

**DECOMPOSITIONS OF 2-GENERATOR FREE KLEINIAN GROUPS
AND HYPERBOLIC DISPLACEMENTS**



BY

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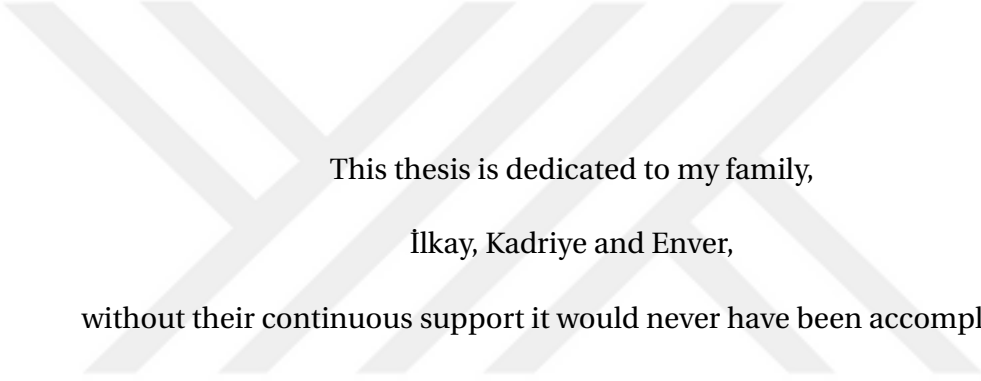
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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Chicago, 2007

Chicago, Illinois



This thesis is dedicated to my family,
İlkay, Kadriye and Enver,
without their continuous support it would never have been accomplished.

ACKNOWLEDGMENTS

I wish to express my sincerest thanks to my thesis advisor Peter B. Shalen. In the process of providing this thesis, I benefited immensely from his experience and knowledge. Without his understanding and consideration, this thesis would likely not have existed. I am also grateful to him for the financial support he generously provided from his grant in the summer terms of 2005, 2006 and 2007. In that matter, I would like to extend my profound thanks to Marc E. Culler and Ian Agol as the other participants of that grant. I also would like to thank Marc E. Culler for his thorough examination of this thesis. His corrections and suggestions made this work more accessible and hence increased its quality. I also would like to express my appreciation to Benson Farb, Steve Hurder and John Wood for their assistance in the thesis committee. I am grateful to them for the time they spared to attend the defense of this thesis in their busy schedule. I also would like to thank Mustafa Korkmaz, Turgut Önder and Yıldıray Ozan for their support that helped me to initiate my graduate education.

Finally, I wish to thank from the bottom of my heart to my beloved family, İlkey, Kadriye and Enver. Without their encouragement and support, I would have never been able to pursue an education in mathematics.

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SUMMARY

In this thesis, I prove that every point of the hyperbolic 3-space $\mathbf{H}^3$ is moved a distance at least $\frac{1}{2} \log(5 + 3\sqrt{2})$ by either ξ , η or their multiplication $\xi\eta$ if $\Gamma = \langle \xi, \eta \rangle$ is a torsion-free, discrete, non-cocompact subgroup of $\text{Isom}^+(\mathbf{H}^3)$ without parabolics. This result, which is also referred to as the $\frac{1}{2} \log(5 + 3\sqrt{2})$ theorem, is a refinement of the log3 theorem proved by Marc E. Culler and Peter B. Shalen (5). Therefore, it requires the use of the same ingredients introduced in the proof of the log3 theorem. In §1.1 and §1.2, I give the necessary definitions and the ingredients used in the proof of the log3 theorem together with a brief summary of the proof of the log3 theorem. The proof of the log3 is considered in two cases either $\Lambda_\Gamma = S_\infty$ or $\Lambda_\Gamma \neq S_\infty$ where Λ_Γ and S_∞ denote the limit set of Γ and the sphere at infinity, respectively. In §1.3, I propose a new calculation technique to evaluate the number log3 in the case $\Lambda_\Gamma = S_\infty$ which leads to a step-by-step method of finding a lower bound for the maximum of displacements under any given set of isometries of $\mathbf{H}^3$. I give an explicit list of these steps in §1.3.

In Ch.2, I complete in detail some of these steps that I outlined in §1.3 for the isometries ξ and η . In particular, I determine the displacement functions f_1 , f_2 , f_3 and f_4 for the paradoxical decomposition $\Gamma_{\mathcal{D}_p}$ of Γ in §2.1. I show that the calculation of the number log3 is a consequence of the statement that requires the calculation of the infimum of maximum of the functions f_1 and f_2 over a 3-dimensional simplex Δ^3 in §2.2. Then I present a slightly different proof of the fact that the conclusion of the log3 theorem holds when every Γ -invariant conformal density is a constant multiple of the area density in §2.3. This concludes the case $\Lambda_\Gamma = S_\infty$ for the log3 theorem.

SUMMARY (Continued)

In Ch. 3, I carry out the details of the steps that I outlined in §1.3 for the isometries ξ , η and $\xi\eta$. I introduce a decomposition $\Gamma_{\mathcal{D}}$ of Γ which concentrates on the isometries ξ , η and $\xi\eta$ in §3.1. Using the group-theoretical relations in the decomposition $\Gamma_{\mathcal{D}}$ studied in §3.2, I determine the displacement functions $f_1, f_2, \dots, f_{14}$ for the isometries ξ , η and $\xi\eta$ in §3.3. Then in §4.1.1-§4.1.5, I analyze the relationships between these functions using their first-order partial derivatives and calculate the infimum of maximum of the displacement functions $f_1, f_2, \dots, f_{14}$ over a 7-dimensional simplex Δ^7 in §4.1.6. In the last section of Ch. 4, I show that the conclusion of the $\frac{1}{2} \log(5 + 3\sqrt{2})$ theorem holds when every every Γ -invariant conformal density is a constant multiple of the area density. This concludes the case $\Lambda_{\Gamma} = S_{\infty}$ for the $\frac{1}{2} \log(5 + 3\sqrt{2})$ theorem.

