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OPTIMAL CONTROL OF PHYSICAL SYSTEMS GOVERNED BY PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

OPTIMAL CONTROL OF PHYSICAL SYSTEMS GOVERNED BY PARTIAL DIFFERENTIAL EQUATIONS

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Ph.D. Thesis

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This thesis consists of two parts: An optimal vibration control of two linear beam models governed by hyperbolic partial differential equations and a bilinear optimal control of heat transfer process governed by a parabolic partial differential equation.
In the first part of the thesis, an optimal boundary control of two different beam models, Mindlin-type beam and second strain gradient theory-based beam, are examined. The beam models to be controlled are described by linear higher order distributed parameter systems. When the Mindlin-type beam is considered, the partial differential equation that governs dynamic behavior of the model must be of the same order with respect to both spatial and time parameters to match up with Einstein's causality. As for the second strain gradient theory-based beam, it captures the size effects of the structures in micro and nanoscale. In this theory, the potential energy is dependent of strains, gradient of strains and also second gradient of strains. The constitutive equation is a hyperbolic partial differential equation of sixth order. For both beam models, controllability and well-posedness of the system are analyzed. Pontryagin's maximum principle is used to obtain an optimal control function and its trajectories. By using Pontryagin's maximum principle, the control problem is turned to solving a new partial differential equation system involving adjoint and state variables providing initial, boundary and terminal conditions. Analytical solutions are formed with the help of an eigenfunction expansion method. Simulations are presented to confirm the theoretical results.
In the second part of the thesis, an optimal control of a parabolic system governed by a bilinear control is discussed. The parabolic system describes heat transfer process. For the heat transfer process, coolant flow rate is a bilinear control variable and temperature is a state variable. As a result of implementation of a bilinear control to a uniform rod with non-uniform temperature, the temperature on the metal rod is homogeneously distributed

when the temperature reaches equilibrium. The performance index to be minimized indicates that the goal is to keep the temperature close to the steady-state value without consuming control effort in large quantities.
The proposed approach uses a reduction of order in the model, Pontryagin's maximum principle and numerical techniques. Using a modal space expansion method the distributed parameter system is transformed into a lumped parameter system. The obtained system corresponds to a bilinear system in the temporal term. Pontryagin's maximum principle is used to obtain the optimal control function that leads to a nonlinear two-point boundary value problem. Two iterative numerical techniques for determining optimal trajectories and optimal control of the system are discussed. Steepest descent and quasilinearization are procedures for solving this nonlinear two-point boundary value problem. The steepest descent method is used as a beginning procedure and quasilinearization to confirm the solution. Comparisons of some features such as what the initial guess is and importance of initial guess, number of iterations and convergence of these two iterative methods in the control problem are presented. Numerical simulation studies show the effectiveness and applicability of the proposed approach.

Keywords: Optimal control, bilinear system, steepest descent method, quasilinearization method, Pontryagin's maximum principle.

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ÖZET

KISMİ DİFERANSİYEL DENKLEMLER TARAFINDAN YÖNETİLEN FİZİKSEL SİSTEMLERİN OPTİMAL KONTROLÜ

Seda GÖKTEPE KÖRPEOĞLU

Matematik Mühendisliği Anabilim Dalı

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Bu tez iki bölümden oluşmaktadır. Bu bölümler, hiperbolik kısmi diferansiyel denklemler tarafından yönetilen iki lineer kiriş modelinin optimal titreşim kontrolü ve parabolik kısmi diferansiyel denklem tarafından yönetilen ısı transfer işleminin bilineer optimal kontrolüdür.

Tezin ilk bölümünde, Mindlin-tipi kiriş modeli ve ikinci gerilme eğimi teorisi tabanlı kiriş modeli olmak üzere iki farklı çeşit kiriş modelinin optimal sınır kontrolü üzerine çalışılmıştır. Kontrol uygulanacak bu kiriş modelleri lineer yüksek mertebeden diferansiyel denklemler ile belirlenmişlerdir. Mindlin-tipi kiriş modeli ele alındığında modelin dinamik davranışını yöneten kısmi diferansiyel sistemde Einstein'ın nedensellik ilkesine bağdaşır şekilde konum ve zaman koordinatları aynı mertebedendir. İkinci gerilme eğimi teorisi tabanlı kiriş modeline baktığımızda, mikro ve nano ölçekte yapıların boyut etkilerini göz önüne aldığı bilinen ikinci gerilme eğimi elastik teorisi dikkate alınmıştır. Bu teoride, potansiyel enerji, gerilmelere, gerilmelerin gradyanına ve ayrıca gerilmelerin ikinci mertebeden gradyanına bağlıdır. Bunlar göz önüne alınarak oluşturulan denklem altıncı mertebeden hiperbolik bir kısmi diferansiyel denklemdir. Her iki kiriş modeli için de sistemin kontrol edilebilirliği ve iyi tanımlılığı analiz edilmiştir. Optimal kontrol fonksiyonu ve eğrisini elde etmek için Pontryagin maksimum prensibi kullanılmıştır. Pontryagin maksimum prensibi yardımıyla kontrol problemi, başlangıç-sınır ve terminal koşulları sağlayan eşlenik ve durum değişkenlerini içeren yeni bir kısmi diferansiyel denklem sisteminin çözümüne dönüştürülmüştür. Analitik çözümler, özfonksiyon genişletme yöntemi kullanılarak oluşturulmuştur. Teorik sonuçları doğrulamak için simülasyonlar sunulmuştur.

Tezin ikinci bölümünde, bilineer kontrol uygulanan bir parabolik sistemin optimal kontrolü ele alınmıştır. Söz konusu parabolik sistem ısı transfer sürecini modellemektedir.

Isı transferi süreci için, soğutucu akış hızı bir bilineer kontrol değişkenidir ve sıcaklık bir durum değişkenidir. Bu çalışmada, sıcaklığı bakımından homojen olmayan düzgün bir metal çubuğa bilineer kontrolün uygulanması sonucunda sıcaklığın denge-durum değerine ulaştığında çubuk üzerinde homojen olarak dağıldığı gösterilmiştir. Minimum düzeye indirilecek olan performans indeksin amacı büyük miktarlarda kontrol eforu tüketmeden, sıcaklığı sabit denge-durum değerine yakın tutmaktır.
Sunulan yaklaşım, mertebe indirgeme modellemesi, Pontryagin maksimum prensibi ve sayısal çözüm tekniklerinden oluşmaktadır. Özfonksiyon genişletme yöntemi kullanılarak dağıtılmış parametre sistemi, toplanmış bir parametre sistemine dönüştürülür. Elde edilen sistem zamansal olarak bir bilineer sisteme karşılık gelir. Pontryagin maksimum prensibi, doğrusal olmayan iki noktalı sınır değer problemine götüren optimal kontrol fonksiyonunu elde etmek için kullanılır. Optimal eğrilerin belirlenmesi ve sistemin optimal kontrolünün bulunması için iki iteratif nümerik teknik tartışılmıştır. En dik iniş yöntemi ve quasilineerizasyon doğrusal olmayan iki noktalı sınır değer probleminin çözümü için kullanılan yöntemlerdir. En dik iniş yöntemi çözüm için bir başlangıç prosedürü, quasilineerizasyon yöntemi de çözümü doğrulamak için bir diğer prosedür olarak kullanılmıştır. Başlangıç tahminin ne olduğu, başlangıç tahmininin önemi, iterasyon sayısı ve bu iki iteratif yöntemin yakınsaklığı gibi bazı özelliklerin karşılaştırılması sunulmuştur. Sayısal simülasyon çalışmaları, tanıtılan yaklaşımın etkinliğini ve uygulanabilirliğini göstermektedir.

Anahtar Kelimeler: Optimal kontrol, bilineer sistem, en dik inis metodu, kuasilinearizasyon metodu, Pontryagin maksimum prensibi.

CHAPTER 1

INTRODUCTION

1.1 Literature Review

Control theory is an interdisciplinary field of research that associates mathematical laws with rich applications in engineering, economics and other sciences. The core subject in control theory is to improve the efficiency of controllers having simple design procedures, which can be employed for both linear and nonlinear systems that achieve the desired system performance. Optimization and optimal control are two main tools used in the development of decision making system for either minimizing a cost or for maximizing a utility. Optimization deals with problems in finite-dimensional spaces whereas optimal control solves problems in infinite-dimensional spaces.

Controls are functions that define admissible external effects on the system. They trigger a system response, and based on this response, an objective function taken as a performance criterion for the behavior of the system, is evaluated. This criterion is minimized by an optimal control. Namely, the aim of optimal control is to determine the control function which will cause a given linear or nonlinear process to satisfy some physical constraints and at the same time maximize or minimize a performance criterion (performance index or cost function) [1], [2]. Robotics, aerospace, optimal guidance of rockets, development of satellite launchers, economics, management, energy policies, biology and medicine, some diseases, radiotherapy, oncology are some of the examples of the application areas of optimal control problems [3], [4], [5], [6], [7].

Although the modern control theory started developing its foundatitons in late 1950s, the classical control was developed much earlier. Bellman developed the dynamic programming technique in 1957 to solve discrete-time optimal control systems. The significant contribution to optimal control systems was the maximum principle in 1956 by Pontryagin and his colleagues. In 1960, Kalman developed linear quadratic regulator (LQR) and linear quadratic Gaussian (LQG) theory to design optimal feedback controls. Kalman has a profound effect on optimal control theory and Kalman filter is one of the most widely used techniques in control theory applications for real-life problems.

In 1724, Riccati published some solutions and results for some types of nonlinear differential equations. After two centuries, these results became famous as the Riccati equations. Matrix Riccati equation is used in Kalman filter techniques and other fields. With all of these, optimal control theory was developed with the roots of the calculus of variations in the 16th and 17th centuries. Bernoulli brothers, Euler and Lagrange essentially contribute to the improvement of the theory of calculus of variations. Hamilton, Leibnitz, Jacobi and Weierstrass developed the theory in the 19th century [8].

The solution methods for optimal control problems are categorized in four groups: Lyapunov function approach, dynamic programming, direct methods and indirect methods. The Lyapunov function approach design relies on an energy type of Lyapunov function for a given nonlinear system and always guarantees control stability. The dynamic programming leads to Hamilton-Jacobi-Bellman equations which are nonlinear partial differential equations. Although Hamilton-Jacobi-Bellman equations or Hamilton-Jacobi-Bellman inequalities are powerful tools for nonlinear optimal control theory, however, they have high computational complexity which makes them less suitable for industrial applications. In direct methods, optimal control problems are transformed into nonlinear optimization problems by using discretization or parametrization techniques. In indirect methods, the Pontryagin's maximum principle is applied to derive the necessary conditions for optimality. Optimality conditions obtained result in nonlinear two-point boundary-value problems [9]. Figure 1.1 shows the classification of four methods examined under modern control theory.

The roots of the Pontryagin's maximum principle are based on engineering problems related to steering aircraft issues in the Soviet Union in 1950s. When the engineers shared their problem with mathematicians in Moscow, Pontryagin and his colleagues were in-

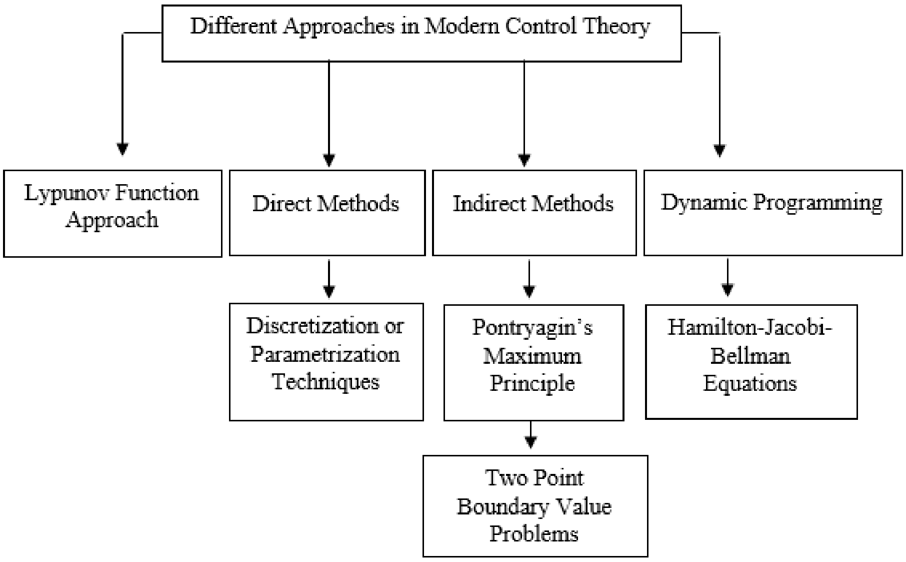

Figure 1.1 Classification of different methods in modern control theory and related solution algorithms

terested to solve the problem and ultimately their investigations resulted in the famous maximum principle [10]. Russian Mathematician Lev Ponrtyagin reformulated the control problem in terms of Hamiltonian. This is not the same Hamiltonian used in classical mechanics, but Ponrtyagin was inspired by it and the method was named after him. It allows us to write down the adjoint equations and optimality conditions in a very compact, general and straightforward form. There is no restriction on the integrand in the cost functional or the differential equation. Therefore, the variational approach of Pontryagin can be applied to nonlinear state equations and non-quadratic cost functionals as well as non-autonomous systems. Publishing "Pontryagin's Mathematical Theory of Optimal Processes", the Pontryagin's maximum principle theory began to attract researchers from other fields as well, such as management and economics fields [11].

Pontryagin's variational approach is the generalization of the necessary conditions of optimality conditions and it is applicable for both linear and nonlinear control systems [1], [12]. Various control systems are modelled with linear partial differential equations by using linear control as active control or boundary control. However, control aspects of linear or nonlinear partial differential equations in a multiplicative way have been seldom discussed. Moreover, multiplicative controls give rise to bilinear systems (BLS). The BLS concept was first introduced by a U.S scientist, Mohler in the 1960s. Bilinear systems are specific types of nonlinear systems. In order to approximate and analyze complex nonlinear systems, the BLS are used due to simplicity of the system. The BLS include products of control and state, namely, they are linear in state and control but not jointly linear in control and state. The terms formed by multiplication of control vector and state vector make these systems nonlinear. Due to the fact that the linearization of nonlinear systems loses real features, bilinear systems are important to understand their natural properties and to improve their performance. Thus, modeling and control of nonlinear systems in a bilinear framework are crucial problems in engineering [13]. Bilinear systems are comprehensively studied mainly for three reasons: First, the BLS are applicable mathematical models for rich classes that have practical importance. Second, the BLS offer higher-order approximations to nonlinear systems compared to the linear systems. Third, the BLS have rich geometric and algebraic properties that opens new horizons in research [14].

This thesis proposes the implementation of the Pontryagin's maximum principle to both linear models and bilinear models. Vibration control of two different beams modelled linearly is discussed as an application of the Pontryagin's maximum principle to linear models. To apply the Pontryagin's maximum principle to the bilinear model, an optimal control of a parabolic system governed by a bilinear control is investigated. The literature review is presented by considering these two models.

Control of vibrations for different types of beam models, e.g., Timoshenko beams, Euler-Bernoulli beams, and Mindlin-type beams are important research fields to control undesirable vibrations leading to instability in structures [15], [16]. The common feature of the research about Timoshenko beams in the literature is that the beams are generally modelled by a set of second order differential equations and the majority of them do not provide the Einstein's causality [17], [18]. Einstein's causality principle states that any of the signals cannot go faster than the velocity of light in vacuum [19]. A partial differential equation that governs the dynamic behavior of a mechanical system must be of the same order in respect of the spatial coordinate and time. In 2006, Metrikine suggested a causal model which is shown to be governed by the equation of completely the same form as the governing equation of the Timoshenko beam model and Mindlin-rod model [19]. Since the suggested model has a fourth order term in time, this term makes the model casual. In [20], the Timoshenko beam is considered under the Einstein's causality principle. The authors presented the vibration control of the Timoshenko beam as a differential equation including the derivatives of the state variable and the fourth order time derivative by using the Pontryagin's maximum principle.

Euler-Bernoulli and Timoshenko beam theories comparing to Mindlin's beam theory have the disadvantage for obtaining the accurate results due to ignoring the microstructural effects in the beam. Due to this observation, Mindlin presented a general advanced elastic theory (1964) to characterize the elastic behavior of isotropic materials taking microstructural effects into account [21], [22], [23]. Mindlin considered that strains and gradient of strains were quadratically formed the potential energy density and kinetic energy function consisting of the quadratic form of velocities [24]. However, using higher order gradients introduced new constants that are difficult to determine. To overcome this difficulty, Mindlin proposed three simpler versions of his theory which are known as Form I, II and III. These versions conclude the same equation of motion. Further, Mindlin suggested a new theory, called the second gradient elastic theory, which can be used to efficiently capture the size effects in micro and nanoscale structures (1965). In his theory, the potential energy is dependent on strains, the gradient of strains and the second gradient of strains. The constitutive equation is a partial differential equation of sixth order [25], [26]. In [27], the sixth order second strain gradient-based beam model is studied to control the free vibrations by implementing an optimal boundary control by means of the Pontryagin's maximum principle.

The studies about bilinear models are reviewed under three categories: The concept of bilinear control, bilinear control of distributed and lumped parameter systems (diffusivity-interior-boundary control) and the optimal control of bilinear systems with quadratic cost functional.

Generally, a reduced order modeling is investigated for the studies related to bilinear control of distributed parameter systems. When the Pontryagin's maximum principle is used, nonlinear two-point boundary-value problems (TPBVPs) that have almost no analytical solutions are obtained. Thus, approximate solution for the problem is studied by researchers [28], [29], [30]. Quasilinearization method, steepest descent method, gradient projection method and shooting methods are some of the iterative numerical solution techniques for solving nonlinear TPBVPs [2]. In this thesis, the steepest descent (SD) method and the quasilinearization (QL) method are chosen as solution techniques. The SD method is one of the oldest and simplest method for minimizing a general nonlinear function. The basis of the method is the simple observation that a continuous function should decrease, at least initially, if one takes a step along the direction of the negative gradient. The only difficulty is to decide how to choose the length of the step. In spite of its simplicity, the SD method has played an important role in the development of the optimization theory [31]. The SD concept is applied to optimal control problems by Kelley [32], Bryson [33] and Denham [34]. The QL method is formed solving a sequence of linearized two-point boundary-value problems. Firstly, the nonlinear differential equations are linearized via Taylor series expansion then linear two-point boundary-value problems are solved [2].

Controllability of the rod equation and a Schrödinger equation with bilinear control is simultaneously discussed by Kime in [35]. Lin et al. [36] studied the linear parabolic system with a bilinear control in the sense of exact controllability. The controllability of a nonlinear parabolic system governed by a bilinear control is studied and an approximate null controllability result is presented in [37]. Controllability of the infinite dimensional bilinear system is discussed in [38]. In [39], Khapalov studied the nonnegative approximate controllability of the parabolic system via a bilinear control. A linear Schrödinger equation with a bilinear control that represents a quantum particle in an electric field is examined in [40]. Heat equation with a bilinear control was studied from the point of exact controllability and explicit control strategy with simulations in [41].

A new procedure to find the solution to a quadratic optimal control problem of a bilinear system is presented in [42], where the optimal control problem is transformed into a nonlinear two-point boundary-value problem by using Pontryagin's maximum principle. The nonlinear TPBVP is solved by using homotopy perturbation method. Hofer and Tibken

[43] considered a bilinear system with quadratic cost functional in which Riccati approach and iterative method are used to solve the optimal control problem. Aganovic and Gajic [44] studied on the successive approximation procedure to make the computations of the optimal solution for a bilinear quadratic optimal control problem simplifier. The presented method in [44] requires only a solution for a sequence of the differential Lyapunov equations while Hofer and Tibken's solution is obtained via a sequence of Riccati differential equations. Kim and Lim [13] searched a robust H_∞ state feedback control technique for continuous time bilinear systems with an additive disturbance input. They studied on a new robust H_∞ control technique based on the successive approximation method to solve the state dependent algebraic Riccati equation (SDARE) by transforming bilinear systems into linear systems. Reduced order modeling (ROM) was proposed to obtain a lumped parameter system corresponds a bilinear system [28], [29]. In both studies, a two-point boundary-value problem was obtained after using the Pontryagin's maximum principle. An iterative method based on the Picard approximation was used [28] and a successive optimal control computation was used to solve the two-point boundary-value problem [29]. Ou and Schuster [30] presented a new successive approximation approach to solve numerically the quadratic optimal control problem subject to a bilinear distributed system connected with the receding horizon control algorithm.

Kucuk et al. [45] considered the optimal vibration control of piezolaminated smart beams via the Pontryagin's maximum principle. Sadek et al. [46] studied the optimal control of a bilinear parabolic equation. A wavelet-based modal space expansion method is proposed to evaluate the optimal control and trajectory. Tang and Hua [47] proposed a semidiscrete finite element method to solve bilinear parabolic optimal control problems. A successive approach consist of finite difference scheme and conjugate gradient method for solving bilinear parabolic PDEs is proposed in [48]. Bichiou et al. [49] proposed a practical approach based on the expansion of the system model on a complete set of orthogonal block pulse functions for solving the problem of time optimal control of bilinear systems. Korpeoglu and Kucuk [50] introduced an approach containing modal space expansion, the Pontryagin's maximum principle and the steepest descent algorithm to solve a parabolic bilinear optimal control problem.

The rest of the thesis is organized as follows. In Chapter 2, an introductory information about an optimal control problem, the calculus of variations, Pontryagin's maximum principle and bilinear models are introduced. Chapter 3 presents an optimal boundary control of a Mindlin-type beam which is enhanced with Einstein's causality principle by using the Pontryagin's maximum principle. In Chapter 4, another linear beam model, the second strain gradient theory based-beam model which is capable of capturing the size effects in micro and nanoscale structures is tackled from the point of optimal boundary control. Chapter 5 presents a parabolic optimal control problem via bilinear control and two iterative solution methods with numerical results. Finally, the results of the presented study are discussed in Chapter 6.

1.2 Objectives of the Thesis

The core objective of this study is to show how the Pontryagin's maximum principle works for linear control systems in new problems and to transfer the mathematical concepts used to bilinear control systems with simulation studies. The objectives of this thesis are as follows:

1. Introduce different types of beam models, the Mindlin-type beam and the second strain gradient theory-based beam,
2. Develop boundary control formulations to damp out excessive vibrations of the introduced beam models,
3. Form the mathematical problems within the scope of the theory of optimal control theory of distributed parameter systems,
4. Define a performance index that involves the physical energy of the system and penalty function to control the expenditure of the control force at a terminal time,
5. Develop the Pontryagin's maximum principle by means of introducing a suitable Hamiltonian for the optimal boundary vibration control of linear problems,
6. Introduce bilinear control concepts and bilinear-quadratic control systems,

7. Formulate an optimal control problem of a parabolic PDE via bilinear control,
8. Use a reduced order modelling and Pontryagin's maximum principle for the bilinear control problem with quadratic cost functional,
9. Apply numerical methods, the SD method and the QL method, to solve the nonlinear TPBVP,
10. Illustrate the introduced approach by applying it to a bilinear parabolic control problem and calculate its effectiveness numerically.

