GREEN CORRESPONDENCE FOR MACKEY FUNCTORS

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

GREEN CORRESPONDENCE FOR MACKEY FUNCTORS

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The Green corespondence for modules of group algebras was introduced by Green in 1964. A version for Mackey functors was introduced by Sasaki in 1982. Sasaki's characterization of Mackey functor correspondence was based on the theory of Green functors. In this thesis, we give Sasaki's characterization and an alternative characterization of the Mackey functor correspondence. Our characterization is closer to Green's original module theoretic approach. We show that the two characterizations are equivalent. This yields a new way of determining vertices, sources and Green correspondents; we shall illustrate this with some examples.

Keywords: Green correspondence, Green functor, Mackey functor, endomorphism Green functor, vertex, source, defect base, defect group.

ÖZET

MACKEY İZLEÇLERİNDE GREEN UYUŞUMU

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Grup cebirlerinin modülleri için Green uyuşumu 1964 yılında Green tarafından ortaya koyuldu. Mackey izleçleri için olan versiyonu 1982 yılında, Sasaki tarafından ortaya koyuldu. Sasaki'nin Mackey izleci uyuşumunu karakterize etmesi, Green izleçleri kuramını baz alır. Bu tezde, Sasaki'nin Mackey izleci uyuşumunu nasıl karakterize ettiğini inceledik ve konuyu diğer bir yoldan nitelendirdik. Bizim yaptığımız nitelendirme Green'in ilk modül kuramsal yaklaşımına daha yakındır. Bu iki nitelendirmenin aynı olduğunu gösterdik. Bunun sayesinde köşeyi, kaynağı ve Green uyuşumlarını tespit etmek için yeni bir yol elde ettik; bunu bazı örneklerle gösterdik.

Anahtar sözcükler: Green uyuşumu, Green izleci, Mackey izleci, özyapı dönüşümü Green izleci, köşe, kaynak, hata bazı, hata grubu.

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Contents

Chapter 1

Introduction

The theory of the Green correspondence is a fundamental theorem in group representation theory. The Green correspondence for Mackey functors is an analogous development.

Green[5] showed that there is a one-to-one correspondence between the isomorphism classes of finitely generated indecomposable $kG-$modules which have vertex Q and isomorphism classes of finitely generated indecomposable $kH-$modules which have vertex Q, where H is a subgroup of G and H contains $N_G(Q)$. Later, in [6], Green established the Green correspondence for Green functors. The Green correspondence for Green functors is a natural application of a general transfer theorem that he established, again in [6]. Sasaki[8] defined the Green correspondence for Mackey functors by adapting the method established for Green functors. After introducing the endomorphism Green functor $End_G(M)$ of a Mackey functor M for G over a complete principle ideal domain, Sasaki defined the vertex of a Mackey functor as a defect group of the endomorphism Green functor of the Mackey functor. Then, he defined the Green correspondent using the method of Green.

In [11], Thévenaz and Webb gave some applications of the Green correspondence for some special Mackey functors: the simple Mackey functors, $S_{H,V}$, projective Mackey functors, $P_{H,V}$, fixed point, $FP(V)$, and fixed quotient $FQ(V)$.

This classic paper substantially laid down the modern notation for the theory of Mackey functors.

We have made our study in historical order of the development of the theory of the Green correspondence and the theory Mackey functors. Part of our task in this thesis has been to provide a modern update of Sasaki's ideas.

In Chapter 2, we give some easy facts to the theory of the Green correspondence for group algebras.

Chapter 3 contains the two equivalent definitions of Mackey functors and the definition of Green functor.

In Chapter 4, we introduce the endomorphism Green functor and show that it is a Green functor with some elegant properties. The definition of the vertex of a Mackey functor, which is crucial in our study, is based on the endomorphism Green functor.

In [6], Green gives some properties of Green functors, defines the defect basis and the defect group for Green functors and gives some methods to identify them. In Chapter 5, we relate these results to the endomorphism Green functor of a Mackey functor.

In Chapter 6 and in Chapter 7, we collect together several results on the Green functors and Mackey functors. We define the correspondence and prove the Green correspondence for Mackey functors.

Finally, in Chapter 8, we define some special Mackey functors and determine the vertices, sources and Green correspondents for them.

To facilitate the reading, important definitions and results have been repeated where necessary.

Chapter 2

Green Correspondence in Group Algebra

The Green correspondence for Mackey functors was introduced by Sasaki. However, the Green correspondence is encumbered with some important technicalities. Our goal in this chapter is to describe group algebras and to introduce the Green correspondence for them. We need some facts about group algebras. The following definitions can be found in [4] and [1]. Consider G, a finite group, and k an algebraically closed field throughout the entire chapter.

2.1 Preliminaries

Free modules, projective modules, related definitions, especially vertices and sources give a great deal of intuition and information about the Green correspondence in group representation theory. Firstly, we give these necessary and useful definitions and results.

Let G be a group and consider the $kG-$modules isomorphic with direct sum of copies of kG. Such $kG-$modules are called *free modules*. Free modules play a central role in ring theory, since any module is the homomorphic image of some free module. The following definition gives the fundamental characterization of summands of free modules.

Definition 2.1 *Let A be an finite dimensional algebra over an algebraically closed field k and U be a $A-$module.The modules satisfying the following equivalent properties are called projective modules*

1. *U is a direct summand of a free module;*
2. *If α is a homomorphism of the $A-$module V onto U then the kernel of α is a direct summand of V;*
3. *If α is a homomorphism of the $A-$module V onto the $A-$module W and β is a homomorphism of U to W then there is a $A-$homomorphism γ of U to V with $\alpha\gamma{=}\beta$.*

So the characterization of the projective modules give us new information with the following definition.

Definition 2.2 *Let A be an finite dimensional algebra over an algebraically closed field k and U be a $A-$module.The modules satisfying the following equivalent properties are called injective modules*

1. *U is a direct summand of a free module;*
2. *If α is one-to-one homomorphism of the $A-$module U into V then $\alpha(U)$ is a direct summand of V;*
3. *If α is a one-to-one homomorphism of the $A-$module W into the $A-$module V and β is a homomorphism of W to U then there is a $R-$homomorphism γ of V to U with $\gamma\alpha{=}\beta$.*

For group algebras, the two notions coincide.

Theorem 2.3 *Let U be a $kG-$module. U is projective if and only if it is injective.*

Now we extend the idea of free modules by relating $kG-$modules and modules for subgroups. By this means, we introduce the induced modules.

Definition 2.4 *Let G be a group and H be a subgroup of G. A $kG-$module U is said to be relatively $H-$free if there is a $kH-$submodule X of U such that any $kH-$ homomorphism of X with respect to any $kG-$module V extends uniquely to a $kG-$ homomorphism of U to V.*

Lemma 2.5 *Let G be a group and H be a subgroup of G and X be a $kH-$module.Then there is a $kG-$module which is relatively $H-$free with respect to X.*

Now we describe these relatively free modules by the tensor product construction.

Definition 2.6 *Let G be a group, H be a subgroup of G and V be a $kH-$module. Consider the tensor product $kG \otimes V$ of two vector spaces and let $kG \otimes_{kH} V$ be the quotient by the subspace spanned by all the elements of the form $ah \otimes v - \otimes hv$ where $a \in kG$, $v \in V$, $h \in H$. We shall give this quotient space the structure of a $kG-$module. For $g \in G$ define $g(a \otimes v) = ga \otimes v$. Obviously, by the bilinearity of this action we have a $kG-$module structure on the vector space $kG \otimes_{kH} V$. Since $g(ah \otimes v - a \otimes hv) = (ga)h \otimes v - (ga) \oplus hv$ the subspace given is preserved by the action of g thus we have a $kG-$module structure on $kG \otimes_{kH} V$.This module is called the $kG-$module induced by the $kH-$module V. For $a \in G$ and $v \in V$ we write $a \otimes V$ for the element of V^G which corresponds to this tensor.*

With these definitions we have the following remark.

Remark 2.7 *Let H be a subgroup of G and V be a $kH-$module. The induced module V^G is relatively $H-$free with respect to V. Furthermore, V^G is the vector space direct sum*

$$V^G = \sum_{s \in G/H} s \otimes V$$

and each subspace $s \otimes V$ has dimension equal to that of V

Now we give a collection of basic facts about induction.

Lemma 2.8 *Let V, V_1, V_2 be $kH-$modules for the subgroup H of G and let U be a $kG-$module.*

1. *If V is free(projective) then V^G is free(projective).*
2. $(V_1 \oplus V_1)^G = V_1^G \oplus V_2^G$.
3. *Let L be a subgroup of H and let W be a $kL-$module. Then $(W^H)^G = W^G$.*
4. $U \otimes V^G \cong (U_H \otimes V)^G$.
5. $Hom_{kG}(V^G, U) \cong Hom_{kH}(V, U_H)$.
6. $Hom_{kG}(U, V^G) \cong Hom_{kH}(U_H, V)$.

The next result, which is Mackey's theorem, allows us to describe the process of inducing and then restricting in terms of restriction followed by induction. Recall that, as a disjoint union, $G = \bigcup_{s \in L\backslash G/H} LsH$ where the notation indicates that s runs over a set of double coset representatives for H and L in G. Now we can state the Mackey's theorem.

Theorem 2.9 *Let H and L be two fixed subgroups of a group G and let V be a $kH-$module.Then the following holds.*

$$(V^G)_L \cong \bigoplus_{s \in L\backslash G/H} (s \otimes (V)_{L \cap sHs^{-1}})^L.$$

To begin to study the Green correspondence in group algebra, lastly we introduce vertex and source. If H and A are subgroups of G then $H \leq_G A$ means that $H^x < A$ and $H =_G A$ means that $H^x = A$ for some $x \in G$. Note that if U and V are $kG-$modules and U is isomorphic with a direct summand of V then we write $U|V$.

Definition 2.10 *Let U be a $kG-$module and H be a subgroup of G. The modules satisfying the following equivalent properties are called relatively $H-$projective or $H-$projective.*

1. *U is a direct summand of a relatively $H-$free module;*
2. *If α is a homomorphism of the $kG-$module V onto U and α split as a $kH-$homomorphism then α is split;*
3. *If α is a homomorphism of the $kG-$module V onto the $kG-$module W and β is a homomorphism of U to W the there is a $kG-$homomorphism γ of U to V with $\alpha\gamma$=β provided there is a $kH-$homomorphism with this property;*
4. *U is a direct summand of $(U_H)^G$.*

Lemma 2.11 *Let Q and H be subgroups of G and let W be an $kQ-$module and V a component of W^G. Suppose that $V_H = U_1 \oplus U_2 \ldots U_t$ where each U_i is an indecomposable $kH-$module. Then for each i there exists $x_i \in G$ such that U_i is $H \cap Q^{x_i}-$projective. In fact $U_i/\{W^{x_i}_{H\cap Q^{x_i}}\}^H$.*

For a $kG-$module V let $B(V)$ be the set of all subgroups H of G such that V is $H-$projective.

Lemma 2.12 *Let V be an indecomposable $kG-$module. Assume that*

1. *Q is a minimal member of $B(V)$.*
2. *W is a $kQ-$module such that $V|W^G$.*
3. *$H \in B(V)$.*

Then $U|V^G$ for at least one indecomposable component U of V_H. Moreover for such U there exist $x \in G$ with $Q^x \leq H$ and $U|(W^x)^H$.

Proof: Since V is $H-$projective $V|(V_H)^G$. Thus $V|U^G$ for some indecomposable component U of V^H as V is indecomposable. By Lemma 2.11 for any such U there exists $x \in G$ such that U is $(H \in Q^x)-$projective. Thus V is $(H^{x^{-1}} \in Q)-$projective. Moreover by the minimality of $A \leq H^{x^{-1}}$. Part (3) is a consequence of Lemma 2.11.

□

Corollary 2.13 *Let V be an indecomposable $kG-$module. Then there exists a subgroup A of G such that V is $Q-$projective and when V is $H-$projective then $Q \leq_G H$ where A is uniquely determined up to conjugation in G.*

Proof: This corollary is immediate by Lemma 2.12.

□

Definition 2.14 *Let U be an indecomposable $kG-$module. The minimal element Q in $B(U)$ is called a vertex of U and the vertex is determined unique up to conjugation in G. Moreover, given an indecomposable $kG-$module U with vertex Q, then Q is a $p-$subgroup of G. The indecomposable $kQ-$module S,unique up to conjugation in $N_G(Q)$, such that $U|S^G$ is called the source of U.*

The idea is that the closer the vertex Q is to the identity, the nearer U is to being projective. The following two results about vertices are useful to understand the Green correspondence in group algebra.

Lemma 2.15 *If U is an indecomposable $kG-$module with vertex Q and H is a subgroup containing Q, then:*

1. *There exists a $kH-$module V such that $V|U_H$ and $U|V^G$;*

2. There exists a $kH-$module V such that $U|V^G$ and V has vertex Q;

3. There exists a $kH-$module V such that $V|U_H$ and V has vertex Q;

Proof:

1. Since U is relatively $H-$projective, we have $U|(U_H)^G$. Therefore there is an indecomposable summand V of U_H such that $U|V^G$.

2. Let S be a $kQ-$module which is a source for U so $U|S^G$. But $S^G|(S_H)^G$ so there is an indecomposable summand V of S^H with U/V^G. We claim that V has vertex Q. By establishing this assertion we will prove (2). Since $V|S_H$ we have that V is relatively $Q-$projective so there is vertex R of V contained in Q. Let W be a $kR-$module such that $V|W_H$. Hence, $V^G|(W_H)^G$, that is, $V^G|W^G$, so $U|W^G$ and U is $R-$projective. Thus, R contains a conjugate of Q. But $R \leq Q$ so $R = Q$ as claimed.

