Center and Trace of the Twisted Heisenberg Category

by

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Dedication

This dissertation is dedicated to my adviser Prof. Aaron Lauda.

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I would like to start by thanking my advisor Prof. Aaron Lauda for introducing the fascinating world of categorification and string diagrams to me. This dissertation is a result of his constant patience, never ending emotional and financial support, insightful guidance, unusual notebook size for drawing string diagrams and ability to hold on to life.

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Finally, I would like to thank my committee members for taking their time to read my work, for attending to my presentation of the dissertation and for their helpful questions and comments.

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Table of Contents

List Of Tables

List Of Figures

Abstract

Heisenberg algebras are fundamental algebraic structures appearing in many areas of mathematics and physics. A categorification program for these algebras have been initiated by Khovanov[20]. A twisted version of Khovanov's Heisenberg category was defined by Cautis and Sussan [8]. By using a decategorification functor, called trace, one can extract interesting information from categories. We describe an isomorphism between the trace of the twisted Heisenberg category and a subalgebra of the W-algebra $W_{1+\infty}$ from conformal field theory. We also describe some combinatorial structures present in the twisted Heisenberg category related to symmetric functions. Namely, we explain how the categorical center of the twisted Heisenberg category is isomorphic to the algebra of symmetric functions, by explicitly identifying certain closed diagrams with non-homogeneous power sum symmetric functions.

Chapter 1

Introduction

Heisenberg algebra $\mathfrak{h}$ is the simplest non-commutative algebra with two generators q, p and relation $qp - pq = 1$ (In the litterature, Heisenberg algebra refers to the Lie algebra generated by p, q and a central element C called central charge such that $[q, p] = C$. Here we are working with its universal envelopping algebra modulo $C - 1$. This version is also known as the Weyl algebra). Despite its simplicity, it appears in many contexts throughout physics and mathematics. To mention a few, it plays a fundamental role in quantum mechanics describing the relation between position and momentum of particles. It is contained in the Virasoro algebra which appears in conformal field theory. It can be identified with its unique irreducible representation, called Fock space representation. Boson-fermion correspondance is expressed in terms of the Fock space representations of the Heisenberg algebra and Clifford algebra. From a mathematical point of view, it is related to Lie algebras through the Fock space representation. Moreover $\mathfrak{h}$ can be realized via induction and restriction on the irreducible representations of the symmetric groups. We will mostly focus on this latter realization of $\mathfrak{h}$.

We are interested in the Heisenberg algebra $\mathfrak{h}$, however our interest is in the framework of categorification. The term *categorification* refers to the process of starting from a basic mathematical theory and building a richer, more powerful one, in such a way that it is possible to recover the original theory via some simplication. We discuss this concept in more detail in chapter 2. As an example, one can keep in mind the study of natural numbers versus the study of vector spaces, where we can recover the natural numbers by taking dimensions of the vector spaces. In this example it is easy to see that categorifying existent theories is a profitable business. Another interesting example is given by the character theory and representation theory. In representation theory, one assigns matrices to the abstract elements of groups or algebras and studies those matrices. In character theory, one studies the traces of those matrices, which are polynomial or numeric values. From this perspective, representation theory is a categorification of character theory.

For the past 25 years there has been a lot work dedicated to the program of categorification of fundamental algebraic objects appearing in mathematics and physics. From the categorification perspective it is natural to ask for a Heisenberg category $\mathcal{H}$, a richer mathematical structure not only with objects, but with morphisms between them, and which will recover the Heisenberg algebra after some simplification. We are primarily interested in a categorification of the Heisenberg algebra.

Representation theory of the symmetric group S_n and its connection to the theory of symmetric functions Λ through Frobenious characteristic is a classical and marvelous story. It has influenced a lot of mathematics in the intersection of representation theory and algebraic combinatorics. By considering all the symmetric groups at once, we allow ourselves to talk about induced representations coming from the standand inclusion $S_n \hookrightarrow S_{n+1}$ and restricted representations coming from the projection $S_n \hookrightarrow S_{n-1}$. Let $S_n - mod$ denote the category of representations of S_n. Then there is an algebra isomorphism

$$\Lambda \simeq \bigoplus_{n \in \mathbb{N}} K_0(S_n - mod)$$

where K_0 is the split Grothendieck group, generated by isomorphism classes of simple objects.

The space Λ is Hopf algebra, where algebra structure is coming from the induction functors $Ind_n^{n+1} : S_n - mod \to S_{n+1} - mod$ and the coalgebra structure is coming from the restriction functors $Res_n^{n+1} : S_n - mod \to S_{n-1} - mod$. These functors satisfy the Heisenberg relations $Res_{n+1}^n \circ {Ind_n}^{n+1} - Ind_{n-1}^n \circ Res_n^{n-1} = Id$, which allows us to identify the Fock space representation $\mathcal{F}$ of $\mathfrak{h}$ with the space of symmetric functions.

One would expect from a Heisenberg category $\mathcal{H}$ to not only lift the structure of the Heisenberg algebra, but also to lift the structure of its Fock space representation. In other words, a Heisenberg category should admit a categorical action on the collection of module categories of symmetric groups.

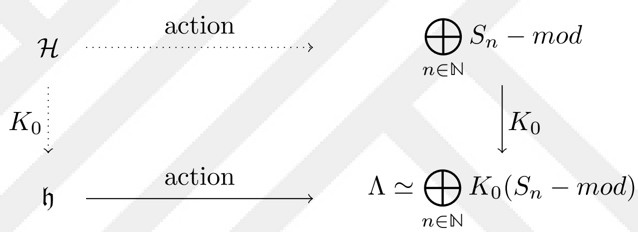


Figure 1.1: Heisenberg categorification

In 2010, Khovanov defined a diagrammatic category [20] whose Grothendieck group contains the Heisenberg algebra (see chapter 3 for more details). This original work started a categorification program for various Heisenberg algebras. Over the past 8 years, various Heisenberg categories and their quantizations have been related to

- Algebraic geometry via Hilbert schemes in [7]
- Categorified quantum groups in [34]
- Vertex operators and quantum affine algebras in [5]
- W-algebras in [6] and [33]
- Elliptic Hall algebra in [4]

- Combinatorics via shifted symmetric functions in [26] , [27]

An overview of the Heienberg categorifications until 2012 can be found in [29]. Here we mention some important cornerstones in this area.

Cautis and Licata gave a categorification of lattice Heisenberg algebras in [7], one for each finite subgroup of $SL_2(\mathbb{C})$. Hill and Sussan gave a similar categorification for twisted lattice Heisenberg algebras in [12]. Licata and Savage defined a quantised version of Khovanov's Heisenberg category by replacing the symmetric group with the Hecke algebra of type A [30]. Cautis and Sussan defined a twisted Heisenberg category [8] as a step towards the categorical twisted Boson-Fermion correspondence. These Heisenberg categorifications didn't fit into the same framework, until Savage and Yacobi [41], Savage and Rosso [39] proposed a unified framework based on a choice of Frobenius algebras. This framework has been extended to the superalgebra setting by Savage and Rosso [38].

More recently, Mackaay and Savage defined the Heisenberg categories of an arbitrary level using cyclotomic quotients of the degenerate affine Hecke algebra [32]. Brundan has showed that some relations in the definition of the Heisenberg categories are redundant [3]. A categorical relation between the Heisenberg categories and categorified quantum groups is provided by Queffelec, Savage and Yacobi. [34]. After describing a Clifford category motivated by contact topology (see [45]), Tian also gave a description of a Heisenberg category using the idea of a categorified Boson-Fermion correspondence and by expressing the Heisenberg generators as complexes in Clifford generators [44].

These Heisenberg categories all came equipped with extra structure of morphisms compared to the classical setting. Naturally, this richer context led us to various connections within mathematics, provided a deeper understanding and new insights. One of the most interesting examples of such a phenomenon occurs when one considers the *trace decategorification*(see section 2.3) of Heisenberg categories. In [6], Lauda et al. computed the trace of Khovanov's Heisenberg category and proved that it is isomorphic to the $W-$algebra $W_{1+\infty}$ with central charge $C = 1$. This is interesting for two reasons. For one, the Grothendieck group of the Heisenberg category is conjectually isomorphic to the heisenberg algebra $\mathfrak{h}$, and the algebra $W_{1+\infty}$ admits an $\mathbb{N}$-filtration whose degree zero part is $\mathfrak{h}$. It wasn't expected for the trace decategorification to unravel such a richer structure compared to the Grothendieck group decategorification. The second reason is that in the literature, it took quite some effort and time to form the bridge from the Heisenberg algebra to $W_{1+\infty}$. One needs to consider generalizations and extensions of the Heisenberg algebra to end up with $W_{1+\infty}$. However it seems that a naive approach to create a category whose generating objects satisfy the Heisenberg relations $QP \simeq PQ \oplus 1$ forces the category to carry the information of the algebra $W_{1+\infty}$ and the correct tool to extract this information is the trace decategorification functor. From this point of view, Khovanov's Heisenberg category is a natural link between the Heisenberg algebra and $W_{1+\infty}$.

This unexpected discovery leads to natural questions such as "What are the traces of the other Heisenberg categories?". Not much is known in the general case. In [4], Lauda et al. asked this question for a quantization of Khovanov's Heisenberg category(see [30]) and the trace in this case turned out to be related to the Elliptic Hall algebra. In a different direction, the combinatorial structure of the categorical center of $\mathcal{H}$ has been studies in [26], where the authors showed an isomorphism between $End_{\mathcal{H}}(\mathbb{1})$ and the algebra of shifted symmetric functions Λ^*.

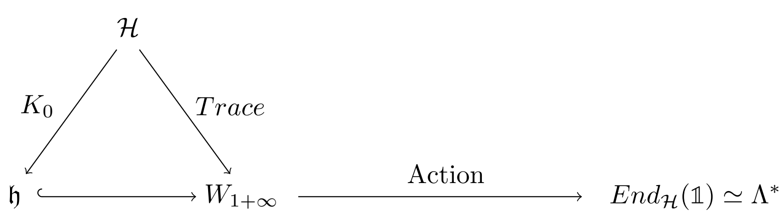


Figure 1.2: Trace of $\mathcal{H}$ and its Fock space representation

Our main object of study in this work is the twisted Heisenberg category $\mathcal{H}_{tw}$ defined by Cautis and Sussan [8]. We will focus on two aspects of $\mathcal{H}_{tw}$. First, we will give a description of the trace decategorification of $\mathcal{H}_{tw}$ and establish its relation to a subalgebra of the W-algebra $W_{1+\infty}$. This requires some new techniques since there is an extra $\mathbb{Z}/2\mathbb{Z}$-grading on the twisted Heisenberg category which wasn't existant for Khovanov's Heisenberg category. The superalgebra structure of the morphism spaces violates the usual isotopy invariance of diagrammatic calculus. The second part of the work consists of discovering the combinatorial structure of the categorical center of $\mathcal{H}_{tw}$. We describe an isomorphism between $End_{\mathcal{H}_{tw}}(\mathbb{1})$ and the algebra odd power sum symmetric functions. In this isomorphism, a natural basis of $End_{\mathcal{H}_{tw}}(\mathbb{1})$ is identified with certain non-homogeneous power sum symmetric functions.

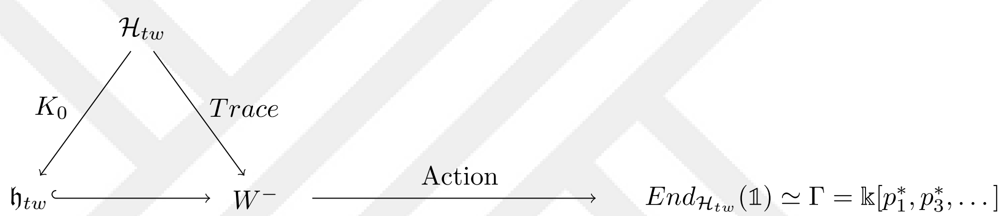


Figure 1.3: Trace of $\mathcal{H}_{tw}$ and its Fock space representation

Organization of the dissertation is as follows. In chapter 2 we start by discussing the philosophy of categorification and provide some examples. Later we explain how categorification can be achieved using diagrammatic algebras. Lastly we recall the trace decategorification functor, which will be the main tool of chapter 4.

In chapter 3, we introduce the objects of interest, namely Heisenberg algebra, Heisenberg category, twisted Heisenberg algebra and twisted Heisenberg category in detail. We study the indecomposable object in the corresponding categories and study their endomorphism spaces.

In chapter 4, we introduce the algebras $W_{1+\infty}$ and W^-, explain their gradings and generating sets. Then we perform many computations in the trace of $\mathcal{H}_{tw}$ using diagrammatic algebra, and finally describe an isomorphism between trace of $\mathcal{H}_{tw}$ and a quotient of the algebra W^-.

Chapter 5 is dedicated to explaining the combinatorics of the categorical center of the twisted Heisenberg category. We establish an explicit isomorphism between the categorical center and the odd power sum subalgebra of the symmetric functions. It follows from general theory that the trace of a category acts on its center. Applying this result to our case, we discuss the action of the trace of $\mathcal{H}_{tw}$ on its center and give an explicit formula on a set of generators, which amounts to describing an action of the W-algebra W^- on the space of symmetric functions.

Finally, this paragraph can be thought of as a check-up for the reliability of the control mechanisms involved in printing a thesis. I have the impression that printed thesis documents are not as reliable as the peer-reviewed journal publications. The main reason being that, despite the careful reading and suggestion of the committee members(which is not a necessary part of the procedure), the final say is left to the owner of the thesis. We will be working with a field $\Bbbk$ of characteristic zero.

Chapter 2

Categorification and Decategorification

2.1 Categorification

The idea of categorification goes back to the 1994 paper of Crane and Frenkel [9]. Their motivation was to increase the dimension of a Topological Quantum Field Theory(TQFT) by one, via increasing the categorical dimension of associated algebraic structures by one (i.e. replace sets with categories, categories with 2-categories...etc).

A common analogy to keep in mind is that by categorifying, one can see the actual object, rather than its shadow. As is the case for shadows and actual objects, the process of categorification is not necessarily unique. To be more precise, categorification is not even a well-defined process since different objects can have the same shadow from a certain angle, and similarly one can enrich a theory in various ways.

A prototypical example of categorification is lifting the structure of natural numbers $\mathbb{N}$ to the category of finite dimensional vector spaces, $\underline{Vect}^{f.d.}$. One can recover $\mathbb{N}$ from $\underline{Vect}^{f.d.}$ simply by taking the dimensions of vector spaces. Clearly this is an enrichment of our theory, since in $\underline{Vect}^{f.d.}$ we have an extra layer of structure which wasn't accesible at the level of $\mathbb{N}$: linear maps between vector spaces. Moreover, addition of two natural numbers lifts to the direct sum of two vector spaces(see Table 2.1). In the same spirit, multiplication lifts to the tensor product. In

$\mathbb{N}$	$\underline{Vect}^{f.d.}$
n	V of dimension n
$n+m$	$V \oplus W$
$n \times m$	$V \otimes W$
Arithmetic	Linear algebra

Table 2.1: Categorification of natural numbers by finite dimensional vector spaces

this sense, by categorifying natural numbers, one gets linear algebra, which has proved to be a marvelous theory in the last centuries.

At the same time, one can think of the category of finite sets as a categorification of natural numbers(see Table 2.2). A crucial difference between sets and categories is the existance of the extra structure given by morphism spaces in a category. Two equal objects at the decategorified level can be isomorphic in many ways at the categorified level. Think of the equality between dimensions of two vector spaces vs the various isomorphisms one has between the two vector spaces. From this perspective, knowing that two sets have cardinality is not very enlightening. Instead, one would like to see the bijections between these sets. This is a common theme in combinatorics where one seeks for "bijective proofs".

The philosophy of enriching algebraic theories at hand by increasing their categorical dimension has been applied to various situations and produced striking new results. For example in knot theory, the construction of Khovanov homology has produced stronger link invariants by categorifying the Jones polynomial. In [16] Jones explaines how to assign a Laurent polynomial to an oriented link, which doesn't depend on the projection of the link. Then in [19], Khovanov explaines how to assign a chain complexes to an oriented link, whose homology gives another link invariant. The homology of this chain complex stores more information about the link compared to a polynomial. Moreover, one can recover the Jones polynomial of a link from its Khovanov

$\mathbb{N}$	Sets
n	set A with cardinanilty n
0	$\varnothing$
$n+m$	$A \sqcup B$
$n \times m$	$A \times B$
2^n	$\mathcal{P}(A)$ power set
$n = m$	$A \simeq B$ maybe in many ways
arithmetic	Set theory

Table 2.2: Categorification of natural numbers by finite sets

homology by taking its graded Euler characterictic. In this sense, Khovanov homology is a categorification of the Jones polynomial. In this example, the coefficients of the Jones polynomial has been replaced by graded vector spaces of the same dimension.

A famous example of categorification is from algebraic topology. It is well known that the Euler characteristic is a topological invariant of a surface. It is possible to recover the Euler characteristic of a surface by taking the alternating sum of dimensions of the homology of an associated complex.

Representation theory and character theory is another example of categorification. In representation theory, say of groups, one assigns a matrix to each group element and studies the related modules. By taking the traces of these matrices, we focus our attention on either numbers or polynomials associated to the group elements. A classical story is that the irreducible representations of the symmetric group S_n are given by Specht modules S^λ, one for each partition of n. Their characters correspond to the famous Schur functions in algebraic combinatorics. Decomposition of tensor product of Specht modules into irreducible components is studied via the multiplication of corresponding Schur functions and expanding them in the Schur function basis(see Table 2.3). Here Specht modules categorify the Schur functions. The classical Schur-Weyl duality can also

Sym	S_n-mod
s_λ Schur polynomial	S^λ Specht module
$s_\lambda + s_\mu$	$S^\lambda \oplus S^\mu$
$s_\lambda \times s_\mu$	$Ind_{S_n \times S_m}^{S_{n+m}} S^\lambda \otimes S^\mu$
$\langle s_\lambda, s_\mu \rangle$	$\mathrm{Hom}(S^\lambda, S^\mu)$
Symmetric polynomials	S_n modules

Table 2.3: Categorification of symmetric functions by symmetric group representations

be seen as a categorification of multiplicities of irreducible representation appearing in a certain decomposition, where numerical values are replaced by representations of the Schur-Weyl group.

In a different direction, categorical structures related to quantum groups $U_q(\mathfrak{g})$ have been introduced in 2008 by Rouquier[40] using abelian categories and by Khovanov-Lauda [21],[23],[22] using diagrammatic categories(see section 2.2 for more details on diagrammatic categories). Khovanov-Lauda showed that the Grothendieck group of their construction is isomorphic to an integral version of the quantum group $U_q(\mathfrak{g})$. Later Brundan showed that the two independent constructions are equivalent, hence they both provide a categorification of $U_q(\mathfrak{g})$. The extra structure obtained by these categorifications are known as KLR-algebras, and they proved to be quite useful in obtaining new results.

2.2 Diagrammatic Categories

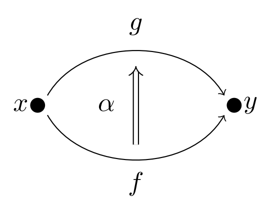
Diagramatic descriptions of mathematical objects can be traced back to the works of Penrose and Feynman. This unconventional notation technique is deceiving at first sight for its simplicity and lack of formalism compared to more conventional algebraic symbolic formalism. However as we will describe in this chapter, diagrammatic calculus requires quite an effort to set up and to justify at the beginning and this is where its formalism and difficulties lie.

In a category $\mathcal{C}$, one has objects and morphisms between objects. The mathematical nature of these objects and morphisms can be mysterious. In its abstract generality, objects are represented by dots and a morphism between two objects is represented by arrow from one dot to another. In a category, what one wants to know is how do the morphisms compose. In particular, when we compose two morphisms, we would like to know if the composition is equal to another morphism. Note that this question doesn't concern with the form of the morphisms, it is not asking for a formula for each of these morphisms. It is simply trying to understand their relation to each other with respect to composition.

It is more natural to consider 2-categories for the discussion of diagrammatics due to the 2-dimensional nature of planar diagrams. Later we will see how a monoidal 1-category can be seen as a 2-category. We will make this transition mainly because the twisted Heisenberg category we will work with is a monoidal 1-category. We will also explain the translation of the diagrammatics from a 2-categorical point of view to a monoidal 1-categorical point of view.

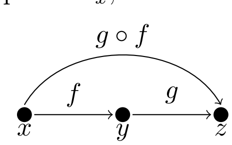
Definition 2.2.1. A two category $\mathcal{D}$ consists of

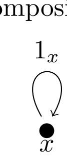
- A class of objects, $Ob(\mathcal{D})$
- A space of 1-morphisms, $1Hom_{\mathcal{D}}(x, y)$, for every pair of objects x, y
- A space of 2-morphisms, $2Hom_D(f, g)$, for every pair of 1-morphisms f, g



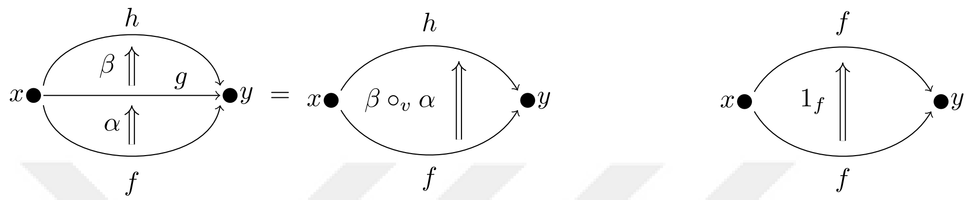
such that

1. There is an associative composition operation $\circ$ for 1-morphisms, and for each object x, there is an identity 1-morphism 1_x, which is the identity of the composition operation.



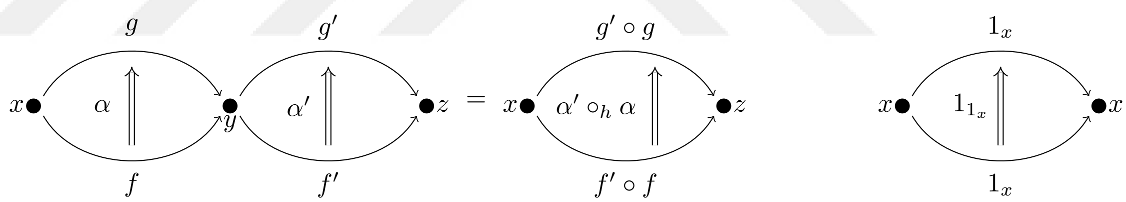


2. There is an associative *vertical composition* operation $\circ_v$ for 2-morphisms $\alpha : f \to g$, $\alpha' : g \to h$ where $\alpha' \circ_v \alpha : f \to h$ and for each 1-morphism f, there is an identity 2-morphism 1_f which is the identity with respect to $\circ_v$:



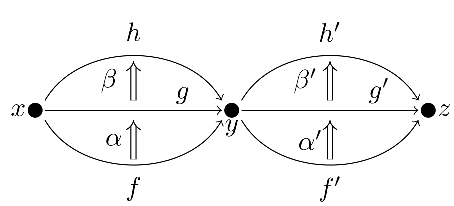
(2.1)

3. There is an associative *horizontal composition* operation $\circ_h$ for 2-morphisms and an identity 2-morphism 1_{1_x} which is the identity with respect to $\circ_h$:



(2.2)

4. The vertical composition and horizontal composition are compatible(interchange law):



$(\beta \circ_v \alpha) \circ_h (\beta' \circ_v \alpha') = (\alpha' \circ_h \alpha) \circ_v (\beta' \circ_h \beta)$

(2.3)

A monoidal category is a triple $(\mathcal{C}, \otimes, \mathbb{1})$ where $\mathcal{C}$ is a category, $\otimes$ denotes the associative tensor product and $\mathbb{1}$ stands for the identity of the left and right tensor product, with similar conditions on the existance of identity morphisms, existance of compositions of morphims and their coherence with the tensor product.

In a category $\mathcal{C}$, we have the composition operation between the morphisms. In a monoidal category, we get an extra operation which allows us to take tensor products of objects and of morphisms. In spirit, this is similar to the two operations we have in a 2-category $\mathcal{D}$ on the 2-morphisms: horizontal composition $\circ_h$ and vertical composition$\circ_v$.

A monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ can be seen as a 2-category $\mathcal{D}$ with a single object as follows:

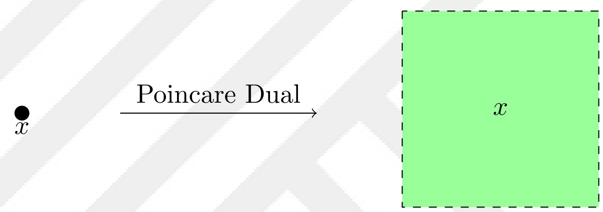
- $\mathcal{D}$ has a single object $*$.
- 1-morphisms of $\mathcal{D}$ are objects of $\mathcal{C}$, where the identity morphism 1_* of $\mathcal{D}$ corresponds to the monoidal identity object of $\mathcal{C}$
- Composition of 1-morphisms of $\mathcal{D}$ is given by tensor product of objects of $\mathcal{C}$
- 2-morphisms of $\mathcal{D}$ are the morphisms of $\mathcal{C}$, where vertical composition corresponds to the composition of morphisms in $\mathcal{C}$
- Vertical composition of 2-morphisms of $\mathcal{D}$ is given by tensoring of morphisms in $\mathcal{C}$

Monoidal category $(C, \otimes, \mathbb{1})$	2-category $\mathcal{D}$
	Single object $*$
$f, g, h \in Ob(C)$	$f, g, h \in 1Hom(*, *)$
$\mathbb{1} \in Ob(C)$	$1_* \in 1Hom(*, *)$
$g \otimes_{\mathcal{C}} f \in Ob(C)$ tensor of objects	composition of 1-morphisms $g \circ f$
$\alpha : f \to g, \beta : g \to h$ morphisms	$\alpha : f \to g, \beta : g \to h$ $2-$morphisms
$\alpha \circ \beta$ composition	$\alpha \circ_h \beta$ horizontal composition
$\alpha \otimes_{\mathcal{C}} \beta$ tensor of morphisms	$\alpha \circ_v \beta$ vertical composition

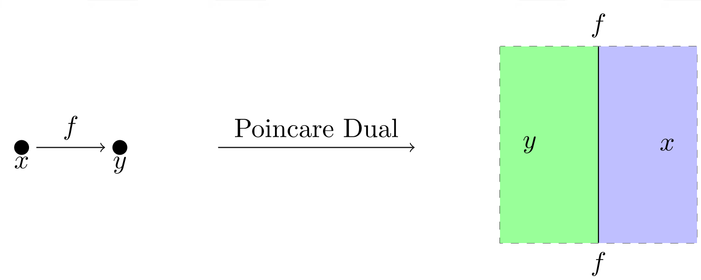
Table 2.4: Monoidal categories as 2-categories with a single object

Therefore the 2-dimensional diagrams we used in the description of a 2-category can be thought of as diagrams for a monoidal category. We just need to restrict ourselves to a single object. Now we need one last modification to end up with the setup for string diagrams for a monoidal category: Poincare dual of our 2-dimensional diagrams. This amounts to depicting 0-dimensional objects as 2-dimensional regions, and 2-dimensional 2-morphisms as 0-dimensional points. The resulting diagrams are known as string diagrams. We read the string diagrams from right to left, and from bottom to top. These diagrams are well defined up to isotopy.

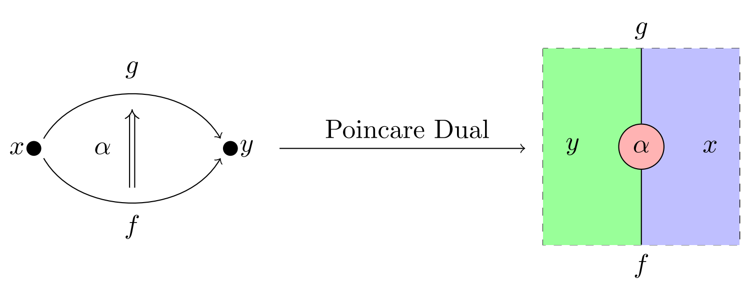
Objects become regions:



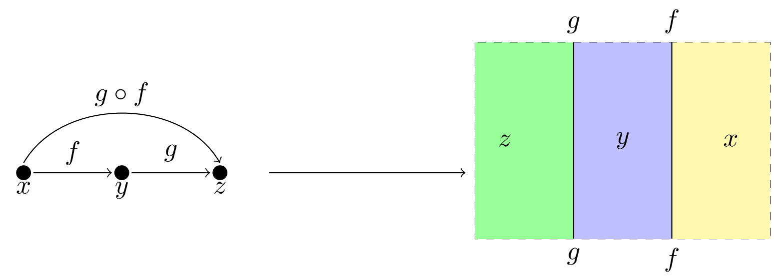
1-morphisms stay one dimensional, but now we draw them vertically:



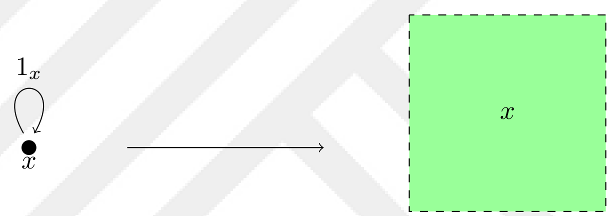
2-morphisms turn into points, but we draw them a bit larger in order to label them:



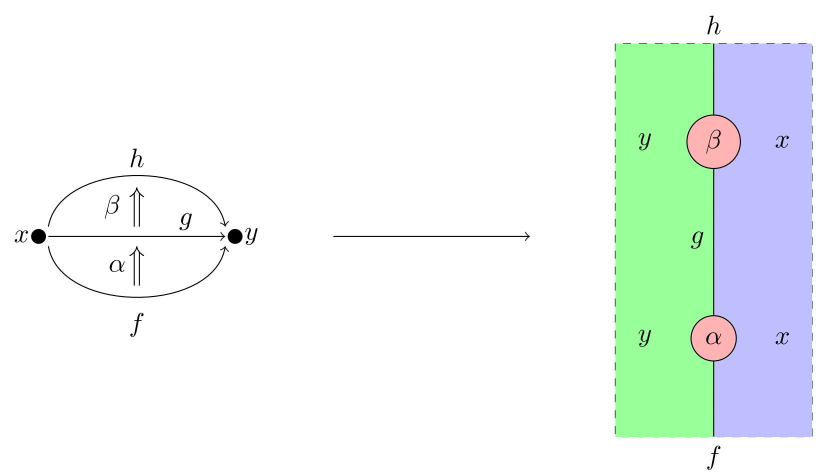
Let's see how to depict the various composition operations we have in string diagrams. The composition of two 1-morphisms is concatenation of string diagrams horizontally:



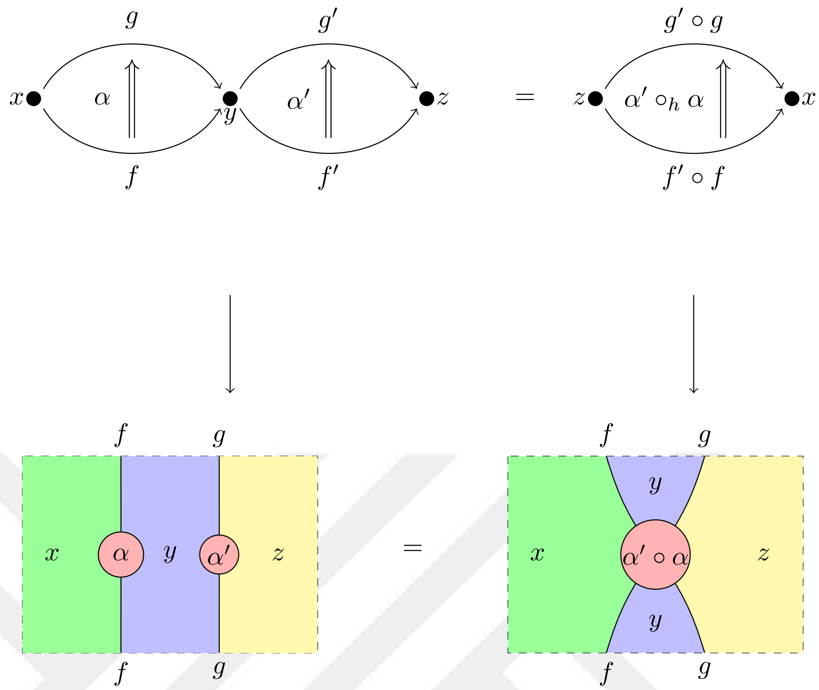
We don't draw anything for the identity 1-morphism 1_x(or monoidal identity object in a monoidal category), so that concatenating its string diagram to the left or to the right doesn't have an effect.