1.3 Hypothesis of the Thesis

This thesis deals with the optimal control of different physical systems via the Pontryagin's maximum principle. Boundary control characterization for two linear beam models governed by hyperbolic partial differential equations are first derived to suppress undesirable vibrations. Second, a bilinear optimal control of heat conduction process governed by parabolic partial differential equations is considered and the results are obtained taking the advantage of two different numerical solution methods. As a result of implementation of a bilinear control to a uniform rod with non-uniform temperature, the temperature on the metal rod is homogeneously distributed when the temperature reaches equilibrium and the controlled solutions reach equilibrium temperature faster than uncontrolled solutions. Consequently, optimality conditions are obtained for both linear and bilinear models in the form of Pontryagin's maximum principle.

CHAPTER 2

OPTIMAL CONTROL: A REVIEW OF BASIC CONCEPTS AND BILINEAR MODELS

In this chapter, a mathematical background is presented for better understanding of the concepts explored in subsequent sections. Firstly, general description of optimal control problem formulation, the properties of optimal control systems, the calculus of variations and Pontryagin's approach are presented. Furthermore, the standard terminologies are described and the fundamental framework for general bilinear control systems is introduced.

2.1 Formulation of an Optimal Control Problem

The ultimate goal of designing the control systems is to design a controller that will cause a system to perform in the desired way. The three essential elements of an optimal control problem are as follows [2]:

1. Mathematical model of the system to be controlled,
2. An expression of the physical constraints,
3. Description of a performance measure.

2.1.1 Mathematical Model

Developing a mathematical description of a truth model for any control problem is extremely difficult and time consuming. Alternatively, instead of designing a complicated truth model, a simplified model can be designed to effectively predict the response of the physical system for the given inputs. The simplified model is called the design model labeled by Friedland [51]. The design model should capture the essential features of the process.

A general model of a system in state variable form is

$$\dot{u}(t) = a(t, u(t), p(t)), \tag{2.1}$$

where $u(t) = (u_1(t), u_2(t), ..., u_n(t)) : [t_0, t_f] \to \mathbb{R}^n$ is state variable, $p(t) = (p_1(t), p_2(t), ..., p_m(t)) : [t_0, t_f] \to \mathbb{R}^m$ is control input to the process at time t and $a : [t_0, t_f] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. In general, $a(t, x(t), p(t))$ is a nonlinear time-varying function of the states and controls for $t \in [t_0, t_f]$, $\dot{u}(t)$ denotes the derivative with respect to t.

A history of control values in the interval $[t_0, t_f]$ is denoted by p and is called a control history or a control. A history of state values in the interval $[t_0, t_f]$ is denoted by u and is called a state trajectory. The terms "history", "curve", "function" and "trajectory" are used interchangeably [2].

2.1.2 Physical Constraints

After the model is selected, the next step is to identify the physical constraints on the control and state variables. The control or states are either bounded or unbounded depending on the problem statement.

- A control that satisfies the control constraints throughout the time interval $[t_0, t_f]$ is called an admissible control. If the admissible control set is denoted by P, $p \in P$ means that the control p is admissible [2].
- A state that satisfies the state constraints during the entire time interval $[t_0, t_f]$ is called an admissible trajectory. If the set of admissible state trajectories set is denoted by U, $u \in U$ means that the state trajectory u is admissible [2].

The admissibility concept is important since it allows us to handle only those trajectories and controls that are admissible instead of considering all values.

2.1.3 Performance Measure

After selecting the model and the state and control constraints, the next step is to evaluate the performance of the system quantitatively. Choosing a performance measure is a process of translation of the physical requirements of the system into mathematical terms. In most cases, various performance measures are tested to evaluate the best optimal performance measure. However, the selection of a performance measure may differ from one problem to another according to its objective. Performance measure examples in optimal control problems are such as minimum control effort problems, minimum fuel problems, terminal control problems and minimum-time problems.

The general form of a measure to evaluate the performance of a system is given below

$$\mathscr{J} = h(t_f, u(t_f)) + \int_{t_0}^{t_f} g(t, u(t), p(t))dt, \tag{2.2}$$

where h and g are scalar functions, t_0 and t_f are the initial and terminal time, respectively. t_f is specified or free depending on the problem.

In light of the information collected, the general optimal control problem is as follows:

If an admissible control p^* minimizes the performance measure

$$\mathscr{J} = h(t_f, u(t_f)) + \int_{t_0}^{t_f} g(t, u(t), p(t))dt, \tag{2.3}$$

subject to the state equations

$$\begin{cases} \dot{u} = a(t, u(t), p(t)), \\ u(t_0) = u_0, \\ u(t_f) \quad \text{could be free or fixed,} \end{cases} \tag{2.4}$$

p^* is called an optimal control and u^* is called an optimal trajectory [2]. Furthermore, it is understood that p^* satisfies the performance measure to be minimized,

$$\mathscr{J}^* = h(t_f, u^*(t_f)) + \int_{t_0}^{t_f} g(t, u^*(t), p^*(t))dt \leq h(t_f, u(t_f)) + \int_{t_0}^{t_f} g(t, u(t), p(t))dt \tag{2.5}$$

for all $p \in P$, $u \in U$. This inequality indicates that performance measure value for an optimal control and its trajectories is less than for all other admissible controls and trajectories.

In most cases, it is tried to find an optimal control rather than prove its existence since existence theorems are quite inadequate. Even though an optimal control exists, it may not be unique. Nonunique optimal controls result in higher computational complexity of the system and finding the solution becomes extremely difficult.

Open-loop and Closed-loop Controls

- If the optimal control is determined as a function of time for a specified initial state value, namely,

  $$p^*(t) = e(t, u(t_0)), \tag{2.6}$$

  then the optimal control is called open-loop control [2].

- If a relation of the form

  $$p^*(t) = f(t, u(t)), \tag{2.7}$$

  is found for the optimal control at time t, then the function f is called the optimal control law [2].

Therefore, whereas open-loop control is optimal only for a particular initial state value, the optimal control history is known for any initial state value when the optimal control law is clear.
The terms optimal feedback control, closed-loop optimal control, optimal control law and optimal control strategy are used interchangeably [2].

2.2 Properties of Systems for Control

Consider the system

$$\dot{u}(t) = a(t, u(t), p(t)), \tag{2.8}$$

with initial state $u(t_0) = u_0$ for $t \geq t_0$.

- **Controllability:** Controllability is the ability to transfer a system from an arbitrary initial state to an arbitrary final state in finite time. If there is a control $p(t), t \in [t_0, t_f]$ and a finite time t_1 , which transfers the initial state $u(t_0)$ to an arbitrary final state $u(t_1)$ at time t_1, the state u_0 is said to be controllable at time t_0. If all values of u_0 are controllable for all t_0 the system is controllable [1].
  Controllability of the system is a necessary, but not sufficient condition for the ex-

istence of a control solution of the system.

- **Observability:** Observability is the ability to uniquely determine all states u_0 for every t_0 of the system by observing the outputs of the system in finite time.
  If the state $u(t_0) = u_0$ is determined by observing the output during the finite time interval $t \in [t_0, t_1]$, the state u_0 is said to be observable at time t_0. If all states u_0 are observable for every t_0, the system is called observable [1].
- **Stabilizability:** Stabilizability is the ability in the event of finding a control input for every initial condition that transfers the state to the stationary point of the system in finite time [1].
  - A controllable system is stabilizable.
  - Even though a system is not controllable, it is stabilizable if all uncontrollable states are stable and all unstable states are controllable.

Once the definitions and properties of an optimal control system are built, the next task is to specify a control function which minimizes or maximizes performance measure.

2.3 The Calculus of Variations

A subject of applied mathematics in solving optimal control problems is the calculus of variations. Queen Dido of Carthage is the first person known to have solved a problem by using variational calculus [52]. In the seventeenth century, it started to become more popular in Western Europe inspite of being traced back to the ancient Greeks. In 1687, Newton used variational principles for determining the shape of a body moving in air which meets the minimum resistance. The Brachistochrone problem is investigated using variational approach by Johann and Jacob Bernoulli, Leibniz, Newton and L'Hospital in 1696. In 1744, Euler succeed in solving minimal surface shapes problems [1], [8].

The calculus of variations is used in various applications such as optics, static equilibrium, fluid mechanics, modern physics, electromagnetics, dynamical systems, optimization and control.

In the variational calculus, it is dealt with a functional that can be regarded as a "function of a function", namely, a function which depends on an other function. More specifically, it is a definite integral whose integrand consists of a function that is not determined.

2.3.1 Functionals of One Independent Variable

A functional generalizes the concept of a function. It is a special function in which the independent variable is a function itself.

Consider the functional $\mathscr{J}[p(t)]$ of the function $p(t)$ of the form

$$\mathscr{J}[p(t)] = \int_{t_0}^{t_f} F[t, p(t), \dot{p}(t)]dt, \tag{2.9}$$

where $p(t)$ is a smooth function, t_0 and t_f are fixed and the end points are specified as $p(t_0) = p_0$ and $p(t_f) = p_f$.

The main aim of the calculus of the variations is to find the functions that minimize a given functional. It is desired to obtain the function $p(t)$ for which the functional $\mathscr{J}[p(t)]$ is an extremum, more generally, for which $p(t)$ is a stationary function of $\mathscr{J}[p(t)]$. In order to achieve this, the following mathematical concepts are needed to know.

2.3.2 The Increment and the Variation of a Functional

- If p and $p+\delta p$ are functions for which the functional $\mathscr{J}$ is defined, then the increment of $\mathscr{J}$, denoted by $\Delta\mathscr{J}$, is

  $$\Delta\mathscr{J} \triangleq \mathscr{J}(p+\delta p) - \mathscr{J}(p). \tag{2.10}$$

  δp is called the variation of the function p [2].

The variation does the same task in determining extreme values of functionals as the differential does in finding maximum and minimum of functions. Differential (d or ∂) is a change of a function from one point to another while variational (δ) is a change from one function to another function.

- The increment of a functional is written as

  $$\Delta\mathscr{J}(p,\delta p) = \delta\mathscr{J}(p,\delta p) + g(p,\delta p) \parallel \delta p \parallel, \tag{2.11}$$

where $\delta\mathscr{J}$ is linear in δp. If

$$\lim_{\|\delta p\|\to 0}\{g(p,\delta p)\}=0, \tag{2.12}$$

then $\mathscr{J}$ is said to be differentiable on p and $\delta\mathscr{J}$ is the variation of $\mathscr{J}$ for the function p [2].

- When an increment of a functional is extended using the Taylor series, the following expression is obtained

$$\begin{aligned}
\Delta\mathscr{J} &= \mathscr{J}(p+\delta p)-\mathscr{J}(p)\\
\Delta\mathscr{J} &= \mathscr{J}(p)+\frac{\partial\mathscr{J}}{\partial p}\delta p+\frac{1}{2!}\frac{\partial^2\mathscr{J}}{\partial p^2}(\delta p)^2+\ldots-\mathscr{J}(p)\\
&= \frac{\partial\mathscr{J}}{\partial p}\delta p+\frac{1}{2!}\frac{\partial^2\mathscr{J}}{\partial p^2}(\delta p)^2+\ldots\\
&= \delta\mathscr{J}+\delta^2\mathscr{J}+\ldots
\end{aligned} \tag{2.13}$$

$\delta\mathscr{J}=\frac{\partial\mathscr{J}}{\partial p}\delta p$ and $\delta^2\mathscr{J}=\frac{1}{2!}\frac{\partial^2\mathscr{J}}{\partial p^2}(\delta p)^2$ is said to be the first variation and second variation of functional $\mathscr{J}$, respectively [53].

$\delta\mathscr{J}$ is the linear approximation to the difference in the functional $\mathscr{J}$ caused by two comparison curves. If $\|\ \delta p\ \|$ is small, the variation is a good approximation to the increment. Figure 2.1 shows the the increment and the variation of a functional [53].

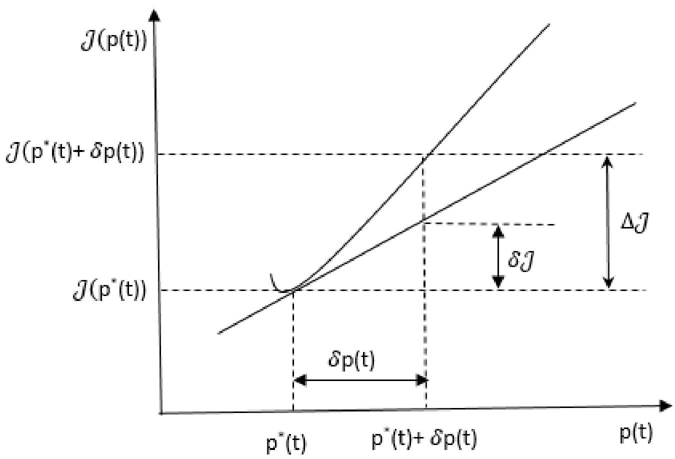

Figure 2.1 The increment and the variation of functional $\mathscr{J}$.

The rate of change of a functional with respect to the function is done using the functional derivative or Fréchet derivative which is denoted by $\frac{\delta \mathscr{J}}{\delta p}$ for the functional $\mathscr{J}[p(t)]$. The Fréchet derivative is the variational analog of the partial derivative and is a special case of the Gâteux derivative which is the variational analog of the directional derivative.

2.3.2.1 Gâteaux Variations

Gâteaux variations is used to find the extremal values of a functional. The functional $\delta \mathscr{J}(p)$ is called the Gâteaux variation of $\mathscr{J}$ at point p. The limit at point p is obtained as follows:

$$\delta \mathscr{J}(p;h) := \lim_{\varepsilon \to 0} \frac{\mathscr{J}(p+\varepsilon h) - \mathscr{J}(p)}{\varepsilon} \tag{2.14}$$

where h is any vector in a vector space V [54].

2.3.2.2 Fréchet Derivatives

If the following condition holds and $d\mathscr{J}(p;h)$ is a linear, continuous functional of h, then $\mathscr{J}$ is said to be Fréchet differentiable at p with increment h.

$$\lim_{\|h\| \to 0} \frac{\mathscr{J}(p+h) - \mathscr{J}(p) - d\mathscr{J}(p;h)}{\| h \|} = 0. \tag{2.15}$$

$d\mathscr{J}(p;h)$ is called the Fréchet differential. If $\mathscr{J}$ is differentiable at each $p \in D$ then $\mathscr{J}$ is Fréchet differentiable in D.

If a functional $\mathscr{J}$ is Fréchet differentiable at p then the Gâteaux variation of $\mathscr{J}$ at p exists and is equal to the Fréchet differential. This can be expressed mathematically as,

$$\delta \mathscr{J}(p;h) = d\mathscr{J}(p;h) \tag{2.16}$$

for all $h \in V$ [54].

2.3.3 Maxima and Minima of a Functional

A functional $\mathscr{J}$ with a specified domain has a relative extremum at p^* if there is an $\varepsilon > 0$ such that for all functions p which satisfy $\| p - p^* \| < \varepsilon$ the increment of $\mathscr{J}$ has the same sign. If following condition is satisfied

$$\Delta \mathscr{J} = \mathscr{J}(p) - \mathscr{J}(p^*) \geq 0, \tag{2.17}$$

$\mathscr{J}(p^*)$ is a relative minimum; if

$$\Delta \mathscr{J} = \mathscr{J}(p) - \mathscr{J}(p^*) \leq 0, \tag{2.18}$$

$\mathscr{J}(p^*)$ is a relative maximum.

If Eq. (2.17) is satisfied for arbitrarily large ε, then $\mathscr{J}(p^*)$ is a global, absolute, minimum. Similarly, if Eq. (2.18) holds for arbitrarily large ε, then $\mathscr{J}(p^*)$ is a global, absolute maximum. p^* is called an extremal and $\mathscr{J}(p^*)$ is an extremum [2].

The following theorem is used to extreme values of functionals.

2.3.4 Fundamental Theorem of the Calculus of Variations

If p^* is an extremal, the variation of $\mathscr{J}$ must vanish on p^*, that is, for all admissible δp the following condition must hold,

$$\delta \mathscr{J}(p^*, \delta p) = 0. \tag{2.19}$$

This is the necessary condition for extremal values and sufficient condition is $\delta^2 \mathscr{J} > 0$ for minimum and $\delta^2 \mathscr{J} < 0$ for maximum [53].

2.3.5 Fundamental Lemma of the Calculus of Variations

If $g(t)$ is a continuous function in $[t_0, t_1]$, and if

$$\int_{t_0}^{t_1} g(t)h(t)dt = 0, \tag{2.20}$$

where $h(t)$ is an arbitrary function in the same interval with $h(t_0) = h(t_1) = 0$, then $g(t) = 0$ at every point in the interval $t_0 \leq t \leq t_1$ [53].

In order to determine extrema of the functional (2.9) depending on a single function, the fundamental theorem and fundamental lemma of the calculus of variations are used. Then, a necessary condition for p^* is obtained. This condition is called Euler, Euler-Lagrange equation.

In general, the Euler equation for (2.9) is a nonlinear, ordinary, time-varying, and second-order differential equation. As the equation usually cannot be solved using analytical approach, therefore, numerical integration methods are used. Unfortunately, split boundary conditions and nonlinearity of the differential equation cause complications from simply solving the Euler equation numerically.

2.4 Pontryagin's Maximum Principle

Pontryagin's maximum principle is a generalization of the fundamental theorem of the calculus of variations. Before stating the Pontryagin's maximum principle, let us consider a generalization of the optimal control problem reformulated using the Pontryagin's Hamiltonian. The general optimal control problem is to minimize the cost functional

$$\mathscr{J}[u,p] = \int_{t_0}^{t_f} F(t,u(t),p(t))dt + h(u(t_f)), \tag{2.21}$$

with respect to $p : [t_0,t_f] \to P$, subject to the state equations

$$\dot{u}(t) = a(t,u(t),p(t)), \quad t_0 < t < t_f, \tag{2.22}$$

and initial conditions

$$u(t_0) = u_0 \tag{2.23}$$

being satisfied. In the classical variational approach, the augmented cost functional is formed as follows

$$\mathscr{J}^*[u,p,\Lambda] := \int_{t_0}^{t_f} \{F(t,u(t),p(t)) - \Lambda^T(t)(\dot{u}(t) - a(t,u(t),p(t))\}dt + h(u(t_f)). \tag{2.24}$$

Pontryagin reformulated the augmented functional in the form

$$\mathscr{J}^*[u,p,\Lambda] = \int_{t_0}^{t_f} [\mathscr{H}(t,u(t),p(t),\Lambda(t)) - \Lambda^T(t)\dot{u}(t)]dt + h(u(t_f)), \tag{2.25}$$

where the Pontryagin's Hamiltonian contains all of the nonderivative terms and is given by

$$\mathscr{H}(t,u(t),p(t),\Lambda(t)) = F(t,u(t),p(t)) + \Lambda^T(t)[a(t,u(t),p(t))]. \tag{2.26}$$

Taking the first variation of $\mathscr{J}^*$ and setting equal to zero yields

$$\delta\mathscr{J}^* = \int_{t_0}^{t_f} [\frac{\partial\mathscr{H}}{\partial u}\delta u + \frac{\partial\mathscr{H}}{\partial p}\delta p + \frac{\partial\mathscr{H}}{\partial\Lambda}\delta\Lambda - \delta(\Lambda^T(t)\dot{u}(t))]dt + \delta(h(u(t_f))) = 0. \tag{2.27}$$

Integrating the fourth term by parts and collecting terms produce

$$\delta \mathscr{J}^* = \int_{t_0}^{t_f} [(\mathscr{H}_u + \dot{\Lambda}^T)\delta u + \mathscr{H}_p \delta p + (\mathscr{H}_\Lambda - \dot{u})\delta \Lambda] dt + h_u \cdot \delta u(t_f). \tag{2.28}$$

Requiring $\delta \mathscr{J}^* = 0$ and noting that $\delta u(t_0) = 0$ yields

$$0 = \int_{t_0}^{t_f} [(\mathscr{H}_u + \dot{\Lambda}^T)\delta u + \mathscr{H}_p \delta p + (\mathscr{H}_\Lambda - \dot{u})\delta \Lambda] dt + (h_u - \Lambda(t_f))\delta u(t_f). \tag{2.29}$$

In terms of the Pontryagin's Hamiltonian, therefore, the adjoint equation, optimality condition, terminal time initial condition are of the form [1]

$$\begin{cases} \dot{\Lambda}^T(t) = -\mathscr{H}_u(t, u(t), p(t), \Lambda(t)), \\ \qquad 0 = \mathscr{H}_p(t, u(t), p(t), \Lambda(t)), \\ \quad \dot{u}(t) = \mathscr{H}_\Lambda(t, u(t), p(t), \Lambda(t)), \\ \Lambda(t_f) = h_u(u(t_f)). \end{cases} \tag{2.30}$$

Consequently, the Pontryagin's maximum principle makes it possible to write the adjoint equations and optimality conditions in a compact form. Besides, there is not any restrictions on the integrand $F(t, u(t), p(t))$ in the performance measure or the differential equation $a(t, u(t), p(t))$. In this way, Pontryagin's approach is workable even if the system contains nonlinear state equations or nonquadratic cost functionals. Only one thing required is that the state equation(s) is written in state-space form, namely, in a system of first order differential equations.

Theorem 2.1 (Pontryagin's Maximum Principle) *If $p^*(t)$ is an optimal control, which causes the system*

$$\dot{u}(t) = a(t, u(t), p(t)) \tag{2.31}$$

to follow an admissible trajectory that minimizes the performance measure

$$\mathscr{J}(p) = \int_{t_0}^{t_f} F(t, u(t), p(t)) dt + h(u(t_f)), \tag{2.32}$$

then necessary conditions for $p^*(t)$ to be an optimal control are

$$\begin{cases} \dot{u}^*(t) = \mathscr{H}_\Lambda(t, u^*(t), p^*(t), \Lambda^*(t)), \\ \dot{\Lambda}^*(t) = -\mathscr{H}_u(t, u^*(t), p^*(t), \Lambda^*(t)), \\ \mathscr{H}(t, u^*(t), p^*(t), \Lambda^*(t)) \leq \mathscr{H}(t, u^*(t), p(t), \Lambda^*(t)) \end{cases} \tag{2.33}$$

for all admissible $p(t)$ and $t \in [t_0, t_f]$ in terms of Hamiltonian [55]

$$\mathscr{H}(t, u(t), p(t), \Lambda(t)) = F(t, u(t), p(t)) + \Lambda^T(t)[a(t, u(t), p(t))]. \tag{2.34}$$

Pontryagin's approach is both equivalent to variational approach and its generalization. Variational approach is in need of all functions, involving the state and control variables, must be continuously differentiable whereas Pontryagin's approach gives opportunity for piecewise-continuous solutions for controls.

If Pontryagin's approach for optimal control is summarized briefly, the Hamiltonian is formed (2.26) using given performance measure (2.24) to be minimized subject to the state and initial conditions (2.22). The optimality condition, adjoint equations and terminal-time initial conditions are expressed by (2.30) if the control is unconstrained.

Notice that the established conditions are necessary but not, in general, sufficient condition for optimality. An optimal control must satisfy the Pontryagin's maximum principle, but there may be controls that satisfy the maximum principle that are not optimal.

2.5 Bilinear Control Processes

Research about bilinear control processes is reviewed under three headings in the literature. The first heading is the concept of bilinear control. This control is seen as multiplicative control which can be applied to linear or nonlinear systems. The second one is bilinear control of distributed and lumped parameter systems with the concepts of diffusivity-interior and boundary control. For this type of control problem, a reduced order model is applied and the corresponding model is a bilinear system. The third heading is about optimal control of bilinear systems with quadratic cost functional.

