REPUBLIC OF TURKEY
YILDIZ TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

A SURVEY ON THE EXISTENCE OF G-FACTORIZATIONS OF λK_N

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A thesis submitted by Esra ÇELİK in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE** is approved by the committee on 06.07.2019 in Department of Mathematics, Mathematics Program.

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Esra ÇELİK

Signature

Dedicated to my family

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Esra ÇELİK

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LIST OF SYMBOLS

$K_{s,t}$	Complete bipartite graph with vertex set partitioned into two subsets of s and t vertices.
K_n	Complete graph with n vertices
C_k	Cycle with k vertices
deg(n)	Degree of a vertex n
$\{v_1, v_2\}$	Edge between vertices v_1 and v_2
E(G)	Edge set of a graph G
$\emptyset$	Empty set
$K_k \cup K_k$	Graph obtained by comprising two disjoint complete graphs with k vertices
$P_k \cup P_k$	Graph obtained by comprising two disjoint paths with k vertices
K_k+e	Graph obtained from a complete graph with k vertices by adding one edge
K_k−e	Graph obtained from a complete graph with k vertices by removing one edge
B$_{i,j}$	j^{th} block of i^{th} parallel class
Q_k	k-dimensional cube
λK_n	$\lambda-$fold complete graph with n vertices
M_k	Matching with k vertices
$\|E(G)\|$	Number of edges of G
$\|V(G)\|$	Number of vertices of G
P_k	Path with k vertices
$\mathbb{Z}^+$	Set of positive integers
S_k	Star with k vertices

$K_{1,k-1}$	Star with k vertices
T_k	Tree with k vertices
V(G)	Vertex set of a graph G

LIST OF ABBREVIATIONS

BIBD	Balanced Incomplete Block Design
OP	Oberwolfach Problem
RBIBD	Resolvable Balanced Incomplete Block Design

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ABSTRACT

A Survey on the Existence of G-factorizations of λK_n

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A G-decomposition of a graph K is a set of subgraphs of K (called blocks), each isomorphic to G, whose edge set partition the edge set of K. A G-decomposition of λK_n is called a λ-fold G-design of order n. A λ-fold G-design is said to be a G-factorization of λK_n, if it is resolvable. So, the blocks of a G-factorization is partitioned into parallel classes P_i, such that every vertex of λK_n appears in exactly one block of each P_i. The factorization problem of λK_n has been investigated by many researchers for years. In this thesis, we survey the current state of information on the existence of G-factorizations of λK_n for complete graphs, cycles, matchings, paths, stars, trees, cubes, complete bipartite graphs and graphs with six or fewer vertices. We give the basic definitions and brief information about the necessary conditions to construct G-factorizations of λK_n and we compile theorems for each mentioned graph. Finally, results are supported with examples.

Keywords: graph factorization, resolvable G-design, resolvable graph decomposition, resolvable λ-fold G-design

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ÖZET

λK_n' nin G-faktorizasyonlarının Varlığı Üzerine Bir Araştırma

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Bir K grafının G-ayrışması, her biri G'ye izomorfik olan K'nın alt graflarının (bloklarının) kümesi olup, G'nin kenar kümesi K'nın kenar kümesini ayırır. λK_n'nin G-ayrışmasına mertebesi n olan λ katlı G-tasarım denir. Yeniden çözülebilen λ katlı G-tasarımlarına λK_n'nin G-faktorizasyonları denir. Dolayısıyla, bir G-faktorizasyonunun blokları P_i paralel sınıflarına ayrılır öyle ki λK_n'nin her köşesi tam olarak P_i paralel sınıflarının her birinin bir bloğunda görülür. λK_n için faktorizasyon problemi yıllardır birçok araştırmacı tarafından araştırılmıştır. Bu tezde, tam graflar, çevrimler, eşlemeler, yollar, yıldızlar, ağaçlar, küpler, tam iki parçalı graflar ve altı veya altıdan daha az köşeli graflar için mevcut bilgi durumu incelenecektir. λK_n'nin G-faktorizasyonlarını kurmak için gerek şartlara kısaca değinilecek, temel kavramlar tanımlanacak ve bahsi geçen graflar için mevcut teoremler derlenecektir. Son olarak, sonuçlar örnekler ile desteklenecektir.

Anahtar Kelimeler: graf faktorizasyonu, yeniden çözülebilir G-tasarım, yeniden çözülebilir graf ayrışımı, λ katlı G-tasarım

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YILDIZ TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

1
INTRODUCTION

1.1 Literature Review

The pair $G = (V,E)$ is an undirected *graph* which consists of a nonempty set of vertices V and a set of edges E. The number of elements in V indicates the *order* of G, while the number of elements in E indicates the *length* of G. The edge containing v_1 and v_2 is written v_1v_2 or $\{v_1,v_2\}$; v_1 and v_2 are called its *endpoints*. If vertices v_1 and v_2 are endpoints of an edge in any graph, then v_1 and v_2 are said to be *adjacent* to each other, and denoted by $v_1 \sim v_2$. The *degree* of a vertex v is the number of vertices that are adjacent to that vertex and is denoted by *deg(v)*. A *simple graph* is a graph where no two edges connect the same pair of vertices and in which each edge is formed by connecting two distinct vertices. A graph with multiple edges connecting the same pair of vertices is called a *multigraph*.

Let $G = (V,E)$ and $H = (W,F)$ be two graphs. If $W \subseteq V$ and $F \subseteq E$, then H is a *subgraph* of G. H is called a *spanning subgraph* of G when H contains all the vertices of G and the degree of all vertices of H is at least one. A *complete graph* K_n with n vertices is a simple graph in which every pair of vertices is connected by only one edge. So, if a graph contains a pair of distinct vertices which are not connected by an edge, then it is called *non-complete*. A *complete multigraph* λK_n, is a graph that has exactly λ edges between each pair of distinct vertices.

A *path* is a graph with distinct vertices $v_1, v_2, ..., v_n$ and edges $\{v_{i-1}, v_i\}$ for $2 \leq i \leq n$. A *connected* graph is an undirected graph which contains at least one vertex and a path between any two distinct vertices. If a graph is not connected, then it is called *disconnected*. A *bipartite graph* is a simple graph whose vertex set can be partitioned into subsets V_1 and V_2 such that the endpoints of each edge of the graph is in V_1 and V_2, respectively. If every vertex in V_1 is connected to every vertex in V_2, then the graph is called a *complete bipartite graph*. A complete bipartite graph is denoted by $K_{s,t}$ when $|V_1| = s$ and $|V_2| = t$.

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be *isomorphic graphs*, if there exists an isomorphism f between G_1 and G_2. So, the function f satisfies the following two conditions. At first, f must be one-to-one and onto. Second, for all $a, b \in V_1$, $\{a, b\} \in E_1$ if and only if $\{f(a), f(b)\} \in E_2$.

A *G-decomposition* of a graph K is a set of subgraphs of K (called *blocks*), each isomorphic to G, whose edge set partition the edge set of K. A G-decomposition of λK_n is called a *λ—fold G-design* of order n. A 1—fold G-design is simply called a *G-design*.

Suppose that we want to get a G-decomposition of λK_n for a simple graph G with vertex degrees $d_1, d_2, ..., d_k$. It is a trivial fact that if $n > 1$ then $n \geq |V(G)|$. The number of edges of λK_n is

$$\lambda \binom{n}{2} = \frac{\lambda n(n-1)}{2} \tag{1.1}$$

Firstly, in order to partition the edges of λK_n into graphs isomorphic to G, $|E(G)|$ must divide $|E(\lambda K_n)|$. So, we get the following equivalence:

$$\frac{\lambda n(n-1)}{2} \equiv 0 \pmod{|E(G)|} \tag{1.2}$$

If the above equivalence is reorganized then we get the following condition:

$$\lambda n(n-1) \equiv 0 \pmod{2\,|E(G)|} \tag{1.3}$$

Secondly, assume that the degrees in G have a greatest common divisor d which is denoted by $d = \gcd\,(d_1, d_2, ..., d_k)$. It is also known that, the degree of a vertex in λK_n, which is $\lambda(n-1)$, must be comprised of the sum of the degrees in G. In other words, d must divide $\lambda(n-1)$. Hence, we get the following condition:

$$\lambda(n-1) \equiv 0 \pmod{d} \tag{1.4}$$

Therefore, for any graph G, if there exists a λ—fold G-design of order n then the followings are satisfied:

(i) If $n > 1$ then $n \geqslant |V(G)|$

(ii) $\lambda n(n-1) \equiv 0 \pmod{2\,|E(G)|}$

(iii) $\lambda(n-1) \equiv 0 \pmod{d}$ where d is the greatest common divisor of degrees of vertices in G.

These conditions are called the necessary conditions. The *spectrum* for a graph G is the set $S = \{n \in \mathbb{Z}^+ : \text{there exists a } G\text{-design of order n}\}$. Also, there exists a survey of Adam et al. [1] which contains the results about the existence of G-designs.

If the blocks of a G-decomposition of λK_n can be partitioned into parallel classes P_i such that every vertex of λK_n is in exactly one block of each P_i, then a G-decomposition of λK_n is said to be resolvable. Every parallel class P_i can be called a G-factor of λK_n, since each P_i is also a spanning subgraph of λK_n. If all vertices in a G-factor have degree k, then it is called a *k-factor*. If the edge set of λK_n can be represented as the edge-disjoint union of G-factors, then λK_n has a G-factorization. So, a resolvable G-decomposition of λK_n is also called a G-factorization of λK_n.