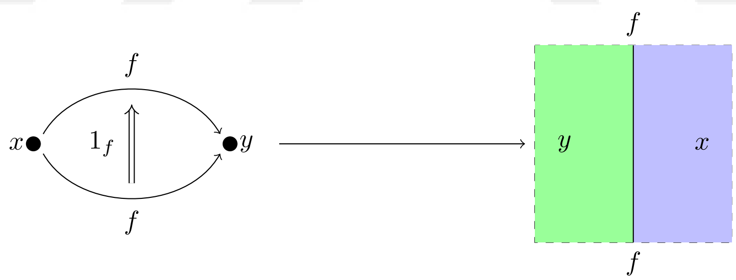
Vertical composition of 2-morphisms(or composition of morphisms in a monoidal category) is given by vertically stacking two diagrams on top of each other:



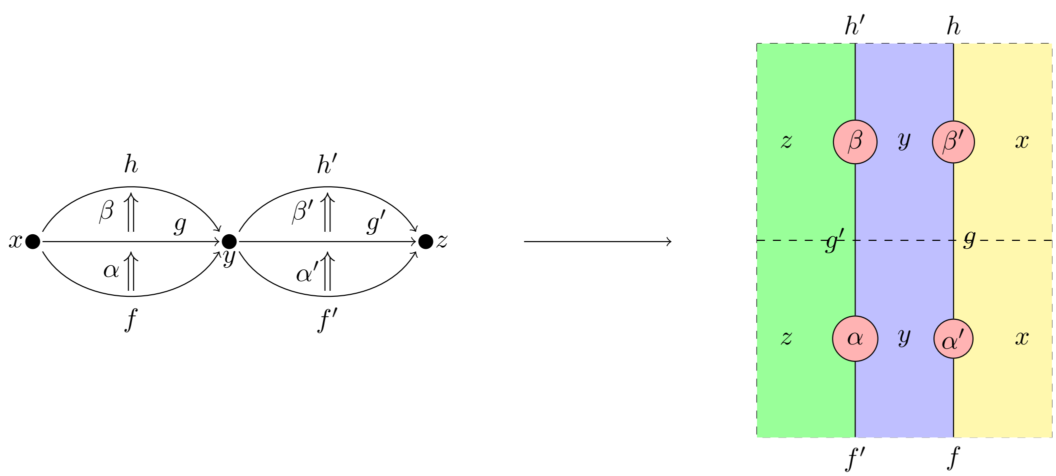
Horizontal composition of 2-morphisms(or tensor of morphisms in a monoidal category) is given by horizontal concatenation:

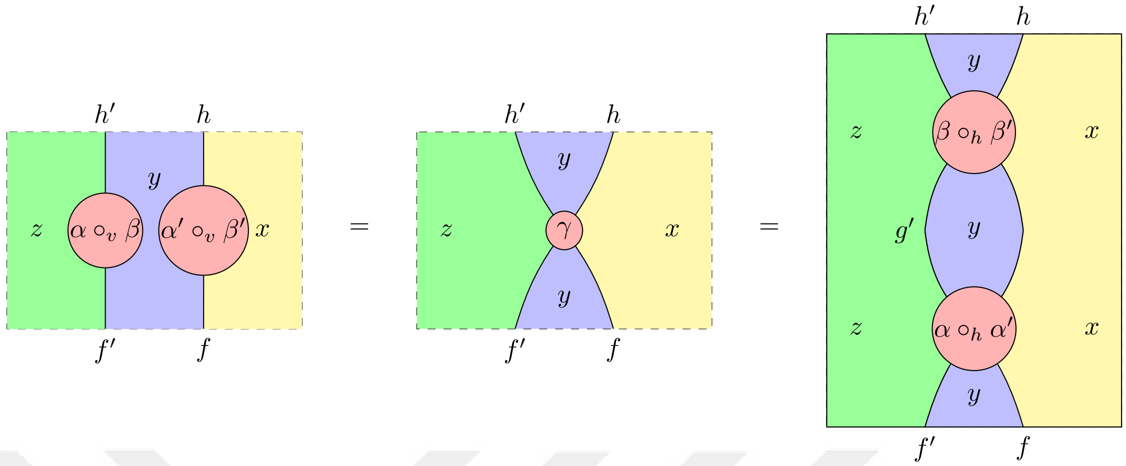


We don't draw the identity 2-morphism for vertical composition, and it is easy to see that this diagram doesn't have an effect with respect to vertical composition.



The interchange law translates to





where $\gamma = (\alpha' \circ_v \beta') \circ_h (\alpha \circ_v \beta)$

In order to use string diagrams for a monoidal category $\mathcal{C}$, recall that one needs to see $\mathcal{C}$ as a 2-category with a single object. Therefore all the regions will be labelled with the same object. In this case, usually one wants to leave the regions blank instead of coloring all the regions with the same object. This will be the case for the Heisenberg category $\mathcal{H}$ and the twisted Heisenberg category $\mathcal{H}_{tw}$ which we will introduce in chapter 3.

2.3 Trace Decategorification

We say that a monoidal additive category $\mathcal{C}$ is a categorification of an algebra A if there is a decategorification functor $\mathcal{F}$ such that

$$\mathcal{F}(\mathcal{C}) \longrightarrow A$$

is an algebra isomorphism. The functor $\mathcal{F}$ sends direct sums of objects to sums of elements of the algebra, and tensor products of objects to the multiplication of elements in the algebra.

Generally one choses the functor $\mathcal{F}$ to be the split Grothendieck group of the category.

Definition 2.3.1. Split Grothendieck group of an additive category $\mathcal{C}$, denoted as $\mathcal{K}_0(\mathcal{C})$, is the abelian group generated by isomorphism classes $[X]$ of objects of $\mathcal{C}$, modulo the relation $[X] = [Y] + [Z]$ whenever $X \simeq Y \oplus Z$ in $\mathcal{C}$.

If there are exact sequences in $\mathcal{C}$, then one can talk about the Grothendieck group where the defining relation is replaced by "$[X] = [Y] + [Z]$ whenever there is a short exact sequence $0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$".

Moreover if $\mathcal{C}$ has the additional structure of a monoidal category, tensoring of objects in $\mathcal{C}$ descends to a multiplication on $\mathcal{K}_0(\mathcal{C})$ which provides it with a ring structure. Therefore the Grothendieck group of an additive monoidal category is a ring. When one says "the category $\mathcal{C}$ categorifies an algebra A", the claim is that $A \simeq \mathcal{K}_0(\mathcal{C}) \otimes \Bbbk$. Thus categorification of an algebra with the Gorthendieck group decategorification in mind requires having an *integral* form of that algebra. This integral form, when tensored with the ground field, becomes isomorphic to the algebra in question.

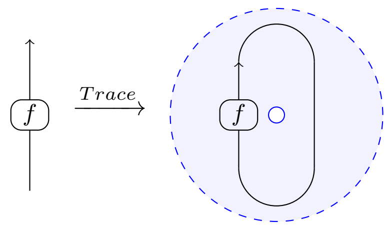
In this dissertation, we will focus on another decategorification functor, called trace decategorification, which was advertised in [2] as an alternative to the Grothendieck group. Trace will have a natural algebra structure if one enriches the space of morphisms of $\mathcal{C}$ via vector spaces.

Definition 2.3.2. Let $\mathcal{C}$ be a $\Bbbk$-linear additive category. Then trace of $\mathcal{C}$, denoted as $Tr(\mathcal{C})$ is defined as

$$Tr(\mathcal{C}) := \left(\bigoplus_{x \in Ob(\mathcal{C})} End_{\mathcal{C}}(x) \right) \Big/ I$$

where I is the $\Bbbk-$span of $\{f \circ g - g \circ f\}$ for $f : x \to y$ and $g : y \to x$ for all $x, y \in Ob(\mathcal{C})$.

In linear algebra, trace is the unique additive map satisfying $tr(ab) = tr(ba)$ up to a scalar. By definition, trace of a category satisfies this relation and this is where the name originates from. Trace of a category is also called the 0-th Hochschild-Mitchell homology, which is a generalization of the classical Hochschild homology to categories. Recall that the 0-th Hochschild homology of an algebra A is given by $A/[A, A]$.



(2.4)

Figure 2.1: Graphical interpretation of elements in the trace of a category

Trace of a $\Bbbk$-linear additive category $\mathcal{C}$ is a $\Bbbk$-vector space. If one equips $\mathcal{C}$ also with a monoidal structure, then by changing the definition of I in the definition of the trace to the *ideal* generated by $\{f \circ g - g \circ f\}$, one obtains an algebra structure on $Tr(\mathcal{C})$ where multiplication descends from tensoring of morphisms.

Just by comparing the definitions of the Grothendieck group and trace of a category, one sees that Grothendieck group is defined through the objects of the category, whereas the trace is defined through the morphisms of $\mathcal{C}$.

There are certain advantages to working with the trace of a category rather than its Grothendieck group.

- The direct sum in the definition of the trace can be taken only over the simple objects of the category.

- Trace of a category is unchanged under idempotent completion. Therefore, for the purposes of decategorifying using the trace functor, one can work with the non-idempotent complete versions of the categories. The diagrammatic descriptions of idempotent completions is usually more difficult to work with(see [18] for more details).

- Trace has a natural diagrammatic interpretation. Given a diagram in a category, its image under the trace functor is the same diagram considered on an annulus. It suffices to connect the top and bottom of the diagram with strands going around the hole of the annulus.

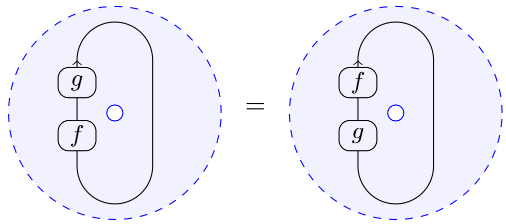


Figure 2.2: Graphical interpretation of the trace relation $fg = gf$

The annulus recaptures the trace relation $fg = gf$ diagrammatically since we can slide f or g around the annulus to change their composition order.

- There is a homomorphism from the Grothendieck group to the trace of a category $\mathcal{C}$ given by the Chern class map

$$\mathcal{K}_0(\mathcal{C}) \longrightarrow Tr(\mathcal{C}) \tag{2.5}$$

$$[X] \longmapsto [id_X] \tag{2.6}$$

In general, this homomorphisms is neither injective, nor surjective.

The proofs of these properties can be found in [2].

The trace decategorification has been computed only for a handful of examples. In some cases, it agrees with the Grothendieck group, and in some cases it has a richer structure than the Grothendieck group. For example, the trace of the categorified quantum group is shown to be isomorphic to the current algebra of the associated lie algebra, hence has a richer structure than the Grothendieck group. The trace of the Hecke category has been computed in [10]. The trace of Khovanov's Heisenberg category was shown to be isomorphic the W-algebra $W_{1+\infty}$ with central charge 1 in [6]. The W-algebra(see chapter 4) is an infinite dimensional, $\mathbb{N}$-filtered algebra, whose degree zero is isomorphic to the Heisenberg algebra. Therefore trace of the Heisenberg category is much richer than its Grothendieck group. We will show the type B analogue of this result in chapter 4 concerning the twisted Heisenberg category. More recently the trace of the quantized Heisenberg category has been related to the elliptic Hall algebra in [4].

As these examples demonstrate, calculating traces of categories of interest can yield new connections and interesting mathematics.

Chapter 3

Heisenberg Algebras and Heisenberg Categories

3.1 Heisenberg Algebra

Definition 3.1.1. Heisenberg Lie algebra is the Lie algebra over the field $\Bbbk$ generated by p, q, C such that

$$[q, p] = C$$
$$[C, p] = [C, q] = 0$$

The central element C in the definition of a Heisenberg algebra is usually referred to as the central charge, motivated from physics. We will mostly focus our attention on the universal envelopping algebra of the Heisenberg Lie algebra with central charge 1.

Definition 3.1.2. Heisenberg algebra $\mathfrak{h}$ is the unital associative algebra over a field $\Bbbk$ with generators p, q such that

$$qp - pq = 1$$

Remark. The Heisenberg algebra $\mathfrak{h}$, as defined above, also appears as the Weyl algebra in the litterature. It can be seen as the quotient of the universal envelopping algebra of the Heisenberg Lie algebra by the two sided ideal generated by $C - 1$.

If one wants to study the representations of $\mathfrak{h}$, by a simple calculation one immediately gets that

$$0 = tr(qp - pq) = tr(qp) - tr(pq) = tr(1) \neq 0$$

Therefore $\mathfrak{h}$ cannot have any finite dimensional representations. An infinite dimensional irreducible representation of $\mathfrak{h}$ is given by the multiplication and differentiation operators on the ring of polynomials in one variable, $\Bbbk[x]$, where q acts as multiplication by x and p acts as $\frac{d}{dx}$, differention with respect to x. These two operators satisfy the relation

$$x\frac{d}{dx} - \frac{d}{dx}x = 1$$

Heisenberg Lie algebra is the Lie algebra of the Heisenberg Lie group $\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} | a, b, c \in \Bbbk\}$, and unitary irreducible representations of the Heisenberg group are classified by

Theorem 3.1.1. (Stone von-Neumann) After choosing a scalar by which the central element C acts, the Heisenberg group has a unique strongly continous unitary irreducible representation.

Therefore, $\mathfrak{h}$ has a unique irreducible representation which can be exponentiated.

Another place where one encounters $\mathfrak{h}$ is when one considers induction and restriction on the irreducible representations of the symmetric groups S_n:

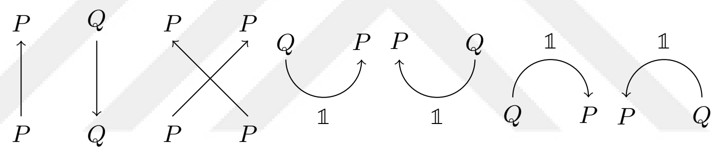
$$Res^n_{n+1} \circ Ind^{n+1}_n - Ind^{n+1}_n \circ Res^n_{n+1} = Id$$

Khovanov's Heisenberg category will be a categorificaton of these induction and restriction functors.

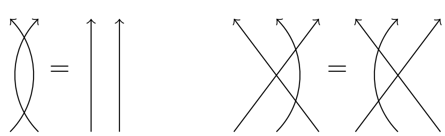
3.2 Heisenberg Category

Khovanov describes an additive $\Bbbk$-linear monoidal category $\mathcal{H}$ in [20] with generating objects P, Q and morphisms given by planar diagrams modulo certain local relations. Therefore the objects of $\mathcal{H}$ are direct sums of finite sequences on letters P and Q such as $PPQQPQPQ$, where by PQ we represent the object $P \otimes Q$. We will denote the sequence with n many P's by P^n. The empty sequence corresponds to the monoidal identity $\mathbb{1}$. The morphisms of $\mathcal{H}$ that we describe below are chosen so that there is an isomorphisms $QP \simeq PQ \oplus \mathbb{1}$.

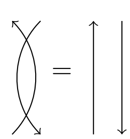
Morphisms of $\mathcal{H}$ are certain compact oriented one dimensional manifolds immersed into $\mathbb{R} \times [0, 1]$, modulo boundary fixing isotopies, generated by



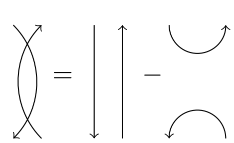
where the first two morphisms are respectively the identity morphisms on P and Q. The orientation of the morphisms determines the source and the target, therefore we will stop writing them from now on. These generators satisfy the following six local relations:



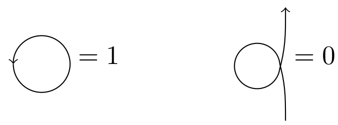
(3.1)



(3.2)

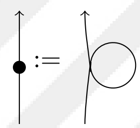


(3.3)



(3.4)

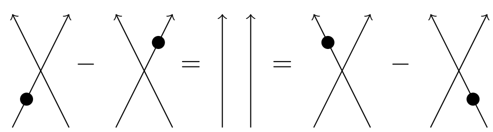
As is the case in algebraic objects described by generators and relations, there are many implications of the above relations one needs to study carefully. For example the second equation in relation 3.4 tells us that left twist curls are zero in $\mathcal{H}$, however we can also have right twist curls. They play an important role in the story, and they will be denoted by a solid dot on a strand:



(3.5)

A solid dot with a label $k \in \mathbb{N}$ will be used to represent the composition of k right twist curls.

The relation of a right twist curl and a crossing is interesting:



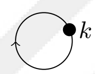
(3.6)

The above relation can be read as $s_i x_i - x_{i+1} s_i = 1 = x_i s_i - s_i x_{i+1}$. We will see in the next section that this is a defining relation of the degenerate affine Hecke algebra.

3.2.1 Endomorphism spaces of $\mathcal{H}$

For the purposes of understanding the trace of a category, it is enough to study the endomorphism spaces rather than morphisms between different objects. In fact, it is enough to understand the endomorphism spaces of simple objects. For the Heisenberg category $\mathcal{H}$, the simple objects are all of the form $P^n Q^m$ for $n, m \geq 0$. We will see that there is a description of $End_{\mathcal{H}}(P^n Q^m)$ in terms of $End_{\mathcal{H}}(\mathbb{1})$, $End_{\mathcal{H}}(P^n)$ and $End_{\mathcal{H}}(Q^m)$, so we will start by explaining the structures of these spaces.

Diagrammatically, the elements of $End_{\mathcal{H}}(\mathbb{1})$ are $\Bbbk$-linear combinations of possibly intersecting or nested closed diagrams, which may have dots. We can always separate the nested pieces, and resolve any crossing that occurs between different closed diagrams using the defining relations of morphisms in $\mathcal{H}$. As a result, we will end up with non intersecting, not nested closed oriented diagrams. Each one can be deformed into an oriented circle with possibly dots on it via an isotopy. A single closed, oriented, non self intersecting diagram is called a bubble. They are the building blocks of endomorphisms of the identity object in $\mathcal{H}$. In [20], Khovanov shows that $End_{\mathcal{H}}(\mathbb{1})$ is isomorphic to the polynomial algebra over $\Bbbk$ with infinitely many variables $\{b_i\}_{i\in\mathbb{N}}$ where

$b_k :=$  (3.7)

We will use the notation $\Pi := End_{\mathcal{H}}(\mathbb{1}) \simeq \Bbbk[b_0, b_1, b_2, \ldots]$.

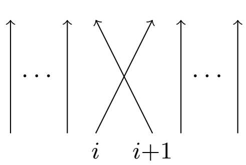
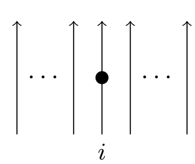
In the study of $End_{\mathcal{H}}(P^n)$, $End_{\mathcal{H}}(Q^m)$ and consequently $End_{\mathcal{H}}(P^nQ^m)$, the degenerate affine Hecke algebra plays an important role .

Definition 3.2.1. The degenerate affine Hecke algebra of type A over $\Bbbk$ is the unital associative algebra generated by transpositions $\{s_i\}_{i=1\ldots n-1}$ and polynomial generators $\{x_i\}_{i=1\ldots n}$ subject to the relations

$$s_i^2 = 1, \qquad s_is_j = s_js_i \quad \text{for } |i-j|>1, \qquad s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \tag{3.8}$$

$$x_ix_j = x_jx_i, \qquad x_js_i = s_ix_j \quad \text{for } |i-j|>1, \qquad x_is_i = s_ix_{i+1}+1 \tag{3.9}$$

The relations (3.1) and (3.6) imply that there is an algebra homomorphism from the degenerate affine Hecke algebra DAH_n to $End_{\mathcal{H}}(P^n)$. The morphism $f : DAH_n \mapsto End_{\mathcal{H}}(P^n)$ is given by

$s_i \mapsto$  $x_i \mapsto$  (3.10)

The morphism f above is in fact injective. Note that $P^n \otimes \mathbb{1} \simeq P^n$ by the definition of a monoidal identity. Therefore by appending endomorphisms of $\mathbb{1}$ to the images of degenerate affine Hecke algebra elements under the morphism f, one obtains an endomorphism of P^n again. The following results tells us that all the endomorphisms of P^n can be obtained this way.

Proposition 3.2.1. ([19], Proposition 4) There is an algebra isomorphism $End_{\mathcal{H}}(P^n) \simeq DAH_n \otimes \Pi$.

Elements of $End_{\mathcal{H}}(Q^m)$ can be obtained in a similar manner to $End_{\mathcal{H}}(P^m)$, where we change the direction of arrows from up to down. In this way, we get an algebra isomorphism $End_{\mathcal{H}}(Q^m) \simeq DAH_n^{op} \otimes \Pi$, where DAH_n^{op} stands for the opposite algebra of DAH_n.

Now we have a good understanding of endomorphisms of objects $\mathbb{1}, P^n, Q^m$. However objects of $\mathcal{H}$ can consist of a mixture of P's and Q's, such as $PPQP$. Note that we can use the isomoprhism $QP = PQ \oplus \mathbb{1}$ to obtain $PPQP \simeq PPPQ \oplus PPP$. In this manner, we can write any object as a direct sum of objects of the form P^nQ^m. In fact, all simple objects of $\mathcal{H}$ are P^nQ^m for $n, m \geq 1$ [6, Lemma 34] and the following result describes the endomorphisms of P^nQ^m.

Proposition 3.2.2. ([19], Equation (19)) There is a short exact sequence

$$0 \longrightarrow J_{n,m} \longrightarrow End_{\mathcal{H}}(P^nQ^m) \longrightarrow DAH_n \otimes DAH_n^{op} \otimes \Pi \longrightarrow 0$$

where $J_{n,m}$ is the ideal generated by endomorphisms of P^nQ^m containing at least one arc connecting two upper points. This short exact sequence splits.

3.2.2 $\mathcal{H}$ as Categorified Induction and Restriction

Recall that one could think of the generators of $\mathfrak{h}$ as induction and restriction functors on the symmetric group representations. Therefore one can think of the morphism spaces in $\mathcal{H}$, which is the extra structure coming from categorification, as natural transformations between induction and restriction functors. More precisely, let $\mathscr{S}_n$ be the category whose objects are compositions of induction and restriction functors between symmetric groups with the standard embedding

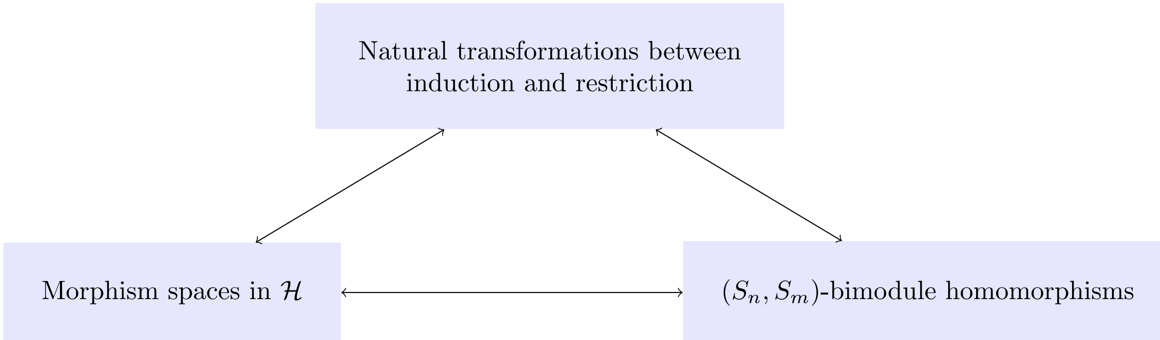


Figure 3.1: Different interpretations of the categorical Fock space action of $\mathcal{H}$

$S_k \subset S_{k+1}$, starting from S_n. Morphisms of $\mathscr{S}_n$ are given by natural transformations between these induction and restriction functors. Then there is a functor $F_n : \mathcal{H} \mapsto \mathscr{S}_n$ for all $n \geq 0$.

We will see the symmetric group S_n as a (S_n, S_n)-bimodule. It is also possible to see S_n as a (S_k, S_n)-bimodule or as a (S_n, S_k)-bimodule for $k \leq n$. It follows that induction and restriction can be thought of as tensoring with certain bimodules:

$$Ind_n^{n+1} \longrightarrow {}_{n+1}S_{n+1} \otimes_n -_n \tag{3.11}$$

$$Res_n^{n-1} \longrightarrow {}_{n-1}S_n \otimes_n -_n \tag{3.12}$$

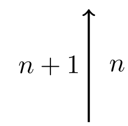
Therefore natural transformations between induction and restriction functors can be seen as bimodule morphisms.

One can discover the relations on the morphisms of $\mathcal{H}$ and their compositions by studying these bimodule morphisms.

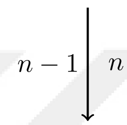
For example let's consider the category $\mathscr{S}_2$. $Ind_3^2 \circ Ind_1^2$ is not an object of $\mathscr{S}_2$ since the first functor doesn't apply to S_2. However $Res_4^3 \circ Ind_3^4 \circ Ind_2^3$ and $Ind_1^2 \circ Res_2^1$ are objects in $\mathscr{S}_2$.

An upward strand with label n on the right and $n+1$ on the left will stand for the identity endomorphism of the induction functor Ind_n^{n+1}, and a downward strand with label n on the right and $n-1$ on the left will stand for the identity endomorphism of the restriction functor Res_n^{n-1}:

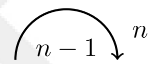
$$Id : Ind_n^{n+1} \to Ind_n^{n+1} \qquad \longrightarrow \tag{3.13}$$



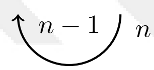
$$Id : Res_n^{n-1} \to Res_n^{n-1} \qquad \longrightarrow \tag{3.14}$$



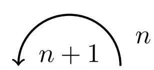
$$\begin{array}{c} {}_nS_n \otimes_{n-1} S_n \to_n S_n \\ g \otimes h \mapsto gh \end{array} \qquad \longrightarrow \tag{3.15}$$



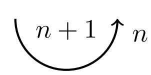
$$q_n :_n S_n \to_n S_n \otimes_{n-1} S_n \qquad \longrightarrow \tag{3.16}$$



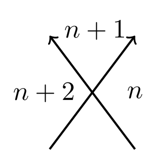
$$p_n :_n (S_{n+1})_n \to_n (S_n)_n \qquad \longrightarrow \tag{3.17}$$



$$\begin{array}{c} i_n :_n (S_n)_n \to_n (S_{n+1})_n \\ g \mapsto g \end{array} \qquad \longrightarrow \tag{3.18}$$

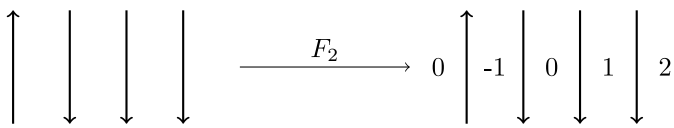


$$\begin{array}{c} {}_{n+2}(S_n)_n \to_{n+2} (S_n)_n \\ g \mapsto gs_{n+1} \end{array} \qquad \longrightarrow \tag{3.19}$$



where $p_n :_n (S_{n+1})_n \to_n (S_n)_n$ is the projection map given by $p_n(g) = g$ if $g \in S_n$, $p_n(g) = 0$ if $g \notin S_n$, and $q_n :_n S_n \to_n S_n \otimes_{n-1} S_n$ is the bimodule map determined by $q_n(1) = \sum_{i=i}^{n} s_i s_{i+1} \ldots s_{n-1} \otimes s_{n-1} s_{n-2} \ldots s_i$.

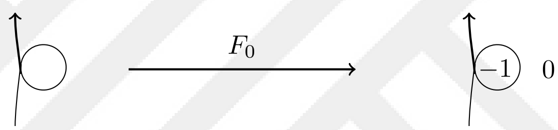
With the above notation, it is easy to see that the effect of the functor F_n on a morphism of $\mathcal{H}$ is to label the rightmost region by n. Upward strands increase the label by one, downward strands decrease the label by one as one reads from right to left. We also declare diagrams with negative label to be zero. For example



(3.20)

will be the zero morphism in $\mathscr{S}_2$.

Similarly,

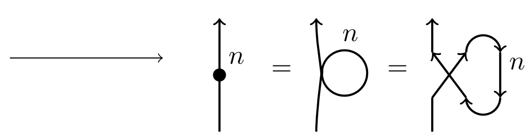


(3.21)

will be the zero morphism in $\mathscr{S}_0$.

Since we know the images of generating morphisms under F_n, we can calculate the image of a particular morphism that plays an important in the story, namely an upward strand with a solid dot.

$$\begin{gathered} {}_{n+1}(S_{n+1})_n \to {}_{n+1}(S_{n+1})_n \\ g \mapsto gJ_n \end{gathered}$$



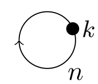
(3.22)

where J_n is the Jucys-Murphy's element

$$J_n = \sum_{i=1}^{n} s_i \dots s_n \dots s_i = (1, n+1) + (2, n+1) + \cdots + (n, n+1).$$

Another interesting set of morphisms is the endomorphisms of $\mathbb{1}$. Under the functor F_n, elements of $End_{\mathcal{H}}(\mathbb{1})$ are mapped to the (S_n, S_n)-bimodule homomorphisms of multiplication by a central element of $\mathbb{C}[S_n]$.

$$
\begin{aligned}
{}_n(S_n)_n &\to_n (S_n)_n \\
1 &\mapsto \sum_{i=1}^{n} s_i \dots s_{n-1} J_{n-1}^{k} s_{n-1} \dots s_i
\end{aligned}
\longrightarrow \tag{3.23}
$$



3.3 Twisted Heisenberg Algebra

There is a twisted version of the Heisenberg algebra:

Definition 3.3.1. The twisted Heisenberg algebra $\mathfrak{h}_{tw}$ is the unital associative $\Bbbk$ algebra with generators $\{h_{\frac{n}{2}}\}_{n\in 2\mathbb{Z}+1}$ and relations $[h_{\frac{n}{2}}, h_{\frac{m}{2}}] = \frac{n}{2}\delta_{n,-m}$.

