

ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

**STATIC ANALYSIS OF STIFFENED LAMINATED COMPOSITE PLATES BY
USING GDQM**



M.Sc. THESIS

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**Department of Aeronautical and Astronautical Engineering
Aeronautics and Astronautics Engineering Master's Program**

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

**GÜÇLENDİRİLMİŞ LAMİNE KOMPOZİT LEVHALARIN DQEM İLE
STATİK ANALİZİ**

YÜKSEK LİSANS TEZİ

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To my family,



FOREWORD

I sincerely express my gratitude and admiration to Prof. Dr. Aytaç ARIKOĞLU, my supervisor, for his invaluable guidance, support, and motivation during all stages of my research.

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June 2023

İhsan Berk GÜNEYTEPE
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ABBREVIATIONS

GDQM	: Generalized Differential Quadrature Method
ANSYS	: Analysis System
FEA	: Finite Element Analysis
SEM	: Spectral Element Method
FDM	: Finite Difference Method
BEM	: Boundary Element Method





SYMBOLS

u	: Axial displacement along x axis of the plane
$u_{s,t}$: Axial displacement along x axis of the stiffener associated with torsion
v	: Axial displacement along y axis of the plane
$v_{s,t}$: Axial displacement along y axis of the stiffener associated with torsion
w	: Axial displacement along z axis of the plane
$w_{s,t}$: Axial displacement along z axis of the stiffener associated with torsion
ϵ_{ij}	: Normal strains of plate
$\epsilon_{jk,b}^s$: Normal strains of stiffener associated with bending
$\bar{Q}_{jk}^{(i)}$: Reduced stiffness coefficients of the i th layer
γ_{ij}	: Shear stains of plate
$\gamma_{jk,b}^s$: Shear strains of stiffener associated with bending
σ_{jk}^i	: Normal stresses of plate i th layer
τ_{jk}^i	: Shear stresses of plate i th layer
$\sigma_{jk,b}^s$: Normal stresses of stiffener associated with bending
$\tau_{jk,b}^s$: Shear stresses of stiffener associated with bending
E	: Modulus of elasticity
G	: Shear modulus of elasticity
ρ	: Density
ρ_s	: Stiffener density
I_0	: Normal Kuvvet Bileşenleri
I_0^s	: Normal Kuvvet Bileşenleri
I_2	: Normal Kuvvet Bileşenleri
I_2^s	: Normal Kuvvet Bileşenleri
M_{ij}	: Section moments of plate
Q_{ij}	: Section forces of plate
M_{xx}^s	: Section moments of stiffener

- Q_{xz}^s : Section forces of stiffener
 J_0 : Polar moment of inertia of the stiffener cross section
 J_s : Equivalent moment of inertia of the cross section due to torsion



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STATIC ANALYSIS OF STIFFENED LAMINATED COMPOSITE PLATES BY USING GDQM

SUMMARY

Due to their extensive use in the aerospace and maritime industries, stiffened laminated composite plates represent an important study topic in engineering. Due to their superior mechanical qualities and high strength-to-weight ratio, stiffened laminated composite plates are frequently employed in a variety of technical applications. This study examines an eccentric stiffener-equipped laminated composite plate's static analysis. First order shear deformation theory is used to mathematically model the main plate under investigation, which includes four layers of carbon/epoxy, while Timoshenko beam theory is used to mathematically describe the isotropic stiffener. A torsional coefficient is also employed and included in the mathematical beam model for isotropic stiffeners. Also, for modelling isotropic plate, Mindlin plate theory is used. The theoretical investigations seek to determine how the insertion of stiffeners impacts the lamination of composite plates' static behavior. The stiffened laminated composite plate's motion equations are derived in the theoretical analysis using Hamilton's principle. A numerical method known as the Generalized Differential Quadrature Method (GDQM) has demonstrated considerable promise in resolving challenging engineering issues. The GDQM approach, which discretizes the partial differential equations into a set of algebraic equations, is used in this study to solve the governing equation that represents the free vibration of eccentrically stiffened laminated composite plates. For each edge of the plate, the clamped boundary conditions are taken into account. The Wolfram Mathematica application is an effective suite of tools for numerical analysis and scientific computing. It is used for numerical problem solving and result visualization. This study investigates the natural frequencies and mode shapes of isotropic stiffened plate for isotropic plate, [0,90,90,0] and [45,60,60,45] fiber angle carbon fiber laminated composite. The findings demonstrate that the GDQM method is a reliable and accurate approach for determining the free vibrations of such plates. The results also show that changing the fiber angle orientation significantly affects the mode shapes and can vary the natural frequencies. These findings are applicable in a variety of engineering disciplines for the design and analysis of stiffened laminated composite plates.



GÜÇLENDİRİLMİŞ LAMİNE KOMPOZİT LEVHALARIN DQEM İLE STATİK ANALİZİ

ÖZET

Havacılık ve denizcilik endüstrilerindeki yaygın kullanımları nedeniyle, sertleştirilmiş lamine kompozit levhalar, mühendislikte önemli bir çalışma konusunu temsil etmektedir. Üstün mekanik kaliteleri ve yüksek mukavemet-ağırlık oranları nedeniyle, sertleştirilmiş lamine kompozit levhalar, çeşitli teknik uygulamalarda sıklıkla kullanılmaktadır. Bu çalışma, eksantrik takviye donanımlı lamine kompozit levhanın statik analizini incelemektedir. Birinci mertebe kayma deformasyonu teorisi, dört karbon/epoksi tabakası içeren, araştırılmakta olan ana levhayı matematiksel olarak modellemek için kullanılırken, Timoshenko kiriş teorisi izotropik sertleştiriciyi matematiksel olarak tanımlamak için kullanılır. İzotropik sertleştiriciler için bir burulma katsayısı da kullanılır ve matematiksel kiriş modeline dahil edilir. Ayrıca, izotropik plakanın modellenmesi için Mindlin plaka teorisi kullanılmıştır. Teorik araştırmalar, sertleştiricilerin yerleştirilmesinin kompozit plakaların statik davranışını nasıl etkilediğini belirlemeye çalışır. Sertleştirilmiş lamine kompozit levhanın hareket denklemleri teorik analizde Hamilton prensibi kullanılarak türetilmiştir. Diferansiyel Quadrature Eleman Metodu (DQEM) olarak bilinen sayısal bir yöntem, zorlu mühendislik sorunlarının çözümünde önemli bir umut vaat etmiştir. Kısmi diferansiyel denklemleri bir dizi cebirsel denklem halinde ayıklaştıran DQEM yaklaşımı, bu çalışmada eksantrik olarak sertleştirilmiş lamine kompozit plakaların serbest titreşimini temsil eden yönetici denklemi çözmek için kullanılmıştır. Plakanın her bir kenarı için kenetlenmiş sınır koşulları dikkate alınır. Wolfram Mathematica uygulaması, sayısal analiz ve bilimsel hesaplama için etkili bir araç paketidir. Sayısal problem çözme ve sonuç görselleştirme için kullanılır. Bu çalışma, izotropik plaka, $[0,90,90,0]$ ve $[45,60,60,45]$ fiber açılı karbon fiber lamine kompozit için izotropik takviyeli levhanın doğal frekanslarını ve mod şekillerini araştırmaktadır. Bulgular, DQEM yönteminin bu tür plakaların serbest titreşimlerini belirlemek için güvenilir ve doğru bir yaklaşım olduğunu göstermektedir. Sonuçlar aynı zamanda fiber açılı oryantasyonunun değiştirilmesinin mod şekillerini önemli ölçüde etkilediğini ve doğal frekansları değiştirebileceğini göstermektedir. Bu bulgular, sertleştirilmiş lamine kompozit levhaların tasarımı ve analizi için çeşitli mühendislik disiplinlerinde uygulanabilir.



1. INTRODUCTION

In the introduction, the main topics of stiffened composite laminated plates, modeling of stiffened composite laminated plates, Hamilton's principle application, and GDQM and other numerical method applications are discussed in detail.

1.1 Stiffened Composite Laminated Plates

In the field of structural engineering, stiffened laminated composite plates have attracted a lot of interest as a result of their remarkable mechanical qualities and diverse range of uses. These plates are made of several different materials—usually fiber-reinforced polymers—layered and then joined together to create a composite structure. Their load-carrying capacity and rigidity are further enhanced by the inclusion of stiffeners, making them useful for a variety of sectors, including civil engineering, automotive, marine, and aerospace.

Stiffened laminated composite plates' special material composition enables the adjustment of their qualities to certain design specifications. Engineers may maximize the mechanical strength, stiffness, and lightweight properties of the composite by determining the right fiber orientation, fiber type, and resin matrix. By enhancing overall load distribution, decreasing buckling and vibration, and raising bending and torsional stiffness, the use of stiffeners like stringers or ribs further enhances the structural performance.

Stiffened laminated composite plate design and analysis provide a number of issues, including the choice of the best layup configurations, effective modeling methods, precise failure mode prediction, and efficient production procedures. To solve these issues and guarantee accurate and affordable designs, researchers and engineers have created a variety of analytical, numerical, and experimental methodologies.

1.2 Modelling of Stiffened Laminated Composite Plates

The design and analysis of these structures heavily rely on the modeling of stiffened laminated composite plates. Engineers may improve their performance, guarantee structural integrity, and lower the danger of failure by precisely simulating their mechanical behavior and reaction to external stresses. There are several different modeling methods for stiffened laminated composite plates, including analytical formulations, numerical simulations, and experimental validation.

It is required to take into account a number of parameters, including material anisotropy, geometric nonlinearity, and the interaction between the stiffeners and the plate, in order to appropriately describe the complicated behavior of stiffened laminated composite plates. The behavior of these structures may be better understood using analytical formulations based on traditional plate theories, such as the Kirchhoff plate theory or the first-order shear deformation theory. They could be constrained, nevertheless, by challenging loading scenarios or asymmetric geometries.

Modeling stiffened laminated composite plates requires the use of numerical simulations, such as finite element analysis (FEA). The plate and stiffener geometries may be discretized into finite elements using FEA, which enables precise description of the material characteristics and boundary conditions. Delamination phenomena and through-thickness changes in material characteristics may be captured using advanced computational approaches, such as the layer-wise approach, which are essential in composite constructions. To forecast the beginning and spread of deterioration in the structure, numerical models may also include progressive damage and failure criteria.

To guarantee the correctness and dependability of the modeling methodologies, experimental validation is crucial. Physical testing at full or reduced scales might provide useful information for calibration and validation needs. For comparison with the numerical predictions, these tests entail applying controlled loads and observing the behavior of the stiffened laminated composite plates.

In this research study, we address analytical formulations and numerical simulations to provide an overview of the modeling approaches used for stiffened laminated composite plates.

1.3 Hamilton's Principle

In order to analyze stiffened laminated composite plates' dynamic behavior and reaction to external loads, the equation of motion for these plates must be solved. A methodical and effective way to find the governing equations of motion for these intricate systems is provided by the Hamilton's principle. The Hamilton's principle enables the development of equations that precisely describe the plate's dynamic behavior while taking into account the interaction between the plate and the stiffeners by framing the issue in terms of variational principles.

The concept of least action, on which the Hamilton's principle is founded, argues that the motion of a system that minimizes the action functional is its genuine motion. The action functional for stiffened laminated composite plates is the difference between kinetic energy and potential energy. The potential energy denotes the work done by outside forces operating on the structure, while the kinetic energy accounts for the inertia of the plate.

It is required to take into account the plate's kinematics, strain-displacement connections, and constitutive equations that regulate the behavior of the material in order to apply the Hamilton's principle to stiffened laminated composite plates. The total potential and kinetic energy of the system may be expressed by taking into account the stiffness and mass contributions from the stiffeners and including the relevant plate theory, such as the Kirchhoff plate theory or the first-order shear deformation theory.

A set of partial differential equations that reflect the equations of motion for the stiffened laminated composite plate may be constructed by modifying the action functional with regard to the permissible changes in the deflection field. With the use of suitable methods, such as finite element analysis or modal analysis, these equations may then be numerically solved to determine how the structure will respond to various loading scenarios.

One of the goals of this research study is to provide a general overview of applying Hamilton's principle to solve the equation of motion for laminated composite plates that have been stiffened. The theoretical underpinnings of the Hamilton's principle, the formulation of the action functional for stiffened laminated composite plates, and

the numerical methods used to resolve the ensuing equations of motion will all be covered in this work.

1.4 GDQM and Other Numerical Methods

An essential first step in examining the dynamic behavior of diverse structures is the solution of the equations of motion. Numerous numerical methods, such as the Generalized Differential Quadrature Method (GDQM) and other numerical techniques, provide effective ways to acquire precise answers when it comes to stiffened laminated composite plates. With the use of these techniques, the equations of motion may be approximated and discretized, allowing for the prediction of the dynamic response of stiffened laminated composite plates under various loading scenarios.

By evaluating a function at discrete places, the Generalized Differential Quadrature Method (GDQM) is a numerical approach that approximates the derivatives of a function. GDQM discretizes the spatial domain of the plate into a set of grid points in order to solve equations of motion, and it depicts the deflection field as a collection of unknowns. The governing equations may be converted into a set of algebraic equations using the GDQM, which can then be solved numerically to determine the plate's deflection and response.

Other numerical techniques, in addition to GDQM, are often used to solve the equations of motion for stiffened laminated composite plates. One of these often used numerical techniques is the finite element method (FEM). FEM approximates the deflection field inside each element by discretizing the plate into smaller finite elements, each with a set of nodes. The equations of motion may be calculated and computationally solved to determine the dynamic response of the structure by building the stiffness and mass matrices.

In order to solve the equations of motion, other numerical approaches, including as the Spectral Element Method (SEM), the Finite Difference Method (FDM), and the Boundary Element Method (BEM), are also used in the study of stiffened laminated composite plates. When dealing with unbounded domains, addressing unusual geometries, or capturing high-frequency events, these approaches have benefits.

2. LITERATURE REVIEW

Understanding the structural behavior of stiffened laminated composite plates and guaranteeing their safe and effective design depend heavily on static analysis. The Generalized Differential Quadrature Method (GDQM), one of several numerical techniques used to solve the governing equations, is showing promise. The current developments in the static analysis of stiffened laminated composite plates using various numerical approaches are summarized in this review of the literature, along with the benefits, drawbacks, and potential uses of each numerical method.

The analysis of stiffened laminated composite plates has been successfully completed by many scholars. For instance, Zhang et al. (2018) looked at the static behavior of composite plates that had been stiffened and loaded under different circumstances. The variational technique was used to study the static deformation of eccentrically stiffened plates with partial composite action. The governing differential equations were developed using the variational technique based on the concept of the minimal potential energy, taking into account the strain energy of the connections between the plate and the stiffeners as well as the accompanying boundary conditions.

The stiffened laminated composite plate's equations of motion are developed using the virtual work concept in another research by Demet Balkan (2020). Galerkin's approach is used to discretize the equations of motion into the time domain.

The natural frequencies and mode forms of a freely vibrating, integrally stiffened and/or stepped plate were also studied by [2] Ahmad and Kapania (2016). Instead of the more common rib stiffeners, plate-strip stiffeners were utilized in this case. The first-order shear deformation theory was used to examine the plate as well as the stiffeners. According to the specified boundary conditions, it was assumed that the deflections and rotations were a tensor product of Timoshenko beam functions. The method described in this study may be used to completely clamped, free, and cantilever supported stiffened plates, unlike Navier and Levy solution approaches. The Rayleigh-Ritz technique was used to resolve the governing differential equations.

