Advances in the Combinatorics of Shifted Tableaux

by

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This work is dedicated to

my parents Emel + Ersan Kantarcı

and my husband Can Ozan Oğuz,

for never supporting my musical endeavours:

thanks to them I got a PhD in Mathematics.

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Abstract

In this work we explore shifted combinatorics, making new constructions and proving results about existing ones. Chapters 1 and 2 are expository in nature and discuss partitions, Young tableaux and symmetric functions in the contexts of regular and shifted combinatorics. In Chapter 3, we consider the type B quasisymmetric Schur functions defined by Jing and Li in 2015. We prove their conjecture that these functions have a positive, integral and unitriangular expansion into peak functions, and refine their combinatorial model to give explicit expansions in monomial, fundamental and peak bases. We also show that these functions are not quasisymmetric Schur, Young quasisymmetric Schur or dual immaculate positive, and do not have a positive multiplication rule. In Chapter 4, we introduce a shifted analogue of the ribbon tableaux defined by James and Kerber [16]. For any positive integer k, we give a bijection between the k-ribbon fillings of a shifted shape and regular fillings of a $\lfloor k/2 \rfloor$-tuple of shapes called its k-quotient. We define the corresponding generating functions, prove that they are symmetric, Schur positive and Schur Q-positive, and introduce a Schur Q-positive q-refinement. In Chapter 5, we introduce crystal operators on semistandard shifted tableaux, giving a new proof that Schur P-functions are Schur positive. We define a queer crystal operator that constructs a connected queer crystal graph on semistandard shifted tableaux of a given shape, providing a new proof that products of Schur P-functions are Schur P-positive.

Chapter 1

Preliminaries

1 Compositions

A *composition* α of n is a sequence of positive numbers $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ that add up to n, called its *parts*. Here, k is called the *length* of α, denoted $\ell(\alpha)$, and n is called the *size* of α, denoted $|\alpha|$. If a composition satisfies $\alpha_i \geq \alpha_j$ whenever $i \leq j$, we call it a *partition*. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is called *strict* if all its parts are distinct. For every composition α, ordering the parts gives a unique partition, which we denote by $\mathrm{sort}(\alpha)$. For example, the composition $\alpha = (4, 1, 2, 6, 2)$ has $\mathrm{sort}(\alpha) = (6, 4, 2, 2, 1)$.

For $n \leq m$, a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is said to be a *refinement* of a composition $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ (denoted by $\alpha \preceq \beta$) if there exist numbers $i_1, i_2, \ldots, i_{n-1}$ such that we have $\beta_j = \sum_{k=i_{j-1}+1}^{i_j} \alpha_k$ for all $j \in [n]$, with the convention that $i_0 = 0$ and $i_n = m$. For example, $\alpha = (4, 1, 2, 6, 2)$ is a refinement of $\beta = (1, 3, 1, 1, 1, 3, 1, 2, 2)$ as we have $\alpha = (1 + 3, 1, 1 + 1, 3 + 1 + 2, 2)$.

We can extend the refinement order on compositions of a number n to a total order called the *lexicographic order* by setting $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) > \beta = (\beta_1, \beta_2, \ldots, \beta_m)$ if there exists some k satisfying $\alpha_i = \beta_i$ for all $i < k$ and $\alpha_k > \beta_k$.

The compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ of n with no parts except possibly the last one equal to 1 are called the *peak compositions*. We will use the notation $\mathcal{PC}(n)$ for the set of peak compositions of n and set $\mathcal{PC} = \bigcup_n \mathcal{PC}(n)$. Note that strict partitions of n form a subset of $\mathcal{PC}(n)$.

For a given composition $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of n we can obtain a subset of $[n-1]$ as follows: $Set(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots \alpha_{k-1}\}$. For example, $Set(4, 1, 2, 6, 2) = \{4, 5, 7, 13\} \subset [15]$. The map Set is

Compositions	$(4),(3,1),(2,2),(2,1,1),(1,3),(1,2,1),(1,1,2),(1,1,1,1)$
Peak Compositions	$(4),(3,1),(2,2)$
Partitions	$(4),(3,1),(2,2),(2,1,1),(1,1,1,1)$
Strict Partitions	$(4),(3,1)$

Table 1.1: Compositions and partitions of 4

invertible by taking the differences of consecutive entries and gives a bijection between compositions of n and subsets of $n-1$. For a a subset S of $[n-1]$, $Set^{-1}(S)$ is a peak composition if and only if S does not contain any consecutive entries or 1.

2 Descents and Peaks

We let S_n denote the symmetric group on n, formed by permutations of $[n] = 1, 2, \ldots, n$. We will write our permutations in list notation $\sigma = [a_1 a_2 \cdots a_n]$ where $a_i = \sigma(i)$. The descent set of a permutation σ is defined as follows:

$$\text{Des}(\sigma) \quad = \quad \{i \mid i \text{ is to the left of } i+1 \text{ in the reading word of } T\} \subset [n-1]$$

In general, for any set $D \subset [n-1]$, the peak and spike sets of D are given by:

$$\begin{aligned} \text{Peak}(D) &= \{i \mid i \in \text{D and } i-1 \notin \text{D}\} \\ \text{Spike}(D) &= \{i \mid i \in \text{D and } i-1 \notin \text{D or } i \notin \text{D and } i-1 \in \text{D}\}. \end{aligned}$$

Throughout this work, we will mainly be interested in the case when D is the descent set of a permutation. For example, $\sigma = 748261359$ has the descent set $\{1, 3, 5, 6\} \subset [8]$. We have $\text{Peak}(\text{Des}(\sigma)) = \{3, 5\}$, and $\text{Spike}(\text{Des}(\sigma)) = \{2, 3, 4, 5, 7\}$.

Note that as descent, spike and peak sets are subsets of $[n-1]$ the map Set^{-1} maps them to compositions. The set $\text{desc}(\sigma) = Set^{-1}(\text{Des}(\sigma))$ is a composition of n, which is called its *descent composition*. For example, the descent composition of $\sigma = 748261359$ is $(1, 2, 2, 1, 3)$. Similarly, we set: $\text{peakc}(\sigma) = Set^{-1}(\text{Peak}(\text{Des}(\sigma))$, $\text{spikec}(\sigma) = Set^{-1}(\text{Spike}(\text{Des}(\sigma))$. Note that peak sets of permutations correspond exactly to peak compositions of n.

3 Signed Permutations

A *signed permutation* is a bijection σ' from the marked alphabet $\{1', 1, 2, 2', \ldots, n', n\}$ onto itself satisfying $\sigma'(i) = j \Leftrightarrow \sigma'(i') = j'$ and $\sigma'(i) = j' \Leftrightarrow \sigma'(i') = j$. Note that a signed permutation is determined by the image of $\{1, 2, \ldots, n\}$, so it makes sense to use the list notation to denote signed permutations as well: $\sigma' = [a_1 a_2 \cdots a_n]$ where $a_i = \sigma(i)$.

For a given signed permutation, we obtain a permutation by first reading the primed entries from right to left, and then reading the unprimed entries left to right. We will calculate the descent set of a signed permutation by calculating the descent set of the permutation obtained this way. For example, for $\sigma' = 74'6'8123'5$, we have $\mathrm{Des}(\sigma') = \mathrm{Des}(36478125) = \{2, 5\}$.

We will end this chapter by introducing two lemmas about descents of signed permutations that will be useful later on. The first is about how the descent set of a marked permutation relates to the descent set of the permutation obtained by deleting the markings.

Lemma 1.3.1. *Let σ' be a signed permutation, and σ be the permutation obtained by deleting the markings. If $i \in Des(\sigma)$, then $i \in Des(\sigma')$ if and only if i is unmarked in σ'. If $i \notin Des(\sigma)$, then $i \in Des(\sigma')$ if and only if $i + 1$ is marked in σ.*

Proof. This follows directly from the definition of descents on signed permutations. □

For the second lemma, note that for a signed permutation σ', if i comes after $i + 1$, then i is a descent if and only if it is unmarked. If i comes before $i + 1$, then i is a descent if and only if $i + 1$ is marked. We can generalize this to get the following:

Lemma 1.3.2. *Assume, i, $i+1$, $\ldots$, $i+k-1$ are not descents. If i comes before $i+k$ in the reading word, then $i + k$ is not marked and if i comes after $i + k$, then i is marked.*

Proof. If $k = 1$, the result follows directly from the definition of descent in marked words. Assume the statements hold for all $k' < k$, and let $k > 1$. If $i + k$ is marked, by our induction hypothesis $i + 1$ comes after $i + k$, and it is marked. Then as i is not a descent, it cannot come before $i + 1$, so it comes after $i + 1$ and $i + k$. For the second part, assume i is unmarked. Then by our induction hypothesis i must come before $i + k - 1$, and $i + k - 1$ must be unmarked, which in turn implies $i + k$ must come after $i + k - 1$ and i. □

Chapter 2

Symmetric Functions and Tableaux

1 The Ring of Symmetric Functions

A symmetric function is a formal power series of bounded degree in variables $x_1, x_2, \ldots$ invariant under exchanging any two variables. We will denote the ring of symmetric functions by Λ. Note that Λ has the structure of a graded ring:

$$\Lambda = \bigoplus_n \Lambda^n,$$

where Λ^n consists of homogeneous symmetric functions of degree n along with the function 0.

We let X denote the infinite set of variables $x_1, x_2, \ldots$, and use the notation X^w to denote the monomial $x_1^{w_1} x_2^{w_2} \ldots x_n^{w_n}$, where w is a possibly infinite list of nonnegative integers with $w_k = 0$ for all $k > n$. Given a symmetric function $f(X)$, sending all variables x_i, $i > n$ to zero gives us a symmetric polynomial on n variables which we will denote by $f(x_1, x_2, \ldots, x_n)$. Conversely for a symmetric function of $f(X)$ degree $\leq n$, the symmetric polynomial $f(x_1, x_2, \ldots, x_n)$ uniquely determines $f(X)$. When working on specific examples, we will often limit ourselves to a finite number of variables which will allow us to work with polynomials instead of formal power series while preserving the structure.

Let μ a partition of n of size k. The symmetric function defined by:

$$m_\mu(X) = \sum_{i_1, i_2, \ldots, i_k \text{distinct}} x_{i_1}^{\mu_1} x_{i_2}^{\mu_2} x \cdots x_{i_k}^{\mu_k} \tag{1.1}$$

is called a *monomial symmetric function*. Monomial symmetric functions form a $\mathbb{Z}$ basis of Λ. Some other commonly used bases for symmetric functions are elementary symmetric functions, complete homogeneous symmetric functions, power sum symmetric functions and Schur functions. We will briefly describe the former three in this section before moving on to Schur functions, the interested reader can refer Table 2.1 for examples, and to Macdonald's wonderful book on symmetric functions [31] for a detailed exposition.

Elementary Symmetric Functions

For any $n \in \mathbb{N}$, we define the *elementary symmetric function* e_n by setting:

$$e_n(X) = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}. \tag{1.2}$$

We extend this construction multiplicatively to partitions by:

$$e_\mu(X) = e_{\mu_1}(X) e_{\mu_2}(X) \cdots e_{\mu_{\ell(\mu)}}(X). \tag{1.3}$$

Complete homogeneous functions have the generating function

$$E(X,t) = \sum_{n \in \mathbb{N}} e_n(X) t^n = \prod_{i \in \mathbb{N}} (1 + x_i t). \tag{1.4}$$

The functions $\{e_\mu\}_\mu$ form a $\mathbb{Z}$ basis for Λ. Furthermore, the functions $e_1, e_2, \ldots$ are algebraically independent and $\Lambda = \mathbb{Z}[e_1, e_2, \ldots]$.

Complete Homogeneous Symmetric Functions

As in the case of elementary symmetric functions, we define *complete homogeneous symmetric functions* multiplicatively by:

$$h_n(X) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}, \qquad h_\mu(X) = h_{\mu_1}(X) h_{\mu_2}(X) \cdots h_{\mu_{\ell(\mu)}}(X). \tag{1.5}$$

Complete homogeneous functions have the generating function

$$H(X,t) = \sum_{n\in\mathbb{N}} h_n(X)t^n = \prod_{i\in\mathbb{N}}(1 + x_it + x_i^2t^2\cdots) = \prod_{i\in\mathbb{N}}(1 - x_it)^{-1}. \tag{1.6}$$

Note that we have $H(X,t)E(X,-t) = \prod_{i\in\mathbb{N}}(1-x_it)(1-x_it)^{-1} = 1$, implying for all $n \geq 1$:

$$\sum_{k\leq n}(-1)^k e_k h_{n-k} = 0. \tag{1.7}$$

The functions $\{h_\mu\}_\mu$ form a $\mathbb{Z}$ basis for Λ, and the functions $h_1, h_2, \ldots$ form a multiplicative basis $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$.

Power Sum Symmetric Functions

We define the *power sum symmetric functions* by extending $\{m_n\}_{n\in\mathbb{N}}$ multiplicatively:

$$p_n(X) = m_n(X) = \sum_{i\in\mathbb{N}} x_i^n, \qquad P_\mu(X) = p_{\mu_1}(X)p_{\mu_2}(X)\cdots p_{\mu_{\ell(\mu)}}(X). \tag{1.8}$$

Power sum symmetric functions have the generating function:

$$P(X,t) = \sum_{n\in\mathbb{N}} p_n(X)t^{n-1} = \sum_{i\in\mathbb{N}} \frac{x_i}{1-x_it} = \sum_{i\in\mathbb{N}} \frac{d}{dt} log\frac{1}{1-x_it}. \tag{1.9}$$

Here the choice of t^{n-1} instead of t^n is so that $P(X,n)$ satisfies the following identities:

$$P(X,t) = H'(X,t)/H(X,t), \tag{1.10}$$

$$P(X,-t) = E'(X,t)/E(X,t), \tag{1.11}$$

where $H'(X,t)$ and $E'(X,t)$ denote the partial derivatives with respect to t.

The functions $\{p_\mu\}_\mu$ form a $\mathbb{Z}$ basis for Λ, and the functions $p_1, p_2, \ldots$ form a multiplicative basis for symmetric functions over rationals $\Lambda_\mathbb{Q} = \Lambda \otimes_\mathbb{Z} \mathbb{Q} = \mathbb{Q}[p_1, p_2, \ldots]$.

2 Young Diagrams and Schur Functions

With every partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, we associate a *Young diagram*, an array with μ_i boxes on row i.

	$\mu=(2)$	$\mu=(2,1)$
m_μ	$x_1^2+x_2^2+x_3^2$	$x_1^2x_2+x_1^2x_3+x_2^2x_1+x_2^2x_3+x_3^2x_1+x_3^2x_2$
e_μ	$x_1x_2+x_1x_3+x_2x_3$	$x_1^2x_2+x_1^2x_3+x_2^2x_1+x_2^2x_3+x_3^2x_1+x_3^2x_2+3x_1x_2x_3$
h_μ	$x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_1x_2$	$x_1^3+x_2^3+x_3^3+2x_1^2x_2+2x_1^2x_3+2x_2^2x_1+2x_2^2x_3+2x_3^2x_1+2x_3^2x_2+3x_1x_2x_3$
p_μ	$x_1^2+x_2^2+x_3^2$	$x_1^3+x_2^3+x_3^3+x_1^2x_2+x_1^2x_3+x_2^2x_1+x_2^2x_3+x_3^2x_1+x_3^2x_2$

Table 2.1: Examples of symmetric functions on 3 variables

A *semi-standard Young tableau* of shape μ is a filling of its boxes with positive integers such that each row is weakly increasing from left to right and each column is strictly increasing from bottom to top. A semi-standard tableau that contains each of the numbers from 1 to n exactly once is called *standard*. We denote the set of semi-standard tableaux of shape μ by $SSYT(\mu)$, and the set of standard ones by $SYT(\mu)$. For any tableaux T, the *weight* of T is the list $\{\mathrm{wt}(T)_i\}i$ where $\mathrm{wt}(T)_i$ is given by the number of times i occurs on T.

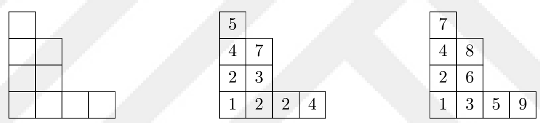


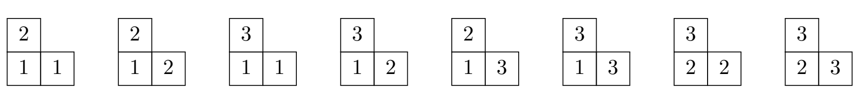
Figure 2.1: The diagram $\mu=(4,2,2,1)$ with corresponding semi-standard and standard fillings.

The semi-standard filling in Figure 2.1 has weight $(1,3,1,2,1,0,1)$, and it corresponds to the monomial $X^{\mathrm{wt}(T)}=x_1x_2^3x_3x_4^2x_5x_7$. Summing up the corresponding monomials for all semi-standard Young tableaux of a given shape gives us a symmetric function, called the *Schur function*:

$$s_\mu(X)=\sum_{T\in SSYT(\mu)} X^{\mathrm{wt}(T)}. \tag{2.1}$$

It is a nontrivial exercise to show that Schur functions are indeed symmetric.

Example 2.2.1. *There are eight semi-standard fillings of $\mu=(2,1)$ by numbers ≤ 3, given below.*



Summing up the corresponding monomials gives us:

$$s_{(2,1)}(x_1,x_2,x_3)=x_1^2x_2+x_1x_2^2+x_1^2x_3+2x_1x_2x_3+x_1x+3^2+x_2^2x_3+x_2x_3^2.$$

n=1	$s_1 = m_1$
n=2	$s_2 = m_2 + m_{11}$ $s_{11} = m_{11}$
n=3	$s_3 = m_3 + m_{21} + m_{111}$ $s_{21} = m_{21} + 2m_{111}$ $s_{111} = m_{111}$
n=4	$s_4 = m_4 + m_{31} + m_{22} + m_{211} + m_{1111}$ $s_{31} = m_{31} + m_{22} + 2m_{211} + 3m_{1111}$ $s_{22} = m_{22} + m_{211} + 2m_{1111}$ $s_{211} = m_{211} + 3m_{1111}$ $s_{1111} = m_{1111}$
n=5	$s_5 = m_5 + m_{41} + m_{32} + m_{311} + m_{221} + m_{2111} + m_{11111}$ $s_{41} = m_{41} + m_{32} + 2m_{311} + 2m_{221} + 3m_{2111} + 4m_{11111}$ $s_{32} = m_{32} + m_{311} + 2m_{221} + 3m_{2111} + 5m_{11111}$ $s_{311} = m_{311} + m_{221} + 3m_{2111} + 6m_{11111}$ $s_{221} = m_{221} + 2m_{2111} + 5m_{11111}$ $s_{2111} = m_{2111} + 4m_{11111}$ $s_{11111} = m_{11111}$
n=6	$s_6 = m_6 + m_{51} + m_{42} + m_{411} + m_{33} + m_{321} + m_{3111} + m_{222} + m_{2211} + m_{21111} + m_{111111}$ $s_{51} = m_{51} + m_{42} + 2m_{411} + m_{33} + 2m_{321} + 3m_{3111} + 2m_{222} + 3m_{2211} + 4m_{21111} + 5m_{111111}$ $s_{42} = m_{42} + m_{411} + m_{33} + 2m_{321} + 3m_{3111} + 3m_{222} + 4m_{2211} + 6m_{21111} + 9m_{111111}$ $s_{411} = m_{411} + m_{321} + 3m_{3111} + m_{222} + 3m_{2211} + 6m_{21111} + 10m_{111111}$ $s_{33} = m_{33} + m_{321} + m_{3111} + m_{222} + 2m_{2211} + 3m_{21111} + 5m_{111111}$ $s_{321} = m_{321} + 2m_{3111} + 2m_{222} + 4m_{2211} + 8m_{21111} + 16m_{111111}$ $s_{3111} = m_{3111} + m_{2211} + 4m_{21111} + 10m_{111111}$ $s_{222} = m_{222} + m_{2211} + 2m_{21111} + 5m_{111111}$ $s_{2211} = m_{2211} + 3m_{21111} + 9m_{111111}$ $s_{21111} = m_{21111} + 5m_{111111}$ $s_{111111} = m_{111111}$

Table 2.2: Schur functions of size $n \leq 6$ in the monomial basis

As $(2,1)$ *has only* 3 *boxes and can contain at most* 3 *distinct entries, and as the Schur function is symmetric, this list is representative of all fillings and helps us calculate the Schur function in terms of monomial symmetric functions:*

$$s_{(2,1)}(X) = m_{(2,1)}(X) + 2m_{(1,1,1)}(X).$$

The list of all Schur functions with size $n \leq 6$ given in terms of monomial symmetric functions can be found in Table 2.2.

Given two partitions μ and ν with $\nu \subset \mu$, the *skew-diagram* μ/ν is given by the cells in μ that are not in ν, as seen in Figure 2.2, left.

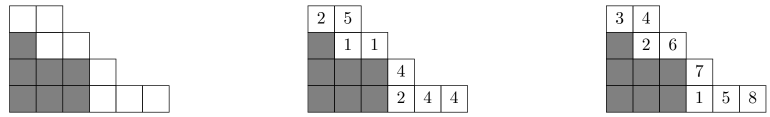


Figure 2.2: The skew diagram $(6,4,3,2)/(3,3,1)$ with a corresponding semi-standard and standard filling.

A semi-standard skew-tableaux of shape μ/ν is a filling of the skew diagram with positive integers such that each row is weakly increasing from left to right and each column is strictly increasing from bottom to top. The set of all semi-standard skew-tableaux is denoted $SST(\mu/\nu)$. A semi-standard skew-tableaux where each number $1, 2, \ldots n$ occurs exactly once is called standard, with $ST(\mu/\nu)$ denoting the set of standard skew-tableaux of shape μ/ν. We can extend the definition of Schur functions to the skew case, by defining the weight of a skew-diagram to be given by the number of occurrences of each i just as in the regular case. For example, the semi-standard skew tableaux in Figure 2.2, center has weight $(2,2,0,3,1)$, and corresponds to the monomial $x_1^2 x_2^2 x_4^3 x_5^2$. Summing up the monomials for all possible standard fillings gives us the skew-Schur function:

$$s_{\mu/\nu}(X) = \sum_{S \in SSShT(\lambda/\gamma)} X^{\mathrm{wt}(S)}. \tag{2.2}$$

Multiplication of Schur functions and skew Schur functions are *Schur positive*, meaning they expand in the Schur basis with non-negative integer coefficients:

$$\begin{aligned} s_{\mu/\nu}(X) &= \sum_{\eta} c^{\mu}_{\nu\eta} s_{\eta}(X) \\ s_{\nu}(X) s_{\eta}(X) &= \sum_{\mu} c^{\mu}_{\nu\eta} s_{\mu}(X) \end{aligned}$$

The coefficients $c^{\mu}_{\nu\eta}$ that come up are the same in both cases and are called the Littlewood-Richardson coefficients. The interested reader is referred to Fulton's book on Young tableaux [7] for more details regarding the Littlewood-Richardson coefficients, as well as the combinatorics of unshifted tableaux. We will now move on to shifted tableaux and related functions, which are the main focus of this work.

3 Shifted Diagrams and Schur's P- and Q-functions

With every strict partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we associate a *shifted diagram*, an array with λ_i boxes on row i, where row i is shifted $k - i$ steps to the right, forming a staircase shape as seen in Figure 2.3. For any cell C on a shifted diagram, we define its diagonal value to be $\text{diag}(C) = \text{col}(C) - \text{row}(C) + 1$, where $\text{col}(C)$ is the column index of C counted from left to right and $\text{row}(C)$ is the row index of C counted from bottom to top. Note that the smallest diagonal value is 1 and is attained only at the leftmost diagonal which we call the *main diagonal* of λ.

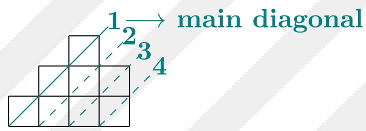


Figure 2.3: The shifted diagram for $\lambda = (4, 3, 1)$ with diagonals labelled with diagonal values.

A *semi-standard shifted tableau* of shape λ is a filling of its boxes with elements from the marked alphabet $1' < 1 < 2' < 2 < 3' < 3 < \cdots$ such that each row is increasing from left to right with no repeated marked numbers, and each column is increasing from bottom to top with no repeated unmarked numbers. A semi-standard shifted tableau of shape λ that contains each of the numbers $1, 2, \ldots, |\lambda|$ exactly once (possibly marked), it is called *marked standard*, and if they are all unmarked it is called *standard*. We will denote the set of semi-standard shifted tableaux of shape λ by $SSShT(\lambda)$, the set of marked standard ones by $SShT \pm (\lambda)$ and the set of the standard ones by $SShT(\lambda)$ (See Figure 2.4).

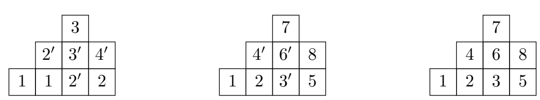


Figure 2.4: Examples of semi-standard (left), marked standard (middle) and standard (right) shifted tableaux for $\lambda = (4, 3, 1)$. The tableaux given here are related by a standardization algorithm, which will be introduced in Section 4.

The weight of a shifted tableaux T is a list $\{\text{wt}(T)_i\}i$ of non-negative integers where $\text{wt}(T)_i$ is given by the total number of occurrences of i and i' on T. The semi-standard shifted tableau given in Figure 2.4 has weight $(2, 3, 2, 1)$ and the marked standard and standard ones both have weight $(1, 1, 1, 1, 1, 1, 1)$. Summing up the monomial corresponding to the weights of all semi-standard shifted tableaux of a given shape gives us Schur's Q-function:

$$Q_\lambda(X) \quad = \sum_{T \in SSShT(\lambda)} X^{\mathrm{wt}(T)} \tag{3.1}$$

We denote the subset of semi-standard shifted tableaux of shape λ with no marked entries on the main diagonal by $SSShT^*(\lambda)$. Limiting ourselves to this subset (or equivalently dividing Schur's Q-function by 2 to the number of rows of λ) gives us Schur's P-function:

$$P_\lambda(X) \quad = \quad 2^{-\ell(\lambda)} \sum_{T \in SSShT(\lambda)} X^{\mathrm{wt}(T)} \quad = \sum_{S \in SSShT^*(\lambda)} X^{X^{\mathrm{wt}(S)}} \tag{3.2}$$

Schur's Q- and P-functions are the analogues of Schur functions associated to lie groups of types C and B respectively. As they are related only by a scalar, studying one is often enough to get information about both. Schur's Q- and P-functions are Schur positive: they expand in the basis of Schur functions with non-negative coefficients. The list of all Schur's P-functions of size $n \leq 6$ given in terms of Schur functions can be found in Table 2.3.

n=1	$P_1 = s_1$
n=2	$P_2 = s_2 + s_{11}$
n=3	$P_3 = s_3 + s_{21} + s_{111}$ $P_{21} = s_{21}$
n=4	$P_4 = s_4 + s_{31} + s_{211} + s_{1111}$ $P_{31} = s_{31} + s_{22} + s_{211}$
n=5	$P_5 = s_5 + s_{41} + s_{311} + s_{2111} + s_{11111}$ $P_{41} = s_{41} + s_{32} + s_{311} + s_{221} + s_{2111}$ $P_{32} = s_{32} + s_{311} + s_{221}$
n=6	$P_6 = s_6 + s_{51} + s_{411} + s_{3111} + s_{21111} + s_{111111}$ $P_{51} = s_{51} + s_{42} + s_{411} + s_{321} + s_{3111} + s_{2211} + s_{21111}$ $P_{42} = s_{42} + s_{411} + s_{33} + 2s_{321} + s_{3111} + s_{222} + s_{2211}$ $P_{321} = s_{321}$
n=7	$P_7 = s_7 + s_{61} + s_{511} + s_{4111} + s_{31111} + s_{211111} + s_{1111111}$ $P_{61} = s_{61} + s_{52} + s_{511} + s_{421} + s_{4111} + s_{3211} + s_{31111} + s_{221111} + s_{211111}$ $P_{52} = s_{52} + s_{511} + s_{43} + 2s_{421} + s_{4111} + s_{331} + s_{322} + 2s_{3211} + s_{31111} + s_{2221} + s_{22111}$ $P_{43} = s_{43} + s_{421} + s_{4111} + s_{331} + s_{322} + s_{3211} + s_{2221}$ $P_{421} = s_{421} + s_{331} + s_{322} + s_{3211}$

Table 2.3: Schur's P-functions of size $n \leq 7$ in the Schur basis.

Schur's P-functions (or alternatively Schur's Q-functions as they only differ by constants) generate a graded subalgebra of symmetric functions denoted Γ.

Given two strict partitions γ and λ with $\gamma \subset \lambda$, the *skew-shifted diagram* λ/γ consists of the cells in λ that are not in γ.

Figure 2.5: The skew-shifted diagram $(4,3,1)/(3,1)$ with a corresponding semi-standard, marked standard and standard filling.

We can apply the above definitions to the skew-shifted diagrams to get skew-shifted tableaux. More precisely, the set of semi-standard shifted tableaux of shape λ/γ, denoted $SSShT(\lambda/\gamma)$ is given by all the fillings of λ/γ from the marked alphabet with non-decreasing columns and rows such that we have no unmarked numbers repeated along columns and no marked numbers repeated along rows. We will denote the marked standard fillings of λ/γ (where we use each number from 1 to n once, possibly marked) by $SShT \pm (\lambda/\gamma)$ and its standard fillings (where we use each number from 1 to n once, unmarked) by $SShT(\lambda/\gamma)$. This gives rise to a skew analogue for Schur's Q-function:

$$Q_{\lambda/\gamma}(X) = \sum_{S \in SSShT(\lambda/\gamma)} X^{\mathrm{wt}(S)}. \tag{3.3}$$

Multiplications of two Schur's Q- and P- functions are Schur Q- and P-positive respectively, meaning they can be written in terms of Schur's Q- and P-functions with non-negative coefficients. It was shown by Stembridge that the skew-shifted Q- and P- functions also expand positively into Schur's Q- and P-functions respectively, and coefficients are related in a way similar to the unshifted case:

Theorem 2.3.1. (Stembridge [40]) *There exist coefficients* $f^{\lambda}_{\gamma,\phi} \in \mathbb{N}$ *satisfying:*

$$Q_{\lambda/\gamma}(X) = \sum_{\phi} f^{\lambda}_{\gamma,\phi} Q_{\phi}(X) \qquad P_{\gamma}(X)P_{\phi}(X) = \sum_{\lambda} f^{\lambda}_{\gamma,\phi} P_{\lambda}(X)$$

where $f^{\lambda}_{\gamma,\phi} = 0$ *unless* $|\gamma| + |\phi| = |\lambda|$.

Chapter 3

Quasisymmetric Functions and the Peak Algebra

1 The Ring of Quasisymmetric Functions

Quasisymmetric functions are formal power series of bounded degree in variables $x_1, x_2, \ldots$ where for a given composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ the coefficient of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_t}^{a_t}$ is the same for any $i_1 < i_2 < \cdots < i_t$. They form a graded algebra $\mathrm{QSYM} = \bigoplus_n \mathrm{QSYM}^{(n)}$ where $\mathrm{QSYM}^{(n)}$ stands for the homogeneous quasisymmetric functions of degree n. Note that symmetric functions form a graded subalgebra of QSYM. As in the case of the symmetric functions, sending all variables x_i, $i > n$ gives us a polynomial on n variables which we call a quasisymmetric polynomial.

The subspace $\mathrm{QSYM}^{(n)}$ is generated by the *monomial quasisymmetric functions* of degree n, defined for compositions $\beta = (\beta_1, \beta_2 ... \beta_m)$ of size n as follows:

$$M_\beta(X) = \sum_{i_1 < i_2 < \cdots < i_m} x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_m}^{\beta_m} \tag{1.1}$$

For example, we have:

$$\begin{aligned} M_{3,2}(x_1, x_2, x_3) &= x_1^3 x_2^2 + x_1^3 x_3^2 + x_2^3 x_3^2, \\ M_{2,3}(x_1, x_2, x_3) &= x_1^2 x_2^3 + x_1^2 x_3^3 + x_2^2 x_3^3. \end{aligned}$$

Note that $M_{3,2}(x_1, x_2, x_3) + M_{2,3}(x_1, x_2, x_3) = m_{3,2}(x_1, x_2, x_3)$. In general, we can get monomial symmetric functions from monomial quasisymmetric functions by summing up over the possible compositions that sort to a given partition.

$$m_\mu(X) = \sum_{\mathrm{sort}(\alpha)=\mu} M_\alpha(X). \tag{1.2}$$

1.1 Gessel's Fundamental Basis and the Schur Functions

Another basis for QSYM that we will make use of is the fundamental basis defined by Gessel [8], which will give us an easier way of calculating Schur polynomials.

For a given standard or shifted tableaux T, we define its *row reading word* denoted $\mathrm{rw}(T)$ to be a reading of all its entries row by row, left to right, top to bottom. If T is a standard Young tableau S, its reading word gives us a permutation of $[n]$, so we can talk about the descent set of T (denoted $\mathrm{Des}(T)$) as well as the corresponding composition (denoted $\mathrm{desc}(T)$). The standard Young tableaux of shape $(2, 2, 1)$ are illustrated in Table 3.2, along with their reading words, descent sets and descent compositions.

For a given composition α of n, the corresponding fundamental quasisymmetric functions is defined as follows:

$$F_\alpha(X) = \sum_{\beta \preceq \alpha} M_\beta(X). \tag{1.3}$$

Schur functions expand positively into fundamental quasisymmetric functions as follows:

Theorem 3.1.1 (Gessel [8])**.** $s_\mu(X) = \sum_{S \in \mathrm{SSYT}(\mu)} F_{\mathrm{desc}(S)}(X)$

The list of all Schur functions of size ≤ 6 in fundamental quasisymmetric basis is given in Table 3.1.

In Example 2.2.1, we calculated $s_{(2,1)}(x_1, x_2, x_3)$ using its eight semi-standard fillings. By contrast, if we calculate it using descent functions, we only need to consider the two standard fillings given below:

$$\begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \qquad \begin{array}{|c|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

The first filling has descent composition $(2, 1)$ while the second one has $(1, 2)$, so we get:

$$s_{(2,1)}(X) = F_{21}(X) + F_{12}(X).$$

n=1	$s_1 = F_1$
n=2	$s_2 = F_2$
	$s_{11} = F_{11}$
n=3	$s_3 = F_3$
	$s_{21} = F_{12} + F_{21}$
	$s_{111} = F_{111}$
n=4	$s_4 = F_4$
	$s_{31} = F_{13} + F_{22} + F_{31}$
	$s_{22} = F_{22} + F_{121}$
	$s_{211} = F_{112} + F_{121} + F_{211}$
	$s_{1111} = F_{1111}$
n=5	$s_5 = F_5$
	$s_{41} = F_{15} + F_{24} + F_{33} + F_{42} + F_{51}$
	$s_{32} = F_{23} + F_{32} + F_{122} + F_{131} + F_{221}$
	$s_{311} = F_{113} + F_{122} + F_{131} + F_{212} + F_{221} + F_{311}$
	$s_{221} = F_{122} + F_{212} + F_{221} + F_{1121} + F_{1211}$
	$s_{2111} = F_{1112} + F_{1121} + F_{1211} + F_{2111}$
	$s_{11111} = F_{11111}$
n=6	$s_6 = F_6$
	$s_{51} = F_{15} + F_{24} + F_{33} + F_{42} + F_{51}$
	$s_{42} = F_{24} + F_{33} + F_{42} + F_{123} + F_{132} + F_{141} + F_{222} + F_{231} + F_{321}$
	$s_{411} = F_{114} + F_{123} + F_{132} + F_{141} + F_{213} + F_{222} + F_{231} + F_{312} + F_{321} + F_{411}$
	$s_{33} = F_{33} + F_{132} + F_{222} + F_{231} + F_{1221}$
	$s_{321} = F_{123} + F_{132} + F_{213} + 2F_{222} + F_{231} + F_{312} + F_{321} + F_{1122} + F_{1131} + F_{1212}$ $+2F_{1221} + F_{1311} + F_{2121} + F_{2211}$
	$s_{3111} = F_{1113} + F_{1122} + F_{1131} + F_{1212} + F_{1221} + F_{1311} + F_{2112} + F_{2121} + F_{2211} + F_{3111}$
	$s_{222} = F_{222} + F_{1212} + F_{1221} + F_{2121} + F_{11211}$
	$s_{2211} = F_{1122} + F_{1212} + F_{1221} + F_{2112} + F_{2121} + F_{2211} + F_{11121} + F_{11211} + F_{12111}$
	$s_{21111} = F_{11112} + F_{11121} + F_{11211} + F_{12111} + F_{21111}$
	$s_{111111} = F_{111111}$

Table 3.1: Schur functions of size $n \leq 6$ in the fundamental quasisymmetric basis.

