MACKEY FUNCTORS AND FUSION SYSTEMS

by

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ABSTRACT

MACKEY FUNCTORS AND FUSION SYSTEMS

In this thesis, we present a survey for the paper [1]. We study with the biset functors representation rings and superclass functions. We define representation rings of a saturated fusion system $\mathcal{F}$ and $\mathcal{F}$-stable elements of superclass functions, then, we investigate the dimension homomorphism between those sets. We consider arbitrary saturated fusion system $\mathcal{F}$ and finite group G. Our aim is to describe the image of the $\mathcal{F}$-stable elements of real representation ring of G, under the dimension function which is the set of Borel-Smith functions which satisfy an extra condition.

ÖZET

MACKEY İZLEÇLERİ VE FÜZYON SİSTEMLERİ

Bu tezde [1] makalesi üzerine bir araştırma ortaya konulmaktadır. İki etki izleci olan grup temsil halkası ve süpersınıf fonksiyonları çalışılmaktadır. Doymuş bir füzyon sistem $\mathcal{F}$ için grup temsil halkası ile $\mathcal{F}$-kararlı süpersınıf fonksiyonları tanımlanmakta ve sonrasında belirtilen iki küme arasındaki boyut homomorfizması incelenmektedir. Herhangi bir doymuş füzyon sistem $\mathcal{F}$ ve sonlu grup G ele alınsın. Bu tezdeki amaç G grubunun gerçel grup temsil halkasındaki $\mathcal{F}$-kararlı elemanları kümesinin boyut fonksiyonu altındaki görüntüsünü tarif etmektir. Bu görüntü kümesi Borel-Smith fonksiyonlarına ilave bir koşul ekleyerek elde ediliyor.

TABLE OF CONTENTS

LIST OF SYMBOLS

A_4	alternating group of order 4
$A(H,G)$	double Burnside ring of (H,G)-bisets
$\mathbb{C}$	Complex numbers
C_n	the cyclic group of order n
$C_G(H)$	centralizer of the subgroup H in G
$\dim_k$	dimension over the field k
$\lvert G\rvert$	order of the group G
$\mathrm{Gal}(\mathbb{L}/\mathbb{K})$	Galois group of $\mathbb{L}$ over $\mathbb{K}$
$\mathrm{GL}(n,\mathbb{C})$	$n \times n$-invertible matrices with coefficients from $\mathbb{C}$
$\mathrm{Hom}(G,H)$	set of group homomorphisms from the group G to H
$\mathrm{Hom}_{\mathcal{C}}(P,Q)$	class of morphisms in the category $\mathcal{C}$ from the object P to Q
ker	kernel
$\mathbb{N}$	natural numbers, including 0
$N_G(H)$	normaliser of the subgroup H in G
$\mathrm{Ob}(\mathcal{C})$	class of objects in the category $\mathcal{C}$
$\mathbb{R}$	Real numbers
R-mod	category of left R-modules
$\mathrm{Sub}(G)$	set of all subgroups of the group G
Q_8	the quaternion group
V_4	the Klein four-group
X^H	the set of fixed point of the group H on the set X
$S \wr H$	wreath product of S and H
$\mathrm{Tr}(A)$	trace of the matrix A
$\overline{z}$	the complex conjugate of the complex number z
$\mathbb{Z}$	Integers
$\mathbb{Z}/n$	the additive group of the integers modulo n
Σ_n	the symmetric group on $\{1,\ldots,n\}$

1. INTRODUCTION

The notion fusion system has arisen from the finite group theory. Mainly, the theory of fusion system imitates notions from finite group theory; however, it also uses the representation theory and topological techniques. Although Brauer has introduced the term fusion in the 1950s, the term has appeared in the late 19th century. For a finite group G and a Sylow p-subgroup S, if two subsets of S are G-conjugates then two subsets are said to be fused. A subgroup H of G is said to control fusion if any two G-conjugated elements of S are also H-conjugated. The fusion system $\mathcal{F}_S(G)$ is the category whose objects are subgroups of S and whose morphisms are appropriate conjugations in G. The axiomatization of the notion fusion system has been started around 1990 with Puig's works. Puig abstracts this definition by eliminating the ambient group G. He used the term Frobenius Category and he actually defined the notion of saturated fusion system. Later, Alperin and Broue convey the notion fusion system to homotopy theory. Fusion systems connect algebra and topology; they have usage in modular representation theory, homotopy theory, and finite group theory. [2] [3]

Bouc describes the theory of Mackey functors as an axiomatization of the induction and restriction process in [4]. The notion biset functor appears in order to consider the deflation and inflation process along with inductions and restrictions. With the biset structure of the representation ring, the biset theory assists to understand the problems in the representation theory. [5]

In this thesis we follow the article [1]. For the necessary definitions and theorems, we mainly refer to the books [5], [6], [3] and [2]. This thesis is organized as follows. In Chapter 2, we give brief information about the representation theory and the biset theory. Moreover, we give the definitions of biset functor, Mackey functor and fusion system. In Chapter 3, we define $\mathcal{F}$-stable elements of a biset functor for a saturated fusion system $\mathcal{F}$ and study the group of the $\mathcal{F}$-stable elements of a representation ring. Later, we give the definition of superclass function and we show that it has a biset functor structure. After describing biset functor structures of a representation ring and superclass functions, we define the dimension function which can be seen as a natural transformation between the representation ring and superclass functions. Finally, we study the surjectivity of the dimension function between the sets of $\mathcal{F}$-stable elements of the mentioned two biset functors.

1.1. Notations

For a group G and an element $g \in G$, the conjugation homomorphism by g is denoted by $c_g : G \to G$ where $x \mapsto g^{-1}xg$. Moreover, for a subgroup H of G, by H^g we mean the image of H under the map c_g, i.e. $H^g = g^{-1}Hg$. Similarly, ${}^gH$ denote the image of H under the inverse of conjugation map c_g, i.e. ${}^gH = gHg^{-1}$. Finally, we denote the inclusion function with domain H and codomain G by ι_H^G.

In this thesis, $\mathbb{K}$ denotes a subfield of $\mathbb{C}$. We exclusively consider fields of characteristic 0.

2. CHAPTER 2

2.1. Representation ring

This section is a quick and incomplete review of representation theory of finite groups. Although most of the definitions, theorems and notations are from [6], we also benefit from [7] and [8].

Definition 2.1. *Let V be a vector space over a field k of characteristic 0.*

(i) The automorphisms of V will be denoted by $\mathrm{GL}(V)$, i.e.

$$\mathrm{GL}(V) = \{T : V \to V \,|\, T \text{ is bijective linear mapping}\}.$$

Note that when V has finite dimension n, we can identify $\mathrm{GL}(V)$ by $\mathrm{GL}(n,k)$ by fixing a basis for V.

(ii) A linear representation of a finite group G in V is a group homomorphism $\rho : G \to \mathrm{GL}(V)$. We will write ρ_g instead of $\rho(g)$. Expressly, to each $g \in G$, we assign an element ρ_g of $\mathrm{GL}(V)$ (which can be seen as a matrix from $\mathrm{GL}(n,k)$ whenever the dimension V is finite) such that $\rho_{gh} = \rho_g\rho_h$ for any $g, h \in G$. If the group homomorphism ρ is clear from the context, we just say V is a representation space of G.

From now on, we will only consider representation spaces of finite dimension.

(iii) Let ρ be a linear representation of a finite group G in a vector space V. The degree of the representation is defined as the dimension of V.

(iv) Let ρ, ρ' be two linear representations of a finite group G in vector spaces V and V', respectively. We say that ρ and ρ' are equivalent (isomorphic) if there is a linear isomorphism $T : V \to V'$ such that $T\rho_g = \rho'_g T$ for any $g \in G$. It is easy to observe that equivalence of representations induces an equivalence relation on

the class of all representations of G.

Definition 2.2. *Let $\rho^1 : G \to \mathrm{GL}(V_1)$ and $\rho^2 : G \to \mathrm{GL}(V_2)$ be two linear representations of a group G.*

(i) We define the direct sum of ρ^1 and ρ^2 as $\rho : G \to \mathrm{GL}(V_1 \oplus V_2)$ where $\rho_g(v_1, v_2) = (\rho_g^1(v_1), \rho_g^2(v_2))$ for all $v_1 \in V_1$, $v_2 \in V_2$ and $g \in G$. This representation is denoted by $\rho^1 \oplus \rho^2$.

(ii) We define the tensor product of ρ^1 and ρ^2 as $\rho : G \to \mathrm{GL}(V_1 \otimes V_2)$ where $\rho_g(v_1 \otimes v_2) = \rho_g^1(v_1) \otimes \rho_g^2(v_2)$ for all $v_1 \in V_1$, $v_2 \in V_2$ and $g \in G$. This representation is denoted by $\rho^1 \otimes \rho^2$.

Definition 2.3. *Let k be a commutative ring and G be a finite group. The group algebra of G over k, denoted by $k[G]$, is a k-algebra whose basis is the set of elements of G. More explicitly, any element of $k[G]$ is of the form*

$$\sum_{g\in G} \lambda_g g \qquad \text{where } \lambda_g \in k$$

and for two elements $u = \sum_{g\in G} \lambda_g g$, $v = \sum_{g\in G} \mu_g g$ in $k[G]$ and $\lambda \in k$, we have

$$\begin{aligned} u + v &= \sum_{g\in G} (\lambda_g + \mu_g)\, g, \\ ku &= \sum_{g\in G} k\lambda_g g, \\ uv &= \left(\sum_{g\in G} \lambda_g g\right)\left(\sum_{g\in G} \mu_g g\right) = \sum_{g,h\in G} \lambda_g \mu_h (gh) \end{aligned}$$

where the multiplication of the basis elements is the multiplication of G.

Similarly, for a finite set X, we define $k[X]$ as k-module with basis X.

Furthermore, for $H, N \leqslant G$, the group algebra $k[G]$ has a $(k[H], k[N])$-bimodule structure with its own multiplication.

Remark 2.4. *Let k be a field and G be a finite group.*

(i) *Let V be a k-vector space and $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G in V. Consider the action*

$$f \cdot v = \sum_{g \in G} a_g \rho_g(v)$$

for any $f = \sum_{g \in G} a_g g \in k[G]$ and $v \in V$. Via this action, V becomes a $k[G]$-module.

(ii) *Let V be a $k[G]$-module. Define $\rho : G \to \mathrm{GL}(V)$ by*

$$\rho_g(v) = g \cdot v$$

for any $g \in G$ and $v \in V$. Then, ρ becomes a linear representation of G in V as a result of the module structure of V.

Therefore, we get a correspondence between representation of G and $k[G]$-modules. From now on, by abuse of notation, taking V as a representation of G will also mean that V is a representation space for some linear representation $\rho : G \to \mathrm{GL}(n, k)$ and that V is a $k[G]$-module. Since we assumed the representations are finite dimensional, so are $k[G]$-modules.

Definition 2.5. *Let V be a vector space over a field of characteristic 0, W be a vector subspace of V and $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G in V.*

(i) *We say that W is stable under the action of G if $\rho_g(W) = W$ for any $g \in G$.*

(ii) *Suppose W is stable under the action of G. For each $g \in G$, by restricting ρ_g to W, we obtain a linear representation $\rho^W : G \to \mathrm{GL}(W)$. In this case, W is called a subrepresentation of V.*

(iii) *If $V \neq 0$ and V has no subrepresentation other than 0 and itself, then we say that V is an irreducible representation. Equivalently, V is an irreducible representation if V is not a direct sum of non-zero representations.*

Notation. *Let k be a field, G be a finite group and V, W be $k[G]$-modules. The set of $k[G]$-homomorphisms from V to W is denoted by* $\mathrm{Hom}_{k[G]}(V, W)$.

Proposition 2.6. *Let V_i, W_j be $k[G]$-modules for $i = 1, \ldots, r$ and $j = 1, \ldots, s$. Then*

$$\dim_k \mathrm{Hom}_{k[G]}(V_1 \oplus \cdots \oplus V_r, W_1 \oplus \cdots \oplus W_s) = \sum_{i=1}^{r} \sum_{j=1}^{s} \dim_k \mathrm{Hom}_{k[G]}(V_i, W_j).$$

Theorem. *(Mashke's Theorem) Let G be a finite group and V be a $k[G]$-module. If U is a $k[G]$-submodule of V, then there is a $k[G]$-submodule W of V such that $V = U \oplus W$.*

Proposition 2.7. *Let G be a finite group and V, W be a $k[G]$-modules such that V is of the form $U_1 \oplus \cdots \oplus U_r$ where $U_1, \ldots U_r$ are irreducible $k[G]$-modules. If W is irreducible, then $\dim_k \mathrm{Hom}_{k[G]}(V, W)$ is equal to the number of $k[G]$-modules U_i such that $U_i \cong W$.*

Definition 2.8. *Let k be a field, G be a group and $H \leqslant G$.*

(i) Suppose V is a $k[H]$-module. The group algebra $k[G]$ has a $(k[G], k[H])$-bimodule structure. Therefore,

$$k[G] \otimes_{k[H]} V$$

is a $k[G]$-module which is called the induced representation of V and denoted by $\mathrm{Ind}_H^G V$.

(ii) Suppose V is a $k[G]$-module. By considering only elements of $k[H]$, we can see V as a $k[H]$-module. This $k[H]$-module structure of V is called the restriction of the representation V and denoted by $\mathrm{Res}_H^G V$.

Definition 2.9. *Let V be a $k[G]$-vector space and $\rho : G \to \mathrm{GL}(V)$ be a representation. Define the character of the representation ρ as the map*

$$\chi_\rho \colon G \to k$$
$$g \mapsto \mathrm{Tr}(R_g)$$

where R_g is a matrix form of ρ_g. Note that this definition is independent from the choice of the matrix R_g representing ρ_g. We may also write χ_V instead of χ_ρ.

Theorem 2.10. *Two representations over a field of characteristic 0 are isomorphic if and only if they have the same character.*

Definition 2.11. *Let G be a group and $\mathbb{K}$ be a subfield of $\mathbb{C}$.*

(i) Let $R^+_{\mathbb{K}}(G)$ be the set of all isomorphism classes of linear representations of G over $\mathbb{K}$. With the operations direct sum and tensor products, $R^+_{\mathbb{K}}(G)$ forms a semiring which is called the representation semiring of G over $\mathbb{K}$. By abuse of notation, the isomorphism class of a representation V will also be denoted by V.

(ii) We consider the group completion of $R^+_{\mathbb{K}}(G)$. With tensor product, this group becomes a ring. This structure $R_{\mathbb{K}}(G)$ is called the representation ring of G over $\mathbb{K}$. The elements of the representation ring $R_{\mathbb{K}}(G)$ are called virtual representations and they are of the form $U - V$ for some representations $U, V \in R^+_{\mathbb{K}}(G)$. Note that we could also define $R_{\mathbb{K}}(G)$ as the Grothendieck ring of $R^+_{\mathbb{K}}(G)$.

Remark 2.12. *(i) Let χ_{V_1}, χ_{V_2} be two characters of V_1 and V_2, respectively. By using basic properties of matrices, it is easy to see that*

$$\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$$

and

$$\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}.$$

(ii) For a finite group G and a field k, similar to representations, the set characters of representations of G over $\mathbb{K}$ forms a semiring with the above operations. Moreover, by taking Grothendieck ring of this semiring we obtain the character ring. Note that, by using Theorem 2.10, we can identify the character ring with the representation ring $R_{\mathbb{K}}(G)$.

Definition 2.13. *Let G be a finite group. We define an inner product on the complex representation ring of G as follows:*

$$\langle V, W\rangle_G := \dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}[G]}(V, W)$$

for any $V, W \in R_{\mathbb{C}}(G)$.

Remark. *If we consider characters of G, then the inner product $\langle -, -\rangle_G$ becomes the usual inner product of the complex G-representations. I.e. for any two complex G-representations V, W, we have*

$$\langle V, W\rangle_G = \langle \chi_V, \chi_W\rangle$$

where

$$\langle \chi_V, \chi_W\rangle = \frac{1}{|G|} \sum_{g\in G} \chi_V(g)\overline{\chi_W(g)}.$$

Proposition 2.14. *Let G be a finite group. For any $k[G]$-module V and $H \leqslant G$ we have*

$$\dim_k V^H = \langle V, k[G/H]\rangle_G\,.$$

Proof. Take an arbitrary $V \in R_k^+(G)$ and $H \leqslant G$. Let I_H denote the trivial $k[H]$-module. The dimension of the fixed points of H on V is the number of I_H contained in V. So, by Proposition 2.7 we conclude that $\dim_k V^H = \langle \operatorname{Res}^G_H V, \mathrm{I}_H\rangle_H$. Therefore, applying Frobenius Reciprocity gives that $\dim_k V^H = \langle V, \operatorname{Ind}^G_H \mathrm{I}_H\rangle_G$. On the other hand, we have that $\operatorname{Ind}^G_H \mathrm{I}_H \cong k[G/H]$ as $k[G]$-modules and thus, we obtain that $\dim_k V^H = \langle V, k[G/H]\rangle_G$. □

Theorem 2.15. *Let G be a finite group and V be the regular $\mathbb{K}[G]$-module. Suppose the irreducible characters of G are $\chi_1, \dots \chi_k$. Then $\chi_V = n_1\chi_1 + \cdots + n_k\chi_k$ where $n_i = \chi_i(1)$ for $i = 1, \dots, k$.*

2.2. Bisets

For this section, we follow [5].

Definition 2.16. *Let G be a finite group and X be a finite set.*

(i) A (left) group action of G on X is a mapping $G \times X \longrightarrow X$, $(g, x) \longmapsto g \cdot x$ such that $1_G \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$, for all $g, h \in G$ and $x \in X$. If there is an action of G on X, we say that X is a (left) G-set.

(ii) Similarly, a right group action of G on X is a mapping $X \times G \longrightarrow X$ with $(x, g) \longmapsto x \cdot g$ such that $x \cdot 1_G = x$ and $(x \cdot g) \cdot h = x \cdot (gh)$, for all $g, h \in G$ and $x \in X$. In this case we say that X is a right G-set.

(iii) Let $\alpha : X \to Y$ be a function between two G-sets. The function α is called a morphism of G-sets (or a G-equivariant map) if $\alpha(g \cdot x) = g \cdot \alpha(x)$ for any $g \in G$ and $x \in X$.

(iv) A bijective morphism of G-sets $\alpha : X \to Y$ is called an isomorphism of G-sets.