2.5.1 Concept of Bilinear Control

Image and signal processing, channel equalization, echo cancellation, multiplicative disturbance tracking, and many other areas of engineering, biology and, socioeconomics are some example applications where bilinear control is used [56].

In addition, the bilinear control is also used in the areas of flow control. The velocity of the fluid pumped into the tube is considered as the control which appears as the convective coefficient in the convective-diffusive PDE system governing the contamination concentration [57].

In various chemical and biological reactions, bilinear controls are used like catalysts that can accelerate or decelerate the reaction [39].

- Linear parabolic PDE system governed by a bilinear control is in the following form [58]:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + p(t)u + \phi(x,t),\ 0 \leq x \leq 1,\ 0 < t \leq T, \tag{2.35}$$

with initial condition

$$u(x,0) = f(x),\ 0 \leq x \leq 1, \tag{2.36}$$

and boundary conditions

$$\begin{aligned} u(0,t) &= g_0(t),\ 0 < t \leq T, \\ u(1,t) &= g_1(t),\ 0 < t \leq T, \end{aligned} \tag{2.37}$$

where f, g_0, g_1, ϕ are known functions and the functions u and p are unknown.

- Nonlinear parabolic PDE system via bilinear control is as follows [37] :

$$\frac{\partial u}{\partial t} = u\frac{\partial^2 u}{\partial x^2} + p(t)u + \phi(x,t),\ 0 \leq x \leq 1,\ 0 < t \leq T, \tag{2.38}$$

with initial condition

$$u(x,0) = f(x),\ 0 \leq x \leq 1, \tag{2.39}$$

and boundary conditions

$$u(0,t) = g_0(t),\ 0 < t \leq T,$$
$$u(1,t) = g_1(t),\ 0 < t \leq T, \tag{2.40}$$

where f, g_0, g_1, ϕ are known functions and the functions u and p are unknown.

- Let A, $N_1, ..., N_n$ constant matrices in $\mathbb{R}^{n\times n}$, $p \in \mathbb{R}^n$ control variable with the system state variable $u \in \mathbb{R}^n$ and $u_0 \in \mathbb{R}^n$ is the initial state vector. Bilinear control for ODE is explained as in the following definition [59]:

$$\dot{u}(t) = Au(t) + \sum_{j=1}^{n} u_j N_j p(t), \tag{2.41}$$

with initial condition

$$u(t_0) = u_0. \tag{2.42}$$

2.5.2 Bilinear Control of Distributed and Lumped Parameter Systems

Parabolic equations are used to model many physical systems. Physical actuation appears in parabolic partial differential equations (PDEs) in three different ways: source terms (interior control), boundary conditions (boundary control) and diffusivity coefficients (diffusivity control). Studies about interior and boundary control have been studied comprehensively but, control aspects of PDEs via diffusivity actuator have been rarely discussed [28]. A special type of bilinear control, diffusivity control, improves the controllability obtained by just using either interior or boundary control.

Xu et al. [28] presented an optimal control problem for a parabolic system with three types of actuations. Their problem takes its source from the current profile control of magnetically confined fusion plasmas. Three physical actuators are used to lead the plasma to get a desired profile in a designated time period. In their other study, Xu et al. [29] studied a finite-time optimal control problem for a parabolic system with diffusivity and interior actuation mechanisms motivated by the control of the current density profile in tokamak plasmas.

- In the following distributed parameter system over $\Omega = \{(x,t) : 0 \leq x \leq \ell, t_0 \leq t \leq$

$t_f\}$, diffusivity-interior and boundary control was seen [29]:

$$\frac{\partial u}{\partial t} = f_D(x)p_D(t)\frac{1}{x}\frac{\partial}{\partial x}\left(xf(x)\frac{\partial u}{\partial x}\right) + f_I(x)p_I(t), \tag{2.43}$$

with boundary conditions

$$\frac{\partial u}{\partial x}|_{x=0} = 0, \qquad \frac{\partial u}{\partial x}|_{x=\ell} = p_B(t) \tag{2.44}$$

where $p_D(t)$, $p_I(t)$ $p_B(t)$ denote the diffusivity, interior and boundary control inputs respectively, f, f_D, f_I are spatial functions, and x represents the normalized spatial coordinate indexing the magnetic flux surfaces.

The authors state an optimal control problem for the parabolic system (2.43) with the following cost functional

$$\min_{p_D,p_I,p_B(t)} \mathscr{J} = \frac{1}{2}\int_0^\ell \mathscr{S}(x)u^2(x,t_f)dx + \frac{1}{2}\int_\Omega \mathscr{Q}(x)u^2(x,t)dxdt + \frac{1}{2}\int_{t_0}^{t_f}(r_Ip_I^2 + r_Dp_D^2 + r_Bp_B^2)dt \tag{2.45}$$

where $\mathscr{S}(x)$ and $\mathscr{Q}(x)$ are positive weight functions; r_I, r_D and r_B are positive definite control weighting factors.

By using reduced order modeling which is based on the proper orthogonal decomposition (POD) method and Galerkin projection, a lumped parameter bilinear system with quadratic cost functional is obtained.

2.5.3 Optimal Control of Bilinear Systems with Quadratic Cost Functional

Particle accelerators [60], nuclear power plants [61], plasma [62], quantum devices [63] and biomedicine [64] are some of the many research areas where control problems of bilinear systems are studied. Bilinear systems may also simulate a large class of communication systems [14].

For nuclear fission, neutron level or power is regarded as a state variable and reactivity or neutron multiplication is regarded as a control variable. For heat transfer, coolant flow rate is a control variable and temperature is a state variable. The multiplication of state and control is produced between a solid wall and moving coolant fluid. The generation of poison products in nuclear reactors is described by a bilinear model [64], [65], [66].

Chemical and biological reactions with a catalyst as a control is also modelled with bilinear models. For cellular biochemical reactions, enzyme concentration is the catalyst. Besides, in the discussion of population, energy production and steel production, bilinear systems are found [64].

Consider the optimal control problem of a bilinear system with quadratic cost functional [43]:

$$\begin{aligned} &\dot{u}(t) = Au(t) + Bp(t) + \{uN\}p(t) \\ &u(t_0) = u_0, \qquad uN = \sum_{j=1}^{n} u_j N_j \end{aligned} \tag{2.46}$$

where $u \in \mathbb{R}^n$ are the system state variables, $p \in \mathbb{R}^m$ are the control inputs, and A, B and N_j are constant matrices of appropriate dimensions with $N_j \in \mathbb{R}^{n\times m}$ and $u_0 \in \mathbb{R}^n$ is the initial state vector. The objective is to find the optimal control law $p^*(t)$ that minimizes the following quadratic cost functional:

$$\mathscr{J} = \frac{1}{2}u(t_f)^T F u(t_f) + \frac{1}{2}\int_{t_0}^{t_f} (u^T(t)Qu(t) + p^T(t)Rp(t))dt \tag{2.47}$$

where Q and F are positive semidefinite symmetric $n\times$ n matrices and R is a positive definite symmetric $m \times m$ matrix.

CHAPTER 3

OPTIMAL BOUNDARY CONTROL OF A MINDLIN-TYPE BEAM VIA MAXIMUM PRINCIPLE

In this chapter, optimal boundary control for damping the undesirable vibrations in a Mindlin-type beam is introduced. Performance index functional is defined in terms of the dynamic response of the beam which is defined as a weighted quadratic functional of the displacement and the velocity at terminal time and expenditure of the control energy added as a penalty term.

An adjoint variable satisfying the adjoint equation corresponding to the state equation is introduced to obtain the optimal control function. In addition, the control problem is turned into solving a system of partial differential equations by means of the Pontryagin's maximum principle. This system involves adjoint and state variables including initial, boundary and terminal conditions. The obtained numerical results are presented in table and graphical forms.

The importance of the study in this chapter is that it deals with the vibration control of a Timoshenko beam modeled as an equation including the derivatives of the state variable with respect to the time variable at the fourth order by means of the Pontryagin's maximum principle.

3.1 Mindlin-Type Beam Model

Beams are important structural components for a rich array of applications including several different areas as engineering, nano-techonology, meteorology, aerospace etc. There are two well-known beam models in the literature: Euler-Bernoulli and Timoshenko beam models. If the cross-sectional dimension of the beam is not important by the side of its length, the transversal vibration of an elastic beam is described by the Euler-Bernoulli beam equation [67]. However, since some effects such as rotary inertia and shear deformation are ignored in the Euler-Bernoulli beam models, they are unsuitable. The equations of motion for a thick beam which include the effects of rotary inertia and shear deformation were first introduced in two research by Timoshenko in 1921. Two important aspects of Timoshenko's beam theory are the improvement of shear deformation by the introduction of a mid-plane rotation variable and the usage of shear correction factor [68].

Control of vibrations in a structure is a research area that needs to be studied in detail because undesirable vibration control increases the lifespan of the structure and it prevents the instability in the structure. The studies related to the vibration control and control strategies exist in the literature but the typical feature of the research is that the studies dealing with the Timoshenko beam are modelled by a set of second order differential equations. But, there is no study dealing with the boundary control of vibration for Mindlin-type beam via maximum principle.

Timoshenko beam models are generally expressed in the system of PDEs that composes of the rotation angle, displacement and their derivatives with regard to the space and time variables. However, the models introduced in the greater part of the research which are related to these beams do not satisfy the Einstein's causality [17], [18]. Einstein's causality principle indicates that any of the signals can not go faster than the velocity of light in vacuum [69]. A partial differential equation governing dynamic behavior of a model must be of the same order with respect to the spatial coordinate and with respect to time to coincide with Einstein's causality. In 2006, Metrikine introduced a causal model and this model was shown to be governed by equation of exactly the same form as the governing equation of the Timoshenko-beam model and the Mindlin-rod model [69]. In [70], the model of Metrikine is presented and the model is casual.

3.2 Mathematical Formulation of the Problem

In this part, simply supported Timoshenko beam is considered for suppressing the undesirable vibrations. The motion equation of the considered beam is given by [26]:

$$u_{xx} - \ell^2 L u_{xxxx} = \frac{1}{c^2}(u_{tt} - \ell^2 H u_{xxtt}) + \frac{1}{c^4}\ell^2 N u_{tttt} \tag{3.1}$$

in which $u = u(x,t)$ is the transversal displacement, $(x,t) \in \mathscr{S} = (0,\ell) \times (0,t_f)$ is the space variable and the time variable, respectively, ℓ is the length of the beam, t_f is the predetermined terminal time, $c^2 = \frac{E}{\rho}$, ρ is the linear density, E is the Young modulus, L is a nondimensional constant, N and H are constants. Eq. (3.1) is subjected to the following boundary conditions

$$\begin{aligned} &u(0,t) = u(\ell,t) = 0 \\ &u_{xx}(0,t) = u_{xx}(\ell,t) = p(t) \end{aligned} \tag{3.2}$$

where $p(t)$ is the control function to be stated. Initial conditions are as follows:

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad u_{tt}(x,0) = u_2(x), \quad u_{ttt}(x,0) = u_3(x). \tag{3.3}$$

The beam system, initially, is undeformed and vibrations in the beam are induced by initial effects, which are represented by initial conditions.

3.3 Wellposedness and Controllability of the Optimal Control Problem

Let us assume the following conditions on the solution;

$$u, \frac{\partial^i u}{\partial t^i}, \frac{\partial^i u}{\partial x^i}, \frac{\partial^4 u}{\partial t^2 \partial x^2} \in L^2(\mathscr{S}), \quad i = 0,1,...,4,$$

$$p(t) \in L^2(0,t_f), \quad u_0(x) \in H^1(0,\ell),$$

$$u_i(x) \in L^2(0,\ell), \quad i = 1,2,3.$$

where $L^2(\mathscr{S})$ denote the Hilbert space of real-valued square-integrable functions defined in the domain $\mathscr{S}$ in the Lebesque sense with usual inner product and norm defined by

$$\parallel \zeta \parallel^2 = <\zeta,\zeta>, \quad <\zeta,\eta>_{\mathscr{S}} = \int_{\mathscr{S}} \zeta\eta \, d\mathscr{S}.$$

Under the above assumptions, the equation system (3.1)-(3.3) has a solution in the class of analytic functions [71]. In addition, Eqs. (3.1)-(3.3) can be reduced to ordinary differential equation and therefore it is shown by means of the linear Picard-Lindelöf existence-uniqueness theorem that Eqs. (3.1)-(3.3) have a solution. For the uniqueness of the solution to Eqs. (3.1)-(3.3), let us introduce the following lemma.

Lemma 3.1 *$u_1(x,t), u_2(x,t)$ are state functions which satisfy the system given by Eqs.* (3.1)-(3.3) *corresponding to the control functions $p_1(t)$ and $p_2(t)$, respectively. Difference function is defined by $\Delta u(x,t) = u_1(x,t) - u_2(x,t)$. Note that $\Delta u(x,t)$ satisfies the following equation*

$$\Delta u_{xx} - \ell^2 L \Delta u_{xxxx} = \frac{1}{c^2}(\Delta u_{tt} - \ell^2 H \Delta u_{xxtt}) + \frac{1}{c^4}\ell^2 N \Delta u_{tttt} \tag{3.4}$$

and following boundary conditions

$$\Delta u(x,t) = \Delta u_{xx}(x,t) = 0 \quad at \quad x = 0, \ell \tag{3.5}$$

also, zero initial conditions

$$\begin{aligned} &\Delta u(x,0) = \Delta u_t(x,0) = 0 \\ &\Delta u_{tt}(x,0) = \Delta u_{ttt}(x,0) = 0. \end{aligned} \tag{3.6}$$

Then,

$$\int_0^\ell \Delta u_{ttt}^2(x,t_f)dx = \int_0^\ell \Delta u_{tt}^2(x,t_f)dx = \int_0^\ell \Delta u_t^2(x,t_f)dx = \int_0^\ell \Delta u^2(x,t_f)dx = o(\varepsilon) \tag{3.7}$$

and

$$\int\int_{\mathscr{S}} \Delta u^2(x,t) d\mathscr{S} = o(\varepsilon) \tag{3.8}$$

in which $o(\varepsilon)$ is a quantity such that $\lim_{\varepsilon \to 0}(o(\varepsilon)/|\varepsilon|) = 0$.

Proof 1 *Let the energy integral be given by [72], [73],*

$$E(t) = \frac{1}{2}\int_0^\ell \left\{ \frac{\ell^2 N}{c^4}\Delta u_{ttt}^2 + \frac{1}{c^2}\Delta u_{tt}^2 - \frac{\ell^2 H}{c^2}\frac{\partial^2}{\partial x^2}\Delta u_{tt}^2 \right\} dx. \tag{3.9}$$

After differentiating $E(t)$ with respect to t, the following equality is obtained by using integration by parts and homogeneous boundary conditions Eq. (3.5),

$$\begin{aligned} \frac{dE(t)}{dt} &= \int\int_{\mathscr{S}} \left\{ \frac{\ell^2 N}{c^4}\Delta u_{tttt} + \frac{1}{c^2}\Delta u_{tt} - \frac{\ell^2 H}{c^2}\Delta u_{ttxx} \right\} \Delta u_{ttt} d\mathscr{S} \\ &= \int\int_{\mathscr{S}} \left\{ \Delta u_{xx} - \ell^2 L \Delta u_{xxxx} \right\} \Delta u_{ttt} d\mathscr{S} \le \int\int_{\mathscr{S}} \Delta u_{xx} \Delta u_{ttt} d\mathscr{S}. \end{aligned} \tag{3.10}$$

Applying the Cauchy-Schwartz inequality to the space integral, the following inequality is observed

$$\int\int_{\mathscr{S}} \Delta u_{xx} \Delta u_{ttt} d\mathscr{S} \leq \int_0^t \left[\int_0^\ell \Delta u_{ttt}^2 dx \right]^{1/2} \left[\int_0^\ell \Delta u_{xx}^2 dx \right]^{1/2} dt. \tag{3.11}$$

Taking the sup of both sides of the last integral leads to

$$\sup E(t) \leq \sup E^{1/2}(t) \int\limits_0^t \left[\int\limits_0^\ell \Delta u_{xx}^2 dx \right]^{1/2} dt = \sup E^{1/2}(t) \sum_{i=1}^{P} o(\varepsilon_i^{5/4}) \tag{3.12}$$

where $o(r)$ is a quantity such that

$$\lim_{r \to 0^+} (o(r)/r) = constant. \tag{3.13}$$

By means of Eq. (3.12) *, it is observed as follows for each $t \in [0, t_f]$*

$$0 \leq E^{1/2}(t) \leq o(\varepsilon^{5/4}). \tag{3.14}$$

Because $5/4 > 0$, [74] the following equality is obtained

$$E(t) = o(\varepsilon). \tag{3.15}$$

Since the coefficients of Eq. (3.4) *are bounded away from zero, the following conclusion of Lemma 3.1 is obtained from Eq.* (3.15)*. It is concluded from Lemma 3.1 that*

$$\lim_{\Delta p(t) \to 0} \Delta u(x,t) = 0. \tag{3.16}$$

That is to say, the system defined by Eqs. (3.1)-(3.3) *has a unique solution.*

By taking Lemma 3.1 into consideration, u is a unique solution to the system defined by Eqs. (3.1)-(3.3). In the light of this information, the corresponding control function p must be unique for preserving the uniqueness of u. Because the system has a unique solution and a unique control function, the system is called observable. In accordance with the Hilbert uniqueness method, observable is equal to the controllable [75], [76]. This means that the system is controllable.

3.4 Optimal Control Problem

The goal of the optimal control problem is to determine an optimal control function $p^o(t)$ which minimizes the dynamic response of the beam at a predetermined terminal time t_f.

The performance index is defined in terms of the weighted dynamic response of the beam and the expenditure of the control voltage spent in control duration. The set of admissible control functions is given as follows

$$P_{ad} = \{p(t) | p \in L^2(0,t_f), \quad |p(t)| \leq c_0 < \infty, \ c_0 \text{ is a constant}\} \tag{3.17}$$

and the performance index of the controlled system is defined by

$$\mathscr{J}(p(t)) = \int_0^1 [\mu_1 u^2(x,t_f) + \mu_2 u_t^2(x,t_f)]dx + \int_0^{t_f} \mu_3 p^2(t)dt \tag{3.18}$$

where $\mu_1, \mu_2 \geq 0$, $\mu_1 + \mu_2 \neq 0$ and $\mu_3 > 0$ are weighting constants. The first integral in Eq. (3.18) is the modified dynamic response of the beam and the last integral represents the measure of the total voltage energy that accumulates over $(0,t_f)$. The optimal control of the beam is stated as

$$\mathscr{J}(p^\circ(t)) = \min_{p(t)\in P_{ad}} \mathscr{J}(p(t)) \tag{3.19}$$

subject to Eqs. (3.1)-(3.3).

3.5 Boundary Control Characterization

In order to achieve the Pontryagin's maximum principle, let us define the adjoint variable, $v \in \mathscr{S}'$, which is dual to $\mathscr{S}$ and has the same inner product and norm like in $\mathscr{S}$. In addition, v satisfies the adjoint system corresponding to (3.1)-(3.3) as follows,

$$v_{xx} - \ell^2 L v_{xxxx} = \frac{1}{c^2}(v_{tt} - \ell^2 H v_{ttxx}) + \frac{1}{c^4}\ell^2 N v_{tttt} \tag{3.20}$$

and subject to the following boundary conditions

$$\begin{aligned} v(0,t) = v(\ell,t) \quad &= 0, \\ v_{xx}(0,t) = v_{xx}(\ell,t) &= 0 \end{aligned} \tag{3.21}$$

and terminal conditions

$$\begin{aligned} &\frac{\ell^2 N}{c^4} v_{ttt}(x,t_f) + \frac{1}{c^2}(v_t(x,t_f) - \ell^2 H v_{xxt}(x,t_f)) = 2\mu_1 u(x,t_f), \\ &\frac{\ell^2 N}{c^4} v_{tt}(x,t_f) + \frac{1}{c^2}(v(x,t_f) - \ell^2 H v_{xx}(x,t_f)) = -2\mu_2 u_t(x,t_f), \\ &v(x,t_f) = 0, \quad v_t(x,t_f) = 0. \end{aligned} \tag{3.22}$$

The Pontryagin's maximum principle in terms of Hamiltonian functional is derived as a necessary condition for the optimal control function. It is proved in [77] that under some convexity assumption, which are satisfied by Eq. (3.18) on performance index function, the Pontryagin's maximum principle is also sufficient condition for the optimal control function. The Pontryagin's maximum principle gives an explicit expression for the optimal control function and relates the optimal control to the state variable implicitly. Then, the Pontryagin's maximum principle can be given as follows:

Theorem 3.1 *If the optimal control $p^\circ(t) \in P_{ad}$ satisfies the following maximization problem*

$$\max \mathscr{H}(t;\nu,p) = \mathscr{H}(t;\nu^\circ,p^\circ)$$

where the Hamiltonian is given by the equation

$$\mathscr{H}(t;\nu,p) = -p(t)R(t) - \mu_3 p^2(t)$$

in which

$$R(t) = \ell^2 L(\nu_x(0,t) - \nu_x(\ell,t))$$

then

$$\mathscr{J}(p^\circ) \leq \mathscr{J}(p).$$

Proof 2 *Let us introduce an operator and deviations as follows, respectively;*

$$\Psi(u) = \frac{\ell^2 N}{c^4} u_{tttt} + \frac{1}{c^2}(u_{tt} - \ell^2 H u_{xxtt}) + \ell^2 L c^2 u_{xxxx} - u_{xx}, \tag{3.23}$$

$$\begin{aligned} \Delta u &= u(x,t) - u^\circ(x,t), \\ \Delta p &= p(x,t) - p^\circ(x,t). \end{aligned} \tag{3.24}$$

The operator satisfies the equality $\Psi(\Delta u) = 0$ and is subjected to following boundary conditions

$$\begin{aligned} &\Delta u(0,t) = \Delta u(\ell,t) = 0, \\ &\Delta u_{xx}(0,t) = \Delta u_{xx}(\ell,t) = \Delta p(t) \end{aligned} \tag{3.25}$$

and initial conditions

$$\Delta u(x,0) = \Delta u_t(x,0) = 0,$$
$$\Delta u_{tt}(x,0) = \Delta u_{ttt}(x,0) = 0. \tag{3.26}$$

Consider the following relation:

$$\begin{aligned}\int_0^\ell \int_0^{t_f} (\Delta u\Psi(\nu) - \nu\Psi(\Delta u))dtdx = &\underbrace{\int_0^\ell \int_0^{t_f} \frac{\ell^2 N}{c^4}\{\Delta u[\nu_{tttt} - \nu\Delta u_{tttt}]\}dtdx}_{I} + \underbrace{\int_0^\ell \int_0^{t_f} \frac{1}{c^2}[\Delta u\nu_{tt} - \nu\Delta u_{tt}]dtdx}_{II} \\ &+ \underbrace{\int_0^\ell \int_0^{t_f} \frac{\ell^2 H}{c^2}[\nu\Delta u_{xxtt} - \Delta u\nu_{xxtt}]dtdx}_{III} + \underbrace{\int_0^\ell \int_0^{t_f} \ell^2 L[\Delta u\nu_{xxxx} - \nu\Delta u_{xxxx}]dtdx}_{IV} \\ &+ \underbrace{\int_0^\ell \int_0^{t_f} [\nu\Delta u_{xx} - \Delta u\nu_{xx}]dtdx}_{V} = 0.\end{aligned}$$

