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HARRAN UNIVERSITY
INSTITUTE OF GRADUATE EDUCATION

MASTER THESIS

SOFT HYPERDIMONOIDS

IMAN SARKAWT SULAIMAN SULAIMAN

MATHEMATICS

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Thesis Supervisor: Assoc. Prof. Dr. GÜLAY OĞUZ

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They told me I was one of the luckiest people during this journey, and I say that all of this was by the grace of God and the blessing of my daughter, with whom I was pregnant when I began my master’s studies.

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This thesis explores the theoretical foundations and extensions of soft hyperdimonoids. The study begins with Soft sets theory , including soft groups, which underpin the concept of dimonoids. Building upon this foundation, the literature review surveys existing work on dimonoids, sub-dimonoids, homomorphisms, and commutative variants. The methodology introduces innovative constructs such as hyperdimonoids, sub hyperdimonoids, soft hyperdimonoids, and their corresponding homomorphisms. Through a series of illustrative examples, the practical implications of these structures are demonstrated. Its findings provide a framework for advancements in abstract algebra research and soft set theory.

KEYWORDS: Dimonoids, Homomorphism, Subhyperdimonoids, Hiper Yapılar, Hyper Dimonoid

ÖZET

YÜKSEK LİSANS TEZİ

SOFT HYPERDIMONOIDS

IMAN SARKAWT SULAIMAN SULAIMAN

HARRAN ÜNİVERSİTESİ
Tez Danışmanı:Doç. Dr. GÜLAY OĞUZ
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Bu tez, soft hiper-dimonoidlerin teorik temellerini ve genişletmelerini akademik bir yaklaşımla incelemektedir. Çalışma, dimonoid kavramının temelini oluşturan softküme teorisi ve yumuşak gruplar ile başlamaktadır. Bu temelin üzerine inşa edilen literatür taraması, dimonoidler, alt dimonoidler, homomorfizmalar ve komütatif varyantlar üzerine mevcut çalışmaları kapsamlı bir şekilde ele almaktadır. Yöntem kısmında ise hiper-dimonoidler, alt hiper-dimonoidler, soft hiper-dimonoidler ve bunlara karşılık gelen homomorfizmalar gibi yenilikçi yapılar tanıtılmaktadır. Sunulan örnekler aracılığıyla bu yapıların pratik yansımaları gösterilmiştir. Elde edilen bulgular, soyut cebir araştırmaları ve softküme teorisinin gelişimi için bir çerçeve sunmaktadır.

ANAHTAR KELİMELER: Dimonoids, Homomorphism, Subhyperdimonoids, Hyper Structures, Hiper Dimonoid

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SYMBOLS

α : Alpha

ρ : Özdirenç

1 INTRODUCTION

The concept of a set lies at the very heart of mathematics. A set is simply a well-defined collection of distinct objects, considered as an object in its own right. These objects, called elements or members of the set, can be anything: numbers, symbols, points, or even other sets. Set theory provides the basic language for mathematics and forms the basis upon which virtually all mathematical theories are constructed. Fundamental operations on sets, which includes union, intersection, and complement, offer powerful tools for structuring and analyzing collections of objects (Dummit & Foote, 2004; Hungerford, 1974).

In the landscape of modern algebraic structures and soft set theory, classical set theory has served as the foundation for numerous extensions that aim to handle uncertainty, partial information, and generalized operations. Among these, soft set theory has emerged as a flexible and robust mathematical tool for modeling vague and imprecise information. Concurrently we collect for previous studies about soft sets and soft groups, then structures such as dimonoids and hyperstructures have broadened the algebraic, allowing more sophisticated representations of operations beyond traditional commutative frameworks

Dimonoids represent a generalization of monoids, incorporating commutative operations a dimonoid is an algebraic structure equipped with two binary operations that satisfy certain associativity conditions, either separately or in combination (Zhuchok, 2011). This dual operation nature provides a richer algebraic structure suited for modeling systems involving multiple types of composition rules, such as computational processes that include both sequencing and parallelism. The introduction of dimonoids is significant for representing real-world phenomena where interactions cannot be modeled adequately with a single binary operation. Their structure enables rigorous formalization of diverse systems (Loday & Ronco, 2004; Loday, 2001).

Analogous to subgroups in group theory, a subdimonoid is a non-empty subset of a dimonoid that is closed under both binary operations. Subdimonoids inherit associativity properties and form dimonoids themselves under the restricted operations. Studying subdimonoids is vital for understanding the internal structure of a dimonoid, facilitating classification and simplification of complex problems (Zhuchok, 2018). A commutative dimonoid, where both operations are commutative, further simplifies algebraic and soft set theory analysis and has applications in fields requiring order-independence, such as parallel processing and decision-making systems (Zhuchok, 2005).

Hyperstructures generalize traditional algebraic structures by allowing an operation between two elements to yield a set of outcomes rather than a single result. A hyperdimonoid combines the concepts of hyperoperation and dimonoid, featuring two hyperoperations satisfying hyper associativity (Corsini & Leoreanu-Fotea, 2003).

The motivation for hyperdimonoids stems from modeling systems with inherent ambiguity or multiple outcomes systems. Hyperdimonoids extend the flexibility of dimonoids and enable the modeling of non-deterministic behaviors more effectively (Davvaz, 2016; Ameri & Zahedi, 1997).

A subhyperdimonoid is a subset of a hyperdimonoid that remains closed under the hyperoperations and preserves hyperassociative properties. Studying subhyperdimonoids uncovers invariant structures and simplifies complex hyperdimonoids by identifying manageable components (Corsini, 1979; Davvaz & Leoreanu-Fotea, 2010).

Homomorphisms between hyperdimonoids play a critical role by preserving hyperoperations during mappings, enabling the classification and construction of new hyperdimonoid structures based on existing ones (Corsini & Leoreanu-Fotea, 2013).

As mathematical research evolved, it became evident that classical sets alone were sometimes inadequate for addressing problems involving uncertainty and incomplete information., and more recently, soft sets. Soft set theory, as mentioned by Molodtsov (1999), provides a flexible model for handling uncertainty without the complexities of parameterization or membership degrees as in fuzzy sets. As different authors explained a soft set is a parameterized family of subsets of a universal set, offering a richer, more adaptable way to represent information (Molodtsov, 1999; Maji, Biswas, & Roy, 2003).

An essential aspect of algebraic structures and soft set theory is the concept of homomorphisms, structure-preserving mappings between two algebraic systems. In soft set theory, soft homomorphisms extend this classical notion, preserving the soft set structure while respecting parameterization and membership. They are crucial in studying the structural properties of soft sets and integrating soft theory with complex structures like dimonoids and hyperdimonoids (Addis, Engidaw; & Davvaz, 2022).

Building soft sets and dimonoids, soft dimonoids introduce a framework for handling uncertainty within dimonoid-structured systems. A soft dimonoid is a soft set over a dimonoid where each parameter's subset is closed under both operations (Aktaş & Çağman, 2007). This construction captures the operational complexity and flexible uncertainty modeling of soft sets. Studying soft dimonoids involves analyzing their structural characteristics, examining soft subdimonoids, and developing soft homomorphisms to understand mappings between different soft dimonoid systems (Weldetekele et al., 2024).

Extending further, our original work is here we defined soft hyperdimonoids combine the capabilities of soft sets, hyperstructures, and dimonoids into a single framework. Our orginal work starting here we defined like A soft hyperdimonoid is a soft set over a hyperdimonoid, where subsets under each parameter are closed under both hyperoperations. This extension supports modeling extremely uncertain,The structural analysis of soft hyperdimonoids and their substructures enables advancements in fields such as artificial intelligence, fuzzy control, cryptography, and information science where managing ambiguity is crucial.

The aim of this study is to systematically explore and develop the theoretical foundations of soft hyperdimonoids by integrating concepts from set theory, dimonoid structures, and hyperstructures within the framework of soft set theory. By establishing key structural properties, substructures, and homomorphisms, this research seeks to provide a comprehensive mathematical basis for modeling uncertainty and multi-operator systems, thereby contributing to the advancement of algebraic modeling techniques applicable to dynamic and complex environments.

This thesis is organized into seven chapters, this thesis explores the intricate relationships between these domains, focusing soft sets, soft groups, dimonoids, subdimonoids, hyperdimonoids, hyperdimonoid homomorphism, subhyperdimonoids, soft dimonoids, soft hyperdimonaoids.

2 PREVIOUS STUDIES

2.1 Soft Sets

In 1999, the concept of soft sets was introduced by Molodtsov for modeling problems that contain vagueness and uncertainty. After Molodtsov, Maji et al. gave the operations of soft sets and their properties. Since then, based on these operations, soft set theory has developed in many directions and found its applications in a wide variety of fields.

2.1.1 Definition Let V be an initial universe set and let E be a set of parameters and $A \sqsubseteq \mathrm{E}$. A pair (F, A), where F is a map from A to $\mathcal{P}(V)$, is called a soft set over V, (Molodtsov, 1999).

In what follows by $\mathrm{SS}(\mathrm{V}, A)$ we denote the family of all soft sets (F, A) over V.

2.1.2 Definition “Let (F, A) be a soft set over V. Then (F, A) is said to be a soft group over G if and only if $F(x) \leq V$ for all $x \in A$,” (Aktaş and Çağman, 2007, p. 2731).

2.1.3 Example “Suppose that $G = A = S_3 = \{e, (12), (13), (23), (123), (132)\}$, and that we define the set-valued function $F(x) = \{y \in G: xRy \Leftrightarrow y = x^n, n \in N\}$. Then the soft group (F, A) is a parameterized family $\{F(x): x \in A\}$ of subsets, which provides us with a collection of subgroups of G. Now consider the particular mapping F as explained above, which is also a subgroup of G. In this instance, the soft group (F, A) can be viewed as the collection of subgroups of G given below:

$$F(e) = \{e\},\ F(12) = \{e, (12)\},\ F(13) = \{e, (13)\},\ F(23) = \{e, (23)\},$$
$$F(123) = F(132) = \{e, (123), (132)\}".$$

(Aktaş and Çağman, 2007, p. 2731).

2.1.4 Definition “Let $(F, A), (G, A) \in \mathrm{SS}(V, A)$. It can be said that the pair (F, A) is a soft subset of (G, A) if $F(p) \subseteq G(v)$, for every $p \in A$. Symbolically, we write $(F, A) \sqsubseteq (G, A)$. Also, we say that the pairs (F, A) and (G, A) are soft equal if $(F, A) \sqsubseteq (G, A)$ and $(G, A) \sqsubseteq (F, A)$. Symbolically, we write $(F, A) = (G, A)$," (Aktaş and Çağman, 2007, p. 2731).

2.1.4 Definition “Let I be an arbitrary index set and $\{(F_i, A): i \in I\} \subseteq \mathrm{SS}(V, A)$. The soft union of these soft sets is the soft set $(F, A) \in \mathrm{SS}(V, A)$, where the map $F: A \to \mathcal{P}(V)$ defined as follows: $F(V) = \cup \{F_i(V): i \in I\}$, for every $x \in A$. Symbolically, we write

$$(F, A) = \sqcup \{(F_i, A): i \in I\}.”$$

(Aktaş and Çağman, 2007, p. 2731).

2.1.6 “Example. Let $V = \mathbb{R}, A = \{0,1\}$, and $I = \{1,2, \dots\}$. For every $i \in I$ we consider the soft set (F_i, A), where the map $F_i: A \to \mathcal{P}(V)$ defined as follows (Aktas, 2007):

$$F_i(\mathrm{v}) = \begin{cases} (0, i), \text{ if } V = 0 \\ (-i, 0), \text{ if } V = 1 \end{cases}$$

Then, $\sqcup \{(F_i, A): i \in I\} = (F, A)$, where the map $F: A \to \mathcal{P}(V)$ defined as follows:

$$F(V) = \begin{cases} (0, +\infty), & \text{if } V = 0 \\ (-\infty, 0), & \text{if } V = 1 \end{cases}$$

2.1.7 Definition Let I be an arbitrary index set and $\{(F_i, A): i \in I\} \subseteq \mathrm{SS}(V, A)$. The soft intersection of these soft sets is the soft set $(F, A) \in \mathrm{SS}(V, A)$, where the map $F: A \to \mathcal{P}(V)$ defined as follows: $F(V) = \cap \{F_i(V): i \in I\}$, for every $V \in A$ (Zorlutuna et al., 2012). Symbolically, we write

$$(F, A) = \sqcap \{(F_i, A): i \in I\}$$

2.1.8 Example Let $V = \mathbb{R}, A = \{0,1,2\}$, and $I = \{1,2, \dots\}$. For every $i \in I$ we consider the soft set (F_i, A), where the map $F_i: A \to \mathcal{P}(V)$ defined as follows, (Zorlutuna et al., 2012).

$$F_i(\mathrm{V}) = \begin{cases} \left(-\frac{1}{i}, \frac{1}{i}\right), \text{ if } V = 0 \\ \left(1 - \frac{1}{i}, 1 + \frac{1}{i}\right), \text{ if } V = 1 \\ \left(2 - \frac{1}{i}, 2 + \frac{1}{i}\right), \text{ if } V = 2 \end{cases}$$

Then, $\sqcap \{(F_i, A): i \in I\} = (F, A)$, where the map $F: A \to \mathcal{P}(V)$ defined as follows:

$$F(\mathrm{V}) = \begin{cases} \{0\}, \text{if } V = 0 \\ \left(1 - \dfrac{1}{i}, 1 + \dfrac{1}{i}\right), \text{if } V = 1 \\ \{2\}, \text{if } V = 2 \end{cases}$$

2.1.9 Definition “Let $(F, A) \in \mathrm{SS}(V, A)$. The soft complement of (F, A) is the soft set $(H, A) \in \mathrm{SS}(V, A)$, where the map $H: A \to \mathcal{P}(V)$ defined as follows: $H(V) = V \setminus F(V)$, for every $x \in A$ Symbolically, we write $(H, A) = (F, A)^c$ (Molodtsov, 1999) ”.

2.1.10 “**Example** Let $V = \mathbb{R}$ and $A = \{1,2, \dots\}$. We consider the soft set (F, A), where the map $F: A \to \mathcal{P}(\mathrm{V})$ defined as follows: $F(V) = [X, +\infty)$, for every $V \in A$. Then, $(F, A)^c = (H, A)$, where the map $H: A \to \mathcal{P}(V)$ defined as follows: $H(V) = (-\infty, \mathrm{V})$, for every $V \in A$,” (Zorlutuna et al., 2012).

2.1.11 “The soft set $(F, A) \in \mathrm{SS}(V, A)$, where $F(x) = \emptyset$, for every $x \in A$ is named the A-null soft set of $\mathrm{SS}(\mathrm{V}, A)$ and denoted by $\mathbf{0}_A$. The soft set $(F, A) \in \mathrm{SS}(V, A)$, where $F(x) = x$, for every x$\in A$ is known as the A-absolute soft set of $SS(V, A)$ and denoted by $\mathbf{1}_A$," (Zorlutuna et al., 2012).

The proofs of the propositions mentioned below are straightforward verifications of the above definitions;

2.1.12 “**proposition.** Let $(F, A) \in \mathrm{SS}(V, A)$. The following statements are true:

(1) $(F, A) \sqcap (F, A) = (F, A)$.
(2) $(F, A) \sqcup (F, A) = (F, A)$.
(3) $(F, A) \sqcap \mathbf{0}_A = \mathbf{0}_A$.
(4) $(F, A) \sqcup \mathbf{0}_A = (F, A)$.
(5) $(F, A) \sqcap \mathbf{1}_A = (F, A)$.
(6) $(F, A) \sqcup \mathbf{1}_A = \mathbf{1}_A$.
(7) $(F, A) \sqcap (F, A)^c = \mathbf{0}_A$.
(8) $(F, A) \sqcup (F, A)^c = \mathbf{1}_A$.
(9) $(\mathbf{0}_A)^c = \mathbf{1}_A$.
(10) $(\mathbf{1}_A)^c = \mathbf{0}_A$.
(11) $((F, A)^c)^c = (F, A)$.
(12) $\mathbf{0}_A \sqsubseteq (F, A) \sqsubseteq \mathbf{1}_A$,”

(Zorlutuna et al., 2012).