Definition 2.17. *Let G be a finite group and X be a finite G-set. With the action*

$$\begin{aligned} X \times G &\to X \\ (x, g) &\mapsto g^{-1} \cdot x \end{aligned}$$

the set X itself turns to be a right G-set. This right G-set structure is called the opposite of X and denoted by X^{op}. For a right G-set X, the opposite (left) group action X^{op} is similarly defined.

Lemma 2.18. *Let G be a finite group and $(X_i)_{i \in I}$ be a finite family of G-sets. Then the disjoint union $\bigsqcup_{i \in I} X_i$ is also a G-set.*

Definition 2.19. *Let G be a group, X be a G-set and $H \leqslant G$.*

(i) For $x \in X$, the H-orbit of x is the set

$$Hx = \{h \cdot x \,|\, h \in H\}.$$

(ii) *The set of H-orbits in X is denoted by $H\backslash X$ and $[H\backslash X]$ denotes a set of representatives of H-orbits in X.*

(iii) *For $x \in X$, the set*

$$G_x = \{g \in G \mid g \cdot x = x\}$$

is called the stabilizer of x in G.

(iv) *The set of fixed points of H on X is*

$$X^H = \{x \in X \mid h \cdot x = x \text{ for all } h \in H\}.$$

(v) *The G-set X is called transitive, if for any $x \in X$ the G-orbit of x is the whole X. That means for any $x, y \in X$, there exists $g \in G$ such that $x = g \cdot y$.*

The following example is important because it will turn out by Proposition 2.21 that any transitive G-set is isomorphic to the following form.

Example 2.20. *Let G be a group and $H \leqslant G$. The set G/H of cosets of H in G is a G-set with the multiplication from left. For any $xH, yH \in G/H$ by taking $g = yx^{-1} \in G$ we get $g \cdot xH = (gx)H = yH$. So G/H is a transitive G-set.*

Proposition 2.21. *Let G be a group and X be a G-set.*

(i) *Suppose $H, K \leqslant G$. The G-sets G/H and G/K are isomorphic if and only if H and K are conjugate in G.*

(ii) *The G-set X is transitive if and only if X is isomorphic to G/H for some $H \leqslant G$.*

(iii) *There is G-set isomorphism between $\bigsqcup_{x \in [G\backslash X]} G/G_x$ and X by $gG_x \mapsto g \cdot x$.*

Definition 2.22. *Let G, H be finite groups. A set X is called an (H, G)-biset if X has a group action of H and a right group action of G which commute with each other; i.e. for all $h \in H$, $g \in G$, $x \in X$ we have $(h \cdot x) \cdot g = h \cdot (x \cdot g)$.*

The followings are fundamental examples of bisets. Later, we will define composition of bisets (Definition 2.27). The importance of the five types of bisets given in the following example is clear by the fact that any transitive biset is a composition of such bisets.

Example 2.23. *Let G be a finite group.*

(i) *Let $H \leqslant G$. Then the set G itself is an (H, G)-biset with the left and right multiplications of G. I.e. $h \cdot x \cdot g = hxg$ for any $h \in H$ and $x, g \in G$. This (H, G)-biset is denoted by Res^G_H.*

(ii) *Again let $H \leqslant G$. Then the set G itself is a (G, H)-biset with the left and right multiplications of G. I.e. $g \cdot x \cdot h = gxh$ for any $g, x \in G$ and $h \in H$. This (G, H)-biset is denoted by Ind^G_H.*

(iii) *Let $N \trianglelefteq G$ and $H \coloneqq G/N$. Then the set H is a (G, H)-biset where the right action is given by right multiplication of H and the left action is given by the projection onto H and left multiplication of H. I.e. $g \cdot x \cdot h = (gH)xh$ for any $g \in G$ and $x, h \in H$. This (G, H)-biset is denoted by Inf^G_H.*

(iv) *Again let $N \trianglelefteq G$ and $H \coloneqq G/N$. Then the set H is an (H, G)-biset where the left action is given by left multiplication of H and the right action is given by the projection onto H and right multiplication of H. I.e. $h \cdot x \cdot g = hx(gH)$ for any $h, x \in H$ and $g \in G$. This (H, G)-biset is denoted by Def^G_H.*

(v) *Let $f \colon G \to H$ be a group isomorphism for a group H. Then the set H is an (H, G)-biset where the left action is given by the left multiplication of H and the right action is given by the right multiplication of the image of f in H. I.e. $h \cdot x \cdot g = hxf(g)$ for any $h, x \in H$ and $g \in G$. This (H, G)-biset is denoted by Iso_f.*

Definition 2.24. *Let G, H be finite groups and X be an (H, G)-biset.*

(i) *The action of G is said to be free if for any $x \in X$, $g \in G$, $g \cdot x = x$ implies g is the identity of G.*

(ii) *The (H, G)-biset X is said to be a free biset if both H and G actions are free.*

Remark 2.25. *(i) Let H, G be finite groups. Then, any (H, G)-biset X is also an $H \times G$-set via the following action*

$$(h, g) \cdot x = h \cdot x \cdot g^{-1}$$

for all $h \in H$, $g \in G$ and $x \in X$. Similarly, any $(H \times G)$-set can be seen as an (H, G)-biset. Therefore, all notions which are defined for G-sets are also valid for (H, G)-bisets. For instance, a function $\alpha : X \to Y$ between two (H, G)-bisets is called a morphism of (H, G)-bisets if $\alpha(h \cdot x \cdot g) = h \cdot \alpha(x) \cdot g$ for any $h \in H$, $g \in G$ and $x \in X$. Moreover, any theorem or proposition for G-sets have versions for bisets and vice versa.

(ii) For any finite group G, the set G itself is a (G, G)-biset via the left and right multiplications, that means, for any $x, h, g \in G$ the action is $h \cdot x \cdot g = hxg$. We denote this biset by Id_G. Moreover, such bisets are called identity biset which will make sense after Proposition 2.28.

Definition 2.26. *Let X be an (H, G)-biset and $x \in X$. The (H, G)-orbit of x is the set*

$$H \cdot x \cdot G = \{h \cdot x \cdot g \,|\, g \in G,\, h \in H\}.$$

The set of all (H, G)-orbits in X is denoted by $H \backslash X / G$. The (H, G)-biset X is called transitive if the cardinality of $H \backslash X / G$ is equal to 1.

Definition 2.27. *Let G, H and K be groups. Suppose V is a (K, H)-biset and U is an (H, G)-biset. The composition of V and U is the set of H-orbits on the Cartesian product $V \times U$, where the right action of H is given by*

$$(v, u) \cdot h = (v \cdot h, h^{-1} \cdot u)$$

for any $(v, u) \in V \times U$ and $h \in H$. The composition is denoted by $V \times_H U$ and its elements are denoted by $(v,_H u)$. The set $V \times_H U$ has a (K, G)-biset structure with the *following actions:*

$$k \cdot (v,_H u) \cdot g = (k \cdot v,_H u \cdot g)$$

for any $k \in K$, $g \in G$ *and* $(v,_H u) \in V \times_H U$.

Proposition 2.28. *(i) Let* U, V, W *be* (H,G), (K,H), (L,K)*-bisets, respectively. There is an isomorphism of* (L,G)*-bisets:*

$$W \times_K (V \times_H U) \cong (W \times_K V) \times_H U.$$

(ii) Let U, U' *be* (H,G)*-bisets and* V, V' *be* (K,H)*-bisets. There are isomorphisms of* (K,G)*-bisets:*

$$V \times_H (U \sqcup U') \cong (V \times_H U) \sqcup (V \times_H U'),$$
$$(V \sqcup V') \times_H U \cong (V \times_H U) \sqcup (V' \times_H U).$$

(iii) Let U *be an* (H,G)*-biset. There is an isomorphism of* (H,G)*-bisets:*

$$\mathrm{Id}_H \times_H U \cong U \quad \text{and} \quad U \times_G \mathrm{Id}_G \cong U.$$

Definition 2.29. *Let* H *and* G *be groups. By Lemma 2.18, the isomorphism classes of* (H,G)*-bisets together with disjoint union form a monoid. The group completion of this monoid is denoted by* $A(H,G)$ *and* $A(H,G)$ *has a unitary ring structure via the direct product. The ring* $A(H,G)$ *is called the double Burnside ring and its elements are called virtual bisets. Actually, the double Burnside ring can be seen as the Grothendieck ring of the mentioned monoid. By abuse of notation, we denote the isomorphism class of the* (H,G)*-biset* X *by* X *itself. Note that any virtual* (H,G)*-biset is of the form* $X-Y$ *for some actual* (H,G)*-bisets* X *and* Y*. Additionally, for a group* G*, by Remark 2.25, we also have the definition of the Burnside ring* $A(G)$*.*

Proposition 2.30. *Let G, H, K be finite groups and let*

$$\times'_H : A(K,H) \times A(H,G) \to A(K,G)$$

be the bilinear map such that

$$V \times'_H U = V \times_H U$$

where, in the left hand side, we see (K,H)-biset V and (H,G)-biset U as elements of Burnside rings. The map $\times'_H$ satisfies the properties of Proposition 2.28 when we see bisets as elements of Burnside rings. By abuse of notation, we will also denote this map by $\times_H$.

Notation. *Let P be a finite group. For $R \leqslant P$ and a group homomorphism $\varphi : R \to Q$, we define the following transitive (P,Q)-biset:*

$$[R,\varphi]_P^Q := (P \times Q)/\Delta_\varphi(R)$$

where $\Delta_\varphi(R) = \{\big(r, \varphi(r^{-1})\big) \mid r \in R\}$.

Proposition. *(Frobenius Reciprocity) Let G, H be finite groups and the set U be an (H,G)-biset. For any $\mathbb{C}[G]$-module V and $\mathbb{C}[H]$-module W, we have*

$$\big\langle W, \mathbb{C}[U] \otimes_{\mathbb{C}[G]} V \big\rangle_H = \big\langle \mathbb{C}[U^{\mathrm{op}}] \otimes_{\mathbb{C}[H]} W, V \big\rangle_G .$$

2.3. Mackey Functors

In this section, we mostly keep the notation of [1]; moreover we also refer to [5]. Proposition 2.30 allows us to give the following definition:

Definition 2.31. *The biset category of finite groups is the category whose objects are finite groups and morphisms are $A(H,G)$ for the source object G, the target object H, i.e., $\mathrm{Hom}_{\mathcal{A}}(G,H) = A(H,G)$. We will denote this category by $\mathcal{A}$. Moreover, for*

$V \in \mathrm{Hom}_{\mathcal{A}}(G,H)$ and $U \in \mathrm{Hom}_{\mathcal{A}}(H,K)$, their composition $U \circ V \in \mathrm{Hom}_{\mathcal{A}}(G,K)$ is given by $U \times_H V$.

Definition 2.32. *Let R be a commutative ring with unity. The category $R\mathcal{A}$ is the category whose objects are finite groups and $\mathrm{Hom}_{R\mathcal{A}}(G,H) = R \otimes_{\mathbb{Z}} A(H,G)$. The composition of morphisms in $R\mathcal{A}$ is the R-linear extension of the composition in $\mathcal{A}$.*

Notation. *Let $\mathcal{X}$ be a collection of finite groups closed under subgroups and quotients. The full subcategory of $R\mathcal{A}$, whose object class is $\mathcal{X}$, will be denoted by $R\mathcal{A}_{\mathcal{X}}$. Moreover, $R\mathcal{A}_{\mathcal{X}}^{bifree}$ will denote the subcategory of $R\mathcal{A}_{\mathcal{X}}$ with the same objects but whose morphisms are just bifree virtual bisets.*

Definition 2.33. *Let R be a commutative ring with unity.*

(i) A category $\mathcal{C}$ is called R-linear if $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is an R-module for any two objects A, B and the compositions of any two morphisms is R-bilinear.

(ii) A functor Φ between two R-linear categories $\mathcal{C}$ and $\mathcal{D}$ is said to be R-linear if Φ gives an R-linear map between $\mathrm{Hom}_{\mathcal{C}}(A,B)$ and $\mathrm{Hom}_{\mathcal{D}}\big(\Phi(A),\Phi(B)\big)$ for any objects A, B in $\mathcal{C}$.

Definition 2.34. *Let R be a commutative ring with unity and let $\mathcal{X}$ be a collection of finite groups closed under subgroups and quotients.*

(i) A biset functor on $\mathcal{X}$ over R is an R-linear (covariant) functor from $R\mathcal{A}_{\mathcal{X}}$ to R-mod.

(ii) A global Mackey functor on $\mathcal{X}$ over R is an R-linear (covariant) functor from $R\mathcal{A}_{\mathcal{X}}^{bifree}$ to R-mod.

(iii) A global Mackey functor (or a biset functor) is said to be defined on a p-group S, if S is contained in $\mathcal{X}$.

Notation 2.35. *Let M be a global Mackey functor (or a biset functor) on $\mathcal{X}$ over R. For any groups Q, P in $\mathcal{X}$, by definition, M defines R-module homomorphisms for each virtual (P,Q)-biset. Suppose $Q \leqslant P$ in $\mathcal{X}$. We will denote the restriction and* *induction maps given by M as follows:*

$$\mathrm{res}^P_Q : M(P) \xrightarrow{[Q,incl]^P_Q} M(Q),$$
$$\mathrm{ind}^P_Q : M(Q) \xrightarrow{[Q,\mathrm{id}]^Q_P} M(P).$$

Moreover, consider a homomorphism $\varphi : S \to P$ for S in $\mathcal{X}$. Then the restriction along φ is denoted by

$$\mathrm{res}_\varphi : M(P) \xrightarrow{[S,\varphi]^P_S} M(S).$$

Remark 2.36. *Let X be a (G, H)-biset. Define $X^*\colon R_{\mathbb{K}}(H) \to R_{\mathbb{K}}(G)$ by*

$$U - V \mapsto \big(\mathbb{K}[X] \otimes_{\mathbb{K}[H]} U\big) - \big(\mathbb{K}[X] \otimes_{\mathbb{K}[H]} V\big)$$

where U, V are actual H-representations.
Now, take an (H, K)-biset Y. Then, for any actual $\mathbb{K}[K]$-module V, we have

$$\begin{aligned}
\big(X^* \circ Y^*\big)V &= X^*\big(\mathbb{K}[Y] \otimes_{\mathbb{K}[K]} V\big) \\
&= \mathbb{K}[X] \otimes_{\mathbb{K}[H]} \big(\mathbb{K}[Y] \otimes_{\mathbb{K}[K]} V\big) \\
&= \big(\mathbb{K}[X] \otimes_{\mathbb{K}[H]} \mathbb{K}[Y]\big) \otimes_{\mathbb{K}[K]} V \\
&= \mathbb{K}[X \times_H Y] \otimes_{\mathbb{K}[K]} V \\
&= \big(X \times_H Y\big)^* V \\
&= (X \circ Y)^* V.
\end{aligned}$$

This equation also holds for virtual bisets and hence, we get $(X \circ Y)^* = X^* \circ Y^*$ for any $X \in \mathrm{Hom}_{\mathcal{A}}(H, G)$ and $Y \in \mathrm{Hom}_{\mathcal{A}}(K, H)$. Moreover, we also have $\big([S, \mathrm{id}]^S_S\big)^* = \mathrm{id}$. Thus, we observed that $R_{\mathbb{K}}(-)$ is a biset functor over $\mathbb{K}$ on all finite groups.

Proposition 2.37. *[9] Let R be a ring with unity and let $\mathcal{X}$ be a collection of finite groups closed under subgroups and quotients. Then a global Mackey functor on $\mathcal{X}$ over R is a structure M which defines an R-module $M(G)$ for each group $G \in \mathcal{X}$.*

Moreover, for each injective homomorphism $\alpha : G \to K$ where $G, K \in \mathcal{X}$, M gives R-module homomorphisms $\alpha_* : M(G) \to M(K)$ and $\alpha^* : M(K) \to M(G)$ which satisfy the following conditions:

(i) *For all injective group homomorphisms α, β, γ, δ we have $(\alpha\gamma)_* = \alpha_*\gamma_*$ and $(\beta\delta)^* = \delta^*\beta^*$ whenever the composition of homomorphisms makes sense.*

(ii) *If $\alpha : G \to G$ is an inner automorphism, then $\alpha_* = 1 = \alpha^*$.*

(iii) *For group isomorphisms α, β, suppose that the following diagram commutes where $\iota_1 : K \to H$ and $\iota_2 : \beta^{-1}(K) \to G$ are inclusions.*

$$\begin{array}{ccc} \beta^{-1}(K) & \xrightarrow{\alpha} & K \\ \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota_1} \\ G & \xrightarrow{\beta} & H \end{array}$$

Then, we have ${\iota_1}^*\beta_* = \alpha_*{\iota_2}^*$ and $\beta^*\iota_{1*} = \iota_{2*}\alpha^*$.

(iv) *For all group isomorphisms α, β, γ, δ if the diagram*

$$\begin{array}{ccc} H & \xrightarrow{\alpha} & K \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \delta} \\ G & \xrightarrow{\gamma} & H \end{array}$$

commutes, then we have $\beta_*\alpha^* = \gamma^*\delta_*$.

(v) *For every group G and $H, K \leqslant G$, we have*

$$(\iota_H^G)^*(\iota_K^G)_* = \sum_{g \in [H\backslash G/K]} (\iota^H_{H \cap {}^gK})_*(c_{g^{-1}})_*(\iota^K_{H^g \cap K})^*$$

where $c_g : H \cap {}^gK \to H^g \cap K$.

2.4. Fusion Systems

We mostly use [2] and [3].

If G is a finite group, then for any prime p, we denote the highest power of p in $|G|$ by $|G|_p$.

For a group G and subgroups $H, K \leqslant G$ we define the following sets:

- $\mathrm{Hom}_G(H, K) := \{\phi \in \mathrm{Hom}(H, K) \,|\, \phi = c_g$ *for some* $g \in G$ *such that* $H^g \leqslant K\}$
- $\mathrm{Iso}_G(H, K) := \{\phi \in \mathrm{Hom}_G(H, K) \,|\, \phi$ *is an isomorphism*$\}$
- $\mathrm{Aut}_G(H) := \mathrm{Iso}_G(H, H)$

Definition 2.38. *[10] Let $\mathcal{I}$ be the subcategory of the finite group category, whose objects are finite groups but its morphisms are just injective group homomorphisms. A subcategory $\mathcal{F}$ of the category $\mathcal{I}$ is called a fusion system if it satisfies the following conditions:*

(i) $\mathrm{Ob}(\mathcal{F})$ *is closed under taking subgroups.*
(ii) Suppose $G \xrightarrow{\phi} G'$ is a morphism in $\mathcal{F}$. If $H \leqslant G$ and $H' \leqslant G'$ with $\phi(G) \subseteq H'$, then $H \xrightarrow{\phi} G'$ and $G \xrightarrow{\phi} H'$ are also morphisms in $\mathcal{F}$.
(iii) If $G \xrightarrow{\phi} G'$ is a morphism in $\mathcal{F}$, then so is $\phi(G) \xrightarrow{\phi^{-1}} G$.
(iv) For an object G of $\mathcal{F}$ and $g \in G$, $G \xrightarrow{c_g} G$ is a morphism in $\mathcal{F}$.