In addition to the necessary conditions for the existence of a G-decomposition, there exist extra resolvability conditions for the existence of a G-factorization. The number of vertices in G must divide the number of vertices in λK_n to get a single parallel class. Hence, we must have that:

$$n \equiv 0 \pmod{|V(G)|} \tag{1.5}$$

Moreover, the number of parallel classes is

$$\frac{\frac{\lambda n(n-1)}{2|E(G)|}}{\frac{n}{|V(G)|}} = \frac{\lambda\,|V(G)|\,(n-1)}{2\,|E(G)|} \tag{1.6}$$

Since the number of parallel classes must be a positive integer, the following equivalence is obtained.

$$\lambda\,|V(G)|\,(n-1) \equiv 0 \pmod{2\,|E(G)|} \tag{1.7}$$

Consequently, the necessary conditions to construct a G-factorization of λK_n are that:

(i) $n \equiv 0 \pmod{|V(G)|}$

(ii) $\lambda\,|V(G)|\,(n-1) \equiv 0 \pmod{|2E(G)|}$

(iii) $\lambda(n-1) \equiv 0 \pmod{\text{d}}$ where d is the greatest common divisor of degrees of vertices in G.

1.2 Objective of the Thesis

This thesis is a survey about the necessary and sufficient conditions for the existence of G-factorizations of λK_n. The main objective of the thesis is providing useful and detailed information to researchers about studies related to the G-factorizations of λK_n from past to present. The thesis consists of several chapters containing solutions for some special graphs. In Section 2.1, known results about complete graphs are discussed. In Section 2.2, 2.3 and 2.4, the complete solution for the existence problem of factorizations of λK_n is given for cycles,matchings and paths, respectively. Star, tree, cube, complete bipartite graph factorizations are respectively showed in Sections 2.5, 2.6, 2.7, and 2.8 for some special cases. In Section 2.9, the solutions of some graphs with six or fewer vertices are explained.

1.3 Hypothesis

The existence problem of G-factorizations of λK_n is an important subject of graph theory. Many researchers have worked on this problem, but there are still many questions waiting to be answered. So, this thesis will help to determine the unsolved problems about G-factorizations of λK_n.

2
GRAPH FACTORIZATIONS

2.1 Complete Graphs

A complete graph with k vertices is denoted by K_k. A λ-fold K_k-design is also known as an (n,k,λ)-BIBD. (Balanced Incomplete Block Design). If an (n,k,λ)-BIBD is resolvable, then it is denoted by (n,k,λ)-RBIBD.

There are two necessary conditions to construct an (n,k,λ)-RBIBD. These are given below:

$$\lambda(n-1) \equiv 0 \pmod{k-1} \tag{2.1}$$

$$n \equiv 0 \pmod{k} \tag{2.2}$$

Ray-Chaudhuri and Wilson [2] gave a general theorem about the existence of an $(n,k,1)$-RBIBD.

Theorem 2.1. [2] There is an integer $n_2(k)$ such that if $n \geqslant n_2(k)$, then there exists an (n,k,λ)-RBIBD if and only if $n \equiv k \pmod{k(k-1)}$.

For $k=2$, the solution of (n,k,λ)-RBIBDs are given in the book of Wallis [3].

Theorem 2.2. [3] λK_{2n} has a K_2-factorization for all n and all λ.

In 1847, the first question about the existence of RBIBDs arose in Kirkman [4]. This question is also known as the Kirkman schoolgirl problem. Three years later, he published the same question in the Lady's and Gentleman's Diary [5] again. Then, Cayley [6] found the first solution for this problem in 1850. After Cayley, Kirkman [7] had a solution for $(9,3,1)$ and $(15,3,1)$-RBIBDs. At the same year, the method was generalized by Kirkman in order to construct $(3^m,3,1)$-RBIBDs and $(q^m,q,1)$-RBIBDs with a prime q and an integer m. Lastly, Wilson and Ray-Chaudhuri [8] gave a solution for $(n,3,1)$-RBIBDs. The existence of (n,k,λ)-RBIBDs were proven for $k=4$ and $\lambda=1$ by Hanani et.al [9]; $k=3$ and $\lambda=2$ by Hanani [10]; $k=4$ and $\lambda=3$ by a combination of works of Baker, Hanani and Wilson [9], [11], [12] . Also, several authors have discussed the existence problem of $(n,3,\lambda)$-RBIBDs and $(n,4,\lambda)$-RBIBDs (see [13], [14], [15], [16]).

Theorem 2.3. [8], [10] There exists an $(n,3,\lambda)$-RBIBD if and only if $\lambda \equiv 0 \pmod 2$ and $n \equiv 0 \pmod 3$, $n \neq 6$, or $\lambda \geqslant 1$ and $n \equiv 3 \pmod 6$.

Theorem 2.4. [9], [11], [12] There exists an $(n,4,\lambda)$-RBIBD if and only if $\lambda \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 4$ or $\lambda \geqslant 1$ and $n \equiv 4 \pmod{12}$.

Example 2.1 A $(15,3,1)$-RBIBD is given below.
Let $\{a_1,a_2,a_3\}$ denote the complete graph K_3. In all examples, $B_{i,j}$ denotes the j^{th} block of the i^{th} parallel class.

$B_{1,1}=\{0,5,10\}$
$B_{1,2}=\{1,6,11\}$
$B_{1,3}=\{2,7,12\}$
$B_{1,4}=\{3,8,13\}$
$B_{1,5}=\{4,9,14\}$
$B_{2,1}=\{0,1,4\}$
$B_{2,2}=\{2,3,6\}$
$B_{2,3}=\{7,8,11\}$
$B_{2,4}=\{9,10,13\}$
$B_{2,5}=\{12,14,5\}$
$B_{3,1}=\{1,2,5\}$
$B_{3,2}=\{3,4,7\}$
$B_{3,3}=\{8,9,12\}$
$B_{3,4}=\{10,11,14\}$
$B_{3,5}=\{13,0,6\}$
$B_{4,1}=\{4,5,8\}$
$B_{4,2}=\{6,7,10\}$
$B_{4,3}=\{11,12,0\}$
$B_{4,4}=\{13,14,2\}$
$B_{4,5}=\{1,3,9\}$

$B_{5,1}=\{2,4,10\}$
$B_{5,2}=\{3,5,11\}$
$B_{5,3}=\{6,8,14\}$
$B_{5,4}=\{7,9,0\}$
$B_{5,5}=\{12,13,1\}$
$B_{6,1}=\{4,16,12\}$
$B_{6,2}=\{5,7,13\}$
$B_{6,3}=\{8,10,1\}$
$B_{6,4}=\{9,11,2\}$
$B_{6,5}=\{14,0,3\}$
$B_{7,1}=\{10,12,3\}$
$B_{7,2}=\{11,13,4\}$
$B_{7,3}=\{14,1,7\}$
$B_{7,4}=\{0,2,8\}$
$B_{7,5}=\{5,6,9\}$

Example 2.2 [17] A $(16,4,1)$-RBIBD is given below. Let $\{a_1,a_2,a_3,a_4\}$ denote the complete graph K_4.

$B_{1,1}=\{16,1,6,11\}$
$B_{1,2}=\{2,3,5,9\}$
$B_{1,3}=\{7,8,10,14\}$
$B_{1,4}=\{12,13,15,4\}$
$B_{2,1}=\{16,2,7,12\}$
$B_{2,2}=\{3,4,6,10\}$
$B_{2,3}=\{8,9,11,15\}$
$B_{2,4}=\{13,14,1,5\}$
$B_{3,1}=\{16,3,8,13\}$
$B_{3,2}=\{4,5,7,11\}$
$B_{3,3}=\{9,10,12,1\}$
$B_{3,4}=\{14,15,2,6\}$

$B_{4,1}=\{16,4,9,14\}$
$B_{4,2}=\{5,6,8,12\}$
$B_{5,1}=\{16,5,10,15\}$
$B_{5,2}=\{6,7,9,13\}$

$B_{4,3} = \{10, 11, 13, 2\}$ $\quad B_{5,3} = \{11, 12, 14, 3\}$
$B_{4,4} = \{15, 1, 3, 7\}$ $\quad B_{5,4} = \{1, 2, 4, 8\}$

Example 2.3 [18] An $(8, 4, 3)$-RBIBD is given below. Let $\{a_1, a_2, a_3, a_4\}$ denote the complete graph K_4.