	$\begin{array}{cc} 5 & \\ 3 & 4 \\ 1 & 2 \end{array}$	$\begin{array}{cc} 4 & \\ 3 & 5 \\ 1 & 2 \end{array}$	$\begin{array}{cc} 5 & \\ 2 & 4 \\ 1 & 3 \end{array}$	$\begin{array}{cc} 4 & \\ 2 & 5 \\ 1 & 3 \end{array}$	$\begin{array}{cc} 3 & \\ 2 & 5 \\ 1 & 4 \end{array}$
Row reading word	53412	43512	52413	42513	32514
Descents	$\{2,4\}$	$\{2,3\}$	$\{1,3,4\}$	$\{1,3\}$	$\{1,2,4\}$
Descent composition	$(2,2,1)$	$(2,1,2)$	$(1,2,1,1)$	$(1,2,2)$	$(1,2,1,1)$

Table 3.2: Standard tableaux of shape $(2,2,1)$.

Similarly, we can calculate the Schur function for $(2,2,1)$ using Table 3.2:

$$s_{(2,2,1)}(X) = F_{221}(X) + F_{212}(X) + F_{1211}(X) + F_{122}(X) + F_{1211}(X).$$

1.2 Quasisymmetric Schur Functions

With every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, we associate a *composition diagram*, an array with α_i boxes on row $k-i$. Note that if α is a partition, this is just its Young diagram drawn upside down. For a filling T of a composition diagram with positive integers, the *weight* of T is a list of non-negative integers $\{\mathrm{wt}(T)_i\}_i$ where $\mathrm{wt}(T)_i$ is the number of occurrences of i on T.

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & & \\ \hline 3 & 3 & \\ \hline 4 & 4 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline 2 & 2 & 1 \\ \hline 3 & & \\ \hline 5 & 4 & \\ \hline 9 & 7 & 5 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline 5 & 3 & 2 \\ \hline 4 & & \\ \hline 8 & 7 & \\ \hline 9 & 9 & 9 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline 3 & 3 & 3 \\ \hline 5 & & \\ \hline 7 & 6 & \\ \hline 8 & 8 & 7 \\ \hline \end{array}$$

Figure 3.1: Examples of reverse composition tableaux of shape (3,1,2,3).

An *reverse composition tableau* of shape α is a filling of the composition diagram of α with positive integers such that:

1. first column strictly increases from top to bottom,

2. rows weakly decrease from left to right,

n=1	$\check{\mathscr{S}}_1 = F_1$	
n=2	$\check{\mathscr{S}}_2 = F_2$	$\check{\mathscr{S}}_{11} = F_{11}$
n=3	$\check{\mathscr{S}}_3 = F_3$	$\check{\mathscr{S}}_{21} = F_{21}$
	$\check{\mathscr{S}}_{12} = F_{12}$	$\check{\mathscr{S}}_{111} = F_{111}$
n=4	$\check{\mathscr{S}}_4 = F_4$	$\check{\mathscr{S}}_{31} = F_{31}$
	$\check{\mathscr{S}}_{22} = F_{22} + F_{121}$	$\check{\mathscr{S}}_{211} = F_{211}$
	$\check{\mathscr{S}}_{13} = F_{13} + F_{22}$	$\check{\mathscr{S}}_{121} = F_{121}$
	$\check{\mathscr{S}}_{112} = F_{112}$	$\check{\mathscr{S}}_{1111} = F_{1111}$
n=5	$\check{\mathscr{S}}_5 = F_5$	$\check{\mathscr{S}}_{41} = F_{41}$
	$\check{\mathscr{S}}_{32} = F_{32} + F_{221} + F_{131}$	$\check{\mathscr{S}}_{311} = F_{311}$
	$\check{\mathscr{S}}_{23} = F_{23} + F_{122}$	$\check{\mathscr{S}}_{221} = F_{221} + F_{1211}$
	$\check{\mathscr{S}}_{212} = F_{212}$	$\check{\mathscr{S}}_{2111} = F_{2111}$
	$\check{\mathscr{S}}_{14} = F_{14} + F_{32} + F_{23}$	$\check{\mathscr{S}}_{131} = F_{131} + F_{221}$
	$\check{\mathscr{S}}_{122} = F_{122} + F_{1121}$	$\check{\mathscr{S}}_{1211} = F_{1211}$
	$\check{\mathscr{S}}_{113} = F_{113} + F_{122} + F_{212}$	$\check{\mathscr{S}}_{1121} = F_{1121}$
	$\check{\mathscr{S}}_{1112} = F_{1112}$	$\check{\mathscr{S}}_{11111} = F_{11111}$
n=6	$\check{\mathscr{S}}_6 = F_6$	$\check{\mathscr{S}}_{51} = F_{51}$
	$\check{\mathscr{S}}_{42} = F_{42} + F_{321} + F_{231} + F_{141}$	$\check{\mathscr{S}}_{411} = F_{411}$
	$\check{\mathscr{S}}_{33} = F_{33} + F_{132} + F_{222} + F_{231} + F_{1221}$	$\check{\mathscr{S}}_{321} = F_{321} + F_{2211} + F_{1311}$
	$\check{\mathscr{S}}_{312} = F_{312}$	$\check{\mathscr{S}}_{3111} = F_{3111}$
	$\check{\mathscr{S}}_{24} = F_{24} + F_{33} + F_{123} + F_{132} + F_{222}$	$\check{\mathscr{S}}_{231} = F_{231} + F_{1221}$
	$\check{\mathscr{S}}_{222} = F_{222} + F_{1212} + F_{1221} + F_{2121} + F_{11211}$	$\check{\mathscr{S}}_{2211} = F_{2211} + F_{12111}$
	$\check{\mathscr{S}}_{213} = F_{213} + F_{222} + F_{1212}$	$\check{\mathscr{S}}_{2121} = F_{2121}$
	$\check{\mathscr{S}}_{2112} = F_{2112}$	$\check{\mathscr{S}}_{21111} = F_{21111}$
	$\check{\mathscr{S}}_{15} = F_{15} + F_{24} + F_{33} + F_{42}$	$\check{\mathscr{S}}_{141} = F_{141} + F_{231} + F_{321}$
	$\check{\mathscr{S}}_{132} = F_{132} + F_{222} + F_{1131} + F_{1221} + F_{2121}$	$\check{\mathscr{S}}_{1311} = F_{1311} + F_{2211}$
	$\check{\mathscr{S}}_{123} = F_{123} + F_{1122}$	$\check{\mathscr{S}}_{1221} = F_{1221} + F_{11211}$
	$\check{\mathscr{S}}_{1212} = F_{1212}$	$\check{\mathscr{S}}_{12111} = F_{12111}$
	$\check{\mathscr{S}}_{114} = F_{114} + F_{123} + F_{132} + F_{213} + F_{222} + F_{312}$	$\check{\mathscr{S}}_{1131} = F_{1131} + F_{1221} + F_{2121}$
	$\check{\mathscr{S}}_{1122} = F_{1122} + F_{11121}$	$\check{\mathscr{S}}_{11211} = F_{11211}$
	$\check{\mathscr{S}}_{1113} = F_{1113} + F_{1122} + F_{1212} + F_{2112}$	$\check{\mathscr{S}}_{11121} = F_{11121}$
	$\check{\mathscr{S}}_{11112} = F_{11112}$	$\check{\mathscr{S}}_{111111} = F_{111111}$

Table 3.3: Quasisymmetric Schur functions of size $n \leq 6$ in the fundamental basis.

3. If there exist two cells $x = (i, k+1)$ and $y = (j, k+1)$ with for $i > j$, and the label of y is less than or equal to the label of x, then there exists a cell z on row i column $k+1$ whose label is greater than that of y.

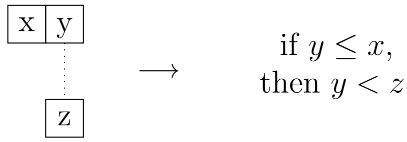


Figure 3.2: Illustration of Condition 3.

For a composition α we define the corresponding *quasisymmetric Schur function* [13] as follows:

$$\check{\mathscr{S}}_\alpha(X) = \sum_T X^{\mathrm{wt}(T)}, \tag{1.4}$$

where the sum is over all reverse composition tableaux T of shape α.

A reverse composition tableau that contains each of the numbers from 1 to n exactly once is called *standard*. For a given standard reverse composition tableau T, we define its descent set to be the set of i such that $i+1$ appears weakly to the right of i. We denote the corresponding composition by $\mathrm{desc}_{\mathrm{rc}}(T)$ to differentiate it from the regular case where we calculate descents using the row reading word.

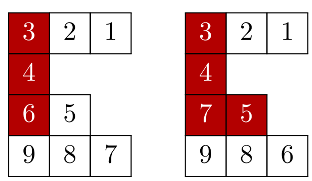


Figure 3.3: Standard reverse composition tableaux of shape (3,1,2,3) with descents shown in red.

Quasisymmetric Schur functions expand positively into fundamental quasisymmetric functions. In fact, we have:

$$\check{\mathscr{S}}_\alpha(X) = \sum_S F_{\mathrm{desc}_{\mathrm{rc}}(T)}(X), \tag{1.5}$$

where the sum is over the standard reverse composition tableaux of shape α.

Figure 3.3 shows the two standard reverse composition tableaux of shape $(3, 1, 2, 3)$ with their descents marked in red. They have the descent sets: $\{3, 4, 6\}$ and $\{3, 4, 5, 7\}$ which correspond to descent compositions $(3, 1, 2, 3)$ and $(3, 1, 1, 2, 2)$, so we have:

$$\check{\mathscr{S}}_{(3,1,2,3)}(X) = F_{3123}(X) + F_{31122}(X).$$

The list of quasisymmetric Schur functions up to size 6 is given in Table 3.3.

Quasisymmetric Schur functions refine Schur functions [13] in the sense that:

$$s_{\mu}(X) = \sum_{\mathrm{sort}(\alpha)=\mu} \check{\mathscr{S}}_{\alpha}(X). \tag{1.6}$$

1.3 Young Quasisymmetric Schur Functions

An *Young composition tableau* of shape α is a filling of composition diagram with positive integers such that:

1. first column increases from top to bottom,
2. rows weakly increase from left to right,
3. If for $i > j$ there exists two cells $x = (i, k+1)$ and $y = (j, k+1)$ with the label of y greater than or equal to the label of x, then there exists a cell z on row i column $k+1$ whose label is less than that of y.

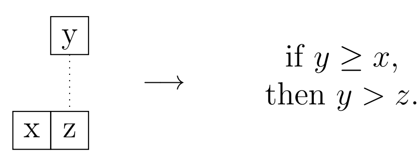


Figure 3.4: Illustration of Condition 3.

For a composition α we define the corresponding *Young quasisymmetric Schur function* [13] as follows:

$$\hat{\mathscr{S}}_{\alpha}(X) = \sum_{T} X^{\mathrm{wt}(T)}, \tag{1.7}$$

where the sum is over all Young composition tableaux of shape α.

n=1	$\hat{\mathscr{S}}_1 = F_1$	
n=2	$\hat{\mathscr{S}}_2 = F_2$	$\hat{\mathscr{S}}_{11} = F_{11}$
n=3	$\hat{\mathscr{S}}_3 = F_3$	$\hat{\mathscr{S}}_{21} = F_{21}$
	$\hat{\mathscr{S}}_{12} = F_{12}$	$\hat{\mathscr{S}}_{111} = F_{111}$
n=4	$\hat{\mathscr{S}}_4 = F_4$	$\hat{\mathscr{S}}_{31} = F_{22} + F_{31}$
	$\hat{\mathscr{S}}_{13} = F_{13}$	$\hat{\mathscr{S}}_{22} = F_{22} + F_{121}$
	$\hat{\mathscr{S}}_{211} = F_{211}$	$\hat{\mathscr{S}}_{121} = F_{121}$
	$\hat{\mathscr{S}}_{112} = F_{112}$	$\hat{\mathscr{S}}_{1111} = F_{1111}$
n=5	$\hat{\mathscr{S}}_5 = F_5$	$\hat{\mathscr{S}}_{41} = F_{23} + F_{32} + F_{41}$
	$\hat{\mathscr{S}}_{14} = F_{14}$	$\hat{\mathscr{S}}_{32} = F_{32} + F_{221}$
	$\hat{\mathscr{S}}_{23} = F_{23} + F_{122} + F_{131}$	$\hat{\mathscr{S}}_{311} = F_{212} + F_{221} + F_{311}$
	$\hat{\mathscr{S}}_{131} = F_{122} + F_{131}$	$\hat{\mathscr{S}}_{113} = F_{113}$
	$\hat{\mathscr{S}}_{221} = F_{221} + F_{1211}$	$\hat{\mathscr{S}}_{212} = F_{212}$
	$\hat{\mathscr{S}}_{122} = F_{122} + F_{1121}$	$\hat{\mathscr{S}}_{2111} = F_{2111}$
	$\hat{\mathscr{S}}_{1211} = F_{1211}$	$\hat{\mathscr{S}}_{1121} = F_{1121}$
	$\hat{\mathscr{S}}_{1112} = F_{1112}$	$\hat{\mathscr{S}}_{11111} = F_{11111}$
n=6	$\hat{\mathscr{S}}_6 = F_6$	$\hat{\mathscr{S}}_{51} = F_{24} + F_{33} + F_{42} + F_{51}$
	$\hat{\mathscr{S}}_{15} = F_{15}$	$\hat{\mathscr{S}}_{42} = F_{33} + F_{42} + F_{222} + F_{231} + F_{321}$
	$\hat{\mathscr{S}}_{24} = F_{24} + F_{123} + F_{132} + F_{141}$	$\hat{\mathscr{S}}_{411} = F_{213} + F_{222} + F_{231} + F_{312} + F_{321} + F_{411}$
	$\hat{\mathscr{S}}_{141} = F_{123} + F_{132} + F_{141}$	$\hat{\mathscr{S}}_{114} = F_{114}$
	$\hat{\mathscr{S}}_{33} = F_{33} + F_{132} + F_{222} + F_{231} + F_{1221}$	$\hat{\mathscr{S}}_{321} = F_{321} + F_{2211}$
	$\hat{\mathscr{S}}_{312} = F_{222} + F_{312} + F_{2121}$	$\hat{\mathscr{S}}_{231} = F_{222} + F_{231} + F_{1212} + F_{1221} + F_{1311}$
	$\hat{\mathscr{S}}_{213} = F_{213}$	$\hat{\mathscr{S}}_{132} = F_{132} + F_{1221}$
	$\hat{\mathscr{S}}_{123} = F_{123} + F_{1122} + F_{1131}$	$\hat{\mathscr{S}}_{3111} = F_{2112} + F_{2121} + F_{2211} + F_{3111}$
	$\hat{\mathscr{S}}_{1311} = F_{1311} + F_{1221} + F_{1311}$	$\hat{\mathscr{S}}_{1131} = F_{1122} + F_{1131}$
	$\hat{\mathscr{S}}_{1113} = F_{1113}$	$\hat{\mathscr{S}}_{222} = F_{222} + F_{1212} + F_{1221} + F_{2121} + F_{11211}$
	$\hat{\mathscr{S}}_{2211} = F_{2211} + F_{12111}$	$\hat{\mathscr{S}}_{2121} = F_{2121}$
	$\hat{\mathscr{S}}_{2112} = F_{2112}$	$\hat{\mathscr{S}}_{1221} = F_{1221} + F_{11211}$
	$\hat{\mathscr{S}}_{1212} = F_{1212}$	$\hat{\mathscr{S}}_{1122} = F_{1122} + F_{11121}$
	$\hat{\mathscr{S}}_{21111} = F_{21111}$	$\hat{\mathscr{S}}_{12111} = F_{12111}$
	$\hat{\mathscr{S}}_{11211} = F_{11211}$	$\hat{\mathscr{S}}_{11121} = F_{11121}$
	$\hat{\mathscr{S}}_{11112} = F_{11112}$	$\hat{\mathscr{S}}_{111111} = F_{111111}$

Table 3.4: Young quasisymmetric Schur functions of size $n \leq 6$ in the fundamental basis

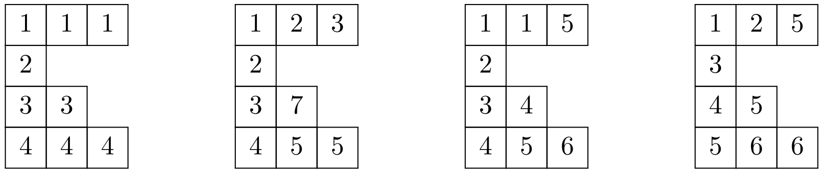


Figure 3.5: Examples of Young composition tableaux of shape (3,1,2,3).

A Young composition tableau that contains each of the numbers from 1 to n exactly once is called *standard*. The descent set of a standard Young composition tableau is the set of i such that $i+1$ appears weakly to the left of i. We denote the corresponding composition by $\mathrm{desc}_{\mathrm{yc}}(T)$ to differentiate it from the regular case. The standard Young composition tableaux of shape $(2,3,1)$ are given in Figure 3.6 below, with their descents shown in red.

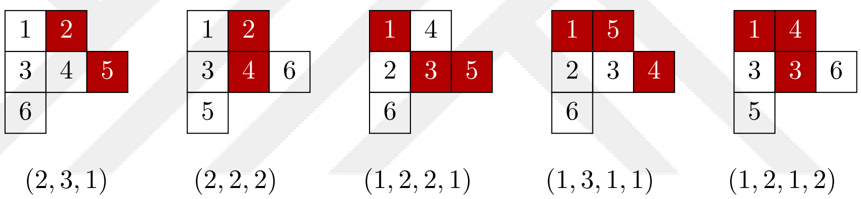


Figure 3.6: Standard Young composition tableaux of shape (2,3,1) with corresponding descent compositions.

Young quasisymmetric Schur functions have similar properties to quasisymmetric Schur functions. They also expand positively into fundamental quasisymmetric functions as follows:

$$\hat{\mathscr{S}}_\alpha(X) = \sum_S F_{\mathrm{Des}_{\mathrm{yc}}(S)}(x), \tag{1.8}$$

where the sum is over the standard Young composition tableaux of shape α. The list of Young quasisymmetric functions up to size 6 is given in Table 3.4.

Additionally, they satisfy the same refinement formula: Quasisymmetric Schur functions refine Schur functions [13]:

$$s_\mu(X) = \sum_{\mathrm{sort}(\alpha)=\mu} \hat{\mathscr{S}}_\alpha(X). \tag{1.9}$$

1.4 Dual Immaculate Tableaux

An *immaculate tableau* of shape α is a filling of its boxes with positive integers such that the first column will be strictly increasing from top to bottom, and each row will be weakly increasing from left to right. An immaculate tableau that contains each of the numbers from 1 to n exactly once is called *standard*.

1	2	2	4
2	7		
4	5	5	

1	3	4	9
2	8		
5	6	7	

Figure 3.7: The composition diagram for $\alpha = (4, 2, 3)$ with corresponding immaculate and standard immaculate filling examples.

For a partition α we define the corresponding *dual immaculate function* [5] [1] as follows:

$$\mathfrak{S}^*_\alpha(X) = \sum_T X^{\mathrm{wt}(T)}, \tag{1.10}$$

where the sum is over all immaculate tableaux of shape X.

Dual immaculate functions are quasisymmetric, and they expand positively into fundamental quasisymmetric functions. The list of dual immaculate functions up to size 5 is given in Table 3.5.

Theorem 3.1.2 (Allen, Hallam, Mason [1]). *Dual immaculate quasisymmetric functions expand positively into Young quasisymmetric Schur functions.*

Table 3.5 also includes the expansions of dual immaculate functions of size ≤ 5 in the Young quasisymmetric Schur basis.

2 Peak Algebra

For any $P \subseteq [n-1]$ with $\mathrm{comp}(P) \in \mathcal{PC}(n)$ we associate a quasisymmetric function called its *peak function*, defined in [41] by:

$$G_P(X) = \sum_{\substack{D \subseteq [n-1] \\ P \subseteq D \triangle D+1}} F_D(X), \tag{2.1}$$

[1] The word dual in the definition comes from the fact that this is the dual of the immaculate basis of non-symmetric functions, which will not be discussed in this work. The interested writer is advised to refer to [5], [6] for more details.

n=1	$\mathfrak{S}^*_1 = F_1 = \hat{\mathscr{S}}_1$
n=2	$\mathfrak{S}^*_2 = F_2 = \hat{\mathscr{S}}_2$ $\mathfrak{S}^*_{11} = F_{11} = \hat{\mathscr{S}}_{11}$
n=3	$\mathfrak{S}^*_3 = F_3 = \hat{\mathscr{S}}_3$ $\mathfrak{S}^*_{21} = F_{21} + F_{12} = \hat{\mathscr{S}}_{21} + \hat{\mathscr{S}}_{12}$ $\mathfrak{S}^*_{12} = F_{12} = \hat{\mathscr{S}}_{12}$ $\mathfrak{S}^*_{111} = F_{111} = \hat{\mathscr{S}}_{111}$
n=4	$\mathfrak{S}^*_4 = F_4 = \hat{\mathscr{S}}_4$ $\mathfrak{S}^*_{31} = F_{31} + F_{22} + F_{13} = \hat{\mathscr{S}}_{31} + \hat{\mathscr{S}}_{13}$ $\mathfrak{S}^*_{22} = F_{22} + F_{121} + F_{13} = \hat{\mathscr{S}}_{22} + \hat{\mathscr{S}}_{13}$ $\mathfrak{S}^*_{211} = F_{211} + F_{112} + F_{121} = \hat{\mathscr{S}}_{211} + \hat{\mathscr{S}}_{112} + \hat{\mathscr{S}}_{121}$ $\mathfrak{S}^*_{13} = F_{13} = \hat{\mathscr{S}}_{13}$ $\mathfrak{S}^*_{121} = F_{121} + F_{112} = \hat{\mathscr{S}}_{121} + \hat{\mathscr{S}}_{112}$ $\mathfrak{S}^*_{112} = F_{112} = \hat{\mathscr{S}}_{112}$ $\mathfrak{S}^*_{1111} = F_{1111} = \hat{\mathscr{S}}_{1111}$
n=5	$\mathfrak{S}^*_5 = F_5 = \hat{\mathscr{S}}_5$ $\mathfrak{S}^*_{41} = F_{41} + F_{32} + F_{23} + F_{14} = \hat{\mathscr{S}}_{41} + \hat{\mathscr{S}}_{14}$ $\mathfrak{S}^*_{32} = F_{32} + F_{23} + F_{14} + F_{221} + F_{122} + F_{131} = \hat{\mathscr{S}}_{32} + \hat{\mathscr{S}}_{23} + \hat{\mathscr{S}}_{14}$ $\mathfrak{S}^*_{311} = F_{311} + F_{221} + F_{131} + F_{212} + F_{122} + F_{113} = \hat{\mathscr{S}}_{311} + \hat{\mathscr{S}}_{131} + \hat{\mathscr{S}}_{113}$ $\mathfrak{S}^*_{23} = F_{23} + F_{14} + F_{122} + F_{131} = \hat{\mathscr{S}}_{23} + \hat{\mathscr{S}}_{14}$ $\mathfrak{S}^*_{221} = F_{221} + F_{212} + F_{113} + 2F_{122} + F_{131} + F_{1121} + F_{1211} = \hat{\mathscr{S}}_{221} + \hat{\mathscr{S}}_{212} + \hat{\mathscr{S}}_{113} + \hat{\mathscr{S}}_{131} + \hat{\mathscr{S}}_{122}$ $\mathfrak{S}^*_{212} = F_{212} + F_{122} + F_{113} + F_{1121} = \hat{\mathscr{S}}_{212} + \hat{\mathscr{S}}_{122} + \hat{\mathscr{S}}_{113}$ $\mathfrak{S}^*_{2111} = F_{2111} + F_{1211} + F_{1121} + F_{1112} = \hat{\mathscr{S}}_{2111} + \hat{\mathscr{S}}_{1211} + \hat{\mathscr{S}}_{1121} + \hat{\mathscr{S}}_{1112}$ $\mathfrak{S}^*_{14} = F_{14} = \hat{\mathscr{S}}_{14}$ $\mathfrak{S}^*_{131} = F_{131} + F_{122} + F_{113} = \hat{\mathscr{S}}_{131} + \hat{\mathscr{S}}_{113}$ $\mathfrak{S}^*_{122} = F_{122} + F_{113} + F_{1121} = \hat{\mathscr{S}}_{122} + \hat{\mathscr{S}}_{1121}$ $\mathfrak{S}^*_{1211} = F_{1211} + F_{1121} + F_{1112} = \hat{\mathscr{S}}_{1211} + \hat{\mathscr{S}}_{1121} + \hat{\mathscr{S}}_{1112}$ $\mathfrak{S}^*_{113} = F_{113} = \hat{\mathscr{S}}_{113}$ $\mathfrak{S}^*_{1121} = F_{1121} + F_{1112} = \hat{\mathscr{S}}_{1121} + \hat{\mathscr{S}}_{1112}$ $\mathfrak{S}^*_{1112} = F_{1112} = \hat{\mathscr{S}}_{1112}$ $\mathfrak{S}^*_{11111} = F_{11111} = \hat{\mathscr{S}}_{11111}$

Table 3.5: Dual immaculate functions of size $n \leq 5$ in the fundamental basis and in the Young quasisymmetric Schur basis.

where $D+1$ is the subset of $[n]\backslash\{1\}$ defined by adding 1 to the elements of D, and $D\triangle D+1$ denotes the symmetric difference of the two sets.

Like in the case of Schur functions, Schur's Q-functions can be expanded in terms of the fundamental quasisymmetric functions:

$$Q_\lambda(X) = \sum_{T'\in SShT\pm(\lambda)} F_{Des(T')}(X)$$

For this expansion, we only look at the marked standard tableaux of shape λ. An expansion that also eliminates the markings and only considers the standard fillings was given by Stembridge[41]:

$$Q_\lambda(X) = \sum_{T\in SShT(\lambda)} 2^{|Peak(T)|+1} G_{Peak(T)}(X)$$

Here, the functions G_P, where P is a subset of $[2,3,..,n-1]$ with no consecutive entries, are the *peak functions* are defined in [41] by:

$$G_P(X) = \sum_{\substack{D\in[n-1]\\ Spike(D)\supset P}} F_D(X)$$

n=1	$P_1 = G_1$
n=2	$P_2 = G_2$
n=3	$P_3 = G_3$ $P_{21} = G_{21}$
n=4	$P_4 = G_4$ $P_{31} = G_{31} + G_{22}$
n=5	$P_5 = G_5$ $P_{41} = G_{41} + G_{32} + G_{23}$ $P_{32} = G_{32} + 2G_{221}$
n=6	$P_6 = G_6$ $P_{51} = G_{51} + G_{42} + G_{33} + G_{24}$ $P_{42} = G_{42} + G_{33} + 2G_{321} + 2G_{231} + 2G_{222}$ $P_{321} = G_{321} + G_{222}$
n=7	$P_7 = G_7$ $P_{61} = G_{61} + G_{52} + G_{43} + G_{34} + G_{25}$ $P_{52} = G_{52} + G_{43} + G_{34} + 2G_{421} + 2G_{331} + 2G_{322} + 2G_{241} + 2G_{232} + 2G_{223}$ $P_{43} = G_{43} + 2G_{331} + 2G_{322} + 2G_{232} + 4G_{2221}$ $P_{421} = G_{421} + G_{331} + 2G_{322} + G_{232} + G_{223} + 2G_{2221}$

Table 3.6: Schur's P-functions of size $n \leq 7$ in the peak basis

These identities also extend to the skew case:

$$Q_{\lambda\backslash\mu}(X) = \sum_{S \in SSShT(\lambda\backslash\mu)} X^{|S|} = \sum_{T' \in SShT\pm(\lambda\backslash\mu)} F_{Des(T')}(X) = \sum_{T \in SShT(\lambda\backslash\mu)} 2^{|Peak(T)|+1} G_{Peak(T)}(X). \quad (2.2)$$

Chapter 4

A Shifted Analogue to Quasisymmetric Schur Functions

The quasisymmetric Schur and Young quasisymmetric Schur bases for quasisymmetric functions were described in Chapter 3. In addition to refining Schur functions in a natural way, these functions exhibit several nice properties of the Schur functions, including having Pieri and Littlewood-Richardson rules [29]. A natural question to ask is if a type B analogue exists.

In 2015, a candidate basis $\widehat{P}$ was proposed by Jing and Li [17], that refined Schur's P-functions as an alternating sum. They conjectured that these quasisymmetric Schur P-functions have an integral, unitriangular and positive expansion in terms of the peak functions.

In this chapter, we follow the constructions in [20] to prove this conjecture, as well as give an explicit formula. We refine the peak composition tableaux defined in [17] using the marked alphabet, giving precise expansions in terms of the monomial, fundamental and peak bases. We also provide counterexamples to show that those functions do not expand positively into Young's quasisymmetric Schur functions, and that their multiplication with Schur's P-functions is not $\widehat{P}$-positive.

1 Peak Composition Tableaux and Quasisymmetric Schur Q-functions

For a given peak composition α, a *peak composition tableau* [17] of shape α is a filling of the composition diagram of α with positive integers satisfying the following properties:

1. Each row is weakly increasing from left to right.

2. First column is strictly increasing from top to bottom.

3. For any k, the subdiagram of α labeled with numbers $\{1, 2, \ldots, k\}$ is a peak composition.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 4 & 4 & 5 \\ \hline 5 \\ \cline{1-1}\end{array}$$

Figure 4.1: A peak composition tableau of shape (3,2,3,1) and weight (1,1,1,3,2,1).

We denote by $PCT(\alpha)$ the set of peak composition tableaux of shape α whose weight is a composition (equivalently, no number is skipped in the labelling).

$$T_1 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad T_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad T_3 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Figure 4.2: The elements of $PCT(2, 1)$.

Remark 4.1.1. *For a peak composition tableau T of shape α, we have $|\mathrm{wt}(T)| = |\alpha|$. Furthermore, $\mathrm{wt}(T)$ is less than or equal to α with respect to the lexicographical order.*

To facilitate our proofs, we will replace the original definition of quasisymmetric Schur P-functions with the defining Proposition 4.16 from [17].

Definition 4.1.2 ([17])**.** *For a peak composition α, the corresponding quasisymmetric Schur P- and Q-functions are defined as follows:*

$$\widehat{P}_\alpha(X) = \sum_{T \in PCT(\alpha)} 2^{p(T)-m(T)} M_{\mathrm{wt}(T)}(X), \qquad (1.1)$$

$$\widehat{Q}_\alpha(X) = 2^{\ell(\alpha)} \widehat{P}_\alpha(X). \qquad (1.2)$$

Here $p(T) = \sum_{i \leq \ell(\alpha)} p_i(T)$ where $p_i(T) + 1$ is the number of distinct integers in row i and $m(T)$ denotes the number of boxes in the first column whose bottom and right adjacent boxes have the same number, as illustrated below:

$$\begin{array}{|c|c|}\hline x & y \\ \hline y \\ \cline{1-1}\end{array}$$

For the example given in Figure 4.2, we have:

$$\widehat{P}_{(2,1)}(X) = M_{(2,1)}(X) + M_{(1,2)}(X) + 2M_{(1,1,1)}(X).$$

Note that the multiplicity of the monomials corresponding to T_1 and T_2 are 1, and the multiplicity of the monomial corresponding to T_3 is 2.

It is shown in [17] that Schur's Q-functions can be written as an alternating sum of quasisymmetric Schur Q-functions.

Theorem 4.1.3 ([17])**.** *For any strict partition λ we have:*

$$Q_\lambda(X) = \sum_{\substack{\alpha \in \mathcal{PC} \\ \mathrm{sort}(\alpha)=\lambda}} (-1)^{\mathrm{length}(\sigma_\alpha)} \widehat{Q}_\alpha(X), \tag{1.3}$$

where $\mathrm{sort}(\alpha)$ *is the unique rearrangement of α into a partition, and σ_α is a corresponding permutation of minimal length.*

It was conjectured in [17] that the quasisymmetric Schur P-functions have a unitary, unitriangular and positive expansion in terms of the peak functions. The main result of this chapter is to provide a proof for this conjecture, and give this expansion explicitly (Theorem 4.2.8).

2 Marked Peak Composition Tableaux

We introduce a variation on peak tableaux that will be a combinatorial model for the expansions in monomial, fundamental and peak bases.

Definition 4.2.1. *For a peak composition α, a* marked peak composition tableau *of shape α is a filling of its diagram with numbers from the alphabet $1' < 1 < 2' < 2...$ satisying the following properties:*

1. *Each row is weakly increasing from left to right, with no repeated marked numbers.*
2. *First column is strictly increasing from top to bottom.*
3. *For any positive number k, the subdiagram of α labeled with numbers $\{1', 1, 2', 2...k', k\}$ is a peak composition.*
4. *If any box in the first column has an unprimed number i in the box right adjacent to it, then it cannot have i or i' in the box adjacent below.*

Note that erasing the marks from a marked peak composition tableau gives a peak composition tableau, so the definition of weight naturally extends to marked peak composition tableaux as in the case of shifted tableaux, with $\mathrm{wt}(T)_i$ given by the total number of occurrences of i and i' in T.

We will denote the set of the marked peak composition tableaux of shape α with weight given by a strong composition by $MPCT(\alpha)$. We will use the notation $MPCT^*(\alpha)$ to denote the subset of $MPCT(\alpha)$ where we have no marked numbers in the first column.

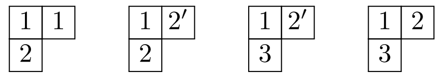


Figure 4.3: The elements of $MPCT^*(2,1)$.

The elements of $MPCT^*(2,1)$ are listed in Figure 4.3. Note that by forgetting the markings, we have one element that maps to each of T_1 and T_2 each and two elements that map to T_3 from Figure 4.2, so that we have:

$$\widehat{P}_{(2,1)}(X) = \sum_{T \in MPCT^*(2,1)} M_{\mathrm{wt}(T)}(X).$$

We will now see that this holds true in general.

Proposition 4.2.2. *For a peak composition α we have:*

$$\widehat{P}_{\alpha}(X) = \sum_{T \in MPCT^*(\alpha)} M_{\mathrm{wt}(T)}(X), \tag{2.1}$$

$$\widehat{Q}_{\alpha}(X) = \sum_{T \in MPCT(\alpha)} M_{\mathrm{wt}(T)}(X). \tag{2.2}$$

Proof. To prove the statement for the P-functions, it suffices to show that for each peak composition tableau T, there are exactly $2^{p(T)-m(T)}$ ways of marking it to obtain a marked peak composition tableau with no marks in the first column, where $p(T)$ and $m(T)$ are defined as in Definition 4.1.2.