Three times integration by parts of I gives

$$I = \int_0^\ell \frac{\ell^2 N}{c^4}[\nu_{ttt}(x,t_f)\Delta u(x,t_f) - \nu_{tt}(x,t_f)\Delta u_t(x,t_f) + \nu_t(x,t_f)\Delta u_{tt}(x,t_f) - \nu(x,t_f)\Delta u_{ttt}(x,t_f)]dx. \tag{3.27}$$

Similarly twice integration by parts of II yields

$$II = \int_0^\ell \int_0^{t_f} \frac{1}{c^2}[\Delta u\nu_{tt} - \nu\Delta u_{tt}]dtdx = \int_0^\ell \frac{1}{c^2}[\Delta u(x,t_f)\nu_t(x,t_f) - \Delta u_t(x,t_f)\nu(x,t_f)]dx. \tag{3.28}$$

Three times integration by parts of III reveals following relation

$$III = \int_0^\ell \int_0^{t_f} \frac{\ell^2 H}{c^2}[\nu\Delta u_{xxtt} - \Delta u\nu_{xxtt}]dtdx = \int_0^\ell \frac{\ell^2 H}{c^2}[\nu_{xx}(x,t_f)\Delta u_t(x,t_f) - \nu_{xxt}(x,t_f)\Delta u(x,t_f)]dx. \tag{3.29}$$

Four times integration by parts of IV leads to

$$IV = \int_0^{t_f} \int_0^\ell \ell^2 L[\Delta u\nu_{xxxx} - \nu\Delta u_{xxxx}]dxdt = \int_0^{t_f} \ell^2 L[\nu_x(\ell,t)\Delta u_{xx}(\ell,t) - \nu_x(0,t)\Delta u_{xx}(0,t)]dt. \tag{3.30}$$

Twice integration by parts of V gives the following

$$V = \int_0^\ell \int_0^{t_f} [\nu\Delta u_{xx} - \nu_{xx}\Delta u]dtdx = 0. \tag{3.31}$$

Combining these five parts, after using the terminal conditions given by Eq. (3.22) *and*

Eqs. (3.25)-(3.26)*, the following equation is obtained*

$$\int_0^{\ell}\{2\mu_1\Delta u(x,t_f)u(x,t_f)+2\mu_2\Delta u_t(x,t_f)u_t(x,t_f)\}dx=\int_0^{t_f}\ell^2L\{\nu_x(0,t)-\nu_x(\ell,t)\}\Delta p(t)dt.$$

Now focus on the difference of the performance index functional

$$\begin{aligned}\Delta\mathscr{J}(p)&=\mathscr{J}(p)-\mathscr{J}(p^\circ)\\&=\int_0^{\ell}\{\mu_1[u^2(x,t_f)-u^{\circ^2}(x,t_f)]+\mu_2[u_t^2(x,t_f)-u_t^{\circ^2}(x,t_f)]\}dx+\mu_3\int_0^{t_f}[p^2(t)-p^{\circ^2}(t)]dt\end{aligned}\tag{3.32}$$

Expanding the terms $u^2(x,t_f)$ *and* $u_t^2(x,t_f)$ *by using Taylor series about* $u^\circ(x,t_f)$ *and* $u_t^\circ(x,t_f)$ *leads to*

$$\begin{aligned}u^2(x,t_f)-u^{\circ^2}(x,t_f)&=2u^\circ(x,t)\Delta u(x,t_f)+r_1,\\u_t^2(x,t_f)-u_t^{\circ^2}(x,t_f)&=2u_t^\circ(x,t)\Delta u_t(x,t_f)+r_2,\end{aligned}\tag{3.33}$$

where

$$r_1=2(\Delta u)^2+...>0,\ \ r_2=2(\Delta u_t)^2+...>0.\tag{3.34}$$

Substituting Eqs. (3.33) *into Eq.* (3.32) *yields*

$$\begin{aligned}\Delta\mathscr{J}(p)=&\int_0^{1}\{2\mu_1[u^\circ(x,t_f)\Delta u^\circ(x,t_f)+r_1]+2\mu_2[u_t^\circ(x,t_f)\Delta u_t^\circ(x,t_f)+r_2]\}dx\\&+\mu_3\int_0^{t_f}[p^2(t)-p^{\circ^2}(t)]dt.\end{aligned}$$

Since $2\mu_1r_1+2\mu_2r_2\geq 0$*, the following is obtained*

$$\Delta\mathscr{J}(p)\geq\int_0^{t_f}\ell^2L\{v_x(0,t)-v_x(\ell,t)\}\Delta p(t)dt+\int_0^{t_f}\mu_3\{p^2(t)-p^{\circ^2}(t)\}dt,$$

and

$$\int_0^{t_f}\{p(t)R(t)+\mu_3p^2(t)-(p^\circ(t)R(t)+\mu_3p^{\circ^2}(t))\}dt\geq 0.$$

Hence,

$$\max\mathscr{H}(t;\nu,p)=\mathscr{H}(t;\nu^\circ,p^\circ)\tag{3.35}$$

and

$$\mathscr{J}(p)\geq\mathscr{J}(p^\circ).\tag{3.36}$$

Taking the first variation of $\mathscr{H}(t;\nu^\circ, p^\circ)$ gives the optimal control function as follows;

$$p^\circ(t) = \frac{\ell^2 L}{2\mu_3}\{\nu_x(\ell,t) - \nu_x(0,t)\}. \tag{3.37}$$

3.6 Solution Method

The solution of problem (3.19) is outlined as follows:

Step 1: Conversion of nonhomogenous boundary conditions to homogenous ones.

Nonhomogenous boundary conditions in (3.2) can be converted to homogenous boundary conditions by defining a new variable

$$\bar{u}(x,t) = u(x,t) - \gamma(x)p(t), \tag{3.38}$$

where

$$\gamma(x) = \frac{1}{2}(x(x-\ell)). \tag{3.39}$$

Then, the partial differential equation in (3.1) becomes

$$\begin{aligned}\bar{u}_{xx} - \ell^2 L\bar{u}_{xxxx} - \frac{1}{c^2}(\bar{u}_{tt} - \ell^2 H\bar{u}_{xxtt}) - \frac{1}{c^4}\ell^2 N\bar{u}_{tttt} = -p(t) + \frac{1}{c^2}\gamma(x)p''(t) - \frac{1}{c^2}\ell^2 Hp''(t)\\ + \frac{1}{c^4}\ell^2 N\gamma(x)p^{(4)}(t)\end{aligned} \tag{3.40}$$

with the new boundary conditions defined as

$$\begin{aligned}&\bar{u}(0,t) = 0 \quad \text{and} \quad \bar{u}(\ell,t) = 0,\\ &\bar{u}_{xx}(0,t) = 0 \quad \text{and} \quad \bar{u}_{xx}(\ell,t) = 0\end{aligned} \tag{3.41}$$

and initial conditions

$$\begin{aligned}&\bar{u}(x,0) = u_0(x) - \gamma(x)p(0),\\ &\bar{u}_t(x,0) = u_1(x) - \gamma(x)p'(0),\\ &\bar{u}_{tt}(x,0) = u_2(x) - \gamma(x)p''(0),\\ &\bar{u}_{ttt}(x,0) = u_3(x) - \gamma(x)p'''(0).\end{aligned} \tag{3.42}$$

Step 2: Approximating the solution of the adjoint system.

Solve the distributed parameter adjoint system (3.20) and (3.21) by approximating in terms of Nth-terms of sine Fourier series

$$v^{\circ}(x,t) = \sum_{n=1}^{N} Q_n(t) \sin \lambda_n x, \tag{3.43}$$

where

$$Q_n(t) = e^{at}(e_{1n} \cos bt + e_{2n} \sin bt) + e^{-at}(e_{3n} \cos bt + e_{4n} \sin bt). \tag{3.44}$$

The values a and b are known constants and they depend on c, ℓ, H, N and λ_n.

Eq. (3.43) becomes

$$v_x^{\circ}(x,t) = \sum_{n=1}^{N} \lambda_n Q_n(t) \cos \lambda_n x, \tag{3.45}$$

in which $\lambda_n = n\pi/\ell$.

Step 3: Computing the optimal control force $p^{\circ}(t)$.

Substituting Eq. (3.43) into Eq. (3.37), one obtains

$$p^{\circ}(t) = \frac{\ell^2 L}{2\mu_3} \sum_{n=1}^{N} \lambda_n [(-1)^n - 1] Q_n(t). \tag{3.46}$$

Step 4: Solve the distributed parameter state system (3.40) and (3.41).

Similarly, the solution of Eq. (3.40) can be obtained by using the Nth-terms of the sine Fourier series

$$\bar{u}(x,t) = \sum_{n=1}^{N} Z_n(t) \sin \lambda_n x, \tag{3.47}$$

where

$$Z_n(t) = e^{at}(d_{1n} \cos bt + d_{2n} \sin bt) + e^{-at}(d_{3n} \cos bt + d_{4n} \sin bt). \tag{3.48}$$

with $\lambda_n = n\pi/\ell$.

Substituting Eq. (3.47) in Eq. (3.40), one obtains

$$\begin{aligned} &\sum_{n=1}^{N} \{\frac{1}{c^4}\ell^2 N Z_n^{(4)}(t) + \frac{1}{c^2} Z_n''(t) + \frac{1}{c^2}\ell^2 H \lambda_n^2 Z_n''(t) + \ell^2 L \lambda_n^4 Z_n(t) + \lambda_n^2 Z_n(t)\} \sin \lambda_n x \\ &\quad = \frac{1}{c^2}\ell^2 H p''(t) - \frac{1}{c^2}\gamma(x) p''(t) + p(t) - \frac{1}{c^4}\ell^2 N \gamma(x) p^{(4)}(t). \end{aligned} \tag{3.49}$$

By applying the orthogonality of the Fourier sine series in Eq. (3.49), this leads to

$$\begin{aligned} &\frac{1}{c^4}\ell^2 N Z_n^{(4)}(t) + (\frac{1}{c^2} + \frac{\ell^2 H \lambda_n^2}{c^2}) Z_n''(t) + (\ell^2 L \lambda_n^4 + \lambda_n^2) Z_n(t) \\ &= 2\int_0^{\ell} \{\frac{1}{c^2}\ell^2 H p''(t) - \frac{1}{c^2}\gamma(x) p''(t) + p(t) - \frac{1}{c^4}\ell^2 N \gamma(x) p^{(4)}(t)\} \sin \lambda_n x dx. \end{aligned} \tag{3.50}$$

Solution of Eq. (3.50) is obtained by using the method of variation of parameters. The constants d_{1n}, d_{2n}, d_{3n} and d_{4n} in the solution of Eq. (3.50) can be determined by the initial conditions (3.42).

Step 5: Finding the undetermined constants e_{1n}, e_{2n}, e_{3n} and e_{4n}.

Inserting the expansions (3.43) and (3.47) into the mixed terminal conditions (3.22) leads to a system of linear modal equations

$$\begin{aligned}
&\frac{\ell^2 N}{c^4}\frac{d^3}{dt^3}Q_n(t_f;e_{1n},e_{2n},e_{3n},e_{4n})+\frac{1}{c^2}\Big(\frac{d}{dt}Q_n(t_f;e_{1n},e_{2n},e_{3n},e_{4n})+\ell^2 H\frac{d}{dt}Q_n(t_f;e_{1n},e_{2n},e_{3n},e_{4n})\Big)\\
&=2\mu_1 Z_n(t_f;e_{1n},e_{2n},e_{3n},e_{4n}),\\
&\frac{\ell^2 N}{c^4}\frac{d^2}{dt^2}Q_n(t_f;e_{1n},e_{2n},e_{3n},e_{4n})+\frac{1}{c^2}\big(Q_n(t_f;a_{1n},b_{1n},c_{1n},d_{1n})+\ell^2 H Q_n(t_f;e_{1n},e_{2n},e_{3n},e_{4n})\big)\\
&=-2\mu_2\frac{d}{dt}Z_n(t_f;e_{1n},e_{2n},e_{3n},e_{4n}),\\
&Q_n(t_f;e_{1n},e_{2n},e_{3n},e_{4n})=0,\\
&\frac{d}{dt}Q_n(t_f;e_{1n},e_{2n},e_{3n},e_{4n})=0.
\end{aligned} \tag{3.51}$$

Substituting the modal time solutions $Q_n(t)$ and $Z_n(t)$ into the lumped parameter system (3.50) leads to four linear equations in e_{1n}, e_{2n}, e_{3n} and e_{4n}. Solving for e_{1n}, e_{2n}, e_{3n} and e_{4n}, the adjoint function $v^\circ(x,t)$ is determined from Eq. (3.43), the optimal control $p^\circ(t)$ is given explicitly by Eq. (3.46) and finally the optimal response of the beam is obtained explicitly

$$u^\circ(x,t)=\bar{u}^\circ(x,t)+\gamma(x)p^\circ(t) \tag{3.52}$$

and optimal performance index is expressed by

$$\mathscr{J}(p^\circ)=\int_0^\ell[\mu_1 u^{\circ^2}(x,t_f)+\mu_2 u_t^{\circ^2}(x,t_f)]dx+\int_0^{t_f}\mu_3 p^{\circ^2}(t)dt. \tag{3.53}$$

3.7 Numerical Simulations

In this section, the effectiveness and capability of the introduced boundary control algorithm are simulated by means of computer codes produced in MATLAB ©. In the simulations, the predetermined terminal time is taken as $t_f=3$ and weighted coefficients μ_1,μ_2,μ_3 in the performance index functional are taken into account as $1,1$ and 10^{-3}, respectively. The length of the beam is considered as 1m. The size of the unit cell is chosen as 1 and constants $N = H = 1$. Also, for convenience $c = 1$. But, introduced control algorithm is valid in case coefficients can be chosen as desired. The values of the displacement and velocity of the Timoshenko beam are calculated at the midpoint of the beam $x = 0.5$. The response of the Timoshenko beam is examined subject to the following initial conditions;

$u(x,0) = 0, \quad u_t(x,0) = \sqrt{2}\sin(\pi x),$

$u_{tt}(x,0) = \sqrt{2}\cos(\pi x), \quad u_{ttt}(x,0) = \sqrt{2}\cos(\pi x).$

Displacement of the beam is plotted in Figure 3.1 and it is easily concluded that undesirable deflections can be suppressed effectively using minimum level of control. Also, controlled and uncontrolled velocities of the beam are given in Figure 3.2 and same observation is valid for understanding the effect of the boundary control.

The two functionals defining the dynamic response of the beam and the accumulated control over $[0,t_f]$ are introduced as

$$\mathscr{J}_D = \int_0^1 [u^2(x,t_f) + u_t^2(x,t_f)]dx, \tag{3.54}$$

and used control accumulates over $(0,t_f)$

$$\mathscr{J}_C = \int_0^{t_f} p^2(t)dt, \tag{3.55}$$

respectively. Note that Eq. (3.54) is corresponding to Eq. (3.18) in case of $\mu_1 = \mu_2 = 1$. The dynamic response $\mathscr{J}_D$ and the total control $\mathscr{J}_C$ are tabulated for different values of μ_3 in Table 3.1. It is observed from Table 3.1 that as the weighting factor μ_3 decreases, the dynamic response of the beam decreases while the corresponding control expenditure increases. In Figures 3.1 and 3.2, uncontrolled and controlled displacement and velocities of the beam are observed at $x = 0.5$ for $t \in [0,3]$.

Table 3.1 The values of $\mathscr{J}_D(u)$ and $\mathscr{J}_C(p)$ for different values of μ_3.

μ_3	$\mathscr{J}_D(u)$	$\mathscr{J}_C(p)$
10^2	0.4521	4.4683
10^1	0.2714e-1	26.8136
10^0	0.3761e-3	36.8842
10^{-1}	0.5898e-5	40.1492
10^{-2}	0.4560e-6	49.4758
10^{-3}	0.1099e-7	57.3350

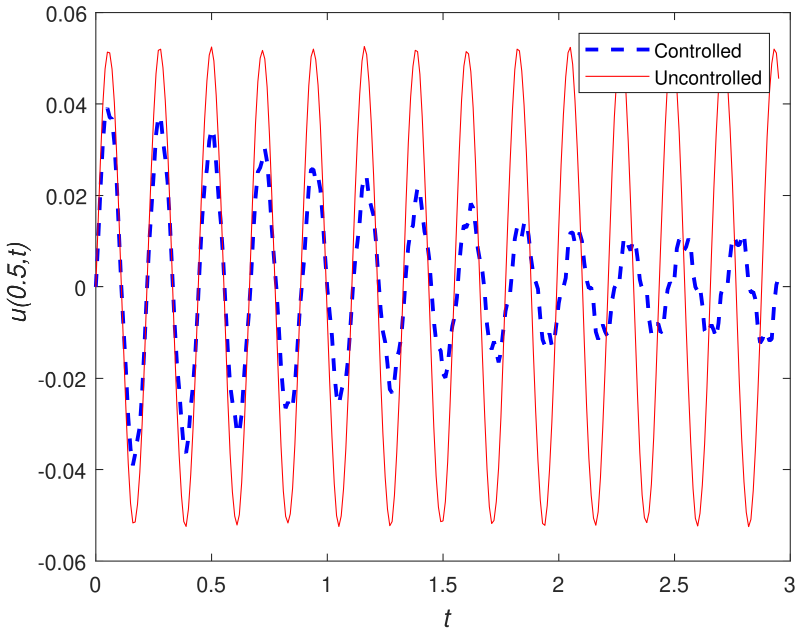

Figure 3.1 Controlled and uncontrolled displacements at $x = 0.5$.

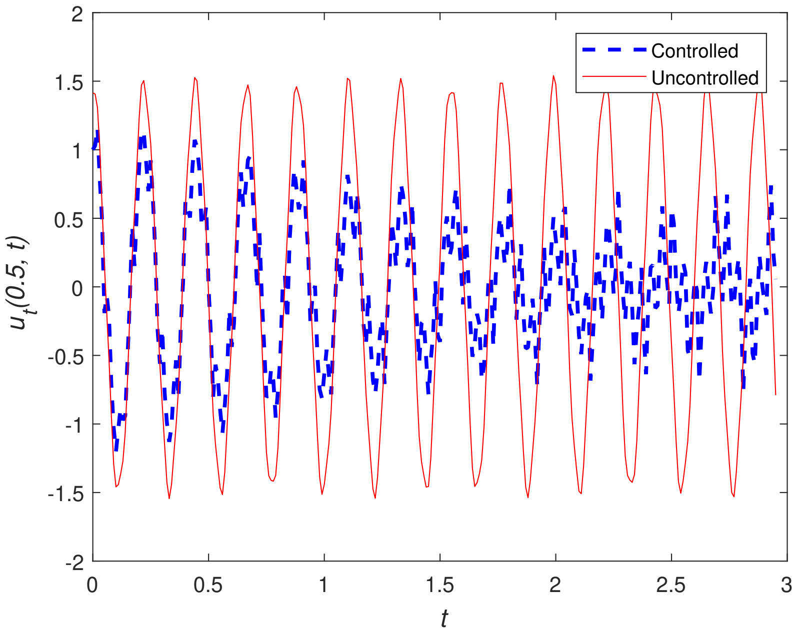

Figure 3.2 Controlled and uncontrolled velocities at $x = 0.5$.

The results presented in figures 3.1, 3.2 and table 3.1. It is concluded that undesirable vibrations in the Mindlin-type beam is suppressed effectively after application of the introduced boundary control algorithm. Also, the boundary control algorithm introduced in this study can be applied to different types of structures, hence undesirable vibrations can be suppressed with minimum control expense.

CHAPTER 4

OPTIMAL BOUNDARY CONTROL FOR A SECOND STRAIN GRADIENT THEORY-BASED BEAM

In this chapter, an optimal boundary control problem is formulated for a second strain gradient theory-based beam model to control free vibrations in the system. A quadratic performance index to be minimized indicates controlling the dynamic response of the system while an affordable control is in use.

An indirect method based on the Pontryagin's maximum principle is used to derive necessary conditions for the optimal control problem. Then, the problem is transformed into a system of partial differential equations including state and costate (adjoint) variables. The effectiveness and competence of the introduced optimal boundary control algorithm are presented in numerical simulations.

The importance of the study in this chapter is that it deals with optimal boundary control of a second strain gradient theory-based beam by means of Pontryagin's maximum principle.

4.1 Second Strain Gradient Theory-Based Beam Model

The second strain gradient theory is very strong non-classical continuum theory that captures the behavior of micrometer and nanometer sized structures. Timoshenko and Euler-Bernoulli beam models are other different types of beam models, but they do not meet the expectations about microscopic effects of the structures when compared to the second strain gradient theory-based beam model. Euler-Bernoulli and Timoshenko beam theories by comparing to Mindlin's beam theory have the disadvantage for obtaining the accurate results due to ignoring the microstructural effects in the beam. Due to this observation, Mindlin presented a general advanced elastic theory to characterise elastic behavior of isotropic materials taking into account microstructural effects [21], [22], [76]. He considered that strains and gradient of strains are quadratically formed the potential energy density and kinetic energy function consisting of quadratic form of both velocities and gradients of velocities [24]. However, using higher order gradients introduces new constants that are difficult to be determined. To overcome this difficulty, Mindlin proposed three simpler versions of his theory. These new versions are known as Form I, II and III which conclude the same equation of motion. Further, Mindlin suggested [25] a new theory, known as the second gradient elastic theory which is outstanding with regard to capturing the size effects of the structures in micro- and nano-scales. In his theory, the potential energy is dependent on strains, gradient of strains and also second gradient of strains. The constitutive equation is a partial differential equation of sixth order [25], [26].

4.2 Mathematical Formulation of the Problem

The beam system under consideration is initially undeformed at rest and vibrations in the beam system are alerted by initial effects, which are shown by initial conditions.

Figure 4.1 The scheme of the beam

The beam under consideration is mathematically formulated as follows [26]:

$$u_{xx} - \frac{13}{12}\ell^2 u_{xxxx} + \frac{\ell^4}{72}u_{xxxxxx} - \frac{1}{c^2}u_{tt} + \frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho}u_{xxtt} = 0 \tag{4.1}$$

where $u = u(x,t) \in \mathscr{S} = (0,\ell) \times (0,t_f)$ is the transversal displacement at position x and time t, ℓ is the length of the beam, t_f is the fixed terminal time, $c^2 = \frac{E}{\rho}$, c is a constant, E is Young modulus, ρ' and ρ are the density of the microstructural effects. Equation of motion Eq. (4.1) is subjected to the following boundary conditions,

$$\begin{aligned} u(0,t) = u(\ell,t) &= 0 \\ u_{xx}(0,t) = u_{xx}(\ell,t) &= 0 \\ u_{xxxx}(0,t) = u_{xxxx}(\ell,t) &= p(t) \end{aligned} \tag{4.2}$$

in which $p(t)$ is the control function to be computed optimally and the initial conditions,

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \tag{4.3}$$

in which $u_0(x)$ and $u_1(x)$ are known functions.