$B_{1,1} = \{1, 2, 3, 4\}$ $\quad B_{2,1} = \{1, 5, 2, 6\}$ $\quad B_{3,1} = \{1, 7, 8, 2\}$
$B_{1,2} = \{5, 6, 7, 8\}$ $\quad B_{2,2} = \{3, 8, 7, 4\}$ $\quad B_{3,2} = \{6, 5, 4, 3\}$

$B_{4,1} = \{1, 3, 7, 5\}$ $\quad B_{5,1} = \{1, 4, 5, 8\}$ $\quad B_{6,1} = \{1, 6, 4, 7\}$
$B_{4,2} = \{2, 4, 8, 6\}$ $\quad B_{5,2} = \{2, 3, 6, 7\}$ $\quad B_{6,2} = \{2, 5, 3, 8\}$

$B_{7,1} = \{1, 8, 6, 3\}$
$B_{7,2} = \{2, 7, 5, 4\}$

For $(n, 5, \lambda)$-RBIBDs, there are three basic cases; $\lambda = 1, 2, 4$. The necessary conditions are given below:

$$n \equiv 5 \pmod{20} \text{ for } \lambda \equiv 1 \pmod{2} \tag{2.3}$$

$$n \equiv 5 \pmod{10} \text{ for } \lambda \equiv 2 \pmod{4} \tag{2.4}$$

$$n \equiv 0 \pmod{5} \text{ for } \lambda \equiv 0 \pmod{4} \tag{2.5}$$

Ray-Chaudhuri and Wilson [2] conjectured that $n \equiv 5 \pmod{20}$ is sufficient for the existence of an $(n, 5, 1)$-RBIBD. In [19], [20], the existence of an $(n, 5, 1)$-RBIBD was proved for $n \equiv 5 \pmod{20}$ with 147 possible exceptions of n, where the largest one is 23085. Then, the number of possible exceptions was reduced to 113, where 7845 becomes the largest by Zhu et al. [21]. Abel and Greig [22] constructed $(n, 5, 1)$-RBIBDs for all but six possible exceptions $n \notin \{45, 185, 225, 345, 465, 645\}$. Finally, the existence of a $(185, 5, 1)$-RBIBD and a $(225, 5, 1)$-RBIBD was proved in [23], [24], respectively.

Miao and Zhu [25] showed that the necessary conditions are sufficient for the existence of $(n, 5, 2)$-RBIBDs for $n \geqslant 50722395$. Then, Abel et al. [23] proved the existence of $(n, 5, 2)$-RBIBDs with 12 possible exceptions largest of which is 395. On the other hand, the nonexistence of a $(15, 5, 2)$-RBIBD was showen by Furino [14] in

1996.

Abel et al. [23] proved the existence of a $(110, 5, 4)$-RBIBD and a $(140, 5, 4)$-RBIBD. Moreover, he stated that for $(n, 5, 4)$-RBIBDs, the necessary conditions are sufficient when $n \neq 10$ and there exists 7 possible exceptions $n \notin \{15, 70, 90, 135, 160, 190, 195\}$. Then, Kaski and Östergård [26] proved the nonexistence of a $(15, 5, 4)$-RBIBD. Finally, a $(70, 5, 4)$-RBIBD and a $(90, 5, 4)$-RBIBD were constructed in [27] and [24], respectively.

Theorem 2.5. [24] An $(n, 5, \lambda)$-RBIBD exists if any of the following conditions are satisfied:

(i) $n \equiv 5 \pmod{20}$ for $\lambda \equiv 1 \pmod{2}$, except possibly for $n \in \{45, 345, 465, 645\}$,

(ii) $n \equiv 5 \pmod{10}$ for $\lambda \equiv 2 \pmod{4}$, except for $n = 15$ and possibly for $n \in \{45, 115, 135, 195, 215, 235, 295, 315, 335, 345, 395\}$,

(iii) $n \equiv 0 \pmod{5}$ for $\lambda \equiv 0 \pmod{4}$, except for $n \in \{10, 15\}$ and possibly for $n \in \{135, 160, 190, 195\}$.

For $k = 6$, there exist two basic cases; $\lambda = 1, 5$. The necessary conditions are as follows:

$$n \equiv 6 \pmod{30} \text{ for } \lambda \equiv 1, 2, 3, 4 \pmod{5} \tag{2.6}$$

$$n \equiv 0 \pmod{6} \text{ for } \lambda \equiv 0 \pmod{5} \tag{2.7}$$

Abel et al. [28] proved the sufficiency of the necessary conditions for the existence of an $(n, 6, 5)$-RBIBD. Costa et al. [29] studied the $(n, 6, 1)$-RBIBDs. Furthermore, there exist studies of Baker [13] about $(n, 6, 10)$-RBIBDs. For $k = 6$ and $\lambda = 10$, he proved that the necessary conditions are sufficient. But, there are some situations where the necessary conditions are not sufficient for some values of k. For example, the theorem of Bruck-Ryser [30] showed that there aren't $(k^2, k, 1)$-RBIBDs for many values of k, the smallest one is $k = 6$.

Theorem 2.6. [28] There is an $(n, 6, 5)$-RBIBD if and only if $n \equiv 0 \pmod{6}$.

Theorem 2.7. [29] There is a $(125p + 1, 6, 1)$-RBIBD for any prime $p \equiv 7 \pmod{12}$ and $p > 43$.

For $(n, 7, \lambda)$-RBIBDs, there are four basic cases; λ=1, 2, 3, 6. The necessary conditions are as follows:

$$n \equiv 7 \text{ (mod 42) for } \lambda \equiv 1, 5 \text{ (mod 6)} \tag{2.8}$$

$$n \equiv 7 \text{ (mod 21) for } \lambda \equiv 2, 4 \text{(mod 6)} \tag{2.9}$$

$$n \equiv 7 \text{ (mod 14) } for\ \lambda \equiv 3 \text{ (mod 6)} \tag{2.10}$$

$$n \equiv 0 \text{ (mod 7) for } \lambda \equiv 0 \text{ (mod 6)} \tag{2.11}$$

First case was constructed for $n \geqslant 294427$ by Greig [31]. Last case was showed by Abel et al. [32]. But, Furino [14] has also constructed $(n, 7, 6)$-RBIBDs for $n \equiv 0$ (mod 7) with $n \geqslant 33943$ before Abel.

Theorem 2.8. [31] An $(n, 7, 1)$-RBIBD exists if $n \equiv 7$ (mod 42) and $n \geqslant 294427$.

Theorem 2.9. [32] An $(n, 7, 6)$-RBIBD exists if and only if $n \equiv 0$ (mod 7) and $n \neq 14$ with 18 possible exceptions $n \notin \{84, 119, 126, 133, 175, 182, 189, 210, 231, 238, 259, 266, 287, 413, 420, 427, 434, 462\}$.

Example 2.4 [33] A (49, 7, 1)-RBIBD is given below. Let $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ denote the complete graph K_7.

$B_{1,1} = \{1, 2, 3, 4, 5, 6, 7\}$
$B_{1,2} = \{8, 9, 10, 11, 12, 13, 14\}$
$B_{1,3} = \{15, 16, 17, 18, 19, 20, 21\}$
$B_{1,4} = \{22, 23, 24, 25, 26, 27, 28\}$
$B_{1,5} = \{29, 30, 31, 32, 33, 34, 35\}$
$B_{1,6} = \{36, 37, 38, 39, 40, 41, 42\}$
$B_{1,7} = \{43, 44, 45, 46, 47, 48, 49\}$

$B_{2,1} = \{1, 8, 15, 22, 29, 36, 43\}$
$B_{2,2} = \{2, 9, 16, 23, 30, 37, 44\}$
$B_{2,3} = \{3, 10, 17, 24, 31, 38, 45\}$
$B_{2,4} = \{4, 11, 18, 25, 32, 39, 46\}$
$B_{2,5} = \{5, 12, 19, 26, 33, 40, 47\}$
$B_{2,6} = \{6, 13, 20, 27, 34, 41, 48\}$
$B_{2,7} = \{7, 14, 21, 28, 35, 42, 49\}$

$B_{3,1} = \{1, 9, 17, 25, 33, 41, 49\}$
$B_{3,2} = \{2, 10, 18, 26, 34, 42, 43\}$
$B_{3,3} = \{3, 11, 19, 27, 35, 36, 44\}$
$B_{3,4} = \{4, 12, 20, 28, 29, 37, 45\}$
$B_{3,5} = \{5, 13, 21, 22, 30, 38, 46\}$
$B_{3,6} = \{6, 14, 15, 23, 31, 39, 47\}$

$B_{4,1} = \{1, 10, 19, 28, 30, 39, 48\}$
$B_{4,2} = \{2, 11, 20, 22, 31, 40, 49\}$
$B_{4,3} = \{3, 12, 21, 23, 32, 41, 43\}$
$B_{4,4} = \{4, 13, 15, 24, 33, 42, 44\}$
$B_{4,5} = \{5, 14, 16, 25, 34, 36, 45\}$
$B_{4,6} = \{6, 8, 17, 26, 35, 37, 46\}$

$B_{3,7} = \{7,8,16,24,32,40,48\}$ $B_{4,7} = \{7,9,18,27,29,38,47\}$

$B_{5,1} = \{1,11,21,24,34,37,47\}$ $B_{6,1} = \{1,12,16,27,31,42,46\}$
$B_{5,2} = \{2,12,15,25,35,38,48\}$ $B_{6,2} = \{2,13,17,28,32,36,47\}$
$B_{5,3} = \{3,13,16,26,29,39,49\}$ $B_{6,3} = \{3,14,18,22,33,37,48\}$
$B_{5,4} = \{4,14,17,27,30,40,43\}$ $B_{6,4} = \{4,8,19,23,34,38,49\}$
$B_{5,5} = \{5,8,18,28,31,41,44\}$ $B_{6,5} = \{5,9,20,24,35,39,43\}$
$B_{5,6} = \{6,9,19,22,32,42,45\}$ $B_{6,6} = \{6,10,21,25,29,40,44\}$
$B_{5,7} = \{7,10,20,23,33,36,46\}$ $B_{6,7} = \{7,11,15,26,30,41,45\}$