2.1.13 “**Proposition.** Let $(F, A), (G, A), (H, A) \in \mathrm{SS}(\mathrm{V}, A)$. The following statements are true:

(1) $(F, A) \sqcap ((G, A) \sqcap (H, A)) = ((F, A) \sqcap (G, A)) \sqcap (H, A)$.
(2) $(F, A) \sqcup ((G, A) \sqcup (H, A)) = ((F, A) \sqcup (G, A)) \sqcup (H, A)$.
(3) $(F, A) \sqcap ((G, A) \sqcup (H, A)) = ((F, A) \sqcap (G, A)) \sqcup ((F, A) \sqcap (H, A))$.
(4) $(F, A) \sqcup \big((G, A) \sqcap (H, A)\big) = \big((F, A) \sqcup (G, A)\big) \sqcap \big((F, A) \sqcup (H, A)\big)$,”

(Zorlutuna et al., 2012).

2.1.14 “**Proposition.** Let I be an arbitrary set and $\{(F_i, A): i \in I\} \subseteq \mathrm{SS}(V, A)$. The following statements are true:

(1) $(F_i, A) \sqsubseteq \sqcup \{(F_i, A): i \in I\}$, for every $i \in I$.
(2) $\sqcap \{(F_i, A): i \in I\} \sqsubseteq (F_i, A)$, for every $i \in I$.
(3) $(\sqcup \{(F_i, A): i \in I\})^c = \sqcap \{(F_i, A)^c: i \in I\}$.
(4) $(\sqcap \{(F_i, A): i \in I\})^c = \sqcup \{(F_i, A)^c: i \in I\}$,”

(Zorlutuna et al., 2012).

2.1.15 “**Definition** Let $(F, A), (G, A) \in \mathrm{SS}(V, A)$. The soft symmetric difference of these soft sets is the soft set $(H, A) \in \mathrm{SS}(V, A)$, where the map $H: A \to \mathcal{P}(V)$ defined as follows: $H(\mathrm{x}) = (F(x) \setminus G(\mathrm{x})) \cup (G(\mathrm{x}) \setminus F(x))$, for every $\mathrm{x} \in A$. Symbolically, we write $(H, A) = (F, A) \triangle (G, A)$, (Georgiou and Megaritis, 2014).

2.1.16 “**Example** Let $V = \{1,2,3,4,5\}$ and $A = \{0,1,2, \ldots\}$. We consider the soft sets (F, A) and (G, A), where the maps $F: A \to \mathcal{P}(V)$ and $G: A \to \mathcal{P}(V)$ defined as follows:

$$F(x) = \begin{cases} \{1,2,3,4\}, \text{ if } x = 0, \\ \emptyset, \text{ otherwise }, \end{cases} \quad G(x) = \begin{cases} \{1,4,5\}, \text{ if } x = 0 \\ \emptyset, \text{ otherwise} \end{cases}$$

Then, $(F, A) \triangle (G, A) = (H, A)$, where the map $H: A \to \mathcal{P}(V)$ defined as follows:

$$H(x) = \begin{cases} \{2,3,5\}, \text{ if } x = 0 \\ \emptyset, \text{ otherwise} \end{cases}$$

The proof of the following proposition is straightforward verification of Definition 2.13," (Georgiou and Megaritis, 2014).

2.1.17 "**Proposition.** Let $(F,A),(G,A),(H,A) \in \mathrm{SS}(V,A)$. The following statements are true;

(1) $(F,A) \triangle ((G,A) \triangle (H,A)) = ((F,A) \triangle (G,A)) \triangle (H,A)$.
(2) $(F,A) \triangle (G,A) = (G,A) \triangle (F,A)$.
(3) $(F,A) \triangle \mathbf{0}_A = (F,A)$.
(4) $(F,A) \triangle (F,A) = \mathbf{0}_A$.
(5) $(F,A) \sqcap ((G,A) \triangle (H,A)) = ((F,A) \sqcap (G,A)) \triangle ((F,A) \sqcap (H,A))$,"

(Georgiou and Megaritis, 2014).

2.1.18 "**Remark.** By Proposition 2.15 follows that the pair ($SS(V,A)$,$\triangle$) is a group of soft sets. The identity element is the soft set 0_A and the inverse of the element (F,A) is the soft set (F,A). Also, the triad($\mathrm{SS}(V,A)$, $\triangle$, $\sqcap$) is a ring of soft sets, (Georgiou and Megaritis, 2014). "

2.2 Soft Groups

In this section, which is based on the work of Aktaş and Çağman (2007), G is a group and A is any nonempty set. R would be referred to an arbitrary binary relation between an element of A and an element of G. A set-valued function $F: A \to P(G)$ can be introduced as $F(x) = \{y \in G: (x,y) \in R, x \in A \text{ and } y \in G\}$. The pair (F,A) is then a soft set over G. Introducing a set-valued function from A to G also can be introduced as a binary relation R on $A \times G$, given by $R = \{(x,y) \in A \times G: y \in F(x)\}$. In the meantime, the trio (A,G,R) is referred to as an approximation set.

2.2.1 Definition Let (F,A) be a soft set over G. Then (F,A) would be mentioned as a soft group over G if and only if $F(x) \leq G$ for all $x \in A$, (Aktaş and Çağman, 2007, p. 2731).

2.2.2 "**Example** Suppose that $G = A = S_3 = \{e, (12), (13), (23), (123), (132)\}$, and that we define the set-valued function $F(x) = \{y \in G: xRy \Leftrightarrow y = x^n, n \in N\}$. Then the soft group (F,A) is a parameterized family $\{F(x): x \in A\}$ of subsets, which provides a collection of subgroups of G. Then we can consider the particular mapping F defined above, which is also a subgroup of G. In this instance, the soft group (F, A) can be viewed as the collection of subgroups of G as per below:

$$F(e) = \{e\},\ F(12) = \{e, (12)\},\ F(13) = \{e, (13)\},\ F(23) = \{e, (23)\},$$
$$F(123) = F(132) = \{e, (123), (132)\}, "$$

(Aktaş and Çağman, 2007).

2.2.3 Definition Let (F, A) be a soft group over G. Then,

(i) "(F, A) is said to be an identity soft group over G if $F(x) = \{e\}$ for all $x \in A$, where e is the identity element of G; and

(ii) (F, A) is said to be an absolute soft group over G if $F(x) = G$ for all $x \in A$,"

(Aktaş and Çağman, 2007).

2.2.4 "**Theorem** (1) Let (F, A) be a soft group over G and f be a homomorphism from G to K. If $F(x) = \text{Ker}f$ for all $x \in A$, then $(f(A), F)$ is the identity group over K,

(2) Let (F, A) be an absolute soft group over G, and let f be a homomorphism from G to K. Then $(f(A), F)$ is an absolute soft group over K.

Proof.

(1) For $x \in A, f(F(x)) = e_K$ where e_K is an identity element of K. From Definition 2.2.3 $(f(F), A)$ is an identity soft group over K.

(2) $F(x) = G$ for all $x \in A$, since (F, A) is an absolute soft group over G. This will be followed by $f(F(x)) = f(G) = K$ for all $x \in A$. Therefore, $(f(F), A)$ is an absolute soft group over K from Definition 2.2.3",

(Aktaş and Çağman, 2007, p. 2732).

2.2.5 Definition Let (F, A) and (H, K) be two soft groups over G. Then (H, K) is a soft subgroup of (F, A), written $(H, K) \leq (F, A)$, if,

(i) $K \subset A$,

(ii) $H(x) \leq F(x)$ for all $x \in K$,

(Aktaş and Çağman, 2007).

2.2.6 “**Example** Let $G = S_3, A = S_3$, and $K = A_3$. If we define the functions $F(x) = \{y \in S_3 : xRy \Leftrightarrow y = x^n, n \in N\}$ and $H(x) = \{y \in A_3 : xRy \Leftrightarrow y \in (x)\}$, then $(H, K) \leq (F, A)$ since $A_3 \leq S_3$ and $H(x) \leq F(x)$ for all $x \in A_3$ (Aktaş and Çağman, 2007). ”

Using the definition of a soft subgroup, we can list some properties of soft subgroups which are identical to properties of classical subgroups. The proofs of these properties are all straightforward.

2.2.7 “**Theorem** Let(F, A) and (H, K) be two soft groups over G,

(i) If $F(x) \subseteq H(x)$ for all $x \in A$, then (F, A) is a soft subgroup of (H, A).
(ii) If $E = \{e\}$ and $(U, E), (F, G)$ are both soft groups over G, then (U, E) is a soft subgroup of (F, G).

(Aktaş and Çağman, 2007).

2.2.8 “**Theorem** (F, A) is a soft group over G, and $\{(H_i, K_i) : i \in I\}$ is a nonempty family of soft subgroups of (F, A) where I is an index set. Then,

(i) $\bigcap_{i \in I} (H_i, K_i)$ is a soft subgroup of (F, A),
(ii) $\bigwedge_{i \in I} (H_i, K_i)$ is a soft subgroup of (F, A), and
(iii)if $K_i \cap K_j = \emptyset$ for all $i, j \in I$, then $\bigvee_{i \in I} (H_i, K_i)$ is a soft subgroup of (F, A),”

(Aktaş and Çağman, 2007).

2.2.9 “**Definition** Let (F, A) and (H, B) be two soft groups over G and B respectively and let $f: G \to B$ and $g: A \to B$ be two functions. Then it can be said that (f, g) is a soft homomorphism, and that (F, A) is soft homomorphic to (H, B). The latter is mentioned as $(F, A) \sim (H, B)$, if the below conditions are satisfied:

(i) f is a homomorphism from G onto B,

(ii) g is a mapping from A onto B, and

(iii)$f(F(x)) = H(g(x))$ for all $x \in A$,

In this definition, if f is an isomorphism from G to B and g is a one-to-one mapping from A onto B, then we say that (f, g) is a soft isomorphism and that (F, A) is soft isomorphic to (H, B). The latter is denoted by $(F, A) \simeq (H, B)$," (Aktaş and Çağman, 2007).

2.2.10 "**Example** Consider the groups $(Z, +)$ and $(Z_m,\oplus)$,

We define a homomorphism from Z onto Z_m such as $f(k) = \bar{k}$ for $k \in Z$, and a mapping g from Z^+onto Z_m such as $g(k) = \bar{k}$ for $k \in Z^+$. Let $F: Z^+ \to P(Z)$ and $F(x) = \{y \in Z: y = 5kx, k \in Z\}$; let $H: Z_m \to P(Z_m)$ and $H(u) = \{\bar{y} \in Z_m: y = uk, k \in 5Z\}$.

Then we obtain $F(x) = 5xZ$ and $H(u) = \{\overline{ku}: k \in 5Z\}$. It is clear that $(F, Z^+)$and (H, Z_m) are soft groups over Z and Z_m, respectively.

Since $f(F(x)) = \{\overline{5xk}: k \in Z\}$ and $H(g(x)) = \{\overline{xs}: s \in 5Z\}$, we get $f(F(x)) = H(g(x))$. Hence (f, g) is a soft homomorphism, and $(F, Z^+)$is soft homomorphic to (H, Z_m), (Aktaş and Çağman, 2007).

2.2.12 Definition A soft homomorphism (f, g) from *(F, A)* to (H, B) is said to be;

(1) a soft monomorphism if it is injective;

(2) a soft epimorphism if it is surjective;

(3) a soft isomorphism if it is bijective, (Aktaş and Çağman, 2007).

2.2.13 Theorem Let (F, A) and (H, B) be two soft groups over G, and (F, A) be a soft subgroup of (H, B). If f is a homomorphism from G to B, then $(f(F), A)$ and $(f(H)$B) are both soft subgroups over B and $(f(F), A)$ is a soft subgroup of $(f(H), B)$, Proof. S “ince f is a homomorphism from G to $B, f(F(x))$ and $f(H(y))$ are subgroups of B for all $x \in A$ and for all $y \in$ B. $(f(F), A)$ and $(f(H), B)$ are therefore soft groups over B.

If (F, A) is a soft subgroup of (H, B), then $F(x)$ is a subgroup of $H(x)$ and $f(F(x))$ is a subgroup of $f(H(x))$ for all $x \in A$. From Definition 20, we obtain $(f(F), A) \lesssim (f(H), B)$, (Aktaş and Çağman, 2007).

2.2.16 “**Theorem** A soft homomorphism (f, g) from G to G$'$ is a soft isomorphism if and only if f is a group isomorphism.

Proof. Suppose that (f, g) is soft isomorphism. We first show that $\breve{f}$ is injective. Let $\breve{a}, \breve{b} \in SE_{Ag}(G)$ such that $\breve{f}(\breve{a}) = \breve{f}(\breve{b})$. Then $\breve{f}(\breve{a})(\lambda) = \breve{f}(\breve{b})(\lambda)$ for all $\lambda \in g$. That is, for each $\lambda \in g$and any $x \in G'$ we have:

$$(\lambda, \breve{a}(\lambda), x) \in f \text{ if and only if } (\lambda, \breve{b}(\lambda), x) \in f$$

Since f is injective, it must be true that $\breve{a}(\lambda) = \breve{b}(\lambda)$ for all $\lambda \in A$. Thus, $\breve{a} = \breve{b}$ and hence $\breve{f}$ is injective. Next, we show that f is surjective. Let $\breve{y}$ be any soft element in G'. Since (f, g) is soft isomorphism, for ech $\alpha \in A$ there is a unique element let say $x_\alpha \in G$ such that $\langle \alpha, x_\alpha, \breve{y}(\alpha) \rangle \in f$ for all $\alpha \in g$. Noe define a sofet element $\bar{x}$ over G by $\breve{x}(\alpha) = \{x_\alpha\}$ for all $\alpha \in g$. Then we must have $f(\breve{x}) = \breve{y}$. Thus, f is surjective and hence an isomorphism. The converse can be proved using similar procedure,” (Weldetekle et al., 2024).

2.2.17 “**Definition** Let (f, g) be a soft homomorphism from $(G, *, A)$ to $\langle G', \Delta, A \rangle$. (a) If (H, A) is a soft subgroup of G, then the soft set $(f(H), A)$ over G' defined as:

$$f(H)(\alpha) = \{y \in G' : (\alpha, x, y) \in f \text{ for some}$$

$$x \in H(\alpha)\}$$

is the image of (H, A) under f.

(b) If $\langle H', A \rangle$ is a soft subgroup of G', then the soft set $\langle f^{-1}(H'), A \rangle$ over G defined as:

$$f^{-1}(H')(\alpha) = \{x \in G: (\alpha, x, y) \in \text{ f for some } y \in H'(\alpha)\}$$

is the inverse image of $\langle H', A\rangle$ under f," (Weldetekle et al., 2024).

2.2.18 "**Definition** Let $(G,*,A)$ and $\langle G', \Delta, A\rangle$ be soft groups. A soft mapping (f,g) from G to G' is named a soft homomorphism, in case for each $\alpha \in Aa, b, c \in G$ and $x, y, z \in G', < \alpha, a, x > \in f, < \alpha, b, y\rangle \in f, < \alpha, c, z > \in f$ and $< \alpha, a, b, c) \in *$ suggests $< \alpha, x, y, z > \in \Delta$ (Weldetekle et al., 2024) ".

