SUPERSYMMETRIC COHERENT STATES AND SUPERQUBIT UNITS OF QUANTUM INFORMATION

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ABSTRACT

SUPERSYMMETRIC COHERENT STATES AND SUPERQUBIT UNITS OF QUANTUM INFORMATION

In this thesis, we study the set of maximally entangled Bell based super-coherent states, involving both fermionic and bosonic components. By extending the supersymmetric annihilation operator introduced by Aragone and Zypmann, we develop four distinct types of supersymmetric coherent states, related to the Bell two-qubit quantum states. These Bell super-qubit states form the basis for the Bell-based supersymmetric coherent states, which are constructed using a displacement operator. When these states are combined with separable bosonic coherent states, represented as points on the super-Bloch sphere, the resulting structure is called Bell-based super-coherent states. The entanglement between the bosonic and fermionic components is analyzed through a displacement bosonic operator, which acts on a super-qubit reference state. For these entangled super-coherent states, uncertainty relations are expressed by concurrence. The monotonic relationship between uncertainty and concurrence C indicates the influence of entanglement on uncertainty relations. Then, we observe quadrature squeezing in the uncertainties of position and momentum. Furthermore, we describe an infinite sequence of super-coherent states, whose uncertainty relations are characterized by the ratio of two Fibonacci numbers.

For generalization of previous results, we introduce the generic super-qubit quantum state, where the single super-particle state is defined by a complex parameter ζ. This leads us to description of PK-super-qubit quantum states, which are characterized by two unit spheres. These states form the basis for what we refer to as PK-supersymmetric coherent states, for which we have analyzed the entanglement properties. The pq-deformed super-coherent states and particular case as q-deformed super-coherent states are studied.

ÖZET

SÜPERSİMETRİK EŞ UYUMLU DURUMLAR VE KUANTUM BİLGİSİNİN SÜPERKÜBİT BİRİMLERİ

Bu tezde, hem fermiyonik hem de bozonik bileşenleri içeren, maksimum dolanık Bell tabanlı süper-eş uyumlu durumlar kümesini inceliyoruz. Aragone ve Zypmann tarafından tanıtılan süpersimetrik yok edici operatörü genişleterek, Bell iki-kübit kuantum durumlarıyla ilişkili dört farklı süpersimetrik eş uyumlu durum geliştiriyoruz. Bu Bell süperkübit durumları, yer değiştirme operatörü kullanılarak inşa edilen Bell tabanlı süpersimetrik eş uyumlu durumların temelini oluşturur. Bu durumlar, süper-Bloch küresi üzerinde noktalar olarak temsil edilen ayrık bozonik eş uyumlu durumlarla birleştirildiğinde, ortaya çıkan yapıyı Bell tabanlı süper-eş uyumlu durumlar olarak adlandırılır.. Bozonik ve fermiyonik bileşenler arasındaki dolanıklık, süper-kübit referans durumu üzerinde etkili olan bir bozonik yer değiştirme operatörü aracılığıyla analiz edilir. Bu dolanık süper-eş uyumlu durumlar için belirsizlik ilişkileri concurrence C ile ifade edilir. Belirsizlik ile concurrence arasındaki monoton ilişki, dolanıklığın belirsizlik ilişkileri üzerindeki etkisini göstermektedir. Daha sonra, konum ve momentum belirsizliklerinde kuadratür sıkışması gözlemliyoruz. Ayrıca, belirsizlik ilişkileri iki Fibonacci sayısının oranı ile karakterize edilen sonsuz bir süper-eş uyumlu durum dizisi tanımlıyoruz.

Önceki sonuçları genelleştirmek amacıyla, tek bir süper-parçacık durumunun karmaşık bir parametre ζ ile tanımlandığı genel bir süper-kübit kuantum durumu tanıtıyoruz. Bu tanımlama, iki birim küre ile karakterize edilen PK-süper-kübit kuantum durumlarına yol açmaktadır. Bu durumlar, PK-süpersimetrik eş uyumlu durumlar olarak adlandırdığımız yapıların temelini oluşturur ve bu durumların dolanıklık özelliklerini inceliyoruz. Son olarak, pq-deforme süper-eş uyumlu durumları ve özel bir durum olarak q-deforme süper-eş uyumlu durumları ele alıyoruz.

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CHAPTER 1

INTRODUCTION

The qubit as a unit of quantum information is described by two level quantum system, basis states of which can be represented by vectors in $\mathbb{C}^2$,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.1}$$

The qubit state is superposition of these states, which after normalization, up to global phase, takes the standard form

$$|\theta, \phi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle, \tag{1.2}$$

where angles $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$ determine points on the unit sphere. This sphere is called the Bloch sphere, so that there is a one-to-one correspondence between points on Bloch sphere and qubit quantum states (Benenti et all, 2019, 100-101).

To realise qubits in quantum optics, it is necessary to have two orthogonal states of photons. photons, which can be constructed from special quantum states of photons called coherent states. These states were first introduced by Schrödinger (Schrödinger, 1926, 664-665) to establish a link between quantum and classical harmonic oscillators. He constructed non-stationary states with probability densities in the form of Gaussians, whose centres oscillate according to the classical equations of motion of harmonic oscillators.It has been shown that these states minimize the coordinate-momentum uncertainty relations and therefore they are considered to be the quantum states that are closest to classical states. Three different approaches to these coherent states, also known as standard coherent states (Peremolov, 1986, 13-23) or Glauber coherent states (Glauber, 1963, 2769-2775), have been proposed. The first one is minimization of the Heinsenberg uncertainty relation. In the second approach, they are formulated as eigenstates of the annihilation operator. In the last one, the application of unitary displacement operator to the vacuum state gives the coherent state. For Glauber states, these three approaches are equivalent.

The direct use of coherent states to describe a qubit is not possible because the Glauber coherent states are not orthogonal. As it was proposed in 1974 (Dodonov et all, 1974, 600-603), even and odd combinations of Glauber coherent states are orthogonal and could be used to describe a qubit. These states are called the Schröndinger cat states. By using finite superposition of coherent states with mod-n symmetry, it is also possible to construct an arbitrary number of orthogonal states (Pashaev and Kocak, 2019, 8-9). They can be used to describe generalizations of the qubit state as the qutrit, the ququad and the qudit quantum states. However, it should be noted that such type of superposition is not minimizing uncertainty relation and they represent non-classical quantum states.

Another type of orthogonal states can be produced by application of displacement operator not only to vacuum state but also to $|1\rangle, |2\rangle, |3\rangle, ..., |n\rangle$ photon number states in the Fock space. This infinite set of states is called the displaced coherent states, but these states are not coherent states because they are not minimizing uncertainty relation. The displaced coherent states are closely related with the so called photon-added states(P.A.C) ((Agarwal and Tara, 1991, 492-493) and (Francis and Tame, 2020, 3-5)), described by adding finite number of photons to the coherent state. As it is known, in contrast to coherent state as maximally classical states, the photon added states are non-classical states with several specific properties as quadrature squeezing, Poisson distribution, etc. (Francis and Tame, 2020, 3-8). As non-classical states, they attract interest in applications to quantum sensing, continuous variable quantum information processing, quantum state engineering and probing fundamental properties of quantum mechanics (Francis and Tame, 2020, 5-11).

To implement transition from classical to non-classical states, the superposition of coherent states with PAC states where studied theoretically and experimentally (see (Zavatta et all, 2004, 660-662) and references therein). Denoting the Glauber coherent state as $|0, \alpha\rangle = D(\alpha)|0\rangle$, the displaced Fock state $|1, \alpha\rangle = D(\alpha)|1\rangle$ can be represented as such type superposition of $|1, \alpha\rangle = \hat{a}^{\dagger}|0, \alpha\rangle + \bar{\alpha}|0, \alpha\rangle$. Then, the generic superposition of the coherent state and the PAC state appears as displaced state $c_0|0, \alpha\rangle + c_1|1, \alpha\rangle = D(\alpha)(c_0|0\rangle + c_1|1\rangle)$ of the one qubit state $c_0|0\rangle + c_1|1\rangle$. For $c_1 \neq 0$, the state is linearly independent of $|0,\alpha\rangle$, so that if we combine the states as two component spinor of Fock states, it becomes descriptive of entangled fermion-boson states of supersymmetric quantum oscillator (Pashaev and Kocak, 2025, 5-13). This allows us to study transition from classical to nonclassical states in framework of supersymmetric quantum mechanics (Cooper et all, 2001, 7-30), and its dependence on entanglement between fermions and bosons.

The purpose of the present thesis is description of coherent states in supersymmetric quantum mechanics and their relations with units of quantum information, which we call as the super-qubit quantum states. By using displacement bosonic operator, the states are generated by acting on a reference state, in addition to traditional vacuum state $|\Psi_0\rangle = |0\rangle_f \otimes |0\rangle_b$, includes superposition with the one super-particle state. This superposition is naturally called as the super-qubit state. The usual qubit state is a superposition of $|0\rangle$ and $|1\rangle$ computational states, as eigenstates of the number operator N_f. Inspired from this, we define the super-qubit state as a superposition

$$|\theta,\phi\rangle_S = \cos\frac{\theta}{2}|0\rangle_S + \sin\frac{\theta}{2}e^{i\phi}|1\rangle_S \tag{1.3}$$

where $|0\rangle_S$ and $|1\rangle_S$ are super-computational basis states, as eigenstates of the super-number operator $\mathcal{N}$, which counts number of superparticles. The states are parametrized by coordinates on the unit sphere, which we call as the super-Bloch sphere. Contribution of superparticles to energy is the same and does not distinguish fermions from bosons, so that the superqubits are the degenerate states, but with different level of fermion-boson entanglement. In this thesis, we first work with superqubit state as a superposition of separable $|0\rangle_S$ state and maximally entangled $|1\rangle_S$ state. For the last one we use the first pair of Bell states in fermion-boson basis. After applying displacement operator to the superqubit state we get the first pair of super-coherent states. The second pair of states is generated from the second pair of fermion-boson Bell states, being not exact eigenstates of the supernumber operator (but only in averages).

We show that the fermion-boson entanglement in super-coherent state is equal to the one in the corresponding super-qubit reference state and does not depend on displacement parameter α. The entangled super-coherent states describe non-classical behavior in non-minimal form of uncertainty relations, quadrature squeezing and sub-Poissonian distribution. The entanglement of states is independent of time evolution and in contrast to Glauber coherent states, which are never orthogonal, the super-coherent states can be orthogonal. Depending on value of concurrence, we have the full circle of the equidistant maximally entangled states (C=1), orthogonal to the given one, and the pair of orthogonal antipodal states for arbitrary $0 < C < 1$. For $C = 1$, three mutually orthogonal states, associated with equilateral triangle in complex plane are found. For separable states with $C = 0$, no orthogonal states are possible. This shows that entanglement of bosons with fermions is required to have orthogonality of coherent states. The entanglement affects also uncertainty relations. The coordinate-momentum uncertainty for supercoherent states, represented in terms of monotonically growing function of the concurrence, allows us to relate the uncertainty with level of boson-fermion entanglement. For the states along equator of the super- Bloch sphere, with $C = \frac{1}{2}$ we find the representation of uncertainty by ratio of two Fibonacci numbers $\hbar F_5/F_6$. Then, by using the sequence of concurrences $C_n = \sqrt{F_{n-2}/F_{n+1}}$, convergent to $C = \varphi^{-3/2}$, we obtain the sequence of uncertainties $\hbar F_n/F_{n+1}$, in the limit $n \to \infty$ convergent to the Golden ratio uncertainty $\hbar/\varphi$.

Similar to superposition of coherent states with PAC states, making the state non-classical, our supercoherent states show the quadrature squeezing - when uncertainty in X variable is lower than $1/2$, by expense of increasing uncertainty in P variable, bigger than $1/2$, and vice versa. This result can improve the measurement limits in SUSY quantum oscillator and can be applied in several fields as quantum communications and quantum sensing, quantum optics and information processing. More general super-qubit states and corresponding super-coherent states appear if we notice that the one super-particle state is not unique and is parametrized by complex number ζ (Pashaev and Kocak, 2025(3, 5-10)). The pq-deformed super-coherent states also provide another type of supersymmetric deformed quantum oscillator states.

Here we briefly describe existing literature in the field (list of which never could be complete) and the differences with our paper. The supersymmetric coherent states were studied in several papers from different points of view. In first and seminal paper (Aragone and Zypman, 1986, 2268-2270), for simplest (N = 1) supersymmetric generalization of the standard quantum mechanical harmonic oscillator, the supersymmetric annihilation operator $A_1 = \hat{I}_f \otimes \hat{a} + \hat{f} \otimes \hat{I}_b$, entangling bosons with fermions was defined. The super-coherent states as the eigenstates of this operator were determined by using

Fock space expansion. The difference with (Aragone and Zypman, 1986, 2271-2275) we have in representation of supersymmetric states, by using fermion number operator $N_f = diag(0, 1)$, which leads to opposite number of fermions in the given state. Though both definitions are correct and a matter of preference is dictated by the goal, our choice is motivated by standard notations in quantum computation and quantum information theory (Benenti et all, 2019, 49-110), where computational qubit basis state $|0\rangle$ we associate with zero fermions, while state $|1\rangle$ - with one fermion. The same definition is used also in (Cooper et all, 2001, 8-20). An extension of the Fock state expansion by application of the displacement operator, acting on different reference states as super-qubit states, has several advantages. It allowed us to identify the reference state in paper (Aragone and Zypman, 1986, 2272-2274) as the fermion-boson Bell state and extend the set of super-coherent states to other Bell states. The approach, together with calculation of concurrence and von Neumann entropy greatly simplifies much the calculations and clarifies meaning of uncertainty relations and the entanglement property. In addition, it allows us to find orthogonality of entangled coherent states, quadrature squeezing, Fibonacci sequences and Golden uncertainty. One more specific is that we have four different super-annihilation operators $A_\pm, A_\pm^T$, which include not only f operator, annihilating $|0\rangle$ state, but also $f^\dagger$, annihilating $|1\rangle$ state. The paper Berube-Lauziere (1993) works with $A_0 = I_f \otimes a$, as another super-annihilation operator, trying to formulate three equivalent definitions of supercoherent states, similar to the Glauber states. More general forms of nonlinear super-annihilation operator were studied in (Kornbluth and Zypman, 2013, 2-5) by Fock space expansion. The group-theoretical approach to supercoherent states with Grassman variables was subject of paper (Fatyga et all, 1991, 1405-1410), and (Nieto, 1991, 95-99) proposed interpretation of the Grassman coherent state as the photino, the superpartner of photon.

The influence of squeezing operator on uncertainty relation for SUSY oscillator was counted in paper (Orszag and Salamo, 1988, 61-64). It was shown that unitary displacement operator for super-coherent states could be in the form $I \otimes D(\alpha)$ only, and this is exactly the one we are using in present paper.The idea to use displacement operator for supersymmetric construction of displaced number states was explored in (Zypman, 2015, 1019-1025), where specific form of displacement operator as the translation operator, written in terms of $A_1^\dagger - A_1$ were considered. In our paper (Pashaev and Kocak, 2025, 15-

19) we use similar idea, but with different super-displacement operator, given by direct product with pure bosonic displacement operator .

The entanglement of bosons with fermions in SUSY, was not much explored. Finite supersymmetry transformations and highly entangled combinations of bosons and fermions, invariant under supertranslations were worked out in (Iliyeva et all, 2004, 119-127). In (Laba and Tkachuk, 2020, 2-7), by exploring the Pauli Hamiltonian, entanglement of spin variables of electron in uniform magnetic field, which exhibit SUSY was examined. They have determined the concurrence by the mean value of spin and calculated it explicitly for SUSY quantum mechanical states. The entanglement entropy in Gaussian states, related by SUSY is subject of discussion in paper (Jonsson et all, 2021, 7-12). In paper (Motamedinasab et all, 2018, 1167-1175) an entanglement of generalized supercoherent states with nonlinearly extended operator A were studied by applying the concurrence formula, given by single determinant of coefficients for two-qubit states, where one of them is chosen as coherent state. In paper (Pashaev and Kocak, 2025, 18-22), by using the reduced density matrix approach, the concurrence formula obtained which includes an infinite number of 2x2 determinants, covering all Fock states. In that paper the set of Bell based supersymmetric coherent states was studied, as well as the several applications.

The thesis is organized as follows.

In Chapter 2, we briefly review main definition and properties of Glauber coherent states. The $pq-$deformation of coherent states and corresponding pq calculus are described in Chapter 3. Chapter 4 is devoted to fermionic oscillator, fermionic- bosonic oscillator and the supersymmetric harmonic oscillator.

For description of supercoherent states in Chapter 5, we use supersymmetric annihilation operator, which was proposed in the first time by Aragone and Zypmann. We called the eigenstates of this operator as the AZ-supersymmetric coherent states. Section 5.2 introduces the Supersymmetric Bloch sphere (Section 5.2.1) and uncertainty relations for the supersymmetric Bloch sphere (Section 5.2.2). The coordinate and momentum representation of these states are described in Section 5.3.

It turns out that supersymmetric coherent states can be seperable or entagled fermion-boson quantum states. For description of entanglement for fermion-boson states in Chapter 6, we introduce characteristic of fermion-boson entanglement in terms of the concurrence, related to the linear entropy, and the Von-Neumann entropy. By using these characteristics, we calculate entanglement of super number states in Section 6.2. Then, the fermion-boson Bell states are constructed in Section 6.2.2. Superposition of these states with the vacuum state in Section 6.2.3 produces the Bell based super-qubit states. In Section 6.3, by applying displacement operator we construct the Bell-based supersymmetric coherent states. Entanglement of these states and orthogonality properties are subject of Section 6.4. In section 6.5., we show that during the time evolution of these states, the entanglement of fermion-boson states is time independent. In Section 6.6, we calculate uncertainty relations and entanglement for the Bell based super coherent states. In Section 6.6.1, we show quadratic squeezing of coordinate and momentum uncertainties. The infinite set of super coherent states, related with Fibonacci numbers and limiting case $n \to \infty$, producing uncertainty relation with Golden ratio are subject of Section 6.6.2.

In Chapter 7, we study entanglement of PK-super-qubit quantum states and super-coherent states. Since in supersymmetric quantum mechanics, one superparticle number state is not unique and can be parametrized by complex number ζ, (Pashaev and Kocak, 2025(3)) introduced the PK-supersymmetric annihilation operator (Section 7.2) for description of corresponding super-qubit states. Application of displacement operator to PK-superqubit states produce PK-super coherent state. Entanglement of such PK-super coherent state is subject of Section 7.2.1 . In Section 7.3, we describe flipped PK-superqubit states. Uncertainty relations for these states and related Fibonacci sequence are derived in Section 7.4.

Chapter 8 is devoted to pq-deformed supersymmetric annilation operator and corresponding coherent states. Uncertainty relations for these states are subject of Section 8.2.

As a particular, but more explicit form of pq-deformed oscillator, in Chapter 9 we describe q-deformed supersymmetric coherent states. In Section 9.2, we discuss q-supersymmetric annihilation operator and corresponding $q-$super-coherent states. Uncertainty relations are derived in Section 9.3. The time evolution of q supersymmetric coherent states is subject of Sections 9.4,9.5 and 9.6.

In Conclusion, we summarize our results. Details of some calculations are given in Appendices A,B and C.

CHAPTER 2

GLAUBER COHERENT STATES

This chapter provides an overview of the definition and fundamental properties of coherent states. For further details, we refer to (Peremolov, 1986, 7-37) and (Wolfgang, 2001, 295-311).

2.1. The Heisenberg-Weyl Algebra and Bosonic Oscillator

In quantum mechanics, the coordinate operator $\hat{x}$ and momentum operator $\hat{p}$ are Hermitian operators. They act in the Hilbert space $\mathcal{H}$ and satisfy Heisenberg commutation relations:

$$[\hat{x}, \hat{p}] = i\hbar\hat{\mathrm{I}} \,, \quad \left[\hat{x}, \hat{\mathrm{I}}\right] = \left[\hat{p}, \hat{\mathrm{I}}\right] = 0\,. \tag{2.1}$$

Here $\hat{\mathrm{I}}$ is the identity operator and $\hbar$ is Planck's constant, and the bracket means the commutator $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$. Instead of operators $\hat{x}$ and $\hat{p}$, another pair of operators as the annihilation operator $\hat{a}$ and creation operator $\hat{a}^\dagger$ is defined ($m = 1$)

$$\hat{a}^\dagger = \frac{\omega\hat{x} - i\hat{p}}{\sqrt{2\hbar\omega}} \quad , \quad \hat{a} = \frac{\omega\hat{x} + i\hat{p}}{\sqrt{2\hbar\omega}}\,. \tag{2.2}$$

Motivation of introducing these operators is to solve harmonic oscillator problem algebraically. The commutation relation for these operators follows from (2.1) and (2.2)

$$\left[\hat{a}, \hat{a}^\dagger\right] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = \hat{\mathrm{I}}\,, \tag{2.3}$$

and is called the bosonic commutation relation. For two vectors $|\phi\rangle$ and $|\Psi\rangle$ describing the quantum states in the Hilbert space, the Hermitian inner product is denoted by $\langle\Psi|\phi\rangle$.

There is a vacuum vector $|0\rangle \in \mathcal{H}$ defined as

$$\hat{a}|0\rangle = 0 \ , \quad \text{where } \langle 0|0\rangle = 1 \ . \tag{2.4}$$

Succesive applications of creation operator to the vacuum state generates n-particle state

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle \ , \qquad n = 0, 1, 2, ... \tag{2.5}$$

The set of vectors $|n\rangle$ form a basis in $\mathcal{H}$, which is called the Fock space. The action of operators on these states are given by

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad \& \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \ . \tag{2.6}$$

The number operator $\widehat{N}$, defined by product of operators $\hat{a}$ and $\hat{a}^\dagger$, is Hermitian and has eigenvalues n, as natural numbers

$$\widehat{N} = \hat{a}^\dagger \hat{a} \quad \Rightarrow \quad \widehat{N}|n\rangle = n|n\rangle \quad , n = 0, 1, 2. \ldots \ . \tag{2.7}$$

The bosonic Hamiltonian $\hat{H}$, written in terms of the annihilation and creation operators, allows one to determine the energy spectrum of $\hat{H}$. The Hamiltonian for the quantum harmonic oscillator is

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\widehat{N} + \frac{1}{2}\right),$$

and corresponding energy levels are quantized as

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2. \ldots$$

Each eigenstate $|n\rangle$ has energy E_n, based on integer values of n, with the ground state energy $E_0 = \frac{1}{2}\hbar\omega$, representing the zero-point energy.

2.2. Coherent States and Complex Plane

In this section, we introduce main properties of coherent states (Peremolov, 1986, 7-37). A coherent state is the specific quantum state introduced by Schrödinger (Schrödinger, 1926, 664-665) for the quantum harmonic oscillator, which has dynamics most close to the behaviour of classical harmonic oscillator.

Definition 2.1 *The Glauber coherent state* $|\alpha\rangle$ *is defined as eigenstate of the annihilation operator* $\hat{a}$ *, with eigenvalue* $\alpha \in \mathbb{C}$,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{2.8}$$

By using the displacement operator $D(\alpha) = e^{\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}}$(see Appendix Eqn.($C$.1)), these coherent states can be generated from the vacuum state(2.4).

Proposition 2.1 *Coherent states are obtained by applying displacement operator* $D(\alpha)$ *to the vacuum state:*

$$|\alpha\rangle = D(\alpha)|0\rangle. \tag{2.9}$$

Proposition 2.2 *The state* $|\alpha\rangle = D(\alpha)|0\rangle$ *satisfies the eigenvalue problem* (2.8) *for coherent states.*

Proof We start with the assumption that the displacement operator $D(\alpha)$ acting on the vacuum state $|0\rangle$ gives the coherent state $|\alpha\rangle$. Then, applying the annihilation operator $\hat{a}$ to both sides of this equation gives $\hat{a}D(\alpha)|0\rangle = \hat{a}|\alpha\rangle$, due to the following relation

$$\hat{a}D(\alpha) = \hat{I}\hat{a}D(\alpha) = D(\alpha)D^\dagger(\alpha)\hat{a}D(\alpha) \overset{(C.4)}{=} D(\alpha)\,(\hat{a} + \alpha) = D(\alpha)\hat{a} + D(\alpha)\alpha\,, \tag{2.10}$$

we obtain eigenvalue equation

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle.$$

□

The coherent state $|\alpha\rangle$ can be written in terms of $|0\rangle$ in a compact form,

$$|\alpha\rangle = \frac{e^{\alpha\hat{a}^\dagger}}{\sqrt{e^{|\alpha|^2}}}|0\rangle \tag{2.11}$$

by Baker−Campbell−Hausdorff formula(See Appendix (C.9)).

Proposition 2.3 *Representation of coherent states in the Fock basis is*

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\,, \tag{2.12}$$

where $|n\rangle$ is the eigenstate of number operator (2.7).

2.2.1. Inner Product of Coherent States

Proposition 2.4 *Inner product of two coherent states is equal to*

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\bar{\alpha}\beta}. \tag{2.13}$$

This implies the following corollary.

Corollary 2.1 *Coherent states are not orthogonal,*

$$|\langle\alpha|\beta\rangle|^2 = \langle\alpha|\beta\rangle\langle\beta|\alpha\rangle = e^{-(|\alpha|^2+|\beta|^2-\overline{\alpha}\beta-\overline{\beta}\alpha)} = e^{-|\alpha-\beta|^2} \neq 0.$$

Since the exponential function is never zero, coherent states are not orthogonal. An additional characteristic for coherent states is that they form an overcomplete basis, spanning the entire space that allows reconstruction of any arbitrary state through integration over coherent state parameters. Mathematically, the completeness relation for a set of coherent states $|\alpha\rangle$ is given by following proposition.

Proposition 2.5 *(Peremolov, 1986, 15) The collection of coherent states* $|\alpha\rangle$, *where* $\alpha \in \mathbb{C}$, *forms an overcomplete set*

$$\frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha|\; d^2\alpha = \hat{I}\,, \tag{2.14}$$

where $d^2\alpha = i d\alpha_1 d\alpha_2$, $\alpha = \alpha_1 + i\alpha_2$.

2.2.2. Heisenberg Uncertainty Relation

Another essential property that defines coherent states is their ability to minimize the uncertainty relation. In quantum mechanics, the Heisenberg uncertainty principle establishes a lower bound on the product of the uncertainties (or dispersions) in position $\hat{x}$ and momentum $\hat{p}$: $(m = 1, \omega = 1)$

$$\Delta\hat{x}\Delta\hat{p} \geq \frac{\hbar}{2}.$$

Coherent states are unique in that they achieve this bound exactly, minimizing the uncertainty product. This minimum uncertainty condition is what gives coherent states their "classical-like" behavior, as they resemble the most localized wave packets allowed by quantum mechanics. For coherent state $|\alpha\rangle$, the uncertainties in position and momentum satisfy:

$$\Delta\hat{x}\Delta\hat{p} = \frac{\hbar}{2}. \tag{2.15}$$

Thus, the condition of satisfying the minimum uncertainty relation provides alternative definition of coherent states. In the proof of Heinsenberg uncertainty relation for the coherent state $|\alpha\rangle$, one uses definition of $\hat{x}$ and $\hat{p}$ operators in terms of $\hat{a}$ and $\hat{a}^\dagger$:

$$\hat{x} = \sqrt{\frac{\hbar}{2}}\left(\hat{a} + \hat{a}^\dagger\right) \qquad , \qquad \hat{p} = -i\sqrt{\frac{\hbar}{2}}\left(\hat{a} - \hat{a}^\dagger\right).$$

The expectation value of $\hat{x}$ and $\hat{p}$ operators in the coherent state $|\alpha\rangle$ are

$$\begin{aligned}\langle\hat{x}\rangle_\alpha &= \langle\alpha|\hat{x}|\alpha\rangle = \langle\alpha|\sqrt{\frac{\hbar}{2}}(\hat{a}+\hat{a}^\dagger)|\alpha\rangle = \sqrt{\frac{\hbar}{2}}(\alpha+\bar{\alpha})\ ,\\ \langle\hat{p}\rangle_\alpha &= \langle\alpha|\hat{p}|\alpha\rangle = \langle\alpha|-i\sqrt{\frac{\hbar}{2}}(\hat{a}-\hat{a}^\dagger)|\alpha\rangle = -i\sqrt{\frac{\hbar}{2}}(\alpha-\bar{\alpha})\ .\end{aligned}$$

Then, the variance for coordinate and momentum operators takes the form,

$$\begin{aligned}\langle\hat{x}^2\rangle_\alpha &= \langle\alpha|\hat{x}^2|\alpha\rangle = \frac{\hbar}{2}\langle\alpha|\left(\hat{a}+\hat{a}^\dagger\right)^2|\alpha\rangle = \frac{\hbar}{2}\left((\alpha+\overline{\alpha})^2+1\right)\ ,\\ \langle\hat{p}^2\rangle_\alpha &= \langle\alpha|\hat{p}^2|\alpha\rangle = -\frac{\hbar}{2}\langle\alpha|\left(\hat{a}-\hat{a}^\dagger\right)^2|\alpha\rangle = -\frac{\hbar}{2}\left((\alpha-\overline{\alpha})^2-1\right)\ .\end{aligned}$$

Ultimately, relationships between the expectation values and variances of the position and momentum operators in the coherent state $|\alpha\rangle$ can be expressed as:

$$\langle\hat{x}^2\rangle_\alpha = \langle\hat{x}\rangle_\alpha^2 + \frac{\hbar}{2} \quad , \quad \langle\hat{p}^2\rangle_\alpha = \langle\hat{p}\rangle_\alpha^2 + \frac{\hbar}{2}.$$

Using these results, along with the definition of uncertainty (see Eq.(B.17)), the uncertainties in position and momentum for coherent states are obtained as follows:

$$\begin{aligned}\sqrt{(\Delta\hat{x})_\alpha^2} \equiv (\Delta\hat{x})_\alpha &= \sqrt{\langle\hat{x}^2\rangle_\alpha - \langle\hat{x}\rangle_\alpha^2} = \sqrt{\frac{\hbar}{2}},\\ \sqrt{(\Delta\hat{p})_\alpha^2} \equiv (\Delta\hat{p})_\alpha &= \sqrt{\langle\hat{p}^2\rangle_\alpha - \langle\hat{p}\rangle_\alpha^2} = \sqrt{\frac{\hbar}{2}}.\end{aligned}$$

These expressions confirm that coherent states $|\alpha\rangle$ satisfy the minimum uncertainty relation. Calculating the deviations of the coordinate and momentum operators in the coherent state $|\alpha\rangle$ reveals that these deviations are independent of the parameter α. Consequently, the results for an arbitrary α align with those obtained when $\alpha = 0$, which corresponds to the vacuum state:

$$(\Delta\hat{x})_\alpha^2 = (\Delta\hat{x})_0^2 = \frac{\hbar}{2} \quad , \quad (\Delta\hat{p})_\alpha^2 = (\Delta\hat{p})_0^2 = \frac{\hbar}{2}\,. \tag{2.16}$$

2.3. Coordinate Representation of Coherent States

Coherent states in coordinate representation give non-stationary wave function of Gaussian form, which is the generating function of Hermite Polynomials. It was shown by Schrödinger (Schrödinger, 1926) that it provides solution of quantum harmonic oscillator, where position of Gaussian function oscillates according to equation of classical harmonic oscillator. To find the coordinate representation of the coherent state, we begin with evaluating the wave function:

$$\psi_\alpha(x) = \langle x|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \langle x|n\rangle, \tag{2.17}$$

where the terms $\langle x|n\rangle$ represent the position space representation of the number states, given by

$$\langle x|n\rangle = \frac{1}{\pi^{1/4}} \frac{e^{-\frac{x^2}{2}}}{2^{n/2}\sqrt{n!}} H_n(x)\,. \tag{2.18}$$

After substituting these expressions, we use the generating function for Hermite polynomials $H_n(x)$:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{-t^2+2tx}\,, \tag{2.19}$$

to simplify the wave function to the Gaussian form

$$\langle x|\alpha\rangle = \frac{1}{\pi^{1/4}} e^{-\frac{|\alpha|^2}{2} - \frac{x^2}{2}} e^{-\frac{\alpha^2}{2} + \sqrt{2}x\alpha}. \tag{2.20}$$

The probability density for this state is

$$|\psi_\alpha(x)|^2 = \frac{1}{\pi^{1/2}} e^{-\left(x-\frac{\alpha_1}{\sqrt{2}}\right)^2} e^{-\frac{3}{2}\alpha_1^2}\,, \tag{2.21}$$

showing a Gaussian distribution centered at $\frac{\alpha_1}{\sqrt{2}}$, where $\alpha = \alpha_1 + i\alpha_2$.

CHAPTER 3

PQ-DEFORMED COHERENT STATES

This chapter is devoted to pq-calculus, which serves as the basis for generalized coherent states for pq-deformed quantum oscillator. We start by outlining the fundamentals of pq-calculus, followed by an exploration of pq-coherent states, extending traditional coherent state concepts.

3.1. *pq* calculus

The $pq-$calculus is the two base quantum calculus, (Arik et all, 1992, 90-94), (Chakrabarti and Jagannathan, 1991, 711) with $pq-$number, defined in terms of two numbers p and q (Nalci and Pashaev, 2014, 75-142).

Definition 3.1 *The pq-number is defined as follows*

$$[n]_{pq} \equiv \frac{p^n - q^n}{p - q},$$

where $p \neq q$. This expression is symmetric in p and q, so that $[n]_{pq} = [n]_{qp}$.

The following addition/substraction and multiplication/division formulas for pq-numbers are valid

$$[n+m]_{pq} = p^n[m]_{pq} + q^m[n]_{pq}, \tag{3.1}$$

$$[n-m]_{pq} = p^n[-m]_{pq} + q^{-m}[n]_{pq}, \tag{3.2}$$

$$[nm]_{pq} = [n]_{pq}[m]_{(pq)^n} = [m]_{pq}[n]_{(pq)^m}, \tag{3.3}$$

$$\left[\frac{n}{m}\right]_{pq} = \frac{[n]_{pq}}{[m]_{p^{\frac{n}{m}},q^{\frac{n}{m}}}} = \frac{[n]_{p^{\frac{1}{m}},q^{\frac{1}{m}}}}{[m]_{p^{\frac{1}{m}},q^{\frac{1}{m}}}}. \tag{3.4}$$

The pq factorial for $n = 1, 2, 3, ...$ is defined as

$$[n]_{pq}! = [1]_{pq}[2]_{pq}...[n]_{pq}$$

and $[0]_{pq}! = 1$.

Definition 3.2 *In pq-calculus, the pq-derivative of a function $f(x)$ is defined as*

$$D_{pq}f(x) = \frac{f(px) - f(qx)}{(p - q)x}\,, \tag{3.5}$$

for $p \neq q$ and $x \neq 0$, where D_{pq} acts on an arbitrary function $f(x)$.

Proposition 3.1 *The pq-analogue of Leibnitz formula is*

$$D_{pq}(f(x)g(x)) = D_{pq}f(x)g(px) + f(qx)D_{pq}g(x), \tag{3.6}$$

$$D_{qp}(f(x)g(x)) = D_{pq}f(x)g(qx) + f(px)D_{pq}g(x). \tag{3.7}$$

Definition 3.3 *The (pq) -Exponential functions are defined in the following form (, Nalci and Pashaev, 2014, 107-110)*

$$e^x_{pq} = e_{pq}(x) \equiv \sum_{n=0}^{\infty} \frac{1}{[n]_{pq}!} x^n, \tag{3.8}$$

$$E^x_{pq} = E_{pq}(x) \equiv \sum_{n=0}^{\infty} \frac{1}{[n]_{pq}!} (pq)^{\frac{n(n-1)}{2}} x^n. \tag{3.9}$$

Convergency region for these functions depend on values of p and q. These functions satisfy following $pq-$difference equations

$$D_{pq}e_{pq}(x) = e_{pq}(x)\,, \quad D_{pq}E_{pq}(x) = E_{pq}(pqx).$$

The pq−calculus for particular choice of p and q can be reduced to several important cases of quantum or q−calculus. We transit through non-symmetrical and symmetrical q-calculus approaches, and eventually arrive with the Fibonacci calculus and the Tamm-Dankoff calculus. In the first case, by choosing $p = 1$ we obtain non-symmetric q-calculus,

$$[n]_q = \frac{1 - q^n}{1 - q}, \qquad (3.10)$$
$$D^x_q f(x) = \frac{f(qx) - f(x)}{(q - 1)x}.$$

In the limit $q \to 1$, we get $[n]_{q=1} = n$. For the symmetrical q-calculus, by setting $p = \frac{1}{q}$, we establish a symmetric quantum calculus,

$$[n]_{q,\frac{1}{q}} = [n]_{\widetilde{q}} = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad (3.11)$$
$$D_{q,\frac{1}{q}} f(x) = D^x_{\widetilde{q}} f(x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}.$$

Next, if we set $p = \frac{1+\sqrt{5}}{2} \equiv \varphi$ and $q = \frac{1-\sqrt{5}}{2} = \varphi' \equiv -\frac{1}{\varphi}$, as the Golden and the Silver ratio, which are the roots of equation

$$\varphi^2 = \varphi + 1,$$

we have pq numbers as Fibonacci numbers(The Binet formula)

$$[n]_{\varphi,-\frac{1}{\varphi}} = \frac{\varphi^n - (-\frac{1}{\varphi})^n}{\varphi + \frac{1}{\varphi}} = F_n\,, \qquad (3.12)$$

and pq derivative as Golden derivative

$$D^x_{\varphi} f(x) = \frac{f(\varphi x) - f(\varphi' x)}{(\varphi - \varphi')x} = \frac{f(\varphi x) - f(-\varphi^{-1}x)}{(\varphi + \varphi^{-1})x}.$$

As a final reduction, when we take limit $p \to q$, the $pq-$number becomes

$$[n]_{p\to q} = \lim_{p\to q} \frac{p^n - q^n}{p - q} = \lim_{\epsilon\to 0} \frac{(q+\epsilon)^n - q^n}{\epsilon} = nq^{n-1}, \tag{3.13}$$

and corresponding derivative is

$$D_{q,q} = q^{x\frac{d}{dx}} \frac{d}{dx}.$$

This type of quantum calculus is called the Tamm-Dankoff calculus.

3.2. The pq-Quantum Harmonic Oscillator

For the pq- harmonic oscillator, we have creation operator a_{pq}^{+}, annihilation operator a_{pq} and Hermitian number operator $N = a^{+}a$, satisfying commutation relations: $[N, a_{pq}^{+}] = a_{pq}^{+}, \quad [N, a_{pq}] = -a_{pq}.$. The algebraic relations between these operators are given by following;

$$a_{pq}a_{pq}^{+} - pa_{pq}^{+}a_{pq} = q^N, \tag{3.14}$$

$$a_{pq}a_{pq}^{+} - qa_{pq}^{+}a_{pq} = p^N. \tag{3.15}$$

Then, by using definition of pq-number operator,

$$[N]_{pq} = a_{pq}^{+}a_{pq}\,, \; [N+1]_{pq} = a_{pq}a_{pq}^{+},$$

we have

$$[N+1]_{pq} - p[N]_{pq} = q^N,$$

$$[N+1]_{pq} - q[N]_{pq} = p^N,$$

and $[N]_{pq} = \frac{p^N - q^N}{p-q}$. The orthonormal basis in the Fock space is defined by eigenstates of this $pq-$number operator

$$|n\rangle_{pq} = \frac{(a^+_{pq})^n}{\sqrt{[n]_{pq}!}}|0\rangle_{pq}\,, \tag{3.16}$$

with $a_{pq}|0\rangle_{pq} = 0$, where $|0\rangle_{pq}-$vacuum state, so that

$$[N]_{pq}|n\rangle_{pq} = [n]_{pq}|n\rangle_{pq}\,,$$

and

$$\begin{aligned} a^+_{pq}|n\rangle_{pq} &= \sqrt{[n+1]_{pq}}|n+1\rangle_{pq}\,, \\ a_{pq}|n\rangle_{pq} &= \sqrt{[n]_{pq}}|n-1\rangle_{pq}. \end{aligned}$$

Here the $pq-$number is $[n]_{pq} = \frac{p^n - q^n}{p-q}$. Now, we can define the $pq-$deformed position and momentum operators in terms of the pq-creation and annihilation operators

$$\begin{aligned} X_{pq} &= \sqrt{\frac{\hbar}{2m\omega}}(a^+_{pq} + a_{pq}), \\ P_{pq} &= i\sqrt{\frac{m\hbar\omega}{2}}(a^+_{pq} - a_{pq}). \end{aligned}$$

The Hamiltonian of the pq-Harmonic oscillator is

$$H_{pq} = \frac{P^2_{pq}}{2m} + \frac{1}{2}m\omega^2 X^2_{pq} = \frac{\hbar\omega}{2}(a_{pq}a^+_{pq} + a^+_{pq}a_{pq}).$$

By action on corresponding eigenstates $|n\rangle_{pq}$,

$$H_{pq}|n\rangle_{pq} = \frac{\hbar\omega}{2}(a_{pq}a^{+}_{pq} + a^{+}_{pq}a_{pq})|n\rangle_{pq} = \frac{\hbar\omega}{2}([N]_{pq} + [N+1]_{pq})|n\rangle_{pq}$$
$$= \frac{\hbar\omega}{2}([n]_{pq} + [n+1]_{pq})|n\rangle_{pq},$$

we get the energy spectrum

$$E_n = \frac{\hbar\omega}{2}([n]_{pq} + [n+1]_{pq}),$$

where $n = 0, 1, 2, ...$ Consequently, the energy levels are not equally spaced for general values of p and q, but the ground state energy remains the same as $\frac{\hbar\omega}{2}$. We can express pq- operators a^{+}_{pq} , a_{pq} in terms of bosonic operators a and a^{+},

$$a^{+}_{pq} = a^{+}\sqrt{\frac{[N+1]_{pq}}{N+1}} = \sqrt{\frac{[N]_{pq}}{N}}\, a^{+}, \tag{3.17}$$

$$a_{pq} = \sqrt{\frac{[N+1]_{pq}}{N+1}}\, a = a\sqrt{\frac{[N]_{pq}}{N}}. \tag{3.18}$$

The commutation relation between a^{+}_{pq} and a_{pq} is

$$[a_{pq}, a^{+}_{pq}] = a_{pq}a^{+}_{pq} - a^{+}_{pq}a_{pq} = [N+1]_{pq} - [N]_{pq}.$$

It can be demonstrated that the same set of eigenvectors $|n\rangle$, spans the Hilbert space for both, the standard harmonic oscillator and the $pq-$deformed one. In order to establish the link between the vacuum state $|0\rangle$ and $pq-$vacuum state $|0\rangle_{pq}$, we apply (3.18) as

$$a_{pq}|0\rangle_{pq} = \sqrt{\frac{[N+1]_{pq}}{N+1}}\, a|0\rangle_{pq} = 0.$$

This gives that $a|0\rangle_{pq} = 0$. From another side , if $a|0\rangle_{pq} = 0$, it implies $a_{pq}|0\rangle_{pq} = 0$. Therefore, the vacuum state $|0\rangle$ for ordinary oscillator is exactly the same as for $pq-$

deformed oscillator vacuum state $|0\rangle \equiv |0\rangle_{pq}$.

To compare n-particle states for both oscillators, we apply $(a_{pq}^{+})^n$ to the vacuum state $|0\rangle_{pq}$ and use relation

$$(a_{pq}^{+})^n = \left(a^{+}\sqrt{\frac{[N+1]_{pq}}{N+1}}\right)^n = (a^{+})^n\sqrt{\frac{[N+n]_{pq}!}{[N]_{pq}!}\frac{N!}{(N+n)!}}, \tag{3.19}$$

so that

$$(a_{pq}^{+})^n|0\rangle_{pq} = (a^{+})^n\sqrt{\frac{[N+n]_{pq}!}{[N]_{pq}!}\frac{N!}{(N+n)!}}|0\rangle_{pq} = \sqrt{\frac{[n]_{pq}!}{n!}}(a^{+})^n|0\rangle.$$

It implies that

$$|n\rangle_{pq} = |n\rangle.$$

The eigenstates of both, the standard and the pq-deformed harmonic oscillators are identical, though their energy eigenvalues differ. In the case of the standard oscillator, the energy eigenvalues are determined by natural number n,

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right),$$

but for deformed oscillator, they are provided by the corresponding equations related to pq-number $[n]_{pq}$,

$$E_n = \frac{\hbar\omega}{2}([n]_{pq} + [n+1]_{pq}).$$

3.2.1. Non-symmetrical q-Oscillator

For non-symmetrical $q-$calculus, the following algebraic relations are valid

$$a_q a_q^{+} - a_q^{+} a_q = q^N, \tag{3.20}$$

$$a_q a_q^{+} - q a_q^{+} a_q = 1, \tag{3.21}$$

where a_q and a_q^+ are annihilation and creation operators of non-symmetric $q-$calculus. The definition of non-symmetrical $q-$ number operator

$$a_q^+ a_q = [N]_q, \qquad a_q a_q^+ = [N+1]_q$$

gives

$$[N+1]_q - q[N]_q = 1,$$
$$[N+1]_q - [N]_q = q^N.$$

In this case, the Fock space basis $|n\rangle_q$ is defined by

$$|n\rangle_q = \frac{(a_q^+)^n|0\rangle_q}{\sqrt{[n]_q!}},$$

and operators act on the basis as following

$$[N]_q|n\rangle_q = [n]_q|n\rangle_q\,,$$
$$a_q^+|n\rangle_q = \sqrt{[n+1]_q}|n+1\rangle_q\,,$$
$$a_q|n\rangle_q = \sqrt{[n]_q}|n-1\rangle_q.$$

The energy levels for the corresponding eigenstates $|n\rangle_q$ are

$$E_n = \frac{\hbar\omega}{2}([n]_q + [n+1]_q)$$

where $n = 0, 1, 2, ..$

3.2.2. Symmetrical q-Oscillator

In symmetric $q-$calculus, the algebraic relations take following form

$$a_{\tilde{q}}a_{\tilde{q}}^{+} - qa_{\tilde{q}}^{+}a_{\tilde{q}} = q^{-N},$$
$$a_{\tilde{q}}a_{\tilde{q}}^{+} - q^{-1}a_{\tilde{q}}^{+}a_{\tilde{q}} = q^{N}.$$

Using the definition of symmetric q-number operator

$$a_{\tilde{q}}^{+}a_{\tilde{q}} = [N]_{\tilde{q}}, \qquad a_{\tilde{q}}a_{\tilde{q}}^{+} = [N+1]_{\tilde{q}}.$$

gives

$$[N+1]_{\tilde{q}} - q[N]_{\tilde{q}} = q^{-N},$$
$$[N+1]_{\tilde{q}} - q^{-1}[N]_{\tilde{q}} = q^{N}.$$

The basis in the Fock space is defined by

$$|n\rangle_{\tilde{q}} = \frac{(a_{\tilde{q}}^{+})^{n}|0\rangle_{\tilde{q}}}{\sqrt{[n]_{\tilde{q}}!}},$$

and the action of the operators on the basis gives

$$[N]_{\tilde{q}}|n\rangle_{\tilde{q}} = [n]_{\tilde{q}}|n\rangle_{\tilde{q}},$$
$$a_{\tilde{q}}^{+}|n\rangle_{\tilde{q}} = \sqrt{[n+1]_{\tilde{q}}}|n+1\rangle_{\tilde{q}},$$
$$a_{\tilde{q}}|n\rangle_{\tilde{q}} = \sqrt{[n]_{\tilde{q}}}|n-1\rangle_{\tilde{q}}.$$

The eigenvalues of the energy are written in symmetrical q-basis as

$$E_n = \frac{\hbar\omega}{2}([n]_{\tilde{q}} + [n+1]_{\tilde{q}}),$$

where $n = 0, 1, 2, ..$

3.2.3. Fibonacci Oscillator

For Fibonacci calculus, algebraic relations have the following form

$$a_F a_F^+ - \varphi a_F^+ a_F = \left(-\frac{1}{\varphi}\right)^N, \tag{3.22}$$

$$a_F a_F^+ + \frac{1}{\varphi} a_F^+ a_F = \varphi^N. \tag{3.23}$$

Fibonacci $q-$number operator satisfies

$$[N+1]_F - \varphi[N]_F = \left(-\frac{1}{\varphi}\right)^N,$$

$$[N+1]_F + \frac{1}{\varphi}^{-1}[N]_F = \varphi^N,$$

where $a_F^+ a_F = [N]_F$, $a_F a_F^+ = [N+1]_F$. The basis in the Fock space is written by

$$|n\rangle_F = \frac{(a_F^+)^n}{\sqrt{[n]_F!}}|0\rangle_F,$$

and actions on $|n\rangle_F$ give

$$[N]_F|n\rangle_F = [n]_F|n\rangle_F\,,$$

$$a_F^+|n\rangle_F = \sqrt{[n+1]_F}|n+1\rangle_F\,,$$

$$a_F|n\rangle_F = \sqrt{[n]_F}|n-1\rangle_F.$$

3.3. The pq-Coherent states

Definition 3.4 *The pq-coherent states are defined as eigenstates of operator* $\hat{a}_{pq}$

$$\hat{a}_{pq}|\alpha\rangle_{pq} = \alpha|\alpha\rangle_{pq}, \tag{3.24}$$

taking the form $|\alpha\rangle_{pq} = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]_{pq}!}}|n\rangle_{pq}$ *(without normalization).*

Proposition 3.2 *Inner product of two pq-coherent states* $|\alpha\rangle_{pq}$ *and* $|\beta\rangle_{pq}$ *is*

$$_{pq}\langle\beta|\alpha\rangle_{pq} = e_{pq}^{\alpha\bar{\beta}},$$

where e_{pq}^{x} *is defined by Eq.*(3.8) *and*

$$_{pq}\langle\alpha|\alpha\rangle_{pq} = e_{pq}^{|\alpha|^2}.$$

Definition 3.5 *The normalized pq-coherent states are*

$$\left(e_{pq}^{|\alpha|^2}\right)^{-1/2}|\alpha\rangle_{pq} = e_{pq}^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{1}{[n]_{pq}!}\alpha^n|n\rangle_{pq} \equiv |0,\alpha\rangle_{pq}.$$

Definition 3.6 *The action of the pq−derivative operator* (3.5) *to the state* $\left|\frac{\alpha}{\lambda}\right\rangle_{pq}$, *we denote as*

$$D_{pq}^{\alpha}\left|\frac{\alpha}{\lambda}\right\rangle_{pq} = \frac{1}{\lambda}\sum_{n=1}^{\infty} \frac{[n]_{pq}\alpha^{n-1}}{\sqrt{[n]_{pq}!}\lambda^{n-1}}|n\rangle_{pq} \equiv \left|\frac{\alpha'}{\lambda}\right\rangle_{pq}. \tag{3.25}$$

For particular values $\lambda = 1$ and $\lambda = pq$, we have respectively

$$D_{pq}^{\alpha}|\alpha\rangle_{pq} = |\alpha'\rangle_{pq} \quad , \quad D_{pq}^{\alpha}\left|\frac{\alpha}{pq}\right\rangle_{pq} = \left|\frac{\alpha'}{pq}\right\rangle_{pq}.$$

Proposition 3.3 *Action of the pq creation and annihilation operators, a^+_{pq} and a_{pq}, on $\left|\frac{\alpha}{\lambda}\right\rangle_{pq}$ state*

$$\begin{aligned} a^+_{pq}\left|\frac{\alpha}{\lambda}\right\rangle_{pq} &= \lambda D^{\alpha}_{pq}\left|\frac{\alpha}{\lambda}\right\rangle_{pq} = \lambda\left|\frac{\alpha'}{\lambda}\right\rangle_{pq}, \\ a_{pq}\left|\frac{\alpha}{\lambda}\right\rangle_{pq} &= \frac{\alpha}{\lambda}\left|\frac{\alpha}{\lambda}\right\rangle_{pq}, \end{aligned}$$

can be represented by following operators

$$a^+_{pq} \rightarrow \lambda D^{\alpha}_{pq}, \tag{3.26}$$

$$a_{pq} \rightarrow \frac{\alpha}{\lambda}. \tag{3.27}$$

Proof Action of a^+_{pq} to the state

$$\begin{aligned} a^+_{pq}\left|\frac{\alpha}{\lambda}\right\rangle_{pq} &= \sum_{n=0}^{\infty}\frac{\alpha^n}{\lambda^n\sqrt{[n]_{pq}!}}a^+_{pq}|n\rangle_{pq} = \sum_{n=0}^{\infty}\frac{\alpha^n}{\lambda^n\sqrt{[n]_{pq}!}}\sqrt{[n+1]_{pq}}|n+1\rangle_{pq} \\ &= \sum_{n=0}^{\infty}\frac{\alpha^n}{\lambda^n\sqrt{[n+1]_{pq}!}}[n+1]_{pq}|n+1\rangle_{pq} \end{aligned}$$

after changing summation index gives

$$a^+_{pq}\left|\frac{\alpha}{\lambda}\right\rangle_{pq} = \sum_{n=1}^{\infty}\frac{[n]_{pq}\alpha^{n-1}}{\lambda^{n-1}\sqrt{[n]_{pq}!}}|n\rangle_{pq} = \lambda\sum_{n=1}^{\infty}\frac{[n]_{pq}\alpha^{n-1}}{\lambda^{n}\sqrt{[n]_{pq}!}}|n\rangle_{pq} = \lambda D^{\alpha}_{pq}\left|\frac{\alpha}{\lambda}\right\rangle_{pq}.$$

or by using (3.25),

$$a^+_{pq}\left|\frac{\alpha}{\lambda}\right\rangle_{pq} = \lambda D^{\alpha}_{pq}\left|\frac{\alpha}{\lambda}\right\rangle_{pq} = \lambda\left|\frac{\alpha'}{\lambda}\right\rangle_{pq}. \tag{3.28}$$

Eqn.(3.27) is evident from definition of the pq−coherent states (3.24) . □

Proposition 3.4 *Action of* a^+_{pq} *operator on* $\left|\frac{\alpha'}{\lambda}\right\rangle_{pq}$ *state gives*

$$a^+_{pq}\left|\frac{\alpha'}{\lambda}\right\rangle_{pq} = \lambda D^{\alpha}_{pq}\left|\frac{\alpha'}{\lambda}\right\rangle_{pq} = \lambda(D^{\alpha}_{pq})^2\left|\frac{\alpha}{\lambda}\right\rangle_{pq} = \lambda\left|\frac{\alpha''}{\lambda}\right\rangle_{pq}$$

or

$$a^+_{pq}\left|\frac{\alpha'}{\lambda}\right\rangle_{pq} = \lambda\left|\frac{\alpha''}{\lambda}\right\rangle_{pq}, \tag{3.29}$$

and for the action of a_{pq}*, we have two equivalent forms*

$$a_{pq}\left|\frac{\alpha'}{\lambda}\right\rangle_{pq} = \frac{q\alpha}{\lambda}\left|\frac{\alpha'}{\lambda}\right\rangle_{pq} + \frac{1}{\lambda}\left|\frac{p\alpha}{\lambda}\right\rangle_{pq}, \tag{3.30}$$

$$= \frac{p\alpha}{\lambda}\left|\frac{\alpha'}{\lambda}\right\rangle_{pq} + \frac{1}{\lambda}\left|\frac{q\alpha}{\lambda}\right\rangle_{pq}. \tag{3.31}$$

Proof The relation (3.29) follows from expanding the state in terms of the number states $|n\rangle_{pq}$

$$\begin{aligned} a^+_{pq}\left|\frac{\alpha'}{\lambda}\right\rangle_{pq} = \sum_{n=1}^{\infty}\frac{[n]_{pq}\alpha^{n-1}}{\lambda^n\sqrt{[n]_{pq}!}}a^+_{pq}|n\rangle_{pq} &= \sum_{n=1}^{\infty}\frac{[n]_{pq}[n+1]_{pq}\alpha^{n-1}}{\lambda^n\sqrt{[n+1]_{pq}!}}|n+1\rangle_{pq} \\ &= \sum_{n=2}^{\infty}\frac{[n-1]_{pq}[n]_{pq}\alpha^{n-2}}{\lambda^{n-1}\sqrt{[n]_{pq}!}}|n\rangle_{pq} \\ &= \lambda(D^{\alpha}_{pq})^2\left(\sum_{n=0}^{\infty}\frac{\alpha^n}{\lambda^n}\frac{|n\rangle_{pq}}{\sqrt{[n]_{pq}!}}\right), \end{aligned}$$

which we can express as

$$a^+_{pq}\left|\frac{\alpha'}{\lambda}\right\rangle_{pq} = \lambda\left|\frac{\alpha''}{\lambda}\right\rangle_{pq}.$$

To establish the second result, the following approach is used from Eqn.(3.25),

$$\begin{aligned}
a_{pq}\left|\frac{\alpha'}{\lambda}\right\rangle_{pq} &= \sum_{n=1}^{\infty}\frac{[n]_{pq}\alpha^{n-1}}{\lambda^n\sqrt{[n]_{pq}!}}a_{pq}|n\rangle_{pq} \\
&= \sum_{n=1}^{\infty}\frac{[n]_{pq}\alpha^{n-1}}{\lambda^n\sqrt{[n]_{pq}!}}\sqrt{[n]_{pq}}|n-1\rangle_{pq} \\
&= \sum_{n=0}^{\infty}\frac{[n+1]_{pq}\alpha^{n}}{\lambda^{n+1}\sqrt{[n]_{pq}!}}|n\rangle_{pq}
\end{aligned}$$

and applying $[n+1]_{pq} = q[n]_{pq} + p^n$,(see Eqn.(3.1)) gives

$$\begin{aligned}
a_{pq}\left|\frac{\alpha'}{\lambda}\right\rangle_{pq} &= \sum_{n=0}^{\infty}\frac{(q[n]_{pq}+p^n)\alpha^{n}}{\lambda^{n+1}\sqrt{[n]_{pq}!}}|n\rangle_{pq} \\
&= \frac{q\alpha}{\lambda}\left(\sum_{n=0}^{\infty}\frac{[n]_{pq}\alpha^{n-1}}{\lambda^n\sqrt{[n]_{pq}!}}|n\rangle_{pq}\right) + \frac{1}{\lambda}\left(\sum_{n=0}^{\infty}\left(\frac{p\alpha}{\lambda}\right)^n\frac{1}{\sqrt{[n]_{pq}!}}|n\rangle_{pq}\right) \\
&= \frac{q\alpha}{\lambda}\left|\frac{\alpha'}{\lambda}\right\rangle_{pq} + \frac{1}{\lambda}\left|\frac{p\alpha}{\lambda}\right\rangle_{pq}.
\end{aligned}$$

As a result of the $p \leftrightarrow q$ exchange symmetry, the second form (3.31) is obtained as

$$a_{pq}\left|\frac{\alpha'}{\lambda}\right\rangle_{pq} = \frac{p\alpha}{\lambda}\left|\frac{\alpha'}{\lambda}\right\rangle_{pq} + \frac{1}{\lambda}\left|\frac{q\alpha}{\lambda}\right\rangle_{pq}.$$

In particular case $\lambda = pq$, we have

$$\begin{aligned}
a_{pq}\left|\frac{\alpha'}{pq}\right\rangle_{pq} &= \frac{\alpha}{p}\left|\frac{\alpha'}{pq}\right\rangle_{pq} + \frac{1}{pq}\left|\frac{\alpha}{q}\right\rangle_{pq}, \\
&= \frac{\alpha}{q}\left|\frac{\alpha'}{pq}\right\rangle_{pq} + \frac{1}{pq}\left|\frac{\alpha}{p}\right\rangle_{pq}.
\end{aligned}$$

□

CHAPTER 4

SUPERSYMMETRIC HARMONIC OSCILLATOR

This chapter examines the fermionic oscillator and then addresses the fermion-boson harmonic oscillator which is characterized by fermionic and bosonic states. Finally, the supersymmetric harmonic oscillator is introduced, which interplays between fermions and bosons within the framework of supersymmetry (Cooper et all, 2001, 7-30).