This twisted version of the Heisenberg algebra comes from a central extension of the affine lie algebra corresponding to $\mathfrak{sl}_2$ as follows:

Consider the complex simple Lie algebra $\mathfrak{sl}_2$ with Chevalley generators e, f, h. It has an involution θ given by $\theta : e \mapsto f$, $\theta : f \mapsto e$ and $\theta : h \mapsto -h$. Now define two Lie subalgebras

$$\mathfrak{sl}_{2,0} = \{g \in \mathfrak{sl}_2 | \theta(g) = g\} \text{ and } \mathfrak{sl}_2, 1 = \{g \in \mathfrak{sl}_2 | \theta(g) = -g\}$$

Then the twisted affine Lie algebra is given by

$$\hat{\mathfrak{sl}_2} = \left(\mathfrak{sl}_{2,0} \otimes \mathbb{C}[t^2, t^{-2}]\right) \oplus \left(\mathfrak{sl}_{2,1} \otimes t\mathbb{C}[t^2, t^{-2}]\right) \oplus \mathbb{C}c$$

with the Lie bracket

$$[x \otimes t^n, y \otimes t^m] = (n+m)[x, y] + \frac{n}{2}\delta_{n,-m}(x, y)c$$

where c is central and (x, y) stands for the Killing form on $x, y \in \mathfrak{sl}_2$

To obtain the twisted Heisenberg algebra, one needs to consider the above construction restricted to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_2$, take its universal enveloping algebra and quotient out by $c-1$:

$$\mathfrak{h}_{tw} = U\big(\mathfrak{h} \otimes t\mathbb{C}[t^2, t^{-2}] \oplus \mathbb{C}c\big)/(c-1)$$

where the Lie bracket restricts to

$$[h \otimes t^n, h \otimes t^m] = \frac{n}{2}\delta_{n,-m}.$$

By letting $h_{\frac{n}{2}} = h \otimes t^n$ one recovers the definition 3.3.1.

3.4 Twisted Heisenberg Category

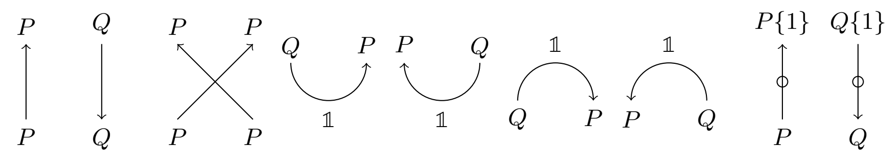
Cautis and Sussan describe a twisted version of Khovanov's Heisenberg category in [8]. The role of the symmetric group S_n played in $\mathcal{H}$ is replaced this time by the Sergeev algebra(a.k.a. Hecke-Clifford algebra). The morphisms of the twisted Heisenberg category correspond to natural transformations between the induction restriction functors on the Sergeev algebra supermodules. For more information on super representation theory and representations of the Sergeev algebra, we refer the reader to [46] and to [24].

Definition 3.4.1. Sergeev algebra is the unital associate $\Bbbk$ superalgebra

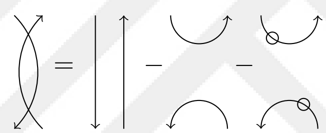
$$\mathbb{S}_n := \mathcal{C}\ell_n \rtimes \mathbb{C}[S_n]$$

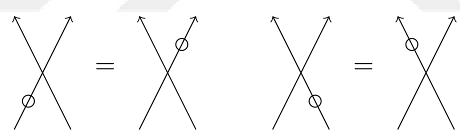
where $\mathcal{C}\ell_n$ stands for the Clifford algebra with generators $c_1, \ldots, c_n$ and relations $c_i^2 = 1$, $c_i c_j = -c_j c_i$ for $i \neq j$. The action of S_n on these generators is by permuting their indices. It becomes a super algebra by declaring the Clifford generators to have degree $\bar{1}$ and the symmetric group generators to have degree $\bar{0}$.

The twisted Heisenberg Heisenberg category $\mathcal{H}_{tw}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $\Bbbk$-linear additive monoidal category with two generating objects P, Q of degree $\bar{0}$. Due to the $\mathbb{Z}/2\mathbb{Z}$-grading, we also have objects $P\{1\}$ and $Q\{1\}$ with degree $\bar{1}$. The morphism spaces of $\mathcal{H}_{tw}$ are generated by the diagrams

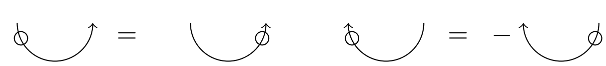


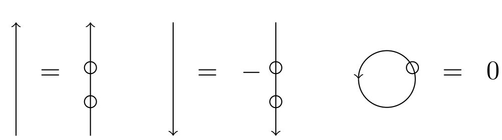
where the last two generators have $\mathbb{Z}/2\mathbb{Z}$-degree $\bar{1}$, others have $\mathbb{Z}/2\mathbb{Z}$-degree $\bar{0}$. The generating morphisms satisfy the relations (3.1), (3.2), (3.4) of the Heisenberg category, and


(3.24)


(3.25)

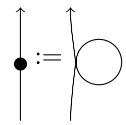

(3.26)


(3.27)


(3.28)

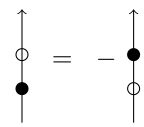

(3.29)

Again by declaring

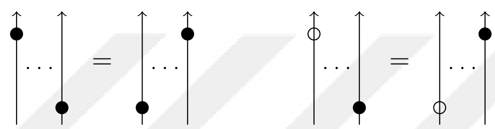


(3.30)

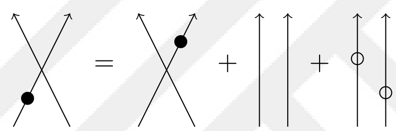
we get the following relations:



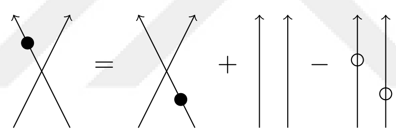
(3.31)



(3.32)



(3.33)



(3.34)

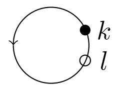
The main difference between $\mathcal{H}$ and $\mathcal{H}_{tw}$ is the hollow dots bringing in the $\mathbb{Z}/2\mathbb{Z}$-grading. The relations (3.26),(3.27),(3.29) indicate that the relative positions of hollow dots with respect to hollow dots on different strands and where they are placed on cup and cap diagrams is important. This means that the diagrams are not well defined up to isotopy, but they are well defined up to a sign up to isotopy. To fix the sign, one needs to work with isotopies that fix the relative heights of the hollow dots.

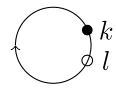
3.4.1 Endomorphism Spaces of $\mathcal{H}_{tw}$

In the same spirit of subsection 3.2.1, first we will study the spaces $End_{\mathcal{H}_{tw}}(\mathbb{1})$, $End_{\mathcal{H}_{tw}}(P^n)$, $End_{\mathcal{H}_{tw}}(Q^m)$ to understand $End_{\mathcal{H}_{tw}}(P^nQ^m)$.

Elements of $End_{\mathcal{H}_{tw}}(\mathbb{1})$ can be expressed as linear combinations of bubbles, but this time we allow hollow dots to appear on the bubbles.

Fefine

$\bar{d}_{k,l} :=$ 

and $d_{k,l} :=$ 

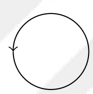
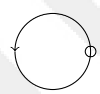
for $k, l \in \mathbb{Z}_{\geq 0}$.

Given any closed diagram with any configuration of dots, it is possible to collect the hollow dots and the solid dots together, possibly after multiplying the diagram by -1, by using the relations (3.31), (3.26) and (3.27). After regrouping, we may assume that the dots are placed on the right middle side of the diagram as above.

Moreover, using the left two equations in relation (3.28), we can erase a pair of hollow dots, possibly by changing the sign of the diagram.

Therefore the set $\{d_{k,l}, \bar{d}_{k,l} | k \in \mathbb{Z}_{\geq 0}, l \in \{0,1\}\}$ is a spanning set for $End_{\mathcal{H}_{tw}}(1)$.

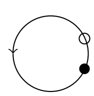
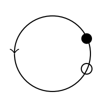
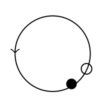
In our defining relations, we have that

$\bar{d}_{0,0} =$  $= 1$ and $\bar{d}_{0,1} =$  $= 0$.

Actually whenever we have hollow dot on a bubble, it will be equal to zero:

Lemma 3.4.1. $\bar{d}_{k,1} = 0$ and $d_{k,1} = 0$ for all non-negative integers k.

Proof. An example computation shows that

$\bar{d}_{1,1} =$  $= -$  $= -$  $= -\bar{d}_{1,1} = 0,$

where in the second equality, negative sign comes from relation (3.31), and the third equality comes from sliding the solid dot around.

Now more generally, if we have k solid dots where k is an even integer, then sliding the hollow dot around the circle and passing it through k solid dots multiplies the diagram by $(-1)^{k+1}$, and being equal to its negative,the diagram is zero. If k is an odd number, sliding a solid dot around the circle and passing it through a hollow dot catches a minus sign, so these diagrams are zero as well.

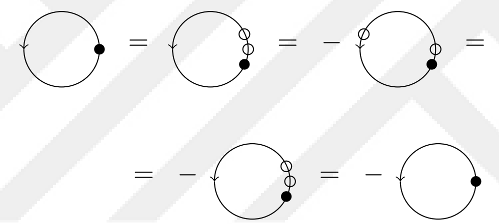
These arguments do not depend on the orientation of the bubble, hence the result follows. ☐

From now on, we will assume that the second index in $\bar{d}_{k,l}$ and $d_{k,l}$ is always zero. We will omit it from our notation and write d_k instead of $d_{k,0}$, and $\bar{d}_k$ instead of $\bar{d}_{k,0}$. But actually the bubbles with an odd number of solid dots are zero.

Lemma 3.4.2. We have that $d_{2n+1} = \bar{d}_{2n+1} = 0$ for all non-negative integers n.

Proof. Note that

$\bar{d}_1 =$



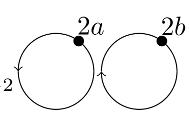
$= 0.$

The same arguments works for any odd number of solid dots and works for clockwise oriented bubbles. ☐

The following lemma shows that it is possible to express counter-clockwise bubbles in terms of clockwise bubbles.

Lemma 3.4.3. We have that

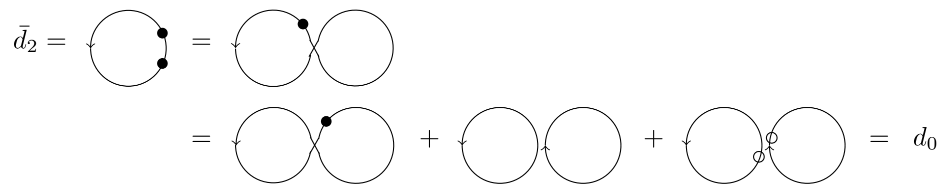
$$\bar{d}_{2n} = \sum_{2a+2b=2n-2}$$



$$= \sum_{2a+2b=2n-2} \bar{d}_{2a} d_{2b}$$

for any positive integer $n \geq 1$ where the sum is over positive integers a, b.

Proof. For the $n = 1$ case, we have the following computation:



where the first diagram on right hand side is zero since it contains a left curl, the second term is $\bar{d}_0 d_0 = d_0$ and the last term is zero by Lemma 3.4.1.

For general n, if you replace one of the solid dots with a right-twist curl, and slide the remaining $2n-1$ dots through the crossings using relations (3.33) and (3.34) repeatedly, we will get many resolution terms, consisting of a sum of product of counterclockwise and clockwise bubbles, some with only solid dots, some with hollow dots as well. The terms with hollow dots are zero, and so are the terms with an odd number of solid dots. Also, the figure eight shape contains a left twist curl, so it is zero as well, which proves the statement. $\square$

Therefore there is a surjective algebra map $\psi : \Bbbk[d_0, d_2, d_4, \dots] \longrightarrow End_{\mathcal{H}_{tw}}(\mathbb{1})$. We will show later that the bubbles $\{d_0, d_2, \dots\}$ are in fact algebraically independent using the action of $\mathcal{H}_{tw}$ on the category of induction and restriction on Sergeev algebras. This will give us an algebra isomorphism

$$End_{\mathcal{H}_{tw}}(\mathbb{1}) \simeq \Bbbk[d_0, d_2, d_4, \dots].$$

Recall that in the untwisted case, the degenerate affine Hecke algebra was crucial to the understanding of the spaces $End_{\mathcal{H}_{tw}}(P^n)$ and $End_{\mathcal{H}}(Q^m)$. Replacing the symmetric group with the Sergeev algebra in the construction of the Heisenberg category has the effect of changing the role of the degenerate affine Hecke algebra played in endomorphism spaces of P^n, Q^m with the degenerate affine Hecke-Clifford algebra.

Definition 3.4.2. The degenerate affine Hecke-Clifford algebra $DAHC_n$ of type A is the unital associative $\Bbbk$ algebra generated by transpositions $\{s_i\}_{i=1,\ldots,n-1}$, polynomial generators $\{x_i\}_{i=1,\ldots,n}$ and Clifford generators $\{c_i\}_{i=1,\ldots,n}$ such that $\Bbbk[x_1,\ldots,x_n]$, $\mathcal{C}\ell_n$ and $\mathbb{C}[S_n]$ are subalgebras and

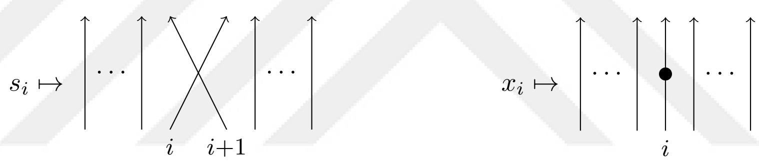
$$x_ic_i = -c_ix_i, \quad x_ic_j = c_jx_i \quad \text{for } i \neq j, \tag{3.35}$$

$$\sigma c_i = c_{\sigma(i)}\sigma \quad \text{for } 1 \leq i \leq n \text{ and } \sigma \in S_n, \tag{3.36}$$

$$x_{i+1}s_i - s_ix_i = 1 - c_{i+1}c_i, \tag{3.37}$$

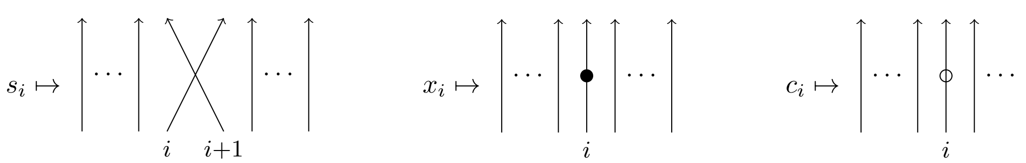
$$x_js_i = s_ix_j \quad \text{for } j \neq i, i+1. \tag{3.38}$$

For the untwisted case, we had a morphism $DAH_n \mapsto End_{\mathcal{H}}(P^n)$ given by



(3.39)

For $\mathcal{H}_{tw}$, we replace this morphism by $g : DAHC_n \mapsto End_{\mathcal{H}_{tw}}(P^n)$ given by



(3.40)

It is a consequence of the defining diagrammatic relations of $\mathcal{H}_{tw}$ that g is an injective algebra homomorphism.

In fact we have algebra isomorphisms

$$End_{\mathcal{H}_{tw}}(P^n) \simeq DAHC_n^{op} \otimes \Bbbk[d_0, d_2, d_4, \ldots]$$

and

$$End_{\mathcal{H}_{tw}} Q^m \simeq DAHC_m \otimes \Bbbk[d_0, d_2, d_4, \dots]$$

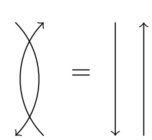
analogous to the untwisted case. Recall that the multiplication in the opposite super-algebra is given by $a * b := (-1)^{|a||b|} ba$. This agrees with the fact that two hollow dots on an upward oriented strand is equal to 1 and on a downward oriented strand equal to -1.

To relate $End_{\mathcal{H}_{tw}}(P^n Q^m)$ to $End_{\mathcal{H}_{tw}}(\mathbb{1})$, $End_{\mathcal{H}_{tw}}(P^n)$ and $End_{\mathcal{H}_{tw}}(Q^m)$, let $n, m \geq 0$ and define $J_{n,m}$ to be the 2-sided ideal in $End_{\mathcal{H}_{tw}}(P^n Q^m)$ generated by diagrams which contain at least one arc connecting a pair of upper points.

Lemma 3.4.4. There exists a split short exact sequence

$$0 \to J_{m,n} \to End_{\mathcal{H}_{tw}}(P^n Q^m) \to (DAHC_n)^{op} \otimes DAHC_m \otimes \Bbbk[d_0, d_2, d_4....] \to 0.$$

Proof. In $End_{\mathcal{H}_{tw}}(P^n Q^m)$, due to the middle diagram in relation (3.2), we can assume our diagrams have no crossing between opposite oriented strands. Taking the quotient $End_{\mathcal{H}_{tw}}(P^n Q^m)/J_{m,n}$ kills diagrams with cups connecting two upper points, and those with caps connecting two lower points. Therefore we are left with diagrams, possibly with bubbles, which have no caps or cups and have crossings only among like-oriented strands. Note that in the quotient $End_{\mathcal{H}_{tw}}(P^n Q^m)/J_{n,m}$, the diagram in relation (3.24) simplifies to



and therefore we can move the bubbles in our diagrams to the rightmost part of our diagrams for free. This gives us a short exact sequence

$$0 \to J_{n,m} \to End_{\mathcal{H}_{tw}}(P^n Q^m) \to End_{\mathcal{H}_{tw}}(P^n) \otimes End_{\mathcal{H}_{tw}}(Q^m) \otimes End_{\mathcal{H}_{tw}}(\mathbb{1}) \to 0.$$

By [8, Proposition 7.1], we have that $End_{\mathcal{H}_{tw}}(P^n)$ without the bubbles is isomorphic to $DAHC_n^{op}$ and that $End_{\mathcal{H}_{tw}}(Q^m)$ without the bubbles is isomoprhic to $DAHC_m$. By Proposition 3.4.7, we have that $End_{\mathcal{H}_{tw}}(\mathbb{1})$ is isomorphic to $\Bbbk[d_0, d_2, d_4....]$. Hence the result follows. □

Lemma 3.4.5. If $f, g \in DAHC_n$ such that $fg = 1$, then $f, g \in \mathcal{C}\ell_n \rtimes \mathbb{C}[S_n] \subset DAHC_n$.

Proof. There is an $\mathbb{N}$-filtration on $DAHC_n$ given by $\deg(x_i) = 1$ for $i \in \{1, ..., n\}$ and other generators have degree zero. Under this filtration, the degree zero part of $DAHC_n$ is the semidirect product $Cl_n \rtimes \mathbb{C}[S_n]$. Therefore, in the associated graded object, we see that if $fg = 1$, $\deg(gr(f)gr(g)) = \deg(gr(f)) + \deg(gr(g)) = \deg(1) = 0$, hence $gr(f), gr(g)$ are in degree zero part. Therefore $f, g \in \mathcal{C}\ell_n \rtimes \mathbb{C}[S_n]$. □

Lemma 3.4.6. The indecomposable objects of $\mathcal{H}_{tw}$ are of the form P^nQ^m for $n, m \in \mathbb{Z}_{\geq 0}$.

Proof. First, note that if QP appears in an object, that object can be decomposed into more components using the diagram in relation (3.24). Hence all indecomposable objects must be of the form P^nQ^m.

On the other hand, to see that every sequence of the form P^nQ^m is an indecomposable object, we will show that any idempotent in $End_{\mathcal{H}_{tw}}(P^nQ^m)$ has to be the identity.

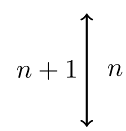
Let f, g be two maps as mentioned in Lemma 3.4.5. Note that gf is an idempotent since $(gf)(gf) = g(fg)f = gf$. Since we had the splitting short exact sequence $0 \to J_{m,n} \to End_{\mathcal{H}_{tw}}(P^nQ^m) \to End_{\mathcal{H}_{tw}}(P^n) \otimes End_{\mathcal{H}_{tw}}(Q^m) \otimes End_{\mathcal{H}_{tw}}(\mathbb{1}) \to 0$ in Lemma 3.4.4, we know that the maps f and g will decompose into (f_1, f_2) and (g_1, g_2) where $f_1, g_1 : P^n \to P^n$ and $(f_2, g_2) : Q^m \to Q^m$. Now g_1f_1 is the identity map in $End_{\mathcal{H}_{tw}}(P^n)$, and by the above lemma $g_1, f_1 \in \mathcal{C}\ell_n \rtimes \mathbb{C}[S_n]$. Similarly, $f_2, g_2 \in \mathcal{C}\ell_n \rtimes \mathbb{C}[S_n]$.

But in $\mathcal{C}\ell_n \rtimes \mathbb{C}[S_n]$, $g_1f_1 = 1$ implies that $f_1g_1 = 1$ as well. To see this, consider the diagrams corresponding to g_1 and f_1 which consist of a permutation and some hollow dots on top. After composing these diagrams, we can collect all the hollow dots on the top since hollow dots can pass through crossing for free, possibly gaining a sign. Furthermore, each strand has an even number of hollow dots, since this composition is the identity map. So, the hollow dots cancel with each other. This shows that the corresponding permutations of f_1 and g_1 are inverses of each other, and in particular they commute. Therefore $f_1 g_1 = 1$. Similarly, $f_2 g_2 = 1$. Thus we have that $fg = 1$. $\square$

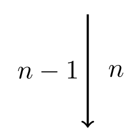
3.4.2 $\mathcal{H}_{tw}$ as Categorified Induction and Restriction

The category $\mathcal{H}_{tw}$ can be seen as a categorification of the induction and restriction functors of Sergeev algebra super-bimodules in an analogus way to the untwisted case. To be more precise, let $\underline{Ser}_n$ be the category whose objects are induction and restriction functors between Sergeev algebras $\mathbb{S}_r$, starting from $\mathbb{S}_n$, and whose morphisms are natural transformations between these induction and restriction functors. Then there is a functor $F_n^{tw} : \mathcal{H}_{tw} \mapsto \underline{Ser}_n$ given by labelling the rightmost region of a diagram by n. In the twisted case, the corresponding super-bimodule morphisms are as follows:

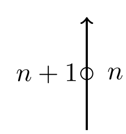
$$Id : Ind_n^{n+1} \to Ind_n^{n+1} \longrightarrow \tag{3.41}$$



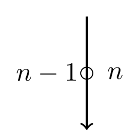
$$Id : Res_n^{n-1} \to Res_n^{n-1} \longleftrightarrow \tag{3.42}$$



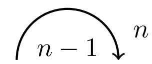
$$\begin{matrix} {}_{n+1}(\mathbb{S}_{n+1})_n \to {}_{n+1}(\mathbb{S}_{n+1})_n \\ g \mapsto (-1)^{|x|} g c_{n+1} \end{matrix} \longleftrightarrow \tag{3.43}$$



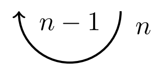
$$\begin{matrix} {}_{n+1}(\mathbb{S}_{n+1})_n \to {}_{n+1}(\mathbb{S}_{n+1})_n \\ g \mapsto c_{n+1} g \end{matrix} \longleftrightarrow \tag{3.44}$$



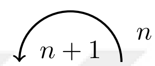
$$
\begin{gathered}
{}_n(\mathbb{S}_n) \otimes_{n-1} \mathbb{S}_n \to_n (\mathbb{S}_n)_n \\
g \otimes h \mapsto gh
\end{gathered}
\quad \longleftrightarrow \tag{3.45}
$$



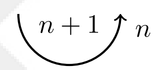
$$
q'_n :_n (\mathbb{S}_n)_n \to_n \mathbb{S}_n \otimes_{n-1} (\mathbb{S}_n)_n \quad \longleftrightarrow \tag{3.46}
$$



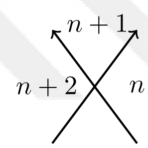
$$
p'_n :_n (\mathbb{S}_{n+1})_n \to_n (\mathbb{S}_n)_n \quad \longleftrightarrow \tag{3.47}
$$



$$
\begin{gathered}
i_n :_n (\mathbb{S}_n)_n \to_n (\mathbb{S}_{n+1})_n \\
g \mapsto g
\end{gathered}
\quad \longleftrightarrow \tag{3.48}
$$



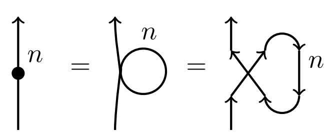
$$
\begin{gathered}
{}_{n+2}(\mathbb{S}_n)_n \to_{n+2} (\mathbb{S}_n)_n \\
g \mapsto g s_{n+1}
\end{gathered}
\quad \longleftrightarrow \tag{3.49}
$$



where $p'_n :_n (\mathbb{S}_{n+1})_n \to_n (\mathbb{S}_n)_n$ is the projection map given by $p_n(g) = g$ if $g \in \mathbb{S}_n$, $p_n(g) = 0$ if $g \notin \mathbb{S}_n$, and $q'_n :_n \mathbb{S}_n \to_n \mathbb{S}_n \otimes_{n-1} \mathbb{S}_n$ is the bimodule map determined by $q'_n(1) = \sum_{i=i}^{n} s_i s_{i+1} \dots s_{n-1} \otimes s_{n-1} s_{n-2} \dots s_i - s_i s_{i+1} \dots s_{n-1} c_n \otimes c_n s_{n-1} s_{n-2} \dots s_i$.

The image of a solid dot under F_n^{tw} this time corresponds to an even Jucys-Murphy's element:

$$
\begin{gathered}
{}_{n+1}(\mathbb{S}_{n+1})_n \to_{n+1} (\mathbb{S}_{n+1})_n \\
g \mapsto g J_{n+1,odd}
\end{gathered}
\quad \longleftrightarrow \tag{3.50}
$$

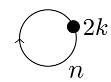


where $J_{n+1,even}$ is the odd Jucys-Murphy's element

$$
J_{n+1,even} = \sum_{i=1}^{n} (1 - c_{n+1} c_i)(i, n+1).
$$

The functor F_n^{tw} sends elements of $End_{\mathcal{H}_{tw}}(\mathbb{1})$ to the $(\mathbb{S}_n, \mathbb{S}_n)$-super bimodule morphisms of multiplication by an element in the even center of $\mathbb{S}_n$:

$$\begin{array}{c} {}_n(\mathbb{S}_n)_n \to {}_n(\mathbb{S}_n)_n \\ 1 \mapsto \sum_{i=1}^{n} s_i \dots s_{n-1} J_{n,even}^{2k} s_{n-1} \dots s_i - c_n s_i \dots s_{n-1} J_{n,even}^{2k} s_{n-1} \dots s_i c_1 \end{array} \longrightarrow$$



(3.51)

Now to prove the algebraic independence of the bubbles $\{d_0, d_2, d_4, \dots\}$, we can apply the filtration argument on the number of disturbances of permutations as done in [19, Section 4].

Proposition 3.4.7. The elements $\{d_{2k}\}_{k \geq 0}$ are algebraically independent and there is an isomorphism

$$End_{\mathcal{H}_{tw}}(\mathbb{1}) \cong \Bbbk[d_0, d_2, d_4, ...].$$

Chapter 4

Trace of the twisted Heisenberg Category

In this chapter, we will establish an isomorphism between the trace decategorification of the twisted Heisenberg category and a subalgebra of the vertex operator algebra $W_{1+\infty}$ fixed under an anti-involution.

4.1 W-algebras $W_{1+\infty}$ and W^-

Let $\mathfrak{D}^a$ denote the unital associative $\mathbb{C}$ algebra of differential operators on the circle, where a stands for associative. That is $\mathfrak{D}^a$ is the algebra generated by the operators of multiplication by t and differetiation with respect to t, $\dfrac{d}{dt}$ on $\mathbb{C}[t, t^{-1}]$. $\mathfrak{D}^a$ has a natural vector space basis $\{t^k(\frac{d}{dt})^l\}_{k\in\mathbb{Z}, l\in\mathbb{Z}_{\geq 0}}$. Define $D = t\frac{d}{dt}$. Then another basis for $\mathfrak{D}^a$ is given by $\{w_{k,l} := t^k D^l\}_{k\in\mathbb{Z}, l\in\mathbb{Z}_{\geq 0}}$. Multiplication is given by composition of operators. $\mathfrak{D}^a$ is a $\mathbb{Z}$-graded algebra via

$$\mathfrak{D} = \bigoplus_{i\in\mathbb{Z}} t^i\mathbb{C}[D]$$

.

We call this grading the principal gradation.

Denote by $\mathfrak{D}$ the Lie algebra corresponding to $\mathfrak{D}^a$ by defining the usual commutator as the Lie bracket. Principal gradation of $\mathfrak{D}^a$ turns $\mathfrak{D}$ into a $\mathbb{Z}$-graded Lie algebra(i.e. $\mathfrak{D} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{D}_i$ where $[\mathfrak{D}_i, \mathfrak{D}_j] \subset \mathfrak{D}_{i+j}$). On the basis elements, the bracket is given by

$$[t^r D^k, t^s D^l] = t^{r+s}((D+s)^k D^l - D^k(D+r)^l)$$

.

A central extension $\hat{\mathfrak{D}}$ of $\mathfrak{D}$ is described in [17]. Here we recall its definition. $\hat{\mathfrak{D}}$ is the Lie algebra generated by C and by $w_{k,l} = t^k D^l$ for $k \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq 0}$, subject to relations that C and $w_{0,0}$ are central, and:

$$[t^r f(D), t^s g(D)] = t^{r+s}\big(f(D+s)g(D) - f(D)g(D+r)\big) + \psi\big(t^r f(D), t^s g(D)\big)C, \tag{4.1}$$

where the 2-cocycle is given by

$$\psi\big(t^r f(D), t^s g(D)\big) = \begin{cases} \sum_{-r\leq j\leq -1} f(j)g(j+r) & r = -s \geq 0 \\ 0 & r+s \neq 0 \end{cases} \tag{4.2}$$

for f, g polynomials.

Note that the central extension has drastic effects on the Lie algebra structure. For example powers of t don't necessarily commute, since $[t^r, t^{-r}] = \psi(t^r, t^{-r})C = rC$.

The W-algebra $W_{1+\infty}$ is the universal envelopping algebra of $\hat{\mathfrak{D}}$.

We will identify trace of the twisted Heisenberg category with the universal enveloping algebra of a central extension of a Lie subalgebra of $\mathfrak{D}$, fixed by a degree preserving anti-involution. Consider the map σ given by

$$
\begin{array}{rcccc}
 & \mathfrak{D} & \xrightarrow{\text{central extension}} & \widehat{\mathfrak{D}} & \xrightarrow{\text{enveloping algebra}} & W_{1+\infty} \\
\text{fixed by -}\sigma & \cup & & \cup & & \cup \\
 & \mathfrak{D}^- & \xrightarrow{\text{central extension}} & \widehat{\mathfrak{D}^-} & \xrightarrow{\text{enveloping algebra}} & W^-
\end{array}
$$

Figure 4.1: Relation of algebra of differential operators and the W-algebras $W_{1+\infty}$ and W^-

$$
\begin{aligned}
\sigma : \quad \mathfrak{D} &\longrightarrow \mathfrak{D} \\
1 &\mapsto \sigma(1) = -1 \\
t &\mapsto \sigma(t) = -t \\
D &\mapsto \sigma(D) = -D.
\end{aligned}
$$

Then $-\sigma$ is a degree preserving anti-involution of $\mathfrak{D}$, and the Lie subalgebra fixed by $-\sigma$ is

$$\mathfrak{D}^- := \{a \in \mathfrak{D} | \sigma(a) = -a\}.$$

The 2-cocycle ψ of $\mathfrak{D}$ restricts to a 2-cocycle of $\mathfrak{D}^-$. Let $\hat{\mathfrak{D}}^-$ denote the central extension of $\mathcal{D}^-$ by the restriction of ψ. Therefore $\hat{\mathfrak{D}}^-$ is a Lie subalgebra of $\hat{\mathfrak{D}}$.