In addition, [1] Wang (2015)'s paper introduces two improvements to help solve the problem of employing the GDQM to solve differential equations using Dirac-delta

functions. The governing equation is numerically integrated before it is discretized in terms of the differential quadrature, and the moving point load is work-equivalent to loads applied at all grid points. With these adjustments, it is now possible to study the static behavior and forced vibration of beams under a fixed or moving point load directly using the GDQM. It is shown that the improved GDQM may provide very precise results. A time-dependent Dirac-delta function is used to solve the partial differential equations while retaining the compactness and computing effectiveness of the GDQM.



3. HYPOTHESIS

In comparison to other numerical approaches, the Generalized Differential Quadrature Method (GDQM) will provide precise and effective findings when used to the static analysis of stiffened laminated composite plates.

Due to its intrinsic correctness and computational economy, it can be said that the GDQM will be a viable numerical approach for the static analysis of stiffened laminated composite plates. We anticipate being able to forecast the structural response—including deflections, stresses, and strains—exactly using the GDQM under a range of loading scenarios.

GDQM's precision in approximating derivatives is what gives it its accuracy. When working with stiffened laminated composite plates, which display complicated mechanical behavior as a result of the interaction between the plate and the stiffeners, this property is very helpful. We expect to achieve credible findings that closely resemble the real plate behavior by precisely capturing the derivatives and including suitable plate theories, such as the Kirchhoff plate theory or the first-order shear deformation theory.

Additionally, compared to certain other numerical techniques, including the Finite Element Method (FEM), GDQM's computing efficiency enables quicker calculations. GDQM's spectrum convergence qualities make it possible to achieve precise solutions with fewer grid points, which saves time and computing resources. When working with large-scale or parametric studies using stiffened laminated composite plates, where the effectiveness of the numerical technique is critical, this benefit is essential.

By employing GDQM and other numerical approaches, such as FEM or BEM (Boundary Element Method), to solve the static analysis issue of stiffened laminated composite plates, a comparison study may be done to verify this theory. The accuracy, computational effectiveness, and convergence behavior of the findings may be compared. The hypothesis may also be confirmed or disproven by statistical analysis and validation against experimental evidence.

In conclusion, it can be mentioned that GDQM will be a viable numerical approach for the static analysis of stiffened laminated composite plates, providing accurate and effective results.



4. THEORY

4.1 Modelling of Laminated Composite Plate

The governing equations and boundary conditions that explain the behavior of thin plates while including the effects of transverse shear deformation are established as part of the development of the Mindlin plate theory. An summary of the formulation is given below.

The three components $u(x, y, z, t)$, $v(x, y, z, t)$, and $w(x, y, z, t)$, which correspond to the in-plane displacements in the x and y directions and the out-of-plane displacement, respectively, characterize the displacement field of the plate. It is presumed that the displacement field varies linearly with plate thickness.

The kinematic relations for the plate are,

$$\begin{aligned}u(x, y, z, t) &= z\phi_y(x, y, t) \\v(x, y, z, t) &= -z\phi_x(x, y, t) \\w(x, y, z, t) &= w(x, y, t)\end{aligned}\tag{4.1}$$

where ϕ_x and ϕ_y are rotations of the plate section about x and y axes respectively and w is the transverse displacement.

The displacement field is used to calculate the plate's stresses. The strains in the Mindlin plate theory are the transverse shear strains γ_{xz} and γ_{yz} , the shear strain γ_{xy} , and the in-plane normal strains ϵ_x and ϵ_y . Through the use of the proper derivatives, these stresses are connected to the displacement components.

Strain displacement relations are shown below.

$$\begin{aligned}
\varepsilon_{xx} &= z \frac{\partial \phi_y}{\partial x} \\
\varepsilon_{yy} &= -z \frac{\partial \phi_x}{\partial y} \\
\gamma_{xy} &= -z \frac{\partial \phi_x}{\partial x} + z \frac{\partial \phi_y}{\partial y} \\
\gamma_{xz} &= \phi_y + \frac{\partial w}{\partial x} \\
\gamma_{yz} &= -\phi_x + \frac{\partial w}{\partial y}
\end{aligned} \tag{4.2}$$

These equations link the stresses and plate strains via the plate's constitutive relationship. For isotropic materials, the Young's modulus (E), Poisson's ratio (ν), and shear modulus (G) of the plate are all factors in the stress-strain equations. For orthotropic composite material, modulus of elasticity for principle axis E1, E2, poisson ratio ν12, ν21 and shear modulus G12, G23 and G13 used to define stiffness matrices.

The reduced stiffness matrix [Q] is calculated for each layer.

$$\begin{aligned}
Q_{11}(i) &= \frac{E_1}{1-\nu_{12}\nu_{21}} & Q_{12}(i) &= \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & Q_{22}(i) &= \frac{E_2}{1-\nu_{12}\nu_{21}} \\
Q_{66}(i) &= G_{12} & Q_{44}(i) &= G_{23} & Q_{55}(i) &= G_{13}
\end{aligned} \tag{4.3}$$

By using this reduced stiffness matrix, the transformed reduced stiffness matrix [\bar{Q}] is calculated for each layer.

$$\begin{aligned}
\bar{Q}_{11}^{(i)} &= Q_{11}^{(i)} \cos^4 \beta_i + 2(Q_{12}^{(i)} + 2Q_{66}^{(i)}) \sin^2 \beta_i \cos^2 \beta_i + Q_{22}^{(i)} \sin^4 \beta_i \\
\bar{Q}_{12}^{(i)} &= (Q_{11}^{(i)} + Q_{22}^{(i)} - 4Q_{66}^{(i)}) \sin^2 \beta_i \cos^2 \beta_i + Q_{12}^{(i)} (\sin^4 \beta_i + \cos^4 \beta_i) \\
\bar{Q}_{22}^{(i)} &= Q_{11}^{(i)} \sin^4 \beta_i + 2(Q_{12}^{(i)} + 2Q_{66}^{(i)}) \sin^2 \beta_i \cos^2 \beta_i + Q_{22}^{(i)} \cos^4 \beta_i \\
\bar{Q}_{16}^{(i)} &= (Q_{11}^{(i)} - Q_{12}^{(i)} - 2Q_{66}^{(i)}) \sin \beta_i \cos^3 \beta_i \\
&\quad + (Q_{12}^{(i)} - Q_{22}^{(i)} + 2Q_{66}^{(i)}) \sin^3 \beta_i \cos \beta_i \\
\bar{Q}_{26}^{(i)} &= (Q_{11}^{(i)} - Q_{12}^{(i)} - 2Q_{66}^{(i)}) \sin^3 \beta_i \cos \beta_i \\
&\quad + (Q_{12}^{(i)} - Q_{22}^{(i)} + 2Q_{66}^{(i)}) \sin \beta_i \cos^3 \beta_i \\
\bar{Q}_{66}^{(i)} &= (Q_{11}^{(i)} + Q_{22}^{(i)} - 2Q_{12}^{(i)} - 2Q_{66}^{(i)}) \sin^2 \beta_i \cos^2 \beta_i \\
&\quad + Q_{66}^{(i)} (\sin^4 \beta_i + \cos^4 \beta_i) \\
\bar{Q}_{44}^{(i)} &= Q_{44}^{(i)} \cos^2 \beta_i + Q_{55}^{(i)} \sin^2 \beta_i \\
\bar{Q}_{45}^{(i)} &= (Q_{55}^{(i)} - Q_{44}^{(i)}) \sin \beta_i \cos \beta_i \\
\bar{Q}_{55}^{(i)} &= Q_{55}^{(i)} \cos^2 \beta_i + Q_{44}^{(i)} \sin^2 \beta_i
\end{aligned} \tag{4.4}$$

And the stress strain relations for the orthotropic lamina are demonstrated below.

$$\begin{Bmatrix} \sigma_{xx}^{(i)} \\ \sigma_{yy}^{(i)} \\ \tau_{xy}^{(i)} \\ \tau_{yz}^{(i)} \\ \tau_{xz}^{(i)} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11}^{(i)} & \bar{Q}_{12}^{(i)} & \bar{Q}_{16}^{(i)} & 0 & 0 \\ \bar{Q}_{21}^{(i)} & \bar{Q}_{22}^{(i)} & \bar{Q}_{26}^{(i)} & 0 & 0 \\ \bar{Q}_{16}^{(i)} & \bar{Q}_{26}^{(i)} & \bar{Q}_{66}^{(i)} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44}^{(i)} & \bar{Q}_{45}^{(i)} \\ 0 & 0 & 0 & \bar{Q}_{45}^{(i)} & \bar{Q}_{55}^{(i)} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{4.5}$$

4.2 Modelling of Isotropic Stiffener

A traditional beam theory that offers a sophisticated mathematical model for examining the behavior of thin beams is the Timoshenko beam theory, sometimes referred to as the

Timoshenko-Bernoulli beam theory. Timoshenko beam theory takes into account the effects of both axial and shear deformations, in contrast to the traditional Euler-Bernoulli beam theory, which assumes that beams only experience minor shear deformation.

Timoshenko beam theory makes the following assumptions. The beam is narrow, which means that its length is much larger than its cross-sectional dimensions. The beam experiences slight deformations but maintains its linear elastic properties. Throughout deformation, the beam's cross sections stay level and perpendicular to the longitudinal axis.

Axial displacement is represented by $u_{s,b}(x, z, t)$ and $w_{s,b}(x, z, t)$, which represents the transverse deflection, make up the displacement field of the beam. It is presumed that the displacement field varies linearly over the beam's cross-section.

For the bending of stiffener in x direction, the kinematic relations are shown below.

$$\begin{aligned} u_{s,b}(x, z, t) &= -z\varphi(x, t) \\ w_{s,b}(x, z, t) &= w(x, y_s, t) \end{aligned} \quad (4.6)$$

where $u_{s,b}$ and $w_{s,b}$ are displacements of stiffeners in x and z directions, φ is the shear deformation, θ_s is the angle of twist of the cross-section and y_s is the y location of the stiffener.

And for the torsion, the kinematic relations are indicated below.

$$\theta_s(x, t) = \phi_x(x, y_s, t) \quad (4.7)$$

where θ_s is the angle of twist of the cross-section about stiffener centroid that is assumed identical as the rotation of plate section about x axis. Neglecting the warping deformation, the displacements associated with torsion are:

$$\begin{aligned}
v_{s,t}(x, y, z, t) &= -z\phi_x(x, y_s, t) \\
w_{s,t}(x, y, z, t) &= y\phi_x(x, y_s, t) \\
u_{s,t} &\cong 0
\end{aligned}
\tag{4.8}$$

The displacement field is used to calculate the stresses in the beam. The axial strain $\varepsilon_{xx,b}^s$, the bending curvature κ , and the shear strain $\gamma_{xz,b}^s$ are the strains in the Timoshenko beam theory. Through the use of the proper derivatives, these stresses are connected to the displacement components.

The strain displacement relations for the stiffener in bending are demonstrated below.

$$\begin{aligned}
\varepsilon_{xx,b}^s &= -z \frac{\partial \phi}{\partial x} \\
\gamma_{xz,b}^s &= \frac{\partial w}{\partial z} - \phi
\end{aligned}
\tag{4.9}$$

The strain displacement relations for the stiffener in torsion are demonstrated below.

$$\begin{aligned}
\gamma_{xy,t}^s &= -z \frac{\partial \phi_x}{\partial x} \\
\gamma_{xz,t}^s &= y \frac{\partial \phi_x}{\partial x}
\end{aligned}
\tag{4.10}$$

The beam's stresses and strains are related by the constitutive equations. For isotropic materials, the Young's and shear moduli of the beam (E and G) have a role in the stress-strain equations.

The stiffener is assumed isotropic and the stress strain relations in bending are shown below.

$$\begin{Bmatrix} \sigma_{xx,b}^s \\ \tau_{xz,b}^s \end{Bmatrix} = \begin{bmatrix} E & 0 \\ 0 & \kappa G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx,b}^s \\ \gamma_{xz,b}^s \end{Bmatrix} \quad (4.11)$$

The stiffener is assumed isotropic and the stress strain relations in torsion are shown below.

$$\begin{Bmatrix} \tau_{xy,t}^s \\ \tau_{xz,t}^s \end{Bmatrix} = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \begin{Bmatrix} \gamma_{xy,t}^s \\ \gamma_{xz,t}^s \end{Bmatrix} \quad (4.12)$$

4.3 Equation of Motion of Stiffened Laminated Composite Plate

Hamilton's principle can be written for the symmetrically laminated composite plate and the isotropic stiffener as follows:

$$\begin{aligned} & \sum_{i=1}^n \rho_i \int_{t_1}^{t_2} \int_0^b \int_0^a \int_{z_i}^{z_{i-1}} \left(\frac{\partial^2 u}{\partial t^2} \delta u + \frac{\partial^2 v}{\partial t^2} \delta v + \frac{\partial^2 w}{\partial t^2} \delta w \right) dz dx dy dt \\ & - \sum_{i=1}^n \int_{t_1}^{t_2} \int_0^b \int_0^a \int_{z_i}^{z_{i-1}} \left(\sigma_{xx}^{(i)} \delta \varepsilon_{xx} + \sigma_{yy}^{(i)} \delta \varepsilon_{yy} + \tau_{xy}^{(i)} \delta \gamma_{xy} + \tau_{yz}^{(i)} \delta \gamma_{yz} + \tau_{xz}^{(i)} \delta \gamma_{xz} \right) dz dx dy dt \\ & + b_s \rho_s \int_{t_1}^{t_2} \int_0^a \int_{z_s}^{z_s+h_s} \left(\frac{\partial^2 u_{s,b}}{\partial t^2} \delta u_{s,b} + \frac{\partial^2 w_{s,b}}{\partial t^2} \delta w_{s,b} \right) dz dx dt - b_s \int_{t_1}^{t_2} \int_0^a \int_{z_s}^{z_s+h_s} \left(\sigma_{xx,b}^s \delta \varepsilon_{xx,b}^s + \tau_{xz,b}^s \delta \gamma_{xz,b}^s \right) dz dx dt \quad (4.13) \\ & - \int_{t_1}^{t_2} \int_0^a \int_{A_s} \left(\tau_{xy,t}^s \delta \gamma_{xy,t}^s + \tau_{xz,t}^s \delta \gamma_{xz,t}^s \right) dA_s dx dt + \rho_s \int_{t_1}^{t_2} \int_0^a \int_{A_s} \left(\frac{\partial^2 v_{s,t}}{\partial t^2} \delta v_{s,t} + \frac{\partial^2 w_{s,t}}{\partial t^2} \delta w_{s,t} \right) dA_s dx dt = 0 \end{aligned}$$

where n is the number of plies of the composite plate, ρ_i are the density of these plies and ρ_s and h_s are the stiffener density and thickness respectively.

The first and second terms are the virtual work of inertia forces and the virtual strain energy of the plate. The third and fourth terms are the virtual work of inertia forces and the virtual strain energy of the stiffener due to bending and fifth and sixth terms are the virtual work of inertia forces and the virtual strain energy of the stiffener due to torsion. Also, z_s is the distance

between the bottom of stiffener and the stiffened-plate neutral axis and z_i is the location of ply bottom surface that can be calculated as follows:

$$z_s = \frac{bh^2 - b_s h_s^2}{2bh + 2b_s h_s} \quad (4.14)$$

$$z_i = \frac{h}{2} - \sum_{j=1}^i h_j \quad (4.15)$$

where h and h_j are the total thickness of the plate and the thickness of each ply of the symmetrically laminated plate respectively.