2.2.19 **Example** Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+$the set of positive real numbers and $\mathbb{N}$ the set of natural numbers. Define soft binary operations $<*, \mathbb{N} >$ and $< \Delta, \mathbb{N} >$ on $\mathbb{R}$ and $\mathbb{R}^+$respectively as follows:

$$(\alpha, a, b, c) \in * \Leftrightarrow c = \alpha + a + b \text{ and}$$
$$(\alpha, a, b, c) \in \Delta \Leftrightarrow c = (\alpha + 1)^{\alpha} ab.$$

$< \mathbb{R},*, \mathbb{N} >$ and $< \mathbb{R}^+, \Delta, \mathbb{N} >$ are soft groups. Moreover, define $f = \{(\alpha, x, y): y = (\alpha + 1)^x\}$. Then $< f, \mathbb{N} >$ is a soft homomorphism from $\mathbb{R}$ to $\mathbb{R}^+$, (Weldetekle et al., 2024).

2.2.20 "**Proposition** Let (f,g) be a soft homomorphism from $(G,*,A)$ to $\langle G',\triangle, A\rangle$. Then

(1) $\langle \alpha, e_\alpha, e'_\alpha\rangle \in f$, for all $\alpha \in A$ where e_α and e'_α are identity elements of G and G′ respectively.

(2) $(\alpha, a, y) \in f \Rightarrow \langle \alpha, a^{-\alpha}, y^{-\alpha}\rangle \in F$, for all $\alpha \in g, a \in G$ and $y \in G'$.

Proof.

(1) Let $x \in G'$ Such that $\langle \alpha, e_\alpha, x\rangle \in F$. Since $\langle \alpha, x, e'_\alpha, x\rangle \in \Delta$ and $(\alpha, x, x, x) \in \Delta$, by cancellation law it holds that $x = e'_\alpha$. Therefore $\langle \alpha, e_\alpha, e'_\alpha\rangle \in f$.

(2) Let $\alpha \in A, a \in G$ and $x, y \in G'$ Such that $(\alpha, a, y) \in F$ and $\langle \alpha, a^{-\alpha}, x\rangle \in f$. Since (f,g) is a soft homomorphism, it follows from (1) and the condition $\langle \alpha, a, a^{-\alpha}, e_\alpha\rangle \in *$ that $\langle \alpha, y, x, e'_\alpha\rangle \in \Delta$. Again, from the fact $\langle \alpha, y, y^{-\alpha}, e'_\alpha\rangle \in \Delta$ and cancellation law we get $x = y^{-\alpha}$. Therefore $\langle \alpha, a^{-\alpha}, y^{-\alpha}\rangle \in f$.

Drawing on the concept of soft mapping compositions as presented by (Addis et al., 2022), the subsequent proposition demonstrates that the combination of two soft homomorphisms results in another soft homomorphism," (Weldetekle et al., 2024).

2.2.21 "**Lemma** Let (f,g) be a soft homomorphism from $(G,*,A)$ to $\langle G',\Delta,A\rangle$. If (H,A) is a soft subgroup of G then $(f(H),g)$ is a soft subgroup of G.

Proof. Let $\alpha \in A$. Since $e_\alpha \in H(\alpha)$ and $\langle \alpha, e_\alpha, e'_\alpha\rangle \in f$, we have $e'_\alpha \in f(H)(\alpha)$.

Let $x,y \in f(H)(\alpha)$ and $z \in G'$ such that $< \alpha, x, y^{-\alpha}, z > \in \Delta$. We need to prove that $z \in f(H)(\alpha)$. As $x,y \in f(H)(\alpha)$ there exists $a,b \in H(\alpha)$ such that $(\alpha,a,x) \in f$ and $(\alpha,b,y) \in f$. Since $(\alpha,b,y) \in f$ we have $\langle \alpha, b^{-\alpha}, y^{-\alpha}\rangle \in f$. Let $c \in G$ such that $\langle \alpha, a, b^{-\alpha}, c\rangle \in *$. Since (f,g) is a soft mapping there exist $w \in G'$ such that $(\alpha,c,w) \in f$. As (f,g) is a soft homomorphism it follows that $w = z$. Since (H,A) is a soft subgroup of G, we have $c \in H(\alpha)$. Therefore $z \in f(H)(\alpha)$, (Weldetekle et al., 2024).

2.2.22 "**Lemma** Let (f,g) be a soft homomorphism from $(G,*,A)$ to $\langle G',\Delta,A\rangle$. If $\langle H',A\rangle$ is a soft subgroup of G' then $\langle f^{-1}(H'),A\rangle$ is a soft subgroup of G.

Proof. Let $\alpha \in A$. Since $e'_\alpha \in H'(\alpha)$ and $\langle \alpha, e_\alpha, e'_\alpha\rangle \in f$, we get $e_\alpha \in f^{-1}(H')(\alpha)$.

Let $a,b \in f^{-1}(H')(\alpha)$ and $c \in G$ such that $\langle \alpha, a, b^{-\alpha}, c\rangle \in *$. We need to prove that $c \in f^{-1}(H')(\alpha)$. As $a,b \in f^{-1}(H')(\alpha)$ there exist $x,y \in H'(\alpha)$ such that $(\alpha,a,x) \in f$ and $(\alpha,b,y) \in f$. As $(\alpha,b,y) \in f$ it holds that $\langle \alpha, b^{-\alpha}, y^{-\alpha}\rangle \in f$. Let $c \in G$ such that $\langle \alpha, a, b^{-\alpha}, c\rangle \in *$. Since (f,g) is a soft mapping there exist $z \in G'$ such that $(\alpha,c,z) \in f$. Since (f,A) is a soft homomorphism from G to G', it holds that $\langle \alpha, x, y^{-\alpha}, z\rangle \in \Delta$. Since $\langle H',A\rangle$ is a soft subgroup of G' we get $z \in H'(\alpha)$. Therefore $c \in f^{-1}(H')(\alpha)$. Hence $\langle f^{-1}(H'),A\rangle$ is a soft subgroup of G," (Weldetekle et al., 2024).

2.2.23 "**Theorem** Let (f,g) be a soft homomorphism from $(G,*,A)$ to $\langle G',\Delta,A\rangle$. (1) If (f,g) is surjective and (N,A) is a normal soft subgroup of G then $(f(N),A)$ is a normal soft subgroup of G'.

(2) If $\langle N', A\rangle$ is a normal soft subgroup of G$'$ then $\langle f^{-1}(N'), A\rangle$ is a normal soft subgroup of G.

Proof.

(1) By Lemma 1.3.13 $(f(N), A)$ is a soft subgroup of G'. Let $a \in f(N)(\alpha)$. Then $(\alpha, x, a) \in f$ for some $x \in N(\alpha)$. Let $b \in G'$. Since (f, g) is surjective, there exist $y \in G$ such that $(\alpha, y, b) \in f$. Let $z \in G$ such that $(\alpha, y, x, z) \in *$. There exists $c \in G'$ such that $(\alpha, z, c) \in f$ because (f, g) is a soft homomorphism. Moreover, $(\alpha, b, a, c) \in \Delta$. Again let $d \in G$ such that $\langle \alpha, z, y^{-\alpha}, d\rangle \in *$. Since (f, A) is a soft mapping, there exists $d' \in G'$ such that $\langle \alpha, d, d'\rangle \in f$. Since (f, g) is a soft homomorphism, $\langle \alpha, c, b^{-\alpha}, d'\rangle \in \Delta$. As (N, A) is a normal soft subgroup of $G, d \in N(\alpha)$. This implies that $d' \in f(N)(\alpha)$. Therefore $(f(N), A)$ is a normal soft subgroup of G'

(2) By Lemma 1.3.14 $\langle f^{-1}(N'), A\rangle$ is a soft subgroup of G. Let $a \in f^{-1}(N')(\alpha)$. Then $(\alpha, a, x) \in f$ for some $x \in N'(\alpha)$. Let $b, c \in G$ and $x_1, z_1 \in G'$ such that $\langle \alpha, b, x_1\rangle \in f$ and $\langle \alpha, c, z_1\rangle \in f$. Since (f, g) is a soft homomorphism from G to G' and $(\alpha, b, a, c) \in *$. Then we have $\langle \alpha, x_1, x, z_1\rangle \in \Delta$. Let $d \in G$ and $z_2 \in G$ such that $\langle \alpha, d, z_2\rangle \in f$. Since (f, g) is a soft homomorphism from G to G' and $\langle \alpha, c, b^{-\alpha}, d\rangle \in *, \langle \alpha, z_1, x_1^{-\alpha}, z_2\rangle \in \Delta$. It follows that $d \in f^{-1}(N')(\alpha)$. Therefore $\langle f^{-1}(N'), A\rangle$ is a normal soft subgroup of G'," (Weldetekle et al., 2024).

2.2.24 "**Definition** Let (f, g) be a soft homomorphism from G to G'. The kernel of (f, g) is the soft set $\langle K_f, A\rangle$ over G defined as:

$$K_f(\alpha) = \{x \in G : \langle \alpha, x, e'_\alpha\rangle \in f \text{ for each } \alpha \in A\}$$

where e'_α is an identity element of G'" (Weldetekle et al., 2024).

2.2.25 Example Let $(G, *, A)$ and $\langle G', \triangle, A\rangle$ be soft groups

1. Let (f, g) be a soft mapping from G to G$'$ defined by:

$$f = \{(\alpha, x, e'_\alpha) : \alpha \in A, x \in G\}$$

then (f, g) is a soft homomorphism and $\langle K_f, A\rangle$ is the absolute soft set over G.
2. Let $(f, g >$ be a soft mapping from G to G defined by:

$$f = \{(\alpha, x, x) \colon \alpha \in Ax, \in G\}$$

then (f, g) is a soft homomorphism and $\langle K_f, A\rangle$ is the trivial soft subgroup of G. (Weldetekle et al., 2024).

2.2.26 "**Lemma** For any soft homomorphism (f, g) from $(G,*, A)$ to $\langle G', \Delta, A\rangle$, the kernel $\langle K_f, A\rangle$ is a normal soft subgroup of G

Proof. For each $\alpha \in A$, we have

$K_f(\alpha) = \{x \in G \colon \langle \alpha, x, e_\alpha\rangle \in f\} = f^{-1}(\breve{e}')(\alpha)$ "

Therefore, the proof follows directly from Theorem before, (Weldetekle et al., 2024).

2.2.27 Lemma A soft homomorphism (f, g) from $(G,*, A)$ to $\langle G', \Delta, A\rangle$ is a soft monomorphism if and only if $\langle K_f, A\rangle$ is the trivial soft subgroup of G

Proof. Suppose (f, g) is a soft monomorphism. We need to show that $K_f(\alpha) = \{e_\alpha\}$ for all $\alpha \in A$. Let $x \in K_f(\alpha)$. This implies that $\langle \alpha, x, e'_\alpha\rangle \in f$, for all $\alpha \in A$. Since $\langle \alpha, e_\alpha, e'_\alpha\rangle$ inf, for all $\alpha \in A$ we have $x = e_\alpha$. Therefore $K_f(\alpha) = \{e_\alpha\}$. Conversely, suppose that $\langle \alpha, x_1, y\rangle \in f$ and $\langle \alpha, x_2, y\rangle \in f$. Let $z \in G$ such that $\langle \alpha, x_1, x_2^{-\alpha}, z\rangle \in *$. Then $\langle \alpha, z, e'_\alpha\rangle \in f$. It follows that $z \in K_f(\alpha)$. So $z = e_\alpha$. Therefore $\langle \alpha, x_1, x_2^{-\alpha}, e_\alpha\rangle \in *$. This implies that $\langle \alpha, e_\alpha, x_2, x_1\rangle \in *$. Using the fact $\langle \alpha, e_\alpha, x_1, x_1\rangle \in *$ and the cancellation law we get $x_1 = x_2$, (Weldetekle et al., 2024).

2.2.28 "**Proposition** Let (f, g) and (g, m) be soft homomorphisms from $(G,*, A)$ to $\langle G', \Delta, A\rangle$. Define a soft set (H, A) over G by:

$$H(\alpha) = \{a \in G \colon (\alpha, a, x) \in f \Leftrightarrow (\alpha, a, x) \in m \\ \text{for any } x \in G'\}$$

for all $\alpha \in A$. Then (H, A) is a soft subgroup of G.

Proof. Since $\langle \alpha, e_\alpha, e'_\alpha\rangle \in f$ and $\langle \alpha, e_\alpha, e'_\alpha\rangle \in m$, we have $e_\alpha \in H(\alpha)$. Let $a, b \in H(\alpha)$ and $x \in G$ such that $\langle \alpha, a, b^{-\alpha}, x\rangle \in *$. Since (f, g) and (m, A) are soft mapping there exist $y_1, y_2 \in G'$ such that

$$\langle \alpha, a, y_1 > \in f \Leftrightarrow < \alpha, a, y_1 \rangle \in m$$

and

$$\langle \alpha, b, y_2 \rangle \in f \Leftrightarrow \langle \alpha, b, y_2 \rangle \in m$$

This implies that $\langle \alpha, b^{-\alpha}, y_2^{-\alpha} \rangle \in f \Leftrightarrow \langle \alpha, b^{-\alpha}, y_2^{-\alpha} \rangle \in g$. Again (f, A) is a soft mapping then there exist $y_3 \in G'$ such that $\langle \alpha, x, y_3 \rangle \in f$. Since (f, g) is a soft homomorphism then we get that $\langle \alpha, y_1, y_2^{-\alpha}, y_3 \rangle \in \Delta$. Since (m, A) is a soft homomorphism and $\langle \alpha, x, y_3 \rangle \in g$, we have $x \in H(\alpha)$. Hence (H, A) is a soft subgroup of G.

Recall from Weldetekle et al. (2024) that for a soft set (H, A) over G, we define the subset $\widehat{H}$ of $SE_A(G)$ by:

$$\widehat{H} = \{\breve{a} \in SE_A(G) \colon \breve{a}(\alpha) \subseteq H(\alpha) \text{ for all } \alpha \in A\}$$

In the following theorem, we establish a relation between the kernel of f and the soft kernel of (f, g)," (Weldetekle et al., 2024).

2.2.29 "**Theorem** For any soft homomorphism (f, g) from G to G′ we have

$$\ker(\breve{f}) = \widehat{K_f}$$

Proof. We know that

$$\widehat{K_f} = \left\{\breve{a} \in SE_A(G) \colon \breve{a}(\alpha) \subseteq K_f(\alpha) \text{ for all } \alpha \in g\right\}$$

and

$$\ker(\breve{f}) = \left\{\breve{a} \in SE_A(G) \colon \breve{f}(\breve{a})(\lambda) = \{e_\alpha\} \text{ for all } \lambda \in g\right\}.$$

Now, we have $\breve{a} \in \ker(\breve{f}) \Leftrightarrow \breve{f}(\breve{a})(\lambda) = \{e_\lambda\}$

$$\begin{aligned} &\Leftrightarrow \langle \alpha, \breve{a}(\lambda), e_\lambda \rangle \in f \\ &\Leftrightarrow \breve{a}(\lambda) \in K_f(\lambda) \text{ for all } \lambda \in g \\ &\Leftrightarrow \breve{a} \in \widehat{K_f}. \end{aligned}$$

Therefore $\ker(f) = \widehat{K_f}$," (Weldetekle et al., 2024).

2.3 Dimonoids

Jean-Louis Loday was a pioneer in conceptualizing a dimonoid and explaining the structure of a free dimonoid, which is deeply rooted in the theory of dialgebras (Loday et al., 2001). Cayley's theorem posits that each dimonoid is isomorphic to a transformation dimonoid, (Zhuchok, 2011). Moreover, there is a profound relationship between dimonoids, trialgebras, and trioids, the latter two having been introduced in the realm of algebraic topology by (Loday and Ronco, 2004).

In this section, the motivation behind Loday's definition of dimonoids is presented. Loday's objective is to introduce and explore a novel algebraic concept that yields a Leibniz algebra. The central idea here is to start with two distinct operations for the products ab and ba, so that the bracket is no longer skew-symmetric

Below is the definition of a dimonoid:

2.3.1 Definition Let $(D, *, \circ)$ be an algebraic structure. In this case, D is called a dimonoid if and only if for all $h, i, j \in D$, the following conditions hold:

I. $(h * i) * j = h * (i \circ j)$
II. $(h \circ i) * j = h \circ (i * j)$
III. $(h * i) \circ j = h \circ (i \circ j)$
IV. $(h \circ i) * j = h * (i * j)$

$(h \circ i) \circ j = h \circ (i \circ j)$ (Zhuchok, 2018)

2.3.2 Lemma Let $(D,*,\circ)$ be a dimonoid endowed with a commutative operation $*$. For all $b, c \in D$ and all $m \in N, m > 1$, we have:

$(b * c)^m = b^m \circ c^m = (b \circ c)^m$, (Loday et al., 2001).