3. Let S be an indecomposable $kQ-$module with $S|U_Q$ and $U|S^G$. Hence, there is an indecomposable $kH-$module V with $V|U_H$ and $S|V_Q$. We will prove that V has vertex Q. $V|U_H$ so $V|(V^G)_H$ and by Mackey's theorem there is $s \in G$ with
$$V|(s \otimes (S)_{H \cap sQs^{-1}})^H.$$
Hence, V has vertex R with $R \leq H \cap sQs^{-1}$. It suffices to prove that R is conjugate to Q in H. However, V is a summand of a module induced from R to H and $S|V_Q$ so Mackey's theorem implies that S is relatively $Q \cap hRh^{-1}$ projective for some $h \in H$. But S has vertex Q(or else, as $U|S^G$, U would be relatively projective for a proper subgroup of Q) so $Q \cap hRh^{-1}$ can not be a proper subgroup of Q, that is $Q \leq hRh^{-1}$. However, $R \leq sQs^{-1}$ so $|R| \leq |Q|$ and we have $Q = hRh^{-1}$. Thus, V satisfies (3).

□

In Corollary 2.22, below, we shall show that there exists a $kG-$module V satisfying all three conditions of the lemma.

Lemma 2.16 *Let Q be a subgroup of the subgroup H of G. If V is a relatively $Q-$projective $kH-$module, then*

$$(V^G)_H \cong V \oplus W$$

where every indecomposable summand of W is relatively projective for a subgroup of the form $sQs^{-1} \cap H$, $s \in G$, $s \notin H$.

Proof: Since V is relatively $Q-$projective, there is a $kQ-$module U with $V|U^H$. Thus $U^H \cong V \oplus T$ for some $kH-$module T, so $U^G \cong V^G \oplus T^G$ and

$$(U^G)_H \cong V \oplus W \oplus T \oplus X$$

where $(V^G)_H \cong V \oplus W$ and $(T^G)_H \cong T \oplus X$ for suitable $kH-$modules W and X. However, by Mackey's theorem,

$$(V^G)_L \cong \bigoplus_{s \in H \ G/Q} (s \otimes (U)_{H \cap sQs^{-1}})^L \cong U^G \oplus Y$$

where the summand $s \in H$ gives U^H, Y is the direct sum of all terms for $s \in H$ and so each indecomposable summand of Y is relatively projective for a subgroup of the form $sQs^{-1} \cap H$, $s \notin H$. So it is immediate that $W \oplus X \cong Y$ so W is as claimed.

□

2.2 Green Correspondence in Group Algebra

Now we can begin to study Green correspondence in group algebra.

Firstly let us fix some notation. Let Q be a $p-$subgroup of G and let M be a subgroup of G containing $N_G(Q)$, which is the *normalizer* of Q.Remember that if P and R are subgroups of G, $P \leq_G R$ means that a conjugate of P in G is contained in R. Let $\mathcal{H}$ is a collection of subgroups of G then $P \leq_G \mathcal{H}$ means $P \leq_G H$ for some $H \in \mathcal{H}$.

Secondly we fix some collections of $p-$subgroup of G. Let

$$\mathfrak{X} = \{sQs^{-1} \cap Q | s \in G, s \notin M\}$$

$$\mathfrak{Y} = \{sQs^{-1} \cap M | s \in G, s \notin M\}$$

$$\mathfrak{Z} = \{R | R \leq Q, R \not\leq_G \mathfrak{X}\}$$

$\mathfrak{X}$ consists of the proper subgroups of Q because $N_G(Q) \leq H$. Thus $Q \in \mathfrak{Z}$. Clearly $\mathfrak{X} \leq \mathfrak{Y}$.

Finally, if $\mathcal{H}$ is any collection of subgroups of G we shall say that the $kG-$module U is relatively $\mathcal{H}-$projective if U is direct sum of modules each of which is relatively projective for a subgroup of $\mathcal{H}$. Now we can state the fundamental theorem.

Theorem 2.17 *There is a one-to-one correspondence between isomorphism classes of indecomposable $kG-$modules with vertex in $\mathfrak{Z}$ and isomorphism classes of indecomposable $kM-$modules with vertex in $\mathfrak{Z}$. If U and V are such modules for G and M, respectively, which corresponds then U and V have the same vertex and*

$$U_M \cong V \oplus Y$$

$$V^G \cong U \oplus X$$

where Y is a relatively $\mathfrak{Y}-$projective $kM-$module and X is a relatively $\mathfrak{X}-$projective $kG-$module.

The next four results are necessary preliminaries.

Lemma 2.18 *Suppose that R is a subgroup of Q. Then the following are equivalent.*

1. $R \leq_G \mathfrak{X}$;
2. $R \leq_M \mathfrak{X}$;
3. $R \leq_M \mathfrak{Y}$.

Proof: If (1) is valid then there is $g \in G$ with $gRg^{-1} \leq Q \cap sQs^{-1}$, where $s \in G$, $s \notin M$. Then if $g \in M$ then certainly (2) holds. If $g \notin M$ then $R \leq g^{-1}Qg$ yields $R \leq Q \cap g^{-1}Qg$, that is R in $\mathfrak{X}$ so certainly $R \leq_M \mathfrak{X}$. So (2) holds.
If (2) holds then there is $x \in M, s \in G, s \notin M$ such that $xRx^{-1} \leq Q \cap sQs^{-1}$. Thus, we get $xRx^{-1} \leq M \cap sQs^{-1}$ and $R \leq_M \mathfrak{Y}$. So (3) holds.
If (3) holds then there is $x \in M, s \in G, s \notin M$ with $xRx^{-1} \leq M \cap sQs^{-1}$. Hence $R \leq M \cap (x^{-1}s)Q(s^{-1}x)$. However $x^{-1}s \notin M$. So clearly we have $R \leq Q \cap (x^{-1}s)Q(s^{-1}x)$ and $R \leq_G \mathfrak{X}$. So (1) holds.

□

The next lemma gives the reason why the statement of the theorem is not symmetrical in $\mathfrak{X}$ and $\mathfrak{Y}$, but is quite symmetrical in G and M, restriction and induction.

Lemma 2.19 *If U is a relatively $\mathfrak{X}-$projective $kG-$module then U_M is relatively $\mathfrak{Y}-$ projective. If V is a relatively $\mathfrak{Y}-$projective $kM-$module then V_G is relatively $\mathfrak{X}-$ projective.*

Proof: Let W be an indecomposable summand of U so W is relatively projective for a subgroup of the form $sQs^{-1}, s \notin M$. Hence, by Mackey's theorem, W_M is relatively projective for the collection of subgroups of the form

$$t(sQs^{-1} \cap Q)t^{-1} \cap M = tsQs^{-1}t^{-1} \cap tQt^{-1} \cap M.$$

But either $t \notin M$ or $ts \notin M$ so such a subgroup is contained in an element of $\mathfrak{Y}$ so W_M and U_L are relatively $\mathfrak{Y}-$projective.

If W is an indecomposable summand of V and W has vertex P then $P \leq_M \mathfrak{Y}$ so W^G is relatively $P-$projective and $P \leq_G \mathfrak{X}$, by Lemma 2.18 , so W^G is relatively $\mathfrak{X}-$projective and we proved the lemma.

□

The next two results contains the bulk of the theorem.

Lemma 2.20 *If U is an indecomposable $kG-$module with vertex R in $\mathfrak{Z}$ then $U_M \cong V \oplus W$, where V is an indecomposable $kM-$module with vertex R, $U|V^G$ and W is a relatively $\mathfrak{Y}-$projective $kM-$module.*

Proof: By Lemma 2.15, there is an indecomposable $kM-$module V with vertex R and $U|V^G$. Now $(V^G)_M \cong V \oplus W_1$, where W_1 is relatively $\mathfrak{Y}-$projective, by Lemma 2.16. Thus U_M is either isomorphic with $V \oplus Y$ or W for some summand W of W_1. But, again by Lemma 2.15, U_M has an indecomposable summand W with vertex R. Now W cannot be isomorphic with summand of W, or else $R \leq_M \mathfrak{X}$, by Lemma 2.18, so $R \notin \mathfrak{Z}$. Thus, $W \cong V$ and $U_M \cong V \oplus W$.

□

Lemma 2.21 *If V is an indecomposable $kM-$module with vertex R in $\mathfrak{Z}$ then $V^G \cong U \oplus V$ where U is an indecomposable $kG-$module with vertex R, $V|U_M$ and X is a relatively $\mathfrak{X}-$projective $kG-$module.*

Proof: Let $V^G = U_1 + \ldots + U_r$ be a direct sum of indecomposable $kG-$modules.Since $(V^G)_M \cong V \oplus Y$, where Y is a relatively $\mathfrak{Y}-$projective $kM-$module, by Lemma 2.16, we have, after renumbering, that $(U_1)_M \cong V + Y_1$, $(U_i)_M \cong Y_i$, $2 \leq i \leq r$, where the Y_i are $kM-$modules and $Y \cong Y_1 \oplus \ldots \oplus Y_r$. We claim that U_i has a vertex in $\mathfrak{Z}$ and that $U_1, \ldots, U_r$ are relatively $\mathfrak{X}-$projective. Indeed, U_1 has a vertex in Q, as $U_1|V^G$, and U_1 cannot be relatively $\mathfrak{X}-$projective. Then Lemma 2.19 would imply that $(U_1)_M \cong V + Y_1$ is relatively $\mathfrak{Y}-$projective which is not the case. Hence, U_1 does not have a vertex in $\mathfrak{Z}$. Moreover, if U_i,

$i \leq 2$, was not relatively $\mathfrak{X}-$projective. Then Lemma 2.20 would imply to it and $(U_i)_M$ would not be relativle $\mathfrak{Y}-$projective, which it is. Thus, U_i is relatively $\mathfrak{X}-$projective.
Setting $U = U_1$, $X = U_2 + \ldots + U_r$ we have that $U|V^G$, $V^G \cong U \oplus X$ where X is relatively $\mathfrak{X}-$projective. It remains only to prove that U has vertex R. However, the vertex of U is in $\mathfrak{Z}$. Thus Lemma 2.20 applies and there is a unique summand, in any decomposition of U_M into the direct sum of indecomposable modules, which is not relatively $\mathfrak{Y}-$projective and that module has a vertex equal to a vertex of U. However $U_M \cong V \oplus Y$. So U has vertex R as V has vertex R. This completes the proof.

□

Proof: Obviously in the last two lemmas almost everything of the Theorem 2.17 established. We only need to show the one-to-one property. We need to show two things:
If U is an indecomposable $kG-$module with vertex R in $\mathfrak{Z}$, V is as in Lemma 2.20. Then $V^G = U' \oplus Y$ as in Lemma 2.21. Thus $U \cong U'$. But in Lemma 2.20 we proved that $U|V^G$. If we start with V, a similar result holds as proved in Lemma 2.21 that $V|U_M$. This shows the one-to-one property and Green correspondence is established which completes the proof of theorem of Theorem 2.17.

□

Corollary 2.22 *If U is an indecomposable $kG-$module with vertex Q and H is a subgroup containing Q then there is an indecomposable $kH-$module V satisfying the following*
$V|U_H$ and $U|V^G$ and V has vertex Q.

Chapter 3

Green Functors and Mackey Functors

Our goal in this chapter is to define Mackey and Green functors. Actually we shall give two equivalent definitions of a Mackey functor.

3.1 Mackey Functors

3.1.1 Definition of Mackey functors in terms of subgroups

Let G be a group, k be a commutative ring with unity element. A Mackey functor M for G over k consists of the following:

A. A $k-$module $M(H)$ for all $H \leq G$.

B. $k-$linear maps $r_K^H : M(H) \longrightarrow M(K)$ and $t_K^H : M(K) \longrightarrow M(H)$ for all $K \leq H \leq G$. We call r_K^H and t_K^H a *restriction map* and a *transfer map*, respectively.

C. A $k-$linear map $c_g^H : M(H) \longrightarrow M({}^gH)$ for all $g \in G$ and $H \leq G$. Here ${}^gH = gHg^{-1}$. We call c_g^H a *conjugation map.*

Furthermore, the following relations are to be satisfied for all $g, h \in G$ and $H, K, L \leq G$.

1. $r^K_L r^H_K = r^H_L$ and $t^H_K t^K_L = t^H_L$ if $L \leq K \leq H$,
2. $r^H_H = t^H_H = id_{M(H)}$,
3. $c^H_{gh} = c_{g,H^{h^{-1}}} c_{h,H}$,
4. $c_h : M(H) \longrightarrow M(H)$ is the identity if $h \in H$,
5. $c^K_g r^H_K = r^{H^{g^{-1}}}_{K^{g^{-1}}} c^H_g$ and $c^H_g t^H_K = t^{H^{g^{-1}}}_{K^{g^{-1}}} c^K_g$ if $K \leq H$,
6. $(MackeyAxiom)$ If $L, K \leq H$,

$$r^H_L t^H_K = \sum_{h \in L \backslash H / K} t^H_{L \cap K^{h^{-1}}} r^{K^{h^{-1}}}_{L \cap K^{h^{-1}}} c^K_h$$

   where $L \leq hH \geq K$ and $L \backslash H / K$ denotes a set of representatives of the $(L, K)-$double cosets LhK

It is not hard to show that the formula in the Mackey axiom does not depend on the choice of the double cosets representatives.