4.1. Fermionic Oscillator

The fermionic oscillator is a fundamental quantum system that describes particles obeying Fermi-Dirac statistics, such as electrons, protons, and neutrons. Unlike bosonic oscillator, which allows multiple particles to occupy the same quantum state, fermionic oscillator is governed by the Pauli exclusion principle, which restricts each quantum state to be occupied by at most one fermion. The mathematical structure for fermionic oscillator is built upon the algebra of fermionic creation $\hat{f}^\dagger$ and annihilation $\hat{f}$ operators, that satisfy anticommutation relations

$$\left\{\hat{f}, \hat{f}^\dagger\right\}_+ = \hat{f}\hat{f}^\dagger + \hat{f}^\dagger\hat{f} = \widehat{\mathrm{I}}, \quad \left\{\hat{f}, \hat{f}\right\}_+ = 0, \quad \left\{\hat{f}^\dagger, \hat{f}^\dagger\right\}_+ = 0.$$

These operators can be represented by 2×2 matrices, given by

$$\hat{f} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{f}^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow \widehat{N}_f = \hat{f}^\dagger\hat{f} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The operator $\widehat{N}_f$ represents the number operator for fermions, and its eigenvalues correspond to the fermionic occupation numbers 0 and 1,

$$\widehat{N}_f \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \widehat{N}_f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The action of the fermionic operators $\hat{f}$ and $\hat{f}^\dagger$ on these states is defined by

$$\begin{aligned} \hat{f}^\dagger|0\rangle = |1\rangle \quad & \quad \hat{f}^\dagger|1\rangle = 0\,, \\ \hat{f}|1\rangle = |0\rangle \quad & \quad \hat{f}|0\rangle = 0\,. \end{aligned} \tag{4.1}$$

This structure reflects the essential properties of fermions, where each state can be either occupied or unoccupied, corresponding to the fermionic occupation numbers 0 and 1.

The Hamiltonian for fermionic oscillator is defined as

$$\hat{H}_f = \frac{\hbar\omega_f}{2}(\hat{f}^\dagger\hat{f} - \hat{f}\hat{f}^\dagger) = \frac{\hbar\omega_f}{2}(2\hat{f}^\dagger\hat{f} - 1) \Rightarrow \hat{H}_f = \hbar\omega_f\left(\widehat{N}_f - \frac{1}{2}\right),$$

or in the matrix representation

$$\hat{H}_f = \omega_f\left(\widehat{N}_f - \frac{1}{2}\right) = -\frac{\omega_f}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\frac{\omega_f}{2}\sigma_3\,,$$

where $\hbar$ is set to 1. The energy eigenstates are defined by the occupation numbers,

$$\begin{aligned} \widehat{N}_f|0\rangle = 0|0\rangle \quad & \Rightarrow \quad \hat{H}_f|0\rangle = -\frac{\omega_f}{2}|0\rangle\,, \\ \widehat{N}_f|1\rangle = 1|1\rangle \quad & \Rightarrow \quad \hat{H}_f|1\rangle = +\frac{\omega_f}{2}|1\rangle. \end{aligned}$$

providing associated energies, E_0 and E_1 for two level quantum system,

$$E_n = \omega_f \left(n - \frac{1}{2}\right), \quad n = 0, 1.$$

4.2. Fermion-Boson Harmonic Oscillator

The Hamiltonian $\hat{H}$ of the Fermion-Boson Harmonic Oscillator is expressed as $\hat{H} = \hat{H}_B + \hat{H}_F$, where $\hat{H}_B$ represents the bosonic part (the usual harmonic oscillator Hamiltonian) and $\hat{H}_F$ represents the fermionic part , which includes the creation and annihilation operators for fermions. This can be expressed as:

$$\hat{H} = \widehat{I}_F \otimes \widehat{H}_B + \widehat{H}_F \otimes \widehat{I}_B = \frac{\omega_B}{2}\left\{\hat{a}, \hat{a}^\dagger\right\} + \frac{\omega_F}{2}\left[\hat{f}^\dagger, \hat{f}\right] = \omega_B\left(\widehat{N}_B + \frac{1}{2}\right) + \omega_F\left(\widehat{N}_F - \frac{1}{2}\right).$$

In the matrix form, it can be written as

$$\widehat{H} = \widehat{I}_F \otimes \widehat{H}_B + \widehat{H}_F \otimes \widehat{I}_B = \begin{pmatrix} \widehat{H}_B & 0 \\ 0 & \widehat{H}_B \end{pmatrix} + \frac{\omega_F}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \omega_B \widehat{N}_B + \frac{\omega_B - \omega_F}{2} & 0 \\ 0 & \omega_B \widehat{N}_B + \frac{\omega_B + \omega_F}{2} \end{pmatrix},$$

or equivalently,

$$\widehat{H} = \widehat{H}_B - \frac{\omega_F}{2}\sigma_3 = \begin{pmatrix} \omega_B \hat{a}^\dagger \hat{a} + \frac{\omega_B - \omega_F}{2} & 0 \\ 0 & \omega_B \hat{a}\hat{a}^\dagger - \frac{\omega_B - \omega_F}{2} \end{pmatrix}. \tag{4.2}$$

For this composition, the number operator can be formally represented in a matrix-like form:

$$\widehat{N} = \widehat{I}_F \otimes \widehat{N}_B + \widehat{N}_F \otimes \widehat{I}_B = \begin{pmatrix} \widehat{N}_B & 0 \\ 0 & \widehat{N}_B + 1 \end{pmatrix},$$

and its eigenstates are defined as $|n_F, n_B\rangle = |n_F\rangle \otimes |n_B\rangle$, where $|n_B\rangle$ represents the bosonic states and $|n_F\rangle$ represents the fermionic states. These eigenstates satisfy the eigenvalue problems

$$\widehat{N}_B|n_B\rangle = n_B|n_B\rangle\,,$$
$$\widehat{N}_F|n_F\rangle = n_F|n_F\rangle\,,$$

with $n_B = 0, 1, 2, \ldots$ for bosons and $n_F = 0, 1$ for fermions. The energy levels of the system are given by:

$$\widehat{H}|n_F, n_B\rangle = \left[\omega_F\left(n_F - \frac{1}{2}\right) + \omega_B\left(n_B + \frac{1}{2}\right)\right]|n_F, n_B\rangle = E_{n_F,n_B}|n_F, n_B\rangle,$$

where the energy eigenvalues are

$$E_{n_F,n_B} = \omega_F\left(n_F - \frac{1}{2}\right) + \omega_B\left(n_B + \frac{1}{2}\right).$$

The total number of particles, combining bosonic and fermionic particles in the system, is given by

$$\widehat{N}|n_F, n_B\rangle = (n_F + n_B)|n_F, n_B\rangle,$$

where the eigenstates for the combined system are represented as

$$|0_F, n_B\rangle = \begin{pmatrix} |n_B\rangle \\ 0 \end{pmatrix}, \quad |1_F, n_B\rangle = \begin{pmatrix} 0 \\ |n_B\rangle \end{pmatrix}.$$

These expressions define the energy structure and state composition of the Supersymmetric harmonic oscillator.

4.3. The Supersymmetric Harmonic Oscillator

The supersymmetry implies that the frequencies of the harmonic oscillators are equal to each other $\omega_B = \omega_F = \omega$ for both bosonic and fermionic particles. This ensures that the energy levels of bosonic and fermionic states are degenerate. The Hamiltonian is then given by($\hbar = 1$)

$$\widehat{H} = \omega \begin{pmatrix} \hat{a}^\dagger \hat{a} & 0 \\ 0 & \hat{a}\hat{a}^\dagger \end{pmatrix} = \{Q, Q^\dagger\},$$

where the supercharges Q and $Q^\dagger$ are defined as

$$Q = \sqrt{\omega}\, \hat{a} \otimes \hat{f}^\dagger, \quad Q^\dagger = \sqrt{\omega}\, \hat{a}^\dagger \otimes \hat{f}.$$

The corresponding eigenstates become

$$\widehat{H} \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} = n\omega \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix}, \quad \widehat{H} \begin{pmatrix} 0 \\ |n-1\rangle \end{pmatrix} = n\omega \begin{pmatrix} 0 \\ |n-1\rangle \end{pmatrix}.$$

with energies $E_n = n\omega$ for $n > 0$ and $E_0 = 0$ for $n = 0$. This degeneracy implies that any arbitrary superposition of these states will also have the same energy. Therefore, the solution to the equation $\widehat{H}|\Psi_n\rangle = E_n|\Psi_n\rangle$ can be written as

$$|\Psi_n\rangle = \alpha \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ |n-1\rangle \end{pmatrix}, \quad E_{n>0} = n\omega,$$

where α and β are constants. The ground state of the system, with energy $E_0 = 0$, is

$$|\Psi_0\rangle = \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \tag{4.3}$$

For normalized states, the coefficients α and β must satisfy $|\alpha|^2 + |\beta|^2 = 1$.

CHAPTER 5

THE SUPERSYMMETRIC ANNIHILATION OPERATOR

In the previous section, we have introduced supersymmetric harmonic oscillator. Thus, a natural question arises: if a supersymmetric harmonic oscillator exists, does a supersymmetric extension of standard coherent states exist as well? This question becomes important due to role of supersymmetry in classification the spectra of various nucleons.

5.1. Aragone-Zypmann AZ supersymmetric annihilation operator

The first response to the question comes from the work of Aragone and Zypman. They introduced the supersymmetric annihilation operator, as referenced in (Aragone and Zypman, 1986, 2271-2272), associated with the vacuum state (4.3)

$$\widehat{A} = \hat{a} \otimes \widehat{\mathrm{I}}_F + \widehat{\mathrm{I}}_B \otimes \hat{f} = \begin{pmatrix} \hat{a} & 1 \\ 0 & \hat{a} \end{pmatrix} \tag{5.1}$$

ensuring that $\widehat{A}|\Psi_0\rangle = 0$. The operator satisfies the commutation relation $[\widehat{A}, \widehat{H}] = \omega\widehat{A}$, similar to $[\hat{a}, \widehat{H}_B] = \omega\hat{a}$ for the bosonic harmonic oscillator. This operator provides a basis for defining supersymmetric coherent states as eigenstates of this operator. Consequently, we have

$$\widehat{H}|\Psi_n\rangle = E_n|\Psi_n\rangle \Rightarrow \widehat{H}(\widehat{A}|\Psi_n\rangle) = E_{n-1}(\widehat{A}|\Psi_n\rangle),$$

demonstrating that the operator $\widehat{A}$ reduces the number of quanta in the state $|\Psi_n\rangle$ by one. The $AZ-$supersymmetric coherent states $|\alpha\rangle_{AZ}$ are introduced, as the eigenstates of the supersymmetric annihilation operator $\widehat{A}$ from (5.1):

$$\widehat{A}|\alpha\rangle_{AZ} = \alpha|\alpha\rangle_{AZ}, \tag{5.2}$$

which can be expressed in terms of standard (bosonic) coherent states. To solve this equation, we expand $|\alpha\rangle_{AZ}$ in terms of the basis eigenstates

$$\left\{|0\rangle = \begin{pmatrix}|0\rangle\\ 0\end{pmatrix},\, |b_n\rangle = \begin{pmatrix}|n\rangle\\ 0\end{pmatrix},\, |f_n\rangle = \begin{pmatrix}0\\ |n-1\rangle\end{pmatrix}\right\},$$

leading to

$$|\alpha\rangle_{AZ} = a_0|0\rangle + \sum_{n=1}^{\infty} a_n|b_n\rangle + \sum_{n=1}^{\infty} c_n|f_n\rangle. \tag{5.3}$$

Substituting this expansion into (5.2) yields the following relations

$$c_{n+1} = c_1\frac{\alpha^n}{\sqrt{n!}}, \quad a_n = \frac{1}{\sqrt{n!}}\left(a_0\alpha^n - c_1 n\alpha^{n-1}\right),$$

resulting in

$$|\alpha\rangle_{AZ} = a_0\begin{pmatrix}|\alpha\rangle\\ 0\end{pmatrix} + c_1\begin{pmatrix}-|\alpha'\rangle\\ |\alpha\rangle\end{pmatrix},$$

where we define

$$|\alpha\rangle_b \equiv \begin{pmatrix}|\alpha\rangle\\ 0\end{pmatrix},\, |\widetilde{\alpha}\rangle_s \equiv \begin{pmatrix}-|\alpha'\rangle\\ |\alpha\rangle\end{pmatrix}. \tag{5.4}$$

Thus, we have

$$|\alpha\rangle_{AZ} = a_0|\alpha\rangle_b + c_1|\widetilde{\alpha}\rangle_s,$$

where $|\alpha'\rangle = \frac{\partial}{\partial\alpha}\{|\alpha\rangle\} = \sum_{n=1}^{\infty}\frac{n\alpha^{n-1}}{\sqrt{n!}}|n\rangle$ and $|\alpha\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$ are not normalized bosonic coherent states. The action of $\hat{a}$ and $\hat{a}^\dagger$ on $|\alpha\rangle$ and $|\alpha'\rangle$ gives

$$\begin{aligned}&\hat{a}|\alpha'\rangle = |\alpha\rangle + \alpha|\alpha'\rangle\,,\\ &\hat{a}^\dagger|\alpha\rangle = |\alpha'\rangle \Rightarrow \hat{a}^\dagger|\alpha'\rangle = |\alpha''\rangle \equiv \frac{\partial^2}{\partial\alpha^2}|\alpha\rangle.\end{aligned}$$

From these relations, we obtain the inner products

$$\langle\alpha|\alpha\rangle = e^{|\alpha|^2} \Rightarrow \langle\alpha|\alpha^{'}\rangle = \frac{\partial}{\partial\alpha}\{\langle\alpha|\alpha\rangle\} = \bar{\alpha}e^{|\alpha|^2}, \tag{5.5}$$

$$\langle\alpha|\alpha^{''}\rangle = \frac{\partial^2}{\partial\alpha^2}\{\langle\alpha|\alpha\rangle\} = \bar{\alpha}^2 e^{|\alpha|^2}, \tag{5.6}$$

$$\langle\alpha^{'}|\alpha^{'}\rangle = \frac{\partial}{\partial\bar{\alpha}}\left\{\langle\alpha|\alpha^{'}\rangle\right\} = (1+|\alpha|^2)e^{|\alpha|^2}, \tag{5.7}$$

$$\langle\alpha^{'}|\alpha^{''}\rangle = \frac{\partial}{\partial\bar{\alpha}}\frac{\partial^2}{\partial\alpha^2}\{\langle\alpha|\alpha\rangle\} = \bar{\alpha}(2+|\alpha|^2)e^{|\alpha|^2}. \tag{5.8}$$

It is evident from these inner products that ${}_b\langle\alpha|\widetilde{\alpha}\rangle_s \neq 0$, so $|\alpha\rangle_b$ and $|\widetilde{\alpha}\rangle_s$ are not orthogonal. However, we can define a new state $|\alpha\rangle_s$ as a linear combination of the states in (5.4)

$$|\alpha\rangle_s \equiv \frac{\bar{\alpha}}{\sqrt{2}}|\alpha\rangle_b + \frac{1}{\sqrt{2}}|\widetilde{\alpha}\rangle_s = \frac{1}{\sqrt{2}}\begin{pmatrix}\bar{\alpha}|\alpha\rangle - |\alpha^{'}\rangle \\ |\alpha\rangle\end{pmatrix},$$

which is orthogonal to $|\alpha\rangle_b$. The states $|\alpha\rangle_b$ and $|\alpha\rangle_s$ are orthogonal, each with the same norm

$${}_b\langle\alpha|\alpha\rangle_b =_s \langle\alpha|\alpha\rangle_s = \langle\alpha|\alpha\rangle = e^{|\alpha|^2}. \tag{5.9}$$

The AZ-supersymmetric coherent state can then be written as a superposition of two orthogonal states $|\alpha\rangle_b$ and $|\alpha\rangle_s$,

$$|\alpha\rangle_{AZ} = \gamma|\alpha\rangle_b + \beta|\alpha\rangle_s\,, \tag{5.10}$$

where γ and β are complex coefficients, with

$$|\alpha\rangle_b \equiv \begin{pmatrix}|\alpha\rangle \\ 0\end{pmatrix}, \quad |\alpha\rangle_s \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}\bar{\alpha}|\alpha\rangle - |\alpha^{'}\rangle \\ |\alpha\rangle\end{pmatrix}.$$

The norm of this supersymmetric coherent state is then ${}_{AZ}\langle\alpha|\alpha\rangle_{AZ} = (|\gamma|^2 + |\beta|^2)e^{|\alpha|^2}$.

5.2. Uncertainty for Supersymmetric AZ-Coherent States

Equation (5.9) allows us to define the normalized orthogonal states in the following form

$$|A\rangle_b = e^{-\frac{|\alpha|^2}{2}}|\alpha\rangle_b = e^{-\frac{|\alpha|^2}{2}}\begin{pmatrix} |\alpha\rangle \\ 0 \end{pmatrix}, \tag{5.11}$$

$$|A\rangle_s = e^{-\frac{|\alpha|^2}{2}}|\alpha\rangle_s = \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{2}}\begin{pmatrix} \bar{\alpha}|\alpha\rangle - |\alpha^{'}\rangle \\ |\alpha\rangle \end{pmatrix}, \tag{5.12}$$

so that ${}_b\langle A|A\rangle_b = 1 = {}_s\langle A|A\rangle_s, {}_b\langle A|A\rangle_s = 0$. Now, we can introduce bosonic and fermionic eigenstates of coordinate and momentum operators

$$|x\rangle_b = |0\rangle_F \otimes |x\rangle_B = \begin{pmatrix} |x\rangle \\ 0 \end{pmatrix}, \quad |x\rangle_f = |1\rangle_F \otimes |x\rangle_B = \begin{pmatrix} 0 \\ |x\rangle \end{pmatrix} \tag{5.13}$$

and

$$|p\rangle_b = |0\rangle_F \otimes |p\rangle_B = \begin{pmatrix} |p\rangle \\ 0 \end{pmatrix}, \quad |p\rangle_f = |1\rangle_F \otimes |p\rangle_B = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix}. \tag{5.14}$$

Since the state $|A\rangle_b$ has only bosonic component $|\alpha\rangle$, the following relations are equivalent to the usual bosonic coherent state case,

$$\langle\hat{x}\rangle_b \equiv {}_b\langle A|\hat{x}|A\rangle_b = \sqrt{2}\,\mathrm{Re}(\alpha), \tag{5.15}$$

$$\langle\hat{p}\rangle_b \equiv {}_b\langle A|\hat{p}|A\rangle_b = \sqrt{2}\,\mathrm{Im}(\alpha), \tag{5.16}$$

$$\langle(\Delta\hat{x})^2\rangle_b \equiv \langle\hat{x}^2\rangle_b - \langle\hat{x}\rangle_b^2 = \frac{1}{2}, \tag{5.17}$$

$$\langle(\Delta\hat{p})^2\rangle_b \equiv \langle\hat{p}^2\rangle_b - \langle\hat{p}\rangle_b^2 = \frac{1}{2}, \tag{5.18}$$

and it corresponds to minimal uncertainity relation

$$(\Delta\hat{x})_b\,(\Delta\hat{p})_b = \frac{1}{2}. \tag{5.19}$$

Similar calculations for the state $|A\rangle_s$ yield

$$\langle\hat{x}\rangle_s \equiv {}_s\langle A|\hat{x}|A\rangle_s = \sqrt{2}\,\mathrm{Re}(\alpha), \tag{5.20}$$

$$\langle\hat{p}\rangle_s \equiv {}_s\langle A|\hat{p}|A\rangle_s = \sqrt{2}\,\mathrm{Im}(\alpha), \tag{5.21}$$

$$\langle\hat{x}^2\rangle_s \equiv {}_s\langle A|\hat{x}^2|A\rangle_s = 1 + 2(\mathrm{Re}(\alpha))^2, \tag{5.22}$$

$$\langle\hat{p}^2\rangle_s \equiv {}_s\langle A|\hat{p}^2|A\rangle_s = 1 + 2(\mathrm{Im}(\alpha))^2, \tag{5.23}$$

so that the uncertainty relation for $|A\rangle_s$ is

$$(\Delta\hat{x})_s\,(\Delta\hat{p})_s = 1. \tag{5.24}$$

Comparing the uncertainty relations (5.19) and (5.24) reveals that the state $|A\rangle_b$ is a minimal uncertainty state, making it the closest to classical behavior. In contrast, $|A\rangle_s$ does not minimize the uncertainty, indicating it is less classical than $|A\rangle_b$.

5.2.1. Supersymmetric Bloch Sphere

We can introduce a generic normalized supersymmetric coherent state $|\alpha\rangle_{AZ}$ in the form

$$|\alpha\rangle_{AZ} = c_b|A\rangle_b + c_s|A\rangle_s,$$

where the normalization condition is $|c_b|^2 + |c_s|^2 = 1$. This state can be parametrized, up to a global phase, as

$$c_b = \cos\frac{\theta}{2}, \quad c_s = e^{i\phi}\sin\frac{\theta}{2},$$

leading to the expression

$$|\alpha;\theta,\phi\rangle_{AZ} = \cos\frac{\theta}{2}|A\rangle_b + e^{i\phi}\sin\frac{\theta}{2}|A\rangle_s, \tag{5.25}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. This parametrization allows $|\alpha\rangle_{AZ}$ to be represented by points on a unit sphere, referred to as the supersymmetric Bloch sphere (Pashaev and Kocak, 2025, 14-18). In this framework, the bosonic state $|A\rangle_b$ maps to the north pole, and $|A\rangle_s$ to the south pole. The probabilities of collapsing to each component state in (5.25) are:

$$P_b = |_b\langle A|\alpha;\theta,\phi\rangle_{AZ}|^2 = \cos^2\frac{\theta}{2}, \quad P_s = |_s\langle A|\alpha;\theta,\phi\rangle_{AZ}|^2 = \sin^2\frac{\theta}{2}$$

and $P_b + P_s = 1$.

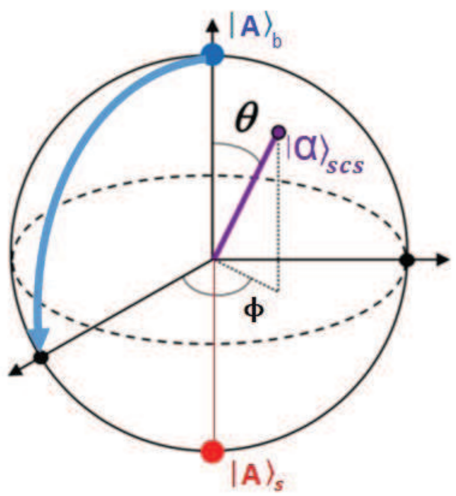

Figure 5.1. Supersymmetric Bloch Sphere

Figure (5.1) displays the supersymmetric Bloch sphere. Note that the state $|\alpha;\theta,\phi\rangle_{AZ}$ also depends on the complex parameter α, which defines the photon number $|\alpha|^2$ for the $|A\rangle_b$ state.

5.2.2. The uncertainty relations for Supersymmetric Bloch Sphere

The Supersymmetric Bloch Sphere provides a geometric framework for representing and analyzing uncertainty in supersymmetric quantum states. By mapping the combined bosonic and fermionic components onto the sphere's coordinates, (θ, ϕ), we can easily calculate average values and variations. Before moving to uncertainty calculations, we need to obtain the mean values of $\{\hat{x}, \hat{p}, \hat{x}^2, \hat{p}^2\}$ in the following form for the state(5.25),

$$\begin{aligned}
\langle \hat{x} \rangle_{AZ} &\equiv {}_{AZ}\langle \alpha; \theta, \phi | \hat{x} | \alpha; \theta, \phi \rangle_{AZ} = \sqrt{2}\,\mathrm{Re}(\alpha) - \frac{1}{2}\cos\phi \sin\theta, \\
\langle \hat{p} \rangle_{AZ} &\equiv {}_{AZ}\langle \alpha; \theta, \phi | \hat{p} | \alpha; \theta, \phi \rangle_{AZ} = \sqrt{2}\,\mathrm{Im}(\alpha) - \frac{1}{2}\sin\phi \sin\theta, \\
\langle \hat{x}^2 \rangle_{AZ} &= 1 + 2(\mathrm{Re}(\alpha))^2 - \frac{1}{2}\cos^2\frac{\theta}{2} - \sqrt{2}\,\mathrm{Re}(\alpha)\cos\phi\sin\theta, \\
\langle \hat{p}^2 \rangle_{AZ} &= 1 + 2(\mathrm{Im}(\alpha))^2 - \frac{1}{2}\cos^2\frac{\theta}{2} - \sqrt{2}\,\mathrm{Im}(\alpha)\sin\phi\sin\theta.
\end{aligned}$$

In the limiting case, $\theta = 0$ simplifies equation (5.15) for the $|A\rangle_b$ state, while the case $\theta = \pi$ yields (5.20) for the $|A\rangle_s$ state. For dispersions, we have

$$\begin{aligned}
(\Delta\hat{x})^2_{scs} &\equiv \langle \hat{x}^2 \rangle_{scs} - \langle \hat{x} \rangle^2_{scs} \equiv 1 - \frac{1}{2}\cos^2\frac{\theta}{2} - \frac{1}{4}\cos^2\phi\sin^2\theta, \\
(\Delta\hat{p})^2_{scs} &\equiv \langle \hat{p}^2 \rangle_{scs} - \langle \hat{p} \rangle^2_{scs} \equiv 1 - \frac{1}{2}\cos^2\frac{\theta}{2} - \frac{1}{4}\sin^2\phi\sin^2\theta.
\end{aligned}$$

The uncertainty relation for the $AZ-$supersymmetric coherent states can then be expressed as

$$(\Delta\hat{x})^2_{AZ}\,(\Delta\hat{p})^2_{AZ} \equiv \frac{1}{4}\left(1 + \sin^4\frac{\theta}{2} + 2\sin^6\frac{\theta}{2} + \sin^2 2\phi\sin^4\frac{\theta}{2}\cos^4\frac{\theta}{2}\right), \tag{5.26}$$

or equivalently,

$$(\Delta\hat{x})_{AZ}\,(\Delta\hat{p})_{AZ} \equiv \frac{1}{2}\sqrt{\left(1 + \sin^4\frac{\theta}{2} + 2\sin^6\frac{\theta}{2} + \sin^2 2\phi\sin^4\frac{\theta}{2}\cos^4\frac{\theta}{2}\right)}. \tag{5.27}$$

This relation is bounded between $\frac{1}{2}$ and 1, corresponding to the values for the states $|A\rangle_b$ and $|A\rangle_s$

$$\frac{1}{2} \leqslant (\Delta\hat{x})_{scs}\,(\Delta\hat{p})_{scs} \leqslant 1\,,$$

for $\theta = 0$ and $\theta = \pi$, respectively. For a given θ, small oscillations in the angle ϕ can be observed, as illustrated in Fig. (5.2).

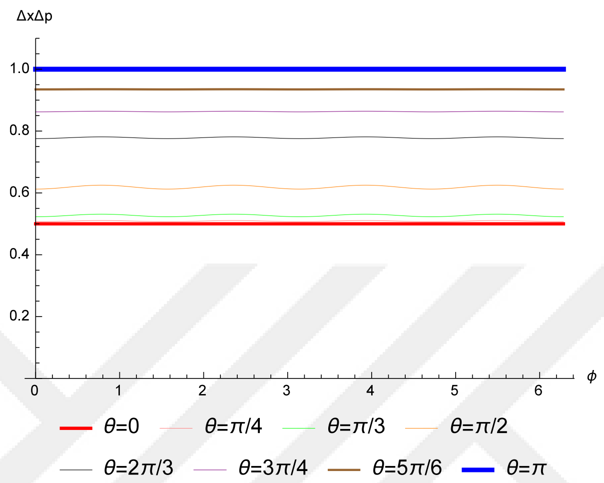

Figure 5.2. Uncertainity relation for supersymmetric coherent states on Bloch sphere

5.3. Coordinate and Momentum Representation for the Supersymmetric AZ-Coherent States

Now, we can find coordinate and momentum representation of AZ-supersymmetric coherent states. The coordinate $\hat{x}$ and momentum $\hat{p}$ operators for bosons are defined in usual form as

$$\hat{x} = \frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger)\,,\ \hat{p} = i\sqrt{\frac{\omega}{2}}(\hat{a}^\dagger - \hat{a})\,,$$

with corresponding eigenstates given by

$$\hat{x}|x\rangle = x|x\rangle \ \&\ \hat{p}|p\rangle = p|p\rangle.$$

To proceed, we use the representation for the supersymmetric bosonic state $|A\rangle_b$, which is written as

$$|A\rangle_b = e^{-\frac{|\alpha|^2}{2}} \begin{pmatrix} |\alpha\rangle \\ 0 \end{pmatrix} = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes |n\rangle.$$

This form allows us to compute the position and momentum wave functions by projecting $|A\rangle_b$ onto the position and momentum eigenstates and we use the position and momentum states in the bosonic representation, which is given in (5.13),

$$|x\rangle_b = |0\rangle_F \otimes |x\rangle_B = \begin{pmatrix} |x\rangle \\ 0 \end{pmatrix} \quad \& \quad |p\rangle_b = |0\rangle_F \otimes |p\rangle_B = \begin{pmatrix} |p\rangle \\ 0 \end{pmatrix}.$$

Using these definitions, we can compute the coordinate wave function $A_b(x)$ as follows

$${}_b\langle x|A\rangle_b = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \langle x|n\rangle.$$

Expanding this expression and using the form of the harmonic oscillator eigenstates in the position representation, we get

$$\begin{aligned} {}_b\langle x|A\rangle_b &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \langle x|n\rangle \\ &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{1}{\pi^{1/4}} \frac{e^{-\frac{x^2}{2}}}{2^{n/2}\sqrt{n!}} H_n(x) = \frac{1}{\pi^{1/4}} e^{-\frac{|\alpha|^2}{2}} e^{\frac{x^2}{2}} e^{-\left(x-\frac{\alpha}{\sqrt{2}}\right)^2}. \end{aligned} \tag{5.28}$$

Thus, we obtain the coordinate wave function

$$A_b(x) = {}_b\langle x|A\rangle_b = \frac{1}{\pi^{1/4}} e^{-\frac{|\alpha|^2}{2}} e^{\frac{x^2}{2}} e^{-\left(x-\frac{\alpha}{\sqrt{2}}\right)^2}.$$

In a Similar way, we calculate the momentum wave function $A_b(p)$ as

$$
\begin{aligned}
{}_b\langle p|A\rangle_b &= e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\langle p|n\rangle \\
&= e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{(-i)^n}{\pi^{1/4}}\frac{e^{-\frac{p^2}{2}}}{2^{n/2}\sqrt{n!}}H_n(p) = \frac{1}{\pi^{1/4}}e^{-\frac{|\alpha|^2}{2}}e^{\frac{p^2}{2}}e^{-\left(p+i\frac{\alpha}{\sqrt{2}}\right)^2}
\end{aligned} \tag{5.29}
$$

and this can be written as

$$
A_b(p) = {}_b\langle p|A\rangle_b = \frac{1}{\pi^{1/4}}e^{-\frac{|\alpha|^2}{2}}e^{\frac{p^2}{2}}e^{-\left(p+i\frac{\alpha}{\sqrt{2}}\right)^2}.
$$

These wave functions allow us to derive the probability distributions for position and momentum, similar to those found in a standard harmonic oscillator

$$
\begin{aligned}
|{}_b\langle x|A\rangle_b|^2 &= \frac{1}{\sqrt{\pi}}e^{-\left(x-\sqrt{2}\,\mathrm{Re}(\alpha)\right)^2}, \\
|{}_b\langle p|A\rangle_b|^2 &= \frac{1}{\sqrt{\pi}}e^{-\left(p-\sqrt{2}\,\mathrm{Im}(\alpha)\right)^2}.
\end{aligned} \tag{5.30}
$$

This result indicates that the probability distributions for AZ-supersymmetric coherent states follow Gaussian forms in both position and momentum spaces, with peaks centered around the real and imaginary parts of α, respectively. The state $|A\rangle_s$ is composed of both bosonic and fermionic components, which distinguishes it from purely bosonic states. Then, we can introduce coordinate representations for the bosonic and fermionic parts, defining them as eigenstates of the coordinate operator $\hat{x}$ and represent the bosonic and fermionic coordinate eigenstates as

$$
|x\rangle_b = |0\rangle_F \otimes |x\rangle_B = \begin{pmatrix} |x\rangle \\ 0 \end{pmatrix}, \quad |x\rangle_f = |1\rangle_F \otimes |x\rangle_B = \begin{pmatrix} 0 \\ |x\rangle \end{pmatrix}. \tag{5.31}
$$

These eigenstates, $|x\rangle_b$ for the bosonic part and $|x\rangle_f$ for the fermionic part, allow us to calculate the wave functions of the state $|A\rangle_s$ by taking inner products with these respective coordinate eigenstates. First, let us calculate the wave function for the bosonic component of the state $|A\rangle_s$. This is done by projecting $|A\rangle_s$ onto the bosonic coordinate eigenstate $|x\rangle_b$ as

$$\begin{aligned} {}_b\langle x|A\rangle_s &= e^{-\frac{|\alpha|^2}{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} \langle x| & 0 \end{pmatrix}\begin{pmatrix} \bar{\alpha}|\alpha\rangle - |\alpha'\rangle \\ |\alpha\rangle \end{pmatrix} \\ &= \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{2}}\left(\bar{\alpha}\langle x|\alpha\rangle - \langle x|\alpha'\rangle\right) = \frac{e^{-\frac{|\alpha|^2}{2}}}{\pi^{1/4}}e^{-\left(x-\frac{\alpha}{\sqrt{2}}\right)^2}e^{\frac{x^2}{2}}(\sqrt{2}\,\mathrm{Re}(\alpha) - x). \end{aligned} \tag{5.32}$$

We can also express this as

$${}_b\langle x|A\rangle_s = \frac{e^{-\frac{|\alpha|^2}{2}}}{\pi^{1/4}}e^{-\left(x-\frac{\alpha}{\sqrt{2}}\right)^2}e^{\frac{x^2}{2}}(\sqrt{2}\,\mathrm{Re}(\alpha) - x).$$

Next, we calculate the wave function for the fermionic component of $|A\rangle_s$ by projecting onto the fermionic coordinate eigenstate $|x\rangle_f$,

$$\begin{aligned} {}_f\langle x|A\rangle_s &= e^{-\frac{|\alpha|^2}{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \langle x| \end{pmatrix}\begin{pmatrix} \bar{\alpha}|\alpha\rangle - |\alpha'\rangle \\ |\alpha\rangle \end{pmatrix} \\ &= \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{2}}\langle x|\alpha\rangle = \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{2}\pi^{1/4}}e^{-\left(x-\frac{\alpha}{\sqrt{2}}\right)^2}e^{\frac{x^2}{2}}. \end{aligned} \tag{5.33}$$

Thus, we have the fermionic wave function

$${}_f\langle x|A\rangle_s = \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{2}\pi^{1/4}}e^{-\left(x-\frac{\alpha}{\sqrt{2}}\right)^2}e^{\frac{x^2}{2}}. \tag{5.34}$$

These wave functions allow us to compute the probability distributions for the bosonic and fermionic components of the $|A\rangle_s$ state. The corresponding probability distributions are as follows. For the fermionic component of $|A\rangle_s$, the probability distribution is given by

$$|{}_f\langle x|A\rangle_s|^2 = \frac{1}{2\sqrt{\pi}}e^{-\left(x-\sqrt{2}\,\mathrm{Re}(\alpha)\right)^2}. \tag{5.35}$$

For the bosonic component of $|A\rangle_s$, the probability distribution takes the form

$$|_b\langle x|A\rangle_s|^2 = \frac{1}{\sqrt{\pi}} e^{-\left(x-\sqrt{2}\,\mathrm{Re}(\alpha)\right)^2} (x - \sqrt{2}\,\mathrm{Re}(\alpha))^2. \tag{5.36}$$

Lastly, for the bosonic component of $|A\rangle_b$, the probability distribution is

$$|_b\langle x|A\rangle_b|^2 = \frac{1}{\sqrt{\pi}} e^{-\left(x-\sqrt{2}\,\mathrm{Re}(\alpha)\right)^2}. \tag{5.37}$$

These distributions characterize the probability densities with peaks centered at $\sqrt{2}\,\mathrm{Re}(\alpha)$, showing where each component is most likely to be found based on the real part of α. From these formulas, we observe that the probability distribution in (5.35) is half the value of that in (5.37). Additionally, for (5.36), the distribution has a zero at $x = \sqrt{2}\,\mathrm{Re}(\alpha)$, which is the center of the Gaussian distribution. This indicates that at $x = \sqrt{2}\,\mathrm{Re}(\alpha)$, the probability in (5.36) reaches a local minimum of zero.

For the momentum representation of $|A\rangle_s$, the momentum eigenstates are defined as

$$|p\rangle_b = \begin{pmatrix} |p\rangle \\ 0 \end{pmatrix} \quad \& \quad |p\rangle_f = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix}.$$

Using these eigenstates, we find the following expressions for the components of $|A\rangle_s$ in the momentum representation

$$_f\langle p|A\rangle_s = \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{2}} \langle p|\alpha\rangle = \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{2}\pi^{1/4}} e^{-\left(p+i\frac{\alpha}{\sqrt{2}}\right)^2} e^{\frac{p^2}{2}}, \tag{5.38}$$

for the fermionic component, and

$$_b\langle p|A\rangle_s = \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{2}} (\bar{\alpha}\langle p|\alpha\rangle - \langle p|\alpha'\rangle) = i\frac{e^{-\frac{|\alpha|^2}{2}}}{\pi^{1/4}} e^{-\left(p+i\frac{\alpha}{\sqrt{2}}\right)^2} e^{\frac{p^2}{2}} (p - \sqrt{2}\,\mathrm{Im}(\alpha)) \tag{5.39}$$

for the bosonic component. The corresponding probability distributions in the momentum representation are given by:

$$|_f\langle p|A\rangle_s|^2 = \frac{1}{2\sqrt{\pi}} e^{-\left(p-\sqrt{2}\,\mathrm{Im}(\alpha)\right)^2}, \tag{5.40}$$

$$|_b\langle p|A\rangle_s|^2 = \frac{1}{\sqrt{\pi}} e^{-\left(p-\sqrt{2}\,\mathrm{Im}(\alpha)\right)^2} (p - \sqrt{2}\,\mathrm{Im}(\alpha))^2. \tag{5.41}$$

These distributions in the momentum representation have the same form as those in (5.35) and (5.36) for the coordinate representation.

5.3.1. Coordinate and Momentum Representation for SuperBloch states

The coordinate representation of the $AZ-$supersymmetric coherent states (5.25) is defined by combining the bosonic and fermionic components. We start with the fermionic part of the coordinate representation:

$${}_f\langle x|\alpha;\theta,\phi\rangle_{AZ} = \cos\frac{\theta}{2}\, {}_f\langle x|A\rangle_b + e^{i\phi}\sin\frac{\theta}{2}\, {}_f\langle x|A\rangle_s,$$

which can be simplified to

$${}_f\langle x|\alpha;\theta,\phi\rangle_{AZ} = \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{2}\pi^{1/4}} e^{i\phi}\sin\frac{\theta}{2} e^{\frac{x^2}{2}} e^{-\left(x-\frac{\alpha}{\sqrt{2}}\right)^2}. \tag{5.42}$$

Similarly, for the bosonic component in the coordinate representation, we have

$${}_b\langle x|\alpha;\theta,\phi\rangle_{AZ} = \cos\frac{\theta}{2}\, {}_b\langle x|A\rangle_b + e^{i\phi}\sin\frac{\theta}{2}\, {}_b\langle x|A\rangle_s,$$

which expands to

$$ {}_b\langle x|\alpha;\theta,\phi\rangle_{AZ} = \frac{e^{-\frac{|\alpha|^2}{2}}}{\pi^{1/4}} e^{\frac{x^2}{2}} \left(\cos\frac{\theta}{2} + e^{i\phi}\sin\frac{\theta}{2}(\sqrt{2}\,\mathrm{Re}(\alpha) - x) \right) e^{-\left(x-\frac{\alpha}{\sqrt{2}}\right)^2}. \tag{5.43} $$

In the momentum space, the fermionic and bosonic components similarly based on the parameters θ and ϕ. For the fermionic component, we write

$$ {}_f\langle p|\alpha;\theta,\phi\rangle_{AZ} = \cos\frac{\theta}{2}\, {}_f\langle p|A\rangle_b + e^{i\phi}\sin\frac{\theta}{2}\, {}_f\langle p|A\rangle_s, $$

leading to

$$ {}_f\langle p|\alpha;\theta,\phi\rangle_{AZ} = \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{2}\pi^{1/4}} e^{i\phi}\sin\frac{\theta}{2} e^{\frac{p^2}{2}} e^{-\left(p+i\frac{\alpha}{\sqrt{2}}\right)^2}. \tag{5.44} $$

For the bosonic component in the momentum representation, we obtain

$$ {}_b\langle p|\alpha;\theta,\phi\rangle_{AZ} = \cos\frac{\theta}{2}\, {}_b\langle p|A\rangle_b + e^{i\phi}\sin\frac{\theta}{2}\, {}_b\langle p|A\rangle_s $$

which can be expanded as

$$ {}_b\langle p|\alpha;\theta,\phi\rangle_{AZ} = \frac{e^{-\frac{|\alpha|^2}{2}}}{\pi^{1/4}} e^{\frac{p^2}{2}} \left(\cos\frac{\theta}{2} - ie^{i\phi}\sin\frac{\theta}{2}(\sqrt{2}\,\mathrm{Im}(\alpha) - p) \right) e^{-\left(p+i\frac{\alpha}{\sqrt{2}}\right)^2}. \tag{5.45} $$

Setting $\alpha = \frac{x_0+ip_0}{\sqrt{2}}$, we derive the probability distributions for the state $|\alpha\rangle_{scs}$ in the coordinate representation as follows

$$ |{}_f\langle x|\alpha;\theta,\phi\rangle_{AZ}|^2 = \frac{e^{-(x-x_0)^2}}{2\sqrt{\pi}} \sin^2\frac{\theta}{2}, \tag{5.46} $$

$$ |{}_b\langle x|\alpha;\theta,\phi\rangle_{AZ}|^2 = \frac{e^{-(x-x_0)^2}}{\sqrt{\pi}} \Big[\cos^2\frac{\theta}{2} + (x-x_0)^2\sin^2\frac{\theta}{2} - (x-x_0)\sin\theta\cos\phi \Big], \tag{5.47} $$

and in the momentum representation as

$$|_f\langle p|\alpha;\theta,\phi\rangle_{AZ}|^2 = \frac{e^{-(p-p_0)^2}}{2\sqrt{\pi}}\sin^2\frac{\theta}{2}, \tag{5.48}$$

$$|_b\langle p|\alpha;\theta,\phi\rangle_{AZ}|^2 = \frac{e^{-(p-p_0)^2}}{\sqrt{\pi}}\Big[\cos^2\frac{\theta}{2} + (p-p_0)^2\sin^2\frac{\theta}{2} -(p-p_0)\sin\theta\sin\phi\Big]. \tag{5.49}$$

Analyzing these equations, we find that the probability distribution (5.47) reaches zero at the point

$$x = x_0 + \sqrt{2}\cot\left(\frac{\theta}{2}\right)\cos\left(\phi \mp \frac{\pi}{4}\right),$$

which acts as the center of the Gaussian distribution. This implies that at this central location, the probability attains a minimum value of zero.

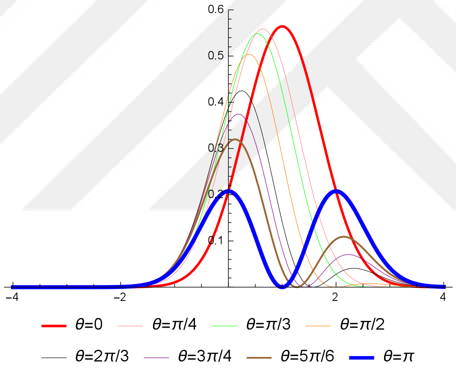

Figure 5.3. Probability for $|\alpha;\theta,\phi=0\rangle_{AZ}$ state when $\sqrt{2}\,\mathrm{Re}(\alpha) = 1$

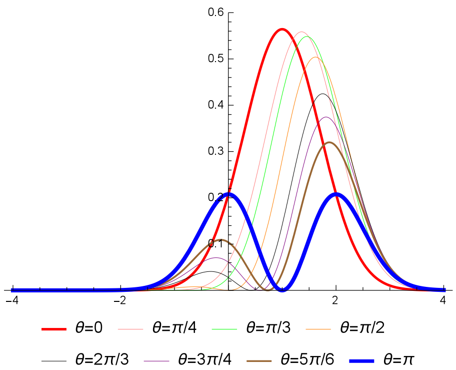

Figure 5.4. Probability for $|\alpha;\theta,\phi=\pi\rangle_{AZ}$ state when $\sqrt{2}\,\mathrm{Re}(\alpha) = 1$

In Fig.(5.3) and Fig.(5.4), the zeros are moving while θ is changing with corresponding $\phi = 0$ and $\phi = \pi$, respectively. For probabiliy distribution (5.46), we can see that there is no zero as in Fig.(5.5).