Note that if the ith row of T contains more than one occurrences of a number j, only the leftmost can be marked. So for each distinct j in row i, we get two choices, with two exceptions. First is, if the leftmost box labeled j is in the first column, we are not allowed to mark it. So far we have $p_i(T) - 1$ choices to make for each row. The second exception is, if the leftmost box in row i has boxes labeled j adjacent to the right and below, the one adjacent to the right cannot be marked- this situation occurs $m(T)$ times. Consequently we get exactly $2^{p(T)-m(T)}$ marked peak composition tableaux.

Case of Q-functions follows, as there are $2^{\ell(\alpha)}$ ways of marking the boxes in the first column, and the conditions of Definition 4.2.1 is independent of the markings of the first column. □

Definition 4.2.3. *A marked peak composition tableau of shape $\alpha \in \mathcal{PC}(n)$ is called* marked standard *if it contains each number from* 1 *to* n*, possibly marked. A marked standard tableau with no marked entries is called* standard.

$$S_1 = \begin{array}{|c|c|c|} \hline 1 & 2 & 8' \\ \hline 3 & 4' \\ \cline{1-3} 5 & 6 & 7' \\ \hline 9' \\ \cline{1-1} \end{array}$$

Figure 4.4: A marked standard peak composition tableau of shape $(3, 2, 3, 1)$.

We will denote the set of marked standard peak composition tableaux of shape α by $SMPCT(\alpha)$, with $SMPCT^*(\alpha)$ denoting the set of marked standard tableaux with no marked entries in the first column. We will denote the set of standard peak composition tableaux of shape α by $SPCT(\alpha)$.

For a peak composition tableau S, consider its row reading word $\mathrm{rw}(S)$. For example, the tableau in Figure 4.4 has $\mathrm{rw}(S_1) = 1359'24'68'7'$. If S is a marked standard tableau, the reading word gives us a marked permutation of n, so we can talk about its descent set $\mathrm{Des}(S)$ and the corresponding composition $\mathrm{desc}(S)$. For example, forS_1 from Figure 4.4, we have $\mathrm{Des}(S_1) = \mathrm{Des}(784913526) = \{2, 3, 6\}$.

Definition 4.2.4. *Let T be a marked peak composition tableaux of shape α. Then $St'(T)$ is marked filling of the diagram of α given by assigning numbers $1, 2, 3...n$ to the boxes of T in the following order, and then marking k if and only if the corresponding box is marked in T:*

- *We assign numbers to the boxes in the order $1' < 1 < 2' < 2...$*
- *If there is more than one box of label i, the numbers increase following the order of the reading word.*
- *If there is more than one box of label i', the numbers increase in the reverse order of the reading word.*

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 5' \\ \hline 3 & 4' \\ \cline{1-3} 4 & 4 & 5' \\ \hline 5' \\ \cline{1-1} \end{array} \quad \xrightarrow{St'} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 8' \\ \hline 3 & 4' \\ \cline{1-3} 5 & 6 & 7' \\ \hline 9' \\ \cline{1-1} \end{array}$$

Proposition 4.2.5. *The filling $St'(T)$ defined above is a marked standard peak composition tableau.*

Proof. By definition, $St'(T)$ contains each of the numbers $1' < 1 < 2' < 2...$ once, possibly marked. As each row in T is weakly increasing, and any marked number i' can only occur once in one row, the numbers in $St'(T)$ increase from left to right in each row. As the first column in T is strictly increasing no i or i' is repeated, and the numbers in the first column of $St'(T)$ increase from bottom to top. Lastly, Condition (4) of Definition 4.2.1 ensures that for any box in the first column, the number in its right adjacent box will be lower than the number in its bottom adjacent box, implying that the subdiagrams labeled with numbers less than some $k \leq n$ correspond to peak compositions. □

For marked standard peak composition tableaux S', the corresponding elements $F_{\mathrm{desc}(S')}(X)$ of Gessel's fundamental basis allow us to calculate the quasisymmetric Schur Q- and P-functions.

Theorem 4.2.6. *The quasisymmetric Schur Q- and P-functions have a positive integral expansion in terms of F_D's, given by:*

$$\widehat{P}_{\alpha}(X) = \sum_{S' \in SMPCT^*(\alpha)} F_{\mathrm{desc}(S')}(X), \tag{2.3}$$

$$\widehat{Q}_{\alpha}(X) = \sum_{S' \in SMPCT(\alpha)} F_{\mathrm{desc}(S')}(X). \tag{2.4}$$

Proof. We will prove the case for Q-functions only, the case for P-functions follows by duplication. By the definition of F_D, it suffices to show that for any $S \in SMPCT(\alpha)$ and any $\beta \preceq \mathrm{desc}(S)$ (equivalently any β satisfying $\mathrm{Des}(\beta) \subseteq \mathrm{Des}(S)$) , there is a unique tableau $T_{S,\beta} \in MPCT(\alpha, \beta)$ that has $St'(T_{S,\beta}) = S$, and all peak composition tableaux of shape α are of this form for some S and β.

First observe that as we standardize following the order of the numbers, for any marked standard peak composition tableau S and composition $\beta = (\beta_1, \beta_2 ... \beta_n)$ if there exists a tableau T that standardizes to S with $\mathrm{wt}(T) = \beta$, then T must be equal to $T_{S,\beta}$, obtained by replacing $1, 2..\beta_1$ in S by 1, $\beta_1 + 1, ...\beta_1 + \beta_2$ by 2 and so on, and carrying the marks. We will denote the box labeled k/k' in S by C_k and its label in $T_{S,\beta}$ by c_k, and use the notation $|c_k| = i$ to mean $c_k \in \{i, i'\}$.

The next step is to show that if $Des(\beta) \supseteq \mathrm{Des}(S)$ then $T_{S,\beta} \in MPCT(\alpha, \beta)$ and $St'(T_{S,\beta}) = S$. We first verify that the conditions $(1) - (4)$ of the definition of marked peak composition tableaux (Definition 4.2.1) are satisfied. Note that condition (3) is trivially satisfied as $\bigcup_{a \leq k} C_a$ is a peak composition for all k. Let C_a and C_{a+k} be in the same row. Then, as rows increase from left to right in S, C_{a+k} is to the right of C_a, and by our construction, $|c_a| \leq |c_{a+k}|$. Furthermore if $|c_a| = |c_{a+k}|$, then $a, a+1...a+k-1 \notin \mathrm{Des}(T_{S,\beta}) \subseteq \mathrm{Des}(S)$. Also as C_{a+k} is to the right of C_a, $a + k$ comes after a in the reading word, so by Lemma 1.3.2 $a + k$ is not marked in S, implying c_{a+k} is not marked in $T_{S,\beta}$. This takes care of (1). The second condition is analogous to the first. For (4), assume that we have a box in the first column with C_a right adjacent and C_b bottom adjacent to it with $|c_a| = |c_b| = i$. We want to show that in this case we mush have $c_a = i'$. First note that as $\bigcup_{k<r} C_k$ is a peak composition, it cannot include two parts of size 1, so we must have $b > a$. That means, if $c_a = i$ then c_b is also equal to i, implying $a, a + 1, \ldots, b - 1$ are not descents. As C_b comes before C_a in the reading word, by Lemma 1.3.2 c_a must be marked, contradicting the assumption $c_a = i$. This shows $T_{S,\beta}$ is a valid marked peak composition tableau, its standardization is well defined. Let us show that its standardization is indeed $T_{S,\beta}$. Assume $c_a = c_{a+1} = i$. Then $a \notin \mathrm{Des}(T_{S,\beta}) \subseteq \mathrm{Des}(S)$, so by Lemma 1.3.2 C_a comes after C_{a+1} in the reading word. Similarly if $c_a = c_{a+1} = i'$ then C_a comes before C_{a+1} in the reading word- so S is indeed the labeling given by the map St'.

Lastly we will show that if $T_{S,\beta}$ is a marked peak composition tableau that satisfies $St'(T_{S,\beta}) = S$, then $\mathrm{Des}(\beta) \supseteq \mathrm{Des}(S)$. Assume there exists an element a in $\mathrm{Des}(S) \backslash Des(\beta)$. As $a \notin \mathrm{Des}(\beta)$, we have $|c_a| = |c_{a+1}| = i$ for some i. Assume C_a comes before C_{a+1} in the reading word. Then, as a is a descent of S, $a+1$ is marked in S, and therefore $c_{a+1} = i'$. As we count all the i' before counting any i in the standardization map, $c_a = i'$. As we count primed numbers in the reverse reading order in standardization, we need $a > a+1$ which is a contradiction. The case of C_a coming after C_{a+1} is symmetrical. □

We will now turn our attention to peak sets corresponding to marked peak composition tableaux.

Lemma 4.2.7. *Let S^0 denote the standard marked peak composition tableau of shape α that has labels $1, 2, \ldots, \alpha_1$ in the first row; labels $\alpha_1+1, \alpha_1+2, \ldots, \alpha_1+\alpha_2$ on the second row and so on. Then* $\mathrm{peakc}(S_0) = \alpha$. *Furthermore, if S is any other standard marked peak composition tableau of shape α , then* $\mathrm{peakc}(S) < \alpha$ *with respect to the lexicographic order.*

Proof. First note that by its definition, the peaks of the reading word of S_0 are given by the rightmost number in each row except the last, so $\mathrm{Peak}(S_0) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}\}$ and $\mathrm{peakc}(S_0) = \alpha$. Any change in this ordering results in a tableau S with decreased peak values, so
$\mathrm{peakc}(S) < \mathrm{peakc}(S_0)$ for any $S \neq S_0$. □

Now we are ready to state and prove our main theorem.

Theorem 4.2.8. *The quasisymmetric Schur P-functions expand into peak functions as:*

$$\widehat{P}_\alpha(X) \quad = \quad 2^{-\ell(\alpha)} \sum_{S \in SPCT(\alpha)} 2^{|\mathrm{Peak}(S)|+1} G_{\mathrm{peakc}(S)}(X).$$

In particular, they are positive, integral and unitriangular.

Proof. This identity itself follows from the following observation: Let w be an unmarked word of length k with $\mathrm{Peak}(\mathrm{Des}(w)) = P$. Denote by $M(w)$ the 2^k element set of possible marked versions of w. Then we have:

$$G_P(X) = 2^{-|P|-1} \sum_{w' \in M(w)} F_{\mathrm{desc}(w')}(X) \tag{2.5}$$

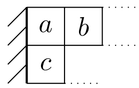
A detailed proof of this can be found in [21]. Plugging in Equation 2.3 we get:

$$\sum_{S \in SPCT(\alpha)} 2^{|\mathrm{Peak}(S)|+1} G_{\mathrm{peakc}(S)}(X) = \sum_{\substack{S \in SPCT(\alpha) \\ w' \in M(\mathrm{rw}(S))}} \sum F_{\mathrm{desc}(w')}(X) =$$

$$\sum_{S' \in SMPCT(\alpha)} F_{\mathrm{desc}(S')}(X) = 2^{\ell(\alpha)} \widehat{P}_\alpha(X).$$

The fact that the expansion is unitriangular follows from Lemma 4.2.7. For integrality, we need to show that for any standard peak composition tableaux S of shape α, the number of peaks of S is at least $\ell(\alpha) - 1$. Let

S be such a tableaux of length k. It follows from Definition 4.2.1 Condition (3) that as peak compositions can not have a part of size one except at the last row, for any box in the first column, the box right adjacent has a larger label than the box directly below, so they are ordered as follows:



$$\Rightarrow a < b < c$$

Note that b comes after both a and c in the reading word. If b is not a peak, either $b+1$ or $b-1$ comes after b in the reading word, and is also between a and c. Repeating this process, we can find a peak of S between a and c. As we can do this for any pair of consecutive entries in the first column, we have at least $\ell(\alpha) - 1$ peaks. □

3 Further properties of quasisymmetric Schur Q-functions

Equation (1.3) gives an alternating expansion of Schur's P-functions into $\widehat{P}$ functions. A consequence of the negative signs appearing in this expansion is that the quasisymmetric Schur P-functions lack some of the positivity properties of their symmetric analogues.

First we will look at the expansions into quasisymmetric Schur functions $\{\check{\mathscr{S}}_\alpha\}_\alpha$ [29], Young quasisymmetric functions $\{\hat{\mathscr{S}}_\alpha\}_\alpha$ [29] and dual immaculate quasisymmetric functions$\{\mathfrak{S}_\alpha\}_\alpha$ [5].

Theorem 4.3.1. *Quasisymmetric Schur Q-functions do not expand positively into quasisymmetric Schur functions or Young quasisymmetric Schur functions.*

Proof. Two examples of minimum size where negative coefficients come up are listed below.

$$\widehat{P}_{(2,3)} = \check{\mathscr{S}}_{(1,4)} - \check{\mathscr{S}}_{(3,2)} + \check{\mathscr{S}}_{(1,2,2)} + \check{\mathscr{S}}_{(2,1,2)} + 2\check{\mathscr{S}}_{(1,3,1)} - \check{\mathscr{S}}_{(1,1,2,1)} + \check{\mathscr{S}}_{(1,2,1,1)} + \check{\mathscr{S}}_{(2,1,1,1)},$$

$$\begin{aligned}\widehat{P}_{(2,3,1)} = \hat{\mathscr{S}}_{(2,3,1)} + \hat{\mathscr{S}}_{(2,2,2)} + \hat{\mathscr{S}}_{(2,2,1,1)} + \hat{\mathscr{S}}_{(2,1,3)} + \hat{\mathscr{S}}_{(2,1,2,1)} + \hat{\mathscr{S}}_{(2,1,1,2)} \\ + \hat{\mathscr{S}}_{(1,4,1)} + \hat{\mathscr{S}}_{(1,3,2)} - \hat{\mathscr{S}}_{(1,2,2,1)} - \hat{\mathscr{S}}_{(1,2,1,2)} - \hat{\mathscr{S}}_{(1,2,1,1,1)}.\end{aligned}$$

□

Corollary 4.3.2. *Quasisymmetric Schur functions do not expand positively into dual immaculate quasisymmetric functions.*

Proof. This follows as dual immaculate quasisymmetric functions expand positively into young quasisymmetric schur functions, as shown by Allen, Hallam and Mason in [1]. A minimal example with negative coefficients is given below:

$$\widehat{P}_{(3,1)} = \mathfrak{S}_{(3,1)} + \mathfrak{S}_{(2,2)} + \mathfrak{S}_{(2,1,1)} - \mathfrak{S}_{(1,3)}.$$

□

We will now consider the multiplication of quasisymmetric Schur P-functions.

Theorem 4.3.3. *The multiplication of a Schur's P-function with a a quasisymmetric Schur P-function does not necessarily have a positive expansion into quasisymmetric Schur P-functions.*

Proof. Positivity fails even in the simplest case of $\widehat{P}_{(1)}$:

$$\widehat{P}_{(1)}(X)P_{(3,1)}(X) = \widehat{P}_{(4,1)}(X) + \widehat{P}_{(3,2)}(X) - \widehat{P}_{(2,3)}(X).$$

□

Note that we do not even have a set inclusion between the diagrams of $(2,3)$ and $(3,1)$ so we do not have a natural rule of calculating the expansion of $\widehat{P}_{(1)}P_\lambda$ by adding boxes to the diagram of λ and assigning signs. Same statement applies to the multiplication of $\widehat{P}$ functions, as $\widehat{P}_{(3,1)} = P_{(3,1)}$.

Corollary 4.3.4. *The multiplication of quasisymmetric Schur P-functions is not $\widehat{P}$-positive.*

Chapter 5

Shifted Ribbon Tableaux

The study of ribbon tableaux on shifted shapes combines two existing areas of work: the theory of ribbon tableaux and Schur's Q-functions. Ribbon tableaux introduced by James and Kerber [16] have applications to the representations of the symmetric group over a field of finite characteristic. Schur's Q-functions come up as the symmetric functions that correspond to the shifted diagrams. They have a connection to the irreducible spin characters of the symmetric group, analogous to that of Schur functions and irreducible characters of linear representations [31]. Since their introduction in [37], applications to diverse mathematical fields have been discovered, including the cohomology of isotropic Grassmannians [18] and polynomial solutions to the BKP equation in hydrodynamics.

In this chapter, we are following the work in [21] to merge these two ideas to initiate a combinatorial theory of ribbon tableaux for shifted shapes. The k-quotients and k-cores for shifted shapes were previously studied by Morris and Yaseen in 1986 [32] in the context of modular representations. We expand upon their work, reformulating it in a more explicit way that is analogous to the ribbon tilings of unshifted shapes due to James and Kerber [16]. We also look at standard and semi-standard fillings of these shapes, and define shifted k-ribbon functions. There is a bijective correspondence between standard and semi-standard fillings of the k-ribbon tableaux with fillings of its k-quotient, analogous to the unshifted case, which gives a positive expansion in terms of Schur's Q-functions.

This chapter was partly motivated by the possiblity of providing a shifted analogue for the LLT polynomials of Lascoux, Leclerc and Thibon which extend the theory of ribbon tableaux in the unshifted case, and have connections to the Fock space representation of the universal enveloping algebra of quantum affine $\mathfrak{sl}_n$ [26]. While we are able to provide a Schur Q-positive q-analogue derived from the unshifted LLT polynomials using the quotient, this analogue unfortunately does not lift the underlying combinatorial structure to ribbon shifted tableaux. In fact this paper provides a negative result in that the spin statistic does not have a natural analogue for shifted ribbons, along with counterexamples that should prove valuable for further research.

1 Standard Ribbon Tableaux

A *k-ribbon* on an unshifted diagram is a connected skew-diagram of size k that contains no 2×2 square. A k-ribbon R is removable from diagram μ if $\mu/\nu = R$ for some $\nu \subset \mu$. A diagram with no removable k-ribbon is called a k-core.

On a given k-ribbon R, the rightmost lowest cell is called the *head* of the R. A set of disjoint ribbons form a *horizontal strip* if their disjoint union is a (not necessarily connected) skew-shape and their heads lie on different columns. A *semi-standard k-ribbon tableau* of shape μ is a sequence of shifted diagrams $\mu_0 \subset \mu_1 \subset \cdots \subset \mu_n = \mu$ where μ_0 is a k-core, and each μ_i/μ_{i-1} is a horizontal k-ribbon strip, the ribbon on which we label by i. A semi-standard k-ribbon tableau is called *standard* if all labels from 1 to n occur exactly once. The generating function for the k-ribbon tableaux of shape μ is given by:

$$GF^{(k)}_{\mu/\mu_0}(X) = \sum_{T \in SSRT_k(\mu)} X^T = \sum_{S \in SRT_k(\mu)} F_{Des(S)} X$$

where $SSRT_k(\mu)$ denotes the set of semi-standard k-ribbon tableaux of shape μ, and $SRT_k(\mu)$ denotes the set of standard ones.

Figure 5.1: Standard(left) and semi-standard(right) 3-ribbon tableaux of shape $\mu = (6, 3, 3, 2)$.

James and Kerber [16] showed that there is a weight-preserving bijection between semi-standard ribbon tableaux of shape μ, and semi-standard fillings of a k-tuple of unshifted shapes $(\mu^0, \mu^1, \ldots, \mu^{k-1})$ called the *k-quotient of μ*. This shows that:

$$GF^{(k)}_{\mu/\mu_0}(X) = s_{\mu^0}(X)s_{\mu^1}(X)\cdots s_{\mu^{k-1}}(X) \tag{1.1}$$

The *spin* statistic for a ribbon R, defined by Lascoux, Leclerc and Thibon [26] is given by $(|R| - ht(R) - 1)/2$, which is not necessarily an integer. For a semi-standard k-ribbon tableaux T of shape μ, we define the *spin* of T to be the sum of the spins of all ribbons on T. The *cospin* of T is given by $spin(T*) - spin(T)$ where $T*$ is a semi-standard k-ribbon tableaux of shape μ with maximum spin. The cospin is an integer for every tiling T.

Multiplying the fundamental quasisymmetric function for each tableau by a variable q raised to the tableau's cospin gives us the LLT-polynomial:

$$GF^{(k)}_{\mu/\mu_0}(X; q) = \sum_{T \in SRT_k(\mu)} q^{cospin(T)} F_{\mathrm{Des}(T)}(X). \tag{1.2}$$

The LLT-polynomials can be written as a sum of Schur polynomials with coefficients from $\mathbb{Z}^+[q]$ (See [12] for details).

2 Ribbons on Shifted Diagrams

On a shifted diagram, we call the columns strictly to the left of the last row its *shifted region*, and the rest its *unshifted region*. Note that the unshifted region uniquely determines the diagram [1].

The definition of the hook of a cell on a shifted diagram depends on whether the cell falls into the shifted region. For any cell C in the unshifted region, the *hook* of C is the union of C, with the cells above it in its column, and the cells to its right in the row. For a cell in the shifted region, its hook additionally includes the row of cells directly above the highest cell in the column of C. The number of cells in its hook is called the *hook length* of C.

[1] We deviate from convention in defining the shifted region, to define hook lengths more easily.

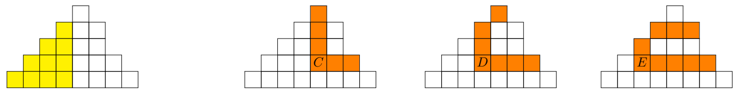


Figure 5.2: Shifted region of $(8, 6, 4, 3, 1)$ and hooks of the cells C, D and E.

Definition 5.2.1. *We define a* single-ribbon *on λ to be a connected skew-shifted diagram with each cell on a different diagonal.*

Some important notations we will use about ribbons throughout this chapter are heads and tails of ribbons. Analogous to the unshifted case, the rightmost lowest cell (the cell with the highest diagonal value) will be called the *head* of R, denoted by $H(R)$, and the one with the lowest diagonal value will be called the *tail* of R, denoted $T(R)$. We will also use $|R|$ for the size of a ribbon (the number of cells it contains) and $ht(R)$ for the height of a ribbon (the number of rows of λ that it intersects).

Note that for a single-ribbon R, $|R| =$diag$(H(R))-$diag$(T(R)) + 1$

Definition 5.2.2. *A* double-ribbon *of size k is a union of two disjoint single-ribbons R and S of sizes $r \geq s$ with $r + s = k$ such that the tail of R is on the main diagonal of λ, and the tail of S is on the main diagonal of $\lambda - R$, and their union forms a skew-shifted shape.*

The head of $Q = (R, S)$ is the head of R, and its tail is the tail of s.

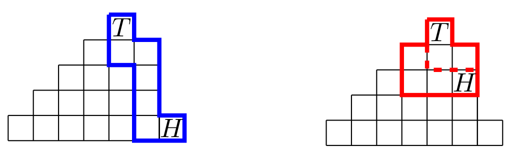


Figure 5.3: A single ribbon(left) and a double ribbon (right).

Definition 5.2.3. *A k-*ribbon *is a single or double ribbon of size k. A k-ribbon R is* removable *if $\lambda - R$ is also a shifted diagram.*

Proposition 5.2.4. *For any removable k-ribbon R on λ, there is no cell on R strictly to the right or strictly below $H(R)$.*

Proposition 5.2.5. *A shifted diagram λ has a removable single k-ribbon with diag$(H(R)) = m$ if and only if it has a part of size m, no part of size $m-k$ and $m \geq k$. If it exists, it is the unique ribbon R where $\lambda - R$ has the part m replaced with $m - k$. Furthermore, λ has a removable double k-ribbon with diag$(H(R)) = a$ if and only if $a < k$ and λ has parts of sizes a and $k - a < a$. It is the unique ribbon R where $\lambda - R$ is λ with the parts of size a and $k - a$ removed.*

Proof. If λ has a part of size m and no part of size $m-k$, then removing the outermost boxes from diagonals m to $m-k+1$ gives us a removable ribbon of size k, with $diag(H(R)) = m$. In particular, if $lambda_i$ is the smallest part greater than $m-k$ and $\lambda_j = m$, this ribbon decreases the length of λ_j to λ_{j-1}, λ_{j-1} to λ_{j-2} and so on, ending up with reducing the length of λ_i to $m-k$. Conversely, if R is a removable ribbon with $diag(H(R)) = m$, $\lambda \backslash R$ is itself a skew-shifted diagram, which means there is no box on λ above $T(R)$ or to the left of $H(R)$, and R contains the outermost box of each diagonal from m to $m-k+1$. As $H(R)$ is at the end of a row, and has diagonal value m, λ contains a part of size m. Similarly, as $T(R)$ is on a row with at least $m+1$ boxes, and as the row above has no box above $T(R)$ it has less than m boxes, so λ has no part of size m. The case for the double ribbon follows as the double ribbon is a union of two single ribbons, and a ribbon of size a with $diag(H(R)) = a$ will have its tail on the main diagonal. □

A corollary of this proposition is that no shape can have a removable double ribbon of size (t, t).

Theorem 5.2.6. *A shifted diagram λ admits no removable k-ribbon iff it has no cells with hook length equal to k.*

In this case, we call λ a *k-core*.

Proof. We claim that there is a bijection between removable k-ribbons and cells with hook length equal to k, where cells in the unshifted part correspond to single ribbons and cells in the shifted part correspond to double ribbons. Under this bijection it is clear that if a diagram admits no k ribbons, it can not have a cell with hook length k and vice versa.

Let C have hook length equal to k. First let us look at the case when C is in the unshifted part. Let R be the single k-ribbon consisting of the outermost cell on each diagonal that the hook of C passes. As the head and tail of R are the endpoints of the hook of C, R is removable. Conversely, if R is a single k-ribbon, the cell on the row of $H(R)$ and the column of $T(R)$ has hook length k and is on the unshifted part.

Now let us assume C is a cell in the shifted part, so that its hook includes the row above the column of C. Assume this row has length t. This means C is on a row of size $t-k$, and by Proposition 5.2.4 the shape has a unique removable double ribbon of size $(t, k-t)$. Conversely, if R is a removable double ribbon of size $(t, k-t)$ with $t < t-k$, the diagram has rows i and j with sizes t and $t-k$. The cell on row j and column right below row i falls on the shifted part and has hook length k. □

2.1 The k-Abacus Correspondence

In this section, we will show that the ribbons on shifted tableaux can be expressed using the k-abacus notation of James and Kerber [16]. A k-abacus consists of runners labeled by $1, 2, 3, \ldots, k$, and numbers placed on these runners as follows:

1	2	3	$\cdots$	k
1	2	3	$\cdots$	k
k+1	k+2	k+3	$\cdots$	2k
2k+1	2k+2	2k+3	$\cdots$	3k
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

Two runners i and j are called *k-conjugate* if $i + j = k$. To any shifted diagram $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ we will associate the k-abacus with beads on positions $\lambda_1, \lambda_2, \dots, \lambda_n$. For example, for the diagram $\lambda = (16, 11, 10, 9, 8, 7, 4, 3, 1)$ we get the 5-abacus:

a_1	a_2	a_3	a_4	a_5
●	○	●	●	○
○	●	●	●	●
●	○	○	○	○
●	○	○	○	○
○	○	○	○	○
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

Given a strict partition λ, we identify each runner a_i in its abacus with a shifted shape $\alpha^{(i)}$, by treating the runners as the abacus of a shifted 1-core. More precisely, $\alpha^{(i)}$ will be the shifted shape $(\alpha_1^{(i)}, \alpha_2^{(i)}, \dots, \alpha_t^{(i)})$ where $k(\alpha_1^{(i)} - 1) + i, \dots, k(\alpha_t^{(i)} - 1) + i$ are the parts of λ that are equal to i modulo k. We will call the k-tuple $(\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(n)})$ the abacus representation λ. The 5-abacus representation of the example $(16, 11, 10, 9, 8, 7, 4, 3, 1)$ with the abacus above is $((4, 3, 1), (2), (2, 1), (2, 1), (2))$.

There are two types of moves allowed on the k-abacus of λ [33]:

- *Type I Move:* Sliding one bead one position higher in its runner if that position is unoccupied, or removing a bead from the top row of column k

- *Type II Move:* Removing two beads from the first row, if they are on two conjugate runners.

After a move on the k-abacus, we get a new shifted diagram $\lambda* \subset \lambda$. A *Type I* move corresponds to replacing a part of size $m+k$ with one of size m, whereas a *Type II* move corresponds to removing two parts of sizes adding up to k. By Proposition 5.2.5, we have the following correspondence:

Theorem 5.2.7. *Making a move on the k-abacus of λ is equivalent to removing a k-ribbon from λ. In particular,*

1. Single-Ribbon Correspondence: *Making the Type I move from position $m+k$ to position m is equivalent to removing a single-ribbon A with diag$(H_A) = m+k$ and diag$(T_A) = m+1$ (where the case $m = 0$ is removing a bead from the top row of column k) .*

2. Double-Ribbon Correspondence: *Making the Type II move removing top beads t and $k-t$ from conjugate runners a_t and a_{k-t} equivalent to removing a double-ribbon of size $(t, k-t)$.*

This theorem implies that the k-core of a shifted diagram corresponds to the k-core of the abacus. As the final state of the abacus is independent of the order of the moves [16], we have the following corollary.

Corollary 5.2.8. *The k-core of a shifted diagram is unique.*

2.2 Shifted Ribbon Tableaux

Definition 5.2.9. *A* standard k-ribbon tableau *of shape λ is a sequence of shifted diagrams $\lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(n)} = \lambda$, where each $A_i = \lambda^{(i)}/\lambda^{(i-1)}$, $i = 1, 2..n$ is a k-ribbon, and $\lambda^{(0)}$ is a k-core. For each i, we label ribbon A_i with i.*

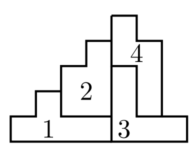


Figure 5.4: A 5-Ribbon Tableau with no 5-core of shape $\lambda = (7, 5, 4, 3, 1)$.

Definition 5.2.10. *A skew-shifted diagram S on a shifted diagram λ is called a* horizontal k-ribbon strip *(resp.* vertical k-ribbon strip*) if there exists a sequence of shifted diagrams $\lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(t)} = \lambda$, where:*

1. *Each $R_i := \lambda^{(i)} \setminus \lambda^{(i-1)}$ is a k-ribbon.*

2. *$H(R_i)$ is strictly to the right of (resp. strictly above) $H(R_{i-1})$ for each i.*

3. *$S = \bigcup_{i=1}^{n} R_i = \lambda \setminus \lambda^{(0)}$.*

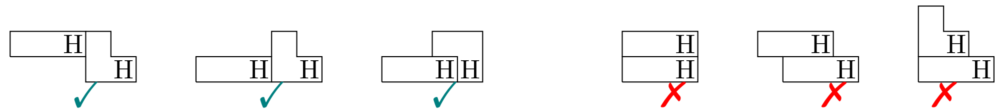


Figure 5.5: Horizontal strip examples and non-examples.

Definition 5.2.11. *A* semi-standard shifted k-ribbon tableau *is given by a sequence* $\lambda^{(0)} \subset \lambda^{(1')} \subset \lambda^{(1)} \subset \lambda^{(2')} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(n)} = \lambda$*, where:*

- $\lambda^{(0)}$ *is the k-core of* λ*.*
- *Each* $\lambda^{(i)} \setminus \lambda^{(i')}$ *is a (possibly empty) horizontal k-strip.*
- *Each* $\lambda^{(i')} \setminus \lambda^{(i-1)}$ *is a (possibly empty) vertical k-strip.*

We number the ribbons on the strip $\lambda^{(i)} \setminus \lambda^{(i')}$ *by i and the ribbons forming the strip* $\lambda^{(i')} \setminus \lambda^{(i-1)}$ *by i' for each* $i = 1, 2, \ldots, n$*.*

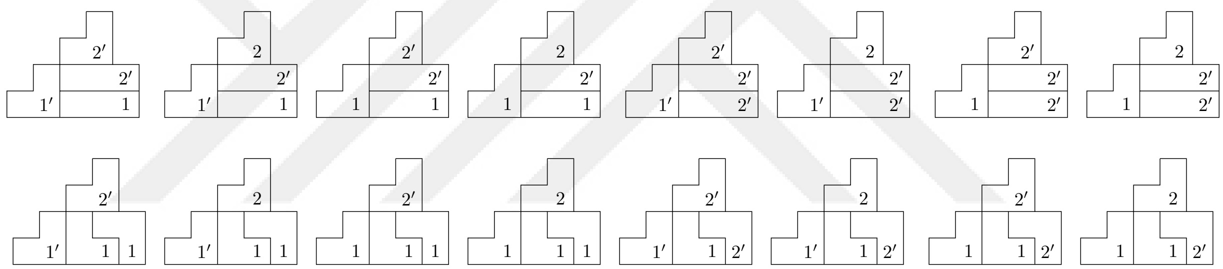


Figure 5.6: Semi-standard 5-ribbon fillings of $(5, 4, 2, 1)$ with numbers ≤ 2.

Definition 5.2.12. *For a shifted shape λ, we define its k-ribbon Q- and P-functions as follows:*

$$RQ_\lambda^{(k)}(X) = \sum_{S \in SSShT^{(k)}(\lambda)} X^{|S|}, \qquad RP_\lambda^{(k)}(X) = \sum_{S \in SSShT^{*(k)}(\lambda)} X^{|S|}$$

where $SSShT^{(k)}(\lambda)$ is the set of semi-standard shifted k-ribbon tableaux of shape λ, and $SSShT^{*(k)}(\lambda)$ is its subset consisting of tableaux with no marked entries on the ribbons that have boxes on the main diagonal.

The example illustrated in Figure 5.6 gives the following ribbon Q- and P-functions restricted to two variables:

$$\begin{aligned} RQ^{(3)}_{(5,4,2,1)}(x_1, x_2) &= 4x_1^3x_2 + 8x_1^2x_2^2 + 4x_1x_2^3 = Q_{3,1}(x_1, x_2) \\ RP^{(3)}_{(5,4,2,1)}(x_1, x_2) &= x_1^3x_2 + 2x_1^2x_2^2 + x_1x_2^3 = P_{3,1}(x_1, x_2) \end{aligned}$$

Note that in this example, the ribbon Q- and P-functions are Schur Q and P-positive respectively. One of our main results in this paper will be to show that the positivity holds true in general, and give a formula for the ribbon Q-functions in terms of Schur's Q-functions.

Proposition 5.2.13. *All the shifted k-ribbon tableux of shape λ have the same number of ribbons that have a box on the main diagonal, called the* k-length *of λ, denoted $\ell^{(k)}(\lambda)$. Consequentally, the Q and P k-ribbon functions for λ are related by a scalar:*

$$RQ_{\lambda}^{(k)}(X) = 2^{\ell^{(k)}(\lambda)} RP_{\lambda}^{(k)}(X).$$

Proof. By Theorem 5.2.7, the ribbons that have a box on the main diagonal correspond to the moves on the abacus where beads are removed. The total number is independent of the order of the moves. In fact, if we denote the number of beads on runner i by $|a_i|$ then:

$$\ell^{(k)}(\lambda) = |a_k| + \sum_{i<(k/2)} max\{|a_i|, |a_{k-i}|\}.$$

This means that for every S in $sSShT^{*(k)}(\lambda)$, we have $2^{\ell^{(k)}(\lambda)}$ ways of marking the $\ell^{(k)}(\lambda)$ ribbons with boxes on the main diagonal to get tableaux in $sSShT^{(k)}(\lambda)$. □

3 Folded Tableaux

In this section, we will introduce an operation combining two shifted shapes to get an unshifted shape with a specialized diagonal, which we will call a folded diagram. We will later use the folded diagrams, along with their corresponding tableaux in describing the k-quotient of a shifted shape. We will use the notation δ_n to denote the staircase partition $(n, n-1, \ldots, 1)$ of size n.