If V denotes the space $PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$ with the complex topology then the set $\mathcal{D}$ of all points $(\xi', \eta') \in V$ such that the group $\langle \xi', \eta' \rangle$ generated by ξ' and η' is a free Kleinian group of rank 2 is a closed subset of V (4). If $\mathfrak{G}\mathfrak{F}$ denotes the set of all points $(\xi', \eta') \in \mathcal{D}$ such that $\Lambda_{\Gamma'} \neq S_{\infty}$ where $\Gamma' = \langle \xi', \eta' \rangle$, then $\mathfrak{G}\mathfrak{F}$ is an open subset of V ((7), Theorem 8.1). The frontier of $\mathfrak{G}\mathfrak{F}$ in V will be denoted by $\mathfrak{B} = \overline{\mathfrak{G}\mathfrak{F}} - \mathfrak{G}\mathfrak{F}$. If (ξ', η') is in $\mathfrak{B}$, then the limit set of the group $\langle \xi', \eta' \rangle$ is the sphere at infinity. In Ch. 5, I show that the function

$$\begin{aligned} f_{z_0} &: \mathcal{D} \longrightarrow \mathbf{R} \\ (\xi', \eta') &\mapsto \max(\text{dist}(z_0, \xi' \cdot z_0), \text{dist}(z_0, \eta' \cdot z_0), \text{dist}(z_0, \xi'\eta' \cdot z_0)). \end{aligned}$$

has no local minimum on the set $\mathfrak{G}\mathfrak{F}$ which concludes the proof of the $\frac{1}{2} \log(5 + 3\sqrt{2})$ theorem in the case $\Lambda_{\Gamma} \neq S_{\infty}$ since there exists a dense subset $\mathfrak{C}$ of $\mathfrak{B}$ such that every Γ' -invariant conformal density is a constant multiple of the area density for every $(\xi', \eta') \in \mathfrak{C}$.

CHAPTER 1

INTRODUCTION

Let $U^3 = \{(x, y, t) \in \mathbb{R}^3 | t > 0\}$ be the upper half 3-space and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The set denoted by $\hat{\mathbb{C}}$ is called the sphere at infinity when regarded as lying on the boundary of U^3 as $t = 0$. Let $\text{PSL}(2, \mathbb{C})$ be the quotient of the group of all 2×2 matrices with complex entries and determinant 1 by its center $\{\pm I\}$. Then, any element γ of $\text{PSL}(2, \mathbb{C})$ has the form

$$\gamma = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$$

for some $a, b, c, d \in \mathbb{C}$ so that $ad - bc = 1$. Each element $\gamma \in \text{PSL}(2, \mathbb{C})$ acts on $\hat{\mathbb{C}}$ when it is identified with the linear fractional transformation

$$z \mapsto \frac{az + b}{cz + d}.$$

The action of each γ on the sphere at infinity extends to an action on U^3 as follows: γ is a product of an even number of inversions in circles and lines in $\mathbb{C}$. Each circle C and line ℓ in $\mathbb{C}$ has a unique hemisphere $\hat{C}$ and a plane $\hat{\ell}$ in U^3 , orthogonal to C and meeting $\mathbb{C}$ in C or ℓ . The extension of γ is the corresponding product of inversions in $\hat{C}$ and reflections in $\hat{\ell}$. This extension is called the Poincaré extension of γ . A subgroup of $\text{PSL}(2, \mathbb{C})$ is said to be *non-elementary* if it has no finite orbit in its action on the union of the upper half 3-space U^3 and the sphere at infinity $\hat{\mathbb{C}}$.

The upper half 3-space U^3 is called the *upper half-space model* for the hyperbolic 3-space $\mathbf{H}^3$ (i.e., the unique three-dimensional connected and simply connected Riemannian manifold with constant sectional curvature -1) if it is equipped with the hyperbolic metric induced from the line element

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}.$$

The line element induces a distance metric on U^3 so that every Cauchy sequence in U^3 converges with respect to this metric. The inversions in hemispheres and reflections in planes are isometries of U^3 with the hyperbolic metric and generate the full group of orientation-preserving isometries denoted by $\text{Isom}^+(U^3)$. Under the Poincaré extension, $\text{PSL}(2, \mathbb{C})$ is isomorphic to $\text{Isom}^+(U^3)$, hence, is isomorphic to the group $\text{Isom}^+(\mathbf{H}^3)$ of orientation-preserving isometries of $\mathbf{H}^3$.

A subgroup Γ of $\text{Isom}^+(\mathbf{H}^3)$ is called a *Kleinian group* if it is isomorphic to a discrete, nonelementary subgroup of $\text{PSL}(2, \mathbb{C})$. A Kleinian group Γ acts discontinuously on $\mathbf{H}^3$ which means that the set $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$ is finite for any compact subset K of $\mathbf{H}^3$ ((9), Ch.5). Since Γ acts discontinuously on $\mathbf{H}^3$, it is possible to construct a closed subset F of $\mathbf{H}^3$ for the action of Γ on $\mathbf{H}^3$ so that $\cup_{\gamma \in \Gamma} \gamma \cdot F = \mathbf{H}^3$ and $\text{int}(F) \cap \gamma \cdot \text{int}(F) = \emptyset$ for every non-identity $\gamma \in \Gamma$ and the boundary of F has measure zero ((6), Ch.1). The set F is called a *fundamental domain* for the action of Γ on $\mathbf{H}^3$ ((9), Ch.6). The group Γ is said to be *co-compact* if Γ has a compact fundamental domain F .

An orientable hyperbolic 3-manifold M may be regarded as the quotient of the hyperbolic 3-space $\mathbf{H}^3$ by a discrete group Γ of orientation-preserving isometries of $\mathbf{H}^3$. If Γ is a torsion free

Kleinian group and $M = \mathbf{H}^3/\Gamma$, then it is called *topologically tame* if M is homeomorphic to the interior of a compact 3-manifold.