Moreover, if $\mathrm{Ob}(\mathcal{F})$ *consists of all subgroups of S for some finite p-group S, then $\mathcal{F}$ is called a fusion system on S.*

The general definition of a fusion system is given in Definition 2.38. However, generally fusion system on a finite group is studied. In the rest of the study, we will be interested in fusion systems on p-groups. It is easily observable that the following definition is equivalent to Definition 2.38.

Definition 2.39. *A fusion system on a finite p-group S is a category $\mathcal{F}$ whose objects consist of all subgroups of S and morphisms satisfy the following conditions:*

(i) $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ *for all $P, Q \leqslant S$ where* $\mathrm{Inj}(P,Q)$ *is the set of all injective group homomorphisms from P to Q.*

(ii) Every morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ can be factored as $P \xrightarrow{\phi} \phi(P) \xrightarrow{\iota} Q$ in $\mathcal{F}$ where ι is the inclusion map. Moreover, the inverse homomorphism ϕ^{-1} is in $\mathrm{Hom}_{\mathcal{F}}\big(\phi(P), P\big)$.

Example 2.40. *Let S be a p-subgroup of a finite group G. Let $\mathcal{F}_S(G)$ denote the category whose objects consist of all subgroups of S and, for any $P, Q \leqslant S$, we have* $\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)$. *Clearly, $\mathcal{F}_S(G)$ is a fusion system on S.*

Definition 2.41. *Let $\mathcal{F}$ be a fusion system on a finite p-group S. Two subgroups $P, Q \leqslant S$ are $\mathcal{F}$-conjugate if there is an isomorphism $\phi : P \to Q$ in $\mathcal{F}$. In this case, we write $P \underset{\mathcal{F}}{\sim} Q$. Moreover, we denote the set of all subgroups of S which are $\mathcal{F}$-conjugate to P by $P^{\mathcal{F}}$.*

Definition 2.42. *Let $\mathcal{F}$ be a fusion system on a finite p-group S. A subgroup $P \leqslant S$ is called*

- *fully $\mathcal{F}$-normalized if for all $Q \in P^{\mathcal{F}}$, $|N_S(P)| \geqslant |N_S(Q)|$,*
- *fully $\mathcal{F}$-centralized if for all $Q \in P^{\mathcal{F}}$, $|C_S(P)| \geqslant |C_S(Q)|$.*

Proposition 2.43. *Let S be a Sylow p-subgroup of the finite group G.*

(i) A subgroup $Q \leqslant S$ is fully $\mathcal{F}_S(G)$-normalized if and only if $N_S(Q) \in \mathrm{Syl}_p(N_G(Q))$.

(ii) A subgroup $Q \leqslant S$ is fully $\mathcal{F}_S(G)$-centralized if and only if $C_S(Q) \in \mathrm{Syl}_p(C_G(Q))$.

Proof. Suppose that $Q \leqslant S$ is fully $\mathcal{F}_S(G)$-normalized. Note that by construction of $\mathcal{F}_S(G)$, $\mathcal{F}_S(G)$-conjugates means G-conjugates, i.e. $Q^{\mathcal{F}_S(G)} = \{Q^g \,|\, g \in G,\, Q^g \leqslant S\}$. So, we assumed that for all $g \in G$, $|N_S(Q^g)| \leqslant |N_S(Q)|$. Take $R \in \mathrm{Syl}_p(N_G(Q))$ which contains $N_S(Q)$. Then, there exists $x \in G$ such that $R^x \leqslant S$. Therefore, we have $R^x \leqslant S \cap N_G(Q)^x = S \cap N_G(Q^x) = N_S(Q^x)$ and this implies $|R| \leqslant |N_S(Q^x)|$. By our assumption, we conclude that $|R| \leqslant |N_S(Q)|$ and hence, $N_S(Q)$ is a Sylow p-subgroup of $N_G(Q)$.

For the converse, suppose $N_S(Q) \in \mathrm{Syl}_p(N_G(Q))$. Let $g \in G$ be arbitrary. By the fact that conjugation map is an isomorphism, we have $|N_G(Q)| = |N_G(Q)^g|$ and so $|N_G(Q)|_p = |N_G(Q)^g|_p$. Therefore, since $N_G(Q)^g = N_G(Q^g)$, we get the equation $|N_G(Q)|_p = |N_G(Q^g)|_p$. Hence, $|N_S(Q)|_p = |N_G(Q^g)|_p$ by our assumption. Moreover, we have $|N_S(Q^g)| \leqslant |N_G(Q^g)|_p$ since $N_S(Q^g)$ is a p-subgroup of $N_G(Q^g)$. Thus, $|N_S(Q^g)| \leqslant |N_S(Q)|$ and this means that Q is fully $\mathcal{F}_S(G)$-normalized.

The second part is similar to the first one. We only need to consider centralizer instead of normalizer. □

Definition 2.44. *A fusion system $\mathcal{F}$ on a finite p-group S is called saturated if it satisfies the following conditions:*

(i) For $P \leqslant S$, P is fully $\mathcal{F}$-centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ whenever P is fully $\mathcal{F}$-normalized.

(ii) For any $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, if $\phi(P)$ is fully $\mathcal{F}$-centralized, then there is an extension $\widetilde{\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_\phi, S)$ of ϕ where

$$N_\phi := \{x \in N_S(P) \,|\, \exists y \in N_S(\phi(P)) : \phi \circ c_x = c_y \circ \phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)\}.$$

The main example of saturated fusion systems is $\mathcal{F}_S(G)$ where S is a Sylow p-subgroup of a finite group G.

Proposition 2.45. *Let S be a Sylow p-subgroup of a finite group G. The fusion system $\mathcal{F}_S(G)$ is saturated.*

Proof. For the first condition, take a fully $\mathcal{F}_S(G)$-normalized subgroup $P \leqslant S$. Notice that $\mathrm{Aut}_S(P) = \{c_s : P \to P | \, s \in N_S(P)\}$ and c_s is the identity map if and only if $s \in C_S(P)$. Therefore, we have $\mathrm{Aut}_S(P) \cong N_S(P)/C_S(P)$ and since morphisms of

$\mathcal{F}_S(G)$ are conjugation maps by elements of G, with the same argument we also have $\mathrm{Aut}_{\mathcal{F}_S(G)}(P) \cong N_G(P)/C_G(P)$. Thus, we obtain

$$\frac{|N_G(P)|}{|N_S(P)|} = \frac{|\,\mathrm{Aut}_{\mathcal{F}_S(G)}(P)|}{|\,\mathrm{Aut}_S(P)|}\frac{|C_G(P)|}{|C_S(P)|}. \tag{2.1}$$

By Proposition 2.43, we have $N_S(Q) \in \mathrm{Syl}_p\big(N_G(Q)\big)$. Therefore, the left hand side of Equation (2.1) is coprime to p and so is the right hand side. This result implies that $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p\big(\,\mathrm{Aut}_{\mathcal{F}}(P)\big)$ and $C_S(Q) \in \mathrm{Syl}_p\big(C_G(Q)\big)$. Again by Proposition 2.43, we conclude that P is fully $\mathcal{F}$-centralized.

For the second condition, take $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ such that $\phi(P)$ is fully $\mathcal{F}_S(G)$-centralized. So $\phi = c_g$ for some $g \in G$ such that $P^g \leqslant S$. This gives

$$\begin{aligned}
N_\phi &= \{x \in N_S(P) \,|\, \exists y \in N_S(P^g) \text{ such that } c_g \circ c_x = c_y \circ c_g \in \mathrm{Hom}_{\mathcal{F}}(P,S)\}\\
&= \{x \in N_S(P) \,|\, \exists y \in N_S(P^g) \text{ such that } c_{xg} = c_{gy} \in \mathrm{Hom}_{\mathcal{F}}(P,S)\}\\
&= \{x \in N_S(P) \,|\, \exists y \in N_S(P^g) \text{ such that } g^{-1}x^{-1}zxg = y^{-1}g^{-1}zgy \text{ for all } z \in P\}\\
&= \{x \in N_S(P) \,|\, \exists y \in N_S(P^g) \text{ such that } gyg^{-1}x^{-1}zxgy^{-1}g^{-1} = z \text{ for all } z \in P\}\\
&= \{x \in N_S(P) \,|\, \exists y \in N_S(P^g) \text{ such that } gyg^{-1}x^{-1} \in C_G(P)\}\\
&= \{x \in N_S(P) \,|\, \exists y \in N_S(P^g) \text{ such that } yg^{-1}x^{-1}g \in C_G(P^g)\}\\
&= \{x \in N_S(P) \,|\, \exists y \in N_S(P^g) \text{ such that } g^{-1}xgy^{-1} \in C_G(P^g)\}\\
&= \{x \in N_S(P) \,|\, \exists y \in N_S(P^g) \text{ such that } g^{-1}xgy^{-1} \in C_G(P^g)\}\\
&= \{x \in N_S(P) \,|\, \exists y \in N_S(P^g),\, h \in C_G(P^g) \text{ such that } x^g = hy\}.
\end{aligned}$$

Therefore, $(N_\phi)^g \leqslant C_G(P^g)N_S(P^g)$. Note that $N_S(P^g)$ normalizes $C_G(P^g)$ and $C_S(P^g)$ is a Sylow p-subgroup of $C_G(P^g)$ as S is a Sylow p-subgroup of G. Therefore, $N_S(P^g)$ is a Sylow p-subgroup of $C_G(P^g)N_S(P^g)$. As $(N_\phi)^g$ is a p-group, there is $\alpha \in C_G(P^g)N_S(P^g)$ such that $(N_\phi)^{g\alpha} \leqslant N_S(P^g)$. Write α as xy where $x \in C_G(P^g)$ and $y \in N_S(P^g)$. Then $(N_\phi)^{gx} \leqslant N_S(P^{gy^{-1}}) = N_S(P^g)$. Hence, $\widetilde{\phi} = c_{gx}$ works. □

There are some equivalent conditions for a fusion system to be saturated. To get one of these results, we need the following concepts:

Definition 2.46. *Let $\mathcal{F}$ be a fusion system on a finite p-group S.*

(i) A subgroup $P \leqslant S$ is called fully $\mathcal{F}$-automized if $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p\big(\mathrm{Aut}_{\mathcal{F}}(P)\big)$.

(ii) A subgroup $P \leqslant S$ is called receptive in $\mathcal{F}$ if for each $Q \leqslant S$ and for each isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$, there is an extension $\widetilde{\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_\phi, S)$ of ϕ where

$$N_\phi := \{x \in N_S(P) \mid \exists y \in N_S\big(\phi(P)\big) : \phi \circ c_x = c_y \circ \phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)\}.$$

The following theorem gives an equivalent condition for being saturated fusion system.

Theorem 2.47. *Let $\mathcal{F}$ be a fusion system on a finite p-group S. Then $\mathcal{F}$ is saturated if and only if for any $P \leqslant S$, there is $Q \in P^{\mathcal{F}}$ such that Q is fully $\mathcal{F}$-automized and receptive.*

The next result describes a structure of an arbitrary saturated fusion system on a finite p-group.

Theorem 2.48. *[11] For every saturated fusion system $\mathcal{F}$ on a finite p-group S, there is a finite group G having S as a (not necessarily Sylow) subgroup such that $\mathcal{F} = \mathcal{F}_S(G)$.*

For a saturated fusion system $\mathcal{F}$ on a finite p-group S, even though there is a finite group G such that S is a Sylow p-subgroup of G and $\mathcal{F} = \mathcal{F}_S(G)$, Theorem 2.48 usually does not give the group G which contains S as its Sylow p-subgroup. Instead, the construction gives a larger group G' such that S is not a Sylow p-subgroup of G'.

Example 2.49. *([11] Example 4) Let S be a finite p-group and $E \leqslant \mathrm{Aut}(S)$ with the possible maximum order n that is coprime to p. Consider $H = S \rtimes E$ and suppose $\mathcal{F} = \mathcal{F}_S(H)$. Since $(n, p) = 1$, the group S is a Sylow p-subgroup of H. Therefore, by Proposition 2.45, $\mathcal{F}$ is saturated and then, Theorem 2.48 is valid for $\mathcal{F}$. With the construction that has been done in [11] for the proof of Theorem 2.48, we obtain a* *group G and a p-subgroup $P \cong S$ of G such that $\mathcal{F} = \mathcal{F}_P(G)$. Explicitly, the group $G = S \wr \Sigma_n$ where*

$$S \wr \Sigma_n = (\underbrace{S \times \cdots \times S}_{n\text{-times}}) \rtimes \Sigma_n$$

and P is the image of the injection

$$\begin{aligned} S &\hookrightarrow S \wr \Sigma_n \\ s &\mapsto \big(\alpha(s); id\big)_{\alpha \in E} \end{aligned}$$

We need the following definitions in order to prove Proposition 3.8.

Definition 2.50. *Let S_1 and S_2 be two groups. An (S_1, S_2)-pair is a pair (P, φ) where $P \leqslant S_1$ and $\varphi : P \to S_2$ is a group homomorphism.*

Definition 2.51. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two fusion systems on finite p-groups S_1 and S_2, respectively. Suppose (P, ψ) and (Q, ρ) are two (S_1, S_2)-pairs.*

(i) The (S_1, S_2)-pair (Q, ρ) is $(\mathcal{F}_1, \mathcal{F}_2)$-subconjugate to (P, ψ) if the diagram

$$\begin{array}{ccc} Q & \xrightarrow{\ \rho\ } & \rho(Q) \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} \\ P & \xrightarrow{\ \psi\ } & \psi(P) \end{array}$$

commutes for some $\varphi_1 \in \mathrm{Hom}_{\mathcal{F}_1}(Q, P)$ and $\varphi_2 \in \mathrm{Hom}_{\mathcal{F}_2}\big(\rho(Q), \psi(P)\big)$. In this case, we write $(Q, \rho) \underset{(\mathcal{F}_1, \mathcal{F}_2)}{\precsim} (P, \psi)$.

(ii) The (S_1, S_2)-pair (Q, ρ) is $(\mathcal{F}_1, \mathcal{F}_2)$-conjugate to (P, ψ) if both $(Q, \rho) \underset{(\mathcal{F}_1, \mathcal{F}_2)}{\precsim} (P, \psi)$ and $(P, \psi) \underset{(\mathcal{F}_1, \mathcal{F}_2)}{\precsim} (Q, \rho)$ hold. In this case, we write $(Q, \rho) \underset{(\mathcal{F}_1, \mathcal{F}_2)}{\sim} (P, \psi)$.

(iii) The (S_1, S_2)-pair (Q, ρ) is strictly $(\mathcal{F}_1, \mathcal{F}_2)$-subconjugate to (P, ψ) if (Q, ρ) is not $(\mathcal{F}_1, \mathcal{F}_2)$-conjugate to (P, ψ) but $(Q, \rho) \underset{(\mathcal{F}_1, \mathcal{F}_2)}{\precsim} (P, \psi)$. In this case, we write $(Q, \rho) \underset{(\mathcal{F}_1, \mathcal{F}_2)}{\precnsim} (P, \psi)$.

Remark 2.52. *Let $\mathcal{F}$ be a fusion system on a finite p-group S. For any (S,S)-pair (P,φ), where φ is a morphism in $\mathcal{F}$, we have $(P,\varphi) \underset{(\mathcal{F}_S(S),\mathcal{F})}{\sim} (P,\mathrm{id})$.*

Proof. Note that φ^{-1} is also a morphism in $\mathcal{F}$. We have the following diagrams which trivially commute.

$$
\begin{array}{ccc}
P & \xrightarrow{\varphi} & \varphi(P) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi^{-1}} \\
P & \xrightarrow{\mathrm{id}} & P
\end{array}
\qquad\qquad
\begin{array}{ccc}
P & \xrightarrow{\mathrm{id}} & P \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi} \\
P & \xrightarrow{\varphi} & \varphi(P)
\end{array}
$$

These diagrams give that $(P,\varphi) \underset{(\mathcal{F}_S,\mathcal{F})}{\precsim} (P,\mathrm{id})$ and $(P,\mathrm{id}) \underset{(\mathcal{F}_S,\mathcal{F})}{\precsim} (P,\varphi)$ $\square$

3. CHAPTER 3

In this chapter, we mainly follow [1]. From now on, when we study with a representation or a biset, we consider them up to isomorphism. By abuse of language, when we mention a representation or a biset we mean their isomorphism classes. Through this section, S will refer to a finite p-group for some fixed prime number p.

3.1. $\mathcal{F}$-stable elements of a biset functor

Notation. *Let S be a finite p-group. For the Burnside ring $A(S,S)$, we denote the p-localization of $A(S,S)$ by $A(S,S)_{(p)} := A(S,S) \times \mathbb{Z}_{(p)}$ where $\mathbb{Z}_{(p)}$ is the p-localization of the integer numbers. Moreover, we denote the p-adic completion of $A(S,S)$ by $A(S,S)^{\wedge}_{p} := A(S,S) \times \mathbb{Z}^{\wedge}_{p}$ where $\mathbb{Z}^{\wedge}_{p}$ is the p-adic integers.*

Definition 3.1. *Let $\mathcal{F}$ be a fusion system on a p-group S and M be a global Mackey functor (or a biset functor) defined on S. An element $V \in M(S)$ is called $\mathcal{F}$-stable if we have*

$$\operatorname{res}_{\varphi} V = \operatorname{res}^{S}_{P} V$$

for any $P \leqslant S$ and for any $\varphi \in \operatorname{Hom}_{\mathcal{F}}(P,S)$. We define $M(\mathcal{F})$ as the set of all $\mathcal{F}$-stable elements of $M(S)$.