Definition 5.3.1. *A* folded diagram $\Gamma = (\gamma, \mathfrak{d})$ *is an unshifted diagram γ called the* underlying shape *of Γ along with a specialized main diagonal $\mathfrak{d}$ which does not necessarily intersect γ.*

Definition 5.3.2. *Let α and β be shifted shapes (or equivalently strict partitions). Denote $n = min\{ht(\alpha), ht(\beta)\}$. Their* combination*, which will be denoted by $\alpha \diamond \beta$ will be the folded diagram we obtain by:*

- *Step I: If one of the shapes has height m larger than n, delete its $m-n$ leftmost columns, so that both shapes have the same number of boxes in their main diagonals.*
- *Step II: Transpose α.*
- *Step III: Paste the two diagrams together along their main diagonals, and label this diagonal $\mathfrak{d}$.*

Example:

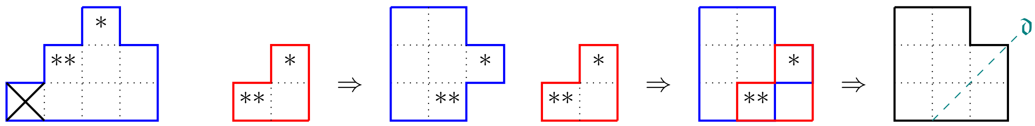


Figure 5.7: The combination of $\alpha = (4,3,1)$ and $\beta = (2,1)$.

Proposition 5.3.3. *Any given folded diagram $(\gamma, \mathfrak{d})$ can be uniquely described as a combination of two shifted shapes.*

Proof. Let $\gamma^T = (\gamma_1^T, \ldots, \gamma_s^T)$ denote the transpose of γ. If $\mathfrak{d}$ is k units to the right of the main diagonal and going through $t > 0$ boxes, then $(\gamma, \mathfrak{d}) = \alpha \diamond \beta$ where $\alpha = \delta_{k+t} + (\gamma_{t+1}, \gamma_{t+2}, \ldots, \gamma_n)^T$ and $\beta = \delta_t + (\gamma_{k+t+1}^T, \gamma_{k+t+2}^T, \ldots, \gamma_s^T)^T$.

If $\mathfrak{d}$ is $k \geq 0$ units to the right of the bottom left corner and outside the shape (the case $k < 0$ is symmetrical) we have $(\gamma, \mathfrak{d}) = \alpha \diamond \beta$ where $\alpha = \delta_k + \gamma^T$ and β is empty.

The process can be inverted by transposing the cells weakly to the left of $\mathfrak{d}$ and completing them to a shifted shape to get α, and completing the cells weakly to the right of $\mathfrak{d}$ to a shifted shape get β (Follow Figure 5.7 right to left). □

Definition 5.3.4. *A* standard folded tableau *of shape $\Gamma = (\gamma, \mathfrak{d})$ is a filling of the cells of γ using numbers $1, 2, \ldots, n$ each exactly once, with numbers increasing left to right and bottom to top.*

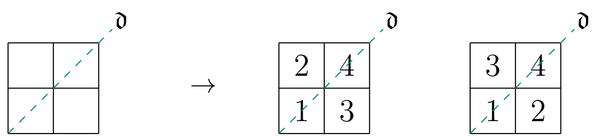


Figure 5.8: Standard Folded Tableaux of shape $\Gamma = ((2,2), \mathfrak{d})$.

Definition 5.3.5. *A* semi-standard folded tableau *of shape $\Gamma = (\gamma, \mathfrak{d})$ is a semi-standard filling of the skew-shifted diagram γ with the rules inverted weakly above the specialized diagonal (above $\mathfrak{d}$ we can have no repeated unmarked numbers on the same row, and no two repeated marked numbers on the same column).*

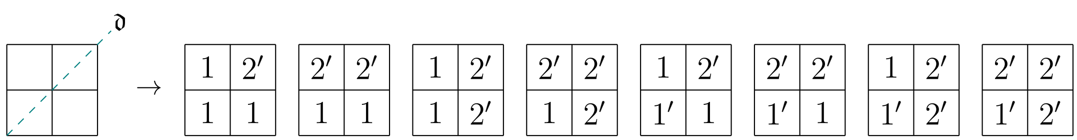


Figure 5.9: Semi-standard folded tableaux of shape $\Gamma = ((2,2), \mathfrak{d})$, using numbers $\leq 2'$.

Definition 5.3.6. *We define the folded P- and Q-functions as follows.*

$$Q^f_{(\gamma,\mathfrak{d})}(X) = \sum_{S \in \mathrm{SSFT}(\gamma,\mathfrak{d})} X^{|S|}$$

$$P^f_{(\gamma,\mathfrak{d})}(X) = 2^{-size(\mathfrak{d})} Q^f_{(\gamma,\mathfrak{d})}(X)$$

where $SSFT(\gamma, \mathfrak{d})$ denotes the set of all semi-standard folded tableaux of shape $\Gamma = (\gamma, \mathfrak{d})$.

The folded shape $((2,2), \mathfrak{d})$ illusturated in Figures 3 and 5.9 has the following folded P- and Q-functions:

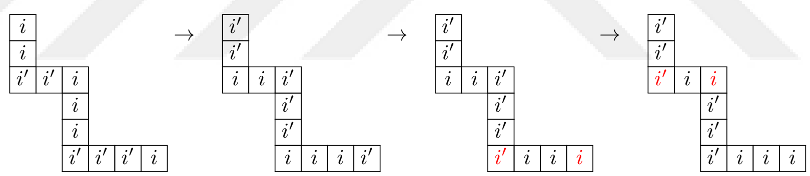
$$P^f_{(22,\mathfrak{d})}(X) = m_{31}(X) + 2m_{22}(X) + 4m_{211}(X) + 8m_{1111}(X) = s_{31}(X) + s_{22}(X) + s_{2111}(X) = P_{(31)}(X)$$

$$Q^f_{(22,\mathfrak{d})}(X) = 2^{ht(\mathfrak{d})} P^f_{(22,\mathfrak{d})}(X) = 4(s_{31}(X) + s_{22}(X) + s_{2111}(X)) = Q_{(31)}(X)$$

Theorem 5.3.7. *The function $Q^f_{(\gamma,\mathfrak{d})}(X)$ is independent of the choice of the main diagonal $\mathfrak{d}$.*

Proof. We will prove this by giving a bijection weight preserving between SSFT$(\gamma, \mathfrak{d})$ with semi-standard fillings of the skew-shifted shape γ.

Let $S \in SSFT(\gamma, \mathfrak{d})$. We start with inverting the markings of all cells weakly above the main diagonal. Each connected piece weakly above the main diagonal containing only i or i's forms a ribbon, with at most one i' on each row (on the rightmost cell), and at most one i on each column (on the lowest cell). The next step is to go from bottom to the top in the ribbon, flipping i' with the leftmost i of each row and flipping the i with the highest i' on each column. As the algorithm corrects the ordering when the process is repeated for all i, we get with a semi-standard skew-shifted filling of γ.



The process can be inverted by inverting the markings weakly above the main diagonal again, and this time working our way from top to bottom on each ribbon, flipping any i' with the lowest i on columns and flipping any i with the rightmost i' in rows. □

Corollary 5.3.8. *The folded Q-functions are Schur Q-positive. In fact*

$$Q^f_{(\gamma,\mathfrak{d})}(X) = Q_{\lambda/\delta_n}(X) = \sum_{\varepsilon} f^{\lambda}_{\varepsilon,\delta_n} Q_{\varepsilon}(X),$$

where λ is a shifted shape with $\lambda/\delta_n = \gamma$ and $f^{\lambda}_{\varepsilon,\delta_n}$ are the non-negative integers defined by:

$$P_{\varepsilon} P_{\delta_n} = \sum_{\lambda} f^{\lambda}_{\varepsilon,\delta_n} P_{\lambda}.$$

Proof. By Theorem 5.3.7, the folded Q-function is independent of the main diagonal, so we can take $\mathfrak{d}$ to be completely above the shape. In this case the semi-standard fillings of $(\gamma, \mathfrak{d})$ are exactly the semi-standard fillings of the diagram γ seen as a skew-shifted shape. □

As the folded P-function depends on the size of the main diagonal, it is not independent of $\mathfrak{d}$. Instead, it tells us that $Q^f_\gamma(X)$ is divisible by 2^d, where d is the size of the longest diagonal on γ.

Corollary 5.3.9. *An unshifted shape and its conjugate have the same folded Q-function. In particular, for two shifted shapes α and β, $Q^f_{\alpha\diamond\beta}(X) = Q^f_{\beta\diamond\alpha}(X)$.*

Proof. For an unshifted shape γ, the conjugation operation gives a bijection of folded tableaux $(\gamma, \mathfrak{d})$ with $\mathfrak{d}$ above the shape and folded tableaux $(\gamma^T, \mathfrak{d}^T)$ with $\mathfrak{d}^T$ below the shape. As the folded Q-function is independent of the placement of the specialized diagonal, we have: $Q^f_\gamma(X) = Q^f_{\gamma^T}(X)$. □

4 Quotients of Ribbon Tableaux

In this section, we will introduce the k-quotient for a shifted diagram, and we will give a bijection between the k-ribbon tableaux and the fillings of its k-quotient. Our definition of the k-quotient extends the one given by Morris and Yaseen in [32] by specialized diagonals which we will use for a direct bijection between semi-standard k-ribbon tableaux and semi-standard fillings of its k-quotient.

Definition 5.4.1. *The k-quotient of a shifted shape λ with k-abacus representation $(\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)})$ will consist of $\lfloor k/2 \rfloor$ folded shapes and one shifted shape, defined as follows:*

$$\overline{\Phi^k}(\lambda) = \begin{cases} (\alpha^{(1)} \diamond \alpha^{(k-1)}, \alpha^{(2)} \diamond \alpha^{(k-2)}, \ldots, \alpha^{((k-1)/2)} \diamond \alpha^{((k+1)/2)}, \alpha^{(k)}) & k \text{ odd} \\ (\alpha^{(1)} \diamond \alpha^{(k-1)}, \alpha^{(2)} \diamond \alpha^{(k-2)}, \ldots, \alpha^{(k/2-1)} \diamond \alpha^{(k/2+1)}, \alpha^{(k/2)} \diamond \varnothing, \alpha^{(k)}) & k \text{ even} \end{cases}$$

The strict partition $\lambda = (16, 11, 10, 9, 8, 7, 5, 4, 3, 1)$ has the 5-quotient:

$$\overline{\Phi^5}(\lambda) = ((4,3,1) \diamond (2,1), (2) \diamond (2,1), (2,1)) = (((3,3,2), \mathfrak{d}_1), ((3), \mathfrak{d}_2), (2,1)).$$

where $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are the specialized diagonals given by the combination operation.

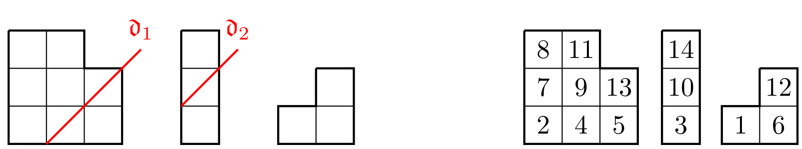


Figure 5.10: The 5-quotient of $\lambda = (16, 11, 10, 9, 8, 7, 4, 3, 1)$ and a standard filling.

We call a simultaneous semi-standard filling of the $\lfloor k/2 \rfloor$ folded shapes and the shifted shape a *semi-standard filling* of the k-quotient. If this filling uses each number from 1 to n exactly once, unmarked, it will be called a *standard filling.* Next, we develop a bijection between the fillings of the ribbon tableaux with the fillings of its quotient.

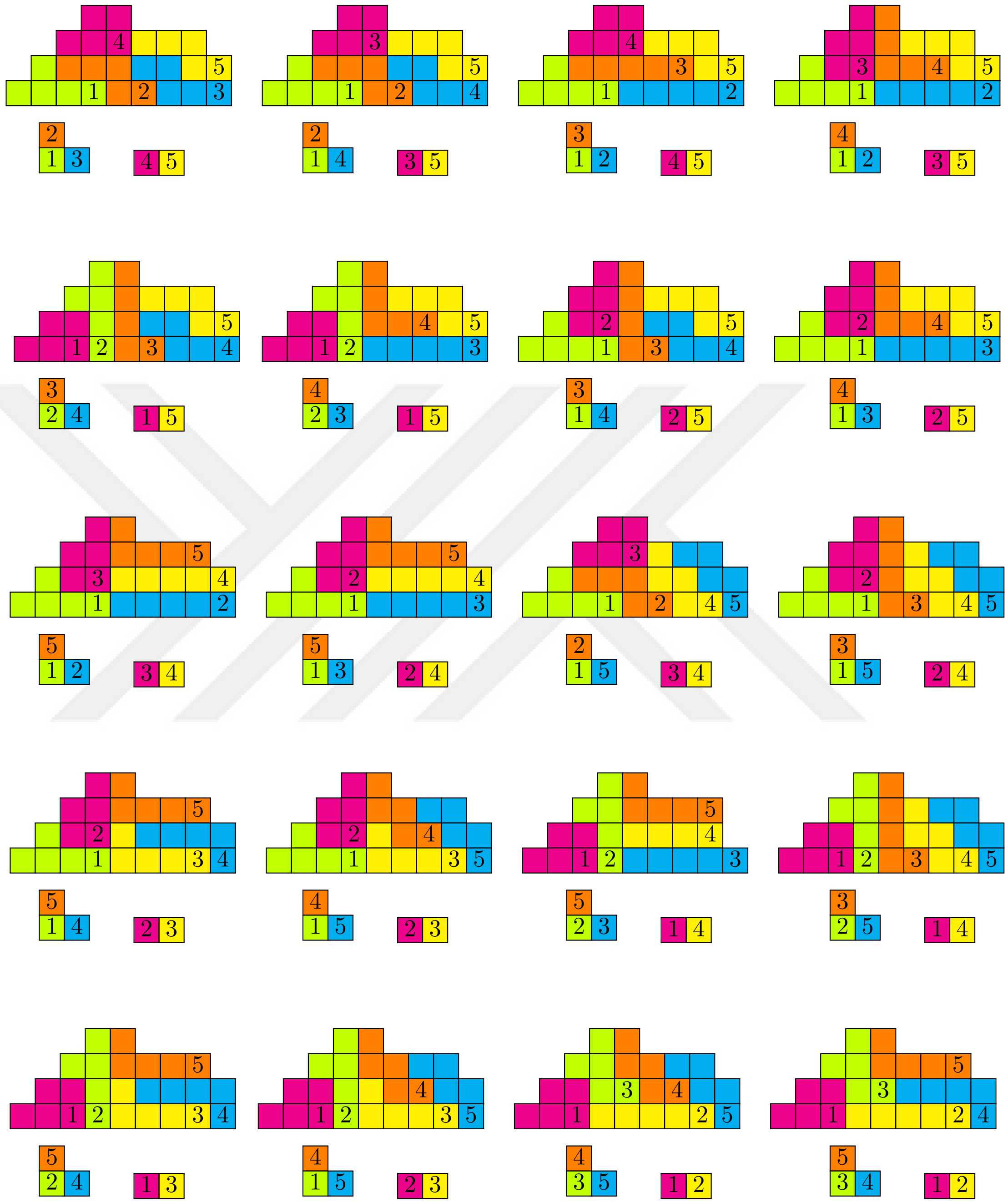


Figure 5.11: Correspondence between the standard 5-ribbon tableaux of shape $(9, 8, 6, 2)$ and the standard fillings of its quotient$\{(2, 1), (2), \varnothing\}$.

Theorem 5.4.2. *There is a bijection Φ_λ^k between standard k-ribbon tableaux of shape λ and standard fillings of its k-quotient preserving diagonal values (two ribbons that have the same diagonal value will be mapped to the same shape and diagonal in the quotient).*

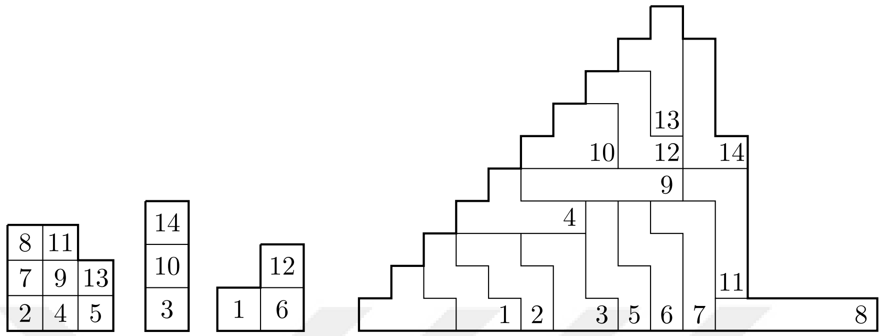


Figure 5.12: A 5-ribbon tableau of shape $\lambda = (16, 11, 10, 9, 8, 7, 4, 3, 1)$ with the corresponding filling of the 5-quotient.

An instance of this bijection can be seen in Figure 5.12. Notice that the ribbons 1 and 12 that are matched to the main diagonal of $\alpha^{(5)}$ are exactly those whose heads lie on diagonal 5, and the second diagonal of $\alpha^{(5)}$ is matched to ribbon 6 with diagonal value 10.

Proof. Consider a k-ribbon tableau T of shape λ with abacus representation $(\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)})$. As each ribbon corresponds to a move in the abacus representation of λ, T uniquely corresponds to a sequence of moves from $(\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)})$ to the k-core of λ. As we can move independently on each runner pair (a_i, a_{k-i}) and on a_k, it will suffice to match the moves on a_k moves to shifted tableaux of shape $\alpha^{(k)}$, and the moves on runner pairs (a_i, a_{k-i}) to moves on $\alpha^{(i)} \diamond \alpha^{(k-i)}$for each i.
Claim: There is a bijection between sequences of moves from a_k to the empty runner and standard shifted tableaux of shape $\alpha^{(k)}$, where a move from row d to $d-1$ on the abacus corresponds to a box on diagonal d.
Proof: A bead on the ith row of runner a_k will make a total of i moves, $i-1$ to one row higher, and one last move to be removed. We map these moves to a row of i boxes so that the move from position j to $j-1$ will correspond to a box on diagonal j, and the removal move will correspond a box on the main diagonal. For example, in Figure 5.13 the top bead (shown in teal) can make only one move which is matched to a row of size one on $\alpha^{(5)}$. The lower bead (shown in magenta) has two moves to make which are matched to a row of size two. In general, if a_k has beads on positions $i_1 > i_2 > \cdots > i_t$ we map the moves to the shifted diagram $\alpha^{(k)} = (i_1, i_2, \ldots, i_t)$. Let us number the moves in decreasing order with numbers from 1 to $|\alpha^{(k)}|$. This will give us a filling of $\alpha^{(k)}$, with the conditions that beads can only move to unoccupied positions (meaning rows need to increase left to right), and a bead can only move higher (meaning columns increase bottom to top). Note that these conditions exactly correspond to the tableaux conditions.

Now, we can turn our attention to runner pairs a_i, a_{k-i}.

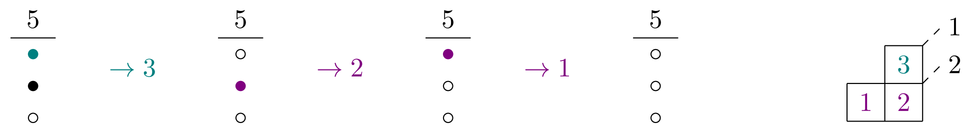


Figure 5.13: Moves on runner a_5 can be matched to a standard filling of the shifted diagram $\alpha^{(5)} = (2, 1)$.

Claim: There is a bijection between sequences of moves from the pair of runners (a_i, a_{k-i}) to the abacus core and standard unshifted tableau of shape $\alpha^{(i)} \diamond \alpha^{(k-i)}$, where moves removing beads are mapped to the specialized diagonal of $\alpha^{(i)} \diamond \alpha^{(k-i)}$, and moves on runner a_i (resp. a_{k-i}) from row r to $r - 1$ are mapped to the diagonal d units to the left (resp. right).

Proof: The move sequences on each runner can be matched to a shifted tableau of corresponding shape as in Claim 1, except now we have an additional constraint: To remove a bead from the first row one runner, we must simultaneously remove a bead from the first row of the second runner. This implies that the main diagonals of the two shapes must contain the same numbers, and they will be on the main diagonal, so it makes sense to transpose on of the shapes and paste them along main diagonals. An example of this can be

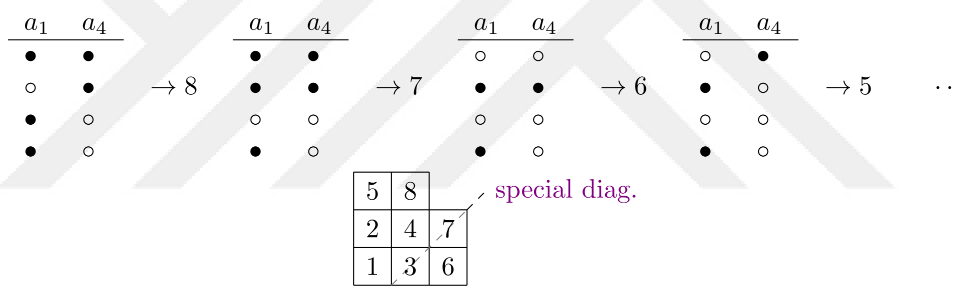


Figure 5.14: Moves on runners a_1 and a_4 give a standard filling of the folded diagram $\alpha^{(1)} \diamond \alpha^{(4)}$.

seen in Figure 5.14. Note that the moves on a_1 are to the left of the special diagonal, and the moves on a_4 are to the right. Move 7 of removing two top row beads simultaneously falls on the special diagonal.

Also note that if one runner has q more beads than the other one, these can not be removed, and the moves which depend on the removal of these beads can not be made, which means the the subdiagram of shape δ_q inside the larger shape will be left empty. Deleting the corresponding boxes gives us a filling of $\alpha^{(i)} \diamond \alpha^{(k-i)}$.

When k is even, we can move ribbons up on runner $a_{k/2}$ but not remove them, as if it has a conjugate runner with no beads, so any remark we made on $\alpha^{(i)} \diamond \alpha^{(k-i)}$ above automatically applies to $\alpha^{(k/2)} \diamond \varnothing$. □

Remark 5.4.3. *The correspondence of diagonals gives us a way of labeling the diagonals of the quotient to match the values of the original shape. This way, the shifted shape* $\alpha^{(k)}$ *will have diagonals* $0, k, 2k \ldots$ *and the folded shapes* $\alpha^{(i)} \diamond \alpha^{(k-i)}$ *will have diagonals:* $\{\ldots i + 2k, i + k, k - i, 2k - i, 3k - i \ldots\}$ *where the specialized diagonal* $\mathfrak{d}_i$ *will have the diagonal value* $k - i$.

Note that the diagonal values $i < (k-1)/2$ do not appear. The reason of this is our convention of setting the head of double ribbon to be the head of the larger piece.

Corollary 5.4.4. *A k-ribbon R has a box on the main diagonal of λ if and only if $diag(R) \leq k$.*

Definition 5.4.5. *Consider a semi-standard k-ribbon tableaux T of shape λ, with $|\lambda| = n$. The* standardization *of T, denoted $St(T)$ is the standard k-ribbon tableaux of shape λ that we obtain by the following numbering:*

- *We number the boxes in the order $1' < 1 < 2' < 2 < \cdots$*
- *If there is more than one box of label i, we order them so that their diagonal values will be increasing.*
- *If there is more than one box of label i', we order them so that their diagonal values will be decreasing.*

Proposition 5.4.6. *$St(T)$ is well defined.*

Proof. As ribbons labeled i form a horizontal strip, they can be removed in the order diagonals are decreasing. Similarly, ribbons labeled i' form a vertical strip and can be removed in the order their diagonals are increasing. □

Using the labeling of the diagonals from Remark 5.4.3, we can also do the same standardization operation on the semi-standard fillings of the k-quotient.

Proposition 5.4.7. *We can extend Φ^k_λ to a weight preserving bijection between semi-standard k-ribbon tableaux of shape λ, and semi-standard fillings of its k-quotient.*

Proof. Let T be a semi-standard k-ribbon tableaux of shape λ, given by the sequence $\lambda_0 \subset \lambda_{1'} \subset \lambda_1 \subset \lambda_{2'} \subset \lambda_2 \subset \cdots \subset \lambda_t = \lambda$ of shifted diagrams. As our definition of standardization respects the inclusion order, $St(T)$ restricted to any λ_i gives a standardization of λ_i. The same is true for $\lambda_{i'}$. Let us apply the Φ^k_λ to the standardization of T. This gives us a bijection ϕ between ribbons of T and the boxes on the k-quotient. This also can be restricted to the subdiagrams λ_i and λ'_i, giving a sequence $\Phi^k(St(\lambda_0)) \subset \Phi^k(St(\lambda_{1'})) \subset \Phi^k(St(\lambda_1)) \subset \cdots \subset \Phi^k(\lambda_t) = \Phi^k(St(\lambda))$. Here, the subset relation is defined pointwise in the $(k+1)$-tuples of quotient diagrams.
Claim: The filling of the k-quotient obtained by this is a semi-standard filling, and is equal to $\Phi^k_\lambda(T)$ if the filling T is standard.
Proof: The second part of the claim follows from the definition of $\Phi^k_\lambda(T)$. For the first part, we need to show that the filling of each $a^{(i)} \diamond a^{(k-i)}$ gives a semi-standard folded shape and the filling of $a^{(k)}$ gives a semi-standard shifted shape. Let us look first at the case of $a^{(k)}$. Let us assume the ribbons R_1 and R_2 are both labeled j and the corresponding boxes B_1 and B_2 in the quotient are on the same column, with B_1 higher. Then we have $diag(B_1) < diag(B_2)$, implying $diag(R_1) < diag(R_2)$, which can not happen as the standardization algorithm would give B_1 a smaller number than B_2. they form a horizontal strip and have different diagonal values. Let us assume $diag(R_1) < diag(R_2)$. Then, if the boxes corresponding to R_1 and

R_2 on the quotient are in the same column, the box for R_2 is higher. Symmetrically, no two boxes marked j' can be on the same row, so we indeed have a semi-standard filling of $a^{(k)}$. Now consider the boxes marked j on $a^{(i)} \diamond a^{(k-i)}$ in some i. As they come from the difference $\Phi^k(St(\lambda_j))/\Phi^k(St(\lambda_{j'}))$, they form a skew shape. Also, the boxes that are labeled j to the right of the main diagonal form a horizontal strip by the same reasoning in the case of λ. The boxes labeled j to the left of the main diagonal form a vertical strip, as we have the inverted version of the same rules. The j' case is again symmetrical.

Now let us define the inverse of this operation. Given a semi-standard filling $\bar{T}$ of the k-quotient, as the k-quotient has the diagonal values induced by λ, we can apply the same standardization algorithm to the quotient, to get a standard filling $St(\bar{T})$ of the quotient. Applying Φ_λ^{k-1} to this filling gives a standard filling of λ. We can use this bijection between boxes of the quotient and ribbons of λ to carry the labels in $\bar{T}$ to the corresponding ribbons in λ. Note that, this inverts the above operation by definition.

Claim: The inverse operation takes $\bar{T}$ to a semi-standard filling of λ.

Proof: Let R and S be two ribbons marked j on λ. We will show that they form a horizontal strip. The case of j' is symmetrical. First note that we can not have $diag(R) = diag(S)$, as that would imply the corresponding boxes in the quotient are both in the same $a^{(i)} \diamond a^{(k-i)}$ (or both in $a^{(k)}$) on the same diagonal, which is not possible. Let us assume, without loss of generality, that $diag(R) > diag(S)$. Then, in the standardization, the label of R will be higher than the label of S, meaning R can be removed before S: $H(R)$ can not be below $H(S)$ in the same column. In this case $H(R)$ is strictly to the right of $H(S)$ implying they form a horizontal strip, as otherwise $H(R)$ would be strictly below $H(S)$ in a row strictly to the left, giving us no possible way to label the ribbon containing the box C in the same row as $H(R)$ and the same column as $H(S)$. □

This bijective relationship shows that the k-ribbon Q-function is equal to the product of the Q-functions of its quotient:

Theorem 5.4.8. *The k-ribbon Q-function of a shifted shape λ with k-abacus representation $(\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)})$ has the following expansion in terms of Schur's Q-functions:*

$$RQ_\lambda^{(k)}(X) = Q_{a^{(k)}}(X) \prod_{i \leq \lfloor k/2 \rfloor} Q_{\mu_i}(X) \tag{4.1}$$

where μ_i is the underlying skew-unshifted shape of $a^{(i)} \diamond a^{(k-i)}$ if $i < k/2$ and $a^{(k/2)} \diamond \varnothing$ if $i = k/2$.

Corollary 5.4.9. *The Q-ribbon functions expand positively into Schur's Q-functions.*

Proof. Follows from the last theorem and the Schur Q-positivity of the skew-shifted Schur Q-functions (Theorem 2.3.1). □

Note that the Schur's Q-functions are themselves k-ribbon Q-functions for any k, as $k\lambda = (k\lambda_1, k\lambda_2, \ldots, k\lambda_n)$ has $RQ_{k\lambda}^{(k)}(X) = Q_\lambda$.

5 Peak Functions of Ribbon Tableaux

The *reading word* of a shifted k-ribbon tableau is a reading of the labels on the heads of the ribbons, left to right, top to bottom.

Definition 5.5.1. *A* marked standard shifted k-ribbon tableau T' *of shape* λ *is defined to be a standard shifted k-ribbon tableau* T *of shape* λ *together with a subset* M *of* $[n]$ *determining the marked coordinates.*

We identify $T' = (T, M)$ with the diagram of T where the label of R_i is replaced with i' for all i in M, as in Figure 5.15. We also let $\mathrm{Mark}(T) = \{(T, M) \mid M \subset [n]\}$ denote the set of the 2^n marked versions of T.

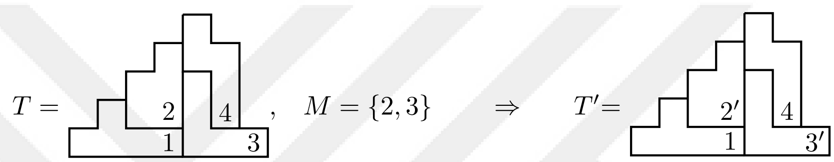


Figure 5.15: The marked 5-ribbon tableau T' has reading word $2'413'$.

Theorem 5.5.2. *The k-ribbon Q-function of a shifted shape* λ *we have can be written in terms of descent functions and peak functions as follows:*

$$RQ^{(k)}_{\lambda}(X) = \sum_{T' \in SShT\pm^{(k)}(\lambda)} F_{\mathrm{desc}(T')}, \qquad RQ^{(k)}_{\lambda}(X) = \sum_{T \in SShT^{(k)}(\lambda)} 2^{|\mathrm{Peak}(T))|+1} G_{\mathrm{peakc}(T)}$$

where $SShT \pm^{(k)} (\lambda)$ *is the set of marked standard shifted k-ribbon tableaux of shape* λ*.*

We postpone the proof of Theorem 5.5.2 to the end of this section to develop some necessary machinery. A *run* of a subset D of $[n]$ is a maximal subset of consecutive numbers. We will denote by $\mathrm{Run}(D)$ the set of the runs of D. Note that D is the disjoint union of its runs. For example, $D = \{2, 3, 5, 8, 9, 10\} = \{2, 3\} \cup \{5\} \cup \{8, 9, 10\}$.

Proposition 5.5.3. *We can calculate the number of the peaks of a tableau* T *from its descent set as follows:*

$$|P| = \begin{cases} |\mathrm{Run}(\mathrm{Des}(T))| & 1 \notin \mathrm{Des}(T) \\ |\mathrm{Run}(\mathrm{Des}(T))| - 1 & 1 \in \mathrm{Des}(T) \end{cases}.$$

Proof. This follows from the fact that the elements j of the peak set satisfy $j \in D$ and $j - 1 \notin D$ for all $j > 1$, so that elements of the peak set are given by the smallest elements of each run, with the exception of the case if there is a run starting with 1. □

Lemma 5.5.4. *For any $T' \in \text{Mark}(T)$, the descent set of T' is independent of whether a given $i \leq n$ is marked if and only if:*

- $i > 1$ *with* $i - 1 \in \text{Des}(T)$, $i \notin \text{Des}(T)$ *or*
- $i = 1 \notin \text{Des}(T)$.

The number of such i is given by $|\text{Peak}(T)| + 1$.

Proof. The first part comes from Lemma 1.3.1. Note that as $\text{Des}(T) \subset [n-1]$, the number 1 larger than the highest descent satisfies this condition. The other numbers $i > 1$ that satisfy these conditions are the i one lower than the lowest number of a run of $\text{Des}(T)$. That means, if $1 \notin \text{Des}(T)$, we have as many as the number of runs of those. If $1 \in \text{Des}(T)$, we have $|\text{Run}(\text{Des}(T))| - 1$ of those and 1 is a descent. In both cases, there are $|\text{Peak}(T)| + 1$ numbers in total that satisfy the conditions. ☐

Proposition 5.5.5. *For any $T' \in \text{Mark}(T)$, we have* $\text{Spike}(T') \supset \text{Peak}(T)$.

Proof. Note that if $i \in \text{Peak}(T)$, we have $i \in \text{Des}(T)$ and $i - 1 \notin \text{Des}(T)$. For any given T' if i is unmarked on T', then $i \in \text{Des}(T')$ and $i - 1 \notin \text{Des}(T')$ so $i \in \text{Spike}(T')$. Otherwise i is marked, so that we have $i \notin \text{Des}(T')$ and $i - 1 \in \text{Des}(T')$, implying again that $i \in \text{Spike}(T')$. ☐

The proposition above shows that the descent map takes the elements of $\text{Mark}(T)$ to subsets D of $[n-1]$ with $\text{Spike}(D) \subset \text{Peak}(T)$. Next, we will show that this map is surjective. In fact, we will prove the stronger statement that the preimage of every element is of the same size.

Lemma 5.5.6. *Assume D is a subset of $[n-1]$ satisfying* $\text{Spike}(D) \supset \text{Peak}(T)$. *Then, there is a marked version T' of T such that* $\text{Des}(T') = D$.

Proof. Let us generate a marked version T' of T as follows: Starting with $i = 1$, at Step i we mark i if $i \in \text{Des}(T)$ and $i \notin D$, and we mark $i + 1$ if $i \notin \text{Des}(T)$ and $i \in D$ (marking the same number a second time has no effect). Then we move on to the next number, until we have gone through all $i \leq n - 1$.

Let us verify that the descent set of T' is indeed equal to D. For a fixed i assume $i \in \text{Des}(T)$. Then by Lemma 1.3.1 i is a descent of T' iff i is unmarked. Therefore, it is sufficent to show i is unmarked iff $i \in D$. If $i \notin D$, then we marked i on Step i, so $i \in \text{Des}(T')$. Otherwise $i \in D$, and we can only have marked i at step $i-1$. This implies $i-1 \notin \text{Des}(T)$ and $i-1 \in D$. This contradicts our assumption $\text{Spike}(D) \supset \text{Peak}(T)$ as i is a peak of T but not a spike of T. The case $i \notin \text{Des}(T)$ is similar. ☐

Proposition 5.5.7. *The descent map taking elements of* $\text{Mark}(T)$ *to subsets D of $[n-1]$ with* $\text{Spike}(D) \supset \text{Peak}(T)$ *is a 2^m to one cover, where* $m = |\text{Peak}(T)| + 1$.