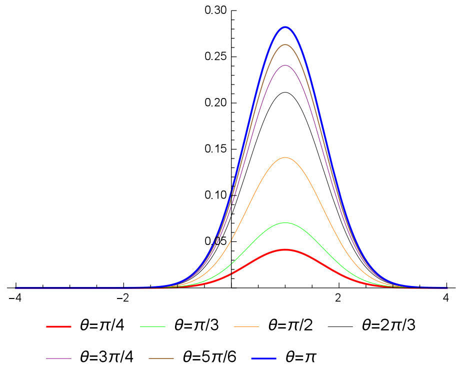

Figure 5.5. Probability for $|\alpha; \theta, \phi\rangle_{AZ}$ state when $\sqrt{2}\,\mathrm{Re}(\alpha) = 1$

CHAPTER 6

THE BELL BASED SUPER COHERENT STATES

6.1. Fermion-Boson States

Let f and $f^\dagger$ are fermionic annihilation and creation operators, $ff^\dagger + f^\dagger f = \mathrm{I}$. The eigenstates $|0\rangle_f$ and $|1\rangle_f$ of $N_f = f^\dagger f$, corresponding to fermionic numbers $n_0 = 0$ and $n_1 = 1$ we denote as the qubit basis states. Normalized linear combination of these states determines the qubit unit of quantum information

$$|\theta, \phi\rangle = \cos\frac{\theta}{2}|0\rangle_f + \sin\frac{\theta}{2}e^{i\phi}|1\rangle_f,$$

parametrized by points on the Bloch sphere S^2: $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$. To address fermionic and bosonic states, we first introduce the qubit-qudit state within the Hilbert space $H_f \otimes H_n$. In order to obtain the Fock space corresponding to bosonic states, we take the limit as $n \to \infty$. The qudit state is characterized by the computational basis vectors $|0\rangle, |1\rangle, \ldots, |n-1\rangle$. The general qubit-qudit state can be formulated as

$$|\Psi\rangle = \sum_{k=0}^{n-1} c_{0k}|0\rangle_f \otimes |k\rangle + \sum_{k=0}^{n-1} c_{1k}|1\rangle_f \otimes |k\rangle.$$

The state can be rewritten in two different forms. The first one

$$|\Psi\rangle = |0\rangle_f \otimes |\psi_0\rangle + |1\rangle_f \otimes |\psi_1\rangle = \begin{pmatrix} |\psi_0\rangle \\ |\psi_1\rangle \end{pmatrix},$$

represents it in terms of the pair of one qudit states

$$|\psi_0\rangle = \sum_{k=0}^{n-1} c_{0k}|k\rangle, \quad |\psi_1\rangle = \sum_{k=0}^{n-1} c_{1k}|k\rangle.$$

In the second one,

$$|\Psi\rangle = |\varphi_0\rangle_f \otimes |0\rangle + |\varphi_1\rangle_f \otimes |1\rangle + ... + |\varphi_{n-1}\rangle_f \otimes |n-1\rangle = \sum_{l=0}^{n-1} |\varphi_l\rangle_f \otimes |l\rangle$$

it is given by n, the one qubit states $|\varphi_l\rangle$, $l = 0, ..., n-1$, defined as

$$|\varphi_l\rangle = \begin{pmatrix} c_{0l} \\ c_{1l} \end{pmatrix} = c_{0l}|0\rangle_f + c_{1l}|1\rangle_f.$$

Now, we send dimension of the qudit state, $n \to \infty$, so that the space of states H_n becomes the Fock space H_b, and the computational basis of qudit states transforms to Fock number states $|k\rangle_\infty \equiv |k\rangle$, $k = 0, 1, 2, ...$ The fermionic-bosonic basis states are formed by tensor product of fermionic (qubit) states with Fock states, $|0\rangle \otimes |k\rangle$, and $|1\rangle \otimes |k\rangle$, $k = 0, 1, 2, ...$ and for arbitrary state

$$|\Psi\rangle = \sum_{k=0}^{\infty} c_{0k}|0\rangle_f \otimes |k\rangle_b + \sum_{k=0}^{\infty} c_{1k}|1\rangle_f \otimes |k\rangle_b, \tag{6.1}$$

from $H_f \otimes H_b$ Hilbert space, we have following two representations. The first one is

$$|\Psi\rangle = |0\rangle \otimes |\psi_0\rangle + |1\rangle \otimes |\psi_1\rangle = \begin{pmatrix} |\psi_0\rangle \\ |\psi_1\rangle \end{pmatrix}, \tag{6.2}$$

where two bosonic states

$$|\psi_0\rangle = \sum_{k=0}^{\infty} c_{0k}|k\rangle, \quad |\psi_1\rangle = \sum_{k=0}^{\infty} c_{1k}|k\rangle. \tag{6.3}$$

are vectors in the Fock space. Here and in some cases, where notations are clear by meaning, we skip bosonic and fermionic indices. The second representation

$$|\Psi\rangle = |\varphi_0\rangle \otimes |0\rangle + |\varphi_1\rangle \otimes |1\rangle + ... + |\varphi_n\rangle \otimes |n\rangle + ... = \sum_{n=0}^{\infty} |\varphi_n\rangle \otimes |n\rangle,$$

is determined by infinite set of qubits $|\varphi_n\rangle$, $n = 0, 1, 2, ...$, defined as

$$|\varphi_n\rangle = \begin{pmatrix} c_{0n} \\ c_{1n} \end{pmatrix} = c_{0n}|0\rangle + c_{1n}|1\rangle. \tag{6.4}$$

6.1.1. Entanglement of Fermion-Boson States

The fermionic-bosonic state from $H_f \otimes H_b$ is separable if $|\Psi\rangle = |\Phi\rangle_f \otimes |\Xi\rangle_b$, where $|\Phi\rangle_f$ is the one qubit or the fermionic state, and $|\Xi\rangle_b$ is bosonic state from the Fock space. If the state $|\Psi\rangle$ is not separable, then it is entangled.

Proposition 6.1 *The state (6.1) is separable if and only if in representation (6.2) two Fock states (6.3) are linearly dependent,* $|\psi_0\rangle = \lambda|\psi_1\rangle$.

If these states are linearly independent, the state (6.1) is entangled. To find the level of entanglement for the generic pure state (6.1), we calculate the reduced density matrices. For normalized state in (6.2) the density matrix is

$$\rho = |\Psi\rangle\langle\Phi| = \begin{pmatrix} |\psi_0\rangle\langle\psi_0| & |\psi_0\rangle\langle\psi_1| \\ |\psi_1\rangle\langle\psi_0| & |\psi_1\rangle\langle\psi_1| \end{pmatrix},$$

and due to normalization condition, $\sum_{n=0}^{\infty}(|c_{0n}|^2 + |c_{1n}|^2) = 1$,

$$tr\rho = \langle\psi_0|\psi_0\rangle + \langle\psi_1|\psi_1\rangle = 1. \tag{6.5}$$

For the reduced bosonic density matrix

$$\rho_b = tr_f\,\rho = |\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|$$

we obtain

$$tr\rho_b^2 = \langle\psi_0|\psi_0\rangle^2 + \langle\psi_1|\psi_1\rangle^2 + 2|\langle\psi_0|\psi_1\rangle|^2,$$

and for the fermionic one

$$\rho_f = tr_b\,\rho = \sum_{n=0}^{\infty} |\varphi_n\rangle\langle\varphi_n| \tag{6.6}$$

the expression is

$$tr\rho_f^2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |\langle\varphi_n|\varphi_m\rangle|^2.$$

As easy to check by direct computation, it coincides with the bosonic one, so that $tr\rho_b^2 = tr\rho_f^2$. The first one we rewrite in the form

$$tr\rho_b^2 = (\langle\psi_0|\psi_0\rangle + \langle\psi_1|\psi_1\rangle)^2 - 2(\langle\psi_0|\psi_0\rangle\langle\psi_1|\psi_1\rangle - \langle\psi_0|\psi_1\rangle\langle\psi_1|\psi_0\rangle)$$

and by taking into account the squared equation (6.5), we get

$$1 - tr\rho_b^2 = 2\begin{vmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_1\rangle \\ \langle\psi_1|\psi_0\rangle & \langle\psi_1|\psi_1\rangle \end{vmatrix}. \tag{6.7}$$

Deviation of the trace from unity gives a simplest characteristics of the level of entanglement. It is known as the linear entropy (Buscemi, 2007, 3-6), appearing in the linear approximation of the von Neumann entropy. In following, for this difference we introduce definition of the concurrence C in the determinant form, normalized as for the two qubit states (Parlakgorur and Pashaev, 2019, 2-3).

Definition 6.1 *The concurrence C of a pure fermion-boson state is defined by reduced* *density matrix ρ_f (or ρ_b) as the number*

$$C = \sqrt{2}\sqrt{1 - tr\rho_f^2},$$

satisfying

$$tr\rho_f^2 + \frac{1}{2}C^2 = 1. \tag{6.8}$$

From (6.8) and (6.7), we find the concurrence square as determinant of the Hermitian inner product metric $g_{ij} = \langle\psi_i|\psi_j\rangle$, (the Gram determinant), of two vectors ($i, j = 0, 1$) in Fock space,

$$C^2 = 4\begin{vmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_1\rangle \\ \langle\psi_1|\psi_0\rangle & \langle\psi_1|\psi_1\rangle \end{vmatrix},$$

and for the generic quantum state (6.1),

$$C = 2\sqrt{det\begin{pmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_1\rangle \\ \langle\psi_1|\psi_0\rangle & \langle\psi_1|\psi_1\rangle \end{pmatrix}}. \tag{6.9}$$

Due to relation

$$tr\rho_f^2 = 1 - \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\begin{vmatrix} \langle\varphi_n|\varphi_n\rangle & \langle\varphi_n|\varphi_m\rangle \\ \langle\varphi_m|\varphi_n\rangle & \langle\varphi_m|\varphi_m\rangle \end{vmatrix},$$

the concurrence can be represented also in another form

$$C^2 = 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\begin{vmatrix} \langle\varphi_n|\varphi_n\rangle & \langle\varphi_n|\varphi_m\rangle \\ \langle\varphi_m|\varphi_n\rangle & \langle\varphi_m|\varphi_m\rangle \end{vmatrix}.$$

By using explicit form of the one qubit states (6.4) it can be rewritten as an infinite sum of modulus squares of all 2×2 minors of the coefficient matrix c_{nm},

$$C^2 = 4 \sum_{0=n<m}^{\infty} \left\| \begin{matrix} c_{0n} & c_{0m} \\ c_{1n} & c_{1m} \end{matrix} \right\|^2 .$$

This provides us two equivalent expressions for the concurrence.

Proposition 6.2 *For generic normalized fermion-boson state (6.1) from Hilbert space $H_f \otimes H_b$, the concurrence is equal*

$$C = 2 \sqrt{det \begin{pmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_1\rangle \\ \langle\psi_1|\psi_0\rangle & \langle\psi_1|\psi_1\rangle \end{pmatrix}} = 2 \sqrt{\sum_{0=n<m}^{\infty} \left\| \begin{matrix} c_{0n} & c_{0m} \\ c_{1n} & c_{1m} \end{matrix} \right\|^2} . \tag{6.10}$$

Corollary 6.1 *The determinant of 2×2 inner product metric in Fock space can be represented by an infinite sum of modulus squares of minors of the infinite matrix from coefficients c_{nm} of the state (6.1),*

$$det \begin{pmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_1\rangle \\ \langle\psi_1|\psi_0\rangle & \langle\psi_1|\psi_1\rangle \end{pmatrix} = \sum_{0=n<m}^{\infty} \left\| \begin{matrix} c_{0n} & c_{0m} \\ c_{1n} & c_{1m} \end{matrix} \right\|^2 .$$

By using the definition and above expressions for the concurrence, now we calculate entanglement in fermion-boson system by the von Neumann entropy.

Proposition 6.3 *The entanglement, as the value of the von Neumann entropy*

$$E_f = -tr(\rho_f \log_2 \rho_f) \tag{6.11}$$

for ρ_f in (6.6) is

$$E_f = -\frac{1+\sqrt{1-C^2}}{2}\log_2\frac{1+\sqrt{1-C^2}}{2} - \frac{1-\sqrt{1-C^2}}{2}\log_2\frac{1-\sqrt{1-C^2}}{2} \tag{6.12}$$

where the concurrence C is given by (6.10). The value of concurrence is bounded between $0 \leq C \leq 1$.

Proof 6.1 *The characteristic equation for matrix ρ_f,*

$$\lambda^2 - \lambda + det\,\rho_f = 0$$

has two real eigenvalues

$$\lambda_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \det\rho_f},$$

where the determinant of ρ_f can be expressed by the concurrence as

$$det\,\rho_f = \sum_{0=n<m}^{\infty} \left\| \begin{matrix} c_{0n} & c_{0m} \\ c_{1n} & c_{1m} \end{matrix} \right\|^2 = \frac{1}{4}C^2$$

We note that for the fermion-boson system the entanglement E_f is function of C only, though the last one includes infinite sum of modulus squares of 2×2 minors.

6.2. Super-Number States

The supersymmetric(SUSY) harmonic oscillator is a composition of fermionic and bosonic harmonic oscillators with equal frequencies (Cooper et all, 2001, 7-30),

$$\mathrm{H} = \mathrm{H}_b + \mathrm{H}_f = \frac{\omega}{2}\left\{a, a^\dagger\right\} + \frac{\omega}{2}\left[f^\dagger, f\right] = \omega\mathcal{N}.$$

Here, the super-number operator $\mathcal{N}$

$$\mathcal{N} = I_f \otimes N + N_f \otimes I_b = \begin{pmatrix} N & 0 \\ 0 & N + I_b \end{pmatrix}, \tag{6.13}$$

has eigenstates $|n_b, n_f\rangle = |n_f\rangle \otimes |n_b\rangle$, where $n_b = 0, 1, 2, ...$ and $n_f = 0, 1$ are eigenvalues of bosonic and fermionuc number operators correspondingly, $N|n_b\rangle = n_b|n_b\rangle$, $N_f|n_f\rangle = n_f|n_f\rangle$. It counts the total number of fermions and bosons $n = n_b + n_f$ in state $|n_b, n_f\rangle$. The eigenstates $|0\rangle \otimes |n\rangle$, and $|1\rangle \otimes |n-1\rangle$ have the energy $E_n = n\omega, n > 0$ and $E_0 = 0$, for $n = 0$. This shows that fermionic and bosonic quanta have the same energy ω, and the states have the same number n of supersymmetric boson-fermion quanta (super-particles or super-quanta). The difference between states is the number of fermions, which is zero in the first case (pure bosonic state) and is one in the second case. Moreover, as was noticed first time in (Aragone and Zypman, 1986, 2271-2272), an arbitrary superposition of these two states is also state with n super-quanta, which after normalization can be written as the super-number state

$$|n, \theta, \phi\rangle = \cos\frac{\theta}{2}\begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\begin{pmatrix} 0 \\ |n-1\rangle \end{pmatrix} \quad . \tag{6.14}$$

This shows that the energy levels with n super-quanta $E_{n>0} = n\omega$ are double degenerate with arbitrary $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$. For $n = 0$ the state $|\Psi_0\rangle = |0\rangle_f \otimes |0\rangle_b$ is the ground state with $E_0 = 0$. The super-number state (6.14) contains n super-quanta, $\mathcal{N}|n, \theta, \phi\rangle = n|n, \theta, \phi\rangle$, in superposition of the zero fermionic state $|0\rangle_f \otimes |n\rangle$ and the one fermionic state $|1\rangle_f \otimes |n-1\rangle$. For this superposition, the probabilities do not depend on n and are equal

$$\langle n, \theta, \phi|P_0|n, \theta, \phi\rangle = \cos^2\frac{\theta}{2} \equiv p_0, \quad \langle n, \theta, \phi|P_1|n, \theta, \phi\rangle = \sin^2\frac{\theta}{2} \equiv p_1, \tag{6.15}$$

where projection operators are $P_0 = (|0\rangle\langle 0|) \otimes I_b$, and $P_1 = (|1\rangle\langle 1|) \otimes I_b$. This allows us to represent the super-number state (6.14) as a state on the Bloch type sphere, which is natural to call as the super-Bloch sphere, where the north pole of the sphere $\theta = 0$, corresponds to the zero fermion state and the south pole $\theta = \pi$ to the one fermion state. The states along the equator $\theta = \frac{\pi}{2}$ are in maximally random superposition of these states. The randomness of the state in given basis is determined by Shannon entropy, as advocated in paper (Deutsch, 1983, 631-633) and explored for geometric probabilities and quantum coins.

In this manner, we obtain a geometrical representation of the degeneracy of the n super-quanta state using the super-Bloch sphere.

6.2.1. Entanglement of Super-Number States

To evaluate level of entanglement between bosons and fermions in the super-number states (6.14), we use the reduced density matrix method. The density matrix for the pure state (6.14) is equal

$$\rho_n = |n,\theta,\phi\rangle\langle n,\theta,\phi| = \begin{pmatrix} \cos^2\frac{\theta}{2}\,|n\rangle\langle n| & \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi}|n\rangle\langle n-1| \\ \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi}|n-1\rangle\langle n| & \sin^2\frac{\theta}{2}\,|n-1\rangle\langle n-1| \end{pmatrix}$$

It satisfies $tr\rho_n = 1$, $tr\rho_n^2 = 1$. By taking partial trace of ρ_n according to fermionic states we get the reduced bosonic density matrix (See Appendix C.4)

$$\rho_b = tr_f\,\rho_n = \sin^2\frac{\theta}{2}\,|n-1\rangle\langle n-1| + \cos^2\frac{\theta}{2}\,|n\rangle\langle n|,$$

as an infinite dimensional matrix with only two nonzero diagonal terms, $\sin^2\frac{\theta}{2}$ and $\cos^2\frac{\theta}{2}$ at positions n and $n+1$, correspondingly. The partial trace according to bosonic states gives fermionic density matrix as 2×2 diagonal matrix

$$\rho_f = tr_b\,\rho = \cos^2\frac{\theta}{2}\,|0\rangle\langle 0| + \sin^2\frac{\theta}{2}\,|1\rangle\langle 1|.$$

For trace of the square of both reduced density matrices (See Appendix C.5), we get

$$tr\rho_f^2 = tr\rho_b^2 = 1 - \frac{1}{2}\sin^2\theta.$$

Then, by using formula (6.8) we obtain that the reduced bosonic, as well as fermionic state is mixed and the generic state $|n, \theta, \phi\rangle$ is entangled with concurrence

$$C = \sin\theta. \tag{6.16}$$

It is bounded $0 \leq C \leq 1$ and does not dependent on n. The north pole state $|n, \theta = 0, \phi\rangle$ (n-bosons state), and the south pole state $|n, \theta = \pi, \phi\rangle$ (n-1 bosons and one fermion state) are separable for any n, and correspond to $C = 0$. Contrary, the states along the equator on super-Bloch sphere, $|n, \theta = \frac{\pi}{2}, \phi\rangle$ with the concurrence $C = 1$ are maximally entangled states. The general form of these states is

$$|n, \frac{\pi}{2}, \phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |n\rangle + e^{i\phi}|1\rangle \otimes |n-1\rangle). \tag{6.17}$$

6.2.2. Fermion-Boson Bell States

For the case of $n = 1$, the maximally entangled states are

$$|L_\phi\rangle \equiv |1, \frac{\pi}{2}, \phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_f|1\rangle_b + e^{i\phi}|1\rangle_f|0\rangle_b),$$

giving the fermion-boson analog of the Bell states ($\phi = 0, \pi$),

$$|L_\pm\rangle \equiv |1, \frac{\pi}{2}, \pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_f|1\rangle_b \pm |1\rangle_f|0\rangle_b). \tag{6.18}$$

Definition 6.2 *The states with n-superparticles are defined as*

$$|n, \pm\rangle \equiv |n, \frac{\pi}{2}, \pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_f |n\rangle_b \pm |1\rangle_f |n-1\rangle_b), \tag{6.19}$$

where $n = 1, 2, ...$ For $n = 1$ the states become just the fermionic-bosonic Bell states $|L_\pm\rangle$ as in (6.18).

The infinite set of these states is maximally entangled, $C = 1$, for any positive integer n and satisfies orthonormality conditions

$$\langle m, \pm|n, \pm\rangle = \delta_{n,m}, \qquad \langle m, \mp|n, \pm\rangle = 0. \quad n, m = 1, 2, ... \tag{6.20}$$

In addition to the pair of Bell states (6.18) we introduce another pair of fermionic-bosonic Bell states

$$|B_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_f |0\rangle_b \pm |1\rangle_f |1\rangle_b). \tag{6.21}$$

The four Bell states (6.18),(6.21) are orthonormal

$$\langle L_+|L_+\rangle = \langle L_-|L_-\rangle = 1, \quad \langle L_+|L_-\rangle = 0, \tag{6.22}$$

$$\langle B_+|B_+\rangle = \langle B_-|B_-\rangle = 1, \quad \langle B_+|B_-\rangle = 0, \tag{6.23}$$

$$\langle B_\pm|L_\pm\rangle = \langle B_\pm|L_\mp\rangle = 0, \tag{6.24}$$

and represent maximally entangled complete set of basis states

$$|L_+\rangle\langle L_+| + |L_-\rangle\langle L_-| + |B_+\rangle\langle B_+| + |B_-\rangle\langle B_-| = I_f \otimes I_b.$$

It is noticed that in contrast with $|L_\pm\rangle$, the states (6.21) are not eigenstates of the supernumber operator. In fact, states $|L_\pm\rangle$ are exact eigenstates of $\mathcal{N}$ with one superparticle $n = 1$, $\mathcal{N}|L_\pm\rangle = |L_\pm\rangle$, while states $|B_\pm\rangle$ are not the eigenstates and only the average number of superparticles in these states is one, $\langle B_\pm|\mathcal{N}|B_\pm\rangle = 1$(See Appendix C.1).

6.2.3. The Bell based Super-qubit States

To generate coherent states, we use the displacement operator $\mathcal{D}(\alpha)$, as defined in (6.40). If this operator is acting on the vacuum ($n = 0$) state, $|\Psi_0\rangle = |0\rangle_f \otimes |0\rangle_b$, annihilated by operator $A_0 = I_f \otimes a$, so that, $A_0|\Psi_0\rangle = 0$, the corresponding coherent state as the eigenstate of this operator, would be separable. Another state, annihilated by this operator $|\Psi_1\rangle = |1\rangle_f \otimes |0\rangle_b$, is the one particle state with $n = 1$, and it is also separable. Moreover, any superposition of these two, the vacuum and one particle states, $\alpha(|0\rangle_f \otimes |0\rangle_b) + \beta(|1\rangle_f \otimes |0\rangle_b) = (\alpha|0\rangle_f + \beta|1\rangle_f) \otimes |0\rangle_b$ is separable. To create an entangled fermionic-bosonic coherent state, instead of this, we have to choose the reference state as the entangled state. To proceed in this direction, we first describe the maximally entangled states ($C = 1$) and then take superposition of these states with the separable ones ($C = 0$). This way we get entangled states, depending on the concurrence parameter C and implementing transition from separable to maximally entangled state. The natural choice for maximally entangled states is the set of four fermionic-bosonic Bell states (6.18), (6.21). Due to entanglement of bosons with fermions, these states are not annihilated by pure bosonic annihilation operator A_0 and require a mixture of bosonic and fermionic operators. In fact, for every Bell state we have its own annihilation operator, which in addition to bosonic annihilation operator a includes the fermionic annihilation or creation operators, f and $f^\dagger$. We define four operators

$$A_{\pm 1} = \begin{pmatrix} a & \pm 1 \\ 0 & a \end{pmatrix} = I_f \otimes a \pm f \otimes I_b, \tag{6.25}$$

$$A^T_{\pm 1} = \begin{pmatrix} a & 0 \\ \pm 1 & a \end{pmatrix} = I_f \otimes a \pm f^\dagger \otimes I_b, \tag{6.26}$$

annihilating the following Bell states (See Appendix C.2)

$$A_1|L_-\rangle = 0, \qquad A_{-1}|L_+\rangle = 0, \tag{6.27}$$

$$A_1^T|B_-\rangle = 0, \qquad A_{-1}^T|B_+\rangle = 0, \tag{6.28}$$

and acting as quantum gates, transforming the states to each other,

$$A_1|B_\pm\rangle = \pm|L_+\rangle, \qquad A_{-1}|B_\pm\rangle = \mp|L_-\rangle, \tag{6.29}$$

$$A_1^T|L_\pm\rangle = |B_+\rangle, \qquad A_{-1}^T|L_\pm\rangle = |B_-\rangle. \tag{6.30}$$

The annihilation operator A_{+1}, entangling bosons with fermions was first introduced in (Aragone and Zypman, 1986, 2268-2270). After recognition of their super-coherent state as one of the super-Bell states, and generalization of construction to four super-Bell states, we found four specific annihilation operators $A_{\pm1}, A_{\pm1}^T$, corresponding to every state.

The above supersymmetric annihilation operators include also creation operator $f^\dagger$. It should be not surprising, since action of this operator on one fermion state gives zero $f^\dagger|1\rangle_f = 0$. This is why, the set of the annihilation operators become richer and it is valid for any two level system or any qubit state.

The first pair of states $|L_\pm\rangle$ can be generated from the vacuum state $|\Psi_0\rangle$ and vice versa (See Appendix C.3)

$$|L_\pm\rangle = \frac{1}{\sqrt{2}}A_{\pm1}^\dagger|\Psi_0\rangle, \qquad |\Psi_0\rangle = \frac{1}{\sqrt{2}}A_{\pm1}|L_\pm\rangle,$$

and the second pair of states $|B_\pm\rangle$ from the one fermion state $|\Psi_1\rangle$ by

$$|B_\pm\rangle = \pm\frac{1}{\sqrt{2}}(A_{\pm1}^T)^\dagger|\Psi_1\rangle, \qquad |\Psi_1\rangle = \pm\frac{1}{\sqrt{2}}A_{\pm1}^T|B_\pm\rangle.$$

The vacuum state is annihilated by two operators

$$A_{\pm 1}|\Psi_0\rangle = 0,$$

and it is orthogonal to the pair of Bell states $|L_+\rangle$ and $L_-\rangle$. By taking superposition of the state with these Bell states we get two normalized reference states,

$$|0, C, \phi\rangle_{L_\pm} \quad = \quad \sqrt{1-C}|\Psi_0\rangle + \sqrt{C}e^{i\phi}|L_\pm\rangle, \tag{6.31}$$

which are annihilated by operators

$$A_{\mp 1}|0, C, \phi\rangle_{L_\pm} = 0. \tag{6.32}$$

The states are parametrized by real number C, bounded between $0 \leq C \leq 1$. It represents the concurrence, calculated from formula (6.10) and showing the level of fermion-boson entanglement in the reference state.

The parametrization allows us to give two physical interpretations of concurrence C. In the first one, it shows probability to measure the one superparticle state $|L_+\rangle$ or $|L_-\rangle$

$$C = \langle 0, C, \phi|P_1|0, C, \phi\rangle = p_1$$

in the superposition (6.31) of vacuum (zero superparticle state) and $|L_\pm\rangle$ (one superparticle state). The second meaning of C is the average value of supernumber operator in the superposition state

$$C =_{L_\pm} \langle 0, C, \phi|\mathcal{N}|0, C, \phi\rangle_{L_\pm}.$$

To calculate the second pair of reference states we notice that application of $f^\dagger$

operator on the vacuum state $|\Psi_0\rangle$ generates one fermion state

$$f^\dagger|\Psi_0\rangle = |\Psi_1\rangle = \begin{pmatrix} 0 \\ |0\rangle \end{pmatrix}, \tag{6.33}$$

annihilated by operators

$$A^T_{\mp 1}|\Psi_1\rangle = 0$$

and orthogonal to the second pair of Bell states $|B_+\rangle$ and $|B_-\rangle$. Superposition of the state with these Bell states gives another pair of reference states,

$$|0, C, \phi\rangle_{B_\pm} = \sqrt{1-C}|\Psi_1\rangle + \sqrt{C}e^{i\phi}|B_\pm\rangle, \tag{6.34}$$

which are annihilated by operators

$$A^T_{\mp 1}|0, C, \phi\rangle_{B_\pm} = 0. \tag{6.35}$$

As a result, we have constructed four, the Bell type reference states

$$|0, C, \phi\rangle_{L_\pm} = \sqrt{1-C}|\Psi_0\rangle + \sqrt{C}e^{i\phi}|L_\pm\rangle, \tag{6.36}$$

$$|0, C, \phi\rangle_{B_\pm} = \sqrt{1-C}|\Psi_1\rangle + \sqrt{C}e^{i\phi}|B_\pm\rangle, \tag{6.37}$$

with the inner products

$${}_{L_+}\langle 0, C, \phi|0, C, \phi\rangle_{L_-} = 1 - C, \quad {}_{B_+}\langle 0, C, \phi|0, C, \phi\rangle_{B_-} = 1 - C$$

and corresponding fidelity $F = (1-C)^2$, expressed in terms of the concurrence C. The reference states are characterized by real number C, bounded as $0 \le C \le 1$, and the angle $0 \le \phi \le 2\pi$. This is why geometrically, every state represents the point on surface of circular cylinder with radius one and height C. Another geometrical image of these states is associated with the unit disk $|z| \leq 1$ in complex plane $z = \sqrt{C}e^{i\phi}$. One more representation is given by points on unit sphere, parametrized by two angles $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, related to concurrence by $\sin\frac{\theta}{2} = \sqrt{C}$, $\cos\frac{\theta}{2} = \sqrt{1-C}$.

Definition 6.3 *The reference states*

$$|0, \theta, \phi\rangle_{L_\pm} = \cos\frac{\theta}{2}|\Psi_0\rangle + \sin\frac{\theta}{2}e^{i\phi}|L_\pm\rangle, \tag{6.38}$$

as superposition of zero super-particle state and one super-particle state are called the super-qubit states. Every state is represented by point on the unit sphere, which we call the super-Bloch sphere.

The north pole of the sphere corresponds to separable vacuum state, while the south pole to maximally entangled Bell state. Similarly to the usual qubit state, the north pole state $|\Psi_0\rangle \equiv |0\rangle_S$ is $n = 0$ superparticle state, $\mathcal{N}|0\rangle_S = 0\,|0\rangle_S$, and the south pole state $|L_\pm\rangle \equiv |1\rangle_S$ is $n = 1$ superparticle state, $\mathcal{N}|1\rangle_S = 1\,|1\rangle_S$. However, the state is fermion-boson entangled and the computational basis for this super-qubit state is made from $|0\rangle_S$ and $|1\rangle_S$ eigenstates of super-number operator $\mathcal{N}$.

The second pair of reference states is defined as

$$|0, \theta, \phi\rangle_{B_\pm} = \cos\frac{\theta}{2}|\Psi_1\rangle + \sin\frac{\theta}{2}e^{i\phi}|B_\pm\rangle, \tag{6.39}$$

but basis states are not eigenstates of $\mathcal{N}$ operator. In next section, by applying the displacement operator to these states, we generate four orthogonal super-coherent states.

6.3. The Bell based Supersymmetric Coherent States

To construct supersymmetric coherent state we follow the displacement operator approach. Specific form of displacement operator in $A_1^\dagger - A_1$ was explored in paper (Zypman, 2015, 1019-1025).

6.3.1. Displacement Operator

We introduce the bosonic displacement operator as the direct product

$$\mathcal{D}(\alpha) = \begin{pmatrix} D(\alpha) & 0 \\ 0 & D(\alpha) \end{pmatrix} = I_f \otimes D(\alpha) = I_f \otimes e^{\alpha a^\dagger - \bar{\alpha} a}, \tag{6.40}$$

satisfying unitarity condition $\mathcal{D}(\alpha)\mathcal{D}^\dagger(\alpha) = I$. Applying this operator to vacuum state $|\Psi_0\rangle$ and the one fermion state (6.33) we get corresponding supersymmetric coherent states

$$\mathcal{D}(\alpha)|\Psi_0\rangle = \begin{pmatrix} D(\alpha)|0\rangle \\ 0 \end{pmatrix} = \begin{pmatrix} |0,\alpha\rangle \\ 0 \end{pmatrix}, \tag{6.41}$$

$$\mathcal{D}(\alpha)|\Psi_1\rangle = \begin{pmatrix} 0 \\ D(\alpha)|0\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ |0,\alpha\rangle \end{pmatrix}. \tag{6.42}$$

The commutator

$$\mathcal{D}^\dagger(\alpha) A_0 \mathcal{D}(\alpha) = A_0 + \alpha I \qquad \rightarrow \qquad [A_0, \mathcal{D}(\alpha)] = \alpha \mathcal{D}(\alpha)$$

applied to state $|\Psi\rangle$

$$A_0(\mathcal{D}(\alpha)|\Psi\rangle) = \alpha(\mathcal{D}(\alpha)|\Psi\rangle) + \mathcal{D}(\alpha)A_0|\Psi\rangle,$$

gives the eigenvalue problem

$$A_0(\mathcal{D}(\alpha)|\Psi\rangle) = \alpha(\mathcal{D}(\alpha)|\Psi\rangle),$$

if the reference state $|\Psi\rangle$ is annihilated by operator A_0: $A_0|\Psi\rangle = 0$. Therefore, the coherent states, created from reference states $|\Psi_0\rangle$ and $|\Psi_1\rangle$ and their superposition $|\Psi\rangle = c_0|\Psi_0\rangle +$

$c_1|\Psi_1\rangle$, satisfy eigenvalue problem

$$A_0(c_0|0\rangle_f + c_1|1\rangle_f) \otimes |0,\alpha\rangle = \alpha\,(c_0|0\rangle_f + c_1|1\rangle_f) \otimes |0,\alpha\rangle, \tag{6.43}$$

and are separable. To create entangled super-coherent state we have to choose different reference state with entangled bosons and fermions. In present work we consider the set of maximally entangled four Bell reference states (6.18), (6.21), as super-qubit states.

Definition 6.4 *The Bell super-coherent states are defined as*

$$|\alpha, L_\pm\rangle \equiv \mathcal{D}(\alpha)|L_\pm\rangle, \quad |\alpha, B_\pm\rangle \equiv \mathcal{D}(\alpha)|B_\pm\rangle. \tag{6.44}$$

Proposition 6.4 *The Bell super-coherent states are eigenstates of corresponding super-symmetric annihilation operators*

$$A_1|\alpha, L_-\rangle = \alpha|\alpha, L_-\rangle, \quad A_{-1}|\alpha, L_+\rangle = \alpha|\alpha, L_+\rangle, \tag{6.45}$$

$$A_1^T|\alpha, B_-\rangle = \alpha|\alpha, B_-\rangle, \quad A_{-1}^T|\alpha, B_+\rangle = \alpha|\alpha, B_+\rangle. \tag{6.46}$$

The states are orthonormal and maximally entangled. In explicit form the states are expressed as

$$|\alpha, L_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_f|1,\alpha\rangle \pm |1\rangle_f|0,\alpha\rangle), \tag{6.47}$$

$$|\alpha, B_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_f|0,\alpha\rangle \pm |1\rangle_f|1,\alpha\rangle), \tag{6.48}$$

in terms of the displaced Fock states(See Appendix C.3)

$$|0,\alpha\rangle = D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2}|\alpha\rangle,$$

$$|1,\alpha\rangle = D(\alpha)|1\rangle = e^{-\frac{1}{2}|\alpha|^2}(\frac{d}{d\alpha}|\alpha\rangle - \bar{\alpha}|\alpha\rangle).$$

Here $|\alpha\rangle$ is the Glauber coherent state (not normalized).

The proof is given in Appendix C.11. The linear combination of these, maximally entangled states with orthogonal separable states produces the set of four supercoherent states. These states are created by displacement operator, acting on super-qubit reference states.

Proposition 6.5 *The states (6.31) annihilated by A_1 and A_{-1} operators correspondingly, as in (6.32), determine the pair of super-coherent states*

$$|\alpha, C, \phi\rangle_{L_\pm} \equiv \mathcal{D}(\alpha)|0, C, \phi\rangle_{L_\pm},$$

which are eigenstates of super annihilation operators

$$A_1|\alpha, C, \phi\rangle_{L_-} = \alpha|\alpha, C, \phi\rangle_{L_-}, \quad A_{-1}|\alpha, C, \phi\rangle_{L_+} = \alpha|\alpha, C, \phi\rangle_{L_+}$$

Proposition 6.6 *The pair of reference states (6.34), annihilated by Eq. (6.35), gives the pair of super-coherent states*

$$|\alpha, C, \phi\rangle_{B_\pm} \equiv \mathcal{D}(\alpha)|0, C, \phi\rangle_{B_\pm},$$

which are eigenstates of operators

$$A_1^T|\alpha, C, \phi\rangle_{B_-} = \alpha|\alpha, C, \phi\rangle_{B_-}, \quad A_{-1}^T|\alpha, C, \phi\rangle_{B_+} = \alpha|\alpha, C, \phi\rangle_{B_+}$$

Then we have following definition.

Definition 6.5 *The super-coherent states as displaced super-qubit states*

$$|\alpha, C, \phi\rangle_{L_-} = \sqrt{1-C}|0\rangle_f \otimes |0,\alpha\rangle + \sqrt{C}e^{i\phi}|\alpha, L_-\rangle, \quad (6.49)$$

$$|\alpha, C, \phi\rangle_{L_+} = \sqrt{1-C}|0\rangle_f \otimes |0,\alpha\rangle + \sqrt{C}e^{i\phi}|\alpha, L_+\rangle, \quad (6.50)$$

$$|\alpha, C, \phi\rangle_{B_-} = \sqrt{1-C}|1\rangle_f \otimes |0,\alpha\rangle + \sqrt{C}e^{i\phi}|\alpha, B_-\rangle, \quad (6.51)$$

$$|\alpha, C, \phi\rangle_{B_+} = \sqrt{1-C}|1\rangle_f \otimes |0,\alpha\rangle + \sqrt{C}e^{i\phi}|\alpha, B_+\rangle. \quad (6.52)$$

We call as the super-Bell based states.

On the super-Bloch sphere these states take form

$$|\alpha, \theta, \phi\rangle_{L_\mp} = \cos\frac{\theta}{2}\begin{pmatrix} |0,\alpha\rangle \\ 0 \end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\frac{1}{\sqrt{2}}\begin{pmatrix} |1,\alpha\rangle \\ \mp|0,\alpha\rangle \end{pmatrix}, \quad (6.53)$$

$$|\alpha, \theta, \phi\rangle_{B_\mp} = \cos\frac{\theta}{2}\begin{pmatrix} 0 \\ |0,\alpha\rangle \end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\frac{1}{\sqrt{2}}\begin{pmatrix} |0,\alpha\rangle \\ \mp|1,\alpha\rangle \end{pmatrix}, \quad (6.54)$$

or explicitly

$$|\alpha, \theta, \phi\rangle_{L_\mp} = \cos\frac{\theta}{2}e^{-\frac{|\alpha|^2}{2}}\begin{pmatrix} |\alpha\rangle \\ 0 \end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{2}}\begin{pmatrix} |\alpha\rangle' - \bar{\alpha}|\alpha\rangle \\ \mp|\alpha\rangle \end{pmatrix},$$

$$|\alpha, \theta, \phi\rangle_{B_\mp} = \cos\frac{\theta}{2}e^{-\frac{|\alpha|^2}{2}}\begin{pmatrix} 0 \\ |\alpha\rangle \end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{2}}\begin{pmatrix} |\alpha\rangle \\ \mp|\alpha\rangle' \pm \bar{\alpha}|\alpha\rangle \end{pmatrix}.$$

The states are eigenstates of super-annihilation operators

$$A_{\pm 1}|\alpha, \theta, \phi\rangle_{L_\mp} = \alpha|\alpha, \theta, \phi\rangle_{L_\mp},$$

$$A^T_{\pm 1}|\alpha, \theta, \phi\rangle_{B_\mp} = \alpha|\alpha, \theta, \phi\rangle_{B_\mp},$$

with inner products

$$_{L_+}\langle\alpha,\theta,\phi|\alpha,\theta,\phi\rangle_{L_-} = \cos^2\frac{\theta}{2} = {}_{B_+}\langle\alpha,\theta,\phi|\alpha,\theta,\phi\rangle_{B_-}.$$

It is noted that supercoherent state $|\alpha,\theta,\phi\rangle_{L_-}$, for angle $\phi' = \phi + \pi$ coincides with the one, derived early in (Aragone and Zypman, 1986, 2272-2274)).

6.4. Entanglement of Supercoherent States

In this section, we will calculate the entanglement of the supercoherent states. The first step is to determine the concurrence for the reference states (6.36) and (6.37). Then we show that concurrence is independent of action of the displacement operator on the states, and as follows it is independent of α. Consequently, we find that the concurrence of the super-qubit reference state is identical to that of the corresponding super-coherent state.

6.4.1. Entanglement of Super-qubit States

First,we start by computing the entanglement of the superqubit states (6.38). For these states

$$|0,\theta,\phi\rangle_{L_\pm} = |0\rangle_f \otimes \left(\cos\frac{\theta}{2}|0\rangle_b + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}e^{i\phi}|1\rangle_b\right) \pm |1\rangle_f \otimes \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}e^{i\phi}|0\rangle_b$$

the reduced density matrices are expressed by the same form, but in fermionic $|0\rangle_f, |1\rangle_f$ (two-component) and bosonic $|0\rangle_b, |1\rangle_b$ (infinite component) states,

$$\rho_b = \rho_f = (\cos^2\frac{\theta}{2} + \frac{1}{2}\sin^2\frac{\theta}{2})|0\rangle\langle 0| + \frac{1}{2}\sin^2\frac{\theta}{2}|1\rangle\langle 1| + \frac{1}{2\sqrt{2}}\sin\theta(e^{-i\phi}|0\rangle\langle 1| + e^{i\phi}|1\rangle\langle 0|)$$

so that

$$tr\rho_b^2 = tr\rho_f^2 = 1 - \frac{1}{2}\sin^4\frac{\theta}{2}.$$

Comparing with (6.8), we obtain the concurrence for the reference states (6.38)

$$C = \sin^2\frac{\theta}{2}. \tag{6.55}$$

The result can be obtained also from general formula (6.9) by identification with reference states (6.36), written in terms of C,

$$|\psi_0\rangle = \sqrt{1-C}|0\rangle + \frac{1}{\sqrt{2}}\sqrt{C}e^{i\phi}|1\rangle, \qquad |\psi_1\rangle = \pm\frac{1}{\sqrt{2}}\sqrt{C}e^{i\phi}|0\rangle$$

so that

$$\langle\psi_0|\psi_0\rangle = 1 - \frac{1}{2}C, \qquad \langle\psi_1|\psi_1\rangle = \frac{1}{2}C,$$
$$\langle\psi_0|\psi_1\rangle = \overline{\langle\psi_1|\psi_0\rangle} = \pm\frac{1}{\sqrt{2}}\sqrt{C(1-C)}e^{i\phi}.$$

By calculating determinant (6.9), we obtain formula (6.55). The formula implies that on the super-Bloch sphere the concurrence is monotonically increasing function of θ, so that the minimal value $C = 0$ at the north pole ($\theta = 0$) corresponds to separable state $|\Psi_0\rangle$, while the maximally entangled state with $C = 1$ relates to the south pole ($\theta = \pi$). On the equator ($\theta = \frac{\pi}{2}$) concurrence of the states is equal $C = \frac{1}{2}$. Equation (6.55) justifies representation (6.36) of reference states by concurrence C and shows that entanglement is independent of angle ϕ.

The same results for concurrence we obtain in case of the second couple of reference states (6.37) or (6.39). Thus, we have following proposition.

Proposition 6.7 *The concurrence C, $0 \le C \le 1$, for four reference states (6.38) and (6.39) is equal*

$$C = \sin^2\frac{\theta}{2}.$$

The states can be parametrized by this concurrence as in (6.36) and (6.37).

The proof is given in Appendix C.12. In the following section, we demonstrate that the same formula for concurrence applies to the supersymmetric coherent states (6.53).

6.4.2. Entanglement for Displaced States

An arbitrary normalized state $|\Phi\rangle$ from $H_f \otimes H_b$

$$|\Phi\rangle = \sum_{i=0}^{1}\sum_{n=0}^{\infty} c_{in}|i\rangle_f \otimes |n\rangle,$$

where

$$\sum_{i=0}^{1}\sum_{n=0}^{\infty} |c_{in}|^2 = 1,$$

after application of the displacement operator $\mathcal{D}(\alpha)$ becomes

$$|\Phi,\alpha\rangle = \mathcal{D}(\alpha)|\Phi\rangle = \sum_{i=0}^{1}\sum_{n=0}^{\infty} c_{in}|i\rangle_f \otimes D(\alpha)|n\rangle = \sum_{i=0}^{1}\sum_{n=0}^{\infty} c_{in}|i\rangle_f \otimes |n,\alpha\rangle,$$

where $|n,\alpha\rangle = D(\alpha)|n\rangle$ are displaced Fock states. This can be rewritten in two forms according to following propositions.

Proposition 6.8 *For an arbitrary state from $H_f \otimes H_b$, represented as*

$$|\Phi\rangle = |0\rangle_f \otimes |\psi_0\rangle + |1\rangle_f \otimes |\psi_1\rangle$$

by two states in Fock space

$$|\psi_0\rangle = \sum_{n=0}^{\infty} c_{0n}|n\rangle, \qquad |\psi_1\rangle = \sum_{n=0}^{\infty} c_{1n}|n\rangle,$$

the displaced state is

$$|\Phi,\alpha\rangle = \mathcal{D}(\alpha)|\Phi\rangle = |0\rangle_f \otimes |\psi_0,\alpha\rangle + |1\rangle_f \otimes |\psi_1,\alpha\rangle$$

where

$$|\psi_0,\alpha\rangle = \sum_{n=0}^{\infty} c_{0n}|n,\alpha\rangle, \qquad |\psi_1,\alpha\rangle = \sum_{n=0}^{\infty} c_{1n}|n,\alpha\rangle,$$

and the displaced Fock states are $|n,\alpha\rangle = D(\alpha)|n\rangle$. The last states satisfy orthonormality conditions

$$\langle m,\alpha|n,\alpha\rangle = \langle m|D^\dagger(\alpha)D(\alpha)|n\rangle = \langle m|n\rangle = \delta_{mn}$$

and completeness relation

$$\sum_{n=0}^{\infty} |n,\alpha\rangle\langle n,\alpha| = D(\alpha)\sum_{n=0}^{\infty} |n\rangle\langle n|D^\dagger(\alpha) = D(\alpha)D^\dagger(\alpha) = I.$$

Proposition 6.9 *For arbitrary state from $H_f \otimes H_b$, represented by sum of infinite number of qubits*

$$|\Phi\rangle = \sum_{n=0}^{\infty} |\varphi_n\rangle \otimes |n\rangle = \sum_{n=0}^{\infty} \begin{pmatrix} c_{0n} \\ c_{1n} \end{pmatrix} \otimes |n\rangle$$

the displaced state is

$$|\Phi\rangle = \sum_{n=0}^{\infty} |\varphi_n\rangle \otimes |n,\alpha\rangle = \sum_{n=0}^{\infty} \begin{pmatrix} c_{0n} \\ c_{1n} \end{pmatrix} \otimes |n,\alpha\rangle.$$

As we have seen in (6.9) the concurrence of a state depends on inner products of two bosonic states. By calculating the inner product for the displaced states

$$|\psi_i,\alpha\rangle = D(\alpha)|\psi_i\rangle, \qquad i = 0,1,$$

we find that it is invariant under displacement operation and independent of α,

$$\langle\psi_i,\alpha|\psi_j,\alpha\rangle = \langle\psi_i|D^\dagger(\alpha)D(\alpha)|\psi_j\rangle = \langle\psi_i|\psi_j,\rangle.$$

This suggests that entanglement for generic state $|\Phi\rangle$ and the displaced one $|\Phi, \alpha\rangle = \mathcal{D}(\alpha)|\Phi\rangle$ is the same. Indeed, from density matrix for displaced state

$$\rho(\alpha) = |\Phi, \alpha\rangle\langle\Phi, \alpha| = \mathcal{D}(\alpha)|\Phi\rangle\langle\Phi|\mathcal{D}^\dagger(\alpha) = \mathcal{D}(\alpha)\rho\mathcal{D}^\dagger(\alpha)$$

we get reduced density matrix

$$\rho_b(\alpha) = |\psi_0, \alpha\rangle\langle\psi_0, \alpha| + |\psi_1, \alpha\rangle\langle\psi_1, \alpha| = D(\alpha)(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|)D^\dagger(\alpha),$$

so that $\rho_b(\alpha) = D(\alpha)\rho_b D^\dagger(\alpha)$ and $\rho_b^2(\alpha) = D(\alpha)\rho_b^2 D^\dagger(\alpha)$. By taking trace from both sides we find $tr\rho_b^2(\alpha) = tr\rho_b^2$. This shows that the concurrence $C^2 = 2(1-tr\rho_b^2)$ and entanglement for both states is the same and don't depends on complex parameter α. Therefore, we present following proposition.

Proposition 6.10 *The concurrences (entanglement) for state $|\Phi\rangle$ and the displaced state $|\Phi, \alpha\rangle = \mathcal{D}(\alpha)|\Phi\rangle$ are equal.*

Corollary 6.2 *For supersymmetric coherent states $|\alpha, C, \phi\rangle_{L_\mp}$, $|\alpha, C, \phi\rangle_{B_\mp}$, defined in (6.49)-(6.52), the concurrence is independent of α and is equal*

$$C = \sin^2\frac{\theta}{2}.$$

For these states, the concurrence $C = p_1$ coincides with the probability of transition to maximally entangled states and represents the geometric probability, as relative area of spherical cup on super-Bloch sphere $C = A_\theta/A$.

In Fig.(6.1), the concurrence C and the von Neumann entropy E are shown as functions of the angle θ on the super-Bloch sphere.

As an example, by using determinant formula (6.9) for Hermitian metric, with two states

$$|\psi_0, \alpha\rangle = \sqrt{1-C}|0, \alpha\rangle + \frac{1}{\sqrt{2}}\sqrt{C}e^{i\phi}|1, \alpha\rangle, \quad |\psi_1, \alpha\rangle = \pm\frac{1}{\sqrt{2}}\sqrt{C}e^{i\phi}|0, \alpha\rangle$$

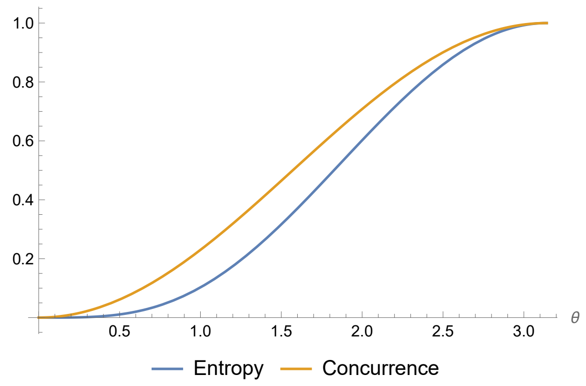

Figure 6.1. Concurrence and Entanglement versus angle θ on super-Bloch sphere

we find the same concurrence for states (6.49), (6.50). The second state is the Glauber coherent state, while the first one is superposition of Glauber state with the one photon added coherent state. The last one is adding non-classical property to the coherent state, and as we can see it is responsible for entanglement between fermions and bosons in supercoherent state.

6.4.3. Orthogonality of Super Coherent States

Here, we evaluate the inner product of two super-coherent states at the same position on the super-Bloch sphere (θ, ϕ) and show that, in contrast to Glauber coherent states, they can exhibit orthogonality. The product formulas for our displacement operators

$$\mathcal{D}(\alpha)\mathcal{D}(\beta) = e^{2i\,\mathrm{Im}(\alpha\bar{\beta})}\mathcal{D}(\beta)\mathcal{D}(\alpha) = e^{i\,\mathrm{Im}(\alpha\bar{\beta})}\mathcal{D}(\alpha+\beta)$$

give

$$\begin{aligned}\langle\beta, \theta, \phi|\alpha, \theta, \phi\rangle &= \langle 0, \theta, \phi|\mathcal{D}^{\dagger}(\beta)\mathcal{D}(\alpha)|0, \theta, \phi\rangle \\ &= e^{-i\,\mathrm{Im}(\beta\bar{\alpha})}\langle 0, \theta, \phi|\mathcal{D}(\alpha-\beta)|0, \theta, \phi\rangle \\ &= e^{-i\,\mathrm{Im}(\beta\bar{\alpha})}\langle 0, \theta, \phi|\alpha-\beta, \theta, \phi\rangle.\end{aligned}$$

By using matrix elements (See Appendix C.9)

$$\begin{aligned} \langle 0|D(\alpha)|0\rangle &= e^{-\frac{1}{2}|\alpha|^2}, \qquad \langle 1|D(\alpha)|0\rangle = \alpha e^{-\frac{1}{2}|\alpha|^2}, \\ \langle 0|D(\alpha)|1\rangle &= -\bar{\alpha} e^{-\frac{1}{2}|\alpha|^2}, \qquad \langle 1|D(\alpha)|1\rangle = (1-|\alpha|^2)\alpha e^{-\frac{1}{2}|\alpha|^2}, \end{aligned}$$

we find

$$_{L_\pm}\langle\beta,\theta,\phi|\alpha,\theta,\phi\rangle_{L_\pm} = e^{-i\,\mathrm{Im}(\beta\bar{\alpha})} e^{-\frac{1}{2}|\alpha-\beta|^2}\left(1 - \frac{\sin\theta}{2\sqrt{2}}((\bar{\alpha}-\bar{\beta})e^{i\phi} - (\alpha-\beta)e^{-i\phi}) - \frac{|\alpha-\beta|^2}{2}\sin^2\frac{\theta}{2}\right)$$

(See Appendix Defn.C.4). In the limiting case $\theta = 0$, (separable state at the north pole) we have the usual inner product formula for bosonic coherent states

$$\langle\beta,0,\phi|\alpha,0,\phi\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\beta|^2} e^{\bar{\beta}\alpha},$$

which is never zero. For another limit $\theta = \pi$, (maximally entangled state at the south pole) it becomes

$$_{L_\pm}\langle\beta,\pi,\phi|\alpha,\pi,\phi\rangle_{L_\pm} = \left(1 - \frac{1}{2}|\alpha-\beta|^2\right) e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\beta|^2} e^{\bar{\beta}\alpha}.$$

In contrast with pure bosonic coherent states, in this case the states can be orthogonal. The set of orthogonal maximally entangled states satisfies condition

$$|\alpha-\beta|^2 = 2,$$

and belongs to the circle in complex plane with radius $r = \sqrt{2}$ around point α. Then, every state on the circle, parametrized by $\beta = \alpha + \sqrt{2}\,e^{it}$, $0 \le t \le 2\pi$ is orthogonal to state α. From this set it is always possible to choose the pair of states β_1 and β_2 at distance $|\beta_1 - \beta_2| = \sqrt{2}$ and as a result, orthogonal to each other. So, we have three mutually orthogonal states $\alpha, \beta_1 = \alpha + \sqrt{2}e^{it_1}$ and $\beta_2 = \alpha + \sqrt{2}e^{i(t_1+\frac{\pi}{3})}$, located at vertices of equilateral triangle.