Proposition 3.2. *Let $\mathcal{F}$ be a fusion system on a p-group S and M be a global Mackey functor (or a biset functor) defined on S. Then $M(\mathcal{F})$ is a submodule of $M(S)$.*

Proof. The statement comes from the fact that $\operatorname{res}_{\varphi}$ and $\operatorname{res}^{S}_{P}$ are module homomorphisms for any $P \leqslant S$ and $\varphi \in \operatorname{Hom}_{\mathcal{F}}(P,S)$. □

Definition 3.3. *Let S be a finite p-group and $\mathcal{F}$ be a fusion system on S. A virtual (S,S)-biset $X \in A(S,S)$ (or $X \in A(S,S)_{(p)}$ or $X \in A(S,S)^{\wedge}_{p}$) is called $\mathcal{F}$-characteristic if it satisfies the following conditions.*

(i) The virtual biset X is a linear combination of the bisets of the form $[P,\varphi]_S^S$ with $P \leqslant S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$. i.e., X is of the form

$$\sum_{\substack{P\leqslant S,\\ \varphi\in\mathrm{Hom}_{\mathcal{F}}(P,S)}} \alpha_{\varphi,P}[P,\varphi]_S^S$$

where $\alpha_{\varphi,P} \in \mathbb{Z}$.

(ii) The virtual biset X is $\mathcal{F}$-stable with respect to both left and right actions of S.

(iii) The number p does not divide $|X|/|S|$.

Proposition 3.4. *If a fusion system $\mathcal{F}$ on a p-group S has characteristic elements in $A(S,S)_p^{\wedge}$, then it is saturated ([12], Corollary 6.7). The converse also holds. Indeed, any saturated fusion system $\mathcal{F}$ on a p-group S has a unique minimal $\mathcal{F}$-characteristic biset which is denoted by $\Omega_{\mathcal{F}}$ ([13], Corollary 5.5). That is, for any $\mathcal{F}$-characteristic biset Λ, we have $\Omega_{\mathcal{F}} \subseteq \Lambda$ up to isomorphism. Moreover, $\Omega_{\mathcal{F}} - [S,\mathrm{id}]_S^S$ is an actual biset ([13], Theorem 5.3). Furthermore, there is a unique $\mathcal{F}$-characteristic idempotent element in $A(S,S)_p$ which is denoted by $\omega_{\mathcal{F}}$ ([14], Proposition 5.6). Any of these characteristic elements gives the structure of $\mathcal{F}$ because, from an $\mathcal{F}$-characteristic element, we can reconstruct the saturated fusion system $\mathcal{F}$ ([12], Proposition 6.5).*

Remark 3.5. *Let $\mathcal{F}$ be a saturated fusion system on a p-group S. Therefore, it has a $\mathcal{F}$-characteristic idempotent $\omega_{\mathcal{F}}$ which has to be of the form*

$$\sum_{\substack{P\leqslant S,\\ \varphi\in\mathrm{Hom}_{\mathcal{F}}(P,S)}} \alpha_{\varphi,P}[P,\varphi]_S^S.$$

Lemma 3.6. *([14], Lemma 5.5) For a saturated fusion system $\mathcal{F}$ on S and its unique $\mathcal{F}$-characteristic idempotent $\omega_{\mathcal{F}}$, we have*

$$\sum_{(P,\varphi)\underset{(\mathcal{F}_S(S),\mathcal{F})}{\sim}(P,\iota_P)} \alpha_{\varphi,P} = \begin{cases} 0, & \text{if } P \lneq S, \\ 1, & \text{if } P = S. \end{cases}$$

Lemma 3.7. *Let $\mathcal{F}$ be a saturated fusion system on S and M be a global Mackey functor (or a biset functor) defined on S. Then, for any $X \in M(\mathcal{F})_{(p)}$, we have*

$$\omega_{\mathcal{F}} X = X.$$

Proof. Firstly, by Remark 2.52, we conclude that

$$\begin{aligned}\sum_{\substack{P \leqslant S, \\ \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)}} \alpha_{\varphi,P} &= \sum_{P \leqslant S} \sum_{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)} \alpha_{\varphi,P} \\ &= \sum_{P \leqslant S} \sum_{(P,\varphi) \underset{(\mathcal{F}_S(S),\mathcal{F})}{\sim} (P,\iota_P)} \alpha_{\varphi,P}.\end{aligned}$$

Therefore,

$$\begin{aligned}\omega_{\mathcal{F}} X &= \Big(\sum_{\substack{P \leqslant S, \\ \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)}} \alpha_{\varphi,P} [P,\varphi]_S^S \Big) X \\ &= \sum_{P \leqslant S} \sum_{(P,\varphi) \underset{(\mathcal{F}_S(S),\mathcal{F})}{\sim} (P,\iota_P)} \alpha_{\varphi,P} [P,\varphi]_S^S X \\ &= \sum_{P \leqslant S} \sum_{(P,\varphi) \underset{(\mathcal{F}_S(S),\mathcal{F})}{\sim} (P,\iota_P)} \alpha_{\varphi,P} \operatorname{ind}_P^S \operatorname{res}_\varphi X \\ &= \sum_{P \leqslant S} \Big(\sum_{(P,\varphi) \underset{(\mathcal{F}_S(S),\mathcal{F})}{\sim} (P,\iota_P)} \alpha_{\varphi,P} \Big) \operatorname{ind}_P^S \operatorname{res}_P^S X\end{aligned}$$

where the last equality comes from the fact that X is $\mathcal{F}$-stable. Then, we obtain

$$\begin{aligned}\omega_{\mathcal{F}} X &= \Big(\sum_{(S,\varphi) \underset{(\mathcal{F}_S(S),\mathcal{F})}{\sim} (S,\mathrm{id})} \alpha_{\varphi,S} \Big) \operatorname{ind}_S^S \operatorname{res}_S^S X + \sum_{P \lneq S} \Big(\sum_{(P,\varphi) \underset{(\mathcal{F}_S(S),\mathcal{F})}{\sim} (P,\iota_P)} \alpha_{\varphi,P} \Big) \operatorname{ind}_P^S \operatorname{res}_P^S X \\ &\overset{(*)}{=} \operatorname{ind}_S^S \operatorname{res}_S^S X \\ &= X\end{aligned}$$

where $(*)$ comes from Lemma 3.6. $\square$

Proposition 3.8. *Let $\mathcal{F}$ be a saturated fusion system on S and M be a global Mackey functor (or a biset functor) defined on S. Then, we have*

$$M(\mathcal{F})_{(p)} = \omega_{\mathcal{F}} M(S)_{(p)}.$$

Moreover, we have

$$tr_S^{\mathcal{F}} \circ \operatorname{res}_S^{\mathcal{F}} = \operatorname{id}$$

where $tr_S^{\mathcal{F}} : M(S) \xrightarrow{\omega_{\mathcal{F}}} M(\mathcal{F})$, $X \mapsto \omega_{\mathcal{F}} X$ *and* $\operatorname{res}_S^{\mathcal{F}} : M(\mathcal{F}) \hookrightarrow M(S)$ *is the inclusion.*

Proof. By Lemma 3.7, we get $M(\mathcal{F})_{(p)} = \omega_{\mathcal{F}} M(\mathcal{F})_{(p)} \subseteq M(S)_{(p)}$. For the other side, let $X \in M(S)_{(p)}$ be arbitrary. If we consider $\omega_{\mathcal{F}} X \in M(S)_{(p)}$, then for any $P \leqslant S$ and $\varphi \in \operatorname{Hom}_{\mathcal{F}}(P, S)$, we have

$$\operatorname{res}_{\varphi}(\omega_{\mathcal{F}} X) = (\operatorname{res}_{\varphi} \omega_{\mathcal{F}})X = (\operatorname{res}_P^S \omega_{\mathcal{F}})X = \operatorname{res}_P^S(\omega_{\mathcal{F}} X).$$

Note that the second equality comes from the $\mathcal{F}$-stability of $\omega_{\mathcal{F}}$ and the rest are just associativity of ring homomorphisms. Therefore, we showed that $\omega_{\mathcal{F}}$ is $\mathcal{F}$-stable and hence, $\omega_{\mathcal{F}} X \in M(\mathcal{F})_{(p)}$. Finally, by Lemma 3.7, we conclude that $tr_S^{\mathcal{F}} \circ \operatorname{res}_S^{\mathcal{F}} = \operatorname{id}$. □

Proposition 3.9. *Let $\mathcal{F}$ be a saturated fusion system on S. Suppose M_1, M_2, M_3 are global Mackey functors (or biset functors) defined on S and*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is a short exact sequence. Then the following induced sequence is also exact.

$$0 \longrightarrow M_1(\mathcal{F})_{(p)} \longrightarrow M_2(\mathcal{F})_{(p)} \longrightarrow M_3(\mathcal{F})_{(p)} \longrightarrow 0.$$

Proof. The exact sequence $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ gives us another exact sequence $0 \longrightarrow (M_1)_{(p)} \xrightarrow{\alpha_1} (M_2)_{(p)} \xrightarrow{\alpha_2} (M_3)_{(p)} \longrightarrow 0$ where α_1

and α_2 are extensions from the given sequence. Note that α_1 (and α_2) is actually a natural transformation, so it is of the from $\big(\alpha_Q\colon (M_1(Q))_{(p)} \to (M_2(Q))_{(p)}\big)_{Q\leqslant S}$; however, instead of α_Q, we will only write α_1. For an $\mathcal{F}$-stable $X \in M_1(S)$, since α_1 is a natural transformation, we have

$$\operatorname{res}_\varphi \alpha_1(X) = \alpha_1(\operatorname{res}_\varphi X) = \alpha_1(\operatorname{res}^S_P X) = \operatorname{res}^S_P \alpha_1(X)$$

for any $P \leqslant S$ and $\varphi : P \to S$ in $\mathcal{F}$. Similar result holds for α_2. Therefore, we get $\alpha_1\big(M_1(\mathcal{F})_{(p)}\big) \subseteq M_2(\mathcal{F})_{(p)}$ and $\alpha_2\big(M_2(\mathcal{F})_{(p)}\big) \subseteq M_3(\mathcal{F})_{(p)}$. Moreover, by Proposition 3.2, for each $j \in \{1, 2, 3\}$, we get that $M_j(S)_{(p)}$ is a subgroup of $M_j(\mathcal{F})_{(p)}$ and so we have the inclusion maps from $M_j(\mathcal{F})_{(p)}$ to $M_j(S)_{(p)}$. Hence, we obtain the following sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1(\mathcal{F})_{(p)} & \xrightarrow{\beta_1} & M_2(\mathcal{F})_{(p)} & \xrightarrow{\beta_2} & M_3(\mathcal{F})_{(p)} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & M_1(S)_{(p)} & \xrightarrow{\alpha_1} & M_2(S)_{(p)} & \xrightarrow{\alpha_2} & M_3(S)_{(p)} & \longrightarrow & 0 \end{array} \tag{3.1}$$

where β_1 and β_2 are the restriction maps of α_1 and α_2, respectively.

Note that as α_1 is injective, so is β_1. Now, in order to show the surjectivity of β_2, take an arbitrary $X \in M_3(\mathcal{F})_{(p)}$. Consider X as an element of $M_3(S)_{(p)}$ via inclusion. Since α_2 is surjective, there exists $Y \in M_2(S)_{(p)}$ whose image is X. By Proposition 3.8, we have $\omega_{\mathcal{F}} Y \in M_2(\mathcal{F})_{(p)}$ and therefore, we obtain $\beta_2(\omega_{\mathcal{F}} Y) = \omega_{\mathcal{F}} \beta_2(Y) = \omega_{\mathcal{F}} X$ because $\omega_{\mathcal{F}}$ is the unique $\mathcal{F}$-characteristic idempotent. This yields the surjectivity of β_2, since $\beta_2(\omega_{\mathcal{F}} Y) = X$ by Lemma 3.7.

It remains to show that $\ker \beta_2 = \operatorname{Im} \beta_1$. Let $X \in M_2(\mathcal{F})_{(p)}$ whose image is 0 under β_2. Considering X as an element of $M_2(S)_{(p)}$ gives an element $Y \in M_1(S)_{(p)}$ such that $\alpha_2(Y) = X$. With same arguments of above, we get $\beta_1(\omega_{\mathcal{F}} Y) = \omega_{\mathcal{F}} X = X$ and hence $\ker \beta_2 \subseteq \operatorname{Im} \beta_1$. Finally, the set inclusions

$$\operatorname{Im} \beta_1 \subseteq \operatorname{Im} \alpha_1 \cap M_2(\mathcal{F})_{(p)} = \ker \alpha_2 \cap M_2(\mathcal{F})_{(p)} \subseteq \ker \beta_1$$

completes the equality $\ker \beta_2 = \operatorname{Im} \beta_1$. Thus, the upper row of (3.1) is an exact sequence. □

3.2. $\mathcal{F}$-stable virtual representations

We recall the definition of being $\mathcal{F}$-stable (Definition 3.1) by restricting the concept to virtual representations.

Definition 3.10. *Let $\mathcal{F}$ be a saturated fusion system on a p-group S. We say that a virtual representation $V \in R_{\mathbb{K}}(S)$ is $\mathcal{F}$-stable if*

$$\operatorname{res}_{\varphi} V = \operatorname{res}_P^S V$$

for any $P \leqslant S$ and for any $\varphi \in \operatorname{Hom}_{\mathcal{F}}(P, S)$. We define $R_{\mathbb{K}}(\mathcal{F})$ as the set of all $\mathcal{F}$-stable virtual S-representations and $R_{\mathbb{K}}^{+}(\mathcal{F})$ as the set of $\mathcal{F}$-stable actual S-representations.

By Remark 2.36, as a particular case of Proposition 3.2, we obtain that $R_{\mathbb{K}}(\mathcal{F})$ is a subgroup of $R_{\mathbb{K}}(S)$. Actually, $R_{\mathbb{K}}(\mathcal{F})$ is a subring of $R_{\mathbb{K}}(S)$ as $\operatorname{res}_{\varphi}$ and res_P^S respect tensor products. We will show in Proposition 3.11, that $R_{\mathbb{K}}(\mathcal{F})$ could also be defined as Grothendieck group of $R_{\mathbb{K}}^{+}(\mathcal{F})$.

Proposition 3.11. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then we have*

$$R_{\mathbb{K}}(\mathcal{F}) = \left\langle R_{\mathbb{K}}^{+}(\mathcal{F}) \right\rangle$$

where $\left\langle R_{\mathbb{K}}^{+}(\mathcal{F}) \right\rangle$ is the subgroup generated by $R_{\mathbb{K}}^{+}(\mathcal{F}) \leqslant R_{\mathbb{K}}^{+}(S)$.

Proof. We extend the group homomorphism given by the inclusion $\iota : R_{\mathbb{K}}^{+}(\mathcal{F}) \to R_{\mathbb{K}}^{+}(S)$ to the injection $f : \left\langle R_{\mathbb{K}}^{+}(\mathcal{F}) \right\rangle \to R_{\mathbb{K}}(S)$. Since the elements of $R_{\mathbb{K}}(S)$ are of the form $U - V$ for some $U, V \in R_{\mathbb{K}}^{+}(S)$, we conclude that f is a homomorphism. Clearly, we have $\left\langle R_{\mathbb{K}}^{+}(\mathcal{F}) \right\rangle \leqslant R_{\mathbb{K}}(\mathcal{F})$. So, we need to show that $R_{\mathbb{K}}(\mathcal{F}) \leqslant \left\langle R_{\mathbb{K}}^{+}(\mathcal{F}) \right\rangle$. Take any

$X \in R_{\mathbb{K}}(\mathcal{F})$ which has to be of the form $U - V$ for some actual S-representations U and V. Thus, we have

$$\begin{aligned} X &= U - V \\ &= U - V - \big(\Omega_{\mathcal{F}} - [S, \mathrm{id}]\big)^* V + \big(\Omega_{\mathcal{F}} - [S, \mathrm{id}]\big)^* V \\ &= U - V + [S, \mathrm{id}]^* V - \Omega_{\mathcal{F}}^* V + \big(\Omega_{\mathcal{F}} - [S, \mathrm{id}]\big)^* V \\ &= U - V + V - \Omega_{\mathcal{F}}^* V + \big(\Omega_{\mathcal{F}} - [S, \mathrm{id}]\big)^* V \\ &= U - \Omega_{\mathcal{F}}^* V + \big(\Omega_{\mathcal{F}} - [S, \mathrm{id}]\big)^* V \end{aligned}$$

and therefore,

$$X + \Omega_{\mathcal{F}}^* V = U + \big(\Omega_{\mathcal{F}} - [S, \mathrm{id}]\big)^* V. \tag{3.2}$$

Note that applying any functor to an $\mathcal{F}$-stable element gives another $\mathcal{F}$-stable element. So, $\Omega_{\mathcal{F}}^* V$ is $\mathcal{F}$-stable since $\Omega_{\mathcal{F}}$ is $\mathcal{F}$-stable. Additionally, by the stability of X, we get $X + \Omega_{\mathcal{F}}^* V$ is $\mathcal{F}$-stable. Therefore, the right side of equation (3.2) is $\mathcal{F}$-stable. Recall that, by Proposition 3.4, $\Omega_{\mathcal{F}} - [S, \mathrm{id}]$ is an actual biset. Thus, we conclude that X can be written as a difference of two representations, $U + \big(\Omega_{\mathcal{F}} - [S, \mathrm{id}]\big)^* V$ and $\Omega_{\mathcal{F}}^* V$ which are also $\mathcal{F}$-stable. Hence, $X \in \big\langle R_{\mathbb{K}}^{+}(\mathcal{F})\big\rangle$. □

Definition 3.12. *Let G be a group. We define $\overline{R_{\mathbb{K}}(G)}$ as the set of all complex characters of G which takes its values from $\mathbb{K}$. I.e.,*

$$\overline{R_{\mathbb{K}}(G)} = \{\chi \in R_{\mathbb{C}}(G) \,|\, \chi(g) \in \mathbb{K} \text{ for any } g \in G\}$$

Trivially, $R_{\mathbb{K}}(G)$ is a subring of $\overline{R_{\mathbb{K}}(G)}$. Moreover, we can easily observe that $\overline{R_{\mathbb{K}}(G)}$ is a subring of $R_{\mathbb{C}}(G)$.

Definition 3.13. *Let G be a finite group and $n \geq 1$ be such that $g^n = \mathrm{id}$ for all $g \in G$. Take a primitive n-th root of unity ξ and consider the field extension $\mathbb{L}$ of $\mathbb{K}$ by ξ. Then, all irreducible characters of G take their values in $\mathbb{L}$.*

(i) For an irreducible character $\chi \in R_{\mathbb{C}}(G)$ *and an automorphism* $\sigma \in \mathrm{Gal}(\mathbb{L}/\mathbb{K})$*, we define* ${}^{\sigma}\chi \colon G \to \mathbb{L}$ *by*

$$\left({}^{\sigma}\chi\right)(g) = \sigma\left(\chi(g)\right)$$

for any $g \in G$*. Note that* ${}^{\sigma}\chi$ *is also an irreducible character.*

(ii) The map defined as

$$\begin{aligned} \mathrm{tr} : R_{\mathbb{C}}(G) &\to \overline{R_{\mathbb{K}}(G)} \\ \chi &\mapsto \sum_{\sigma \in \mathrm{Gal}(\mathbb{L}/\mathbb{K})} {}^{\sigma}\chi \end{aligned}$$

is called a transfer map.