It is possible to classify each non-identity element γ of Γ using the set of fixed points of γ . The isometry γ is said to be *elliptic* if and only if it has a finite orbit in $\mathbf{H}^3$. It is said to be *parabolic* if and only if γ fixes a point of $\hat{\mathbb{C}}$ in its action on $\hat{\mathbb{C}}$ and has no other finite orbits in $U^3 \cup \hat{\mathbb{C}}$. The isometry γ is called *loxodromic* or *hyperbolic* if it is neither elliptic or parabolic. With the terminology set as above, I prove the following result in this thesis

Let ξ and η be non-commuting isometries of $\mathbf{H}^3$. Suppose that ξ and η generate a torsion-free discrete group which is topologically tame, is not co-compact and contains no parabolics. Then for any $z_0 \in \mathbf{H}^3$ we have

$$\max(\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0), \text{dist}(z_0, \xi\eta \cdot z_0)) \geq \frac{1}{2} \log(5 + 3\sqrt{2}),$$

which is stated as Theorem 5.1.1 in Chapter 5. Theorem 5.1.1 is a refinement of the log3 theorem, proved by Culler and Shalen in their 'Paradoxical Decompositions, 2-Generator Kleinian Groups and Volumes of Hyperbolic 3-Manifolds' paper ((5), Theorem 9.1) which states that

Let ξ and η be non-commuting isometries of $\mathbf{H}^3$. Suppose that ξ and η generate a torsion-free discrete group which is topologically tame, is not co-compact and contains no parabolics. Then for any $z_0 \in \mathbf{H}^3$ we have

$$\max(\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0)) \geq \log 3.$$

It is natural that the proof of Theorem 5.1.1 requires the use of the same ingredients introduced in (5) to prove the $\log 3$ theorem. In the following sections of this chapter, I review these ingredients. In particular, I summarize the proof of the $\log 3$ theorem in §1.2 with an emphasis on the calculations required to obtain the number $\log 3$. In §1.3, I propose an alternative technique to perform these calculations which leads to completion of one of the key steps in the proof of Theorem 5.1.1. Inspired by the $\log 3$ theorem, Theorem 5.1.1 will also be referred to as the $\frac{1}{2} \log(5 + 3\sqrt{2})$ theorem.

I would like to point out that the assumption 'topologically tame' may be dropped from the statements of Theorem 5.1.1 and the $\log 3$ theorem due to recent developments. Agol (1) and Calegari-Gabai (2) proved that every finitely generated Kleinian group is topologically tame. Therefore, the group $\Gamma = \langle \xi, \eta \rangle$ given in the statements of Theorem 5.1.1 and the $\log 3$ theorem already satisfies that assumption.

In the following, the sphere at infinity of $\mathbf{H}^3$ will be denoted by S_∞ and the canonical compactification of $\mathbf{H}^3$ will be denoted by $\overline{\mathbf{H}}^3$. Note that S_∞ is the boundary of $\overline{\mathbf{H}}^3$. Any isometry $\gamma : \mathbf{H}^3 \rightarrow \mathbf{H}^3$ extends to a conformal automorphism $\bar{\gamma} : \overline{\mathbf{H}}^3 \rightarrow \overline{\mathbf{H}}^3$. The conformal automorphism of S_∞ obtained by restricting $\bar{\gamma}$ will be denoted by γ_∞ .

1.1 The Patterson-Sullivan measure

Let B^3 be the open unit ball in $\mathbf{R}^3$. B^3 is called the *Poincaré model* for the hyperbolic 3-space $\mathbf{H}^3$ when it is equipped with the hyperbolic metric induced from the line element

$$ds^2 = \frac{4(dx^2 + dy^2 + dz^2)}{(1 - r^2)^2}$$

where r is the Euclidean distance from the base point to the origin. For any given point $z \in \mathbf{H}^3$, there exists an isometry $h : \mathbf{H}^3 \rightarrow B^3$ which is unique modulo composition with orthogonal linear transformations of $\mathbf{R}^3$ such that $h(z) = 0$. The pull back of the metric on $S^2 = \partial B^3$ under $\bar{h}$ gives a metric on S_∞ which is uniquely determined by the point z . This metric is called the *round metric centered at z* .

If $\gamma : \mathbf{H}^3 \rightarrow \mathbf{H}^3$ is an isometry, then it induces a conformal diffeomorphism γ_∞ of S_∞ . Let z_0 be a given point in $\mathbf{H}^3$. If S_∞ is equipped with the round metric centered at z_0 , then at each point ζ in S_∞ , the tangent map $d\gamma_\infty : T_\zeta S_\infty \rightarrow T_{\gamma_\infty(\zeta)} S_\infty$ satisfies $|d\gamma_\infty(v)| = \lambda|v|$ for every tangent vector v at ζ where $|\cdot|$ is the length in the round metric at z_0 . The constant λ is called the *conformal expansion factor* of γ_∞ at the point ζ and will be denoted by $\lambda_{\gamma, z_0}(\zeta)$. The expression λ_{γ, z_0} gives a smooth function on S_∞ . In fact, it is possible to show that

$$\lambda_{\gamma, z_0}(\zeta) = \mathcal{P}(z_0, \gamma^{-1}z_0, \zeta) \quad (1.1)$$

for every $\zeta \in S_\infty$ ((5), Proposition 2.4) where $\mathcal{P}$ is defined as follows: let $\Pi : \mathbf{H}^3 \times \mathbf{H}^3 \times \mathbf{H}^3 \rightarrow \mathbf{R}$ be the continuous function defined by $\Pi(z, z', w) = e^{\text{dist}(z, w) - \text{dist}(z', w)}$. The map Π has a unique positive continuous extension $\bar{\Pi} : \mathbf{H}^3 \times \mathbf{H}^3 \times \bar{\mathbf{H}}^3 \rightarrow \mathbf{R}$. The restriction $\mathcal{P}$ of $\bar{\Pi}$ is a function on $\mathbf{H}^3 \times \mathbf{H}^3 \times S_\infty$ with the formula

$$\mathcal{P}(z, z', \zeta) = (\cosh \text{dist}(z, z') - \sinh \text{dist}(z, z') \cos \angle z' z \zeta)^{-1} \quad (1.2)$$

where $\angle z' z \zeta$ is the angle between the ray from z to z' and the ray from z to ζ .

The map $\mathcal{P}$ is called the *Poisson kernel* for $\mathbf{H}^3$. Since the function $\phi_\zeta : \mathbf{H}^3 \rightarrow \mathbf{R}$ defined by $\phi_\zeta(z) = \mathcal{P}(z_0, z, \zeta)^r$ satisfies the Laplace Equation

$$\Delta \phi_\zeta(z) = -r(3-r-1)\phi_\zeta(z) \quad (1.3)$$

for any $z_0 \in \mathbf{H}^3$, $\zeta \in \mathbf{H}^3$ and $r \in \mathbf{R}$ ((5), Proposition 2.5), the Poisson kernel $\mathcal{P}$ for $\mathbf{H}^3$ provides a connection between the study of conformal expansion factors and the study of *D-conformal densities* for S_∞ , which is a more general concept, introduced by Sullivan, of constructing solutions to the equation in (Equation 1.3).