Proof. The number of subsets D of $[n-1]$ with $\text{Spike}(D) \supset \text{Peak}(T)$ is given by $2^{n-1-|\text{Peak}(T)|}$. By Lemma 5.5.6, we know that the descent map is surjective. By Lemma 5.5.4, the preimage of each element under the descent map contains at least 2^m elements. $2^m \times 2^{n-1-|\text{Peak}(T)|} = 2^n$ which is the cardinality of $\text{Mark}(T)$, so the preimage of each element must contain exactly 2^m elements. ☐

Now we are ready to prove Theorem 5.5.2 from the beginning of the section.

Proof of Theorem 5.5.2. Let S be a semi-standard k-ribbon tableaux of shape λ. We have already defined the standardization of S (Definition 5.4.5). Let us denote by $St'(S)$ the marked standardization of S, which is simply standardization while keeping the marked boxes marked. We will show that there is a bijection $\phi_{T'}$ between semi-standard k-ribbon tableaux S that standardize to T' and the compositions C refining $\mathrm{Des}(T')$, satisfying $x^{|C|} = x^{|\phi_{T'}(S)|}$. This will imply:

$$\sum_{T' \in SShT\pm^{(k)}(\lambda)} F_{\mathrm{desc}(T')}(X) = \sum_{T'} \sum_{C \preceq (\mathrm{desc}(T'))} X^C = \sum_{T'} \sum_{C} x^{|\phi_{T'}^{-1}(C)|} = \sum_{S \in sSShT^{(k)}(\lambda)} X^{|S|} = RQ_\lambda^{(k)}(X)$$

where for a combination $C = (c_1, c_2, \ldots, c_t)$ we use X^C to denote $x_1^{c_1} x_2^{c_2} \cdots x_t^{c_t}$.

Assume S satisfies $St'(S)$=T'. We define $\phi_{T'}(S) = (i_1, i_2, \ldots)$ where i_m stands for the total number of boxes labelled m or m' on S. As S has n ribbons, $\phi_{T'}(S)$ will be a combination of n that satisfies $x^{|S|} = x^{|\phi_{T'}(S)|}$.

Claim: S refines $\mathrm{Des}(T')$.

Proof: Let S be a semi-standard filling with $St'(S) = T'$. Consider the pre-image of ribbon R_i (the unique ribbon labeled i or i' on T') under St'. We will denote the label of this ribbon in S by $St'^{-1}(i)$. To prove that S refines $\mathrm{Des}(T')$, it is sufficient to show that if i is a descent of T', then $St'^{-1}(i)$ and $St'^{-1}(i+1)$ are not both elements of $\{m, m'\}$ for any m (Note that, by the standardization algorithm, we will have $St'^{-1}(i) \leq St'^{-1}(i+1)$ in any case).

Let i be a descent of T'. By Lemma 1.3.1, there are two possiblities:

- Case 1, $i \in \mathrm{Des}(T)$ and i is not marked in T': Then $St'^{-1}(i)$ is an unmarked number m. $St'^{-1}(i) \geq m$, so it can not be m'. Assume it is also m. Then, we have two ribbons labeled m, but $diag(R_i) > diag(R_{i+1})$ by the definition of standardization (Definition 5.4.5).

- Case 2, $i \notin \mathrm{Des}(T)$ and $i+1$ is marked in T': This means $St'^{-1}(i+1)$ is a marked number m', and $St'^{-1}(i) \geq m'$ can not be m. It can not be m' either, because as in the first case, we get two ribbons labeled m' but $diag(R_i) < diag(R_{i+1})$, which as in Case 1, contradicts the definition of standardization.

Claim: For any combination C refining $\mathrm{Des}(T)$ there is a unique S that standardizes to T' with $\phi_{T'}(S) = C$

Proof: Let $C = (c_1, c_2, \ldots, c_T)$ be a combination of n that refines $\mathrm{Des}(T')$. We will define S by labeling the ribbons R_1 to R_{c_1} with 1, ribbons R_{c_1+1} to $R_{c_1+c_2}$ with 2, $R_{c_1+c_2+1}$ to $R_{c_1+c_2+c_3}$ with 3 and so on, and then marking the image of R_i iff i is marked in T. We need to show that this S is semi-simple, and it standardizes to T'. Uniqueness then, comes from the fact that the placement of the markings are preserved.

Assume ribbons R_i and R_{i+1} have the same unmarked label m. Then, $i \notin \mathrm{Des}(T')$ and i and $i+1$ are both not labeled in T', so we must have $i \notin \mathrm{Des}(T)$ by Lemma 1.3.1. That means, $diag(R_i) < diag(R_{i+1})$, so unmarked numbers are ordered so that their diagonals will increase in T'. Similarly, if R_i and R_{i+1} both have the same marked label m', then $i \in \mathrm{Des}(T')$ and $diag(R_i) > diag(R_{i+1})$. These mean that we have $St'(S) = T$.

Additionally, we can remove ribbon R_{i+1} before R_i. This implies that if both ribbons are labeled m, $H(R_{i+1})$ is going to be strictly to the right of $H(R_i)$ as $diag(R_i) < diag(R_{i+1})$. If they are both labeled m', $H(R_{i+1})$ is going to be strictly above $H(R_i)$ as $diag(R_i) > diag(R_{i+1})$.

This proves the expansion of the k-ribbon function in terms of descent functions. The peak function expansion follows by Proposition 5.5.7:

$$RQ_\lambda^{(k)}(X) = \sum_{T' \in SShT\pm^{(k)}(\lambda)} F_{\mathrm{Des}(T')} = \sum_{T} \sum_{T' \in \mathrm{Mark}(T)} F_{\mathrm{Des}(T')} = \sum_{T} 2^{|\mathrm{Peak}(T)|+1} G_{\mathrm{Peak}(T)}.$$

□

6 Shifted LLT polynomials

In [26], Lascoux, Leclerc and Thibon give a q-analogue for the k-ribbon that is Schur positive functions for the unshifted case. For this, they use the spin statistic on ribbon tableaux which depends on the total height of its ribbons. In this section, we show that there is no direct way of extending the concept of height to double ribbons that will give positive structure coefficients in the shifted case. Nevertheless, we are able to give a non-trivial q-analogue for the shifted ribbon functions and prove its Schur Q-positivity.

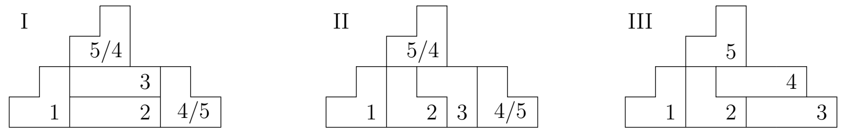
Reading word: $4, 3, 1, 2$ Reading word: $4, 1, 2, 3$

Peak function: $4G_2$ Peak function: $4G_3$

Figure 5.16: The two 3-ribbon tableaux of shape $\{5, 4, 2, 1\}$ and their corresponding peak functions

Theorem 5.6.1. *There is no intrinsic definition for the height of a double ribbon, which, along with the usual definition of heights for the single ribbon, gives a Schur Q-positive or even a Schur positive function. Here by intrinsic, we mean there is no definition that only comes from the shape of the double ribbon, and is independent of its placement or the other ribbons in the shape.*

Proof. If we consider the example in Figure 5.16, we can see that the only difference between the two fillings is the placement of ribbons 2 and 3. For any intrinsic definition, the heights of the double ribbons 4 and 1 would match in the two fillings. The total height of 2 and 3 being higher on the shape on the right, we get a function $4q^cG_2 + 4q^dG_3$ with $c \neq d$ which is not Schur P-positive. In fact, G_2 and G_3 by themselves are not even Schur positive or symmetric functions. □



Reading words: $5,3,1,2,4$
$4,3,1,2,5$
Peak function: $8G_{\{2,4\}} + 4G_{\{2\}}$

Reading words: $5,1,2,3,4$
$4,1,2,3,5$
Peak function: $4G_{\{4\}} + 4G_{\{3\}}$

Reading word: $5,4,1,2,3$
Peak function: $4G_{\{3\}}$

Figure 5.17: Standard 3-ribbon tableaux of shape $(7,5,2,1)$ and their corresponding peak functions

Another example where all the tableaux need to have the same cospin to obtain Schur Q-positivity is given in Figure 5.17. A common point of these two examples with trivial cospin is that both have only one piece in their 3-quotient, which motivates a slightly technical q-analogue defined through the quotient.

Definition 5.6.2. *For a shifted shape λ with k-abacus representation $[\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(k)}]$, we define the q-analogue of the shifted k-ribbon Q-function as follows:*

$$RQ_\lambda^{(k)}(X;q) = Q_{\alpha^{(k)}}(X) \sum_{T \in SRT_{\lfloor k/2 \rfloor}(\mu)} q^{cospin(T)} 2^{|\mathrm{Peak}(T)|+1} G_{\mathrm{peakc}(T)}(X) \tag{6.1}$$

where μ is the unshifted partition corresponding to the $\lfloor k/2 \rfloor$-quotient $(\mu^1, \mu^2, \dots, \mu^{\lfloor k/2 \rfloor})$, with $\mu^i = a^{(i)} \diamond a^{(k-i)}$ if $i < k/2$ and $\mu_{k/2} = a^{(k/2)} \diamond \varnothing$ when k is even.

Note that when $q = 1$, we get the formulation of $RQ_\lambda^{(k)}(X)$ given in Equation 4.1, so $RQ_\lambda^{(k)}(X;1) = RQ_\lambda^{(k)}(X)$ as desired.

Theorem 5.6.3. *The function $RQ_\lambda^{(k)}(X;q)$ has an expansion into Schur's Q-functions with coefficients from $\mathbb{Z}^+[q]$.*

Proof. Let $[\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(k)}]$ be the k-abacus representation for λ, and μ is the be the unshifted partition corresponding to the $\lfloor k/2 \rfloor$-quotient $(\mu^1, \mu^2, \dots, \mu^{\lfloor k/2 \rfloor})$ as defined in Theorem 5.6.3. The LLT polynomial for μ satisfies

$$GF_{\mu/\mu_0}^{(\lfloor k/2 \rfloor)}(X;q) = \sum_{T \in SRT_{\lfloor k/2 \rfloor}(\mu)} q^{cospin(T)} F_{\mathrm{desc}(T)}(X) = \sum f_\gamma^\mu(q) s_\gamma(X).$$

where γ are unshifted shapes and $f_\gamma^\mu(q)$ have positive integer coefficients. As any unshifted γ can be seen as a skew-shifted shape $\gamma^+/\delta_{\ell(\gamma)}$. Multiplying by $2^{|\mathrm{Peak}(T)|+1} G_{\mathrm{peakc}(T)}(X)$ instead $F_{\mathrm{Des}(T)}(X)$ for each γ tableau T corresponds to calculating $Q_{\gamma^+/\delta_{\ell(\gamma)}}(X)$ instead of $s_\gamma(X)$. So we have:

$$RQ_\lambda^{(k)}(X;q) := Q_{\alpha^{(k)}}(X) \left(\sum_{T \in SRT_k(\mu)} q^{cospin(T)} 2^{|\mathrm{Peak}(T)|+1} G_{\mathrm{peakc}(T)}(X) \right) = Q_{\alpha^{(k)}}(X) \left(\sum f_\gamma^\mu(q) Q_{\gamma^+/\delta_{\ell(\gamma)}}(X) \right).$$

As multiplication and skewing operations on Schur's Q-functions give positive expansions into Schur's Q-functions the result follows. □

Let us finish with calculating an example.

Consider the shape $(9,8,6,2)$ with the 5-ribbon quotient $\{(2,1),(2),\varnothing\}$ with standard fillings given in Figure 4. It has the Q 5-ribbon function:

$$RQ^{(5)}_{(9,8,6,2)}(X) = Q_{(3,1)/(1)}(X)\ Q_{(3)/(1)}(X) = 2Q_{(5)}(X) + 4Q_{(4,1)}(X) + 3Q_{(3,2)}(X).$$

Viewed as a 2-quotient, $\{(2,1),(2)\}$ corresponds to unshifted shape $(4,4,1,1)$ with the LLT polynomial:

$$GF^{(2)}_{(4,4,1,1)}(X;q) = \sum_{T\in SRT_2(4,4,1,1)} q^{cospin(T)} F_{\mathrm{desc}(T)}(X) = q^2 s_{(2,2,1)}(X) + q s_{(3,1,1)}(X) + q s_{(3,2)}(X) + s_{(4,1)}(X).$$

Viewing $(2,2,1)$, $(3,1,1)$, $(3,2)$ and $(4,1)$ as skew shifted shapes, we get the following:

$$\begin{aligned} RQ^{(5)}_{(9,8,6,2)}(X;q) &= P_{\varnothing}(X)\big(q^2 Q_{(5,3,2)/(2,1)}(X) + q Q_{(5,4,1)/(2,1)}(X) + q Q_{(4,2)/(1)}(X) + Q_{(5,1)/(1)}(X)\big) \\ &= (q+1)Q_{(5)}(X) + (q^2+2q+1)Q_{(4,1)}(X) + (q^2+2q)Q_{(3,2)}(X). \end{aligned}$$

The Schur Q-positivity of $RQ^{(k)}_{\lambda}(X;q)$ implies a possible connection to representation theory. It might interesting to explore that connection in future work, as well as find a way to calculate the powers of q directly from the ribbons.

Chapter 6

A Kashiwara Crystal Structure on Shifted Tableaux

This chapter is based on joint work with Assaf [3] [4]. Crystal bases were introduced independently by Kashiwara [23, 22] and Lusztig [30], to study the representations of q-deformations of enveloping algebras. They have the structure of colored oriented graphs, and contain key information about representations including highest weights, characters and tensor products. We start with introducing the theory of crystal graphs for type A Lie groups. Then, we focus our attention on the case of quantum queer superalgebras the characters of whose irreducible representations correspond to Schur's P-functions. We define explicit crystal operators on shifted tableaux and show they form a queer crystal. We also give a local characterization of queer crystals, and provide a new combinatorial formula for the Schur decomposition of Schur's P-functions.

1 Crystals for the quantum general linear Lie algebra

In this section, we introduce crystal bases through the case of the Lie group $\mathfrak{gl}(n)$, using the language of root systems. The representation theory of the quantized envelopping algebra $U_q(\mathfrak{gl}(n))$ is the same as that of $\mathfrak{gl}(n)$ in the sense that the $\mathfrak{gl}(n)$-modules in category $\mathcal{O}_{\text{int}}$ can be deformed into $U_q(\mathfrak{gl}(n))$-modules in category $\mathcal{O}^q_{\text{int}}$ with same weight space decomposition for any q. This allows us to focus on the case $q = 0$, where we get crystal bases. We go on to we review the combinatorics of Young tableaux, describe their crystal graph structure and give Stembridge's local axiomatization.

1.1 Crystal bases and crystal graphs

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{r+1}$ be the standard basis for $V = \mathbb{R}^{r+1}$ with the usual inner product. Consider the *root system* $\Phi = \{\mathbf{e}_i - \mathbf{e}_j \mid i \neq j\}$. We refer to the *positive roots* as the subset $\Phi^+ = \{\mathbf{e}_i - \mathbf{e}_j \mid i < j\}$. Let $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ for $i = 1, \ldots, r$ denote the *simple roots*. The *weight lattice* is $\Lambda = \mathbb{Z}^{r+1}$, and the *dominant weights* $\Lambda^+ \subset \Lambda$ are those $\lambda \in \Lambda$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r+1} \geq 0$.

Definition 6.1.1. *A* crystal *of dimension $r+1$ is a nonempty set $\mathcal{B}$ not containing 0 together with* crystal operators $e_i, f_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$ *for $i = 1, 2, \ldots, r$ and a* weight map $\mathrm{wt} : \mathcal{B} \to \Lambda$ *satisfying the conditions*

1. *for $b, b' \in \mathcal{B}$, $e_i(b) = b'$ if and only if $f_i(b') = b$, and in this case we have $\mathrm{wt}(b') = \mathrm{wt}(b) + \alpha_i$;*
2. *for $b \in \mathcal{B}$ and $i = 1, \ldots, r$, we have $\varepsilon_i(b) = (b_i - b_{i+1}) + \varphi_i(b)$, where $\varphi_i, \varepsilon_i : \mathcal{B} \to \mathbb{Z}$ are*

$$\varphi_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid e_i^k(b) \neq 0\}, \tag{1.1}$$

$$\varepsilon_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid f_i^k(b) \neq 0\}. \tag{1.2}$$

We call $\varphi_i(b), \varepsilon_i(b)$ the string lengths *through b, with $\varphi_i(b)$ the i-tail and $\varepsilon_i(b)$ the i-head.*

Note that it is enough to define the f_i's and the weight map, and, abusing notation, we denote the crystal $(\mathcal{B}, \{e_i, f_i\}_{1 \leq i \leq r}, \mathrm{wt})$ simply by $\mathcal{B}$.

As a first example, the *standard crystal* $\mathcal{B}(n)$, for $n \in \mathbb{Z}_{>0}$, has basis $\left\{\boxed{i} \mid i = 1, \ldots, n\right\}$, crystal operators f_j that act by incrementing the entry if $i = j$ or 0 otherwise, and weight map $\mathrm{wt}\left(\boxed{i}\right) = \mathbf{e}_i$.

Definition 6.1.2. *The* character *of a crystal $\mathcal{B}$ is the polynomial*

$$\mathrm{ch}(\mathcal{B}) = \sum_{b \in B} x_1^{\mathrm{wt}(b)_1} x_2^{\mathrm{wt}(b)_2} \cdots x_{r+1}^{\mathrm{wt}(b)_{r+1}}. \tag{1.3}$$

For example, the character of the standard crystal $\mathcal{B}(n)$ is $\mathrm{ch}(\mathcal{B}(n)) = x_1 + x_2 + \cdots + x_n$. Condition (1) of Definition 6.1.1 ensures that the characters of connected crystals are homogeneous of a given degree.

A *crystal graph* is a directed, colored graph with vertex set given by the crystal basis $\mathcal{B}$ and directed edges given by the crystal operators e_i and f_i, where if $b' = e_i(b)$ (resp. $b'' = f_i(b)$), then we write $b \xleftarrow{i} b'$ (resp. $b \xrightarrow{i} b''$) and all edges to 0 are omited. The *i-string through b* is the maximal path

$$e_i^{\varphi_i} x \xrightarrow{i} \cdots \xrightarrow{i} e_i x \xrightarrow{i} x \xrightarrow{i} f_i x \xrightarrow{i} \cdots \xrightarrow{i} f_i^{\varepsilon_i} x.$$

For example, the standard crystal graph $\mathcal{B}(n)$ is shown in Figure 6.1.

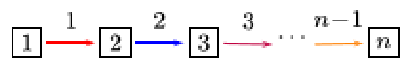


Figure 6.1: The standard crystal $\mathcal{B}(n)$ for $U_q(\mathfrak{gl}(n))$.

Definition 6.1.3. *Given two crystals $\mathcal{B}_1$ and $\mathcal{B}_2$, the* tensor product $\mathcal{B}_1 \otimes \mathcal{B}_2$ *is the set $\mathcal{B}_1 \otimes \mathcal{B}_2$ together with crystal operators e_i, f_i are defined on the tensor product $\mathcal{B}_1 \otimes \mathcal{B}_2$ by*

$$f_i(b_1 \otimes b_2) = \begin{cases} f_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes f_i(b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \tag{1.4}$$

and weight function $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$*, where addition is taken coordinatewise.*

We use the combinatorial structure of a crystals to encode essential information for studying the corresponding representations of the quantum group, with the crystal operators corresponding to deformations of the Chevalley generators. To do so, we must restrict our attention to *normal crystals.* A direct definition of normal crystals is quite involved, but the following lemma gives a complete characterization.

Proposition 6.1.4. *Given two normal crystals $\mathcal{B}_1$ and $\mathcal{B}_2$, the tensor product $\mathcal{B}_1 \otimes \mathcal{B}_2$ is a normal crystal. Further, every connected normal crystal of dimension $r+1$ and degree k arises as a connected component in $\mathcal{B}(r+1)^{\otimes k}$, the k-fold tensor product of the standard crystal $\mathcal{B}(r+1)$.*

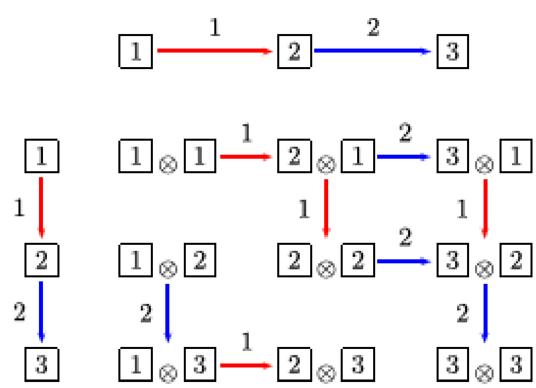


Figure 6.2: The tensor product of two standard crystals $\mathcal{B}(3)$ for $U_q(\mathfrak{gl}(3))$.

Connected normal crystals are in one-to-one correspondence with dominant weights, which in turn index irreducible representations. Given a dominant weight $\lambda \in \Lambda^+$, let $\mathcal{B}(\lambda)$ denote the connected normal crystal with highest weight λ. Then $\text{ch}(\mathcal{B}(\lambda))$ is precisely the character of the irreducible representation indexed by λ, which corresponds to the Schur polynomial $s_\lambda(x_1, \ldots, x_{r+1})$. Even more compelling is the remarkable fact that the following combinatorial procedure on crystals corresponds to the tensor product of the corresponding representations. For example, Figure 6.2 computes that the tensor product of two copies of the standard crystal $\mathcal{B}(3) = \mathcal{B}((1,0,0))$ is given by $\mathcal{B}((2,0,0))$ and $\mathcal{B}((1,1,0))$.

Definition 6.1.5. *An element $b \in \mathcal{B}$ of a crystal is a* highest weight element *if $e_i(b) = 0$ for all $i = 1, 2, \ldots, r$.*

For example, the highest weight element of $\mathcal{B}(n)$ is $\boxed{1}$, which has weight $\mathbf{e}_1 \in \Lambda^+$. Highest weights give an efficient way to categorize normal crystals.

Proposition 6.1.6. *A connected, normal crystal $\mathcal{B}$ has a unique highest weight b, and we call* $\text{wt}(b) \in \Lambda^+$ *the* highest weight of $\mathcal{B}$*. Moreover, two connected normal crystals are isomorphic as colored directed graphs if and only if they have the same highest weight.*

A consequence of this proposition is that normal crystals are determined completely by the underlying graph structure, and we do not need to be concerned with the weight maps. Even so, it is not a straightforward process to understand which crystals are normal, but as we shall see, there is a local characterization of that will prove quite useful. Before presenting this, we give an explicit combinatorial realization of normal crystals using tableaux.

1.2 Crystals on Young tableaux

As Schur polynomials are the irreducible characters for polynomial representations of the general linear group, semi-standard Young tableaux naturally index a basis for the irreducible representations. Therefore it is natural to seek a crystal structure with semi-standard Young tableaux as the underlying basis. This was done by Kashiwara and Nakashima [24] and Littelman [28], though our presentation is again simplified for the case of the general linear group.

For a word w of length k, a positive integer $r \leqslant k$, and a positive integer i, define

$$m_i(w, r) = \text{wt}(w_1 w_2 \cdots w_r)_i - \text{wt}(w_1 w_2 \cdots w_r)_{i+1}, \tag{1.5}$$

where $\mathrm{wt}(w)$ is the weak composition whose jth part is the number of j's in w. Set $m_i(w) = \max_r(m_i(w,r), 0)$. Observe that if $m_i(w) > 0$ and w_p is the leftmost occurrence of this maximum, then $w_p = i$, and if q is the rightmost occurrence of this maximum, then either $q = k$ or $w_{q+1} = i + 1$.

Recall that for a Young tableau T, the row reading word $\mathrm{rw}(T)$ is the word obtained by reading the entries of T left to right along rows, from the top row down.

Definition 6.1.7. *Given a positive integer i, define* lowering operators*, denoted by f_i, on semistandard Young tableaux as follows: if $m_i(\mathrm{rw}(T)) = 0$, then $f_i(T) = 0$; otherwise, let p be the smallest index such that $m_i(\mathrm{rw}(T), p) = m_i(\mathrm{rw}(T))$, and set $f_i(T)$ to change the entry in T corresponding to w_p to $i + 1$.*

Another way to state Definition 6.1.7 is in terms of blocked and free entries in the row reading word of T. For this, we say that a pair of letters w_a, w_b are *i-paired* if $w_a = i + 1$, $w_b = i$, and $a < b$ such that for any $a < c < b$, either $w_c \neq i, i + 1$ or w_c is i-paired with some w_d between w_a and w_b. The entries that are i-paired are called i-blocked, and any letter i or $i + 1$ that is not i-blocked is called *i-free*. With this terminology, f_i changes the rightmost i-free entry i to become $i + 1$.

For example, Figure 6.3 shows the lowering operators applied to the left and middle tableaux, where the changed cells are indicated with a circle. This is, in fact, the complete 4-string through these tableaux.

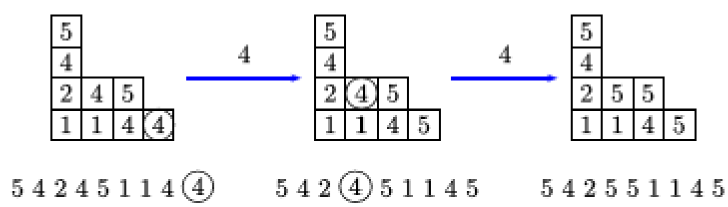


Figure 6.3: A complete 4-string on semi-standard Young tableaux of shape $(4, 3, 1, 1)$, with the row reading word indicated below.

Definition 6.1.8. *Given a positive integer i, define* raising operators*, denoted by e_i, on semistandard Young tableaux as follows: let q be the largest index such that $m_i(\mathrm{rw}(T), q) = max_r(m_i(\mathrm{rw}(T), r))$. If $q = k$, then $e_i(T) = 0$; otherwise, set $e_i(T)$ to change the entry in T corresponding to w_{q+1} to i.*

Note that $m_i(T)$ is precisely the string length $\varepsilon_i(T)$, which also coincides with the number of i-free entries equal to i. The string length $\varphi_i(T)$ is given by

$$\max\{r \mid \mathrm{wt}(w_r w_{r+1} \cdots w_n)_{i+1} - \mathrm{wt}(w_r w_{r+1} \cdots w_n)_i\}, \tag{1.6}$$

which also coincides with the number of i-free entries equal to $i+1$. Moreover, the largest index to attain the maximum in (1.6), if positive, is the entry on which e_i acts, which is the leftmost i-free entry $i+1$.

Theorem 6.1.9 ([24, 28])**.** *The raising and lowering operators e_i, f_i for $i = 1, \ldots, n-1$ are well-defined maps from $\mathrm{SSYT}_n(\lambda)$ to $\mathrm{SSYT}_n(\lambda) \cup \{0\}$ that determine a normal crystal $(\mathrm{SSYT}_n(\lambda), \{e_i, f_i\}_{1 \le i < n}, \mathrm{wt})$ that is isomorphic to $\mathcal{B}(\lambda)$.*

For example, Figure 6.4 depicts the crystal structures on semistandard Young tableaux of shapes $(3,1)$, $(2,2)$, and $(2,1,1)$ with entries $\{1,2,3\}$. These crystals are isomorphic to $\mathcal{B}((3,1,0))$, $\mathcal{B}((2,2,0))$, and $\mathcal{B}((2,1,1))$, respectively.

In particular, on the level of polynomials we have

$$s_\lambda(x_1, \ldots, x_n) = \mathrm{ch}(\mathcal{B}(\lambda)).$$

Using crystal theory, this means that whenever we have a crystal structure on a set of objects, the objects are in weight-preserving bijection with highest weight crystals $\mathcal{B}(\lambda)$), and so the generating polynomial is Schur positive. Specifically, we have the following.

Corollary 6.1.10. *For $\mathcal{B}$ be a normal crystal, we have*

$$\mathrm{ch}(\mathcal{B}) = \sum_{\substack{b \in \mathcal{B} \\ e_i(b) = 0 \forall i}} s_{\mathrm{wt}(b)}. \tag{1.7}$$

In particular, the character of $\mathcal{B}$ is symmetric and Schur positive.

1.3 Local characterization of crystals

Establishing that a given directed graph structure on a combinatorial set is a normal crystal is difficult. To simplify this greatly, Stembridge [42] gave a local characterization of crystals that arise from representations for simply-laced types.

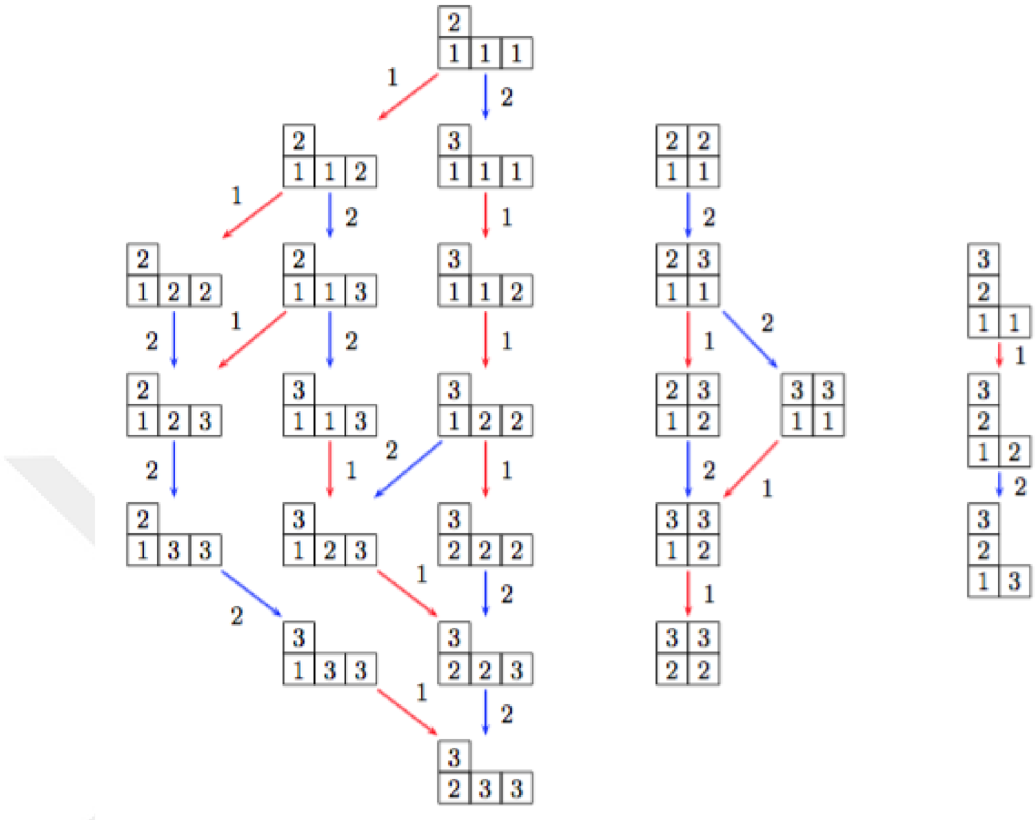


Figure 6.4: The crystal structure on semi-standard Young tableaux of shape $(3,1)$ with entries $\{1,2,3\}$, which is isomorphic to $\mathcal{B}((3,1,0))$.

In order to define Stembridge's axioms for type A_{r+1}, we first introduce notation associated with a directed, colored graph. A *graph of dimension* $r+1$ will mean a directed, colored graph $\mathcal{X}$ with directed edges $e_i(x)\xrightarrow{i}x\xrightarrow{i}f_i(x)$ for $i=1,2,\ldots,r$. Extending the string lengths φ_i,ε_i to this case, we also have the following differences whenever e_i, f_i is defined at x,

$$\begin{aligned} \Delta_i\varphi_j(x) &= \varphi_j(x)-\varphi_j(e_i(x)), & \nabla_i\varphi_j(x) &= \varphi_j(f_i(x))-\varphi_j(x),\\ \Delta_i\varepsilon_j(x) &= \varepsilon_j(e_ix)-\varepsilon_j(x), & \nabla_i\varepsilon_j(x) &= \varepsilon_j(x)-\varepsilon_j(f_ix). \end{aligned}$$

Definition 6.1.11 ([42])**.** *A directed, colored graph $\mathcal{X}$ is* regular *if the following hold:*

(A1) all monochromatic directed paths have finite length;

(A2) for every vertex x, there is at most one edge $x\xleftarrow{i}y$ and at most one edge $x\xrightarrow{i}z$;

(A3) *assuming* $e_i x$ *is defined,* $\Delta_i\varphi_j(x) + \Delta_i\varepsilon_j(x) = \begin{cases} 2 & \text{if } j = i \\ -1 & \text{if } j = i \pm 1 \\ 0 & \text{if } |i-j| \geq 2 \end{cases}$;

(A4) *assuming* $e_i x$ *is defined,* $\Delta_i\varphi_j(x), \Delta_i\varepsilon_j(x) \leq 0$ *for* $j \neq i$;

(A5) $\Delta_i\varphi_j(x) = 0 \Rightarrow e_i e_j x = e_j e_i x = y$ *and* $\nabla_j\varepsilon_i(y) = 0$;
$\nabla_i\varepsilon_j(x) = 0 \Rightarrow f_i f_j x = f_j f_i x = y$ *and* $\Delta_j\varphi_i(y) = 0$;

(A6) $\Delta_i\varphi_j(x) = \Delta_j\varphi_i(x) = -1 \Rightarrow e_i E_j^2 e_i x = e_j E_i^2 e_j x = y$ *and* $\nabla_i\varepsilon_j(y) = \nabla_j\varepsilon_i(y) = -1$;
$\nabla_i\varepsilon_j(x) = \nabla_j\varepsilon_i(x) = -1 \Rightarrow f_i F_j^2 f_i x = f_j F_i^2 f_j x = y$ *and* $\Delta_i\varphi_j(y) = \Delta_j\varphi_i(y) = -1$.

Axiom A1 ensures that the crystal operators have finite order when applied to a given basis element, though it does not force the graph itself to be finite. Axiom A2 ensures that the edges of the graph correspond to well-defined operators, which we assume map a basis element to 0 in the absence of an edge.

Axioms A3 and A4 dictate how the length of a j-string differs between x and $e_i x$; see Figure 6.5. When $|i-j| \geq 2$, there is no change (left case), but when $j = i \pm 1$, the length changes by 1, either by decreasing the tail when $\Delta_i\varphi(x,j) = -1$ (middle case) or by increasing the head when $\Delta_i\varepsilon(x,j) = -1$ (right case).

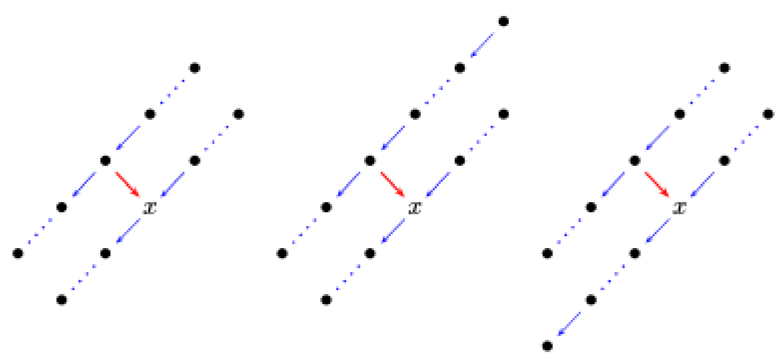


Figure 6.5: An illustration of axioms A3 and A4, where $f_j \swarrow$, $f_i \searrow$.

Axioms A5 and A6 give information about how edges with different labels interact; see Figure 6.6. When $|i-j| \geq 2$, the conditions of A5 will always be satisfied, though when $j = i \pm 1$, either A5 or A6 could hold, the former when $\Delta_i\varphi(x,j) = 0$ and the latter when $\Delta_i\varphi(x,j) = \Delta_j\varphi(x,i) = -1$.