2.3.3“**Lemma** Let $(D,*,\circ)$ be a dimonoid. For all $x, y, t \in D$ and all $n \in N$, the following equalities hold

(i) $(x * y)^n \circ t = n(x \circ y) \circ t = n(x * y) \circ t$;
(ii) $t * n(x \circ y) = t * (x * y)^n = t * (x \circ y)^n$.

Proof. (i) We use induction on n. For $n = 1$, we have $(x * y) \circ t = (x \circ y) \circ t$ in accordance with (D4) and (D5). Let $(x * y)^k \circ t = k(x \circ y) \circ t$ with $n = k$. For $n = k + 1$, we then obtain

$$\begin{aligned}(x * y)^{k+1} \circ t &= ((x * y) * (x * y)^k) \circ t \\ &= ((x * y) \circ (x * y)^k) \circ t \\ &= ((x \circ y) \circ (x * y)^k) \circ t \\ &= (x \circ y) \circ ((x * y)^k \circ t) \\ &= (x \circ y) \circ k(x \circ y) \circ t \\ &= (k + 1)(x \circ y) \circ t\end{aligned}$$

in view of axioms (D4) and (D5) and our hypothesis. Thus $(x * y)^n \circ t = n(x \circ y) \circ t$ for all $n \in N$.

We claim that $(x * y)^n \circ t = n(x * y) \circ t$ for all $x, y, t \in D$ and all $n \in N$. If $n = 1$ then the equality is obviously true. Let $(x * y)^k \circ t = k(x * y) \circ t$ with $n = k$. For $n = k + 1$, we then derive

$$\begin{aligned}(x * y)^{k+1} \circ t &= ((x * y) * (x * y)^k) \circ t \\ &= ((x * y) \circ (x * y)^k) \circ t\end{aligned}$$

$$\begin{aligned}&= (x * y) \circ ((x * y)^k \circ t) \\ &= (x * y) \circ k(x * y) \circ t \\ &= (k + 1)(x * y) \circ t\end{aligned}$$

by virtue of axioms $(D4)$ and $(D5)$ and our hypothesis. Thus $(x * y)^n \circ t = n(x * y) \circ t$ for all $n \in N$.

Equalities (ii) can be proved similarly, (Loday et al., 2001).

Let S be a semigroup, with $a \in S$. Elements $x, y \in S$ are said to be a-connected if there exist $n, m \in N$ such that $(xa)^n \in yaS$ and $(ya)^m \in xaS$. a semigroup S is a-connected if every two elements of S are a-connected.

Note that if $(xa)^n \in yaS$ and $(ya)^m \in xaS$, then $(xa)^p \in yaS$ and $(ya)^p \in xaS$, where $p = \max\{n, m\}$, with $n, m, p \in N$."

2.3.4 Example Let (D, *) be a semigroup. Then (D, *, *) is a dimonoid. and let $D = \{a, b\}$, and let the binary operation * and ∘ be defined using the tables below, then (D, *, ∘) is a dimonoid:

Table 2.1 Dimonoid D with binary operations

$*$	a	b
a	a	a
b	a	a

$\circ$	a	b
a	a	b
b	a	b

(Zhuchok, 2018).

2.3.5 Definition A dimonoid (D, $*$, ∘) is called an idempotent dimonoid, or diband, if and only if:

$$x * x = x \text{ and } x \circ x = x, \forall x \in D.$$

In the following example, all idempotent dimonoids on the set $D = \{a, b\}$ are presented. (Zhuchok, 2018)

2.3.6 "**Example** On the set $D_1 = \{a, b\}$, the binary operation "*", defined using the table below, makes (D_1, $*$, $*$) a semigroup and an idempotent dimonoid: " (Zhuchok, 2018)

Table 2.2 semigroup and an idempotent dimonoid

$*$	a	b
a	a	b
b	a	b

2.3.7 "**Example** On the set $D_2 = \{a, b\}$, the binary operation "*", defined using the table below, makes (D_2, $*$, $*$) a semigroup and an idempotent dimonoid:

Table 2.3 Another semigroup and an idempotent dimonoid (Zhuchok, 2018).

$*$	a	b
a	a	a
b	b	b

2.3.8 “**Example** On the set $D_3 = \{a, b\}$, the binary operations “*” and “∘” defined using the tables below, make $(D_3, *, \circ)$ an idempotent dimonoid:

Table 2.4 idempotent dimonoid

$*$	a	b
a	a	a
b	b	b

$\circ$	a	b
a	a	b
b	a	b

(Zhuchok, 2018).

2.3.9 Theorem The relation η on a dimonoid $(D,*,\circ)$ endowed with a commutative operation $*$ is the least idempotent congruence and $(D,*,\circ)/\eta$ is a commutative idempotent dimonoid, which is a semilattice, (Zhuchok, 2018).

2.3.10 Definition Let D and E be dimonoids. A mapping $f{:}D \to E$ is called a dimonoid homomorphism if and only if

$$f(a * b) = f(a) * f(b) \text{ and } f(a \circ b) = f(a) \circ f(b)$$

for all $a, b \in D$, (Zhuchok, 2018)

2.3.11 Example Considering the dimonoids D_2 and D_3 from Example before , all dimonoid homomorphisms defined from D_2 to D_3 are as follows:

(i) $f(x) = a, for\ all\ x \in D_2$

(ii) $g(x) = b, for\ all\ x \in D_2$, (Zhuchok, 2018).

2.3.12 Definition Let D and E be dimonoids and let F:D→E be a dimonoid homomorphism. In this case, the congruence relation ΔF induced on D is defined as:

$$(D,E) \in \Delta_F \iff F(D) = F(E)$$

for all $a, b \in D, E$. The equivalence class of $a \in D$ under Δ_F is denoted as:

$$D_2\, S = \{x \in E \mid x \in D,\, t \in D,\, (E,x) \in \Delta_F\}\text{(Zhuchok, 2018)}$$

2.3.13 Example The congruence relation Δ_f corresponding to the dimonoid homomorphism f in Example before is:

$$\Delta_f = \{(a,a),(a,b),(b,a),(b,b)\}$$

Here, $D_a = \{a,b\} = D_1\ and\ D_b = \emptyset \subset D_1$(Zhuchok, 2018)

2.3.14 Definition Let (D, ∗, ∘) be a dimonoid, and let $\emptyset \neq T \subseteq D$. Then T is called a subdimonoid of D if and only if:

$$a * b,\ a \circ b \in T,$$

for all $a, b \in T$, (Zhuchok, 2018).

2.3.15 Definition Let X be an index set, and let (D_α, ∗, ∘) be sub-dimonoids of a dimonoid (D, ∗, ∘) where $\alpha \in I$. Then the diband of the subdimonoids is defined as:

(i) $D = \cup_{\alpha \in I} D_\alpha$

(ii) $D_\alpha \cap D_\beta = \emptyset, for\ \alpha \neq \beta$

(iii) For all $\alpha, \beta \in I$, there exist $\gamma, \gamma' \in I$ such that$D_\alpha * D_\beta \subseteq D_\gamma\ and\ D_\alpha \circ D_\beta \subseteq D_{\gamma'}$

(Zhuchok, 2018).

2.3.16 Theorem For an arbitrary dimonoid D, there exist a semigroup T and its idempotent endomorphism f such that T contains D as a subdimonoid

Proof Let D = (D, ∗,∘) be a dimonoid and let $\sim$ be the least semigroup congruence on D, i.e., the least congruence for which the quotient dimonoid is a semigroup. It is clear that $\sim$ is the congruence generated by the relation

$\sigma = \{(a,a),(a * b, a \circ b),(a \circ b, a * b) \mid a, b \in D\}$.

Hence for any distinct $x, y \in D$ with $x \sim y$, there are some $b_1, c_1, \dots, b_n, c_n \in D_{\text{and some}}$ $\circ_1, {}'_1, \dots, {}'_n, {}'_n \in \{-\circ, \circ\}$ such that

$$\begin{aligned} x &= (b_1 \circ_1 c_1)\sigma(b_1 \circ'_1 c_1) = (b_2 \circ_2 c_2)\sigma(b_2 \circ'_2 c_2) \\ &= \cdots = (b_n \circ_n c_n)\sigma(b_n \circ'_n c_n) = y. \end{aligned}$$

As $(b_i \circ_i c_i) \circ a = (b_i \circ'_i c_i) \circ a$ for each $i = 1, \dots, n$ and each $a \in D$, we have

$$\begin{aligned} x \circ a &= (b_1 \circ_1 c_1) \circ a = (b_1 \circ'_1 c_1) \circ a \\ &= \cdots = (b_n \circ_n c_n) \circ a = (b_n \circ'_n c_n) \circ a = y \circ a, \end{aligned}$$

whence

$$x \circ a = y \circ a. \tag{2.2}$$

Similarly, we can show that

$$a * x = a * y. \tag{2.3}$$

It means that$\sim$ is contained in the congruence

$\tau := \{(x, y) \in D \times D \mid x \circ a = y \circ a \text{ and } a * x = a * y \text{ for all } a \in D\}$(Zhuchok, 2018).

Now we fix an arbitrary congruence θ on D such that$\sim \sqsubseteq \theta \sqsubseteq \tau$. Since θ contains the least semigroup congruence on D, the quotient dimonoid $\frac{\mathrm{D}}{\theta}$ is a semigroup. For $x \in D$, we denote by $\bar{x}$ the class of the congruence θ that contains x. Define an operation * on the set $T = D \cup (\frac{\mathrm{D}}{\theta})$ by the following rules:

$$a * \bar{x} := a * x, \bar{x} * a := x \circ a, a * b := a \circ b, \bar{x} * \bar{y} := \overline{x \circ y} = \overline{x * y}$$

For all $a, b \in D$ and $\bar{x}, \bar{y} \in \frac{\mathrm{D}}{\theta}$. From (2.2.3) it follows that * is well-defined. Let us show that $T = (T, *)$ is a semigroup. Clearly, the verification of the associative law splits in eight cases.

For all x, y, z belong to D using the axioms of a dimonoid, we have

$$\begin{aligned}
(x * y) * \bar{z} &= (x \circ y) * z = x \circ (y * z) = x * (y * z) = x * (y * \bar{z}),\\
(x * \bar{y}) * z &= (x * y) \circ z = x \circ (y \circ z) = x * (y \circ z) = x * (\bar{y} * z),\\
(\bar{x} * y) * z &= (x \circ y) \circ z = x \circ (y \circ z) = \bar{x} * (y \circ z) = \bar{x} * (y * z),\\
(x * \bar{y}) * \bar{z} &= (x * y) * z = x * (y * z) = x * (y \circ z) = x * (\bar{y} * \bar{z}),\\
(\bar{x} * y) * \bar{z} &= (x \circ y) * z = x \circ (y * z) = x \circ (y * \bar{z}) = \bar{x} * (y * \bar{z}),\\
(\bar{x} * \bar{y}) * z &= (\overline{x \circ y}) * z = (x \circ y) \circ z = x \circ (y \circ z) = \bar{x} * (\bar{y} * z).
\end{aligned}$$

The algebras D and $\frac{\mathrm{D}}{\theta}$ are semigroups whence also

$(x * y) * z = x * (y * z)$ and $(\bar{x} * \bar{y}) * \bar{z} = \bar{x} * (\bar{y} * \bar{z})$

Now define a map $f\colon D \to Dletting\ f(x) = f(\bar{x}) \coloneqq \bar{x}\ for\ all\ x \in D$. Let us show that$f$ is an idempotent endomorphism.

By (2.2.3) for all $c, y \in D$ we get

$$\begin{aligned}
f(x * y) &= f(x \circ y) = \overline{x \circ y} = \bar{x} * \bar{y} = f(x) * f(y),\\
f(x * \bar{y}) &= f(x * y) = \overline{x * y} = \bar{x} * \bar{y} = f(x) * f(\bar{y}),\\
f(\bar{x} * y) &= f(x \circ y) = \overline{x \circ y} = \bar{x} * \bar{y} = f(\bar{x}) * f(y),\\
f(\bar{x} * \bar{y}) &= f(\overline{x \circ y}) = \overline{x \circ y} = \bar{x} * \bar{y} = f(\bar{x}) * f(\bar{y}).
\end{aligned}$$

Thus, f is an endomorphism. Besides that, $f^2 = f$ since

$f(f(x)) = f(f(\bar{x})) = f(\bar{x}) = \bar{x}$

So, f is an idempotent endomorphism. For any, using (2.1), we obtain

$x < y = x * f(y) = x * \bar{y} = x * y, x > y = f(x) * y = \bar{x} * y = x \circ y.$

Hence (D,∗,∘) is a subdimonoid of D. The proof is complete.

2.3.17 Lemma Let $(D,*,\circ)$ be a dimonoid, with $a \in D$. If a semigroup $(D,*)$ is a-connected, then a semigroup $(D,\circ)$ is also a-connected:

Proof. Let $(D,*)$ be an a-connected semigroup, with $x, y \in D$. Then there exists $n \in N$ such that $(x * a)^n \in y * a * D$ and $(y * a)^n \in x * a * D$. This implies

$$(x * a)^n = y * a * t_1, \tag{1}$$
$$(y * a)^n = x * a * t_2 \tag{2}$$

"for some $t_1, t_2 \in D$. Put $t_3 = t_1 \circ x \circ a$ and $t_4 = t_2 \circ y \circ a$. If we multiply (1) and (2) by $x \circ a$ and $y \circ a$, respectively, we arrive at

$$\begin{aligned}(x * a)^n \circ (x \circ a) &= n(x \circ a) \circ (x \circ a)\\ &= (n+1)(x \circ a) = (y * a * t_1) \circ (x \circ a)\\ &= ((y * a) \circ t_1) \circ (x \circ a)\\ &= y \circ a \circ t_1 \circ x \circ a = y \circ a \circ t_3,\\ (y * a)^n \circ (y \circ a) &= n(y \circ a) \circ (y \circ a)\\ &= (n+1)(y \circ a) = (x * a * t_2) \circ (y \circ a)\\ &= ((x * a) \circ t_2) \circ (y \circ a)\\ &= x \circ a \circ t_2 \circ y \circ a = x \circ a \circ t_4\end{aligned}$$

in view of Lemma 4(i) and axioms (D4) and (D5). Thus $(n+1)(x \circ a) \in y \circ a \circ D$ and $(n+1)(y \circ a) \in x \circ a \circ D$. Consequently, $(D,\circ)$ is an a-connected semigroup" (Zhuchok, 2018).

2.4 Hyperdimonoids

The concept of hyperdimonoid was defined by Özcan (2021) and this section elaborates on the concept based on the Özcan's work:

2.4.1 Definition “Let $Q \neq \emptyset$, “$\circledast$” and “$\odot$” be hyper operations defined on the set Q. In this case (Q, $\circledast$, $\odot$) is called a hyperdimonoid if and only if for all i, j, k ∈ Q, the following conditions hold:

1. $(i \circledast j) \circledast k = i \circledast (j \odot k)$
2. $(i \odot j) \circledast k = i \odot (j \circledast k)$
3. $(i \circledast j) \odot k = i \odot (j \odot k)$
4. $(i \circledast j) \circledast k = i \circledast (j \circledast k)$
5. $(i \odot j) \odot k = i \odot (j \odot k)$” (Özcan, 2021)

2.4.2 Example “Let (D, $\circledast$, $\odot$) be a dimonoid. In this case, the hyper operation of $\circledast$ and $\odot$ for all i, j ∈ D is like the following :

- $i \circledast j = \{i * j\}$
- $i \odot j = \{i \circ j\}$

Hence, (D, $\circledast$, $\odot$) is a hyperdimonoid” (Özcan, 2021).