$B_{7,1} = \{1,13,18,23,35,40,45\}$ $B_{8,1} = \{1,14,20,26,32,38,44\}$
$B_{7,2} = \{2,14,19,24,29,41,46\}$ $B_{8,2} = \{2,8,21,27,33,39,45\}$
$B_{7,3} = \{3,8,20,25,30,42,47\}$ $B_{8,3} = \{3,9,15,28,34,40,46\}$
$B_{7,4} = \{4,9,21,26,31,36,48\}$ $B_{8,4} = \{4,10,16,22,35,41,47\}$
$B_{7,5} = \{5,10,15,27,32,37,49\}$ $B_{8,5} = \{5,11,17,23,29,42,48\}$
$B_{7,6} = \{6,11,16,28,33,38,43\}$ $B_{8,6} = \{6,12,18,24,30,36,49\}$
$B_{7,7} = \{7,12,17,22,34,39,44\}$ $B_{8,7} = \{7,13,19,25,31,37,43\}$

For $(n,8,\lambda)$-RBIBDs, there are two basic cases; $\lambda = 1,7$ and the necessary conditions are as follows:

$$n \equiv 8 \pmod{56} \text{ for } \lambda \equiv 1,2,3,4,5,6 \pmod{7} \tag{2.12}$$

$$n \equiv 0 \pmod{8} \text{ for } \lambda \equiv 0 \pmod{7} \tag{2.13}$$

Greig and Abel [34] have shown that the necessary conditions are enough for constructing an $(n,8,1)$-RBIBD with the possible exception of 66 values. Then Costa et al. [29] constructed $(n,8,1)$-RBIBDs for $n \in \{624,1576,2976,5720,5776,10200,14176,24480\}$. Rokowska [35] has also shown that an $(n,8,7)$-RBIBD exists whenever $n \equiv 0 \pmod{8}$ and n is not divisible by $3,5$ or 7. Then, Furino et al. [36] proved that the necessary conditions are enough for constructing an $(n,8,7)$-RBIBD with at most 36 possible exceptions. This 36 possible exceptions were reduced to 27 possible exceptions by Furino [14] et al. and Abel [37] et al. proved the existence of $(n,8,7)$-RBIBDs for $n \in \{504,624,1104,1120,1200\}$. For $n = 616$, the existence of $(n,8,7)$-RBIBD was proved in [38]. Finally, the number of possible exceptions was reduced to 14, without any published proof, in [39].

Theorem 2.10. [29], [34] There exists an $(n,8,1)$-RBIBD if and only if $n \equiv 8$

(mod 56) with 58 possible exceptions given in Table 2.2.

Theorem 2.11. [35], [40], [14], [37], [39] There exists an $(n, 8, 7)$-RBIBD if and only if $n \equiv 0 \pmod 8$ with 14 possible exceptions given in Table 2.1.

For $k = 12$, there exist studies of Lee [15] showing that the necessary condition for constructing a $(12m, 12, 11)$-RBIBD is sufficient when $m > 9064$. Also, Greig's [41] studies proved that the necessary condition for constructing a $(16m, 16, 15)$-RBIBD is sufficient when $m > 5240$.

Table 2.1 Solutions of complete graphs

Graphs	n	λ	**Possible Exceptions**
K_2	$n \equiv 0 \pmod 2$	$\lambda \geqslant 1$	$\emptyset$
K_3	$n \equiv 3 \pmod 6$	$\lambda \geqslant 1$	$\emptyset$
	$n \equiv 0 \pmod 3$, $n \neq 6$	$\lambda \equiv 0 \pmod 2$	$\emptyset$
K_4	$n \equiv 4 \pmod{12}$	$\lambda \geqslant 1$	$\emptyset$
	$n \equiv 0 \pmod 4$	$\lambda \equiv 0 \pmod 3$	$\emptyset$
K_5	$n \equiv 5 \pmod{20}$	$\lambda \geqslant 1$	$45, 345, 465, 645$
	$n \equiv 5 \pmod{10}$, $n \neq 15$	$\lambda \equiv 0 \pmod 2$	$45, 115, 135, 195, 215, 235, 295, 315, 335, 345, 395.$
	$n \equiv 0 \pmod 5$ $n \neq 10, 15$	$\lambda \equiv 0 \pmod 4$	$135, 160, 190, 195$
K_6	$n = 125q + 1$ for any prime $q \equiv 7 \pmod{12}$ and $q > 43$	$\lambda \geqslant 1$	$n \equiv 6 \pmod{30}$ that does not satisfy the condition at left.
	$n \equiv 0 \pmod 6$	$\lambda \equiv 0 \pmod 5$	$\emptyset$
K_7	$n \equiv 7 \pmod{42}$	$\lambda \geqslant 1$	$n < 294427$
	$n \equiv 0 \pmod 7$, $n \neq 14$	$\lambda \equiv 0 \pmod 6$	$84, 119, 126, 133, 175, 182$ $189, 210, 231, 238, 259, 266,$ $287, 413, 420, 427, 434, 462$
K_8	$n \equiv 8 \pmod{56}$	$\lambda \geqslant 1$	see Table 2.2
	$n \equiv 0 \pmod 8$	$\lambda \equiv 0 \pmod 7$	$24, 160, 168, 192, 224, 240, 312, 336, 440, 552, 560, 744, 1464, 1488$
K_{12}	$n \equiv 0 \pmod{12}$	$\lambda \equiv 0 \pmod{11}$	$n \leqslant 108768$
K_{16}	$n \equiv 0 \pmod{16}$	$\lambda \equiv 0 \pmod{15}$	$n \leqslant 83840$

Table 2.2 Possible exception values of n for $(n,8,1)$-RBIBD

176	736	1128	1240	1296	1408	1464	1520
1744	2136	2416	2640	2920	3256	3312	3424
3760	3872	4264	4432	5216	6224	6280	6448
6896	6952	7008	7456	7512	7792	7848	8016
9752	10704	10760	10928	11040	11152	11376	11656
11712	11824	11936	12216	12328	12496	12552	12720
12832	12888	13000	13280	13616	13840	13896	14008
14032	21904						

2.2 Cycles

A *cycle* is a graph with vertices $a_1, a_2, ..., a_k$ and edges $\{a_i, a_{i+1}\}$ for $1 \leqslant i \leqslant k-1$ and $\{a_k, a_1\}$. It is denoted by C_k for $k \geqslant 3$. A C_k-factorization of λK_n is the uniform case of Oberwolfach problem which was posed by Ringel when several mathematicians interested in graph theory gathered in Oberwolfach, Germany for a conference and was first mentioned by Guy [42]. There exist many research about Oberwolfach problem, see [43], [44], [45], [46], [47], [48], [49], [50], [51], [52].

For the existence of C_k-factorizations of λK_n, there are two necessary conditions which are given below:

$$n \equiv 0 \pmod{k} \tag{2.14}$$

$$\lambda(n-1) \equiv 0 \pmod{2} \tag{2.15}$$

Alspach, et al. [52] gave a proof about the sufficiency of these necessary conditions when $\lambda = 1$. Then this result was extended to multigraphs λK_n by Gvozdjak [43] in 1997.

Theorem 2.12. [43] Let λ, k and n be positive integers with $k \geqslant 3$. Then λK_n has a C_k-factorization if and only if k divides n, $\lambda(n-1) \equiv 0 \pmod{2}$ and the following is not the case.

$$\lambda \equiv 2 \pmod{4},\ n = 6,\ k = 3 \tag{2.16}$$

Example 2.5 A C_7-factorization of K_7 is given below.
Let $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ denote the cycle C_7 with edges $\{\{a_1, a_2\}, \{a_2, a_3\},$

$\{a_3, a_4\}$, $\{a_4, a_5\}$, $\{a_5, a_6\}$, $\{a_6, a_7\}$, $\{a_7, a_1\}\}$.

$B_{1,1} = \{1, 2, 3, 4, 5, 6, 7\}$
$B_{2,1} = \{1, 3, 5, 7, 2, 4, 6\}$
$B_{3,1} = \{1, 4, 7, 3, 6, 2, 5\}$

Example 2.6 A C_3-factorization of K_9 is given below.
Let $\{a_1, a_2, a_3\}$ denote the cycle C_3 with edges $\{\{a_1, a_2\}$, $\{a_2, a_3\}\}$, $\{a_3, a_1\}\}$.

$B_{1,1} = \{1, 2, 3\}$ $B_{2,1} = \{1, 4, 7\}$ $B_{3,1} = \{1, 5, 9\}$ $B_{4,1} = \{1, 6, 8\}$
$B_{1,2} = \{4, 5, 6\}$ $B_{2,2} = \{2, 5, 8\}$ $B_{3,2} = \{2, 6, 7\}$ $B_{4,2} = \{2, 4, 9\}$
$B_{1,3} = \{7, 8, 9\}$ $B_{2,3} = \{3, 6, 9\}$ $B_{3,3} = \{3, 4, 8\}$ $B_{4,3} = \{3, 5, 7\}$

Example 2.7 A C_4-factorization of $2K_4$ is given below.
Let $\{a_1, a_2, a_3, a_4\}$ denote the graph C_4 with edges $\{\{a_1, a_2\}$, $\{\{a_2, a_3\}$, $\{\{a_3, a_4\}$, $\{\{a_4, a_1\}\}$.