Since every conjugation map c^H_g has the inverse $c^{H^{g^{-1}}}_{g^{-1}}$ it is an isomorphism. We allow G to act as $k-$linear automorphisms on $\bigoplus_{H \in S(G)} M(H)$ such that, by defining;

$$^g m = c^H_g(m) \ for \ m \in M(H) \ and \ g \in G.$$

Furthermore the subgroup $N_G(H)$ stabilizes $M(H)$. By axiom (4), the quotient $\overline{N_G}(H) = N_G(H)/H$ acts on $M(H)$ as a group of $k-$linear automorphims. So we can say $M(H)$ is a $\overline{kN_G}(H)-$module. In particular M(1) is a $kG-$module. Thus, a Mackey functor can be considered as a family of modules related to one another by restriction, transfer and conjugation maps.

A morphism of Mackey functor $f : M \longrightarrow N$ is a family $f^G_H : H \leq G$ of maps $f_H : M(H) \longrightarrow N(H)$ which commute with restriction, transfer and conjugation.

By a subfunctor N of a Mackey functor M for *S(G)* we mean a family of $k-$submodules $N(H) \leq M(H)$, which is stable under restriction, transfer and conjugation.

3.1.2 Definition of Mackey functors as modules over the Mackey algebra

The second definition of a Mackey functor is by means of the Mackey algebra $Mack_R(G)$ which we shall define in a moment. First, we define Mack(G) to be the algebra over $\mathbb{Z}$ generated by elements generated by the elements t_K^H, r_K^H and c_g^H. For $K \leq H \leq G$ and $g, h \in G$ we impose the following relations on the generators:

1. $t_K^H t_L^K = t_L^H$ if $L \leq K \leq H$,
2. $r_L^K r_K^H = r_L^H$ if $L \leq K \leq H$,
3. $c_{gh}^H = c_g^{H^{h^{-1}}} c_h^H$,
4. $r_H^H = t_H^H = c_h^H = id_{M(H)}$,
5. $c_g^H t_K^H = t_{K^{g^{-1}}}^{H^{g^{-1}}} c_g^K$,
6. $c_g^K r_K^H = r_{K^{g^{-1}}}^{H^{g^{-1}}} c_g^H$,
7. $\sum_H t_H^H = \sum_H r_H^H = 1$,
8. $(MackeyAxiom)$ if $L, K \leq H$,

$$r_L^H t_K^H = \sum_{h \in L \backslash H/K} t_{L \cap K^{h^{-1}}}^H r_{L \cap K^{h^{-1}}}^{K^{h^{-1}}} c_h^K$$

any other product of r_K^H, t_K^H and c_g^H being zero.

Let $Mack_k(G) = k \otimes_{\mathbb{Z}} Mack(G)$. Given a Mackey functor M for G over k, as defined above, then we can define a $Mack_k(G)-$module $\tilde{M} = \bigoplus_{H \leq G} M(H)$. When we pass from Mackey functors M to $Mack_k(G)-$modules $\tilde{M}$, morphisms of

Mackey functors $f : M \to N$ give rise to maps of $Mack_k(G)-$modules $\tilde{f} : \tilde{M} \to \tilde{N}$ such that $\tilde{f}$ and f_H agree as maps $M(H) \to N(H)$.

So it is possible to define a Mackey functor as a $Mack_R(G)-$module, and a morphism of Mackey functors as a morphism of $Mack_R(G)-$modules. If M is a $Mack_R(G)-$module, then M corresponds to a Mackey functor M_1 in the first sense, defined by $M_1(H) = t_H^H M$, the maps $t_K^H, r_K^H M$ and c_H^g being defined as the multiplications by the corrsponding elements of the Mackey algebra.

Note that for $H \leq G$ a Mackey functor for G over over k is naturally considered to be a Mackey functor for H over k. Such a Mackey functor is called the *restriction* of M to H and is denoted by $\downarrow_H^G M$.

Definition 3.1 *Let M and N be Mackey functors for G over k. Then a morphism of Mackey functors $\theta : M \to N$ is defined to be a family $(\theta_H))_{H \leq G}$ of $k-$homomorphisms $\theta_H : M(H) \to N(H)$ such that*

$$\begin{aligned} t_H^K(\theta_H(m)) &= \theta_K(t_H^K(m)) \ (m \in M(H)), \\ r_H^K(\theta_K(m')) &= \theta(H)(r_H^K m') \ (m' \in M(H)), \\ c^g(\theta_H(m)) &= \theta(H^g)(c^g(m)) \ (M \in M(H)) \end{aligned}$$

for all $H \leq K \leq G$ and $g \in G$, where k is an complete principle ideal domain. We denote by $Mack_k(G)$ the category whose objects are the Mackey functors for G over k and with morphisms as just defined. $Mack_k(G)$ is an abelian category.

3.2 Green Functors

With the notation of the previous section, A *Green functor* for G over R is a Mackey functor A such that $A(H)$ is endowed with an associative $R-$algebra structure with a unity element for every $H \in S(G)$ and the following axioms are satisfied:

1. All restriction maps $r_K^H : A(H) \to A(K)$ and the conjugation maps $c_H^g : A(H) \to A(K)$ are unitary homomorphisms of R-algebras.

2. (Frobenius Axiom) For all $K \leq H$, $\alpha \in A(K), \beta \in A(H)$, then

$$t_K^H(\alpha.r_K^H(\beta)) = t_K^H(\alpha).\beta$$

$$t_K^H(r_K^H(\beta).\alpha) = \beta.t_K^H(\alpha)$$

We emphasize that t_K^H need not to be a ring homomorphism. In fact, the Frobenius axiom implies that the image of t_K^H is a two sided ideal of $A(H)$. The formulas in the Frobenius axiom are also called *projection formulas.*

Since the conjugation maps are unitary homomorphisms of $R-$algebras, G acts on $\prod_{H \in S(G)} A(H)$ as a group of algebra automorhisms, and in particular $\bar{N}(H)$ acts on $A(H)$ as a group of algebra automorphisms. So $A(H)$ is an $\bar{N}(H)-$algebra, and in particular $A(1)$ is endowed with an action of G by $R-$algebra automorphisms.

Moreover, there is an evident notion of morphism of Green functors: a morphism ϕ from the Green functor A to the Green functor B is a morphism of Mackey functors such that, for any subgroup H of G, the morphism $\phi_H : A(H) \to B(H)$ is a morphism of rings. The morphism ϕ is said to be unitary if the morphism ϕ_H preserves the unit for all H. It is actually enough that morphism ϕ_G preserves the unit, since

$$\phi_G(1_{A(H)}) = \phi_G(r_H^G)(1_{A(G)}) = r_H^G \phi_G(1_{A(G)}) = r_H^G 1_{B(G)} = 1_{B(H)}$$

A module over the Green functor A, or $A-$module, is defined to be a Mackey functor M for the group G, such that for any subgroup H of G, the module $M(H)$ has a structure of $A(H)-$module with unit. Furthermore, the structure must be compatible with the Mackey structure, in the following sense:

- If $x \in G$ and $K \leq G$, let $m \mapsto^x m$ be the conjugation by x from $M(K)$ to $M(^xK)$. If $a \in A(K)$ and $M \in m(K)$, then $^x(a.m) = {}^x(a).{}^a(m)$.
- If $H \leq K$ are subgroups of G, if $a \in A(K)$ and $m \in M(K)$, then $r_H^K(a.m) = r_H^K(a).r_H^K(m)$.

- In the same conditions, if $a \in A(K)$ and $m \in M(H)$, then

$$a.t_H^K(m) = t_H^K(r_H^K(a).m)$$

and if $a \in A(H)$ and $m \in M(K)$, then

$$t_H^K(a).m = t_H^K(a.r_H^K(m))$$

A morphism ϕ from the $A-$module M to the $A-$module N is a morphism of Mackey functors from M to N such that any subgroup H of G, the morphism ϕ_H is a morphism of $A(H)-$modules.

Now we will define another category $\mathcal{A}_k(G)$ as subcategory of $Mack_k(G)$ as the following:

Definition 3.2 *Let A, B be Green functors where G is a group. Then a ring homomorphism $\psi = (\theta_H)_{H \leq G} : A \to B$ is a morphism between Green functors such that each θ_H is an algebra of $Mack_k(G)$ whose objects are all Green functors and morphisms are ring homomorphisms.*

An important example of a Green functor is $End(M)$, obtained for any Mackey functor M by,

$$End(M)(H) = End_H(M) = Hom_{Mack(H)}(\downarrow_H^G M, \downarrow_H^G M)$$

we will study $End(M)$ in the next chapter.

Chapter 4

Endomorphism Green Functors

Let G be a finite group and M a Mackey functor. In this chapter, we will study a Green functor, defined by Sasaki, called *endomorphism Green functor* of a Mackey functor M. The role of the endomorphism Green functor in the Green correspondence for Mackey functors is crucial because we define the vertex of a Mackey functor M to be the by vertex of the endomorphism Green functor of M.

Firstly, we will give the definition of endomorphism Green functor. Then we will show that the endomorphism Green functor satisfies the axioms for Green functors.

Definition 4.1 *Let M be a Mackey functor for G over k. Then the Green functor $End_H(M) = (End_H(M), T, R, C)$ is defined as follows. For each $H \leq G$ we define*

$$End_H(M) = Hom_{Mack_k(H)}(\downarrow_H^G M, \downarrow_H^G M),$$

the set of morphisms from $\downarrow_H^G M$ to $\downarrow_H^G M$ in $Mack_k(H)$. Let $H \leq K \leq G$ and $g \in G$.
Define the transfer map;

$$T_H^K : End_H(M) \longrightarrow End_K(M) : \theta \mapsto T_H^K(\theta)$$

as follows. Writing

$$(\theta_Y)_{Y\leq H} \mapsto ((T_H^K(\theta))_L)_{L\leq K}$$

then, for $L \leq K$*, the map* $(T_H^K(\theta))_L : M(L) \longrightarrow M(L)$ *is such that, for* $x \in M(L)$*, we have*

$$x \mapsto \sum_{HgL\leq K} t^H_{H\cap {}^gL} c^g \theta_{H^g\cap L} r^{H^g}_{H^g\cap L} c_H^{g-1}(x).$$

Define the restriction map;

$$R_H^K : End_K(M) \longrightarrow End_H(M) : \psi \mapsto R_H^K(\psi)$$

as follows. Writing

$$(\psi_Y)_{Y\leq K} \mapsto ((R_H^K(\psi))_D)_{D\leq H}$$

then for $E \leq K$*, the map* $(R_H^K(\psi))_D : M(E) \longrightarrow M(D)$ *is such that, for* $y \in M(D)$*, we have*

$$y \mapsto \psi_D(y).$$

Define the conjugation map;

$$C_H^g : End_H(M) \longrightarrow End_{{}^gH}(M) : \varphi \mapsto C_H^g(\varphi)$$

as follows. Writing

$$(\varphi_Y)_{Y\leq K} \mapsto ((C_H^g(\varphi))_E)_{E\leq {}^gH}$$

then for $E \leq K$*, the map* $(C_H^G(\varphi))_E : M(E) \longrightarrow M(E)$ *is such that, for* $z \in M(E)$*, we have*

$$z \mapsto C_{{}^gE}^{g^{-1}} \varphi_{{}^gE} C_E^g(z).$$

Now we will show that the endomorphism Green functor $End_G(M)$ satisfies the axioms given in Chapter 3.

Proposition 4.2 *If* M *is a Mackey functor for the group* G*, then* $(End_G(M), T, R, C)$ *is a Green functor.*

Proof: Letting M be a Mackey functor for G, then
(1) For $L \leq K \leq H$, $R_L^K R_K^H = R_L^H$. Indeed, given $D \leq K$ and $m \in M(D)$ and $\varphi \in End_H(M)$ then $R_K^H(\varphi) \in End_K(M)$ satisfies

$$(R_K^H(\varphi))(m) = (\varphi_D)_{D\leq K}(m).$$

For $E \leq L$ and $R^H_K \in End_K(M)$, we have

$$R^K_L(R^H_K(\varphi))(m) = (((\varphi)_D)_E)_{D\leq K, E\leq L}(m) = (\varphi_E)_{E\leq L}(m) = (R^H_L(\varphi))(m)$$

for $\varphi \in End_H(M)$.
(2) For $L \leq K \leq H$, $T^K_H T^K_L = T^L_H$.
Indeed, for given $S \leq K$ and $m \in M(S)$ and $\varphi \in End_L(M)$ then $T^K_L(\varphi) \in End_K(M)$ satisfies

$$(T^K_L(\varphi)(m))_S = \sum_{SkL\leq K, k\in K} t^S_{S\cap^k L} c^k \varphi_{S^k\cap L} r^{S^k}_{S^k\cap L} c^{k^{-1}}_S (m).$$

Given $R \leq K$ and $n \in M(R)$ and $\psi \in End_K(M)$ then $T^H_K(\psi) \in End_H(M)$ satisfies;

$$(T^H_K(\psi)(n))_R = \sum_{RhK\leq H, h\in H} t^R_{R\cap^h K} c^h \psi_{R^h\cap K} r^{R^h}_{R^h\cap K} c^{k^{-1}}_R (n).$$

Given $R \leq H$ and $n \in M(R)$ and $\varphi \in End_L(M)$ then;

$$\begin{aligned}
(T^H_K(T^K_L(\psi)(n))_L &= \sum_{RhK\leq H, h\in H} t^R_{R\cap^h K} c^h (t^K_L)_{R^h\cap K} r^{R^h}_{R^h\cap K} c^{k^{-1}}_R (n) \\
(T^K_L(\psi)(n))_{R^h\cap K} &= \sum_{(R^h\cap K)kL\leq K, h\in K} t^{R^h\cap K}_{R^h\cap K\cap^k L} c^k \psi_{R^{hk}\cap K^k\cap L} r^{R^{hk}\cap K^k}_{R^{hk}\cap K^k\cap L} c^{k^{-1}} (n)
\end{aligned}$$

So

$$\begin{aligned}
(T^H_K(T^K_L(\psi)) &= \sum_{(R^h\cap K)kL\leq K, RhK\in H} t^R_{R\cap^h K} c^h t^{R^h\cap K}_{R^h\cap K\cap^k L} c^k \psi_{R^{hk}\cap K^k\cap L} r^{R^{hk}\cap K^k}_{R^{hk}\cap K^k\cap L} c^{k^{-1}} r^{R^h}_{R^h\cap K} c^{h^{-1}}_R \\
&= \sum_{(R^h\cap K)kL\leq K, RhK\in H} t^R_{R\cap^h K} t^{R\cap^h K}_{R\cap^x K\cap^x L} c^x \psi_{R^x\cap K\cap L} c^{x^{-1}} r^{R\cap^h K}_{R\cap^h K\cap^x L} r^R_{R\cap^h K} \\
&= \sum_{RxL\leq H} t^L_{R\cap^x(H\cap L)} c^x \psi_{R^x\cap K\cap L} c^{x^{-1}} r_{R\cap^x(H\cap L)}
\end{aligned}$$

where $x = hk$.
The result is what we claimed because as k and h runs over double coset representatives $RkK \leq H$ and $(R^h \cap K)hL \leq H$ then $x = hk$ runs over double coset representatives $RkK \leq H$.
(3) For a finite group H, $R^H_H = id_{End_H(M)}$.