A *D-conformal density* for S_∞ is a family $\mathcal{M} = (\mu_z)_{z \in \mathbf{H}^3}$ of finite Borel measures on S_∞ such that the equality $d\mu_{z'} = \mathcal{P}(z, z', \cdot)^D d\mu_z$ holds for any two points $z, z' \in \mathbf{H}^3$ where D , called the *degree* of $\mathcal{M}$, is a number in $[0, 2]$. The equation $d\mu_{z'} = \mathcal{P}(z, z', \cdot)^D d\mu_z$ means that $\mu_{z'}(E) = \int_E \mathcal{P}(z, z', \cdot)^D d\mu_z$ for every measurable set $E \subset S_\infty$. A family $\mathcal{M} = (\mu_z)_{z \in \mathbf{H}^3}$ of finite Borel measures on S_∞ will be called a *conformal density* if $\mathcal{M} = (\mu_z)_{z \in \mathbf{H}^3}$ is a *D-conformal* for some $D \in [0, 2]$. If $\mathcal{M} = (\mu_z)_{z \in \mathbf{H}^3}$ is a conformal density, then all the measures $\mu_z, z \in \mathbf{H}^3$ have the same support since $\mathcal{P}(z, z', \cdot)$ is a strictly positive function. The common support of the measures $\mu_z, z \in \mathbf{H}^3$ is denoted by $\text{supp } \mathcal{M}$. The family $\mathcal{A} = (A_z)_{z \in \mathbf{H}^3}$, for example, is a 2-conformal density such that $\text{supp } \mathcal{A} = S_\infty$ where A_z denotes a normalized area measure determined by the round metric at z . The density $\mathcal{A}$ is called the *area density* on S_∞ .

It is possible to find a lower bound for the hyperbolic displacement $\text{dist}(z_0, \gamma \cdot z_0)$ using the measure $A_{z_0} \in \mathcal{A}$ as follows: let $d = \text{dist}(z_0, \gamma \cdot z_0)$, $c = \cosh d$ and $s = \sinh d$. Let us identify $\overline{\mathbf{H}}^3$ confor-

mally with the unit ball in $\mathbf{R}^3$ so that z_0 is mapped to the origin and $\gamma^{-1} \cdot z_0$ is on the positive vertical axis. Then the equalities in (Equation 1.1) and (Equation 1.2) give that $\lambda_{\gamma, z_0}(\zeta) = (c - s \cos \phi)^{-1}$ where $\phi = \phi(\zeta)$ is the angle between the positive vertical axis and the ray from the origin through ζ . Note that ϕ is the polar angle of ζ in spherical coordinates. Since S_∞ has the round metric at z_0 , $A_{z_0} \in \mathcal{A}$ is obtained by dividing the area measure on the unit sphere by the area 4π of the sphere. In other words, the area element is given as

$$dA_{z_0} = \frac{1}{4\pi} \sin \phi d\phi d\theta$$

in spherical coordinates ϕ and θ . If C denotes the spherical cap defined by the inequality $\phi \geq \phi_0$ where $\phi_0 = \arccos(1 - 2a)$ and $0 < a < 1$, the evaluation of an elementary iterated integral shows that $A_{z_0}(C) = a$.

Let us assume that there exists a measure ν on S_∞ such that $\nu \leq A_{z_0}$, $\nu(S_\infty) \leq a$ and $\int_{S_\infty} \lambda_{\gamma, z_0}^2 d\nu \geq b$ for some $0 < b < 1$. The assumption $\nu \leq A_{z_0}$ gives that $A_{z_0}(C) \geq \nu(S_\infty)$ and $\nu(S_\infty - C) = \nu(S_\infty) - \nu(C) \leq A_{z_0}(C) - \nu(C) = (A_{z_0} - \nu)(C)$ where $A_{z_0} - \nu$ is again a measure on S_∞ . The inequality $\nu(S_\infty - C) \leq (A_{z_0} - \nu)(C)$ combined with the fact that λ_{γ, z_0} is positive and monotone decreasing for $0 < \phi < \pi$ implies that $\inf \lambda_{\gamma, z_0}^2(C) \geq \sup \lambda_{\gamma, z_0}^2(S_\infty - C)$ and consequently

$$\begin{aligned} \int_{S_\infty - C} \lambda_{\gamma, z_0}^2 d\nu &\leq \sup(\lambda_{\gamma, z_0}^2(C)) \nu(S_\infty - C) \\ &\leq (\inf(\lambda_{\gamma, z_0}^2(C))) (A_{z_0} - \nu)(C) \\ &\leq \int_C \lambda_{\gamma, z_0}^2 d(A_{z_0} - \nu). \end{aligned}$$

It follows from the assumption $\int_{S_\infty} \lambda_{\gamma, z_0}^2 d\nu \geq b$ and the inequalities above that

$$\begin{aligned}
 b &\leq \int_{S_\infty} \lambda_{\gamma, z_0}^2 d\nu \\
 &= \int_C \lambda_{\gamma, z_0}^2 d\nu + \int_{S_\infty - C} \lambda_{\gamma, z_0}^2 d\nu \\
 &\leq \int_C \lambda_{\gamma, z_0}^2 d\nu + \int_C \lambda_{\gamma, z_0}^2 d(A_{z_0} - \nu) \\
 &= \int_C \lambda_{\gamma, z_0}^2 dA_{z_0} \\
 &= \int_C (c - s \cos \phi)^{-2} dA_{z_0} \\
 &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\phi_0} (c - s \cos \phi)^{-2} \sin \phi d\phi d\theta \\
 &= \frac{a}{(c-s)(c-s+2as)}.
 \end{aligned} \tag{1.4}$$

Rewriting c and s as $\frac{1}{2}(e^h + e^{-h})$ and $\frac{1}{2}(e^h - e^{-h})$ and simplifying the inequality in (Equation 1.4) provides the conclusion of the following lemma:

Lemma 1.1.1. ((5), Lemma 5.5) *Let a and b be numbers such that $0 < a < 1$ and $0 < b < 1$, let γ be an isometry of $\mathbf{H}^3$ and let z_0 be a point in $\mathbf{H}^3$. Suppose that ν is a measure on S_∞ such that*

$$(i) \quad \nu \leq A_{z_0},$$

$$(ii) \quad \nu(S_\infty) \leq a; \text{ and}$$

$$(iii) \quad \int_{S_\infty} \lambda_{\gamma, z_0}^2 d\nu \geq b.$$

Then

$$\text{dist}(z_0, \gamma \cdot z_0) \geq \frac{1}{2} \log \frac{b(1-a)}{a(1-b)}.$$

The existence of such a measure ν used in Lemma 1.1.1 requires the investigation of the decompositions of the Patterson density and the Patterson-Sullivan measure.