$B_{1,1} = \{1, 2, 3, 4\}$ $B_{2,1} = \{1, 2, 4, 3\}$ $B_{3,1} = \{1, 3, 2, 4\}$

2.3 Matchings

A *matching* on $2k$ vertices is a set of k disjoint edges and is denoted by M_k. If λK_n has an M_k-factorization, then it provides the following condition:

$$n \equiv 0 \pmod{2k} \tag{2.17}$$

A matching that saturates all the vertices of a graph is called a 1-factor of G or a perfect matching. An M_1-factorization of G is also known as a 1-factorization of G. We have $M_1 \cong K_2$ and this graph is covered in Section 2.1.

Theorem 2.13. [3] λK_{2n} has a 1-factorization for all n and all λ.

If 2k divides n, then n is even and λK_n has an M_1-factorization, where each parallel class has $n/2$ blocks. Then, we can get an M_k-factorization by partitioning each factor of an M_1-factorization into $\frac{n}{2k}$ M_k's.

Theorem 2.14. λK_n has an M_k-factorization if and only if $n \equiv 0 \pmod{2k}$.

Example 2.8 K_8 has an M_1-factorization. Parallel classes are given below:

$Class_1$	$Class_2$	$Class_3$	$Class_4$	$Class_5$	$Class_6$	$Class_7$
$\{1,5\}$	$\{5,4\}$	$\{6,5\}$	$\{5,2\}$	$\{6,3\}$	$\{7,5\}$	$\{8,6\}$
$\{2,6\}$	$\{3,2\}$	$\{4,3\}$	$\{6,1\}$	$\{7,2\}$	$\{8,2\}$	$\{5,3\}$
$\{4,8\}$	$\{8,1\}$	$\{1,2\}$	$\{7,4\}$	$\{4,1\}$	$\{1,3\}$	$\{7,1\}$
$\{3,7\}$	$\{7,6\}$	$\{7,8\}$	$\{8,3\}$	$\{5,8\}$	$\{4,6\}$	$\{4,2\}$

In addition, K_8 has an M_2-factorization. Parallel classes are given below:

	$Class_1$	$Class_2$	$Class_3$	$Class_4$	$Class_5$	$Class_6$	$Class_7$
Block 1	$\{1,5\}$	$\{5,4\}$	$\{6,5\}$	$\{5,2\}$	$\{6,3\}$	$\{7,5\}$	$\{8,6\}$
	$\{2,6\}$	$\{3,2\}$	$\{4,3\}$	$\{6,1\}$	$\{7,2\}$	$\{8,2\}$	$\{5,3\}$
Block 2	$\{4,8\}$	$\{8,1\}$	$\{1,2\}$	$\{7,4\}$	$\{4,1\}$	$\{1,3\}$	$\{7,1\}$
	$\{3,7\}$	$\{7,6\}$	$\{7,8\}$	$\{8,3\}$	$\{5,8\}$	$\{4,6\}$	$\{4,2\}$

Example 2.9 An M_2-factorization of K_{12} is given below.
Let $\{a_1, a_2; a_3, a_4\}$ denote the graph M_2 with the edges $\{\{a_1, a_2\}, \{a_3, a_4\}\}$.

$B_{1,1} = \{1,2;3,4\}$ $B_{1,2} = \{5,6;7,8\}$ $B_{1,3} = \{9,10;11,12\}$
$B_{2,1} = \{1,3;2,4\}$ $B_{2,2} = \{5,7;6,8\}$ $B_{2,3} = \{9,11;10,12\}$
$B_{3,1} = \{1,4;2,3\}$ $B_{3,2} = \{5,8;6,7\}$ $B_{3,3} = \{9,12;10,11\}$
$B_{4,1} = \{1,5;2,6\}$ $B_{4,2} = \{3,9;4,10\}$ $B_{4,3} = \{7,11;8,12\}$
$B_{5,1} = \{1,6;2,5\}$ $B_{5,2} = \{3,10;4,9\}$ $B_{5,3} = \{7,12;8,11\}$
$B_{6,1} = \{1,7;2,8\}$ $B_{6,2} = \{3,11;4,12\}$ $B_{6,3} = \{5,9;6,10\}$
$B_{7,1} = \{1,8;2,7\}$ $B_{7,2} = \{3,12;4,11\}$ $B_{7,3} = \{5,10;6,9\}$
$B_{8,1} = \{1,9;2,10\}$ $B_{8,2} = \{3,7;4,8\}$ $B_{8,3} = \{5,11;6,12\}$
$B_{9,1} = \{1,10;2,9\}$ $B_{9,2} = \{3,8;4,7\}$ $B_{9,3} = \{5,12;6,11\}$
$B_{10,1} = \{1,11;2,12\}$ $B_{10,2} = \{3,5;4,6\}$ $B_{10,3} = \{7,9;8,10\}$
$B_{11,1} = \{1,12;2,11\}$ $B_{11,2} = \{3,6;4,5\}$ $B_{11,3} = \{7,10;8,9\}$

2.4 Paths

A *path*, denoted by P_k, is a graph with distinct vertices $a_1, a_2, ..., a_k$ and edges $\{a_{i-1}, a_i\}$ for $2 \leqslant i \leqslant k$.

If λK_n has a P_k-factorization, then it satisfies the following necessary conditions:

$$n \equiv 0 \pmod{k} \tag{2.18}$$

$$\lambda k(n-1) \equiv 0 \pmod{2(k-1)} \tag{2.19}$$

For $k > 3$, the sufficiency of the above conditions was showed by J.C. Bermond et al. [53], in 1990. For $k = 3$, the necessary and sufficient conditions were obtained by Horton [54]. So the existence of path factorization problem of λK_n is completely solved. Moreover, the solution of P_3 and P_4 has also been provided in the article of Hell and Rosa [55].

Theorem 2.15. [53], [54] When $k \geqslant 3$, λK_n has a P_k-factorization if and only if $n \equiv 0 \pmod{k}$ and $\lambda k(n-1) \equiv 0 \pmod{2(k-1)}$.

Example 2.10 [55] A P_3-factorization of K_9 is given below.
Let $\{a_1, a_2, a_3\}$ denote the path P_3 with edges $\{\{a_1, a_2\}, \{a_2, a_3\}\}$.

$B_{1,1} = \{1,2,3\}$ $B_{2,1} = \{3,7,5\}$ $B_{3,1} = \{5,9,1\}$
$B_{1,2} = \{4,5,6\}$ $B_{2,2} = \{6,1,8\}$ $B_{3,2} = \{8,3,4\}$
$B_{1,3} = \{7,8,9\}$ $B_{2,3} = \{9,4,2\}$ $B_{3,3} = \{2,6,7\}$

$B_{4,1} = \{7,2,5\}$ $B_{5,1} = \{9,7,1\}$ $B_{6,1} = \{2,9,3\}$
$B_{4,2} = \{3,1,4\}$ $B_{5,2} = \{5,3,6\}$ $B_{6,2} = \{1,5,8\}$
$B_{4,3} = \{9,6,8\}$ $B_{5,3} = \{2,8,4\}$ $B_{3,1} = \{7,4,6\}$

Example 2.11 A P_4-factorization of K_4 is given below.
Let $\{a_1, a_2, a_3, a_4\}$ denote the graph P_4 with vertices $\{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}\}$.

$B_{1,1} = \{1,2,4,3\}$ $B_{2,1} = \{4,1,3,2\}$

2.5 Stars

A *k-star* or a complete bipartite graph $K_{1,k-1}$ is denoted by S_k. It has k vertices and $k-1$ edges. If λK_n has an S_k-factorization, then it provides the following necessary conditions:

$$n \equiv 0 \pmod{k} \tag{2.20}$$

$$\lambda k(n-1) \equiv 0 \pmod{2(k-1)} \tag{2.21}$$

In 1976, Huang [56] proved the existence of an S_k-factorization of λK_n for $k \geqslant 4$ and k even. In the same article, he studied on the S_k-factorizations of λK_n for some special cases when k is odd.

Theorem 2.16. [56] If k is even, an S_k-factorization of λK_n exists if and only if there is a divisor x of $k-1$ such that

i) $n \equiv k \pmod{\frac{k(k-1)}{x}}$ and

ii) $\lambda \equiv 0 \pmod{2x}$

Theorem 2.17. [56] Let k be odd and let q be a positive integer. An S_k-factorization of λK_n exists if the following conditions are satisfied:

i) $n = k^{2q}$, or

ii) λ is even and $n = k^q$, or

iii) There is a divisor x of $k-1$ such that x is even, $n \equiv k \pmod{\frac{2k(k-1)}{x}}$, and $\lambda \equiv 0 \pmod{x}$.

Yu [57] proved that the necessary conditions are enough to construct an S_k-factorization of K_n for odd k.

Theorem 2.18. [57] If k is odd, then there exist an S_k-factorization of K_n if and only if $n \equiv 0 \pmod{k}$ and $n-1 \equiv 0 \pmod{2(k-1)}$.

The existence of an S_3-factorization of K_n is discussed in Section 2.4.