Lemma 3.14. *Let* G *be a finite group.*

(i) There is a finite field extension $\mathbb{L}$ *of the field* $\mathbb{K}$ *such that* $\overline{R_{\mathbb{K}}(G)} \leq R_{\mathbb{L}}(G)$*.*

(ii) Let $\mathbb{L}$ *be a finite field extension of* $\mathbb{K}$ *such that* $\overline{R_{\mathbb{K}}(G)} \leq R_{\mathbb{L}}(G)$*. Then we have*

$$d\,\overline{R_{\mathbb{K}}(G)} \subseteq R_{\mathbb{K}}(G)$$

where d *is the degree of the extension* $\mathbb{L}$ *of* $\mathbb{K}$*.*

Definition 3.15. *Let* G *be a group.*

(i) A function f *on* G *is called a class function if it takes the same values on the conjugacy classes of* G*, that is for any* $g, h \in G$*, if* $g = \sigma(h)$ *for some inner automorphism* σ *of* G*, then* $f(g) = f(h)$*.*

(ii) The set of all class function on G *with values in* $\mathbb{K}$ *is denoted by* $\mathrm{c}(G, \mathbb{K})$*.*

The following remark argues that $\mathrm{c}(-,\mathbb{C})$ is a biset functor.

Remark 3.16. *Let X be a (G,H)-biset. Define*

$$\begin{aligned} X^*\colon\ \mathrm{c}(H,\mathbb{C}) &\to \mathrm{c}(G,\mathbb{C}) \\ f &\mapsto X^*(f) \end{aligned}$$

where

$$\begin{aligned} X^*(f)\colon G &\to \mathbb{C} \\ g &\mapsto \frac{1}{|H|} \sum_{\substack{h\in H,\, x\in X \\ s.t.\ \ g\cdot x = x\cdot h}} f(h) \end{aligned}$$

Firstly, take two elements $g_1, g_2 \in G$ which are conjugate in G. Say $g_1 = g^{-1}g_2g$ for some $g \in G$. Fix an element $h \in H$ and suppose x is an element of X such that $g_1 \cdot x = x \cdot h$. Therefore, we obtain $g^{-1}g_2g \cdot x = x \cdot h$ and so $g_2 \cdot (g \cdot x) = (g \cdot x) \cdot h$. Hence, for each $x \in X$ which satisfy $g_1 \cdot x = x \cdot h$, there is an element $y \in X$ which also satisfy $g_2 \cdot y = y \cdot h$. Symmetrically, for each $x \in X$ which satisfy $g_2 \cdot x = x \cdot h$, there is an element $y \in X$ which also satisfy $g_1 \cdot y = y \cdot h$. Thus, by considering the fact that f is a class function, we conclude that $X^*(f)$ is also a class function.

Now, take an (H,K)-biset Y. Then for any $f \in \mathrm{c}(K,\mathbb{C})$ and $g \in G$, we have

$$\begin{aligned} (X^* \circ Y^*)(f)(g) &= X^*\big(Y^*(f)\big)(g) \\ &= \frac{1}{|H|} \sum_{h\in H} \sum_{\substack{x\in X\ s.t. \\ g\cdot x = x\cdot h}} \big(Y^*(f)\big)(h) \\ &= \frac{1}{|H|} \sum_{h\in H} \sum_{\substack{x\in X\ s.t. \\ g\cdot x = x\cdot h}} \Bigg(\frac{1}{|K|} \sum_{k\in K} \sum_{\substack{y\in Y\ s.t. \\ h\cdot y = y\cdot k}} f(k)\Bigg). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}(X \circ Y)^*(f)(g) &= \frac{1}{|K|} \sum_{k \in K} \sum_{\substack{(x,_H y) \in X \times_H Y \ s.t. \\ g \cdot (x,_H y) = (x,_H y) \cdot k}} f(k) \\ &= \frac{1}{|K|} \sum_{k \in K} \sum_{\substack{(x,_H y) \in X \times_H Y \ s.t. \\ (g \cdot x,_H y) = (x,_H y \cdot k)}} f(k) \\ &= \frac{1}{|K|} \sum_{k \in K} \sum_{\substack{(x,_H y) \in X \times_H Y \ s.t. \\ (g \cdot x,_H y) = (xh,_H h^{-1} y \cdot k) \ for \ some \ h \in H}} f(k) \\ &= \frac{1}{|K|} \sum_{k \in K} \sum_{\substack{(x,_H y) \in X \times_H Y, h \in H \ s.t. \\ g \cdot x = x \cdot h \ and \ h \cdot y = y \cdot k}} f(k) \\ &= \frac{1}{|K|} \sum_{k \in K} \frac{1}{|H|} \sum_{\substack{x \in X, y \in Y, h \in H \ s.t. \\ g \cdot x = x \cdot h \ and \ h \cdot y = y \cdot k}} f(k) \\ &= \frac{1}{|K||H|} \sum_{k \in K} \sum_{h \in H} \sum_{\substack{x \in X \ s.t. \\ g \cdot x = x \cdot h}} \sum_{\substack{y \in Y \ s.t. \\ h \cdot y = y \cdot k}} f(k).\end{aligned}$$

Therefore, we get $(X \circ Y)^* = X^* \circ Y^*$. *By linearly extending the operator* *, *we obtain* $(X \circ Y)^* = X^* \circ Y^*$ *for any* $X \in A(G,H)$ *and* $Y \in A(H,K)$. *Note also that if we take* X *as* 1_G, *for* $f \in \mathrm{c}(G,\mathbb{C})$ *and* $g \in G$ *we obtain*

$$X^*(f)(g) = \frac{1}{|G|} \sum_{\substack{h \in G, \, x \in G \\ s.t. \ gx = xh}} f(h) = \frac{1}{|G|} \sum_{x \in G} f(x^{-1}gx) = \frac{1}{|G|} \sum_{x \in G} f(g) = f(g).$$

Thus, $\mathrm{c}(-,\mathbb{C})$ *is a biset functor.*

Lemma 3.17. *Let* G, H *be groups and* $\varphi\colon G \to H$ *be a homomorphism. Then, we have* $\mathrm{res}_\varphi \cong H$ *as* (G,H)*-bisets where the action on* H *is given by* $g \cdot x \cdot h = \varphi(g)xh$ *for any* $g \in G$, *and* $x, h \in H$.

Proof. Firstly, recall that $\mathrm{res}_\varphi = G \times H/\Delta$ where $\Delta = \{\big(g, \varphi(g^{-1})\big) \in G \times H \mid g \in G\}$. Consider the map

$$\begin{aligned}\Phi\colon H &\to G \times H/\Delta \\ x &\mapsto (1, x)\Delta\end{aligned}$$

which is a morphism of bisets because for any $g \in G$ and $x, h \in H$ we have

$$\Phi(g \cdot x \cdot h) = \Phi\big(\varphi(g)xh\big) = (1, \varphi(g)xh)\Delta = (g, xh)\Delta = g(1, x)\Delta h = g \cdot \Phi(x) \cdot h\,.$$

In order to show the injectivity, take two elements x, y from H with same image. Therefore, $(1, x)\Delta = (1, y)\Delta$ and so $(1, xy^{-1}) \in \Delta$. This means $\varphi(1) = xy^{-1}$ and hence $x = y$. Finally, the surjectivity comes from the fact that $(g, h)\Delta = (1, \varphi(g)h)\Delta$ for any $g \in G$ and $h \in H$. □

Lemma 3.18. *Let G, H be groups and $\varphi\colon G \to H$ be an homomorphism. Then we have*

$$\mathrm{res}_\varphi(f) = f \circ \varphi$$

for any $f \in \mathrm{c}(H, \mathbb{C})$.

Proof. We have $\mathrm{res}_\varphi \cong H$ by Lemma 3.17. In order to distinguish the biset structure of H from its group structure, we denote the biset by X. Now, we take an arbitrary $f \in \mathrm{c}(H, \mathbb{C})$ and $g \in G$. Therefore,

$$\begin{aligned}\Big(\mathrm{res}_\varphi(f)\Big)(g) = \frac{1}{|H|} \sum_{\substack{h \in H,\, x \in X \\ s.t.\ g \cdot x = x \cdot h}} f(h) &= \frac{1}{|H|} \sum_{\substack{h \in H,\, x \in X \\ s.t.\ \varphi(g)x = xh}} f(h) \\ &= \frac{1}{|H|} \sum_{x \in X} \sum_{\substack{h \in H\ s.t. \\ x^{-1}\varphi(g)x = h}} f(h) \\ &= \frac{1}{|H|} \sum_{x \in X} f(x^{-1}\varphi(g)x).\end{aligned}$$

Since f is a class function on H, we conclude that

$$\Big(\operatorname{res}_\varphi(f)\Big)(g) = \frac{1}{|H|}\bigg(\sum_{x\in X} f(\varphi(g))\bigg) = \frac{1}{|H|}\Big(|H| f\big(\varphi(g)\big)\Big) = f\big(\varphi(g)\big).$$

□

Lemma 3.19. *Let $\mathcal{F}$ be a fusion system on a p-group S and $\chi \in \mathrm{c}(H,\mathbb{C})$. Then the class function χ is $\mathcal{F}$-stable if and only if for all $s,t \in S$, we have $\chi(s) = \chi(t)$ whenever $s = \varphi(t)$ for some $\varphi : P \to S$ in $\mathcal{F}$.*

Proof. Suppose the class function χ is $\mathcal{F}$-stable. Take two elements $s,t \in S$ such that $s = \varphi(t)$ for some $\varphi \in \operatorname{Hom}_{\mathcal{F}}(P,S)$. The stability of χ gives $\operatorname{res}_\varphi \chi = \operatorname{res}_P^S \chi$ and this means that $\chi \circ \varphi = \chi \restriction_P$ by Remark 3.18. Thus, by considering $t \in P$, we conclude that

$$\chi(t) = \chi \restriction_P (t) = \chi \circ \varphi(t) = \chi(s).$$

For the other direction, we take an arbitrary $\varphi \in \operatorname{Hom}_{\mathcal{F}}(P,S)$ with $P \leqslant S$ and assume $\chi(\varphi(t)) = \chi(t)$ for any $t \in P$. This means $\chi \restriction_P = \chi \circ \varphi$ which yields $\operatorname{res}_P^S \chi = \operatorname{res}_\varphi \chi$ by Remark 3.18. Therefore, χ is $\mathcal{F}$-stable. □

Proposition 3.20. *Let $\mathcal{F}$ be a fusion system on a p-group S and $\mathbb{L}$ be a field extension of $\mathbb{K}$ by a primitive n-th root of unity such that all irreducible characters are in $R_{\mathbb{K}}(S)$. Then for any $\chi \in R_{\mathbb{C}}(\mathcal{F})$ and $\sigma \in \operatorname{Gal}(\mathbb{L}/\mathbb{K})$, the character ${}^\sigma\chi$ is $\mathcal{F}$-stable. Moreover, if we restrict the transfer map $\operatorname{tr}\colon R_{\mathbb{C}}(S) \to \overline{R_{\mathbb{K}}(S)}$ to $R_{\mathbb{C}}(\mathcal{F})$, then we get the map $\operatorname{tr}\colon R_{\mathbb{C}}(\mathcal{F}) \to \overline{R_{\mathbb{K}}(\mathcal{F})}$.*

Proof. Suppose $\chi \in R_{\mathbb{C}}(\mathcal{F})$ and $\sigma \in \operatorname{Gal}(\mathbb{L}/\mathbb{K})$. Let $s,t \in S$ be such that $t = \varphi(s)$ for some φ in $\mathcal{F}$. Note that by Lemma 3.19, χ is $\mathcal{F}$-stable gives $\chi(s) = \chi(t)$. Therefore, we conclude that

$$^{\sigma}\chi(t) = \sigma(\chi(t)) = \sigma(\chi(s)) = {}^{\sigma}\chi(s)$$

and again, by Lemma 3.19, this yields that $^{\sigma}\chi$ is $\mathcal{F}$-stable.

Now, consider the transfer map $\mathrm{tr}(\chi) = \sum_{\sigma \in \mathrm{Gal}(\mathbb{L}/\mathbb{K})} {}^{\sigma}\chi$ whose summands are $\mathcal{F}$-stable characters. Since $\mathrm{tr}(\chi)$ is sum of finitely many $\mathcal{F}$-stable characters, it is also $\mathcal{F}$-stable. Thus, $\mathrm{tr}\left(R_{\mathbb{C}}(\mathcal{F})\right) \subseteq \overline{R_{\mathbb{K}}(\mathcal{F})}$. □

3.3. Superclass functions

Definition 3.21. *Let G be a finite group. A function f from the set of subgroups of G to $\mathbb{Z}$ is called a superclass function defined on G if it takes the same values on the conjugacy classes of subgroups of G, that is for any $H_1, H_2 \leqslant G$ if $H_1 = H_2^g$ for some $g \in G$, then $f(H_1) = f(H_2)$. The set of superclass functions defined on G is denoted by $\mathrm{C}(G)$, i.e.*

$$\mathrm{C}(G) = \left\{ f : \mathrm{Sub}(G) \to \mathbb{Z} \,|\, \text{for any } H \leqslant G,\ g \in G,\ f(H) = f(H^g) \right\}.$$

Example 3.22. *Let G be a finite group and $H \leqslant G$ be fixed. Define*

$$\begin{aligned} \varepsilon_H \colon \mathrm{Sub}(G) &\to \mathbb{Z} \\ K &\mapsto \begin{cases} 1 & \text{if } K = H^g \text{ for some } g \in G, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly, the function ε_H is a superclass function defined on G.

With the usual function addition and multiplication, $\mathrm{C}(G)$ becomes a unitary ring. The group structure of $\mathrm{C}(G)$ is just a free group with basis $\{\varepsilon_H \,|\, H \in \mathrm{Cl}(G)\}$ where $\mathrm{Cl}(G)$ is a set of representatives of conjugacy classes of subgroups of G.

Definition 3.23. *For a finite group G, the dual of the Burnside group is*

$$A^*(G) \coloneqq \operatorname{Hom}\big(A(G), \mathbb{Z}\big).$$

Remark 3.24. *Let G be a finite group.*

(i) We identify the set of superclass functions defined on G with the dual of the Burnside group of G by

$$\begin{aligned} \mathrm{C}(G) &\to A^*(G) \\ f &\mapsto \tilde{f} \end{aligned} \qquad \text{where} \quad \begin{aligned} \tilde{f}\colon A(G) &\to \mathbb{Z} \\ X &\mapsto \sum_{x\in[G\backslash X]} f(G_x) \end{aligned}$$

The function $\tilde{f}$ is meaningful by Proposition 2.21 and the same proposition also assures that the correspondence above is indeed an identification. Notice that the inverse of this identification is given by $\tilde{f} \mapsto f$ where $f(L) = \tilde{f}(G/L)$ for $L \leqslant G$. Usually, we will regard a superclass function as an element of the dual of the Burnside group.

(ii) Let U be a (G, H)-biset. Define

$$\begin{aligned} \mathrm{C}(U)\colon\ \mathrm{C}(H) &\to \mathrm{C}(G) \\ \tilde{f} &\mapsto U \cdot \tilde{f} \end{aligned}$$

where

$$(U \cdot \tilde{f})(X) = \tilde{f}(U^{\mathrm{op}} \times_G X)$$

for $X \in A(G)$.

Now, take an (H,K)-biset V. Then, for any $f \in \mathrm{C}(K)$ and $X \in A(G)$ we have

$$\Big(\big(\mathrm{C}(U)\circ \mathrm{C}(V)\big)(f)\Big)X = \big(U\cdot(V\cdot f)\big)X$$
$$= (V\cdot f)(U^{\mathrm{op}} \times_G X) = f\big(V^{\mathrm{op}} \times_H (U^{op}\times X)\big)$$
$$= f\big((U\times_H V)^{\mathrm{op}} \times X\big) = \big(\mathrm{C}(U\times_H V)(f)\big)X$$

Therefore, by linearly extending this structure to virtual bisets, $\mathrm{C}(-)$ becomes a biset functor.

Remark 3.25. *Let H be a finite group, $K \leqslant H$ and $U = \mathrm{Res}_K^H$ be the (K,H)-biset with the left and right multiplications of K and H, respectively. Then for any $L \leqslant K$ and $f \in \mathrm{C}(K)$ we have*

$$(U\cdot f)(K/L) = f\big(U^{\mathrm{op}} \times_K K/L\big) = f\big(H/L\big). \tag{3.3}$$

If we see f as a superclass function on H, then (3.3) means $(U\cdot f)(L) = f(L)$. So, $U\cdot f$ is actually the function restriction of the superclass function f to K. In this case, $U\cdot f$ is denoted by $\mathrm{res}_K^H f$ which coincides with Notation 2.35.

Definition 3.26. *Let $\mathcal{F}$ be a fusion system on a finite p-group S. A superclass function defined on S is said to be defined on $\mathcal{F}$ if it takes same values on the $\mathcal{F}$-conjugacy classes of subgroups of S. The set of superclass function defined on $\mathcal{F}$ is denoted by $\mathrm{C}'(\mathcal{F})$, i.e.*

$$\mathrm{C}'(\mathcal{F}) = \{f \in \mathrm{C}(S)\,|\, \textit{for any } P,R \leqslant S,\ f(P) = f(R) \textit{ if } P \underset{\mathcal{F}}{\sim} R\}.$$

Similar to $\mathrm{C}(S)$, with addition and multiplication of functions $\mathrm{C}'(\mathcal{F})$ has also a ring structure, in fact $\mathrm{C}'(\mathcal{F})$ is a subring of $\mathrm{C}(S)$. Recall that $\mathrm{C}(-)$ is a biset functor by Remark 3.24. So, we can apply Definition 3.1 to the biset functor $\mathrm{C}(-)$ and obtain the following definition.

Definition 3.27. *Let $\mathcal{F}$ be a fusion system on a finite p-group S. The set of $\mathcal{F}$-stable elements of* $\mathrm{C}(S)$ *is denoted by* $\mathrm{C}(\mathcal{F})$. *More precisely,*

$$\mathrm{C}(\mathcal{F}) = \{f \in \mathrm{C}(S) \mid \mathrm{res}_\varphi f = \mathrm{res}_P^S f \text{ for all } P \leqslant S \text{ and } \varphi \in \mathrm{Hom}_\mathcal{F}(P, S)\}.$$

Recall that as $\mathrm{C}(-)$ is a biset functor, $\mathrm{C}(\mathcal{F})$ is a subring of by Proposition 3.2. Actually Definition 3.26 and Definition 3.27 coincide, that is $\mathrm{C}'(\mathcal{F})$ and $\mathrm{C}(\mathcal{F})$ are the same ring.

