 - \lambda_3 + \lambda_5 + 2\lambda_6 \leq 6$	(26354)	(123456789)
$-2\lambda_1 - \lambda_2 + \lambda_4 - 3\lambda_6 \leq -4$	(14235)	(123456789)

Bibliography

[1] Altunbulak, M., Klyachko, A.: *The Pauli principle revisited.* Commun. Math. Phys. **282**(2), 287–322 (2008)

[2] Bell, R. J., Borland, R. E., and Dennis, K.: *The* $N+2$ *spin-orbital approximation to the* N*-body antisymmetric wave function.* J. Phys. B: Atom. Molec. Phys. **3**, 1047–1051 (1970)

[3] Berenstein, A. and Sjamaar, R.: *Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion.* J. Amer. Math. Soc. **13**(2), 433–466 (2000)

[4] Bernstein, I., Gelfand, I., and Gelfand, S.: *Schubert cells and cohomology of the space* G/P. Russian Math. Survey **28**(3), 1–26 (1973)

[5] Borland, R. E. and Dennis, K.: *The conditions on the one-matrix for three-body fermion wavefunctions with one-rank equal to six.* J. Phys. B: Atom Molec. Phys. **5**, 7–15 (1972)

[6] Christandl, M. and Mitchison, G.: *The spectra of density operators and the Kronecker coefficients of the symmetric group*. Commun. Math. Phys. **261**(3), 789–797 (2005)

[7] Christandl, M., Harrow, A. W., and Mitchison, G.: *Nonzero Kronecker Coefficients and What They Tell us about Spectra*. Commun. Math. Phys. **270**(3), 575–585 (2007)

[8] Cohen, A. M., van Leeuwen. M., and Lisser, B.: *LiE, a software package for Lie group theoretical computations*, avaiable at `wwwmathlabo.univ-poitiers.fr/~maavl/LiE/`

[9] Coleman, A. J.: *Structure of Fermion Density Matrices.* Rev. Mod. Phys. **35**, 668–686 (1963)

[10] Coleman, A. J., and Yukalov, V. I.: *Reduced density matrices: Coulson's challenge.* (Berlin: Springer) (2000)

[11] Dadok, J. and Kac, V.: *Polar representations.* J. Algebra **92**(2), 504–524 (1985)

[12] Duck, I. and Sudarshan, E.C.G.: *Pauli and the spin-statistics theorem.* (Singapore: World Scientific) (1997)

[13] Franz, M.: *Moment polytopes of projective G-varieties and tensor products of symmetric group representations.* J. Lie Theory **12**, 539–549 (2002)

[14] Franz, M.: *Convex, a Maple package for convex geometry*, available at `http://www-fourier.ujf-grenoble.fr/~franz/convex/`

[15] Fulton, W. and Harris, J.: *Representation theory.* (Springer: New York) (1991)

[16] Fulton, W.: *Young Tableaux, with Applications to Representation Theory and Geometry.* (Cambridge University Press) (1997)

[17] Fulton, W.: *Schubert varieties and degeneracy loci.* (Berlin: Springer) (1998)

[18] Klyachko, A.: *Stable bundles, representation theory, and Hermitian operators.* Selecta Math. **4**, 419–445 (1998)

[19] Klyachko, A.: *Vector bundles, Linear representations, and Spectral problems.* Proc. Int. Congress of Math. Beijing 2002, Invited Lectures, Vol. II, p. 599-614 (Higher Education Press: Beijing) (2003)

[20] Klyachko, A.: *Quantum marginal problem and representations of the symmetric group.* Preprint quant-ph/0409113 (2004)

[21] Klyachko, A.: *Quantum marginal problem and N-representability*, Journal of Physics: Conf. Series **36**, 72–86 (2006)

[22] Klyachko, A.: *Dynamic symmetry approach to entanglement*, in Proc. NATO Advanced Study Inst., Cargese, Corsica, France, 2005, J.-P. Gazeau et. el. eds., 25–54, (Amsterdam: IOS Press, 2007)

[23] Lascoux, A. and Schützenberger, M.-P.: *Symmetry and flag manifolds.* Lecture Notes in Mathematics **25**, 159–198 (1974)

[24] Lascoux, A.: *Classes de Chern d'un produit tensoriel.* C. R. Acad. Sci. Paris **286**, 385–387 (1978)

[25] Lascoux, A. and Schützenberger, M.-P.: *Polyôme de Schubert.* C. R. Acad. Sci. Paris **294**, 447–450 (1982)

[26] Macdonald, I. G.: *Schubert polynomials.* London Math. Soc. Lecture Notes **166**, 73–99 (1991)

[27] Macdonald, I. G.: *Symmetric functions and Hall polynomials.* (Oxford: Clarendon Press) (1995)

[28] Ness, L.: *A stratification of the null cone via moment map.* Amer. J. Math. **106**, 1281–1329 (1984)

[29] Müller, C. W.: *Sufficient conditions for pure state N-representability.* J. Phys. A: Math. Gen. **32**, 4139–4148 (1999)

[30] Pauli, W.: *About the relationship of the closing of electron shells in atoms with the complex structure of the spectrum.* Zeitschrift für Physik. **31**, 765 (1925)

[31] Peltzer, C. P. and Brandstatter, J. J.: *Studies in the theory of the generalized density operators. III. The N-fermion representability problem for the first order generalized density operator.* J. Math. Anal. Appl. **33**, 263–277 (1971)

[32] Perelomov, A. M.: *Generalized coherent states and their applications.* (Springer: Berlin) (1986)

[33] Ruskai, M. B.: *N-representability problem: Particle-hole equivalence.* J. Math. Phys. **11**, 3218–3224 (1970)

[34] Ruskai, M. B.: *Comments on Peltzer–Brandstatter papers: Two counterexamples.* J. Math. Anal. Appl. **44**, 131–135 (1973)

[35] Ruskai, M. B.: *Connecting N-representability to Weyl's problem: The one particle density matrix for $N = 3$ and $R = 6$.* J. Phys. A: Math. Theor. **40**, F961–F967 (2007)

[36] Stillinger, F. H. *et al*: *Mathematical challenges from theoretical/computational chemistry.* (Washington, DC: National Academy Press) (1995)

[37] Vinberg, E. and Popov, V.: *Invariant theory*, in "Algebraic Geometry IV", A.N. Parshin and I. Shafarevich, eds., (Springer: Berlin) (1992)

[38] Weyl, H.: *The theory of groups and quantum mechanics.* (Dover: New York) (1931)