The governing equations are obtained as follows:

$$\begin{aligned} \delta\phi_x : I_2 \frac{\partial^2 \phi_x}{\partial t^2} - \frac{\partial M_{yy}}{\partial y} - \frac{\partial M_{xy}}{\partial x} + Q_{xz} + \delta_D (y - y_s) \left[J_0 \rho_s \frac{\partial^2 \phi_x}{\partial t^2} - GJ_s \frac{\partial^2 \phi_x}{\partial x^2} \right] &= 0 \\ \delta\phi_y : I_2 \frac{\partial^2 \phi_y}{\partial t^2} + \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_{yz} &= 0 \\ \delta w : I_0 \frac{\partial^2 w}{\partial t^2} + \frac{\partial Q_{xz}}{\partial y} + \frac{\partial Q_{yz}}{\partial x} + \delta_D (y - y_s) \left[I_0^s \frac{\partial^2 w}{\partial t^2} + \frac{\partial Q_{xz}^s}{\partial x} \right] &= 0 \\ \delta\phi : I_2^s \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial M_{xx}^s}{\partial x} + Q_{xz}^s &= 0 \end{aligned} \quad (4.16)$$

where δ_D is the Dirac-Delta function, J_0 is the polar moment of inertia of the stiffener cross section, J_s is the equivalent moment of inertia of the cross section due to torsion, I 's are the inertia terms and M 's and Q 's are section moments and forces that can be calculated as:

$$\begin{aligned}
I_0 &= \sum_{i=1}^n \rho_i \int_{z_i}^{z_{i-1}} dz = \sum_{i=1}^n \rho_i (z_{i-1} - z_i) = \sum_{i=1}^n \rho_i h_i \\
I_2 &= \sum_{i=1}^n \rho_i \int_{z_i}^{z_{i-1}} z^2 dz = \sum_{i=1}^n \frac{\rho_i}{3} (z_{i-1}^3 - z_i^3) \\
I_0^s &= b_s \rho_s \int_{z_s}^{z_s+h_s} dz = \rho_s A_s \\
I_2^s &= b_s \rho_s \int_{z_s}^{z_s+h_s} z^2 dz = \frac{b_s \rho_s}{3} [(z_s + h_s)^3 - z_s^3]
\end{aligned} \tag{4.17}$$

$$M_{xx} = \sum_{i=1}^n \int_{z_i}^{z_{i-1}} \sigma_{xx}^{(i)} z dz$$

$$M_{yy} = \sum_{i=1}^n \int_{z_i}^{z_{i-1}} \sigma_{yy}^{(i)} z dz$$

$$M_{xy} = \sum_{i=1}^n \int_{z_i}^{z_{i-1}} \tau_{xy}^{(i)} z dz$$

$$Q_{xz} = \sum_{i=1}^n \int_{z_i}^{z_{i-1}} \tau_{xz}^{(i)} dz \tag{4.19}$$

$$Q_{yz} = \sum_{i=1}^n \int_{z_i}^{z_{i-1}} \tau_{yz}^{(i)} dz$$

$$M_{xx}^s = b_s \int_{z_s}^{z_s+h_s} \sigma_{xx,b}^s z dz$$

$$Q_{xz}^s = b_s \int_{z_s}^{z_s+h_s} \tau_{xz,b}^s dz$$

In addition, J_s for the rectangular cross section is obtained as follows [2]:

$$J_s = \begin{cases} \text{If } a_s > b_s & \frac{1}{3} a_s^3 b_s \left(1 - \frac{192 a_s}{\pi^5 b_s} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^5} \tanh \left[\frac{(2n-1)\pi b_s}{2a_s} \right] \right) \\ \text{If } a_s < b_s & \frac{1}{3} b_s^3 a_s \left(1 - \frac{192 b_s}{\pi^5 a_s} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^5} \tanh \left[\frac{(2n-1)\pi a_s}{2b_s} \right] \right) \end{cases} \quad (4.20)$$

4.4 GDQM Solution

A numerical method used to solve differential equations is called the Generalized Differential Quadrature Method (GDQM). It works especially effectively for boundary value issues and situations involving ordinary or partial differential equations. The GDQM method uses a weighted sum of function values at discrete places, referred to as quadrature points, to estimate the derivative terms in the governing equations. An outline of the GDQM formulation and solution is provided below.

Discretization: In GDQM, the problem domain is discretized into a collection of grid points, where an approximation of the solution is found. Although the grid points are normally perfectly spaced, they may potentially be dispersed unevenly to increase accuracy in certain areas.

Weighted Approximation: A weighted sum of the function values at other grid locations is used to approximation the unknown function at each grid point. By selecting suitable basis functions and corresponding weights, the weights are established. Polynomial or trigonometric functions are frequent options for basis functions.

Derivatives Approximation: By adding the weighted sum of the function values at nearby grid points, one may estimate the derivatives of an unknown function. The differentiation matrix connected to the GDQM scheme and the selected basis functions are used to derive the weights. The coefficients needed to approximate the derivatives are provided by the differentiation matrix.

Construction of Algebraic Equations: Algebraic equations are created by inserting approximations into the governing differential equations. These equations connect the derivatives of the function values at the grid points.

Boundary Conditions: The algebraic equations are subjected to the all edges clamped boundary conditions. These requirements may be directly applied at the border grid points or included through extra equations.

The resultant system of algebraic equations is solved using numerical methods such matrix inversion, iterative approaches, or direct solution techniques. A full solution throughout the whole domain may be reconstructed using the values of the unknown function at the grid points provided by the solution.

For this problem, integration of the governing equations are required to remove the Dirac delta function. A similar approach to the one presented by [1] for the GDQM solution of moving load problems is proposed.

Let's consider the first governing differential equation:

$$I_2 \frac{\partial^2 \phi_x}{\partial t^2} - \frac{\partial M_{yy}}{\partial y} - \frac{\partial M_{xy}}{\partial x} + Q_{xz} = \delta_D (y - y_s) \left(GJ_s \frac{\partial^2 \phi_x}{\partial x^2} - J_0 \rho_s \frac{\partial^2 \phi_x}{\partial t^2} \right) \quad (4.21)$$

This equation of motion when discretized at GDQM grid points that have $y \neq y_s$ give:

$$I_2 \frac{\partial^2 \phi_x}{\partial t^2} - \frac{\partial M_{yy}}{\partial y} - \frac{\partial M_{xy}}{\partial x} + Q_{xz} = 0 \quad (4.22)$$

Let's take the integral of both sides of the equation of motion over the y domain:

$$\int_0^b \left(I_2 \frac{\partial^2 \phi_x}{\partial t^2} - \frac{\partial M_{yy}}{\partial y} - \frac{\partial M_{xy}}{\partial x} + Q_{xz} \right) dy = \int_0^b \delta_D (y - y_s) \left(GJ_s \frac{\partial^2 \phi_x}{\partial x^2} - J_0 \rho_s \frac{\partial^2 \phi_x}{\partial t^2} \right) dy \quad (4.23)$$

$$\int_0^b \left(I_2 \frac{\partial^2 \phi_x}{\partial t^2} - \frac{\partial M_{yy}}{\partial y} - \frac{\partial M_{xy}}{\partial x} + Q_{xz} \right) dy = GJ_s \frac{\partial^2 \phi_x(x, y_s, t)}{\partial x^2} - J_0 \rho_s \frac{\partial^2 \phi_x(x, y_s, t)}{\partial t^2} \quad (4.24)$$

Now integration of the left hand side of this equation can be performed by Gauss quadrature rule, with the quadrature points selected identical with the GDQM grid points.

$$\begin{aligned} & b \sum_{j=1}^m w_j \left[I_2 \frac{\partial^2 \phi_x(x, y_j, t)}{\partial t^2} - \frac{\partial M_{yy}(x, y_j, t)}{\partial y} - \frac{\partial M_{xy}(x, y_j, t)}{\partial x} + Q_{xz}(x, y_j, t) \right] \\ & = GJ_s \frac{\partial^2 \phi_x(x, y_s, t)}{\partial x^2} - J_0 \rho_s \frac{\partial^2 \phi_x(x, y_s, t)}{\partial t^2} \end{aligned} \quad (4.25)$$

From Eq. (4.25), it is obvious that the summation will give zero value for points other than $y = y_s$. Then, Eq. (4.25) simplifies to:

$$\begin{aligned} & I_2 \frac{\partial^2 \phi_x(x, y_s, t)}{\partial t^2} - \frac{\partial M_{yy}(x, y_s, t)}{\partial y} - \frac{\partial M_{xy}(x, y_s, t)}{\partial x} + Q_{xz}(x, y_s, t) \\ & = \frac{1}{bw_s} \left[GJ_s \frac{\partial^2 \phi_x(x, y_s, t)}{\partial x^2} - J_0 \rho_s \frac{\partial^2 \phi_x(x, y_s, t)}{\partial t^2} \right] \end{aligned} \quad (4.26)$$

Similar approach applied to the third governing equation we have:

If $y \neq y_s$:

$$I_0 \frac{\partial^2 w}{\partial t^2} + \frac{\partial Q_{xz}}{\partial y} + \frac{\partial Q_{yz}}{\partial x} = 0 \quad (4.27)$$

If $y = y_s$:

$$I_0 \frac{\partial^2 w(x, y_s, t)}{\partial t^2} + \frac{\partial Q_{xz}(x, y_s, t)}{\partial y} + \frac{\partial Q_{yz}(x, y_s, t)}{\partial x} + \frac{1}{bw_s} \left[I_0^s \frac{\partial^2 w(x, y_s, t)}{\partial t^2} + \frac{\partial Q_{xz}^s(x, t)}{\partial x} \right] = 0 \quad (4.28)$$

Boundary conditions for the stiffened plate model are defined as below.

x=0	x=a	y=0	y=b
$M_{xy} + GJ_s \delta_D (y - y_s) \frac{\partial \phi_x}{\partial x} = 0$ or $\phi_x = 0$	$M_{xy} + GJ_s \delta_D (y - y_s) \frac{\partial \phi_x}{\partial x} = 0$ or $\phi_x = 0$	$M_{yy} = 0$ $\phi_x = 0$	$M_{yy} = 0$ or $M_{yy} = 0$ or $\phi_x = 0$
$-M_{xx} = 0$ or $\phi_y = 0$	$-M_{xx} = 0$ or $\phi_y = 0$	$-M_{xy} = 0$ or $\phi_y = 0$	$-M_{xy} = 0$ or $-M_{xy} = 0$ or $\phi_y = 0$
$-Q_{yz} - \delta_D (y - y_s) Q_{xz}^s = 0$ or $w = 0$	$-Q_{yz} - \delta_D (y - y_s) Q_{xz}^s = 0$ or $w = 0$	$-Q_{xz} = 0$ or $w = 0$	$-Q_{xz} = 0$ or $-Q_{xz} = 0$ or $w = 0$
$M_{xx}^s = 0$ or $\varphi = 0$	$M_{xx}^s = 0$ or $\varphi = 0$		

(4.29)

5. RESULT

5.1 Material Mechanical Properties and Dimensions of the Model

For isotropic plates and isotropic stiffeners, mechanical properties of structural steel are considered. For [0/90/90/0] and [45/60/60/45] laminated composite plates, mechanical properties of CFRP are considered.

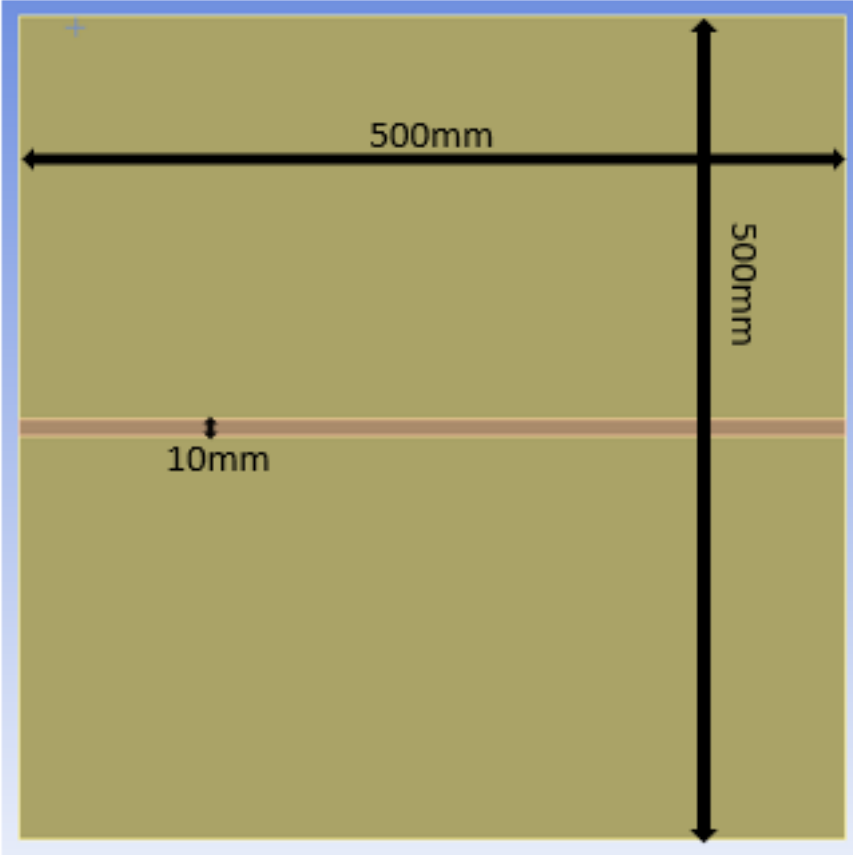


Figure 5.1 : Dimensions of the model



Figure 5.2 : Dimensions of the model

5.2 Mesh Statistics and Resolution Time Comparison

GDQM is renowned for its potential computational efficiency as a result of its direct integration of the governing equations without the need to generate a mesh, in contrast to

FEA, which requires mesh generation and refinement. This efficiency can be especially advantageous for problems with expansive and intricate domains. For the GDQM model, 35x35 equispaced grid distribution are used separately for plate and beam model.

Table 5.1 : ANSYS mesh statistics

ANSYS Mesh Statistics	
Nodes	589065
Elements	309728

Table 5.2 : Resolution time comparison

Resolution Time Comparison	
ANSYS	12m 50s
GDQM	48s

The resolution time of GDQM is significantly less than that of ANSYS, and the equispaced grid distribution employed by GDQM has fewer nodes than ANSYS.

5.3 Isotropic Plate With Isotropic Stiffener Results

The first six natural frequency and their related mode shapes for the defined boundary conditions in accordance with the modelled isotropic plate with an isotropic stiffener are obtained by using ANSYS software. The figures are shown below.