Lemma 3.28. *Let $\mathcal{F}$ be a fusion system on a finite p-group S. Then the subrings* $\mathrm{C}(\mathcal{F})$ *and* $\mathrm{C}'(\mathcal{F})$ *are the same.*

Proof. Firstly, suppose that $f \in \mathrm{C}'(\mathcal{F})$. Take any $P \leqslant S$ and $\varphi \in \mathrm{Hom}_\mathcal{F}(P, S)$. We need to show that $\mathrm{res}_\varphi f = \mathrm{res}_P^S f$. Notice that $(\mathrm{res}_\varphi)^{\mathrm{op}} = [\varphi(P), \varphi^{-1}]_S^P$ as $\mathrm{res}_\varphi = [P, \varphi]_P^S$. We take an arbitrary transitive P-set which has to be of the form P/L for some $L \leqslant P$ by Proposition 2.21. So, we have $f\big(S/\varphi(L)\big) = f(S/L)$ since $f \in \mathrm{C}'(\mathcal{F})$. Therefore, we obtain

$$\begin{aligned}
(\mathrm{res}_\varphi f)(P/L) &= f\big((\mathrm{res}_\varphi)^{\mathrm{op}} \times_P P/L\big) \\
&= f\big((\mathrm{Iso}_{\phi^{-1}} \times_{\varphi(P)} \mathrm{res}^S_{\varphi(P)})^{\mathrm{op}} \times_P P/L\big) \\
&= f\big(\,\mathrm{ind}^S_{\varphi(P)} \times_{\varphi(P)} \mathrm{Iso}_\phi \times_P P/L\big) \\
&= f\big(\,\mathrm{ind}^S_{\varphi(P)} \times_{\varphi(P)} \varphi(P)/\varphi(L)\big) \\
&= f\big(S/\varphi(L)\big) \\
&= f\big(S/L\big) \\
&= f\big(\,\mathrm{ind}^S_P P/L\big) \\
&= (\mathrm{res}^S_P f)(P/L)
\end{aligned} \tag{3.4}$$

where $\phi\colon P \to \varphi(P)$ such that $\varphi = \iota^S_{\varphi(P)} \circ \phi$. Since res_φ and $\mathrm{res}^S_P f$ are linear, Proposition 2.21 and Equation (3.4) yields that for any virtual $X \in A(P)$, we have $(\mathrm{res}_\varphi f)X = (\mathrm{res}^S_P f)X$.

Now, suppose that $f \in \mathrm{C}(\mathcal{F})$. We need to show that $f(P) = f(R)$ for any $P, R \leqslant S$ with $P \underset{\mathcal{F}}{\sim} R$. which is equivalent to show that $f\big(S/\varphi(P)\big) = f(S/P)$ for any $P \leqslant S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$. For any $P \leqslant S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ as $f \in \mathrm{C}(\mathcal{F})$ we have $\mathrm{res}^S_P f = \mathrm{res}_\varphi f$ and therefore, similar to (3.4) we obtain

$$\begin{aligned} f(S/P) &= f\big(\mathrm{ind}^S_P \times_P P/P\big) = (\mathrm{res}^S_P f)(P/P) \\ &= (\mathrm{res}_\varphi f)(P/P) = f\big(\mathrm{ind}^S_{\varphi(P)} \times_{\varphi(P)} \mathrm{Iso}_\phi \times_P P/P\big) = f\big(S/\varphi(P)\big) \end{aligned}$$

where $\phi\colon P \to \varphi(P)$ such that $\varphi = \iota^S_{\varphi(P)} \circ \phi$. Thus, $f \in \mathrm{C}'(\mathcal{F})$. Hence, we obtain $\mathrm{C}'(\mathcal{F}) = \mathrm{C}(\mathcal{F})$. Since they are subrings of $\mathrm{C}(S)$, they have same ring structures. □

3.4. Dimension function

Definition 3.29. *Let G be a finite group and $X \in R_{\mathbb{K}}(G)$. The dimension function of the virtual representation X is defined as*

$$\begin{aligned} \mathrm{Dim}\, X\colon \ \mathrm{Sub}(G) &\to \mathbb{Z} \\ H &\mapsto \dim_{\mathbb{K}}(V^H) - \dim_{\mathbb{K}}(W^H) \end{aligned}$$

where $X = V - W$.

Lemma 3.30. *For any finite group G, the map*

$$\begin{aligned} \mathrm{Dim}\colon R_{\mathbb{K}}(G) &\to C(G) \\ X &\mapsto \mathrm{Dim}\, X \end{aligned}$$

is a group homomorphism.

Proof. Firstly, we will show that $\mathrm{Dim}\, X$ is a superclass function defined on G for any $X \in R_{\mathbb{K}}(G)$. Let V be any actual representation of G. Take $H \leq G$ and $g \in G$. We need to show that $\mathrm{Dim}(V)(H) = \mathrm{Dim}(V)(H^g)$. Let $v \in V^{H^g}$ be arbitrary. Then for any $h^g \in H^g$, we have $h^g \cdot v = v$. This means $(g^{-1}hg) \cdot v = v$ and so, $h \cdot (g \cdot v) = g \cdot v$. Therefore, $g \cdot v \in V^H$. Thus, we get the following map

$$\Phi \colon V^{H^g} \to V^H$$
$$v \mapsto g \cdot v$$

Clearly, Φ is a vector space homomorphism. Also, for any $w \in V^H$ similar to above arguments we have $g^{-1} \cdot w \in V^{H^g}$ and this means Φ is surjective. Moreover, for any $v, w \in V^{H^g}$, $g \cdot v = g \cdot w$ implies that $(gg^{-1}) \cdot v = w$. So, Φ is injective and hence, Φ is an isomorphism. Since V^H and V^{H^g} are isomorphic, their dimension are equal. Thus, for any $X = V - W \in R_{\mathbb{K}}(G)$ with $V, W \in R^+_{\mathbb{K}}(G)$, as both $\mathrm{Dim}(V), \mathrm{Dim}(W) \in \mathrm{C}(G)$ we have $\mathrm{Dim}(X) = \mathrm{Dim}(V) - \mathrm{Dim}(W) \in \mathrm{C}(G)$.

Now, we will show that Dim is a group homomorphism. Take actual representations V, W and $H \leqslant G$. Clearly, for any $v \in V^H$ and $w \in W^H$, we have $(v, w) \in (V \oplus W)^H$. So, we get the equality

$$V^H \oplus W^H = (V \oplus W)^H.$$

Therefore, we obtain

$$\begin{aligned}\mathrm{Dim}(V + W)(H) &= \dim_{\mathbb{K}}((V \oplus W)^H) \\ &= \dim_{\mathbb{K}}(V^H \oplus W^H) = \dim_{\mathbb{K}}(V^H) + \dim_{\mathbb{K}}(W^H) \\ &= \mathrm{Dim}(V)(H) + \mathrm{Dim}(W)(H) \quad (3.5)\end{aligned}$$

Note that by definition, $\mathrm{Dim}(V - W) = \mathrm{Dim}(V) - \mathrm{Dim}(W)$. By this fact and (3.5), we conclude that Dim is a group homomorphism.

□

Note that $R_{\mathbb{C}}(G)$ is a free abelian group with basis $\{\chi \in R^{+}_{\mathbb{C}}(G) \,|\, \chi \textit{ is irreducible } \}$ and its rank is equal to the number of conjugacy classes of G. On the other hand, the rank of the free abelian group $\mathrm{C}(G)$ is the number of conjugacy classes of subgroups of G. So, in general we do not expect that the dimension homomorphism between $R_{\mathbb{C}}(G)$ and $\mathrm{C}(G)$ is injective or surjective.

Remark 3.31. *The dimension homomorphism* $\mathrm{Dim}\colon R_{\mathbb{K}}(G) \to \mathrm{C}(G)$ *for a finite group* G *is not necessarily injective or surjective.*

Example. *Take* $\mathbb{K} = \mathbb{R}$ *and* G *as the cyclic group of order* 5, $C_5 =< a >$. *The real character table of* C_5 *is*

C_5	1	a	a^2	a^3	a^4
χ_1	1	1	1	1	1
χ_2	2	$\omega+\omega^4$	$\omega^2+\omega^3$	$\omega^2+\omega^3$	$\omega+\omega^4$
χ_3	2	$\omega^2+\omega^3$	$\omega+\omega^4$	$\omega+\omega^4$	$\omega^2+\omega^3$

(3.6)

where $\omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{5}}$. *Let* V_2 *and* V_3 *be the representations associated with* χ_2 *and* χ_3, *respectively. Then we have* $V_i^1 = V$ *and* $V_i^{C_5} = 0$ *for* $i = 2, 3$. *So, we get*

$$\big(\mathrm{Dim}(V_i)\big)(1) = \dim_{\mathbb{R}} V_i = 2 \quad \textit{and} \quad \big(\mathrm{Dim}(V_i)\big)(C_5) = \dim_{\mathbb{R}} 0 = 0$$

Hence, we conclude that $\mathrm{Dim}\, V_2 = \mathrm{Dim}\, V_3$ *as subgroups of* C_5 *are* 1 *and* C_5. *Thus,* $\mathrm{Dim}\colon R_{\mathbb{R}}(C_5) \to \mathrm{C}\left(C_5\right)$ *is not injective.*

Moreover, let $f_1 = \mathrm{Dim}\, V_1$ *and* $f_2 = \mathrm{Dim}\, V_2\,(= \mathrm{Dim}\, V_3)$. *Therefore, we have* $\mathrm{Dim}\left(R_{\mathbb{R}}(C_5)\right) = \{nf_1 + mf_2 \,|\, n, m \in \mathbb{Z}\}$. *Recall that,* $\mathrm{C}(C_5)$ *is an additive free group with basis* $\{\varepsilon_1, \varepsilon_{C_5}\}$ *where*

$$\begin{aligned} \varepsilon_1\colon \mathrm{Sub}(C_5) &\to \mathbb{Z} \\ 1 &\mapsto 1 \\ C_5 &\mapsto 0 \end{aligned} \qquad\qquad \begin{aligned} \varepsilon_{C_5}\colon \mathrm{Sub}(C_5) &\to \mathbb{Z} \\ 1 &\mapsto 0 \\ C_5 &\mapsto 1 \end{aligned}$$

However, there is no $n, m \in \mathbb{Z}$ *such that* $nf_1 + mf_2$ *is equal to* ε_1 *or* ε_{C_5}. *Thus,* $\varepsilon_1, \varepsilon_{C_5} \notin \mathrm{Dim}\left(R_{\mathbb{R}}(C_5)\right)$ *that means* $\mathrm{Dim}\colon R_{\mathbb{R}}(C_5) \to \mathrm{C}\left(C_5\right)$ *is not surjective.*

Lemma 3.32. *For each finite group* G*, we know that a dimension homomorphism* $\mathrm{Dim}\colon R_{\mathbb{K}}(G) \to \mathrm{C}(G)$ *exists. These homomorphism induces a natural transformation of biset functors* $\mathrm{Dim}\colon R_{\mathbb{K}}(-) \to \mathrm{C}(-)$.

Proof. Let H, K be finite groups and $U \in A(K, H)$. We will show that the diagram

$$\begin{array}{ccc} R_{\mathbb{K}}(H) & \xrightarrow{U^*} & R_{\mathbb{K}}(K) \\ \big\downarrow{\scriptstyle \mathrm{Dim}} & & \big\downarrow{\scriptstyle \mathrm{Dim}} \\ \mathrm{C}(H) & \xrightarrow{\mathrm{C}(U)} & \mathrm{C}(K) \end{array}$$

is commutative where U^* is defined as in Remark 2.36. Let $V \in R_{\mathbb{K}}(H)$. It is enough to consider U to be an actual (K, H)-biset and V be an actual $\mathbb{K}$-linear representation of H. We need to show that for any $L \leqslant K$,

$$(U \cdot \mathrm{Dim}\, V)(L) = \mathrm{Dim}\left(\mathbb{K}[U] \otimes_{\mathbb{K}[H]} V\right)(L)$$

which is equivalent to

$$(\mathrm{Dim}\, V)(U^{\mathrm{op}} \times_K K/L) = \dim_{\mathbb{K}}\left(\mathbb{K}[U] \otimes_{\mathbb{K}[H]} V\right)^L \tag{3.7}$$

We take an arbitrary $L \leqslant K$. Notice that for any $W \in R^+_{\mathbb{K}}(H)$ we have the equality

$$\dim_{\mathbb{K}} W^L = \dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{K}} W)^L$$

and therefore, we may assume that $\mathbb{K} = \mathbb{C}$.

By Proposition 2.21 (iii), the H-set $U^{\mathrm{op}} \times_K K/L$ can be written as $\displaystyle\bigsqcup_{i=1}^{n} H/R_i$ for some $R_i \leqslant H$ with $i = 1, \ldots, n$. Therefore, the left hand side of (3.7) becomes

$(\operatorname{Dim} V)\Big(\bigsqcup_{i=1}^{n} H/R_i\Big)$. Now, by Proposition 2.6 we get

$$\begin{aligned}
(\operatorname{Dim} V)\Big(\bigsqcup_{i=1}^{n} H/R_i\Big) &= \sum_{i=1}^{n} (\operatorname{Dim} V)(H/R_i) \\
&= \sum_{i=1}^{n} \dim_{\mathbb{C}} V_i^R \\
&\overset{*}{=} \sum_{i=1}^{n} \big\langle V, \mathbb{C}[H/R_i] \big\rangle_H \\
&= \big\langle V, \bigoplus_{i=1}^{n} \mathbb{C}[H/R_i] \big\rangle_H \\
&= \sum_{i=1}^{n} \big\langle V, \mathbb{C}\big[\bigsqcup_{i=1}^{n} H/R_i\big] \big\rangle_H \\
&= \big\langle V, \mathbb{C}[U^{\mathrm{op}} \times_K K/L] \big\rangle_H
\end{aligned}$$

where $\overset{*}{=}$ comes from Proposition 2.14.

On the other hand, by applying Proposition 2.14 to the right hand side of (3.7), we obtain

$$\dim_{\mathbb{C}} (\mathbb{C}[U] \otimes_{\mathbb{C}[H]} V)^L = \big\langle \mathbb{C}[U] \otimes_{\mathbb{C}[H]} V, \mathbb{C}[K/L] \big\rangle_K$$

Furthermore, by Frobenius Reciprocity we have

$$\big\langle \mathbb{C}[U] \otimes_{\mathbb{C}[H]} V, \mathbb{C}[K/L] \big\rangle_K = \big\langle V, \mathbb{C}[U^{\mathrm{op}}] \otimes_{\mathbb{C}[K]} \mathbb{C}[K/L] \big\rangle_H = \big\langle V, \mathbb{C}[U^{\mathrm{op}} \times_K K/L] \big\rangle_H$$

Thus, (3.7) holds.

□

Lemma 3.33. *Let $\mathcal{F}$ be a saturated fusion system on a p-group S. Then for any $X \in R_{\mathbb{K}}(\mathcal{F})$, the superclass function* $\operatorname{Dim}(X)$ *is $\mathcal{F}$-stable.*

Proof. Let V be $\mathcal{F}$-stable $\mathbb{K}$-linear S-representation and $P, Q \leqslant S$ be such that $P \underset{\mathcal{F}}{\sim} Q$. So, there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ such that $\varphi(P) = Q$. Recall that $\mathrm{res}_\varphi = P \times S/\Delta_\varphi(P)$ where $\Delta_\varphi(P) = \{(p, \varphi(p^{-1})) \,|\, p \in P\}$. Then, we have $\mathrm{res}_\varphi \cong S$ as bisets where the right S-action is just multiplication and the left P-action is given by

$$P \times S \to S$$
$$(p, x) \mapsto \varphi(p)x$$

Therefore, $\mathrm{res}_\varphi V$ is an $\mathbb{C}[P]$-module where the action of P is given by

$$P \times V \to V$$
$$(p, v) \mapsto \varphi(p) \cdot v$$

Now, let $v \in V^P$ and $q \in Q$. As $Q = \varphi(P)$, there is $p \in P$ such that $q = \varphi(p)$. So, we get $q \cdot v = \varphi(p) \cdot v \overset{*}{=} p \cdot v = v$ where $\overset{*}{=}$ comes from the stability. Thus, $v \in V^Q$. Similarly, let $v \in V^Q$ and $p \in P$. Then, by stability we obtain $p \cdot v = \varphi(p) \cdot v = v$. Hence, $V^P = V^Q$. This implies that $\mathrm{Dim}\, V(P) = \mathrm{Dim}\, V(Q)$ and by Lemma 3.28 we conclude that $\mathrm{Dim}\, V$ is $\mathcal{F}$-stable.

Finally, take another $\mathcal{F}$-stable $\mathbb{K}$-linear S-representation. Then,

$$\mathrm{Dim}(V - W)(P) = \mathrm{Dim}\, V(P) - \mathrm{Dim}\, W(P) = \mathrm{Dim}\, V(Q) - \mathrm{Dim}\, W(Q) = \mathrm{Dim}(V - W)(Q)$$

Thus, for any $X \in R_{\mathbb{K}}(\mathcal{F})$, its image is also $\mathcal{F}$-stable.

□

3.5. Borel-Smith functions

Definition 3.34. *Let G be a finite group. A superclass function $f \in \mathrm{C}(G)$ is said to satisfy the Borel-Smith conditions if it satisfies the following three conditions:*

(i) Suppose $H \leqslant G$ and $L \trianglelefteq H$ is such that $H/L \cong \mathbb{Z}/p$ for some odd prime p. Then, $f(L) - f(H)$ is even.

(ii) Suppose $H \leqslant G$ and $L \trianglelefteq H$ be such that $H/L \cong \mathbb{Z}/p \times \mathbb{Z}/p$ for some prime p. Note that there are $(p+1)$-many subgroups of H/L which has order p. Say H_i/L are all the subgroups of order p for $i = 0, \ldots, p$. Then, we have

$$f(L) - f(H) = \sum_{i=0}^{p} \big(f(H_i) - f(H)\big).$$

(iii) Suppose $H \leqslant G$, $L \trianglelefteq H$, $N \leqslant N_G(L)$ is such that $H \trianglelefteq N$ and $H/L \cong \mathbb{Z}/2$. Then, the following hold.