Given $D \leq H$ and $m \in M(D)$ and $\psi \in End_H(M)$ then $R^H_H(\psi) \in End_H(M)$ satisfies

$$(R^H_H(\psi))(m) = (\psi_D)_{D \leq H}(m).$$

Since $(\psi_D)_{D\leq H}$ is a family of homomorphisms over $D \leq H$ then

$$(\psi_D)_{D\leq H}(m) = \psi(m)$$

as we claimed.
(4) For a finite group H, $T^H_H = id_{End_H(M)}$.
Indeed, for given $S \leq H$ and $m \in M(S)$ and $\psi \in End_H(M)$ then $T^H_H(\psi) \in End_H(M)$ satisfies

$$\begin{aligned}
T^H_H(\psi)(m) &= \sum_{ShH \leq H, h \in S} t^S_{S\cap^h H} c^h \psi_{S^h\cap H} r^{S^h}_{S^h \cap H} c^{h^{-1}}_S (m). \\
&= \sum_{ShH \leq H, h \in S} t^S_S c^h_{S^h\cap H} \psi_{S^h\cap H} r^{S^h}_{S^h \cap H} c^{h^{-1}}_S (m) \\
&= \sum_{ShH \leq H, h \in S} t^S_S c^h_{S^h} \psi_{S^h} r^{S^h}_{S^h} c^{h^{-1}}_S (m) \\
&= \sum_{ShH \leq H, h \in S} \psi_{S^h}(m) \\
&= \sum_{ShH \leq H, h \in H} \psi_{S^h}(m) \\
&= \psi(m)
\end{aligned}$$

as we claimed.
(5) For a finite group G with a subgroup H, we have

$$C^{gh}_H = C^g_{^hH} C^h_H \text{ where } h, g \in G.$$

Indeed, for given $m \in M(E)$, $E \leq^{gh} H$ and $\alpha \in End_H(M)$ then $C^{gh}_H(\alpha) \in End_H(M)$ satisfies;

$$\begin{aligned}
(C^{gh}_H)(m) &= c^{gh}_E \alpha_{E^{gh}} c^{h^{-1}g^{-1}}_{^{gh}E} (m) \\
&= c^g_{^hE}(c^h_E \alpha_{E^{gh}} c^{h^{-1}}_{g^{-1}E}) c^{g^{-1}}_E (m) \\
&= c^g_{^hE}(C^H_H(\alpha)_{E^{gh}}) c^{g^{-1}}_E (m) \\
&= C^g_{^hH} C^h_H(m).
\end{aligned}$$

(6) For a finite group H and $h \in H$, $C^h : End_H(M) \longrightarrow End_H(M)$ is the identity. Indeed, for given $m \in M(E)$ and $E \leq^h H = H$ and $\theta \in End_H(M)$ then;

$$\begin{aligned} (C^h_H(\theta))(m) &= c^{h^{-1}}_{^hE}(\theta_{^hE})_{^hE \leq H} c^h_E(m) \\ &= \theta(m) \end{aligned}$$

where we consider the family of homomorphisms $(\theta_{^hE})_{^hE \leq H}$.

(7) Let G be finite group and $K \leq H \leq G$ and $g \in G$. Then we have

$$C^g_K R^H_K = R^{^gH}_{^gK} C^g_H.$$

Indeed, for given $D \leq K$, $E \leq^g K$ and $m \in M(D)$ and $\alpha \in End_H(M)$;

$$\begin{aligned} C^g_K R^H_K(\alpha)(m) &= C^g_K(\alpha_D)(m) \\ &= C^{g^{-1}}_{^gE}(\alpha_D)_{^gE} C^g_E(m) \\ &= C^{g^{-1}}_{^gE} \alpha_{^gE} C^g_E(m) \\ &= (C^{g^{-1}}_{^gY} \alpha_{^gY} C^g_Y)_E(m) \\ &= R^{^gK}_{^gK}(C^{g^{-1}}_{^gY} \alpha_{^gY} C^g_Y)(m) \\ &= R^{^gH}_{^gK} C^g_H(\alpha)(m) \end{aligned}$$

where $Y \in^g H$.

(8) (Mackey axiom): Take $\theta \in End_H(M)$, and if $L, K \leq H$, then we have

$$R^L_H T^K_H(\theta) = \sum_{LgK \leq H, g \in H} T^L_{L \cap^g k} R^{^gK}_{L \cap^g K} c^g_K(\theta).$$

Indeed, for any $X \leq L$ we must show that

$$(R^L_H T^K_H)_X(x) = \sum_{LgK \leq H, g \in H} (T^L_{L \cap^g k} R^{^gK}_{L \cap^g K} C^g_K(\theta))_X(x)$$

where $x \in L$. We show the equality of the left hand side(LHS) and right hand side(RHS).

LHS :

$$\begin{aligned} (R^L_H T^K_H(\theta))_X(x) &= R^L_H(T^K_H(\theta))_X(x) \\ &= (T^H_K(\theta))_X(x) \\ &= \sum_{XgK \leq H} t^X_{X \cap^g K} c^g \theta_{X^g \cap K} r^{X^g}_{X^g \cap K} c^{g^{-1}}_X(x) \end{aligned}$$

$RHS:$

$$
\begin{aligned}
&\sum_{LgK\leq H, g\in H} (T^{L}_{L\cap^{g}k} R^{^gK}_{L\cap^{g}K} C^{g}_{K}(\theta))_X(x) = \\
&= \sum_{LgK\leq H, g\in H} (T^{L}_{L\cap^{g}K} (R^{^gK}_{L\cap^{g}K} C^{g}_{K}(\theta)))_X(x) \\
&= \sum_{LgK\leq H, g\in H} \sum_{Xu(L\cap^{g}K)\leq L, u\in L} t^{X}_{X\cap^{u}(L\cap^{g}K)} c^{u} (R^{^gK}_{L\cap^{g}K}(C^{g}_{K}(\theta)))_{X^u\cap(L\cap^{g}K)} r_{X^u} \cap (L\cap^{g} K)^{X^u} c^{u^{-1}}_{X}(x) \\
&= \sum_{LgK\leq H} \sum_{Xu(L\cap^{g}K)\leq L} t^{X}_{X\cap^{u}(L\cap^{g}K)} c^{u} (C^{g}_{K}(\theta))_{X^u\cap(L\cap^{g}K)} r^{X^u}_{X^u\cap(L\cap^{g}K)} c^{u^{-1}}_{X}(x) \\
&= \sum_{LgK\leq H} \sum_{Xu(L\cap^{g}K)\leq L} t^{X}_{X\cap^{u}(L\cap^{g}K)} c^{u} c^{g}(\theta)_{(X^u\cap(L\cap^{g}K))^g} c^{g^{-1}} r^{X^u}_{X^u\cap(L\cap^{g}K)} c^{u^{-1}}(x) \\
&= \sum_{LgK\leq H, Xu(L\cap^{g}K)\leq L} t^{X}_{X\cap^{u}(L\cap^{g}K)} c^{ug}(\theta)_{(X^{ug}\cap(L\cap^{g}K))^g} r^{X^{ug}}_{X^{ug}\cap(L\cap^{g}K)^g} c^{g^{-1}u^{-1}}(x) \\
&= \sum_{XgK\leq H} t^{X}_{X\cap^{g}K} c^{g} \theta_{X^g\cap K} r^{X^g}_{X^g\cap K} c^{g^{-1}}_{X}(x)
\end{aligned}
$$

because u and g runs over $Xu(L\cap^{g} K)$ and LgK, respectively, then ug runs over $XugK$. Since $X\leq L$ and $u\in L$, then ug runs over XgK. So LHS and RHS are equal.

(9) (Frobenius axiom) If $K\leq H$, $\alpha\in End_K(M), \beta\in End_H(M)$, then the following multiplicative structures are both satisfied.

$$T^{H}_{K}(\alpha.R^{H}_{K}(\beta)) = T^{H}_{K}(\alpha).\beta$$

$$T^{H}_{K}(R^{H}_{K}(\beta).\alpha) = \beta.T^{H}_{K}(\alpha).$$

Indeed, for the first statement of the Frobenius axiom, we show that the left hand side and the right hand side are equal. Let $S\leq H$.

$LHS:$

$$
\begin{aligned}
T^{H}_{K}(\alpha.R^{H}_{K}(\beta)) &= \sum_{ShK\leq H, h\in H} t^{S}_{S\cap^{h}K} c^{h} (\alpha.R^{H}_{K}(\beta))_{S^h\cap K} r^{S^h}_{S^h\cap K} c^{h^{-1}}_{S} \\
&= \sum_{ShK\leq H, h\in H} t^{S}_{S\cap^{h}K} c^{h} \alpha_{S^h\cap K} (R^{H}_{K}(\beta))_{S^h\cap K} r^{S^h}_{S^h\cap K} c^{h^{-1}}_{S} \\
&= \sum_{ShK\leq H, h\in H} t^{S}_{S\cap^{h}K} c^{h} \alpha_{S^h\cap K} \beta_{S^h\cap K} r^{S^h}_{S^h\cap K} c^{h^{-1}}_{S}
\end{aligned}
$$

RHS :

$$\begin{aligned} T_K^H(\alpha).\beta &= \Big(\sum_{ShK \leq H, h \in H} t^S_{S\cap^h K} c^h \alpha_{S^h\cap K} r^{S^h}_{S^h\cap K} c_S^{h^{-1}}\Big).\beta_H \\ &= \sum_{ShK \leq H, h \in H} t^S_{S\cap^h K} c^h \alpha_{S^h\cap K} \beta_{S^h\cap K} r^{S^h}_{S^h\cap K} c_S^{h^{-1}} \end{aligned}$$

by application of conjugation and restriction on β_H. Since LHS and RHS are equal, the first statement of the Frobenius axiom is satisfied.

Similarly, the endomorphism Green functor satisfies the second statement of the Frobenius axiom.

$\square$

As a consequence, we have shown that the endomorphism Green functor satisfies all the axioms defined for a Green functor.

In this chapter, finally we will give a remark which defines a Mackey functor for which the endomorphism Green functor is a special case. The proof is in Bouc [2]. But it can also be proved by an argument similar to the above.

Proposition 4.3 ***(Bouc,[2])*** *Let M and N be Mackey functors for G over k. Then a Mackey functor $Hom_H(M,N) = (Hom_H(M,N), T, R, C)$ is defined as follows.*
For each $H \leq G$ we define

$$Hom_H(M,N) = Hom_{Mack(H)}(\downarrow_H^G M, \downarrow_H^G N),$$

the set of morphisms from $\downarrow_H^G M$ to $\downarrow_H^G N$ in $Mack(H)$.

Chapter 5

Defect Base and Defect Group for Mackey Functors

In this chapter, making use of the defect base and defect group established by Green in [6] , we shall study the vertex of a Mackey functor.

5.1 Some Notation and Canonical Filtration

Firstly, let us remember some definitions concerning sets of subgroups of a group G. Let K and L be subgroups of H which is a given subgroup of G. We write $K \leq_H L$ to mean that there exists some h in H such that $K^h \leq L$. If $\mathcal{H}, \mathcal{B}$ are sets of subgroups of H, we write $\mathcal{H} \leq_H \mathcal{B}$ to mean that for each $K \in \mathcal{H}$ there exists some $L \in \mathcal{B}$ such that $K \leq_H L$. We write $\mathcal{H} =_H \mathcal{B}$ to mean that $\mathcal{H} \leq_H \mathcal{B}$ and $\mathcal{B} \leq_H \mathcal{H}$. Now we define the *closure operation*, j_H.

Definition 5.1 *If $\mathcal{H}$ is a set of subgroups of H, define*

$$j_H\mathcal{H} = \{L \leq H | \exists K \in \mathcal{H} \text{ such that } L \leq_H K\},$$

and we call $j_H\mathcal{H}$, the j_H-closure of $\mathcal{H}$. The set $\mathcal{H}$ is said to be j_H-closed or closed under subconjugation if $j_H\mathcal{H} = \mathcal{H}$. Thus $\mathcal{H}$ is j_H-closed if and only if $\mathcal{H}$

contains, with any subgroup $L \in \mathcal{H}$, all the subgroups of L and all conjugates of $L \leq H$.