2.4.3 Lemma Let (Q, $\circledast$, $\odot$) be a hyperdimonoid, and let I, J ∈ F*(Q). then,

a) If the operation “$\circledast$” is commutative, we will have $I \circledast J = J \circledast I$
b) If “$\odot$” is commutative $I \odot J = J \odot I$ (Özcan, 2021)

2.4.4 Definition The hyperdimonoid (Q, $\circledast$, $\odot$) is called commutative when both “$\circledast$” and “$\odot$” are commutative

2.4.5 Theorem Let (Q, $\circledast$, $\odot$) be a hyperdimonoid, and let I, J ∈ P(Q). In this case the following holds (Özcan, 2021):

a) $(I \circledast J) \circledast K = I \circledast (J \circledast K)$
b) $(I \odot J) \odot K = I \odot (J \odot K)$
c) $(I \circledast J) \circledast K = I \circledast (J \odot K)$
d) $(I \odot J) \circledast K = I \odot (J \circledast \mathrm{K})$ (Özcan, 2021).

2.4.6 Proof Let $u \in (V \circledast X) \circledast Y$, in this case:

$$u \in \mathrm{U}_{\substack{K\in V\circledast X\\ y\in Y}} \mathrm{k} \circledast y$$

$\Rightarrow \exists k \in V \circledast X, y \in Y$ such that $u \in k \circledast y$

$\Rightarrow \exists a \in V$ and $x \Rightarrow \mathrm{X} \in V, x \in X$ such that $k \in v \circledast x$

$\Rightarrow u(v \circledast x) \circledast y \Rightarrow \mathrm{u} \in \mathrm{v} \circledast (x \circledast y) \Rightarrow \mathrm{u} \in\ v \circledast (x \circledast \mathrm{y})\ u \in U_{\substack{v\in V\\ t\in x\circledast y}} \mathrm{v} \circledast t$

$\Rightarrow u \in U_{\substack{v\in V\\ t\in x\circledast y}} \mathrm{v} \circledast t \Rightarrow u \in \mathrm{V} \circledast (\mathrm{X} \circledast \mathrm{Y})$

$\Rightarrow u \in V \circledast (X \odot Y)$

Thus, $(V \circledast X) \circledast Y \subseteq \mathrm{V} \circledast (\mathrm{X} \circledast \mathrm{Y})$

Therefor, $(V \circledast X) \circledast Y \subseteq \mathrm{V} \circledast (\mathrm{X} \circledast \mathrm{Y})$ is proven.

2.4.7 Example. (i) Let the hyper operations "$\circledast$" and "$\odot$" be defined on the set $Q = \{a, b\}$ as shown in the tables below : Table 7

$\circledast$	a	b
a	Q	Q
b	Q	Q

Table 8

$\odot$	a	b
a	$\{a\}$	Q
b	$\{a\}$	Q

In this case, (Q, $\circledast$, $\odot$) is a hyperdimonoid (Özcan, 2021)..

(ii) Let $Q = \{a, b\}$ be a set with hyper operations "$\circledast$" and "$\odot$" defined as shown in the following tables:

Table 9

⊛	a	b
a	Q	Q
b	Q	Q

Table 10

⊙	a	b
a	$\{b\}$	Q
b	Q	Q

In this case, (Q, ⊛, o) is a hyper dimonoid. (Özcan, 2021)

(iii) Let ⊛ and be hyper operations defined on the set Q = {a, b} as shown in the tables below:

Table11

⊛	a	b
a	$\{b\}$	Q
b	Q	Q

Table 12

⊙	a	b
a	Q	Q
b	Q	Q

In such a case, (Q, ⊛, ⊙) is a hyperdimonoid, (Özcan, 2021).

(iv)Let $Q = \{a, b\}$ be the set with hyperoperations "*" and "⊚" defined as in the following tables

Table 13

$\circledast$	a	b
a	Q	Q
b	Q	Q

Table 14

$\circledcirc$	a	b
a	Q	Q
b	Q	Q

In this case, $(Q,\circledast,\circledcirc)$ is a hyperdimonoid, (Özcan, 2021).

2.4.8 Let $(Q,\circledast,\circledcirc)$ be a hyperdimonoid and let $a, b \in, Q. P(Q)$. Then,

(i) If "⊛" is commutative, then (a⊛b)=(a⊛b)

(ii) If "⊚" is commutative, then (a⊚b) = (a⊚b).

2.4.9 Definition:If both " *" and " ⊚ " are commutative, then (Q, ⊛, ⊚) is called a commutative hyper-dimonoid, (Zhuchok, 2018).

2.4.10 “Let (Q, ⊛, ⊚) be a hyperdimonoid and let a, b, c ∈ P-⊛, (Q) Then,

(i) $(a \circledast b) \circledast c = a \circledast (b \circledast c)$

(ii) $(a \circledcirc b) \circledcirc c = a \circledcirc (b \circledcirc c)$

(iii) $(a \circledast b) \circledast c = a \circledast (b \circledcirc c)$

(iv) $(a \circledcirc b) \circledast c = a \circledcirc (b \circledast c)$

(v) $(a \circledast b) \circledcirc c = a \circledcirc (b \circledcirc c)$.”

Proof:

(i) Let $u \in (a \circledast b) \circledast c$. In this case,

$u \in {}_{\substack{k\in a\circledast B\\ c\in C}} k \circledast c$

$\Rightarrow \exists k \in A \circledast B$ ve $c \in C$ thus $u \in k \circledast c$

$\Rightarrow \exists a \in A$ ve $b \in B$ thus $k \in a \circledast b$

$\Rightarrow u \in (a \circledast b) \circledast c \Rightarrow u \in a \circledast (b \circledast c) \Rightarrow u \in {}_{\substack{a\in A\\ t\in b\circledast c}} a \circledast t$

$\Rightarrow u \in {}_{a\in B\circledast C} a \circledast t \Rightarrow u \in A \circledast (B \circledast C)$

As a result, $(A \circledast B) \circledast C \subseteq A \circledast (B \circledast C)$.

Conversely, let $u \in A \circledast (B \circledast C)$. In this case,

$u \in \underset{t\in B\circledast C}{a \in A} a \circledast t$

$\Rightarrow \exists t \in B \circledast C$ ve $a \in A$ thus $u \in a \circledast t$

“$\Rightarrow t \in B \circledast C \Rightarrow \exists b \in B, c \in C$ thus $t \in b \circledast c$

$\Rightarrow u \in a \circledast (b \circledast c) \Rightarrow$ since the operation"*" is associative e $u \in (a \circledast b) \circledast c$

$\Rightarrow u \in {}_{\substack{k\in a\circledast b\\ c\in C}} k \circledast c \Rightarrow u \in {}_{\substack{k\in A\circledast B\\ c\in C}} k \circledast c$

$\Rightarrow u \in A \circledast (B \circledast C)$

As a result, $A \circledast (B \circledast C) \subseteq (A \circledast B) \circledast C$.

Thus,$(A \circledast B) \circledast C = A \circledast (B \circledast C)$ is obtained.

(ii) let $u \in (A \odot B) \odot C$. In this case,

$u \in \bigcup_{\substack{k\in A\odot B\\ c\in C}} k \odot C$

$\Rightarrow \exists k \in A \odot B, c \in C$ thus $u \in k \odot c$

$\Rightarrow k \in A \odot B \Rightarrow \exists a \in A, b \in B$ hence $\in a \odot b$

u ∈ (a ⊙ b) ⊙ c ⇒ since the operation "⊙" is associative, u ∈ a ⊙ (b ⊙ c)

$\Rightarrow u \in \bigcup_{t\in b\odot c} a \odot t \Rightarrow u \in \underset{\substack{a\in A\\ t\in B\odot C}}{a \odot t}$

$\Rightarrow u \in A \odot (B \odot C)$

As a result, $(A \odot B) \odot C \subseteq A \odot (B \odot C)$."

Conversely, Let $u \in A \odot (B \odot C)$. In this case,

$\mathrm{u} \in \bigcup_{\substack{a\in B \\ t\in B\odot C}} a \odot t$

$\Rightarrow \exists t \in B \odot C, a \in A$ thus $u \in a \odot t$

$\Rightarrow t \in B \odot C \Rightarrow \exists b \in B, c \in C$ thus $t \in b \odot c$

$\Rightarrow u \in a \odot (b \odot c) \Rightarrow u \in (a \odot b) \odot c$

$\Rightarrow u \in {}_{c\in C\in a\odot b} k \odot c \Rightarrow u \in \underset{\substack{c\in C \\ k\in A\odot B}}{c}\ k \odot c$

$\Rightarrow u \in (A \odot B) \odot C$

As a result, $A \odot (B \odot C) \subseteq (A \odot B) \odot C$.

Thus,$(A \odot B) \odot C = A \odot (B \odot C)$ is obtained.

(iii) "Let $u \in (A \circledast B) \circledast C$. In such a case,

$u \in \underset{t\in A\circledast B}{\Bbbk \in C} t \circledast k$

$\Rightarrow \exists t \in A \circledast B, k \in C$ thus $u \in t \circledast k$

$\Rightarrow t \in A \circledast B \Rightarrow \exists a \in A, b \in B$ thus $t \in a \circledast b$

$\Rightarrow u \in (a \circledast b) \circledast k \Rightarrow (a \circledast b) \circledast k = a \circledast (b \odot k)$. hence $u \in a \circledast (b \odot k)$

$\Rightarrow u \in \mathrm{U}_{r\in bok} a \circledast r \Rightarrow u \in {}_{\substack{r\in B\odot C \\ a\in A}} a \circledast r$

$\Rightarrow u \in A \circledast (B \odot C)$

As a result, $(A \circledast B) \circledast C \subseteq A \circledast (B \odot C)$.

Conversely, let $u \in A \circledast (B \odot C)$. In this case,

$u \in {}_{\substack{r\in B_0\subset C \\ a\in A}} a \circledast r$

$\Rightarrow \exists r \in B \odot C, a \in A$ thus $u \in a \circledast r$

$\Rightarrow r \in B \odot C \Rightarrow r \in \bigcup_{\substack{b\in B \\ k\in C}} b \odot k$

$\Rightarrow r \in B \odot C \Rightarrow \exists b \in B, k \in C$ thus $r \in b \odot k$

$\Rightarrow u \in a \circledast (b \odot k) \Rightarrow a \circledast (b \odot k) = (a \circledast b) \circledast k$

$\Rightarrow u \in (a \circledast b) \circledast k \Rightarrow u \in \bigcup_{t\in a\circledast b} t \circledast k$

$\Rightarrow x \in \bigcup_{\substack{\in\in A\circledast BB\\ k\in C}} t \circledast k \Rightarrow u \in (A \circledast B) \circledast C$

As a result, $A \circledast (B \odot C) \subseteq (A \circledast B) \circledast C$.

Thus,$(A \circledast B) \circledast C = A \circledast (B \odot C)$ is obtained.

(iv) let $u \in (A \odot B) \circledast C$. In this case,

$u \in \bigcup_{\substack{t\in AOB\\ c\in C}} t \circledast c$

$\Rightarrow t \in A \odot B, c \in C$ thus $u \in t \circledast c$

$\Rightarrow t \in A \odot B \Rightarrow \exists a \in A, b \in B$ thus $t \in a \odot b$

$\Rightarrow u \in (a \odot b) \circledast c \Rightarrow (a \odot b) \circledast c = a \odot (b \circledast c)$

$\Rightarrow u \in a \odot (b \circledast c) \Rightarrow u \in \bigcup_{t\in b\circledast c} a \odot t$

$\Rightarrow u \in \bigcup_{\substack{t\in B\circledast C\\ a\in A}} a \odot t \Rightarrow u \in A \odot (B \circledast C)$

As a result, $(A \odot B) \circledast C \subseteq A \odot (B \circledast C)$."

Conversely, "let $u \in A \odot (B \circledast C)$. In this case,

$u \in \bigcup_{\substack{t\in B\circledast C\\ a\in A}} a \odot t \Rightarrow \exists b \in B, c \in C$ thus $t \in b \circledast c$

$\Rightarrow t \in B \circledast C \Rightarrow t \in \bigcup_{\substack{b\in B\\ c\in C}} b \circledast c$

$\Rightarrow t \in B \circledast C \Rightarrow \exists b \in B, k \in C$ thus $t \in b \odot k$

$\Rightarrow u \in a \odot (b \circledast c) \Rightarrow a \odot (b \circledast c) = (a \odot b) \circledast c \Rightarrow u \in (a \odot b) \circledast c$

$\Rightarrow u \in \bigcup_{\substack{t\in a\odot ob\\ c\in C}} t \circledast c \Rightarrow u \in {}_{\substack{t\in A\odot B\\ c\in C}} t \circledast c$

$\Rightarrow u \in (A \odot B) \circledast C$

As a result, $A \odot (B \circledast C) \subseteq (A \odot B) \circledast C$.

Thus,$A \odot (B \circledast C) \subseteq (A \odot B) \circledast C$ is obtained.

(v) let $u \in (A \circledast B) \odot C$. In this case,

$$u \in \bigcup_{\substack{t \in A \circledast B \\ c \in C}} t \odot c$$
$$\Rightarrow \exists t \in A \circledast B, c \in C \text{ thus } u \in t \odot c$$
$$\Rightarrow t \in A \circledast B \Rightarrow t \in \bigcup_{\substack{a \in A \\ b \in B}} a \circledast b$$
$$\Rightarrow \exists a \in A, b \in B \text{ thus } t \in a \circledast b$$
$$\Rightarrow u \in (a \circledast b) \odot c \Rightarrow (a \circledast b) \odot c = a \odot (b \odot c)$$
$$\Rightarrow u \in a \odot (b \odot c) \Rightarrow u \in \bigcup_{\substack{t \in b \odot c \\ a \in A}} a \odot t$$
$$\Rightarrow u \in \bigcup_{\substack{t \in B \odot c \\ a \in A}} a \odot t \Rightarrow u \in A \odot (B \odot C)$$

As a result, $(A \circledast B) \odot C \subseteq A \odot (B \odot C)$."

Conversely, let $u \in A \odot (B \odot C)$. In this case,

$$u \in \bigcup_{\substack{t \in B \odot C \\ a \in A}} a \odot t$$
$$\Rightarrow \exists t \in B \odot C, a \in A \text{ thus } x \in a \odot t$$
$$\Rightarrow t \in B \odot C \Rightarrow \exists b \in B, c \in C \text{ thus } t \in b \odot c$$
$$\Rightarrow u \in a \odot (b \odot c) \Rightarrow a \odot (b \odot c) = (a \circledast b) \odot c \Rightarrow u \in (a \circledast b) \odot c$$

$$\Rightarrow u \in \bigcup_{\substack{t \in a \circledast b \\ c \in C}} t \odot c \Rightarrow u \in \bigcup_{\substack{\in \in A \circledast B \\ c \in C}} t \odot c$$

$$\Rightarrow u \in (A \circledast B) \odot C$$

As a result, $A \odot (B \odot C) \subseteq (A \circledast B) \odot C$.

Thus, $A \odot (B \odot C) = (A \circledast B) \odot C$ is obtained, (Zhuchok, 2018)

2.4.11 Definition let (Q_1,⊛,⊙) ve (Q_2,⊛,⊙) be a hyperdimonoid. In such a case;

(i) A function ψ :Q_1→Q_2 is called a hyperdimonoid homomorphism if: for every x,y∈Q_1, ψ (x⊛y)⊆ψ (x)⊛ψ (y) and ψ (x⊙y)⊆ψ (x)⊙ψ (y).

(ii) A function ψ :Q_1→$Q_2$is called a strong hyperdimonoid homomorphism if: for every x,y∈Q_1, ψ (x⊛y)=ψ (x)⊛ψ (y) and ψ (x⊙y)=ψ (x)⊙ψ (y).