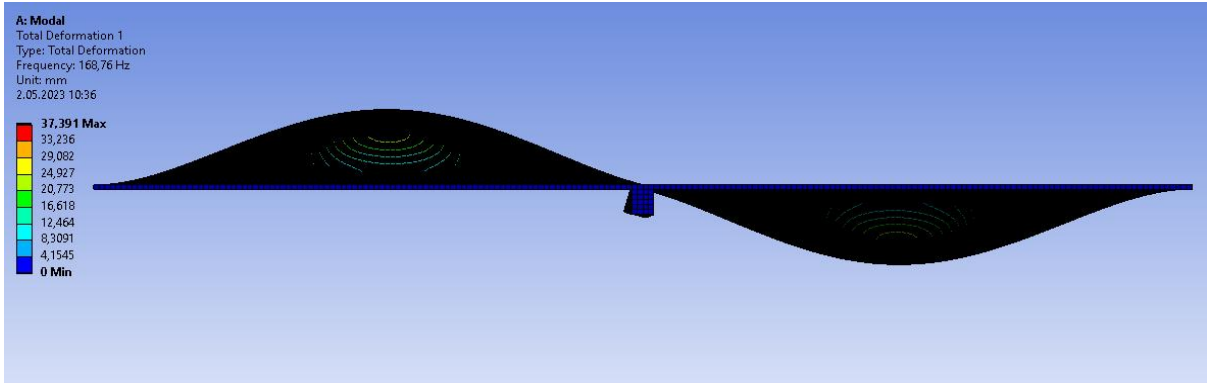


Figure 5.3 : ANSYS first mode shape for isotropic plate with an isotropic stiffener

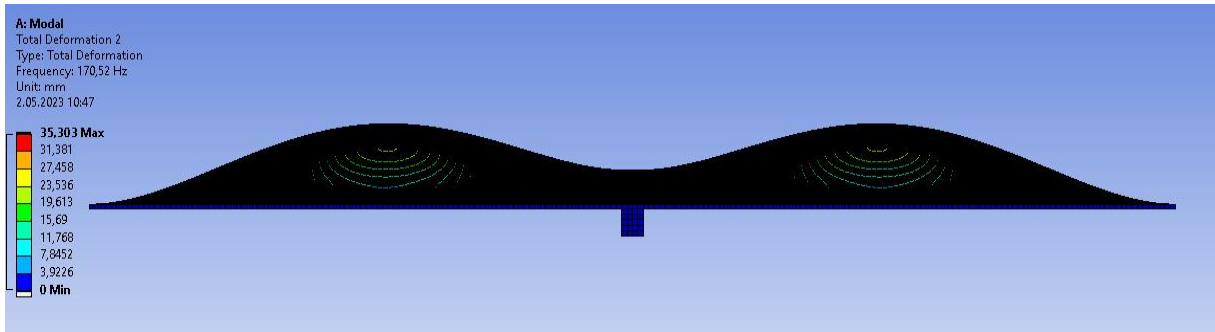


Figure 5.4 : ANSYS second mode shape for isotropic plate with an isotropic stiffener

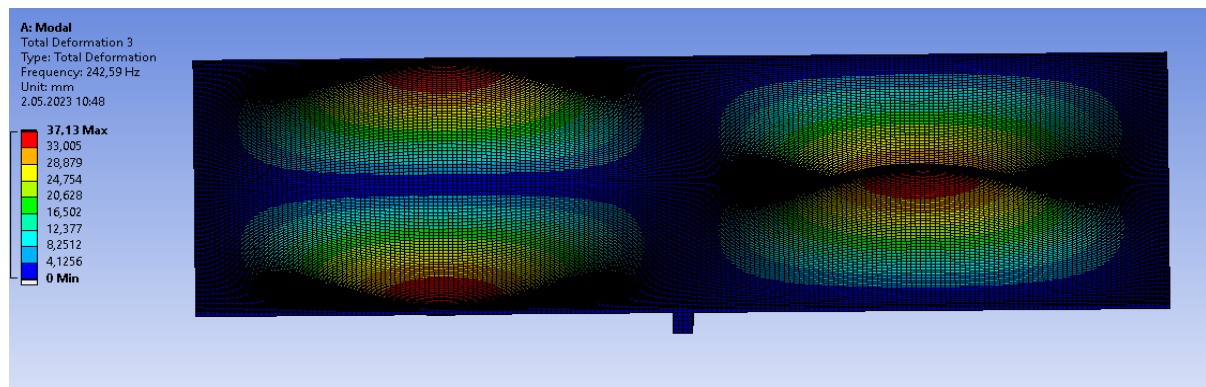


Figure 5.5 : ANSYS third mode shape for isotropic plate with an isotropic stiffener

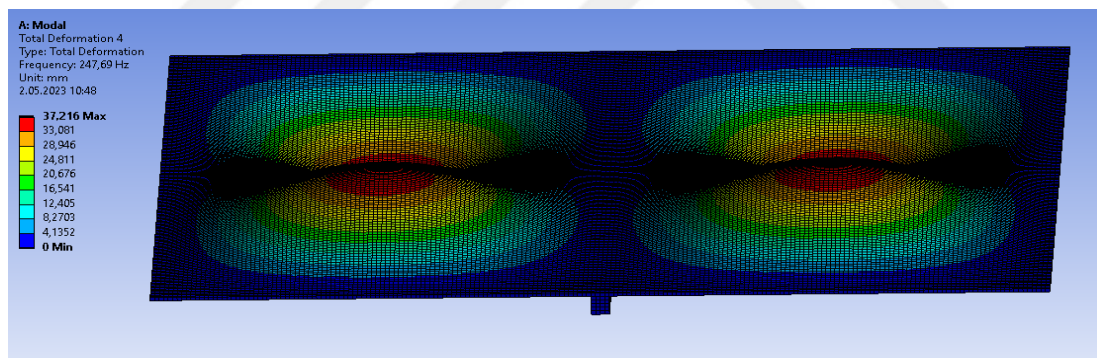


Figure 5.6 : ANSYS fourth mode shape for isotropic plate with an isotropic stiffener

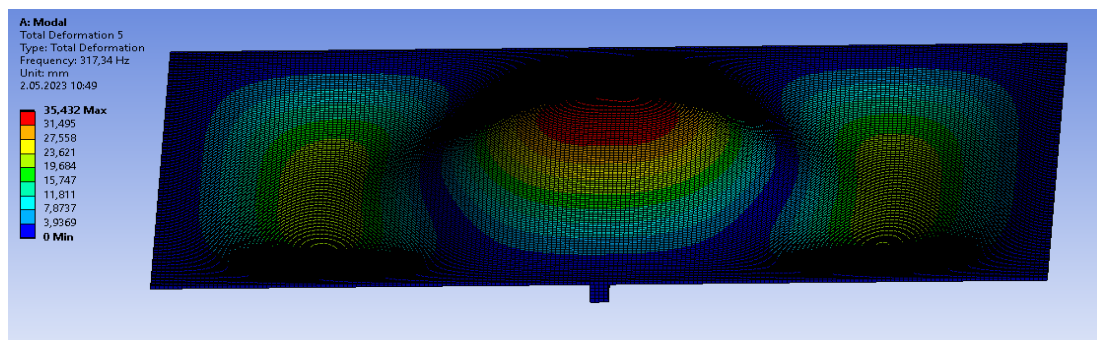


Figure 5.7 : ANSYS fifth mode shape for isotropic plate with an isotropic stiffener

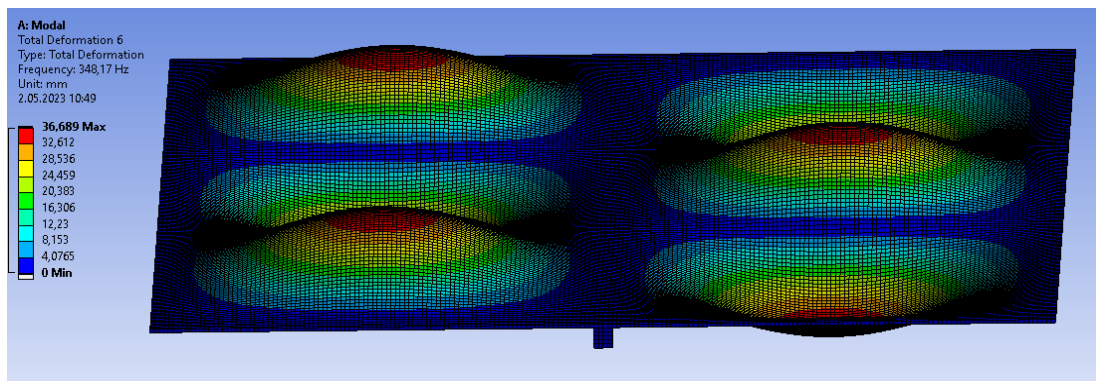


Figure 5.8 : ANSYS sixth mode shape for isotropic plate with an isotropic stiffener

The first six natural frequencies and their associated mode structures for the specified boundary conditions in accordance with the modeled isotropic plate with an isotropic stiffener are obtained using the Generalized Differential Quadrature Method in the Wolfram Mathematica software. The attained figures are displayed below.

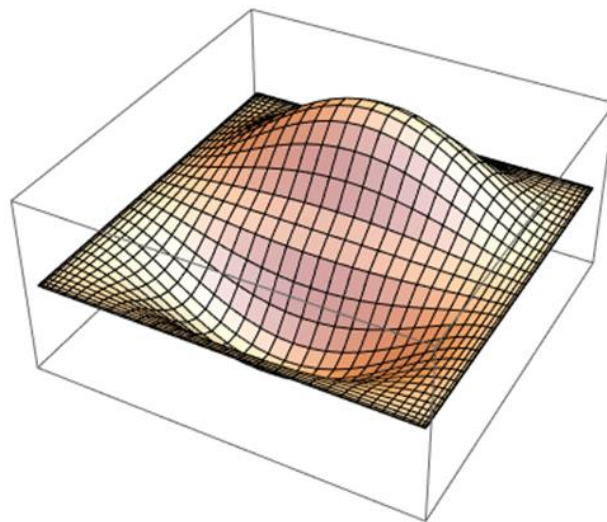


Figure 5.9 : GDQM solution first mode shape for isotropic plate with an isotropic stiffener

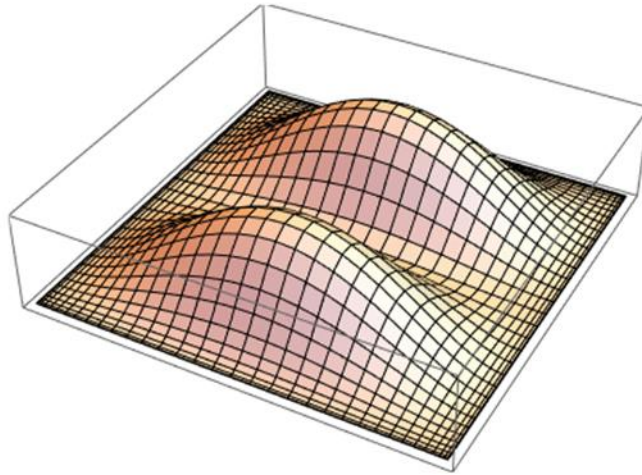


Figure 5.10 : GDQM solution second mode shape for isotropic plate with an isotropic stiffener

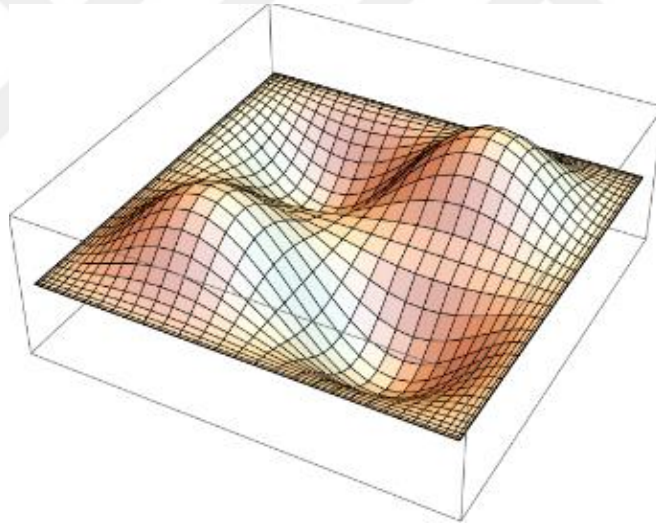


Figure 5.11 : GDQM solution third mode shape for isotropic plate with an isotropic stiffener

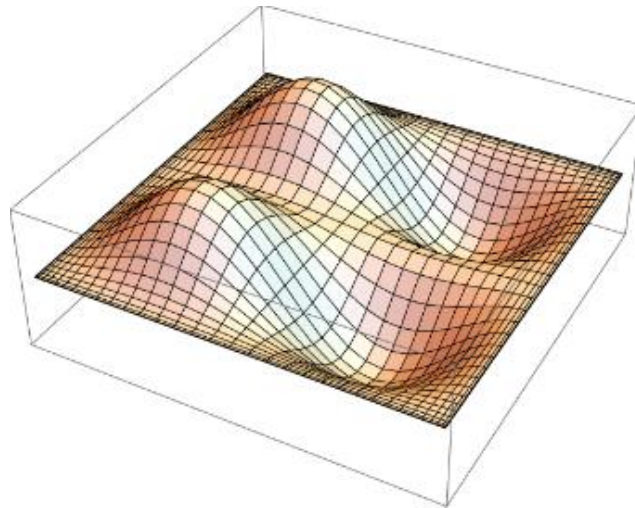


Figure 5.12 : GDQM solution fourth mode shape for isotropic plate with an isotropic stiffener

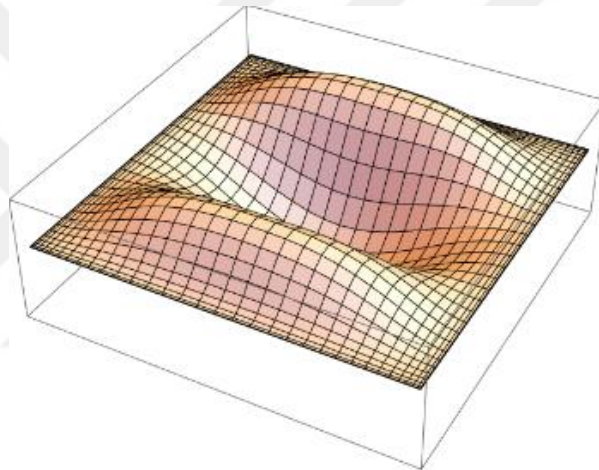


Figure 5.13 : GDQM solution fifth mode shape for isotropic plate with an isotropic stiffener

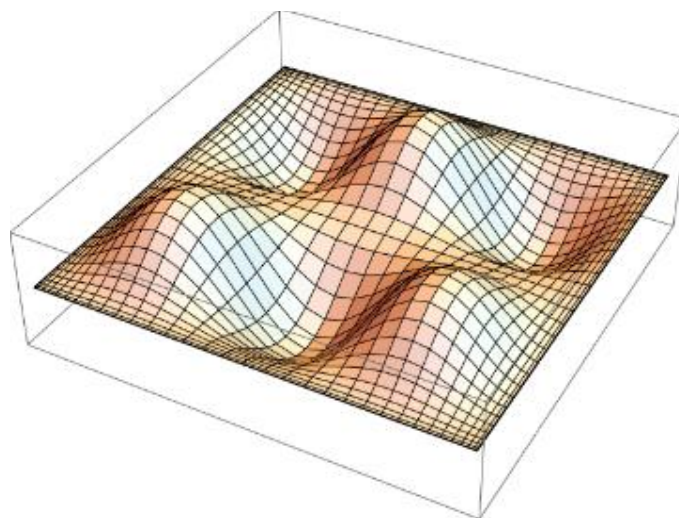


Figure 5.14 : GDQM solution sixth mode shape for isotropic plate with an isotropic stiffener

Table 5.3 : ANSYS and GDQM results comparison for isotropic plate with an isotropic stiffener

Mode	GDQM (Hz)	ANSYS (Hz)	Relative Error %
1	169.62	168.76	0.51
2	176.01	170.52	3.22
3	247.54	242.59	2.04
4	250.80	247.69	1.26
5	340.51	317.34	7.30
6	357.42	348.17	2.66

Using the Generalized Differential Quadrature Method in the Wolfram Mathematica software, the first six natural frequencies and their corresponding mode structures are generated for the stated boundary conditions in line with the model taken from [3] Augustyn (2015) . The FEM model is applied to the geometry defined in the [ref]. Below are the numbers and comparative graphs for the various results.

Table 5.4 : [3] FEA results and GDQM results comparison for isotropic plate with an isotropic stiffener

Mode	GDQM (Hz)	FEA results [3]	Relative Error %
1	724.78	718.10	0.93
2	752.47	751.40	0.14
3	996.24	997.40	0.12
4	1004.65	1007.10	0.24
5	1404.52	1419.80	1.08
6	1421.37	1424.30	0.21

In the [3] Augustyn (2015), the all related mode shapes for the specified model are not obtained. For different ratios of stiffener and plate thickness, the first mode shapes are obtained. Therefore, the mode shapes comparison can not be carried out.