In general case of arbitrary states α and β, the orthogonality condition takes the complex form

$$\frac{1}{2}|w|^2 \sin^2\frac{\theta}{2} + \frac{1}{2\sqrt{2}}(w - \bar{w})\sin\theta - 1 = 0,$$

where $w \equiv (\alpha - \beta)e^{-i\phi}$, equivalent to the pair of real equations

$$\begin{aligned} (w - \bar{w})\sin\theta &= 0, \\ \frac{1}{2}|w|^2 \sin^2\frac{\theta}{2} &= 1. \end{aligned}$$

It has solutions for $\theta = \pi$, considered above. In addition, for arbitrary $0 < \theta < \pi$, such that $\sin\theta \neq 0$, and $w = \bar{w}$ is real, we have two solutions

$$w_{1,2} = \pm\frac{\sqrt{2}}{\sin\frac{\theta}{2}},$$

giving in terms of concurrence C,

$$\alpha - \beta = \pm\sqrt{\frac{2}{C}}\, e^{i\phi}.$$

This implies that for any state α exists two (antipodal) states β_+ and β_-, orthogonal to the state α,

$$\beta_+ = \alpha + \sqrt{\frac{2}{C}}\, e^{i\phi}, \qquad \beta_- = \alpha - \sqrt{\frac{2}{C}}\, e^{i\phi}.$$

These states exist for any level of entanglement $0 < C < 1$ and for separable states with $C = 0$ they move to infinity. For maximally entangled states with $C = 1$, in addition to this pair, appears the circle of states, orthogonal to state α. Similar calculations for second pair of states $|\alpha, \theta, \phi\rangle_{B_\pm}$ give the same conditions of orthogonality.

The above result relates entanglement of super-coherent states with orthogonality, so that to be orthogonal, the states should be necessarily entangled and non-classical.It should be noticed that orthogonality of super coherent states is related with orthogonality of the displaced Fock states $|n, \alpha\rangle$. The displaced Fock states were studied in many papers, and orthogonality property was emphasised in the paper (Baranov, 1991), (see also references in that paper). The super-coherent states in (97), (98) are spinors in displaced vacuum state $|0, \alpha\rangle$ and one particle state $|1, \alpha\rangle$, this is why these states are involved in orthogonality condition for super-coherent states. The level of involvement depends on concurrence, this is why the set of orthogonal super-coherent states depends on boson-fermion entanglement.

6.5. Time Evolution and Time Independence of Entanglement

The time evolution of coherent states is governed by the evolution operator, which describes how the states evolve over time in a given quantum system. By applying the evolution operator, we can explore how these states maintain certain properties, such as minimal uncertainty, or how they transform under different conditions. Time dependence of coherent states is determined by following evolution operator

$$\mathcal{U}(t) = e^{-i\omega Ht} = \begin{pmatrix} e^{-i\omega tN} & 0 \\ 0 & e^{-i\omega t(N+1)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\omega t} \end{pmatrix} \otimes e^{-i\omega tN}.$$

Proposition 6.11 *The concurrence for arbitrary time dependent state $|\Phi(t)\rangle = \mathcal{U}(t)|\Phi\rangle$ is independent of time $C(t) = C$.*

Proof 6.2 *For an arbitrary state (6.1), decomposed as*

$$|\Phi\rangle = |0\rangle_f \otimes |\psi_0\rangle + |1\rangle_f \otimes |\psi_1\rangle,$$

where

$$|\psi_0\rangle = \sum_{n=0}^{\infty} c_{0n}|n\rangle, \quad |\psi_1\rangle = \sum_{n=0}^{\infty} c_{1n}|n\rangle,$$

the time dependent state is

$$|\Phi(t)\rangle = \mathcal{U}(t)|\Phi\rangle = |0\rangle_f \otimes |\psi_0(t)\rangle + e^{-i\omega t}|1\rangle_f \otimes |\psi_1(t)\rangle,$$

where $(a = 0, 1)$,

$$|\psi_a(t)\rangle = e^{-i\omega t N}|\psi_a\rangle = \sum_{n=0}^{\infty} c_{an}(t)|n\rangle = \sum_{n=0}^{\infty} c_{an} e^{-i\omega t n}|n\rangle.$$

By calculating the inner products of these time dependent states and using (6.9), we have time independence of the concurrence

$$\begin{aligned} C(t) &= 2\sqrt{\left| det \begin{pmatrix} \langle\psi_0(t)|\psi_0(t)\rangle & \langle\psi_0(t)|\psi_1(t)\rangle e^{-i\omega t} \\ \langle\psi_1(t)|\psi_0(t)\rangle e^{i\omega t} & \langle\psi_1(t)|\psi_1(t)\rangle \end{pmatrix} \right|} \\ &= 2\sqrt{\left| det \begin{pmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_1\rangle e^{-i\omega t} \\ \langle\psi_1|\psi_0\rangle e^{i\omega t} & \langle\psi_1|\psi_1\rangle \end{pmatrix} \right|} = C. \end{aligned}$$

6.6. Uncertainty Relations and Entanglement on Super-Bloch Sphere

At this point, we determine the uncertainty relations for a quartet of supercoherent states $|\alpha, \theta$ and $\phi\rangle_{L_\pm}$ $|\alpha, \theta, \phi\rangle_{B_\pm}$. Calculations of averages for states $|\alpha, \theta, \phi\rangle_{L_\pm}$ and $|\alpha, \theta, \phi\rangle_{B_+}$ give the same results, this is the reason we omit the index of the states. The sign difference appearing for state $|\alpha, \theta, \phi\rangle_{B_-}$ would be noticed in proper place. The coordinate and momentum operators, given in fermionic-bosonic base $X = I_f \otimes \frac{1}{\sqrt{2}}(a + a^\dagger)$, $P = I_f \otimes \frac{i}{\sqrt{2}}(a^\dagger - a)$, transformed by displacement operator (6.40) to

$$\mathcal{D}^\dagger(\alpha) X \mathcal{D}(\alpha) = X + I_f \otimes \frac{\alpha + \bar{\alpha}}{\sqrt{2}} = X + I_f \otimes \sqrt{2}\,\mathrm{Re}\,\alpha,$$
$$\mathcal{D}^\dagger(\alpha) P \mathcal{D}(\alpha) = X + I_f \otimes i\frac{\bar{\alpha} - \alpha}{\sqrt{2}} = P + I_f \otimes \sqrt{2}\,\mathrm{Im}\,\alpha.$$

The mean values of the operators in supercoherent state

$$|\alpha, C, \phi\rangle = \sqrt{1-C}|\alpha, \Psi_0\rangle + \sqrt{C}e^{i\phi}|\alpha, L_\pm\rangle$$

reduce to the forms

$$\begin{aligned}
\langle\alpha, C, \phi|X|\alpha, C, \phi\rangle &= \langle 0, C, \phi|\mathcal{D}^\dagger(\alpha)X\mathcal{D}(\alpha)|0, C, \phi\rangle = \sqrt{2}\,\mathrm{Re}\,\alpha + \langle 0, C, \phi|X|0, C, \phi\rangle, \\
\langle\alpha, C, \phi|P|\alpha, C, \phi\rangle &= \langle 0, C, \phi|\mathcal{D}^\dagger(\alpha)P\mathcal{D}(\alpha)|0, C, \phi\rangle = \sqrt{2}\,\mathrm{Im}\,\alpha + \langle 0, C, \phi|P|0, C, \phi\rangle,
\end{aligned}$$

which include the mean values in the reference super-qubit state,

$$|0, C, \phi\rangle = \sqrt{1-C}\begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} + \sqrt{C}e^{i\phi}\frac{1}{\sqrt{2}}\begin{pmatrix} |1\rangle \\ \pm|0\rangle \end{pmatrix}.$$

For the final ones, we obtain

$$\begin{aligned}
\langle 0, C, \phi|X|0, C, \phi\rangle &= \sqrt{C(1-C)}\cos\phi, & (6.56) \\
\langle 0, C, \phi|P|0, C, \phi\rangle &= \sqrt{C(1-C)}\sin\phi, & (6.57)
\end{aligned}$$

which are valid for the first three states and including sign minus in the r.h.s. for the state $|\alpha, \theta, \phi\rangle_{B_-}$. The proof is given in Appendix C.13. Then, we have

$$\begin{aligned}
\langle\alpha, C, \phi|X|\alpha, C, \phi\rangle &= \sqrt{2}\,\mathrm{Re}\,\alpha + \sqrt{C(1-C)}\cos\phi, & (6.58) \\
\langle\alpha, C, \phi|P|\alpha, C, \phi\rangle &= \sqrt{2}\,\mathrm{Im}\,\alpha + \sqrt{C(1-C)}\sin\phi. & (6.59)
\end{aligned}$$

In Fig.(6.2), we display average $\bar{X} = \langle\alpha, C, \phi|X|\alpha, C, \phi\rangle$ and in Fig.(6.3), $\bar{P} = \langle\alpha, C, \phi|P|\alpha, C, \phi\rangle$ as functions of the concurrence C and angle ϕ, where $\alpha = (1+i)/\sqrt{2}$.

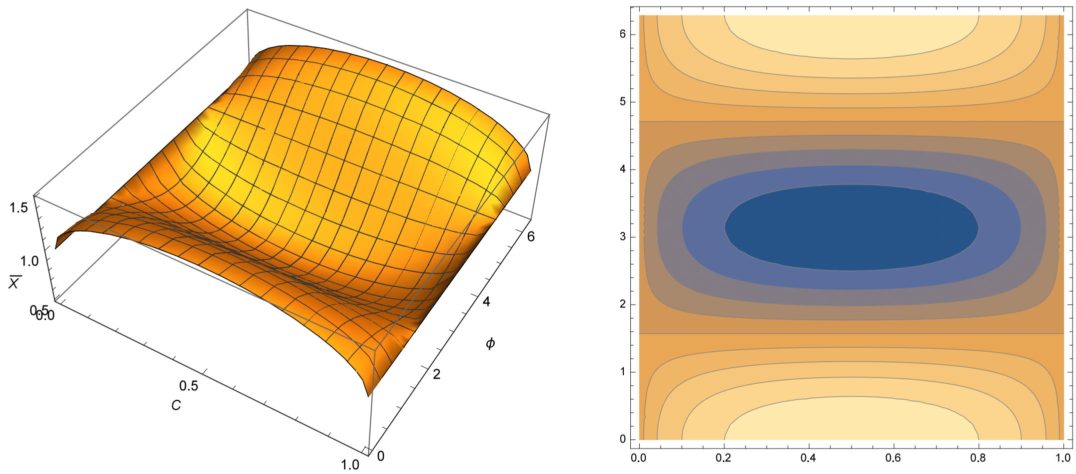

Figure 6.2. The average value $\bar{X}$ as function of C and ϕ, for $\alpha = (1+i)/\sqrt{2}$: a) 3D plot b) Contour Plot

The averages of a and $a^\dagger$ operators are

$$\begin{aligned} \langle 0, C, \phi | I_f \otimes a | 0, C, \phi \rangle &= \sqrt{\frac{C(1-C)}{2}}\, e^{i\phi}, \\ \langle 0, C, \phi | I_f \otimes a^\dagger | 0, C, \phi \rangle &= \sqrt{\frac{C(1-C)}{2}}\, e^{-i\phi}. \end{aligned}$$

and

$$\begin{aligned} \langle \alpha, C, \phi | I_f \otimes a | \alpha, C, \phi \rangle &= \alpha + \sqrt{\frac{C(1-C)}{2}} e^{i\phi}, \\ \langle \alpha, C, \phi | I_f \otimes a^\dagger | \alpha, C, \phi \rangle &= \bar{\alpha} + \sqrt{\frac{C(1-C)}{2}} e^{-i\phi}. \end{aligned}$$

For the states with vanishing average values

$$\langle \alpha, C, \phi | X | \alpha, C, \phi \rangle = 0, \qquad \langle \alpha, C, \phi | P | \alpha, C, \phi \rangle = 0$$

this gives

$$\alpha = -\sqrt{\frac{C(1-C)}{2}} e^{i\phi}$$

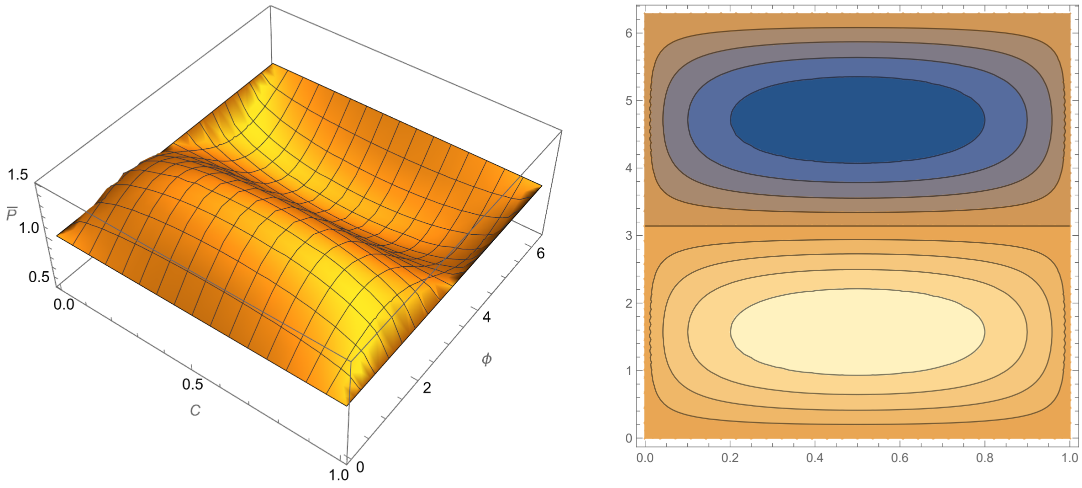

Figure 6.3. The average value $\bar{P}$ as function of C and ϕ for $\alpha = (1+i)/\sqrt{2}$: a) 3D plot b) Contour Plot

so that $|\alpha|^2 = C(1-C)/2$. Equations (6.58), (6.59) show that for states with $C = 0$ and $C = 1$, the average position and momentum are the same as for the bosonic coherent states. It deviates from classical averages, when $0 < C < 1$ and the difference reaches maximal value for $C = \frac{1}{2}$, corresponding to states on the equator of the super-Bloch sphere

$$|\alpha, \frac{1}{2}, \phi\rangle = \frac{1}{\sqrt{2}}(|\alpha, \Psi_0\rangle + e^{i\phi}|\alpha, L_\pm\rangle). \tag{6.60}$$

To calculate the average of X^2 and P^2, we use

$$\begin{aligned}
\langle\alpha, C, \phi|I_f \otimes a^2|\alpha, C, \phi\rangle &= \langle 0, C, \phi|\mathcal{D}^\dagger(\alpha) I_f \otimes a^2 \mathcal{D}(\alpha)|0, C, \phi\rangle \\
&= \langle 0, C, \phi|I_f \otimes (a+\alpha)^2|0, C, \phi\rangle, \\
\langle\alpha, C, \phi|I_f \otimes {a^\dagger}^2|\alpha, C, \phi\rangle &= \langle 0, C, \phi|\mathcal{D}^\dagger(\alpha) I_f \otimes {a^\dagger}^2 \mathcal{D}(\alpha)|0, C, \phi\rangle \\
&= \langle 0, C, \phi|I_f \otimes (a^\dagger + \bar{\alpha})^2|0, C, \phi\rangle,
\end{aligned} \tag{6.61}$$

and

$$\langle 0, C, \phi | I_f \otimes a^\dagger a | 0, C, \phi \rangle = \frac{1}{2} C,$$
$$\langle 0, C, \phi | I_f \otimes a a^\dagger | 0, C, \phi \rangle = \frac{1}{2} C + 1,$$

so that

$$\langle \alpha, C, \phi | X^2 | \alpha, C, \phi \rangle = \frac{1}{2}[(\alpha + \bar{\alpha})^2 + 2\sqrt{2}(\alpha + \bar{\alpha})\sqrt{C(1-C)}\cos\phi + 1 + C], \quad (6.62)$$
$$\langle \alpha, C, \phi | P^2 | \alpha, C, \phi \rangle = \frac{1}{2}[-(\alpha - \bar{\alpha})^2 - 2\sqrt{2}i(\alpha - \bar{\alpha})\sqrt{C(1-C)}\sin\phi + 1 + C] \quad (6.63)$$

(See Appendix C.15). By calculating the dispersions, we derive the following theorem.

Theorem 6.1 *Dispersions of coordinate X and momentum P in all super-coherent states* $|\alpha, C, \phi\rangle_{L_\pm}$ *and* $|\alpha, C, \phi\rangle_{B_\pm}$ *are the same and equal*

$$(\Delta X)^2_\alpha \equiv \langle X^2 \rangle_\alpha - \langle X \rangle^2_\alpha = \frac{1}{2}(1 + C) - C(1 - C)\cos^2\phi, \quad (6.64)$$
$$(\Delta P)^2_\alpha \equiv \langle P^2 \rangle_\alpha - \langle P \rangle^2_\alpha = \frac{1}{2}(1 + C) - C(1 - C)\sin^2\phi. \quad (6.65)$$

They do not depend of α, $(\Delta X)^2_\alpha = (\Delta X)^2_0$, *and* $(\Delta P)^2_\alpha = (\Delta P)^2_0$.

The proof is given in Appendix C.15. The dispersions satisfy "the Pythagoras theorem" in the phase plane

$$(\Delta X)^2 + (\Delta P)^2 = 1 + C^2$$

for the right triangle with sides, ΔX, ΔP and hypotenuse $\sqrt{1 + C^2}$. Then, the uncertainty relation $\Delta X \Delta P = A$ is given by the area of rectangle with diagonal $\sqrt{1 + C^2}$. The sides of the triangle are bounded between

$$\frac{1}{2}(1 - C + 2C^2) \leq (\Delta X)^2 \leq \frac{1}{2}(1 + C),$$
$$\frac{1}{2}(1 - C + 2C^2) \leq (\Delta P)^2 \leq \frac{1}{2}(1 + C).$$

The uncertainty relation for the supersymmetric coherent states are found as monotonically growing function of C,

$$\Delta X \Delta P = \frac{1}{2}\sqrt{1 + C^2 + 2C^3 + C^2(1 - C)^2 \sin^2 2\phi}, \tag{6.66}$$

with small periodic dependence on angle ϕ. It is shown in Fig.(6.4)

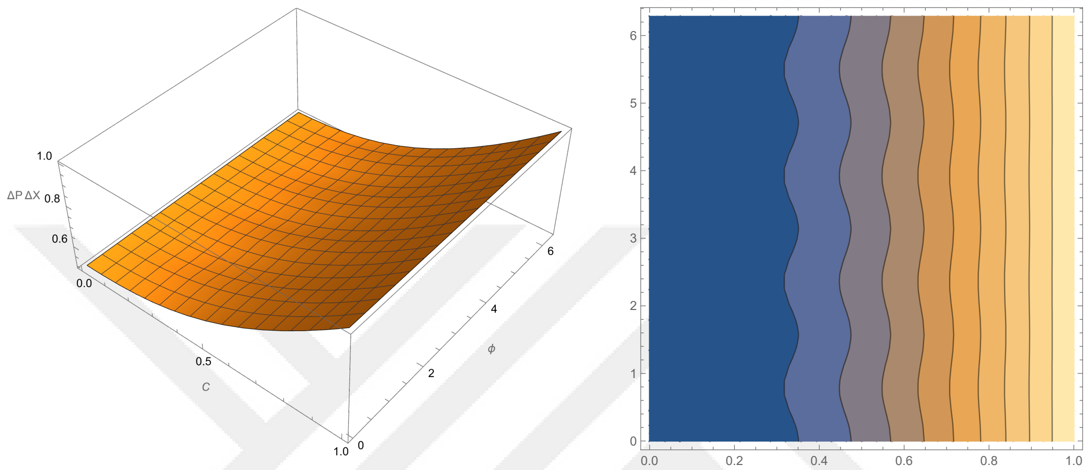

Figure 6.4. Uncertainty relation versus concurrence C and angle ϕ : a) 3D plot b) Contour Plot

This implies inequality

$$\frac{1}{2}\sqrt{1 + C^2 + 2C^3} \leq \Delta X \Delta P \leq \frac{1}{2}(1 + C^2),$$

with minimum value at $\phi = 0$ and the maximal one at $\phi = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. In the last case, the triangle becomes isosceles triangle, so that dispersions are equal,

$$(\Delta X)^2 = (\Delta P)^2 = \frac{1 + C^2}{2}$$

or

$$\Delta X = \Delta P = \sqrt{\frac{1 + C^2}{2}}.$$

The area of the square as the double area of the triangle is maximal for fixed C and gives uncertainty relation

$$\Delta X \Delta P = \frac{1 + C^2}{2}. \tag{6.67}$$

It is noted from (6.64) and (6.65), that similarly to the bosonic coherent states (Klauder, 1985), the dispersions are not dependent on α, but on the reference super-qubit state, corresponding to $\alpha = 0$, (which is not the vacuum state), so that $(\Delta X)^2_\alpha = (\Delta X)^2_0$, and $(\Delta P)^2_\alpha = (\Delta P)^2_0$.

The right hand side of equation (6.66) is monotonically growing function of C, bounded between $\frac{1}{2}$ and 1 ,

$$\frac{1}{2} \leq (\Delta X)(\Delta P) \leq 1.$$

The lower limit

$$(\Delta X)(\Delta P) = \frac{1}{2}$$

corresponds to $C = 0$ and the state $|\alpha, \Psi_0\rangle$, while the upper limit for $C = 1$,

$$(\Delta X)(\Delta P) = 1,$$

to the state $|\alpha, L_\pm\rangle$. Obtained relations show that for zero fermionic state the uncertainty reaches the minimal value, corresponding to pure bosonic coherent state as most classical quantum state and it is separable state with $C = 0$. Then, mixing bosonic and fermionic degrees, due to nonclassical nature of fermions, increases non-classicality of the states and corresponding uncertainty. It reaches maximal value for $C = 1$, which corresponds to maximally entangled bosonic and fermionic states as maximally non-classical states.

6.6.1. Quadratic Squeezing of Coordinate and Momentum Uncertainties

As we have seen from uncertainty relation (6.66), the product $(\Delta X \Delta P)^2$ reaches minimal value $\frac{1}{4}$ for $C = 0$. This suggests that minimal uncertainty as in pure bosonic case of Glauber coherent states, should corresponds to $(\Delta X)^2 = (\Delta P)^2 = \frac{1}{2}$. But, it is not the case. In fact, depending on value of ϕ, and C, $(\Delta X)^2$ reaches local minima, smaller than $\frac{1}{2}$. The uncertainty in X as functions of two variables

$$(\Delta X)^2\,(C,\phi) \equiv f(C,\phi) \;=\; \frac{1}{2}(1+C) - C(1-C)\cos^2\phi,$$

describes two dimensional surface. It is shown in Figure 5. For this surface, we have conditions for first derivatives

$$\begin{aligned} f_C(C,\phi) &= \frac{1}{2} + (2C-1)\cos^2\phi = 0, \\ f_\phi(C,\phi) &= C(1-C)\sin 2\phi = 0, \end{aligned}$$

giving two critical points

$$\phi = 0, \pi, \qquad C = \frac{1}{4}. \tag{6.68}$$

By using second derivatives

$$f_{CC} = 2\cos^2\phi, \quad f_{\phi\phi} = 2C(1-C)\cos 2\phi, \quad f_{C\phi} = f_{\phi C} = (1-2C)\sin 2\phi,$$

we calculate the Gaussian curvature as determinant of the Hessian

$$H = f_{CC} f_{\phi\phi} - f_{C\phi}^2 = 4C(1-C)\cos 2\phi\cos^2\phi - (1-2C)^2\sin^2 2\phi.$$

For the critical points (6.68) we get positive Gaussian curvature $H = \frac{3}{4}$ and due to $f_{CC} = 2 > 0$, the local minimum. The value of dispersions at these critical points is

$$(\Delta X)^2 = f\left(\frac{1}{4}, 0\right) = f\left(\frac{1}{4}, \pi\right) = \frac{7}{16} < \frac{8}{16} = \frac{1}{2}, \quad (\Delta P)^2 = \frac{5}{8} > \frac{4}{8} = \frac{1}{2}. \tag{6.69}$$

The boundary values of $f(C, \phi)$ at $\phi = 0, 2\pi$ and $C = 0, 1$ do not affect the minimum value. The inequalities show that in the super-coherent states

$$|\alpha, \frac{1}{4}, 0\rangle_{L_\pm} = \frac{\sqrt{3}}{2}|\alpha, \Psi_0\rangle + \frac{1}{2}|\alpha, L_\pm\rangle, \qquad |\alpha, \frac{1}{4}, \pi\rangle_{L_\pm} = \frac{\sqrt{3}}{2}|\alpha, \Psi_0\rangle - \frac{1}{2}|\alpha, L_\pm\rangle$$

the X dispersion is maximally squeezed to value $(\Delta X)^2 = \frac{7}{16} < \frac{1}{2}$, while $(\Delta P)^2 = \frac{5}{8} > \frac{1}{2}$. Positions of these states on the super-Bloch sphere are $(\theta = \frac{\pi}{3}, \phi = 0)$ and $(\theta = \frac{\pi}{3}, \phi = \pi)$ correspondingly, and in complex plane representation at $z = \pm\frac{1}{\sqrt{3}}$.

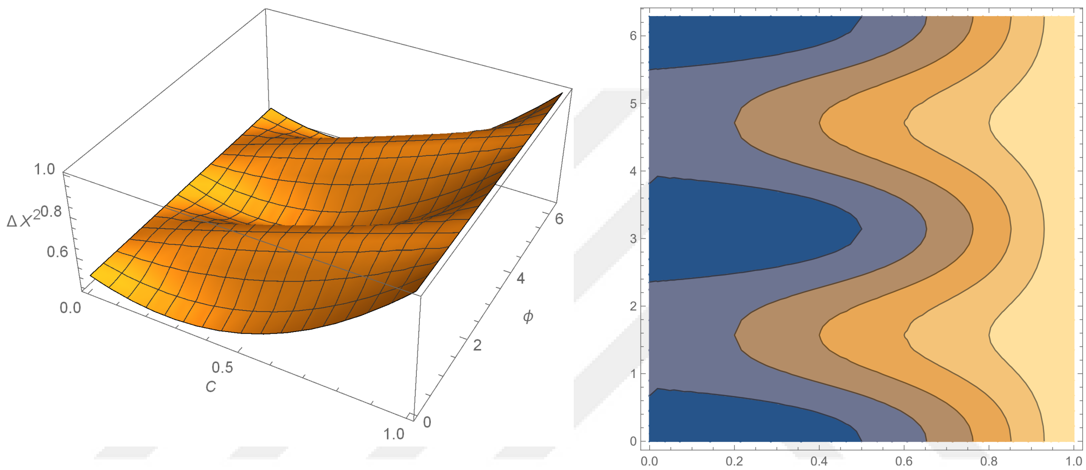

Figure 6.5. Quadrature squeezing for dispersion ΔX^2 versus concurrence C and angle ϕ : a) 3D plot b) Contour Plot

In Fig.(6.5), we plot dispersions versus the concurrence C at angles $\phi = 0$ and $\phi = \pi$. Similar calculations for states

$$|\alpha, \frac{1}{4}, \frac{\pi}{2}\rangle_{L_\pm} = \frac{\sqrt{3}}{2}|\alpha, \Psi_0\rangle + \frac{i}{2}|\alpha, L_\pm\rangle, \qquad |\alpha, \frac{1}{4}, \frac{3\pi}{2}\rangle_{L_\pm} = \frac{\sqrt{3}}{2}|\alpha, \Psi_0\rangle - \frac{i}{2}|\alpha, L_\pm\rangle,$$

at critical points on super-Bloch sphere $(\theta = \frac{\pi}{3}, \phi = \frac{\pi}{2})$, $(\theta = \frac{\pi}{3}, \phi = \frac{3\pi}{2})$ or in complex plane $z = \pm\frac{i}{\sqrt{3}}$, give maximal squeezing for the momentum dispersion, $(\Delta P)^2 = \frac{7}{16} < \frac{1}{2}$, $(\Delta X)^2 = \frac{5}{8} > \frac{1}{2}$. This quadrature squeezing is known for photon added coherent states, as non-classical property, and now we have established it also for boson-fermion entangled super-coherent states.

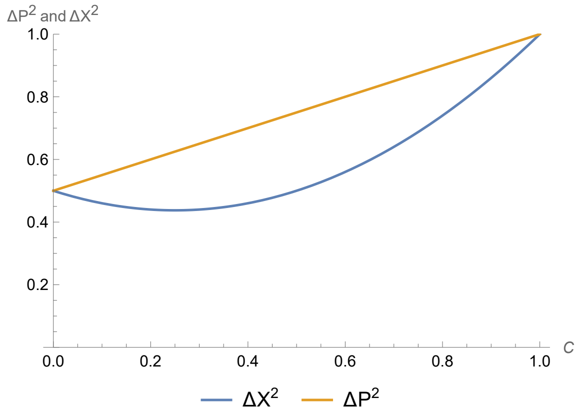

Figure 6.6. Plot of Quadrature squeezing for dispersion ΔX^2 and ΔP^2 versus concurrence for angle $\phi = 0, \pi$

6.6.2. Golden Uncertainty Relation and Fibonacci Numbers

In above calculations (6.69) we have seen that dispersion $(\Delta P)^2 = \frac{5}{8} = \frac{F_5}{F_6}$ is equal to the ratio of two Fibonacci numbers. Depending on angle ϕ, for $C = \frac{1}{4}$ we distinguish two cases: a) $\phi = 0, \pi$, then from (6.64), (6.65) we have $(\Delta X)^2 = \frac{7}{16}$, $(\Delta P)^2 = \frac{5}{8}$, and b) for $\phi = \frac{\pi}{2}$, it is $(\Delta X)^2 = \frac{5}{8}$, $(\Delta P)^2 = \frac{7}{16}$. In fact, we are going to show that the whole sequence of Fibonacci numbers and the Golden Ratio can be involved to uncertainty relations for super-coherent states. For maximally random states (6.60) with $C = \frac{1}{2}$, located on the equator of the super Bloch sphere, the uncertainty relation is

$$\Delta X \Delta P = \frac{1}{8}\sqrt{24 + \sin^2 2\phi}. \tag{6.70}$$

For angle $\phi = \frac{\pi}{4}$ it gives ratio of two Fibonacci numbers

$$\Delta X \Delta P = \frac{5}{8} = \frac{F_5}{F_6}. \tag{6.71}$$

In addition, we notice that the minimal uncertainty $\frac{1}{2} = \frac{F_2}{F_3}$ (corresponding to bosonic coherent states) and the maximum uncertainty $1 = \frac{F_1}{F_2}$ (for maximally entangled boson-fermion states) also represent the ratio of two Fibonacci numbers. In all these cases the uncertainty is equal to the ratio $\frac{F_n}{F_{n+1}}$, where $n = 1, 2, 5$. Now, we define the inverse ratio

$$\varphi_n = \frac{F_{n+1}}{F_n},$$

for arbitrary positive integer n, representing the Golden sequence, satisfying equation

$$\varphi_n = 1 + \frac{1}{\varphi_{n-1}},$$

and having the Golden ratio as the limit $\varphi_n \to \varphi$, when $n \to \infty$. This suggests that for supersymmetric coherent states exist the sequence of uncertainties, equal $1/\varphi_n$ for any positive n, giving in the limit $n \to \infty$ the Golden Ratio uncertainty

$$\Delta X \Delta P = \frac{1}{\varphi} = \frac{2}{1 + \sqrt{5}} = \frac{\sqrt{5} - 1}{2}. \tag{6.72}$$

To determine the Golden sequence for any n, we fix the angle $\phi = \frac{\pi}{4}$, then dispersions are equal $\Delta X_n = \Delta P_n$, and due to (6.67) they can be chosen as

$$(\Delta X_n)^2 = (\Delta P_n)^2 = \Delta X_n \Delta P_n = \frac{1}{\varphi_n} = \frac{F_n}{F_{n+1}} = \frac{1 + C_n^2}{2}. \tag{6.73}$$

This implies the infinite sequence of concurrences, determined by equation

$$C_n^2 + 1 = 2\frac{F_n}{F_{n+1}}.$$

Using properties of Fibonacci numbers, $F_{n+1} = F_n + F_{n-1}$, it can be simplified as

$$C_n^2 = \frac{F_{n-2}}{F_{n+1}},$$

so that

$$C_n = \sqrt{\frac{F_{n-2}}{F_{n+1}}}.$$

In particular case $n = 5$ it gives value $C_5 = \frac{1}{2}$ considered above (6.71), and for $n = 1$ and $n = 2$, the maximal and minimum uncertainties, 1 and $\frac{1}{2}$, correspondingly. For successive n and $n + 1$ terms, dispersions in X

$$(\Delta X_n)^2 = \frac{F_n}{F_{n+1}}, \qquad (\Delta X_{n+1})^2 = \frac{F_{n+1}}{F_{n+2}},$$

relate uncertainties for different n by fractional transformation

$$(\Delta X_{n+1})^2 = \frac{1}{1 + (\Delta X_n)^2}.$$

The product

$$(\Delta X_{n+1})^2 (\Delta X_n)^2 = \frac{F_n}{F_{n+2}},$$

in the limit $n \to \infty$ takes the form of the Silver Ratio

$$(\Delta X_\infty)^2 = (\Delta P_\infty)^2 = \frac{1}{\varphi} = \frac{\sqrt{5} - 1}{2}.$$

This formula shows how the Golden (Silver) Ratio naturally appears in supersymmetric quantum oscillator.

The set of super-coherent states corresponding to the Golden sequence of uncertainties (6.73) is

$$|\alpha, \sqrt{\frac{F_{n-2}}{F_{n+1}}}, \frac{\pi}{4}\rangle_{L_\pm} = \sqrt{1 - \sqrt{\frac{F_{n-2}}{F_{n+1}}}}\, |\alpha, \Psi_0\rangle + \frac{1 + i}{\sqrt{2}} \left(\frac{F_{n-2}}{F_{n+1}}\right)^{1/4} |\alpha, L_\pm\rangle.$$

In the limit $n \to \infty$, the concurrence C_n is represented by the Golden Ratio $C_\infty = \varphi^{-3/2}$

and the sequence of corresponding states converges to the Golden super-coherent state,

$$|\alpha, \frac{1}{\varphi^{3/2}}, \frac{\pi}{4}\rangle_{L_\pm} = \sqrt{1 - \frac{1}{\varphi^{3/2}}}\,|\alpha, \Psi_0\rangle + \frac{1+i}{\sqrt{2}\varphi^{3/4}}|\alpha, L_\pm\rangle.$$

For this state we have the Golden Uncertainty relation (6.72) in the form

$$\Delta X \Delta P = \frac{\hbar}{\varphi}$$

(here we recovered the Planck constant). The relation determines the Golden proportion

$$\varphi = \frac{\hbar}{\Delta x \Delta p}$$

in the phase plane cells, as ratio of Plank constant with area of the cell. Moreover, the uncertainty value $h/2\pi\varphi$ corresponds to the length of the circle $2\pi\varphi$ with radius $r = \varphi$. Inversion of this circle in the unit one gives the circle with radius $1/\varphi$ and the length $2\pi/\varphi$, which determines the Golden Angle. This angle appears in the theory of sunflowers (Newell and Pennybacker, 2013, 90-105) as efficiency model of sunflowers packing, and it would be interesting to see how it can be combined with phase space structure in quantum mechanics.

Another, non-symmetric in X and P sequence appears from (6.64), (6.65), when $\phi = \frac{\pi}{2}$, so that

$$(\Delta X_n)^2 = \frac{1}{2}(1 + C_n), \quad (\Delta P_n)^2 = \frac{1}{2}(1 - C_n + 2C_n^2).$$

For concurrence $C_5 = \frac{1}{4}$ it gives $(\Delta X_5)^2 = \frac{5}{8} = \frac{F_5}{F_6}$. This suggests the sequence of concurrences,

$$C_n = \frac{F_{n-2}}{F_{n+1}}$$

satisfying equation $(1 + C_n)/2 = F_n/F_{n+1}$, and giving uncertainties

$$(\Delta X_n)^2 = \frac{F_n}{F_{n+1}}, \quad (\Delta P_n)^2 = \frac{F_{n+1}F_{n-1} + F_{n-2}^2}{F_{n+1}^2} = \frac{F_n^2 + F_{n-2}^2 + (-1)^n}{F_{n+1}^2},$$

where in the last equation we used Cassini's identity. In the limit $n \to \infty$ we get uncertainties, which includes Golden Ratio and Fibonacci numbers

$$(\Delta X_\infty)^2 = \frac{1}{\varphi}, \quad (\Delta P_\infty)^2 = 3(5 - 3\varphi),$$
$$(\Delta X_\infty)^2(\Delta P_\infty)^2 = 3(5\varphi - 8).$$

Identification of uncertainty relations for supersymmetric coherent states and the squared quadratures, with first few Fibonacci numbers was unexpected and it motivated us to find the sequence of uncertainty relations with Fibonacci oscillations. By introducing an infinite, countable set of super-coherent states, we got the limit state, the Golden super-coherent state, with Golden ratio in uncertainty relations. The uncertainty relations in general, result from operator commutation relations for observables, suggesting to find deformed commutators with Golden ratio as a parameter. Such type of the Golden quantum oscillator, where Fibonacci operator plays the role of *pq*-number operator with Golden and Silver ratios as deformation parameters, has been studied in (Pashaev and Nalci, 2012, 5-18). The spectrum of this oscillator is given by Fibonacci sequence and coherent states are determined by Fibonomials (Pashaev, 2015, 3-11) . It would be interesting to combine such Golden deformed quantum oscillator with supersymmetric quantum states. A more general spectrum in form of Fibonacci divisors and the quantum algebra deformed by powers of the Golden ratio naturally appeared in the infinite hierarchy of Golden deformation for bosons and fermions (Pashaev, 2021, 3-8). Fibonacci numbers and Golden ratio were involved also in entangled qubit quantum states, where the concurrence and transition amplitudes of entangled N-qubit spin coherent states in computational basis are determined by Fibonacci and Lucas numbers (Pashaev, 2012, 4-12) . The Golden ratio and Fibonacci numbers in quantum computation and information theory were encountered also in quantum coin flipping problem with constraints.

CHAPTER 7

ENTANGLEMENT OF PK-SUPER-QUBIT QUANTUM STATES AND SUPER−COHERENT STATES

The super-qubit quantum state introduced in (Pashaev and Kocak, 2024(2, 3-5)) and (Pashaev and Kocak, 2025(3)), is characterized by a superposition of the zero and one super-particle states, which are represented as points on the super-Bloch sphere. As a preliminary step in the creation of the super-qubit, we use states which have same number of particles. The normalized generic n super-number state

$$|n,\zeta\rangle = \frac{1}{\sqrt{1+|\zeta|^2}}\Big(|0\rangle_f \otimes |n\rangle_b + \zeta|1\rangle_f \otimes |n-1\rangle_b\Big) = \frac{1}{\sqrt{1+|\zeta|^2}}\begin{pmatrix} |n\rangle \\ \zeta|n-1\rangle \end{pmatrix}, \tag{7.1}$$

where ζ is an arbitrary complex number, is the eigenstate of the super-number operator $N|n,\zeta\rangle = n|n,\zeta\rangle$. The origin of the complex plane, $\zeta = 0$, corresponds to n pure bosons, while infinity in the extended complex plane, $z = \infty$, corresponds to one fermion and $n-1$ bosons. By stereographic projection, the extended complex plane can be projected to the unit sphere by the formula

$$\zeta = \tan\left(\frac{\theta_1}{2}\right)e^{i\phi_1}, \tag{7.2}$$

so that the state becomes

$$|n,\theta_1,\phi_1\rangle = \cos\left(\frac{\theta_1}{2}\right)\begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} + \sin\left(\frac{\theta_1}{2}\right)e^{i\phi_1}\begin{pmatrix} 0 \\ |n-1\rangle \end{pmatrix}, \tag{7.3}$$

where $0 \le \theta_1 \le \pi$, $0 \le \phi_1 \le 2\pi$ are angles on the sphere.

7.1. Entanglement of ζ-super-number states

Corollary 7.1 *The concurrence of the n-super-particle state* (7.1) *is independent of n and is given by*

$$C = \frac{2|\zeta|}{1+|\zeta|^2}. \tag{7.4}$$

For $\zeta = 0$ and $\zeta = \infty$, the states are separable and $C = 0$. On the unit circle $|\zeta|^2 = 1$, the state is maximally entangled with $C = 1$.

For $n = 1$, we have the state

$$|1,\zeta\rangle = \frac{1}{\sqrt{1+|\zeta|^2}}\Big(|0\rangle_f \otimes |1\rangle_b + \zeta|1\rangle_f \otimes |0\rangle_b\Big) = \frac{1}{\sqrt{1+|\zeta|^2}}\begin{pmatrix}|1\rangle\\ \zeta|0\rangle\end{pmatrix}, \tag{7.5}$$

which is a fermion-boson entangled one super-particle state. The level of entanglement is determined by formula (7.4). In terms of stereographic projection (7.2) and (7.3), the concurrence becomes,

$$C = \sin\theta_1. \tag{7.6}$$

This formula provides a simple geometrical meaning of concurrence on the sphere. The concurrence is equal to the distance from the point (θ_1, ϕ_1) on the sphere, corresponding to the state $|1, \theta_1, \phi_1\rangle$, from the vertical axis. Alternatively, it is equal to the radius of the circle in the horizontal plane, intersecting the vertical axis at: $z = \sqrt{1-C^2} = \cos\theta_1$. The von Neumann entropy as a function of z is given by,

$$E = -\frac{1}{2}\log_2\left(\frac{1-z^2}{4}\right) - \frac{z}{2}\log_2\left(\frac{1+z}{1-z}\right). \tag{7.7}$$

Probabilities of collapse to the states at the poles $|0\rangle_f \otimes |1\rangle_b$ and $|1\rangle_f \otimes |0\rangle_b$ are:

$$p_0 = \cos^2\frac{\theta}{2} = \frac{1+z}{2}, \quad p_1 = \sin^2\frac{\theta}{2} = \frac{1-z}{2}. \tag{7.8}$$

Geometrically, these probabilities correspond to half-distances from the vertical projection of the state to the north and south poles. For the one super-number state, $N|1,\zeta\rangle = 1|1,\zeta\rangle$. In addition, for $n = 0$, we have the separable state:

$$|0,\zeta\rangle = |0\rangle_f \otimes |0\rangle_b = \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix}, \tag{7.9}$$

which satisfies $N|0,\zeta\rangle = 0|0,\zeta\rangle$ and is orthogonal to the first state $\langle 0,\zeta|1,\zeta\rangle = 0$. The states are related by the creation supersymmetric operator:

$$|1,\zeta\rangle = \frac{1}{\sqrt{1+|\zeta|^2}} \begin{pmatrix} |1\rangle \\ \zeta|0\rangle \end{pmatrix} = \frac{1}{\sqrt{1+|\zeta|^2}} \begin{pmatrix} a^\dagger & 0 \\ \zeta & a^\dagger \end{pmatrix} \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix}. \tag{7.10}$$

By taking the superposition of $n = 0$ and $n = 1$ states, we obtain the $PK-$super-qubit state((Pashaev and Kocak, 2024(2, 5-10), (Pashaev and Kocak, 2025(3)).

Definition 7.1 *The super-qubit quantum state is defined by ((Pashaev and Kocak, 2024(2, 6-11), (Pashaev and Kocak, 2025(3))*

$$|\theta,\phi,\zeta\rangle = \cos\frac{\theta}{2}|0,\zeta\rangle + \sin\frac{\theta}{2}e^{i\phi}|1,\zeta\rangle, \tag{7.11}$$

or in explicit form:

$$|\theta,\phi,\zeta\rangle = \cos\frac{\theta}{2}\begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\frac{1}{\sqrt{1+|\zeta|^2}}\begin{pmatrix} |1\rangle \\ \zeta|0\rangle \end{pmatrix}, \tag{7.12}$$

which is characterized by two real parameters θ, ϕ and one complex parameter ζ.

For this state, the first two parameters θ and ϕ are angles on the unit sphere, which we call the super-Bloch sphere. The north pole of the sphere at $\theta = 0$ corresponds to the zero number of super-particles in the state $|0,0,\zeta\rangle$, while the south pole at $\theta = \pi$ corresponds to the one super-particle in the state $|\pi,0,\zeta\rangle$. Any point on the sphere represents a superposition of these two states with varying levels of entanglement. The probabilities of measuring the basis states are:

$$p_0 = |\langle 0,\zeta|\theta,\phi,\zeta\rangle|^2 = \cos^2\frac{\theta}{2}, \quad p_1 = |\langle 1,\zeta|\theta,\phi,\zeta\rangle|^2 = \sin^2\frac{\theta}{2}. \tag{7.13}$$

The super-qubit state is a natural generalization of the fermionic or bosonic one-qubit states.

Proposition 7.1 *In the limiting cases, the PK−super-qubit state (7.12) reduces to separable qubit states: 1. For $\zeta = 0$, the state is a separable one-qubit bosonic state:*

$$|\theta,\phi,0\rangle = \cos\frac{\theta}{2}\begin{pmatrix}|0\rangle\\0\end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\begin{pmatrix}|1\rangle\\0\end{pmatrix} = |0\rangle_f \otimes |\theta,\phi\rangle_b, \tag{7.14}$$

2. For $\zeta = \infty$, the state is a separable one-qubit fermionic state:

$$|\theta,\phi,\infty\rangle = \cos\frac{\theta}{2}\begin{pmatrix}|0\rangle\\0\end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\begin{pmatrix}0\\|0\rangle\end{pmatrix} = |\theta,\phi\rangle_f \otimes |0\rangle_b. \tag{7.15}$$

In general, the state is entangled.

Proposition 7.2 *For $\zeta = e^{i\gamma}$, so that $|\zeta|^2 = 1$, the PK−super-qubit state reduces to the form:*

$$|\theta,\phi,e^{i\gamma}\rangle = \cos\frac{\theta}{2}\begin{pmatrix}|0\rangle\\0\end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\frac{1}{\sqrt{2}}\begin{pmatrix}|1\rangle\\e^{i\gamma}|0\rangle\end{pmatrix}, \tag{7.16}$$

for $\gamma = 0, \pi$, giving the pair of states:

$$|\theta,\phi,\pm\rangle = \cos\frac{\theta}{2}\begin{pmatrix}|0\rangle\\0\end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\frac{1}{\sqrt{2}}\begin{pmatrix}|1\rangle\\\pm|0\rangle\end{pmatrix}, \tag{7.17}$$

which are considered as the reference states in (Pashaev and Kocak, 2025). The corre-

sponding one super-particle states

$$|\pi, \phi, e^{i\gamma}\rangle = |1, e^{i\phi}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} |1\rangle \\ e^{i\gamma}|0\rangle \end{pmatrix}, \tag{7.18}$$

are maximally entangled states with $C = 1$.

Proposition 7.3 *The super-qubit state* (7.11) *is a fermion-boson entangled state with concurrence in the product form:*

$$C = \sin^2 \frac{\theta}{2} \cdot \frac{2|\zeta|}{1 + |\zeta|^2}. \tag{7.19}$$

For $\zeta = 0$ *and* $\zeta = 1$*, the super-qubit state is separable, and* $C = 0$*. For* $|\zeta| = 1$*, the concurrence is:*

$$C = \sin^2 \frac{\theta}{2}, \tag{7.20}$$

as in the special super-qubit case (Pashaev and Kocak, 2025, 5-11).

7.2. PK-supersymmetric annihilation operator

Proposition 7.4 *The super-qubit state* (7.11) *is the reference state annihilated by the super-annihilation operator:*

$$A_{-1/\zeta} = \begin{pmatrix} a & -\frac{1}{\zeta} \\ 0 & a \end{pmatrix}, \quad A_{-1/\zeta}|\theta, \phi, \zeta\rangle = 0. \tag{7.21}$$

This follows from the observation that for the basis states $A_{-1/\zeta}|0, \zeta\rangle = 0$ and $A_{-1/\zeta}|1, \zeta\rangle = 0$. The operators satisfy the following commutation relations with the super-number operator:

$$[N, A_{-1/\zeta}] = -A_{-1/\zeta}, \quad [N, A^{\dagger}_{-1/\zeta}] = A^{\dagger}_{-1/\zeta}, \tag{7.22}$$

$$[A_{-1/\zeta}, A^{\dagger}_{-1/\zeta}] = I + \frac{1}{|\zeta|^2}\sigma_3 \otimes I_b. \tag{7.23}$$

In the limit $\zeta \to \infty$, the operator becomes the direct product $I_f \otimes a$, and the algebra becomes identical to that of the bosonic operators. In a similar manner to previous chapter, we can introduce supersymmetric coherent state.

Definition 7.2 *The PK−supersymmetric coherent state, associated with the PK−super-qubit reference state* (7.11)*, is defined as*

$$|\alpha, \theta, \phi, \zeta\rangle = \mathcal{D}(\alpha)|\theta, \phi, \zeta\rangle \tag{7.24}$$

where $\mathcal{D}(\alpha)$ given in (6.40).

Proposition 7.5 *The PK−super-coherent state can be represented as:*

$$|\alpha, \theta, \phi, \zeta\rangle = \cos\frac{\theta}{2}\begin{pmatrix} |0,\alpha\rangle \\ 0 \end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\frac{1}{\sqrt{1+|\zeta|^2}}\begin{pmatrix} |1,\alpha\rangle \\ \zeta|0,\alpha\rangle \end{pmatrix}, \tag{7.25}$$

where $|0,\alpha\rangle = D(\alpha)|0\rangle$ and $|1,\alpha\rangle = D(\alpha)|1\rangle$ are defined in $(C.17)$ *and* $(C.18)$.