For $k = 5$, the existence problem of an S_k-factorization of λK_n is completely solved by Danziger et al. [58].

Theorem 2.19. [56], [57], [58] λK_n has an S_5-factorization if and only if it satisfies the following conditions:

i) $n \equiv 25 \pmod{40}$ for $\lambda \geqslant 1$,

ii) $n \equiv 5 \pmod{20}$ for $\lambda \equiv 0 \pmod{2}$,

iii) $n \equiv 5 \pmod{10}$ for $\lambda \equiv 0 \pmod{4}$,

iv) $n \equiv 0 \pmod{5}$ for $\lambda \equiv 0 \pmod{8}$.

Example 2.12 [59] An S_4-factorization of $2K_4$ is given below.
Let $\{a_1; a_2, a_3, a_4\}$ denote the graph S_4 with edges $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}\}$.

$B_{1,1} = \{1; 2, 3, 4\}$ $\quad B_{2,1} = \{2; 3, 4, 1\}$ $\quad B_{3,1} = \{3; 4, 1, 2\}$ $\quad B_{4,1} = \{4; 1, 2, 3\}$

Example 2.13 [60] An S_6-factorization of $2K_6$ is given below.
Let $\{a_1; a_2, a_3, a_4, a_5, a_6\}$ denote the graph S_6 with the edges $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_1, a_5\}, \{a_1, a_6\}\}$.

$B_{1,1} = \{1; 2, 3, 4, 5, 6\}$ $\quad B_{2,1} = \{2; 1, 3, 4, 5, 6\}$
$B_{3,1} = \{3; 1, 2, 4, 5, 6\}$ $\quad B_{4,1} = \{4; 1, 2, 3, 5, 6\}$
$B_{5,1} = \{5; 1, 2, 3, 4, 6\}$ $\quad B_{6,1} = \{6; 1, 2, 3, 4, 5\}$

2.6 Trees

A *tree* with k vertices and $k-1$ edges, denoted by T_k, is a connected graph which contains no cycle. In this section, we will study T_k-factorizations of λK_n.

If λK_n has a T_k-factorization, then it provides the following necessary conditions:

$$n \equiv 0 \pmod{k} \tag{2.22}$$

$$\lambda k(n-1) \equiv 0 \pmod{2(k-1)} \tag{2.23}$$

A few years ago, there were some studies about T_k-factorizations. So, the existence problem of T_k-factorizations of λK_n is not a new problem. Firstly, Beineke in 1964 [61] mentioned the existence of a T_k-factorization of λK_n. Later, Lonc [62] proved the existence of tree-factorizations for every graceful tree T_k for odd k. A *graceful tree* is a tree that has a labeling $1, 2, ..., k$ where k is the number of vertices such that $\{|i-j| : ij \text{ is an edge }\} = \{1, 2, ..., k-1\}$.

Theorem 2.20. [62] Let T_k be a graceful tree of odd order k and there exists an integer $n(T)$. If $n \geqslant n(T)$ then there exists a T_k-factorization of K_n if and only if the following is satisfied.

$$n \equiv k^2 \pmod{2k(k-1)} \tag{2.24}$$

Ringel [63] conjectured that all trees are graceful and Kotzig [64] strengthened this. We don't know any tree that are not graceful so far, so we think this class contains most of the trees.

Another study about the existence of T_k-factorizations is done by Min Li Yu in 1990s. He proved the following theorem and corollary.

Theorem 2.21. [65] **i)** If K_k has a T_k-factorization, then λK_n has a T_k-factorization if and only if $n \equiv 0 \pmod{k}$ and $\lambda k(n-1) \equiv 0 \pmod{2(k-1)}$.
ii) If $2K_k$ has a T_k-factorization, then $2\lambda K_n$ has a T_k-factorization if and only if $n \equiv 0 \pmod{k}$ and $\lambda k(n-1) \equiv 0 \pmod{(k-1)}$.

Corollary 2.1. [57] Let T_k be a graceful tree and let λ be even. Then λK_n has a T_k-factorization if and only if $n \equiv 0 \pmod{k}$ and $\lambda k(n-1) \equiv 0 \pmod{2(k-1)}$.

Example 2.14 [66] A T_8-factorization of $2K_8$ is given below.
Let $\{a_1, a_2, a_3, a_4, a_5, a_6; a_7, a_8\}$ denote the graph T_8 with edges $\{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_4, a_5\}, \{a_5, a_6\}, \{a_6, a_7\}, \{a_6, a_8\}\}$.

$B_{1,1} = \{2, 3, 4, 5, 6, 1; 8, 9\}$ $B_{2,1} = \{1, 3, 6, 4, 7, 8; 2, 5\}$
$B_{3,1} = \{8, 4, 1, 5, 2, 7; 3, 6\}$ $B_{4,1} = \{7, 5, 3, 8, 6, 2; 1, 4\}$

2.7 Cubes

A *cube* with order k or the *k-cube*, denoted by Q_k, is the graph whose vertex set consists of all binary strings of length k and edge set consists of all pairs of vertices which differ in exactly one coordinate. The cubes Q_i for i= $1, 2, 3, 4$ are shown in Figure 2.1. Q_k has 2^k vertices and $k \cdot 2^{k-1}$ edges. Firstly, Kotzig [67] mentioned Q_k-factorizations of complete graphs in 1979.

If λK_n has a Q_k-factorization, then it satisfies the following necessary conditions:

$$n \equiv 0 \pmod{2^k} \tag{2.25}$$

$$\lambda(n-1) \equiv 0 \pmod{k} \tag{2.26}$$

First condition implies that n must be even. So, if there exists a Q_k-factorization of K_n, then k must be odd.

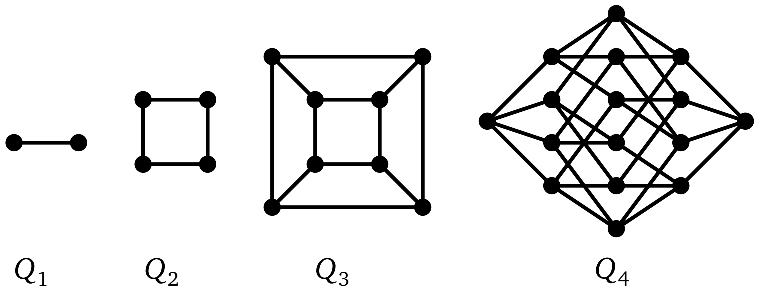

Figure 2.1 Q_1, Q_2, Q_3 and Q_4

The existence problem of a Q_k-factorization of λK_n was settled for $k = 1$ and $k = 2$. For $k = 1$, the graph is a single edge and for $k = 2$ the graph is a 4-cycle. So, solutions are showed in Section 2.1 and Section 2.2, respectively. The Q_3-factorization problem of K_n was settled by Adams, et.al. [68]. In these studies, the sufficiency of the necessary conditions was showed.

Theorem 2.22. [68] K_n has a Q_3-factorization if and only if $n \equiv 16 \pmod{24}$.

This result settles the existence problem of a Q_3-factorization of λK_n for all $\lambda \equiv 1, 2 \pmod 3$. There exist no studies for the case $\lambda \equiv 0 \pmod 3$.

Finally, Q_4-factorization and Q_5-factorization problem of λK_n were entirely solved by Doğan et al. [69], [70], [71]. They showed that the necessary conditions are adequate.

Theorem 2.23. [69], [70] λK_n has a Q_4-factorization if and only if $n \equiv 0 \pmod{16}$ and $\lambda \equiv 0 \pmod 4$.

Theorem 2.24. [69], [71] λK_n has a Q_5-factorization if and only if

i) $n \equiv 0 \pmod{32}$ if $\lambda \equiv 0 \pmod 5$,

ii) $n \equiv 96 \pmod{160}$ if $\lambda \not\equiv 0 \pmod 5$.

The Q_k-factorization problem of λK_n is currently unresolved for $k = 3$, $\lambda \equiv 0 \pmod 3$ and for $k \geqslant 6$.

Table 2.3 Solutions of cubes

Graph	n	λ	**Possible Exceptions**
Q_1	$n \equiv 0 \pmod 2$	$\lambda \geqslant 1$	$\emptyset$
Q_2	$n \equiv 0 \pmod 4$	$\lambda \equiv 0 \pmod 2$	$\emptyset$
Q_3	$n \equiv 16 \pmod{24}$	$\lambda \equiv 1, 2 \pmod 3$	$\emptyset$
	$n \equiv 16 \pmod{24}$	$\lambda \equiv 0 \pmod 3$	$n \equiv 0, 8 \pmod{24}$
Q_4	$n \equiv 0 \pmod{16}$	$\lambda \equiv 0 \pmod 4$	$\emptyset$
Q_5	$n \equiv 0 \pmod{32}$	$\lambda \equiv 0 \pmod 5$	$\emptyset$
	$n \equiv 96 \pmod{160}$	$\lambda \not\equiv 0 \pmod 5$	$\emptyset$

2.8 Complete Bipartite Graphs

The number of edges in $K_{s,t}$ is $\left|E(K_{s,t})\right| = st$ by the definition of complete bipartite graphs. There are three necessary conditions for the existence of a $K_{s,t}$-factorization of λK_n:

$$n \equiv 0 \pmod{s+t} \tag{2.27}$$

$$\lambda(s+t)(n-1) \equiv 0 \pmod{2st} \tag{2.28}$$

$$\lambda(n-1) \equiv 0 \pmod{gcd(s,t)} \tag{2.29}$$

The existence problem of a star factorization of λK_n, that is, a $K_{1,k}$-factorization was discussed for $k \geqslant 1$ in Section 2.5. In 1975, Huang [56] proved some theorems about $K_{s,s}$-factorizations of λK_n.