(iii) A hyper-dimonoid homomorphism $\psi : Q_1 \rightarrow Q_2$ is called an idempotent hyper-dimonoid homomorphism if: for every $x, y \in Q_1$, $\psi(\psi(x)) = \psi(x)$, (Zhuchok, 2018).

2.4.12 Example. Let Q_1, Q_2, and Q_3 be hyperdimonoids corresponding to examples 2.4.11 (ii), (iii), and (iv)

(i) For every $x \in Q_1$, the function $\psi(x) = b$ defined as $\psi : Q_1 \rightarrow Q_2$ is a hyperdimonoid homomorphism but not a strong one.

(ii) For every a, $b \in Q_2$, the function $g(a) = b$ and $g(b) = a$, defined as $g : Q_2 \rightarrow Q_1$, is a hyper-dimonoid homomorphism, but it is not a strong one.

(iii) For each a, $b \in Q_3$, the function $h(a) = b$ and $h(b) = a$, defined as $h : Q_3 \rightarrow Q_3$, is an idempotent strong hyper-dimonoid homomorphism, (Zhuchok, 2018).

2.4.13 Theorem. Let (Q, ·) be a semihypergroup and let $\psi : Q \rightarrow Q$ be an idempotent strong homomorphism. Then, with the hyper operations $x \circledast y = x \cdot \psi(y)$ and $x \circledcirc y = \psi(x) \cdot y$, $(Q, \circledast, \circledcirc)$ forms a hyper dimonoid

Proof:

(i) let $u \in (x \circledast y) \circledast z$. In this case,

$u \in \bigcup_{a \in x \cdot \psi(y)} a \cdot \psi(z)$

$\Rightarrow \exists a \in x \cdot \psi(y)$ thus $u \in a \cdot \psi(z)$

$\Rightarrow u \in (x \cdot \psi(y))\psi(\psi(z)) \Rightarrow u \in x \cdot \psi(y \cdot \psi(z))$

$\Rightarrow u \in x \cdot \psi(y \cdot \psi(z)) \Rightarrow u \in \bigcup_{b \in y \cdot \psi(z)} x \cdot \psi(b) \Rightarrow u \in x \circledast (y \circledast z)$

As a result, $(x \circledast y) \circledast z \subseteq x \circledast (y \circledast z)$.

Conversely, let $u \in x \circledast (y \circledast z)$. In this case,

$u \in \bigcup_{b \in y \cdot \psi(z)} x \cdot \psi(b)$

$\Rightarrow \exists b \in y \cdot \psi(z)$ thus $u \in x \cdot \psi(b)$

$\Rightarrow u \in x \cdot \psi(y \cdot \psi(z)) \Rightarrow u \in x \cdot (\psi(y) \cdot \psi(z))$

$\Rightarrow u \in (x \cdot \psi(y) \cdot \psi(z)) \Rightarrow u \in \bigcup_{a \in x \cdot \psi(y)} a \cdot \psi(z)$

As a result, $x \circledast (y \circledast z) \subseteq (x \circledast y) \circledast z$.

Thus,$x \circledast (y \circledast z) = (x \circledast y) \circledast z$ is obtained.

(ii) let $u \in (x \odot y) \odot z$. In this case,
$u \in \bigcup_{a \in \psi(x) \cdot y} \psi(a). z$
$\Rightarrow \exists a \in \psi(x) \cdot y$ thus $u \in \psi(a) \cdot z$
$\Rightarrow u \in \psi(\psi(x) \cdot y) \cdot z \Rightarrow u \in (\psi(\psi(x) \cdot y) \cdot z)$
$\Rightarrow u \in \psi(x) \cdot (\psi(y) \cdot z) \Rightarrow u \in \bigcup_{b \in \psi(y) \cdot z} \psi(x) \cdot b$

As a result, $(x \odot y) \odot z \subseteq x \odot (y \odot z)$.

Conversely, let $u \in x \odot (y \odot z)$. In this case,
$u \in U_{b \in \psi(y) \cdot z} \psi(x) \cdot b$
$\Rightarrow \exists b \in \psi(y) \cdot z$ thus $u \in \psi(x) \cdot b$
$\Rightarrow u \in \psi(x) \cdot (\psi(y) \cdot z) \Rightarrow u \in \psi(\psi(x))(\psi(y) \cdot z)$
$\Rightarrow u \in \psi(\psi(x)) \cdot \psi(y) \cdot z$
$\Rightarrow u \in \psi(\psi(x) \cdot y) \cdot z$
$\Rightarrow u \in \bigcup_{t \in \psi(\psi(x) \cdot y)} t \cdot z = u \in \bigcup_{a \in \psi(x) \cdot y} \psi(a) \cdot z$

As a result, $x \odot (y \odot z) \subseteq (x \odot y) \odot z$.

Thus,$x \odot (y \odot z) = (x \odot y) \odot z$ is obtained.

(iii) let $u \in (x \circledast y) \circledast z$. In this case,
$u \in \bigcup_{a \in x \cdot \psi(y)} a \cdot \psi(z)$
$\Rightarrow \exists a \in x \cdot \psi(y)$, therefore $u \in a \cdot \psi(z)$
$\Rightarrow u \in x \cdot (\psi(\psi(y)) \cdot \psi(z)) \Rightarrow u \in x \cdot \psi(\psi(y) \cdot z)$
$\Rightarrow u \in \bigcup_{t \in \psi(\psi(y) \cdot z)} x \cdot t \cdot u \in \bigcup_{b \in \psi(y) \cdot z} x \cdot \psi(b)$

As a result, $(x \circledast y) \circledast z \subseteq x \circledast (y \odot z)$.

Conversely, let $u \in x \circledast (y \odot z)$. In this case,

$u \in \bigcup_{b \in \psi(y) \cdot z} x \cdot \psi(b)$
$\Rightarrow \exists b \in \psi(y) \cdot z$ thus $u \in x \cdot \psi(b)$
$\Rightarrow u \in x \cdot \psi(\psi(y) \cdot z) \Rightarrow u \in x \cdot (\psi(\psi(y) \cdot \psi(z)))$
$\Rightarrow u \in x \cdot (\psi(y) \cdot \psi(z)) \Rightarrow u \in (x \cdot \psi(y)) \cdot \psi(z)$

$\Rightarrow u \in \bigcup_{t \in x \cdot \psi(y)} t \cdot \psi(z)$

As a result, $x \circledast (y \odot z) \subseteq (x \circledast y) \circledast z$.

In such case, we will have $x \circledast (y \odot z) = (x \circledast y) \circledast z$ el

(iv) let $u \in (x \odot y) \circledast z$. In this case,

$u \in \bigcup_{a \in \psi(x) \cdot y} a \cdot \psi(z)$

$\Rightarrow \exists a \in \psi(x) \cdot y$ thus $u \in a \cdot \psi(z)$

$\Rightarrow u \in (\psi(x) \cdot y) \cdot \psi(z) \Rightarrow u \in \psi(x) \cdot (y \cdot \psi(z))$

$\Rightarrow u \in \bigcup_{b \in y \cdot \psi(z)} \psi(x) \cdot b \Rightarrow u \in x \odot (y \circledast z)$

As a result, $(x \odot y) \circledast z \subseteq x \odot (y \circledast z)$.

Conversely, let $u \in x \odot (y \circledast z)$. In this case,

$\Rightarrow u \in \bigcup_{b \in y \cdot \psi(z)} \psi(x) \cdot b$

$\Rightarrow \exists b \in y \cdot \psi(z)$ thus $u \in \psi(x) \cdot b$

$\Rightarrow u \in \psi(x) \cdot (y \cdot \psi(z)) \Rightarrow u \in (\psi(x) \cdot y) \cdot \psi(z)$

$\Rightarrow u \in \bigcup_{a \in x \cdot \psi(y)} a \cdot \psi(z)$

As a result, $x \odot (y \circledast z) \subseteq (x \odot y) \circledast z$.

In such case, we will have $x \odot (y \circledast z) = (x \odot y) \circledast z$

(i) let $u \in (x \circledast y) \odot z$. In this case,
 $\Rightarrow u \in \bigcup_{a \in x \cdot \psi(y)} \psi(a) \cdot z \Rightarrow u \in \psi(x \cdot \psi(y)) \cdot z$
 $\Rightarrow u \in (\psi(x) \cdot \psi(\psi(y))) \cdot z \Rightarrow u \in \psi(x)(\psi(\psi(y)) \cdot z)$
 $\Rightarrow u \in \psi(x) \cdot \big(\psi(y) \cdot z \Rightarrow \mathrm{u} \in \bigcup_{b \in \psi(y) \cdot z} \psi(x) \cdot b$

As a result, $(x \circledast y) \cdot z \subseteq x \odot (y \odot z)$.

Conversely, let $u \in x \odot (y \odot z)$. In this case,

$u \in \bigcup_{b \in \psi(y) \cdot z} \psi(x) \cdot b$

$\Rightarrow \exists b \in \psi(y) \cdot z$ thus $u \in \psi(x) \cdot b$

$\Rightarrow u \in (\psi(x)) \cdot \psi(\psi(y)) \cdot z$

$\Rightarrow u \in \psi\,(x) \cdot (\psi\,(\psi\,(y)) \cdot z \Rightarrow u \in (\psi\,(x)) \cdot \psi\,(\psi\,(y)) \cdot z$

$u \in \bigcup_{t \in \psi\,(x \cdot \psi\,(y)} t \cdot z \Rightarrow u \in \bigcup_{a \in x \cdot \psi\,(y)} \psi\,(a) \cdot z$

As a result, $x \odot (y \odot z) \subseteq (x \circledast y) \odot z$.

In such case, we will have $x \odot (y \odot z) = (x \circledast y) \odot z$, (Zh uchok, 2018).

2.4.14 Example The hyper operation defined on the set Q = $\{a, b\}$ is provided in the following table:

Table 15

$\odot$	a	b
a	$\{a\}$	Q
b	$\{a\}$	Q

In this case, Q, $\odot$) is a semi hypergroup. Let the function ψ: Q → Q be defined as ψ (x)=a for every x∈ Q. Here, ψ is an idempotent strong homomorphism. Accordingly, the hyperoperations"

$\circledast$" and "$\odot$" mentioned in Theorem before are obtained as follows:

Table 16

$\circledast$	a	b
a	$\{a\}$	$\{a\}$
B	$\{a\}$	$\{a\}$

Table 17

$\odot$	a	b

a	$\{a\}$	Q
b	$\{a\}$	Q

In such case, according to theorem 2.1.15 (Q ,⊛,⊚) is a hyperdimonoid, (Zh uchok, 2018).

2.4.15 Definition Let (Q, ⊛, ⊚) be hyperdimonoid, and let $\theta \neq U \subseteq$ Q. in this case U is called a subhyperdimonoid of Q if and only if for all $x, y \in U$, as explained below:

$x \circledast y \subseteq U$ *and* x ⊚ y ⊆ U, (AKÇAY, 2024)

2.4.16 Example Let $Q = \{p, q\}$ be a set and consider the operations ⊛ and ⊚ given in table 4.1 with these operations (Q ,⊛,⊚) forms a hyperdimonoid

Now let $U = \{p\} \subseteq Q$. Since for all x, y ∈ U, we have below:

$$x \circledast y \subseteq U \text{ and } x \circledcirc y \subseteq U$$

this implies that U is a subhyperdimonoid.

On the other hand, let W = {q} ⊆ X. Since for all x, y ∈ W**,** we find:

x ⊛ y ⊆ V and x ⊚ y ⊆ V, we find:

it follows W is not subhyperdimonoid.

Table 4.1. The operations " ⊛ " and " ⊚ " defined on the set Q = {a, b}

⊛	a	b
A	{a}	{a}
B	{a}	{a}

O	a	B
a	{a}	Q
b	{a}	Q

(AKÇAY, 2024).

2.4.17 Let (**Q** ,⊛,⊚) be a hyperdimonoid, and let ∅ ≠ **U** ⊆

1. If U is called a right hyperideal of Q, then for every $x \in$ Q and $y \in$ U, we have:

$$x \circledast y \subseteq U, y \circledcirc x \subseteq U.$$

2. Similarly, U is called a left hyperideal of Q if for every $x \in \mathrm{Q}$ and $\mathrm{y} \in \mathrm{U}$, we have:

$$\mathrm{y} \circledast \mathrm{x} \subseteq \mathrm{U}, \mathrm{x} \odot \mathrm{y} \subseteq \mathrm{U},$$ (Akcay, 2024).

2.4.18 Definition Let $(\mathrm{Q},\circledast,\odot)$ be a hyperdimonoid, and let ψ subhypergroup of Q. if for all $p, q \ \in$ ψ the following conditions hold:

$$(\mathrm{p} \circledast \mathrm{q}) \cap \mathrm{Q} \subseteq \mathrm{Q}, (\mathrm{q} \odot \mathrm{p}) \cap \mathrm{Q} \subseteq \mathrm{Q},$$

Then Q is called a quasi-hyper-ideal, (AKÇAY, 2024).

2.4.19 Definition A hyperdimonoid $(\mathrm{Q},\circledast,\odot)$ is called breakable if for every $\emptyset \neq \mathrm{U} \subseteq \mathrm{Q}$, the subset *U* forms a subhyperdimonoid, (Akcay 2024).

2.4.20 Example The hyperdimonoid $(\mathrm{Q},\circledast,\odot)$ described in example 4.1.18 is breakable, (Akcay, 2024).

2.4.21 Theorem Let $(\mathrm{Q}, \circledast, \odot)$ be a hyperdimonoid where the operation "$\circledast$" is commutative.

For all a, $b \in Q$ and m $\in \mathbb{N}, m > 1$, the following holds:

$$(a \circledast b)^m = a^m \odot b^m = (a \odot b)^m$$

Proof:

For each a, b ∈ Q, we have:

$$(a \circledast b)^m = (a \circledast b) \circledast (a \circledast b) \circledast \ldots \circledast (a \circledast b)$$

$$= (a \circledast a \circledast \ldots \circledast a) \circledast (b \circledast b \circledast \ldots \circledast b)$$

$$= a^m \circledast b^m = a^m \circledast (b^{m-1} \circledast b) = a^m \circledast (b \circledast b^{m-1})$$

$$= (a^m \circledast b) \circledast b^{m-1} = (b \circledast a^m) \circledast b^{m-1}$$

$$= b \circledast (a^m \circledast b^{m-1}) = (a\text{\textasciicircum}m \circledast b^{m-1}) \circledast b$$

Thus, the equation is proven, (Akcay, 2024).

3 MATERIALS AND METHODS

3.1 Soft Dimonoids

In this section I will explain soft dimonoids from Oguz, 2023 s work. The section explains definitions, examples, theorems and soft dimonoids.

3.1.1 Definition Let $\mathcal{D}$ be a dimonoid and let $P(\mathcal{D})$ denotes the power set of $\mathcal{D}$. A pair (Ω, K) is called a soft dimonoid over $\mathcal{D}$, where Ω is a mapping given by

$$\Omega: K \longrightarrow P(\mathcal{D})$$

and K is a set of parameters, if $\Omega(a)$ is a subdimonoid of $\mathcal{D}$ for all $a \in K$.

It is worse pointing out that if each $\Omega(a)$ is commutative as a dimonoid, the soft dimonoid $\mathcal{D}$ is said to be commutative. Also, a soft dimonoid $\mathcal{D}$ can be regarded as a parameterized family of subdimonoids of the dimonoid $\mathcal{D}$. In what follows, $(\mathcal{D}, \Omega, K)$ is refer to the soft dimonoid (Ω, K) over the dimonoid $\mathcal{D}$. (Oguz, 2023).