(a) If $N/L \cong \mathbb{Z}/4$, then $f(L) - f(H)$ is even.

(b) If $N/L \cong \mathrm{Q}_8$, then $f(L) - f(H)$ is divisible by 4.

A superclass function which satisfy the Borel-Smith conditions is called a Borel-Smith function. Moreover, the set of Borel-Smith functions defined on G is denoted by $\mathrm{C}_b(G)$.

The Borel-Smith conditions are defined in such a way that the image of the dimension homomorphism satisfies them. It is easy to see that $\mathrm{C}_b(G)$ is a subgroup of $\mathrm{C}(G)$ for any finite group G. More specifically, it gives a subfunctor structure.

Theorem 3.35. *([15], Proposition 3.7) The assignment $G \mapsto \mathrm{C}_b(G)$, where G is finite group, defines a subfunctor of $\mathrm{C}(-)$.*

Theorem 3.36. *([16], Theorem 5.4) Let G be a finite nilpotent group. For any $f \in \mathrm{C}_b(G)$, there exists $V \in R_{\mathbb{R}}(G)$ such that $\mathrm{Dim}\, V = f$. That means the dimension homomorphism $\mathrm{Dim}\colon R_{\mathbb{R}}(G) \to \mathrm{C}_b(G)$ is a surjective.*

Since $\mathrm{C}(-)$ is a biset functor, we can consider its stable elements in a saturated fusion system as in Definition 3.1.

Definition 3.37. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. A Borel-Smith function which is $\mathcal{F}$-stable is said to be a Borel-Smith function for $\mathcal{F}$. The set of Borel-Smith functions for $\mathcal{F}$ is denoted by $\mathrm{C}_b(\mathcal{F})$. I.e.*

$$\mathrm{C}_b(\mathcal{F}) = \{f \in \mathrm{C}_b(G) \mid \mathrm{res}_\varphi f = \mathrm{res}^S_P f \textit{ for all } P \leqslant S \textit{ and } \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)\}.$$

By Lemma 3.28, we also have $\mathrm{C}_b(\mathcal{F}) = \mathrm{C}_b(S) \cap \mathrm{C}'(\mathcal{F})$.

Proposition 3.38. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then the dimension function for the fusion system $\mathcal{F}$ in $\mathbb{Z}_{(p)}$-coefficients*

$$\mathrm{Dim}\colon R_{\mathbb{R}}(\mathcal{F})_{(p)} \to \mathrm{C}_b(\mathcal{F})_{(p)}$$

is surjective.

Proof. We have by Remark 2.36 and Theorem 3.35 that $R_{\mathbb{R}}(-)$ and $\mathrm{C}_b(-)$ are biset functors defined on S. Consider the following sequence

$$0 \longrightarrow \ker \mathrm{Dim} \longrightarrow R_{\mathbb{R}}(-) \xrightarrow{\mathrm{Dim}} \mathrm{C}_b(-) \longrightarrow 0 \tag{3.8}$$

We know by Theorem 3.36 that $\mathrm{Dim}\colon R_{\mathbb{R}}(-) \to \mathrm{C}_b(-)$ is surjective. Therefore, the sequence (3.8) is exact. and hence, by Proposition 3.9, the following induced sequence

$$0 \longrightarrow (\ker \mathrm{Dim})_{(p)} \longrightarrow R_{\mathbb{R}}(\mathcal{F})_{(p)} \xrightarrow{\mathrm{Dim}} \mathrm{C}_b(\mathcal{F})_{(p)} \longrightarrow 0$$

is also exact. In particular, $\mathrm{Dim}\colon R_{\mathbb{R}}(\mathcal{F})_{(p)} \to \mathrm{C}_b(\mathcal{F})_{(p)}$ is surjective. □

Remark 3.39. *For a saturated fusion system $\mathcal{F}$ on S, the dimension homomorphism* $\mathrm{Dim}\colon R_{\mathbb{R}}(\mathcal{F}) \to \mathrm{C}_b(\mathcal{F})$ *may not be surjective.*

Example. *Note that the group* $(\mathbb{Z}/5)^{\times}$ *acts on the group* $\mathbb{Z}/5$ *via the usual multiplication on such a way that the action of every elements of* $\mathbb{Z}/5$ *gives an automorphism of* $\mathbb{Z}/5$*. So, there is the semidirect product* $\mathbb{Z}/5 \rtimes (\mathbb{Z}/5)^{\times}$ *given by this action. More precisely, the multiplication of* $\mathbb{Z}/5 \rtimes (\mathbb{Z}/5)^{\times}$ *is*

$$(n, x)(m, y) = (n + xm, xy)$$

for any $n, m \in \mathbb{Z}/5$ *and* $x, y \in (\mathbb{Z}/5)^{\times}$ *where the addition and multiplication are the addition and multiplication in* $\mathrm{mod}\, 5$*.*

Let $G = \mathbb{Z}/5 \rtimes (\mathbb{Z}/5)^{\times}$ *with the mentioned structure and* $S = \mathbb{Z}/5 \rtimes 1 \leqslant G$*. Also let* $\mathcal{F} = \mathcal{F}_S(G)$ *which is saturated by Proposition 2.45 since* S *is a Sylow* 5*-subgroup of* G*. Recall from (3.6) the real character table of* $\mathbb{Z}/5$ *is of the form*

$\mathbb{Z}/5$	0	1	2	3	4
χ_1	1	1	1	1	1
χ_2	2	α	β	β	α
χ_3	2	β	α	α	β

where $\alpha, \beta \in \mathbb{R}\backslash\{1, 2\}$ *and* $\alpha \neq \beta$*.*

Let V_1, V_2, V_3 *be the representations associated with the characters* χ_1, χ_2, χ_3*, respectively. Firstly, we will try to understand the group* $R_{\mathbb{R}}(\mathcal{F})$*. Note that* $R_{\mathbb{R}}(\mathbb{Z}/5)$ *is the free abelian group generated by* $\{V_1, V_2, V_3\}$*. Since* $S \trianglelefteq G$*, for any morphism* φ *in* $\mathcal{F}$*, we have* $\varphi(S) = S$ *because morphisms are just conjugation homomorphisms by elements of* G*. Also,* S *has only two subgroups; trivial and itself. Therefore, any* $V \in R_{\mathbb{R}}(S)$ *is* $\mathcal{F}$*-stable if the following two conditions hold.*

(i) For any $g \in G$*,* $\mathrm{res}^S_1 V = \mathrm{res}_{\varphi_g} V$ *where* $\varphi_g = c_g \restriction_1 \colon 1 \to S$

(ii) For any $g \in G$*,* $\mathrm{res}^S_S V = \mathrm{res}_{\varphi_g} V$ *where* $\varphi_g = c_g \restriction_S \colon S \to S$*.*

The first condition trivially holds and the second condition is equivalent to

$$\text{For any } g \in G, V = \operatorname{res}_{\varphi_g} V \text{ where } \varphi_g = c_g \restriction_S \colon S \to S. \tag{3.9}$$

Note that for any $(a,b) \in G$ *and* $(s,1) \in S$*, we have*

$$(a,b)^{-1}(s,1)(a,b) = (-ab^{-1}, b^{-1})(s,1)(a,b) = (-ab^{-1}, b^{-1})(s+a,b) = (b^{-1}s, 1)$$

Thus, any such φ_g *is one of the* $\varphi_1(= c_{(0,4)})$, $\varphi_2(= c_{(0,3)})$, $\varphi_3(= c_{(0,2)})$, $\varphi_4(= c_{(0,1)})$ *where*

$$\begin{aligned} \varphi_i \colon S &\to S \\ (s,1) &\mapsto (is,1) \end{aligned}$$

for $i = 1, \ldots 4$*. Notice that Condition 3.9 means that the action of* S *on* $\operatorname{res}_{\varphi_i} V$ *is the same with the usual action of* S *on* V*, for* $i = 1, \ldots, 4$*. In order not to confuse these actions we will denote the action on* $\operatorname{res}_{\varphi_i} V$ *by* $*$*. So we have*

$$s * v = \varphi_i(s) \cdot v \quad \text{and} \quad s \cdot v = \rho_s(v)$$

where $\rho \colon V \to \mathrm{GL}(V)$ *is the associated representation of the* $\mathbb{R}[S]$*-module* V*. If we consider* V_2*, we have*

$$(1,1) * v = \varphi_2(1,1) \cdot v = (2,1) \cdot v \neq (1,1) \cdot v \tag{3.10}$$

because the trace of $\rho_{(1,1)=\alpha}$ *and the trace of* $\rho_{(2,1)=\beta}$ *are different. So,* V_2 *is not* $\mathcal{F}$*-stable. Similarly, we can also see that* V_3 *is not* $\mathcal{F}$*-stable. On the other hand,* V_1 *is trivially* $\mathcal{F}$*-stable. We found that* $R_{\mathbb{R}}(\mathcal{F}) \lneq R_{\mathbb{R}}(S)$*. Let* $V \in R^+_{\mathbb{R}}(S)$ *be arbitrary. Then,*

$V = a_1V_1 + a_2V_2 + a_3V_3$ for some $a_1, a_2, a_3 \in \mathbb{N}$ and the character of V is

$\mathbb{Z}/5$	0	1	2	3	4
χ_V	$a_1 + 2a_2 + 2a_3$	$a_1 + a_2\alpha + a_3\beta$	$a_1 + a_2\beta + a_3\alpha$	$a_1 + a_2\beta + a_3\alpha$	$a_1 + a_2\alpha + a_3\beta$

Therefore, with (3.10) we conclude that if V is $\mathcal{F}$-stable, then $a_2 = a_3$. By considering other elements of S and conjugations in G, with similar computations, we observe that if $a_2 = a_3$, then V is $\mathcal{F}$-stable. Thus, $\mathrm{C}_b(\mathcal{F}) = \{a_1V_1 + a_2V_2 + a_2V_3 \mid a_1, a_2 \in \mathbb{Z}\}$ which is free abelian group with basis $\{V_1, V_2+V_3\}$. Hence, the image of the dimension homomorphism $\mathrm{Dim}\colon R_{\mathbb{R}}(\mathcal{F}) \to \mathrm{C}_b(\mathcal{F})$ is the free abelian group with basis $\{\mathrm{Dim}\, V_1, \mathrm{Dim}(V_2+V_3)\}$ where

$$\begin{aligned} \mathrm{Dim}\, V_1\colon \mathrm{Sub}(S) &\to \mathbb{Z} \\ 1 &\mapsto 1 \\ S &\mapsto 1 \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \mathrm{Dim}(V_1 + V_2)\colon \mathrm{Sub}(S) &\to \mathbb{Z} \\ 1 &\mapsto 4 \\ S &\mapsto 0 \end{aligned}$$

In our case the superclass functions ε_1 and ε_S become

$$\begin{aligned} \varepsilon_1\colon \mathrm{Sub}(S) &\to \mathbb{Z} \\ 1 &\mapsto 1 \\ S &\mapsto 0 \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \varepsilon_S\colon \mathrm{Sub}(S) &\to \mathbb{Z} \\ 1 &\mapsto 0 \\ S &\mapsto 1 \end{aligned}$$

Thus, $\mathrm{Dim}\left(R_{\mathbb{R}}(\mathcal{F})\right) = \left\{(a + 4b)\varepsilon_1 + a\varepsilon_S \mid a, b \in \mathbb{Z}\right\}$.

Now, let us understand $\mathrm{C}_b(\mathcal{F})$. Recall that $\mathrm{C}(S)$ is the free abelian group with basis $\{\varepsilon_1, \varepsilon_S\}$. Let $f \in \mathrm{C}(S)$. So, $f = a\varepsilon_1 + b\varepsilon_S$ for some $a, b \in \mathbb{Z}$. The second and third Borel-Smith conditions hold trivially as there are no such subgroups and the first condition is equivalent to

$$f(1) - f(S) = a - b \textit{ is even.}$$

Furthermore, since $\mathcal{F}$*- (so* G*-)conjugacy classes of* 1 *and* S *are* $\{1\}$ *and* $\{S\}$ *respectively,* f *is trivially in* $\mathrm{C}'(\mathcal{F})$ *which means* f *is* $\mathcal{F}$*-stable by Lemma 3.28. Thus, we obtain that* $\mathrm{C}_b(\mathcal{F}) = \{a\varepsilon_1 + b\varepsilon_S \mid a - b \text{ is even}, a, b \in \mathbb{Z}\}$.

Finally, we conclude $\operatorname{Dim}\big(R_{\mathbb{R}}(\mathcal{F})\big) \lneq \mathrm{C}_b(\mathcal{F})$ *as* $2\varepsilon_1 \in \mathrm{C}_b(\mathcal{F})$ *but* $2\varepsilon_1 \notin \operatorname{Dim}\big(R_{\mathbb{R}}(\mathcal{F})\big)$. *Thus,* Dim *is not surjective.*

3.6. Surjectivity theorem

Notation. *For a given finite group* G*, the set* $\mathcal{P}$ *denote the set of subgroups of* G *that have prime power order.*

Definition 3.40. *Let* G *be a finite group. The set* $\mathcal{D}_{\mathcal{P}}(G)$ *denote the set of functions* $f\colon \mathcal{P} \to \mathbb{Z}$ *which is constant on* G*-conjugacy classes of subgroups of* G*, satisfy the Borel-Smith conditions and the following condition:*

(iv) Let p, q *be prime numbers,* $L, H, M \leqslant G$ *be such that* $L \trianglelefteq H$*,* $H \trianglelefteq M$ *and* $M \leq N_G(L)$ *where* $H/L \cong \mathbb{Z}/p$ *and* H *is a* p*-group. If* $M/H \cong \mathbb{Z}/q^r$ *acts on* H/L *with kernel of prime power order* q^l*, then* $f(L) - f(H)$ *is divisible by* q^{r-l}.

Definition 3.41. *Let* G *be a finite group and* V *be a real representation of* G*. Define the dimension function as*

$$\begin{aligned} \operatorname{Dim}_{\mathcal{P}} V \colon \mathcal{P} &\to \mathbb{Z} \\ H &\mapsto \dim_{\mathbb{R}} V^H \end{aligned}$$

Notice that the dimension function $\operatorname{Dim}_{\mathcal{P}} V$ is the restriction of $\operatorname{Dim} V$ to the set $\mathcal{P}$. Generally, the dimension function $\operatorname{Dim}_{\mathcal{P}} V$ is defined for homotopy representations. However, for any finite dimensional real G-representation V, the unit sphere of V is homotopy representation. So, we can restrict $\operatorname{Dim}_{\mathcal{P}}$ to real G-representations and we get the definition above. [17]

Lemma 3.42. *([18], Proposition 1.2) Let G be a finite group and V be a real representation of G. Then, the function* $\mathrm{Dim}_{\mathcal{P}} V$ *is an element of* $\mathcal{D}_{\mathcal{P}}(G)$.

Theorem 3.43. *(Bauer, [18], Theorem 1.3) Let G be a finite group. Then the dimension homomorphism*

$$\mathrm{Dim}_{\mathcal{P}} \colon R_{\mathbb{R}}(G) \to \mathcal{D}_{\mathcal{P}}(G)$$

is surjective.

Corollary 3.44. *Let S be a Sylow p-group of a finite group G. Let f be a Borel-Smith function on S which satisfy the condition (iv) (Definition 3.40). Then there is a virtual representation $V \in R_{\mathbb{R}}\big(\mathcal{F}_S(G)\big)$ such that* $\mathrm{Dim}\, V = f$.

Proof. We will extend the function f to the set $\mathcal{P}$. For a p-group $P \leqslant G$, there is $g \in G$ such that $P^g \leqslant S$. Define $\tilde{f}(P)$ as $f(P^g)$. For a nontrivial q-group Q where q is a different prime than p, define $\tilde{f}(Q) = 1$. So, we get a map $\tilde{f} \colon \mathcal{P} \to \mathbb{Z}$. By construction $\tilde{f}$ is constant on G-conjugacy classes of subgroups. For any q-group Q where where q is a different prime than p, the Borel-Smith conditions trivially hold as $\tilde{f}$ is equals to 1 on all nontrivial subgroups of Q. For any p-groups P, since f is Borel-Smith function and since $\tilde{f}(P)$ is equal to $f(P^g)$ for appropriate g, the Borel-Smith conditions holds for $\mathcal{P}$. By assumption f and hence $\tilde{f}$ satisfy the condition (iv). Thus, $\tilde{f} \in \mathcal{D}_{\mathcal{P}}(G)$. Therefore, by Theorem 3.43 there exist $V \in R_{\mathbb{R}}(G)$ such that $\mathrm{Dim}\, V = \tilde{f}$. Consider $\mathrm{Res}^G_S V$. Since morphisms in $\mathcal{F}$ are conjugations in G and $\mathrm{Dim}\, V$ is constant on G-conjugacy classes of subgroups, we conclude that $\mathrm{Res}^G_S V$ is $\mathcal{F}$-stable. Hence, $\mathrm{Res}^G_S V$ works. □

Definition 3.45. *Let $\mathcal{F}$ be a fusion system on a p-group S. A Borel-Smith function $f \in \mathrm{C}_b(S)$ is said to satisfy Bauer's Artin relation if the following condition holds.*

- ∗ *Let $H \leqslant S$ and $L \trianglelefteq H$ be such that $H/L \cong \mathbb{Z}/p$.Suppose that $\varphi \in \mathrm{Hom}_{\mathcal{F}}(H, S)$ is an automorphism of H with $\varphi(H) = H$. If the induced auutomorphism of H/L has order m, then $f(L) = f(H)$ is divisible by m.*

The set of Borel-Smith functions which satisfy Bauer's Artin relation is denoted by $\mathrm{C}_{ba}(\mathcal{F})$.

Clearly, $\mathrm{C}_{ba}(F) \leqslant \mathrm{C}_b(\mathcal{F})$. We will observe that the image of the dimension function for $\mathcal{F}$ is contained in $\mathrm{C}_{ba}(F)$.