5.4 [0/90/90/0] Carbon-Fiber Plate With Isotropic Stiffener Results

By using ANSYS software, the first six natural frequencies and their associated mode shapes for the defined boundary conditions of the modeled [0/90/90/0] carbon-fiber plate with an isotropic stiffener are determined. The numbers are listed below.

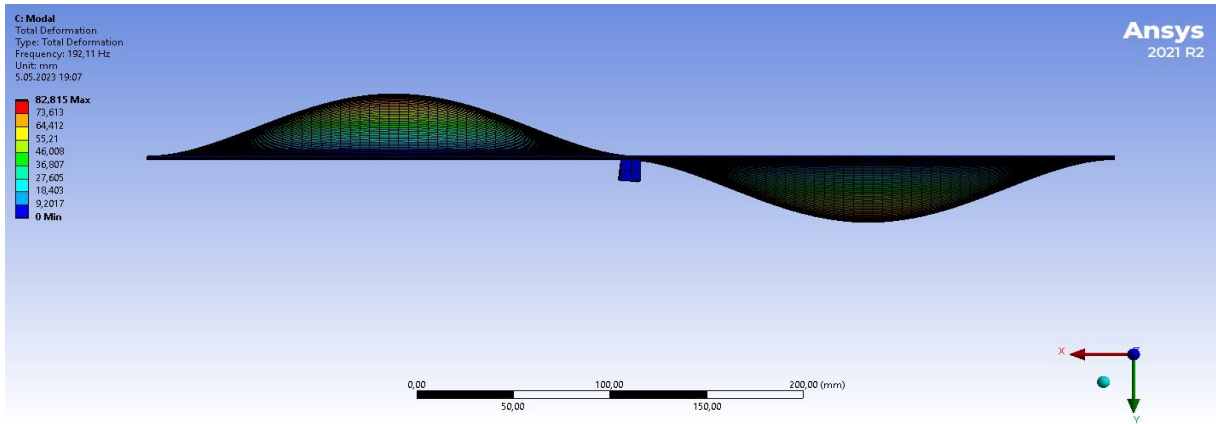


Figure 5.15 : ANSYS first mode shape for [0/90/90/0] carbon-fiber plate with an isotropic stiffener

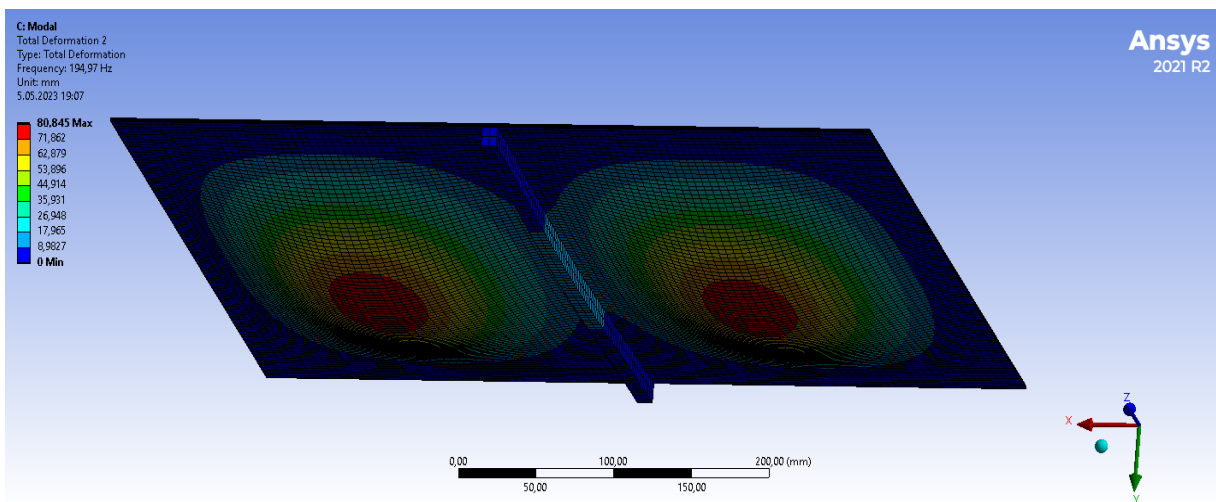


Figure 5.16 : ANSYS second mode shape for [0/90/90/0] carbon-fiber plate with an isotropic stiffener

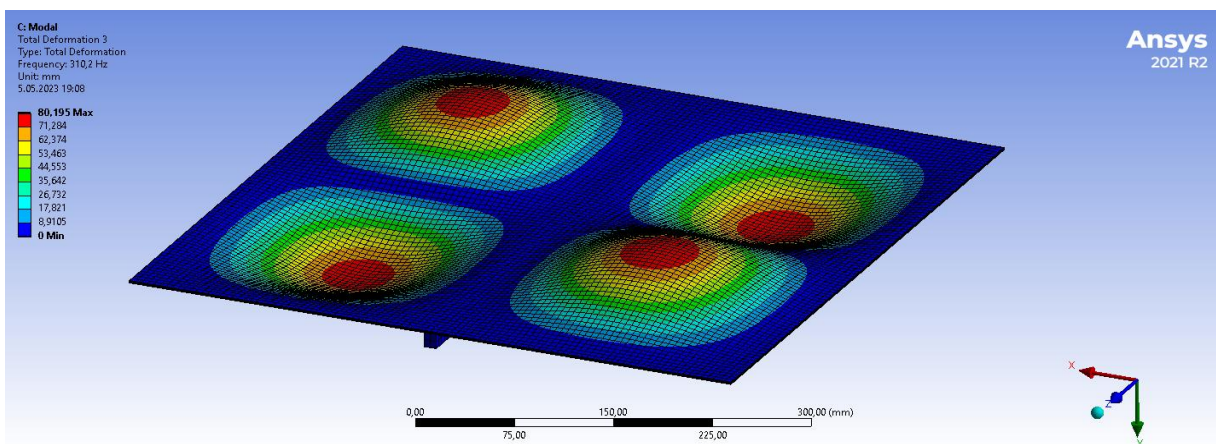


Figure 5.17 : ANSYS third mode shape for [0/90/90/0] carbon-fiber plate with an isotropic stiffener

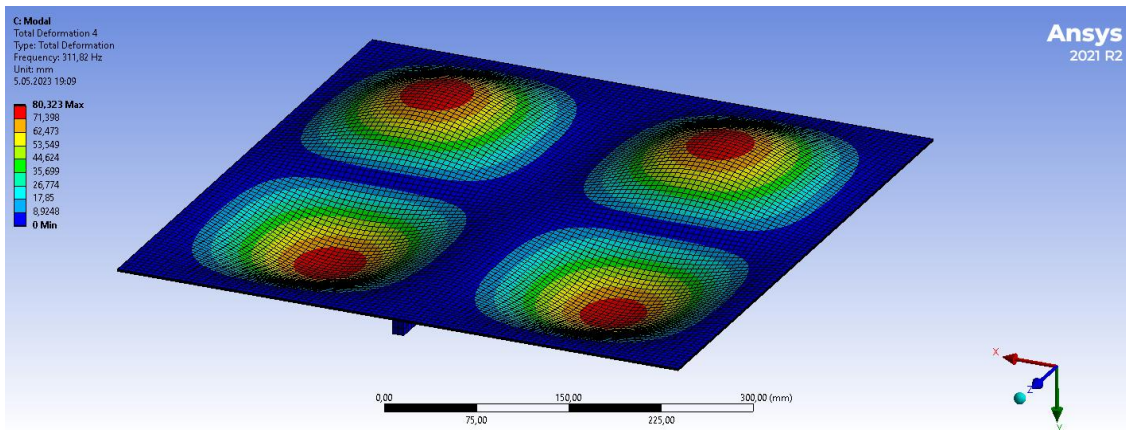


Figure 5.18 : ANSYS fourth mode shape for [0/90/90/0] carbon-fiber plate with an isotropic stiffener

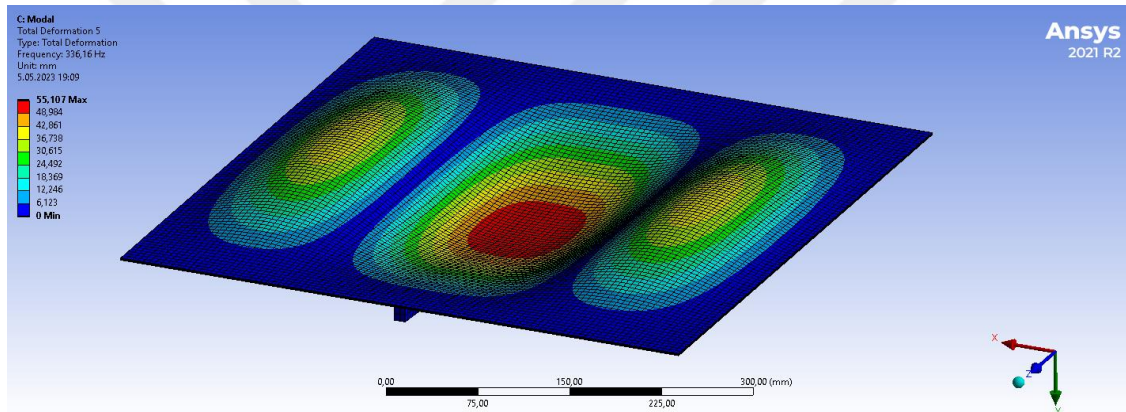


Figure 5.19 : ANSYS fifth mode shape for [0/90/90/0] carbon-fiber plate with an isotropic stiffener

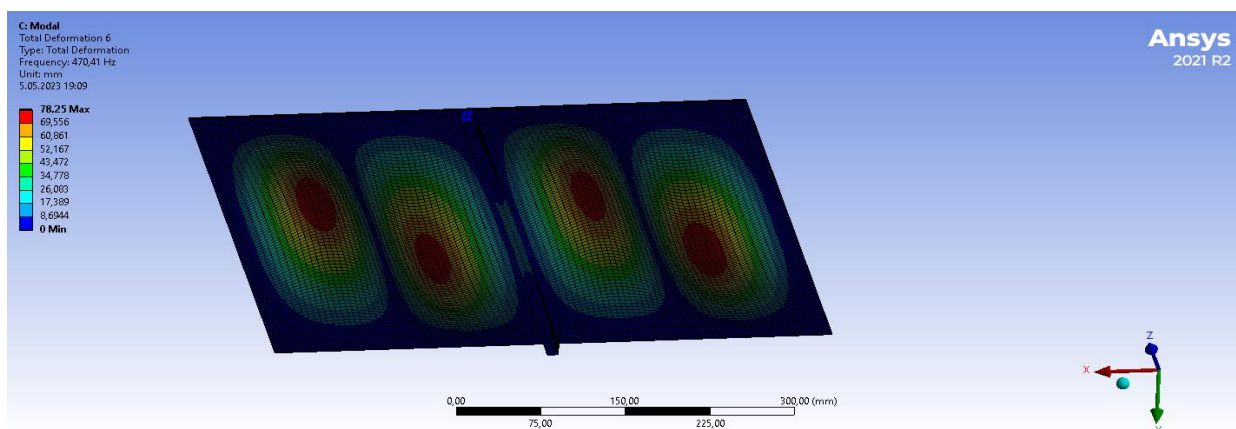


Figure 5.20 : ANSYS sixth mode shape for [0/90/90/0] carbon-fiber plate with an isotropic stiffener

Using the Generalized Differential Quadrature Method in the Wolfram Mathematica software, the first six natural frequencies and their associated mode structures for the specified boundary conditions in accordance with the modeled [0/90/90/0] carbon-fiber plate with an isotropic stiffener are obtained. The achieved figures are shown below.

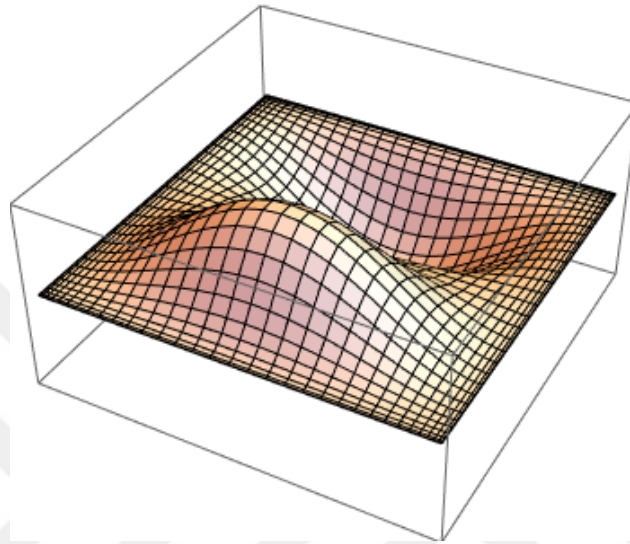


Figure 5.21 : GDQM solution first mode shape for [0/90/90/0] carbon-fiber plate with an isotropic stiffener

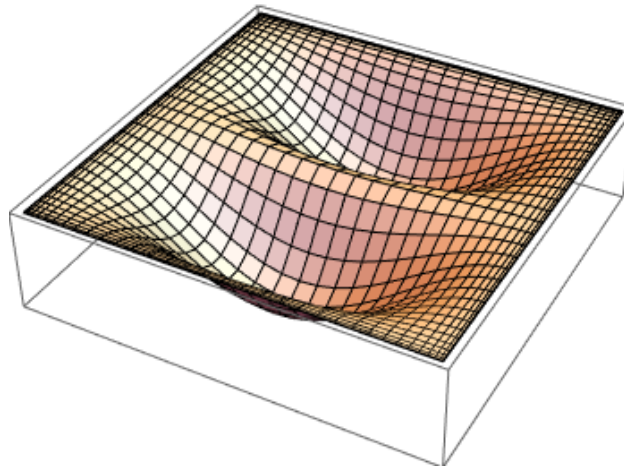


Figure 5.22 : GDQM solution second mode shape for [0/90/90/0] carbon-fiber plate with an isotropic stiffener

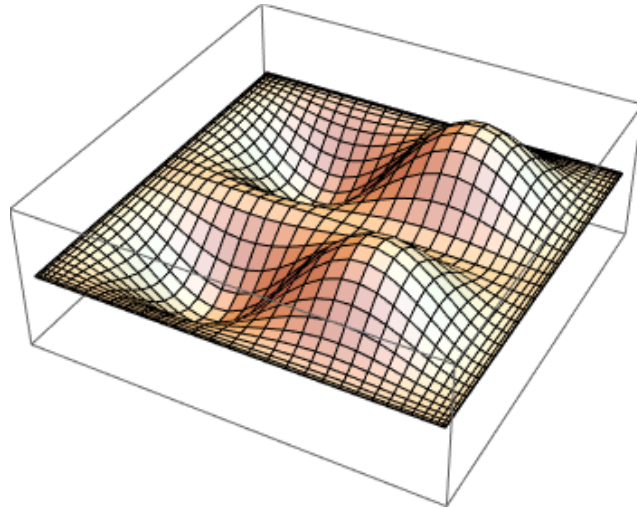


Figure 5.23 : GDQM solution third mode shape for [0/90/90/0] carbon-fiber plate with an isotropic stiffener