The following proposition is obvious.

Proposition 5.2 *Let $\mathcal{H}, \mathcal{B}$ be sets of subgroups of H. Then*

1. $\mathcal{H} \leq_{j_H} \mathcal{H}$.
2. $j_H(j_H\mathcal{H}) = j_H\mathcal{H}$.
3. $\mathcal{H} \leq_H \mathcal{B}$ *if and only if* $j_H\mathcal{H} \leq j_H\mathcal{B}$.
4. $j_H\mathcal{H} =_H \leq_B$ *if and only if* $j_H\mathcal{H} = j_H\mathcal{B}$.
5. *Every j_H-closed $\mathcal{H}$ contains the unit subgroup $\{1\}$ of H.*
6. *The intersection and union of any non-empty set of j_H-closed sets of subgroups of H are both j_H-closed sets of subgroups of H.*

Definition 5.3 *Let H, K, L be subgroups of G such that $H \leq L$ and $K \leq L$. Let $\mathcal{H}, \mathcal{B}$ be sets of subgroups of H, K respectively. Then consider the following set of subgroups.*

$$\mathcal{H} : L : \mathcal{B} = \{U^g \cap V | U \in \mathcal{H}, V \in \mathcal{B}, g \in L\}.$$

This set of subgroups described above is called midrel set.

Proposition 5.4 *With the notation of the definition just above*

$$j_L(\mathcal{H} : L : \mathcal{B}) = j_L\mathcal{H} \cap j_L\mathcal{B}$$

.

Definition 5.5 *Let A be a Green functor for a finite group G in k. For each pair $(\mathcal{H}, H)$, where H is a subgroup of G and $\mathcal{H}$ is a set of sungroups of H($\mathcal{H}$ can consist of single subgroup), consider*

$$A_{\mathcal{H}}(H) = \sum_{U \in \mathcal{H}} t_U^H(A(U)).$$

The family $(A_{\mathcal{H}}(H))$ indexed by the set of pairs $(\mathcal{H}, H)$, is called the canonical filtration on A.

Proposition 5.6 *Let A be a Green functor, and $(A_{\mathcal{H}}(H))$ its canonical filtration. Then each $A_{\mathcal{H}}(H)$ is an ideal of $A(H)$. Also, let D, H, K be subgroups of G, and any set $\mathcal{H}, \mathcal{B}$ of subgroups of H. Then:*

1. $A_H(H) = A(H)$.
2. $\mathcal{H} \leq_H \mathcal{B}$ *implies that* $(A_{\mathcal{H}}(H)) \leq A_{\mathcal{B}}(H))$.
3. $A_{\mathcal{H}}(H).A_{\mathcal{B}}(H) \leq A_{\mathcal{H}:H:\mathcal{B}}(H)$
4. $t_H^K(A_{\mathcal{H}})(H) = A_{\mathcal{H}}(K)$ *for* $H \leq K$.
5. $r_H^D(A_{\mathcal{H}}(H)) \leq A_{\mathcal{H}:H:\mathcal{D}}(D)$ *where* $\mathcal{D}$ *consists of only* D, $D \leq H$.
6. $c_H^g(A_{\mathcal{H}}(H)) = A_{\mathcal{H}^g}(H^g)$ *for all* $g \in G$ *and* $\mathcal{H}^g = \{U^g | U \in \mathcal{H}\}$.

5.2 Defect Bases

Definition 5.7 *Let A be a Green functor for G and $\mathcal{H}$ be a set of subgroups of G. We say that $A(G)$ is $\mathcal{H}-$projective if $(A_{\mathcal{H}}(G)) = A(G)$, which means $A(G)$ is $\mathcal{H}-$projective if*

$$\sum_{U \in \mathcal{H}} t_U^G(A(U)) = A(G).$$

We also say A itself is $\mathcal{H}-$projective in this case.

Lemma 5.8 *With the notation of the Definition 5.7 we have the following:*

1. $A(G)$ *is* $G-$*projective.*
2. $A(G)$ *is* $\mathcal{H}-$*projective if and only if* $A(G)$ *is* $j_G\mathcal{H}-$*projective.*
3. *If* $A(G)$ *is* $\mathcal{H}-$*projective, and if* $\mathcal{H} \leq_G \mathcal{B}$, *then* $A(G)$ *is* $\mathcal{B}-$*projective.*

4. *If $A(G).A(G) = A(G)$ and if $A(G)$ is both $\mathcal{H}-$projective and $\mathcal{B}-$projective, then $A(G)$ is $(j_G\mathcal{H} \cap j_G\mathcal{B})-$projective.*

Proof: (1) follows from Proposition 5.6(1). From Proposition 5.6(2) we deduce that if $\mathcal{H} \leq_G \mathcal{B}$ then $(A_{\mathcal{H}}(G)) = A(G)$ implies $(A_{\mathcal{B}}(G)) = A(G)$, so this proves (3). Since $j_G\mathcal{H} = \mathcal{H}$, we get the result (2). Assuming the hypotheses of (4) then, by Proposition 5.6(3)

$$A(G) = A(G).A(G) = A_{\mathcal{H}}(H).A_{\mathcal{B}}(H) \leq A_{\mathcal{H}:G:\mathcal{B}}(G)$$

which means that $A(G)$ is $(\mathcal{H} : G : \mathcal{B})-$projective. By (2) and Proposition 5.4, $A(G)$ is $(j_G\mathcal{H} \cap j_G\mathcal{B})-$projective.

□

Definition 5.9 *Let A be a Green functor for G and $\mathcal{D}$ be a set of subgroups of G. Then we say that $\mathcal{D}$ is a defect base for $A(G)$ if the following two conditions are both satisfied.*

1. *For all $\mathcal{H}$(any set of subgroups of G), $A(G)$ is $\mathcal{H}-$projective if and only if $\mathcal{D} \leq_G \mathcal{H}$.*
2. *$\mathcal{D}$ is j_G-closed.*

In other words, a defect base of A is a family of subgroups which is closed under subconjugation and which is minimal under the condition

$$\sum_{U \in \mathcal{D}} t_U^G(A(U)) = A(G).$$

Theorem 5.10 ***(Green,[6])*** *Let A be a Green functor such that $A(G).A(G) = A(G)$. Then $A(G)$ has a unique defect base.*

Proof: Let Π be the set of all j_G-closed sets of $\mathcal{H}$ of subgroups of G such that $A(G)$ is $\mathcal{H}-$projective. By Lemma 5.8(1), (2) Π contains $j_G\{G\}$. Thus Π is not empty. We observe that the intersection $\mathcal{D}$ of all the members of Π is itself a

member of $\prod$, by Lemma 5.8(4) and Proposition 5.2(5). If $\mathcal{D} \leq_G \mathcal{H}$ for some $\mathcal{H}$, then $A(G)$ is $\mathcal{H}-$projective, by Lemma 5.8(3).
Conversely, if $A(G)$ is $\mathcal{H}-$projective for some $\mathcal{H}$, then $j_G\mathcal{H} \in \Pi$. Thus $\mathcal{D} \leq j_G\mathcal{H}$, i.e. $\mathcal{D} \leq_G \mathcal{H}$ by Proposition 5.2(3). Therefore $\mathcal{D}$ satisfies the first condition in the definition of the defect base. The uniqueness of $\mathcal{D}$ follows immediately from the definition of the defect base.

□

Remark 5.11 *If $A(G)$ has the identity element 1_G, then the condition $A(G).A(G) = A(G)$ is satisfied. If $\mathcal{H}$ is any set of subgroups of G, then $A(G)$ is $\mathcal{H}-$projective if and only if $1_G \in A_{\mathcal{H}}(G)$.*

The condition $A(G).A(G) = A(G)$ is clearly satisfied with our definition since we always assume the existence of unity elements.

The definition and existence of a defect base for a Green functor A depend only on a small number of properties of the family of ideals $(A_{\mathcal{H}}(G))$ of $A(G)$. In the next definition we isolate these properties and we will be able to speak of the defect base of a $k-$algebra A, relative to a $G-$ family of ideals of A.

Definition 5.12 *Let A be a $k-$algebra, G a finite group, and $\mathcal{S}(G)$ the set of all sets of subgroups of G. Suppose that $(A_{\mathcal{H}})$ is a family of ideals of A, indexed by $\mathcal{H}$ in $\mathcal{S}(G)$ which satisfies the following conditions, for all $\mathcal{H}, \mathcal{B}$ in $\mathcal{S}(G)$:*

1. *$A_G = A$.*

2. *$\mathcal{H} \leq_G \mathcal{B}$ implies $A_{\mathcal{H}} \leq A_{\mathcal{B}}$.*

3. *$A_{\mathcal{H}} * A_{\mathcal{B}} \leq A_{\mathcal{H}:G:\mathcal{B}}$.*

Then $(A_{\mathcal{H}})$ is called a $G-$family on A.

For example, the family $(A_{\mathcal{H}}(G))$ of $A(G)$, where $A_{\mathcal{H}}(G)$ and $A(G)$ are defined by a Green functor A, is a $G-$family on $A(G)$.

If $(A_{\mathcal{H}})$ is a $G-$family on the $k-$algebra A, we say that A is $\mathcal{H}-$projective, relative to $(A_{\mathcal{H}})$, if $A_{\mathcal{H}} = A$. We define a defect base $\mathcal{D}$ relative to $(A_{\mathcal{H}})$ to be a set $\mathcal{D}$ of subgroups of G which satisfies the conditions of the defect base which we stated before.

Proposition 5.13 *Let A be a $k-$algebra such that $A * A = A$, and let $(A_{\mathcal{H}})$ be a $G-$family on A. Then A has a unique defect base $\mathcal{D} = \mathcal{D}(A)$ relative to $(A_{\mathcal{H}})$.*

Now consider the associative $k-$algebra A with the identity element 1 and $(A_{\mathcal{H}})$ be a fixed $G-$family on A. Then we have the following proposition.

Proposition 5.14 *Let S be a subalgebra of A. Then $(S \cap A_{\mathcal{H}})$ is a $G-$family on S and $\mathcal{D}(A) \geq \mathcal{D}(S)$.*

Suppose that e is an idempotent in A. $S = eAe$ is a subalgebra of A; also $eAe \cap I = eIe$, for any ideal I of A. So the $G-$family on subalgebra $eAe = S$ is $(eA_{\mathcal{H}}e)$. For any $\mathcal{H} \in \mathcal{S}(G)$, the condition for eAe to be $\mathcal{H}-$projective is that $e \in eA_{\mathcal{H}}e$, which is equivalent to the condition $e \in A_{\mathcal{H}}$.

Theorem 5.15 ***(Green,[6])*** *If $1 = e_1 + \ldots + e_n$, where $e_1, \ldots, e_n$ are idempotents in A, then $\mathcal{D}(A) = \bigcup_1^n \mathcal{D}(e_i A e_i)$.*

Proof: Let $\mathcal{D}_i = \mathcal{D}(e_i A e_i)$. Then $\mathcal{D}(A) \geq \mathcal{D}_i$, for all i. Therefore, $\mathcal{D}(A) \geq \mathcal{D} = \bigcup_1^n \mathcal{D}_i$. However, we also have $e_i \in A_{\mathcal{D}_i}$ for all i. Therefore

$$1 = \sum e_i \in \sum A_{\mathcal{D}_i} \leq A_{\mathcal{D}},$$

so that A is $\mathcal{D}-$projective. Since $\mathcal{D}$is j_G-closed, $\mathcal{D}(A) \leq \mathcal{D}$. Thus $\mathcal{D}(A) = \mathcal{D}$. This completes the proof.

□

Now consider the idempotents e, f of A, we say that e, f are associated in A if the right $A-$modules eA, fA are isomorphic. Indeed, e, f are associated in A

if and only if there exists elements x, y in A such that

$$exf = x,\ \ fye = y, xy = e,\ \ yx = f.$$

It follows that any ideal $A_{\mathcal{H}}$ of A which contains e, also contains f, and conversely. Thus we have the following.

Proposition 5.16 *If e, f are idempotents which are associated in A, then $\mathcal{D}(eAe) = \mathcal{D}(fAf)$.*

We say that an idempotent e of A is *completely primitive* in A if $eAe/rad(eAe)$ is a division algebra. Here $rad(eAe)$ denotes the Jacabson radical.

In the next proposition and definition, we shall assume that the $G-$family $(A_{\mathcal{H}})$ satisfies that for $\mathcal{H} \in \mathcal{S}(G)$,

$$A_{\mathcal{H}} = \sum_{H \in \mathcal{H}} A(H).$$

Proposition 5.17 *If e is a completely primitive idempotent of A, then $\mathcal{D}(eAe) = j_G D$, for some subgroup D of G, which is determined uniquely up to conjugacy in G.*

Proof: Let $\mathcal{D} = \mathcal{D}(eAe)$. Then $e \in A_{\mathcal{D}} = \sum_{D \in \mathcal{D}} A(D)$ and so $e \in \sum_{D \in \mathcal{D}} eA(D)e$. If all $eA(D)e$ were proper ideals of eAe, they would lie in $rad(eAe)$, giving the contradiction $e \in rad(eAe)$. So there is some D in $\mathcal{D}$ for which $e \in A(D)$. Hence eAe is $D-$projective and so $\mathcal{D} \leq j_G D$. But $D \in \mathcal{D}$, so $j_G D \leq \mathcal{D}$. So it completes the proof. Since $\mathcal{D} = j_G D$, D is determined uniquely up to conjugacy in G.