4.3 Wellposedness and Controllability of the Optimal Control Problem

Assume that the following conditions are provided

$$\begin{aligned} & u,\ \frac{\partial^j u}{\partial t^j}, \frac{\partial^i u}{\partial x^i}, \frac{\partial^{k+j} u}{\partial t^k \partial x^j} \in L^2(\mathscr{S}), \quad i = 0,1,...,6, \quad j = 0,1,2, \quad k = 0,1,2, \\ & p(t) \text{ analytic function, } u_0(x) \in H^1(0,\ell),\ u_1(x) \in L^2(0,\ell), \end{aligned} \tag{4.4}$$

where $H(\mathscr{S})$=$L^2(\mathscr{S})$ is Hilbert space such that

$$P_{ad} = \{p(t) \mid p \in L^2(0,t_f),\ |p(t)| < m_0 < \infty,\ m_0 \text{ is a constant}\}$$

where the inner product for L^2 of two functions q and r on $\mathscr{S}$ with $< q,r >= \int_{\mathscr{S}} q(s)r(s)ds$ and the norm $\|q\|^2 =< q,q >$ for all $q,r \in L^2(\mathscr{S})$. Then, the system defined by Eqs. (4.1)-(4.3) has a solution [71]. In addition, Eqs. (4.1)-(4.3) can be reduced to ordinary differential equation and therefore it is shown by means of linear Picard-Lindelöf existence-uniqueness theorem that Eqs. (4.1)-(4.3) has a solution.
Proving the uniqueness of the solution of the system given by Eqs. (4.1)-(4.3) is important for the uniqueness of the control.

Lemma 4.1 *The solution to the problem* (4.1)-(4.3) *is unique.*

Proof 3 *Suppose that u_1 and u_2 are two solutions to system given by Eqs.* (4.1)-(4.3). *Then, the difference function*

$$w(x,t) = u_1(x,t) - u_2(x,t)$$

will satisfy the homogeneous equation

$$w_{xx} - \frac{13}{12}\ell^2 w_{xxxx} + \frac{\ell^4}{72} w_{xxxxxx} - \frac{1}{c^2} w_{tt} + \frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho} w_{xxtt} = 0 \tag{4.5}$$

subject to the homogeneous boundary conditions and initial conditions, respectively,

$$w(0,t) = w(\ell,t) = w_{xx}(0,t) = w_{xx}(\ell,t) = w_{xxxx}(0,t) = w_{xxxx}(\ell,t) = 0, \tag{4.6}$$

$$w(x,0)=w_t(x,0)=0. \tag{4.7}$$

If $w(x,t)$ is shown that it is identically zero in $\mathscr{S}$, the uniqueness of the solution is obtained. Let us examine the energy integral

$$E(T)=\frac{1}{2}\int_0^{\ell}\left\{\frac{1}{c^2}w_t^2(x,t)+\frac{\ell^2\acute{\rho}}{3c^2\rho}w_{xt}^2(x,t)+w_x^2(x,t)+\frac{13\ell^2}{12}w_{xx}^2(x,t)+\frac{\ell^2}{72}w_{xxx}^2(x,t)\right\}dx, \tag{4.8}$$

and show that it is independent of t. Differentiating $E(t)$ with respect to t, the following equation is obtained

$$\begin{aligned}\frac{dE(T)}{dt}=&\int_0^{\ell}\left\{\frac{1}{c^2}w_{tt}-\frac{\ell^2\acute{\rho}}{3c^2\rho}w_{xxtt}-w_{xx}+\frac{13\ell^2}{12}w_{xxxx}-\frac{\ell^2}{72}w_{xxxxxx}\right\}w_t dx\\&+\left\{w_{xtt}u_t+w_xw_t+w_{xx}w_{xt}-w_{xxx}w_t+w_{xxx}w_{xxt}-w_{xxxx}w_{xt}+w_{xxxxx}w_t\right\}\Big|_0^{\ell}.\end{aligned} \tag{4.9}$$

By using Eq. (4.5) *and boundary conditions* (4.6)*, it follows that* $\frac{E(T)}{dt}=0$*, that is,* $E(t)=constant$*.*

Taking the initial conditions (4.7) *into consideration, the following equality holds*

$$\begin{aligned}E(T)&=constant=E(0)\\&=\frac{1}{2}\int_0^{\ell}\left\{\frac{1}{c^2}w_t^2(x,t)+\frac{\ell^2\acute{\rho}}{3c^2\rho}w_{xt}^2(x,t)+w_x^2(x,t)+\frac{13\ell^2}{12}w_{xx}^2(x,t)+\frac{\ell^2}{72}w_{xxx}^2(x,t)\right\}\Big|_{t=0}dx=0.\end{aligned}$$

Then it follows from Eq. (4.8) *and from the initial conditions* (4.7) *that $w(x,t)$ is identically equal to zero in $\mathscr{S}$, that is, $u_1=u_2$, which completes the proof.*

By considering the uniqueness solution of the beam system, it is determined that the control function is unique by the reason of the uniqueness of u. In this case, the studied system is observable due to having a unique solution and unique control function. Briefly, the system defined by Eqs. (4.1)-(4.3) is controllable according to the Hilbert uniqueness method [75], [76].

4.4 Optimal Control Problem

It is desired to identify an optimal control function $p(t)$ for damping out the undesired vibrations that is placed on the boundary. To this end, the cost functional (performance index) which is minimized over the time interval $0 \leq t \leq t_f$ is defined in two parts: The first term measures the dynamical response of the system at the terminal time t_f and the second term is the penalty function that minimizes control force's expenditure of used in $[0, t_f]$. The performance index is as follows

$$\mathscr{J}(p(t)) = \int_0^{\ell} [\mu_1 u^2(x,t_f) + \mu_2 u_t^2(x,t_f)]dx + \int_0^{t_f} \mu_3 p^2(t)dt \tag{4.10}$$

where $p(t) \in P_{ad}$ is to be determined optimally and μ_1, μ_2 and μ_3 weighting coefficients satisfying $\mu_1, \mu_2 \geq 0$, $\mu_1 + \mu_2 \neq 0$ and $\mu_3 > 0$. Hence, the optimal boundary control problem of our main interest is expressed in the following manner:

$$\mathscr{J}(p^{\circ}(t)) = \min_{p(t) \in P_{ad}} \mathscr{J}(p(t)). \tag{4.11}$$

subject to Eqs. (4.1)-(4.3).

4.5 Boundary Control Characterization

The Pontryagin's maximum principle provides us to obtain the optimal control function analytically and is used to derive necessary condition for optimal control in terms of Hamiltonian. Since the performance index functional satisfies convexity, optimality conditions obtained from results of Pontryagin's maximum principle is also sufficient condition [77]. The Pontryagin's maximum principle enables us to derive the optimal control explicitly and obtain relationship for state and control variables implicitly. For this purpose, let us introduce a Hamiltonian and define an adjoint variable v. The adjoint system related to Eqs. (4.2)-(4.3) is

$$v_{xx} - \frac{13}{12}\ell^2 v_{xxxx} + \frac{\ell^4}{72} v_{xxxxxx} - \frac{1}{c^2} v_{tt} + \frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho} v_{xxtt} = 0. \tag{4.12}$$

The boundary conditions

$$
\begin{aligned}
&v(0,t) = v(\ell,t) = 0,\\
&v_{xx}(0,t) = v_{xx}(\ell,t) = 0, \qquad (4.13)\\
&v_{xxxx}(0,t) = v_{xxxx}(\ell,t) = 0,
\end{aligned}
$$

and terminal conditions are

$$
\begin{aligned}
&\frac{\ell^2\acute{\rho}}{3c^2\rho}v_{xxt}(x,t_f) - \frac{1}{c^2}v_t(x,t_f) = 2\mu_1 u(x,t_f),\\
&\frac{\ell^2\acute{\rho}}{3c^2\rho}v_{xx}(x,t_f) - \frac{1}{c^2}v(x,t_f) = -2\mu_2 u_t(x,t_f).
\end{aligned} \qquad (4.14)
$$

For the control problem, the maximum principle is expressed as follows:

Theorem 4.1 *An optimal control $p^\circ(t) \in P_{ad}$ which causes the maximization problem*

$$\max \mathscr{H}(t;v,p) = \mathscr{H}(t;v^\circ,p^\circ)$$

in which the Hamiltonian is

$$\mathscr{H}(t;v,p) = -p(t)R(t) - \mu_3 p^2(t)$$

where

$$R(t) = \frac{\ell^4}{72}(v_x(0,t) - v_x(\ell,t))$$

satisfies the following inequality

$$\mathscr{J}(p^\circ) \leq \mathscr{J}(p). \qquad (4.15)$$

Proof 4 *First, let us introduce an operator,*

$$\Psi(u) = u_{xx} - \frac{13}{12}\ell^2 u_{xxxx} + \frac{\ell^4}{72}u_{xxxxxx} - \frac{1}{c^2}u_{tt} + \frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho}u_{xxtt}, \qquad (4.16)$$

and deviations,

$$\Delta u = u(x,t) - u^\circ(x,t), \quad \Delta p = p(x,t) - p^\circ(x,t). \qquad (4.17)$$

The operator defined by Eq. (4.16)*, satisfies the following equality*

$$\Psi(\Delta u) = 0, \qquad (4.18)$$

subjected to the following boundary conditions and initial conditions, respectively,

$$\Delta u(x,t) = \Delta u_{xx}(x,t) = 0, \quad \Delta u_{xxxx}(x,t) = \Delta p(t), \quad x = 0, \ell \tag{4.19}$$

$$\Delta u(x,t) = \Delta u_t(x,t) = 0, \quad t = 0. \tag{4.20}$$

Consider the following form:

$$\begin{aligned}\int_0^\ell \int_0^{t_f} (\Delta u \Psi(v) - v\Psi(\Delta u))dtdx &= \int_0^\ell \int_0^{t_f} \{\Delta u(v_{xx} - \frac{13}{12}\ell^2 v_{xxxx} + \frac{\ell^4}{72} v_{xxxxxx} - \frac{1}{c^2} v_{tt} + \frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho} v_{xxtt}) \\ &\quad - v(\Delta u_{xx} - \frac{13}{12}\ell^2 \Delta u_{xxxx} + \frac{\ell^4}{72}\Delta u_{xxxxxx} - \frac{1}{c^2}\Delta u_{tt} + \frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho}\Delta u_{xxtt})\} dtdx \\ &= \underbrace{\int_0^\ell \int_0^{t_f} [\Delta u v_{xx} - v \Delta u_{xx}] dtdx}_{I} + \underbrace{\int_0^\ell \int_0^{t_f} [\frac{13}{12}\ell^2 \Delta u_{xxxx} v - \frac{13}{12}\ell^2 v_{xxxx}\Delta u] dtdx}_{II} \\ &\quad + \underbrace{\int_0^\ell \int_0^{t_f} [\frac{\ell^4}{72}\Delta u v_{xxxxxx} - \frac{\ell^4}{72} v \Delta u_{xxxxxx}] dtdx}_{III} + \underbrace{\int_0^\ell \int_0^{t_f} [\frac{1}{c^2}\Delta u_{tt} v - \frac{1}{c^2} v_{tt}\Delta u] dtdx}_{IV} \\ &\quad + \underbrace{\int_0^\ell \int_0^{t_f} [\frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho}\Delta u v_{xxtt} - \frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho} v \Delta u_{xxtt}] dtdx}_{V} \\ &= 0. \end{aligned} \tag{4.21}$$

By considering Eqs. (4.19)-(4.20)*, let us integrate by parts the terms I, II, III, IV, V in above equality. First, integration by parts twice in term I gives*

$$I = \int_0^\ell \int_0^{t_f} [\Delta u v_{xx} - v \Delta u_{xx}] dtdx = 0. \tag{4.22}$$

Similarly using integration by parts four times in II yields

$$II = \int_0^\ell \int_0^{t_f} [\frac{13}{12}\ell^2 \Delta u_{xxxx} v - \frac{13}{12}\ell^2 v_{xxxx}\Delta u] dtdx = 0. \tag{4.23}$$

By applying integration by parts six times to III, the following relation is obtained

$$\begin{aligned} III &= \int_0^\ell \int_0^{t_f} [\frac{\ell^4}{72}\Delta u v_{xxxxxx} - \frac{\ell^4}{72} v \Delta u_{xxxxxx}] dtdx \\ &= \int_0^{t_f} \frac{\ell^4}{72}(v_x(\ell,t)\Delta u_{xxxx}(\ell,t) - v_x(0,t)\Delta u_{xxxx}(0,t))dt. \end{aligned} \tag{4.24}$$

Similarly, the following equalities are observed:

$$IV = \int_0^\ell \int_0^{t_f} [\frac{1}{c^2}\Delta u_{tt} v - \frac{1}{c^2} v_{tt}\Delta u] dtdx = \int_0^\ell \frac{1}{c^2}(\Delta u_t(x,t_f)v(x,t_f) - \Delta u(x,t_f)v_t(x,t_f))dx, \tag{4.25}$$

$$V = \int_0^\ell \int_0^{t_f} [\frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho}\Delta u v_{xxtt} - \frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho} v \Delta u_{xxtt}] dt dx$$
$$= \int_0^\ell \frac{1}{c^2}\frac{\ell^2}{3}\frac{\acute{\rho}}{\rho}(\Delta u(x,t_f) v_{xxt}(x,t_f) - \Delta u_t(x,t_f) v_{xx}(x,t_f)) dx. \tag{4.26}$$

Substituting these equalities into Eq. (4.21) *and using terminal conditions given by Eq.* (4.14) *gives*

$$\int_0^\ell \{2\mu_1 \Delta u(x,t_f) u(x,t_f) + 2\mu_2 \Delta u_t(x,t_f) u_t(x,t_f)\} dx = \int_0^{t_f} \{\frac{\ell^4}{72}(v_x(0,t) - v_x(\ell,t))\} \Delta p(t) dt \tag{4.27}$$

To analyze the difference performance index functional, the following equality is observed

$$\Delta \mathscr{J}(p) = \mathscr{J}(p) - \mathscr{J}(p^\circ)$$
$$= \int_0^\ell \{\mu_1[u^2(x,t_f) - u^{\circ^2}(x,t_f)] + \mu_2[u_t^2(x,t_f) - u_t^{\circ^2}(x,t_f)]\} dx \tag{4.28}$$
$$+ \mu_3 \int_0^{t_f} [p^2(t) - p^{\circ^2}(t)] dt$$

Using Taylor series for $u^2(x,t_f)$ *and* $u_t^2(x,t_f)$ *about* $u^\circ(x,t_f)$ *and* $u_t^\circ(x,t_f)$*, respectively, leads to the following relation*

$$u^2(x,t_f) - u^{\circ^2}(x,t_f) = 2u^\circ(x,t_f)\Delta u(x,t_f) + r_1,$$
$$u_t^2(x,t_f) - u_t^{o^2}(x,t_f) = 2u_t^\circ(x,t_f)\Delta u_t(x,t_f) + r_2, \tag{4.29}$$

with remainder r_1 *and* r_2*,*

$$r_1 = 2(\Delta u)^2 + ... > 0, \quad r_2 = 2(\Delta u_t)^2 + ... > 0.$$

Substituting Eqs. (4.29) *into Eq.* (4.28) *results in*

$$\Delta \mathscr{J}(f) = \int_0^\ell \{2\mu_1[u^o(x,t_f)\Delta u^\circ(x,t_f) + r_1] + 2\mu_2[u_t^\circ(x,t_f)\Delta u_t^\circ(x,t_f) + r_2]\} dx$$
$$+ \mu_3 \int_0^{t_f} [p^2(t) - p^{\circ^2}(t)] dt \geq 0. \tag{4.30}$$

Since $2\mu_1 r_1 + 2\mu_2 r_2 \geq 0$*, the following inequality is obtained*

$$\Delta \mathscr{J}(p) \geq \int_0^{t_f} \{\frac{\ell^4}{72}(v_x(0,t) - v_x(\ell,t))\} \Delta f(t) dt + \mu_3 \int_0^{t_f} \{p^2(t) - p^{\circ^2}(t)\} dt \geq 0 \tag{4.31}$$

and

$$\Delta \mathscr{J}(p) \geq \int_0^{t_f} \{\frac{\ell^4}{72} p(t)R(t) + \mu_3 p^2(t) - (\frac{\ell^4}{72} p^\circ(t)R(t) + \mu_3 p^{\circ^2}(t))\} \geq 0. \tag{4.32}$$

Hence,

$$\max \mathscr{H}(t;v,p) = \mathscr{H}(t;v^\circ,p^\circ), \ \forall p \in P_{ad} \tag{4.33}$$

and

$$\mathscr{J}(p) \geq \mathscr{J}(p^\circ), \ \forall p \in P_{ad}. \tag{4.34}$$

If the first variation of Hamiltonian $\mathscr{H}(t;v,p)$ vanish at p°,the optimal control function indicates clearly as follows;

$$p^\circ(t) = \frac{\ell^4}{144} \frac{\{v_x(\ell,t) - v_x(0,t)\}}{\mu_3}. \tag{4.35}$$

4.6 Numerical Simulations

In this section, numerical methods introduced in the previous sections are implemented to show the effectiveness of the techniques. In this process, solution method is applied as in Section 3.6. Homogeneous boundary conditions are first obtained by defining a new variable. Then, the adjoint system (4.12)-(4.13) is solved using eigenfunction expansion method in terms of the sine Fourier series and optimal control function (4.35) is computed. Finally, the obtained lumped parameter system with homogeneous boundary conditions are solved by using the Nth-terms of the sine Fourier series, similarly by finding undetermined constants. In the numerical simulations, $t_f = 3$ and other parameters are as follows [69]:

* The size of the unit cell is taken as $\ell = 1$ m,
* Density of the microstructural effects are given by $\rho = \rho' = 6.10^4\ kg/m^3$,
* $c^2 = \frac{E}{\rho}$, $\quad E = 2.10^7\ N/m^2$,
* The displacement and velocity of the beam are given at the midpoint of the beam, $x = 0.5.$,
* Weighting coefficients are taken into account as $\mu_1 = \mu_2 = 1$ and $\mu_3 = 10^{-3}$ for

controlled case.

Also, in the simulations, it is assumed that beam under consideration subjects to the following initial conditions;

$$u(x,0) = \sqrt{2}sin(\pi x), \quad u_t(x,0) = \sqrt{2}cos(\pi x). \tag{4.36}$$

Let us define the dynamical response of the system at the terminal time t_f and control force spend in $[0,t_f]$ as follows, respectively,

$$\mathscr{J}_D(u) = \int_0^1 [u^2(x,t_f) + u_t^2(x,t_f)]dx, \quad \mathscr{J}_C(p) = \int_0^{t_f} p^2(t)dt \tag{4.37}$$

The results obtained for $\mathscr{J}_D(u)$ and $\mathscr{J}_C(p)$ by using the values $\mu_{1,2} = 1$ and different values of μ_3 are summarized in Table 4.1. By observing Table 4.1, it is concluded that, as the weighted coefficient μ_3 on the control function decreases, the dynamic response of the beam decreases corresponding to an increase in the control force. Similar observations can be done by examining Figures 4.2 and 4.3. Displacement and velocity of the beam are plotted for the controlled and uncontrolled cases in Figures 4.2 and 4.3, respectively. These figures show that introduced boundary control damps out the free vibrations such that the amplitude of the vibrations is close to zero at the terminal time t_f. By observing Figure 4.2, the difference between controlled and uncontrolled displacements of the beam means that the introduced boundary control algorithm is very effective and it reaches the objective of the control problem.

Table 4.1 The results obtained for $\mathscr{J}_D(u)$ and $\mathscr{J}_C(p)$ by using the values $\mu_{1,2} = 1$ and different values of μ_3.

μ_3	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
$\mathscr{J}_D(u)$	0.6583e-4	0.5986e-2	0.5766	46.6916	1166.54	3238.67	3717.51
$\mathscr{J}_C(p)$	8151.19	8132.44	7952.54	6440.64	1609.13	44.67	0.51

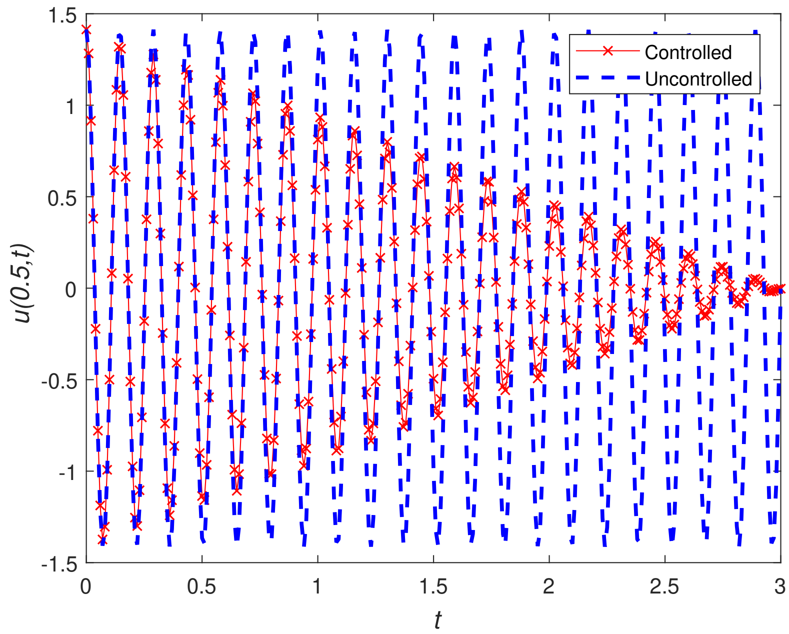

Figure 4.2 Controlled/uncontrolled beam displacement.

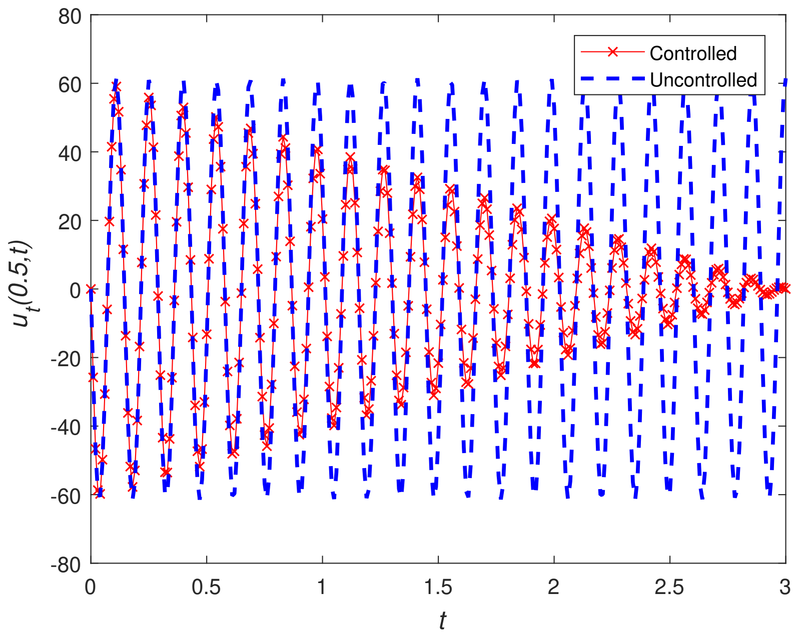

Figure 4.3 Controlled/uncontrolled beam velocity.