More explicitly, $\hat{\mathfrak{D}}^-$ is the Lie algebra over the vector space spanned by $\{C\} \cup \{w_{0,0}\}\, cup \{t^{2k+1} g(D + 2k + \frac{1}{2});\ g \text{ even polynomial function}\} \cup \{t^{2k} f(D+k);\ f \text{ odd polynomial function}\}$ for $k \in \mathbb{Z}$. Its Lie bracket is given by Equation (4.1).

Denote by W^- the universal enveloping algebra of $\hat{\mathfrak{D}}^-$. We will show that trace of twisted Heisenberg category is isomorphic to $W^- / (w_{0,0}, C - 1)$.

Note that not all $w_{k,\ell}$ are contained in W^-.

Example 1.1 When $k - \ell$ is an even integer, $w_{k,\ell} \notin W^-$. Moreover, the difference $k - \ell$ being odd is not sufficient. For example, $t^2 D = w_{2,1} \notin W^-$ since an element starting with t^2

should be followed by $f(D+1)$ where f is an odd polynomial function. Also it is easy to see that $\sigma(t^2D) = -Dt^2 = -tD^2 - 2tD - t$. Hence $t^2D = w_{2,1} \notin W^-$. However $t^2(D+1) = t^2D + t^2 = w_{2,1} + w_{2,0} \in W^-$ (and, indeed, $\sigma(t^2(D+1)) = t^2(-D-1) = -t^2(D+1)$).

Since $-\sigma$ is a degree preserving map, principal gradation of $\mathfrak{D}$ descends to a princical gradation of $\mathfrak{D}^-$. This in turn defines a $\mathbb{Z}$-filtration on $W_{1+\infty}$ and on W^-. Recall that with respect to the principal gradation $deg(w_{k,l}) = k$. It is also natural to consider the exponent of D in the basis $\{w_{k,l}\}$. This doesn't give a grading, however it gives a $\mathbb{Z}_{\geq 0}$ filtration. We will call this filtration the *differential filtration*, and denote it by $|.|_{differential}$. Another filtration comes from considering the difference $k-l$ in the expression t^kD^l. We will call this filtration the *difference filtration*, and denote it by $|.|_{difference}$.

These filtrations are compatible, so we have a $(\mathbb{Z} \times \mathbb{Z}^{\geq 0})$-filtration with with an element $f = t^j g(D - j/2) \in W^-$ in bidegree $\leq (|f|_{difference}, |f|_{differential}) = (\deg(g) - j, \deg(g))$, where $\deg(g)$ is the polynomial degree of $g(w) \in \mathbb{C}[w]$.

If we look at the differential filtration zero part of W^-, it is easy to see that as a vector space this is spanned by $\{C\} \cup \{t^{2n+1}\}_{n\in\mathbb{Z}}$. As an algebra, we have that

$$[t^{2n+1}, t^{2m+1}] = (2n+1)\delta_{n,-m} \tag{4.3}$$

Hence we have an isomorphism between the differential filtration zero part of W^- and the twisted Heisenberg algebra $\mathfrak{h}_{tw}$ given by:

$$\begin{aligned} \phi: \quad \mathfrak{h}_{tw} &\longrightarrow \bigcup_{r\in\mathbb{Z}} W^-_{r,0} \\ h_{\frac{2n+1}{2}} &\mapsto \frac{1}{\sqrt{2}} t^{2n+1} \end{aligned}$$

where $n \in \mathbb{Z}$.

4.1.1 Generators of the algebra W^-

We will describe two generating sets for the algebra W^-. One will be used to identify W^- with the trace of the twisted Heisenberg category. The other will be used to describe the action of W^- on a subalgebra of the symmetric functions, given by the action of the trace of the twisted Heisenberg category on its categorical center.

Lemma 4.1.1. The algebra $W^-/\langle w_{0,0}, C\rangle$ is generated by $w_{1,0}$, $w_{0,3}$, and $w_{\pm 2,1} \pm w_{\pm 2,0}$.

Proof. Let $t^k g(D - k/2)$ be an arbitrary element of W^-. Without loss of generality, we may assume g is a monic monomial of the form $g(w) = w^\ell$ with $\ell - k$ odd, since lower terms in g are just monomials of this form with lower degree, and thus can be generated separately. Therefore, we have

$$t^k g(D - k/2) = \sum_{i=0}^{\ell} \binom{\ell}{i} (-1)^{\ell-i} (k/2)^{\ell-i} t^k D^i. \tag{4.4}$$

The leading term of this element with respect to differential degree is $t^k D^\ell$. We will generate the leading term first, and address lower terms afterwards. There are two cases, depending on the parities of k and ℓ.

First, suppose that $k = 2n$ is even and $\ell = 2m + 1$ is odd (recall that k and ℓ must have opposite parity in W^-). Hence, we must generate $w_{\pm 2n, 2m+1}$. The following calculations are easy, using Formula 4.1:

$$[w_{-2,1} - w_{-2,0}, w_{1,0}] = w_{-1,0},$$

$$[w_{1,0}, w_{0,3}] = -3(w_{1,2} + w_{1,1}) - w_{1,0},$$

$$[w_{1,2b}, w_{0,3}] = -3w_{1,2b+2} + O(w_{1,2b+1}), \tag{4.5}$$

where $O(\omega)$ refers to terms with lower differential degree than ω. Hence, starting with $w_{1,2} - w_{1,1}$, we can use the Equation (4.5) above to generate $w_{1,2b}$ for any b. Now we have:

$$[w_{\pm 2a,1}, w_{1,0}] = w_{\pm 2a+1,0}, \tag{4.6}$$

$$[w_{\pm 2a+1,0}, w_{1,2} - w_{1,1}] = -(4a+2)w_{2a+2,1} - (2a+1)(2a+2)w_{2a+2,0}. \tag{4.7}$$

Thus, starting from $w_{2,1} + w_{2,0}$, we can generate $w_{2a,1}$ for any a. Finally, we have:

$$[w_{-1,0}, w_{1,2b}] = \sum_{i=0}^{2b-1} \binom{2b}{i} (-1)^{2b-i+1} w_{0,i} = w_{0,2b-1} + O(w_{0,2b-2}), \tag{4.8}$$

$$\begin{aligned} [w_{\pm 2a,1}, w_{0,2b-1}] &= -\sum_{i=0}^{2b-2} \binom{2b-1}{i} (\pm 1)^{2b-i} (2)^{2b-2-i} t^{2a} D^{i+1} \\ &= w_{2a,2b-1} + O(w_{2a,2b-2}). \end{aligned}$$

So, we can generate a polynomial with leading term $w_{\pm 2n,2m+1}$.

Next, suppose that $k = 2n+1$ is odd and positive and $\ell = 2m$ is even. Using Formula (4.1), we have:

$$\begin{aligned} [w_{2a+1,0}, w_{0,2b+1}] &= t^{2a+1} \sum_{i=0}^{2b} \binom{2b+1}{i} (2a+1)^{2b+1-i} D^i \\ &= w_{2a+1,2b} + O(w_{2a+1,2b-1}). \end{aligned}$$

Now Equations (4.6) and (4.8) give that we can generate $w_{2a+1,0}$ and $w_{0,2b+1}$. Hence we can generate a polynomial with leading term $w_{2a+1,2b}$.

Finally, assume that $k = -(2n+1)$ is odd and $n = 2m$ is even. Using Formula (4.1), we have:

$$[w_{-2a,1}, w_{1,0}] = w_{1-2a,0}.$$

By Equation (4.7), we can therefore generate $w_{-(2a+1),0}$ for any a. Next, note that:

$$[w_{-1,0}, w_{1,2b}] = -\sum_{i=0}^{2b-1} \binom{2b-1}{i} (-1)^{2b-1-i} D^i = w_{0,2b-1} + O(w_{0,2b-2}).$$

By Equation (4.5), we can generator $w_{0,2b+1}$ for any b. Finally, we have

$$\begin{aligned}[w_{-(2a+1),0}, w_{0,2b-1}] &= t^{-(2a+1)} \sum_{i=0}^{2n-2} \binom{2n-1}{i} (-1)^{2n-i} (2a+1)^{2n-1-i} D^i \\ &= w_{-(2a+1),2b-2} + O(w_{-(2a+1),2b-3}).\end{aligned}$$

Thus, we can generate a polynomial with leading term $w_{-(2n+1),2m}$.

It remains to adjust the lower terms of these equations so that they match those in Equation (4.4). But note that each equation used above to generate the leading term results in lower terms which lie in different filtrations of W^-. Therefore we can adjust the coefficients of lower terms by scaling individual equations above. Since there is no dependency between these equations, we can choose constant coefficients for the generators so that our generated polynomial has the correct lower terms. □

Proposition 4.1.2. The algebra $W^-/\langle \omega_{0,0}, C-1 \rangle$ is also generated by $\omega_{1,0}$, $\omega_{-1,0}$ and $\omega_{0,3}$.

Proof. Let $A := \{\omega_{1,0}, \omega_{-1,0}, \omega_{0,3}\}$. We will show that we can obtain the generators in Proposition 4.1.1 using the elements of A. This amounts to obtaining $\omega_{\pm 2,1} \pm \omega_{\pm 2,0}$ via elements of A. It is a straightforward computation that

$$\omega_{0,1} = -\frac{1}{20}[[\omega_{0,3}, \omega_{-1,0}], \omega_{1,0}] + \frac{1}{5}\omega_{-1,0}\omega_{1,0}$$

and using $\omega_{0,1}$, we can obtain $\omega_{-1,2} - \omega_{-1,1}$ as follows

$$\omega_{-1,2} - \omega_{-1,1} = \frac{1}{6}[\omega_{0,3}, \omega_{-1,0}] + \frac{1}{3}\omega_{-1,0}\omega_{0,1}.$$

Then one of the elements we are looking for is given by

$$\omega_{-2,1} - \omega_{-2,0} = \frac{1}{2}[\omega_{-1,2} - \omega_{-1,1}, \omega_{-1,0}].$$

To obtain $\omega_{2,1} + \omega_{2,0}$, we follow a very similar computation:

$$\omega_{1,2} + \omega_{1,1} = -\frac{1}{6}[\omega_{0,3}, \omega_{1,0}] + \frac{1}{3}\omega_{0,1}\omega_{1,0}$$

and finally

$$\omega_{2,1} + \omega_{2,0} = -\frac{1}{2}[\omega_{1,2} + \omega_{1,1}, \omega_{1,0}].$$

□

4.1.2 Fock space representation of W^-

The algebra W^- inherits a Fock space representation from $W_{1+\infty}$. Let

$$W^{-,>} := \mathbb{C}\langle t^j g(D - j/2) in W^- \,|\, \deg(g) - j \geq 1\rangle$$

$$W^{-,0} := \mathbb{C}\langle g(D) \in W^- \,|\, \deg(g) \text{ odd}\rangle$$

$$W^{-,\geq} := W^{-,0} \oplus W^{-,>}.$$

For parameters $c, d \in \mathbb{C}$, let $\mathbb{C}_{c,d}$ be a one-dimensional module for $W^{-,\geq}$ on which each $w_{k,\ell}$ with $(k, \ell) \neq (0, 0)$ acts as zero, C acts as c, and $w_{0,0}$ acts as d. Let $\mathcal{M}_{c,d} := \mathrm{Ind}_{W^{-,\geq}}^{W^-} \mathbb{C}_{c,d}$. This induced module possesses the following properties:

Proposition 4.1.3. [1, 11] The W^--module $\mathcal{M}_{c,d}$ has a unique irreducible quotient $\mathcal{V}_{c,d}$, which is isomorphic as a vector space to $\mathbb{C}[w_{-1,0}, w_{-2,0}, w_{-3,0}, \ldots]$.

Proposition 4.1.4. [42] The action of $W^-/(C - 1, w_{0,0})$ is faithful on $\mathcal{V}_{1,0}$.

Proof. This follows immmediately from the argument in [42] for $W_{1+\infty}$ because W^- is a subalgebra. □

Proposition 4.1.3 allows us to the compute the action of the generators mentioned in Lemma 4.1.1 on $\mathcal{V}_{1,0}$, which we record for convenience below.

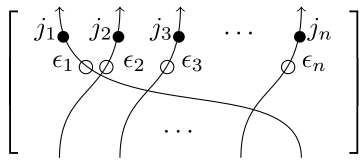
Proposition 4.1.5. Let k be a positive integer. The generators of W^- act on $\mathcal{V}_{1,0}$ as follows:

$$
\begin{aligned}
[w_{1,0}, w_{-k,0}] &= \delta_{1,k}, \\
[w_{-2,1} - w_{-2,0}, w_{-k,0}] &= (k+2) w_{-(k+2),0}, \\
[w_{2,1} + w_{2,0}, w_{-k,0}] &= -(k+2) w_{2-k,0}, \\
[w_{0,3}, w_{-k,0}] &= 3k w_{-k,2} - 3k^2 w_{-k,1} + k^3 w_{-k,0}.
\end{aligned}
$$

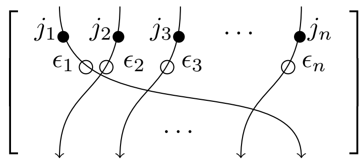
4.2 Diagrammatic computations in $\mathcal{H}_{tw}$ and in $\mathrm{Tr}(\mathcal{H}_{tw})$

This section contains some technical computations to derive relations between diagrams consisting of up and down strands. These relations allow us to find a generating set of $\mathrm{Tr}(\mathcal{H}_{tw})$ in Section 4.3.2.

We will see that any diagram in $\mathrm{Tr}(\mathcal{H}_{tw})$ can be written as a linear combinations of the diagrams we define below. We will mostly work with them, hence we introduce a notation for these diagrams. The brackets indicate that these diagrams are not in $\mathcal{H}_{tw}$, but they are in $\mathrm{Tr}(\mathcal{H}_{tw})$.

$h_n^{(x_1^{j_1}\cdots x_n^{j_n})(c_1^{\epsilon_1}\cdots c_n^{\epsilon_n})} :=$ 

,

$h_{-n}^{(x_1^{j_1}\cdots x_n^{j_n})(c_1^{\epsilon_1}\cdots c_n^{\epsilon_n})} :=$ 

,

where $\epsilon_i \in \{0,1\}$. In both of these elements, we consider the hollow dots to be descending in height from left to right, so that the dot labeled ϵ_1 is the highest.

Remark. These elements are analogues to the ones denoted $h_{\pm n} \otimes (x_1^{j_1} \cdots x_n^{j_n})$ in [6].

Additionally, set

$$h_n^{\sum x_i^{j_i}} = \sum h_n^{x_i^{j_i}}.$$

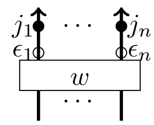
Lemma 4.2.1. For $n \geq 1$ and $1 \leq i \leq n-1$ we have

1. $h_{\pm n}^{x_i} = h_{\pm n}^{x_{i+1}} \pm h_{\pm i} h_{\pm(n-i)}$.

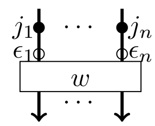
2. $h_{\pm n}^{x_i c_j} = -h_{\pm n}^{x_{i+1} c_{j+1}}$.

Proof. Part (1) is just [6, Lemma 14], except our solid dot sliding relation through crossing involves an extra term with hollow bubbles. But cycles with single hollow dot are zero since sending the hollow dot around the annulus gives us the same diagram with a negative sign. For the above calculations, our n-cycles split into smaller cycles with single hollow dot at least on one of them. The proof of part 2 depends on the relative position of i and j, but is a straightforward computation. □

Let $w \in S_n$, and define the elements:

$$f_{w;j_1,\ldots,j_n;\epsilon_1,\ldots,\epsilon_n} =$$ 

and

$$f_{w;j_1,\ldots,j_n;\epsilon_1,\ldots,\epsilon_n} =$$  .

Lemma 4.2.2. Let $w \in S_n$ and $(n_1, \ldots, n_r)$ be a composition of n. Then

$$[f_{\pm w;j_1,\ldots,j_n;\epsilon_1,\ldots,\epsilon_n}] = \sum d_{n_1,\ldots,n_r} h_{n_1}^{p_{n_1} c_{n_1}} \ldots h_{n_r}^{p_{n_r} c_{n_r}}$$

for constants $d_{n_1,\ldots,n_r} \in \mathbb{C}$, polynomials p_{n_i} in i variables, and elements c_{n_i} consisting of at most i Clifford generators (e.g. $c_{n_3} = \{c_1^{\epsilon_1} c_2^{\epsilon_2} c_3^{\epsilon_3} | \epsilon_i \in \{0, 1\}\}$).

Proof. We proceed by induction on $\sum \epsilon_i$. The base case is $\sum \epsilon_i = 0$; then

$$[f_{\pm w;j_1,\ldots,j_n;\epsilon_1,\ldots,\epsilon_n}] = [f_{\pm w;j_1,\ldots,j_n}]$$

and we apply [6, Lemma 15].

Now assume the statement is true for $\sum \epsilon_i = k$ for all $k < m \leq n$. Take $(\epsilon_1, \ldots, \epsilon_n)$ so that $\sum \epsilon_i = m$. Choose $g \in S_n$ such that $gwg^{-1} = w_\lambda$, where λ is the cycle type of w (so $gwg^{-1} = (s_1 \ldots s_{n_1-1}) \ldots (s_{n_1+\ldots+n_{r-1}} \ldots s_{n_1+\ldots+n_r-1})$). Let $p = x_1^{j_1} \ldots x_n^{j_n}$ and $c = c_1^{\epsilon_1} \ldots c_n^{\epsilon_n}$. Then we have

$$f_{\pm w;j_1,\ldots,j_n;\epsilon_1,\ldots,\epsilon_n} = pcw = (-1)^\epsilon cpw$$

where

$$\epsilon = \sum_{\epsilon_i = 1} j_i.$$

Thus conjugating by g gives that

$$\begin{aligned} gpcwg^{-1} &= (-1)^{\epsilon} gcpwg^{-1} \\ &= (-1)^{\epsilon} (g.c)gpwg^{-1} \\ &= (-1)^{\epsilon} \left[(g.c)(g.p)gwg^{-1} + (g.c)p_L wg^{-1} \right], \end{aligned}$$

where p_L is a polynomial of degree less than $j_1 + \ldots + j_n$. Note that gwg^{-1} is a product of cycles, so the first term in the above expression has the correct form. In the second term, we have $\{i|\epsilon_{g(i)} = 1\} \leq m$ (strict inequality can occur if g has fixed points). If $\{i|\epsilon_{g(i)} = 1\} < m$, we are done by induction, so assume that we have equality.

Now repeat the process on the second term, choosing a $g' \in S_n$ such that $g'(wg^{-1})(g')^{-1}$ is a product of cycles, and conjugating $(g.c)p_L wg^{-1}$. Each application of this process results in one term in which the symmetric group element is a product of cycles (which has the desired form), and one term with the degree of the polynomial part strictly lesser and the degree of the Clifford part weakly lesser.

If the degree of the Clifford part ever strictly decreases, we are done. If not, the conjugation will eventually reduce the degree of the polynomial part to 0, so we have an element of the form $c'\sigma$, $c' \in C\ell_n$ and $\sigma \in S_n$. Choose a $g'' \in S_n$ such that $g''\sigma(g'')^{-1}$ is a product of cycles; then

$$g'' c\sigma(g'')^{-1} = (g''.c)g''\sigma(g'')^{-1}.$$

This now has the desired form. □

Proposition 4.2.3. Let $w \in S_n$ and $(n_1, \ldots, n_r)$ a composition of n. Then

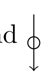
$$[f_{\pm w; j_1,\ldots,j_n;\epsilon_1,\ldots,\epsilon_n}] = \sum d_{n_1,\ldots,n_r} h_{\pm n_1}^{x_1^{\ell_1} c_1^{k_1}} \ldots h_{\pm n_r}^{x_1^{\ell_r} c_1^{k_r}}$$

where $d_{n_1,\dots,n_r} \in \mathbb{C}$ and $\ell_1, \dots, \ell_r, k_1, \dots, k_r \in \mathbb{N}$.

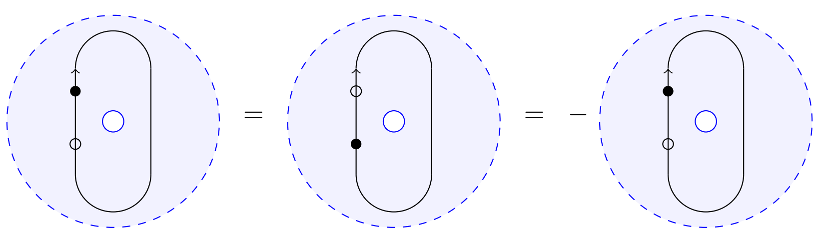
Proof. This follows immediately from the preceding lemmas. □

Proposition 4.2.3 allows us to write any element in $\mathrm{Tr}^{>}(\mathcal{H}_{tw})_{\bar{0}}$ or $\mathrm{Tr}^{<}(\mathcal{H}_{tw})_{\bar{0}}$ as a linear combination of the elements h_n. We will therefore direct our attention to these elements in future computations.

4.2.1 Even part of $\mathrm{Tr}(\mathcal{H}_{tw})$

As described in Section 3.4, $\mathcal{H}_{tw}$ has a $\mathbb{Z}/2\mathbb{Z}$-grading where  and  have degree one, and other generating diagrams have degree zero. We also have supercommutativity relations (3.29) and (3.31) and supercyclicity relations (3.26) and (3.27). These relations have several interesting diagrammatic consequences.

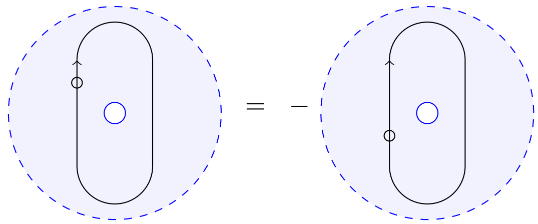
Example 2.1 Working with relation (3.31), we have the following compuation:



$= 0.$

Here the first equality is obtained by sending the solid dot around the annulus using trace relation, and the second equality is a consequence of relation (3.31). Therefore the above diagram is equal to zero in the trace.

Example 2.2 To demonstrate the subtlety with supercyclicity relations (3.26) and (3.27), consider the following situation:



If we denote  by f, with the usual trace relation, we would get $f \circ id = id \circ f$. However, in this case, we gain an extra negative sign from the supercyclicity relations. So, we must replace the usual trace relation $fg = gf$ with the supertrace relation $fg = (-1)^{|f||g|} gf$ in the ideal $\mathcal{I}$, where $|f|, |g|$ are the degrees of f and g with respect to the $\mathbb{Z}/2\mathbb{Z}$ grading. This example can be generalized to show that composition of an an odd morphism with a cycle of odd length is zero in the supertrace, since it will be equal to its negative when a hollow dot travels around the annulus and arrives to its original position.

We wish to restrict our study to the following subalgebra of the trace.

Definition 4.2.1. The *even trace* of $\mathcal{H}_{tw}$ is defined by

$$\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}} \simeq \left(\bigoplus_{x \in \mathcal{O}b(\mathcal{H})} {}_{\bar{0}}(x) \right) / \mathcal{I}_{\bar{0}}$$

where ${}_{\bar{0}}(x)$ consists of even degree endomorphisms and $\mathcal{I}_{\bar{0}}$ is its ideal generated by $\mathrm{span}_{\Bbbk}\{fg - gf\}$ for all $f : x \to y$ and $g : y \to x$, $x, y \in \mathcal{O}b(\mathcal{H}_{tw})$.

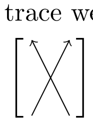
This is the restriction of the trace to only the *even* part (with respect to the $\mathbb{Z}/2\mathbb{Z}$ grading induced by $\deg(c_i) = 1$). The odd part of the trace is not zero (it contains, e.g., $h_1^{c_1}$), but is not interesting from a represenation theoretic viewpoint as explained above. The example of trace functions on the finite Hecke-Clifford algebra in [46, Section 4.1] demonstrates the importance of the even trace.

Wan and Wang study the space of trace functions on the Sergeev algebra $\mathbb{S}_n$: linear functions $\phi : \mathbb{S}_n \to \mathbb{C}$ such that $\phi([h, h']) = 0$ for all $h, h' \in \mathbb{S}_n$, and $\phi(h) = 0$ for all $h \in (\mathbb{S}_n)_{\bar{1}}$. This latter requirement encodes the information that odd elements act with zero trace on any $\mathbb{Z}_2$-graded $\mathbb{S}_n$-module (because multiplication by an odd element results in a shift in degree). The space of such trace functions is clearly canonically isomorphic to the dual of the even cocenter, rather than of the full cocenter. The same observation holds for the trace of the affine Hecke-Clifford algebra, as studied in [37].

We will see in Section 4 that the structure of $\mathrm{Tr}(\mathcal{H}_{tw})$ is largely controlled by the trace of the degenerate affine Hecke-Clifford algebra in type A; we therefore do not lose interesting representation-theoretic information by restricting to $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$, and greatly simplify our calculations by doing so. For instance, the ambiguity in the supercyclicity relations identified in Example 3.2 does not interfere with calculations in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$.

Since $\mathcal{I}_{\bar{0}}$ is an ideal of $\bigoplus_{x \in \mathcal{O}b(\mathcal{H})\ \bar{0}}(x)$, the compositions fg and gf must be even morphisms, even though individually f and g may be odd morphisms. This situation is analogous to the even cocenter of the degenerate affine Hecke-Clifford algebra studied in [37], where Clifford generators c_i do no appear individually (as they are odd generators), but still have an impact on the cocenter via the relation $c_i^2 = -1$.

Diagrammatically, the above definition means that we will have an even number of hollow dots on our diagrams. In a diagram with $2n$ hollow dots, sliding one around the annulus from top to the bottom will multiply the diagram by $(-1)^{2n-1}(-1) = 1$ where $(-1)^{2n-1}$ is a result of changing relative height with the remaining $2n - 1$ hollow dots using relation (3.29) and (-1) is the result of sliding it through a clockwise cup using relation (3.27).

Remark. For the sake of clarity, when working with diagrams in the even trace we will not draw them on an annulus, but will instead draw them inside square brackets, e.g. . This notation refers to the equivalence class of the diagram in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$.

4.2.2 Grading on $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$

Lemma 4.2.4. The algebra $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ is $\mathbb{Z}$-filtered where $\deg\left(h_n^{x_1^{2a}}\right) \leq n$ for any $a \geq 0$.

This is called the rank filtration. Denote by $\mathrm{Tr}^{>}(\mathcal{H}_{tw})_{\bar{0}}$ (resp. $\mathrm{Tr}^{<}(\mathcal{H}_{tw})_{\bar{0}}$) the subalgebra of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ generated by $h_n^{x_1^{2a}}$, $n \geq 1$ (resp. $n \leq 1$).

Lemma 4.2.5. The algebra $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ is $\mathbb{Z}^{\geq 0}$-filtered where $\deg\left(h_n^{x_1^{2a}}\right) \leq a$ for any $a \geq 0$.

Proof. Dots can slide through crossings modulo a correction term containing fewer dots. □

This is called the dot filtration, and corresponds to the differential filtration (given by $\deg(w_{\ell,k}) = k$) in W^-.

These filtrations are compatible, so $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ is $(\mathbb{Z} \times \mathbb{Z}^{\geq 0})$-filtered with $h_n^{x_1^{2a}}$ in bidegree (n, a).

Note that the rank grading and dot gradings are shifted by 1 for clockwise bubbles(so d_2 is in bidegree (1,2) and d_4 is in bidegree (1,3)). This is a consequence of the decomposition formula in Lemma (3.4.3).

Our goal is the rest of chapter 4 is to establish the isomorphism between $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ and W^- as given in the below table:

4.2.3 Counter-clockwise bubble slide moves

In order to describe $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ as a vector space, it would be convenient to have a standard form of writing for our diagrams. We know that a morphism in $\mathcal{H}_{tw}$ is a linear combination of string diagrams. We want to draw these string diagrams so that the part corresponding to a permutation will be on the left. If there are solid or hollow dots on a permutation diagram, we will slide all the dots to the top part of our diagram. Since it is possible that these dots will be sliding through some crossings, we might get new resolution terms. We will apply the same procedure to all the resolution terms as well. In particular, we want to collect all the bubbles appearing in a string diagram on the rightmost part of the diagram. In order to do so, we must describe how bubbles slide through upward and downward strands. Note that since the defining relations of $\mathcal{H}_{tw}$ are

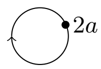
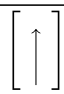
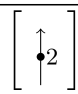
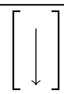
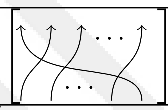
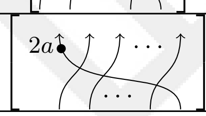
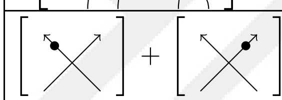
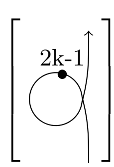
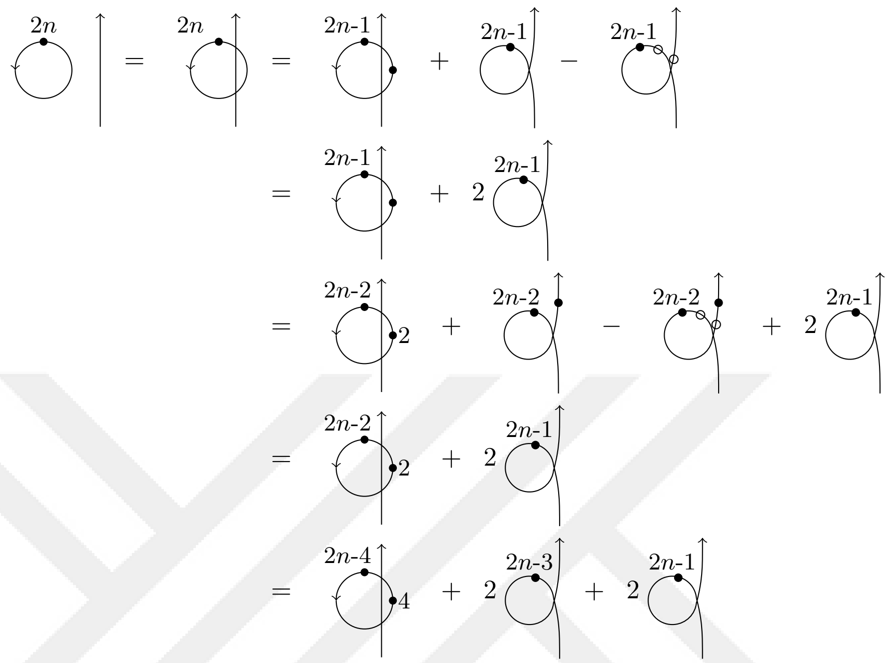
$\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$	bidegree (k-l,k)	values of (l,k)	W^-
	($2a$+1,$2a$+1)	(0,$2a$+1)	$-2w_{0,2a+1}$
	(1,0)	(-1,0)	$\sqrt{2}w_{-1,0}$
	(3,2)	(-1,2)	$\sqrt{2}(w_{-1,2} - w_{-1,1})$
	(-1,0)	(1,0)	$\sqrt{2}w_{1,0}$
	(n,0)	($-n$,0)	$\sqrt{2}w_{-n,0}$
	(n+a,a)	($-n$,a)	$\sqrt{2}(w_{-n,a} + O(w_{-n,a-1}))$
	(3,1)	(-2,1)	$2\sqrt{2}(w_{-2,1} - w_{-2,0})$

Table 4.1: Correspondence between $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ and W^-

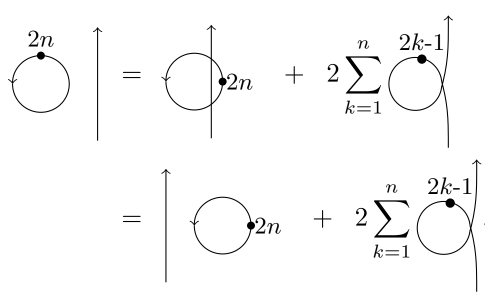
local relations, the bubbles don't have to interact with solid dots or crossings, they can simply slide through under a crossing or under a solid dot.