$\square$

Now we give the definition of the *defect group*.

Definition 5.18 *With the notation of the previous proposition, we say that D is called a defect group of e, or eAe.*

In order to apply this result to a Green functor A, we need the following lemma.

Lemma 5.19 *Let A be a Green functor for a finite group G in k such that each algebra $A(H)$ is associative. Let $e = e_G$ be an idempotent of $A(G)$, and define $e_H = r_H^G e$ and $A^{'}(H) = e_H A(H) e_H$, for all subgroups H of G. Then the family $(A^{'}(H))$ defines a subfunctor $A^{'}$ of A whose canocical filtration is $(e_H A(H) e_H)$.*

For notation, we write eAe for the Green functor $A^{'}$, which is a subfunctor of A.

Proposition 5.20 ***(Green,[6])*** *Let A be Green functor for G over k such that each algebra $A(H)$ is associative and has identity element 1_H. Let*

$$1_G = e_1 + \ldots + e_n$$

be a decomposition of 1_G as sum of mutually orthogonal idempotents $e_1, \ldots, e_n$ of $A(G)$. Then

$$\mathcal{D}(A) = \bigcup_1^n \mathcal{D}(e_i A e_i).$$

and if e_i, e_j are associated in $A(G)$, then $\mathcal{D}(e_i A e_i) = \mathcal{D}(e_j A e_j)$, $(i, j = 1, \ldots, n)$. Moreover, if for each i, e_i is completely primitive in $A(G)$, and if D_i is a defect group of e_i, then

$$\mathcal{D}(A) = j_G\{D_1, \ldots D_n\}.$$

Proof: By Lemma 5.19 for each idempotent e_i, the subfunctor $e_i A e_i$ has its defect basis $\mathcal{D}(e_i A e_i)$. By Theorem 3.15 we have

$$\mathcal{D}(A) = \bigcup_1^n \mathcal{D}(e_i A e_i).$$

And by Proposition 5.16 we have $\mathcal{D}(e_i A e_i) = \mathcal{D}(e_j A e_j)$. Finally by this fact we get

$$\mathcal{D}(A) = j_G\{D_1, \ldots D_n\}.$$

.

□

In the rest of this section we will study the relation between the defect base and defect group of the endomorphism Green functor $End_G(M)$, where M is an indecomposable Mackey functor for G, and the vertex of M.

Theorem 5.21 ***(Green,[6])*** *Let M be a Mackey functor for G and the identity element of $End_G(M)(G)$ be a completely primitive idempotent. Then $End_G(M)(G)$ has a unique defect group up to conjugation.*

Proof: Existence of the unique defect group(up to conjugacy) of $End_G(M)(G)$ follows directly from Proposition 5.17 and Proposition 5.20.

□

Lemma 5.22 ***(Sasaki,[8])***

1. *Let M be a Mackey functor for G and D be a subgroup of G. Then the following are equivalent:*
   (a) M is $D-$projective.
   (b) M is isomorphic to a direct summand of $\uparrow_D^G\downarrow_D^G M$.
   (c) There is a Mackey functor N for D such that M is isomorphic to a direct summand of $\uparrow_D^G N$.
2. *Let $\mathcal{D}$ be a family of subgroups of G. Then a Mackey functor M for G is a $\mathcal{D}-$projective if and only if there exists, for each $D_i \in \mathcal{D}$, a Mackey funtor N_i for D_i such that M is isomorphic to a direct summand of $\sum_i \uparrow_D^G N_i$.*

Proof:

1. Suppose (a) holds. Then there exists an element Φ of $End_G(M)(D)$ such that $\Phi^G = 1$. Put $N = \downarrow_D^G M$. Define morphisms

$$\Lambda : M \to N^G \ \ \textit{and} \ \ \Pi : N^G \to M$$

by for each $H \leq G$.
$\Lambda(H) : M(H) \to N^G(H)$ such that for $\alpha \in M(H)$

$$\alpha \mapsto \sum_{Hg \in H/G} \theta_{H^g \cap D} r^{H^g}_{H^g \cap D} c^{g-1}_H(\alpha) \otimes Hg.$$

and $\Pi : N^G(H) \to M(H)$

$$\sum_{Hg \in H/G} \alpha_g \otimes Hg \mapsto \sum_{g \in H \backslash G/D} t^H_{H \cap {}^g D} c^g \alpha_g$$

respectively. Then $\Lambda(H)\Pi(H) = \Phi^G(H) = 1$ for each $H \leq G$. Namely, we have $\Lambda\Phi = 1$ in $End_G(M)(G)$ and hence M is isomorphic to a direct summand of $\uparrow^G_D \downarrow^G_D M$. (b) implies (c) trivially. Suppose we have (c). Let Θ be an element of $End_G(N^G)(D)$ defined by for each $E \leq D$ $\Theta(E) : N^G(E) \to N^G(E)$ such that for $\beta \in N^G(E)$

$$\sum_{Eg \in E/G} \beta_g \otimes Eg \mapsto \beta_1 \otimes E.$$

Then $\Theta^G = 1$. Let $\Lambda : M \to N^G$ and $\Pi : N^G \to M$ be the injection and the projection, respectively. Let

$$\Phi = \downarrow^G_D \Lambda\Theta \downarrow^G_D \Pi : \downarrow^G_D M \to \downarrow^G_D M.$$

Then $\Phi^G = 1$. Thus (a) holds.

2. If M is $\mathcal{D}-$projective, then there exists, for each $D_i \in \mathcal{D}$, an element Φ_i in $End_G(M)(D_i)$ such that $\sum_i \Phi_i = 1$. Put $N_i = \downarrow^G_{D_i}$ and define morphisms, using Φ_i,

   $$\Lambda_i : M \to N_i^G \quad and \quad \Pi_i : N_i^G \to M$$

   as in the first half of the proof (1). Let, for each D_i, Γ_i and Ξ_i be the injection from N_i^G to $\sum_i N_i^G$ and the projection from $\sum_i N_i^G$ to N_i^G, respectively. Let

   $$\Lambda = \sum_i \Lambda_i \Gamma_i : M \to \sum_i N_i^G$$

   $$\Pi = \sum_i \Xi_i \Pi_i : \sum_i N_i^G \to M.$$

Then we have $\Lambda\Pi = 1$. Namely, M is isomorphic to a direct summand of $\sum_i N_i^G$. Suppose conversely that M is isomorphic to a direct summand of $\sum_i N_i^G$, where N_i is a Mackey functor for D_i for each $D_i \in \mathcal{D}$. Let Γ_i and Ξ_i be the injections from N_i^G to $\sum_i N_i^G$ and the projection from $\sum_i N_i^G$ to N_i^G, respectively. Let Λ and Π be the injection from M to $\sum_i N_i^G$ and the projection from $\sum_i N_i^G$ to M, respectively. Let Θ_i be an element of $End_G(N_i^G)(D_i)$ such that $\Theta_i = 1$ and define

$$\Phi = \downarrow_{D_i}^G \Lambda \downarrow_{D_i}^G \Xi\Theta_i \downarrow_D^G \Gamma_i \downarrow_D^G \Pi_i.$$

Then we have $\sum_i \Phi_i^G = 1$. So M is $\mathcal{D}-$projective.

Definition 5.23 *A Mackey functor M is said to be $\mathcal{H}-$projective if its endomorphism Green functor is $\mathcal{H}-$projective, where $\mathcal{H}$ is a family of subgroups of G.*

Definition 5.24 *Let M be a Mackey functor for G, and $End_G(M)$ be the corresponding endomorphism Green functor. If the identity element of $End_G(M)(G)$ is a completely primitive idempotent, then the defect group of $End_G(M)$ is unique up to conjugacy. The defect group of $End_G(M)$ is called the vertex of the Mackey functor M.*

Proposition 5.25 *Let M be a Mackey functor for G and let $D \leq G$. Then D is a vertex of M if and only if D is minimal such that $M| \uparrow_D^G\downarrow_D^G M$.*

Proof: Let D be a vertex of M. Remember that if D is a vertex of M then D is the defect group of $End_G(M)$. By Definition 5.9 there is $\mathcal{D}$, a set of subgroups of G, such that $End_G(M)$ is $\mathcal{D}-$projective. Thus by Definition 5.23 M is $\mathcal{D}-$projective. Then by Lemma 5.22(1), $M| \uparrow_D^G\downarrow_D^G M$.

Now let $M| \uparrow_D^G\downarrow_D^G M$. Remember that D is minimal with $M| \uparrow_D^G\downarrow_D^G M$. Consider $\mathcal{X}$, the set of subgroups of G, such that D is maximal in it and $\mathcal{X}$ is closed under subconjugation. We need to show that M is $\mathcal{X}-$projective. Indeed, because $\mathcal{X}$ is closed under subconjugation, by Lemma 5.22(2), the sum of induced Mackey functors formed for each element of $\mathcal{X}$ for M, $\sum_i N_i$, is a Mackey functor for D. Since $M| \uparrow_D^G \downarrow_D^G M$, $M| \sum_i N_i$. Thus M is $\mathcal{X}-$projective. By Definition 5.23, $End_G(M)$ is $\mathcal{X}-$projective. So $\mathcal{X}$ is the defect base for $End_G(M)$ and since D is maximal in $\mathcal{X}$, it is a vertex of M.

□

As a summary, let k be a field and M is an indecomposable Mackey functor for a finite group G. There is a unique set of subgroups $\mathcal{H}$ closed under conjugation and taking subgroups minimal with respect to the property that M is $\mathcal{H}-$projective. This set consists of a single conjugacy class of subgroups together with their subgroups. A representative of this single conjugacy class is a vertex of M. Let $D \leq G$ be a vertex of M. Then there is an indecomposable Mackey functor N for D, unique up to conjugacy in $N_G(D)$, such that $M| \uparrow_D^G N$. Note that the vertex of an indecomposable Mackey functor need not to be a $p-$subgroup of G. In fact any $H \leq G$ may be the vertex of M. We use this result to prove the Green correnspondent theorem by analogue proof as in group algebra. Before that we study the Sasaki's proof for Green correspondence for Mackey fucntors.

Chapter 6

Green Correspondence for Mackey Functors(1)

In this chapter we will study the Green correspondence of Mackey functors via Sasaki's proof. Let M be a Mackey functor for G and $\mathcal{H}$ a family of subgroups of $K \leq G$, then,

$$M(\mathcal{H})^K = \sum_{H \in \mathcal{H}} t_H^K M(H).$$

A Green functor A for G is called local if $A(G)$ is a local algebra.

Let k be a complete local principle ideal domain and R be a finitely generated associative $k-$algebra with identity element 1. Then by [6], 1 has at least one primitive decomposition in R, as

$$1 = \epsilon_0 + \ldots + \epsilon_m$$

and $\epsilon_0, \ldots, \epsilon_m$ are mutually orthogonal idempotents and they are completely primitive in R. Let $1 = \alpha_0 + \ldots + \alpha_s$ be another primitive decomposition of 1 in R. So $m = s$ and the α_j can be so numbered that, $\beta^{-1}\epsilon_j\beta = \alpha_j$ for $j = 0, \ldots, m$, for a suitable element β of R. Furthermore, every idempotent of R can be expressed as the sum of mutually orthogonal primitive idempotents of R. In particular every primitive idempotent of R is completely primitive. Note that for easy notation we use ϵ_K as the restriction of ϵ and ϵ^G as the induction of ϵ

where ϵ is an idempotent in $A(H)$.

Definition 6.1 *Let $A \in \mathcal{A}_k(G)$ be a local Green functor and let $M \in Mack_k(G)$ an $A-$module. Let D be a defect group of A and H be a subgroup of G which contains $N_G(D)$. Let*

$$\mathfrak{X} = \{sQs^{-1} \cap D | s \in G, s \notin H\}$$

$$\mathfrak{Y} = \{sQs^{-1} \cap H | s \in G, s \notin H\}$$

which we studied in Chapter 2. Let

$$1_H = \epsilon_0 + \ldots + \epsilon_n$$

be a primitive decomposition of the identity element 1_H of $A(H)$. Then by [6] there is exactly one index i such that D is a defect group of $\epsilon_i \downarrow_H^G A$. Furthermore, again by [6], arranging notation so that $i = 0$, we have

$$1_H \equiv \epsilon_0 \ \ mod A(\mathfrak{Y})^H.$$

where $\epsilon_i \in A(\mathfrak{Y})^H$ for $i \geq 1$. Also

$$\epsilon_0^G \equiv 1_G \ \ mod A(\mathfrak{Y})^G.$$

We put $\epsilon = \epsilon_0$ and note that ϵ^G is a unit in $A(G)$.
We define N, a Mackey functor for H, as a subfunctor of $\downarrow_H^G M$ for each $L \leq H$,

$$N(L) = \epsilon_L M(L).$$

This Mackey functor for H is called a Green correspondent of M with respect to (G, D, H, A).

Remark 6.2 *The definiton of Green correspondents depends on a primitive decomposition of the identity element of $A(H)$. However, all decompositions give rise to isomorphic Green correspondents.*

The following lemma is a result from Sasaki.