Theorem 2.25. [56] Let s be even and $2s-1$ a prime power. λK_n has a $K_{s,s}$-factorization if and only if $n \equiv 0 \pmod{2s}$ and $\lambda \equiv 0 \pmod s$.

At the same time, Huang [56] proved the following theorems about a $K_{s,t}$-factorization of λK_n.

Theorem 2.26. [56] If there is a $K_{s,t}$-factorization of $\lambda_1 K_n$ and a $K_{s,t}$-factorization of $\lambda_2 K_n$, then there exists a $K_{s,t}$-factorization of $\lambda_3 K_n$ where $\lambda_3 = p_1\lambda_1 + p_2\lambda_2$, p_1 and p_2 being non-negative integers not both equal to zero.

Theorem 2.27. [56] If there is a $K_{s,t}$-factorization of λK_n and a $K_{s,t}$-factorization of λK_m, then there exists a $K_{s,t}$-factorization of λK_{mn}.

Theorem 2.28. [56] If there is an (n, k, λ)-RBIBD and a $K_{s,t}$-factorization of K_k, then there exists a $K_{s,t}$-factorization of λK_n.

2.9 Graphs with Small Number of Vertices

In this section, we investigate the existence problem of G-factorizations of λK_n for small graphs that have at most six vertices. Subsection 2.9.1 is about the solution of all graphs on four or fewer vertices. Subsection 2.9.2 deals with graphs with five vertices and Subsection 2.9.3 considers graphs with six vertices.

2.9.1 Graphs With Four or Fewer Vertices

In Figure 2.2., ten non-isomorphic graphs with four or fewer vertices with no isolated vertex are shown.

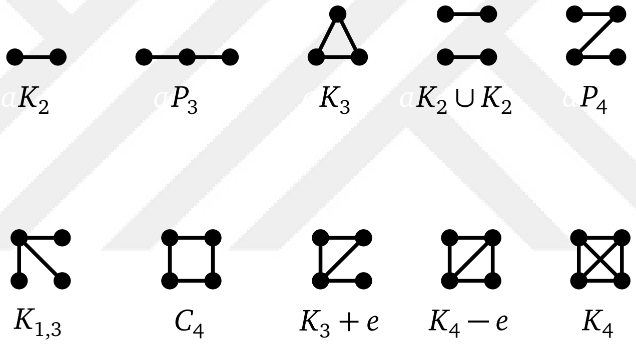

Figure 2.2 Graphs with four or fewer vertices

The complete graphs K_2, K_3 and K_4 are mentioned in Section 2.1. The existence problem of P_3 and P_4-factorizations of λK_n are covered in Section 2.4. $K_2 \cup K_2$, $K_{1,3}$ and C_4 are mentioned in Sections 2.3, 2.5 and 2.2, respectively. The $(K_4 - e)$-factorization of K_n was settled in [72], [73], [74].

Theorem 2.29. [72], [73], [74] K_n has a $(K_4 - e)$-factorization if and only if $n \equiv 16 \pmod{20}$.

The factorization problem for $(K_4 - e)$ and $(K_3 + e)$ when $\lambda \geqslant 1$ was completely solved by Gionfriddo et al.[59]

Theorem 2.30. [59] λK_n has a $(K_4 - e)$-factorization if and only if

i) $n \equiv 0, 4, 8, 12 \pmod{20}$ for $\lambda \equiv 0 \pmod{5}$,

ii) $n \equiv 16 \pmod{20}$ for any positive integer λ.

Theorem 2.31. [59] λK_n has a $(K_3 + e)$-factorization if and only if $n \equiv 0$ (mod 4) and $\lambda \equiv 0 \pmod{2}$.

Table 2.4 Solutions of graphs with four or fewer vertices

Graphs	n	λ	**Possible Exceptions**
K_2	$n \equiv 0 \pmod{2}$	$\lambda \geqslant 1$	$\emptyset$
P_3	$n \equiv 9 \pmod{12}$	$\lambda \geqslant 1$	$\emptyset$
	$n \equiv 3 \pmod{6}$	$\lambda \equiv 0 \pmod{2}$	$\emptyset$
	$n \equiv 0 \pmod{3}$	$\lambda \equiv 0 \pmod{4}$	$\emptyset$
K_3	$n \equiv 3 \pmod{6}$	$\lambda \geqslant 1$	$\emptyset$
	$n \equiv 0 \pmod{3}, n \neq 6$	$\lambda \equiv 0 \pmod{2}$	$\emptyset$
$K_2 \cup K_2$	$n \equiv 0 \pmod{4}$	$\lambda \geqslant 1$	$\emptyset$
P_4	$n \equiv 4 \pmod{12}$	$\lambda \geqslant 1$	$\emptyset$
	$n \equiv 0 \pmod{4}$	$\lambda \equiv 0 \pmod{3}$	$\emptyset$
$K_{1,3}$	$n \equiv 4 \pmod{12}$	$\lambda \equiv 0 \pmod{2}$	$\emptyset$
	$n \equiv 0 \pmod{4}$	$\lambda \equiv 0 \pmod{6}$	$\emptyset$
C_4	$n \equiv 0 \pmod{4}$	$\lambda \equiv 0 \pmod{2}$	$\emptyset$
$K_3 + e$	$n \equiv 0 \pmod{4}$	$\lambda \equiv 0 \pmod{2}$	$\emptyset$
$K_4 - e$	$n \equiv 16 \pmod{20}$	$\lambda \geqslant 1$	$\emptyset$
	$n \equiv 0 \pmod{4}$	$\lambda \equiv 0 \pmod{5}$	$\emptyset$
K_4	$n \equiv 4 \pmod{12}$	$\lambda \geqslant 1$	$\emptyset$
	$n \equiv 0 \pmod{4}$	$\lambda \equiv 0 \pmod{3}$	$\emptyset$

Example 2.15 [59] A $(K_3 + e)$-factorization of $2K_4$ is given below.
Let $\{a_1, a_2, a_3 - a_4\}$ denote graph $K_3 + e$ with edges $\{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_1\}, \{a_3, a_4\}\}$.

$B_{1,1} = \{1, 3, 4 - 2\}$ $\quad$ $B_{2,1} = \{4, 3, 2 - 1\}$ $\quad$ $B_{3,1} = \{3, 2, 1 - 4\}$

Example 2.16 [59] A $(K_4 - e)$-factorization of $5K_4$ is given below.
Let $\{a_1, a_2, a_3; a_4\}$ denote the graph K_4—e with the edges $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}, \{a_1, a_4\}, \{a_2, a_4\}\}$.

$B_{1,1} = \{2, 3, 1; 4\}$ $\quad$ $B_{2,1} = \{4, 1, 3; 2\}$ $\quad$ $B_{3,1} = \{2, 4, 1; 3\}$
$B_{4,1} = \{3, 1, 2; 4\}$ $\quad$ $B_{5,1} = \{2, 1, 3; 4\}$ $\quad$ $B_{6,1} = \{3, 4, 2; 1\}$

2.9.2 Graphs with Five Vertices

The non-isomorphic graphs with five vertices are shown in Figure 2.3. But, the solutions are not completed for all of these graphs.

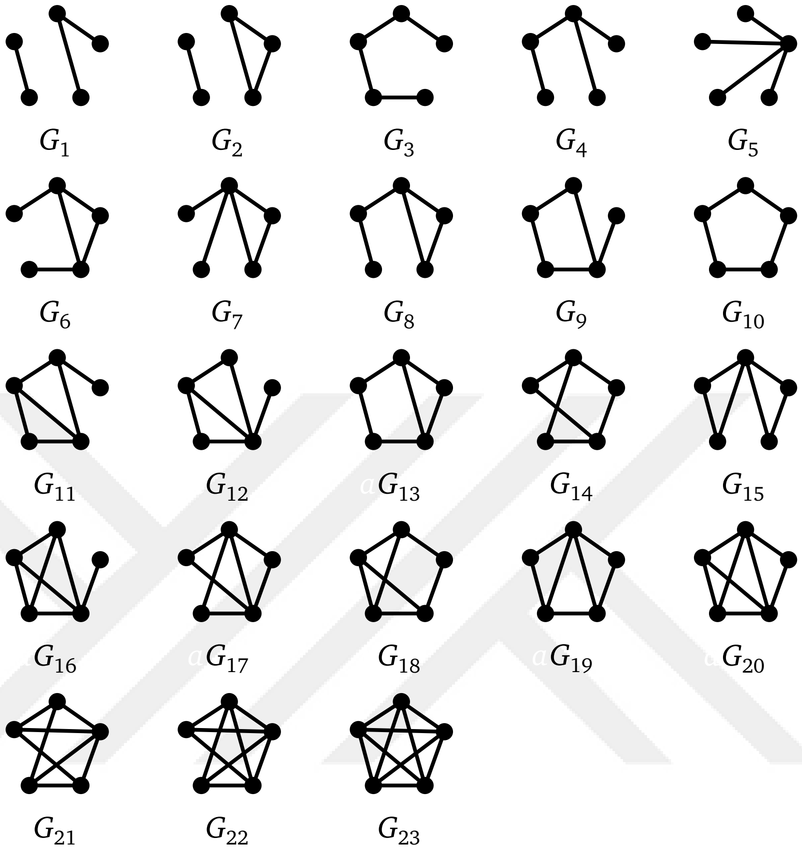

Figure 2.3 Graphs with five vertices

There are few information about the existence of G-factorizations where G is a graph with five vertices. The existence problem of G_i-factorizations of λK_n is completely settled for $i \in \{3,4,5,10\}$ and almost completely settled for $i = 23$. We have $G_3 \cong P_5$, $G_5 \cong K_{1,4}$, $G_{10} \cong C_5$ and $G_{23} \cong K_5$ and the solutions of these graphs are given in Sections 2.4, 2.5, 2.2 and 2.1, respectively. Finally, the solution for G_4-factorizations of λK_n is given by Danziger et al. [58].