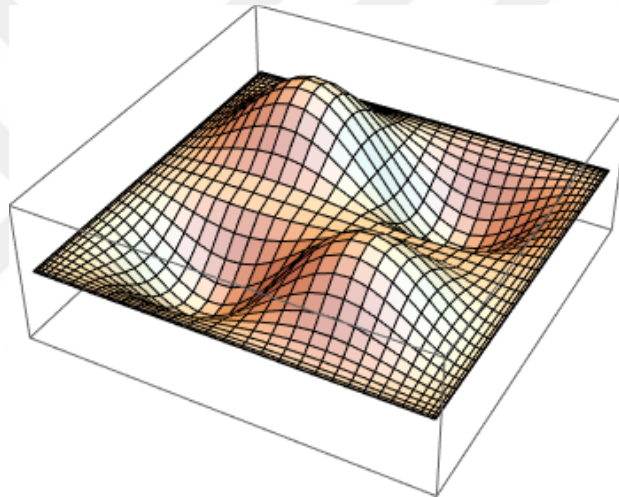


Figure 5.24 : GDQM solution fourth mode shape for [0/90/90/0] carbon-fiber plate with an isotropic stiffener

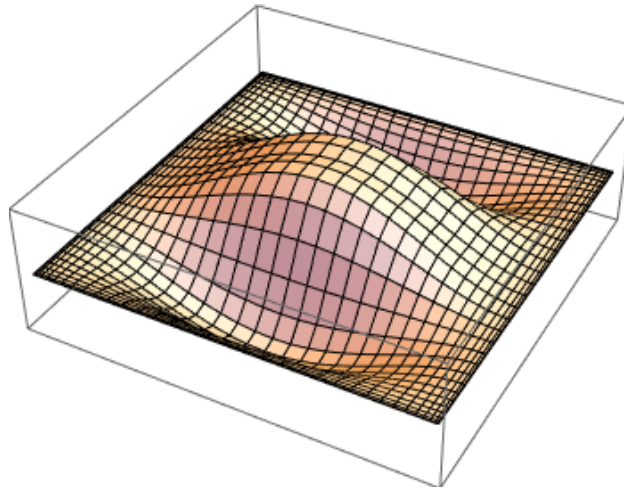


Figure 5.25 : GDQM solution fifth mode shape for [0/90/90/0] carbon-fiber plate with an isotropic stiffener

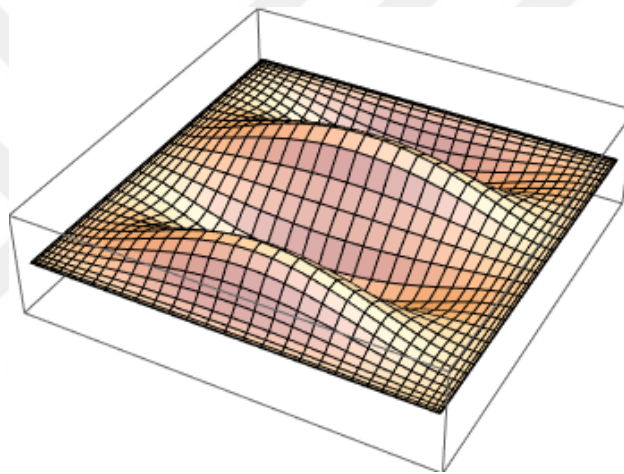


Figure 5.26 : GDQM solution sixth mode shape for [0/90/90/0] carbon-fiber plate with an isotropic stiffener

Table 5.5 : ANSYS and GDQM results comparison for [0/90/90/0] carbon-fiber plate with an isotropic stiffener

Mode	GDQM (Hz)	ANSYS (Hz)	Relative Error %
1	189.07	192.27	1.66
2	191.88	195.43	1.82
3	308.16	310.70	0.82
4	310.09	312.05	0.63
5	369.67	336.05	10.00
6	464.12	469.64	1.17

5.5 [45/60/60/45] Carbon-Fiber Plate With Isotropic Stiffener Results

The first six natural frequencies and their associated mode shapes for the specified boundary conditions of the modeled [45/60/60/45] carbon-fiber plate with an isotropic stiffener are determined using the ANSYS software. The numbers are listed in the table below.

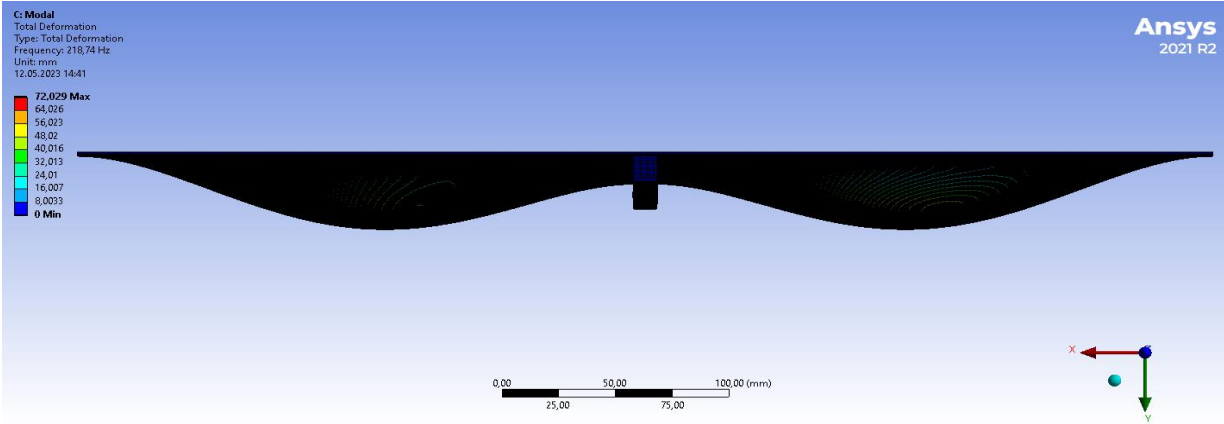


Figure 5.27 : ANSYS first mode shape for [45/60/60/45] carbon-fiber plate with an isotropic stiffener

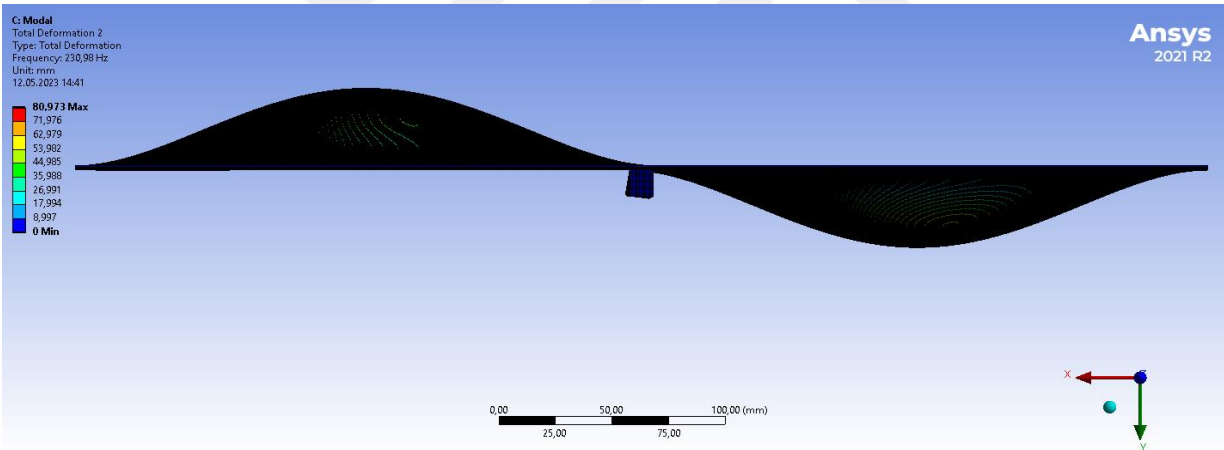


Figure 5.28 : ANSYS second mode shape for [45/60/60/45] carbon-fiber plate with an isotropic stiffener

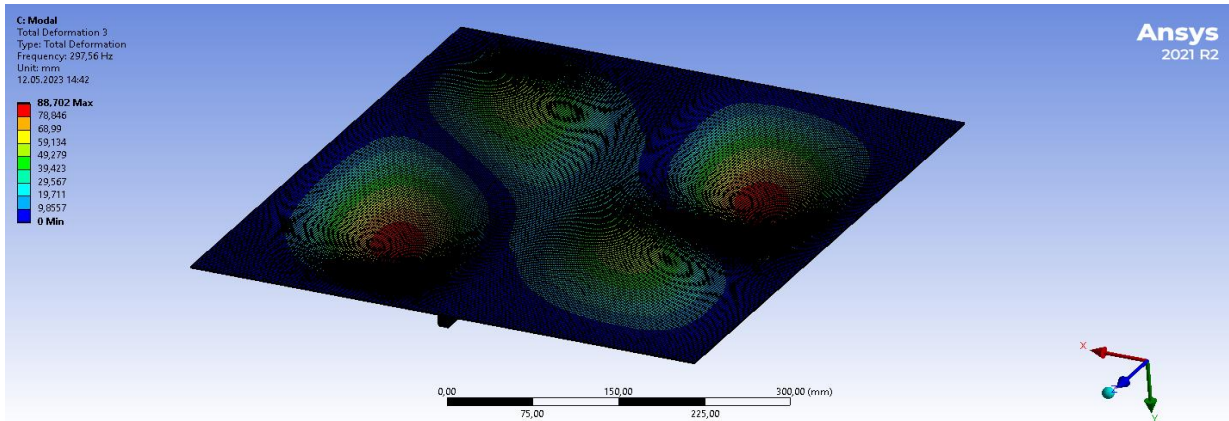


Figure 5.29 : ANSYS third mode shape for [45/60/60/45] carbon-fiber plate with an isotropic stiffener

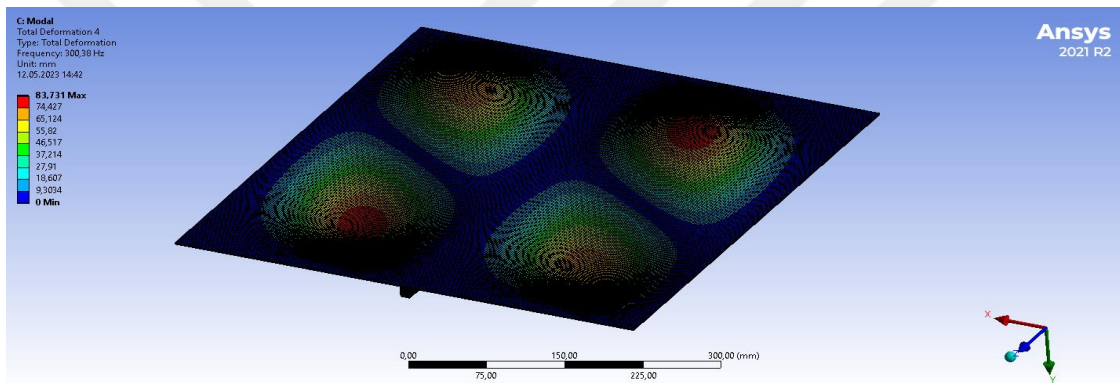


Figure 5.30 : ANSYS fourth mode shape for [45/60/60/45] carbon-fiber plate with an isotropic stiffener

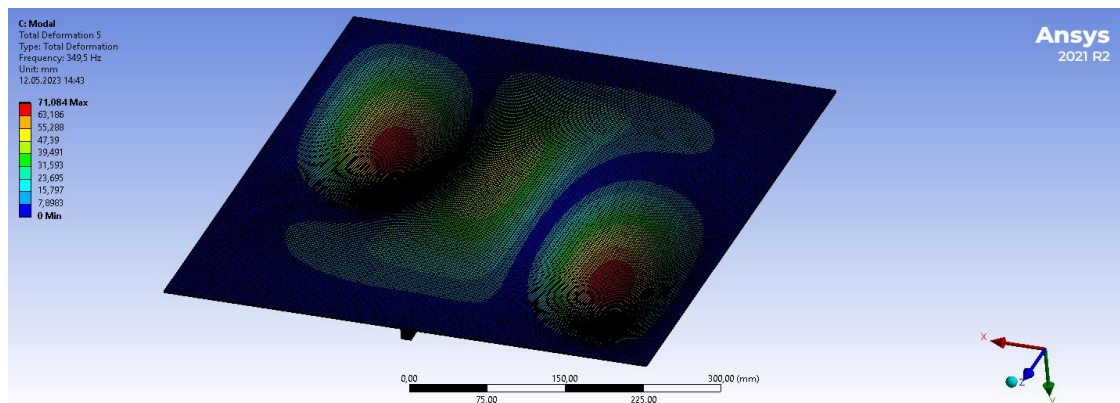


Figure 5.31 : ANSYS fifth mode shape for [45/60/60/45] carbon-fiber plate with an isotropic stiffener

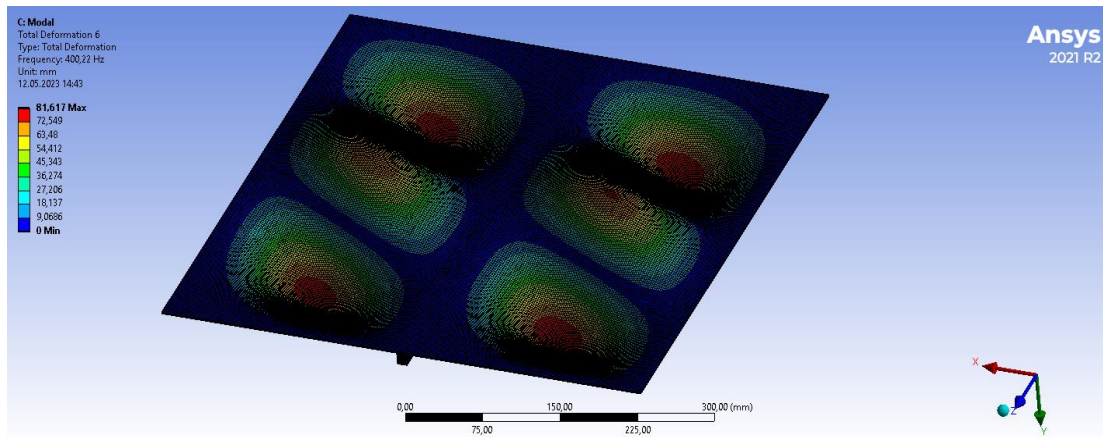


Figure 5.32 : ANSYS sixth mode shape for [45/60/60/45] carbon-fiber plate with an isotropic stiffener

The first six natural frequencies and their associated mode structures are found for the given boundary conditions in accordance with the modeled [45/60/60/45] carbon-fiber plate with an isotropic stiffener using the Generalized Differential Quadrature Method in the Wolfram Mathematica program. The results are shown below.