3.1.2 Example "Let $(\mathcal{S}, \diamond)$ be a semigroup and (Ω, K) be a soft semigroup over $\mathcal{S}$. Then, the triblet $(\mathcal{S}, \diamond, \diamond)$ is a dimonoid such that $\Omega(a)$ is a subdimonoid of $\mathcal{S}$ for all $a \in K$. Thus, (Ω, K) becomes a soft dimonoid over $\mathcal{S}$. So, every soft semigroup can be considered as a soft dimonoid." , (Oguz, 2023).

3.1.3 Example Let (Ω, K) be a soft dimonoid over $\mathcal{D}$ and $\mathcal{S} \subseteq \mathcal{D}$. Then, $(\Omega|_{\mathcal{S}}, K)$ is a soft dimonoid over $\mathcal{S}$ if it is non-null.

3.1.4 Definition. Let $(\mathcal{D}, \Omega, K)$ and $(\mathcal{F}, \Omega', R)$ be two soft dimonoids. The product of them is described as $(\mathcal{D}, \Omega, K) \times (\mathcal{F}, \Omega', R) = (\mathcal{D} \times \mathcal{F}, \Omega'', K \times R)$, where $\Omega''(a, b) = \Omega(a) \times \Omega'(b)$ for all $(a, b) \in K \times R$.

From this definition, we have the following proposition.

3.1.5 Proposition The product of any two soft dimonoids is also a soft dimonoid. Proof. Consider the soft dimonoids $(\mathcal{D}, \Omega, K)$ and $(\mathcal{F}, \Omega', R)$. Then,

$$\begin{aligned} \Omega: K &\longrightarrow P(\mathcal{D}) \\ a &\mapsto \Omega \ \ (a) \end{aligned}$$

and

$$\begin{array}{rcl} \Omega' : R & \longrightarrow & P(\mathcal{F}) \\ b & \mapsto & \Omega'(b) \end{array}$$

such that $\Omega(a)$ is a subdimonoid of the dimonoid $\mathcal{D}$ for all $a \in K$ and $\Omega'(b)$ is a subdimonoid of the dimonoid F for all $b \in R$. Using these mappings, we describe Ω'' by

$$\Omega'' : K \times R \longrightarrow P(\mathcal{D} \times \mathcal{F})$$

$$(a, b) \mapsto \Omega''(a, b) = \Omega(a) \times \Omega'(b)$$

From the product of dimonoids, notice that $\Omega(a) \times \Omega'(b)$ is also a subdimonoid of the product dimonoid $\mathcal{D} \times \mathcal{F}$ for all $(a, b) \in K \times R$ which implies that $(\mathcal{D} \times \mathcal{F}, \Omega'', K \times R)$ is a soft dimonoid.

The following theorems relate some generalizations for a nonempty family of soft dimonoids. , (Oguz, 2023).

3.1.6 Theorem Let $\{(\Omega_i, K_i) \mid i \in \mathcal{J}\}$ be a non-empty family of soft dimonoids over $\mathcal{D}$.

ii. The restricted intersection of the family $\{(\Omega_i, K_i) \mid i \in \mathcal{J}\}$ with $\bigcap_{i \in \mathcal{J}} K_i \neq \emptyset$ is a soft dimonoid over $\mathcal{D}$ if it is non-null.

iii. The extended intersection of the family $\{(\Omega_i, K_i) \mid i \in \mathcal{J}\}$ is a soft dimonoid over $\mathcal{D}$ if it is non-null.

Proof. i. The restricted intersection of the family $\{(\Omega_i, K_i) \mid i \in \mathcal{J}\}$ with $\bigcap_{i \in \mathcal{J}} K_i \neq \emptyset$ given by the soft set $\bigcap^{\sim}{}_{i \in \mathcal{J}} (\Omega_i, K_i) = (\Omega, K)$ such that $\bigcap_{i \in I} \Omega_i(\alpha)$ for all $\alpha \in K$. Take $\alpha \in \text{Supp}(\Omega, K)$. Together with the hypothesis, $\bigcap_{i \in \mathcal{J}} \Omega_i(\alpha) \neq \emptyset$, implies that $\Omega_i(\alpha) \neq \emptyset$ for all $i \in \mathcal{J}$. Since $\{(\Omega_i, K_i \mid i \in \mathcal{J}\}$ is a non-empty family of soft dimonoids over $\mathcal{D}$, it is then easy to see that $\Omega_i(\alpha)$ is a subdimonoid of $\mathcal{D}$ for all $i \in \mathcal{J}$. Morever, $\bigcap_{i \in \mathcal{J}} \Omega_i(\alpha)$ is a subdimonoid of $\mathcal{D}$ too.

Consequently, (Ω, K) is a soft dimonoid over $\mathcal{D}$.

ii. It is similar to the proof of previous case. , (Oguz, 2023).

3.1.7 Theorem Let $\{(\Omega_i, K_i) \mid i \in \mathcal{I}\}$ be a non-empty family of soft dimonoids over $\mathcal{D}$. Then the $\wedge$-intersection $\wedge^{\sim}{}_{i\in\mathcal{I}}(\Omega_i, K_i)$ is a soft dimonoid over $\mathcal{D}$ if it is non-null.

Proof. Assume $(\Omega, K = \wedge^{\sim}{}_{i\in\mathcal{I}}(\Omega_i, K_i, \tau)$ for a non-empty family $\{(\Omega_i, K_i) \mid i \in \mathcal{I}\}$ of soft dimonoids over $\mathcal{D}$. Choose $\alpha \in \mathrm{Supp}(\Omega, K)$. It holds from the the assumption $\bigcap_{i\in\mathcal{I}} \Omega_i(\alpha_i) \neq \emptyset$ that $\Omega_i(\alpha_i) \neq \emptyset$ for all $i \in \mathcal{I}$ and $(\alpha_i)_{i\in\mathcal{I}} \in K_i$. Hence, $\Omega_i(\alpha_i)$ is a subdimonoid of $\mathcal{D}$ for all $i \in \mathcal{I}$ so that their intersection is a subdimonoid of $\mathcal{D}$ too. Hence, (Ω, K) is a soft dimonoid over $\mathcal{D}$. The proof is complete. , (Oguz, 2023).

In this section, we study the soft subdimonoids and describe the notion of soft dimonoid homomorphisms.

3.1.8 Definition Let (Ω, K) and (Ω', R) be any two soft dimonoids over $\mathcal{D}$ and $\mathcal{F}$, respectively. Then, (Ω', R) is said to be a soft subdimonoid of (Ω, K) if $R \subset K$ and $\Omega'(b)$ is a subdimonoid of $\Omega(b)$ for all $b \in S, P$ (Ω', R).

3.1.9 Example For the soft dimonoid (Ω, K) over $\mathcal{D}$, assume (Ω', R) be a soft dimonoid over the idempotent dimonoid $\mathcal{F}$ and $R \subset K$. It is easy to check that (Ω', R) is a subdimonoid of (Ω, K)

From the above definition, it is straightforward to see that.

3.1.10 Theorem Let $(\Omega, K), (\Omega', R)$ and (Ω'', S) be any three soft dimonoids over D_1, D_2 and D_3, respectively. If (Ω'', S) is a soft subdimonoid of (Ω', R) and (Ω', R) is a soft subdimonoid of (Ω, K), then (Ω'', S) is a soft subdimonoid of (Ω, K)

Proof. It is immediate.

3.1.11 Theorem Let (Ω, K) and (Ω', R) be two soft dimonoids over $\mathcal{D}$. Then (Ω', R) is a soft subdimonoid of (Ω, K) if (Ω', R) is a soft subset of (Ω, K).

Proof. Assume that (Ω, K) and (Ω', R) are two soft dimonoids over $\mathcal{D}$. In this case, if (Ω', R) is a soft subset of (Ω, K), we obtain $R \sqsubseteq K$ and $\Omega'(b) \sqsubseteq \Omega(b)$ for all $b \in \mathrm{Supp}(\Omega', R)$. It follows that $\Omega'(b)$ is a subdimonoid of $\Omega(b)$. Therefore, (Ω', R) is a soft subdimonoid of (Ω, K). , (Oguz, 2023).

3.1.12 Theorem Let (Ω, K) be a soft dimonoid over $\mathcal{D}$ and (Ω', R) be a soft subdimonoid of (Ω, K) (Oguz, 2023).

i. The restricted intersection of (Ω, K) and (Ω', R) is a soft subdimonoid of (Ω, K) if it is non-null.

ii. The restricted union of (Ω, K) and (Ω', R) is a soft subdimonoid of (Ω, K) if it is non-null.

Proof. i. Suppose (Ω', R) be a soft subdimonoid of (Ω, K). If it is non-null, then $R \subseteq K$ and $\Omega'(a)$ is a subdimonoid of $\Omega(a)$ for all $a \in \mathrm{Supp}(\Omega', R)$. Thus, it is immediate to check that $R \cap K \subseteq K$ and $\Omega'(a) \cap \Omega(a)$ is a subdimonoid of $\Omega(a)$ for all $a \in \mathrm{Supp}(\Omega', R)$. So, the restricted intersection $(\Omega, K) \tilde{\cap} (\Omega', R)$ is a soft subdimonoid of (Ω, K). , (Oguz, 2023).

ii. It is similar to previous proof.

3.1.13 Definition Let (Ω, K) be a soft dimonoid over $\mathcal{D}$ and (Ω_i, K_i) be soft subdimonoids of (Ω, K) for $\in \mathcal{J}$. Then, (Ω, K) is said to be a soft diband of soft subdimonoids (Ω_i, K_i) for $i \in I$ if the following conditions hold, (Oguz, 2023):

i. $(\Omega, K) = \cup^{\sim}{}_{i \in \mathcal{J}} (\Omega_i, K_i)$

ii. $(\Omega_i, K_i) \cap^{\sim} (\Omega_j, K_j) = \widetilde{\emptyset}$ for all $i \neq j$.

iii. For all $i, j \in I$, there is any one $n, m \in I$ such that $(\Omega_i, K_i) \circ (\Omega_j, K_j) \widetilde{\subseteq} (\Omega_n, K_n)$ and $(\Omega_i, K_i) \diamond (\Omega_j, K_j) \widetilde{\subseteq} (\Omega_m, K_m)$.

Now we would like to study soft homomorphism between soft dimonoids.

3.1.14 Definition Let (Ω, K) and (Ω', R) be soft dimonoids over $\mathcal{D}$ and $\mathcal{D}'$, respectively. Let $\delta: K \longrightarrow R$ and $\lambda: \mathcal{D} \longrightarrow \mathcal{D}'$ be two mappings. In this case, the pair (δ, λ) is called a soft homomorphism if the following conditions are satisfied:

i. δ is a dimonoid homomorphism.

ii. $\lambda(\Omega(a)) = \Omega'(\delta(a))$ for all $a \in \mathrm{Supp}(\Omega, K)$. , (Oguz, 2023).

Also, we establish a new category whose objects are soft dimonoids and whose arrows are soft homomorphisms.

From the above example, we have directly the following consequence:

3.1.15 Theorem Let $(\Omega, K), (\Omega', R)$ and (Ω'', S) be soft topological dimonoids over $\mathcal{D}, \mathcal{D}'$ and $\mathcal{D}''$, respectively. If $(\delta, \lambda): (\Omega, K) \longrightarrow (\Omega', R)$ and $(\delta', \lambda'): (\Omega', R) \longrightarrow (\Omega'', S)$ are two soft homomorphisms, then $(\delta' \circ \delta, \lambda' \circ \lambda): (\Omega, K) \longrightarrow (\Omega'', S)$ is a soft homomorphism, (Oguz, 2023).

Let us draw the final corollary.

3.1.16 Theorem Let the pair (δ, λ) be a soft homomorphism from the soft dimonoids (Ω, K) and (Ω', R) over $\mathcal{D}$ and $\mathcal{D}'$, respectively. Then $(\delta^{-1}(\Omega'), K)$ is a soft dimonoid over $\mathcal{D}$ if it is non-null, (Oguz, 2023).

Proof. Let (Ω, K) and (Ω', R) be two soft dimonoids over $\mathcal{D}$ and $\mathcal{D}'$, respectively. Then it is easy to check that

$$\lambda\big(\mathrm{Supp}\big(\delta^{-1}(\Omega', R) = \lambda^{-1}\big(\mathrm{Supp}(\Omega', R)\big)$$

for all $b \in \mathrm{Supp}(\Omega', R)$. Taking $a \in \mathrm{Supp}(\delta^{-1}(\Omega', K)$, we get $\lambda(a) \in \mathrm{Supp}(\Omega', R)$. So, the nonempty set $\Omega'(\lambda(a))$ is a subdimonoid of $\mathcal{D}'$. Morever, since δ is a dimonoid homomorphism, we conclude that $\delta^{-1}(\Omega'(\lambda(a))) = \delta^{-1}(\Omega'(a))$ is a subdimonoid of $\mathcal{D}$. This implies that $(\delta^{-1}(\Omega'), K)$ is a soft dimonoid over $\mathcal{D}$. , (Oguz, 2023).

4 FINDINGS

4.1 Soft Hyperdimonoids

In this section, soft hyperdimonoids are introduced and the structural features of this concept are examined in detail.

4.1.1 Definition: Let Q be a hyperdimonoid and let $\mathrm{P}(Q)$ denotes the set of all subhyperdimonoids of Q. A pair (T,K) is defined as a soft hyperdimonoid over $\mathcal{H}$, where T is a mapping given by

$$\mathrm{T}\colon K \longrightarrow \mathrm{P}(\mathrm{Q})$$

and K is a set of parameters, if $K(\omega)$ is a subhyperdimonoid of Q for all $\omega \in Kupp\ (\mathrm{T},K)$.

In this instance, it is evident that if each $\mathrm{T}(\omega)$ is commutative as a hyperdimonoid, the soft hyperdimonoid Q is said to be commutative. Furthermore, a soft hyperdimonoid Q can be considered as a parameterized *collection* of subhyperdimonoids of the hyperdimonoid Q.

In the following sections, the notation $(\mathrm{Q},\mathrm{T},K)$ will be used to refer to the soft hyperdimonoid (T,K) over the hyperdimonoid Q.

4.1.2 Example: Take $Q = \{$ ᶍ, ꞗ$\}$. Consider a hyperdimonoid (Q,⊙,⊙) with the binary operation as follows:

Table A:

⊙	ᶍ	ꞗ
ᶍ	Q	Q
ꞗ	Q	Q

Now define the mapping $\mathrm{T}\colon K = \mathbb{N} \longrightarrow \mathcal{P}(\mathrm{Q})$ letting $\mathrm{T}(\omega) = \{$ᶍ$\}$ for all $\omega \in \mathbb{N}$It is a straightforward process to verify that $(\mathrm{Q},\mathrm{T},\mathbb{N})$ is a soft hyperdimonoids.

4.1.3 Defition: Let (T, K) be an soft hyperdimonoid over Q and $\mathcal{G} \subseteq Q$. Then, $(\mathrm{T}|_{\mathcal{G}}, \mathrm{K})$ is a soft hyperdimonoid over $\mathcal{G}$ if it is non-null.

4.1.4 proposition: Let $(Q_1, \mathrm{T}_1, \mathrm{K})$ and (Q_2, T_2, V) be two soft hyperdimonoids. The product of them is defined as $(Q_1, \mathrm{T}_1, \mathrm{K}) \times (Q_2, \mathrm{T}_2, V) = (Q_1 \times Q_2, \mathrm{T}, \mathrm{K} \times V)$, where $\mathrm{T}(ҳ, ե) = \mathrm{T}_1(ҳ) \times \mathrm{T}_2(ե)$ for all $(ҳ, ե) \in \mathrm{K} \times V$.

In light of the aforementioned definition, the following proposition can be put forth. The product of any two soft hyperdimonoids is itself a soft hyperdimonoid.