Theorem 2.32. [58] λK_n has a G_4-factorization if and only if

i) $n \equiv 0, 10, 20, 30 \pmod{40}$ for $\lambda \equiv 0 \pmod{8}$,

ii) $n \equiv 15, 35 \pmod{40}$ for $\lambda \equiv 0 \pmod{4}$,

iii) $n \equiv 5 \pmod{40}$ for $\lambda \equiv 0 \pmod{2}$,

iv) $n \equiv 25 \pmod{40}$ for any λ.

Table 2.5 Solutions of graphs with five vertices

Graph	n	λ	**Exception**
G_3, G_4, G_5	$n \equiv 25 \pmod{40}$	$\lambda \geqslant 1$	$\emptyset$
	$n \equiv 5 \pmod{20}$	$\lambda \equiv 0 \pmod{2}$	$\emptyset$
	$n \equiv 5 \pmod{10}$	$\lambda \equiv 0 \pmod{4}$	$\emptyset$
	$n \equiv 0 \pmod{5}$	$\lambda \equiv 0 \pmod{8}$	$\emptyset$
G_{10}	$n \equiv 5 \pmod{10}$	$\lambda \geqslant 1$	$\emptyset$
	$n \equiv 0 \pmod{5}$	$\lambda \equiv 0 \pmod{2}$	$\emptyset$
G_{23}	$n \equiv 5 \pmod{20}$	$\lambda \geqslant 1$	$45, 345, 465, 645$
	$n \equiv 5 \pmod{10}$, $n \neq 15$	$\lambda \equiv 0 \pmod{2}$	$45, 115, 135, 195, 215, 235,$ $295, 315, 335, 345, 395.$
	$n \equiv 0 \pmod{5}$ $n \neq 10, 15$	$\lambda \equiv 0 \pmod{4}$	$135, 160, 190, 195$

Example 2.17 A G_4-factorization of $2K_5$ is given below.
Let $\{a_1; a_2, a_3, a_4; a_5\}$ denote the graph G_4 with edges $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_4, a_5\}\}$.

$B_{1,1} = \{1; 2, 3, 4; 5\}$ $\quad B_{2,1} = \{2; 3, 4, 5; 1\}$ $\quad B_{3,1} = \{3; 4, 5, 1; 2\}$
$B_{4,1} = \{4; 5, 1, 2; 3\}$ $\quad B_{5,1} = \{5; 1, 2, 3; 4\}$

2.9.3 Graphs with Six Vertices

Results obtained on the existence of a G-factorization of λK_n when G is a graph with six vertices are very few. Solutions are completed for ten of these graphs, and almost completed for one graph.

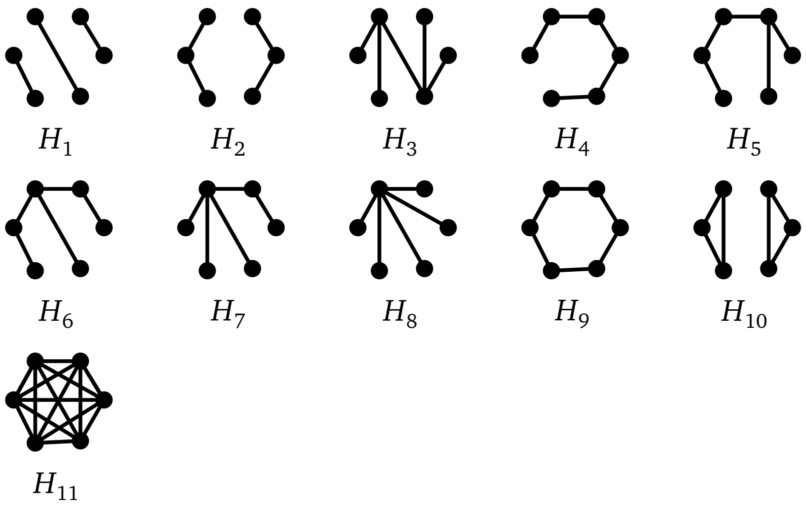

Figure 2.4 Some graphs with six vertices

We know that $H_1 \cong M_3$, $H_4 \cong P_5$, $H_8 \cong K_{1,5}$, $H_9 \cong C_6$, and $H_{11} \cong K_6$, and solutions of these graphs are shown in Section 2.3, 2.4, 2.5, 2.2, and 2.1. For $i = 3,5,6,7$; the graphs H_i are graceful trees and solutions are given in Section 2.6. Also, we have $H_2 \cong P_3 \cup P_3$ and $H_{10} \cong K_3 \cup K_3$.

If λK_n has a $P_3 \cup P_3$-factorization, then it satisfies the necessary conditions which are given below:

$$n \equiv 0 \pmod 6 \tag{2.30}$$

$$\lambda(n-1) \equiv 0 \pmod 4 \tag{2.31}$$

These conditions also provide the necessary and sufficient conditions to construct a P_3-factorization of λK_n. So, if λK_n has a P_3-factorization, where the number of blocks $n/3$ in each parallel class is even, then we can get a $P_3 \cup P_3$-factorization by partitioning each factor of a P_3-factorization into $\frac{n}{6}$ $P_3 \cup P_3$'s. Since, n is a multiple of 6, $(n-1)$ is not divided by 4. So, $\lambda \equiv 0 \pmod 4$. Consequently, the necessary and sufficient condition for the existence of a $P_3 \cup P_3$-factorization of λK_n is $n \equiv 0 \pmod 6$ and $\lambda \equiv 0 \pmod 4$. The operations described above can also be used to get a $K_3 \cup K_3$-factorization of λK_n. In addition, the existence of a $(K_3 \cup K_3)$-factorization of λK_6 has also been proven by Bolstad [75].

Theorem 2.33. λK_n has a $P_3 \cup P_3$-factorization if and only if $n \equiv 0 \pmod 6$ and $\lambda \equiv 0 \pmod 4$.

Theorem 2.34. λK_n has a $K_3 \cup K_3$-factorization if and only if $n \equiv 0 \pmod 6$

and $\lambda \equiv 0 \pmod 2$.

Table 2.6 Solutions of some graphs with six vertices

Graph	n	λ	**Exception**
H_1	$n \equiv 0 \pmod 6$	$\lambda \geqslant 1$	$\emptyset$
H_2	$n \equiv 0 \pmod 6$	$\lambda \equiv 0 \pmod 4$	$\emptyset$
$H_3, H_4, H_5, H_6, H_7, H_8$	$n \equiv 6 \pmod{30}$	$\lambda \equiv 0 \pmod 2$	$\emptyset$
	$n \equiv 0 \pmod 6$	$\lambda \equiv 0 \pmod{10}$	$\emptyset$
H_9, H_{10}	$n \equiv 0 \pmod 6$	$\lambda \equiv 0 \pmod 2$	$\emptyset$
H_{11}	$n =$ 125q+1 for any prime $q \equiv 7 \pmod{12}$ and $q > 43$	$\lambda \geqslant 1$	$n \equiv 6 \pmod{30}$ that does not satisfy the condition at left
	$n \equiv 0 \pmod 6$	$\lambda \equiv 0 \pmod 5$	$\emptyset$

3
RESULTS AND DISCUSSION

In this thesis, we review the existence of G-factorizations of λK_n. Firstly, we give the basic definitions and brief information about the necessary conditions for constructing λ-fold G-designs and G-factorizations of λK_n. Secondly, we introduce some types of graphs that are complete graphs, cycles, matchings, paths, stars, trees, cubes, complete bipartite graphs and small graphs with six or fewer vertices and we compile results about the existence of G-factorizations of λK_n for these graphs. The results are illustrated both in tables and figures. Also, we try to give examples for almost every graph mentioned in the thesis for easily understanding the subject.

The existence problem of G-factorizations of λK_n are completely settled for paths, cycles, matchings and graphs with four or fewer vertices. But, solutions for the other graphs are not completed. So, solutions for these graphs can be completed in the future and research can be started for some special graphs that have not been studied yet such as theta graphs, Petersen graphs, Heawood graphs, octahedrons etc.

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1. "A Survey on the Existence of G-factorizations of λK_n", (2020), 8^{th} ISPEC International Conferences on Engineering and Natural Sciences, 12-14 June, Ankara.