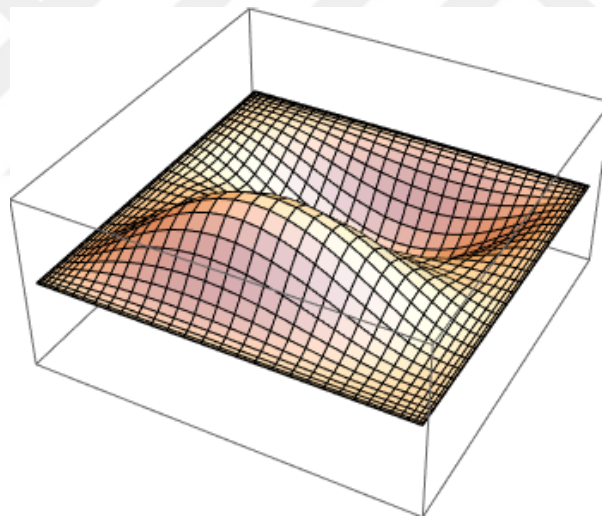


Figure 5.33 : GDQM solution first mode shape for [45/60/60/45] carbon-fiber plate with an isotropic stiffener

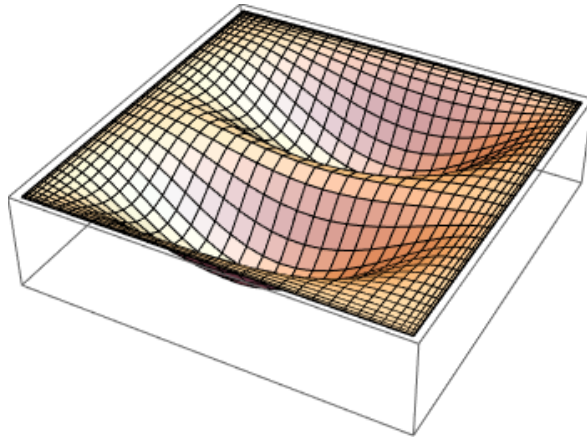


Figure 5.34 : GDQM solution second mode shape for [45/60/60/45] carbon-fiber plate with an isotropic stiffener

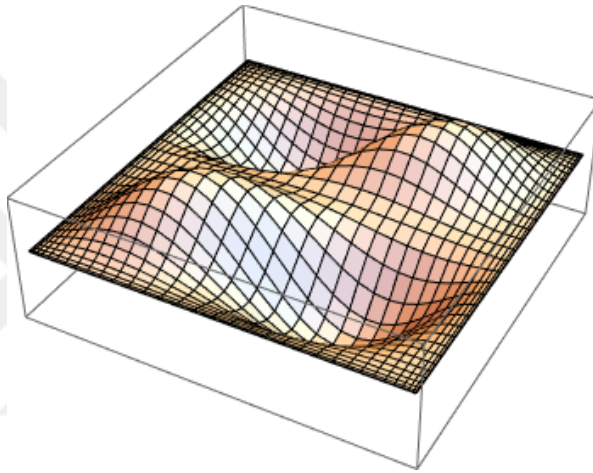


Figure 5.35 : GDQM solution third mode shape for [45/60/60/45] carbon-fiber plate with an isotropic stiffener

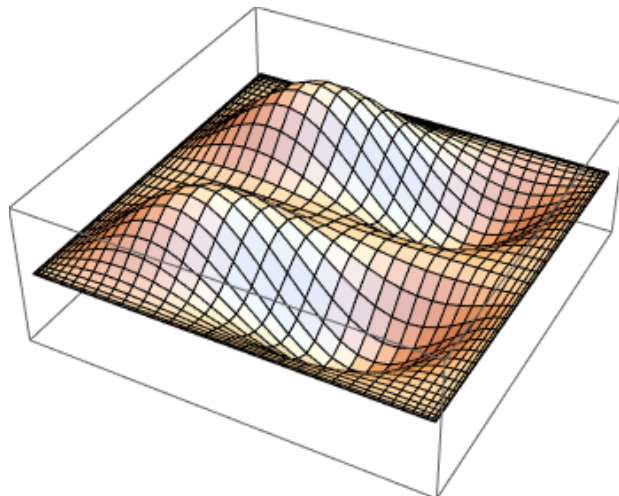


Figure 5.36 : GDQM solution fourth mode shape for [45/60/60/45] carbon-fiber plate with an isotropic stiffener

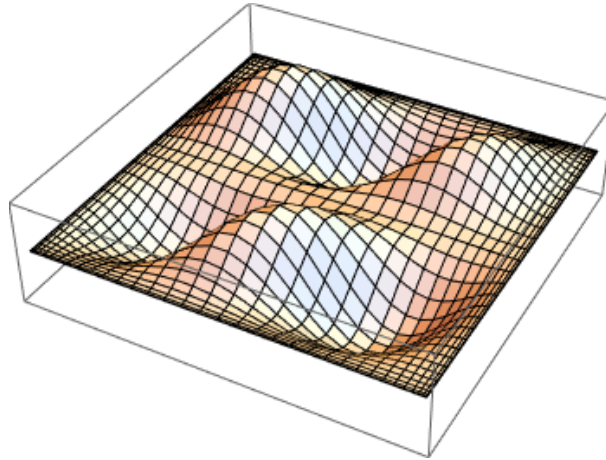


Figure 5.37 : GDQM solution fifth mode shape for [45/60/60/45] carbon-fiber plate with an isotropic stiffener

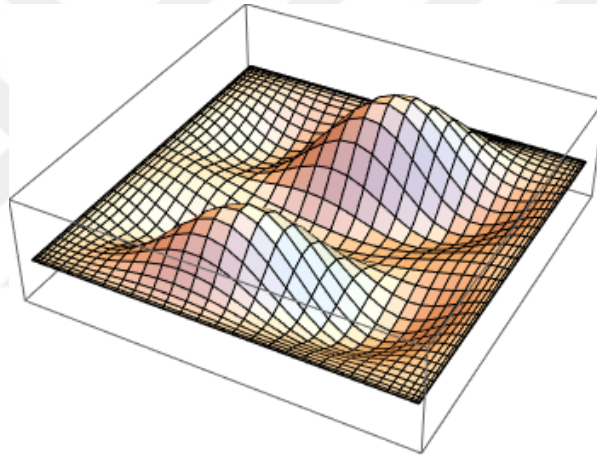


Figure 5.38 : GDQM solution sixth mode shape for [45/60/60/45] carbon-fiber plate with an isotropic stiffener

Table 5.6 : ANSYS and GDQM results comparison for [45/60/60/45] carbon-fiber plate with an isotropic stiffener

Mode	GDQM (Hz)	ANSYS (Hz)	Relative Error %
1	224.19	218.74	2.49
2	225.23	230.98	2.49
3	297.98	297.56	0.14
4	298.90	300.38	0.49
5	397.50	349.50	13.73
6	398.36	400.22	0.47



6. CONCLUSIONS AND DISCUSSIONS

Similar results were obtained using the Generalized Differential Quadrature Method (GDQM) and ANSYS for the analysis of stiffened laminated composite plates, indicating that both methods provide accurate and trustworthy solutions. There are several explanations for the similarity of the results.

The similarity of the results suggests that the displacements and stresses predicted by GDQM and ANSYS are in excellent agreement. This suggests that both techniques accurately predict the behavior of stiffened laminated composite plates under the specified loading and boundary conditions.

The accord between GDQM and ANSYS results validates the use of GDQM to analyze stiffened laminated composite plates. It demonstrates that GDQM can be a suitable numerical method for such analyses when correctly implemented and validated. This finding increases confidence in the precision of GDQM and its applicability to analogous situations.

The similarity of the results suggests that the modeling assumptions and parameters utilized by both methodologies are consistent. This comprises the material properties, laminate construction, geometry of the stiffener, and boundary conditions. Consistency in these factors indicates that both GDQM and ANSYS accurately depict the physical behavior of the stiffened laminated composite plates.

Noting that the similarity of results does not assure complete accuracy or the absence of errors is essential. There may be a need for additional verification and validation studies to establish the results' dependability and validate their agreement with experimental data or analytical solutions. In spite of this, the similarity between GDQM and ANSYS results for stiffened laminated composite plates inspires confidence in the accuracy of both methods and validates their use in engineering applications.

GDQM's advantages over other numerical techniques:

GDQM is renowned for its high precision when approximating solutions to differential equations. GDQM can provide highly accurate solutions by employing a weighted summation of function values at discrete points, particularly when a sufficient number of grid points are employed.

Compared to other numerical methodologies, GDQM is relatively simple to implement. It does not necessitate intricate mesh generation or element connectivity, making it simpler to configure and solve problems. This simplicity can result in shorter development timelines and simplified implementation.

Particularly for problems with regular geometries and well-behaved solutions, GDQM can be computationally efficient. The method entails approximating derivatives at discrete points, which can reduce computational effort in comparison to other methods requiring the solution of complex systems of equations.

GDQM is adaptable and can easily manage irregular domains. The method permits the use of grid points with non-uniform spacing, making it appropriate for problems involving complex geometries or irregular boundaries.

Compared to other numerical techniques, GDQM has the following disadvantages:

GDQM may encounter difficulties when coping with extremely complex geometries or irregular boundary conditions. The method relies on grid-based approximation, and it can be challenging to adapt the grid to complex domains. This can restrict its applicability in particular circumstances.

The precision of GDQM is extremely sensitive to the distribution of grid points and the selected basis functions. Incorrect selection or positioning of grid elements may lead to less precise solutions. High precision may necessitate a dense distribution of grid points, which can increase computational expenses.

Comparatively to well-established commercial software packages such as ANSYS, GDQM may offer less support and fewer resources. This can make it more challenging to gain access to extensive libraries, pre-built models, or technical support, which are typically offered by commercial software vendors.

Notably, the benefits and drawbacks of GDQM should be evaluated in the context of particular applications and problem specifications. Other numerical methods, such as finite element methods and boundary element methods, may offer advantages in certain circumstances; therefore, the selection of the appropriate method should be based on the analysis's specific requirement.

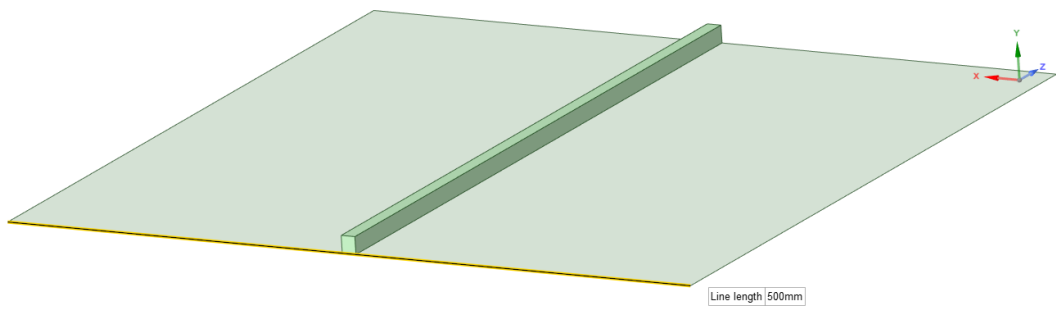
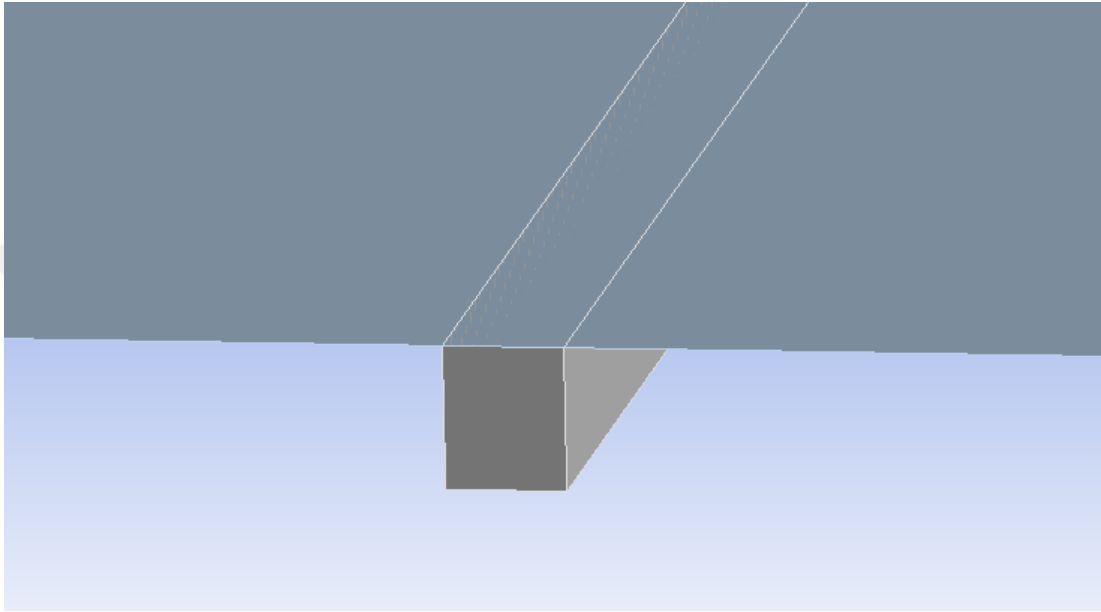
7. REFERENCES