Let Γ be a non-elementary discrete subgroup of $\text{Isom}^+(\mathbf{H}^3)$. Also let $\mathcal{M} = (\mu_z)_{z \in \mathbf{H}^3}$ be a D conformal density and $\gamma : \mathbf{H}^3 \rightarrow \mathbf{H}^3$ be an isometry in Γ . The family of Borel measures $(\gamma_\infty^* \mu_{\gamma z})_{z \in \mathbf{H}^3}$ gives a D -conformal density where $\gamma_\infty^* \mu_{\gamma z}$ is the pull-back of μ_z defined by $(\gamma_\infty^* \mu_{\gamma z})(E) = \mu_z(\gamma_\infty(E))$ for every Borel set $E \subset S_\infty$. The density $(\gamma_\infty^* \mu_{\gamma z})$, denoted by $\gamma^* \mathcal{M}$, is called the *pull-back* of $\mathcal{M} = (\mu)_{z \in \mathbf{H}^3}$. A conformal density $\mathcal{M}$ is said to be Γ -invariant if it satisfies that $\gamma^* \mathcal{M} = \mathcal{M}$ for every $\gamma \in \Gamma$. A point a of S_∞ is called a *limit point* of the group Γ if there is a point x in $\mathbf{H}^3$ and a sequence $\{\gamma_j\}_{j=1}^\infty \subset \Gamma$ such that $\{\gamma_j \cdot x\}_{j=1}^\infty$ converges to a . If Λ_Γ denotes the limit set of Γ , Patterson (8) proves that

Theorem 1.1.2. ((5), Corollary 4.2) *There is a Γ -invariant conformal density $\mathcal{M}_{\Gamma, z_0} = (\mu_z)_{z \in \mathbf{H}^3}$ for $\mathbf{H}^3$ whose support is Λ_Γ .*

In the following, $\mathcal{M}_{\Gamma, z_0}$ will be called the *Patterson density*. The member μ_{z_0} of the Patterson density will be referred as the *Patterson-Sullivan measure at z_0* on S_∞ . Note that if $\mathcal{M} = (\mu_z)_{z \in \mathbf{H}^3}$ is a D -conformal Γ -invariant density, then for any $z \in \mathbf{H}^3$ and $\gamma \in \Gamma$

$$\begin{aligned}
 d(\gamma^* \mu_z) &= d\mu_{\gamma^{-1}z} \\
 &= \mathcal{P}(z, \gamma^{-1}z, \cdot)^D d\mu_z \\
 &= \lambda_{\gamma, z}^D d\mu_z.
 \end{aligned} \tag{1.5}$$

The last equality follows from the equation in (Equation 1.1).

A subset W of $\mathbf{H}^3$ is called *uniformly discrete* if there exists an $\epsilon > 0$ such that $\text{dist}(z, w) > \epsilon$ for any two distinct points z, w of $\mathbf{H}^3$. The closure of W in $\mathbf{H}^3$ is a union of the form $W \cup \Lambda_W$ where Λ_W , the limit set of W , is a closed subset of S_∞ . Note that the orbit $W = \Gamma \cdot z$ is a uniformly discrete set for any $z \in \mathbf{H}^3$ since Γ is discrete and z has compact stabilizer in $\text{Isom}^+(\mathbf{H}^3)$. Let W_0 denote the orbit $\Gamma \cdot z_0$. In (5), Culler and Shalen show that Theorem 1.1.2 is a corollary of the following proposition

Proposition 1.1.3. *((5), Proposition 4.2) Let W be an infinite, uniformly discrete subset of $\mathbf{H}^3$, and let $\mathcal{V}$ be a countable collection of subsets of W . Suppose $W \in \mathcal{V}$. Then there exists a number $D \in [0, 2]$ and a family $(\mathcal{M}_V)_{V \in \mathcal{V}}$ of D -conformal densities for $\mathbf{H}^3$, indexed by the collection $\mathcal{V}$, satisfying the following conditions:*

- (i) $\mathcal{M}_W \neq 0$.
- (ii) For any finite family $(V_i)_{1 \leq i \leq m}$ of disjoint sets in $\mathcal{V}$ such that $V = \sqcup_{i=1}^m V_i \in \mathcal{V}$, we have $\mathcal{M}_V = \sum_{i=1}^m \mathcal{M}_{V_i}$.
- (iii) For any $V \in \mathcal{V}$ and any isometry $\gamma: \mathbf{H}^3 \rightarrow \mathbf{H}^3$ such that $\gamma V \in \mathcal{V}$, we have $\gamma_\infty^*(\mathcal{M}_{\gamma V}) = \mathcal{M}_V$.
- (iv) For any $V \in \mathcal{V}$, the support of $\mathcal{M}_V$ is contained in the limit set Λ_V of V . In particular, for any finite set $V \in \mathcal{V}$ we have $\mathcal{M}_V = 0$,

when W is set as W_0 and $\mathcal{V}$ is chosen as $\{W_0\}$, i.e., $\mathcal{M}_{W_0} = \mathcal{M}_{\Gamma, z_0}$. In the proposition above, $\sum_{i=1}^m \mathcal{M}_{V_i}$ denotes the sum $(\sum_{i=1}^m \mu_{V_i, z})_{z \in \mathbf{H}^3}$ of the D -conformal densities $\mathcal{M}_{V_i} = (\mu_{V_i, z})_{z \in \mathbf{H}^3}$. $\sum_{i=1}^m \mathcal{M}_{V_i}$ is also a D -conformal density. The expression $\mathcal{M}_V = 0$ means that if $\mathcal{M}_V = (\mu_{V, z})_{z \in \mathbf{H}^3}$ then $\mu_{V, z} = 0$ for every $z \in \mathbf{H}^3$. Proposition 1.1.3 allows to construct decompositions of the Patterson density and

consequently decompositions of the Patterson-Sullivan measure corresponding to decompositions of the group Γ .