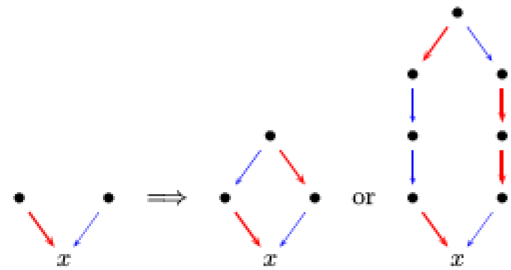


Figure 6.6: An illustration of axioms A5 and A6, where $f_j \swarrow$, $f_i \searrow$.

The following theorem shows that regular graphs correspond precisely to the crystal graphs of representations for the general linear group. This result allows us to combinatorialize the problem of studying representations of the general linear group by instead studying regular graphs of degree n.

Theorem 6.1.12 ([42]). *Every normal crystal is a regular graph and every regular graph is a normal crystal.*

The theorem is proved by showing that Littelmann's path operators [27] generate regular graphs, since the Path Model is known to generate $\mathcal{B}(\lambda)$. This then gives a simple machinery for proving that a given structure is a crystal graph by showing that the graph is regular.

2 A crystal for shifted tableaux

In this section we define crystal operators on semistandard shifted tableaux analogous to those for semistandard Young tableaux. Them use Stembridge's axioms to prove that our crystal is normal, thus giving a new proof of the Schur positivity of Schur's P-polynomials. We note that Hawkes, Paramonov, and Schilling [15] recently constructed a crystal on semistandard shifted tableaux in the context of type B/C Stanley symmetric functions. Their construction uses Haiman's mixed insertion [14] to associate a reduced word for a signed permutation with a pair of shifted tableaux, the left semistandard and the right standard. Using shifted insertion developed independently by Worley [43] and Sagan [35] to associate the left semistandard shifted tableau with a semistandard Young tableauthey are able to put the usual crystal structure reviewed in §1.2 on the semistandard Young tableaux. Therefore, while the Hawkes-Paramonov-Schilling construction gives a desirable crystal interpretation of the Schur positivity of Schur P-polynomials, it does not give a new proof of positivity in the sense that it relies on the positivity that follows from shifted insertion [43, 35].

We developed our crystal independently, unaware of [15], and our presentation below is direct and differs from the derived description in [15]. However, in Proposition 6.2.16 below, we prove that the two constructions are indeed equivalent. Nevertheless, we proceed with our direct description below and give a direct proof, using Stembridge's axioms, that this defines a normal crystal, thus giving a new proof of the Schur positivity of Schur P-polynomials independent of shifted insertion.

2.1 Crystals on shifted tableaux

We define a new reading word for shifted tableaux as follows:

Definition 6.2.1. *For T a shifted tableau, the* hook reading word of T, *denoted by* $\mathrm{w}(T)$, *is the word obtained by reading the marked entries of T up the ith column then the unmarked entries of T along the ith row, left to right, for i from* $\max(\gamma_1, \ell(\gamma))$ *to* 1.

For example, the hook reading words for the three tableaux in Figure 6.7 are shown.

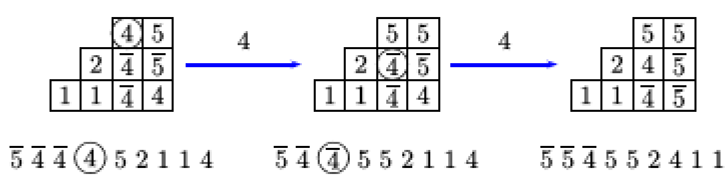


Figure 6.7: A complete 4-string on semistandard shifted tableaux of shape $(4, 3, 2)$, with the hook reading word indicated below.

The row and column conditions for shifted tableaux ensure that the cells with entries $\bar{i}, i$ must form a ribbon, i.e. contain no 2×2 block. This observation helps to make the following well-defined.

Definition 6.2.2. *The* shifted lowering operators, *denoted by $\overline{f}_i$, act on semistandard shifted tableaux by: $\overline{f}_i(T) = 0$ if $m_i(\mathrm{w}(T)) \leqslant 0$; otherwise, letting p be the smallest index such that $m_i(\mathrm{w}(T), p) = m_i(\mathrm{w}(T))$, letting x denote the entry of T corresponding to w_p, and letting y be the entry north of x and z the entry east of x, we have*

(L1) (a) if $x = i$ and $z = \overline{i+1}$, then $\overline{f}_i$ changes x to $\overline{i+1}$ and changes z to $i+1$;

(b) else if $x = i$ and y does not exist or $y > i+1$, then $\overline{f}_i$ changes x to $i+1$;

(c) else if $x = i$ and the northeastern-most cell on the $(i+1)$-ribbon containing y has a marking, then $\overline{f}_i$ removes that marking and changes x to $\overline{i+1}$;

(d) else if $x = i$, then $\overline{f}_i$ changes x to $\overline{i+1}$;

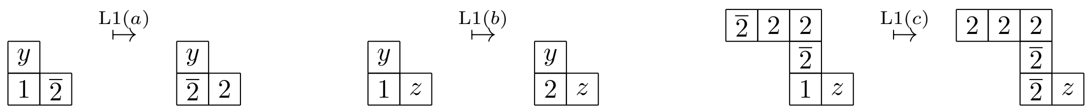


Figure 6.8: An illustration of the shifted lowering operators with $x = i = 1$.

(L2) (a) if $x = \bar{i}$ and $y = i$, then $\overline{f}_i$ changes x to i and changes y to $\overline{i+1}$;

(b) else if $x = \bar{i}$ and z does not exist or $z > \overline{i+1}$, then $\overline{f}_i$ changes x to $\overline{i+1}$;

(c) else if $x = \bar{i}$, then $\overline{f}_i$ changes x to i and changes the first entry i southwest along the i-ribbon containing x that is not followed by i or $\overline{i+1}$ to $\overline{i+1}$.

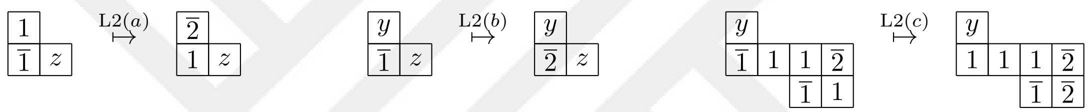


Figure 6.9: An illustration of the shifted lowering operators with $x = \bar{i} = \bar{1}$.

The rules for the shifted lowering operators are illustrated case by case in Figures 6.8 and 6.9. Figure 6.7 shows the lowering operators applied to the left and middle tableaux, where the changed cells are indicated with a circle. This is, in fact, the complete 4-string through these tableaux. A complete example on $\mathrm{SSHT}_3((3,1))$ is shown in Figure 6.10.

Note that the total number of marked cells stays the same except for Case (1)(d), where it increases by one. From the definition, it is not obvious that these operators are well-defined nor that the result is again a semistandard shifted tableau. However, both are indeed the case, as will be shown in Theorem 6.2.5.

In order to analyze how the shifted crystal operators change the lengths of monochromatic paths, we introduce the notion of *blocked* entries of a semistandard shifted tableau T, determined via the hook reading word $w(T)$. Blocking will depend upon a parameter i, for $1 \leq i < n$, and for i-blocking, only entries $\bar{i}, i, \overline{i+1}, i+1$ will be considered.

Definition 6.2.3. *Let $i \geq 1$ be an index and T a semistandard shifted tableau. A pair of entries y, x of T, with $y \in \{\overline{i+1}, i+1\}$ and $x \in \{\bar{i}, i\}$ are i-*paired *if y occurs before x in the hook reading word of T and*

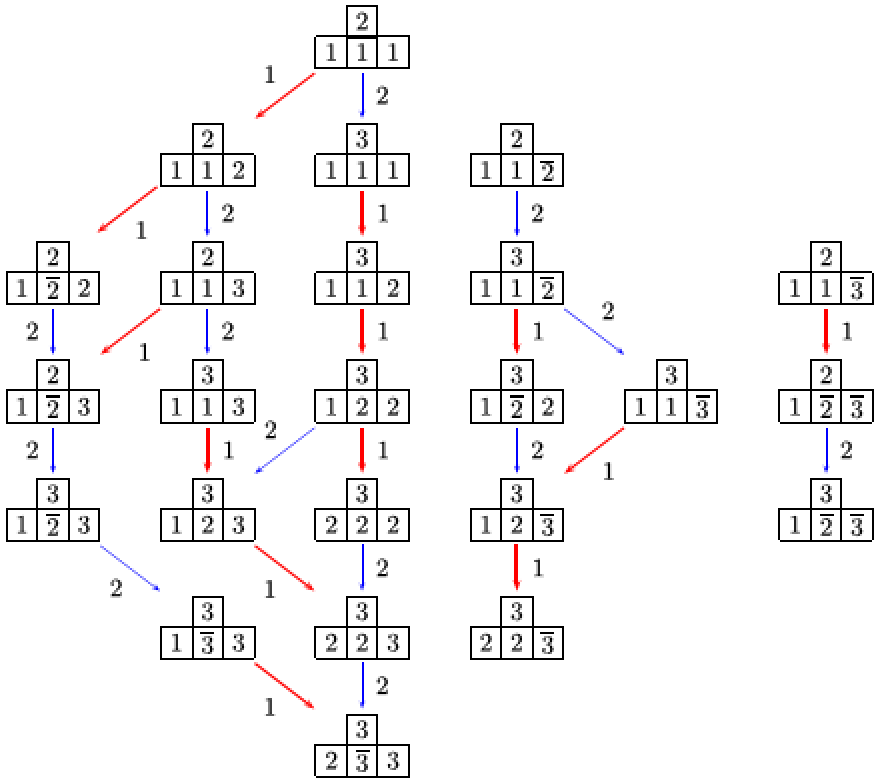


Figure 6.10: The crystal structure on semi-standard shifted tableaux of shape $(3,1)$ with entries $\{\bar{1},1,\bar{2},2,\bar{3},3\}$ and no marks on the main diagonal.

every entry $\bar{i}, i, \overline{i+1}, i+1$ *that lies between them in the hook reading word is* i*-blocked. The* i*-paired entries become* i*-blocked, and we continue the operation recursively till there are no more possible* i*-pairs. An entry* $\bar{i}, i, \overline{i+1}, i+1$ *of* T *is* i-free *if it is not part of an* i*-pair.*

Figure 6.11 shows the 4-blocked entries of the hook reading word paired with under brackets and the 4-free entries are indicated in red. Notice that for each tableau, there are exactly two 4-free entries.

We show below that the i-free entries of T occur with all entries $\bar{i}, i$ preceding all entries $\overline{i+1}, i+1$ in the hook reading word $w(T)$. This implies the following alternative characterization for the entry of T on which $\overline{f}_i$ acts.

Lemma 6.2.4. *For* T *a semistandard shifted tableau and* $i \geq 1$ *an index,* $m_i(w(T))$ *is number of* i*-free entries* $\bar{i}, i$ *of* T*, and if* $m_i(w(T)) > 0$*, then the smallest index* p *for which* $m_i(w(T), p) = m_i(w(T))$ *occurs at the rightmost* i*-free entry* $\bar{i}, i$*.*

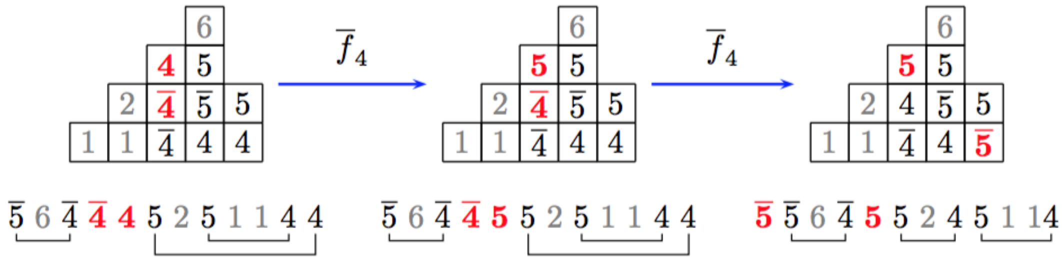


Figure 6.11: A 4-string on $\mathrm{SSHT}_6(5,4,2,1)$, with 4-blocked entries bracketed and 4-free entries indicated in red.

Proof. We claim all i-free entries $\bar{i}, i$ precede all i-free entries $\overline{i+1}, i+1$ in the hook reading word $w(T)$. To see this, note that i-blocked entries are uniquely paired with an entry $\overline{i+1}, i+1$ paired with the nearest entry $\bar{i}, i$ such that the balancing condition of Definition 6.2.3 is met. For example, see Figure 6.11. With this pairing, the pairs of i-blocked entries are nested, and no i-free entry may occur between a pair of i-blocked entries. Therefore, if y, x are i-free entries with $y \in \{\overline{i+1}, i+1\}$, $x \in \{\bar{i}, i\}$ and y preceding x in $w(T)$ such that no intermediate entries are i-free, then all intermediate entries $\bar{i}, i, \overline{i+1}, i+1$ are paired with another entry between y and x, therefore y and x have the same number of entries equal to $\overline{i+1}, i+1$ as equal to $\bar{i}, i$ between them in $w(T)$, contradicting that y, x are free. Thus no i-free entry $\overline{i+1}, i+1$ may precede an i-free entry $\bar{i}, i$ in $w(T)$, establishing the claim.

Recall that $m_i(w(T)) = \max_r(m_i(w(T), r))$ is positive if and only if $\overline{f}_i(T) \neq 0$, and in this case the smallest index p for which $m_i(w(T), p) = m_i(w(T))$ occurs at an entry $\bar{i}$ or i. By the previous claim, if $r < s$ are the indices in $w(T)$ of an i-blocked pair, then $m_i(w(T), r-1) = m_i(w(T), s)$. Therefore the smallest index p for which $m_i(w(T), p) = m_i(w(T))$ must occur at the rightmost i-free entry $\bar{i}$ or i. Furthermore, since $w(T)_p$ cannot lie between any i-blocked pair, we have $m_i(w(T), p)$ is the number of i-free entries $\bar{i}$ or i weakly preceding $w(T)_p$, which by the previous claim, is simply the number of i-free entries $\bar{i}$ or i of T. □

Using blocked and free entries, we now prove the shifted lowering operators are well-defined. The proof is a case by case analysis that acting as prescribed in Definition 6.2.2 results in a semistandard shifted tableau.

Theorem 6.2.5. *For any strict partition γ, the shifted lowering operators $\{\overline{f}_i\}_{1 \le i < n}$ are well-defined maps $\overline{f}_i : \mathrm{SSHT}_n(\gamma) \to \mathrm{SSHT}_n(\gamma) \cup \{0\}$.*

Proof. Let T be a semistandard shifted tableaux and assume $m_i(w(T)) > 0$ so that $\overline{f}_i$ acts non-trivially on T. Let x, y, z be as in Definition 6.2.2, and let their indices in the hook reading word be p_x, p_y, p_z, respectively. By definition, p_x is the smallest number satisfying $m_i(w(T), p) = m_i(w(T))$. By Lemma 6.2.4, every cell labeled $\overline{i+1}$ or $i+1$ that precedes x in $w(T)$ is i-paired with some cell labeled $\bar{i}/i$ that lies between itself and x. Similarly, every cell labeled $\bar{i}/i$ that follows x in $w(T)$ is i-paired with some cell labeled $\overline{i+1}/i+1$ that lies between x and itself. In particular, if x has label i, then z cannot also have label i since it follows immediately after x in $w(T)$, and if x has label $\bar{i}$ then y cannot have label $\bar{i}$ for exactly the same reason.

Case L1(a): We must verify two things to show that the shifted tableaux rules are not violated: x is not on a main diagonal, and the cell above z is not $i+1$ or $\overline{i+1}$. We first show that x can not be on the main diagonal. As $z = \overline{i+1}$, it needs to be matched to a cell labeled i or $\bar{i}$ after it and before x in the reading word, which can only happen in the column of x (Figure 6.12, left). Assume there are t cells labeled $\bar{i}$ in the column of x. The cells left adjacent to those can only be labeled i or $\overline{i+1}$ as columns and rows weakly increase. Furthermore, every $\overline{i+1}$ in the column of z needs to be matched to a $\bar{i}$ between x and itself, so at most $t-1$ of these can be $\overline{i+1}$, and the bottom one most must be labeled i. Let us call this cell x' (Figure 6.12, middle). As $x' = i$ it must be matched to some $i+1$ between x and x' in the reading word ($\overline{i+1}$ never after i). By the tableaux rules, such a cell must be on the row above. If there are t' such cells labeled i, they need to be matched to t' cells labeled $i+1$ on the row above, which means the leftmost i (say x'') is followed by an $\overline{i+1}$. To match this $\overline{i+1}$, the cell under x'' must be labeled $\bar{i}$, which takes us back to our starting point (see Figure 6.12, right). Since our shape finite, this case can not happen. The second step is to show

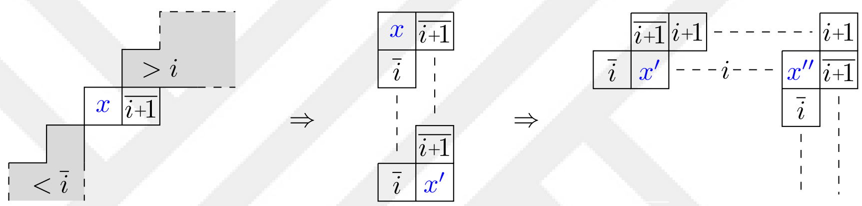


Figure 6.12: Case L1(a), showing x can not be on the main diagonal.

that the cell above z can not be labeled $i+1$ or $\overline{i+1}$. We can eliminate the $\overline{i+1}$ case easily, as x is not on the main diagonal, there is a cell above x and there is no valid way to fill it (Figure 6.13, left). Assume the cell above z is labeled $i+1$. As it is not on the main diagonal, any $\bar{i}$ comes after it in the reading word, so it needs to be matched to an i between x and itself, so there needs to be another cell labeled i to the left of x. If i is not on the main diagonal, then it has a cell above it that is labeled at least $\overline{i+1}$, so the cell above x also must be $i+1$, and we need another i to match it. This continues till we hit the main diagonal, and the $i+1$ on the diagonal still needs to be matched. As there is no more place left for an i, it must be matched to an $\bar{i}$ after it on the reading word, which can only happen in the column to the left. This leads to a contradiction as there is no way to fill the cell to the right of $\bar{i}$ (see Figure 6.13, right).

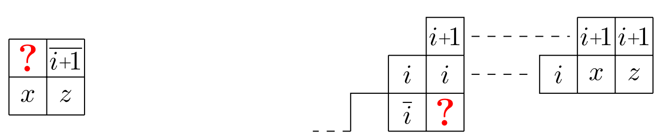


Figure 6.13: Case L1(a), showing the cell above z can not be $i+1$ or $\overline{i+1}$.

Case L1(b): In this case we assume $z \geq i+1$ (since we are not case L1(a)) and either y does not exist or $y > i+1$, so there are no possible row or column violations that can arise from changing x from i to $i+1$.

Cases L1(c) and L1(d): In this case we assume $z \geq i+1$ (since we are not case L1(a)) and either $y = \overline{i+1}$ or $y = i+1$ (since we are not case L1(b)), therefore there are no possible row or column violations that can arise from changing x from i to $\overline{i+1}$. Let w denote the entry at the head of the $(i+1)$-ribbon containing y. If $w = i+1$ (case L1(d)), then no change occurs here, and if $w = \overline{i+1}$ (case L1(c)), then since the $(i+1)$-ribbon terminates at w, the cell above w is larger than $i+1$, removing the marking on w cannot create a row or column violation either.

Case L2(a): As there can be at most one $\bar{i}$ on a row, changing x from $\bar{i}$ to i does not create a row or column violation, therefore we only need to check that no violations arise in changing y from i to $\overline{i+1}$. Note that y can not be on the main diagonal, since otherwise x would immediately precede y, giving $m_i(w(T), p_y) > m_i(w(T), p_x)$. So the only possible problem is if the cell right adjacent to y is labeled $\overline{i+1}$. Assume it is. Then it must be i-paired with a cell labeled $\bar{i}$ that comes before x in $w(T)$. However, this can only happen in the column of x, so the cell below x must be $\bar{i}$. Assume there are k cells labeled $\bar{i}$ below x. Then we can have at most k cells labeled $\overline{i+1}$ in the next column, so the bottom two $\bar{i}$'s need to have i's left adjacent to them, which can not happen by our column rules (Figure 6.14, left).

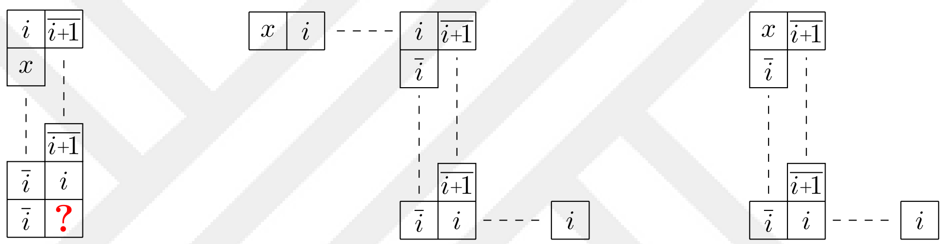


Figure 6.14: Case L2: $x = \bar{i}$

Case L2(b): We must have $y \geq \overline{i+1}$ (since we are not case L2(a)) and either z does not exist or $z > \overline{i+1}$, so there are no possible row or column violations that can arise from changing x from $\bar{i}$ to $\overline{i+1}$.

Case L2(c): Since $y \geq \overline{i+1}$ (since we are not case L2(a)) no row or column violation can arise in changing x from $\bar{i}$ to i. Therefore we only need to verify that, in the i-ribbon containing x, there is an entry i that is not followed by i or $\overline{i+1}$ as described, and that turning it into $\overline{i+1}$ creates no problems. We have two cases based on the possible values of z as i or $\overline{i+1}$ (since we are not in case L2(b)). We first consider the case $z = i$. If the rightmost i in this row is not followed by $\overline{i+1}$, we are done. If it is, then it needs to be matched to a cell marked $\bar{i}$ between x and itself in the reading word, which can only happen in the column of the rightmost i. Assume this column contains k cells labeled $\bar{i}$. Then there can be at most k cells labeled $\overline{i+1}$ in the next column, so the bottom-most one needs to be followed by an i. If the rightmost i is not followed by $\overline{i+1}$, we are done. If it is, we can repeat the same argument till we find such an i (Figure 6.14, middle). The case $z = \overline{i+1}$ is similar, as z needs to be matched to a cell labeled $\bar{i}$ that comes after x in the reading word, and that can only happen in the column of x. As above, the bottom $\bar{i}$ in the column must be followed by an i, and we can continue until we find an i not followed by $\overline{i+1}$ (Figure 6.14, right). ☐

Definition 6.2.6. *The* shifted raising operators*, denoted by* $\bar{e}_i$*, act on semistandard shifted tableaux by:* $\bar{e}_i(T) = 0$ *if* $m_i(\mathrm{w}(T), k) = m_i(\mathrm{w}(T))$*; otherwise, letting* q *be the largest index such that* $m_i(\mathrm{w}(T), q) = m_i(\mathrm{w}(T))$*,letting* x *denote the entry of* T *corresponding to* w_q*, and letting* y *be the entry south of* x *and* z *the entry west of* x*, we have*

(R1) *(a)* *if* $x = i+1$ *and* $z = \overline{i+1}$, *then* $\overline{e}_i$ *changes* x *to* $\overline{i+1}$ *and changes* z *to* i*;*

(b) *else if* $x = i+1$ *and* y *does not exist or* $y < i$, *then* $\overline{e}_i$ *changes* x *to* i

(c) *else if* $x = i+1$ *then* $\overline{e}_i$ *changes* x *to* $\overline{i+1}$ *and changes the first entry* $\overline{i+1}$ *southwest along the* $i+1$*-ribbon containing* x *that is not above an* i *or* $\overline{i+1}$ *to* i*;*

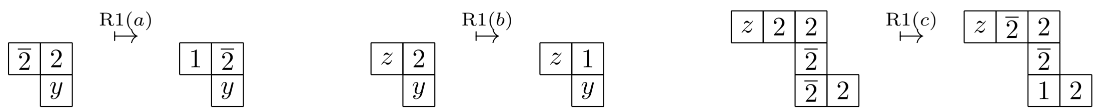


Figure 6.15: An illustration of the shifted raising operators with $x = i = 1$.

(R2) *(a)* *if* $x = \overline{i+1}$ *and* $y = i$, *then* $\overline{e}_i$ *changes* x *to* i *and changes* y *to* $\bar{i}$*;*

(b) *else if* $x = \overline{i+1}$ *and* z *does not exist or* $z < \bar{i}$, *then* $\overline{e}_i$ *changes* x *to* $\bar{i}$*;*

(c) *else if* $x = \overline{i+1}$, *and the northeastern-most cell on the* (i)*-ribbon containing* z *is not on the main diagonal, then* $\overline{e}_i$ *adds a marking to that cell and changes* x *to* $\bar{i}$*;*

(d) *else if* $x = \overline{i+1}$, *then* $\overline{e}_i$ *changes* x *to* i*.*

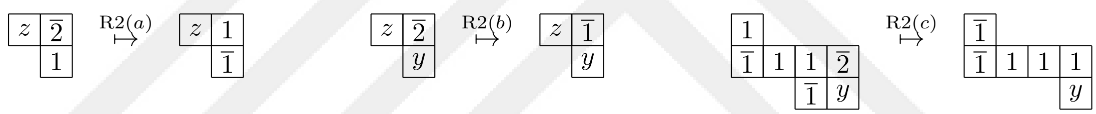


Figure 6.16: An illustration of the shifted raising operators with $x = \bar{i} = \bar{1}$.

The proof that the shifted raising operators are well-defined is completely analogous to that for the shifted lowering operators. Similarly, if $\overline{e}_i(T) \neq 0$, then the largest index q for which $m_i(w(T), q-1) = m_i(w(T))$ occurs at the leftmost i-free entry $\overline{i+1}$ or $i+1$, parallel to Lemma 6.2.4. Details of both proofs are omitted.

2.2 Verification of local axioms

To prove that our operators define a normal crystal, we begin by showing that they satisfy the conditions required to be a crystal.

Theorem 6.2.7. *For any strict partition* γ, *the shifted raising and lowering operators* $\overline{e}_i, \overline{f}_i$ *for* $i = 1, 2, \ldots, r$ *together with the usual weight map define a crystal on* $\mathrm{SSHT}_{r+1}(\gamma)$.

Proof. From the definitions of the shifted operators, we have $\mathrm{wt}(\overline{f}_i(T)) = \mathrm{wt}(T) + \alpha_i$ and $\mathrm{wt}(\overline{e}_i(T)) = \mathrm{wt}(T) - \alpha_i$. Therefore there are two statements to prove based on Definition 6.1.1, first that $\overline{e}_i(T) = T'$ if and only if $\overline{f}_i(T') = T$, and second that $\varepsilon_i(T) - \varphi_i(T) = (\mathrm{wt}(T)_i - \mathrm{wt}(T)_{i+1})$. Note that the second statement follows once we show $\varphi_i(T)$ is the number of i-free entries $\overline{i+1}, i+1$ in T and $\varepsilon_i(T)$ is the number of i-free entries $\overline{i}, i$ of T, since all other entries appear in pairs whose weights cancel. We prove both statements together in cases based on Definition 6.2.2 for the shifted lowering operators.

Case L1(a): We assume $x = 1$ and $z = \overline{i+1}$ in T, therefore $\overline{f}_i(T)$ has $x = \overline{i+1}$ and $z = i+1$. For an example of this case, see Figure 6.17. In the hook reading words, the position of x in $w(T)$ is precisely the position of z in $w(\overline{f}_i(T))$. However, the position of z in $w(T)$ is weakly left of the position of x in $w(\overline{f}_i(T))$, with the offset equal to the number of marked entries strictly above z or strictly below x in T (and in $\overline{f}(T)$). However, since $\overline{f}_i(T)$ is semistandard, there cannot be an entry $\overline{i+1}$ above z, so we only need to be concerned about any entries $\overline{i}$ below x. Note that if $u = \overline{i}$ is below x and v is the cell immediately to its right, then $\overline{i+1} < v$, and since v is below z, we also have $v \leq \overline{i+1}$. Therefore $v = \overline{i+1}$, therefore in this way we pair off each $\overline{i}$ below x with an $\overline{i+1}$ below z. In particular, since marked entries in the column of z are read immediately before marked entries in the column of x, every $\overline{i}$ below x is i-paired with an $\overline{i+1}$ weakly below z in both $w(T)$ and in $w(\overline{f}_i(T))$. Therefore the number of i-blocked pairs is unchanged in passing from $w(T)$ to $w(\overline{f}_i(T))$. Furthermore, the result is that the rightmost i-free entry i becomes the leftmost i-free entry $i+1$, landing us in case R1(a) for the shifted raising operator, which will precisely undo the action.

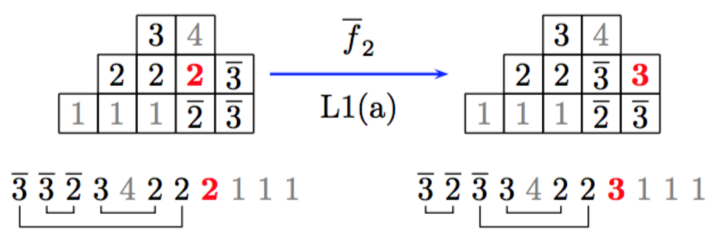


Figure 6.17: Example of i-blocked/free entries when $\overline{f}_i$ acts by case L1(a).

Case L2(a): We assume $x = \overline{i}$ and $y = i$ in T, therefore $\overline{f}_i(T)$ has $x = i$ and $y = \overline{i+1}$. For an example of this case, see Figure 6.18. The position of $x = \overline{i}$ in $w(T)$ is precisely the position of $y = \overline{i+1}$ in $w(\overline{f}_i(T))$. However, the position of $y = i$ in $w(T)$ is weakly right of the position of $x = i$ in $w(\overline{f}_i(T))$, with the offset equal to the number of unmarked entries strictly right of y or strictly left of x in T (and in $\overline{f}(T)$). We claim every cell v with entry $i+1$ in this range is i-blocked. Indeed, in this case v must be right of y, therefore the cell immediately below v, say u, must have entry $\overline{i} < u < i+1$. However, the entry cannot be $\overline{i+1}$, since this forces a downward column of entries $\overline{i+1}$ equal in length to the downward column of entries $\overline{i}$ from x, therefore x would be i-blocked. Therefore u must have entry i, so there are at least as many entries i right of x as entries $i+1$ right of y, proving that all of the latter are indeed i-blocked. Thus the number of i-blocked pairs is unchanged in passing from $w(T)$ to $w(\overline{f}_i(T))$. Furthermore, the result is that the rightmost i-free entry $\overline{i}$ becomes the leftmost i-free entry $\overline{i+1}$, landing us in case R2(a) for the shifted raising operator, which will precisely undo the action.

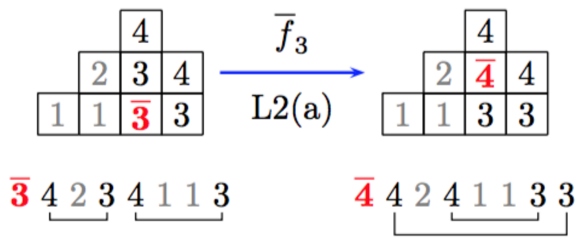


Figure 6.18: Example of i-blocked/free entries when $\overline{f}_i$ acts by case L2(a).

Case L1(b): We have $x = i$ in T becomes $x = i+1$ in $\overline{f}_i(T)$, and by Lemma 6.2.4 this happens at the rightmost i-free entry, so the number of i-blocked pairs is again unchanged and the rightmost i-free entry i becomes the leftmost i-free entry $i+1$ landing in case R1(b) of the shifted raising operator, which precisely toggles back to case L1(b) under $\overline{e}_i$.

Case L2(b): We have $x = \overline{i}$ in T becomes $x = \overline{i+1}$ in $\overline{f}_i(T)$, and by Lemma 6.2.4 this happens at the rightmost i-free entry, so the number of i-blocked pairs is unchanged and the rightmost i-free entry $\overline{i}$ becomes the leftmost i-free entry $\overline{i+1}$ landing in case R2(b) of the shifted raising operator, which precisely toggles back to case L2(b) under $\overline{e}_i$.

Case L1(c): We assume $x = i$, $y = \overline{i+1}$ or $i+1$, and $z > \overline{i+1}$ in T, and, letting u denote the northeastern-most cell of the $(i+1)$-ribbon containing y, $u = \overline{i+1}$. Then $\overline{f}_i(T)$ has $x = \overline{i+1}$ and $u = i+1$. For an example of this case, see Figure 6.19. The situation here is similar to case L1(a), though now both x and u move when passing from $w(T)$ to $w(\overline{f}_i(T))$, so we consider each in turn. First, comparing the position of $x = i$ in $w(T)$ with that of $u = i+1$ in $w(\overline{f}_i(T))$, x moves left past any unmarked entries that lie in a row strictly between that of u and x. We claim that any such entries i are i-blocked. Since no two entries $\overline{i+1}$ may occur in the same row, the number of entries $i+1$ along the $(i+1)$-ribbon is one fewer than the number of columns spanned by the ribbon (since the northwestern-most entry is $\overline{i+1}$). Further, since $i < \overline{i+1}$ with no intermediate values, any i in a row between u and x must be immediately below the $(i+1)$-ribbon, therefore the maximum number of such entries is again the number of columns spanned minus one for the column of x. Since the $(i+1)$-ribbon lies above the i's, all of the i's will be i-paired with one of those $i+1$'s, thus proving the claim. Second, comparing the position of $u = \overline{i+1}$ in $w(T)$ with that of $x = \overline{i+1}$ in $w(\overline{f}_i(T))$, u moves left past any marked entries that lie weakly between u and x in the column reading word (bottom to top along columns, from right to left). By the same analysis of ribbons, where now we count columns instead of rows, any such entries $\overline{i}$ are i-blocked. Therefore the number of i-blocked pairs is once again unchanged. Furthermore, the head of the $(i+1)$-ribbon will become the leftmost i-free entry $i+1$ and satisfy the conditions of case R2(c) for the shifted raising operators, whose action will precisely undo that of the shifted raising operator for this case.

Case L2(c): We assume $x = \overline{i}$, $y > i$, and $z = i$ or $\overline{i+1}$ in T, and let $u = i$ denote the southwestern-most entry i of the i-ribbon containing x not followed by i or $\overline{i+1}$. Then $\overline{f}_i(T)$ has $x = i$ and $u = \overline{i+1}$. For an example of this case, see Figure 6.20. As with case L1(c), both x and u move when passing from $w(T)$ to

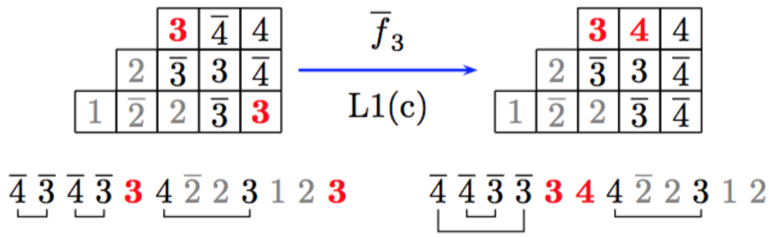


Figure 6.19: Example of i-blocked/free entries when $\overline{f}_i$ acts by case L1(c).