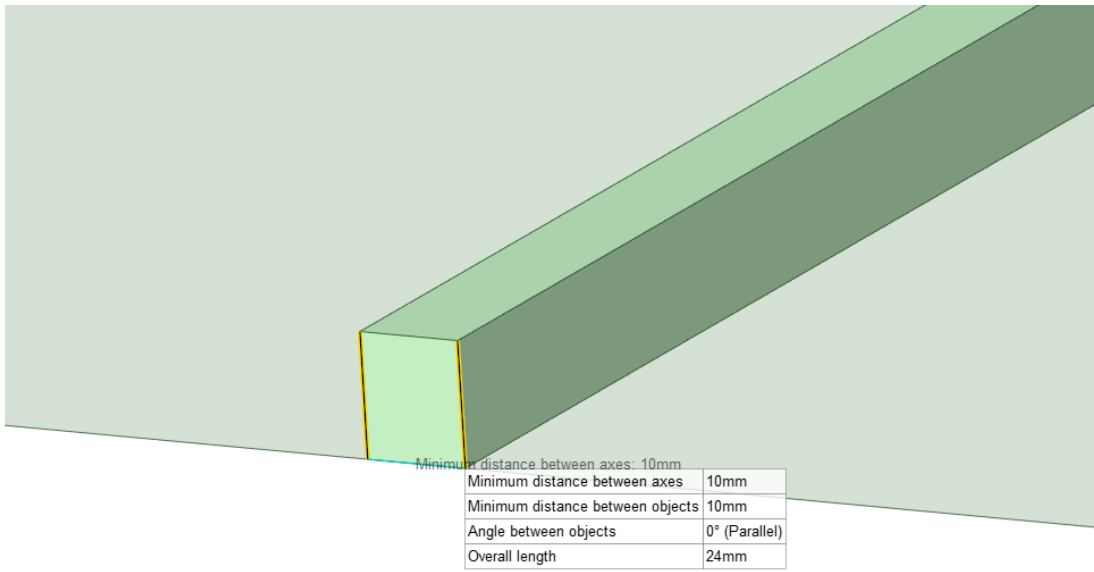
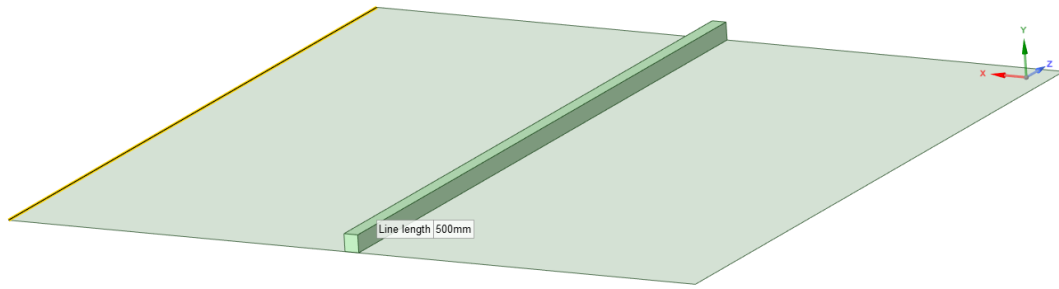
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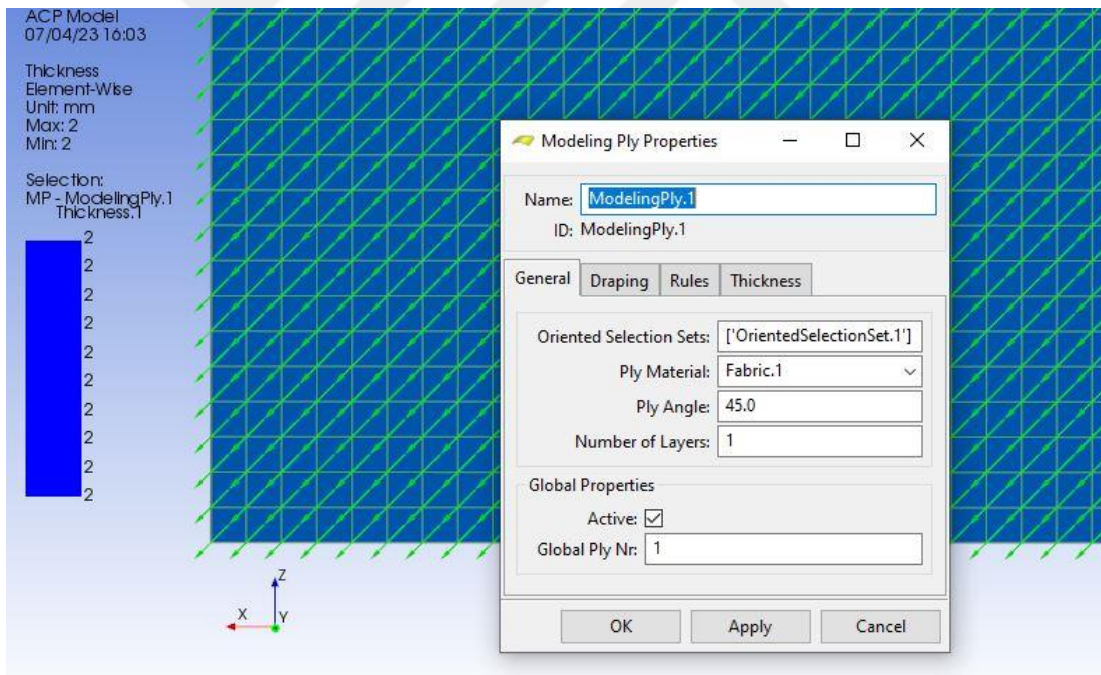
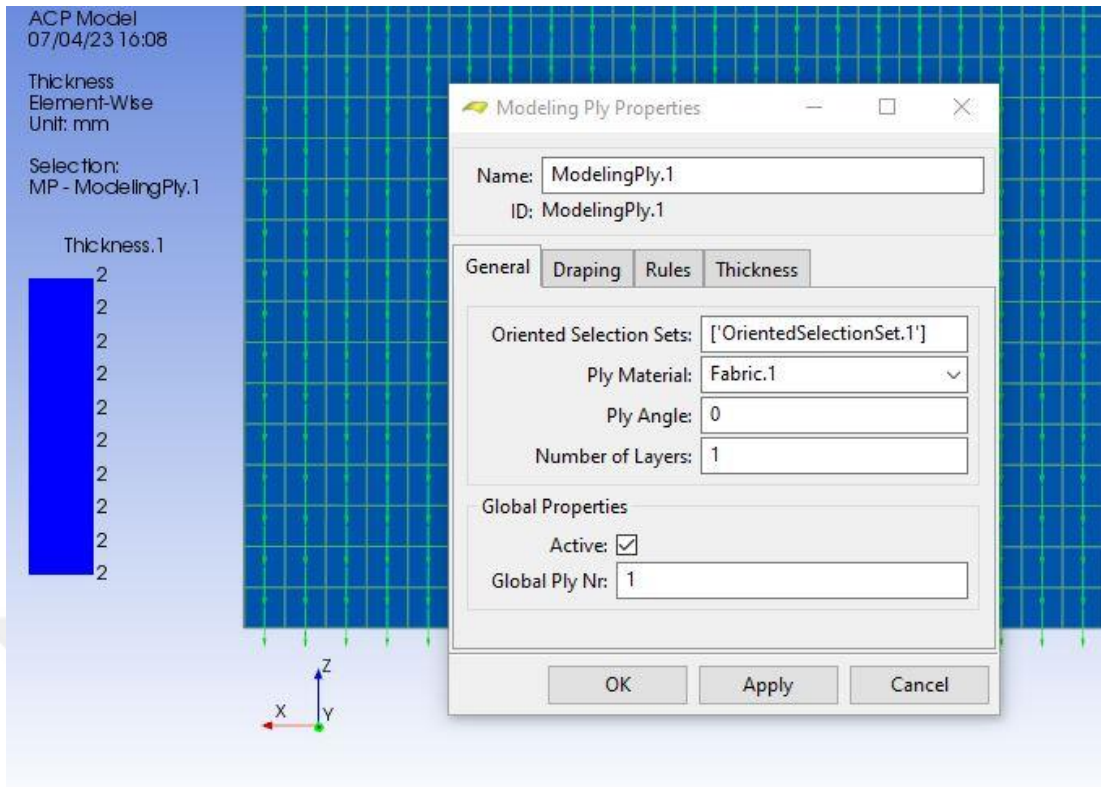
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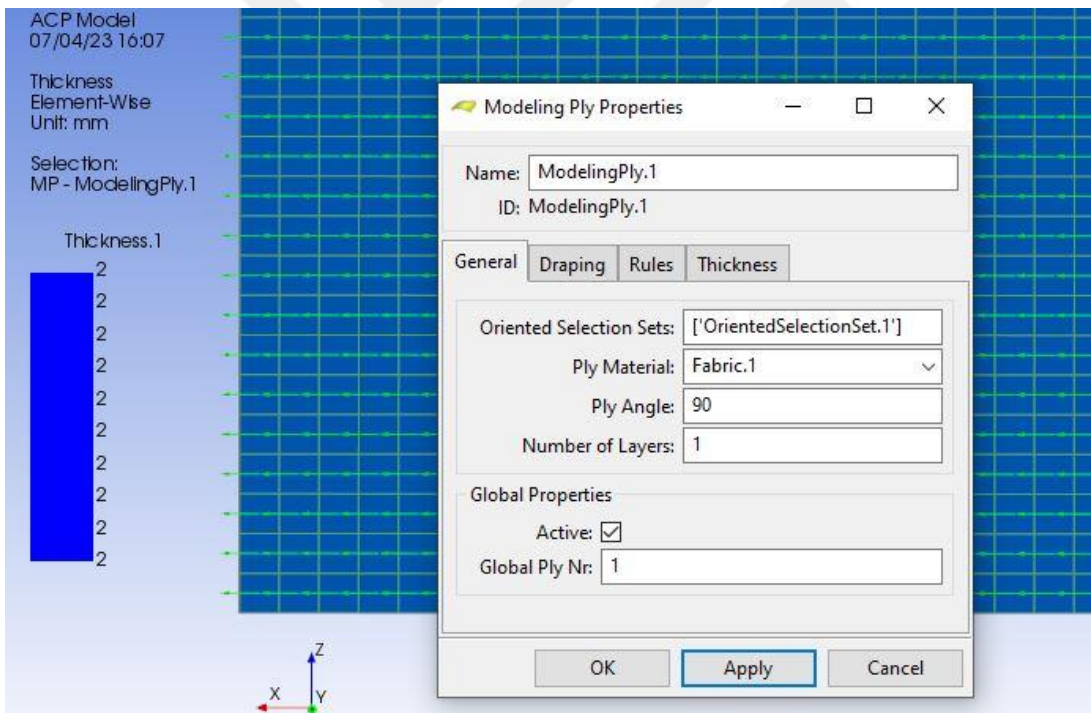
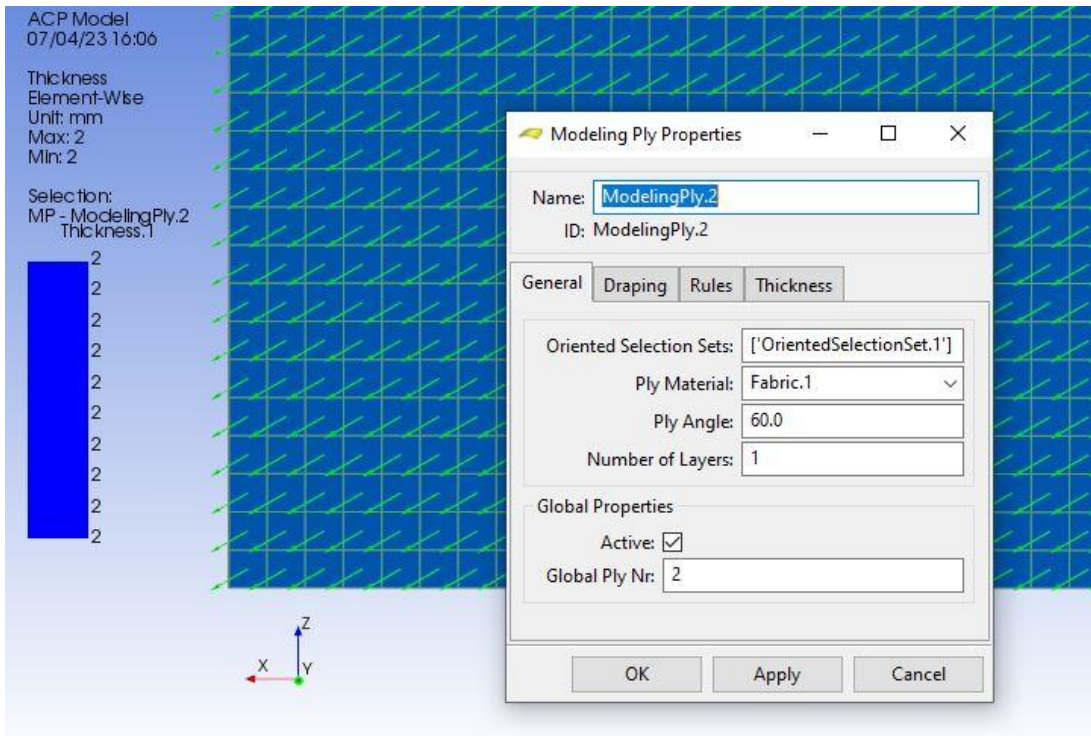
APPENDICES

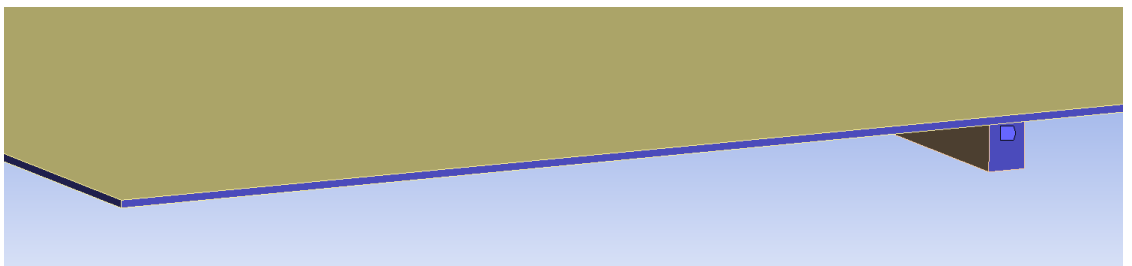
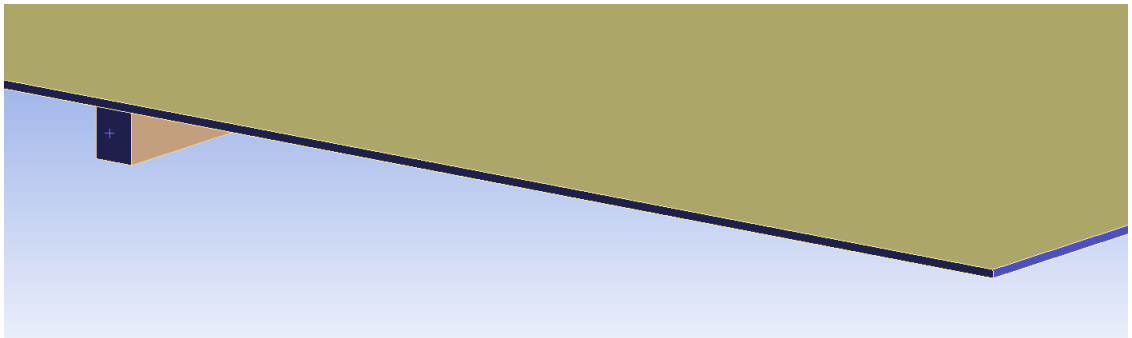
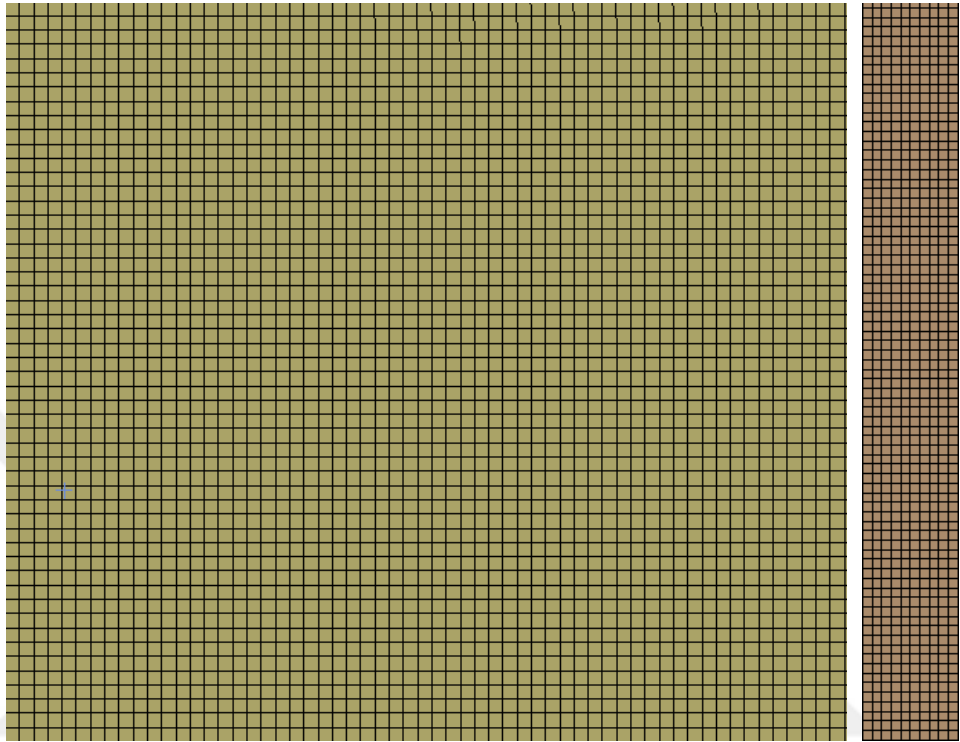
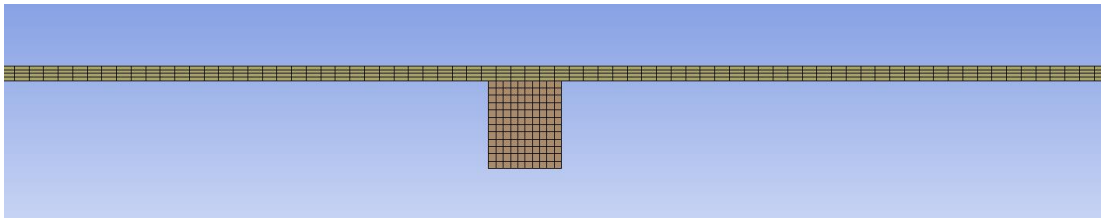
APPENDIX A:













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PROFESSIONAL EXPERIENCE AND REWARDS:

-

PUBLICATIONS, PRESENTATIONS AND PATENTS ON THE THESIS:

-

OTHER PUBLICATIONS, PRESENTATIONS AND PATENTS:

-