Lemma 3.46. *Let $\mathcal{F}$ be a fusion system on a p-group S. For $V \in R_{\mathbb{R}}(\mathcal{F})$, the dimension function* $\operatorname{Dim} V$ *satisfy Bauer's Artin relation.*

Proof. Let $V \in R^+_{\mathbb{R}}(\mathcal{F})$, $H \leqslant S$ and $L \trianglelefteq H$ be such that $H/L \cong \mathbb{Z}/p$. Suppose that φ is an automorphism of H in $\mathcal{F}$ with $\varphi(H) = H$. Suppose also that the induced automorphism of H/L has order m. Let $f = \operatorname{Dim} V$. Therefore, by Proposition 2.14 we have

$$\begin{aligned} f(L) - f(H) &= \dim_{\mathbb{R}} V^L - \dim_{\mathbb{R}} V^H \\ &= \big\langle V, \mathbb{R}[S/L] \big\rangle_S - \big\langle V, \mathbb{R}[S/H] \big\rangle_S \\ &= \big\langle V, \mathbb{R}[S/L] - \mathbb{R}[S/H] \big\rangle_S \\ &= \big\langle V, \mathbb{R}[\operatorname{Ind}_H^S H/L] - \mathbb{R}[\operatorname{Ind}_H^S H/H] \big\rangle_S \\ &= \big\langle V, \operatorname{Ind}_H^S \big(\mathbb{R}[H/L] - \mathbb{R}[H/H]\big) \big\rangle_S \end{aligned}$$

By applying Frobenius Reciprocity, we obtain

$$f(L) - f(H) = \big\langle \operatorname{Res}_H^S V, \mathbb{R}[H/L] - \mathbb{R}[H/H] \big\rangle_H \tag{3.11}$$

The complex character table of $\mathbb{Z}/p$ is

$\mathbb{Z}/p$	0	1	2	$\dots$	$p-1$
χ_0	1	1	1	$\dots$	1
χ_1	1	ω	ω^2	$\dots$	ω^{p-1}
χ_2	1	ω^2	ω^4	$\dots$	$\omega^{2(p-1)}$
$\vdots$	$\vdots$			$\dots$	
$\chi_{(p-1)}$	1	ω^{p-1}	$\omega^{2(p-1)}$	$\dots$	$\omega^{(p-1)^2}$

where $\omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{p}}$. For any character χ of H/L we consider the inflation character $\inf_{H/L}^{H} \chi$ of H which is $\inf_{H/L}^{H} \chi(h) = \chi(hL)$ for $h \in H$. For the sake of clarity we omit $\inf_{H/L}^{H}$. By Theorem 2.15, we have $\mathbb{R}[H/L] = \chi_0 + \cdots + \chi_{p-1}$ and therefore, $\mathbb{R}[H/L] - \mathbb{R}[H/H] = \chi_1 + \cdots + \chi_{p-1} = \sum_{i \in (\mathbb{Z}/p)^\times} \chi_i$. Hence, (3.11) turns into

$$f(L) - f(H) = \left\langle \operatorname{Res}_H^S V, \sum_{i \in (\mathbb{Z}/p)^\times} \chi_i \right\rangle_H \tag{3.12}$$

Moreover, since $\mathbb{F}$ is a fusion system, $\iota_H^G \circ \varphi \in \operatorname{Hom}_{\mathcal{F}}(H, G)$. So, we can consider $\varphi\colon H \to G$. The stability of V gives that $\operatorname{res}_\varphi V = \operatorname{res}_H^S V$. Therefore, for any $i = 1, \dots p-1$ Frobenius Reciprocity implies

$$\left\langle \operatorname{Res}_H^S V, \chi_i \right\rangle_H = \left\langle \operatorname{res}_\varphi V, \chi_i \right\rangle_H = \left\langle V, (\operatorname{res}_\varphi)^{\mathrm{op}} \chi_i \right\rangle_S \tag{3.13}$$

On the other hand, by assumption $\varphi \in \operatorname{Aut}(H)$ with $\varphi(L) = L$. Then the induced homomorphism $\tilde{\varphi}\colon H/L \to H/L$ is an element of $\operatorname{Aut}(H/L)$. Furthermore, $H/L \cong \mathbb{Z}/p$ and $\operatorname{Aut}(\mathbb{Z}/p) \cong (\mathbb{Z}/p)^\times$. Let $J =< \delta >$ where δ is the image of $\tilde{\varphi}$ under the isomorphism $\operatorname{Aut}(H/L) \cong (\mathbb{Z}/p)^\times$. Note that by assumption the order of J is m. By construction J is generated by the image of the induced homomorphism $\tilde{\varphi}$. So, for any $i \in (\mathbb{Z}/p)^\times$ the character $\operatorname{ind}_H^S \chi_{ij}$ is actually $(\operatorname{res}_\varphi)^{\mathrm{op}} \chi_i$ for any $j \in J$. Thus, again by Frobenius Reciprocity (3.13) becomes

$$\left\langle \operatorname{Res}_H^S V, \chi_i \right\rangle_H = \left\langle \operatorname{Res}_H^S V, \chi_i j \right\rangle_H \tag{3.14}$$

Additionally, let $i_1, \ldots, i_n$ be a representatives of the cosets of J in $(\mathbb{Z}/p)^{\times}$. Therefore, combining (3.12) and (3.14) produces

$$\begin{aligned} f(L) - f(H) &= \sum_{k=1}^{n} \sum_{i \in i_n J} \left\langle \operatorname{Res}_H^S V, \chi_i \right\rangle_H \\ &= \sum_{k=1}^{n} \sum_{j \in J} \left\langle \operatorname{Res}_H^S V, \chi_{i_k j} \right\rangle_H \\ &= \sum_{k=1}^{n} \sum_{j \in J} \left\langle \operatorname{Res}_H^S V, \chi_{i_k} \right\rangle_H \\ &= \sum_{k=1}^{n} m \left\langle \operatorname{Res}_H^S V, \chi_{i_k} \right\rangle_H \\ &= m \sum_{k=1}^{n} \left\langle \operatorname{Res}_H^S V, \chi_{i_k} \right\rangle_H \end{aligned}$$

and hence, m divides $f(L) - f(H)$. □

Notation. *(i) For a fixed finite group G, we will denote the set of cyclic subgroups of G by $\mathcal{H}_{cyc}$.*

(ii) We say that a nonempty family $\mathcal{H} \subseteq \mathcal{H}_{cyc}$ is nice if it is closed under conjugation of G under taking subgroups.

(iii) For a nice family $\mathcal{H} \subseteq \mathcal{H}_{cyc}$ by an adjacent family of $\mathcal{H}$ we means the nice family $\mathcal{H}' = \mathcal{H} \cup \operatorname{Sub}\left(\{gHg^{-1} \mid g \in G\}\right)$ for some cyclic group $H \in \mathcal{H}_{cyc} \backslash \mathcal{H}$.

Definition 3.47. *Let $\mathbb{F}$ be a fusion system on a finite p-group S and $\mathcal{H} \subseteq \mathcal{H}_{cyc}$ be a nice family. A function $f \in \mathrm{C}_{ba}(\mathcal{F})$ is said to be realized over the family $\mathcal{H}$ if there exists a virtual representation $V \in R_{\mathbb{R}}(\mathcal{F})$ such that $\operatorname{Dim} V(H) = f(H)$ for all $H \in \mathcal{H}$.*

Lemma 3.48. *Let S be a finite p-group and f be Borel-Smith function on S. Then f is uniquely determined by the values on cyclic subgroups of S.*

Proof. The proof will be done by induction on the orders of subgroups of S. The statement is obvious for the trivial group. Suppose that $P \leqslant S$ is nontrivial and that $f(Q)$ is determined by the values of cyclic subgroups for any $Q \leqslant S$ with $|Q| \lneq |P|$. We need to show that $f(P)$ depends only the values of cyclic groups. If P is cyclic, then the statement is trivial. So, suppose P is non-cyclic. Then, by Corollary 3.50, there is $N \trianglelefteq P$ such that $P/L \cong \mathbb{Z}/p \times \mathbb{Z}/p$. Therefore, by the second Borel-Smith condition

$$f(P) = f(N) + \sum_{i=1}^{p} \big(f(P) - f(P_i)\big)$$

where $P_1/N, \ldots, P_p$ are all the subgroups of P/N of order p. Since the order of P/N is p^2, N and P_i's are proper subgroups of P. Thus, $f(P)$ is determined by the values of f on the proper subgroups of P. However, by inductive hypothesis the image of such groups are determined by the images of cyclic groups. □

Theorem 3.49. *(Burnside Basis Theorem) [19] Let P be a nontrivial finite p-group and D be the intersections of all maximal subgroups of P. Then, the quotient group P/D is an elementary abelian group, i.e. isomorphic to $\mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$. Moreover, if the order of the quotient P/D is p^n, then every set of elements $x_1, \ldots, x_s$ which generates P contains a subset of n elements $x_{k_1}, \ldots, x_{k_n}$ which generates P. In the canonical homomorphism $P \hookrightarrow P/D$, the elements $x_{k_1}, \ldots, x_{k_n}$ are mapped onto a basis $x_{k_1}D, \ldots, x_{k_n}D$ of P/D. Conversely, if a set of n elements of P is mapped to a set of generators of P/D in $P \hookrightarrow P/D$, then the set generates P.*

Corollary 3.50. *For any finite non-cyclic p-group P, there exists $H \trianglelefteq P$ such that $P/H \cong \mathbb{Z}/p \times \mathbb{Z}/p$.*

Proof. Let D be the intersections of all maximal subgroups of P. Note that as P is non-cyclic, it is nontrivial and hence, it has a maximal subgroup and D makes sense. Then by Burnside Basis Theorem, P/D is an elementary abelian group. Say $P/D \cong (\mathbb{Z}/p)^n$ for some $n \in \mathbb{N}$. Since maximal subgroups are proper, $n \neq 0$. Therefore, $n \leqslant 1$. If $n \geq 2$, then P/D has a subgroup H/D such that $(P/D)\big/(H/D) \cong \mathbb{Z}/p \times \mathbb{Z}/p$. Therefore, by isomorphism theorems $P/H \cong (P/D)\big/(H/D) \cong \mathbb{Z}/p \times \mathbb{Z}/p$. So, the subgroup H works for the case $n \geq 2$. Now, we will show that if $n \lneq 2$, then P must be cyclic. Note that if $P/D \cong \mathbb{Z}/p$, then P/D is cyclic and hence, it has an element $xD \in P/D$ which generates P/D. Therefore, by Burnside Basis Theorem, the element $x \in P$ generates P. Thus, the group P is cyclic. □

Although Bauer did not precisely state the following lemma, he showed it in the proof of Theorem 1.3 in [18]. Note that for a nice family $\mathcal{P}' \subseteq \mathcal{D}_{\mathcal{P}}(G)$ we can consider $\mathcal{D}_{\mathcal{P}'}(G)$.

Lemma 3.51. *[18] Let G be a finite group, $\mathcal{P}' \subseteq \mathcal{H}_{cyc}$ be nice and $f \in \mathcal{D}_{\mathcal{P}'}(G)$. Also let $\mathcal{H} \subseteq \mathcal{P}'$ be a nice family and $\mathcal{H}'$ be an adjacent family of $\mathcal{H}$. If f is realizable over $\mathcal{H}$, then for any prime q, there exists a number n_q which is coprime to q such that $n_q f$ is realizable over $\mathcal{H}'$.*

Theorem 3.52. *Let $\mathcal{F}$ be a saturated fusion system on a p-group S. Then the dimension homomorphism*

$$\mathrm{Dim}\colon R_{\mathbb{R}}(\mathcal{F}) \to \mathrm{C}_{ba}(\mathcal{F})$$

is surjective.

Proof. Let $f \in \mathrm{C}_{ba}(\mathcal{F})$ We will show by induction that f is realizable over the family $\mathcal{H}_{cyc}$. Firstly, as a base case we will show that f is realizable over $\{1\}$. Let V be the trivial representation of S and $\chi = f(1)V$. Then, $X \in R_{\mathbb{R}}(S)$ and $\mathrm{Dim}\, X(1) = \dim_{\mathbb{R}}(X) = f(1) \dim_{\mathbb{R}} V = f(1)$. Thus, f is realized at 1. Suppose f is realizable over a nice family $\mathcal{H}$. Let $\mathcal{H}'$ be an adjacent family of $\mathcal{H}$. We need to show that f is also realizable over $\mathcal{H}'$.

For any prime q, we will find a number n_q which is coprime to q such that $n_q f$ is realizable over $\mathcal{H}'$. By Proposition 3.38, there is $V \in R_{\mathbb{R}}(\mathcal{F})$ and $a \in \mathbb{Z}_{(p)}$ such that $\mathrm{Dim}(aV) = f$. Since $a \in \mathbb{Z}_{(p)}$, there is $n_p \in \mathbb{Z}$ which is coprime to p such that $n_p a \in \mathbb{Z}$. Thus, $n_p aV \in R_{\mathbb{R}}(\mathcal{F})$ and $n_p f = \mathrm{Dim}(n_p aV)$. Now, suppose q is a prime number different than p. Let $H \in \mathcal{H}_{cyc} \backslash \mathcal{H}$ and L be a maximal proper subgroup of H. So $H/L \cong \mathbb{Z}/p$. By Theorem 2.48, there is a finite group G such that $S \leqslant G$ and $\mathcal{F} = \mathcal{F}_S(G)$. We want to apply Lemma 3.51 to f, but we need to show that $f \in \mathcal{D}_{\mathcal{H}_{cyc}}(G)$. Since f is Borel-Smith function, it only remains to show that the condition (iv) of Definition 3.40. Suppose $H \trianglelefteq K \leqslant N_H(L)$ and $K/H \cong \mathbb{Z}/q^r$ acts on h/L with kernel of order q^l. In order to satisfy (iv), we need to show that

$f(L)-f(H)$ is divisible by q^{r-l}. By the given action K/H induces a cyclic subgroup of $\mathrm{Aut}(H/L) \cong \mathrm{Aut}(\mathbb{Z}/p)$ and the order of this cyclic group is q^{r-l}. We chose an element $k \in K$ which induces an automorphism of H/L of order q^{r-l}. Since $f \in \mathrm{C}_{ba}(\mathcal{F})$, f satisfies the condition $*$ of Definition 3.45 and therefore, $f(L) - f(H)$ is divisible by q^{r-l}. Thus, f satisfy (iv). Hence, we can apply Lemma 3.51 and we get a number n_q such that $n_q f$ is realized over $\mathcal{H}'$ and n_q is coprime to q.

For prime q, we denote the virtual representation which realized $n_q f$ over $\mathcal{H}'$ by V_q. Suppose $q_1, \ldots, q_t$ are all the prime divisors of n_p. So, the numbers $n_p, n_{q_1}, \ldots, n_{q_t}$ do not have common divisor as n_q is coprime to q for any q. Therefore, Bézout's Identity there are $m_0, m_1, \ldots, m_t \in \mathbb{Z}$ such that $m_0 n_p + m_1 n_{q_1} + \cdots m_t n_{q_t} = 1$. Define $V = m_o V_p + m_1 V_{q_1} + \cdots m_t V_{q_t} \in R_{\mathbb{R}}(\mathcal{F})$. Therefore, for any $P \in \mathcal{H}'$, we get

$$\begin{aligned}
\mathrm{Dim}\, V(P) &= \mathrm{Dim}(m_o V_p)(P) + \mathrm{Dim}(m_1 V_{q_1})(P) + \cdots + \mathrm{Dim}(m_t V_{q_t})(P) \\
&= m_o \,\mathrm{Dim}(V_p)(P) + m_1 \,\mathrm{Dim}(V_{q_1})(P) + \cdots + m_t \,\mathrm{Dim}(V_{q_t})(P) \\
&= m_o n_p f(P) + m_1 n_{q_1} f(P) + \cdots + m_t n_{q_t} f(P) \\
&= (m_o n_p + m_1 n_{q_1} + \cdots + m_t n_{q_t}) f(P) \\
&= f(P)
\end{aligned}$$

Thus, V realized f over $\mathcal{H}'$. By induction we conclude that f is realizable over $\mathcal{H}_{cyc}$. On the other hand, by Lemma 3.48 both f and $\mathrm{Dim}\, V$ are uniquely determined with the values of cyclic subgroups of S. Hence, $f = \mathrm{Dim}\, V$ on all subgroups of S. □

Remark. *By Theorem 2.48, we get a group G such that $\mathcal{F} = \mathcal{F}_S(G)$. However, it can not be said that a Borel-Smith function $f \in \mathrm{C}_{ba}(S)$ has an extension $\tilde{f} \in \mathrm{C}_{ba}(T)$ for the Sylow p-subgroup T of G.*

Example 3.53. *Let $\mathcal{F} = \mathcal{F}_S(H)$ where $H = \mathrm{A}_4$ and $S = \mathrm{V}_4 \leqslant \mathrm{A}_4$. Note that V_4 is a Sylow 2-subgroup of H. So, by Proposition 2.45, $\mathcal{F}$ is a saturated fusion system. On the other hand, $\mathrm{Aut}(\mathrm{V}_4) \cong \Sigma_3$. Moreover, since $|\Sigma_3| = 6$, the possible order of the subgroup of Σ_3 which is coprime to 2 is 3. Let $E = \langle \alpha \rangle \in \Sigma_3$ where $\alpha = (123)$. Also, write $\mathrm{V}_4 = \{a_0, a_1, a_2, a_3\}$ where $a_0 = \mathrm{id}$, $a_1 = (12)(34)$, $a_2 = (13)(24)$ and*

$a_3 = a_1a_2 = (14)(23)$. *Then we have* $\mathrm{A}_4 \cong \mathrm{V}_4 \rtimes E$ *with the action*

$$\begin{aligned} \mathrm{V}_4 \times E &\to E \\ (a_i, \sigma) &\mapsto a_{\sigma(i)} \end{aligned}$$

Thus, we are in the same case with Example 2.49. Therefore, by Theorem 2.48 we have the groups $G = \mathrm{V}_4 \times \mathrm{V}_4 \times \mathrm{V}_4 \rtimes \Sigma_3$ *and* $P = \iota(\mathrm{V}_4)$ *where*

$$\begin{aligned} \iota\colon \mathrm{V}_4 &\to \mathrm{V}_4 \times \mathrm{V}_4 \times \mathrm{V}_4 \rtimes \Sigma_3 \\ a_i &\mapsto \big(a_i, a_{\alpha(i)}, a_{\alpha^2(i)}, \mathrm{id}\big) \end{aligned}$$

However, P *is not a Sylow* 2*-subgroup of* G*. Let* $\beta = (12)$ *and* $T = \mathrm{V}_4 \times \mathrm{V}_4 \times \mathrm{V}_4 \rtimes \langle \beta \rangle$ *which is a Sylow* 2*-subgroups of* G*. Note also that* P *embeds in* T *via the function* ι*.*

4. CONCLUSION

In this thesis, we give the definitions biset functor, representation ring and we study the biset functor structure of the representation ring. Moreover, we explain fusion systems and examine $\mathcal{F}$-stable elements of the representation ring. We investigate the natural transformation $\mathrm{Dim}\colon R_{\mathbb{R}}(-) \to \mathrm{C}(-)$ and determine the image of $R_{\mathbb{R}}(\mathcal{F})$ under the map Dim.

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