Proposition 7.6 *The super-coherent states* (7.25) *are eigenstates of the super-annihilation operator $A_{-1/\zeta}$:*

$$A_{-1/\zeta}|\alpha, \theta, \phi, \zeta\rangle = \alpha|\alpha, \theta, \phi, \zeta\rangle. \tag{7.26}$$

Proof We begin by noting the commutation relation:

$$[A_{-1/\zeta}, \mathcal{D}(\alpha)] = A_{-1/\zeta}\mathcal{D}(\alpha) - \mathcal{D}(\alpha)A_{-1/\zeta} = \alpha\mathcal{D}(\alpha),$$

which will be useful when applying this operator to the state. Now, we apply $A_{-1/\zeta}$ to the state $|\alpha, \theta, \phi, \zeta\rangle$ as follows

$$A_{-1/\zeta}|\alpha, \theta, \phi, \zeta\rangle = A_{-1/\zeta}\mathcal{D}(\alpha)|\theta, \phi, \zeta\rangle,$$

by the commutation relation, this can be rewritten as

$$A_{-1/\zeta}|\alpha, \theta, \phi, \zeta\rangle = \mathcal{D}(\alpha)A_{-1/\zeta}|\theta, \phi, \zeta\rangle + \alpha\mathcal{D}(\alpha)|\theta, \phi, \zeta\rangle.$$

The first term on the right-hand side vanishes due to the property $A_{-1/\zeta}|\theta, \phi, \zeta\rangle = 0$, leaving us with

$$A_{-1/\zeta}|\alpha, \theta, \phi, \zeta\rangle = \alpha|\alpha, \theta, \phi, \zeta\rangle.$$

□

7.2.1. Entanglement of PK-supersymmetric Coherent states

In order to calculate the entanglement of the state (7.25), we note that

$$|\alpha, \theta, \phi, \zeta\rangle = \mathcal{D}(\alpha)|\theta, \phi, \zeta\rangle = |0\rangle_f \otimes D(\alpha)|\psi_0\rangle + |1\rangle_f \otimes D(\alpha)|\psi_1\rangle, \tag{7.27}$$

where for the reference state, we have

$$|\theta, \phi, \zeta\rangle = |0\rangle_f \otimes |\psi_0\rangle + |1\rangle_f \otimes |\psi_1\rangle, \tag{7.28}$$

and

$$|\alpha, \theta, \phi, \zeta\rangle = |0\rangle_f \otimes |\psi_0, \alpha\rangle + |1\rangle_f \otimes |\psi_1, \alpha\rangle. \tag{7.29}$$

Since the Fock states are connected by a unitary transformation

$$|\psi_0, \alpha\rangle = D(\alpha)|\psi_0\rangle, \quad |\psi_1, \alpha\rangle = D(\alpha)|\psi_1\rangle, \tag{7.30}$$

the inner products do not depend on α:

$$\langle\psi_i, \alpha|\psi_j, \alpha\rangle = \langle\psi_i|D^\dagger(\alpha)D(\alpha)|\psi_j\rangle = \langle\psi_i|\psi_j\rangle, \tag{7.31}$$

as well as the Gram determinant (6.9) of the inner products. Consequently, the concurrence C is also independent of α.

Corollary 7.2 *The concurrence for the super-coherent state* (7.25) *coincides with that of the super-qubit state* (7.11) *and is given by:*

$$C = \sin^2\frac{\theta}{2}\cdot\frac{2|\zeta|}{1+|\zeta|^2}. \tag{7.32}$$

7.3. Flipped PK-Super-Qubits and Super-Coherent States

The flipping operator is defined as

$$\mathcal{X} = X\otimes I_b = \sigma_1\otimes I_b.$$

It acts on the fermion number of states as

$$\mathcal{X}(N_f\otimes I_b)\mathcal{X} = \bar{N}_f\otimes I_b, \tag{7.33}$$

where $\bar{N}_f = diag(1,0)$ corresponds to interchanging the number of fermions. Applying operator $\mathcal{X}$ to the n- super-number state (7.1), we get the flipped state

$$\mathcal{X}|n,\zeta\rangle = \frac{|1\rangle_f\otimes|n\rangle_b+\zeta|0\rangle_f\otimes|n-1\rangle_b}{\sqrt{1+|\zeta|^2}} = \frac{1}{\sqrt{1+|\zeta|^2}}\begin{pmatrix}\zeta|n-1\rangle\\ |n\rangle\end{pmatrix}. \tag{7.34}$$

Proposition 7.7 *The flipped one super-particle state is given by:*

$$\mathcal{X}|1,\zeta\rangle = \frac{1}{\sqrt{1+|\zeta|^2}}\begin{pmatrix}\zeta|0\rangle\\ |1\rangle\end{pmatrix} = \frac{\zeta|0\rangle_f\otimes|0\rangle_b+|1\rangle_f\otimes|1\rangle_b}{\sqrt{1+|\zeta|^2}}, \tag{7.35}$$

The flipped PK−super-qubit state is:

$$\mathcal{X}|\theta,\phi,\zeta\rangle = \cos\frac{\theta}{2}\begin{pmatrix} 0 \\ |0\rangle \end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\frac{1}{\sqrt{1+|\zeta|^2}}\begin{pmatrix} \zeta|0\rangle \\ |1\rangle \end{pmatrix}. \tag{7.36}$$

Proposition 7.8 *The concurrence for the state $|\Psi\rangle$ and the flipped state $\mathcal{X}|\Psi\rangle$ is the same.*

Proof If the state $|\Psi\rangle$ is represented as $|\Psi\rangle = |0\rangle_f \otimes |\psi_0\rangle + |1\rangle_f \otimes |\psi_1\rangle$, then the flipped state is

$$\mathcal{X}|\Psi\rangle = \mathcal{X}|0\rangle_f \otimes |\psi_0\rangle + \mathcal{X}|1\rangle_f \otimes |\psi_1\rangle = |1\rangle_f \otimes |\psi_0\rangle + |0\rangle_f \otimes |\psi_1\rangle. \tag{7.37}$$

This implies that the result of flipping is an interchange of the Fock states $|\psi_0\rangle$ and $|\psi_1\rangle$. From the representation of concurrence by the Gram determinant (6.9), it is clear that the determinant is invariant under such interchange, and thus the concurrence remains unchanged. □

Corollary 7.3 *The concurrence for the flipped state* (7.35) *is equal to that of the state* $|1,\zeta\rangle$*:*

$$C = \frac{2|\zeta|}{1+|\zeta|^2}. \tag{7.38}$$

The concurrence for the flipped super-qubit state (7.36) *is equal to that of the super-qubit state:*

$$C = \sin^2\frac{\theta}{2}\cdot\frac{2|\zeta|}{1+|\zeta|^2}. \tag{7.39}$$

Proposition 7.9 *Applying the flipping gate to the super-annihilation operator gives the transposed operator*

$$\mathcal{X}A_{-1/\zeta}\mathcal{X} = A^T_{-1/\zeta} = \begin{pmatrix} a & 0 \\ -\frac{1}{\zeta} & a \end{pmatrix}. \tag{7.40}$$

The flipped super-qubit state is annihilated by this operator

$$A^T_{-1/\zeta}\mathcal{X}|\theta,\phi,\zeta\rangle = 0. \tag{7.41}$$

Definition 7.3 *The flipped PK−super-coherent state is defined by the action of the dis-*

placement operator on the flipped super-qubit state:

$$|\alpha, \theta, \phi, \zeta\rangle_{\mathcal{X}} = \mathcal{D}(\alpha)\mathcal{X}|\theta, \phi, \zeta\rangle. \tag{7.42}$$

Since the operators $\mathcal{D}(\alpha)$ and $\mathcal{X}$ commute, $[\mathcal{D}(\alpha), \mathcal{X}] = 0$, we have the following proposition.

Proposition 7.10 *The flipped PK−super-coherent state can be represented as:*

$$\mathcal{X}|\alpha, \theta, \phi, \zeta\rangle = \cos\frac{\theta}{2}\begin{pmatrix} 0 \\ |0, \alpha\rangle \end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\frac{1}{\sqrt{1+|\zeta|^2}}\begin{pmatrix} \zeta|0, \alpha\rangle \\ |1, \alpha\rangle \end{pmatrix}. \tag{7.43}$$

It is the eigenvalue of the operator $A^T_{-1/\zeta}$:

$$A^T_{-1/\zeta}\mathcal{X}|\alpha, \theta, \phi, \zeta\rangle = \alpha\mathcal{X}|\alpha, \theta, \phi, \zeta\rangle, \tag{7.44}$$

and has the same concurrence (7.39) *as the PK-super-qubit state.*

7.4. Uncertainty Relations and Fibonacci Sequence for PK-supersymmetric Coherent states

Proposition 7.11 *The average values of the X and P operators in super-coherent states are:*

$$\langle\alpha, \theta, \phi, \zeta|X|\alpha, \theta, \phi, \zeta\rangle = \sqrt{2}\,\mathrm{Re}(\alpha) + \frac{\sin\theta\cos\phi}{\sqrt{2}\sqrt{1+|\zeta|^2}}, \tag{7.45}$$

$$\langle\alpha, \theta, \phi, \zeta|P|\alpha, \theta, \phi, \zeta\rangle = \sqrt{2}\,\mathrm{Im}(\alpha) + \frac{\sin\theta\sin\phi}{\sqrt{2}\sqrt{1+|\zeta|^2}}. \tag{7.46}$$

Definition 7.4 *The classical values of coordinate and momentum are denoted as $x_c = \sqrt{2}\,\mathrm{Re}(\alpha)$ and $y_c = \sqrt{2}\,\mathrm{Im}(\alpha)$ in the complex plane α. The spherical coordinates on the* *super-Bloch sphere are:*

$$x = \sin\theta\cos\phi, \quad y = \sin\theta\sin\phi, \quad z = \cos\theta, \tag{7.47}$$

so that $2\sin^2\frac{\theta}{2} = 1 - z$, *and* $x^2 + y^2 + z^2 = 1$.

Corollary 7.4 *The average values of the coordinate and momentum are projections:*

$$\begin{aligned}
\langle\alpha,\theta,\phi,\zeta|X|\alpha,\theta,\phi,\zeta\rangle &= x_c + \frac{x}{\sqrt{2}\sqrt{1+|\zeta|^2}}, \\
\langle\alpha,\theta,\phi,\zeta|P|\alpha,\theta,\phi,\zeta\rangle &= y_c + \frac{y}{\sqrt{2}\sqrt{1+|\zeta|^2}}.
\end{aligned}$$

Proposition 7.12 *The dispersions of the X and P operators in super-coherent states are independent of* α *and are given by:*

$$(\Delta X)^2 = \frac{1}{2}\left(1 + \frac{2\sin^2\frac{\theta}{2} - \sin^2\theta\cos^2\phi}{1+|\zeta|^2}\right), \tag{7.48}$$

$$(\Delta P)^2 = \frac{1}{2}\left(1 + \frac{2\sin^2\frac{\theta}{2} - \sin^2\theta\sin^2\phi}{1+|\zeta|^2}\right), \tag{7.49}$$

or in terms of the Cartesian coordinates on the super-Bloch sphere:

$$(\Delta X)^2 = \frac{1}{2}\left(1 + \frac{1 - z - x^2}{1+|\zeta|^2}\right), \tag{7.50}$$

$$(\Delta P)^2 = \frac{1}{2}\left(1 + \frac{1 - z - y^2}{1+|\zeta|^2}\right). \tag{7.51}$$

For $\phi = \frac{\pi}{4}$, the coordinates $x = y$ and the dispersions are equal. For the corresponding state in the equatorial plane $\theta = \frac{\pi}{2}$, we have:

$$(\Delta X)^2 = (\Delta P)^2 = \frac{1}{2}\left(1 + \frac{1}{2(1+|\zeta|^2)}\right). \tag{7.52}$$

From this formula, for states with $|\zeta| = 1$, we have:

$$(\Delta X)^2 = (\Delta P)^2 = \frac{5}{8} = \frac{F_5}{F_6}. \tag{7.53}$$

This suggests constructing the sequence of states with the ratio of Fibonacci numbers for any n.

Proposition 7.13 *The sequence of circles in the complex plane ζ is given by:*

$$|\zeta_n|^2 = \frac{F_{n-1}}{F_{n-2}} - \frac{1}{2}, \tag{7.54}$$

where F_n denotes the Fibonacci numbers. The dispersions are determined as:

$$(\Delta X_n)^2 = (\Delta P_n)^2 = \frac{F_n}{F_{n+1}}, \tag{7.55}$$

and the uncertainty relations are:

$$\Delta X_n \Delta P_n = \frac{F_n}{F_{n+1}}. \tag{7.56}$$

Corollary 7.5 *The square of the radius of the circles $|\zeta_n|^2$ oscillates around the value corresponding to the limit $n \to \infty$, where:*

$$|\zeta_\infty|^2 = \varphi - \frac{1}{2}, \tag{7.57}$$

with φ being the Golden Ratio. In this limit, we obtain the Golden dispersions:

$$(\Delta X_\infty)^2 = (\Delta P_\infty)^2 = \frac{1}{\varphi}, \tag{7.58}$$

and the Golden uncertainty relation: $\Delta X_\infty \Delta P_\infty = \dfrac{1}{\varphi}$.

CHAPTER 8

PQ-DEFORMED SUPERSYMMETRIC ANNIHILATION OPERATOR

This Chapter introduces the *pq*-deformed supersymmetric annihilation operator and supersymmetric quantum states under *pq*-deformation. First, we define the supersymmetric pq-annihilation operator as

$$\widehat{A}_{pq} = \begin{bmatrix} p\hat{a}_{pq} & 1 \\ 0 & q\hat{a}_{pq} \end{bmatrix} \tag{8.1}$$

and corresponding supersymmetric pq-coherent state as eigenstate of this operator

$$\widehat{A}_{pq}|A\rangle_{pq} = \alpha|A\rangle_{pq}. \tag{8.2}$$

We notice that *pq* numbers are symmetric under $p \leftrightarrow q$ exchange. However, definition (8.1) is not symmetric to $p \leftrightarrow q$ exchange and bosonic, fermionic state definition. The different choice of *p* and *q* parameters gives following particular reductions.

1)*Supersymmetric non-symmetric q-annihilation operator:* When we choose $p = 1$, the supersymmetric pq-annihilation operator becomes in $q-$deformed form

$$\widehat{A}_{q} = \begin{bmatrix} \hat{a}_{q} & 1 \\ 0 & q\hat{a}_{q} \end{bmatrix},$$

where $\hat{a}_q$ is annihilation operator of non-symmetric q-calculus (9.1). If in addition $q \to 1$, then we get $AZ-$supersymmetric annihilation operator (5.1).

2)*Supersymmetric, symmetric q-annihilation operator* : By choosing $p = \frac{1}{q}$, we get supersymmetric annihilation operator in the following form,

$$\widehat{A}_{\widetilde{q}} = \begin{bmatrix} \frac{1}{q}\hat{a}_{\widetilde{q}} & 1 \\ 0 & q\hat{a}_{\widetilde{q}} \end{bmatrix},$$

where $\hat{a}_{\widetilde{q}} = \hat{a}_{\frac{1}{q},q}$, is annihilation operator of q-symmetric oscillator.

3)*Supersymmetric Golden-Fibonacci annihilation operator:* If we choose $p = \varphi$ and $q = \varphi^{'}$, so that $[n]_{\varphi\varphi'} = F_n$, the corresponding Supersymmetric annihilation operator is

$$\widehat{A}_F = \begin{bmatrix} \varphi\hat{a}_F & 1 \\ 0 & \varphi^{'}\hat{a}_F \end{bmatrix},$$

where $\hat{a}_F$ is annihilation operator of Golden-Fibonacci calculus (3.22).

4)*Supersymmetric Tamm-Dankoff annihilation operator:* When $p \to q$, we get Supersymmetric Tamm-Dankoff annihilation operator

$$\widehat{A}_{TD} = \begin{bmatrix} q\hat{a}_{TD} & 1 \\ 0 & q\hat{a}_{TD} \end{bmatrix} = q\begin{bmatrix} \hat{a}_{TD} & \frac{1}{q} \\ 0 & \hat{a}_{TD} \end{bmatrix},$$

By denoting $\epsilon = \frac{1}{q}$, this operator can be represented in form similar to $(A.1)$.

$$\widehat{A}_{TD} = q\begin{bmatrix} \hat{a}_{TD} & \frac{1}{q} \\ 0 & \hat{a}_{TD} \end{bmatrix} = \frac{1}{\epsilon}\begin{bmatrix} \hat{a}_{TD} & \epsilon \\ 0 & \hat{a}_{TD} \end{bmatrix}.$$

Here $\hat{a}_{TD}$ is TD annihilation operator.

8.1. pq-Supersymmetric Coherent States

The coherent state $|A\rangle_{pq}$ can be splitted to the two parts,

$$
|A_b\rangle_{pq} = C_b \begin{bmatrix} |\frac{\alpha}{p}\rangle_{pq} \\ 0 \end{bmatrix} \quad ,
$$

$$
|A_s\rangle_{pq} = C_s \begin{bmatrix} e_{pq}^{\frac{|\alpha|^2}{p^2 q}} \frac{\bar{\alpha}}{p^2} |\frac{\alpha}{p}\rangle_{pq} - e_{pq}^{\frac{|\alpha|^2}{p^2}} q |\frac{\alpha'}{pq}\rangle_{pq} \\ \\ e_{pq}^{\frac{|\alpha|^2}{p^2}} |\frac{\alpha}{q}\rangle_{pq} \end{bmatrix},
$$

where normalization constants(See Appendix B.2.2) are

$$
\begin{aligned}
C_s^{-2} &= \left[\left(e_{pq}^{\frac{|\alpha|^2}{p^2}} \right)^2 \left(e_{pq}^{\frac{|\alpha|^2}{q^2}} + \frac{1}{p^2} e_{pq}^{\frac{|\alpha|^2}{p^2 q}} \right) + \frac{|\alpha|^2}{p^4 q^2} e_{pq}^{\frac{|\alpha|^2}{p^2}} \left(p \left(e_{pq}^{\frac{|\alpha|^2}{p^2}} \right) \left(e_{pq}^{\frac{|\alpha|^2}{p^2 q^2}} \right) - q^2 \left(e_{pq}^{\frac{|\alpha|^2}{p^2 q}} \right)^2 \right) \right] \\
&= \left[\left(e_{pq}^{\frac{|\alpha|^2}{p^2}} \right)^2 \left(e_{pq}^{\frac{|\alpha|^2}{q^2}} + \frac{1}{p^2} e_{pq}^{\frac{|\alpha|^2}{pq^2}} \right) + \frac{|\alpha|^2}{p^4 q^2} e_{pq}^{\frac{|\alpha|^2}{p^2}} \left(q \left(e_{pq}^{\frac{|\alpha|^2}{p^2}} \right) \left(e_{pq}^{\frac{|\alpha|^2}{p^2 q^2}} \right) - q^2 \left(e_{pq}^{\frac{|\alpha|^2}{p^2 q}} \right)^2 \right) \right], \\
C_b^{-2} &= e_{pq}^{\frac{|\alpha|^2}{p^2}} .
\end{aligned}
$$

These states are orthonormal states (See Appendix B.2.1)

$$
{}_{pq}\langle A_b | A_b \rangle_{pq} = {}_{pq}\langle A_s | A_s \rangle_{pq} = 1 \ , \ {}_{pq}\langle A_b | A_s \rangle_{pq} = 0,
$$

so that an arbitrary $pq-$supersymmetric coherent state is

$$
|A\rangle_{pq} = c_0 |A_b\rangle_{pq} + c_1 |A_s\rangle_{pq},
$$

where ${}_{pq}\langle A|A\rangle_{pq} = 1$, and as follows $|c_0|^2 + |c_1|^2 = 1$. By choosing

$$
c_0 = \cos\frac{\theta}{2} \, , \, c_1 = e^{i\phi} \sin\frac{\theta}{2},
$$

we obtain the superbloch sphere representation of supersymmetric $pq-$coherent states

$$|\alpha;\theta,\phi\rangle_{pq} = \cos\frac{\theta}{2}|A_b\rangle_{pq} + e^{i\phi}\sin\frac{\theta}{2}|A_s\rangle_{pq} \ . \tag{8.3}$$

8.2. Uncertainty relations for pq-Supersymmetric Coherent States

The $pq-$deformed coordinate and momentum operators are defined as

$$\hat{x}_{pq} = \left(\frac{\hat{a}^{\dagger}{}_{pq} + \hat{a}_{pq}}{\sqrt{2}}\right)\otimes \hat{I}_F \ , \qquad \hat{p}_{pq} = i\left(\frac{\hat{a}^{\dagger}{}_{pq} - \hat{a}_{pq}}{\sqrt{2}}\right)\otimes \hat{I}_F. \tag{8.4}$$

Proposition 8.1 *The mean values of $\hat{x}_{pq}, \hat{p}_{pq}, \hat{x}^2_{pq}, \hat{p}^2_{pq}$, in $|A_b\rangle_{pq}$ state are*

$$
\begin{aligned}
{}_{pq}\langle A_b|\hat{x}_{pq}|A_b\rangle_{pq} &\equiv \frac{1}{p}\sqrt{2}\,\mathrm{Re}(\alpha),\\
{}_{pq}\langle A_b|\hat{p}_{pq}|A_b\rangle_{pq} &\equiv \frac{1}{p}\sqrt{2}\,\mathrm{Im}(\alpha),\\
{}_{pq}\langle A_b|\hat{x}^2_{pq}|A_b\rangle_{pq} &\equiv \frac{1}{2p^2}\left[\bar{\alpha}^2 + \alpha^2 + (1+q)|\alpha|^2 + p^2\left(e_{pq}^{\frac{|\alpha|^2}{p}}\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^{-1}\right],\\
{}_{pq}\langle A_b|\hat{p}^2_{pq}|A_b\rangle_{pq} &\equiv \frac{1}{2p^2}\left[-\bar{\alpha}^2 - \alpha^2 + (1+q)|\alpha|^2 + p^2\left(e_{pq}^{\frac{|\alpha|^2}{p}}\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^{-1}\right].
\end{aligned}
$$

The proof is given in Appendix B.5.

Proposition 8.2 *Dispersions of $\hat{x}$ and $\hat{p}$ in $|A_b\rangle_{pq}$ state are equal,*

$$
\begin{aligned}
{}_{pq}\langle A_b|\left(\Delta\hat{x}_{pq}\right)^2|A_b\rangle_{pq} &\equiv {}_{pq}\langle A_b|\left(\Delta\hat{p}_{pq}\right)^2|A_b\rangle_{pq}\\
&= \frac{1}{2p^2}\left((q-1)|\alpha|^2 + p^2\left(e_{pq}^{\frac{|\alpha|^2}{p}}\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^{-1}\right),
\end{aligned}
$$

and uncertainty relation is

$$\Delta\hat{x}_{pq}\Delta\hat{p}_{pq} = \frac{1}{2p^2}\left((q-1)|\alpha|^2 + p^2\left(e_{pq}^{\frac{|\alpha|^2}{p}}\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^{-1}\right). \qquad (8.5)$$

Proposition 8.3

$$\begin{aligned}
{}_{pq}\langle A_s|\hat{x}_{pq}^2|A_s\rangle_{pq} \;=\; & |C_s|^2\Big[\left(\frac{-1}{2p^4q^2}(\overline{\alpha}^2|\alpha|^2 + \alpha^2|\alpha|^2) + \left(\frac{q+1}{2p^6} - \frac{p+1}{p^5q}\right)|\alpha|^4\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right)^2 e_{pq}^{\frac{|\alpha|^2}{p^2}} \\
& + \left(\frac{1}{2p^2q^2}(\alpha^2 + \overline{\alpha}^2) + \left(\frac{1}{2p^3q^2}(p+2q+(p+q)^2) - \frac{p+q+1}{p^4}\right)|\alpha|^2\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2 e_{pq}^{\frac{|\alpha|^2}{p^2q}} \\
& + \frac{|\alpha|^2}{2p^4}\left(e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right)^2 e_{pq}^{\frac{|\alpha|^2}{p}} + \left(\frac{|\alpha|^2}{2p^3q^4}(\alpha^2 + \overline{\alpha}^2) + \frac{p+1}{2p^3q^4}|\alpha|^4\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2 e_{pq}^{\frac{|\alpha|^2}{p^2q^2}} \\
& + \frac{p+q+1}{2p^2}\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^3 + \left(\frac{1}{2q^2}(\overline{\alpha}^2 + \alpha^2 + (q+1)|\alpha|^2)\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2 e_{pq}^{\frac{|\alpha|^2}{q^2}} + \frac{1}{2}\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2 e_{pq}^{p\frac{|\alpha|^2}{q^2}},
\end{aligned}$$

$$\begin{aligned}
{}_{pq}\langle A_s|\hat{p}_{pq}^2|A_s\rangle_{pq} \;=\; & |C_s|^2\Big[\left(\frac{1}{2p^4q^2}(\overline{\alpha}^2|\alpha|^2 + \alpha^2|\alpha|^2) + \left(\frac{q+1}{2p^6} - \frac{p+1}{p^5q}\right)|\alpha|^4\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right)^2 e_{pq}^{\frac{|\alpha|^2}{p^2}} \\
& + \left(\frac{-1}{2p^2q^2}(\alpha^2 + \overline{\alpha}^2) + \left(\frac{1}{2p^3q^2}(p+2q+(p+q)^2) - \frac{p+q+1}{p^4}\right)|\alpha|^2\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2 e_{pq}^{\frac{|\alpha|^2}{p^2q}} \\
& + \frac{|\alpha|^2}{2p^4}\left(e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right)^2 e_{pq}^{\frac{|\alpha|^2}{p}} + \left(\frac{-|\alpha|^2}{2p^3q^4}(\alpha^2 + \overline{\alpha}^2) + \frac{p+1}{2p^3q^4}|\alpha|^4\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2 e_{pq}^{\frac{|\alpha|^2}{p^2q^2}} \\
& + \frac{p+q+1}{2p^2}\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^3 + \left(\frac{-1}{2q^2}(\overline{\alpha}^2 + \alpha^2 - (q+1)|\alpha|^2)\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2 e_{pq}^{\frac{|\alpha|^2}{q^2}} + \frac{1}{2}\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2 e_{pq}^{p\frac{|\alpha|^2}{q^2}}.
\end{aligned}$$

Then,

$$\begin{aligned}
{}_{pq}\langle A_s|\left(\Delta\hat{x}_{pq}\right)^2|A_s\rangle_{pq} &= {}_{pq}\langle A_s|\hat{x}_{pq}^2|A_s\rangle_{pq} - \left(\frac{1}{q}\sqrt{2}\,\mathrm{Re}(\alpha)\right)^2, \\
{}_{pq}\langle A_s|\left(\Delta\hat{p}_{pq}\right)^2|A_s\rangle_{pq} &= {}_{pq}\langle A_s|\hat{p}_{pq}^2|A_s\rangle_{pq} - \left(\frac{1}{q}\sqrt{2}\,\mathrm{Im}(\alpha)\right)^2.
\end{aligned}$$

The uncertainity relation for $|A\rangle_s$ takes the form $(\Delta\hat{x})_s\,(\Delta\hat{p})_s = 1$ as $p, q \to 1$,. The proof is given in Appendix B.4.2.

CHAPTER 9

Q-DEFORMED SUPERSYMMETRIC COHERENT STATES

This chapter explores q-deformed supersymmetric coherent states and their dynamics. We begin with the q-deformed quantum oscillator and the associated q-coherent states, followed by an examination of their time evolution. Then, this chapter introduces the q-supersymmetric annihilation operator, which is essential for defining these states within a supersymmetric framework. We also look at the uncertainty relations, specific to q-supersymmetric coherent states and explore how these states change over time.

9.1. The q-deformed quantum Oscillator and q-Coherent states

For non-symmetrical case, the q-number is defined as

$$[n]_q \equiv \frac{q^n - 1}{q - 1}$$

and the following algebraic relations are valid

$$a_q a_q^+ - a_q^+ a_q = q^N, \tag{9.1}$$

$$a_q a_q^+ - q a_q^+ a_q = 1, \tag{9.2}$$

where a_q and a_q^+ are annihilation and creation operators of non-symmetric $q-$calculus. The definition of non-symmetrical $q-$ number operator

$$a_q^+ a_q = [N]_q, \quad a_q a_q^+ = [N+1]_q$$

gives

$$[N+1]_q - q[N]_q = 1, \tag{9.3}$$

$$[N+1]_q - [N]_q = q^N. \tag{9.4}$$

In this case, the Fock space basis $|n\rangle_q$ is defined by

$$|n\rangle_q = \frac{(a_q^+)^n|0\rangle_q}{\sqrt{[n]_q!}},$$

where $a_q|0\rangle_q = 0$ and operators act on the basis as following

$$\begin{aligned} [N]_q|n\rangle_q &= [n]_q|n\rangle_q\,, \\ a_q^+|n\rangle_q &= \sqrt{[n+1]_q}|n+1\rangle_q\,, \\ a_q|n\rangle_q &= \sqrt{[n]_q}|n-1\rangle_q. \end{aligned}$$

The Hamiltonian is

$$H_q = \frac{\hbar\omega}{2}([N]_q + [N+1]_q)\,,$$

with energy levels for the corresponding eigenstates $|n\rangle_q$;

$$E_n = \frac{\hbar\omega}{2}([n]_q + [n+1]_q)\,,$$

where $n = 0, 1, 2, ..$ The limit $n \to \infty$ for $[n]_q$ gives

$$\lim_{n\to\infty}[n]_q = \begin{cases} \infty, & \text{q>1;} \\ \frac{1}{1-q}, & \text{q<1.} \end{cases} \tag{9.5}$$

so that when $|q| < 1$

$$\lim_{n\to\infty} E_n = \frac{\hbar\omega}{1-q}.$$

Theorem 9.1 *The sequence of eigenstates $E_n, n = 0, 1, 2, ...,$ is the Cauchy sequence for $|q| < 1$.*

Corollary 9.1 *The maximum value of the energy spectrum for $|q| < 1$ is*

$$E_\infty = \lim_{n\to\infty} E_n = \frac{\hbar\omega}{1-q}.$$

For the q-deformed quantum oscillator, we can define the q-coherent states as states that generalize the concept of classical coherent states to this deformed case.

Definition 9.1 *The q-coherent states are defined as eigenstates of $\hat{a}_q$ operator*

$$\hat{a}_q|\alpha\rangle_q = \alpha|\alpha\rangle_q, \tag{9.6}$$

where $|\alpha\rangle_q = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]_q!}}|n\rangle_q$*.(not normalized state)*

Proposition 9.1 *The inner product of two q-coherent states $|\alpha\rangle_q$ and $|\beta\rangle_q$ is*

$${}_q\langle\beta|\alpha\rangle_q = e_q^{\alpha\bar{\beta}},$$

where e_q^x is defined by Eq.(3.8) *and*

$${}_q\langle\alpha|\alpha\rangle_q = e_q^{|\alpha|^2}.$$

9.1.1. Time evolution of q-Coherent states

We define Hamiltonian in terms of $q-$Number operator

$$H_q = \frac{i\hbar\omega}{2}\left([N]_q + [N+1]_q\right),$$

then, the time evolution operator becomes

$$U(t) = e^{-i\frac{\hbar\omega}{2}([N]_q+[N+1]_q)t}.$$

It provides the time evolution of q-coherent states, calculated as

$$U(t)|\alpha\rangle_q = |\alpha, t\rangle_q = N_q(t)\sum_{n=0}^{\infty}\alpha^n \frac{e^{-i\frac{\hbar\omega}{2}([n]_q+[n+1]_q)t}}{\sqrt{[n]_q!}}|n\rangle_q,$$

with normalization $(N_q(t))^{-2} = \sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{\sqrt{[n]_q!}}$.

For normalized coherent states with $q = 1$, the average gives $\alpha(t) = \alpha(0)e^{-i\omega t}$, as solution of the classical harmonic oscillator equation, $\ddot{\alpha}(t) + \omega^2\alpha(t) = 0$. For arbitrary q-coherent states, the average is the superposition

$${}_q\langle\alpha, t|a_q|\alpha, t\rangle_q = \alpha_q(t) = \alpha\frac{\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{\sqrt{[n]_q!}}e^{-i\frac{\hbar\omega}{2}(q^n(q+1))t}}{\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{\sqrt{[n]_q!}}} = \sum_{n=0}^{\infty}\alpha_n(t),$$

where we have defined frequency $\omega_n(q) = \omega\frac{q+1}{2}q^n$, and functions

$$\alpha_n(t) = \frac{\alpha}{\sqrt{e_q^{|\alpha|^2}}}\frac{|\alpha|^{2n}}{\sqrt{[n]_q!}}e^{-i\hbar\omega_n(q)t},$$

satisfying

$$\ddot{\alpha}_n(t) + \omega_n^2(q)\alpha_n(t) = 0.$$

They represent the set of harmonic oscillators with frequencies in geometric progression. The normalized time evolved q-coherent states also can be rewritten in following form

$$\begin{aligned}
\alpha_q(t) &= \frac{\alpha}{\sqrt{e_q^{|\alpha|^2}}} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{[n]_q!} e^{-i\frac{\omega}{2}(q^n(q+1))t} \\
&= \frac{\alpha}{\sqrt{e_q^{|\alpha|^2}}} \sum_{k=0}^{\infty} \frac{1}{k!}\left(-i\frac{\omega}{2}(q+1)t\right)^k \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{[n]_q!}(q^k)^n \\
&= \frac{\alpha}{\sqrt{e_q^{|\alpha|^2}}} \sum_{k=0}^{\infty} \frac{1}{k!}\left(-i\frac{\omega}{2}(q+1)t\right)^k e_q^{q^k|\alpha|^2} \\
&= \alpha \sum_{k=0}^{\infty} \frac{1}{k!}\left(-i\frac{\omega}{2}(q+1)t\right)^k \frac{e_q^{q^k|\alpha|^2}}{\sqrt{e_q^{|\alpha|^2}}}.
\end{aligned}$$

Definition 9.2 *Function of two variables $F_q(x,\tau)$, is defined as*

$$F_q(x,\tau) = \sum_{k=0}^{\infty} \frac{1}{k!}\tau^k e_q^{q^k x} = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} x^n e^{q^n \tau}. \tag{9.7}$$

Proposition 9.2 *Function $F_q(x,\tau)$ satisfies the initial value problem for differential-difference equation,*

$$D_q^{\alpha} F_q(x,\tau) = F_q(x, q\tau), \tag{9.8}$$

$$\frac{\partial}{\partial\tau} F_q(x,\tau) = F_q(qx,\tau), \tag{9.9}$$

$$F_q(x,0) = e_q^x \, , F_q(0,\tau) = e^{\tau}. \tag{9.10}$$

Proof We aim to demonstrate the existence and uniqueness of function $F_q(x,\tau)$ that satisfies the initial conditions. To achieve this, we expand the function in two variables

$$F_q(x,\tau) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_{n,m}x^n\tau^m.$$

and substitute to equations (9.8) and (9.9),respectively,

$$\begin{aligned} D_q^{\alpha}F_q(x,\tau) &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} b_{n,m}[n]_q x^{n-1}\tau^m = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_{n,m}x^n(q\tau)^m, \\ \frac{\partial}{\partial\tau}F_q(x,\tau) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_{n,m}x^n m\tau^{m-1} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_{n,m}(qx)^n\tau^m. \end{aligned}$$

These give the recurrence relations for $b_{n,m}$;

$$\begin{aligned} b_{n+1,m} &= \frac{q^m}{[n+1]_q}b_{n,m}, \\ b_{n,m+1} &= \frac{q^n}{m+1}b_{n,m}, \end{aligned}$$

and by (9.10),

$$b_{n,m} = \frac{q^{nm}}{[n]_q!m!}b_{0,0} \ .$$

Therefore, the solution has following form

$$F_q(x,\tau) \ = \ b_{0,0}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{nm}}{[n]_q!m!}x^n\tau^m.$$

Since $F_q(0,0) = 1$, then $b_{0,0} = 1$.Thus, we have it in the form

$$\begin{aligned} F_q(x,\tau) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{nm}}{[n]_q!m!}x^n\tau^m \\ &= \sum_{m=0}^{\infty}\frac{\tau^m}{m!}\sum_{n=0}^{\infty}\frac{(q^m)^n x^n}{[n]_q!} = \sum_{m=0}^{\infty}\frac{1}{m!}\tau^m e_q^{q^k x}, \end{aligned}$$

or

$$F_q(x,\tau) = \sum_{n=0}^{\infty}\frac{1}{[n]_q!}x^n e^{q^n\tau}.$$

□

Proposition 9.3 *The solution $\alpha_q(t)$ can be expressed by the function* (9.7) *as follows:*

$$\alpha_q(t) = \alpha\frac{F_q(|\alpha|^2, -i\frac{\omega}{2}(q+1)t)}{\sqrt{F_q(|\alpha|^2,0)}}. \qquad (9.11)$$

9.2. q-Supersymmetric annihilation operator

In Chapter 5 and Chapter 7, we considered supersymmetric annihilation operator $\widehat{A}$ in terms of bosonic operator $\hat{a}$ and deformation of it by parameter ζ. Here, by using $\hat{a}_q$ operators we propose a new supersymmetric annihilation operator in the form

$$\widehat{A}_q = \begin{bmatrix} q\hat{a}_q & 1 \\ 0 & \hat{a}_q \end{bmatrix}. \qquad (9.12)$$

When $q = 1$ it reduces to the one in Eq.(5.1). This operator determines the $q-$supersymmetric coherent state $|A\rangle_q$ as eigenstate of this operator,

$$\widehat{A}_q|A\rangle_q = \alpha|A\rangle_q. \qquad (9.13)$$

This supersymmetric q-coherent state $|A\rangle_q$ can be in two forms

$$|A_b\rangle_q = C_b \begin{bmatrix} |\frac{\alpha}{q}\rangle_q \\ 0 \end{bmatrix} \quad , \tag{9.14}$$

$$|A_s\rangle_q = C_s \begin{bmatrix} \frac{\bar{\alpha}}{q^2}|\frac{\alpha}{q}\rangle_q - |\frac{\alpha'}{q}\rangle_q \\ \\ |\alpha\rangle_q \end{bmatrix}. \tag{9.15}$$

These states are orthonormal with normalization constants

$$\begin{aligned} C_s^{-2} &= \left(\frac{(q-1)|\alpha|^2 + q^2}{q^4}\right) e_q^{\frac{|\alpha|^2}{q^2}} + e_q^{|\alpha|^2} \\ C_b^{-2} &= e_q^{|\alpha|^2}. \end{aligned}$$

9.3. Uncertainty relations for $q-$supersymmetric coherent state

The q-coordinate and the q-momentum operators are defined as

$$\hat{x}_q = \left(\frac{\hat{a}_q^{\dagger} + \hat{a}_q}{\sqrt{2}}\right) \otimes \hat{I}_F \ , \qquad \hat{p}_q = i\left(\frac{\hat{a}_q^{\dagger} - \hat{a}_q}{\sqrt{2}}\right) \otimes \hat{I}_F. \tag{9.16}$$

For average values of these operators in supersymmetric q-coherent states $|A_s\rangle_q$ and $|A_b\rangle_q$, we have

$$\begin{aligned} {}_q\langle A_b|\hat{x}_q|A_b\rangle_q &\equiv \frac{1}{q}\sqrt{2}\,\mathrm{Re}(\alpha), \\ {}_q\langle A_b|\hat{p}_q|A_b\rangle_q &\equiv \frac{1}{q}\sqrt{2}\,\mathrm{Im}(\alpha). \end{aligned}$$

To calculate uncertainty relations, one needs to find the averages of operators $\hat{X}_q^2$ and $\hat{P}_p^2$,

$$\hat{X}_q^2 \otimes \hat{I}_F = \frac{(\hat{a}_q^\dagger)^2 + (a_q)^2 + (q+1)[N]_q + 1}{2} \otimes \hat{I}_F,$$
$$\hat{P}_q^2 \otimes \hat{I}_F = \frac{-(\hat{a}_q^\dagger)^2 - (a_q)^2 + (q+1)[N]_q + 1}{2} \otimes \hat{I}_F.$$

By using addition formulas in (9.3) and (9.4) for average in state $|A_b\rangle_q$, we get

$$_q\langle A_b|\hat{X}_q^2|A_b\rangle_q \equiv \frac{1}{2}\left[1 + \frac{\bar{\alpha}^2}{q^2} + \frac{\alpha^2}{q^2} + (1+q)\frac{|\alpha|^2}{q^2}\right],$$
$$_q\langle A_b|\hat{P}_q^2|A_b\rangle_q \equiv \frac{1}{2}\left[1 - \frac{\bar{\alpha}^2}{q^2} - \frac{\alpha^2}{q^2} + (1+q)\frac{|\alpha|^2}{q^2}\right].$$

Proposition 9.4 *The dispersions of $\hat{x}$ and $\hat{p}$ operators in $|A_b\rangle_q$ state are equal,*

$$_q\langle A_b|\left(\Delta\hat{x}_q\right)^2|A_b\rangle_q \quad\equiv\quad _q\langle A_b|\left(\Delta\hat{p}_q\right)^2|A_b\rangle_q = \frac{1}{2} + \left(\frac{(q-1)}{q^2}|\alpha|^2\right).$$

Corollary 9.2 *The uncertainty relation in state $|A_b\rangle_q$ is*

$$\Delta\hat{x}_q\Delta\hat{p}_q = \frac{1}{2} + \left(\frac{(q-1)}{q^2}|\alpha|^2\right).$$

Proposition 9.5 *The averages for the state $|A_s\rangle_q$ are*

$$_q\langle A_s|\hat{x}_q|A_s\rangle_q \equiv \sqrt{2}\,\mathrm{Re}(\alpha),$$
$$_q\langle A_b|\hat{p}_q|A_b\rangle_q \equiv \sqrt{2}\,\mathrm{Im}(\alpha),$$

then

$$
\begin{aligned}
{}_q\langle A_s|\hat{X}_q^2|A_s\rangle_q &= |C_s|^2\Bigg[\Bigg(\frac{|\alpha|^2}{2q^4} + ((q+1)(q+2)+1)\,\frac{|\alpha|^2}{2q^3} - (q+2)\frac{|\alpha|^2}{q^4} \\
&+(q+1)\frac{|\alpha|^4}{2q^6} + (q+1)\frac{|\alpha|^4}{2q^3} - (q+1)\frac{|\alpha|^4}{q^5} \\
&+\frac{\bar{\alpha}^2+\alpha^2}{2q^3}|\alpha|^2 - \frac{\bar{\alpha}^2+\alpha^2}{2q^4}|\alpha|^2 + \frac{\bar{\alpha}^2+\alpha^2}{2q^2} + \frac{(q+2)}{2q^2}\Bigg) e_q^{\frac{|\alpha|^2}{q^2}} \\
&+\left(\frac{\bar{\alpha}^2+\alpha^2}{2} + \frac{q+1}{2}|\alpha|^2 + \frac{1}{2}\right) e_q^{|\alpha|^2}\Bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
{}_q\langle A_s|\hat{P}_q^2|A_s\rangle_q &= |C_s|^2\Bigg[\Bigg(\frac{|\alpha|^2}{2q^4} + ((q+1)(q+2)+1)\,\frac{|\alpha|^2}{2q^3} - (q+2)\frac{|\alpha|^2}{q^4} \\
&+(q+1)\frac{|\alpha|^4}{2q^6} + (q+1)\frac{|\alpha|^4}{2q^3} - (q+1)\frac{|\alpha|^4}{q^5} \\
&-\frac{\bar{\alpha}^2+\alpha^2}{2q^3}|\alpha|^2 + \frac{\bar{\alpha}^2+\alpha^2}{2q^4}|\alpha|^2 - \frac{\bar{\alpha}^2+\alpha^2}{2q^2} + \frac{(q+2)}{2q^2}\Bigg) e_q^{\frac{|\alpha|^2}{q^2}} \\
&+\left(-\frac{\bar{\alpha}^2+\alpha^2}{2} + \frac{q+1}{2}|\alpha|^2 + \frac{1}{2}\right) e_q^{|\alpha|^2}\Bigg].
\end{aligned}
$$

These give dispersions in the form

$$
\begin{aligned}
{}_q\langle A_s|\left(\Delta\hat{x}_q\right)^2|A_s\rangle_q &\equiv {}_q\langle A_s|\left(\Delta\hat{p}_q\right)^2|A_s\rangle_q \\
&= |C_s|^2\Bigg[\Bigg(\frac{|\alpha|^2}{2q^4} + ((q+1)(q+2)+1)\,\frac{|\alpha|^2}{2q^3} - (q+2)\frac{|\alpha|^2}{q^4} - \frac{|\alpha|^2}{q^2} \\
&+\left(\frac{1}{2q^6} + \frac{1}{2q^3} - \frac{1}{q^5}\right)(q+1)|\alpha|^4 - \frac{q-1}{q^4}|\alpha|^4 + \frac{q+2}{2q^2}\Bigg) e_q^{\frac{|\alpha|^2}{q^2}} \\
&+\left(\frac{q-1}{2}|\alpha|^2 + \frac{1}{2}\right) e_q^{|\alpha|^2}\Bigg],
\end{aligned}
$$

or

$$
\left(\Delta\hat{x}_q\right)_s^2 \equiv \left(\Delta\hat{p}_q\right)_s^2 = \frac{1}{2} + \frac{q-1}{2}|\alpha|^2 + \frac{\frac{q+1}{2q^2}\left(\frac{q-1}{q^2}|\alpha|^2+1\right)^2 e_q^{\frac{|\alpha|^2}{q^2}}}{\frac{1}{q^2}\left(\frac{q-1}{q^2}|\alpha|^2+1\right) e_q^{\frac{|\alpha|^2}{q^2}} + e_q^{|\alpha|^2}}.
$$

Then, the uncertainty relation for state $|A_s\rangle_q$ is

$$\left(\Delta\hat{x}_q\right)_s\left(\Delta\hat{p}_q\right)_s = \frac{1}{2} + \frac{q-1}{2}|\alpha|^2 + \frac{\frac{q+1}{2q^2}\left(\frac{q-1}{q^2}|\alpha|^2+1\right)^2 e_q^{\frac{|\alpha|^2}{q^2}}}{\frac{1}{q^2}\left(\frac{q-1}{q^2}|\alpha|^2+1\right)e_q^{\frac{|\alpha|^2}{q^2}} + e_q^{|\alpha|^2}}.$$

9.4. Time evolution of $q-$supersymmetric coherent state

To describe time evolution of $q-$supercoherent states we first show time evolution of AZ-supercoherent states.

9.4.1. Time evolution of $AZ-$supersymmetric coherent states

The time evolution operator $U(t)$ for supersymmetric oscillator is defined as

$$U(t) = e^{-i\frac{\hbar\omega}{2}\widehat{H}t} = e^{-i\omega\widehat{N}t} = \begin{pmatrix} e^{-i\omega Nt} & 0 \\ 0 & e^{-i\omega(N+1)t} \end{pmatrix}.$$

Application of operator $U(t)$ to state $|A_b\rangle$ gives

$$|A,t\rangle_b = U(t)|A\rangle_b = e^{-\frac{|\alpha|^2}{2}}\begin{pmatrix} e^{-i\omega Nt}|\alpha\rangle \\ 0 \end{pmatrix} = e^{-\frac{|\alpha(t)|^2}{2}}\begin{pmatrix} |\alpha(t)\rangle \\ 0 \end{pmatrix} \tag{9.17}$$

where $\alpha(t) = \alpha e^{-i\omega t}$ and for state $|A_s\rangle$,

$$|A,t\rangle_s = U(t)|A\rangle_s = \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{2}}\begin{pmatrix} \bar{\alpha}e^{-i\omega Nt}|\alpha\rangle - e^{-i\omega Nt}|\alpha'\rangle \\ e^{-i\omega(N+1)t}|\alpha\rangle \end{pmatrix} \tag{9.18}$$

$$= e^{-i\omega t}e^{-\frac{|\alpha|^2}{2}}\begin{pmatrix} \bar{\alpha}(t)|\alpha\rangle - |\alpha(t)\rangle' \\ |\alpha(t)\rangle \end{pmatrix} \tag{9.19}$$

This shows that time evolution of supersymmetric coherent state $|A, t\rangle_b$ is described by rotation in complex plane $\alpha : \alpha(t) = \alpha e^{-i\omega t}$, with frequency ω. For supersymmetric coherent state $|A, t\rangle_s$,we have an additional phase factor $e^{-i\omega t}$.

9.4.2. Time evolution of $q-$deformed supersymmetric coherent states

For supersymmetric q-oscillator, the Hamiltonian is

$$\hat{H}_{q_{ss}} = \omega \begin{pmatrix} a_q^+ a_q & 0 \\ 0 & a_q a_q^+ \end{pmatrix} = \omega \begin{pmatrix} [\hat{N}]_q & 0 \\ 0 & [\hat{N}+1]_q \end{pmatrix} = [\hat{N}_{ss}]_q \,,$$

and time evolution operator $U(t)$ is $(\omega = 1)$,

$$U(t) = \begin{pmatrix} e^{-i[\hat{N}]_q t} & 0 \\ 0 & e^{-i[\hat{N}+1]_q t} \end{pmatrix}.$$

Lemma 9.1 *For complex number α, the following relation holds*

$$\begin{aligned} q^{\alpha D_q^\alpha} \alpha^N = e^{\ln q \alpha D_q^\alpha} \alpha^N &= \sum_{n=0}^{\infty} \frac{(\ln q)^n}{n!} \left(\alpha D_q^\alpha\right)^n \alpha^N \\ &= \sum_{n=0}^{\infty} \frac{(\ln q)^n}{n!} \left([\hat{N}]_q\right)^n \alpha^N = e^{\ln q [\hat{N}]_q} \alpha^N = q^{[\hat{N}]_q} \alpha^N, \end{aligned}$$

where N is an arbitrary natural number. Then, for arbitrary analytic function $f(\alpha) = \sum_{N=0}^{\infty} C_N \alpha^N$, the relation is valid

$$q^{\alpha D_q^\alpha} f(\alpha) = q^{\alpha D_q^\alpha} \sum_{N=0}^{\infty} C_N \alpha^N = \sum_{n=0}^{\infty} C_N q^{[\hat{N}]_q} \alpha^N = \sum_{N=0}^{\infty} C_N \prod_{s=0}^{N-1} q^{q^s} \alpha^N,$$

Proposition 9.6 *The time evolution of q-supersymmetric coherent states, as given by* *equation* (9.14)*, can be expressed formally as follows*

$$|A_b(t)\rangle_q = U(t)|A_b(t)\rangle_q = \frac{1}{\sqrt{e_q^{|\alpha|^2/q^2}}}\begin{pmatrix} e^{-i[\hat{N}]_q t} & 0 \\ 0 & e^{-i[\hat{N}+1]_q t}\end{pmatrix}\begin{pmatrix} |\frac{\alpha}{q}\rangle_q \\ 0 \end{pmatrix} \tag{9.20}$$

$$= \frac{1}{\sqrt{e_q^{|\alpha|^2/q^2}}}\begin{pmatrix} e^{-i[\hat{N}]_q t}|\frac{\alpha}{q}\rangle_q \\ 0 \end{pmatrix}. \tag{9.21}$$

Proof To begin , we start with formula

$$e^{-i[\hat{N}]_q t}|\alpha\rangle_q = \sum_{n=0}^{\infty}\frac{(-it)^n}{n!}\left([\hat{N}]_q\right)^n|\alpha\rangle_q \tag{9.22}$$

$$= \sum_{n=0}^{\infty}\frac{(-it)^n}{n!}\left(\alpha D_q^{\alpha}\right)^n|\alpha\rangle_q = e^{-i\alpha D_q^{\alpha} t}|\alpha\rangle_q \tag{9.23}$$

where $[\hat{N}]_q|\alpha\rangle_q = \frac{q^N-1}{q-1}|\alpha\rangle_q = \frac{|q\alpha\rangle_q - |\alpha\rangle_q}{q-1}|\alpha\rangle_q = \alpha D_q^{\alpha}|\alpha\rangle_q$. Then, by using the above lemma, we have

$$q^{[\hat{N}]_q} = e^{\ln q(1+q+q^2+...+q^{N-1})} = e^{\ln q}e^{(\ln q)\, q}...e^{(\ln q)\, q^{N-1}} = q.q^q.q^{q^2}...q^{q^{(N-1)}} = \prod_{s=0}^{N-1} q^{q^s}. \tag{9.24}$$

Since q-number can be expressed by Bernoulli polynomials,

$$[\hat{N}]_q = \hat{N} + \sum_{m=1}^{\infty}\left(B_{m+1}(N) - B_{m+1}(0)\right)\frac{(\ln q)^m}{(m+1)!} \tag{9.25}$$

we can write

$$e^{-i[\hat{N}]_q t}|\alpha\rangle_q = e^{-i\alpha D_q^{\alpha} t}\sum_{N=0}^{\infty}\frac{\alpha^N}{\sqrt{[N]_q!}}|N\rangle_q \tag{9.26}$$

$$= \left(\alpha e^{-it}\right)^N\sum_{N=0}^{\infty}\frac{|N\rangle_q}{\sqrt{[N]_q!}}\prod_{k=1}^{\infty}\frac{e^{-it(\ln q)^k\frac{B_{k+1}(N)}{(k+1)!}}}{e^{-it(\ln q)^k\frac{B_{k+1}}{(k+1)!}}}. \tag{9.27}$$

Then,

$$|A_b(t)\rangle_q = U(t)|A_b(t)\rangle_q = \frac{1}{\sqrt{e_q^{|\alpha|^2/q^2}}}\begin{pmatrix} e^{-i[\hat{N}]_q t}|\frac{\alpha}{q}\rangle_q \\ 0 \end{pmatrix} \quad (9.28)$$

$$= \frac{1}{\sqrt{e_q^{|\alpha|^2/q^2}}}\left(\alpha e^{-it}\right)^N \sum_{N=0}^{\infty} \frac{|N\rangle_q}{q^N \sqrt{[N]_q!}} \prod_{k=1}^{\infty} \frac{e^{-it(\ln q)^k \frac{B_{k+1}(N)}{(k+1)!}}}{e^{-it(\ln q)^k \frac{B_{k+1}}{(k+1)!}}} \quad (9.29)$$

□

CHAPTER 10

CONCLUSION

In the present thesis, we have studied new class of coherent states for supersymmetric quantum oscillator and its relations with superqubit unit of quantum information. By generalizing supersymmetric annihilation operator of Aragone and Zypmann, we constructed four different type of supersymmetric coherent states related with the Bell two-qubit quantum states. These Bell super-qubit states determine the Bell-Based supersymmetric coherent states, which we created by using displacement operator. These states are entangled and we quantified the level of entanglement between bosons and fermions by the concurrence characteristics and the Von-Neumann entropy. We studied several properties of these Bell based super coherent states, as orthogonality and time evolution of entanglement. Uncertainty relation for these entangled super coherent states are expressed in terms of the concurrence. Monotonical dependence of uncertainty in concurrence C shows that the uncertainty relations can be also considered as a measure of entanglement. In fact, minimum uncertainty corresponds to seperable coherent stateas and maximal uncertainty corresponds to maximally entagled states. This allowed us to see the influence of entanglement on uncertainty relations. Particularly, we found quadrature squeezing of coordinate and momentum uncertainties. Moreover, we describe infinite sequence of super coherent states with uncertainty relations, determined by ratio of two Fibonacci numbers. The limiting state $n \to \infty$ is the Golden-supercoherent state and corresponding uncertainty relation is determined by the Golden ratio.