Lemma 4.2.6. We have that $[\bar{d}_{2n}, h_1] = 2\sum_{k=1}^{n}$  in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ for any positive integer n.

Proof. The proof is a direct computation, given below:



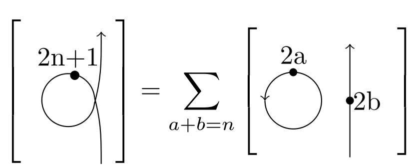
Continuing to slide dots in the first term in this way, we obtain:



To complete the proof, it is enough to consider the images of the above diagrams in Tr($\mathcal{H}_{tw}$).

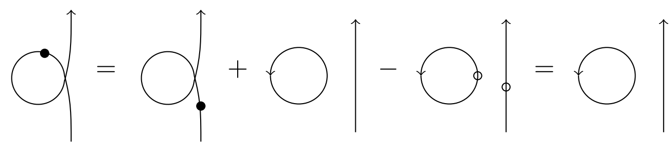
□

Lemma 4.2.7. We have that



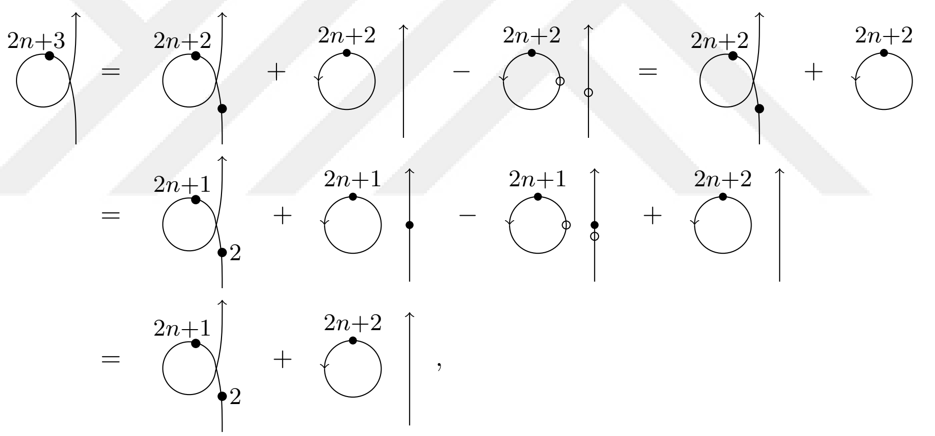
in $\mathrm{Tr}(\mathcal{H}_{tw})_{\overline{0}}$ for any non-negative integer n.

Proof. This is an easy computation using induction on n. The base case is

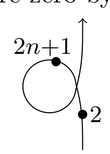


where the first term after the first equality contains a left twist curl, and the last term is zero since a bubble with a hollow dot is zero.

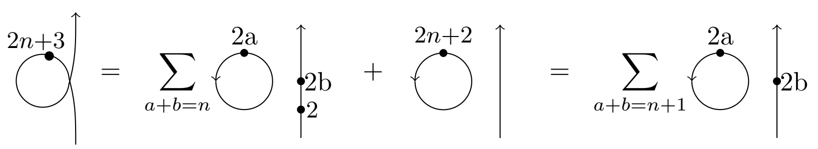
For the induction step, suppose the statement holds for $n \geq 1$. Then



where on the second line, we know that counter-clockwise bubbles with odd number of hollow dots are zero by Proposition 3.4.2, and the terms with hollow dots are zero by Proposition 3.4.1.

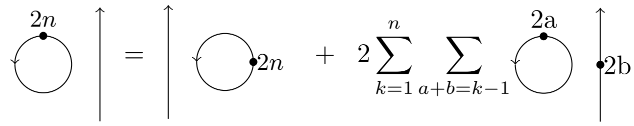
Now we can apply our induction hypothesis to the upper part of 

to get that



The statement follows after passing to the $\mathrm{Tr}(\mathcal{H}_{tw})$. □

Combining the lemmas 4.2.6 and 4.2.7, we get



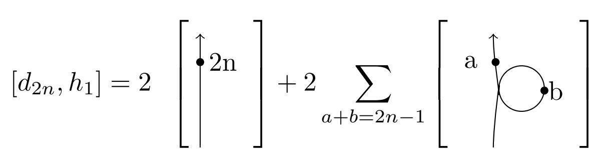
in $\mathcal{H}_{tw}$.

Obtaining an explicit formula for sliding counter-clockwise bubbles through upward strands is difficult since we express their commutator in terms of left twist curls with some dots on the curl, whose resolution terms still leave us with counter-clockwise bubbles on the left side of upward strands. The number of dots on the counter-clockwise bubbles in the resolution terms are less than the original number of dots on our counter-clockwise bubble in the commutator, hence it is possible to iterate this result and get an expression where on the right hand side, there are no bubbles to the left of upward strand, even though the formula is complicated. However, the situation is better with clockwise oriented bubbles.

4.2.4 Clockwise bubble slide moves

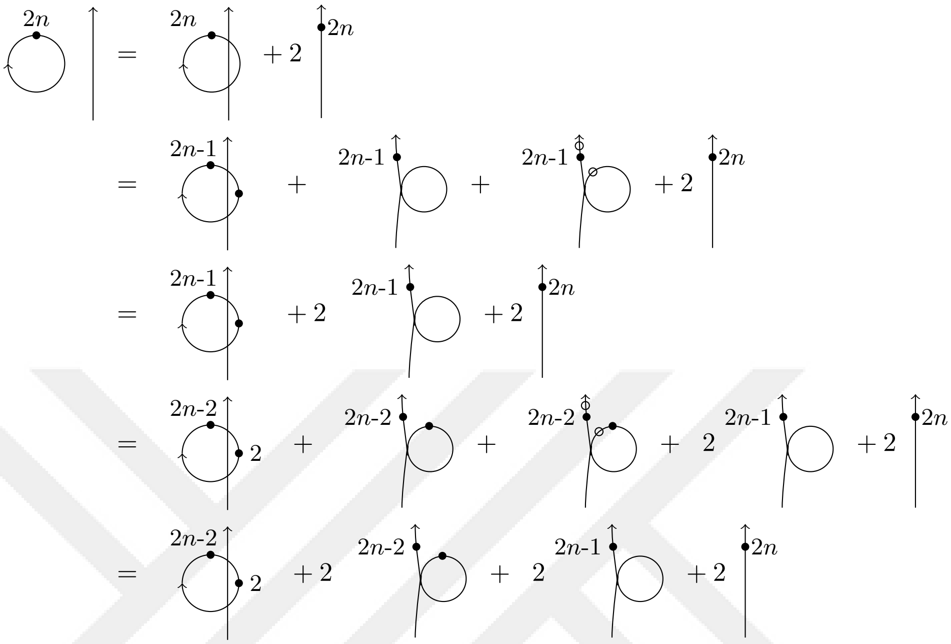
We can compute an explicit formula for clockwise bubble slides.

Lemma 4.2.8. We have

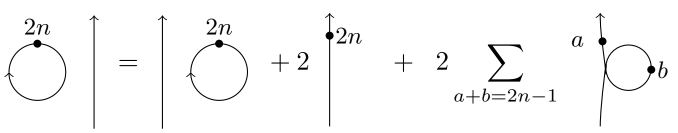


in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ for all $n \geq 0$.

Proof. This follows from a direct computation in $\mathcal{H}_{tw}$, given below:



Continuing to slide dots in the first term in same manner, we obtain:

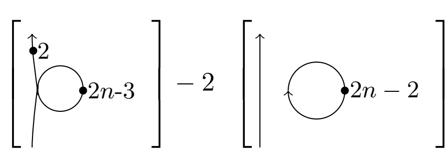


□

In particular, we can refine this statement to obtain the following recursive formula for computing $[d_{2n}, h_1]$.

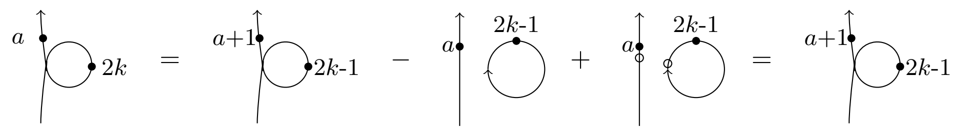
Lemma 4.2.9. We have

$$[d_{2n}, h_1] = [d_{2n-2}, h_1] \circ h_1^{x_1^2} + 4$$



in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ for all $n \geq 0$.

Proof. This lemma follows from the observation that

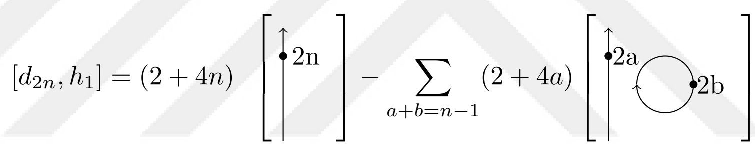


where the second term after the first equality is zero by Lemma 3.4.2, and the third term is zero by Lemma 3.4.1. Applying this result to the summands in the statement of Lemma 4.2.8 yields the result. □

Finally, we obtain an explicit formula for computing the bubble slide through an upward strand.

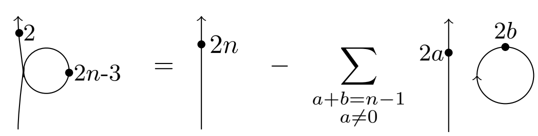
Proposition 4.2.10. We have

$$[d_{2n}, h_1] = (2+4n)\,[\ \cdot\ ] - \sum_{a+b=n-1} (2+4a)\,[\ \cdot\ ]$$

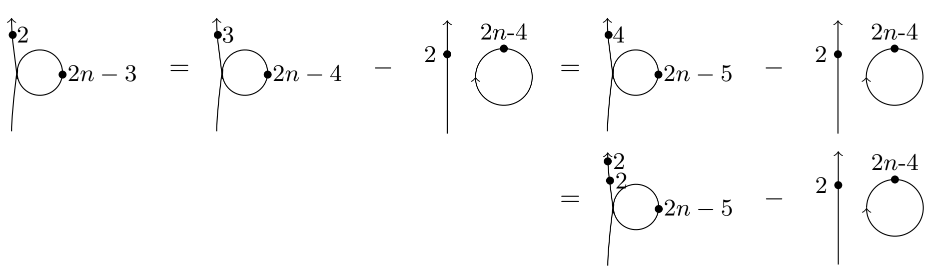


in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ for all $n \geq 0$.

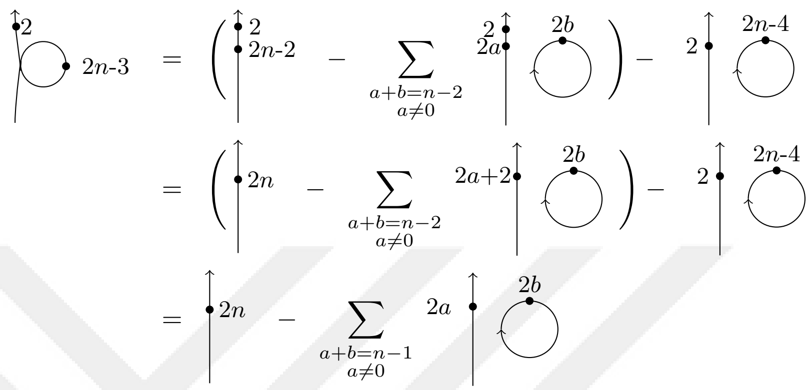
Proof. We claim that



for $n \geq 2$ and prove via induction on n. The base case $n = 2$ is a direct computation. Now suppose the formula holds for some $n \geq 2$. Then



Now we can apply the induction hypothesis to the lower part of the first term in the last expression. This gives us:

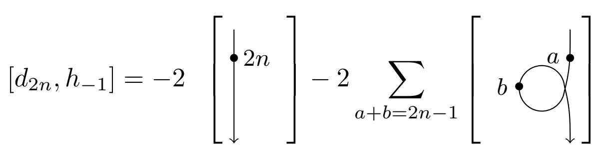


Applying this result to the recursive formula in Lemma 4.2.9 proves the statement.

□

Commutators of bubbles with downward strands are similar to those of bubbles with upward strands.

Lemma 4.2.11. We have



in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ for all $n \geq 0$.

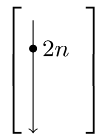
Proof. This follows from a computation similar to those in the proofs of Lemmas 4.2.8 and 4.2.9.

□

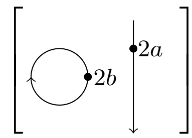
Finally we have an explicit formula for commutators of clockwise oriented bubbles and a single downward strand.

Proposition 4.2.12. We have

$$[d_{2n}, h_{-1}] = -(2+4n)$$



$$+ \sum_{a+b=n-1} (2+4a)$$



in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ for $n \geq 0$.

Proof. This follows from Lemma 4.2.11, using a similar argument as in the proof of Proposition 4.2.10. □

Note that in this formula, we are still left with clockwise bubbles on the left side of a downward strand, but with fewer dots on it. Hence the formula may be applied inductively in order to move all the bubbles to the rightmost part of the diagram.

To summarize, the formulas for sliding clockwise bubbles through downward strands or for sliding counter-clockwise bubbles through upward strands are not explicit, but can be obtained via recursion. On the other hand, we have an explicit bubble slide formula for clockwise bubbles and upward strands.

Also note that the bubble slide formulas obtained in this section also hold in $\mathcal{H}_{tw}$, meanining that their proofs don't use the trace relation. We stated them for $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ since we will use them to identify $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ and W^-.

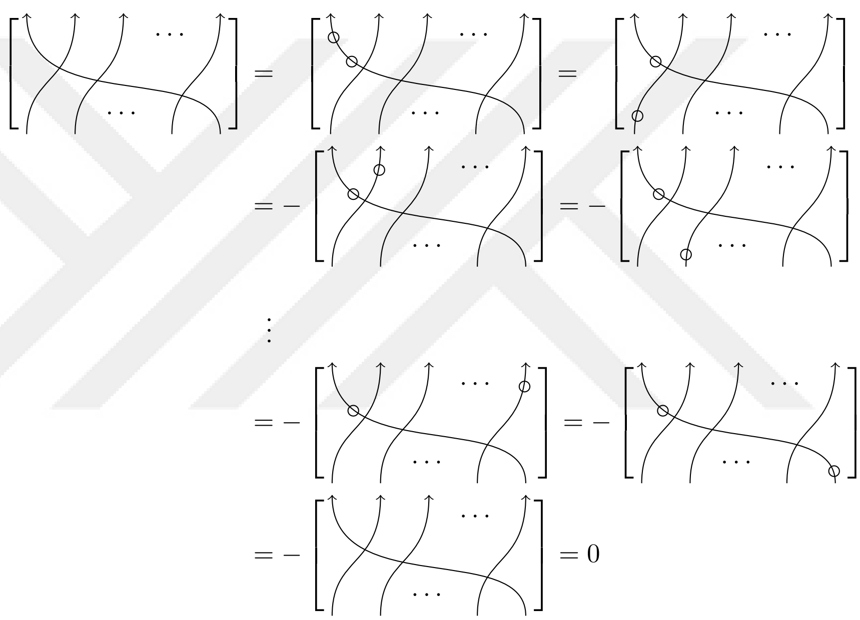
4.2.5 Dot degree zero part of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$

Recall that the dot degree of a diagram was determined by the number of solid dots on it. Therefore the dot degree zero part of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ consists of linear combinations of elements $\{h_n\}_{n\in\mathbb{Z}}$. In fact we also have a counter-clockwise bubble, $\bar{d}_0$, with differential degree zero, however $\bar{d}_0 = 1$ by the defining relations of $\mathcal{H}_{tw}$.

We will try to identify the dot degree zero part of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ with the differential degree zero part of W^{-}, which is isomorphic to the twisted Heisenberg algebra. Therefore in this section, we will show that the elements $\{h_n\}_{n\in\mathbb{Z}}$ satisfy the relations in $\mathfrak{h}_{tw}$.

Proposition 4.2.13. We have $h_{2n} \cong 0$ for any $n \in \mathbb{Z}$.

Proof. The proof follows from a direct computation, using the trace relation and the fact that hollow dots on different strands anti-commute.



when we have an even number of strands. □

Therefore in the dot degree zero part of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$, we are left with the elements $\{h_{2n+1}\}_{n\in\mathbb{Z}}$. These elements satisfy the following relations.

Lemma 4.2.14. The following commutators are zero for all non-negative integers n, m:

1. $[h_{2n+1}, h_{2m+1}] = 0$,

2. $[h_{-2n-1}, h_{-2m-1}] = 0$.

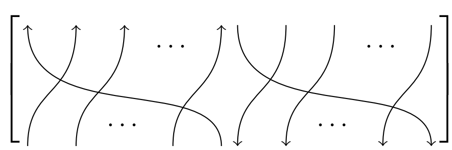
Proof. Parts (1) and (2) follow from the fact that similarly oriented strands can be split apart when they cross twice. □

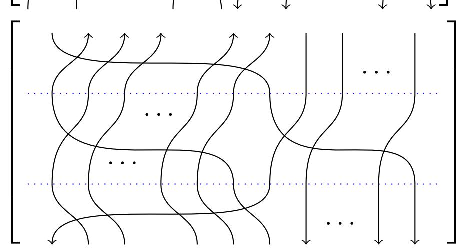
To obtain a copy of the twisted Heisenberg algebra in the $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$, we need to look at commutators between elements with odd numbers of oppositely oriented strands.

Lemma 4.2.15. We have, for any $n, m \in \mathbb{Z}^{\geq 0}$,

$$[h_{(2n+1)}, h_{-(2m+1)}] = -2(2n+1)\delta_{n,-m}.$$

Proof. First note that [6, Lemma 19] and [6, Lemma 20] holds in our twisted case with a small modification, since all the arguments in their proofs use the fact that the resolution terms contain left twist curls, hence are zero. Normally we would have extra resolution terms with hollow dots due to relation (3.3), but hollow dots on a diagram containing a left twist curl still gives zero, since it is a local relation. The only modification comes in the case $m = n$ where we get two copies of counter-clockwise bubbles instead of one, since two hollow dots on a counter-clockwise bubble end up canceling each other without changing the sign of the diagram. We immediately get that when $m \neq n$, our commutator is zero since we have no solid dots. Therefore we have

$h_{2n+1}h_{-(2m+1)} =$ 

$=$ 

$$= h_{-(2m+1)}h_{2n+1} - 2\bar{d}_0(2n+1).$$

Hence $[h_{(2n+1)}, h_{-(2m+1)}] = -2(2n+1)\delta_{n,-m}$. □

Therefore the subalgebra $\langle A\rangle$ of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ with generators $A = \{h_{(2n+1)}\}_{n\in\mathbb{Z}}$ is isomorphic to the twisted Heisenberg algebra via

$$\begin{aligned} \phi: \quad \mathfrak{h}_{tw} &\xrightarrow{\sim} \langle A\rangle \\ h_{\frac{2n+1}{2}} &\mapsto \frac{1}{2}h_{-(2n+1)}. \end{aligned}$$

In the filtration degree zero of the W-algebra W^-, we have an isomophic copy of the twisted Heisenberg algebra as well, given by generators $B = \{\omega_{2n+1,0}\}_{n\in\mathbb{Z}}$. The isomorphism given by

$$\begin{aligned} \psi: \quad \mathfrak{h}_{tw} &\xrightarrow{\sim} \langle B\rangle \\ h_{\frac{2n+1}{2}} &\mapsto \frac{1}{\sqrt{2}}\,\omega_{2n+1,0}. \end{aligned}$$

Therefore we have an isomorphism between the dot degree zero part of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ and the differential degree zero part of W^-:

$$\begin{aligned} \psi\circ\phi^{-1}: \quad \langle A\rangle &\xrightarrow{\sim} \langle B\rangle \\ h_{-(2n+1)} &\mapsto \sqrt{2}\,w_{2n+1,0}. \end{aligned}$$

4.2.6 Nonzero dot degree part of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$

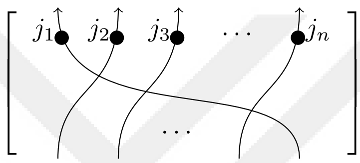
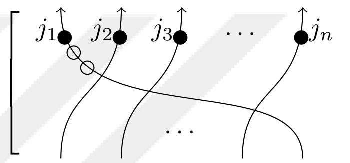
Recall that in W^-, the basis elements $w_{k,\ell}$ are zero unless $k-\ell$ is odd. In other words, k and ℓ had to have the opposite parity. Now we establish the analogous result in $\mathrm{Tr}^{>}(\mathcal{H}_{tw})_{\bar{0}}$.

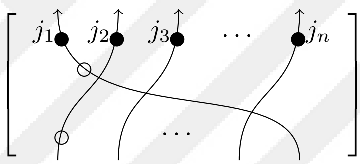
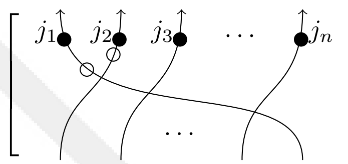
Proposition 4.2.16. Let $n \in \mathbb{Z}$, $j_1, j_2, \ldots, j_n \in \mathbb{Z}_{\geq 0}$ and let $k = j_1 + j_2 + \cdots + j_n$. We have

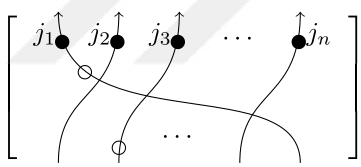
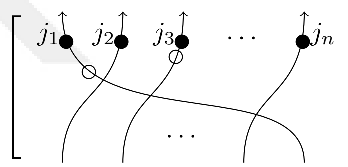
$$h_n^{x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}} \simeq 0$$

when n and k have the same parity.

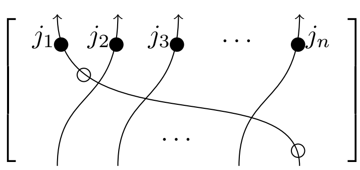
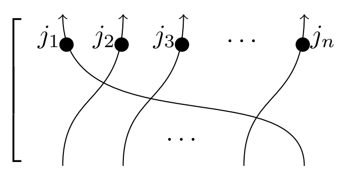
Proof.

$h_n^{x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}} =$  $=$ 

$= (-1)^{j_1}$  $= (-1)^{j_1+1}$ 

$= (-1)^{j_1+j_2+1}$  $= (-1)^{j_1+j_2+2}$ 

$\vdots$

$= (-1)^{k+n-1}$  $= (-1)^{k+n-1}$  $= 0$

when $k + n$ is even, and the proposition follows. Changing the orientation of the n-cycle doesn't change the computation. □

Using commutators of filtration degree one elements, we can increase the number of strands.

Lemma 4.2.17. For $m, n \in \mathbb{Z}$ with $mn > 0$, we have

$$[h_{2m}^{x_1}, h_{2n}^{x_1}] = 2(n - m) h_{2n+2m}^{x_1}.$$

Proof. This is a slight modification of [6, Lemma 23]. By Proposition 4.2.16, if at least one of the indices inside the commutator is odd, the commutator will be zero. Hence we will work with the case where both indices are even numbers. The modification we need in [6, Lemma 23] is a result of us having two resolution terms in our relations (3.33) and (3.34). As a consequence of having even number of strands in both of our elements, canceling the two empty dots in our resolution terms give rise to the same sign as the other resolution term, hence we have a coefficient of two in front of our result.

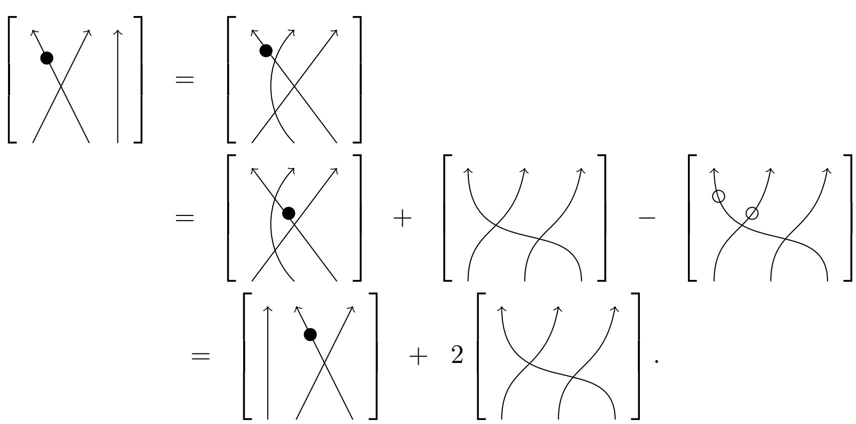
□

Recall that we use the notation $h_n^{x_1+\cdots+x_n}$ for $h_n^{x_1} + h_n^{x_2} + \cdots + h_n^{x_n}$. Using commutators of $h_{2n}^{x_1+\cdots+x_{2n}}$ and h_1, we can increase the number of strands and obtain h_{2n+1}.

Lemma 4.2.18. For $n \in \mathbb{Z}$, we have

$$[h_{2n}^{(x_1+\ldots+x_{2n})}, h_1] = 4nh_{(2n+1)}.$$

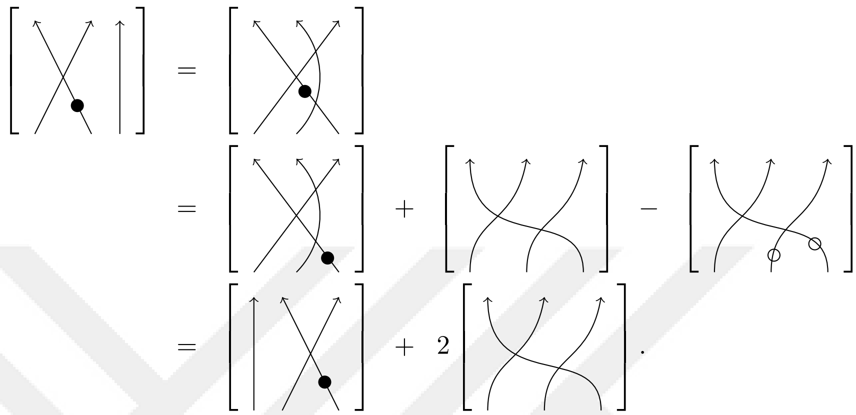
Proof. First note that we have:



(4.9)

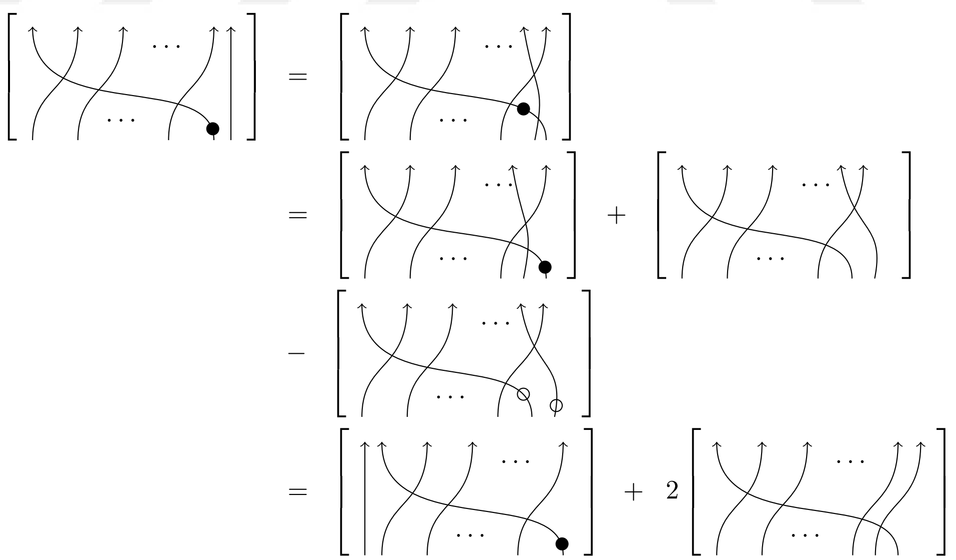
Hence $[h_2^{x_1}, h_1] = 2h_3$.

Next, moving the solid dot in $h_2^{x_2}$ around to the bottom of the crossing using the trace relation gives:



So, $[h_2^{(x_1+x_2)}, h_1] = 4h_3$.

Next, we claim that $[h_{2n}^{x_{2n}}, h_1] = 2h_{2n+1}$ for any n. Indeed, we have:



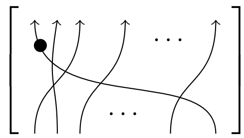
where the last equality is obtained by pushing the crossings at the bottom of the diagrams without dots to the top. Indeed, diagrammatic calculations similar to the above give that

$$[h_{2n}^{x_a}, h_1] = 2h_{2n+1}$$

for any $1 < a \leq 2n$.

Finally, note that

$$h_{2n}^{x_1} h_1 \quad = $$



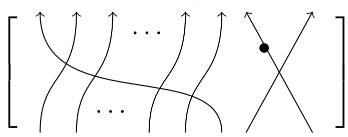
The dot will slide over the top-leftmost crossing in the same manner as in Equation (4.9), meaning the correction terms will cancel out. Hence, we have the desired result. □

Lemma 4.2.19. Let m be an odd integer. We have

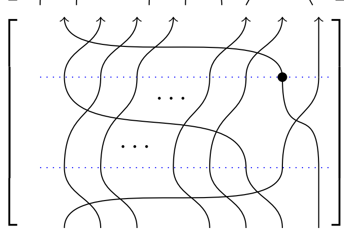
$$[h_2^{(x_1+x_2)}, h_m] = 4m h_{m+2}.$$

Proof. We compute directly:

$$h_m h_2^{x_1} \quad =$$



$=$



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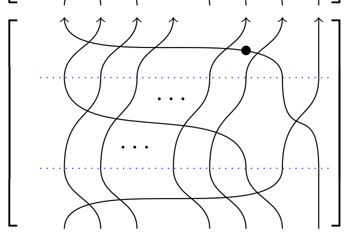

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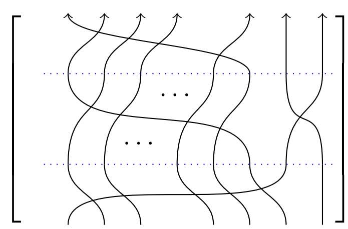
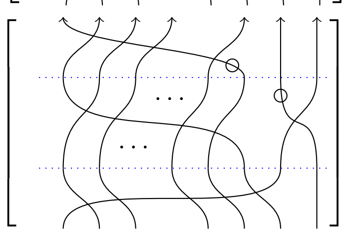
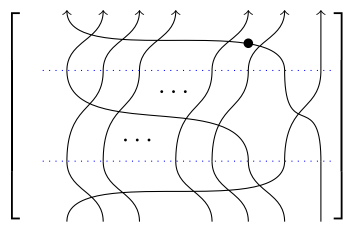


$+$

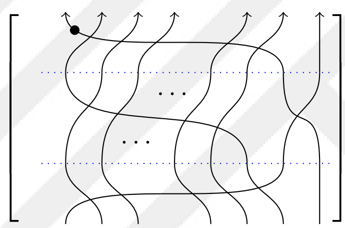


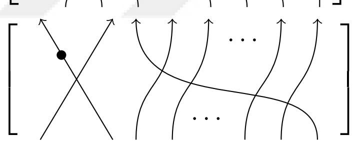
.