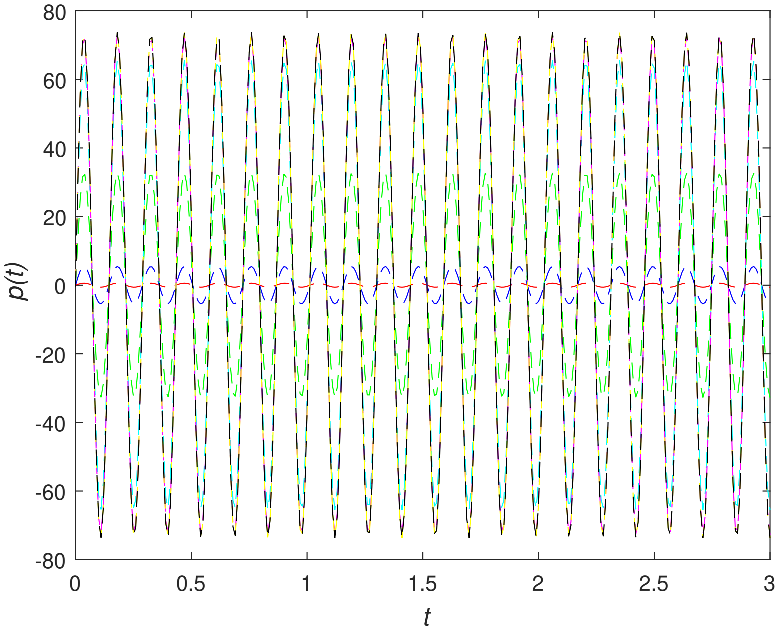

Figure 4.4 Optimal control solution for the case with $t_f = 3$ and different weight coefficients μ_3.

CHAPTER 5

OPTIMAL CONTROL OF A PARABOLIC PDE WITH A BILINEAR CONTROL

This chapter focuses on obtaining optimal control and optimal trajectories to a parabolic equation with a bilinear control. The investigated model is used to describe heat conduction process with a bilinear parameter. The performance index to be minimized indicates controlling the dynamic response of the system while an affordable control is in use.

The presented approach uses order reduction in the model, Pontryagin's maximum principle and numerical techniques. Using a modal space expansion method the distributed parameter system is turned into a lumped parameter system. In the temporal terms, the obtained system corresponds to a bilinear system. The Pontryagin's maximum principle is used to obtain the optimal control function that leads to a nonlinear two-point boundary value problem (TPBVP) with split boundary conditions.

Two iterative numerical techniques for determining optimal trajectories and optimal control of the system are discussed. Steepest descent and quasilinearization are procedures for solving this nonlinear TPBVP. The steepest descent method is used as a beginning procedure and the quasilinearization method to confirm the solution. Comparisons were drawn to evaluate the initial guesses and their importance, number of iterations and convergence of these two iterative methods connecting with control problem. Numerical simulations showed the effectiveness and applicability of the proposed approach.

5.1 Bilinear-Quadratic Optimal Control Problem Setting

Consider a distributed parameter system describing a heat conduction process with a source term [58].

$$u_t = \Delta u + p(t)u + \phi(x,t), \tag{5.1}$$

where $u = u(x,t) \in \Omega = \Omega_x \times \Omega_t$ is the displacement of the system at position x and time t. $p(t)$ is the control function to be determined optimally with external heat source $\phi(x,t)$. The set $\Omega_x = [0,\ell]$ is a subset of Euclidean space $\mathbb{R}^1$ and Ω_t denote a given time interval $(0,t_f)$ where t_f is a predetermined terminal time. Eq. (5.1) is subjected to the following initial and boundary conditions, respectively;

$$\begin{aligned} &u(x,0) = \varphi(x), \\ &u(0,t) = g_0(t), \quad u(\ell,t) = g_1(t) \end{aligned} \tag{5.2}$$

for given functions $g_0(t)$, $g_1(t)$ and $\varphi(x)$. Let the admissible control set be

$$P_{ad} = \{p(t) \mid p(t) \in L^2(\Omega_t), \mid p(t) \mid \leq c_0 < \infty,\ c_0 \text{ is a constant}\}.$$

The performance index functional $\mathscr{J}[p(t)]$ is specified as a weighted quadratic functional of the dynamic response which is to be minimized at the terminal time t_f subject to Eqs. (5.1)-(5.2),

$$\mathscr{J}[p(t)] = \frac{1}{2}\int_{\Omega_x} [r_1 u^2(x,t_f) + r_2 u_t^2(x,t_f)]dx + \frac{1}{2}q\int_{\Omega} u^2(x,t)dxdt + \frac{1}{2}s\int_{\Omega_t} p^2(t)dt, \tag{5.3}$$

where r_1, r_2, q and s are weighting constants satisfying the condition $r_1, r_2, q \geq 0$ and $s > 0$. The last term on the right hand side of Eq. (5.3) is a penalty term which limits expending large amounts of control effort.

It is desired to find optimal control function $p^\circ(t) \in P_{ad}$ that minimizes the performance index

$$\mathscr{J}[p^\circ(t)] = \min_{p(t)\in P_{ad}} \mathscr{J}[p(t)], \tag{5.4}$$

satisfying equations Eqs. (5.1)-(5.2).

5.2 Wellposedness and Controllability of the Optimal Control Problem

Energy methods are one of the most important and most widely used methods to demonstrate the existence and uniqueness of the solution. The starting point of these methods is the physical fact that the kinetic energy of the homogeneous system decreases in time in the absence of external forces. The existence of Eqs. (5.1)-(5.2) are discussed in [78], [79], [80], [81], [82].

The uniqueness of the problem is then proved by means of energy methods, as the uniqueness of solution of the system yields the uniqueness of control. To prove the uniqueness of the solution, a priori bound is obtained on the difference of two solutions with the same initial data [83]. In the following lemma, Gronwall's lemma is introduced and is used in Lemma 5.2 that shows the uniqueness of the solution.

Lemma 5.1 (Gronwall's Inequality) *Let $x(t) : [0,T] \to \mathbb{R}$ be a differentiable function. Suppose that $x(t)$ satisfies the following differential inequality*

$$\frac{d}{dt}x(t) \leq g(t)x(t) + h(t)$$

where g is continuous and h is a locally integrable function. Then the following inequality holds

$$x(t) \leq x(0)e^{G(t)} + \int_0^t e^{G(t)-G(s)}h(s)ds,$$

where

$$G(t) := \int_0^t g(r)dr.$$

Lemma 5.2 *The solution of Eq.* (5.1) *for $u(x,t) \in L^2(\mathbb{R}^N)$ subject to Eq.* (5.2) *is unique.*

Proof 5 *Suppose that there are two distinct smooth solutions u_1 and u_2 satisfying Eqs.* (5.1)-(5.2). *Then their difference, $\bar{u} = u_1 - u_2$, satisfies Eq.* (5.1) *with zero initial-*

boundary conditions, i.e,

$$\begin{cases} \bar{u}_t = \Delta\bar{u} + p(t)\bar{u}, & 0 \le x \le \ell, \quad 0 < t < t_f \\ \bar{u}(x,0) = 0, \\ \bar{u}(0,t) = 0, \quad \bar{u}(\ell,t) = 0. \end{cases} \tag{5.5}$$

Now define the energy integral as follows

$$E[\bar{u}] = \frac{1}{2}\int_0^\ell |\bar{u}(x,t)|^2 dx \tag{5.6}$$

which is always positive and decreasing, if $\bar{u}$ solves Eqs. (5.1)-(5.2)*. The energy increases that can be seen by taking inner product $(.,.)$ of $\bar{u}_t$ with $\bar{u}$ and integrating with respect to x over the spatial interval $[0,\ell]$.*

The L^2 inner product of Eq. (5.5) *with $\bar{u}$ is taken to get*

$$(\bar{u}_t, \bar{u}) = (\Delta\bar{u}, \bar{u}) + (p(t)\bar{u}, \bar{u}). \tag{5.7}$$

In the proof, $||.||_{L^2}$ denotes the L^2 norm on $\mathbb{R}^N$ and is defined as

$$||f||_{L^2} = \left(\int_{\mathbb{R}^N} |f|^2 dx\right)^{1/2}.$$

Differentiating Eq. (5.6) *with respect to t yields*

$$\frac{d}{dt}E = \int_0^\ell \bar{u}\bar{u}_t dx = \int_0^\ell \bar{u}(\bar{u}_{xx} + p(t)\bar{u})dx = \int_0^\ell \bar{u}\bar{u}_{xx} dx + \int_0^\ell p(t)\bar{u}^2 dx. \tag{5.8}$$

Using integrating by parts in the last integral leads to

$$\frac{d}{dt}E = \bar{u}\bar{u}_x|_0^\ell - \int_0^\ell \bar{u}_x^2 dx + \int_0^\ell p(t)\bar{u}^2 dx \tag{5.9}$$

and the following equation is obtained

$$\frac{d}{dt}E = -\int_0^\ell \bar{u}_x^2 dx + \int_0^\ell p(t)\bar{u}^2 dx \le p(t)\|\bar{u}\|_{L^2}^2. \tag{5.10}$$

In that case the inequality

$$\frac{d}{dt}\|\bar{u}\|_{L^2} \le p(t)\|\bar{u}\|_{L^2} \tag{5.11}$$

holds and the result is obtained applying the Gronwall's lemma

$$\|u_1 - u_2\|_{L^2} \le \|u_1 - u_2|_{t=0}\|_{L^2} exp\left(\int_0^{t_f} p(r)dr\right). \tag{5.12}$$

This completes the uniqueness of smooth solutions.

In order to preserve the uniqueness of the solution provided by Lemma 5.2, the corresponding control function $p(t)$ must be unique. Since the system has a unique solution and control function, the system is called observable. By taking the Hilbert uniqueness into account, the observability is equivalent to the controllability [75], [76]. Briefly, the system is controllable.

5.3 Modal Control Space Problem

In this section, the optimal control of distributed parameter system (5.1)-(5.2) is transformed into optimal control of lumped parameter system by implementing modal space expansion technique. New system gives rise to a bilinear system in the temporal term. In order to achieve the transformation, first a new parameter $w(x,t)$ is introduced to convert nonhomogeneous boundary conditions to homogeneous boundary conditions.
By letting

$$w(x,t) = u(x,t) - \frac{(\ell - x)}{\ell} g_0(t) - \frac{x}{\ell} g_1(t), \tag{5.13}$$

in Eq. (5.1) obtained following new distributed parameter system is

$$w_t - w_{xx} = \frac{(x-\ell)}{\ell} g_{0t} - \frac{x}{\ell} g_{1t} + p(t)\big(w + \frac{(\ell - x)}{\ell} g_0 + \frac{x}{\ell} g_1\big) + \phi(x,t), \tag{5.14}$$

subject to

$$\begin{aligned} &w(x,0) = \varphi(x) - \frac{(\ell - x)}{\ell} g_0(0) - \frac{x}{\ell} g_1(0), \\ &w(0,t) = 0, \quad w(\ell,t) = 0. \end{aligned} \tag{5.15}$$

Theorem 5.1 *Any $w(x,t) \in H((0,\ell))$ has a unique representation [84]*

$$w(x,t) = \sum_{n=1}^{\infty} \psi_n(x) z_n(t), \tag{5.16}$$

where $\{\psi_n(x)\}_{n=1}^{\infty}$ is a complete orthonormal basis in $H((0,\ell))$ and $z_n(t)$ is the temporal term.

Having a modal space expansion yields an infinite-dimensional system theoretically which makes the problem physically insurmountable since there will be a large number of modes to control. Hence, a truncated Fourier series expansion of Eq. (5.16) is taken in the computations hereafter

$$w(x,t) \approx \sum_{n=1}^{N} \psi_n(x) z_n(t). \tag{5.17}$$

Let V denote a complete orthonormal basis defined as follows:

$$V = \{v | v, \frac{\partial v}{\partial x} \in H((0,\ell)) \quad \text{and} \quad v|_{\partial(0,\ell)} = 0\}. \tag{5.18}$$

Multiplying both sides of Eq. (5.14) with a basis function v, and using integrating by parts, the solution $w(x,t)$ of the system satisfies

$$\int_0^\ell \frac{\partial w}{\partial t} v dx - \int_0^\ell \frac{\partial^2 w}{\partial x^2} v dx = \int_0^\ell (\frac{x-\ell}{\ell} g_{0t} - \frac{x}{\ell} g_{1t}) v dx + \int_0^\ell p(w + \frac{\ell - x}{\ell} g_0 + \frac{x}{\ell} g_1) v dx$$
$$+ \int_0^\ell \phi(x,t) v dx$$

where $w, v \in V$. If Eq. (5.17) for $w(x,t)$ is used in Eq. (5.14) and $v = \psi_m$, $m = 1,2,...$ the finite dimensional system is got ($D \triangleq \frac{\partial}{\partial x}$):

$$\frac{dz}{dt} = Kz + pz + Gp + L \tag{5.19}$$

where $z(t) = (z_1(t), z_2(t), ..., z_n(t))^T \in \mathbb{R}^N$; M and K are matrices from $\mathbb{R}^{N \times N}$; G and L are vectors from $\mathbb{R}^N$; and $p = p(t)$ is the control function. The vector $z(t)$ is the finite dimensional approximation to the temporal term. The initial values are determined by

$$z_n(0) = (w(x,0), \psi_m), \;\; m,n = 1,2,...,N. \tag{5.20}$$

In Eq. (5.19), the following notations are used:

$$M_{mn} = (\psi_m, \psi_n) = \int_0^\ell \psi_m(x) \psi_n(x) dx = \delta_{mn}, \tag{5.21}$$

$$\delta_{mn} = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{otherwise.} \end{cases} \tag{5.22}$$

$$K_{mn} = (\psi_m, D^2 \psi_n) = \int_0^\ell \psi_m(x) \frac{\partial^2 (\psi_n(x))}{\partial x^2} dx, \tag{5.23}$$

$$F_m = \int_0^\ell (\frac{(x-\ell)}{\ell} g_{0t} - \frac{x}{\ell} g_{1t}) \psi_m(x) dx, \tag{5.24}$$

$$G_m = \int_0^\ell (\frac{(\ell-x)}{\ell} g_0 - \frac{x}{\ell} g_1) \psi_m(x) dx, \tag{5.25}$$

$$H_m = \int_0^\ell \phi(x,t) \psi_m(x) dx, \tag{5.26}$$

$$L_m = F_m + H_m. \tag{5.27}$$

5.4 Derivation of the Pontryagin's Maximum Principle for the Bilinear System

Lemma 5.3 *If Q is a symmetric matrix and $F(x) = x^T Qx$, then $\frac{d}{dx}F(x) = 2x^T Q$.*

Theorem 5.2 *If the Pontryagin's maximum principle is applied to the bilinear system given by Eq's.* (5.19)-(5.20)*, a canonical optimality condition is obtained,*

$$\begin{cases} \dot{z}(t) = Kz(t) - s^{-1}\Lambda^T(t)(z(t)+G)^2 + L, \\ \dot{\Lambda}(t) = -Qz(t) - K^T\Lambda(t) - s^{-1}(z(t)+G)^T\Lambda^2(t), \\ \Lambda(t_f) = R_1 z(t_f) + R_2 \dot{z}(t_f), \\ z(t_0) = z_0 \end{cases} \tag{5.28}$$

which is a nonlinear two-point boundary value problem (TPBVP). In Eq. (5.28)*, Q, R_1 and R_2 are positive semidefinite symmetric $N \times N$ matrices, K, G and L are defined in* (5.23)*,* (5.25) *and* (5.27)*, respectively.*

Proof 6 *Consider the optimal control of the bilinear system Eqs.* (5.19)-(5.20)

$$\frac{dz}{dt} = Kz + pz + Gp + L,$$

where $z(t)$ is the finite dimensional approximation to $w(x,t)$ and $p(t)$ is the control func-

tion. The quadratic cost functional associated with Eqs. (5.19)-(5.20) *is given by*

$$\min_{p(t)} \mathscr{J} = \frac{1}{2}[z(t_f)]^T R_1[z(t_f)] + \frac{1}{2}[\dot{z}(t_f)]^T R_2[\dot{z}(t_f)] + \frac{1}{2}\int_{t_0}^{t_f} z(t)^T Qz(t)dt + \frac{1}{2}\int_{t_0}^{t_f} sp^2(t)dt \tag{5.29}$$

where R_1*,* R_2 *and* Q *are* $M \times N$ *matrices in which* $R_{1mn} = \int_0^\ell r_1 \psi_m(x)\psi_n(x)dx$*,* $R_{2mn} = \int_0^\ell r_2 \psi_m(x)\psi_n(x)dx$ *and* $Q_{mn} = \int_0^\ell q\psi_m(x)\psi_m(x)dx$ *respectively, for* $m,n = 1,...,N$*.*

The cost functional (5.29) *is minimized by using augmented cost functional*

$$\mathscr{J}^*[z,p,\Lambda] = \int_{t_0}^{t_f} \{\frac{1}{2}(z^T Qz + sp^2) - \Lambda^T(t)(\dot{z}(t) - Kz(t) - p(t)z(t) - Gp - L)\}dt + \frac{1}{2}[z(t_f)]^T R_1[z(t_f)] + \frac{1}{2}[\dot{z}(t_f)]^T R_2[\dot{z}(t_f)]. \tag{5.30}$$

Introducing the so-called Hamiltonian, the augmented functional becomes

$$\mathscr{J}^*[z,p,\Lambda] = \int_{t_0}^{t_f} [\mathscr{H}(t,z,p,\Lambda) - \Lambda^T \dot{z}(t)]dt + \frac{1}{2}[z(t_f)]^T R_1[z(t_f)] + \frac{1}{2}[\dot{z}(t_f)]^T R_2[\dot{z}(t_f)]. \tag{5.31}$$

If (z,p,Λ) *is a minimizer of* $\mathscr{J}^*$*, then the following equality holds*

$$\delta\mathscr{J}^* = \int_{t_0}^{t_f} [\frac{\partial\mathscr{H}}{\partial z}\delta z + \frac{\partial\mathscr{H}}{\partial p}\delta p + \frac{\partial\mathscr{H}}{\partial\Lambda}\delta\Lambda - \delta(\Lambda^T(t)\dot{z}(t))]dt + \delta(\frac{1}{2}[z(t_f)]^T R_1[z(t_f)] + \frac{1}{2}[\dot{z}(t_f)]^T R_2[\dot{z}(t_f)]) = 0. \tag{5.32}$$

After variational operations and using integration by parts, the following are obtained

$\mathscr{H}_p = 0,$

$p(t) = -s^{-1}\Lambda^T(t)(z+G).$

Secondly,

$\mathscr{H}_z + \dot{\Lambda}^T = 0,$

$\dot{\Lambda}(t) = -Qz(t) - K^T\Lambda(t) - p^T\Lambda(t).$

Thirdly,

$\mathscr{H}_\Lambda - \dot{z}(t) = 0,$

$\dot{z}(t) = Kz(t) - s^{-1}\Lambda^T(t)(z(t)+G)^2 + L.$

And finally,

$[z(t_f)]^T R_1 + [\dot{z}(t_f)]^T R_2 - \Lambda^T(t_f) = 0,$

$R_1 z(t_f) + R_2 \dot{z}(t_f) = \Lambda(t_f).$

In this proof, Pontryagin's maximum principle leads to a nonlinear two-point boundary-value problem that cannot be solved analytically to obtain the control function. The difficulty of solving this problem is caused by the combination of split boundary values and nonlinear differential equations.

In the following part, two iterative numerical techniques are discussed for determining optimal controls and optimal trajectories. The steepest descent and the quasilinearization methods are presented and the properties of solutions are illustrated on an example.

5.5 Algorithm Based on the Steepest Descent Method

In this part, the steepest descent method is introduced for solving nonlinear two-point boundary-value problems in the context of optimal control theory [2].

Suppose an initial guess for the control function $p^{(i)}(t)$ is known and is used to solve differential equations

$$\begin{aligned}\dot{u}^{(i)}(t) &= a(t, u^{(i)}(t), p^{(i)}(t)),\\ \dot{\Lambda}^{(i)}(t) &= -\frac{\partial \mathscr{H}}{\partial u}(t, u^{(i)}(t), p^{(i)}(t), \Lambda^{(i)}(t))\end{aligned} \tag{5.33}$$

so that the initial state-costate function $u^{(i)}$, $\Lambda^{(i)}$ satisfies the boundary conditions

$$\begin{aligned}u^{(i)}(t_0) &= u_0,\\ \Lambda^{(i)}(t_f) &= \frac{\partial h}{\partial u}(u^{(i)}(t_f)).\end{aligned} \tag{5.34}$$

If the initial control guess also satisfies

$$\frac{\partial \mathscr{H}}{\partial p}(t, u^{(i)}(t), p^{(i)}(t), \Lambda^{(i)}(t)) = 0, \tag{5.35}$$

then $p^{(i)}(t)$, $u^{(i)}(t)$, and $\Lambda^{(i)}(t)$ are extremal. The superscript (i) corresponds to the iteration counter.

Assume that Eq. (5.35) is not satisfied, the variation of the augmented functional $\mathscr{J}^*$ on the initial state-costate-control is

$$\delta \mathscr{J}^* = [\frac{\partial h}{\partial u}(u^{(i)}(t_f)) - \Lambda^{(i)}(t_f)]^T \delta u(t_f) + \int_{t_0}^{t_f} \{[\dot{\Lambda}^{(i)}(t) + \frac{\partial \mathscr{H}}{\partial u}(t, u^{(i)}(t), p^{(i)}(t), \Lambda^{(i)}(t))]^T \delta u(t)$$
$$+ [\frac{\partial \mathscr{H}}{\partial p}(t, u^{(i)}(t), p^{(i)}(t), \Lambda^{(i)}(t))]^T \delta p(t) + [a(t, u^{(i)}(t), p^{(i)}(t)) - \dot{u}^{(i)}(t)]^T \delta\Lambda(t)\} dt$$

where $\delta u(t) \triangleq u^{(i+1)}(t) - u^{(i)}(t)$, $\delta p(t) \triangleq p^{(i+1)}(t) - p^{(i)}(t)$ and $\delta\Lambda(t) \triangleq \Lambda^{(i+1)}(t) - \Lambda^{(i)}(t)$.

If Eqs. (5.33)-(5.34) are satisfied, then

$$\delta \mathscr{J}^* = \int_{t_0}^{t_f} [\frac{\partial \mathscr{H}}{\partial p}(u^{(i)}(t), p^{(i)}(t), \Lambda^{(i)}(t), t)]^T \delta p(t) dt. \tag{5.36}$$

$\delta \mathscr{J}^*$ is the linear part of the increment $\Delta \mathscr{J}^* \triangleq \mathscr{J}^*(p^{(i+1)}) - \mathscr{J}^*(p^{(i)})$ as seen from Eq. (2.11). If the norm of δp is small, the sign of $\Delta \mathscr{J}^*$ is determined by the sign of $\delta \mathscr{J}^*$. $\Delta \mathscr{J}^*$ must be negative as $\mathscr{J}^*$ is to be minimized. If the change in p is selected as following

$$\delta p(t) = p^{(i+1)}(t) - p^{(i)}(t) = -\tau \frac{\partial \mathscr{H}^{(i)}}{\partial p}(t), \tag{5.37}$$

with $\tau > 0$, then

$$\delta \mathscr{J}^* = -\tau \int_{t_0}^{t_f} [\frac{\partial \mathscr{H}^{(i)}}{\partial p}(t)]^T [\frac{\partial \mathscr{H}^{(i)}}{\partial p}(t)] dt \leq 0, \tag{5.38}$$

because the integrand is nonnegative for all $t \in [t_0, t_f]$. The equality occurs if and only if

$$\frac{\partial \mathscr{H}^{(i)}}{\partial p}(t) = 0,$$

for all $t \in [t_0, t_f]$.