As a generalization of previous results, we introduced the generic super-qubit quantum state, where the one super-particle state is determined by complex parameter ζ as stereographic projection of corresponding unit sphere. This allowed us to introduce the concept of PK-super-qubit quantum states, which are parametrized by two unit spheres. These states determine the so called PK-supersymmetric coherent states and we found entanglement of these states. The information content of PK-super-qubit quantum states is twice bigger than the standard qubit state. For PK-supersymmetric coherent states, we constructed corresponding flipped states. Then, Fibonacci sequence of

PK-supersymmetric coherent states with correponding uncertainty relations was derived. Finally, we described $pq-$deformed supersymmetric annihilation operator, corresponding $pq-$supersymmetric coherent states and uncertainty relations. More explicit form of these calculations, including time evoluiton, we did for particular case of $q-$deformed supersymmetric coherent states.

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APPENDIX A

GENERALIZATION OF SUPERSYMMETRIC ANNIHILATION OPERATOR

In Section 7.2, we have introduced the supersymmetric annihilation operator and corresponding $PK-$supercoherent states. By using conformal mapping $\varepsilon = -\frac{1}{\zeta}$, we get annihilation operator in the form

$$\hat{A}_\varepsilon = \begin{pmatrix} \hat{a} & \varepsilon \\ 0 & \hat{a} \end{pmatrix}, \tag{A.1}$$

where $\varepsilon \in \mathbb{C}$ is a complex parameter. When $\varepsilon = 1$, this operator reduces to the supersymmetric annihilation operator defined in equation (5.1). For $\varepsilon = 0$, it simplifies to $\hat{A}_{\varepsilon=0} = \widehat{I} \otimes \hat{a}$. The operator $\hat{A}_\varepsilon$ satisfies the commutation relation $[\hat{A}_\varepsilon, \widehat{H}] = \omega\hat{A}_\varepsilon$ and has an internal commutation structure

$$[\hat{A}_\varepsilon, \hat{A}_\varepsilon^\dagger] = \begin{pmatrix} 1 + |\varepsilon|^2 & 0 \\ 0 & 1 - |\varepsilon|^2 \end{pmatrix} = \widehat{I} + |\varepsilon|^2\sigma_3.$$

In particular, for $\varepsilon = 0$, this reduces to $[\hat{A}_{\varepsilon=0}, \hat{A}_{\varepsilon=0}^\dagger] = \widehat{I}$, while for $\varepsilon = 1$, it yields $[\hat{A}_{\varepsilon=1}, \hat{A}_{\varepsilon=1}^\dagger] = \widehat{I} - \widehat{N}_f$.The supercoherent states $|\alpha\rangle_{scs}$ are defined as the eigenstates of the supersymmetric annihilation operator $\hat{A}_\varepsilon(A.1)$,

$$\hat{A}_\varepsilon|\alpha\rangle_{scs} = \alpha|\alpha\rangle_{scs}. \tag{A.2}$$

These states can be expressed as a linear combination of two basis eigenstates

$$|\alpha\rangle_{scs} = a_0|\alpha\rangle_b + c_1|\widetilde{\alpha}\rangle_s,$$

where the basis states are given by

$$|\alpha\rangle_b \equiv \begin{pmatrix} |\alpha\rangle \\ 0 \end{pmatrix}, \quad |\widetilde{\alpha}\rangle_s \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -\varepsilon|\alpha'\rangle \\ |\alpha\rangle \end{pmatrix}. \tag{A.3}$$

However, these basis states are not orthogonal, a new state can be defined as

$$|\alpha\rangle_s \equiv \lambda|\alpha\rangle_b + \mu|\widetilde{\alpha}\rangle_s,$$

which is orthogonal to $|\alpha\rangle_b$ when $\lambda = \varepsilon\mu\overline{\alpha}$. Thus, the orthogonal set of states consists of $|\alpha\rangle_b$ and $|\alpha\rangle_s = \mu\left(\varepsilon\overline{\alpha}|\alpha\rangle_b + |\widetilde{\alpha}\rangle_s\right)$. After normalization, the orthogonal states are obtained as

$$|A_\varepsilon\rangle_b = e^{-\frac{|\alpha|^2}{2}} \begin{pmatrix} |\alpha\rangle \\ 0 \end{pmatrix}, \quad |A_\varepsilon\rangle_s = \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{1+|\varepsilon|^2}} \begin{pmatrix} \varepsilon\overline{\alpha}|\alpha\rangle - \varepsilon|\alpha'\rangle \\ |\alpha\rangle \end{pmatrix}.$$

It is observed that the state $|A_\varepsilon\rangle_b$ does not depend on ε and is determined solely by the Glauber coherent state $|\alpha\rangle$. Therefore, the index ε is omitted for this state.

A.1. Coordinate and Momentum Representation of ε-Supersymmetric coherent States

Since $|A_\varepsilon\rangle_b \equiv |A\rangle_b$, their representations are identical, it is sufficient to perform calculations for $|A_\varepsilon\rangle_s$.The coordinate representations for the ε-supersymmetric state $|A_\varepsilon\rangle_s$ provide the following wave functions. For the bosonic component, the coordinate wave function is given by

$${}_b\langle x|A_\varepsilon\rangle_s = \frac{\sqrt{2}\varepsilon}{\sqrt{1+|\varepsilon|^2}} \frac{e^{-\frac{|\alpha|^2}{2}}}{\pi^{1/4}} e^{-\left(x-\frac{\alpha}{\sqrt{2}}\right)^2} e^{\frac{x^2}{2}} (\sqrt{2}\,\mathrm{Re}(\alpha) - x), \tag{A.4}$$

while the fermionic component has the form

$$
{}_f\langle x|A_\varepsilon\rangle_s = \frac{1}{\sqrt{1+|\varepsilon|^2}} \frac{e^{-\frac{|\alpha|^2}{2}}}{\pi^{1/4}} e^{-\left(x-\frac{\alpha}{\sqrt{2}}\right)^2} e^{\frac{x^2}{2}}. \tag{A.5}
$$

From these wave functions, the corresponding probability distributions in the coordinate representation can be derived. For the fermionic component, the probability distribution is

$$
|{}_f\langle x|A_\varepsilon\rangle_s|^2 = \frac{1}{1+|\varepsilon|^2} \frac{1}{\sqrt{\pi}} e^{-\left(x-\sqrt{2}\,\mathrm{Re}(\alpha)\right)^2}, \tag{A.6}
$$

and for the bosonic component,

$$
|{}_b\langle x|A_\varepsilon\rangle_s|^2 = \frac{2|\varepsilon|^2}{1+|\varepsilon|^2} \frac{1}{\sqrt{\pi}} e^{-\left(x-\sqrt{2}\,\mathrm{Re}(\alpha)\right)^2} \left(x-\sqrt{2}\,\mathrm{Re}(\alpha)\right)^2. \tag{A.7}
$$

Similarly, the momentum representations of the ε-supersymmetric states $|A_\varepsilon\rangle_s$ can be calculated based on the momentum eigenstates. The bosonic momentum representation for $|A_\varepsilon\rangle_s$ is

$$
{}_b\langle p|A_\varepsilon\rangle_s = i\frac{\sqrt{2}\varepsilon}{\sqrt{1+|\varepsilon|^2}} \frac{e^{-\frac{|\alpha|^2}{2}}}{\pi^{1/4}} e^{-\left(p+i\frac{\alpha}{\sqrt{2}}\right)^2} e^{\frac{p^2}{2}} \left(p-\sqrt{2}\,\mathrm{Im}(\alpha)\right), \tag{A.8}
$$

and the fermionic component in the momentum representation is

$$
{}_f\langle p|A_\varepsilon\rangle_s = \frac{1}{\sqrt{1+|\varepsilon|^2}} \frac{e^{-\frac{|\alpha|^2}{2}}}{\pi^{1/4}} e^{-\left(p+i\frac{\alpha}{\sqrt{2}}\right)^2} e^{\frac{p^2}{2}}. \tag{A.9}
$$

The probability distributions in the momentum representation for these components are

$$|_f\langle p|A\rangle_s|^2 = \frac{1}{1+|\varepsilon|^2}\frac{1}{\sqrt{\pi}}e^{-\left(p-\sqrt{2}\,\mathrm{Im}(\alpha)\right)^2}, \tag{A.10}$$

$$|_b\langle p|A\rangle_s|^2 = \frac{2|\varepsilon|^2}{1+|\varepsilon|^2}\frac{1}{\sqrt{\pi}}e^{-\left(p-\sqrt{2}\,\mathrm{Im}(\alpha)\right)^2}(p-\sqrt{2}\,\mathrm{Im}(\alpha))^2. \tag{A.11}$$

A.2. Uncertainity in Superqubit state

To calculate the uncertainty, it is necessary to determine both the mean $\langle\hat{x}\rangle$ and the mean of its square, $\langle\hat{x}^2\rangle$. Since the operators $\hat{x}$ and $\hat{p}$ are given by

$$\hat{x} = \frac{1}{\sqrt{2\omega}}(\hat{a}+\hat{a}^\dagger)\,,\ \hat{p} = i\sqrt{\frac{\omega}{2}}(\hat{a}^\dagger-\hat{a}),$$

we find that the expectation values in the states $|A_\varepsilon\rangle_s$ and $|A\rangle_b$ are

$$\begin{aligned} {}_s\langle A_\varepsilon|\hat{x}|A_\varepsilon\rangle_s &= {}_b\langle A|\hat{x}|A\rangle_b = \frac{\bar{\alpha}+\alpha}{\sqrt{2\omega}}, \\ {}_s\langle A_\varepsilon|\hat{p}|A_\varepsilon\rangle_s &= {}_b\langle A|\hat{p}|A\rangle_b = i\sqrt{\frac{\omega}{2}}(\bar{\alpha}-\alpha), \end{aligned}$$

where these results are independent of ε. Additionally, we have the following off-diagonal elements:

$$\begin{aligned} {}_s\langle A_\varepsilon|\hat{x}|A\rangle_b &= -\frac{\bar{\varepsilon}}{\sqrt{2\omega(1+|\varepsilon|^2)}}, \\ {}_s\langle A_\varepsilon|\hat{p}|A\rangle_b &= -i\frac{\bar{\varepsilon}}{\sqrt{2\omega(1+|\varepsilon|^2)}}, \\ {}_b\langle A|\hat{x}|A_\varepsilon\rangle_s &= -\frac{\varepsilon}{\sqrt{2\omega(1+|\varepsilon|^2)}}, \\ {}_b\langle A|\hat{p}|A_\varepsilon\rangle_s &= i\frac{\varepsilon}{\sqrt{2\omega(1+|\varepsilon|^2)}}. \end{aligned}$$

Definition A.1 *The state* $|\alpha; \varepsilon, \theta, \phi\rangle_{scs}$ *represents a superqubit state given by:*

$$|\alpha; \varepsilon, \theta, \phi\rangle_{scs} = \cos\frac{\theta}{2}|A\rangle_b + e^{i\phi}\sin\frac{\theta}{2}|A_\varepsilon\rangle_s, \tag{A.12}$$

and is parameterized by points on the super-Bloch sphere $(0 \le \theta \le \pi, 0 \le \phi \le 2\pi)$. *For* $\varepsilon = 0$, *this state is a direct product of a single-qubit state* $|\theta, \phi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ *and the Glauber coherent state:*

$$|\alpha; \varepsilon = 0, \theta, \phi\rangle_{scs} = |\theta, \phi\rangle \otimes |\alpha\rangle e^{-\frac{|\alpha|^2}{2}}.$$

If we calculate for the state in $(A.12)$, then the expectation values of $\hat{x}$ and $\hat{p}$ for the state $|\alpha; \varepsilon, \theta, \phi\rangle_{scs}$ are

$$\begin{aligned}
{}_{scs}\langle\alpha; \varepsilon, \theta, \phi|\hat{x}|\alpha; \varepsilon, \theta, \phi\rangle_{scs} &= \frac{\bar{\alpha} + \alpha}{\sqrt{2\omega}} - \frac{\sin\theta}{\sqrt{2\omega(1 + |\varepsilon|^2)}}\left(\frac{\varepsilon e^{i\varphi} + \bar{\varepsilon}e^{-i\varphi}}{2}\right), \\
{}_{scs}\langle\alpha; \varepsilon, \theta, \phi|\hat{p}|\alpha; \varepsilon, \theta, \phi\rangle_{scs} &= i\sqrt{\frac{\omega}{2}}(\bar{\alpha} - \alpha) + i\sqrt{\frac{\omega}{2}}\frac{\sin\theta}{\sqrt{1 + |\varepsilon|^2}}\left(\frac{\varepsilon e^{i\varphi} - \bar{\varepsilon}e^{-i\varphi}}{2}\right).
\end{aligned}$$

To calculate the mean values of the operator $\hat{x}^2$ in the states $|A\rangle_b$ and $|A_\varepsilon\rangle_s$, we start by representing $\hat{x}^2$ in terms of the creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ as follows:

$$\begin{aligned}
\hat{x}^2 &= \left(\frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger)\right)^2 = \frac{1}{2\omega}\left((\hat{a}^\dagger)^2 + \hat{a}^2 + 2\hat{a}^\dagger\hat{a} + 1\right), \\
\hat{p}^2 &= \left(i\sqrt{\frac{\omega}{2}}(\hat{a}^\dagger - \hat{a})\right)^2 = \frac{-\omega}{2}\left((\hat{a}^\dagger)^2 + \hat{a}^2 - 2\hat{a}^\dagger\hat{a} - 1\right).
\end{aligned}$$

Next, we evaluate the mean values for the terms in these expressions. First, for the state $|A\rangle_b$, we get

$${}_b\langle A|\hat{a}^2|A\rangle_b = \alpha^2 \quad , \quad {}_b\langle A|(\hat{a}^\dagger)^2|A\rangle_b = \bar{\alpha}^2 \quad , \quad {}_b\langle A|\hat{a}^\dagger\hat{a}|A\rangle_b = |\alpha|^2.$$

Similarly, for the state $|A_\varepsilon\rangle_s$, we obtain

$$ {}_s\langle A_\varepsilon|\hat{a}^2|A_\varepsilon\rangle_s = \alpha^2 \,, \quad {}_s\langle A_\varepsilon|(\hat{a}^\dagger)^2|A_\varepsilon\rangle_s = \bar{\alpha}^2 \,, \quad {}_s\langle A_\varepsilon|\hat{a}^\dagger\hat{a}|A_\varepsilon\rangle_s = |\alpha|^2 + \frac{|\varepsilon|^2}{1+|\varepsilon|^2}. $$

We also calculate additional cross terms

$$ \begin{aligned} &{}_s\langle A_\varepsilon|\hat{a}^2|A\rangle_b = 0 \,, \quad {}_b\langle A|(\hat{a^\dagger})^2|A_\varepsilon\rangle_s = 0 \\ &{}_s\langle A_\varepsilon|(\hat{a^\dagger})^2|A\rangle_b = \frac{-2\bar{\alpha}\bar{\varepsilon}}{\sqrt{1+|\varepsilon|^2}} \,, \quad {}_b\langle A|\hat{a}^2|A_\varepsilon\rangle_s = \frac{-2\alpha\varepsilon}{\sqrt{1+|\varepsilon|^2}} \\ &{}_s\langle A_\varepsilon|\hat{a}^\dagger\hat{a}|A\rangle_b = \frac{-\alpha\bar{\varepsilon}}{\sqrt{1+|\varepsilon|^2}} \,, \quad {}_b\langle A|\hat{a}^\dagger\hat{a}|A_\varepsilon\rangle_s = \frac{-\bar{\alpha}\varepsilon}{\sqrt{1+|\varepsilon|^2}}. \end{aligned} $$

By using above relations, we can expand the mean of $\hat{x}^2$ in $|A\rangle_b$ as

$$ \begin{aligned} {}_b\langle A|\hat{x}^2|A\rangle_b &= \frac{1}{2\omega}\left[{}_b\langle A|\hat{a}^2 + (\hat{a^\dagger})^2 + 2\hat{a}^\dagger\hat{a} + 1|A\rangle_b\right], \\ &= \frac{1}{2\omega}\left[{}_b\langle A|\hat{a}^2|A\rangle_b + {}_b\langle A|(\hat{a^\dagger})^2|A\rangle_b + 2\,{}_b\langle A|\hat{a}^\dagger\hat{a}|A\rangle_b +{}_b\langle A|A\rangle_b\right], \\ &= \frac{1}{2\omega}\left[(\bar{\alpha}+\alpha)^2 + 1\right]. \end{aligned} $$

For the cross terms in $|A\rangle_b$ and $|A_\varepsilon\rangle_s$ when calculating the mean value of the $\hat{x}^2$ operator, we find

$$ \begin{aligned} {}_b\langle A|\hat{x}^2|A\rangle_b &= \frac{1}{2\omega}\left[(\bar{\alpha}+\alpha)^2 + 1\right] \\ {}_s\langle A_\varepsilon|\hat{x}^2|A_\varepsilon\rangle_s &= \frac{1}{2\omega}\left[(\bar{\alpha}+\alpha)^2 + 1 + \frac{2|\varepsilon|^2}{1+|\varepsilon|^2}\right] \\ {}_b\langle A|\hat{x}^2|A_\varepsilon\rangle_s &= \frac{-\varepsilon}{\omega}\left(\frac{\bar{\alpha}+\alpha}{\sqrt{1+|\varepsilon|^2}}\right) \\ {}_s\langle A_\varepsilon|\hat{x}^2|A\rangle_b &= \frac{-\bar{\varepsilon}}{\omega}\left(\frac{\bar{\alpha}+\alpha}{\sqrt{1+|\varepsilon|^2}}\right), \end{aligned} $$

and for the mean value of the $\hat{p}^2$ operator, the cross terms are given by

$$
\begin{aligned}
{}_b\langle A|\hat{p}^2|A\rangle_b &= \frac{-\omega}{2}\left[(\alpha-\bar{\alpha})^2-1\right] \\
{}_s\langle A_\varepsilon|\hat{p}^2|A_\varepsilon\rangle_s &= \frac{-\omega}{2}\left[(\alpha-\bar{\alpha})^2-1-\frac{2|\varepsilon|^2}{1+|\varepsilon|^2}\right] \\
{}_b\langle A|\hat{p}^2|A_\varepsilon\rangle_s &= \frac{-\omega}{2}\left(\frac{2\varepsilon(\bar{\alpha}-\alpha)}{\sqrt{1+|\varepsilon|^2}}\right) \\
{}_s\langle A_\varepsilon|\hat{p}^2|A\rangle_b &= \frac{-\omega}{2}\left(\frac{2\bar{\varepsilon}(\alpha-\bar{\alpha})}{\sqrt{1+|\varepsilon|^2}}\right).
\end{aligned}
$$

Then, the mean values of the operator $\hat{x}^2$ for the super qubit state $|\alpha;\varepsilon,\theta,\phi\rangle_{scs}$ is expanded by

$$
\begin{aligned}
{}_{scs}\langle\alpha;\varepsilon,\theta,\phi|\hat{x}^2|\alpha;\varepsilon,\theta,\phi\rangle_{scs} &= \cos^2\frac{\theta}{2}\,{}_b\langle A|\hat{x}^2|A\rangle_b+\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi}\,{}_b\langle A|\hat{x}^2|A_\varepsilon\rangle_s \\
&+ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi}\,{}_s\langle A_\varepsilon|\hat{x}^2|A\rangle_b+\sin^2\frac{\theta}{2}\,{}_s\langle A_\varepsilon|\hat{x}^2|A_\varepsilon\rangle_s \\
&= \frac{1}{2\omega}\left[(\bar{\alpha}+\alpha)^2+1\right]+\frac{1}{\omega}\sin^2\frac{\theta}{2}\left[\frac{|\varepsilon|^2}{1+|\varepsilon|^2}\right] \\
&- \frac{\sin\theta}{2\omega}\frac{\bar{\alpha}+\alpha}{\sqrt{1+|\varepsilon|^2}}(\varepsilon e^{i\phi}+\bar{\varepsilon}e^{-i\phi}).
\end{aligned}
$$

Corollary A.1 *The mean values of the operators $\hat{x}^2$ and $\hat{p}^2$ in the super qubit state $|\alpha;\varepsilon,\theta,\phi\rangle_{scs}$ are given by*

$$
\begin{aligned}
{}_{scs}\langle\alpha;\varepsilon,\theta,\phi|\hat{x}^2|\alpha;\varepsilon,\theta,\phi\rangle_{scs} &= \frac{1}{2\omega}\left[(\bar{\alpha}+\alpha)^2+1\right]+\frac{1}{\omega}\sin^2\frac{\theta}{2}\left[\frac{|\varepsilon|^2}{1+|\varepsilon|^2}\right] \\
&-\frac{1}{2\omega}\sin\theta\frac{\bar{\alpha}+\alpha}{\sqrt{1+|\varepsilon|^2}}\left(\varepsilon e^{i\phi}+\bar{\varepsilon}e^{-i\phi}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
{}_{scs}\langle\alpha;\varepsilon,\theta,\phi|\hat{p}^2|\alpha;\varepsilon,\theta,\phi\rangle_{scs} &= \frac{-\omega}{2}\left[(\alpha-\bar{\alpha})^2-1\right]+\frac{\omega}{2}\sin^2\frac{\theta}{2}\left[\frac{2|\varepsilon|^2}{1+|\varepsilon|^2}\right] \\
&-\frac{\omega}{2}\sin\theta\frac{\bar{\alpha}-\alpha}{\sqrt{1+|\varepsilon|^2}}\left(\varepsilon e^{i\phi}-\bar{\varepsilon}e^{-i\phi}\right).
\end{aligned}
$$

Using the previous calculations, we determine the dispersions for the superqubit state $|\alpha;\varepsilon,\theta,\phi\rangle_{scs}$

$$(\Delta\hat{x})^2_{scs} = \frac{1}{2\omega} + \frac{1}{\omega}\sin^2\frac{\theta}{2}\left[\frac{|\varepsilon|^2}{1+|\varepsilon|^2}\right] - \frac{1}{2\omega}\frac{\sin^2\theta}{1+|\varepsilon|^2}\left[\frac{\varepsilon e^{i\phi}+\bar{\varepsilon}e^{-i\phi}}{2}\right]^2$$
$$(\Delta\hat{p})^2_{scs} = \frac{\omega}{2} + \frac{\omega}{2}\sin^2\frac{\theta}{2}\left[\frac{2|\varepsilon|^2}{1+|\varepsilon|^2}\right] + \frac{\omega}{2}\frac{\sin^2\theta}{1+|\varepsilon|^2}\left[\frac{\varepsilon e^{i\phi}-\bar{\varepsilon}e^{-i\phi}}{2}\right]^2.$$

Corollary A.2 *For the superqubit state $|\alpha;\varepsilon,\theta,\phi\rangle_{scs}$, the uncertainty relation is expressed as*

$$\begin{aligned}(\Delta\hat{x})^2_{scs}(\Delta\hat{p})^2_{scs} &= \left[\frac{1}{2}+\sin^2\frac{\theta}{2}\left(\frac{|\varepsilon|^2}{1+|\varepsilon|^2}\right)\right]^2 - \left[\frac{1}{2}+\sin^2\frac{\theta}{2}\left(\frac{|\varepsilon|^2}{1+|\varepsilon|^2}\right)\right]\frac{\sin^2\theta|\varepsilon|^2}{2(1+|\varepsilon|^2)} \\ &- \frac{\sin^4\theta}{4(1+|\varepsilon|^2)^2}\left[\frac{\varepsilon^2 e^{2i\phi}-(\bar{\varepsilon})^2 e^{-2i\phi}}{4}\right]^2\end{aligned}$$

where $\omega = 1$.

This expression is consistent with the uncertainty formula in Equation (5.27) when $\varepsilon = 1$. For $\theta = 0$ and $\theta = \pi$, representing the states $|A\rangle_b$ and $|A_\varepsilon\rangle_s$ respectively, we obtain

$$(\Delta\hat{x})^2_b(\Delta\hat{p})^2_b = \frac{1}{4},$$
$$(\Delta\hat{x})^2_s(\Delta\hat{s})^2_s = \left(\frac{1}{2}+\frac{|\varepsilon|^2}{1+|\varepsilon|^2}\right)^2.$$

The first result is independent of ε, while the second is bounded by

$$\frac{1}{2} \leq (\Delta\hat{x})_s(\Delta\hat{p})_s = \frac{1}{2}+\frac{|\varepsilon|^2}{1+|\varepsilon|^2} \leq \frac{3}{2},$$

depending on $|\varepsilon| \to 0$ or $|\varepsilon| \to \infty$. In the limit $|\varepsilon| = 0$, the arbitrary superqubit state achieves the minimum uncertainty

$$(\Delta\hat{x})_{scs}(\Delta\hat{p})_{scs} = \frac{1}{2}.$$

A.2.1. Coordinate and Momentum Representation for $\varepsilon-$superqubit state

The coordinate representation of supersymmetric coherent states $|\alpha;\varepsilon,\theta,\phi\rangle_{scs}$ can be expressed in terms of fermionic and bosonic components. The coordinate representation of the fermionic part of the supersymmetric coherent state is given by

$$_f\langle x|\alpha;\varepsilon,\theta,\phi\rangle_{scs} = \cos\frac{\theta}{2}\,{}_f\langle x|A\rangle_b + e^{i\phi}\sin\frac{\theta}{2}\,{}_f\langle x|A_\varepsilon\rangle_s,$$

and expanding this expression, we get

$$_f\langle x|\alpha;\varepsilon,\theta,\phi\rangle_{scs} = \frac{1}{\sqrt{1+|\varepsilon|^2}}\left(\frac{e^{-\frac{|\alpha|^2}{2}}}{\pi^{1/4}}e^{i\phi}\sin\frac{\theta}{2}e^{\frac{x^2}{2}}e^{-\left(x-\frac{\alpha}{\sqrt{2}}\right)^2}\right).$$

The bosonic component of the supersymmetric coherent state in coordinate representation can be expressed as

$$_b\langle x|\alpha;\varepsilon,\theta,\phi\rangle_{scs} = \cos\frac{\theta}{2}\,{}_b\langle x|A\rangle_b + e^{i\phi}\sin\frac{\theta}{2}\,{}_b\langle x|A_\varepsilon\rangle_s,$$

which can be simplified to

$$_b\langle x|\alpha;\varepsilon,\theta,\phi\rangle_{scs} = \frac{e^{-\frac{|\alpha|^2}{2}}}{\pi^{1/4}}e^{\frac{x^2}{2}}\left(\cos\frac{\theta}{2}+\left(\frac{\sqrt{2}\varepsilon}{\sqrt{1+|\varepsilon|^2}}\right)e^{i\phi}\sin\frac{\theta}{2}(\sqrt{2}\,\mathrm{Re}(\alpha)-x)\right)e^{-\left(x-\frac{\alpha}{\sqrt{2}}\right)^2}.$$

The momentum representation of the supersymmetric coherent states $|\alpha;\varepsilon,\theta,\phi\rangle_{scs}$ for the fermionic part is given by

$$_f\langle p|\alpha;\varepsilon,\theta,\phi\rangle_{scs} = \cos\frac{\theta}{2}\,{}_f\langle p|A\rangle_b + e^{i\phi}\sin\frac{\theta}{2}\,{}_f\langle p|A_\varepsilon\rangle_s,$$

which can be written as

$$_{f}\langle p|\alpha;\varepsilon,\theta,\phi\rangle_{scs} = \frac{1}{\sqrt{1+|\varepsilon|^2}}\left(\frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{2}\pi^{1/4}}e^{i\phi}\sin\frac{\theta}{2}e^{\frac{p^2}{2}}e^{-\left(p+i\frac{\alpha}{\sqrt{2}}\right)^2}\right).$$

For the bosonic part, it is expressed by

$$_{b}\langle p|\alpha;\varepsilon,\theta,\phi\rangle_{scs} = \cos\frac{\theta}{2}\,{}_{b}\langle p|A\rangle_{b} + e^{i\phi}\sin\frac{\theta}{2}\,{}_{b}\langle p|A_{\varepsilon}\rangle_{s},$$

which expands further as

$$_{b}\langle p|\alpha;\varepsilon,\theta,\phi\rangle_{scs} = \frac{e^{-\frac{|\alpha|^2}{2}}}{\pi^{1/4}}e^{\frac{p^2}{2}}\left(\cos\frac{\theta}{2}-\left(\frac{\sqrt{2}\varepsilon}{\sqrt{1+|\varepsilon|^2}}\right)ie^{i\phi}\sin\frac{\theta}{2}(\sqrt{2}\,\mathrm{Im}(\alpha)-p)\right)e^{-\left(p+i\frac{\alpha}{\sqrt{2}}\right)^2}.$$

APPENDIX B

PQ-COHERENT STATES

B.1. Inner products of pq-coherent states

In this section, we explore the inner products of *pq*-coherent states, which are crucial for understanding the overlap between different coherent states in a given basis. These inner products form the foundation for calculating various physical observables and expectation values in the *pq*-coherent state. The analysis of these overlaps also provides insight into the behavior of *pq*-deformed systems, revealing how quantum properties such as uncertainty and coherence are affected by the deformation parameters p and q.

Proposition B.1 *The inner product of the states* $\left|\frac{\beta}{\lambda}\right\rangle_{pq}$ *and* $\left|\frac{\alpha'}{\mu}\right\rangle_{pq}$ *is given by the following expression*

$$\left\langle\frac{\beta}{\lambda}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} = \frac{\overline{\beta}}{\lambda\mu} e_{pq}^{\frac{\alpha\overline{\beta}}{\lambda\mu}}. \tag{B.1}$$

Furthermore, for the case $\beta = \alpha$, *the inner product becomes:*

$$\left\langle\frac{\alpha}{\lambda}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} = \frac{\overline{\alpha}}{\lambda\mu} e_{pq}^{\frac{|\alpha|^2}{\lambda\mu}},$$

where λ *and* μ *are real numbers.*

Proof The inner product between the states is computed using the differentiation operator D_{pq}^{α}, which acts on the coherent state overlap. Then, we continue with substituting the known result for the inner product and it results as following

$$\left\langle\frac{\beta}{\lambda}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} = D_{pq}^{\alpha}\left\langle\frac{\beta}{\lambda}\middle|\frac{\alpha}{\mu}\right\rangle_{pq} = D_{pq}^{\alpha} e_{pq}^{\frac{\alpha\overline{\beta}}{\lambda\mu}} = \frac{\overline{\beta}}{\lambda\mu} e_{pq}^{\frac{\alpha\overline{\beta}}{\lambda\mu}}.$$

□

Lemma B.1 *The inner product between the states* $\left|\frac{\beta'}{\lambda}\right\rangle_{pq}$ *and* $\left|\frac{\alpha'}{\mu}\right\rangle_{pq}$ *can be computed as*

$$\left\langle \frac{\beta'}{\lambda} \middle| \frac{\alpha'}{\mu} \right\rangle_{pq} = \frac{1}{\lambda\mu} e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}} + q\frac{\alpha\bar{\beta}}{(\lambda\mu)^2} e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} \tag{B.2}$$

$$= \frac{1}{\lambda\mu} e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} + p\frac{\alpha\bar{\beta}}{(\lambda\mu)^2} e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}. \tag{B.3}$$

Proof The inner product is computed by applying the derivative operators D_{pq}^{α} and $D_{pq}^{\bar{\beta}}$. First, by applying the derivative operator $D_{pq}^{\bar{\beta}}$ to the exponential function, we obtain

$$\left\langle \frac{\beta'}{\lambda} \middle| \frac{\alpha'}{\mu} \right\rangle_{pq} = D_{pq}^{\alpha}\left(D_{pq}^{\bar{\beta}} e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}\right) = D_{pq}^{\alpha}\left(\frac{\alpha}{\lambda\mu} e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}\right).$$

By using equations (3.6) and (3.7) for the action of the differentiation operators, the expression becomes

$$\left\langle \frac{\beta'}{\lambda} \middle| \frac{\alpha'}{\mu} \right\rangle_{pq} = \frac{1}{\lambda\mu} e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}} + q\frac{\alpha\bar{\beta}}{(\lambda\mu)^2} e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}},$$

or equivalently

$$\left\langle \frac{\beta'}{\lambda} \middle| \frac{\alpha'}{\mu} \right\rangle_{pq} = \frac{1}{\lambda\mu} e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} + p\frac{\alpha\bar{\beta}}{(\lambda\mu)^2} e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}.$$

This completes the proof. □

Corollary B.1 *When* $\beta = \alpha$, *the inner product between the states simplifies to the following forms*

$$\left\langle \frac{\alpha'}{\lambda} \middle| \frac{\alpha'}{\mu} \right\rangle_{pq} = \frac{1}{\lambda\mu}\left(\frac{q}{\lambda\mu}|\alpha|^2 e_{pq}^{\frac{|\alpha|^2}{\lambda\mu}} + e_{pq}^{p\frac{|\alpha|^2}{\lambda\mu}}\right), \tag{B.4}$$

$$= \frac{1}{\lambda\mu}\left(\frac{p}{\lambda\mu}|\alpha|^2 e_{pq}^{\frac{|\alpha|^2}{\lambda\mu}} + e_{pq}^{q\frac{|\alpha|^2}{\lambda\mu}}\right). \tag{B.5}$$

B.2. Orthogonality and Normalization of $|A_b\rangle_{pq}$ and $|A_s\rangle_{pq}$ states

To solve the eigenvalue problem in Eq. (8.2), we expand the state $|A\rangle_{pq}$ in terms of the basis eigenstates

$$\left\{|0\rangle_{pq} = \begin{pmatrix} |0\rangle_{pq} \\ 0 \end{pmatrix}, \quad |b_n\rangle_{pq} = \begin{pmatrix} |n\rangle_{pq} \\ 0 \end{pmatrix}, \quad |f_n\rangle_{pq} = \begin{pmatrix} 0 \\ |n-1\rangle_{pq} \end{pmatrix}\right\},$$

then, the state $|A\rangle_{pq}$ can be written as

$$|A\rangle_{pq} = a_0|0\rangle_{pq} + \sum_{n=1}^{\infty} a_n|b_n\rangle_{pq} + \sum_{n=1}^{\infty} c_n|f_n\rangle_{pq}. \tag{B.6}$$

Substituting this expression into Eq. (8.2) gives the following relations among the coefficients

$$a_n = a_n\frac{\alpha^n}{p^n\sqrt{[n]_{pq}!}} - c_1\frac{\alpha^{n-1}}{p^n q^{n-1}\sqrt{[n]_{pq}!}}[n]_{pq}!, \quad c_n = c_1\frac{\alpha^{n-1}}{q^{n-1}\sqrt{[n]_{pq}!}},$$

and this leads to the following expression for $|A\rangle_{pq}$

$$|A\rangle_{pq} = a_0\begin{pmatrix} |\frac{\alpha}{p}\rangle_{pq} \\ 0 \end{pmatrix} + c_1\begin{pmatrix} -q|\frac{\alpha'}{pq}\rangle_{pq} \\ |\frac{\alpha}{q}\rangle_{pq} \end{pmatrix} = a_0|\widetilde{A}_b\rangle_{pq} + c_1|\widetilde{A}_s\rangle_{pq}.$$

Here, we define the states $|\widetilde{A}_b\rangle_{pq}$ and $|\widetilde{A}_s\rangle_{pq}$ as

$$|\widetilde{A}_b\rangle_{pq} \equiv \begin{pmatrix} |\frac{\alpha}{p}\rangle_{pq} \\ 0 \end{pmatrix}, \quad |\widetilde{A}_s\rangle_{pq} \equiv \begin{pmatrix} -q|\frac{\alpha'}{pq}\rangle_{pq} \\ |\frac{\alpha}{q}\rangle_{pq} \end{pmatrix}.$$

This representation allows $|A\rangle_{pq}$ to be expressed in terms of the modified basis states $|\widetilde{A}_b\rangle_{pq}$ and $|\widetilde{A}_s\rangle_{pq}$, which are defined by the parameters α, p, and q. To examine the properties of the states $|\widetilde{A}_b\rangle_{pq}$ and $|\widetilde{A}_s\rangle_{pq}$, we calculate their inner products as follows

$$
\begin{aligned}
{}_{pq}\langle\widetilde{A}_b|\widetilde{A}_b\rangle_{pq} &= e_{pq}^{\frac{|\alpha|^2}{p^2}}, \\
{}_{pq}\langle\widetilde{A}_s|\widetilde{A}_s\rangle_{pq} &= \frac{1}{p^2}\left(\frac{1}{p^2 q}|\alpha|^2 e_{pq}^{\frac{|\alpha|^2}{p^2q^2}} + e_{pq}^{\frac{|\alpha|^2}{pq^2}}\right) + e_{pq}^{\frac{|\alpha|^2}{q^2}},
\end{aligned}
$$

which can also be expressed as

$$
{}_{pq}\langle\widetilde{A}_s|\widetilde{A}_s\rangle_{pq} = \frac{1}{p^2}\left(\frac{1}{pq^2}|\alpha|^2 e_{pq}^{\frac{|\alpha|^2}{p^2q^2}} + e_{pq}^{\frac{|\alpha|^2}{p^2 q}}\right) + e_{pq}^{\frac{|\alpha|^2}{q^2}}.
$$

From these calculations, it is clear that the states $|\widetilde{A}_b\rangle_{pq}$ and $|\widetilde{A}_s\rangle_{pq}$ are neither normalized nor orthogonal, as shown by the non-zero inner product: ${}_{pq}\langle\widetilde{A}_b|\widetilde{A}_s\rangle_{pq} \neq 0$.

B.2.1. Orthogonal $|A_b\rangle_{pq}$ and $|A_s\rangle_{pq}$ states

To construct a state orthogonal to $|\widetilde{A}_b\rangle_{pq}$, we introduce a new state defined as $|A_s\rangle_{pq} \equiv \gamma|\widetilde{A}_b\rangle_{pq} + \beta|\widetilde{A}_s\rangle_{pq}$, where the coefficients γ and β are chosen to satisfy the orthogonality condition ${}_{pq}\langle A_s|\widetilde{A}_b\rangle_{pq} = 0$. Specifically, let

$$
\gamma = c\frac{\bar{\alpha}}{p^2} e_{pq}^{\frac{|\alpha|^2}{p^2 q}}, \quad \beta = c\, e_{pq}^{\frac{|\alpha|^2}{p^2}},
$$

with c being a normalization constant. This yields two orthogonal states:

$$
|A_b\rangle_{pq} \equiv c_b \begin{pmatrix} |\frac{\alpha}{p}\rangle_{pq} \\ 0 \end{pmatrix},
$$

and

$$
|A_s\rangle_{pq} \equiv c_s \begin{pmatrix} \frac{\bar{\alpha}}{p^2} e_{pq}^{\frac{|\alpha|^2}{p^2 q}} |\frac{\alpha'}{pq}\rangle - q e_{pq}^{\frac{|\alpha|^2}{p^2}} |\frac{\alpha'}{pq}\rangle \\ e_{pq}^{\frac{|\alpha|^2}{p^2}} |\frac{\alpha}{q}\rangle \end{pmatrix}.
$$

Thus, these states $|A_b\rangle_{pq}$ and $|A_s\rangle_{pq}$ are orthogonal, with ${}_{pq}\langle A_b|A_s\rangle_{pq} = 0$.

B.2.2. Normalization of $|A_b\rangle_{pq}$ and $|A_s\rangle_{pq}$ states

For normalization, the first step is to compute the following inner product

$$
\begin{aligned}
{}_{pq}\langle A_s|A_s\rangle_{pq} &= |C_s|^2\Bigg[\left(e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right)^2 \frac{|\alpha|^2}{p^4}\left\langle\frac{\alpha}{p}\middle|\frac{\alpha}{p}\right\rangle_{pq} - e_{pq}^{\frac{|\alpha|^2}{p^2q}} e_{pq}^{\frac{|\alpha|^2}{p^2}} \frac{q\alpha}{p^2}\left\langle\frac{\alpha}{p}\middle|\frac{\alpha'}{pq}\right\rangle_{pq} \\
&- e_{pq}^{\frac{|\alpha|^2}{p^2q}} e_{pq}^{\frac{|\alpha|^2}{p^2}} \frac{q\overline{\alpha}}{p^2}\left\langle\frac{\alpha'}{pq}\middle|\frac{\alpha}{p}\right\rangle_{pq} + \left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2 q^2\left\langle\frac{\alpha'}{pq}\middle|\frac{\alpha'}{pq}\right\rangle_{pq} + \left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2\left\langle\frac{\alpha}{q}\middle|\frac{\alpha}{q}\right\rangle_{pq}\Bigg].
\end{aligned}
$$

This expression represents the full inner product in terms of the states involved, using the results of previous calculations for the inner products between states which is derived in $(B.1)$ and $(B.2)$. An alternative form of the inner product can be written as:

$$
\begin{aligned}
{}_{pq}\langle A_s|A_s\rangle_{pq} &= |C_s|^2\Bigg[\left(e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right)^2 \frac{|\alpha|^2}{p^4}\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right) + \left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2\left(e_{pq}^{\frac{|\alpha|^2}{q^2}}\right) \\
&+ \left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2 \frac{1}{p^2}\left(\frac{|\alpha|^2}{pq^2} e_{pq}^{\frac{|\alpha|^2}{p^2q^2}} + e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right) - e_{pq}^{\frac{|\alpha|^2}{p^2q}} e_{pq}^{\frac{|\alpha|^2}{p^2}} \frac{q}{p^2}\left(\frac{2|\alpha|^2}{p^4} e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right)\Bigg].
\end{aligned}
$$

Then, the expression must satisfy the following equation:

$$
\begin{aligned}
{}_{pq}\langle A_s|A_s\rangle_{pq} &= |C_s|^2\Bigg[\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2\left(e_{pq}^{\frac{|\alpha|^2}{q^2}}\right) + \frac{1}{p^2}\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2 e_{pq}^{\frac{|\alpha|^2}{p^2q}} \\
&+ \frac{|\alpha|^2}{p^4q^2} e_{pq}^{\frac{|\alpha|^2}{p^2}}\left(p\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2q^2}}\right) - q^2\left(e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right)^2\right)\Bigg] = 1
\end{aligned}
$$

From this equation, the normalization constant C_s can be computed. Solving for C_s^{-2} is achieved by using different form of inner product given in $(B.4)$ and $(B.5)$,

$$
\begin{aligned}
C_s^{-2} &= \Bigg[\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2\left(e_{pq}^{\frac{|\alpha|^2}{q^2}} + \frac{1}{p^2} e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right) + \frac{|\alpha|^2}{p^4q^2} e_{pq}^{\frac{|\alpha|^2}{p^2}}\left(p\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2q^2}}\right) - q^2\left(e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right)^2\right)\Bigg] \\
&= \Bigg[\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2\left(e_{pq}^{\frac{|\alpha|^2}{q^2}} + \frac{1}{p^2} e_{pq}^{\frac{|\alpha|^2}{pq^2}}\right) + \frac{|\alpha|^2}{p^4q^2} e_{pq}^{\frac{|\alpha|^2}{p^2}}\left(q\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2q^2}}\right) - q^2\left(e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right)^2\right)\Bigg]
\end{aligned}
$$

Definition B.1 *The normalized states $|A_b\rangle_{pq}$ and $|A_s\rangle_{pq}$,*

$$|A_b\rangle_{pq} = C_b \begin{bmatrix} |\frac{\alpha}{p}\rangle_{pq} \\ 0 \end{bmatrix} \quad ,$$

$$|A_s\rangle_{pq} = C_s \begin{bmatrix} e_{pq}^{\frac{|\alpha|^2}{p^2 q}} \frac{\overline{\alpha}}{p^2} |\frac{\alpha}{p}\rangle_{pq} - e_{pq}^{\frac{|\alpha|^2}{p^2}} q |\frac{\alpha'}{pq}\rangle_{pq} \\ \\ e_{pq}^{\frac{|\alpha|^2}{p^2}} |\frac{\alpha}{q}\rangle_{pq} \end{bmatrix}$$

where

$$C_s^{-2} = \left[\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2 \left(e_{pq}^{\frac{|\alpha|^2}{q^2}} + \frac{1}{p^2} e_{pq}^{\frac{|\alpha|^2}{p^2 q}}\right) + \frac{|\alpha|^2}{p^4 q^2} e_{pq}^{\frac{|\alpha|^2}{p^2}} \left(p \left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2 q^2}}\right) - q^2 \left(e_{pq}^{\frac{|\alpha|^2}{p^2 q}}\right)^2\right)\right] \quad \text{(B.7)}$$

$$= \left[\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2 \left(e_{pq}^{\frac{|\alpha|^2}{q^2}} + \frac{1}{p^2} e_{pq}^{\frac{|\alpha|^2}{pq^2}}\right) + \frac{|\alpha|^2}{p^4 q^2} e_{pq}^{\frac{|\alpha|^2}{p^2}} \left(q \left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2 q^2}}\right) - q^2 \left(e_{pq}^{\frac{|\alpha|^2}{p^2 q}}\right)^2\right)\right] \quad \text{(B.8)}$$

$$C_b^{-2} = e_{pq}^{\frac{|\alpha|^2}{p^2}}, \quad \text{(B.9)}$$

form orthonormal basis for $|A\rangle_{pq}$.

B.3. Expectation Value

In quantum mechanics , the expectation value is the probabilistic expected value of the result of an measurement.