1.2 Paradoxical decompositions and Proof of the log3 theorem

Let ξ and η be non-commuting isometries of $\mathbf{H}^3$. Suppose that ξ and η generate a torsion-free discrete group which is topologically tame, is not co-compact and contains no parabolics. It is possible to show that $\langle \xi, \eta \rangle$ is a free group of rank 2 ((5), Proposition 9.2). Let $\Gamma = \langle \xi, \eta \rangle$. Then the proof of the log3 theorem is divided into two parts: if $\Lambda_\Gamma = S_\infty$ or $\Lambda_\Gamma \neq S_\infty$.

Let us assume that $\Lambda_\Gamma = S_\infty$. Since Γ is a free group on a generating set $\Xi = \{\xi, \eta\}$, every element γ of Γ can be written uniquely as a reduced word $\psi_1 \cdots \psi_m$, where $m \geq 0$, each ψ_i is an element of $\Psi = \Xi \amalg \Xi^{-1}$, and $\psi_{i+1} \neq \psi_i^{-1}$ for $i = 1, \dots, m-1$. If $\gamma \neq 1$, ψ_1 is called the *initial letter* of γ . For each $\psi \in \Xi \amalg \Xi^{-1}$, let J_ψ be the set of all non-trivial elements of Γ that have initial letter ψ . Then Γ is a disjoint union $\{1\} \amalg \coprod_{\psi \in \Psi} J_\psi$ which is called the *paradoxical decomposition* of Γ .

Let z_0 be a given point in $\mathbf{H}^3$. If V_ψ denotes the set $\{\gamma \cdot z_0 : \gamma \in J_\psi\}$, the orbit $W_0 = \Gamma \cdot z_0$ is a disjoint union $W_0 = \{z_0\} \amalg \coprod_{\psi \in \Psi} V_\psi$. Observe that W_0 is uniformly discrete since Γ is discrete and torsion-free. Let $\mathcal{V}$ be the finite collection of all subsets of W_0 containing all the sets of the form $\coprod_{\psi \in \Psi'} V_\psi$ or $\{z_0\} \amalg \coprod_{\psi \in \Psi'} V_\psi$ for $\Psi' \subset \Psi$. With these choices of W_0 and $\mathcal{V}$, Proposition 1.1.3 gives a number $D \in [0, 2]$ and a family $(\mathcal{M}_V)_{V \in \mathcal{V}}$ of D -conformal densities so that the conditions (i) – (iv) are satisfied. Part (iii) of Proposition 1.1.3 implies that $\mathcal{M}_{W_0}$ is Γ -invariant. The definition of a D -conformal density and part (i) of Proposition 1.1.3 show that $\mu_{W_0, z_0} \neq 0$. After normalization, it is possible to assume that $\mu_{W_0, z_0}(S_\infty) = 1$. It follows from the equality $W_0 = \{z_0\} \amalg \coprod_{\psi \in \Psi} V_\psi$ and part (ii) of Proposition 1.1.3 that $\mu_{W_0, z_0} = \mu_{\{z_0\}, z_0} + \sum_{\psi \in \Psi} \mu_{V_\psi, z_0}$. Since the set $\{z_0\}$ is finite, the measure $\mu_{\{z_0\}, z_0} = 0$ by Proposition

1.1.3 part (iv). Note that $\mu_{W_0, z_0} = \mu_{z_0}$. Then the decomposition of the Patterson-Sullivan measure μ_{z_0} corresponding to the paradoxical decomposition of Γ is obtained as

$$\mu_{z_0} = \sum_{\psi \in \Psi} \mu_{V_\psi, z_0}.$$

The definition of the sets J_ψ gives the group theoretical identities $\psi J_{\psi^{-1}} = \Gamma - J_\psi$ so that $\psi V_{\psi^{-1}} = W_0 - V_\psi$ for every $\psi \in \Psi$. Since the set $W_0 - V_\psi = \{z_0\} \sqcup \bigsqcup_{\psi' \neq \psi} V_{\psi'}$ is in $\mathcal{V}$, Proposition 1.1.3(iii) implies that $\mathcal{M}_{V_{\psi^{-1}}} = \psi_\infty^*(\mathcal{M}_{W_0 - V_\psi})$. Proposition 1.1.3 part (ii) shows that $\mathcal{M}_{W_0} = \mathcal{M}_{W_0 - V_\psi} + \mathcal{M}_{V_\psi}$. Therefore, the equality $\mathcal{M}_{V_{\psi^{-1}}} = \psi_\infty^*(\mathcal{M}_{W_0 - V_\psi})$ can be rewritten as $\mathcal{M}_{V_{\psi^{-1}}} = \psi_\infty^*(\mathcal{M}_{W_0} - \mathcal{M}_{V_\psi})$ which implies that $\mu_{V_{\psi^{-1}}, \psi(z_0)} = \psi_\infty^*(\mu_{z_0} - \mu_{V_\psi, z_0})$. Then it follows from the equalities given in (Equation 1.4) that

$$\int_{S_\infty} d(\psi_\infty^*(\mu_{z_0} - \mu_{V_\psi, z_0})) = \int_{S_\infty} \lambda_{\psi, z_0}^D d\mu_{V_{\psi^{-1}}, \psi(z_0)} \quad (1.6)$$

Since $\mathcal{M}_{V_{\psi^{-1}}}$ is a D -conformal Γ -invariant density for every $\psi \in \Psi$ by Proposition 1.1.3(iii), the equality in (Equation 1.6) gives that

$$\begin{aligned} \int_{S_\infty} \lambda_{\psi, z_0}^D d\mu_{V_{\psi^{-1}}, \psi(z_0)} &= \int_{S_\infty} d(\psi_\infty^*(\mu_{z_0} - \mu_{V_\psi, z_0})) \\ &= \int_{S_\infty} d(\mu_{z_0} - \mu_{V_\psi, z_0}) \\ &= \int_{S_\infty} d\mu_{z_0} - \int_{S_\infty} d\mu_{V_\psi, z_0} \\ &= \mu_{z_0}(S_\infty) - \int_{S_\infty} d\mu_{V_\psi, z_0} \\ &= 1 - \int_{S_\infty} d\mu_{V_\psi, z_0}. \end{aligned}$$