Lemma 6.3 *Let M be a Mackey functor for G over k and $\epsilon \in End_G(M)(G)$ be an idempotent. Then a subfunctor N of M defined by $N(H) = \epsilon_H M(H)$ $(H \leq G)$ is a direct summand of M. Furthermore $End_H(N)$ is isomorphic to $\epsilon End_G(M)\epsilon$.*

Proposition 6.4 ***(Sasaki)*** *Using the notation of Definition 6.1 we have the following.*

1. *The Green correspondent N is an $\epsilon \downarrow_H^G A\epsilon$-module.*
2. *$\downarrow_H^G M$, which is the Mackey functor for H, is the direct sum of N and a $\mathfrak{Y}-$projective Mackey functor for H. The induced Mackey functor $\uparrow_H^G N$ is the direct sum of M and a $\mathfrak{X}-$ projective Mackey functor for G.*
3. *The factor modules $M(G)/M(\mathfrak{X})^G$ and $N(H)/M(\mathfrak{Y})^H$ are isomorphic by*
$$\phi + M(\mathfrak{X})^G \longmapsto \epsilon(r_H^G \phi) + N(\mathfrak{Y})^H.$$

Proof:

1. Let L be a subgroup of H. For an element $\epsilon_L\phi$ and an element $\epsilon_L\theta\epsilon_L$ of $\epsilon \downarrow_H^G A\epsilon(L)$,
$$(\epsilon_L\phi)(\epsilon \downarrow_H^G A\epsilon(L)) = \phi\epsilon_L\theta\epsilon_L$$
which is in $N(L)$. Thus N is an $\epsilon \downarrow_H^G A\epsilon-$module.
2. Let $\gamma = \epsilon_1 + \ldots + \epsilon_n$. Then γ is in $A(\mathfrak{Y})^G$. We define a subfunctor P of $\downarrow_H^G M$ by $P(L) = \gamma_L M(L)$ for $L \leq H$. Obviously, $\downarrow_H^G M$ is the direct sum of N and P. Moreover P is a $\gamma \downarrow_H^G A\gamma-$module, which can be shown as in (1). Thus P is $\mathfrak{Y}-$projective.
Now we define the following morphisms
$$\Omega : \uparrow_H^G N \to M$$
such that
$$\Omega(K) : \uparrow_H^G N(K) \to M(K) : \sum_{Kg \in K\backslash G} \beta \otimes Kg \longmapsto \sum_{g \in K\backslash G/H} t_{K\cap {}^gH}^K c^{g^{-1}}(\beta)$$

$$\Gamma : M \to \uparrow_H^G N$$

such that

$$\Gamma(K) : M(K) \to \uparrow_H^G N(K) : \alpha \longmapsto \sum_{Kg \in K\backslash G} \epsilon_{K^g \cap H} r^K_{K \cap {}^g H} c^g \otimes Kg$$

by for each $K \leq G$.

Then for $K \leq G$ we have

$$\Gamma(K)\Omega(K) : \alpha \longmapsto \epsilon^G{}_K\ \alpha$$

and this is an isomorphism so that $\Gamma\Omega : M \longrightarrow M$ is an isomorphism. And M is isomorphic to a direct summand of $\uparrow_H^G N$. Let consider $End_G(N^G)$. Obviously, $End_G(N^G)$ is $D-$projective. Let $\Psi = \Omega\Gamma$ and

$$1_G - \Psi = \Theta_1 + \ldots + \Theta_q$$

be a primitive decomposition of $1_G - \Psi$ in $End_G(N^G)(G)$. We need to show that Θ_i is in $End_G(N^G)(\mathfrak{X})^G$. Because $End_G(N^G)$ is $D-$projective, by Theorem 2 of [6] Θ_i is in $End_G(N^G)(\mathfrak{X})^G$ if and only if $(\Theta_i)_H$ is in $End_G(N^G)(\mathfrak{Y})^H$. $\downarrow_H^G\uparrow_H^G N$ is the direct sum of N and a $\mathfrak{Y}-$projective Mackey functor for H by Lemma 5.22. Furthermore N is a direct summand of $\downarrow_G^H M$ and $\downarrow_G^H M$ is isomorphic to a direct summand of $\downarrow_H^G\uparrow_H^G N$. Thus $(\Theta_i)_H$ must lie in $End_G(N^G)(\mathfrak{Y})^H$.

3. M is defined by its transfer map t, restriction map r and conjugation map c. By Theorem 2.3 of [7], r_G^H induces an isomorphism of $M(G)/M(\mathfrak{X})^G$ and $N(H)/M(\mathfrak{Y})^H$. After realizing this isomorphism, we can combine this with the first half of the (2). Clearly we get the result.

$\square$

The following theorem is our main theorem, Green Correspondence for Mackey functors, which is the result what we have done up to now. Since everything is established, the proof of the theorem is easy to follow.

Theorem 6.5 ***(Sasaki)*** *Let k be a field, D be a subgroup of G, and H be a subgroup of G containing $N_G(D)$. Then there exists a one-to-one correspondence between the set of all isomorphism classes of indecomposable Mackey functors for G over k which have vertex D and the set of all isomorphism classes of indecomposable Mackey functors for H over k which have vertex D, in which a Mackey functor M for G and a Mackey functor N for H correspond if, and only if, either N is isomorphic to a direct summand of $\downarrow_H^G M$ or M is isomorphic to a direct summand of $\uparrow_H^G N$.*

Proof: Let M be an indecomposable Mackey functor for G over k. Consider the endomorphism Green functor for M, $End_G(M)$. Let say $A = End_G(M)$. Then $A(G)$ is local because M is indecomposable. Let D be a vertex of M, H a subgroup of G containing $N_G(D)$, and N be a Green correspondent of M, which is as we defined above, with respect to G, D, H, A. Let ϵ be our primitive idempotent of $A(H)$ that defines N as in Definition 6.1. Then by Lemma 6.3 $\epsilon \downarrow_H^G A\epsilon$ is isomorphic to $End_H(N)$. Furthermore, N has vertex D. Hence, the desired one-to-one correspondence is the one that corresponds the isomorphism class of M to isomorphism classes of N.

$\square$

Chapter 7

Green Correspondence for Mackey Functors(2)

In the previous chapter we studied Sasaki's proof for the Green correspondence for indecomposable Mackey functors. Sasaki proved the Green correspondence for an indecomposable Mackey functor M via endomorphism Green functor $End(M)$ after defining the vertex of M via the vertex of $End(M)$. In this chapter we shall prove the theorem by an analogue of the proof for group algebras.

Theorem 7.1 *Let k be a field, D be a subgroup of G, and H be a subgroup of G containing $N_G(D)$. Then there exists a one-to-one correspondence between the set of all isomorphism classes of indecomposable Mackey functors for G over k which have vertex D and the set of all isomorphism classes of indecomposable Mackey functors for H over k which have vertex D, in which a Mackey functor M for G and a Mackey functor N for H correspond if, and only if, either N is isomorphic to a direct summand of $\downarrow_H^G M$ or M is isomorphic to a direct summand of $\uparrow_H^G N$.*

Proof: Remember some collections of subgroups of G. Let

$$\mathfrak{X} = \{sDs^{-1} \cap D | s \in G, s \notin H\}$$

$$\mathfrak{Y} = \{sDs^{-1} \cap H | s \in G, s \notin H\}$$

$$\mathfrak{D} = \{R | R \leq D, R \not\leq_G x\}$$

We have the following results.

Lemma 7.2 *Suppose that R is a subgroup of D. Then the following are equivalent.*

1. $R \leq_G \mathfrak{X}$;
2. $R \leq_H \mathfrak{X}$;
3. $R \leq_H \mathfrak{Y}$.

Proof: If (1) is valid then there is $g \in G$ with $gRg^{-1} \leq D \cap sDs^{-1}$, where $s \in G$, $s \notin H$. Then if $g \in H$ then certainly (2) holds. If $g \notin M$ then $R \leq g^{-1}Dg$ yields $R \leq D \cap g^{-1}Dg$, that is R in $\mathfrak{X}$ so certainly $R \leq_H \mathfrak{X}$. So (2) holds.
If (2) holds then there is $x \in H, s \in G, s \notin M$ such that $xRx^{-1} \leq D \cap sDs^{-1}$. Thus, we get $xRx^{-1} \leq H \cap sDs^{-1}$ and $R \leq_H \mathfrak{Y}$. So (3) holds.
If (3) holds then there is $x \in H, s \in G, s \notin H$ with $xRx^{-1} \leq H \cap sDs^{-1}$. Hence $R \leq H \cap (x^{-1}s)D(s^{-1}x)$. However $x^{-1}s \notin H$. So clearly we have $R \leq D \cap (x^{-1}s)D(s^{-1}x)$ and $R \leq_G \mathfrak{X}$. So (1) holds.

□

Lemma 7.3 *Let M be a Mackey functor for G which is $\mathfrak{X}-$projective and N be a Mackey functor for H which is $\mathfrak{Y}-$projective. Then $\downarrow_H^G M$ is $\mathfrak{Y}-$projective and $\uparrow_H^G N$ is $\mathfrak{X}-$projective.*

Proof: Let T be an indecomposable summand of M so T is projective for a subgroup of the form $sDs^{-1}, s \notin H$. Hence, by Mackey's theorem, $\downarrow_H^G T$ is projective for the collection of subgroups of the form

$$t(sDs^{-1} \cap D)t^{-1} \cap H = tsDs^{-1}t^{-1} \cap tDt^{-1} \cap H.$$

But either $t \notin H$ or $ts \notin H$ so such a subgroup is contained in an element of $\mathfrak{Y}$ so $\downarrow_H^G T$ and $\downarrow_H^G M$ are relatively $\mathfrak{Y}-$projective.
If T is an indecomposable summand of N and T has vertex P then $P \leq_H \mathfrak{Y}$ so $\uparrow_H^G T$ is $P-$projective and $P \leq_G \mathfrak{X}$, by Lemma 7.2 , so $\uparrow_H^G T$ is $\mathfrak{X}-$projective. The lemma is proved.

□

The next two results contain the bulk of the theorem.

Lemma 7.4 *If M is an indecomposable Mackey functor for G with vertex R in $\mathfrak{D}$ then $\downarrow_H^G M \cong N \oplus T$ where N is an indecomposable Mackey functor for H with vertex R, $M | \uparrow_H^G N$ and T is a $\mathfrak{Y}-$projective Mackey functor for H.*

Proof: There is an indecomposable Mackey functor N for H with vertex R and $M | \uparrow_H^G N$. Now $\downarrow_H^G \uparrow_H^G N \cong N \oplus T_1$, where T_1 is relatively $\mathfrak{Y}-$projective, by Lemma 2.16. Thus $\downarrow_H^G M$ is either isomorphic with $N \oplus T$ or T for some summand T of T_1. But, $\downarrow_H^G M$ has an indecomposable summand Z with vertex R. Now Z cannot be isomorphic with summand of T, or else $R \leq_H \mathfrak{X}$, so $R \notin \mathfrak{D}$. Thus, $Z \cong N$ and $U_M \downarrow_H^G M \cong N \oplus T$.

□

Lemma 7.5 *If N is an indecomposable Mackey functor for H with vertex R in $\mathfrak{D}$ then $\uparrow_H^G N \cong M \oplus W$ where M is an indecomposable Mackey functor for G vertex R, $N | \downarrow_H^G M$ and W is a $\mathfrak{X}-$projective Mackey functor for G.*

Proof: Let $\uparrow_H^G N = M_1 + \ldots + M_r$ be a direct sum of indecomposable Mackey functors for G. Since $\downarrow_H^G \uparrow_H^G N \cong N \oplus T$, where Y is a $\mathfrak{Y}-$projective Mackey functor for H.We have, after renumbering, that $\downarrow_H^G M_1 \cong N + T_1$, $\downarrow_H^G M_i \cong T_i$, $2 \leq i \leq r$, where T_i are Mackey functors for H and $T \cong T_1 \oplus \ldots \oplus T_r$. We claim that M_i has a vertex in $\mathfrak{D}$ and that $M_1, \ldots, M_r$ are relatively $\mathfrak{X}-$projective.

Indeed, M_1 has a vertex in D, as $M_1| \uparrow_H^G N$, and M_1 cannot be $\mathfrak{X}-$projective. Then $\downarrow_H^G M_1 \cong N + T_1$ is $\mathfrak{Y}-$projective which is not the case. Hence, M_1 does not have a vertex in $\mathfrak{D}$. Moreover, if M_i, $i \leq 2$, was not $\mathfrak{X}-$projective. Then Lemma 7.4 would imply to it and $\downarrow_H^G M_i$ would not be $\mathfrak{Y}-$projective, which it is. Thus, M_i is relatively $\mathfrak{X}-$projective.
Setting $M = M_1$, $W = M_2 + \ldots + M_r$ we have that $M| \uparrow_H^G N$, $\uparrow_H^G N \cong M \oplus W$ where W is $\mathfrak{X}-$projective. It remains only to prove that M has vertex R. However, the vertex of M is in $\mathfrak{D}$. Thus Lemma 7.4 applies and there is a unique summand, in any decomposition of $\downarrow_H^G M$ into the direct sum of indecomposable modules, which is not $\mathfrak{Y}-$projective and that Mackey functor has a vertex equal to a vertex of M. However $\downarrow_H^G M \cong N \oplus T$. So M has vertex R as N has vertex R.

□

We almost proved the Theorem 7.1. We only need to show the one-to-one property.

If M is an indecomposable Mackey functor for G with vertex R in $\mathfrak{D}$, N is as in Lemma 7.4. Then $\uparrow_H^G N = M' \oplus T$ as in Lemma 7.5. Thus $M \cong M'$. But in Lemma 7.4 we proved that $M| \uparrow_H^G N$. If we start with N, a similar result holds as proved in Lemma 7.5 that $N| \downarrow_H^G M$. This shows the one-to-one property.