Choosing δp in this way makes certain that each value of the performance measure will be at least as small as the preceding value. After all, when $\mathscr{J}$ reaches a minimum the vector $\frac{\partial \mathscr{H}}{\partial p}$ will be zero throughout the time interval $[t_0, t_f]$. It should be noted that a suitable value for the step size τ must also be selected.

Case Study

Let us consider the following control problem

$$u_t = u_{xx} + p(t)u + \phi(x,t) \tag{5.39}$$

subject to initial and boundary conditions

$$u(x,0) = \sin(\pi x),$$
$$u(0,t) = 0, \quad u(\ell,t) = 0. \tag{5.40}$$

A bilinear control is implemented to a uniform rod of length ℓ with non-uniform temperature distribution on the x-axis from $x = 0$ to $x = \ell$. In order to perform the simulation of the system composed by the nonlinear TPBVP and the proposed bilinear control, a finite dimensional approximation $z(t)$ of $w(x,t)$ based on modal space expansion method is used. The state equation can be expanded by a series of orthonormal basis functions $w(x,t) \approx \sum_{n=1}^{N} \psi_n(x) z_n(t)$ where $\psi_n(x) = \sqrt{2}\sin(n\pi x)$, which satisfy the homogeneous boundary conditions. A finite dimensional system is obtained

$$\dot{z}(t) = Kz(t) + p(t)z(t) + Gp(t) + L \tag{5.41}$$

with initial condition $z(0) = \sqrt{2}/2$. In (5.41), K, G and L are defined in (5.23), (5.25) and (5.27), respectively. The performance index to be minimized is

$$\mathscr{J} = \frac{1}{2}[z(t_f)]^T R_1[z(t_f)] + \frac{1}{2}[\dot{z}(t_f)]^T R_2[\dot{z}(t_f)] + \frac{1}{2}\int_{t_0}^{t_f} z^T Qz dt + \frac{1}{2}\int_{t_0}^{t_f} sp^2(t) dt, \tag{5.42}$$

where the elements of R_1, R_2 and Q are weighting factors that defined by $R_{1mn} = \int_0^1 r_1 \sin(m\pi x)\sin(n\pi x)dx$, $R_{2mn} = \int_0^1 r_2 \sin(m\pi x)\sin(n\pi x)dx$ and $Q_{mn} = \int_0^1 q \sin(m\pi x)\sin(n\pi x)dx$ respectively, for $m,n = 1,...,N$. The system (5.1) - (5.2) is simulated over $t_0 = 0 \le t \le t_f = 2$ with following parameters: $\phi(x,t) = 0$, $n = m = 1$, $\ell = 1$, $\psi_n(x) = \sqrt{2}\sin(n\pi x)$, $\psi_m(x) = \sqrt{2}\sin(m\pi x)$, $s = 0.01$, $r_1 = 1$, $r_2 = 1$ and $q = 1$.

5.6 Simulation Study- the Steepest Descent Method

Let us solve the case study given by (5.39) by using the steepest descent method. The costate equation is determined from the Hamiltonian,

$$\mathscr{H}(t,z,p,\Lambda) = \frac{1}{2}(z(t)^T Qz(t) + sp^2(t)) - \Lambda^T(t)(-Kz(t) - p(t)z(t) - Gp(t) - L)$$

as

$$\dot{\Lambda}^T = -\frac{\partial \mathscr{H}}{\partial z} = -Qz(t) - K^T\Lambda(t) - s^{-1}(z(t)+G)^T\Lambda^2(t).$$

The algebraic relation that must be satisfied is

$$\frac{\partial \mathscr{H}}{\partial P} = sp(t) + \Lambda^T(t)[z(t) + G] = 0.$$

The boundary condition at $t = t_f = 2$ is $\Lambda(2) = R_1 z(2) + R_2 \dot{z}(2)$. The method is implemented using computer codes produced in MATLAB© environment for the solution of the system. Numerical solution is carried out using Runge-Kutta method and an integration interval of 0.01 unit. The following norm is used

$$\|\frac{\partial \mathscr{H}}{\partial p}\|^2 = \int_0^2 [\frac{\partial \mathscr{H}}{\partial p}(t)]^2 dt,$$

the iterative procedure is stopped when $\|\frac{\partial \mathscr{H}}{\partial p}\| \leq 10^{-3}$. With $p^0(t) = 1.0$, $t \in [0,2]$ and an initial $\tau = 0.4$, the value of the performance index as a function of the number of iterations is shown in Figure 5.1. The performance index has declined significantly in the first 30 iterations, however, the rest of the iterations yield very slight improvement. This type of progress is typical of the steepest descent method. Spatial temperature profile of the uncontrolled and controlled of the bilinear system at final time $t_f = 2$ is shown in Figure 5.2. The comparison of temperature profiles in time for fixed value of $x_0 = 0.5$ for uncontrolled and controlled of the bilinear system is shown in Figure 5.3. To illustrate the effects of parameters, the results are summarized in Table 5.1.

Table 5.1 The steepest descent solutions for different values of initial step sizes and initial controls.

Initial control	τ	Iteration Number	Minimum value of $\mathscr{J}$
1.0	0.4	42	0.0496
1.0	1	24	0.0497
0.5	0.4	56	0.0501
0.5	1	21	0.0495

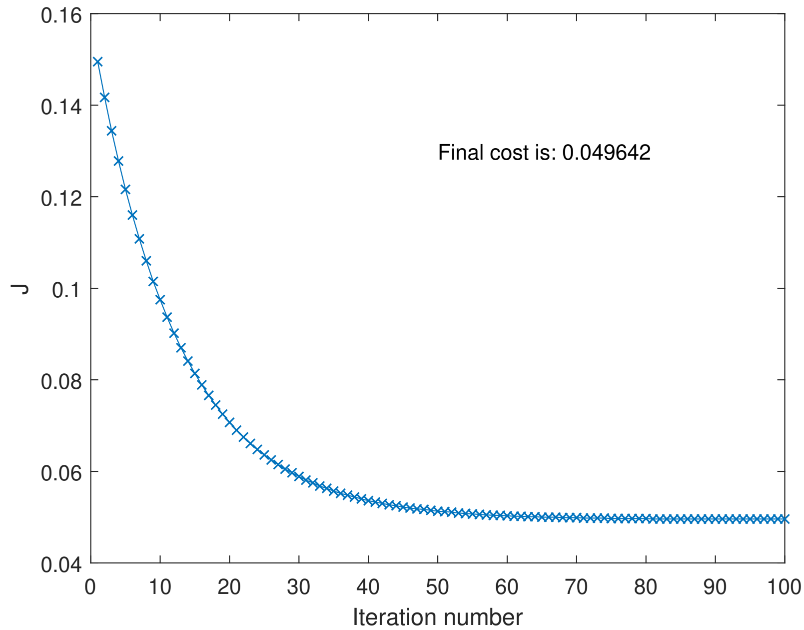

Figure 5.1 Performance index reduction using the steepest descent method.

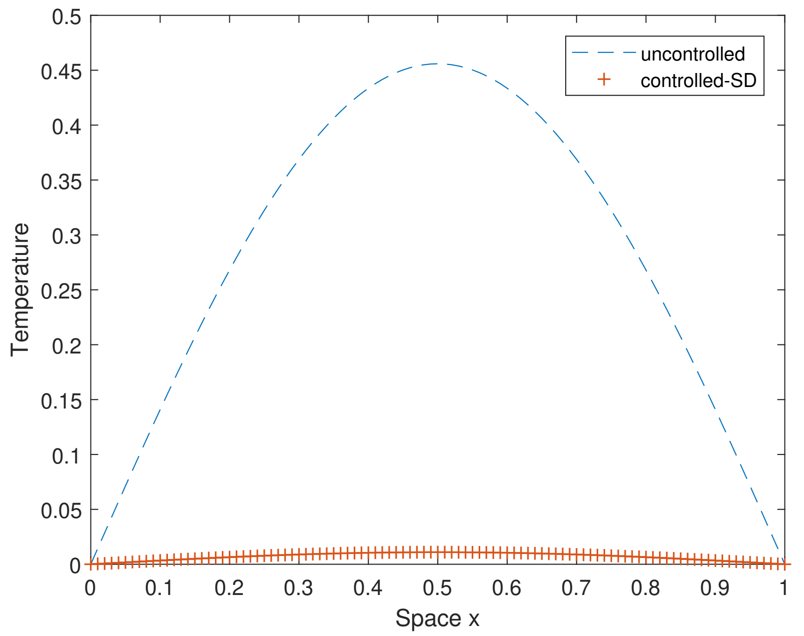

Figure 5.2 Comparison of the spatial temperature profiles for uncontrolled and controlled case using the steepest descent method at $t_f = 2$.

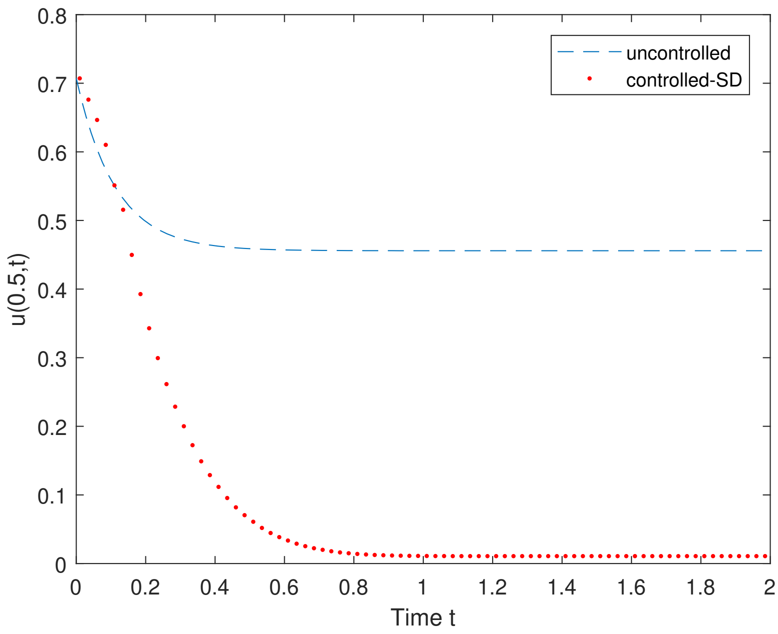

Figure 5.3 Comparison of the temperature profiles in time at $x_0 = 0.5$ for uncontrolled and controlled case using the steepest descent method.

5.7 Algorithm Based on the Quasilinearization Method

In this part, the quasilinearization method in [2] is introduced to solve the nonlinear two-point boundary value problems.

5.7.1 Solution of Linear Two-Point Boundary-Value Problems

Consider the two first-order, linear differential equations

$$\begin{aligned} \dot{u}(t) &= a_{11}(t)u(t) + a_{12}(t)\Lambda(t) + e_1(t), \\ \dot{\Lambda}(t) &= a_{21}(t)u(t) + a_{22}(t)\Lambda(t) + e_2(t) \end{aligned} \tag{5.43}$$

with boundary conditions $u(t_0) = u_0$ and $\Lambda(t_f) = \Lambda_f$.

In Eq. (5.43), a_{11}, a_{12}, a_{21}, a_{22}, e_1 and e_2 are known functions in time. t_0, t_f, u_0 and Λ_f are known constants. It is desired to find solutions for $u(t)$ and $\Lambda(t)$ over $t \in [t_0, t_f]$ that satisfy the split boundary conditions.

First, assume that the solution, $u^H(t)$ and $\Lambda^H(t)$, was formed by numerical integration to the homogeneous differential equations

$$\begin{aligned}\dot{u}(t) &= a_{11}(t)u(t) + a_{12}(t)\Lambda(t)\\ \dot{\Lambda}(t) &= a_{21}(t)u(t) + a_{22}(t)\Lambda(t)\end{aligned} \tag{5.44}$$

with arbitrary assumed initial conditions. Let $u^H(t_0) = 0$ and $\Lambda^H(t_0) = 1$ be as a suitable choice. Afterward, a particular solution, $u^p(t)$, $\Lambda^p(t)$ is determined using $u^p(t_0) = u_0$ and $\Lambda^p(t_0) = 0$ to the non-homogeneous equations (5.43) by numerical integration. Since the differential equations are linear, the principle of superposition applies, and

$$\begin{aligned}u(t) &= c_1 u^H(t) + u^p(t),\\ \Lambda(t) &= c_1 \Lambda^H(t) + \Lambda^p(t)\end{aligned} \tag{5.45}$$

is a solution of Eqs. (5.43) for any value of the constant c_1. It is desired to find the solution that satisfies the specified initial conditions.
This can be achieved by examining that

$$\Lambda(t_f) = \Lambda_f = c_1 \Lambda^H(t_f) + \Lambda^p(t_f). \tag{5.46}$$

If Eq. (5.46) is solved for c_1,

$$c_1 = \frac{\Lambda_f - \Lambda^p(t_f)}{\Lambda^H(t_f)} \tag{5.47}$$

is obtained. The required solution of Eq. (5.43) is given by Eq. (5.45) with this value of c_1. The initial condition $u(t_0) = u_0$ is satisfied for any choice of c_1 because of $u^H(t_0) = 0$ and $u^p(t_0) = u_0$.
The solution of the linear two-point boundary-value problem is obtained in terms of the solution of a linear algebraic solution with the help of superposition principle.

5.7.2 Linearization of the State-Costate Equations

Consider the following nonlinear state and costate differential equations

$$\begin{aligned}\dot{u}(t) &= a(t, u(t), \Lambda(t)),\\ \dot{\Lambda}(t) &= d(t, u(t), \Lambda(t))\end{aligned} \tag{5.48}$$

where a and d are nonlinear functions of $u(t)$ and $\Lambda(t)$. Let $u^{(0)}(t)$, $\Lambda^{(0)}(t)$ be a known trajectory and $u^{(1)}(t)$, $\Lambda^{(1)}(t)$ be any other trajectory. By using a Taylor series expansion of the differential equations (5.48) about $u^{(0)}(t)$, $\Lambda^{(0)}(t)$ and using only terms of up to first order gets

$$\begin{aligned}
\dot{u}^{(1)}(t) &\approx \dot{u}^{(0)}(t) + [\frac{\partial a}{\partial u}(t,u^{(0)}(t),\Lambda^{(0)}(t))][u^{(1)}(t)-u^{(0)}(t)] \\
&\quad + [\frac{\partial a}{\partial \Lambda}(t,u^{(0)}(t),\Lambda^{(0)}(t))][\Lambda^{(1)}(t)-\Lambda^{(0)}(t)] \\
\dot{\Lambda}^{(1)}(t) &\approx \dot{\Lambda}^{(0)}(t) + [\frac{\partial d}{\partial u}(t,u^{(0)}(t),\Lambda^{(0)}(t))][u^{(1)}(t)-u^{(0)}(t)] \\
&\quad + [\frac{\partial d}{\partial \Lambda}(t,u^{(0)}(t),\Lambda^{(0)}(t))][\Lambda^{(1)}(t)-\Lambda^{(0)}(t)]
\end{aligned} \tag{5.49}$$

and substituting $a(u^{(0)}(t),\Lambda^{(0)}(t),t)$ for $\dot{u}^{(0)}(t)$ and $d(u^{(0)}(t),\Lambda^{(0)}(t),t)$ for $\dot{\Lambda}^{(0)}(t)$

$$\begin{aligned}
\dot{u}^{(1)}(t) &\approx a(t,u^{(0)}(t),\Lambda^{(0)}(t)) + [\frac{\partial a}{\partial u}(t,u^{(0)}(t),\Lambda^{(0)}(t))][u^{(1)}(t)-u^{(0)}(t)] \\
&\quad + [\frac{\partial a}{\partial \Lambda}(t,u^{(0)}(t),\Lambda^{(0)}(t))][\Lambda^{(1)}(t)-\Lambda^{(0)}(t)] \\
\dot{\Lambda}^{(1)}(t) &\approx d(t,u^{(0)}(t),\Lambda^{(0)}(t)) + [\frac{\partial d}{\partial u}(t,u^{(0)}(t),\Lambda^{(0)}(t))][u^{(1)}(t)-u^{(0)}(t)] \\
&\quad + [\frac{\partial d}{\partial \Lambda}(t,u^{(0)}(t),\Lambda^{(0)}(t))][\Lambda^{(1)}(t)-\Lambda^{(0)}(t)].
\end{aligned} \tag{5.50}$$

Using the fact that $u^{(0)}$ and $\Lambda^{(0)}$ are known functions of time, Eqn (5.50) are re-written as

$$\begin{aligned}
\dot{u}^{(1)}(t) &= a_{11}(t)u^{(1)}(t) + a_{12}(t)\Lambda^{(1)}(t) + e_1(t) \\
\dot{\Lambda}^{(1)}(t) &= a_{21}(t)u^{(1)}(t) + a_{22}(t)\Lambda^{(1)}(t) + e_2(t)
\end{aligned} \tag{5.51}$$

where

$$\begin{aligned}
a_{11}(t) &\triangleq \frac{\partial a}{\partial u}(t,u^{(0)}(t),\Lambda^{(0)}(t)), \\
a_{12}(t) &\triangleq \frac{\partial a}{\partial \Lambda}(t,u^{(0)}(t),\Lambda^{(0)}(t)), \\
a_{21}(t) &\triangleq \frac{\partial d}{\partial u}(t,u^{(0)}(t),\Lambda^{(0)}(t)), \\
a_{22}(t) &\triangleq \frac{\partial d}{\partial \Lambda}(t,u^{(0)}(t),\Lambda^{(0)}(t)), \\
e_1(t) &\triangleq a(t,u^{(0)}(t),\Lambda^{(0)}(t)) - [\frac{\partial a}{\partial u}(t,u^{(0)}(t),\Lambda^{(0)}(t))]u^{(0)}(t) \\
&\quad -[\frac{\partial a}{\partial \Lambda}(t,u^{(0)}(t),\Lambda^{(0)}(t))]\Lambda^{(0)}(t)
\end{aligned}$$

and

$$\begin{aligned}
e_2(t) &\triangleq d(t,u^{(0)}(t),\Lambda^{(0)}(t)) - [\frac{\partial d}{\partial u}(t,u^{(0)}(t),\Lambda^{(0)}(t))]u^{(0)}(t) \\
&\quad -[\frac{\partial d}{\partial \Lambda}(t,u^{(0)}(t),\Lambda^{(0)}(t))]\Lambda^{(0)}(t)
\end{aligned}$$

are all known functions of time.

A set of ordinary, linear, time-varying and nonhomogeneous differential equations are obtained by expanding the differential equations given by (5.48) about a trajectory $u^{(0)}$, $\Lambda^{(0)}$. These linear differential equations are solved by using the procedure introduced in Section 5.7.1.

If the method of quasilinearization for solving nonlinear two-point boundary-value problems are used , the following two steps are performed:

1. Linearize nonlinear differential equations,

2. Solve linear two-point boundary-value problems.

Let us demonstrate how these two steps go together with one iteration of the quasilinearization algorithm. The generalization of these steps to systems of $2n$ differential equations leads to similar equations but with matrices replacing scalar quantities.

Suppose that $\frac{\partial \mathscr{H}}{\partial p} = 0$ has been solved for $p(t)$ and substituted in the state and costate equations to obtain the reduced differential equations

$$\begin{aligned} \dot{u}(t) &= a(t, u(t), \Lambda(t)), \\ \dot{\Lambda}(t) &= -\frac{\partial \mathscr{H}}{\partial u}(t, u(t), \Lambda(t)). \end{aligned} \tag{5.52}$$

The boundary conditions are $u(t_0) = u_0$ and $\Lambda(t_f) = \Lambda_f$, in which Λ_f is an $n \times 1$ matrix of constants, t_f is specified, and $u(t_f)$ is free.

The first step is to linearize the reduced differential equations. The differential equations are expanded in a Taylor series about a known trajectory $u^{(i)}(t)$, $\Lambda^{(i)}(t)$, $t \in [t_0, t_f]$ to do this. Then, the linearized reduced differential equations are as follows:

$$\begin{aligned} \dot{u}^{(i+1)}(t) \approx\ & a(t, u^{(i)}(t), \Lambda^{(i)}(t)) + [\frac{\partial a}{\partial u}(t, u^{(i)}(t), \Lambda^{(i)}(t))][u^{(i+1)}(t) - u^{(i)}(t)] \\ & + [\frac{\partial a}{\partial \Lambda}(t, u^{(i)}(t), \Lambda^{(i)}(t))][\Lambda^{(i+1)}(t) - \Lambda^{(i)}(t)] \\ \dot{\Lambda}^{(i+1)}(t) \approx & -\frac{\partial \mathscr{H}}{\partial u}(t, u^{(i)}(t), \Lambda^{(i)}(t)) - [\frac{\partial^2 \mathscr{H}}{\partial u^2}(t, u^{(i)}(t), \Lambda^{(i)}(t))][u^{(i+1)}(t) - u^{(i)}(t)] \\ & - [\frac{\partial^2 \mathscr{H}}{\partial u \partial \Lambda}(t, u^{(i)}(t), \Lambda^{(i)}(t))][\Lambda^{(i+1)}(t) - \Lambda^{(i)}(t)], \end{aligned} \tag{5.53}$$

where the jk^{th} elements of the indicated matrices are

$[\frac{\partial a}{\partial u}]_{jk} = \frac{\partial a_j}{\partial u_k}$, $[\frac{\partial a}{\partial \Lambda}]_{jk} = \frac{\partial a_j}{\partial \Lambda_k}$, $[\frac{\partial^2 \mathscr{H}}{\partial u^2}]_{jk} = \frac{\partial^2 \mathscr{H}}{\partial u_j \partial u_k}$ and $[\frac{\partial^2 \mathscr{H}}{\partial u \partial \Lambda}]_{jk} = \frac{\partial^2 \mathscr{H}}{\partial u_j \partial \Lambda_k}$.
The differential equations (5.53) are written

$$\begin{aligned} \dot{u}^{(i+1)}(t) &= A_{11}(t)u^{(i+1)}(t) + A_{12}(t)\Lambda^{(i+1)}(t) + e_1(t), \\ \dot{\Lambda}^{(i+1)}(t) &= A_{21}(t)u^{(i+1)}(t) + A_{22}(t)\Lambda^{(i+1)}(t) + e_2(t) \end{aligned} \tag{5.54}$$

or, in partitioned matrix form,

$$\begin{aligned} \begin{bmatrix} \dot{u}^{(i+1)}(t) \\ \dot{\Lambda}^{(i+1)}(t) \end{bmatrix} &= \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{bmatrix} u^{(i+1)}(t) \\ \Lambda^{(i+1)}(t) \end{bmatrix} + \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} \\ &\triangleq A(t) \begin{bmatrix} u^{(i+1)}(t) \\ \Lambda^{(i+1)}(t) \end{bmatrix} + \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} \end{aligned} \tag{5.55}$$

where the matrices
$A_{11}(t) \triangleq \frac{\partial a}{\partial u}$, $A_{12}(t) \triangleq \frac{\partial a}{\partial \Lambda}$, $A_{21}(t) \triangleq -\frac{\partial^2 \mathscr{H}}{\partial u^2}$,
$A_{22}(t) \triangleq -\frac{\partial^2 \mathscr{H}}{\partial u \partial \Lambda}$, $e_1(t) \triangleq -A_{11}(t)u(t) - A_{12}(t)\Lambda(t) + a$ and
$e_2(t) \triangleq -A_{21}(t)u(t) - A_{22}(t)\Lambda(t) - \frac{\partial \mathscr{H}}{\partial u}$ are evaluated at $u^{(i)}(t)$, $\Lambda^{(i)}(t)$. Therefore, they are known functions of time.