Proof. Consider the soft hyperdimonoids (Q_1, T_1, K) and $(\mathrm{Q}_2, \mathrm{T}_2, V)$. Then,

$$\mathrm{T}_1 : K \longrightarrow \mathcal{P}(Q_1)$$

$$\varpi \mapsto \mathrm{T}_1(\varpi)$$

and

$$\mathrm{T}_2 : V \longrightarrow \mathcal{P}(Q_2)$$

$$\omega \mapsto \mathrm{T}_2(\omega)$$

such that $\mathrm{T}_1(\varpi)$ is a subhyperdimonoid of the hyperdimonoid Q_1 for all $\varpi \in K$ and $\mathrm{T}_2(\omega)$ is a subhyperdimonoid of the hyperdimonoid Q_2 for all $\omega \in V$. Using these mappings, we define T by

$$\mathrm{T} : K \times V \longrightarrow P(\mathrm{Q}_1 \times \mathrm{Q}_2)$$

$$(\varpi, \omega) \mapsto \mathrm{T}(\varpi, \omega) = \mathrm{T}_1(\varpi) \times \mathrm{T}_2(\omega)$$

It can be observed that $\mathrm{T}_1(\varpi) \times \mathrm{T}_2(\omega)$ is also a subhyperdimonoid of the product hyperdimonoid $Q_1 \times \mathrm{Q}_2$ for all $(\varpi, \omega) \in K \times V$. This implies that $(\mathrm{Q}_1 \times \mathrm{Q}_2, \mathrm{T}, K \times V)$ is a soft hyperdimonoid.

Let us give the following theorem, whose proofs are easily demonstrated:

4.1.5 Defition : Let $(Q, \mathrm{T}, \mathrm{K})$ be a soft hyperdimonoid with the hyperoperation $\circledcirc$ and $\diamond$, and $\mathrm{G}, \text{Ꞙ}, \text{Ư} \in \mathcal{P}(\mathrm{T}(\omega)$ for all $\omega \in \mathrm{K}$. Then

i. $\mathrm{G} \diamond (\text{Ꞙ} \diamond \text{Ư}) = (\mathrm{G} \circledcirc \text{Ꞙ}) \diamond \text{Ư}$

ii. $\mathrm{G} \diamond (\text{Ꞙ} \circledcirc \text{Ư}) = (\mathrm{G} \diamond \text{Ꞙ}) \circledcirc \text{Ư}$

iii. $\mathrm{G} \diamond (\text{Ꞙ} \diamond \text{Ư}) = (\mathrm{G} \diamond \text{Ꞙ}) \diamond \text{Ư}$

iv. $\mathrm{G} \circledcirc (\text{Ꞙ} \diamond \text{Ư}) = (\mathrm{G} \circledcirc \text{Ꞙ}) \circledcirc \text{Ư}$

v. $\mathrm{G} \circledcirc (\text{Ꞙ} \circledcirc \text{Ư}) = (\mathrm{G} \circledcirc \text{Ꞙ}) \circledcirc \text{Ư}$

Some generalisations for a non-empty family of soft hyperdimonoids are given in the following theorems.

4.1.6 Theorm : *Let* $\{(T_i, \mathcal{S}_i) \mid i \in \dot{\mathrm{X}}\}$ *be a non-empty collection of soft hyperdimonoids over* Q.

ii. The restricted intersection of the *collection* $\{(\mathrm{T}_i, \mathcal{S}_i) \mid i \in \dot{\mathrm{X}}\}$ with $\bigcap_{i \in \dot{\mathrm{X}}} \mathcal{S}_i \neq \emptyset$ is a soft hyperdimonoid over Q if it is non-null.

iii. The extended intersection of the *collection* $\{(\mathrm{T}_i, \mathcal{S}_i) \mid i \in \dot{\mathrm{X}}\}$ is a soft hyperdimonoid over Q if it is non-null.

Proof. **i.** The restricted intersection of the collection $\{(\mathrm{T}_i, \mathrm{K}_i) \mid i \in \dot{\mathrm{X}}\}$ with $\bigcap_{i \in \dot{\mathrm{X}}} \mathrm{K}_i \neq \emptyset$ defined by the soft set $\widetilde{\bigcap_{\iota \in \dot{\mathrm{X}}}} (\mathrm{T}_i, \mathrm{K}_i) = (\mathrm{T}, \mathrm{K})$ such that $\bigcap_{i \in \dot{\mathrm{X}}} \mathrm{T}_i(\omega)$ for all $\omega \in \mathcal{S}$. Choose $\omega \in \mathrm{K}upp(\mathrm{T}, \mathrm{K})$. Together with the hypothesis, $\bigcap_{i \in \dot{\mathrm{X}}} \mathrm{T}_i(\omega) \neq \emptyset$, implies that $\mathrm{T}_i(\omega) \neq \emptyset$ for all $i \in \dot{\mathrm{X}}$. Since $\{(\mathrm{T}_i, \mathrm{K}_i \mid i \in \dot{\mathrm{X}}\}$ is a non-empty family of soft hyperdimonoids over Q, it is then straightforward to ascertain that $\mathrm{T}_i(\omega)$ is a subhyperdimonoid of Q for all $i \in \dot{\mathrm{X}}$. In addition, $\bigcap_{i \in \dot{\mathrm{X}}} \mathrm{T}_i(\omega)$ is a subhyperdimonoid of Q too. Hence, (T, K) is a soft hyperdimonoid over Q.

ii. The proof is analogous to that of the previous case

4.2 Soft Subhyperdimonoids

4.2.1 Let (T, K) be a soft hyperdimonoid over Q. Then, (T_1, V) is defined as a soft subhyperdimonoid of (T, K) if $V \subseteq \mathrm{K}$ and $\mathrm{T}_1(\omega)$ is a subhyperdimonoid of $\mathrm{T}(\omega)$ for all $\omega \in Supp(\mathrm{T}_1, V)$.

4.2.2 Example : Take $Q = \{$ӽ, ꞗ$\}$. Consider a hyperdimonoid $(Q, \circledcirc, \circledcirc)$ with the binary operation as follows: Table B:

$\circledcirc$	ӽ	ꞗ
ӽ	Q	Q
ꞗ	Q	Q

Now define the mapping $\mathrm{T}: K = \mathbb{Z} \longrightarrow \mathcal{P}(\mathrm{Q})$ letting $\mathrm{T}(\omega) = \mathcal{H}$ for all $\omega \in \mathbb{Z}$. Also define $\mathrm{T}_1: V = \mathbb{N} \longrightarrow \mathcal{P}(\mathrm{Q})$ letting $\mathrm{T}_1(\omega) = \{$ӽ$\}$ for all $\omega \in \mathbb{N}$. It is straightforward to check that $(\mathrm{Q}, \mathrm{T}, \mathbb{Z})$ and $(\mathrm{T}_1, \mathrm{T}_2, \mathbb{N})$ are soft hyperdimonoids. Besides $(\mathrm{Q}, \mathrm{T}_1, \mathbb{N})$ is a soft subhyperdimonoid of $(\mathrm{Q}, \mathrm{T}, \mathbb{Z})$.

4.2.3 Theorem: Let (T, K), (T_1, R) and (T_2, V) be any three soft hyperdimonoids over Q. If (T_1, R) is a soft subhyperdimonoid of (T, K) and (T_2, V) is a soft subhyperdimonoid of (T_1, R), then (T_2, V) is a soft subhyperdimonoid of (T, K).

Proof. Straightforward.

4.2.4 Theorem: Let (T, K) and (T_1, V) be two soft hyperdimonoids over Q. Then (T_1, V) is a soft subhyperdimonoid of (T, K) if (T_1, V) is a soft subset of (T, K).

Proof. Assume that (T, K) and (T_1, V) are two soft hyperdimonoids over Q. Then, if (T_1, R) is a soft subset of (T, K), we have $V \subseteq K$ and $\mathrm{T}_1(\omega) \subseteq \mathrm{T}(\omega)$ for all $\omega \in Supp(\mathrm{T}_1, V)$. It follows that $\mathrm{T}_1(\omega)$ is a subhyperdimonoid of $\mathrm{T}(\omega)$. Hence, (T_1, V) is a soft subhyperdimonoid of (T, K).

The soft homomorphism between soft hyperdimonoids is now going to be defined.

4.2.5 Definition : Let (T, K) and (T_1, V) be soft hyperdimonoids over Q and Q_1, respectively. Let $\lambda: \mathrm{K} \longrightarrow V$ and ꞁ: $Q \longrightarrow Q_1$ be two mappings. In this case, the pair (ꞁ, λ) is defined as a soft homomorphism if the subsequent conditions hold:

i. ꞁ is a strong hyperdimonoid homomorphism;

ii. $\mathrm{T}_1(\lambda(\omega)) =$ ꞁ$(\mathrm{T}(\omega))$ for all $\omega \in Supp\ (\mathrm{T}, K)$.

Note that a soft homomorphism $(Ղ, \lambda)$ is a mapping of soft hyperdimonoids. A new category is thus established, whose objects are soft hyperdimonoids and whose arrows are soft homomorphisms.

4.2.5 Example : Choose $Q = \{ӽ, ե\}$. Consider a hyperdimonoid $(Q, \odot, \odot)$ with the binary operation as follows:

Table C:

$\odot$	ӽ	ե
ӽ	Q	Q
ե	Q	Q

Now define the mapping $\mathrm{T}: K = \mathbb{N} \longrightarrow \mathcal{P}(\mathrm{Q})$ letting $\mathrm{T}(\omega) = \mathrm{Q}$ for all $\omega \in \mathbb{N}$. Clearly, (T, K) is a soft hayperdimonoid on Q. Define the mapping $Ղ: \mathrm{Q} \longrightarrow \mathrm{Q}$, where $Ղ(ӽ) =$ ե and $Ղ($ ե$) =$ ӽ so that ψ is a strong hyperdimonoid homomorphism. Further, define the identity mapping $\mathcal{J}d: K \longrightarrow K$. It can easily be checked that the pair $(Ղ, \mathcal{J}d)$ is a soft homomorphism from (T, K) to $(\mathbb{N}, V)$.

4.2.6 Example: Let (T_1, V) be a soft subhyperdimonoid of (T, K) over Q. Considering the inclusion map $\lambda: V \longrightarrow \mathrm{K}$ and the identity mapping $\mathcal{J}d: Q \longrightarrow Q$, it can be reasonably deduced that pair $(\mathcal{J}d, \lambda)$ is a soft homomorphism from (T_1, V) to (T, K).

The aforementioned definition leads directly to the following consequence:

4.2.7 Theorem : Let (T, K), (T_1, V) and $(\mathrm{T}_2, \mathrm{K}_2)$ be soft hyperdimonoids over Q, Q_1 and Q_2, respectively. If $(Ղ, \lambda): (\mathrm{T}, \mathrm{K}) \longrightarrow (\mathrm{T}_1, V)$ and $(Ղ_1, \lambda_1): (\mathrm{T}_1, V) \longrightarrow (\mathrm{T}_2, \mathrm{K}_2)$ are two soft homomorphisms, then $(Ղ_1 \odot Ղ, \lambda_1 \odot \lambda): (\mathrm{T}, \mathrm{K}) \longrightarrow (\mathrm{T}_2, \mathrm{K}_2)$ is a soft homomorphism.

Let us conclude by presenting the final corollary.

4.2.8 Theorem: Let the couple $(Ղ, \lambda)$ be a soft homomorphism from the soft hyperdimonoids (T, K) and (T_1, V) over Q and Q_1, respectively. Then $(Ղ^{-1}(\mathrm{T}_1), \mathrm{K})$ is a soft hyperdimonoid over Q if it is non-null.

Proof. Let (T, K) and (T_1, V) be two soft hyperdimonoids over Q and Q_1, respectively. Then it is easy to check that

$$\lambda(Supp\ (\daleth^{-1}(\mathrm{T}_1, R) = \lambda^{-1}(Supp\ (\mathrm{T}_1, V))$$

Taking $\omega \in Kupp\ (\daleth^{-1}(\mathrm{T}_1, K)$, we get $\lambda(\omega) \in Supp\ (\mathrm{T}_1, V)$. So, the nonempty set $\mathrm{T}_1(\lambda(\omega))$ is a subhyperdimonoid of Q_1. Also, since $\daleth$ is a strong hyperdimonoid homomorphism, we conclude that $\daleth^{-1}(\mathrm{T}_1(\lambda(\omega))) = \daleth^{-1}(\mathrm{T}_1(\omega))$ is a subhyperdimonoid of Q. This implies that $(\daleth^{-1}(\mathrm{T}_1), K)$ is a soft hyperdimonoid over Q.

5 DISCUSSION

This work advances abstract algebra by unifying three previously distinct strands soft set theory, operations, dimonoids and hyperoperations into the single notion of a soft hyperdimonoid. Theoretical coherence is achieved by anchoring every new object in well-established categorical ideas: each definition immediately yields closure, associativity and homomorphic images, ensuring that the enlarged algebra behaves predictably with respect to familiar constructions

The illustrative examples, though modest in scale, are strategically chosen to expose edge cases where traditional binary algebra falters, particularly in modelling ambiguity. For instance, allowing a product to return a set of elements captures nondeterminism that arises in fuzzy control or concurrent computation; layering soft set parameters over these operations then localizes uncertainty to context-dependent “slices” of the structure. This dual flexibility positions soft hyperdimonoids as promising tools for information systems that juggle both vagueness and multiple composition rules. Limitations stem from the study’s purely theoretical scope. No computational complexity analysis, algorithmic representation or empirical application is offered.

Moreover, while homomorphism theory is explored, categorical relationships to other soft algebraic objects (e.g., neutrosophic or intuitionistic counterparts) remain open. Addressing these gaps would solidify the framework’s relevance beyond pure mathematics and facilitate its uptake in applied domains.

6 CONCLUSION

“This thesis has developed and explored the foundations of advanced algebraic structures, beginning with soft sets and extending through dimonoids, hyperdimonoids, and their soft counterparts. This piece tried to integrate the flexibility of soft set theory with the dual operation nature of dimonoids and the generalized outcomes of hyperstructures, we established a comprehensive framework capable of addressing uncertainty, nondeterminism, and parameter variation within algebraic systems.

Starting from fundamental concepts, we built the theory step-by-step, introducing soft hyperdimonoids as natural extensions. The study emphasized not only the internal structure of these systems through detailed definitions, propositions, and examples but also their interrelationships via soft homomorphisms. These mappings revealed critical structure-preserving properties and enabled a deeper understanding of how soft and hyperdimonoid frameworks behave under transformations.

Soft hyperdimonoids, with their set-valued operations, provided a means to model nondeterministic processes that traditional binary operations cannot fully capture. The integration of soft sets allowed us to incorporate parameterized variation, thus offering an even richer, more adaptable algebraic model. Throughout the work, theoretical consistency was demonstrated rigorously, supported by carefully chosen examples that illustrated complex behaviors within these new structures.

While the focus of this thesis was theoretical, the implications extend toward practical fields such as fuzzy logic, decision support systems, information science, and computational algebra, where handling uncertainty and variability is essential. The concepts developed here not only expand the theoretical landscape of algebraic structures but also pave the way for future applications and deeper investigations into soft and hyper algebraic systems.

We defined soft hyperdimonoids and soft subhyperdimonoids, with founding the relation between softs and dimonoids

In conclusion, the research presented establishes a strong theoretical foundation for soft hyperdimonoids, opening promising avenues for further study and practical innovation in both pure and applied mathematical domains.”

7 RECOMMENDATIONS

“Further studies can be conducted to explore deeper algebraic properties of soft hyperdimonoids, such as identities, inverses, extending known results from classical and soft algebra to soft hyperdimonoids can enrich the theory.

The development of a categorical framework for soft hyperdimonoids would be a valuable direction. Defining objects, morphisms, and functors in this context can help unify various algebraic structures under a formal mathematical setting.

The study of homomorphisms between soft hyperdimonoids can be expanded by classifying soft monomorphisms, epimorphisms, and isomorphisms. This could provide insight into the structure-preserving properties and equivalence of soft hyperdimonoid systems.

Further investigation into substructures such as soft subhyperdimonoids, and their intersections/unions can enhance understanding of the internal composition of soft hyperdimonoids.”

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