Cancelling the empty dots in the last term results in a change in sign, and both of the latter diagrams are $(m+2)$-cycles. Hence we have:

$=$  $-\ 2h_{m+2}$

Sliding the solid dot in the first diagram all the way to the left results in m total crossing resolutions, each of which yieds a term of $-2h_{m+2}$. So,

$=$  $-\ 2mh_{m+2}$

$=$  $-\ 2mh_{m+2}$

Hence we have

$$[h_2^{x_1}, h_m] = 2mh_{m+2}.$$

A similar computation gives that

$$[h_2^{x_2}, h_m] = 2mh_{m+2},$$

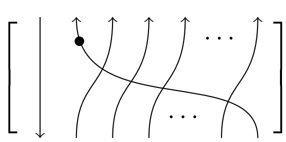
giving the desired result. □

Lemma 4.2.20. We have

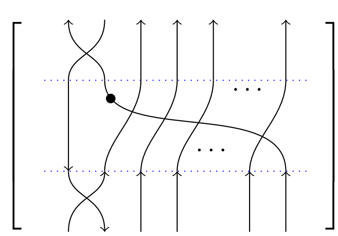
$$[h_{2n}^{(x_1+x_2+\ldots+x_{2n})}, h_{-(2m+1)}] = \begin{cases} -4(2m+1)h_{2n-(2m+1)} & \text{if } n > m \geq 1 \\ 0 & \text{if } n = m \geq 1 \\ -2(2m+1)h_{2n-(2m+1)} & \text{if } 1 \leq n < m. \end{cases}$$

Proof. We follow the methods of [6, Lemma 26], substituting our new relations as necessary.

As in that case, let $\beta_n = h_{2n}^{x_1}$ and $\alpha_m = h_{2m+1}^{x_1}$, and proceed by induction on m. When $m = 1$, we can compute directly:

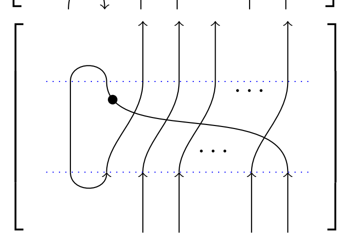


$\overset{(3.2)}{=}$



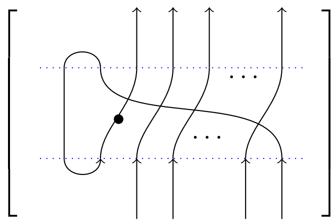
(4.10)

$+ \quad 2$



(4.11)

where the trailing terms arising from relation (3.2) have the same sign after cancelling the empty dots, and thus add together. We claim that the diagram in the second term is h_{2n-1}. Indeed, sliding the dot gives:

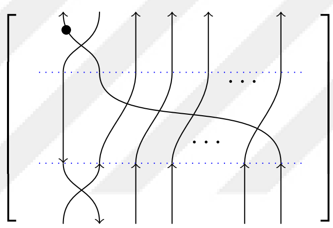


$$\overset{(3.33)}{=} \quad \overline{d}_{0,0}h_{2n-1} + \overline{d}_{0,1}h_{2n-1} = h_{2n-1}$$

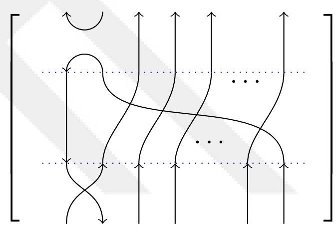
by relations (3.4) and (3.28).

Now, sliding the solid dot over the crossing on the right hand side of Equation (4.10) gives:

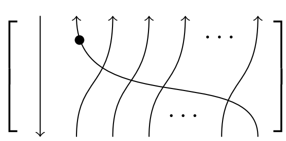
$\overset{(3.33)}{=}$



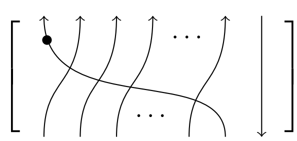
$+ \quad 2$



where the trailing terms arising from relation (3.33) have the same sign after canceling the empty dots, and thus add together. We can use the trace relation to slide the top cup in the second term to the bottom; after simplication, this term is therefore equal to h_{n-1}. The first term is equal to $\beta_n\alpha_{-1}$ as in [6, Lemma 26]. Thus,



$=$



$+ \quad 4h_{2n-1}$

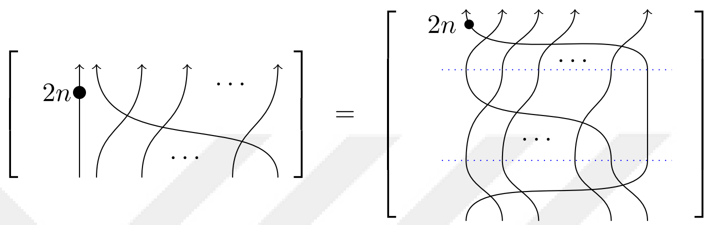
as desired. The base case of the induction is proved. The induction step follows from examination of the Jacobi identity, exactly as in [6, Lemma 26], using our Lemma 4.2.18 in place of [6, Lemma 24].

□

Lemma 4.2.21. Let $n \in \mathbb{Z}$. We have

$$[h_1^{x_1^2}, h_{2n-1}] = 2h_{2n}^{x_1+\ldots+x_{2n}} + 2h_{2n}^{x_2+\ldots+x_{2n-1}}.$$

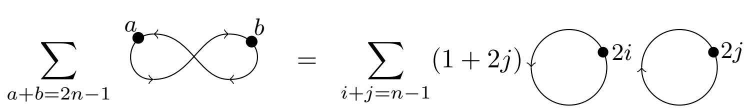
Proof. This is a straightforward diagrammatic calculation similar to Lemmas 4.2.19 and 4.2.20. We have



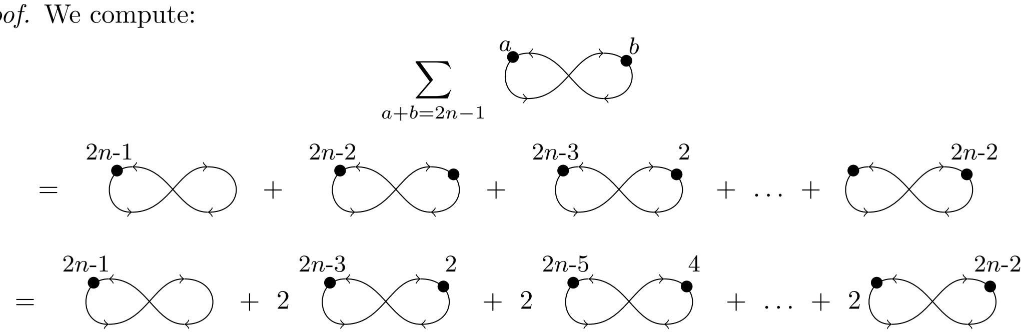
Sliding the dots all the way to the right side of the diagram results in $2(2n-1)$ resolution terms. Each of these resolution terms contains a $2n$-cycle and a single solid dot - there are 2 resolution terms containing a solid dot on the first strand and 2 containing a solid dot on the last strand, and 4 resolution terms with a dot on each other strand. All empty dots cancel in such a way that no resolution terms cancel with each other. The result follows. □

The following lemmas will allow us to generate bubbles with arbitrary numbers of dots using just $h_{\pm 1}^{x_1^2}$.

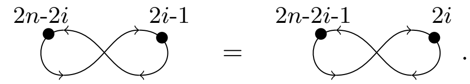
Lemma 4.2.22. We have



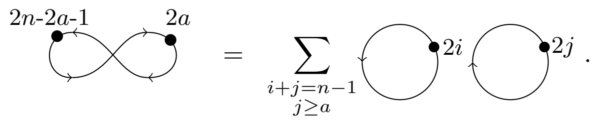
Proof. We compute:



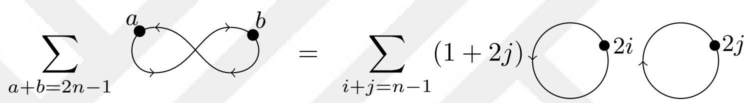
because we have



Moreover, we can decompose these figure eights into a linear combination of products of two bubbles using dot slide relations 3.33 and 3.34 as follows:



Combining these results, we get that



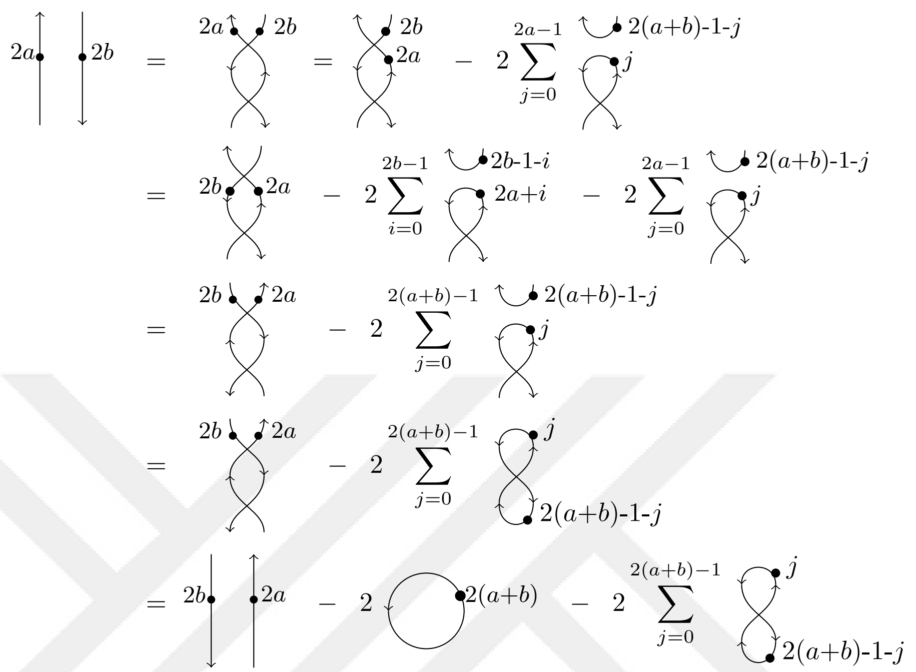
∎

Lemma 4.2.23. We have

$$[h_1^{x_1^{2a}}, h_{-1}^{x_1^{2b}}] = -2\bar{d}_{2(a+b)} - \sum_{i+j=2(a+b)-1} (2+4j)\bar{d}_{2i}d_{2j}$$

for $a, b \in \mathbb{Z}_{\geq 0}$.

Proof. We compute:



Therefore $[h_1^{x_1^{2a}}, h_{-1}^{x_1^{2b}}] = -2\bar{d}_{2(a+b)} - \sum_{i+j=2(a+b)-1} (2+4j)\bar{d}_{2i}d_{2j}.$ □

4.3 Algebra isomorphism between $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ and W^-

In this section, we will study the structure of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$, first as a vector space and then as an algebra. We show that $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ has a triangular decomposition into two copies of the trace of $DAHC_n$ and a polynomial algebra. We then describe a generating set for $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$, which allows us to define the algebra homomorphism to W^-. Finally, we prove that this homomorphism is an isomorphism.

4.3.1 Trace of $\mathcal{H}_{tw}$ as a vector space

Proposition 4.3.1. We have the triangular decomposition of $\mathrm{Tr}(\mathcal{H}_{tw})$:

$$\mathrm{Tr}(\mathcal{H}_{tw}) \cong \bigoplus_{n,m\in\mathbb{Z}_{\geq 0}} \mathrm{Tr}((DAHC_n)^{op} \otimes DAHC_m \otimes \Bbbk[d_0, d_2, d_4....]).$$

Proof. As shown in [2], to find $\mathrm{Tr}(\mathcal{H}_{tw})$, it is enough to consider the direct sum over indecomposable objects of endomorphism spaces of objects of $\mathcal{H}_{tw}$. Let $I = Span_{\Bbbk}\{fg - gf\}$ where $f : x \to y$ and $g : y \to x$ for x, y objects of $\mathcal{H}_{tw}$. By Lemma 3.4.6 we have

$$\mathrm{Tr}(\mathcal{H}_{tw}) \cong \Big(\bigoplus_{n,m\in\mathbb{Z}_{\geq 0}} End_{\mathcal{H}}(P^n Q^m)\Big)/I.$$

By Lemma 3.4.4, this gives us

$$\mathrm{Tr}(\mathcal{H}_{tw}) \cong \Big(\bigoplus_{n,m\in\mathbb{Z}_{\geq 0}} ((DAHC_n)^{op} \otimes DAHC_m \otimes \Bbbk[d_0, d_2, d_4....]) \oplus J_{n,m}\Big)/I.$$

Recall that the ideal $J_{n,m}$ is generated by diagrams containing at least one cup connecting two upper points. Therefore, the diagrams in $J_{n,m}$ must also contain caps, since they are dealing with endomorphisms. Using the trace relation and the relations in $\mathcal{H}_{tw}$, we can express the elements of $J_{m,n}$ as direct sum of endomorphisms of $P^{m'}Q^{n'}$ for $m' \leq m$ and $n' \leq n$ in . Hence we have

$$\begin{aligned}\mathrm{Tr}(\mathcal{H}_{tw}) &\cong \bigoplus_{m,n\in\mathbb{Z}_{\geq 0}} Tr(DAHC_n^o p \otimes DAHC_m \otimes \Bbbk[d_0, d_2, d_4....]) \\ &\cong \Big(\bigoplus_{m,n\in\mathbb{Z}_{\geq 0}} Tr(DAHC_n^o p \otimes DAHC_m)\Big) \otimes \Bbbk[d_0, d_2, d_4....].\end{aligned}$$

□

4.3.2 Generators of the algebra $\mathrm{Tr}(\mathcal{H}_{tw})_{\overline{0}}$

The following gives a generating set for $\mathrm{Tr}(\mathcal{H}_{tw})_{\overline{0}}$ as an algebra.

Lemma 4.3.2. The algebra $\mathrm{Tr}(\mathcal{H}_{tw})_{\overline{0}}$ is generated by h_{-1}, $h_{\pm 2}^{(x_1+x_2)}$, and $d_0 + d_2$.

Proof. First, Proposition 4.2.12 implies that h_1 and $(d_0 + d_2)$ allow us generate a dot degree two element $h_1^{x_1^2}$; since all relations in $\mathcal{H}_{tw}$ are local, we can evaluate the commutator $[h_1^{x_1^2}, (d_0 + d_2)]$ by moving the dot to the bottom of the upward strand and sliding the bubbles over the upper portion. We can therefore apply Lemma 4.2.12 repeatedly to show that $\mathrm{ad}(d_0 + d_2)^n h_1$ has a leading term of $h_1^{x_1^{2n}}$.

By Lemma 4.2.19, the elements h_{-1} and $h_2^{x_1+x_2}$ are sufficient to generate h_{2m+1} for all integers $m > 0$. Then we can generate $h_{2n}^{x_1+\ldots+x_n}$ from $h_1^{x_1^2}$ and h_{2m+1} by using Lemma 4.2.21. Lemma 4.2.20, h_{-1} and $h_{2n}^{x_1+x_2+\ldots+x_n}$ allow us to generate h_{2r+1} for all integers r.

Proposition 4.2.3 implies that all elements with nonzero rank degree can be written as a sum of elements of the form $h_{\pm n}^{x_1^\ell c_1^k}$. By Propositions 4.2.13 and 4.2.16, all elements of this form except for the ones generated in the preceding paragraphs are 0 in $\mathrm{Tr}(\mathcal{H}_{tw})_{\overline{0}}$, so we have generated all of $\mathrm{Tr}^{>}(\mathcal{H}_{tw})_{\overline{0}}$ and $\mathrm{Tr}^{<}(\mathcal{H}_{tw})_{\overline{0}}$.

Finally, Lemma 4.2.23 allows us to generate the elements d_{2n}, applying Lemma 3.4.3 to split up the $\overline{d}_{2n}$ terms. □

4.3.3 The isomorphism

There is an obvious isomorphism of vector spaces between the Fock space representations of $\mathrm{Tr}(\mathcal{H}_{tw})_{\overline{0}}$ and W^-:

$$\phi : V = \mathbb{C}[h_1, h_2, \ldots] \to \mathbb{C}[w_{-1,0}, w_{-2,0}, \ldots] = \mathcal{V}_{1,0}.$$

Recall that each algebra acts faithfully on its Fock space representation.

Lemma 4.3.3. The map ϕ in Equation (4.3.3) commutes with the action of the twisted Heisenberg subalgebras in V and $\mathcal{V}_{1,0}$, i.e.:

$$\phi(h_r v) = \sqrt{2} w_{-r,0} \phi(v).$$

Proof. The vector space realizations of V and $\mathcal{V}_{1,0}$ in Equation (4.3.3) imply that the action of h_r on V is simply the adjoint action of h_r on the subalgebra $\mathrm{Tr}^{>}(\mathcal{H}_{tw})_{\overline{0}}$, and the action of $w_{-r,0}$ on $\phi(v)$ is the adjoint action of $w_{-r,0}$ on $(W^{-})^{-}$. The Lemma follows from our computation of these twisted Heisenberg relations in Propositions 4.1.5 and 4.2.15. □

Lemma 4.3.4. For any $v \in V$ we have $\phi((d_0 + d_2)v) = -2w_{0,3}\phi(v)$.

Proof. Propositions 4.1.5 and 4.2.10 give that $w_{0,3}$ maps $w_{-1,0}$ to an element with leading term $w_{-1,2}$, and $(d_0 + d_2)$ maps h_1 to an element with leading term $h_1^{x_1^2}$. Comparision of the actions of these terms on the twisted Heisenberg subalgebras on either side gives that that their images in the endomorphisms of the Fock space are identical. □

Lemma 4.3.5. For any $v \in V$ we have $\phi(h_{\pm 2}^{(x_1+x_2)} v) = 2\sqrt{2}(w_{\mp 2,1} + w_{\mp 2,0})\phi(v)$.

Proof. This follows from comparision of Lemma 4.2.19 and Proposition 4.1.5. □

Now extend ϕ to a map

$$\Phi : \mathrm{Tr}(\mathcal{H}_{tw})_{\overline{0}} \longrightarrow W^{-}/\langle w_{0,0}, C-1 \rangle$$

by mapping

$$h_1 \mapsto \sqrt{2} w_{-1,0} \qquad h_{\pm 2}^{(x_1+x_2)} \to 2\sqrt{2} w_{\mp 2,1} + w_{\mp 2,0} \qquad d_2 + d_0 \mapsto -2w_{0,3}$$

and extending algebraically, i.e.

$$\Phi(a_1 \dots a_k) = \Phi(a_1) \dots \Phi(a_k)$$

for generators $a_1, \ldots, a_k$ of $\mathrm{Tr}(\mathcal{H}_{tw})_{\overline{0}}$.

Lemma 4.3.6. The map Φ above is well defined.

Proof. Suppose $A \in \mathrm{Tr}(\mathcal{H}_{tw})_{\overline{0}}$ has two representations in terms of generators, $A = a_{i_1} \ldots a_{i_k} = a_{j_1} \ldots a_{j_\ell}$. Then $a_{i_1} \ldots a_{i_k}.V = a_{j_1} \ldots a_{j_\ell}.V$, so applying Φ gives $\Phi(a_{i_1} \ldots a_{i_k}).\mathcal{V}_{1,0} = \Phi(a_{j_1} \ldots a_{j_\ell}).\mathcal{V}_{1,0}$. Hence $\Phi(a_{i_1} \ldots a_{i_k}) = \Phi(a_{j_1} \ldots a_{j_\ell})$ by the faithfulness of the Fock space representation for W^-. □

Theorem 4.3.7. The map Φ is an isomorphism of algebras.

Proof. We immediately have that Φ is surjective, because it maps generators to generators. Thus, it remains to show that Φ is injective. Let $A := a_{i_1} \ldots a_{i_k} \in \mathrm{Tr}(\mathcal{H}_{tw})_{\overline{0}}$ and assume that $\Phi(A).\mathcal{V}_{1,0} = 0$. Then $\Phi(A) = 0$ by the faithfulness of the representation. But then $\Phi(a_{i_1}) \ldots \Phi(a_{i_k}).\mathcal{V}_{1,0} = 0$. Then, by Lemmas 4.3.3, 4.3.4, and 4.3.5, we have $\Phi(a_{i_1}) \ldots \Phi(a_{i_k}).\mathcal{V}_{1,0} = \phi(a_{i_1} \ldots a_{i_k}.V) = \phi(A.v) = 0$. But ϕ is an isomorphism, so this implies that $A.V = 0$. Hence $A = 0$ by the faithfulness of the Fock space representation of $\mathrm{Tr}(\mathcal{H}_{tw})_{\overline{0}}$. □

Chapter 5

Center of the Twisted Heisenberg Category

In this chapter, we will describe the combinatorial structure of the categorical center of $\mathcal{H}_{tw}$, which is $End_{\mathcal{H}_{tw}}(\mathbb{1})$. In 3.4.1, we saw that $End_{\mathcal{H}_{tw}}(\mathbb{1})$ is isomorphic to a polynomial algebra in infinitely many variables. Just as an algebra, it is not that interesting. However it admits interesting basis, just like the ring of symmetric functions.

We will describe a new basis for $End_{\mathcal{H}_{tw}}(\mathbb{1})$, and show that the multiplication of these new basis elements give rise to non-homogeneous power sum symmetric functions. The correspondance between closed diagrams and symmetric functions will be through their identification with the space of functions on strict partitions. It is not very surprising that strict partitions play a role in this story, since $\mathcal{H}_{tw}$ is built on the Sergeev algebra super-modules and super-representation theory of the Sergeev algebra is super-equivalent to the projective representation theory of the symmetric group. Irreducible projective representations of the symmeric group are indexed by strict partitions.

For the super-representation theory of $\mathbb{S}_n$, we will follow the exposition in [46] and use the relation of $\mathbb{S}_n$ to the type B Weyl group(a.k.a. hyperoctahedral group) to study its super- representations. $\mathbb{S}_n$ can be seen as the spin algebra of the hyperoctahedral group (i.e. a quotient of the group algebra of a double cover of the hyperoctahedral group).

5.1 Sergeev algebra and Type B Weyl group

The hyperoctahedral group is

$$B_n = C_2^n \rtimes S_n$$

where C_2 is the cyclic group of order 2, and the symmetric group acts via permuting the n-copies of it.

We would like to use tools from the group theory to study the representation theory of the Sergeev algebra. However the anti-commutativity of the Clifford generators in $\mathbb{S}_n$ makes it impossible to see the Sergeev algebra as a group algebra, since in general groups don't have an element corresponding to the scalar -1. Note that if one replaces the anti-commutativity condition for the Clifford generators with a commutativity condition, one recovers the hyperoctahedral group B_n. To overcome the issue about working in a group theoretic setting and having anti-commutativity relations, we will introduce a central element z to B_n whose square will be the identity element. We will quotient the corresponding group algebra with the relation $z+1$. This is a way to obtain an element in the group which plays the role of "-1".

Define the group

$$\Pi_n := \langle z, a_1, \ldots, a_n \mid a_i^2 = z, a_i a_j = z a_j a_i, z a_i = a_i z, z^2 = 1 \rangle.$$

Π_n is a double cover of C_2^n via the short exact sequence:

$$1 \longrightarrow C_2 \longrightarrow \Pi_n \longrightarrow C_2^n \longrightarrow 1 \tag{5.1}$$

in which C_2 is mapped to the subgroup $\{1, z\} \subset \Pi_n$ and $z \in \Pi_n$ is mapped to 1.

Definition 5.1.1. The *twisted hyperoctahedral group* is defined as $\widehat{B_n} := \Pi_n \rtimes S_n$ where S_n acts on $\{a_i\}$ by permuting their indices, and acts trivially on z.

Just like the Sergeev algebra, $\Bbbk[\widehat{B_n}]$ is also a superalgebra via the $\mathbb{Z}/2\mathbb{Z}$ grading which sets $\deg(a_j) = 1$ for $1 \leq j \leq n$ and $\deg(z) = \deg(s_i) = 0$ for $1 \leq i \leq n-1$. Using (5.1) one can show that $\widehat{B_n}$ is a double cover of the hyperoctahedral group $B_n = C_2^n \rtimes S_n$ via the short exact sequence:

$$1 \longrightarrow C_2 \longrightarrow \widehat{B_n} \xrightarrow{f} B_n \longrightarrow 1$$

where f sends z to 1. On the other hand from a comparison of generators and relations it is clear that

$$\mathbb{S}_n \simeq \Bbbk[\widehat{B_n}]/\langle z+1\rangle.$$

We denote the corresponding projection by $\pi_n : \Bbbk[\widehat{B_n}] \to \mathbb{S}_n$.

Since z is central and $z^2 = 1$, for any $\Bbbk[\widehat{B_n}]$-supermodule L, we have that z must act by multiplication by either 1 or -1. Hence studying $\mathbb{S}_n$-supermodules is equivalent to studying $\Bbbk[\widehat{B_n}]$-supermodules where z acts as multiplication by -1 (these are commonly referred to as spin representations of $\widehat{B_n}$). Furthermore, via the super Wedderburn Theorem it follows that

$$\Bbbk[\widehat{B_n}] \;\cong\; \Bbbk[\widehat{B_n}]/\langle z-1\rangle \oplus \Bbbk[\widehat{B_n}]/\langle z+1\rangle \;\cong\; \Bbbk[B_n] \oplus \mathbb{S}_n. \tag{5.2}$$

The group algebras $\Bbbk[\widehat{B_n}]$ also form a tower of algebras with the embedding $\Bbbk[\widehat{B_{n-1}}] \hookrightarrow \Bbbk[\widehat{B_n}]$ which sends $s_i \mapsto s_i$, $a_i \mapsto a_i$, and $z \mapsto z$. Note that this embedding maps the subalgebra $\Bbbk[S_{n-1}]$ into the subalgebra $\Bbbk[S_n]$ in the usual way, and projected down to $\mathbb{S}_{n-1}$ and $\mathbb{S}_n$ this becomes the standard embedding of Sergeev algebras. We set $\mathbb{C}[\widehat{B_0}]$ to be the subalgebra generated by z.

Lemma 5.1.1. For $n \geq 2$,

$$\{\, s_i \dots s_{n-1} a_n^{\epsilon} \mid 1 \leq i \leq n, \epsilon \in \{0,1\}\} \tag{5.3}$$

is a collection of left coset representatives of $\widehat{B_{n-1}}$ in $\widehat{B_n}$.

Note that we follow the convention that the elements corresponding to $i = n$ are a_n^ϵ for $\epsilon \in \{0, 1\}$ in Lemma 5.1.1.

Proof. The set $\{ s_i \dots s_{n-1} \mid 1 \leq i \leq n \}$ forms a collection of minimal length left coset representatives of S_{n-1} in S_n. It follows from this and the fact that $\widehat{B_n} := \Pi_n \rtimes S_n$ that any element $g \in \widehat{B_n}$ can be written as $g = s_i \dots s_{n-1} \omega a_n^\epsilon a_J z^\beta$ where $1 \leq i \leq n$, $\omega \in S_{n-1}$, $a_J = a_{j_1} \dots a_{j_t}$ for some $J = \{j_1, \dots, j_t\} \subseteq \{1, 2, \dots, n-1\}$, and $\epsilon, \beta \in \{0, 1\}$. Since a_n commutes with S_{n-1} we have $x = s_i \dots s_{n-1} a_n^\epsilon \omega a_J z^\beta$. Since $\omega a_J z^\beta \in \widehat{B_{n-1}}$, the set (5.3) contains a set of left coset representatives. The result then follows from the observation that the size of (5.3) is $2n$ while $|\widehat{B_n}| = 2^{n+1} n!$ and $|\widehat{B_{n-1}}| = 2^n (n-1)!$. □

Remark. When $g = s_i \dots s_{n-1} a_n^\epsilon$ for $\epsilon \in \{0, 1\}$ then $g^{-1} = a_n^\epsilon z^\epsilon s_{n-1} \dots s_i$ and consequently while $\pi_n(g) = s_i \dots s_{n-1} c_n^\epsilon$,

$$\pi_n(g^{-1}) = (-1)^\epsilon c_n^\epsilon s_{n-1} \dots s_i = (-1)^{|g|} c_n^\epsilon s_{n-1} \dots s_i.$$

We use the inclusions $\widehat{B_1} \subset \widehat{B_2} \subset \dots \subset \widehat{B_{n-1}} \subset \widehat{B_n} \subset \dots$ to iterate Lemma 5.1.1 to get that for all $1 \leq k < n$,

$$\widehat{\mathcal{LC}}_k^n := \{ (s_{i_n} \dots s_{n-1} a_n^{\epsilon_n})(s_{i_{n-1}} \dots s_{n-2} a_{n-1}^{\epsilon_{n-1}}) \dots (s_{i_{k+1}} \dots s_k a_{k+1}^{\epsilon_{k+1}})$$

$$\mid 1 \leq i_j \leq j,\ \epsilon_j \in \{0, 1\} \}$$

is a collection of left coset representatives of $\widehat{B_k}$ in $\widehat{B_n}$. Note in particular that

$$|\widehat{\mathcal{LC}}_k^n| = \frac{|\widehat{B_n}|}{|\widehat{B_k}|} = n^{\downarrow k} 2^{n-k} \tag{5.4}$$

where $n^{\downarrow k}$ is the falling factorial

$$n^{\downarrow k} := \frac{n!}{(n-k)!} = n(n-1)\dots(n-k+1)$$

for $1 \leq k < n$. The projection $\pi_n : \Bbbk[\widehat{B_n}] \to \mathbb{S}_n$ sends the elements of $\widehat{\mathcal{LC}}_k^n$ to distinct non-zero elements of $\mathbb{S}_n$ and we set

$$\mathcal{LC}_k^n := \pi_n(\widehat{\mathcal{LC}}_k^n).$$

The conjugacy classes of $\widehat{B_n}$ are determined by cycle types, just like the symmetric group. The difference is that now we have a $\mathbb{Z}/2\mathbb{Z}$ grading, so we will have even and odd parts in cycles. We will use the notation ρ_+ for the even part, and ρ_- for the odd part of a cycle. Then the set of conjugacy classes of $\widehat{B_n}$ are indexed by pairs of partitions (ρ_+, ρ_-) such that $|\rho_+|+|\rho_-| = n$, plus an additional parameter $\epsilon \in \{0,1\}$ when either $(\rho_+, \rho_-) = (\mu, \varnothing)$ with $\mu \in \mathcal{OP}_n$ or $(\rho_+, \rho_-) = (\varnothing, \lambda)$ with $\lambda \in \mathcal{SP}_n$. We denote this indexing set by Conj. A detailed description of the conjugacy class structure of $\widehat{B_n}$ can be obtained by analyzing the conjugacy class structure of B_n (which follows from the basic theory for the conjugacy class structure of wreath products [31, Appendix B]) and investigating how the inverse image of these sets under the map $\gamma : \widehat{B_n} \twoheadrightarrow B_n$ split into new conjugacy classes [35]. The additional parameter $\epsilon \in \{0,1\}$ appears precisely when a conjugacy class in B_n splits into two conjugacy classes in $\widehat{B_n}$.