Let us say $\mu_{V_\psi, z_0} = \nu_\psi$ for each $\psi \in \Psi$. With the convention that every integral in this text, unless otherwise is stated, will be calculated over the sphere at infinity, the discussion given above proves the following lemma

Lemma 1.2.1. ((5), Lemma 5.3) *Let Γ be a Kleinian group which is free on a generating set $\Xi = \{\xi, \eta\}$. Set $\Psi = \Xi \amalg \Xi^{-1} \subset \Gamma$. Let z_0 be any point of $\mathbf{H}^3$. Then there exists a number $D \in [0, 2]$, a Γ -invariant D -conformal density $\mathcal{M}_{\Gamma, z_0} = (\mu_z)_{z \in \mathbf{H}^3}$ for $\mathbf{H}^3$ and a family $(\nu_\psi)_{\psi \in \Psi}$ of Borel measures on S_∞ such that*

- (i) $\mu_{z_0}(S_\infty) = 1$;
- (ii) $\mu_{z_0} = \sum_{\psi \in \Psi} \nu_\psi$; and
- (iii) for each $\psi \in \Psi$ we have

$$\int (\lambda_{\psi, z_0})^D d\nu_{\psi^{-1}} = 1 - \int d\nu_\psi.$$

If Ξ is a two-element set $\{\xi, \eta\}$ and if $z_0 \in \ell(\xi, \eta)$ where $\ell(\xi, \eta)$ denotes the common perpendicular of the isometries ξ and η , then we also have

(iv)

$$\int d\nu_{\xi^{-1}} = \int d\nu_\xi \text{ and } \int d\nu_{\eta^{-1}} = \int d\nu_\eta.$$

The last part of Lemma 1.2.1 follows from the fact that there is a unique geodesic line $\ell \subset \mathbf{H}^3$ such that the 180° rotation τ_ℓ about ℓ satisfies $\tau_\ell \xi \tau_\ell = \xi^{-1}$ and $\tau_\ell \eta \tau_\ell = \eta^{-1}$ when ξ_∞ and η_∞ have no common fixed point in S_∞ ((5), Proposition 1.2). This line is the common perpendicular $\ell(\xi, \eta)$ to

the axis of ξ and η . Note that $\tau_\ell \cdot z = z$ for every $z \in \ell$. Let z_0 be a point on ℓ and $\psi \in \Psi$. The group theoretical identity $\tau_\ell J_\psi \tau_\ell = J_{\psi^{-1}}$ implies that $\tau_\ell V_\psi = V_{\psi^{-1}}$ where $V_\psi = \{\gamma \cdot z_0 : \gamma \in J_\psi\}$ and $\tau_\ell V_\psi = \{\tau_\ell \gamma \tau_\ell \cdot z_0 : \gamma \in J_\psi\}$. Then part (iii) of Proposition 1.1.3 gives that $(\tau_\ell)_\infty^* \mathcal{M}_{V_\psi} = \mathcal{M}_{V_{\psi^{-1}}}$ which means $(\tau)_\infty^* \mu_{V_\psi, z_0} = \mu_{V_{\psi^{-1}}, z_0}$ or $(\tau)_\infty^* \nu_\psi = \nu_{\psi^{-1}}$. Since $\mathcal{M}_{V_\psi}$ is a D -conformal Γ -invariant density by part (iii) of Proposition 1.1.3, integrating both sides of the last equality provides the conclusion (iv) of Lemma 1.2.1

$$\int d\nu_\psi = \int d\nu_{\psi^{-1}}.$$

For the rest of the section, it is possible to assume without loss of generality that $z_0 \in \ell(\xi, \eta)$ since the inequality $\max(\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0)) \geq \max(\text{dist}(z_1, \xi \cdot z_1), \text{dist}(z_1, \eta \cdot z_1))$ holds for z_0 and z_1 where z_1 denotes the orthogonal projection of z_0 onto $\ell(\xi, \eta)$.

Let us consider the Patterson density $\mathcal{M}_{\Gamma, z_0} = (\mu_z)_{z \in \mathbf{H}^3}$ and the Patterson-Sullivan measure μ_{z_0} introduced in Lemma 1.2.1. It is possible to show that $\mathcal{M}_{\Gamma, z_0}$ has a strictly positive degree, i.e., $D > 0$ (5, Proposition 3.8). Since $\mathcal{M}_{\Gamma, z_0}$ is a D -conformal Γ -invariant density, the non-negative real-valued function $u = u_{\mathcal{M}_{\Gamma, z_0}}$ defined by $u(z) = \mu_z(S_\infty)$ is Γ -invariant, i.e., $u(\gamma \cdot z) = u(z)$ for every $\gamma \in \Gamma$, and can be written in the following form

$$\begin{aligned} u(z) &= \mu_z(S_\infty) \\ &= \int d\mu_z \\ &= \int \mathcal{P}(z_0, z, \cdot)^D d\mu_{z_0} \\ &= \int \phi_\zeta(z) d\mu_{z_0}. \end{aligned} \tag{1.7}$$

Observe that u is a C^∞ function because ϕ_ζ is a C^∞ function. Since the function $\phi_\zeta(z) = \mathcal{P}(z_0, z, \zeta)^D$ satisfies the Laplace Equation given in (Equation 1.3), the differentiation of the equation in (Equation 1.7) under the integral sign shows that u also satisfies the Laplace Equation

$$\Delta u = -D(3 - D - 1)u.$$

Note that $-D(3 - D - 1) < 0$ since $D > 0$. Therefore u is a superharmonic function on $\mathbf{H}^3$. A C^∞ function f on a Riemannian manifold is called *superharmonic* if it satisfies the inequality $\Delta f < 0$. It is known that every superharmonic function on $M = \mathbf{H}^3/\Gamma$ is constant when Γ is a topologically tame, Kleinian group which contains no parabolics with $\Lambda_\Gamma = S_\infty$ ((5), Proposition 6.8 and (3), Theorem 7.2). This implies that every Γ -invariant superharmonic function on $\mathbf{H}^3$ is constant which means that $u(z) = C$, or equivalently $\mu_z(S_\infty) = C$ for every $z \in \mathbf{H}^3$ for some constant C . On the other hand, the conformal density $C\mathcal{A} = (CA_z)_{z \in \mathbf{H}^3}$ gives the same superharmonic function $u_{C\mathcal{A}}(z) = C$ for every $z \in \mathbf{H}^3$. In other words, $u_{\mathcal{M}_{\Gamma, z_0}} = u_{C\mathcal{A}}$. The last equality implies that $\mathcal{M}_{\Gamma, z_0} = C\mathcal{A}$, consequently, gives that $D = 2$ ((5), Proposition 3.7). Part (i) of Lemma 1.2.1 provides that $C = 1$ since $\mu_{z_0}(S_\infty) = CA_{z_0}(S_\infty)$ where $A_{z_0}(S_\infty) = 1$. Then the Paterson density is equal to the area density $\mathcal{M}_{\Gamma, z_0} = \mathcal{A}$ which implies that the Patterson-Sullivan measure is equal to the area measure, i.e., $\mu_{z_0} = A_{z_0}$.