$w(\overline{f}_i(T))$. Comparing the position of $x = \bar{i}$ in $w(T)$ with that of $u = \overline{i+1}$ in $w(\overline{f}_i(T))$, x moves left past any marked entries that lie in a column strictly between that of x and u. We claim that any such entries $\bar{i}$ are already i-blocked in T. Starting from x, the first time the ribbon descends there must be an entry $\overline{i+1}$ at the end of that row, else the last i in that row would have been u, therefore all entries immediately right of the descending portion must be $\overline{i+1}$ except for the right turn of the i-ribbon, which must have an i, and the argument repeats. Therefore each step down of the i-ribbon has an $\bar{i}$, and in the column immediately right of this there are equally many $\overline{i+1}$'s offset one row higher, so all entries $\bar{i}$ are i-blocked as claimed. Comparing the position of $u = i$ in $w(T)$ with that of $x = i$ in $w(\overline{f}_i(T))$, u moves left past any unmarked entries that lie in a row between that of x and u. We claim that any such entries $i+1$ are already i-blocked in T and not by pairing with u. The number of i's in this range is the number of columns spanned by the i-ribbon below the top row. Since the $(i+1)$-ribbon(s) in this range must lie immediately on top of the i-ribbon, the number of $i+1$'s is bounded by the number of columns spanned by the i-ribbon excluding the top row minus one for the entry $\overline{i+1}$ that must end the top row, therefore there are strictly more i's including u. Therefore all entries $i+1$ remain i-blocked in $\overline{f}_i(T)$.

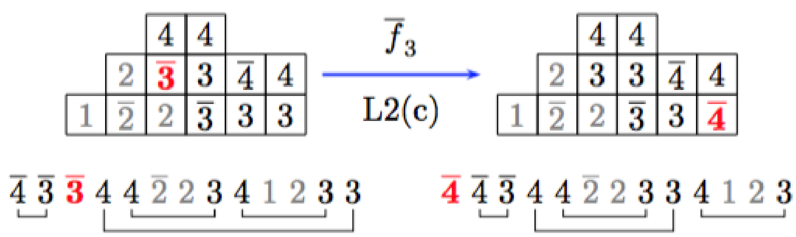


Figure 6.20: Example of i-blocked/free entries when $\overline{f}_i$ acts by case L2(c).

Case L1(d): We assume $x = i$, $y = \overline{i+1}$ or $i+1$, and $z > \overline{i+1}$ in T, and, letting u denote the northeastern-most cell of the $(i+1)$-ribbon containing y, $u = i+1$ lies on the main diagonal. Then $\overline{f}_i(T)$ has $x = \overline{i+1}$. Since u lies on the main diagonal, it occurs in $w(T)$ and in $w(\overline{f}_i(T))$ after all marked entries $\bar{i}, \overline{i+1}$ and before all unmarked entries $i, i+1$ on the two relevant ribbons, therefore the analysis of case L1(c) resolves this case as well, with inverse also given by case R1(c). □

From the proof of Theorem 6.2.7, we have the following characterization of the lengths of the heads and tails of i-strings in terms of the number of i-free entries, and this is equivalent to applying (1.6) and (1.5) to the hook reading word.

Corollary 6.2.8. *For T a semistandard shifted tableau and $i \geq 1$ an index, $\varphi_i(T)$ is the number of i-free entries $\overline{i+1}, i+1$ in T, and $\varepsilon_i(T)$ is the number of i-free entries $\overline{i}, i$ of T, and we have*

$$\varphi_i(T) = \max\{r \mid \mathrm{wt}(w_r w_{r+1} \cdots w_n)_{i+1} - \mathrm{wt}(w_r w_{r+1} \cdots w_n)_i\}, \quad (2.1)$$

$$\varepsilon_i(T) = \max\{r \mid \mathrm{wt}(w_1 w_2 \cdots w_r)_i - \mathrm{wt}(w_1 w_2 \cdots w_r)_{i+1}\}. \quad (2.2)$$

We use Stembridge's characterization of regular graphs [42] to establish that the shifted lowering operators determine a normal crystal on shifted tableaux. To begin, we show that the notation used in Definition 6.1.11 is well-defined for the graph on semistandard shifted tableaux with edges given by the shifted crystal operators.

Lemma 6.2.9. *The graph on $\mathrm{SSHT}_n(\gamma)$ with edges $x \xleftarrow{i} y$ whenever $\overline{e}_i(x) = y$ and $x \xrightarrow{i} z$ whenever $\overline{f}_i(x) = z$ is a directed, colored graph satisfying Stembridge axioms A1 and A2.*

Proof. For axiom A1, all monochromatic directed paths have finite length since $\overline{f}_i$ changes the weight of a semistandard shifted tableau by $\mathrm{wt}(\overline{f}_i(T)) = \mathrm{wt}(T) + \alpha_i$, where α_i is the simple root $\mathbf{e}_i - \mathbf{e}_{i+1}$, and the weight of a semistandard shifted tableau has nonnegative parts. Axiom A2, stating for every vertex x, there is at most one edge $x \xleftarrow{i} y$ and at most one edge $x \xrightarrow{i} z$, follows from $\overline{f}_i$ being well-defined, proved in Theorem 6.2.5, and from $\overline{f}_i$ and $\overline{e}_i$ being inverses to one another, proved in Theorem 6.2.7. □

Given the local nature of the shifted operators, meaning that $\overline{f}_i$ looks only at the positions of entries $\overline{i}, i, \overline{i+1}, i+1$, axioms A3–A6 for $|i-j| \geq 2$ are easy to establish. Corollary 6.2.8 helps to establish axioms A3 and A4 in the case $j = i$ (and axioms A5 and A6 are vacuous in this case). For the remaining cases, $j = i \pm 1$, axioms A3–A6 are resolved with the help of the following lemma.

Lemma 6.2.10. *Let T be a semistandard shifted tableau. Then for $i > 1$, $\overline{f}_i(T)$ either has one more $i-1$-free entry $\overline{i-1}, i-1$ or one fewer $i-1$-free entry $\overline{i}, i$ and not both, and for $i \geq 1$, $\overline{f}_i(T)$ either has one more $i+1$-free entry $\overline{i+1}, i+1$ or one fewer $i+1$-free entry $\overline{i+2}, i+2$ and not both.*

Proof. The net effect of $\overline{f}_i$ on the weight is to remove an entry $\overline{i}$ or i and create an entry $\overline{i+1}$ or $i+1$. If the removed entry $\overline{i}, i$ was $i-1$-blocked, then some entry $\overline{i-1}, i-1$ that was $i-1$-blocked becomes $i-1$-free; otherwise the removed entry $\overline{i}, i$ was $i-1$-free. Similarly, If the created entry $\overline{i+1}, i+1$ pairs with an otherwise $i+1$-free entry $\overline{i+2}, i+2$, then both entries become $i+1$-blocked; otherwise, the created entry $\overline{i+1}, i+1$ is $i+1$-free. Therefore we need only show that the traveling entries in the cases of L1(a),(c),(d) and L2(a),(c) of Definition 6.2.2 do not otherwise change the number of $i \pm 1$-free entries. This is a case by case analysis similar to that in the proof of Theorem 6.2.7. □

Theorem 6.2.11. *The shifted raising and lowering operators* $\overline{e}_i, \overline{f}_i$ *for* $i = 1, 2, \ldots, r$ *define a normal crystal on* $\mathrm{SSHT}_{r+1}(\gamma)$.

Proof. We show directly that the graph on $\mathrm{SSHT}_n(\gamma)$ with edges given by the shifted crystal operators define a regular graph (Definition 6.1.11). Lemma 6.2.9 proves axioms A1 and A2. For axioms A3 and A4, we have three cases based on j. For $j = i$, axiom A4 is vacuous, and the characterization of φ_i and ε_i in Corollary 6.2.8 combined with Lemma 6.2.4 shows that $\varphi_i(\overline{e}_i(T)) = \varphi_i(T) - 1$ and $\varepsilon_i(\overline{e}_i(T)) = \varepsilon_i(T) + 1$, which implies

$$\Delta_i \varphi_i(T) = \varphi_i(T) - \varphi_i(\overline{e}_i(T)) = 1 = \varepsilon_i(\overline{e}_i(T)) - \varepsilon_i(T) = \Delta_i \varepsilon_i(T),$$

thereby proving axiom A3 for $j = i$. For $|i - j| \geq 2$, $\varphi_j(\overline{e}_i(T)) = \varphi_j(T)$ and $\varepsilon_j(\overline{e}_i(T)) = \varepsilon_j(T)$, therefore $\Delta_i \varphi_j(T) = 0 = \Delta_i \varepsilon_j(T)$, establishing axioms A3 and A4. For $|i - j| = 1$, Lemma 6.2.10 ensures that either $\varphi_i(\overline{e}_i(T)) = \varphi_i(T) + 1$ or $\varepsilon_i(\overline{e}_i(T)) = \varepsilon_i(T) - 1$ but not both, again establishing axioms A3 and A4.

For axioms A5 and A6, we have the same three cases based on j. For $i = j$, both axioms are vacuous. For $|i - j| \geq 2$, axiom A6 is vacuous. For axiom A5, note that $|i - j| \geq 2$ implies $\{i, i+1\} \cap \{j, j+1\} = \varnothing$, therefore $\overline{e}_i, \overline{f}_i$ do not alter the relative positions of $j, j+1$, and neither do $\overline{e}_j, \overline{f}_j$ alter the relative positions of $i, i+1$. Therefore i-operators and j-operators commute as required. Finally, consider the case $|i - j| = 1$.

For axiom A5, note that by Lemma 6.2.10, we have

$$\nabla_i \varepsilon_j(T) = 0 \;\Rightarrow\; \varepsilon_j(\overline{f}_i(T)) = \varepsilon_j(T) \;\Rightarrow\; \begin{cases} \overline{f}_i \text{ removes a } j\text{-free entry } i & \text{if } j = i-1, \\ \overline{f}_i \text{ creates a } j\text{-blocked entry } i+1 & \text{if } j = i+1. \end{cases}$$

Applying this, we have $\overline{f}_i$ removes an $(i-1)$-free entry i therefore the rightmost $(i-1)$-free entry $i-1$ is the same in T and in $\overline{f}_i(T)$. Similarly, $\overline{f}_{i-1}$ will create an i-blocked entry i, ensuring that the leftmost i-free entry i is the same for T and $\overline{f}_{i-1}(T)$. Combining these, we see that $\overline{f}_{i-1}\overline{f}_i(T) = \overline{f}_i(T)\overline{f}_{i-1}(T)$ as desired. The case $j = i+1$ is identical, and the analysis for $\Delta_i \varphi_j(T) = 0$ is analogous.

Finally, for axiom A6, by Lemma 6.2.10 we have

$$\nabla_i \varepsilon_j(T) = -1 \;\Rightarrow\; \varepsilon_j(\overline{f}_i(T)) = \varepsilon_j(T) + 1 \;\Rightarrow\; \begin{cases} \overline{f}_i \text{ removes a } j\text{-blocked entry } i & \text{if } j = i-1, \\ \overline{f}_i \text{ creates a } j\text{-free entry } i+1 & \text{if } j = i+1. \end{cases}$$

Applying this to T, we have $\overline{f}_i$ removes an $(i-1)$-blocked entry i, allowing another $(i-1)$-free entry $i-1$ to manifest. Applying $\overline{f}_{i-1}^2$ changes the newly created $(i-1)$-free entry $i-1$ of $\overline{f}_i(T)$ and the original rightmost $(i-1)$-free entry $i-1$ of T to i so that a final application of $\overline{f}_i$ changes the latter to $i+1$. In the other direction, $\overline{f}_{i-1}$ creates an i-free entry i. Applying $\overline{f}_i^2$ changes the newly created i-free entry i of $\overline{f}_{i-1}(T)$ and the original rightmost i-free entry i of T to $i+1$ so that a final application of $\overline{f}_{i-1}$ yields the same result, and we have $\overline{f}_i\overline{f}_{i-1}^2\overline{f}_i(T) = \overline{f}_{i-1}\overline{f}_i^2(T)\overline{f}_{i-1}(T)$ as desired. The case $j = i+1$ is identical, and the analysis for $\Delta_i\varphi_j(T) = -1$ is analogous. $\square$

Using Theorem 6.2.11, we give a new proof of the Schur positivity of Schur's P-functions. Note that this characterization of the Schur coefficients of a Schur's P-function is more explicit than Sagan's and Worley's shifted insertion rule [35, 43] or Assaf's dual equivalence characterization [2].

Definition 6.2.12. *For γ a strict partition, the set of* Yamanouchi shifted tableaux of shape γ*, denoted by* Yam(γ)*, is the set of semistandard shifted tableaux T of shape γ such that $\varphi_i(T) = 0$ for all i.*

For example, the Yamanouchi shifted tableaux of shape $(4,3,1)$ are shown in Figure 6.21. The Yamanouchi shifted tableaux are precisely the highest weights of the normal crystal $(\mathrm{SSHT}_n(\gamma), \{\overline{e}_i, \overline{f}_i\}_{1\leq i<n}, \mathrm{wt})$. By Corollary 6.1.10, this gives an explicit characterization of the Schur expansion of a Schur's P-polynomial.

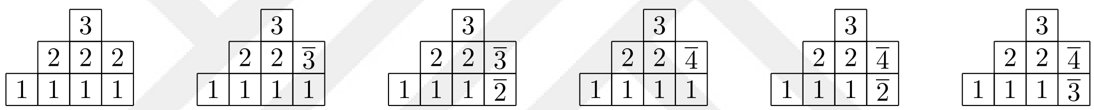


Figure 6.21: The Yamanouchi shifted tableaux of strict shape $(4,3,1)$.

Corollary 6.2.13. *For γ a strict partition, we have*

$$P_\gamma(x_1,\ldots,x_n) = \sum_{T\in\mathrm{Yam}(\gamma)} s_{\mathrm{wt}(T)}(x_1,\ldots,x_n). \tag{2.3}$$

In particular, Schur's P-polynomial is Schur positive with coefficients $g_{\gamma,\lambda} = \#\{T \in \mathrm{Yam}(\gamma) \mid \mathrm{wt}(T) = \lambda\}$.

For example, from Figure 6.21 we compute

$$P_{(4,3,1)}(x_1,\ldots,x_n) = s_{(4,3,1)}(x_1,\ldots,x_n) + s_{(4,2,2)}(x_1,\ldots,x_n) + s_{(3,3,2)}(x_1,\ldots,x_n)$$
$$+ s_{(4,2,1,1)}(x_1,\ldots,x_n) + s_{(3,3,1,1)}(x_1,\ldots,x_n) + s_{(3,2,2,1)}(x_1,\ldots,x_n).$$

In order to ease computations of the Schur expansion of a Schur's P-polynomial using Corollary 6.2.13, we have the following necessary condition for a semistandard shifted tableau to be Yamanouchi.

Lemma 6.2.14. *If T is a Yamanouchi shifted tableaux, then any unmarked entry on row i is equal to i. In particular the leftmost box in row i is labeled i.*

Proof. This is true for the first row, as the unmarked cells on the first row come last in the reading word and anything bigger than 1 can not be paired off, violating the Yamanouchi condition. Now assume row k has a box labeled $k+i$, $i>0$ and no row below has an unmarked entry larger than the row index. As there is no cell labeled $k+i-1$on a row strictly below, this $k+i$ needs to be matched to $\overline{k+i-1}$, which can only happen if the leftmost box of the row below is labeled $k-1+i$, violating our assumption. □

To demonstrate the utility of our formula and the description in Lemma 6.2.14, for $k>1$ and integer, consider $\delta_k = (k-1, k-2, \ldots, 1)$, the *staircase partition*, which is, in particular, strict. Then we have the following result for the coincidence of Schur polynomials and Schur's P-polynomials.

Corollary 6.2.15. *For $k>1$, we have $P_{\delta_k}(x_1,\ldots,x_n) = s_{\delta_k}(x_1,\ldots,x_n)$. Moreover, if γ is a strict partition such that $\gamma \neq \delta_k$ for any k, then $P_{\delta_k}(x_1,\ldots,x_n)$ has more than one term in its Schur expansion.*

Proof. Let $T \in \text{Yam}(\delta_n)$. Note that by Lemma 6.2.14 the highest row of T contains n, so all the entries on T are bounded by n. Also, as the leftmost entry is equal to the row index, any marked number k is on a row of index less than or equal to k. So if T contains a marked entry, T has a row i such that row i contains an entry greater than i, and row $i+1$ only contains $i+1$. By column rules, this row has is except for the rightmost cell which contains $\overline{i+1}$, as seen in Figure 6.22, left. It needs to be paired off with some $\bar{i}$, which can only be below the rightmost i on row i. Consider the bottom-most $\bar{i}$ on that column. The cell right adjacent to it can not be equal to i by Lemma 6.2.14. It can not be $\overline{i+1}$ either, as there is no way to pair off that $\overline{i+1}$. There are no other options by row and column rules, so such a tableaux does not exist.

Now assume γ is not a staircase. Then there exists some i such that $\gamma_i \geq \gamma_{i+1}+2$. The shifted tableaux that contains only ks on each row k except for the rightmost cell of row i which is labeled $\overline{i+1}$ is a Yamanouchi shifted tableaux(Figure 6.22, right), so there are at least two elements of $\text{Yam}(\gamma)$. □

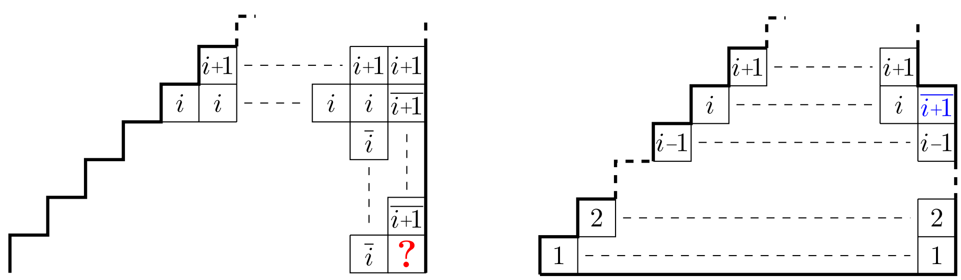


Figure 6.22: There is no shifted Yamanouchi tableaux of staircase shape that contains a marked entry(left), but such a tableaux can be found for any other shape(right).

Finally, we conclude this section with the following proof that our shifted crystal operators, while seemingly different, in fact coincide with the crystal operators defined recently by Hawkes, Paramonov, and

Schilling [15], hereafter termed HPS operators. Sagan defined the shifted insertion algorithm [35], also developed independently by Worley [43], to generalize the Robinson-Schensted insertion algorithm developed by Schensted [36] based on work of Robinson [34] and later generalized by Knuth [25]. Haiman [14] generalized shifted insertion to *mixed insertion* to prove a conjecture of Shor that rectification commutes with shifted insertion, thus resolving a question of Sagan [35]. Hawkes, Paramonov, and Schilling use Haiman's mixed insertion to define crystal operators on shifted tableaux by letting the raising and lowering operators act on the recording tableau under the mixed insertion correspondence. Rather than recall details of these algorithms, we refer the interested reader to the papers [35, 14] for details on shifted and mixed insertion, and to [15] for the explicit definition of the HPS operators.

Proposition 6.2.16. *The shifted crystal operators agree with the HPS operators.*

Proof. In the discussion to follow, we alter the presentation of Hawkes, Paramonov, and Schilling [15] only in switching their notation from English to French to coincide with ours.

The HPS reading word [15](p.13) differs from our hook reading word (Definition 6.2.1). The HPS reading word of a semistandard shifted tableau first reads all marked entries up columns from right to left, then reads all unmarked entries right to left along rows from top to bottom. They then use a bracketing rule equivalent to Definition 6.2.3 and select the rightmost i-free entry $\bar{i}$ or i on which to act. While our reading words differ, the choice of entry of the tableau on which to act is the same since any primed entry $\bar{i}$ or $\overline{i+1}$ must occur in the hook reading word before any unmarked entry i or $i+1$, with the possible exception that an entry $i+1$ on the main diagonal coming before the *sti* on the column to the left in which case both are i-blocked.

The HPS crystal operators act by first transposing the shape and promoting entries one step along the total order $\bar{1} < 1 < \bar{2} < 2 < \cdots$ if the selected entry on which to act is primed, then transposing and demoting after the action. With this caveat in mind, there are four cases, numbered 1, 2(a), 2(b), 2(c), for the HPS operators. We provide a dictionary between their cases and ours in Definition 6.2.2 and provide details only in the one nontrivial case. The correspondence is:

$$\begin{array}{lcl} \text{HPS 1} & \leftrightarrow & L1(a)/L2(a) \\ \text{HPS 2}(a) & \leftrightarrow & L1(b)/L2(b) \\ \text{HPS 2}(b) & \leftrightarrow & L1(d) \\ \text{HPS 2}(c) & \leftrightarrow & L1(c)/L2(c) \end{array}$$

where we match with L1 when an unmarked entry is selected and with L2 when a primed entry is selected (note that HPS 2(b) is vacuous in this case). The cases are direct translations of one another with the exception of case HPS 2(c) when $\bar{i}$ is selected. For this case, HPS operators first transpose and promote entries, then follow the $\overline{i+1}, i+1$-ribbon. Back in the original tableau, this does not correspond to the

$\bar{\imath}, i$-ribbon that we following in case L2(c) of Definition 6.2.2. However, the terminal points for those ribbons, different in the two cases, in fact correspond. Therefore the operators agree in their resulting actions. □

3 Crystals for the quantum queer Lie superalgebra

Lie superalgebras are algebras that, in addition, admit a $\mathbb{Z}/2\mathbb{Z}$ grading, allowing for two families of variables, one commuting and one not, to interact. Originally arising from mathematical physics in connection with supersymmetry, Lie superalgebras were formalized mathematically and classified by Kac [19]. There are two well-studied superalgebra generalization of $\mathfrak{gl}(n)$, namely $\mathfrak{gl}(m,n)$ and the queer superalgebra $\mathfrak{q}(n)$. Quantized universal enveloping algebras were developed for $\mathfrak{q}(n)$ by Sergeev [38], with the corresponding crystal theory developed by Grantcharov, Jung, Kang, Kashiwara, and Kim [11, 9]. In this section, we review the queer crystal theory arising from $U_q(\mathfrak{q}(n))$ from the combinatorial viewpoint. In § 3.1, we review queer crystal bases and define normal queer crystals as those arising from tensor products of the standard queer crystal. In § 3.2, we augment our crystal operators on semistandard shifted tableaux with an additional operator that results in a connected, normal queer crystal. In § 3.3, we formulate an alternative local axiomatization of normal queer crystals, analogous to that of Stembridge [42] for the classical case, to provide a means to prove that a given queer crystal structure is normal.

3.1 Queer crystals

Using notation and terminology from § 1.1, the *dominant weights* $\Gamma^+ \subset \Lambda$ are those $\lambda \in \Lambda$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r+1} \geq 0$ and $\lambda_i = \lambda_{i+1}$ implies $\lambda_i = \cdots = \lambda_{r+1} = 0$. In other words, Λ^+ is to partitions as Γ^+ is to *strict* partitions. We have the following combinatorial definition for queer crystals, augmenting Definition 6.1.1.

Definition 6.3.1. *A* queer crystal *of dimension $r+1$ is crystal of dimension $r+1$ together with additional* queer crystal operators $\overline{e}_0, \overline{f}_0 : \mathcal{B} \to \mathcal{B} \cup \{0\}$ *satisfying the conditions*

1. *for $b, b' \in \mathcal{B}$, $\overline{e}_0(b) = b'$ if and only if $\overline{f}_0(b') = b$, and in this case we have* $\mathrm{wt}(b') = \mathrm{wt}(b) + \alpha_1$*;*
2. *for $i = 3, 4, \ldots, r$, the operators $\overline{e}_0$ and $\overline{f}_0$ commute with e_i and f_i, and if $\overline{e}_0(b) \neq 0$ then $\varepsilon_i(\overline{e}_0(b)) = \varepsilon_i(b)$ and $\varphi_i(\overline{e}_0(b)) = \varphi_i(b)$.*

For example, the *standard queer crystal* $\mathcal{Q}(n)$, for $n \in \mathbb{Z}_{>0}$, is the standard crystal $\mathcal{B}(n)$ together with queer crystal operator $\overline{f}_0$ that acts on $\boxed{i}$ by incrementing the entry if $i = 1$ or 0 otherwise. The standard queer crystal is represented diagrammatically in Figure 6.23 by its queer crystal graph.

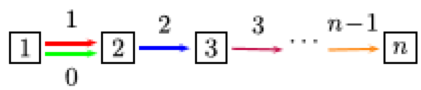


Figure 6.23: The standard queer crystal.

The notion of highest weight elements is still vital to classifying queer crystals, though now the concept is not as straightforward. Given a queer crystal $\mathcal{Q}$ of dimension $r+1$, we define automorphisms S_i, for $i = 1, 2, \ldots, r$ by

$$S_i = \begin{cases} f_i^{\mathrm{wt}(b)_i - \mathrm{wt}(b)_{i+1}}(b) & \text{if } \mathrm{wt}(b)_i \geq \mathrm{wt}(b)_{i+1}, \\ e_i^{\mathrm{wt}(b)_{i+1} - \mathrm{wt}(b)_i}(b) & \text{if } \mathrm{wt}(b)_{i+1} \geq \mathrm{wt}(b)_i. \end{cases}$$

Kashiwara [22] showed these operators satisfy the braid relations for the symmetric group, and so to any permutation w we may define S_w by $S_{i_1} S_{i_2} \cdots S_{i_k}$ whenever $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression for w. For $i = 1, 2, \ldots, r$, define the *odd crystal operators* $e_{\overline{i}}, f_{\overline{i}}$ by

$$e_{\overline{1}} = \overline{e}_0 \qquad f_{\overline{1}} = \overline{f}_0 \qquad e_{\overline{i}} = S_{w_i^{-1}} e_0 S_{w_i} \qquad f_{\overline{i}} = S_{w_i^{-1}} f_0 S_{w_i} \text{ for } i > 1 \tag{3.1}$$

where $w_i = s_2 \cdots s_i s_1 \cdots s_{i-1}$ is the shortest (in coxeter length) permutation such that $w_i \cdot \alpha_i = \alpha_1$.

Definition 6.3.2. *An element $b \in \mathcal{Q}$ of a queer crystal is a* highest weight element *if $e_i(b) = 0 = e_{\overline{i}}(b)$ for all $i = 1, 2, \ldots, r$.*

Again, one of the motivating goals of crystal theory is to use these combinatorial objects to study tensor representations of the queer superalgebra, and so again we must restict our attention to *normal queer crystals*. Connected normal queer crystals are in one-to-one correspondence with dominant weights Γ^+, which in turn index irreducible representations for the queer superalgebra. Given a dominant weight $\gamma \in \Gamma^+$, let $\mathcal{Q}(\gamma)$ denote the connected normal crystal with highest weight γ. Then $\mathrm{ch}(\mathcal{Q}(\gamma))$ is precisely the character of the irreducible representation indexed by γ, which corresponds to the Schur's P- polynomial $P_\gamma(x_1, \ldots, x_{r+1})$.

As in the classical case, we have the remarkable fact that the following combinatorial procedure on queer crystals corresponds to the tensor product of the corresponding representations.

Definition 6.3.3. *Given two queer crystals $\mathcal{Q}_1$ and $\mathcal{Q}_2$, the* tensor product $\mathcal{Q}_1 \otimes \mathcal{Q}_2$ *is the set $\mathcal{Q}_1 \otimes \mathcal{Q}_2$ together with crystal operators e_i, f_i are defined on the tensor product $\mathcal{Q}_1 \otimes \mathcal{Q}_2$ by*

$$f_i(b_1 \otimes b_2) = \begin{cases} f_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \epsilon_i(b_2), \\ b_1 \otimes f_i(b_2) & \text{if } \varphi_i(b_1) \leq \epsilon_i(b_2), \end{cases} \tag{3.2}$$

for $i = 1, 2, \ldots, r$ and for $i = 0$, we have the additional rule

$$\overline{f}_0(b_1 \otimes b_2) = \begin{cases} \overline{f}_0(b_1) \otimes b_2 & \text{if } \mathrm{wt}(b_2)_1 = \mathrm{wt}(b_2)_2 = 0, \\ b_1 \otimes \overline{f}_0(b_2) & \text{otherwise.} \end{cases} \tag{3.3}$$

For example, Figure 6.2 computes the tensor product of two copies of the standard queer crystal $\mathcal{Q}(3)$.

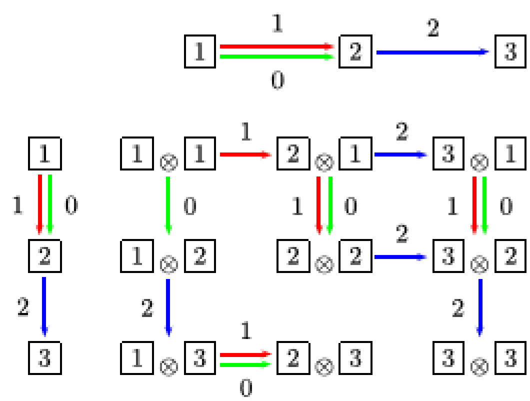


Figure 6.24: The tensor product of two standard queer crystals for $U_q(\mathfrak{q}(3))$.

Proposition 6.3.4. *Given two normal queer crystals $\mathcal{Q}_1$ and $\mathcal{Q}_2$, the tensor product $\mathcal{Q}_1 \otimes \mathcal{Q}_2$ is a normal crystal. Further, every connected normal queer crystal of dimension $r+1$ and degree k arises as a connected component in $\mathcal{Q}(r+1)^{\otimes k}$, the k-fold tensor product of the standard crystal $\mathcal{Q}(r+1)$.*

For example, Figure 6.24 constructs the unique normal queer crystal of dimension 3 and degree 2, namely $\mathcal{Q}((2,0,0))$, from the tensor product of two copies of the standard queer crystal $\mathcal{Q}(3) = \mathcal{Q}((1,0,0))$.

3.2 Queer crystals for shifted tableaux

Sergeev [38] established that the characters of irreducible tensor representations for the queer superalebra are Schur's P-functions. Grantcharov, Jung, Kang, Kashiwara, and Kim [10] developed crystal bases for the quantum queer superalgebra and gave an explicit construction of the queer crystal on semistandard decomposition tableaux, the latter being another combinatorial model for Schur's P-polynomials introduced by Serrano [39]. Grantcharov, Jung, Kang, Kashiwara, and Kim raised the question of whether an explicit queer crystal could be defined directly on semistandard shifted tableaux. To answer this affirmatively we have the following construction building on the crystal graph defined in §2.1.

Note that in a semistandard shifted tableau, both 1 and $\overline{2}$ can only be used in the first row, and any cell on the second row is at least 2. This observation ensures the shifted queer lowering operator below is well-defined and acts only in the first row.

Definition 6.3.5. *The* queer lowering operator*, denoted by $\overline{f}_0$, acts on semistandard shifted tableaux by: if T has no cell labelled* 1 *or if T has a cell labelled $\overline{2}$, then $\overline{f}_0(T) = 0$; otherwise $\overline{f}_0(T)$ changes the rightmost* 1 *in the first row of T to* 2 *if it is on the main diagonal and to $\overline{2}$ otherwise.*

For examples of the queer lowering operator on semi-standard shifted tableaux, see Figure 6.25.

The $\overline{f}_0$ edges may be identified from the normal crystal by connecting $T_1 \to T_2$ if T_1 and T_2 differ in only one, which is labelled 1 in T_1 and $\overline{2}$ in T_2 if the cell is not on the main diagonal, 2 if it is.

Definition 6.3.6. *The* queer raising operator*, denoted by $\overline{e}_0$, acts on semistandard shifted tableaux by: if T has no cell labelled $\overline{2}$ and the leftmost entry of the first row is not* 2*, then $\overline{e}_0(T) = 0$; otherwise $\overline{e}_0(T)$ changes the leftmost $\overline{2}$ in the first row of T, if it exists, or the leftmost entry in the first row, otherwise, to* 1*.*

As required for a queer crystal, the queer raising and lowering operators are inverse to one another.

Proposition 6.3.7. *The queer raising and lowering operators satisfy $\overline{e}_0(T) = T'$ if and only if $\overline{f}_0(T') = T$.*

Proof. Suppose $\overline{e}_0(T) = T'$. Since T is a semistandard shifted tableau, it has at most one $\overline{2}$ in its first row. There are two disjoint cases: either (i) T has a $\overline{2}$ in its first row, or, since $\overline{e}_0(T) \neq 0$, (ii) the leftmost entry in the first row is 2. For case (i), $\overline{e}_0$ changes the $\overline{2}$ in T to become the rightmost 1 in T', and so $\overline{f}_0$ will act nontrivially on T' by changing this entry back to $\overline{2}$. For case (ii), $\overline{e}_0$ changes the leftmost entry in the first row to a 1 in T', and so $\overline{f}_0$ will act nontrivially on T' by changing this entry back to 2. Thus $\overline{f}_0(T') = T$.

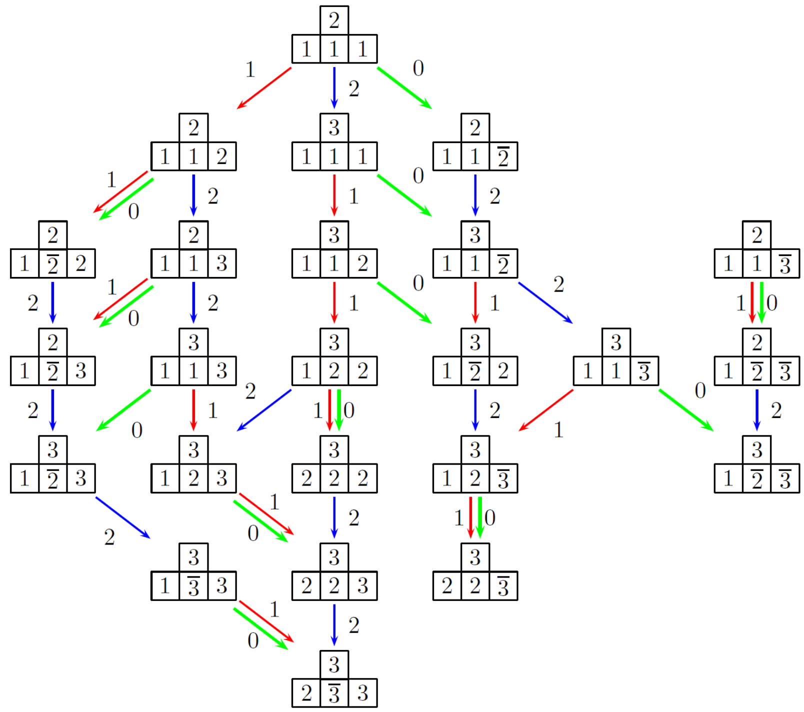


Figure 6.25: The queer crystal structure on semi-standard shifted tableaux of shape $(3, 1)$ with entries $\{\overline{1}, 1, \overline{2}, 2, \overline{3}, 3\}$ and no marks on the main diagonal.

Suppose $\overline{f}_0(T') = T$. Since $\overline{f}_0(T') \neq 0$, T' must have a 1 and no $\overline{2}$ in the first row. Again, we have two disjoint cases: either (i) T has a unique 1 in its first row on the main diagonal, or (ii) the rightmost 1 in the first row of T is not on the main diagonal. For case (i), $\overline{f}_0$ changes the unique 1 to a 2 in T, and so $\overline{e}_0$ will act nontrivially on T by changing this entry back to 1. For case (ii), $\overline{f}_0$ changes the rightmost 1 in T' to a $\overline{2}$ in T, and so $\overline{e}_0$ will act nontrivially on T by changing this entry back to 1. Thus $\overline{e}_0(T) = T'$. $\square$

As we saw in the example for $\gamma = (3, 1)$, the normal crystal $(\mathrm{SSHT}_n(\gamma), \{\overline{e}_i, \overline{f}_i\}_{1\leq i<n}, \mathrm{wt})$ is not always connected. However, the queer crystal obtained by augmenting this with the queer raising and lowering operators is connected. To prove this, we have the following result.