We are ultimately interested in the even part of $\mathbb{S}_n$, which can be described using information about the conjugacy classes indexed by $(\mu, \varnothing, \epsilon)$ for $\mu \in \mathcal{OP}_n$ and $\epsilon \in \{0,1\}$. Therefore from now on we will limit ourselves to considering these classes. We call this set of conjugacy classes $\mathrm{Conj}_{odd} \subset \mathrm{Conj}$. For $\beta \in \mathrm{Conj}$, we write $\mathrm{Conj}(\beta)$ for the corresponding conjugacy class.

Now we introduce a family of elements of B_n which will be useful for constructing representatives for the conjugacy classes from Conj_{odd}. For $\mu = (\mu_1, \dots, \mu_r) \in \mathcal{OP}_k$, set $\pi_\mu = id$ if $\mu = (1^k)$ and otherwise

$$\begin{aligned}\pi_\mu &:= (s_{k-1} \dots s_{k-\mu_r+1}) \dots (s_{\mu_1+\mu_2-1} \dots s_{\mu_1+1})(s_{\mu_1-1} \dots s_2 s_1) \\ &= (k, k-1, \dots, k-\mu_r+1) \dots (\mu_1+\mu_2, \dots, \mu_1+1)(\mu_1, \dots, 2, 1) \in S_k.\end{aligned}$$

For $n \geq k$ we define $\sigma_{\mu;n} := \tau_0 \pi_\mu \tau_0^{-1}$, where τ_0 is the longest element of S_n by Coxeter length. Notice that $\sigma_{\mu;n}$ has cycle type $(\mu, 1^{n-k}) \in \mathcal{OP}_n$ and fixes $1, 2, \dots, n-k$ pointwise.

Example 1.1 Let $\mu = (5,3) \in \mathcal{OP}_8$, then

$$\pi_\mu = (s_7 s_6)(s_4 s_3 s_2 s_1) = (8,7,6)(5,4,3,2,1)$$

and π_μ has cycle type μ. For $n = 12$,

$$\sigma_{\mu;12} = (s_5 s_6)(s_8 s_9 s_{10} s_{11}) = (5,6,7)(8,9,10,11,12)$$

while for $n = 15$,

$$\sigma_{\mu;15} = (s_8 s_9)(s_{11} s_{12} s_{13} s_{14}).$$

Proposition 5.1.2. The elements $\{ \sigma_{\mu;n},\ z\sigma_{\mu;n} \mid \mu \in \mathcal{OP}_n \}$ form a complete set of conjugacy class representatives for the conjugacy classes Conj_{odd} in $\widehat{B_n}$ with $\sigma_{\mu;n}$ corresponding to $(\mu, \varnothing, 0) \in \mathrm{Conj}_{odd}$ and $z\sigma_{\mu;n}$ corresponding to $(\mu, \varnothing, 1) \in \mathrm{Conj}_{odd}$.

Proof. This follows from the description of the conjugacy classes of B_n and results on conjugacy class splitting of $\widehat{B_n}$ in [35] (see also [46, Section 2.5] for an overview). $\square$

Note that under the projection map $\pi_n : \Bbbk[\widehat{B_n}] \to \mathbb{S}_n$, the two sets of conjugacy classes $\{\sigma_{\mu;n}\}_{\mu\in\mathcal{OP}_n}$ and $\{z\sigma_{\mu;n}\}_{\mu\in\mathcal{OP}_n}$ are identified since $\pi_n(z) = -1$.

The size of the conjugacy classes $\mathrm{Conj}(\mu, \varnothing, \epsilon)$ will be important to us later. For $\rho \in \mathcal{P}_n$, we denote by z_ρ the size of the stabilizer of an element of S_n of cycle type ρ under the conjugation action. Recall that

$$z_\rho = \prod_{i\in\mathbb{Z}_{\geq 0}} i^{m_i(\rho)} m_i(\rho)!$$

where $m_i(\rho)$ is the number of parts of size i in ρ.

Lemma 5.1.3. [13] For $\mu \in \mathcal{OP}_n$, $\epsilon \in \{0, 1\}$

$$|\mathrm{Conj}(\mu, \varnothing, \epsilon)| = \frac{n!}{z_\mu} 2^{n-\ell(\mu)}.$$

5.1.1 The super representation theory of $\mathbb{S}_n$ and $\widehat{B_n}$

In this section we will review basic facts about the super representation theory of $\mathbb{S}_n$. Recall that any $\mathbb{S}_n$-supermodule is by definition a spin representation of $\widehat{B_n}$, so all statements about $\mathbb{S}_n$-supermodules also hold for $\widehat{B_n}$ spin representations. We refer the reader to [25] and [46] for thorough accounts of these topics as well as a review of super representation theory.

Let $\delta : \mathcal{SP} \to \{0, 1\}$ be defined by

$$\delta(\lambda) := \begin{cases} 0 & \ell(\lambda) \text{ is even} \\ 1 & \ell(\lambda) \text{ is odd.} \end{cases}$$

The function δ will be useful for describing quantities related to the representation theory of $\mathbb{S}_n$.

Theorem 5.1.4. [43]

1. The set of simple $\mathbb{S}_n$-supermodules are indexed by $\mathcal{SP}_n$, and the simple $\mathbb{S}_n$-supermodule L^λ corresponding to $\lambda \in \mathcal{SP}_n$ is of type M if $\ell(\lambda)$ is even and of type Q if $\ell(\lambda)$ is odd.

2. Let $\lambda \in \mathcal{SP}_n$, then

$$\dim(L^\lambda) = 2^{n - \frac{\ell(\lambda)-\delta(\lambda)}{2}} g_\lambda.$$

The algebras $\{\mathbb{S}_n\}_{n\geq 0}$ are semisimple. When N and M are $\mathbb{S}_n$-supermodules we write

$$[M : N] := \dim(\mathrm{Hom}_{\mathbb{S}_n}(M, N)).$$

For $\lambda \in \mathcal{SP}_n$, we denote by $\widehat{\chi}^\lambda$ the character corresponding to simple $\mathbb{C}[\widehat{B_n}]$-supermodule $\widehat{L}^\lambda$. This descends to a character χ^λ for simple $\mathbb{S}_n$-supermodule L^λ with

$$\chi^\lambda(\pi_n(g)) = \widehat{\chi}^\lambda(g).$$

The *normalized character* $\widetilde{\chi}^\lambda$ is defined such that for $x \in \mathbb{S}_n$

$$\widetilde{\chi}^\lambda(x) := \frac{\chi^\lambda(x)}{\dim(L^\lambda)} = \frac{\chi^\lambda(x)}{\chi^\lambda(1)}.$$

Proposition 5.1.5. For $\lambda \in \mathcal{SP}_n$, the character χ^λ is uniquely determined by its value on the elements $\{\sigma_{\mu;n} \mid \mu \in \mathcal{OP}_n\}$.

Proof. This follows from a similar statement [13, Proposition 1.9] where each element $\sigma_{\mu;n}$ is replaced by an element of $\mathbb{S}_n$ that is conjugate to it. Since characters are constant across conjugacy classes, the result follows. □

Given Proposition 5.1.5, for $\mu \in \mathcal{OP}_k$ with $k \leq n$, it is convenient to write $\chi^\lambda(\mu \cup 1^{n-k}) := \chi^\lambda(\sigma_{\mu;n})$.

5.1.2 The centers of $\mathbb{S}_n$ and $\mathbb{C}[\widehat{B_n}]$

As a superalgebra the center of $\mathbb{S}_n$ breaks up into even and odd components of super-commutative elements, $Z(\mathbb{S}_n) = Z(\mathbb{S}_n)_{\bar{0}} \oplus Z(\mathbb{S}_n)_{\bar{1}}$. In this paper we will focus on $Z(\mathbb{S}_n)_{\bar{0}}$, which corresponds to the center of $\mathbb{S}_n$ after the $(\mathbb{Z}/2\mathbb{Z})$-grading has been forgotten. It will later be important that, $Z(\mathbb{S}_n)_{\bar{0}}$ is exactly those elements that act on all simple $\mathbb{S}_n$-modules as multiplication by a scalar. Following [13] we will construct a basis for $Z(\mathbb{S}_n)_{\bar{0}}$ via the surjection $\pi : \mathbb{C}[\widehat{B_n}] \twoheadrightarrow \mathbb{S}_n$.

Recall that the set Conj indexes the conjugacy classes of $\widehat{B_n}$. For $\beta \in \text{Conj}$ we will write the conjugacy class sum as

$$\widehat{C}_\beta := \sum_{g \in \text{Conj}(\beta)} g.$$

It is clear that $\{\widehat{C}_\beta\}_{\beta \in \text{Conj}}$ is a basis for the ungraded center of $\mathbb{C}[\widehat{B_n}]$. In [13], Ivanov uses the subset of this basis corresponding to elements of Conj_{odd} of the form $(\mu, \varnothing, 0)$ to construct a basis for $Z(\mathbb{S}_n)_{\bar{0}}$. For $\mu \in \mathcal{OP}_n$ let

$$C_\mu := \pi_n(\widehat{C}_{(\mu,\varnothing,0)}).$$

Proposition 5.1.6. [13] The set $\{C_\mu \mid \mu \in \mathcal{OP}_n\}$ is a linear basis for $Z(\mathbb{S}_n)_{\bar{0}}$.

We now define a scaled version of Ivanov's basis of $Z(\mathbb{S}_n)_{\bar{0}}$ which naturally appears from the center of the twisted Heisenberg category.

Definition 5.1.2. For $k \leq n$ and $\mu \in \mathcal{OP}_k$, define

$$\widehat{B}_{\mu;n} := \sum_{g \in \widehat{\mathcal{LC}}^n_{n-k}} g\sigma_{\mu;n}g^{-1}$$

and

$$B_{\mu;n} := \pi(\widehat{B}_{\mu;n}).$$

Proposition 5.1.7. Let $k \leq n$ and $\mu \in \mathcal{OP}_k$ then:

1. $\widehat{B}_{\mu;n} \in Z(\mathbb{C}[\widehat{B_n}])$ and $B_{\mu;n} \in Z(\mathbb{S}_n)_{\bar{0}}$.

2. $\widehat{B}_{\mu;n} = 2^{k-n+\ell(\mu)} \dfrac{z_{\mu\cup 1^{n-k}}}{(n-k)!} \widehat{C}_{(\mu\cup 1^{n-k},\varnothing,0)}$.

3. Let $h \in \widehat{B_n}$ be an element not belonging to the same conjugacy class as $\sigma_{\mu;n}$ or $z\sigma_{\mu;n}$ for some $\mu \in \mathcal{OP}_n$ (i.e. h does not belong to a conjugacy class indexed by $(\mu, \varnothing, \epsilon)$, $\epsilon \in \{0,1\}$). Then

$$\pi_n\Big(\sum_{g\in\widehat{\mathcal{LC}}^n_{n-k}} ghg^{-1}\Big) = 0.$$

Proof. 1. Recall that we defined $\sigma_{\mu;n}$ as a distinguished element from the conjugacy class of $\widehat{B_n}$ indexed by $(\mu \cup 1^{n-k}, \varnothing, 0)$. Since $\sigma_{\mu;n}$ is by definition a product of $s_{n-1}, \ldots, s_{n-k+1}$ it commutes with $\widehat{B_{n-k}}$. Since $\widehat{\mathcal{LC}}^n_{n-k}$ is a collection of left coset representatives of $\widehat{B_{n-k}}$ in $\widehat{B_n}$ any element $g \in \widehat{B_n}$ can be written uniquely as $g = \sigma h$ for $\sigma \in \widehat{\mathcal{LC}}^n_{n-k}$ and $h \in \widehat{B_{n-k}}$. Thus $g\sigma_{\mu;n}g^{-1} = \sigma h \sigma_{\mu;n} h^{-1}\sigma^{-1} = \sigma\sigma_{\mu;n}\sigma^{-1}$ and hence $g\sigma_{\mu;n}g^{-1}$ is completely determined by the left coset to which g belongs. It follows that

$$\sum_{g\in\widehat{B_n}} g\sigma_{\tilde{\mu};n}g^{-1} = |\widehat{B_{n-k}}| \sum_{g\in\widehat{\mathcal{LC}}^n_{n-k}} g\sigma_{\mu;n}g^{-1} = |\widehat{B_{n-k}}|\widehat{B}_{\mu;n} \tag{5.5}$$

and $\widehat{B}_{\mu;n} \in Z(\mathbb{C}[\widehat{B_n}])$ since $\widehat{B}_{\mu;n}$ is a scalar multiple of a central element.

Finally, note that π_n is a degree-preserving homomorphism and $\widehat{B}_{\mu;n}$ is even, so $\pi(\widehat{B}_{\mu;n}) = B_{\mu;n} \in Z(\mathbb{S}_n)_{\overline{0}}$.

2. It follows from Lemma 5.1.3 and the orbit stabilizer theorem that

$$\sum_{g\in\widehat{B_n}} g\sigma_{\mu;n}g^{-1} = 2^{\ell(\mu)+1} z_{\mu\cup 1^{n-k}} \widehat{C}_{(\mu\cup 1^{n-k},\varnothing,0)}.$$

Then (5.5) implies that

$$|\widehat{B_{n-k}}|\widehat{B}_{\mu;n} = 2^{\ell(\mu)+1} z_{\mu\cup 1^{n-k}} \widehat{C}_{(\mu\cup 1^{n-k},\varnothing,0)}.$$

The result follows.

3. We show that for any element $h \in \widehat{B_n}$ which belongs to a conjugacy class indexed by $\beta \neq (\mu, \varnothing, \epsilon)$, for $\epsilon \in \{0,1\}$, $\mu \in \mathcal{OP}_n$, $\pi_n(h) = x$ is zero in the trace of $\mathbb{S}_n$, which is the algebra defined by

$$\overline{\mathbb{S}_n} = \frac{\mathbb{S}_n}{[\mathbb{S}_n, \mathbb{S}_n]}.$$

If $x \equiv 0 \mod [\mathbb{S}_n, \mathbb{S}_n]$, then x is either 0 or conjugate to its negative. In $\overline{\mathbb{S}_n}$, each element is equal to its conjugates, so $x \equiv 0 \mod [\mathbb{S}_n, \mathbb{S}_n]$ implies that for every $h' \in \mathrm{Conj}(\beta)$ with $\pi_n(h') = x'$ either $x' = 0$ or there is another element $h'' \in \mathrm{Conj}(\beta)$ such that $\pi_n(h'') = -x'$. Either case implies that $\pi_n(\widehat{C}_\beta) = 0$, and hence by the same argument as 2,

$$\pi_n\Big(\sum_{g \in \widehat{\mathcal{LC}}^n_{n-k}} ghg^{-1}\Big) = 0. \tag{5.6}$$

When $h \in S_n \subset \widehat{B_n}$ and h has cycle type $\rho \notin \mathcal{OP}_n$, so that h is in the conjugacy class labeled $(\rho, \varnothing)$ then by [36, Proposition 3.9] $\pi_n(h) \equiv 0 \mod [\mathbb{S}_n, \mathbb{S}_n]$. On the other hand when $h \in \widehat{B_n}$ is a member of the conjugacy class labeled (ρ, η) for $\eta \neq \varnothing$ then by [36, Proposition 3.4] $\pi_n(h) \equiv 0 \mod [\mathbb{S}_n, \mathbb{S}_n]$. The result then follows from the previous paragraph.

□

It follows from Proposition 5.1.7.2 that $\{B_{\mu;n} \mid \mu \in \mathcal{OP}_n\}$ is also a linear basis of $Z(\mathbb{S}_n)_{\bar{0}}$.

For a spin representation $\widehat{L}^\lambda$ of $\widehat{B_n}$, the corresponding character $\widehat{\chi}^\lambda$ is a homomorphism when restricted to $Z(\mathbb{C}[\widehat{B_n}])_{\bar{0}}$ and χ^λ is a homomorphism on $Z(\mathbb{S}_n)_{\bar{0}}$.

Proposition 5.1.8. Let $\lambda \in \mathcal{SP}_n$ and $\mu \in \mathcal{OP}_k$. Then

$$\widehat{\chi}^\lambda(B_{\mu;n}) = 2^k n^{\downarrow k} \frac{\chi^\lambda(\sigma_{\mu;n})}{\chi^\lambda(1)}.$$

Proof. This follows from the fact that characters are invariant under conjugation and (5.4).

$\square$

Another basis for $Z(\mathbb{S}_n)_{\bar{0}}$ is given by the set of central idempotents of $\mathbb{S}_n$ corresponding to the simple $\mathbb{S}_n$-supermodules. We denote these central idempotents by $\{e_\lambda \mid \lambda \in \mathcal{SP}_n\}$.

Lemma 5.1.9. For $\lambda \in \mathcal{SP}_n$, the central idempotent $e_\lambda \in \mathbb{S}_n$ corresponding to simple $\mathbb{S}_n$-supermodule L^λ can be written as

$$e_\lambda = 2^{\frac{-\ell(\lambda)-\delta(\lambda)}{2}} \frac{g_\lambda}{n!} \sum_{\mu \in \mathcal{OP}_n} \chi^\lambda(\mu) C_\mu.$$

Proof. The definition of $\mathbb{S}_n$ implies that e_λ is the image of the corresponding central idempotent $\widehat{e_\lambda}$ in $\widehat{B_n}$ under the projection map π_n. There are two cases to consider: that in which L^λ is of type M (i.e. $\delta(\lambda) = 0$) and that in which L^λ is of type Q (i.e. $\delta(\lambda) = 1$). Consider the case where L^λ is of type M. Then $\widehat{L}^\lambda$ is of type M also and $\widehat{L}^\lambda$ viewed as an ungraded $\mathbb{C}[\widehat{B_n}]$-module remains simple. Since $\widehat{B_n}$ is a finite group, the central idempotent corresponding to $\widehat{L}^\lambda$ is

$$\widehat{e_\lambda} = \frac{\dim(\widehat{L}^\lambda)}{|\widehat{B_n}|} \sum_{\beta \in \mathrm{Conj}} \widehat{\chi}^\lambda(\beta) \widehat{C}_\beta.$$

In the case where L^λ is of type Q, $\widehat{L}^\lambda$ viewed as an ungraded $\mathbb{C}[\widehat{B_n}]$-module breaks into the direct sum of two simple $\mathbb{C}[\widehat{B_n}]$-modules $\widehat{L}^\lambda_0$ and $\widehat{L}^\lambda_1$ of equal dimension. Then the central idempotent corresponding to $\widehat{L}^\lambda$ is given by

$$\widehat{e_\lambda} = \frac{\dim(\widehat{L}^\lambda)}{2|\widehat{B_n}|} \sum_{\beta \in \mathrm{Conj}} \widehat{\chi}^\lambda(\beta) \widehat{C}_\beta$$

where $\widehat{\chi}^\lambda(g) = \widehat{\chi}^{\lambda_0}(g) + \widehat{\chi}^{\lambda_1}(g)$. Thus in general

$$\widehat{e_\lambda} = \frac{\dim(\widehat{L}^\lambda)}{2^{\delta(\lambda)}|\widehat{B_n}|} \sum_{\beta \in \mathrm{Conj}} \widehat{\chi}^\lambda(\beta) \widehat{C}_\beta.$$

Applying π_n to $\widehat{e_\lambda}$, Proposition 5.1.7.3 implies that most terms go to zero and we are left with

$$e_\lambda = 2^{\frac{-\ell(\lambda)-\delta(\lambda)}{2}-1}\frac{g_\lambda}{n!}\sum_{\beta\in \mathrm{Conj}_{odd}} \chi^\lambda(\beta)\pi_n(\widehat{C}_\beta).$$

Recall that $\beta \in \mathrm{Conj}_{odd}$ contains pairs $\beta = (\mu, \varnothing, 0)$ and $\bar{\beta} = (\mu, \varnothing, 1)$ for $\mu \in \mathcal{OP}_n$ such that if $x \in \mathrm{Conj}(\beta)$ then $zx \in \mathrm{Conj}(\bar{\beta})$. It follows that $\pi_n(\widehat{C}_{\bar{\beta}}) = -\pi_n(\widehat{C}_\beta)$. At the same time, since z acts as multiplication by -1 on $\widehat{L}^\lambda$ then $\chi^\lambda(z\sigma_{\mu;n}) = -\chi^\lambda(\sigma_{\mu;n})$. It follows that

$$e_\lambda = 2^{\frac{-\ell(\lambda)-\delta(\lambda)}{2}-1}\frac{g_\lambda}{n!}\sum_{\beta\in \mathrm{Conj}_{odd}} \chi^\lambda(\beta)\pi_n(\widehat{C}_\beta) = 2^{\frac{-\ell(\lambda)-\delta(\lambda)}{2}}\frac{g_\lambda}{n!}\sum_{\mu\in\mathcal{OP}_n} \chi^\lambda(\mu)C_\mu.$$

□

5.2 The Subalgebra Γ of Symmetric Functions

We recall relevant facts about the algebra Γ following [31]. Let p_k be the kth power sum symmetric polynomial,

$$p_k(x_1, \ldots, x_n) = x_1^k + \ldots + x_n^k$$

and recall that for $\rho \in \mathcal{P}$,

$$p_\rho(x_1, \ldots, x_n) := \prod_{k=1}^{\ell(\rho)} p_{\rho_k}(x_1, \ldots, x_n).$$

Define Γ_n to be the subalgebra of the symmetric polynomials in $x_1, x_2, \ldots, x_n$ generated by $\{p_\lambda \mid \lambda \in \mathcal{OP}_n\}$.

For each n, there is a surjective homomorphism

$$\Gamma_{n+1} \twoheadrightarrow \Gamma_n$$

given by setting $x_{n+1} = 0$. Define Γ to be the projective limit of these algebras with respect to these homomorphisms

$$\Gamma := \varprojlim \Gamma_n.$$

Alternatively, Γ can be described as the subalgebra of the symmetric functions generated by the odd power sum symmetric functions

$$\Gamma = \mathbb{C}[p_1, p_3, p_5, \ldots].$$

Elements of Γ can be evaluated on partitions in the following way. Let $f \in \Gamma$ and $\rho \in \mathcal{P}$, and define

$$f(\rho) = f(\rho_1, \rho_2, \ldots, \rho_{\ell(\rho)}, 0, \ldots). \tag{5.7}$$

Let $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$ denote the algebra of functions from $\mathcal{SP}$ to $\mathbb{C}$ with pointwise multiplication.

Proposition 5.2.1. [15, Proposition 6.2]) The algebra Γ embeds into $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$ via the evaluation map (5.7).

Using the above embedding, we will start to see p_μ's as functions, evaluated at strict partitions.

Example 2.1 For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathcal{SP}_n$ we have

$$p_1(\lambda) = \lambda_1 + \lambda_2 + \cdots + \lambda_r = n.$$

We recall an important linear basis of Γ, the Schur Q-functions [31, Section III.8]. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathcal{SP}$, then

$$Q_{\lambda|N}(x_1, \ldots, x_N) = \frac{2^{\ell(\lambda)}}{(N-r)!} \sum_{\omega \in S_N} \omega \left(x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_r^{\lambda_r} \prod_{\substack{1 \le i \le r \\ i < j \le N}} \frac{x_i + x_j}{x_i - x_j} \right),$$

where we say $Q_{\lambda|N} = 0$ if $\ell(\lambda) > N$. The sequence $(Q_{\lambda|N})_{N=1,2,\ldots}$ defines an element $Q_\lambda \in \Gamma$ known as the *Schur Q-functions*. The set $\{Q_\lambda\}_{\lambda \in \mathcal{SP}}$ forms a linear basis of Γ.

Define numbers X^λ_μ for $\lambda \in \mathcal{SP}_n$, $\mu \in \mathcal{OP}_n$, via

$$p_\mu = \sum_{\lambda \in \mathcal{SP}_n} 2^{-\ell(\lambda)} X^\lambda_\mu Q_\lambda. \tag{5.8}$$

There is a "factorial" version of the Schur Q-functions, defined in [14]. For $\lambda \in \mathcal{SP}$, the *factorial Schur Q-polynomial* corresponding to λ is defined as:

$$Q^*_{\lambda|N}(x_1, \ldots, x_N) := \frac{2^{\ell(\lambda)}}{(N-l)!} \sum_{\omega \in S_N} \omega \left(x_1^{\downarrow \lambda_1} x_2^{\downarrow \lambda_2} \ldots x_l^{\downarrow \lambda_l} \prod_{\substack{1 \le i \le l \\ i < j \le N}} \frac{x_i + x_j}{x_i - x_j} \right). \tag{5.9}$$

If $\ell(\lambda) > N$, then $Q^*_{\lambda|N}$ is defined to be 0. The collection $(Q^*_{\lambda|N})_{N=1,2,\ldots}$ defines an element of Γ, the *factorial Schur Q-function* Q^*_λ. Factorial Schur Q-functions have the following useful properties.

Proposition 5.2.2. [13] Let $\lambda, \nu \in \mathcal{SP}$.

1. There exists $g \in \Gamma$ of degree less than $|\lambda|$ such that

$$Q^*_\lambda = Q_\lambda + g.$$

2. The collection $\{Q^*_\lambda\}_{\lambda \in \mathcal{SP}}$ is a linear basis of Γ.

3. If $\nu \in \mathcal{SP}_k$, $\lambda \in \mathcal{SP}_n$ for $k \le n$ and $\nu \not\subseteq \lambda$, $Q^*_\lambda(\nu) = 0$.

Let $\psi : \Gamma \to \Gamma$ be the linear map that sends $Q_\lambda \mapsto Q^*_\lambda$. For any $\mu \in \mathcal{OP}$, define the inhomogeneous analogue of the power sum $\mathfrak{p}_\mu := \psi(p_\mu) \in \Gamma$. Applying ψ to both sides of (5.8) gives

$$\mathfrak{p}_\mu = \sum_{\lambda \in \mathcal{SP}_k} 2^{-\ell(\lambda)} X^\lambda_\mu Q^*_\lambda.$$

It also follows from the fact that $X^{\lambda}_{\mu} = 2^{-\ell(\mu)+\frac{\ell(\lambda)-\delta(\lambda)}{2}}\chi^{\lambda}(\mu)$ [13, Proposition 3.3] and

$$Q_{\lambda} = \sum_{\mu\in\mathcal{OP}_n} \frac{2^{\ell(\mu)}}{z_{\mu}} X^{\lambda}_{\mu} p_{\mu}$$

that

$$Q^{*}_{\lambda} = 2^{\frac{\ell(\lambda)-\delta(\lambda)}{2}} \sum_{\mu\in\mathcal{OP}_n} \frac{\chi^{\lambda}(\mu)}{z_{\mu}} \mathfrak{p}_{\mu}. \tag{5.10}$$

The elements $\{\mathfrak{p}_{\mu}\}_{\mu\in\mathcal{SP}}$ were first studied in [13], where Ivanov proves that they satisfy the following properties.

Proposition 5.2.3. [13] Let $\mu \in \mathcal{OP}_k$ and $\lambda \in \mathcal{SP}_n$.

1. There exists $g \in \Gamma$ of degree less than $|\mu|$ such that

$$\mathfrak{p}_{\mu} = p_{\mu} + g.$$

2. The family $(\mathfrak{p}_{\mu})_{\mu\in\mathcal{OP}}$ is a linear basis of Γ.

3.

$$\mathfrak{p}_{\mu}(\lambda) = \begin{cases} n^{\downarrow k} \cdot \dfrac{X^{\lambda}_{\mu\cup(1^{n-k})}}{g_{\lambda}} & \text{if } |\lambda| \geq |\mu|, \\ 0 & \text{otherwise} \end{cases}$$

where in particular $g_{\lambda} = X^{\lambda}_{1^{|\lambda|}}$.

4. Let $\gamma \in \mathcal{OP}$. Define $\mu \cup \gamma$ to be the partition formed by taking the disjoint union of parts of μ and γ and rearranging them in decreasing order. Then there exists $g \in \Gamma$ of degree less than $|\mu \cup \gamma|$ such that

$$\mathfrak{p}_{\mu} \cdot \mathfrak{p}_{\gamma} = \mathfrak{p}_{\mu\cup\gamma} + g.$$

As a Corollary to part 3 of the above Proposition, we have another formula for the value of $\mathfrak{p}_\rho$.

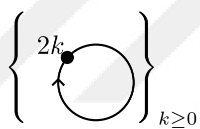
Corollary 5.2.3.1. [13] Let $\mu \in \mathcal{OP}_k$ and $\lambda \in \mathcal{SP}_n$. We have

$$\mathfrak{p}_\mu(\lambda) = 2^{k-\ell(\mu)} n^{\downarrow k} \frac{\chi^\lambda(\mu \cup 1^{n-k})}{\chi^\lambda(1^n)}.$$

Corollary 5.2.3.2. The elements $\{\mathfrak{p}_{2k+1}\}_{k\geq 0}$ are algebraically independent and generate Γ.

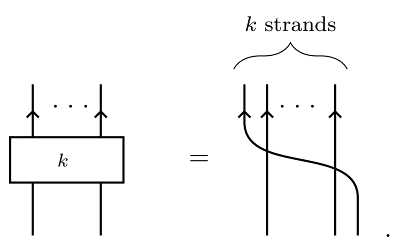
We will see the diagrammatic interpretations of the basis $\{\mathfrak{p}_{2k+1}\}_{k\geq 0}$ and $\{Q^*_\lambda\}_{\lambda \in \mathcal{SP}}$ in $End_{\mathcal{H}_{tw}}(\mathbb{1})$.

5.3 A new basis of $End_{\mathcal{H}_{tw}}(\mathbb{1})$

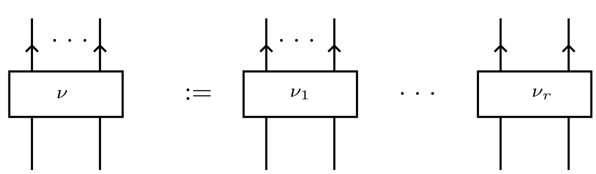
We already saw that 

is a basis of $End_{\mathcal{H}_{tw}}(\mathbb{1})$.

Another natural set of diagrams in $End_{\mathcal{H}_{tw}}(\mathbb{1})$ come from the closure of permutations. We define

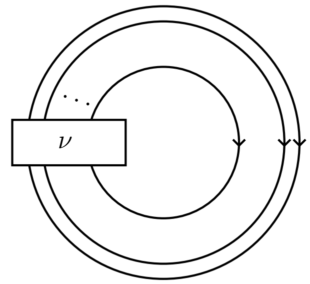


For $\nu = (\nu_1, \dots, \nu_r) \in \mathcal{P}_k$, let



(5.11)

then we define

$\alpha_\nu \quad :=$ 

The diagram on the right doesn't depend on the choice of representative from a conjugacy class. Therefore we can index these diagrams with cycle types. We also set $\alpha_k := \alpha_{(k)}$.