Initial guesses, $u^{(0)}(t)$ and $\Lambda^{(0)}(t)$, are used to evaluate $u(t)$ and $\Lambda(t)$ at the beginning of the first iteration. The next step is to generate solutions for $2n$ homogeneous differential equations

$$\begin{aligned} \dot{u}^{(i+1)}(t) &= A_{11}(t)u^{(i+1)}(t) + A_{12}(t)\Lambda^{(i+1)}(t), \\ \dot{\Lambda}^{(i+1)}(t) &= A_{21}(t)u^{(i+1)}(t) + A_{22}(t)\Lambda^{(i+1)}(t) \end{aligned} \tag{5.56}$$

by numerical integration. These solutions will be denoted by $u^{H_1}, \Lambda^{H_1}; u^{H_2}, \Lambda^{H_2}; \ldots; u^{H_n}, \Lambda^{Hn}$.
The following boundary conditions are chosen to generate the solutions,

$$\begin{aligned} u^{H_1}(t_0) &= 0, \qquad \Lambda^{H_1}(t_0) = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T, \\ u^{H_2}(t_0) &= 0, \qquad \Lambda^{H_2}(t_0) = \begin{bmatrix} 0 & 1 & \ldots & 0 \end{bmatrix}^T, \\ &\vdots \\ u^{H_n}(t_0) &= 0, \qquad \Lambda^{H_n}(t_0) = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \end{bmatrix}^T. \end{aligned} \tag{5.57}$$

Afterwards, one particular solution for u^p and Λ^p is computed by numerically integrating Eq. (5.55) over (t_0, t_f) and using the boundary conditions $u^p(t_0) = u_0$, $\Lambda^p(t_0) = 0$. By using the principle of superposition, the complete solution of Eq. (5.55) is given in the following form

$$\begin{aligned} u^{(i+1)}(t) &= c_1 u^{H_1}(t) + c_2 u^{H_2}(t) + \ldots + c_n u^{H_n}(t) + u^p(t) \\ \Lambda^{(i+1)}(t) &= c_1 \Lambda^{H_1}(t) + c_2 \Lambda^{H_2}(t) + \ldots + c_n \Lambda^{H_n}(t) + \Lambda^p(t) \end{aligned} \tag{5.58}$$

where the values of $c_1, c_2, \ldots, c_n$ are to be determined via $\Lambda^{(i+1)}(t_f) = \Lambda_f$. To find the appropriate values of the c's, $t = t_f$ and

$$\Lambda_f = [\Lambda^{H_1}(t_f) \vdots \Lambda^{H_2}(t_f) \vdots \ldots \vdots \Lambda^{H_n}(t_f)]c + \Lambda^p(t_f), \tag{5.59}$$

where only $c \triangleq \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}^T$ is unknown. Solving for c yields

$$c = [\Lambda^{H_1}(t_f) \vdots \Lambda^{H_2}(t_f) \vdots \ldots \vdots \Lambda^{H_n}(t_f)]^{-1} [\Lambda_f - \Lambda^p(t_f)]. \tag{5.60}$$

Substituting c given by Eq. (5.60) into Eq. (5.58) gives the $(i+1)^{th}$ trajectory. This completes one iteration of the quasilinearization algorithm. $(i+1)^{th}$ trajectory is used to begin another iteration.

If $t = t_0$ is used in Eq. (5.58) and substitute the boundary conditions (5.57), then

$$\begin{aligned} u^{(i+1)}(t_0) &= u^p(t_0) = u_0 \\ \Lambda^{(i+1)}(t_0) &= c. \end{aligned} \tag{5.61}$$

Consequently, the solution $u^{(i+1)}$, $\Lambda^{(i+1)}$ satisfies the initial condition $u^{(i+1)}(t_0) = u_0$ regardless of the value of c.

In deriving Eq. (5.60) from Eq. (5.58) it is assumed that the final costate $\Lambda(t_f)$ is specified constant.

5.8 Simulation Study- Quasilinearization Method

In this section, we solve the case study given by (5.39) by using the quasilinearization method. The reduced differential equations are

$$\begin{aligned} \dot{z}(t) &= Kz(t) - s^{-1}\Lambda^T(t)(z(t)+G)^2 + L, \\ \dot{\Lambda}(t) &= -Qz(t) - K^T\Lambda(t) - s^{-1}(z(t)+G)^T\Lambda^2(t). \end{aligned} \tag{5.62}$$

Linearizing these nonlinear differential equations using Eq. (5.55) leads

$$\begin{bmatrix} \dot{z}^{(i+1)}(t) \\ \dot{\Lambda}^{(i+1)}(t) \end{bmatrix} = A(t) \begin{bmatrix} z^{(i+1)}(t) \\ \Lambda^{(i+1)}(t) \end{bmatrix} + \begin{bmatrix} a(z^i(t), \Lambda^{(i)}(t), t) \\ -\frac{\partial \mathscr{H}}{\partial z}(z^i(t), \Lambda^{(i)}(t), t) \end{bmatrix} - A(t) \begin{bmatrix} z^{(i)}(t) \\ \Lambda^{(i)}(t) \end{bmatrix}. \tag{5.63}$$

$A(t)$ denotes 2×2 matrix

$$A(t) = \begin{bmatrix} K - 2s^{-1}\Lambda(t)z(t) - 2s^{-1}\Lambda(t)G & s^{-1}(z(t)+G)^2 \\ -Q - s^{-1}\Lambda^2(t) & -K - 2s^{-1}(z(t)+G)\Lambda(t) \end{bmatrix}. \tag{5.64}$$

To begin the iterative procedure, the initial state-costate guess

$$\begin{bmatrix} z(t) \\ \Lambda(t) \end{bmatrix} = 0$$

for $t \in [0,1]$ was selected. For this initial guess, the performance index has decreased significantly in the first 20 iterations and converged to a minimum cost of $\mathscr{J} = 0.033454$. Numerical solution is carried out using Runge-Kutta method and an integration interval of 0.01 unit. The norm used is

$$\left\| \begin{bmatrix} z^{(i+1)} \\ \Lambda^{(i+1)} \end{bmatrix} - \begin{bmatrix} z^{(i)} \\ \Lambda^{(i)} \end{bmatrix} \right\| = \sum_{j=1}^{n} \left\{ \max_t \mid z_j^{i+1}(t) - z_j^i(t) \mid + \max_t \mid \Lambda_j^{i+1}(t) - \Lambda_j^i(t) \mid \right\} \leq \gamma \tag{5.65}$$

and the iterative procedure is stopped when $\|\gamma\| \leq 10^{-3}$. The value of the performance index as a function of the number of iterations is shown in Figure 5.4. Performance index has decreased significantly in the first 20 iterations, but the rest of the iterations has made little improvement. Spatial temperature profile of the controlled/uncontrolled of the bilinear system at the final time $t_f = 2$ is shown in Figure 5.5. The temperature profiles in time for fixed value of $x_0 = 0.5$ for controlled/uncontrolled of the bilinear system is shown in Figure 5.6.

Let us introduce two functionals that define the dynamic responce of the system

$$\mathscr{J}_D(u) = \frac{1}{2}\int_{\Omega_x} [u^2(x,t_f) + u_t^2(x,t_f)]dx + \frac{1}{2}\int_{\Omega} u^2(x,t)dxdt, \tag{5.66}$$

and use control throughout the time interval Ω_t

$$\mathscr{J}_C(p) = \int_{\Omega_t} p^2(t)dt. \tag{5.67}$$

Table 5.2 shows that as s on the penalty function decreases, the dynamic response decreases corresponding to an increase in the control if the quasilinearization method is performed.

Table 5.2 The values of $\mathscr{J}_D(u)$ and $\mathscr{J}_C(p)$ for different values of s using the quasilinearization method.

s	$\mathscr{J}_D(u)$	$\mathscr{J}_C(p)$
10	24	$6e^{-5}$
10^0	1.6	$3e^{-4}$
10^{-1}	0.03324	0.0009
10^{-2}	0.02273	0.01072
10^{-3}	0.0031	0.1136

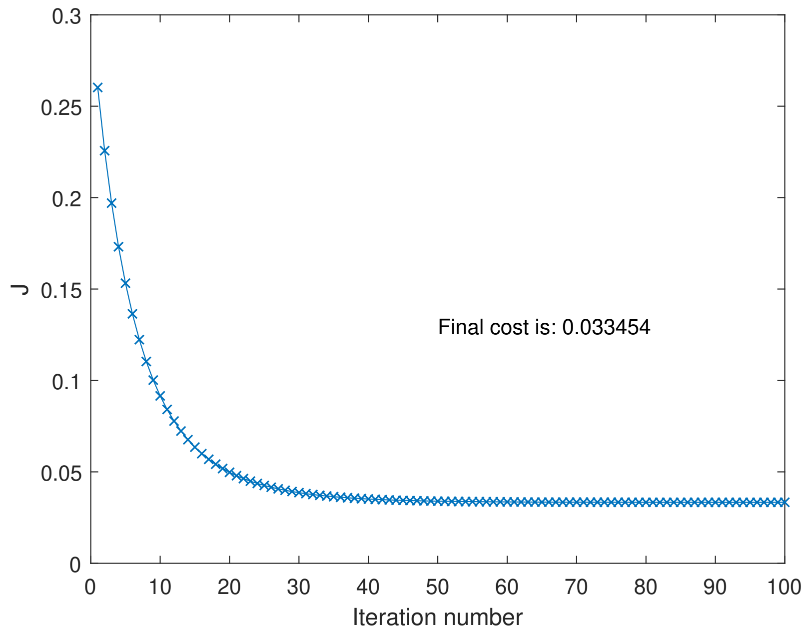

Figure 5.4 Performance index reduction using the quasilinearization method.

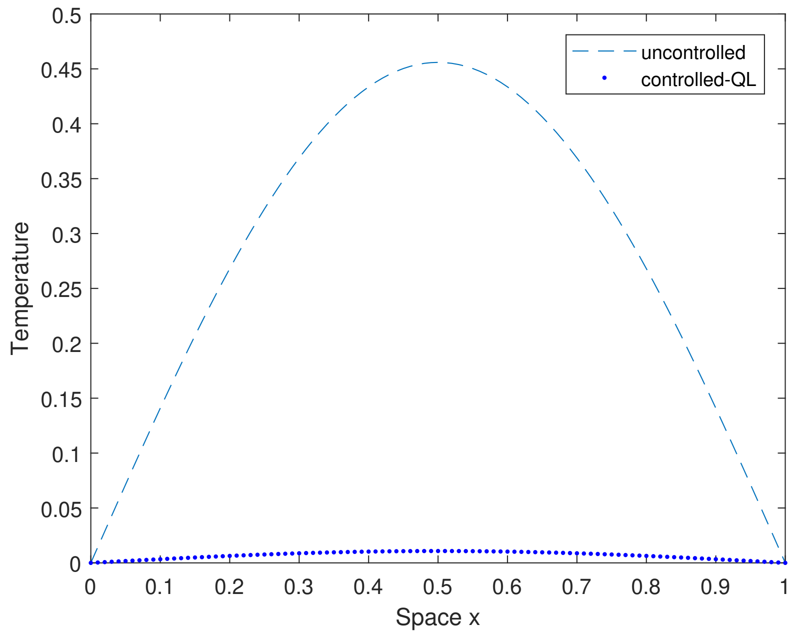

Figure 5.5 Comparison of the spatial temperature profiles for uncontrolled and controlled case using the quasilinearization method at $t_f = 2$.

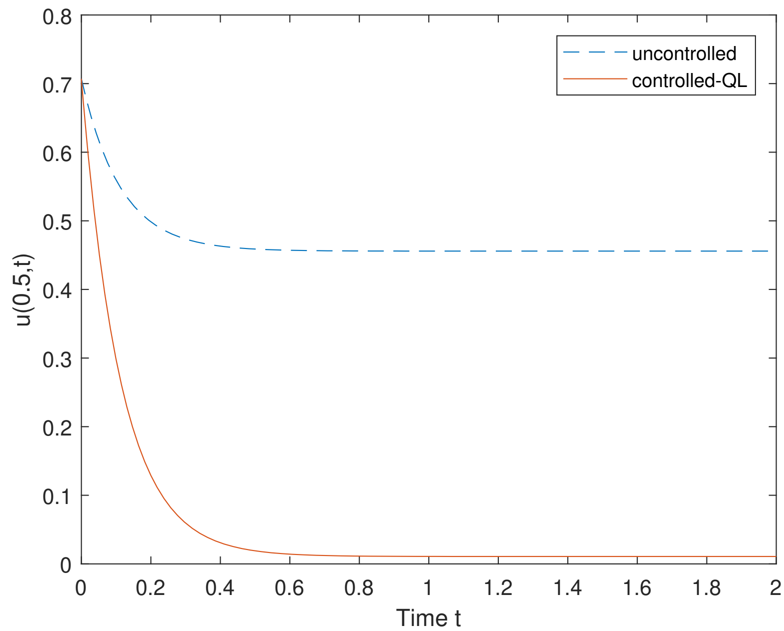

Figure 5.6 Comparison of the temperature profiles in time at $x_0 = 0.5$ for uncontrolled and controlled case using the quasilinearization method.

5.9 The Produced Simulations of the Steepest Descent and the Quasilinearization Algorithms

In this chapter, two iterative numerical methods, the steepest descent and the quasilinearization method for solving the nonlinear two point boundary value problem are considered. The steepest descent method is used as a beginning procedure and the quasilinearization method to confirm the solution. The solution of the steepest descent method is converged for two different initial guesses, while good initial guess for state and costate trajectories are required in the quasilinearization method. The steepest descent method converges slowly when compared to the quasilinearization method. However, poor initial guesses may result in divergence for the quasilinearization method. The performance index value obtained by using the steepest descent method is quite close with the results obtained using the quasilinearization method. Figures 5.8 and 5.9 are shown to present controlled and uncontrolled features of the solution. The temperature eventually reaches 0°C since there are no heat sources or sinks in the rod and the ends of the rod are held at 0°C. The solutions below in the figures confirm this statement. Since the bilinear control act as coolant flow rate, the controlled solutions reaches equilibrium (steady-state) temperature (0°C) solution faster than uncontrolled solutions. It is shown in Figure 5.9 that the center of the rod, the location of the maximum temperature, cools slower for uncontrolled model than controlled model at terminal time t_f .

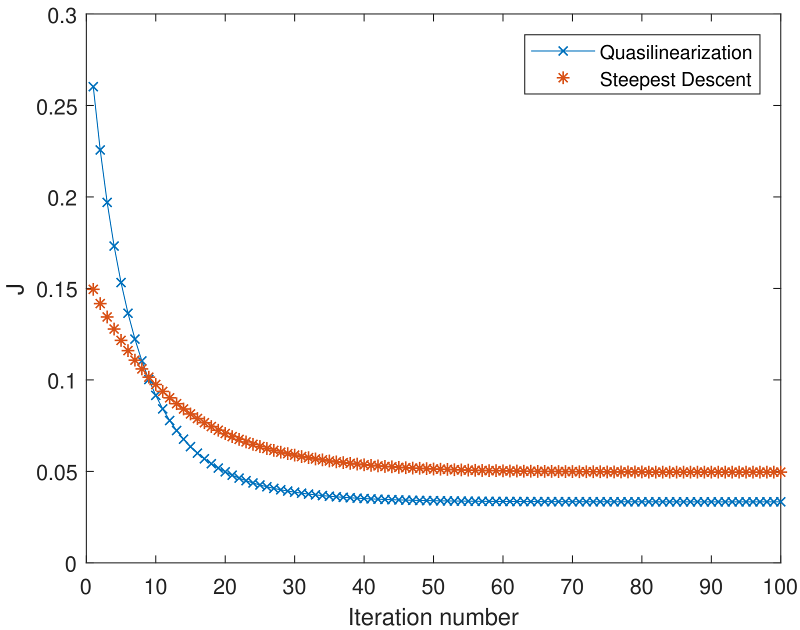

Figure 5.7 Comparison of performance index reduction with the steepest descent and the quasilinearization method.

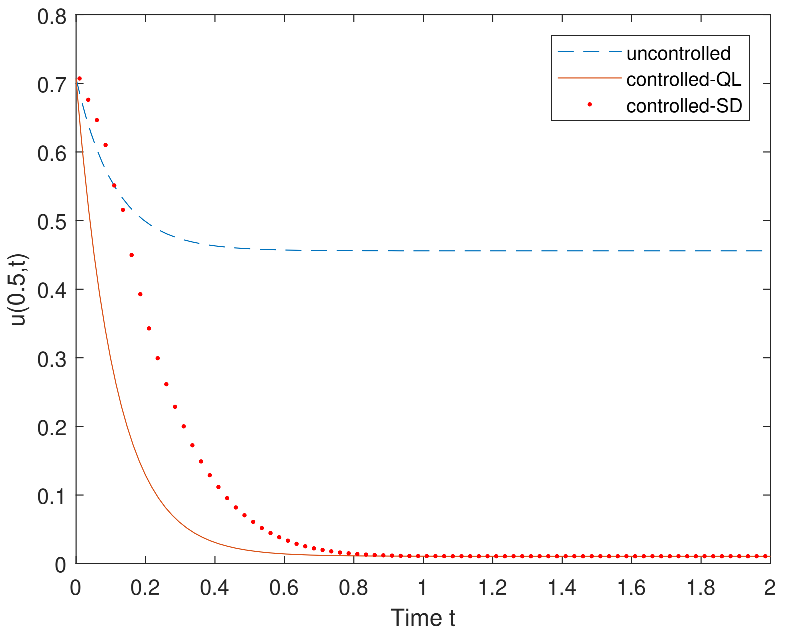

Figure 5.8 Comparison of the temperature profiles in time at $x_0 = 0.5$ for uncontrolled and controlled case using the steepest descent and the quasilinearization method.

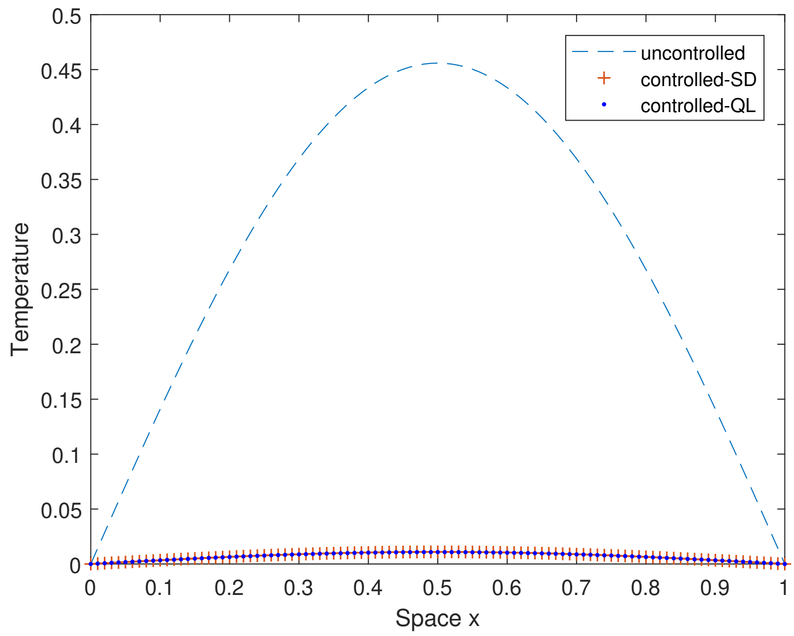

Figure 5.9 Comparison of the spatial temperature profiles for uncontrolled and controlled case using the steepest descent and the quasilinearization method at $t_f = 2$.

CHAPTER 6

RESULTS AND DISCUSSION

In this thesis, an optimal vibration control of two linear beam models governed by hyperbolic partial differential equations has been presented in the first part. The beam models to be controlled are the Mindlin-type beam model and the second strain gradient theory-based beam model described by linear higher order distributed parameter systems. The boundary control has been implemented to the two beam models via Pontryagin's maximum principle. By using the Pontryagin's maximum principle, note that the control problem is turned to solving a new partial differential equation system involving adjoint and state variables providing initial, boundary and terminal conditions. It has been concluded that analytical solutions are formed with the help of an eigenfunction expansion method. Numerical results are obtained using the computer codes produced in MATLAB© and presented in graphical and table forms. It has been observed that the boundary control is effective in reducing the vibrations of the beams.

In the second part of the thesis, a bilinear optimal control of heat transfer process governed by a parabolic partial differential equation has been considered. The performance index to be minimized has indicated that the goal is to keep the temperature close to the steady-state value without consuming control effort in large quantities.

The proposed approach uses a reduction of order in the model, Pontryagin's maximum principle and numerical techniques. Two iterative numerical techniques, the steepest descent and the quasilinearization, have been used to solve the obtained nonlinear two point boundary value problem. The steepest descent method has been used as a beginning procedure and the quasilinearization method to confirm the solution. The solution of the steepest descent method has been converged for two different initial guesses, while poor initial guesses may result in divergence for the quasilinearization method. However, the steepest descent method converges slowly compared to the quasilinearization method. The performance index value obtained by using the steepest descent method is quite close with the results obtained using the quasilinearization method. Temperature profiles in time at the center of the rod and spatial temperature profiles at terminal time have been shown in figures. Since the bilinear control act as coolant flow rate, the controlled solutions reach equilibrium temperature (0°C) faster than uncontrolled solutions. It has been shown in figures that the location of the maximum temperature cools slower for uncontrolled model than controlled model at the terminal time. Numerical simulations show the effectiveness and applicability of the introduced approach.

To further this research, the optimal bilinear control for different models are planned by adopting the presented control algorithms in this thesis. Different numerical techniques are to be implemented to solve the TPBVP. It is also worth noting that the applications of Pontryagin's maximum principle in nonlinear control problems are still to be explored.

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PUBLISHMENTS

Papers

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2. Acikgoz. A., Goktepe. S., (2012). "On Some Strongly Irresolute Multifunctions", International Congress in Honour of Professor H.M.Srivastava , Bursa, 2012.

3. Acikgoz. A., Goktepe. S., (2012). "On Upper and Lower Faintly Beta-star-g Continuous Multifunctions", International Conference on Applied Analysis and Algebra (ICAAA2012), Istanbul, 2012.

4. Acikgoz. A., Goktepe. S., (2012). "On Some Types of Strongly Continuous Multifunctions", First International Eurasian Conference on Mathematical Sciences and Applications (IECMSA), Prishtine, KOSOVO, 2012.

AWARDS

1. Son Sınıf Lisans Öğrencileri için Lisansüstü Burs Programı, 2228 A, TUBITAK, Eylül 2011/1.

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