□

Chapter 8

Some Further Applications

In this chapter we will assume that the coefficient ring is a field k. We will study the proofs of Thévenaz-Webb, results on Green correspondents, vertices, sources of the projective Makcey functor, the fixed point and the fixed quotient Mackey functors and the simple Mackey functor. Moreover, to avoid some further knowledge about the Mackey functors we sometimes give only the theorems and propositions.

Recall that every indecomposable Mackey functor M for a finite group G over k has a vertex $H \leq G$, a unique minimal subgroup up to conjugacy, relatively to which it is projective. The Green correspondence works in this situation, such that if H is vertex of M and K is a subgroup containing $N_G(H)$ then $\downarrow_H^G M$ has a unique summand, let say it $f(M)$, with vertex H, and if L is a Mackey functor for K with vertex H then $\uparrow_K^G M$ has unique summand, let say it $g(L)$, with vertex H, the correspondence f and g being mutually inverse. This idea consists of the method to determine the Green correspondents, vertices, sources of the projective Makcey functor, the fixed point and the fixed quotient Mackey functors and the simple Mackey functor.

8.1 Vertices, Sources and Green Correspondence of Simple Mackey Functors

In this section, firstly we define the simple Mackey functor, the projective Makcey functor, the fixed point and the fixed quotient Mackey functor. Then we give some results for the Green correspondence of these functors. Recall that by a subfunctor N of a Mackey functor M for a finite group G on $\mathcal{H}$, which is a family of subgroups of G, one means a family of $k-$submodules $N(H) \leq M(H)$ where $H \in \mathcal{H}$ is stable under restriction, transfer and conjugation. If N is a subfunctor of M, then the *quotient functor* M/N is defined by $(M/N)(H) = M(H)/N(H)$.

Definition 8.1

- *A mackey functor M is called simple if the only subfunctors of M are M itself and zero(i.e. the Mackey functor which is zero on each subgroup) and is denoted by $S_{H,V}$ for a subgroup H of G and for a $kG-$module V.*
- *A mackey functor is projective and indecomposable, if it is the projective cover of a simple Mackey functor $S_{H,V}$.Then it is denoted by $P_{H,V}$.*
- *For a $kG-$module V, the functor, which is denoted by FP_V, is called the fixed point Mackey functor and satisfies that $FP_V = V^H$ for a subgroup H of G.*
- *For a $kG-$module V, the functor, which is denoted by FQ_V, is called the fixed quotient Mackey functor and satisfies that $FP_V = V_H$ for a subgroup H of G.*

Throughout this section we use superscripts to indicate for which group a Mackey functor is considered. Thus for instance $S^K_{H,V}$ is the simple Mackey functor for K corresponding to the pair (H, V).

Proposition 8.2 ***(Thévenaz-Webb,[11])*** *Let $H \leq G$ and let V be a simple $\overline{kN_G}(H)-$module. Then the indecomposable projective $P_{H,V}$ has vertex H.*

The source of the projective module is defined by some further theorems on Burnside Mackey functors and for more information review [2]

Now we consider Green correspondents of projective Mackey functors.

Theorem 8.3 ***(Thévenaz-Webb,[11])*** *Let K be a subgroup of G which contains $N_G(H)$. The Green correspondent of $P^G_{H,V}$ is $P^K_{H,V}$.*

Proof: We may write $\uparrow^G_K P^K_{H,V} \cong g(P^K_{H,V}) \oplus M$ where M is a Mackey functor all of whose summands have smaller vertex than H. Since induction preseves projectives, these summands have the dorm $P_{L,V}$ with $L < H$. Apllying induction to the epimorphism $P^K_{H,V} \to S^K_{H,V}$ gives an epimorphism $g(P^K_{H,V}) \oplus M \cong \uparrow^G_K P^K_{H,V} \to S^G_{H,V}$, using the fact that $S^G_{H,V}$ is an induced functor. The only way we can have such an epimorphism to a simple Mackey functor is if $P^G_{H,V}$ is a summand, and since this has vertex H we must have $g(P^K_{H,V}) = P^G_{H,V}$.

□

In order to study the vertices and sources of simple Mackey functors, firstly we need the statement of the following lemma.

Lemma 8.4 *Suppose that M is a Mackey functor which is projective relative to a set of subgroups $\mathcal{H}$, and let S(G) be a set of subgroups of G which is closed under taking subgroups and conjugation. Consider the Mackey functor N whose value at a subgroup H is defined to be*

$$N(H) = \sum_{J \leq H, J \in \mathcal{H}} t^H_J M(J).$$

Then N is also projective relative to $\mathcal{H}$.

We need to extend the notation $S_{H,V}$ to non-simple modules V. If V is an arbitrary $\overline{kN}(H)-$module, define $S^{\overline{N(H)}}_{1,V}(K) = t^K_1(V) \leq V^K = FP_V(K)$. Thus $S_{1,V}$ is a subfunctor of FP_V. Then define

$$S_{H,V} = S^G_{H,V} = \uparrow^G_{N(H)} (Inf^{N(H)}_{\overline{N(H)}} S^{\overline{N(H)}}_{1,V})$$

And it is easy to see that $S_{H,V}(H) = V$. As a consequence, $S_{H,V}$ is indecomposable if and only if V is an indecomposable $\overline{kN}(H)-$module.

Here it is necessary to give the definition of inflation Mackey functor.

Remark 8.5 *Whenever we have a normal subgroup $N \lhd G$ and a Mackey functor L for $Q = G/N$ we can form the inflation $Inf_Q^G L$ which is a Mackey functor defined by*

$$Inf_Q^G L(K) = \begin{cases} 0 & \text{if } K \gneqq N \\ L(K/N) & \text{if } K \geq N \end{cases}$$

with zero restriction and induction morphisms r_H^K, t_H^K unless $N \leq H \leq K$ in which case they are the mappings $r_{H/N}^{K/N}, t_{H/N}^{K/N}$, and similarly with conjugations.

The next proposition gives the notion of the vertices of the fixed point and fixed quotient Mackey functors functors.

Proposition 8.6 ***(Thévenaz- Webb,[11])*** *For any indecomposable $kG-$module W, the vertices of FP_W, FQ_W and W are the same.*

Proof: We rely on the fact that induction of Mackey functors commutes with all of FP, FQ and evaluation at 1. Let K be a vertex of W. Then since W is a summand of W_K^G we have that FP_W is a summand of $FP_{W_K^G} \cong \uparrow_K^G FP_W$. Hence K contains a vertex L of FP_W. On the other hand, the split epimorphism $\uparrow_L^G FP_W \to FP_W$ on evaluation at the identity subgroup is a split epimorphism $W_L^G \to W$, and hence L must be K. The proof for FQ_W is similar.

□

Finally we show that the vertex, source and Green correspondent of simple Mackey functors are determined by the Green correspondents of the corresponding modules. Firstly, consider the following lemma.

Lemma 8.7 *Let K be a normal subgroup of G and let $K \leq J \leq G$. Let M be an indecomposable Mackey functor for J with vertex K. Then every indecomposable summand of $\uparrow_J^G M$ has vertex K.*

Proof: Let N be a source of M. Since M is a summand of $\uparrow_K^J N$, the functor $\downarrow_J^K M$ is a summand of

$$\uparrow_K^J \downarrow_K^J = \bigoplus_{h \in J/K}^{h} N.$$

Thus $\downarrow_K^J M$ is a dirrect sum of conjugates of N.
Now let L be an indecomposable summand of $\uparrow_J^G$. Then $\downarrow_G^K L$ is a summand of

$$\uparrow_J^G \downarrow_K^G M = \bigoplus_{g \in K\backslash G/J} \downarrow_K^{{}^gJ} ({}^gM) = \bigoplus_{g \in G/J} (\downarrow_K^J M)^{g^{-1}}$$

and so every summand of $\downarrow_G^K L$ is a conjugate of N. After conjugation, it follows that N is a summand of $\downarrow_G^K L$. Since we also have that L is a summand of $\uparrow_K^G N$ and N is its own source, N must be a source of L. In particular K is a vertex of L. So this completes the proof the lemma.

□

Theorem 8.8 ***(Thévenaz-Webb,[11])***

1. *Let $S_{H,V}^G$ be a simple Mackey functor, and let K be a subgroup of G with $H \leq K \leq N_G(H)$ such that K/H is a vertex of V. Then K is a vertex of $S_{H,V}^G$.*

2. *Let moreover U be a source of V. Then the indecomposable Mackey functor $S_{H,U}^K$ is a source of $S_{H,V}^G$.*

3. *Let $S_{H,V}^G$ be a simple Mackey functor for G and let K/H be a vertex of V. Let W be the Green correspondent of V, a module for $N_{N(H)/H}(K/H) = N(H,K)/H$ where $N(H,K) = N(H) \cap N(K)$. Then the Mackey functor for $N(K)$ which is the Green correspondent of $S_{H,V}^G$ is equal to $\uparrow_{N(H,K)^{N(K)}}$ $S_{H,W}^{N(H,K)}$.*

Proof:

1. Since $S^G_{H,V} = \uparrow^G_{N(H)} (Inf^{N(H)}_{\overline{N(H)}} S^{\overline{N(H)}}_{1,V})$ is an induced functor, it follows that a vertex of $Inf^{N(H)}_{\overline{N(H)}} S^{\overline{N(H)}}_{1,V}$ as a Mackey functor for $N(H)$ is also a vertex of $S^G_{H,V}$. Evidently this will be the preimage in $N(H)$ of a vertex of $S^{\overline{N(H)}}_{1,V}$. By this means we reduce to the case that $H = 1$, so we consider a simple Mackey functor $S_{1,V}$.

   A vertex L of $S_{1,V}$ must always contain a vertex K of V since the split epimorphism $\uparrow^G_L \downarrow^G_L S_{1,V} \to S_{1,V}$ on avaluation at 1 gives a split epimorphism $(V_L)^G \to V$. On the other hand FP_V is projective relative to K by Proposition 8.6 and $S_{1,V}$ is construced as in Lemma 8.4 on taking $M = FP_V$, and *S(G)* $= 1$ so it follows by Lemma 8.4 that $S_{1,V}$ is also projective relative to K. Thus $L = K$.

2. By the same reduction argument as in (1), we are left with the case $H = 1$. Since V is a summand of U^G, FP^G_V is a summand of $FP^G_{U^G} \cong \uparrow^K_U FP^K_U$. By the definition of $S^G_{1,V}$, it is clear that $S^G_{1,V}$ is a summand of $S^G_{1,U^G} = \uparrow^G_K S^K_{1,U}$. Since we already know that K is a vertex of $S^G_{1,V}$, the argument suffices to guarantee that $S^K_{1,U}$ is a source of $S^G_{1,V}$. So this completes the proof of (2).

3. The functor $V \to S_{1,V}$ from modules to Mackey functors is additive and the proof of Proposition 11.4 shows that it preserves vertices. It follows easily that $S^{N(H,K)/H}_{1,W}$ is the Green correspondent of $S^{N(H)/H}_{1,V}$. Applying inflation from $N(H)/H$ to $N(H)$, one obtains that $S^{N(H,K)}_{H,W}$ is the Green correspondent of $S^{N(H)}_{H,V}$. Therefore

   $$\uparrow^{N(H)}_{N(H,K)} S^{N(H,K)}_{H,W} \cong S^{N(H)}_{H,V} \oplus T$$

   where each summand of T has vertex smaller than K. Let $M = \uparrow^{N(H)}_{N(H,K)} S^{N(H,K)}_{H,W}$. Then

   $$\uparrow^G_{N(K)} M = \uparrow^G_{N(H,K)} S^{N(H,K)}_{H,W} \cong S^G_{H,V} \oplus \uparrow^G_{N(H)} T.$$

   By Lemma 8.7, every indecomposable summand of M has vertex K. By Green correspondence, each such summand corresponds to an indecomposable summand of $\uparrow^G_{N(K)} M$ with vertex K. However, we have seen that $S^G_{H,V}$

is the only summand of $\uparrow_{N(K)}^{G} M$ with vertex K. Hence M is indecomposable and M is the Green correspondent of $S_{H,V}^{G}$.

$\square$

Bibliography

[1] J.L. Alperin, "Local Representation Theory", (Cambridge University Press, 1982).

[2] Serge Bouc, "Green Functors and G-sets, Chapters 1-6", (Springer-Verlag, Berlin, Heidelberg 1997).

[3] C.W. Curtis, I. Reiner, "Methods of Representation Theory with Applications to Finite Groups and Orders Vol.1", (Wiley, New York 1987).

[4] W. Feit, "The Representation Theory of Finite Groups, Chapters 1-4", (North-Holland, Amsterdam 1982).

[5] J.A. Green, *A transfer theorem for modular representations*, J. Algebra **1**, 73-84 (1964).

[6] J.A. Green, *Axiomatic representation theory for finite groups*, J. Pure Appl. Algebra **1**, 41-77 (1971).

[7] J.A. Green, *Relative module categories for finite groups*, J. Pure Appl. Algebra **2**, 371-393 (1972).

[8] H. Sasaki, *Green correspondence and transfer theorems of Wielandt Type for G-functors*, J. Algebra **79**, 98-120 (1982).

[9] J. Thévenaz, *Some remarks on G-functors and the Brauer morphism*, J. reine angew. Math. **384**, 24-56 (1988).

[10] J. Thévenaz, "G-Algebras and Modular Representation Theory", (Clarendon Press, Oxford 1995).

[11] J. Thévenaz, P. Webb, *The structure of Mackey functors*, Trans. Amer. Math. Soc. **347**, 1865-1963 (1995).

[12] T. Yoshida, *On G-functors (1): Transfer theorems for chomological G-functors*, Hokkaido Math. J. **9**, 222-357 (1980).