One can impose a grading on $End_{\mathcal{H}_{tw}}(\mathbb{1})$ by setting:

$$\deg(d_0) = 0 \qquad \text{and} \qquad \deg(d_{2k}) = 2k+1. \tag{5.12}$$

Lemma 5.3.1. In terms of the grading defined by (5.12),

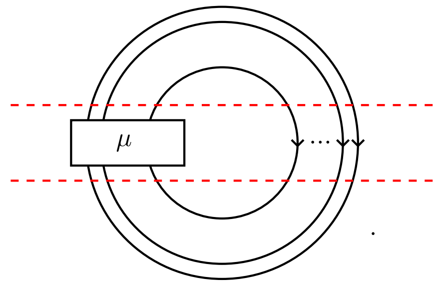
$$\alpha_{2k+1} = d_{2k} + \text{l.o.t.}$$

Proof. We can reduce the diagram α_{2k+1} to a polynomial in $d_0, d_2, d_4, \ldots$ via repeated application of the dot sliding moves (3.33)-(3.34) and clockwise bubble sliding move from Lemma 4.2.10. The goal of each move is to increase the number of crossings coming from solid dots and separate nested diagrams. Each application of these rules will result in a single connected diagram D whose total number of crossings is $2k$ (including those from solid dots), plus additional terms whose total number of crossings (including those from solid dots) is strictly less than $2k$ (this can be seen by examining (3.33)-(3.34) and (4.2.10)). At the end of this process we have a single bubble with $2k$ dots plus additional terms each of which has total number of dots strictly less that $2k$. □

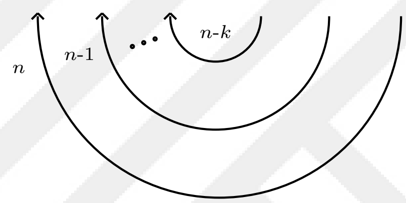
Corollary 5.3.1.1. $End_{\mathcal{H}}(\mathbb{1})$ is generated by $\{\alpha_{2k+1}\}_{k\geq 0}$ and these elements are algebraically independent.

Now we shall investigate the images of α_{2k+1} under the functor F_n^{tw} from section 3.4.2. We do this by seeing α_{2k+1} as a composition of 3 diagrams:

$\alpha_\mu \quad :=$



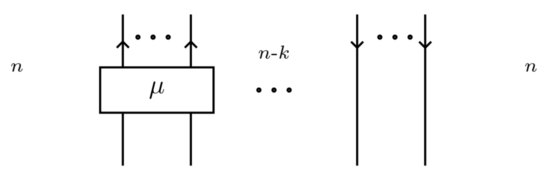
Lemma 5.3.2. 1. The diagram



corresponds to the $(\mathbb{S}_n, \mathbb{S}_n)$-bimodule homomorphism $(n) \to (n)_{n-k}(n)$ which sends

$$1 \;\mapsto \sum_{x \in \mathcal{LC}^n_{n-k}} x \otimes x^{-1}.$$

2. For $\mu \in \mathcal{OP}_k$ with $k \leq n$, the diagram



corresponds to the $(\mathbb{S}_n, \mathbb{S}_n)$-bimodule homomorphism $(n)_{n-k}(n) \to (n)_{n-k}(n)$ which for $x, y \in \mathbb{S}_n$ sends

$$x \otimes y \;\; \mapsto \;\; x\sigma_{\mu;n} \otimes y.$$

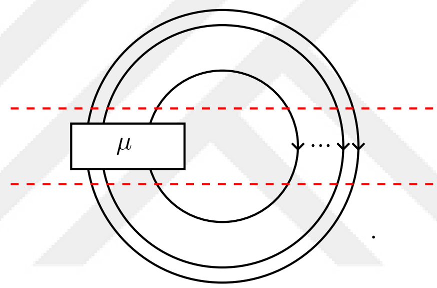
Proof. Both 1 and 2 follow from calculations using the definitions of cups (3.46) and crossings (3.49). □

Proposition 5.3.3. For $\mu \in \mathcal{OP}_k$,

$$F_n^{tw}(\alpha_\mu) = \begin{cases} B_{\mu;n} & \text{if } k \leq n \\ \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The diagram for α_μ can be broken into three components

$\alpha_\mu \quad :=$



Reading from bottom to top, the first component corresponds to Lemma 5.3.2.1, and the second corresponds to Lemma 5.3.2.2 The composition of these two maps sends

$$1 \;\mapsto\; \sum_{x \in \mathcal{LC}^n_{n-k}} x\sigma_{\mu;n} \otimes x^{-1}.$$

The top component of k nested caps is the multiplication map which sends

$$\sum_{x \in \mathcal{LC}^n_{n-k}} x\sigma_{\mu;n} \otimes x^{-1} \;\mapsto\; \sum_{x \in \mathcal{LC}^n_{n-k}} x\sigma_{\mu;n}x^{-1} = B_{\mu;n}.$$

□

5.4 An isomorphism between $End_{\mathcal{H}_{tw}}(\mathbb{1})$ and Γ

In this section we establish an isomorphism between $End_{\mathcal{H}_{tw}}(\mathbb{1})$ and Γ. The key step in the construction of this map will be identifying the elements of $End_{\mathcal{H}_{tw}}(\mathbb{1})$ with functions on $\mathcal{SP}$. To achieve this let $\lambda \in \mathcal{SP}_n$ and $x \in End_{\mathcal{H}_{tw}}(\mathbb{1})$. Then we evaluate x on λ by

$$x(\lambda) := \widetilde{\chi}^{\lambda}(F_n^{tw}(x)).$$

Because F_n^{tw} is a homomorphism on $End_{\mathcal{H}_{tw}}(\mathbb{1})$ which maps into $Z(\mathbb{S}_n)_{\overline{0}}$ and $\widetilde{\chi}^{\lambda}$ is a homomorphism when restricted to $Z(\mathbb{S}_n)_{\overline{0}}$, this defines a homomorphism $End_{\mathcal{H}_{tw}}(\mathbb{1}) \to \mathrm{Fun}(\mathcal{SP}, \mathbb{C})$.

Proposition 5.4.1. For $\mu \in \mathcal{OP}_k$ and $\lambda \in \mathcal{SP}_n$ we have

$$\alpha_\mu(\lambda) = \begin{cases} 2^k n^{\downarrow k} \frac{\chi^{\lambda}(\mu \cup 1^{n-k})}{\chi^{\lambda}(1^n)} & \text{if } k \leq n \\ 0 & \text{otherwise} \end{cases}$$

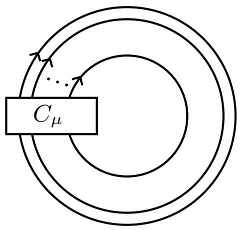
Proof. This follows from Proposition 5.1.8 and Proposition 5.3.3. □

Theorem 5.4.2. There is an algebra isomorphism $\Phi : End_{\mathcal{H}_{tw}}(\mathbb{1}) \to \Gamma$ which for any $\mu \in \mathcal{OP}$, sends

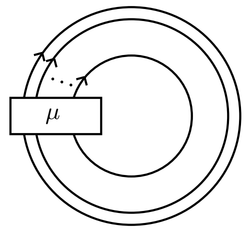
$$\alpha_\mu \mapsto 2^{\ell(\mu)} \mathfrak{p}_\mu.$$

Proof. It is clear from Proposition 5.4.1 and Corollary 5.2.3.1 that $2^{-\ell(\mu)}\alpha_\mu$ and $\mathfrak{p}_\mu$ map to the same function in $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$. Furthermore the collection of functions which are the image of $\{\mathfrak{p}_{2k+1}\}_{k \geq 0}$ are algebraically independent by Proposition 5.2.1 and Corollary 5.2.3.2. By Proposition 5.3.1.1 $End_{\mathcal{H}_{tw}}(\mathbb{1})$ is generated by the algebraically independent elements $\{\alpha_{2k+1}\}_{k \geq 0}$. It then follows that the map that sends $\alpha_\mu \mapsto 2^{\ell(\mu)}\mathfrak{p}_\mu$ is an isomorphism. □

So we identified the basis of $End_{\mathcal{H}_{tw}}(\mathbb{1})$ obtained by closure of permutations with $\{\mathfrak{p}_{2k+1}\}_{k\geq 0}$. Now for the Schur-$Q$ functions, let $\mu \in \mathcal{OP}_n$. It follows from Lemma 5.1.3 and Theorem 5.4.2 that



$$= \frac{n!}{z_\mu} 2^{n-\ell(\mu)}$$



$$\overset{\Phi}{\longmapsto} \frac{n!}{z_\mu} 2^n \mathfrak{p}_\mu. \tag{5.13}$$

We will denote by $h(\lambda)$ the number of paths in the Schur graph of B_n from $\varnothing$ to λ.

Theorem 5.4.3. Let $\lambda \in \mathcal{SP}_n$. Under the isomorphism $\Phi : End_{\mathcal{H}_{tw}}(\mathbb{1}) \to \Gamma$, the closure of the central idempotent e_λ of $\mathbb{S}_n$ maps to $h(\lambda)Q^*_\lambda$.

Proof. Recall from Lemma 5.1.9 that

$$e_\lambda = 2^{\frac{-\ell(\lambda)-\lambda(\lambda)}{2}} \frac{g_\lambda}{n!} \sum_{\mu \in \mathcal{OP}_n} \chi^\lambda(\mu) C_\mu$$

while by (5.10)

$$Q^*_\lambda = 2^{\frac{\ell(\lambda)-\lambda(\lambda)}{2}} \sum_{\mu \in \mathcal{OP}_n} \frac{\chi^\lambda(\mu)}{z_\mu} \mathfrak{p}_\mu.$$

Combining these facts with Theorem 5.4.2 and (5.13) it follows that the closure of e_λ is equal to $2^{n-\ell(\lambda)} g_\lambda Q^*_\lambda = h(\lambda)Q^*_\lambda$. □

Remark. Recall that the Schur Q-functions are related to the Schur P-functions by $P_\lambda = 2^{-\ell(\lambda)}Q_\lambda$. Ivanov also studied factorial Schur P-functions $\{P^*_\lambda\}_{\lambda\in\mathcal{SP}_n}$ where $P^*_\lambda = 2^{-\ell(\lambda)}Q^*_\lambda$ [13]. Then one alternative description of the closure of e_λ in Γ is as $2^n g_\lambda P^*_\lambda$.

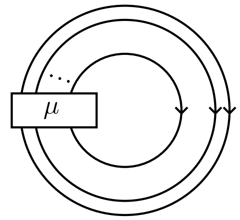
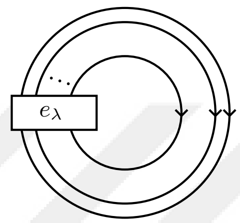
Γ	diagram in $End_{\mathcal{H}_{tw}}(\mathbb{1})$
$\mathfrak{p}_\mu$, $\mu \in \mathcal{OP}$	$\frac{1}{2^{\ell(\mu)}}$ 
Q^*_λ, $\lambda \in \mathcal{SP}$	$\frac{1}{h(\lambda)}$ 

Table 5.1: A dictionary between Γ and diagrams in $End_{\mathcal{H}_{tw}}(\mathbb{1})$. The notation $h(\lambda)$ denotes the number of paths in the Schur graph from $\varnothing$ to λ
.

5.5 An action of $\mathrm{Tr}(\mathcal{H}_{tw})_{\overline{0}}$ on Γ

We saw in chapter 2 that the trace of a diagrammatic category can be realized as the algebra of closed diagrams on an annulus. There is a natural action of $\mathrm{Tr}(\mathcal{C})$ on the center of the category $End_{\mathcal{C}}(\mathbb{1})$, where diagrammatically a closed diagram on an annulus acts on a closed diagram on a disk by plugging the hole of the annulus with the disk, resulting in a new diagram on a disk.

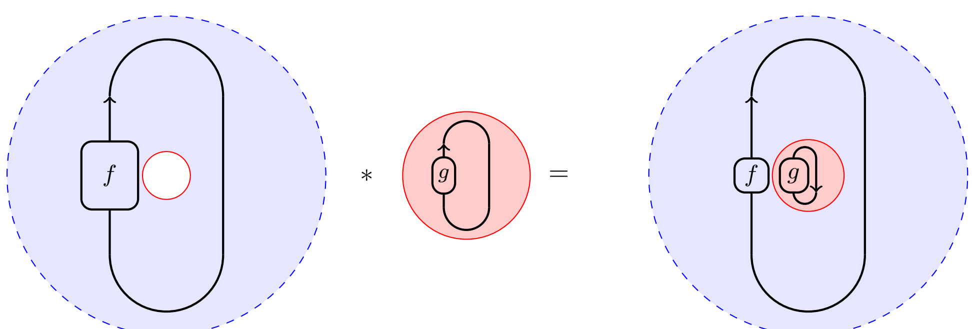
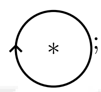
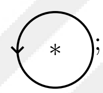


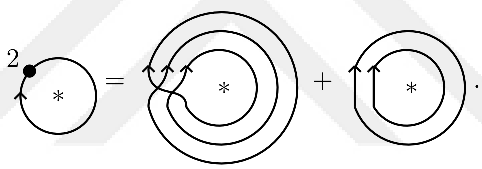
Figure 5.1: Graphical interpretation of the action of $Tr(\mathcal{C})$ on $_{\mathcal{C}}(\mathbb{1})$

The results of chapter 4 along with Theorem 5.4.2 imply that W^- acts on Γ. This action is similar to the action of $W_{1+\infty}$ on the centers of symmetric group algebras described in [28]. In this section we will describe the action of the generators of W^- on basis elements of Γ.

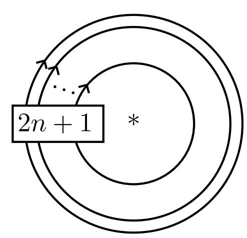
Recall that the images of the generators of the W-algebra under the isomorphism $W^-/\langle \omega_{0,0}, C-1\rangle \to \mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ are as follows:

$\sqrt{2}\omega_{-1,0} \longmapsto$ 

$\sqrt{2}\omega_{1,0} \longmapsto$ 

$-2\omega_{0,3} \longmapsto$ 

Additionally we will use the elements $\omega_{-(2n+1),0} \in W^-$ and their images in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$:

$\sqrt{2}\omega_{-(2n+1),0} \longmapsto$  (5.14)

where τ is a $2n+1$ cycle.

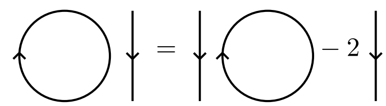
5.5.1 Description of the action

We describe the action of the generating set $\{\omega_{1,0}, \omega_{-1,0}, \omega_{0,3}\}$ of W^- on the vector space basis $\{\mathfrak{p}_\mu\}_{\mu\in\mathcal{OP}}$ of Γ. We achieve this using diagrammatics, by describing the action of the corresponding generators of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ on the basis $\{\alpha_\mu\}_{\mu\in\mathcal{OP}}$ of $End_{\mathcal{H}_{tw}}(\mathbb{1})$.

Lemma 5.5.1. We have

$$\alpha_{(\mu,1)} = \alpha_\mu \alpha_1 - 2|\mu|\alpha_\mu.$$

Proof. This simply follows from the local bubble sliding relation

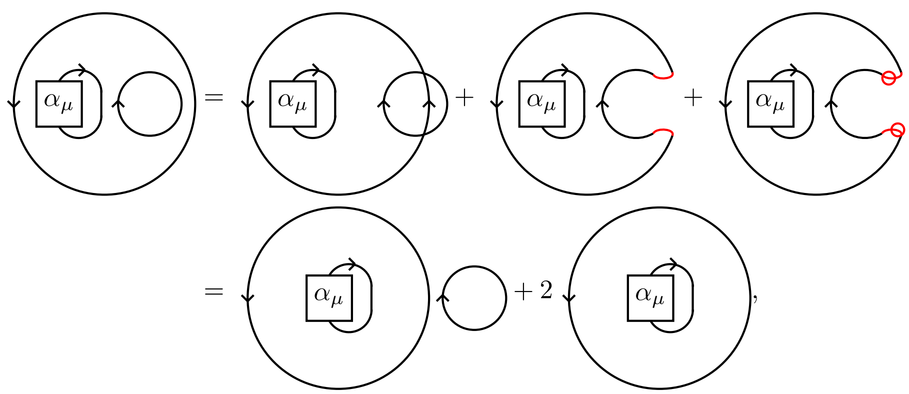


applied $|\mu|$ times to the diagram $\alpha_{(\mu,1)}$, as we pull the clockwise bubble α_1 from within α_μ. □

Lemma 5.5.2. We have

$$\omega_{1,0} \cdot \alpha_\mu \alpha_1 = (\alpha_1 + 2)\omega_{1,0} \cdot \alpha_\mu.$$

Proof. We compute:



as desired. □

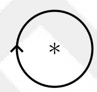
Theorem 5.5.3. The generators $\mathrm{Tr}(\mathcal{H}_{tw})_{\overline{0}}$ act on the basis elements $\{\mathfrak{p}_\mu\}_{\mu \in \mathcal{OP}}$ of Γ as follows:

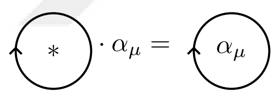
$$\omega_{-1,0} \cdot \mathfrak{p}_\mu = \sqrt{2}\mathfrak{p}_{(\mu,1)},$$

$$\omega_{1,0} \cdot \mathfrak{p}_\mu = \frac{1}{\sqrt{2}}\mathfrak{p}_\mu + \frac{k}{\sqrt{2}}\mathfrak{p}_{\hat{\mu}},$$

$$\omega_{0,3} \cdot \mathfrak{p}_\mu = -\mathfrak{p}_3\mathfrak{p}_\mu - 2\mathfrak{p}_{(1,1)}\mathfrak{p}_\mu$$

where k is the number of parts of size 1 of μ and $\hat{\mu}$ stands for the partition obtained by removing one part of size 1 from μ if this is possible. When $\mu = (1)$ then $\mathfrak{p}_{\widehat{(1)}} = 1$.

Proof. For the action of $\omega_{-1,0}$, note that the action of  on α_μ is diagrammatically just enclosing the diagram of α_μ by a clockwise oriented strand:



(5.15)

and the resulting diagram is the diagram of $\alpha_{(\mu,1)}$. Replacing α_μ by $2^{\ell(\mu)}\mathfrak{p}_\mu$ and the clockwise bubble by $\sqrt{2}\omega_{-1,0}$, we get

$$\omega_{-1,0} \cdot \mathfrak{p}_\mu = \sqrt{2}\mathfrak{p}_{(\mu,1)}.$$

We also know that $\omega_{-(2n+1),0} \cdot \mathfrak{p}_\mu = \sqrt{2}\mathfrak{p}_{(\mu,2n+1)}$ from (5.14). To calculate the action of $\omega_{1,0}$, we will use the the commutator relations

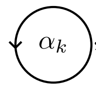
$$[\omega_{-1,0}, \omega_{1,0}] = -1$$

$$[\omega_{-(2n+1),0}, \omega_{1,0}] = 0 \quad \text{for } n \geq 0$$

and (5.15).

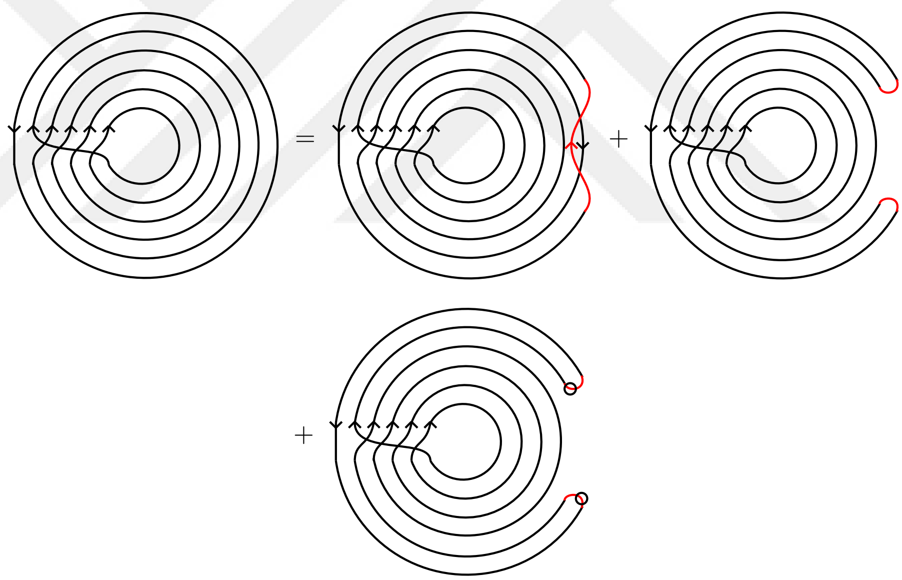
To simplify the notation in the following computations, we will use $\omega_+ := \sqrt{2}\omega_{1,0}$.

We start by showing that if the partition μ doesn't contain any parts of size one, then $\omega_+ \cdot \alpha_\mu = \alpha_\mu$ by induction on $\ell(\mu)$. We provide a diagrammatic proof for the base case $\ell(\mu) = 1$ (i.e. $\alpha_\mu = \alpha_k$ for $k \neq 1$ odd).

In the diagram 

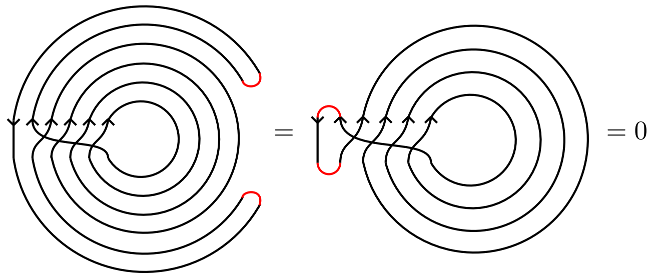
, we claim that we can pass α_k through the outer strand for free, meaning that all the resolution terms that appear as a result of relation (3.24) are zero.

We provide the computation for the case of $\alpha_k = \alpha_5$, and explain how the arguments generalize to any α_k. We have



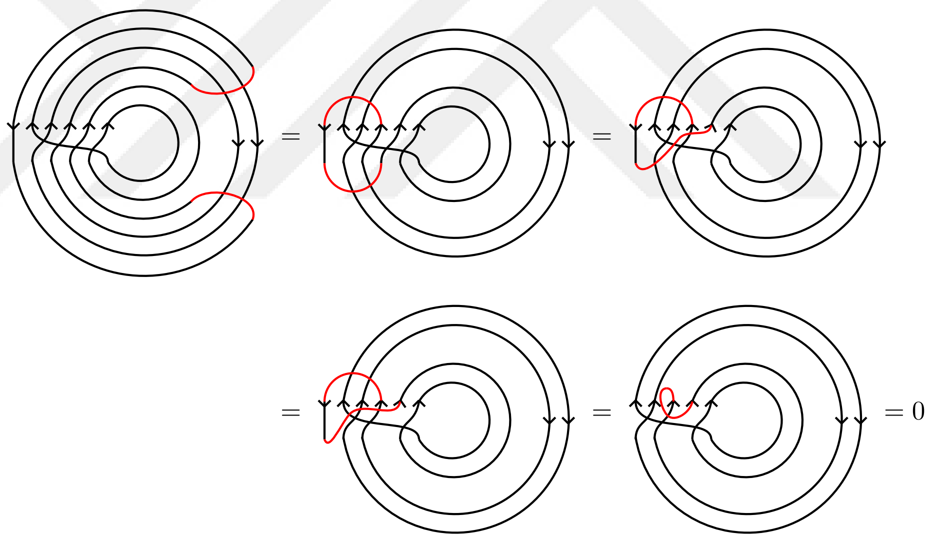
and the two hollow dots appearing in the last term cancel with each other if we slide them along the outermost strand. This observation will hold for the rest of the computation, so we will omit drawing the second resolution term and instead write the first resolution term with coefficient 2. We will show that all resolution terms coming from crossings on the: outermost strand, innermost strand, and intermediate strands are zero.

For the resolution term coming from the crossing of outermost strands, we have



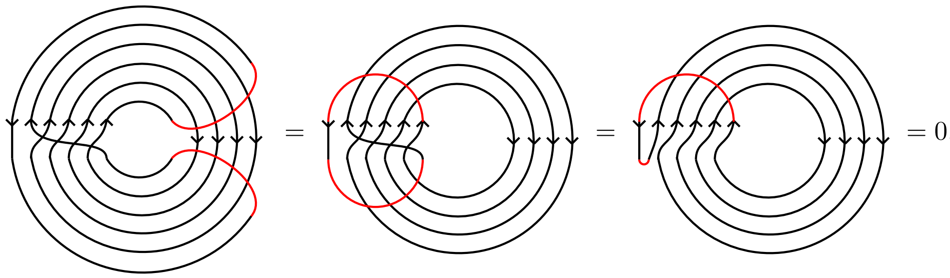
where the last equality follows from the relation (3.4).

For the resolution term coming from the crossing of intermediate strands, consider a generic intermediate strand. We have

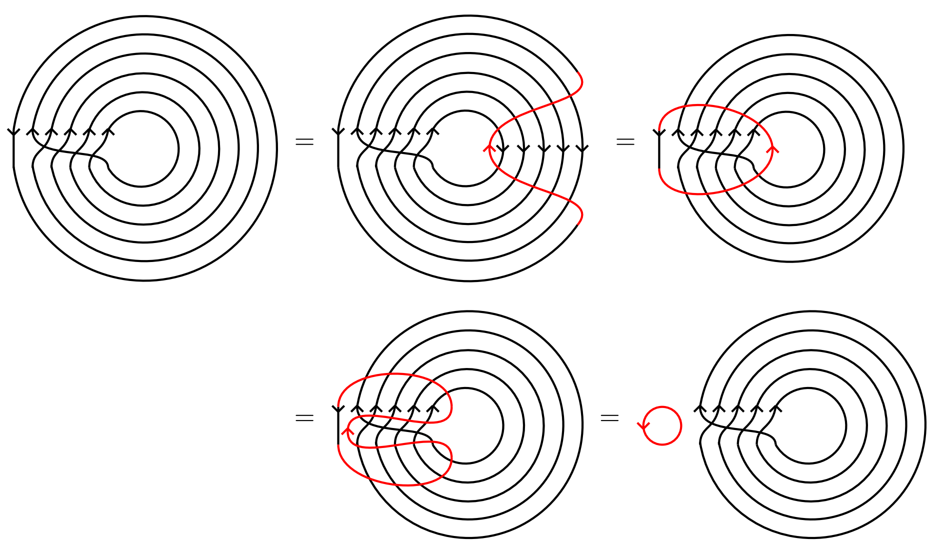


where the second and third equalities follow from a Reidemeister 3 move, and the fourth is a result of relation (3.2). Hence these resolution terms are zero as well. In general, for a resolution term coming from a crossing of intermediate strands, we can first pull the red string above the permutation using Reidemeister 3 moves, and then pull the red string into the permutation using relation (3.2) to get a left twist curl.

Finally, for the resolution term coming from the crossing of intermediate strands the situation is simpler:



Hence all the resolution terms are zero. This leaves us with



and a counter-clockwise oriented bubble is equal to 1 by the defining relation (3.4). These diagrammatic arguments clearly hold for arbitrary $k > 1$. Hence the action of $\omega_{(1,0)}$ on α_k for $k \neq 1$ is trivial.

This concludes the proof of the base case $\ell(\mu) = 1$. Now suppose $\omega_+ \cdot \alpha_\mu = \alpha_\mu$ for some $\mu \in \mathcal{OP}$ such that $\ell(\mu) = m - 1$, and let n be a positive integer. Then

$$\begin{aligned}0 &= [\sqrt{2}\omega_{-(2n+1),0}, \omega_+] \cdot \alpha_\mu = \sqrt{2}\omega_{-(2n+1),0} \cdot (\omega_+ \cdot \alpha_\mu) - \omega_+ \cdot (\sqrt{2}\omega_{-(2n+1),0} \cdot \alpha_\mu) \\ &= \sqrt{2}\omega_{-(2n+1),0} \cdot \alpha_\mu - \omega_+ \cdot (\sqrt{2}\omega_{-(2n+1),0} \cdot \alpha_\mu) \\ &= \alpha_{(\mu,2n+1)} - \omega_+ \cdot \alpha_{(\mu,2n+1)}\end{aligned}$$

and the result follows by induction.

Hence if μ doesn't contain any parts of size 1, then

$$\omega_+ \cdot \alpha_\mu = \alpha_\mu.$$

Now suppose γ is an odd partition without parts of size 1. We will prove that

$$\omega_+ \cdot \alpha_{(\gamma,1^k)} = \alpha_{(\gamma,1^k)} + 2k\alpha_{(\gamma,1^{k-1})}$$

by induction on k. The base case $k = 0$ was proved above. Suppose the formula holds for $\alpha_{(\gamma,1^k)}$.

$$\begin{aligned}\omega_+ \cdot \alpha_{(\gamma,1^{k+1})} &= \omega_+ \cdot \left(\alpha_{(\gamma,1^k)}\alpha_1 - 2|(\gamma,1^k)|\alpha_{(\gamma,1^k)}\right) \quad \text{by Lemma 5.5.1} \\ &= \alpha_1\omega_+ \cdot \alpha_{(\gamma,1^k)} + 2\omega_+ \cdot \alpha_{(\gamma,1^k)} - 2|(\gamma,1^k)|\omega_+ \cdot \alpha_{(\gamma,1^k)} \\ &= (\alpha_1 + 2 - 2|(\gamma,1^k)|)\omega_+ \cdot \alpha_{(\gamma,1^k)} \\ &= (\alpha_1 + 2 - 2|(\gamma,1^k)|)(\alpha_{(\gamma,1^k)} + 2k\alpha_{(\gamma,1^{k-1})}) \quad \text{by the inductive hypothesis} \\ &= (\alpha_1 + 2 - 2|(\gamma,1^k)|)\alpha_{(\gamma,1^k)} + 2k(\alpha_1 + 2 - 2|(\gamma,1^k)|)\alpha_{(\gamma,1^{k-1})} \\ &= \alpha_{(\gamma,1^{k+1})} + 4\alpha_{(\gamma,1^k)} + 2k\alpha_{(\gamma,1^k)} \quad \text{by Lemma 5.5.1} \\ &= \alpha_{(\gamma,1^{k+1})} + 2(k+1)\alpha_{(\gamma,1^k)}\end{aligned}$$

and the result follows after the identification $\alpha_\mu \to 2^{\ell(\mu)}\mathfrak{p}_\mu$.

For the action of $\omega_{0,3}$, note that this element acts on the center as multiplication by itself. Therefore

$$\omega_{0,3} \cdot \alpha_\mu = \alpha_3\alpha_\mu + \alpha_{(1,1)}\alpha_\mu$$

$$-2\omega_{0,3} \cdot 2^{\ell(\mu)}\mathfrak{p}_\mu = 2^{\ell(\mu)+1}\mathfrak{p}_3\mathfrak{p}_\mu + 2^{\ell(\mu)+2}\mathfrak{p}_{(1,1)}\mathfrak{p}_\mu$$

□

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