Definition B.2 *Let $\hat{A}$ be an operator on a Hilbert space and $|\varphi\rangle$ is a normalized state, then the expectation value of $\hat{A}$ in the state $|\varphi\rangle$ is defined as*

$$\langle \hat{A} \rangle = \langle \hat{A} \rangle_{\varphi} = \langle \varphi | \hat{A} | \varphi \rangle \quad \text{(B.10)}$$

Proposition B.2 *For all $\alpha, \beta \in \mathbb{C}$, the transition elements involving the annihilation op-*

erator $\hat{a}_{pq}$ between the states are given by the following relations:

$$
\begin{aligned}
{}_{pq}\left\langle\frac{\beta}{\lambda}\middle|\hat{a}_{pq}\middle|\frac{\alpha}{\mu}\right\rangle_{pq} &= \frac{\alpha}{\mu}e_{pq}^{\frac{\bar{\beta}\alpha}{\lambda\mu}} && \text{(B.11)}\\
\left\langle\frac{\beta'}{\lambda}\middle|\hat{a}_{pq}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} &= \frac{\alpha}{\lambda\mu^2}(p+q)e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}} + q^2\frac{\alpha^2\overline{\beta}}{\lambda^2\mu^3}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} && \text{(B.12)}\\
&= \frac{\alpha}{\lambda\mu^2}(p+q)e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} + p^2\frac{\alpha^2\overline{\beta}}{\lambda^2\mu^3}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} && \text{(B.13)}\\
\left\langle\frac{\beta}{\lambda}\middle|\hat{a}_{pq}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} &= \frac{1}{\mu}e_{pq}^{p\frac{\bar{\beta}\alpha}{\lambda\mu}} + \bar{\beta}\alpha\frac{q}{\lambda\mu^2}e_{pq}^{\frac{\bar{\beta}\alpha}{\lambda\mu}} && \text{(B.14)}\\
&= \frac{1}{\mu}e_{pq}^{q\frac{\bar{\beta}\alpha}{\lambda\mu}} + \bar{\beta}\alpha\frac{p}{\lambda\mu^2}e_{pq}^{\frac{\bar{\beta}\alpha}{\lambda\mu}} && \text{(B.15)}\\
\left\langle\frac{\beta'}{\lambda}\middle|\hat{a}_{pq}\middle|\frac{\alpha}{\mu}\right\rangle_{pq} &= \frac{\alpha^2}{\lambda\mu^2}e_{pq}^{\frac{\bar{\beta}\alpha}{\lambda\mu}}. && \text{(B.16)}
\end{aligned}
$$

For the special case where $\beta = \alpha$, $\lambda = pq$, and $\mu = p$, the relations become:

$$
\begin{aligned}
\left\langle\frac{\alpha'}{pq}\middle|\hat{a}_{pq}\middle|\frac{\alpha'}{pq}\right\rangle_{pq} &= \frac{\alpha}{p^5q^3}\left(p^2(q+p)e_{pq}^{\frac{|\alpha|^2}{pq^2}} + |\alpha|^2 e_{pq}^{\frac{|\alpha|^2}{p^2q^2}}\right)\\
&= \frac{\alpha}{p^3q^5}\left(q^2(q+p)e_{pq}^{\frac{|\alpha|^2}{p^2q}} + |\alpha|^2 e_{pq}^{\frac{|\alpha|^2}{p^2q^2}}\right)\\
\left\langle\frac{\alpha}{p}\middle|\hat{a}_{pq}\middle|\frac{\alpha'}{pq}\right\rangle_{pq} &= \frac{1}{pq}e_{pq}^{\frac{|\alpha|^2}{pq}} + \frac{|\alpha|^2}{p^3q}e_{pq}^{\frac{|\alpha|^2}{p^2q}}\\
&= \frac{1}{pq}e_{pq}^{\frac{|\alpha|^2}{p^2}} + \frac{|\alpha|^2}{p^2q^2}e_{pq}^{\frac{|\alpha|^2}{p^2q}}\\
\left\langle\frac{\alpha'}{pq}\middle|\hat{a}_{pq}\middle|\frac{\alpha}{p}\right\rangle_{pq} &= \frac{\alpha^2}{p^3q^2}e_{pq}^{\frac{|\alpha|^2}{p^2q}}.
\end{aligned}
$$

Proof The first relation follows directly from the definition of the $pq-$annihilation operator. We now compute matrix element denoted as equation in (B.12). Using the action of the annihilation operator from relation (3.30) gives following

$$
\begin{aligned}
\left\langle\frac{\beta'}{\lambda}\middle|\hat{a}_{pq}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} &= \left\langle\frac{\beta'}{\lambda}\middle|\left(\frac{q\alpha}{\mu}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} + \frac{1}{\mu}\left|\frac{p\alpha}{\mu}\right\rangle_{pq}\right)\\
&= \frac{q\alpha}{\mu}\left\langle\frac{\beta'}{\lambda}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} + \frac{1}{\mu}\left\langle\frac{\beta'}{\lambda}\middle|\frac{p\alpha}{\mu}\right\rangle_{pq}.
\end{aligned}
$$

and after putting the inner product relation from equation (B.2), the matrix element becomes

$$\left\langle \frac{\beta'}{\lambda} \middle| \hat{a}_{pq} \middle| \frac{\alpha'}{\mu} \right\rangle_{pq} = \frac{q\alpha}{\mu}\left(\frac{1}{\lambda\mu}\left(e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}} + q\frac{\alpha\overline{\beta}}{\lambda\mu}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}\right)\right) + \frac{1}{\lambda\mu}\left(\frac{\alpha p}{\lambda\mu}e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}}\right)$$
$$= \frac{\alpha}{\lambda\mu^2}(p+q)e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}} + q^2\frac{\alpha^2\overline{\beta}}{\lambda^2\mu^3}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}.$$

Next, by symmetry, the matrix element in equation (B.13) can be derived similarly. Applying the action of the annihilation operator from equation (3.31) and alternative form of inner product, we obtain

$$\left\langle \frac{\beta'}{\lambda} \middle| \hat{a}_{pq} \middle| \frac{\alpha'}{\mu} \right\rangle_{pq} = \left\langle \frac{\beta'}{\lambda} \middle| \left(\frac{p\alpha}{\mu} \middle| \frac{\alpha'}{\mu} \right\rangle_{pq} + \frac{1}{\mu} \middle| \frac{q\alpha}{\mu} \right\rangle_{pq} \Bigg)$$
$$= \frac{p\alpha}{\mu}\left\langle \frac{\beta'}{\lambda} \middle| \frac{\alpha'}{\mu} \right\rangle_{pq} + \frac{1}{\mu}\left\langle \frac{\beta'}{\lambda} \middle| \frac{p\alpha}{\mu} \right\rangle_{pq}$$
$$\overset{(B.3)}{=} \frac{p\alpha}{\mu}\left(\frac{1}{\lambda\mu}\left(e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} + p\frac{\alpha\overline{\beta}}{\lambda\mu}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}\right)\right) + \frac{1}{\lambda\mu}\left(\frac{\alpha q}{\lambda\mu}e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}}\right)$$
$$= \frac{\alpha}{\lambda\mu^2}(p+q)e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} + p^2\frac{\alpha^2\overline{\beta}}{\lambda^2\mu^3}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}$$

Then, the third expression can be written as following sum by using expansion of the states,

$$\left\langle \frac{\beta}{\lambda} \middle| \hat{a}_{pq} \middle| \frac{\alpha'}{\mu} \right\rangle_{pq} = \left(\sum_{n=0}^{\infty} \frac{\overline{\beta}^n}{\lambda^n\sqrt{[n]_{pq}!}}\right)\left(\frac{1}{\mu}\sum_{m=0}^{\infty}\frac{\alpha^m[m+1]_{pq}}{\mu^m\sqrt{[m]_{pq}!}}\right)\, {}_{pq}\langle n|m\rangle_{pq},$$
$$= \frac{1}{\mu}\sum_{n=0}^{\infty}\frac{(\bar{\beta}\alpha)^n}{(\lambda\mu)^n[n]_{pq}!}[n+1]_{pq}.$$

and by applying the relation $[n+1]_{pq} = p^n + q[n]_{pq}$, this becomes

$$\left\langle \frac{\beta}{\lambda} \middle| \hat{a}_{pq} \middle| \frac{\alpha'}{\mu} \right\rangle_{pq} = \frac{1}{\mu}\left(\sum_{n=0}^{\infty}\left(p\frac{\bar{\beta}\alpha}{\lambda\mu}\right)^n \frac{1}{[n]_{pq}!}\right) + \bar{\beta}\alpha\frac{q}{\lambda\mu^2}\left(\sum_{n=1}^{\infty}\left(\frac{\bar{\beta}\alpha}{\lambda\mu}\right)^{n-1}\frac{1}{[n-1]_{pq}!}\right)$$
$$= \frac{1}{\mu}e_{pq}^{p\frac{\bar{\beta}\alpha}{\lambda\mu}} + \bar{\beta}\alpha\frac{q}{\lambda\mu^2}e_{pq}^{\frac{\bar{\beta}\alpha}{\lambda\mu}}.$$

If the alternative relation $[n+1]_{pq} = q^n + p[n]_{pq}$ is used instead, the expression becomes

$$\left\langle \frac{\beta}{\lambda} \middle| \hat{a}_{pq} \middle| \frac{\alpha'}{\mu} \right\rangle_{pq} = \frac{1}{\mu} e_{pq}^{q\frac{\bar{\beta}\alpha}{\lambda\mu}} + \bar{\beta}\alpha \frac{p}{\lambda\mu^2} e_{pq}^{\frac{\bar{\beta}\alpha}{\lambda\mu}},$$

providing a symmetric alternative form for the matrix element. For the last equation, while using the action of the $pq-$annihilation operator, the expression is written as

$$\left\langle \frac{\beta'}{\lambda} \middle| \hat{a}_{pq} \middle| \frac{\alpha}{\mu} \right\rangle_{pq} = \frac{\alpha}{\mu} \left\langle \frac{\beta'}{\lambda} \middle| \frac{\alpha}{\mu} \right\rangle_{pq} = \frac{\alpha}{\mu} \left(\frac{\alpha}{\lambda\mu} e_{pq}^{\frac{\bar{\beta}\alpha}{\lambda\mu}} \right)$$

by applying the inner product result from equation (B.1). This concludes the proof. □

Proposition B.3 *The expectation value of the annihilation operator $\hat{a}_{pq}$ in the state $|A_s\rangle_{pq}$ is given by the following expression:*

$$_{pq}\langle A_s | \hat{a}_{pq} | A_s \rangle_{pq} = \frac{\alpha}{q}.$$

Proof First, while using the expansion for the state $|A_s\rangle_{pq}$, the expectation value can be expressed as

$$\begin{aligned}
_{pq}\langle A_s | \hat{a}_{pq} | A_s \rangle_{pq} &= |C_s|^2 \Bigg[\left(e_{pq}^{\frac{|\alpha|^2}{p^2 q}} \right)^2 \frac{|\alpha|^2}{p^4} \left\langle \frac{\alpha}{p} \middle| \hat{a}_{pq} \middle| \frac{\alpha}{p} \right\rangle_{pq} - e_{pq}^{\frac{|\alpha|^2}{p^2 q}} e_{pq}^{\frac{|\alpha|^2}{p^2}} \frac{q\alpha}{p^2} \left\langle \frac{\alpha}{p} \middle| \hat{a}_{pq} \middle| \frac{\alpha'}{pq} \right\rangle_{pq} \\
&- e_{pq}^{\frac{|\alpha|^2}{p^2 q}} e_{pq}^{\frac{|\alpha|^2}{p^2}} \frac{q\overline{\alpha}}{p^2} \left\langle \frac{\alpha'}{pq} \middle| \hat{a}_{pq} \middle| \frac{\alpha}{p} \right\rangle_{pq} + \left(e_{pq}^{\frac{|\alpha|^2}{p^2}} \right)^2 q^2 \left\langle \frac{\alpha'}{pq} \middle| \hat{a}_{pq} \middle| \frac{\alpha'}{pq} \right\rangle_{pq} \\
&+ \left(e_{pq}^{\frac{|\alpha|^2}{p^2}} \right)^2 \left\langle \frac{\alpha}{q} \middle| \hat{a}_{pq} \middle| \frac{\alpha}{q} \right\rangle_{pq} \Bigg].
\end{aligned}$$

The matrix elements of the annihilation operator $\hat{a}_{pq}$ used in this expression are derived from the proposition (B.2), where each term of the expectation value corresponds to specific transitions between states. Alternatively, simplifying the terms, the expression can be written as

$$
\begin{aligned}
{}_{pq}\langle A_s|\hat{a}_{pq}|A_s\rangle_{pq} &= |C_s|^2\Big[\left(e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right)^2 \frac{|\alpha|^2}{p^4}\left(\frac{\alpha}{p}e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)+\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2\left(\frac{\alpha}{q}e_{pq}^{\frac{|\alpha|^2}{q^2}}\right) \\
&+ \left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2 q^2\left(\frac{\alpha}{p^3q^5}\left(q^2(p+q)e_{pq}^{\frac{|\alpha|^2}{p^2q}} + |\alpha|^2 e_{pq}^{\frac{|\alpha|^2}{p^2q^2}}\right)\right) \\
&- e_{pq}^{\frac{|\alpha|^2}{p^2q}} e_{pq}^{\frac{|\alpha|^2}{p^2}} \frac{q}{p^2}\left(\overline{\alpha}\frac{\alpha^2}{p^3q}e_{pq}^{\frac{|\alpha|^2}{p^2q}} + \alpha\left(\frac{1}{pq}e_{pq}^{\frac{|\alpha|^2}{p^2}} + \frac{|\alpha|^2}{p^3q}e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right)\right)\Big]. \\
&= |C_s|^2\alpha\Big[\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2\left(\left(\frac{p+q}{p^3q}\right)e_{pq}^{\frac{|\alpha|^2}{p^2q}} + \frac{1}{q}e_{pq}^{\frac{|\alpha|^2}{q^2}} - \frac{1}{p^3}e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right) \\
&+|\alpha|^2\Big(\frac{1}{p^5}\left(e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right)^2 e_{pq}^{\frac{|\alpha|^2}{p^2}} + \frac{1}{p^3q^3}\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2 e_{pq}^{\frac{|\alpha|^2}{p^2q^2}} \\
&-\frac{1}{p^4q}\left(e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right)^2 e_{pq}^{\frac{|\alpha|^2}{p^2}} - \frac{1}{p^5}\left(e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right)^2 e_{pq}^{\frac{|\alpha|^2}{p^2}}\Big)\Big].
\end{aligned}
$$

To simplify the expectation value of the annihilation operator $\hat{a}_{pq}$ in the state $|A_s\rangle_{pq}$, we begin by combining terms as follows:

$$
{}_{pq}\langle A_s|\hat{a}_{pq}|A_s\rangle_{pq} = |C_s|^2\alpha\left(g_1(|\alpha|^2) + |\alpha|^2 g_2(|\alpha|^2)\right),
$$

where the functions $g_1(|\alpha|^2)$ and $g_2(|\alpha|^2)$ are defined as:

$$
\begin{aligned}
g_1(|\alpha|^2) &= \frac{1}{p^2q}\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^2\left(e_{pq}^{\frac{|\alpha|^2}{p^2q}} + p^2 e_{pq}^{\frac{|\alpha|^2}{q^2}}\right), \\
g_2(|\alpha|^2) &= \frac{1}{p^4q^3}e_{pq}^{\frac{|\alpha|^2}{p^2}}\left(p e_{pq}^{\frac{|\alpha|^2}{p^2}} e_{pq}^{\frac{|\alpha|^2}{p^2q^2}} - q^2\left(e_{pq}^{\frac{|\alpha|^2}{p^2q}}\right)^2\right).
\end{aligned}
$$

By using the normalization condition from equation (B.7), the resulting expression for the expectation value simplifies significantly. Applying this normalization condition to the calculation leads to:

$$
{}_{pq}\langle A_s|\hat{a}_{pq}|A_s\rangle_{pq} = \frac{\alpha}{q}.
$$

Thus, the expectation value is simplified to the final form. □

Corollary B.2 *The expectation values of the operators $\hat{x}_{pq}$ and $\hat{p}_{pq}$, defined in equation*

(8.4), in the state $|A_s\rangle_{pq}$ are given by the following expressions:

$$
\begin{aligned}
{}_{pq}\langle A_s|\hat{x}_{pq}|A_s\rangle_{pq} &= \frac{1}{q}\sqrt{2}\,\mathrm{Re}(\alpha),\\
{}_{pq}\langle A_s|\hat{p}_{pq}|A_s\rangle_{pq} &= \frac{1}{q}\sqrt{2}\,\mathrm{Im}(\alpha).
\end{aligned}
$$

These results show that the expectation values of the position operator $\hat{x}_{pq}$ and the momentum operator $\hat{p}_{pq}$ are proportional to the real and imaginary parts of the parameter α, respectively, scaled by $\frac{1}{q}\sqrt{2}$.

B.4. Uncertainty(Deviation)

Definition B.3 *The uncertainty of the observable A is a measure of the spread of results around the mean $\langle \hat{A} \rangle$. It is defined in the usual way, that is the difference between each measured result and the mean is calculated.*

$$(\Delta A)^2_{\varphi} = \langle \hat{A}^2 \rangle_{\varphi} - \langle \hat{A} \rangle^2_{\varphi} \tag{B.17}$$

B.4.1. Uncertainty for $|A_b\rangle_{pq}$ state

In order to find uncertainty for state $|A_b\rangle_{pq}$, we need to have following proposition

Proposition B.4 *Let $\hat{a}_{pq}$ and $\hat{a^{\dagger}}_{pq}$ represent the $pq-$annihilation and $pq-$creation operators, respectively, and λ, μ, α, and β be complex parameters. The following identity* *holds:*

$$
\begin{aligned}
{}_{pq}\left\langle \frac{\beta}{\lambda} \middle| \hat{a}_{pq}^{2} \middle| \frac{\alpha}{\mu} \right\rangle_{pq} &= \frac{\alpha^2}{\mu^2} e_{pq}^{\frac{\bar{\beta}\alpha}{\lambda\mu}}, && \text{(B.18)}\\
{}_{pq}\left\langle \frac{\beta}{\lambda} \middle| \hat{a^{\dagger}}^{2}_{pq} \middle| \frac{\alpha}{\mu} \right\rangle_{pq} &= \frac{\overline{\beta}^2}{\lambda^2} e_{pq}^{\frac{\bar{\beta}\alpha}{\lambda\mu}}, && \text{(B.19)}\\
{}_{pq}\left\langle \frac{\beta}{\lambda} \middle| \hat{a^{\dagger}}_{pq}\hat{a}_{pq} \middle| \frac{\alpha}{\mu} \right\rangle_{pq} &= \frac{\bar{\beta}\alpha}{\lambda\mu} e_{pq}^{\frac{\bar{\beta}\alpha}{\lambda\mu}} && \text{(B.20)}\\
{}_{pq}\left\langle \frac{\beta}{\lambda} \middle| \hat{a}_{pq}\hat{a^{\dagger}}_{pq} \middle| \frac{\alpha}{\mu} \right\rangle_{pq} &= e_{pq}^{p\frac{\bar{\beta}\alpha}{\lambda\mu}} + p\frac{\bar{\beta}\alpha}{\lambda\mu} e_{pq}^{\frac{\bar{\beta}\alpha}{\lambda\mu}}, && \text{(B.21)}\\
&= e_{pq}^{q\frac{\bar{\beta}\alpha}{\lambda\mu}} + q\frac{\bar{\beta}\alpha}{\lambda\mu} e_{pq}^{\frac{\bar{\beta}\alpha}{\lambda\mu}}. && \text{(B.22)}
\end{aligned}
$$

Proof The first two cases follow directly from the definition. For the third case, we may proceed to prove the final case, as it can be derived by taking the Hermitian conjugate.Then, we start by evaluating the matrix element by applying the inner product from equation (B.2),

$$
\begin{aligned}
{}_{pq}\left\langle \frac{\beta}{\lambda} \middle| \hat{a}_{pq}\hat{a^{\dagger}}_{pq} \middle| \frac{\alpha}{\mu} \right\rangle_{pq} &= \lambda\mu \left\langle \frac{\beta'}{\lambda} \middle| \frac{\alpha'}{\mu} \right\rangle_{pq}\\
&\overset{(B.2)}{=} \lambda\mu \left(\frac{1}{\lambda\mu} e_{pq}^{p\frac{\alpha\overline{\beta}}{\lambda\mu}} + q\frac{\alpha\overline{\beta}}{(\lambda\mu)^2} e_{pq}^{\frac{\alpha\overline{\beta}}{\lambda\mu}} \right)\\
&= e_{pq}^{p\frac{\bar{\beta}\alpha}{\lambda\mu}} + p\frac{\bar{\beta}\alpha}{\lambda\mu} e_{pq}^{\frac{\bar{\beta}\alpha}{\lambda\mu}}.
\end{aligned}
$$

and we can derive an alternative form of the same expression by using $(B.3)$,

$$
{}_{pq}\left\langle \frac{\beta}{\lambda} \middle| \hat{a}_{pq}\hat{a^{\dagger}}_{pq} \middle| \frac{\alpha}{\mu} \right\rangle_{pq} = e_{pq}^{q\frac{\bar{\beta}\alpha}{\lambda\mu}} + q\frac{\bar{\beta}\alpha}{\lambda\mu} e_{pq}^{\frac{\bar{\beta}\alpha}{\lambda\mu}}
$$

□

Proposition B.5 *Let $\hat{x}_{pq}^2$ and $\hat{p}_{pq}^2$ represent the pq−position and pq−momentum squared operators. For the state $|A_b\rangle_{pq}$, the expectation values of $\hat{x}_{pq}^2$ and $\hat{p}_{pq}^2$ are given by:*

$$
{}_{pq}\langle A_b|\hat{x}_{pq}^2|A_b\rangle_{pq} = \frac{1}{2}\left(\frac{\overline{\alpha}^2}{p^2} + \frac{\alpha^2}{p^2} + (q+1)\frac{|\alpha|^2}{p^2} + e_{pq}^{\frac{|\alpha|^2}{p}} \left(e_{pq}^{\frac{|\alpha|^2}{p^2}} \right)^{-1} \right),
$$

and

$$ {}_{pq}\langle A_b|\hat{p}^2_{pq}|A_b\rangle_{pq} = \frac{1}{2}\left(-\frac{\overline{\alpha}^2}{p^2} - \frac{\alpha^2}{p^2} + (q+1)\frac{|\alpha|^2}{p^2} + e_{pq}^{\frac{|\alpha|^2}{p}}\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^{-1}\right). $$

Proof To compute the expectation value of $\hat{x}^2_{pq}$ for the state $|A_b\rangle_{pq}$, we first express it in terms of the creation and annihilation operators. The matrix element can be written as:

$$ \begin{aligned} {}_{pq}\langle A_b|\hat{x}^2_{pq}|A_b\rangle_{pq} &= \left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^{-1}\begin{pmatrix} {}_{pq}\langle\frac{\alpha}{p}| & 0\end{pmatrix}\begin{pmatrix}\left(\frac{\hat{a}^{\dagger}{}_{pq}+\hat{a}_{pq}}{\sqrt{2}}\right)^2 & 0 \\ 0 & \left(\frac{\hat{a}^{\dagger}{}_{pq}+\hat{a}_{pq}}{\sqrt{2}}\right)^2\end{pmatrix}\begin{pmatrix}|\frac{\alpha}{p}\rangle_{pq} \\ 0\end{pmatrix} \\ &= \left\langle\frac{\alpha}{p}\right|\left(\frac{\hat{a}^{\dagger}{}_{pq}+\hat{a}_{pq}}{\sqrt{2}}\right)^2\left|\frac{\alpha}{p}\right\rangle_{pq} \\ &= \frac{1}{2}\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^{-1}\left(\left\langle\frac{\alpha}{p}\right|\hat{a}^2_{pq}\left|\frac{\alpha}{p}\right\rangle_{pq} + \left\langle\frac{\alpha}{p}\right|\hat{a}^{\dagger}{}^2_{pq}\left|\frac{\alpha}{p}\right\rangle_{pq}\right. \\ &+ \left.\left\langle\frac{\alpha}{p}\right|\hat{a}^{\dagger}{}_{pq}\hat{a}_{pq}\left|\frac{\alpha}{p}\right\rangle_{pq} + \left\langle\frac{\alpha}{p}\right|\hat{a}_{pq}\hat{a}^{\dagger}{}_{pq}\left|\frac{\alpha}{p}\right\rangle_{pq}\right) \\ &= \frac{1}{2}\left(\frac{\overline{\alpha}^2}{p^2} + \frac{\alpha^2}{p^2} + (q+1)\frac{|\alpha|^2}{p^2} + e_{pq}^{\frac{|\alpha|^2}{p}}\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^{-1}\right), \end{aligned} $$

and substituting the given relations from Prop.(B.4) to the expression yields

$$ {}_{pq}\langle A_b|\hat{x}^2_{pq}|A_b\rangle_{pq} = \frac{1}{2}\left(\frac{\overline{\alpha}^2}{p^2} + \frac{\alpha^2}{p^2} + (q+1)\frac{|\alpha|^2}{p^2} + e_{pq}^{\frac{|\alpha|^2}{p}}\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^{-1}\right). $$

By following similar steps for the momentum operator $\hat{p}^2_{pq}$, we find:

$$ {}_{pq}\langle A_b|\hat{p}^2_{pq}|A_b\rangle_{pq} = \frac{1}{2}\left(-\frac{\overline{\alpha}^2}{p^2} - \frac{\alpha^2}{p^2} + (q+1)\frac{|\alpha|^2}{p^2} + e_{pq}^{\frac{|\alpha|^2}{p}}\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^{-1}\right) $$

□

Proposition B.6 *The uncertainty relation for the state* $|A_b\rangle_{pq}$ *is given by*

$$ \begin{aligned} {}_{pq}\langle A_b|\left(\Delta\hat{x}_{pq}\right)^2|A_b\rangle_{pq} &\equiv {}_{pq}\langle A_b|\left(\Delta\hat{p}_{pq}\right)^2|A_b\rangle_{pq} \\ &= \frac{1}{2p^2}\left((q-1)|\alpha|^2 + p^2\left(e_{pq}^{\frac{|\alpha|^2}{p}}\right)\left(e_{pq}^{\frac{|\alpha|^2}{p^2}}\right)^{-1}\right) \end{aligned} $$

B.4.2. Uncertainty for $|A_s\rangle_{pq}$ state

Before proceeding with the calculation of the uncertainty for the state $|A_s\rangle_{pq}$, it is necessary to first determine the transition of the operator $\hat{x}^2_{pq}$. This requires deriving the following relations

Proposition B.7 *For all $\alpha, \beta \in \mathbb{C}$, the following transition relations hold for the operators $\hat{a}^\dagger_{pq}$ and $\hat{a}_{pq}$ in terms of the parameters p, q, λ, and μ*

$$\left\langle \frac{\beta}{\lambda}' \middle| \hat{a^\dagger}^2_{pq} \middle| \frac{\alpha}{\mu} \right\rangle_{pq} = \frac{\bar{\beta}}{\lambda^2}\left((p+q)e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}}\right) + q^2\frac{\bar{\beta}^2\alpha}{\lambda^3\mu}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} \tag{B.23}$$

$$= \frac{\bar{\beta}}{\lambda^2}\left((p+q)e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}}\right) + p^2\frac{\bar{\beta}^2\alpha}{\lambda^3\mu}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} \tag{B.24}$$

$$\left\langle \frac{\beta}{\lambda}' \middle| \hat{a}^2_{pq} \middle| \frac{\alpha}{\mu} \right\rangle_{pq} = \frac{\beta\alpha^2}{\lambda\mu^3}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} \tag{B.25}$$

$$\left\langle \frac{\beta}{\lambda}' \middle| \hat{a^\dagger}_{pq}\hat{a}_{pq} \middle| \frac{\alpha}{\mu} \right\rangle_{pq} = q\frac{\alpha^2\overline{\beta}}{(\lambda\mu)^2}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} + \frac{\alpha}{\lambda\mu}e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}} \tag{B.26}$$

$$= p\frac{\alpha^2\overline{\beta}}{(\lambda\mu)^2}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} + \frac{\alpha}{\lambda\mu}e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} \tag{B.27}$$

$$\left\langle \frac{\beta}{\lambda}' \middle| \hat{a}_{pq}\hat{a^\dagger}_{pq} \middle| \frac{\alpha}{\mu} \right\rangle_{pq} = \frac{\alpha}{\lambda\mu}(q+p)e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}} + q^2\frac{\alpha^2\bar{\beta}}{(\lambda\mu)^2}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} \tag{B.28}$$

$$= \frac{\alpha}{\lambda\mu}(q+p)e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} + p^2\frac{\alpha^2\bar{\beta}}{(\lambda\mu)^2}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}. \tag{B.29}$$

Moreover, for the special case where $\beta = \alpha$, $\lambda = pq$, and $\mu = p$, the relations become:

$$\left\langle \frac{\alpha}{pq}' \middle| \hat{a^\dagger}^2_{pq} \middle| \frac{\alpha}{p} \right\rangle_{pq} = \frac{\overline{\alpha}|\alpha|^2}{p^4 q}e_{pq}^{\frac{|\alpha|^2}{p^2 q}} + \frac{\overline{\alpha}}{p^2q^2}(p+q)e_{pq}^{\frac{|\alpha|^2}{pq}} \tag{B.30}$$

$$\left\langle \frac{\alpha}{pq}' \middle| \hat{a}^2_{pq} \middle| \frac{\alpha}{p} \right\rangle_{pq} = \frac{\alpha^3}{p^4 q}e_{pq}^{\frac{|\alpha|^2}{p^2 q}} \tag{B.31}$$

$$\left\langle \frac{\alpha}{pq}' \middle| \hat{a^\dagger}_{pq}\hat{a}_{pq} \middle| \frac{\alpha}{p} \right\rangle_{pq} = \frac{\alpha|\alpha|^2}{p^4 q}e_{pq}^{\frac{|\alpha|^2}{p^2 q}} + \frac{\alpha}{p^2 q}e_{pq}^{\frac{|\alpha|^2}{pq}} \tag{B.32}$$

$$\left\langle \frac{\alpha}{pq}' \middle| \hat{a}_{pq}\hat{a^\dagger}_{pq} \middle| \frac{\alpha}{p} \right\rangle_{pq} = \frac{\alpha|\alpha|^2}{p^4}e_{pq}^{\frac{|\alpha|^2}{p^2 q}} + \frac{\alpha}{p^2 q}(q+p)e_{pq}^{\frac{|\alpha|^2}{pq}}. \tag{B.33}$$

Proof We begin by considering the first expression, which can be expanded as follows by using relation (3.30)

$$\begin{aligned}\left\langle\frac{\beta'}{\lambda}\middle|\hat{a}^{\dagger 2}_{pq}\middle|\frac{\alpha}{\mu}\right\rangle_{pq} &= \left\langle\frac{\beta'}{\lambda}\middle|\hat{a}^{\dagger}_{pq}\hat{a}^{\dagger}_{pq}\middle|\frac{\alpha}{\mu}\right\rangle_{pq}\\ &= q\frac{\bar{\beta}\mu}{\lambda}\left\langle\frac{\beta'}{\lambda}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} + \frac{\mu}{\lambda}\left\langle\frac{p\beta}{\lambda}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq},\end{aligned}$$

if we substitute the inner product from equation (B.2) into the above expression , then this substitution results

$$\begin{aligned}\left\langle\frac{\beta'}{\lambda}\middle|\hat{a}^{\dagger 2}_{pq}\middle|\frac{\alpha}{\mu}\right\rangle_{pq} &= q\frac{\bar{\beta}\mu}{\lambda}\left(\frac{1}{\lambda\mu}e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}} + q\frac{\alpha\bar{\beta}}{(\lambda\mu)^2}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}\right) + \frac{\mu}{\lambda}\left(p\frac{\bar{\beta}}{\lambda\mu}e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}}\right)\\ &= \frac{\bar{\beta}}{\lambda^2}\left((p+q)e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}}\right) + q^2\frac{\alpha\bar{\beta}^2}{\lambda^3\mu}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}.\end{aligned}$$

Applying a different formulation, we obtain

$$\begin{aligned}\left\langle\frac{\beta'}{\lambda}\middle|\hat{a}^{\dagger 2}_{pq}\middle|\frac{\alpha}{\mu}\right\rangle_{pq} &= \left\langle\frac{\beta'}{\lambda}\middle|\hat{a}^{\dagger}_{pq}\hat{a}^{\dagger}_{pq}\middle|\frac{\alpha}{\mu}\right\rangle_{pq}\\ &\overset{(3.31)}{=} p\frac{\bar{\beta}\mu}{\lambda}\left\langle\frac{\beta'}{\lambda}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} + \frac{\mu}{\lambda}\left\langle\frac{q\beta}{\lambda}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq}\\ &\overset{(B.3)}{=} p\frac{\bar{\beta}\mu}{\lambda}\left(\frac{1}{\lambda\mu}e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} + p\frac{\alpha\bar{\beta}}{(\lambda\mu)^2}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}\right) + \frac{\mu}{\lambda}\left(q\frac{\bar{\beta}}{\lambda\mu}e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}}\right)\\ &= \frac{\bar{\beta}}{\lambda^2}\left((p+q)e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}}\right) + p^2\frac{\alpha\bar{\beta}^2}{\lambda^3\mu}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}\end{aligned}$$

For the second expression, we start by using the properties of the operator $\hat{a}^2_{pq}$, which allows us to rewrite this expression as

$$\left\langle\frac{\beta'}{\lambda}\middle|\hat{a}^2_{pq}\middle|\frac{\alpha}{\mu}\right\rangle_{pq} = \frac{\alpha^2}{\mu^2}\left\langle\frac{\beta'}{\lambda}\middle|\frac{\alpha}{\lambda}\right\rangle_{pq} = \frac{\beta\alpha^2}{\lambda\mu^3}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}$$

In the case of the third formula, we begin by evaluating the matrix element. First, we rewrite the expression using the definition of the annihilation operator and action of the creation operator which is defined in equation (3.30) as

$$\begin{aligned}\left\langle\frac{\beta'}{\lambda}\middle|\hat{a}^{\dagger}{}_{pq}\hat{a}_{pq}\middle|\frac{\alpha}{\mu}\right\rangle_{pq} &= \frac{\alpha}{\mu}\left\langle\frac{\beta'}{\lambda}\middle|\hat{a}^{\dagger}{}_{pq}\middle|\frac{\alpha}{\lambda}\right\rangle_{pq}\\ &= q\frac{\alpha\bar{\beta}}{\lambda\mu}\left\langle\frac{\beta'}{\lambda}\middle|\frac{\alpha}{\mu}\right\rangle_{pq}+\frac{\alpha}{\lambda\mu}\left\langle p\frac{\beta}{\lambda}\middle|\frac{\alpha}{\mu}\right\rangle_{pq}\\ &\overset{(B.1)}{=} q\frac{\alpha^2\overline{\beta}}{(\lambda\mu)^2}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}+\frac{\alpha}{\lambda\mu}e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}}.\end{aligned}$$

Alternatively, applying relation from equation (3.31) follows

$$\begin{aligned}\left\langle\frac{\beta'}{\lambda}\middle|\hat{a}^{\dagger}{}_{pq}\hat{a}_{pq}\middle|\frac{\alpha}{\mu}\right\rangle_{pq} &= \frac{\alpha}{\mu}\left\langle\frac{\beta'}{\lambda}\middle|\hat{a}^{\dagger}{}_{pq}\middle|\frac{\alpha}{\lambda}\right\rangle_{pq}\\ &= p\frac{\alpha\bar{\beta}}{\lambda\mu}\left\langle\frac{\beta'}{\lambda}\middle|\frac{\alpha}{\mu}\right\rangle_{pq}+\frac{\alpha}{\lambda\mu}\left\langle q\frac{\beta}{\lambda}\middle|\frac{\alpha}{\mu}\right\rangle_{pq}\\ &\overset{(B.1)}{=} p\frac{\alpha^2\overline{\beta}}{(\lambda\mu)^2}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}+\frac{\alpha}{\lambda\mu}e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}}.\end{aligned}$$

Next, we turn to the last expression as applying the relation from equation (B.2) and this yields

$$\begin{aligned}\left\langle\frac{\beta'}{\lambda}\middle|\hat{a}_{pq}\hat{a}^{\dagger}{}_{pq}\middle|\frac{\alpha}{\mu}\right\rangle_{pq} &= \mu\left\langle\frac{\beta'}{\lambda}\middle|\hat{a}_{pq}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq}\\ &\overset{(3.30)}{=} q\alpha\left\langle\frac{\beta'}{\lambda}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq}+\left\langle\frac{\beta'}{\lambda}\middle|p\frac{\alpha}{\mu}\right\rangle_{pq}\\ &= q\alpha\left(\frac{1}{\lambda\mu}\left(e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}}+q\frac{\alpha\bar{\beta}}{\lambda\mu}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}\right)\right)+p\frac{\alpha}{\lambda\mu}e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}}\\ &= \frac{\alpha}{\lambda\mu}(q+p)e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}}+q^2\frac{\alpha^2\bar{\beta}}{(\lambda\mu)^2}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}.\end{aligned}$$

As another option, by using the inner product result from equation ($B.3$) after applying (3.31), we get

$$
\begin{aligned}
(B.29): \quad \left\langle \frac{\beta'}{\lambda}\middle| \hat{a}_{pq}\hat{a}^{\dagger}{}_{pq} \middle|\frac{\alpha}{\mu}\right\rangle_{pq} &= \mu\left\langle \frac{\beta'}{\lambda}\middle|\hat{a}_{pq}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} \\
&= p\alpha\left\langle \frac{\beta'}{\lambda}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} + \left\langle \frac{\beta'}{\lambda}\middle|q\frac{\alpha}{\mu}\right\rangle_{pq} \\
&\overset{(B.3)}{=} p\alpha\left(\frac{1}{\lambda\mu}\left(e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} + p\frac{\alpha\bar{\beta}}{\lambda\mu}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}\right)\right) + q\frac{\alpha}{\lambda\mu}e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} \\
&= \frac{\alpha}{\lambda\mu}(q+p)e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} + p^2\frac{\alpha^2\bar{\beta}}{(\lambda\mu)^2}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}
\end{aligned}
$$

□

Proposition B.8 *The expectation values of $\hat{x}^2_{pq}$ for the states are given as follows*

$$
\left\langle \frac{\alpha}{q}\middle|\hat{x}^2_{pq}\middle|\frac{\alpha}{q}\right\rangle_{pq} = \frac{1}{2}e_{pq}^{\frac{|\alpha|^2}{q^2}}\left(\frac{\alpha^2}{q^2} + \frac{\overline{\alpha}^2}{q^2} + \frac{|\alpha|^2}{q^2} + \frac{|\alpha|^2}{q} + \left(e_{pq}^{\frac{|\alpha|^2}{q^2}}\right)^{-1} e_{pq}^{\frac{p|\alpha|^2}{q^2}}\right), \tag{B.34}
$$

$$
\left\langle \frac{\alpha}{p}\middle|\hat{x}^2_{pq}\middle|\frac{\alpha}{p}\right\rangle_{pq} = \frac{1}{2}\left(\left(\frac{\overline{\alpha}^2}{p^2} + \frac{\alpha^2}{p^2} + (q+1)\frac{|\alpha|^2}{p^2}\right)e_{pq}^{\frac{|\alpha|^2}{p^2}} + e_{pq}^{\frac{|\alpha|^2}{p}}\right), \tag{B.35}
$$

$$
\left\langle \frac{\alpha'}{pq}\middle|\hat{x}^2_{pq}\middle|\frac{\alpha}{p}\right\rangle_{pq} = \frac{1}{2}\left(\frac{1}{p^4q}(\overline{\alpha}|\alpha|^2 + \alpha^3 + \alpha|\alpha|^2 + q\alpha|\alpha|^2\right)e_{pq}^{\frac{|\alpha|^2}{p^2q}} \tag{B.36}
$$

$$
+ \frac{1}{p^2q^2}\left((p+q)\overline{\alpha} + q\alpha + (q^2+pq)\alpha)e_{pq}^{\frac{|\alpha|^2}{pq}}\right). \tag{B.37}
$$

Proof The first two results follow easily from Eqs.($B.18$) − ($B.22$). For the third expression, we proceed as using the decomposition of $\hat{x}^2_{pq}$ in terms of the creation and annihilation operators

$$
\begin{aligned}
\left\langle \frac{\alpha'}{pq}\middle|\hat{x}^2_{pq}\middle|\frac{\alpha}{p}\right\rangle_{pq} &= \frac{1}{2}\Bigg(\left\langle \frac{\alpha'}{pq}\middle|\hat{a}^{\dagger 2}{}_{pq}\middle|\frac{\alpha}{p}\right\rangle_{pq} + \left\langle \frac{\alpha'}{pq}\middle|\hat{a}^2_{pq}\middle|\frac{\alpha}{p}\right\rangle_{pq} \\
&+ \left\langle \frac{\alpha'}{pq}\middle|\hat{a}^{\dagger}{}_{pq}\hat{a}_{pq}\middle|\frac{\alpha}{p}\right\rangle_{pq} + \left\langle \frac{\alpha'}{pq}\middle|\hat{a}_{pq}\hat{a}^{\dagger}{}_{pq}\middle|\frac{\alpha}{p}\right\rangle_{pq}\Bigg) \\
&= \frac{1}{2}\Bigg(\frac{1}{p^4q}(\overline{\alpha}|\alpha|^2 + \alpha^3 + \alpha|\alpha|^2 + q\alpha|\alpha|^2)e_{pq}^{\frac{|\alpha|^2}{p^2q}} \\
&+ \frac{1}{p^2q^2}((p+q)\overline{\alpha} + q\alpha + (q^2+pq)\alpha)e_{pq}^{\frac{|\alpha|^2}{pq}}\Bigg).
\end{aligned}
$$

Next, taking the Hermitian conjugate of this expression, we get

$$\left\langle \frac{\alpha}{p} \middle| \hat{x}^2_{pq} \middle| \frac{\alpha'}{pq} \right\rangle_{pq} = \frac{1}{2}\Big(\frac{1}{p^4 q}(\alpha|\alpha|^2 + \overline{\alpha}^3 + \overline{\alpha}|\alpha|^2 + q\overline{\alpha}|\alpha|^2) e_{pq}^{\frac{|\alpha|^2}{p^2 q}} + \frac{1}{p^2 q^2}((p+q)\alpha + q\overline{\alpha} + (q^2 + pq)\overline{\alpha}) e_{pq}^{\frac{|\alpha|^2}{pq}} \Big). \tag{B.38}$$

□

Proposition B.9 *Let $\hat{a}_{pq}$ and $\hat{a^{\dagger}}_{pq}$ represent the $pq-$annihilation and $pq-$creation operators, respectively. The matrix elements involving these operators for the states $\big|\frac{\alpha'}{\lambda}\rangle_{pq}$ and $\big|\frac{\alpha'}{\mu}\rangle_{pq}$ are given as follows:*

$$\left\langle \frac{\beta'}{\lambda} \middle| \hat{a}^2_{pq} \middle| \frac{\alpha'}{\mu} \right\rangle_{pq} = \frac{\alpha^2}{\lambda\mu^3}(p^2 + pq + q^2) e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} + \frac{p^3\alpha^3\bar{\beta}}{\lambda^2\mu^4} e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} \tag{B.39}$$

$$= \frac{\alpha^2}{\lambda\mu^3}(p^2 + pq + q^2) e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}} + \frac{q^3\alpha^3\bar{\beta}}{\lambda^2\mu^4} e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} \tag{B.40}$$

$$\left\langle \frac{\beta'}{\lambda} \middle| \hat{a^{\dagger}}_{pq}\hat{a}_{pq} \middle| \frac{\alpha'}{\mu} \right\rangle_{pq} = p^3 \frac{\bar{\beta}^2\alpha^2}{(\lambda\mu)^3} e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} + \frac{\alpha\bar{\beta}}{(\lambda\mu)^2}(p^2 + 2pq) e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} + \frac{1}{\lambda\mu} e_{pq}^{q^2\frac{\alpha\bar{\beta}}{\lambda\mu}} \tag{B.41}$$

$$= q^3 \frac{\bar{\beta}^2\alpha^2}{(\lambda\mu)^3} e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} + \frac{\alpha\bar{\beta}}{(\lambda\mu)^2}(q^2 + 2pq) e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}} + \frac{1}{\lambda\mu} e_{pq}^{p^2\frac{\alpha\bar{\beta}}{\lambda\mu}}. \tag{B.42}$$

When $\alpha = \beta$ and $\lambda = \mu = pq$, the following holds:

$$\left\langle \frac{\alpha'}{pq} \middle| \hat{a}^2_{pq} \middle| \frac{\alpha'}{pq} \right\rangle_{pq} = \frac{\alpha^2}{(pq)^4}(q^2 + pq + p^2) e_{pq}^{\frac{|\alpha|^2}{p^2 q}} + p^3 \frac{\alpha^2|\alpha|^2}{(pq)^6} e_{pq}^{\frac{|\alpha|^2}{p^2 q^2}}$$

$$= \frac{\alpha^2}{(pq)^4}(q^2 + pq + p^2) e_{pq}^{\frac{|\alpha|^2}{pq^2}} + q^3 \frac{\alpha^2|\alpha|^2}{(pq)^6} e_{pq}^{\frac{|\alpha|^2}{p^2 q^2}}$$

$$\left\langle \frac{\alpha'}{pq} \middle| \hat{a^{\dagger}}_{pq}\hat{a}_{pq} \middle| \frac{\alpha'}{pq} \right\rangle_{pq} = \frac{|\alpha|^4}{p^3 q^6} e_{pq}^{\frac{|\alpha|^2}{p^2 q^2}} + \frac{|\alpha|^2}{p^3 q^4}(p + 2q) e_{pq}^{\frac{|\alpha|^2}{p^2 q}} + \frac{1}{p^2 q^2} e_{pq}^{\frac{|\alpha|^2}{p^2}}$$

$$= \frac{|\alpha|^4}{p^6 q^3} e_{pq}^{\frac{|\alpha|^2}{p^2 q^2}} + \frac{|\alpha|^2}{p^4 q^3}(q + 2p) e_{pq}^{\frac{|\alpha|^2}{pq^2}} + \frac{1}{p^2 q^2} e_{pq}^{\frac{|\alpha|^2}{q^2}}.$$

Proof For the first expression, we can rewrite the matrix element by using action of $\hat{a}_{pq}$ as in Eqn.(3.31),

$$\begin{aligned}\left\langle\frac{\beta^{'}}{\lambda}\middle|\hat{a}^2_{pq}\middle|\frac{\alpha^{'}}{\mu}\right\rangle_{pq} &= \frac{p\alpha}{\mu}\left\langle\frac{\beta^{'}}{\lambda}\middle|\hat{a}_{pq}\middle|\frac{\alpha^{'}}{\mu}\right\rangle_{pq} + \frac{1}{\mu}\left\langle\frac{\beta^{'}}{\lambda}\middle|\hat{a}_{pq}\middle|\frac{q\alpha}{\mu}\right\rangle_{pq} \\ &\overset{(3.31)}{=} \frac{p^2\alpha^2}{\mu^2}\left\langle\frac{\beta^{'}}{\lambda}\middle|\frac{\alpha^{'}}{\mu}\right\rangle_{pq} + \frac{p\alpha}{\mu^2}\left\langle\frac{\beta^{'}}{\lambda}\middle|\frac{q\alpha}{\mu}\right\rangle_{pq} + \frac{q\alpha}{\mu^2}\left\langle\frac{\beta^{'}}{\lambda}\middle|\frac{q\alpha}{\mu}\right\rangle_{pq}.\end{aligned}$$

Now, using the inner product results from (B.1) and (B.3), we substitute the known forms of the inner products:

$$\begin{aligned}\left\langle\frac{\beta^{'}}{\lambda}\middle|\hat{a}^2_{pq}\middle|\frac{\alpha^{'}}{\mu}\right\rangle_{pq} &= \frac{p^2\alpha^2}{\mu^2}\left(\frac{1}{\lambda\mu}e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} + p\frac{\alpha\bar{\beta}}{(\lambda\mu)^2}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}\right) + \frac{\alpha^2}{\lambda\mu^3}(pq+q^2)e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} \\ &= \frac{\alpha^2}{\lambda\mu^3}(p^2+pq+q^2)e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} + \frac{p^3\alpha^3\bar{\beta}}{\lambda^2\mu^4}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}.\end{aligned}$$

For the second expression, we follow a similar approach, but now using the alternative form of the action of the annihilation operator (3.30):

$$\begin{aligned}\left\langle\frac{\beta^{'}}{\lambda}\middle|\hat{a}^2_{pq}\middle|\frac{\alpha^{'}}{\mu}\right\rangle_{pq} &= \frac{q\alpha}{\mu}\left\langle\frac{\beta^{'}}{\lambda}\middle|\hat{a}_{pq}\middle|\frac{\alpha^{'}}{\mu}\right\rangle_{pq} + \frac{1}{\mu}\left\langle\frac{\beta^{'}}{\lambda}\middle|\hat{a}_{pq}\middle|\frac{p\alpha}{\mu}\right\rangle_{pq} \\ &\overset{(3.30)}{=} \frac{q^2\alpha^2}{\mu^2}\left\langle\frac{\beta^{'}}{\lambda}\middle|\frac{\alpha^{'}}{\mu}\right\rangle_{pq} + \frac{q\alpha}{\mu^2}\left\langle\frac{\beta^{'}}{\lambda}\middle|\frac{p\alpha}{\mu}\right\rangle_{pq} + \frac{p\alpha}{\mu^2}\left\langle\frac{\beta^{'}}{\lambda}\middle|\frac{p\alpha}{\mu}\right\rangle_{pq} \\ &\overset{(B.1)(B.2)}{=} \frac{q^2\alpha^2}{\mu^2}\left(\frac{1}{\lambda\mu}e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}} + q\frac{\alpha\bar{\beta}}{(\lambda\mu)^2}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}\right) + \frac{\alpha^2}{\lambda\mu^3}(pq+p^2)e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}} \\ &= \frac{\alpha^2}{\lambda\mu^3}(p^2+pq+q^2)e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}} + \frac{q^3\alpha^3\bar{\beta}}{\lambda^2\mu^4}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}.\end{aligned}$$

Also, the hermition conjugate above equation gives following

$$\begin{aligned}\left\langle\frac{\alpha^{'}}{\mu}\middle|\hat{a^{\dagger}}^2_{pq}\middle|\frac{\beta^{'}}{\lambda}\right\rangle_{pq} &= \frac{\overline{\alpha}^2}{\lambda\mu^3}(p^2+pq+q^2)e_{pq}^{q\frac{\overline{\alpha}\beta}{\lambda\mu}} + \frac{p^3\alpha^3\bar{\beta}}{\lambda^2\mu^4}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} \\ &= \frac{\overline{\alpha}^2}{\lambda\mu^3}(p^2+pq+q^2)e_{pq}^{p\frac{\overline{\alpha}\beta}{\lambda\mu}} + \frac{q^3\alpha^3\bar{\beta}}{\lambda^2\mu^4}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}}.\end{aligned}$$

For the last expression, we begin by evaluating the matrix element $\left\langle \frac{\beta'}{\lambda}\middle| \hat{a^{\dagger}}_{pq}\hat{a}_{pq}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq}$, which involves the product of the creation and annihilation operators. By applying the action of the annihilation operator from equation (3.31), we express the matrix element as

$$\begin{aligned}
\left\langle \frac{\beta'}{\lambda}\middle| \hat{a^{\dagger}}_{pq}\hat{a}_{pq}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} &= \left({}_{pq}\left\langle \frac{\beta'}{\lambda}\middle|\frac{p\overline{\beta}}{\lambda} + {}_{pq}\left\langle \frac{q\beta}{\lambda}\middle|\frac{1}{\lambda}\right)\right.\right.\left(\frac{p\alpha}{\mu}\left|\frac{\alpha'}{\mu}\right\rangle_{pq} + \frac{1}{\mu}\left|\frac{q\alpha}{\mu}\right\rangle_{pq}\right) \\
&= p^2\frac{\bar{\beta}\alpha}{\lambda\mu}\left\langle\frac{\beta'}{\lambda}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} + \frac{p\overline{\beta}}{\lambda\mu}\left\langle\frac{\beta'}{\lambda}\middle|\frac{q\alpha}{\mu}\right\rangle_{pq} \\
&+ \frac{p\alpha}{\lambda\mu}\left\langle\frac{q\beta}{\lambda}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} + \frac{1}{\lambda\mu}\left\langle\frac{q\beta}{\lambda}\middle|\frac{q\alpha}{\mu}\right\rangle_{pq}.
\end{aligned}$$

Then, we apply the corresponding inner product results from equations (B.1) and (B.3), which simplifies to the following result

$$\begin{aligned}
\left\langle \frac{\beta'}{\lambda}\middle| \hat{a^{\dagger}}_{pq}\hat{a}_{pq}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} &= p^2\frac{\bar{\beta}\alpha}{(\lambda\mu)^2}\left(\frac{p}{\lambda\mu}\bar{\beta}\alpha e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} + e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}}\right) + pq\frac{\alpha\bar{\beta}}{(\lambda\mu)^2}e_{pq}^{q\frac{\bar{\alpha}\beta}{\lambda\mu}} \\
&+ pq\frac{\alpha\bar{\beta}}{(\lambda\mu)^2}e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} + \frac{1}{\lambda\mu}e_{pq}^{q^2\frac{\alpha\bar{\beta}}{\lambda\mu}} \\
&= p^3\frac{\bar{\beta}^2\alpha^2}{(\lambda\mu)^3}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} + \frac{\alpha\bar{\beta}}{(\lambda\mu)^2}(p^2+2pq)e_{pq}^{q\frac{\alpha\bar{\beta}}{\lambda\mu}} + \frac{1}{\lambda\mu}e_{pq}^{q^2\frac{\alpha\bar{\beta}}{\lambda\mu}}.
\end{aligned}$$

For an alternative formulation, by applying the action of the annihilation operator from equation (3.30), we can express the matrix element in a similar form

$$\begin{aligned}
\left\langle \frac{\beta'}{\lambda}\middle| \hat{a^{\dagger}}_{pq}\hat{a}_{pq}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} &= \left({}_{pq}\left\langle \frac{\beta'}{\lambda}\middle|\frac{q\overline{\beta}}{\lambda} + {}_{pq}\left\langle \frac{p\beta}{\lambda}\middle|\frac{1}{\lambda}\right)\right.\right.\left(\frac{q\alpha}{\mu}\left|\frac{\alpha'}{\mu}\right\rangle_{pq} + \frac{1}{\mu}\left|\frac{p\alpha}{\mu}\right\rangle_{pq}\right) \\
&= q^2\frac{\bar{\beta}\alpha}{\lambda\mu}\left\langle\frac{\beta'}{\lambda}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} + \frac{q\overline{\beta}}{\lambda\mu}\left\langle\frac{\beta'}{\lambda}\middle|\frac{p\alpha}{\mu}\right\rangle_{pq} \\
&+ \frac{q\alpha}{\lambda\mu}\left\langle\frac{p\beta}{\lambda}\middle|\frac{\alpha'}{\mu}\right\rangle_{pq} + \frac{1}{\lambda\mu}\left\langle\frac{p\beta}{\lambda}\middle|\frac{p\alpha}{\mu}\right\rangle_{pq} \\
&\overset{(B.1)(B.2)}{=} q^2\frac{\bar{\beta}\alpha}{(\lambda\mu)^2}\left(\frac{q}{\lambda\mu}\bar{\beta}\alpha e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} + e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}}\right) + pq\frac{\alpha\bar{\beta}}{(\lambda\mu)^2}e_{pq}^{p\frac{\bar{\alpha}\beta}{\lambda\mu}} \\
&+ pq\frac{\alpha\bar{\beta}}{(\lambda\mu)^2}e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}} + \frac{1}{\lambda\mu}e_{pq}^{p^2\frac{\alpha\bar{\beta}}{\lambda\mu}} \\
&= q^3\frac{\bar{\beta}^2\alpha^2}{(\lambda\mu)^3}e_{pq}^{\frac{\alpha\bar{\beta}}{\lambda\mu}} + \frac{\alpha\bar{\beta}}{(\lambda\mu)^2}(q^2+2pq)e_{pq}^{p\frac{\alpha\bar{\beta}}{\lambda\mu}} + \frac{1}{\lambda\mu}e_{pq}^{p^2\frac{\alpha\bar{\beta}}{\lambda\mu}}.
\end{aligned}$$

Thus, all transitions are derived, completing the proof. □

Proposition B.10 *The expectation value of the squared position operator $\hat{x}^2_{pq}$ for the state can be expressed as*

$$\left\langle \frac{\alpha'}{pq} \middle| \hat{x}^2_{pq} \middle| \frac{\alpha'}{pq} \right\rangle_{pq} = \frac{1}{2}\Bigg(\left(\frac{|\alpha|^2}{p^3q^6}(\alpha^2 + \overline{\alpha}^2) + \frac{|\alpha|^4}{p^3q^6} + \frac{|\alpha|^4}{p^2q^6} \right) e_{pq}^{\frac{|\alpha|^2}{p^2q^2}} + \left(\frac{p^2 + pq + q^2}{p^4q^4}(\alpha^2 + \overline{\alpha}^2) + \frac{|\alpha|^2}{p^3q^4}(p + 2q + (p+q)^2) \right) e_{pq}^{\frac{|\alpha|^2}{p^2q}} + \left(\frac{1 + p + q}{p^2q^2} \right) e_{pq}^{\frac{|\alpha|^2}{p^2}} \Bigg). \tag{B.43}$$

Proof Now, we compute the expectation value of the squared position operator $\hat{x}^2_{pq}$ for the state $\left|\frac{\alpha'}{pq}\right\rangle$. The transitions for each of these terms are derived from Proposition(B.9), and substituting the relevant expressions gives

$$\begin{aligned}
\left\langle \frac{\alpha'}{pq} \middle| \hat{x}^2_{pq} \middle| \frac{\alpha'}{pq} \right\rangle_{pq} &= \frac{1}{2}\Bigg(\left\langle \frac{\alpha'}{pq} \middle| \hat{a^{\dagger}}^2_{pq} \middle| \frac{\alpha'}{pq} \right\rangle_{pq} + \left\langle \frac{\alpha'}{pq} \middle| \hat{a}^2_{pq} \middle| \frac{\alpha'}{pq} \right\rangle_{pq} \\
&+ \left\langle \frac{\alpha'}{pq} \middle| \hat{a^{\dagger}}_{pq}\hat{a}_{pq} \middle| \frac{\alpha'}{pq} \right\rangle_{pq} + \left\langle \frac{\alpha'}{pq} \middle| \hat{a}_{pq}\hat{a^{\dagger}}_{pq} \middle| \frac{\alpha'}{pq} \right\rangle_{pq} \Bigg) \\
&= \frac{1}{2}\Bigg(\left(\frac{|\alpha|^2}{p^3q^6}(\alpha^2 + \overline{\alpha}^2) + \frac{|\alpha|^4}{p^3q^6} + \frac{|\alpha|^4}{p^2q^6} \right) e_{pq}^{\frac{|\alpha|^2}{p^2q^2}} \\
&+ \left(\frac{p^2 + pq + q^2}{p^4q^4}(\alpha^2 + \overline{\alpha}^2) + \frac{|\alpha|^2}{p^3q^4}(p + 2q + (p+q)^2) \right) e_{pq}^{\frac{|\alpha|^2}{p^2q}} \\
&+ \left(\frac{1 + p + q}{p^2q^2} \right) e_{pq}^{\frac{|\alpha|^2}{p^2}} \Bigg)
\end{aligned}$$

□

After all, the expectation value of the squared position operator $\hat{x}^2_{pq}$ for the state $|A_s\rangle_{pq}$ is given by:

$$\begin{aligned}
{}_{pq}\langle A_s|\hat{x}^2_{pq}|A_s\rangle_{pq} &= |C_s|^2\Bigg(\left(e_{pq}^{\frac{|\alpha|^2}{p^2q}} \right)^2 \frac{|\alpha|^2}{p^4} \left\langle \frac{\alpha}{p} \middle| \hat{x}^2_{pq} \middle| \frac{\alpha}{p} \right\rangle_{pq} - e_{pq}^{\frac{|\alpha|^2}{p^2q}} e_{pq}^{\frac{|\alpha|^2}{p^2}} \frac{q\alpha}{p^2} \left\langle \frac{\alpha}{p} \middle| \hat{x}^2_{pq} \middle| \frac{\alpha'}{pq} \right\rangle_{pq} \\
&- e_{pq}^{\frac{|\alpha|^2}{p^2q}} e_{pq}^{\frac{|\alpha|^2}{p^2}} \frac{q\overline{\alpha}}{p^2} \left\langle \frac{\alpha'}{pq} \middle| \hat{x}^2_{pq} \middle| \frac{\alpha'}{pq} \right\rangle_{pq} + \left(e_{pq}^{\frac{|\alpha|^2}{p^2}} \right)^2 q^2 \left\langle \frac{\alpha'}{pq} \middle| \hat{x}^2_{pq} \middle| \frac{\alpha'}{pq} \right\rangle_{pq} \\
&+ \left(e_{pq}^{\frac{|\alpha|^2}{p^2}} \right)^2 \left\langle \frac{\alpha}{q} \middle| \hat{x}^2_{pq} \middle| \frac{\alpha}{q} \right\rangle_{pq} \Bigg).
\end{aligned}$$

This result follows from the previous calculations of the expectation values for the terms involving $\hat{a}_{pq}$, $\hat{a}^{\dagger}_{pq}$, and their corresponding matrix elements, as derived earlier. The final expression incorporates those results to evaluate the expectation value of $\hat{x}^2_{pq}$ for the state $|A_s\rangle_{pq}$, where the exponential factors and inner products have been computed from earlier transitions. This calculation is important as it will help to determine the uncertainty in the position operator $\hat{x}_{pq}$. By combining this with the expectation value of the momentum operator, the uncertainty can be calculated using the standard quantum mechanical uncertainty relations.