Lemma 6.3.8. *For a strict partition γ, let $T \in \mathrm{Yam}(\gamma)$ have the smallest primed entry given by $\overline{k+1}$ for some $k \geq 2$. Then $S_{w_k}(T)$ has a coordinate labeled $\overline{2}$.*

Proof. Let $(a_1, a_2, \ldots, a_n)$ denote the weight of T. Consider boxes on T with labels $\leq k$. As $T \in \mathrm{Yam}(\gamma)$, we have $m_i(T) = 0$ for all i, and so these boxes form a subdiagram θ of size $(a_1, a_2, \ldots, a_k)$ with all the boxes on row i are labeled i, as illustrated in the left tableau of Figure 6.26. Note that we have $a_1 > a_2 > \ldots > a_k \geq a_{k+1}$.

Let T' denote $S_{k-1}S_{k-1}\cdots S_1(T)$. Then, T' has weight $(a_k, a_1, a_2, \ldots a_{k-1}, a_{k+1}, a_{k+2}, \ldots a_n)$ and outside θ it matches T exactly. On θ, the ith row has a_k cells labeled i, one cell labeled $\overline{i+1}$ and $a_i - a_k - 1$ cells labeled $i+1$ for all $i < k$, and the kth row contains a_k cells labeled k. In particular, the $k+1$st southwest to northeast diagonal is formed by the primed entries $\overline{2}, \overline{3}, \ldots, \overline{k}$. The operation $S_iS_{i+1}\ldots S_k(T')$ preserves the cell labeled $\overline{i}$ on the said diagonal, as well as all entries with labels less than i. This implies that $S_{w_k}(T) = S_2S_3\ldots S_k(T')$ contains an entry marked $\overline{2}$. $\square$

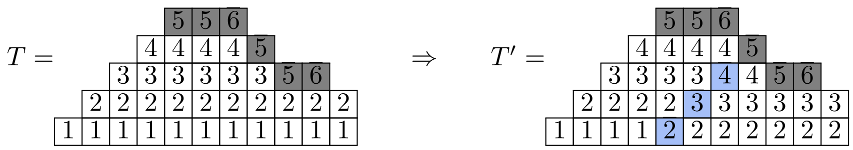


Figure 6.26: An example for Lemma 6.3.8, with the outside of θ shown in gray.

Theorem 6.3.9. *For γ a strict partition, the shifted and queer raising and lowering operators $\overline{e}_i, \overline{f}_i$ for $i = 1, 2, \ldots, r$ and $\overline{e}_0, \overline{f}_0$ define a connected, queer crystal on* $\mathrm{SSHT}_{r+1}(\gamma)$.

Proof. Note that without the shifted raising and lowering operators, we have a collection of crystals connected to highest weights $T \in A(\gamma)$. We will show that with the added shifted operators, the shifted tableaux T with weight γ is the unique highest weight element, implying the crystal is connected. Let $T \in A(\gamma)$ be a Yamanouchi shifted tableau with $\mathrm{wt}(T) \neq \gamma$. By Lemma 6.2.14, any unmarked entry in row i must equal i, so T must contain a marked entry. If T contains a cell labeled $\overline{2}$, $\overline{e}_1(T) \neq 0$ is an element of higher weight. Otherwise let $k > 2$ denote the smallest primed entry in T. By Lemma 6.3.8, $S_{w_k}(T)$ has a coordinate labeled $\overline{2}$, so $\overline{e}_0 S_{w_k}(T) \neq 0$ and consequentially $\overline{e}_k(T) \neq 0$. Therefore the graph is connected.

Definition 6.3.1 holds for the shifted raising and lowering operators by Theorem 6.2.7 and for the queer raising and lowering operators by Proposition 6.3.7. To see that the queer crystal is normal, by Proposition 6.2.16, we know that applying Sagan's shifted insertion to the word $b_n \ldots b_2 b_1$ results in a shifted tableau that can be identified with the type A crystal basis $b_1 \otimes b_2 \otimes \cdots \otimes b_n$. We show by induction that our queer operator defined directly on shifted tableaux agrees with Definition 6.3.3, noting that the base case follows immediately from the standard queer crystal. Consider, then, a shifted tableau T of degree $n-1 \geq 1$. By Definition 6.3.3, we have

$$f_0\left(\boxed{i} \otimes T\right) = \begin{cases} \boxed{2} \otimes T & \text{if } i = 1 \text{ and } \mathrm{wt}(T)_1 = \mathrm{wt}(T)_2 = 0, \\ 0 & \text{if } i \geq 2 \text{ and } \mathrm{wt}(T)_1 = \mathrm{wt}(T)_2 = 0, \\ 0 & \text{if } f_0(T) = 0 \text{ and either } \mathrm{wt}(T)_1 > 0 \text{ or } \mathrm{wt}(T)_2 > 0, \\ \boxed{i} \otimes f_0(T) & \text{if } f_0(T) \neq 0. \end{cases} \tag{3.4}$$

Consider the shifted insertion of i into T, denoted by $T \leftarrow i$. If $\mathrm{wt}(T)_1 = \mathrm{wt}(T)_2 = 0$, then inserting 1 into T has the same bumping path as inserting 2 into T, since all letters of T are larger than both 1 and 2. Therefore $T \leftarrow 2$ is precisely $T \leftarrow 1$ with the 1 changed to a 2. In particular,

$$f_0(T \leftarrow 1) = T \leftarrow 2 \text{ whenever } \mathrm{wt}(T)_1 = \mathrm{wt}(T)_2 = 0.$$

Similarly, if $\mathrm{wt}(T)_1 = \mathrm{wt}(T)_2 = 0$, then $T \leftarrow i$ has no entry equal to 1 for $i \geq 2$, and so

$$f_0(T \leftarrow i) = 0 \text{ whenever } \mathrm{wt}(T)_1 = \mathrm{wt}(T)_2 = 0 \text{ and } i \geq 2.$$

If $f_0(T) = 0$ and either $\mathrm{wt}(T)_1 > 0$ or $\mathrm{wt}(T)_2 > 0$, then either T has an entry $\overline{2}$ in the first row, or T has only entries weakly greater than 2 in the first row. In these cases, $T \leftarrow i$ will have the same property for $i \geq 2$. For $T \leftarrow 1$, the 1 will either bump the $\overline{2}$, if it exists, or will bump a 2 in the first row, if it doesn't, with the result that $T \leftarrow 1$ will have a $\overline{2}$ in the first row. Therefore, in all cases,

$$f_0(T \leftarrow i) = 0 \text{ whenever } f_0(T) = 0 \text{ and either } \mathrm{wt}(T)_1 > 0 \text{ or } \mathrm{wt}(T)_2 > 0.$$

Finally, if $f_0(T) \neq 0$ and either $\mathrm{wt}(T)_1 > 0$ or $\mathrm{wt}(T)_2 > 0$, then we must in fact have a 1 in the first row and no $\overline{2}$ in T. In this case, $T \leftarrow i$ will not affect any entries 1, $\overline{2}$, or 2 in T for $i \geq 3$, and for $i \geq 2$ will at most insert an additional 2. In these cases, the rightmost 1 of T changing to 2 does not alter the insertion path, so $f_0(T \leftarrow i) = f_0(T) \leftarrow i$. For the case $T \leftarrow 1$, the inserted 1 might bump a 2, but insodoing pushes it to a higher row since the 2 cannot be on the diagonal in the first row (since T has a 1). Similarly, the insertion $f_0(T) \leftarrow 1$ will have the 1 bump the newly created $\overline{2}$ which will then follow the bumping path of $T \leftarrow 1$. Thus again we have $f_0(T \leftarrow 1) = f_0(T) \leftarrow 1$, and so

$$f_0(T \leftarrow i) = f_0(T) \leftarrow i \text{ whenever } f_0(T) \neq 0.$$

Therefore we have shown the following,

$$f_0\left(T \leftarrow i\right) = \begin{cases} T \leftarrow 2 & \text{if } i = 1 \text{ and } \mathrm{wt}(T)_1 = \mathrm{wt}(T)_2 = 0, \\ 0 & \text{if } i \geq 2 \text{ and } \mathrm{wt}(T)_1 = \mathrm{wt}(T)_2 = 0, \\ 0 & \text{if } f_0(T) = 0 \text{ and } \mathrm{wt}(T)_1 > 0 \text{ or } \mathrm{wt}(T)_2 > 0, \\ f_0(T) \leftarrow i & \text{if } f_0(T) \neq 0. \end{cases} \tag{3.5}$$

The result follows by comparison of cases between (3.4) and (3.5) and induction on n. □

Using the odd crystal operators $e_{\overline{i}}$, we may characterize our normal queer crystals on semistandard shifted tableaux by their highest weights as in Definition 6.3.2. For example, removing the $\overline{f}_0$ edges and inserting edges $f_{\overline{1}} = f_0$ and $f_{\overline{1}} = S_1 S_2 f_0 S_2 S_1$ for the queer crystal for $\mathrm{SSHT}_3(3,1)$ results in the crystal shown in Figure 6.27, which clearly has a unique highest weight.

By Proposition 6.3.4, the tensor product of two normal queer crystals is again a normal queer crystal. This gives an explicit formula for the Schur P-expansion of a product of Schur's P-polynomials.

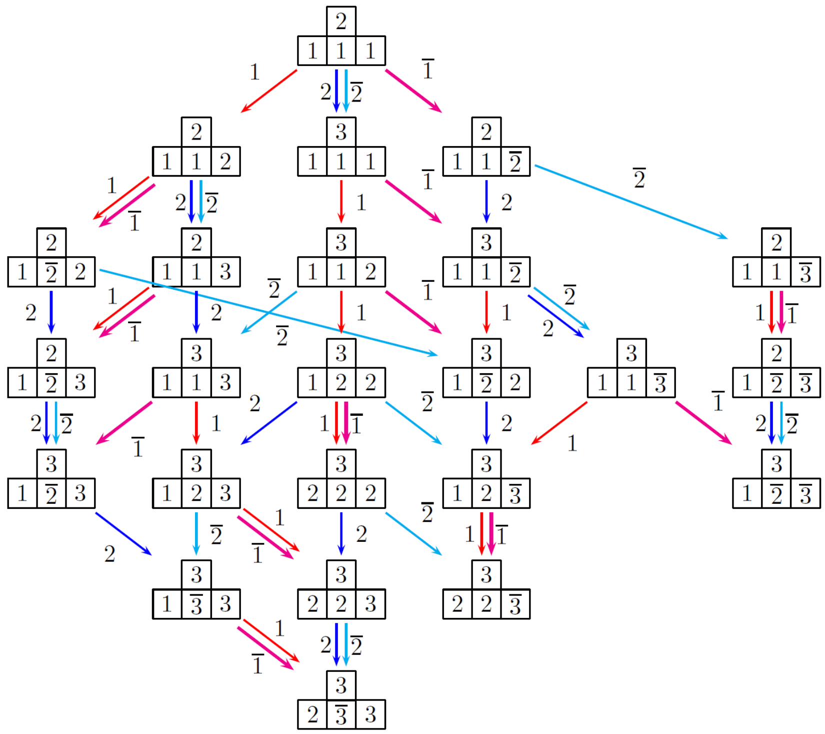


Figure 6.27: The queer crystal $(\mathrm{SSHT}_3(3,1), \{\overline{f}_1, \overline{f}_{\overline{1}}, \overline{f}_2, \overline{f}_{\overline{2}}\}, \mathrm{wt})$.

Corollary 6.3.10. *For γ, δ strict partitions, we have*

$$P_\gamma(x_1, \ldots, x_n) P_\delta(x_1, \ldots, x_n) = \sum_{\substack{(S,T) \in \mathrm{SSHT}(\gamma) \times \mathrm{SSHT}(\delta) \\ \overline{e}_i(S \otimes T) = 0 = \overline{e}_{\overline{i}}(S \otimes T) \forall i}} P_{\mathrm{wt}(S) + \mathrm{wt}(T)}(x_1, \ldots, x_n), \tag{3.6}$$

where the sum of weights is coordinatewise. In particular, the product of Schur's P-polynomials is Schur P-positive with coefficients given by

$$f^{\epsilon}_{\gamma,\delta} = \#\{(S,T) \in \mathrm{SSHT}(\gamma) \times \mathrm{SSHT}(\delta) \mid \mathrm{wt}(S) + \mathrm{wt}(T) = \epsilon,\ \overline{e}_i(S \otimes T) = 0 = \overline{e}_{\overline{i}}(S \otimes T) \forall i\}. \tag{3.7}$$

3.3 Local characterization for queer crystals

Following Stembridge [42], we desire a local characterization of normal queer crystals to aide in proving that a given queer crystal is, in fact, normal.

To this end, a *queer graph of dimension* $r+1$ will mean a directed, colored graph $\mathcal{Y}$ with directed edges $e_i(x)\xrightarrow{i}x\xrightarrow{i}f_i(x)$ for $i = 0, 1, 2, \ldots, r$, and we adopt notation from §1.3. Every queer crystal gives us a queer graph.

Definition 6.3.11. *A queer crystal's graph* $\mathcal{Y}$ *is a* queer regular graph *if the following hold:*

(B0) The subgraph $\mathcal{Y}_+$ *generated by edges with non-zero labels is a regular graph.*

(B1) all 0 *paths have length* 1*, and* $\varphi_0(x) + \varepsilon_0(x) = 1$ *if and only if* $wt_1(x) + wt_2(x) > 0$*;*

(B2) for every vertex x*, there is at most one edge* $x\xleftarrow{0}y$ *and at most one edge* $x\xrightarrow{0}z$*;*

(B3) assuming e_0x *is defined,* $\Delta_0\varphi_i(x) + \Delta_0\varepsilon_i(x) = \begin{cases} 2 & \text{if } i \le 1 \\ -1 & \text{if } i = 2 \\ 0 & \text{if } i \ge 3 \end{cases}$ *;*

(B4) assuming e_0x *is defined,* $\begin{array}{ll} \Delta_0\varphi_i(x) \ge 0, \Delta_0\varepsilon_i(x) > 0 & \text{if } i = 1 \\ \Delta_0\varphi_i(x) \le 0, \Delta_0\varepsilon_i(x) \le 0 & \text{if } i = 2 \\ \Delta_0\varphi_i(x) = 0, \Delta_0\varepsilon_i(x) = 0 & \text{if } i \ge 3 \end{array}$

(B5) For $i \ge 2$ $e_ix = e_0y = z \Rightarrow f_if_0z = f_0f_iz$*;*
For $i = 1$ *or* $i \ge 3$ $f_ix = f_0y = z, x \ne y \Rightarrow e_ie_0z = e_0e_iz$*;*

(B6) assuming e_0x *is defined,* $\begin{array}{l} \Delta_0\varphi_1(x) = 1 \Rightarrow \varepsilon_1(x) = 0 \text{ and } e_1x = e_0x \\ \Delta_0\varepsilon_2(x) = 0 \Leftrightarrow \varepsilon_2(x) = 0 \end{array}$.

Axiom B0 relies on Stembridge's characterization of regular graphs (Definition 6.1.11). The other six axioms in Definition 6.3.11 give the analog of the corresponding axioms in Definition 6.1.11 for the queer raising and lowering operators. To begin to justify our definition, we have the following result.

Theorem 6.3.12. *Every normal queer crystal is a regular queer graph.*

Proof. Axioms B0 and B2 follow directly from the definition of queer crystals. Also, for any $i \ge 3$, the operators $\overline{e}_i$ and $\overline{f}_i$ only affect cells labeled i, $\overline{i}$, $i+1$ or $\overline{i+1}$ so they are fully independent of the queer operators $\overline{e}_0$ and $\overline{f}_0$. This is enough to establish the statements for $i \ge 3$, therefore we need only look at how 0 moves interact with 1 and 2 paths.

For axiom B1, assume $f_0(x) = y$. Then either y contains a $\overline{2}$ or the leftmost box on its first row is labeled 2. In either case we have $f_0y = 0$, so all 0 strings have length 1, and $\varphi_0(x) + \varepsilon_0(x) \le 1$. For the second part, assume $\varphi_0(x) + \varepsilon_0(x) = 0$. Then $e_0(x) = 0$, so x contains no $\overline{2}$, and $f_0(x) = 0$ implies x contains no 1. Also the leftmost box of the first row of x can not be 2 as $e_0(x) = 0$, so x contains no 2 either.

For axioms B3, B4, and B6, recall $\varepsilon_i(T) = m_i(T)$ and $\varphi_i(T)$ is equal to the difference between the number of $i+1$s and the number of is to the right of w_q, where q is the largest index such that $m_i(w(T), q) =$

$m_i(w(T)) > 0$ (if $m_i(w(T)) \leq 0$, $\varphi_i(T)$ is the difference between the total number of $i+1$s and the total number of is). First consider how e_0 affects the 1-string. Assume $e_0(x) = y$. Then y contains no $\overline{2}$, and at least one 1. If it contains a single 1, it is on the main diagonal, and f_0 changes to a 2, so that we have $e_1(x) = e_0(x) = y$, and (B1) implies $\Delta_0\varphi_1(x) = \Delta_0\varepsilon_1(x) = 1$. Note that in this case $m_1(w(x)) = \varepsilon_1(x) = 0$. If y contains $k > 1$ cells labeled 1, e_0 acts by changing the rightmost 1 to a $2'$. The length of the φ_1 string is given by the number of 2s on the first row and remains unchanged. The m_1 value is decreased by 2 as the rightmost 1 is deleted, and replaced with a 2 that comes to the left of all other 1s in the reading word. In this case we have $\Delta_0\varphi_1(x) = 2$, $\Delta_0\varepsilon_1(x) = 0$.

Now let us look at the possibilities for $\Delta_0\varepsilon_2(x)$ and $\Delta_0\varphi_2(x)$. Assume $e_0(x) = y$. Note that the difference between the reading words of x and y is that we have one less 2 and an extra 1. If $f_2(x) = 0$, then $f_2(y) = 0$ as well, and the difference between the total number of 3s and 2s is increased by one, which means $\Delta_0\varepsilon_2(x) = 0$ and $\Delta_0\varphi_2(x) = -1$. This deals with the case $\varepsilon_2(x) = 0$. Let us now assume $\varepsilon_2(x) = k > 0$. Assume q is the largest index where m_2 is achieved. If x has a $2'$, that comes before all the other 2 in the reading word. Otherwise, x has no cells labeled 1, so the second row contains no 2, and the leftmost 2 on the first row comes before all other 2 on the reading word. In both cases, the 2 that turns in to 1 with the e_0 move has an index $\leq q$, so the e_0 move increases m_2 by 1, and does not change φ_2. We have $\Delta_0\varepsilon_2(x) = -1$ and $\Delta_0\varphi_2(x) = 0$.

Finally, for axiom B5, let us first assume that we have $e_2(x) = e_0(y) = z$. The action f_0 on z creates a new 2 or $\overline{2}$ that comes before all the other 2 on the reading word, so $f_2(y)$ is defined, and the algorithm selects the same cell labeled 2 as in $f_2(z)$. Furthermore as the moves (L1) are independent of the changes happening strictly to the left and weakly below the selected cell, it commutes with the action of (f_0): $f_0(f_2(z)) = f_2(f_0(z))$. Now let us assume $f_1(x) = f_0(y) = z$ with $x \neq y$. As there can be no cells labeled $\overline{2}$ on the second row, and no cells labeled $\overline{1}$ anywhere on a shifted tableaux, f_1 acts on x by (L1) a,b or d of Definition 6.2.2. We can further eliminate (L1) b and (L1) d, as $x \neq y$. So the first row of z contains a $\overline{2}$ adjacent to a 2. $e_1(e_0(z)) = e_0(e_1(z))$ gives the tableaux where both these entries are replaced with 1. □

Similar to Figure 6.6 giving a graphical illustration of the local connected component of a regular crystal when considering only two string colors, the following two lemmas give graphical illustrations of the local structure of a regular queer crystal for two color components involving 0 edges.

Lemma 6.3.13. *Connected components of the subgraph generated by $\overline{f}_0$ and f_1 of a regular queer graph are of the form shown in Figure 6.28.*

Proof. By (B3), every connected component will have at least one 1 edge. Consider a maximal 1 string of length $k \geq 1$ on a connected component. Let x be the on this string with $\varepsilon_1(x) = 0$.

If $\overline{f}_0 x = y$ for some y, then as $\varepsilon_1(y)$ can not be less than 0, $\Delta_0\varepsilon_1(x) = 0$ by (B4). This can not happen as $\Delta_0\varphi_1(x) = 2$ would imply y is on a longer 1 string. So $\overline{f}_0 x = 0$, and by (B1) there exists y such that $\overline{e}_0 x = y$. If $e_1 x \neq y$, by (B5) there exists z that $z = \overline{e}_0 e_1 x = e_1(\overline{e}_0(x))$. This can not happen either, as in this case we would have $\varepsilon_1(\overline{e}_0(x)) \neq 0$,$\Delta_0\varphi_1(\overline{e}_0(x)) = 2$ implying $\varepsilon_1(x) \geq 1$.

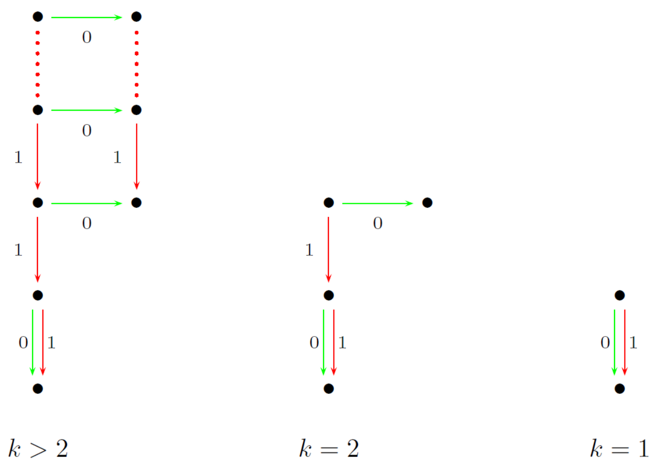


Figure 6.28: Possible connected components for $\overline{f}_0$ and $\overline{f}_1$ in a regular queer graph.

So, we must have $\overline{e}_0(x) = e_1(x)$. If $k = 1$, we are done. If $k > 1$, consider $z = e_1(e_1(x))$. By (B1), either $\overline{e}_0(z)$ or $\overline{f}_0(z)$ exists. If $\overline{e}_0(z)$ existed, by (B6) we would have $\Delta_0\varphi_1(\overline{e}_0(x)) = 2$, $\Delta_0\varepsilon_1(\overline{e}_0(x)) = 0$, contradicting the maximality of the 1 string. Then $\overline{f}_0(z)$ exists and is not equal to $f_1(x)$ as 0 strings have length 1. $\overline{f}_0(z)$ satisfies $\varepsilon_1(\overline{f}_0(x)) = k - 2$ and $\varphi_1(\overline{f}_0(z)) = 0$, so it is on a 1 string of size $k - 2$, and the strings are connected as shown in the diagram by (B5). □

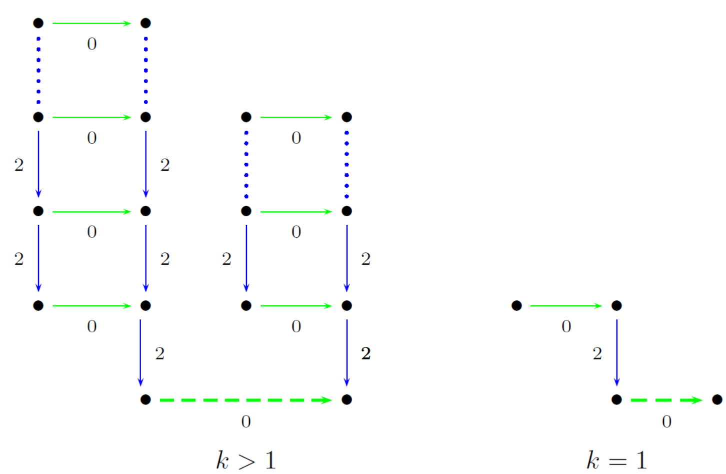


Figure 6.29: Possible connected components for $\overline{f}_0$ and $\overline{f}_2$ in a regular queer graph.

Lemma 6.3.14. *Connected components of the subgraph generated by f_0 and f_2 of a regular queer graph are of the form shown in Figure 6.29, with optional f_0 edge presented by dashed lines.*

Proof. For a connected component, let x be on a maximal 2 string with $\varphi_2(x) = 0$, $\varepsilon_2(x) = k > 0$. By (B1), either $f_0(x)$ or $e_0(x)$ is non-zero. The first is not possible, as $z = f_0(x)$ and $\varepsilon_2(z) \leq k$ imply $\Delta_0\varphi_2(z) = \varphi_2(z) = -1$, which is impossible. So we must have some $z = e_0(x)$. By (B6), $\Delta_0\varepsilon_2(x) = -1$, so that $\varepsilon_2(y) = k - 1$. (B4) implies that all the edges on the 2 string of y will commute with 0 edges.

As any 0 string is of length 1, if $w := f_2^k(x)$ is not connected to any 0 edges, we are done. There can not be an edge $e_0(w)$ by maximality of k. If there is a vertex $f_0(w)$, then by maximality of k, $\varphi_2(f_0(w)) = k-1$. As $f_0(f_0(w)) = 0$, if $k > 1$, the 2 string of $f_0(w)$ needs to be connected to a $k-2$ string by 0 edges that commute with 2 edges, completing the connected component. ☐

Combining Lemmas 6.3.13 and 6.3.14, we have the following simple characterization of regular queer graphs.

Corollary 6.3.15. *A regular crystal with* 0 *strings of length* 1 *is a regular queer graph if and only if connected* $0-1$ *components are characterized as in Figure 6.28, connected* $0-2$ *components are characterized as in Figure 6.29 and* 0 *edges commute with* i *strings for* $i > 2$.

Finally, we believe that the converse of Theorem 6.3.12 holds as well. For example, Figure 6.30 shows the two connected normal queer crystals of degree and dimension 3, namely $\mathcal{Q}(2,1,0)$ and $\mathcal{Q}(3,0,0)$. With this as a reference, we have the following result.

Proposition 6.3.16. *Every regular queer graph of degree* 3 *is a normal queer crystal.*

Proof. Let v be a highest weight vertex (a priori not necessarily unique) of a connected, regular queer graph of degree 3. By axiom (B5), all edges f_i for $i \geq 3$ commute with f_0, so it is enough to consider dimension 3 as well. In this case, v has two possible weights, $(2,1,0)$ or $(3,0,0)$.

Consider first the case of $\mathrm{wt}(v) = (2,1,0)$. As every regular queer graph is a regular graph when f_0 is ignored, we must have f_1 and f_2 as shown on the left side of Figure 6.31. Note that $e_0(v) = 0$ as it is a highest weight, and $\varepsilon_1(v) = 1$, so we must be in case $k = 1$ of Figure 6.28. Therefore $f_0(v) = f_1(v)$. Similarly, as an e_0 move from a vertex of weight $(1,0,2)$ is not possible, that vertex also has an f_0 edge that commutes with the f_1 edge, as shown in the center of Figure 6.31. By axiom (B5) of Definition 6.3.11, the vertex with weight $(2,0,1)$ has an f_0 edge satisfying $f_0(f_2(v)) = f_2(f_0(v))$. As this f_0 edge decreases the 1-head by 2, we must be in case $k = 2$ of Figure 6.28 which gives us the final f_0 edge as shown on the right side of Figure 6.31. Note that as all 0-strings are of length 1, no more edges are possible. Therefore the graph is exactly the normal queer crystal with highest weight $(2,1,0)$ seen in Figure 6.30.

Consider last the case of $\mathrm{wt}(v) = (3,0,0)$. Again, since every regular queer graph is a regular graph when f_0 is ignored, we must have f_1 and f_2 as shown on the left side of Figure 6.32. Since $e_0(v) = 0$ and $\varepsilon_1(v) = 3$, we must be in case $k = 3$ of Figure 6.28. Therefore we must have another vertex, say w, not on this component, of weight $(2,1,0)$ such that $f_0(v) = w$ and $e_1(w) = 0$. Since w is on a regular crystal, it must be a highest weight, and so we have the vertices depicted on the right of Figure 6.32. Applying Figure 6.28,

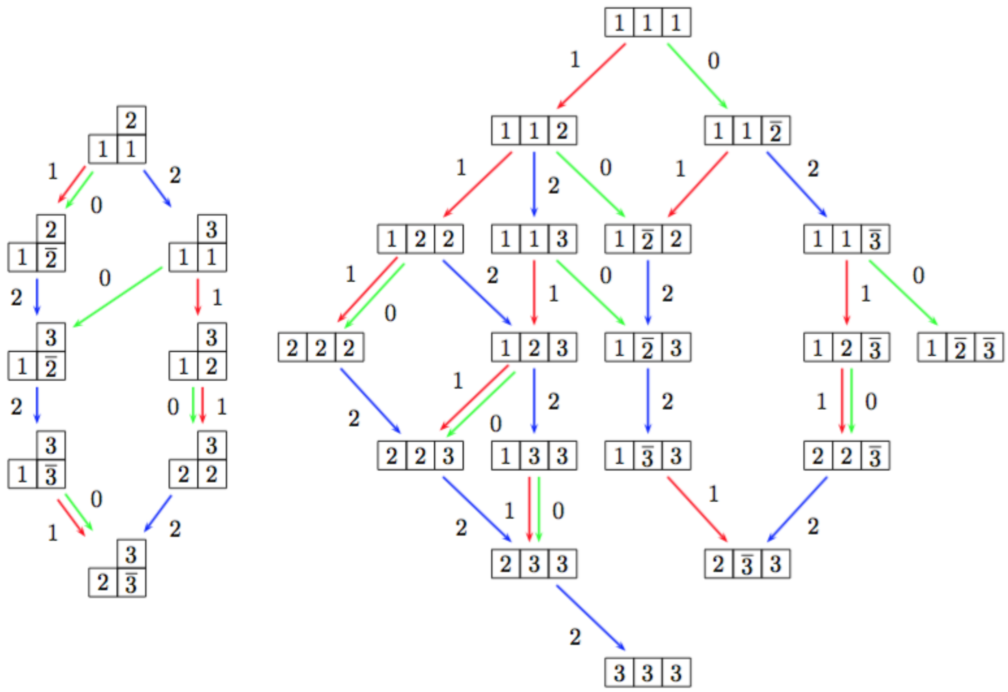


Figure 6.30: The queer crystal structures on semistandard shifted tableaux of shape $(2,1)$ and 3 with entries $\{\overline{1},1,\overline{2},2,\overline{3},3\}$ and no marks on the main diagonal.

we have $f_0(f_1(v)) = f_1(f_0(v)) = f_1(w)$ and $f_0 f_1^3(v) = f_1 f_0 f_1^2(v)$. Applying Figure 6.29 to force $f_0 f_2 = f_2 f_0$ whenever both are defined at a vertex results in the situation depicted on the right side of Figure 6.32.

By axiom (B5), we must have $e_0(f_2(w)) = 0$. Since $\varepsilon_1(f_2(w)) = 2$, we must be in case $k = 2$ of Figure 6.28, and so we must have another vertex, say x, not yet in the picture, of weight $(1,1,1)$ such that $f_0(f_2(w)) = x$ and $e_1(x) = 0$. By Figure 6.29, we must also have $e_2(x) = 0$. Therefore, since x is on a regular crystal of dimension 3, it is a highest weight, and so $f_1(x) = f_2(x) = 0$. This completes the picture, and we have a graph isomorphic to the normal queer crystal with highest weight $(3,0,0)$ seen in Figure 6.30. □

We conclude with the following general conjecture.

Conjecture 6.3.17. *Every regular queer graph is a normal queer crystal.*

As demonstrated in our use of Stembridge's axioms to prove our shifted crystal operators form a crystal, a proof of Conjecture 6.3.17 will provide a powerful tool in the study of Schur P-positive polynomials.

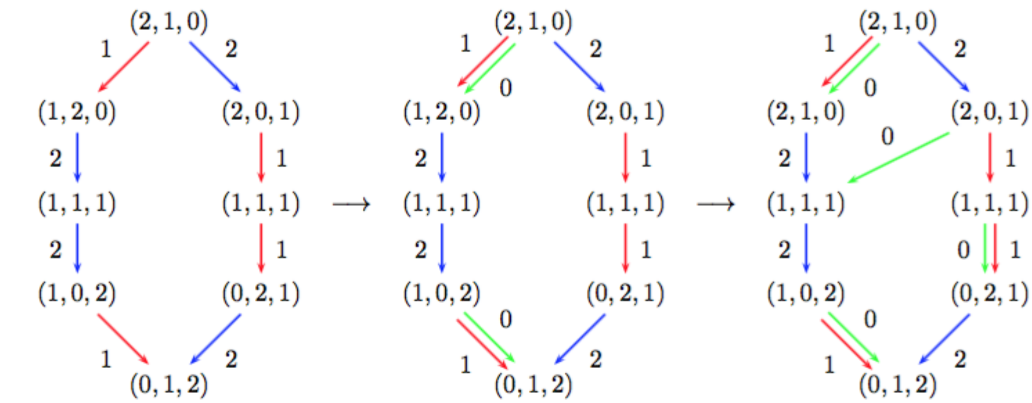


Figure 6.31: Constructing the unique regular queer graph with highest weight $(2, 1, 0)$, where the weights of the vertices are indicated.

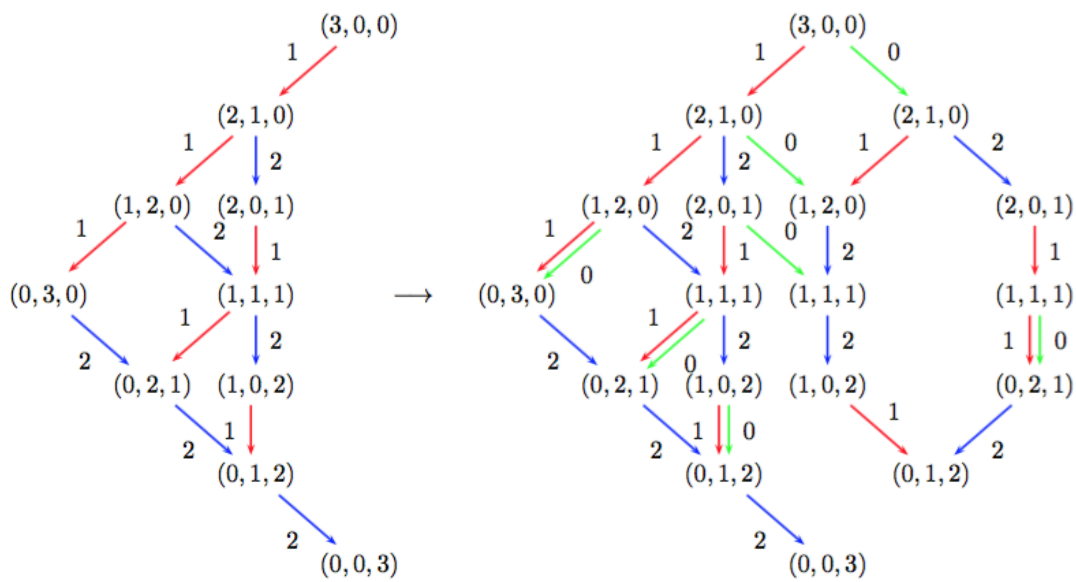


Figure 6.32: Constructing the unique regular queer graph with highest weight $(3, 0, 0)$, where the weights of the vertices are indicated.

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