APPENDIX C

FERMION-BOSON BELL STATES

Bell states are a specific set of maximally entangled quantum states of two qubits. Bell states are maximally entangled. This means that if you measure one qubit in a Bell state, the state of the other qubit becomes instantly determined, regardless of the distance between them. The four Bell states form an orthonormal basis for the two-qubit Hilbert space.

To express the Bell states in terms of fermion-boson states, we use the tensor product notation, where the fermionic states are denoted by $|0\rangle_f = |0\rangle$ and $|1\rangle_f = |1\rangle$, and the bosonic states are denoted by $|0\rangle_b$ and $|1\rangle_b$.

$$\begin{aligned}
|B_+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\left(|0\rangle_f \otimes |0\rangle_b + |1\rangle_f \otimes |1\rangle_b\right) \\
|B_-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}}\left(|0\rangle_f \otimes |0\rangle_b - |1\rangle_f \otimes |1\rangle_b\right) \\
|L_+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\left(|0\rangle_f \otimes |1\rangle_b + |1\rangle_f \otimes |0\rangle_b\right) \\
|L_-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}\left(|0\rangle_f \otimes |1\rangle_b - |1\rangle_f \otimes |0\rangle_b\right)
\end{aligned}$$

If we take $n = 1$ and choose $\theta = \frac{\pi}{2}$ as in (6.17), we have the maximally entangled state

$$|1, \frac{\pi}{2}, \phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_f|1\rangle_b + e^{i\phi}|1\rangle_f|0\rangle_b),$$

providing the fermion-boson analog of the Bell states for $\phi = 0$ and $\phi = \pi$,

$$|L_\pm\rangle \equiv |1, \frac{\pi}{2}, \pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_f|1\rangle_b \pm |1\rangle_f|0\rangle_b).$$

Proposition C.1 *The states $|L_\pm\rangle$ are exact eigenstates of the operator $\mathcal{N}$ with one superparticle ($n = 1$), such that $\mathcal{N}|L_\pm\rangle = |L_\pm\rangle$. In contrast, the states $|B_\pm\rangle$ are not eigenstates of $\mathcal{N}$; however, the average number of superparticles in these states is one, given by*

$\langle B_\pm|\mathcal{N}|B_\pm\rangle = 1$.

Proof To prove this, let us consider that the operator $\mathcal{N}$ is defined in such a way that it counts the number of superparticles in a given state. Since $|L_\pm\rangle$ represents a state with exactly one superparticle ($n = 1$), applying $\mathcal{N}$ to $|L_\pm\rangle$ yields:

$$\mathcal{N}|L_\pm\rangle = n|L_\pm\rangle = 1 \cdot |L_\pm\rangle = |L_\pm\rangle.$$

The expectation value of $\mathcal{N}$ in the state $|B_+\rangle$ is given by

$$\begin{aligned}
\langle B_+|\mathcal{N}|B_+\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} {}_b\langle 0| & {}_b\langle 1| \end{pmatrix}\begin{pmatrix} \mathcal{N} & 0 \\ 0 & \mathcal{N}+1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} |0\rangle_b \\ |1\rangle_b \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} {}_b\langle 0| & {}_b\langle 1| \end{pmatrix}\begin{pmatrix} 0 \\ 2|1\rangle_b \end{pmatrix} = \frac{1}{2}(0 +_b \langle 1|1\rangle_b) = 1
\end{aligned}$$

Parallel calculations can similarly be performed for B_-.This shows that, while $|B_\pm\rangle$ are not eigenstates of $\mathcal{N}$, the expected average number of superparticles in the states(the mean value) is one. □

Proposition C.2 *For each Bell state, there is an associated annihilation operator that combines the bosonic annihilation operator a with the fermionic annihilation or creation operators, f and $f^\dagger$. We define four such operators as follows:*

$$\begin{aligned}
A_{\pm 1} &= \begin{pmatrix} a & \pm 1 \\ 0 & a \end{pmatrix} = I_f \otimes a \pm f \otimes I_b, \\
A^T_{\pm 1} &= \begin{pmatrix} a & 0 \\ \pm 1 & a \end{pmatrix} = I_f \otimes a \pm f^\dagger \otimes I_b,
\end{aligned}$$

These operators annihilate the corresponding Bell states as follows:

$$\begin{aligned}
A_1|L_-\rangle &= 0, \qquad A_{-1}|L_+\rangle = 0, \\
A^T_1|B_-\rangle &= 0, \qquad A^T_{-1}|B_+\rangle = 0.
\end{aligned}$$

Proof Let's start by proving each statement one by one. In following expressions, the action of the bosonic annihilation operator a on the states $|0\rangle$ and $|1\rangle$ yields, $a|0\rangle_b = 0, \quad a|1\rangle_b = |0\rangle_b$. Also, the fermionic annihilation operator f and creation operator $f^\dagger$ act as $f|0\rangle_f = 0, \quad f|1\rangle_f = |0\rangle_f, \quad f^\dagger|0\rangle_f = |1\rangle_f \quad f^\dagger|1\rangle_f = 0$. Applying operator A_1 to the state $L_-\rangle$ with the following actions of a and f gives

$$\begin{aligned}
A_1|L_-\rangle &= \Big(I_f \otimes a + f \otimes I_b\Big)|L_-\rangle \\
&= \frac{1}{\sqrt{2}}\Big(I_f \otimes a + f \otimes I_b\Big)\Big(|0\rangle_f \otimes |1\rangle_b - |1\rangle_f \otimes |0\rangle_b\Big) \\
&= \frac{1}{\sqrt{2}}\Big((|0\rangle_f \otimes a|1\rangle_b) - (|1\rangle_f \otimes a|0\rangle_b) + (f|0\rangle_f \otimes |1\rangle_b) - (f|1\rangle_f \otimes |0\rangle_b)\Big) \\
&= \frac{1}{\sqrt{2}}\Big(|0\rangle_f \otimes |0\rangle_b - 0 + 0 - |0\rangle_f \otimes |0\rangle_b\Big) = 0
\end{aligned}$$

The same type of calculations can be applied to $|L_+\rangle$ for the operator A_{-1}. Now consider the action of the transposed operator A_1^T to the state $|B_-\rangle$,

$$\begin{aligned}
A_1^T|B_-\rangle &= \Big(I_f \otimes a + f^\dagger \otimes I_b\Big)|B_-\rangle \\
&= \frac{1}{\sqrt{2}}\Big(I_f \otimes a + f^\dagger \otimes I_b\Big)\Big(|0\rangle_f \otimes |0\rangle_b - |1\rangle_f \otimes |1\rangle_b\Big) \\
&= \frac{1}{\sqrt{2}}\Big((|0\rangle_f \otimes a|0\rangle_b) - (|1\rangle_f \otimes a|1\rangle_b) + f^\dagger|0\rangle_f \otimes |0\rangle_b - f^\dagger|1\rangle_f \otimes |1\rangle_b\Big) \\
&= \frac{1}{\sqrt{2}}\Big(0 - |1\rangle_f \otimes |0\rangle_b + |1\rangle_f \otimes |0\rangle_b - 0\Big) = 0.
\end{aligned}$$

After these steps, it is easy to find $A_{-1}^T|B_+\rangle = 0$. We have shown that the operators annihilate their respective Bell states. □

Proposition C.3 *The states $|L_\pm\rangle$ can be obtained from the vacuum state $|\Psi_0\rangle = |0\rangle_f \otimes |0\rangle_b$ by applying the creation operators $A_{\pm 1}^\dagger$, and conversely. Explicitly, these relations are given by:*

$$|L_\pm\rangle = \frac{1}{\sqrt{2}}A_{\pm 1}^\dagger|\Psi_0\rangle, \qquad |\Psi_0\rangle = \frac{1}{\sqrt{2}}A_{\pm 1}|L_\pm\rangle.$$

Proof Recall the definitions of the operators $A_{\pm 1}$ and their Hermitian conjugates $A^{\dagger}_{\pm 1}$,

$$A_{\pm 1} = \begin{pmatrix} a & \pm 1 \\ 0 & a \end{pmatrix}, \quad A^{\dagger}_{\pm 1} = \begin{pmatrix} a^{\dagger} & 0 \\ \pm 1 & a^{\dagger} \end{pmatrix}.$$

Applying the operator $A^{\dagger}_{\pm 1}$ to the state $|\Psi_0\rangle$ provides

$$\begin{aligned} \frac{1}{\sqrt{2}} A^{\dagger}_{\pm 1} |\Psi_0\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} a^{\dagger} & 0 \\ \pm 1 & a^{\dagger} \end{pmatrix} \begin{pmatrix} |0\rangle_b \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} a^{\dagger}|0\rangle_b \\ \pm|0\rangle_b \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|0\rangle_f \otimes |1\rangle_b \pm |1\rangle_f \otimes |0\rangle_b \right) = |L_{\pm}\rangle. \end{aligned}$$

Comparable computations can be carried out for other formula. □

Since vacuum state is annihilated by two operators $A_{\pm 1}|\Psi_0\rangle = 0$, and orthogonal to the pair of Bell states $|L_{\pm}\rangle$, then we can use them to express the following state.

Definition C.1 *The normalized reference states can be written as*

$$|0, C, \phi\rangle_{L_{\pm}} = \sqrt{1 - C}|\Psi_0\rangle + \sqrt{C}e^{i\phi}|L_{\pm}\rangle,$$

combination of the vacuum state and Bell states $L_{\pm}$, which are annihilated by operators $A_{\mp 1}|0, C, \phi\rangle_{L_{\pm}} = 0$.The states are parametrized by real number C, bounded between $0 \leq C \leq 1$.

C.1. Entanglement of Super-Number States

Proposition C.4 *For super number state*

$$|n, \theta, \phi\rangle = \cos\frac{\theta}{2} \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} + \sin\frac{\theta}{2} e^{i\phi} \begin{pmatrix} 0 \\ |n-1\rangle \end{pmatrix} \quad ,$$

the density matrix for pure state is

$$\rho_n = |n,\theta,\phi\rangle\langle n,\theta,\phi| = \begin{pmatrix} \cos^2\frac{\theta}{2}\,|n\rangle\langle n| & \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi}|n\rangle\langle n-1| \\ \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi}|n-1\rangle\langle n| & \sin^2\frac{\theta}{2}\,|n-1\rangle\langle n-1| \end{pmatrix}.$$

Then, reduced bosonic and fermionic density matrices are following

$$\begin{aligned} \rho_b &= tr_f\,\rho_n = \sin^2\frac{\theta}{2}\,|n-1\rangle\langle n-1| + \cos^2\frac{\theta}{2}\,|n\rangle\langle n|, \\ \rho_f &= tr_b\,\rho_n = \cos^2\frac{\theta}{2}\,|0\rangle\langle 0| + \sin^2\frac{\theta}{2}\,|1\rangle\langle 1|. \end{aligned}$$

Proof To prove the expressions for the reduced density matrices , we can rewrite ρ_n by explicitly showing the fermionic state contributions $|0\rangle$ and $|1\rangle$

$$\begin{aligned} \rho_n &= |n,\theta,\phi\rangle\langle n,\theta,\phi| = \begin{pmatrix} \cos^2\frac{\theta}{2}\,|n\rangle\langle n| & \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi}|n\rangle\langle n-1| \\ \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi}|n-1\rangle\langle n| & \sin^2\frac{\theta}{2}\,|n-1\rangle\langle n-1| \end{pmatrix} \\ &= \left(\cos^2\frac{\theta}{2}|0\rangle\langle 0|\right)|n\rangle\langle n| + \left(\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi}|0\rangle\langle 1|\right)|n\rangle\langle n-1| \\ &+ \left(\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi}|1\rangle\langle 0|\right)|n-1\rangle\langle n| + \left(\sin^2\frac{\theta}{2}|1\rangle\langle 1|\right)|n-1\rangle\langle n-1| \end{aligned}$$

In order to compute the reduced bosonic density matrix, we need to trace out the fermionic basis states and evaluating each term separately, which gives

$$\begin{aligned} \rho_b &= tr_f\,\rho_n = \langle 0|\rho_n|0\rangle + \langle 1|\rho_n|1\rangle \\ &= \langle 0|\left(\cos^2\frac{\theta}{2}|0\rangle\langle 0\,|n\rangle\langle n|\right)|0\rangle + \langle 1|\left(\sin^2\frac{\theta}{2}|1\rangle\langle 1||n-1\rangle\langle n-1|\right)|1\rangle \\ &= \sin^2\frac{\theta}{2}\,|n-1\rangle\langle n-1| + \cos^2\frac{\theta}{2}\,|n\rangle\langle n|, \end{aligned}$$

infinite dimensional matrix with only two nonzero diagonal terms, $\sin^2\frac{\theta}{2}$ and $\cos^2\frac{\theta}{2}$ at positions n and $n+1$. The same strategy is valid for reduced fermionic density matrix, we compute the partial trace over the bosonic states so that it becomes 2×2 diagonal matrix,

$$\rho_f \quad = \quad tr_b \rho_n = \cos^2 \frac{\theta}{2} |0\rangle\langle 0| + \sin^2 \frac{\theta}{2} |1\rangle\langle 1|.$$

□

Proposition C.5 *The squares of both reduced density matrices have equal traces, given by*

$$tr\rho_f^2 = tr\rho_b^2 = 1 - \frac{1}{2} \sin^2 \theta.$$

Proof First, we start with computing the square of ρ_f

$$\rho_f^2 = \left(\cos^2 \frac{\theta}{2}|0\rangle\langle 0| + \sin^2 \frac{\theta}{2}|1\rangle\langle 1|\right)^2 = \cos^4 \frac{\theta}{2}|0\rangle\langle 0| + \sin^4 \frac{\theta}{2}|1\rangle\langle 1|.$$

where $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ are orthogonal projectors, their cross terms vanish. Then, taking the trace of ρ_f^2 gives

$$\mathrm{tr}\rho_f^2 \quad = \quad \mathrm{tr}\left(\cos^4 \frac{\theta}{2}|0\rangle\langle 0| + \sin^4 \frac{\theta}{2}|1\rangle\langle 1|\right) = \cos^4 \frac{\theta}{2} + \sin^4 \frac{\theta}{2}.$$

We can express the result as

$$\mathrm{tr}\rho_f^2 = 1 - \frac{1}{2} \sin^2 \theta.$$

Similarly calculations can be done for $tr\rho_b^2$. □

By using formula (6.8) we obtain that the reduced bosonic, as well as fermionic state is mixed and the generic state $|n, \theta, \phi\rangle$ is entangled with concurrence

$$C = \sin \theta.$$

C.2. Displacement Operator

Now, we want to define displacement operator which generates the coherent states from vacuum state. The displacement operator $D(\alpha)$, where $\alpha \in \mathbb{C}$, is defined by

$$D(\alpha) = e^{\alpha\hat{a}^{\dagger} - \bar{\alpha}\hat{a}}. \tag{C.1}$$

Proposition C.6 *Properties of displacement operator is given by the following relations*

- $D^{\dagger}(\alpha) = D^{-1}(\alpha) = D(-\alpha)$ (C.2)
- $D^{\dagger}(\alpha)\hat{a}D(\alpha) = \hat{a} + \alpha$ (C.3)
- $D^{\dagger}(\alpha)\hat{a}^{\dagger}D(\alpha) = \hat{a}^{\dagger} + \bar{\alpha}$ (C.4)
- $D(\alpha + \beta) = D(\alpha)D(\beta)e^{-i\, Im(\alpha\bar{\beta})}$ (C.5)

Proof In order to prove the second relation, we will use the Eq.(C.8) by choosing $\hat{A} = \alpha\hat{a}^{\dagger} - \bar{\alpha}\hat{a}$ and $\hat{B} = \hat{a}$ so that

$$D^{\dagger}(\alpha)\hat{a}D(\alpha) = e^{-\left(\alpha\hat{a}^{\dagger} - \bar{\alpha}\hat{a}\right)}\hat{a}e^{\alpha\hat{a}^{\dagger} - \bar{\alpha}\hat{a}} = \hat{a} + \alpha. \tag{C.6}$$

with commutator $[\bar{\alpha}\hat{a} - \alpha\hat{a}^{\dagger}, \hat{a}] = \alpha$. To compute $D^{\dagger}(\alpha)\hat{a}^{\dagger}D(\alpha)$, we will use the Baker-Campbell-Hausdorff formula by letting $A = \bar{\alpha}\hat{a} - \alpha\hat{a}^{\dagger}$ and $B = \hat{a}^{\dagger}$. Then, we start computing the commutator $[A, B]$ which can be separated into two commutators

$$[A, B] = [\bar{\alpha}\hat{a} - \alpha\hat{a}^{\dagger}, \hat{a}^{\dagger}] = \bar{\alpha}[\hat{a}, \hat{a}^{\dagger}] - \alpha[\hat{a}^{\dagger}, \hat{a}^{\dagger}] = \bar{\alpha} \tag{C.7}$$

where $[\hat{a}, \hat{a}^{\dagger}] = 1$.After applying the formula and since higher commutators vanish, the series terminates after the first term. Therefore, the desired relation is

$$D^{\dagger}(\alpha)\hat{a}^{\dagger}D(\alpha) = \hat{a}^{\dagger} - \bar{\alpha}.$$

□

Lemma C.1 *For operators $\hat{A}$ and $\hat{B}$ with commutator $[\hat{A}, \hat{B}]$, The Baker-Campbell-Hausdorff formula which is in its exponential form, states*

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{3!}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \cdots , \tag{C.8}$$

with involving nested commutators $[\hat{A}, \hat{B}]$.

Definition C.2 *The Baker−Campbell−Hausdorff formula for the product of the exponentials of two operators $\hat{A}$ and $\hat{B}$ is*

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+[\hat{A},\hat{B}]/2+...} \tag{C.9}$$

which involves nested commutators of $\hat{A}$ and $\hat{B}$.

Corollary C.1 *An important special case where an exact formula exists is*

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}e^{[\hat{A},\hat{B}]/2} \quad , [\hat{A}, \hat{B}] = c \tag{C.10}$$

where c is a constant (or $[c, \hat{A}] = [c, \hat{B}] = 0$).

Proposition C.7 *The displacement operator $D(\alpha)$, where $\alpha \in \mathbb{C}$, can be written in the form*

$$D(\alpha) = e^{\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}} = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger}e^{-\bar{\alpha}\hat{a}} \tag{C.11}$$

Proof We will prove this equation by using $(C.10)$. The commutator of $\hat{A} = \alpha\hat{a}^\dagger$ and $\hat{B} = -\bar{\alpha}\hat{a}$ can be calculated as

$$
\begin{aligned}
[\hat{A}, \hat{B}] = [\alpha\hat{a}^\dagger, -\bar{\alpha}\hat{a}] &= [\alpha, -\bar{\alpha}\hat{a}]\hat{a}^\dagger + \alpha[\hat{a}^\dagger, -\bar{\alpha}\hat{a}] && \text{(C.12)}\\
&= \alpha\left(-\hat{a}^\dagger\bar{\alpha}\hat{a} + \bar{\alpha}\hat{a}\hat{a}^\dagger)\right) && \text{(C.13)}\\
&= -\alpha\bar{\alpha}\left(\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger\right) = -|\alpha|^2[\hat{a}^\dagger, \hat{a}] = |\alpha|^2\,. && \text{(C.14)}
\end{aligned}
$$

□

By substituting this result into $(C.10)$, we get

$$
e^{\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}} = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger} e^{-\bar{\alpha}\hat{a}} \quad . \tag{C.15}
$$

Proposition C.8 *The displacement operator for fermion-boson states can be written as the direct product form*

$$
\mathcal{D}(\alpha) = \begin{pmatrix} D(\alpha) & 0 \\ 0 & D(\alpha) \end{pmatrix} = I_f \otimes D(\alpha) = I_f \otimes e^{\alpha a^\dagger - \bar{\alpha} a}. \tag{C.16}
$$

Definition C.3 *The displaced Fock states is defined by using* $(C.1)$.

$$
\begin{aligned}
|0, \alpha\rangle &= D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2}|\alpha\rangle, && \text{(C.17)}\\
|1, \alpha\rangle &= D(\alpha)|1\rangle = e^{-\frac{1}{2}|\alpha|^2}(\frac{d}{d\alpha}|\alpha\rangle - \bar{\alpha}|\alpha\rangle). && \text{(C.18)}
\end{aligned}
$$

Proof We can prove first relation by following steps

$$
\begin{aligned}
|0, \alpha\rangle &= D(\alpha)|0\rangle = e^{\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}}|0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger} e^{-\bar{\alpha}\hat{a}}|0\rangle \\
&= e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger} \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!}\,(\hat{a})^n\,|0\rangle \\
&= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle = e^{-\frac{1}{2}|\alpha|^2}|\alpha\rangle.
\end{aligned}
$$

For the second case, the application of $D(\alpha)$ gives

$$\begin{aligned} D(\alpha)|1\rangle &= e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger} e^{-\bar{\alpha}\hat{a}} \hat{a}^\dagger|0\rangle \\ &\overset{(C.8)}{=} e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger}(-\bar{\alpha} + \hat{a}^\dagger) e^{-\bar{\alpha}\hat{a}}|0\rangle = (\hat{a}^\dagger - \bar{\alpha}) D(\alpha)|0\rangle. \end{aligned}$$

Since application of $\hat{a}^\dagger$ to the state $|\alpha\rangle$ provides derivative relation,

$$\begin{aligned} \hat{a}^\dagger|\alpha\rangle &= \hat{a}^\dagger \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n+1}|n+1\rangle \\ &= \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} \sqrt{n}|n\rangle = \sum_{n=1}^{\infty} n\frac{\alpha^{n-1}}{\sqrt{n!}}|n\rangle = \frac{d}{d\alpha}|\alpha\rangle \end{aligned}$$

we can get

$$|1, \alpha\rangle = D(\alpha)|1\rangle = e^{-\frac{1}{2}|\alpha|^2}(\frac{d}{d\alpha}|\alpha\rangle - \bar{\alpha}|\alpha\rangle).$$

□

Proposition C.9 *The matrix elements of the displacement operator are defined as*

$$\begin{aligned} \langle 0|D(\alpha)|0\rangle &= e^{-\frac{1}{2}|\alpha|^2}, \qquad \langle 1|D(\alpha)|0\rangle = \alpha e^{-\frac{1}{2}|\alpha|^2}, \\ \langle 0|D(\alpha)|1\rangle &= -\bar{\alpha} e^{-\frac{1}{2}|\alpha|^2}, \qquad \langle 1|D(\alpha)|1\rangle = (1 - |\alpha|^2) e^{-\frac{1}{2}|\alpha|^2}, \end{aligned} \tag{C.19}$$

Proof The relations can be proven step-by-step process. For the first one, we will use expansion of not normalized Glauber coherent state,

$$\langle 0|D(\alpha)|0\rangle \overset{(C.17)}{=} e^{-\frac{1}{2}|\alpha|^2}\langle 0|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\langle 0|n\rangle = e^{-\frac{1}{2}|\alpha|^2}.$$

Then, the second relation requires to use derivative relation

$$\begin{aligned}\langle 0|D(\alpha)|1\rangle &\overset{(C.18)}{=} e^{-\frac{1}{2}|\alpha|^2}\langle 0|\left(\frac{d}{d\alpha}|\alpha\rangle - \bar{\alpha}|\alpha\rangle\right)\\ &= e^{-\frac{1}{2}|\alpha|^2}\left(\sum_{n=1}^{\infty} n\frac{\alpha^{n-1}}{\sqrt{n!}}\langle 0|n\rangle - \bar{\alpha}\langle 0|\alpha\rangle\right) = -\bar{\alpha}e^{-\frac{1}{2}|\alpha|^2}.\end{aligned}$$

After that, we can easily have

$$\langle 1|D(\alpha)|0\rangle \overset{(C.17)}{=} e^{-\frac{1}{2}|\alpha|^2}\langle 1|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\langle 1|n\rangle = \alpha e^{-\frac{1}{2}|\alpha|^2}$$

As a final step, we obtain

$$\langle 1|D(\alpha)|1\rangle \overset{(C.18)}{=} e^{-\frac{1}{2}|\alpha|^2}\left(\sum_{n=1}^{\infty} n\frac{\alpha^{n-1}}{\sqrt{n!}}\langle 1|n\rangle - \bar{\alpha}\langle 1|\alpha\rangle\right) = e^{-\frac{1}{2}|\alpha|^2}\left(1 - \bar{\alpha}\langle 1|\alpha\rangle\right) = e^{-\frac{1}{2}|\alpha|^2}(1 - |\alpha|^2)$$

□

Proposition C.10 *The homogeneous and non-homogeneous problem for annihilation operator $\hat{a}$ gives*

$$\begin{aligned}\hat{a}|0,\alpha\rangle &= \alpha|0,\alpha\rangle,\\ \hat{a}|1,\alpha\rangle &= \alpha|1,\alpha\rangle + |0,\alpha\rangle\end{aligned}$$

Proof The first relation comes from definition by $|0,\alpha\rangle = D(\alpha)|0\rangle$. For the next, we have following

$$\begin{aligned}
\hat{a}|1,\alpha\rangle = \hat{a}D(\alpha)|1\rangle &= \hat{a}D(\alpha)\hat{a}^\dagger|0\rangle \\
&= D(\alpha)D^\dagger(\alpha)\hat{a}D(\alpha)\hat{a}^\dagger|0\rangle \\
&\overset{(C.3)}{=} D(\alpha)\,(\hat{a}+\alpha)\,\hat{a}^\dagger|0\rangle \\
&= \left(D(\alpha)\hat{a}\hat{a}^\dagger + \alpha D(\alpha)\hat{a}^\dagger\right)|0\rangle \\
&= D(\alpha)(1+\hat{a}^\dagger\hat{a})|0\rangle + \alpha D(\alpha)\hat{a}^\dagger|0\rangle \\
&= D(\alpha)|0\rangle + \alpha D(\alpha)|1\rangle
\end{aligned}$$

so that

$$\hat{a}|1,\alpha\rangle = \alpha|1,\alpha\rangle + |0,\alpha\rangle$$

□

Proposition C.11 *The Bell super-coherent states are eigenstates of corresponding super-symmetric annihilation operators*

$$\begin{aligned}
A_1|\alpha, L_-\rangle &= \alpha|\alpha, L_-\rangle, \quad A_{-1}|\alpha, L_+\rangle = \alpha|\alpha, L_+\rangle, \\
A_1^T|\alpha, B_-\rangle &= \alpha|\alpha, B_-\rangle, \quad A_{-1}^T|\alpha, B_+\rangle = \alpha|\alpha, B_+\rangle.
\end{aligned}$$

The states are orthonormal and maximally entangled. In explicit form the states are expressed as

$$\begin{aligned}
|\alpha, L_\pm\rangle &= \frac{1}{\sqrt{2}}(|0\rangle_f|1,\alpha\rangle \pm |1\rangle_f|0,\alpha\rangle), \\
|\alpha, B_\pm\rangle &= \frac{1}{\sqrt{2}}(|0\rangle_f|0,\alpha\rangle \pm |1\rangle_f|1,\alpha\rangle).
\end{aligned}$$

Proof First,let's consider how the displacement operator $D(\alpha)$ acts in the tensor product form and substituting the expression for $A_{\pm 1}$,

$$
\begin{aligned}
\mathcal{D}^{\dagger}(\alpha)A_{\pm 1}\mathcal{D}(\alpha) &= \mathcal{D}^{\dagger}(\alpha)(I_f \otimes a \pm f \otimes I_b)\mathcal{D}(\alpha) \\
&= (I_f \otimes D^{\dagger}(\alpha))(I_f \otimes a \pm f \otimes I_b)(I_f \otimes D(\alpha)) \\
&= I_f \otimes (D^{\dagger}(\alpha)aD(\alpha)) \pm (I_f f I_f) \otimes (D^{\dagger}(\alpha)I_b D(\alpha)) \\
&= I_f \otimes (D^{\dagger}(\alpha)aD(\alpha)) \pm f \otimes (D^{\dagger}(\alpha)I_b D(\alpha)) \\
&\overset{(C.3)}{=} I_f \otimes (D^{\dagger}(\alpha)aD(\alpha)) \pm f \otimes I_b \\
&= I_f \otimes (a + \alpha I_b) \pm f \otimes I_b \\
&= (I_f \otimes a) + \alpha(I_f \otimes I_b) \pm (f \otimes I_b) \\
&= A_{\pm 1} + \alpha(I_f \otimes I_b)
\end{aligned}
$$

□

This shows how the displacement operator modifies the operator $A_{\pm 1}$ by adding a scalar multiple of the identity matrix to it. In this proposition, the proof of the eigenvalue problems (6.45) and (6.46) follows from unitary displacement transformation of operators

$$\mathcal{D}^{\dagger}(\alpha)A_{\pm 1}\mathcal{D}(\alpha) = A_{\pm 1} + \alpha I, \quad \mathcal{D}^{\dagger}(\alpha)A^{T}_{\pm 1}\mathcal{D}(\alpha) = A^{T}_{\pm 1} + \alpha I,$$

or

$$A_{\pm 1}\mathcal{D}(\alpha) = \mathcal{D}(\alpha)A_{\pm 1} + \alpha\mathcal{D}(\alpha), \quad A^{T}_{\pm 1}\mathcal{D}(\alpha) = \mathcal{D}(\alpha)A^{T}_{\pm 1} + \alpha\mathcal{D}(\alpha).$$

By applying these transformations to the Bell states, we get the result

$$
\begin{aligned}
A_{\pm 1}|\alpha, L_{\mp}\rangle = A_{\pm 1}\mathcal{D}(\alpha)|L_{\mp}\rangle = \mathcal{D}(\alpha)\left(A_{\pm 1}|L_{\mp}\rangle\right) + \alpha\mathcal{D}(\alpha)|L_{\mp}\rangle = \alpha|\alpha, L_{\mp}\rangle, \\
A^{T}_{\pm 1}|\alpha, B_{\mp}\rangle = A^{T}_{\pm 1}\mathcal{D}(\alpha)|B_{\mp}\rangle = \mathcal{D}(\alpha)\left(A^{T}_{\pm 1}|B_{\mp}\rangle\right) + \alpha\mathcal{D}(\alpha)|B_{\mp}\rangle = \alpha|\alpha, B_{\mp}\rangle,
\end{aligned}
$$

where annihilation of Bell states by (6.27) and (6.28) were used. Orthogonality of the states follows from definition of states (6.44) and the displacement of orthogonality conditions (6.22), (6.23), (6.24). The maximal entanglement of Bell super-coherent states is shown, by observation that the concurrence for these states is independent of α, this is why it is equal $C = 1$, as for the Bell states itself. Explicit form of the states (6.47), (6.48) in terms of displaced Fock states results from calculation

$$\begin{aligned}
|\alpha, L_\pm\rangle = \mathcal{D}(\alpha)|L_\pm\rangle &= (I \otimes D(\alpha))\frac{1}{\sqrt{2}}(|0\rangle_f|1\rangle_b \pm |1\rangle_f|0\rangle_b) \\
&= \frac{1}{\sqrt{2}}(|0\rangle_f D(\alpha)|1\rangle_b \pm |1\rangle_f D(\alpha)|0\rangle_b) \\
&= \frac{1}{\sqrt{2}}(|0\rangle_f|1, \alpha\rangle_b \pm |1\rangle_f|0, \alpha\rangle_b),
\end{aligned}$$

and the similar on for $|\alpha, B_\pm\rangle$. This concludes the proof of Proposition (C.11).

C.3. Entanglement of Supercoherent States

Proposition C.12 *The concurrence C, $0 \leq C \leq 1$, for four reference states (6.38) and (6.39) is equal*

$$C = \sin^2\frac{\theta}{2}.$$

The states can be parametrized by this concurrence as in (6.36) and (6.37).

Proof As a first step, we find concurrence for the reference state in (6.38). In order to calculate the entanglement level of the reference state, we can write it as fermion boson form;

$$\begin{aligned}
|0, \theta, \phi\rangle_{L_\pm} &= \cos\frac{\theta}{2}|\Psi_0\rangle + \sin\frac{\theta}{2}e^{i\phi}|L_\pm\rangle \\
&= \cos\frac{\theta}{2}\begin{pmatrix} |0\rangle_b \\ 0 \end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\begin{pmatrix} |1\rangle_b \\ \pm|0\rangle_b \end{pmatrix} \\
&= |0\rangle_f \otimes \left(\cos\frac{\theta}{2}|0\rangle_b + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}e^{i\phi}|1\rangle_b\right) \pm |1\rangle_f \otimes \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}e^{i\phi}|0\rangle_b
\end{aligned}$$

Now, we can express the density matrix in terms of the tensor product:

$$\begin{aligned}\rho &= \left(|0\rangle_f \otimes \left(\cos\frac{\theta}{2}|0\rangle_b + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}e^{i\phi}|1\rangle_b\right) \pm |1\rangle_f \otimes \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}e^{i\phi}|0\rangle_b\right) \\ &\otimes \left(\langle 0|_f \otimes \left(\cos\frac{\theta}{2}\langle 0|_b + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}e^{-i\phi}\langle 1|_b\right) \pm \langle 1|_f \otimes \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}e^{-i\phi}\langle 0|_b\right).\end{aligned}$$

Expanding the terms gives

$$\begin{aligned}\rho &= \left(|0\rangle_f\langle 0|_f\right) \otimes \left(\cos^2\frac{\theta}{2}|0\rangle_b\langle 0|_b + \frac{1}{\sqrt{2}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi}|0\rangle_b\langle 1|_b\right. \\ &+ \left.\frac{1}{\sqrt{2}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi}|1\rangle_b\langle 0|_b + \frac{1}{2}\sin^2\frac{\theta}{2}|1\rangle_b\langle 1|_b\right) \\ &\pm \left(|0\rangle_f\langle 1|_f\right) \otimes \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}e^{-i\phi}\left(\cos\frac{\theta}{2}|0\rangle_b\langle 0|_b + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}e^{i\phi}|0\rangle_b\langle 1|_b\right) \\ &\pm \left(|1\rangle_f\langle 0|_f\right) \otimes \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}e^{i\phi}\left(\cos\frac{\theta}{2}|0\rangle_b\langle 0|_b + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}e^{-i\phi}|1\rangle_b\langle 0|_b\right) \\ &+ \left(|1\rangle_f\langle 1|_f\right) \otimes \frac{1}{2}\sin^2\frac{\theta}{2}|0\rangle_b\langle 0|_b.\end{aligned}$$

Thus, the density matrix ρ is expressed as a tensor product of the states in fermionic-space and bosonic-space. To find the reduced density matrix ρ_b for the subsystem b, we take the partial trace over the subsystem f, $\rho_b = \mathrm{Tr}_f(\rho)$. This involves summing over the basis states of the subsystem f:

$$\rho_b =_f \langle 0|\rho|0\rangle_f +_f \langle 1|\rho|1\rangle_f.$$

Substitute the expanded form of ρ and compute

$$\begin{aligned}\rho_b &= \cos^2\frac{\theta}{2}|0\rangle_{b\,b}\langle 0| + \frac{1}{\sqrt{2}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi}|0\rangle_b\langle 1|_b \\ &+ \frac{1}{\sqrt{2}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi}|1\rangle_b\langle 0|_b + \frac{1}{2}\sin^2\frac{\theta}{2}|1\rangle_b\langle 1|_b + \frac{1}{2}\sin^2\frac{\theta}{2}|0\rangle_b\langle 0|_b.\end{aligned}$$

Combine like terms

$$\begin{aligned}\rho_b &= \left(\cos^2\frac{\theta}{2}+\frac{1}{2}\sin^2\frac{\theta}{2}\right)|0\rangle_b\langle 0|_b+\frac{1}{2\sqrt{2}}\sin\theta\left(e^{-i\phi}|0\rangle_b\langle 1|_b+e^{i\phi}|1\rangle_b\langle 0|_b\right)\\ &+ \frac{1}{2}\sin^2\frac{\theta}{2}|1\rangle_b\langle 1|_b.\end{aligned}$$

When we calculate the reduced density matrix ρ_f, it is clear that it can be expanded in the same form but in different dimension. Then, computing square of these gives

$$\begin{aligned}\rho_b^2=\rho_f^2 &= \left(\cos^2\frac{\theta}{2}+\frac{1}{2}\sin^2\frac{\theta}{2}\right)^2|0\rangle\langle 0|+\left(\frac{1}{2\sqrt{2}}\sin\theta\right)^2|0\rangle\langle 0|\\ &+ \left(e^{-i\phi}\frac{1}{2\sqrt{2}}\sin\theta\right)|0\rangle\langle 1|+\left(e^{i\phi}\frac{1}{2\sqrt{2}}\sin\theta\right)|1\rangle\langle 0|\\ &+ \left(\frac{1}{2\sqrt{2}}\sin\theta\right)^2|1\rangle\langle 1|+\left(\frac{1}{2}\sin^2\frac{\theta}{2}\right)^2|1\rangle\langle 1|\end{aligned}$$

so that

$$\begin{aligned}tr\rho_b^2=tr\rho_f^2 &= \left(\cos^2\frac{\theta}{2}+\frac{1}{2}\sin^2\frac{\theta}{2}\right)^2+\frac{1}{4}\sin^4\frac{\theta}{2}+\frac{1}{4}\sin^2\theta\\ &= \left(\cos^2\frac{\theta}{2}+\frac{1}{2}\sin^2\frac{\theta}{2}\right)^2+\frac{1}{4}\sin^4\frac{\theta}{2}+\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\\ &= \cos^4\frac{\theta}{2}+\frac{1}{2}\sin^4\frac{\theta}{2}+2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\mp\frac{1}{2}\sin^4\frac{\theta}{2}\\ &= 1-\frac{1}{2}\sin^4\frac{\theta}{2}\end{aligned}$$

Comparing with (6.8), we find the concurrence for the reference states (6.38)

$$C=\sin^2\frac{\theta}{2}.$$

□

C.4. Inner product of Super Coherent States

Definition C.4 *The inner product between two supercoherent states is provided by the following expression*

$$_{L_\pm}\langle\beta,\theta,\phi|\alpha,\theta,\phi\rangle_{L_\pm} = e^{-i\,\mathrm{Im}(\beta\bar{\alpha})}e^{-\frac{1}{2}|\alpha-\beta|^2}\left(1-\frac{\sin\theta}{2\sqrt{2}}\left((\bar{\alpha}-\bar{\beta})e^{i\phi}-(\alpha-\beta)e^{-i\phi}\right)-\frac{|\alpha-\beta|^2}{2}\sin^2\frac{\theta}{2}\right).$$

Proof By using definition of 6.53, the inner product can be calculated as following for L_+ and the result will be same for L_-

$$\begin{aligned}
_{L_+}\langle\beta,\theta,\phi|\alpha,\theta,\phi\rangle_{L_+} &= \langle 0,\theta,\phi|\mathcal{D}^\dagger(\beta)\mathcal{D}(\alpha)|0,\theta,\phi\rangle \\
&\overset{(C.5)}{=} e^{-i\,\mathrm{Im}(\beta\bar{\alpha})}\langle 0,\theta,\phi|\mathcal{D}(\alpha-\beta)|0,\theta,\phi\rangle \\
&= e^{-i\,\mathrm{Im}(\beta\bar{\alpha})}\langle 0,\theta,\phi|\begin{pmatrix} D(\alpha-\beta) & 0 \\ 0 & D(\alpha-\beta)\end{pmatrix}|0,\theta,\phi\rangle \\
&= e^{-i\,\mathrm{Im}(\beta\bar{\alpha})}\Big(\cos^2\frac{\theta}{2}\langle 0|D(\alpha-\beta)|0\rangle+\frac{e^{i\phi}}{\sqrt{2}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\langle 0|D(\alpha-\beta)|1\rangle \\
&+ \frac{e^{-i\phi}}{\sqrt{2}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\langle 1|D(\alpha-\beta)|0\rangle \\
&+ \frac{1}{2}\sin^2\frac{\theta}{2}\left(\langle 0|D(\alpha-\beta)|0\rangle+\langle 1|D(\alpha-\beta)|1\rangle\right)\Big)
\end{aligned}$$

with expressions for matrix elements in $(C.9)$,

$$\begin{aligned}
_{L_\pm}\langle\beta,\theta,\phi|\alpha,\theta,\phi\rangle_{L_\pm} &= e^{-i\,\mathrm{Im}(\beta\bar{\alpha})}\Big(\cos^2\frac{\theta}{2}e^{-\frac{1}{2}|\alpha-\beta|^2}-\frac{e^{i\phi}}{\sqrt{2}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}(\overline{\alpha}-\overline{\beta})e^{-\frac{1}{2}|\alpha-\beta|^2} \\
&+ \frac{e^{-i\phi}}{\sqrt{2}}(\overline{\alpha}-\overline{\beta})e^{-\frac{1}{2}|\alpha-\beta|^2}++\frac{1}{2}\sin^2\frac{\theta}{2}\left(2-|\alpha-\beta|^2\right)e^{-\frac{1}{2}|\alpha-\beta|^2}\Big)
\end{aligned}$$

we obtain the inner product

$$_{L_\pm}\langle\beta,\theta,\phi|\alpha,\theta,\phi\rangle_{L_\pm} = e^{-i\,\mathrm{Im}(\beta\bar{\alpha})}e^{-\frac{1}{2}|\alpha-\beta|^2}\left(1-\frac{\sin\theta}{2\sqrt{2}}\left((\bar{\alpha}-\bar{\beta})e^{i\phi}-(\alpha-\beta)e^{-i\phi}\right)-\frac{|\alpha-\beta|^2}{2}\sin^2\frac{\theta}{2}\right).$$

□

C.5. Average values for supercoherent state

Proposition C.13 *The average values for the reference state is*

$$\langle 0, C, \phi | X | 0, C, \phi \rangle \;=\; \sqrt{C(1-C)}\cos\phi, \tag{C.20}$$

$$\langle 0, C, \phi | P | 0, C, \phi \rangle \;=\; \sqrt{C(1-C)}\sin\phi. \tag{C.21}$$

Proof First, we start with calculating action of operator X to the reference state for L_- in (6.31),

$$\begin{aligned}
X|0, C, \phi\rangle &= I_f \otimes \frac{1}{\sqrt{2}}(a + a^\dagger)\left(\sqrt{1-C}|0\rangle_f|0\rangle_b + \sqrt{\frac{C}{2}}e^{i\phi}\left(|0\rangle_f|1\rangle_b - |1\rangle_f|0\rangle_b\right)\right) \\
&= \sqrt{\frac{1-C}{2}}|0\rangle_f(a + a^\dagger)|0\rangle_b + \frac{\sqrt{C}}{2}e^{i\phi}\left(|0\rangle_f(a + a^\dagger)|1\rangle_b - |1\rangle_f(a + a^\dagger)|0\rangle_b\right) \\
&= \sqrt{\frac{1-C}{2}}|0\rangle_f|1\rangle_b + \frac{\sqrt{C}}{2}e^{i\phi}\left(|0\rangle_f(|0\rangle_b + \sqrt{2}|2\rangle_b) - |1\rangle_f|1\rangle_b\right)
\end{aligned}$$

then average value of X become

$$\begin{aligned}
\langle 0, C, \phi | X | 0, C, \phi \rangle &= \sqrt{1-C}\,{}_f\langle 0|_b\langle 0| + \sqrt{\frac{C}{2}}e^{-i\phi}\left({}_f\langle 0|_b\langle 1| - {}_f\langle 1|_b\langle 0|\right) \\
&\quad \sqrt{\frac{1-C}{2}}|0\rangle_f|1\rangle_b + \frac{\sqrt{C}}{2}e^{i\phi}\left(|0\rangle_f(|0\rangle_b + \sqrt{2}|2\rangle_b) - |1\rangle_f|1\rangle_b\right) \\
&= \frac{\sqrt{C(1-C)}}{2}e^{i\phi} + \frac{\sqrt{C(1-C)}}{2}e^{-i\phi} = \sqrt{C(1-C)}\cos\phi.
\end{aligned} \tag{C.22}$$

Similar steps gives the average for momentum operator. The results will be same for three reference state, just B_- includes minus. □

The following proposition allows us to find average of X^2 in L_-.

Proposition C.14 *The average for $a^2, a, a^\dagger a$ and $aa^\dagger$ is given by*

$$\begin{aligned}
\langle 0, C, \phi | I_f \otimes a^2 | 0, C, \phi \rangle &= 0 \\
\langle 0, C, \phi | I_f \otimes a | 0, C, \phi \rangle &= \sqrt{\frac{C(1-C)}{2}}\, e^{i\phi} \\
\langle 0, C, \phi | I_f \otimes a^\dagger | 0, C, \phi \rangle &= \sqrt{\frac{C(1-C)}{2}}\, e^{-i\phi} \\
\langle 0, C, \phi | I_f \otimes a^\dagger a | 0, C, \phi \rangle &= \frac{1}{2} C \\
\langle 0, C, \phi | I_f \otimes a a^\dagger | 0, C, \phi \rangle &= \frac{1}{2} C + 1
\end{aligned} \tag{C.23}$$

Proof Let's start with acting a^2 to the reference state and it gives

$$I_f \otimes a^2 |0, C, \phi\rangle \;=\; \sqrt{1-C}|0\rangle_f a^2 |0\rangle_b + \sqrt{\frac{C}{2}} e^{i\phi} \left(|0\rangle_f a^2 |1\rangle_b - |1\rangle_f a^2 |0\rangle_b \right) = 0.$$

Then,it is easy to show the others comes from the proof of previous proposition. The next expression gives

$$\begin{aligned}
I_f \otimes a^\dagger a |0, C, \phi\rangle &= I_f \otimes a^\dagger a \sqrt{1-C}|0\rangle_f |0\rangle_b + \sqrt{\frac{C}{2}} e^{i\phi} \left(|0\rangle_f |1\rangle_b - |1\rangle_f |0\rangle_b \right) \\
&= \sqrt{1-C}|0\rangle_f a^\dagger a |0\rangle_b + \sqrt{\frac{C}{2}} e^{i\phi} \left(|0\rangle_f a^\dagger a |1\rangle_b - |1\rangle_f a^\dagger a |0\rangle_b \right) \\
&= \sqrt{\frac{C}{2}} e^{i\phi} \left(|0\rangle_f a^\dagger a |1\rangle_b \right) = \sqrt{\frac{C}{2}} e^{i\phi} \left(|0\rangle_f |1\rangle_b \right)
\end{aligned}$$

so that

$$\langle 0, C, \phi | I_f \otimes a^\dagger a | 0, C, \phi \rangle \;=\; \frac{1}{2} C$$

and by using commutation relation $[a, a^\dagger] = 1$, the last one results

$$\langle 0, C, \phi | I_f \otimes a a^\dagger | 0, C, \phi \rangle \;=\; \frac{1}{2} C + 1.$$

□

Proposition C.15 *The average value of X^2 and P^2 for Bell-Super coherent states*

$$\langle\alpha, C, \phi|X^2|\alpha, C, \phi\rangle = \frac{1}{2}[(\alpha+\bar{\alpha})^2 + 2\sqrt{2}(\alpha+\bar{\alpha})\sqrt{C(1-C)}\cos\phi + 1 + C],$$
$$\langle\alpha, C, \phi|P^2|\alpha, C, \phi\rangle = \frac{1}{2}[-(\alpha-\bar{\alpha})^2 - 2\sqrt{2}i(\alpha-\bar{\alpha})\sqrt{C(1-C)}\sin\phi + 1 + C].$$

Proof Due to the definition of operator X^2, the average can be provided by the following expression

$$\begin{aligned}
\langle\alpha, C, \phi|X^2|\alpha, C, \phi\rangle &= \langle 0, C, \phi|\mathcal{D}^\dagger(\alpha)X^2\mathcal{D}(\alpha)|0, C, \phi\rangle \\
&= \langle 0, C, \phi|I_f \otimes D^\dagger(\alpha)X^2 D(\alpha)|0, C, \phi\rangle \\
&= \frac{1}{2}\langle 0, C, \phi|I_f \otimes D^\dagger\left(a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a\right)D(\alpha)|0, C, \phi\rangle \\
&\overset{(C.3)}{=} \frac{1}{2}\langle 0, C, \phi|I_f \otimes \Big((a+\alpha)^2 + (a^\dagger + \bar{\alpha})^2 \\
&\quad + (a+\alpha)(a^\dagger + \bar{\alpha}) + (a^\dagger + \bar{\alpha})(a+\alpha)\Big)|0, C, \phi\rangle \\
&= \frac{1}{2}\langle 0, C, \phi|I_f \otimes \Big[(\alpha+\bar{\alpha})^2 + a^2 + a^{\dagger 2} + 2a(\alpha+\bar{\alpha}) \\
&\quad + 2a^\dagger(\alpha+\bar{\alpha}) + aa^\dagger + a^\dagger a\Big]|0, C, \phi\rangle
\end{aligned}$$

so that

$$\langle\alpha, C, \phi|X^2|\alpha, C, \phi\rangle = \frac{1}{2}[(\alpha+\bar{\alpha})^2 + 2\sqrt{2}(\alpha+\bar{\alpha})\sqrt{C(1-C)}\cos\phi + 1 + C]$$

for the reference states $L_\pm$ and B_+. A corresponding calculation may similarly be performed for the momentum operator. □

In order to find the dispersion (or variance) of the operators X, we substitute the values for (6.58) and (6.62)

$$\begin{aligned}
(\Delta X)^2 &= \langle \alpha, C, \phi | X^2 | \alpha, C, \phi \rangle - (\langle \alpha, C, \phi | X | \alpha, C, \phi \rangle)^2 \\
&= \frac{1}{2}\left[(\alpha + \bar{\alpha})^2 + 2\sqrt{2}(\alpha + \bar{\alpha})\sqrt{C(1-C)}\cos\phi + 1 + C\right] \\
&- \left(\frac{\alpha + \bar{\alpha}}{\sqrt{2}} + \sqrt{C(1-C)}\cos\phi\right)^2 \\
&= \frac{1}{2}\left[(\alpha + \bar{\alpha})^2 + 2\sqrt{2}(\alpha + \bar{\alpha})\sqrt{C(1-C)}\cos\phi + 1 + C\right] \\
&- \left[\frac{(\alpha + \bar{\alpha})^2}{2} + \sqrt{2}(\alpha + \bar{\alpha})\sqrt{C(1-C)}\cos\phi + C(1-C)\cos^2\phi\right] \\
&= \frac{1}{2}(1 + C) - C(1-C)\cos^2\phi,
\end{aligned}$$

and it helps to represent the uncertainty in position operator in the super-coherent states.

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