Conclusions (ii) and (iv) of Lemma 1.2.1 give that $1 = 2\nu_\xi(S_\infty) + 2\nu_\eta(S_\infty)$ so that either $\nu_\xi(S_\infty)$ or $\nu_\eta(S_\infty)$ is less than or equal to $\frac{1}{4}$. Let us assume that $\nu_\xi(S_\infty) \leq \frac{1}{4}$. Parts (iii) and (iv) of Lemma 1.2.1 imply that

$$\int \lambda_{\xi^{-1}, z_0}^2 d\nu_\xi = 1 - \nu_\xi \geq \frac{3}{4}.$$

Let $a = \frac{1}{4}$, $b = \frac{3}{4}$, $\nu = \nu_\xi$ and $\gamma = \xi^{-1}$. Since $\nu_\xi \leq A_{z_0}$ by Lemma 1.2.1 (i), it is possible to apply Lemma 1.1.1 with these choices and conclude that

$$\text{dist}(z_0, \xi^{-1} \cdot z_0) \geq \frac{1}{2} \log 9.$$

If $\nu_\eta(S_\infty) \leq \frac{1}{4}$, similar calculations with $a = \frac{1}{4}$, $b = \frac{3}{4}$, $\nu = \nu_\eta$ and $\gamma = \eta^{-1}$ show that

$$\text{dist}(z_0, \eta^{-1} \cdot z_0) \geq \frac{1}{2} \log 9.$$

Note that $\text{dist}(z_0, \xi \cdot z_0) = \text{dist}(z_0, \xi^{-1} \cdot z_0)$ and $\text{dist}(z_0, \eta \cdot z_0) = \text{dist}(z_0, \eta^{-1} \cdot z_0)$. Therefore, the following proposition completes the proof of the log3 theorem in the case $\Lambda_\Gamma = S_\infty$:

Proposition 1.2.2. ((5), Proposition 5.2) *Let ξ and η be two orientation-preserving isometries of $\mathbf{H}^3$. Suppose that the group Γ generated by ξ and η is discrete and is free on the generators ξ and η . Suppose in addition that every Γ -invariant conformal density for $\mathbf{H}^3$ is a constant multiple of the area density. Then for any $z_0 \in \mathbf{H}^3$ we have*

$$\max(\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0)) \geq \log 3.$$

For the second part of the proof, let us assume that $\Lambda_\Gamma \neq S_\infty$ where $\Gamma = \langle \xi, \eta \rangle$. Let $V = PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$. V will be considered with the complex topology. Let us denote by $\mathfrak{D}$ the set of all points $(\xi', \eta') \in V$ such that the group $\langle \xi', \eta' \rangle$ generated by ξ' and η' is a free Kleinian group of rank 2. $\mathfrak{D}$ is

a closed subset of $V((4))$. If $\mathfrak{G}\mathfrak{F}$ denotes the set of all points $(\xi', \eta') \in \mathfrak{D}$ such that $\Lambda_{\Gamma'} \neq S_\infty$ where $\Gamma' = \langle \xi', \eta' \rangle$, then $\mathfrak{G}\mathfrak{F}$ is an open subset of $V((7)$, Theorem 8.1). Note that $(\xi, \eta) \in \mathfrak{G}\mathfrak{F}$. The frontier of $\mathfrak{G}\mathfrak{F}$ in V will be denoted by $\mathfrak{B} = \overline{\mathfrak{G}\mathfrak{F}} - \mathfrak{G}\mathfrak{F}$. There exists a dense subset $\mathfrak{C}$ of $\mathfrak{B}$ such that every superharmonic function on $M = \mathbf{H}^3/\Gamma'$ is constant for every $(\xi', \eta') \in \mathfrak{C}$ ((5), Theorem 8.2). If (ξ', η') is in $\mathfrak{B}$, then the limit set of the group $\langle \xi', \eta' \rangle$ is the sphere at infinity. Let z_0 be a fixed point in $\mathbf{H}^3$. Let us define the function

$$\begin{aligned} f_{z_0} : \mathfrak{D} &\longrightarrow \mathbf{R} \\ (\xi', \eta') &\longmapsto \max(\text{dist}(z_0, \xi' \cdot z_0), \text{dist}(z_0, \eta' \cdot z_0)). \end{aligned}$$

Since the set $f_{z_0}^{-1}([0, B]) = \{(\xi', \eta') \in V : \max(\text{dist}(z_0, \xi' \cdot z_0), \text{dist}(z_0, \eta' \cdot z_0)) \leq B\}$ is compact for every $B \geq 0$, f_{z_0} is a continuous, proper, non-negative valued function on V . Therefore, f_{z_0} must take a minimum value on $\overline{\mathfrak{G}\mathfrak{F}}$. Let $(\xi_0, \eta_0) \in \overline{\mathfrak{G}\mathfrak{F}}$ be a point at which f_{z_0} takes a minimum value. The fact that every superharmonic function on $M = \mathbf{H}^3/\Gamma'$ is constant for every $(\xi', \eta') \in \mathfrak{C}$ implies that $\mathcal{M}_{\Gamma', z_0} = \mathcal{A}$ for every $(\xi', \eta') \in \mathfrak{C}$ where $\Gamma' = \langle \xi', \eta' \rangle$. Then Proposition 1.5 gives that $f_{z_0}(\xi', \eta') \geq \log 3$ for every $(\xi', \eta') \in \mathfrak{C}$. If (ξ_0, η_0) is in $\mathfrak{B}$, then $f_{z_0}(\xi_0, \eta_0) \geq \log 3$ since $\mathfrak{C}$ is a dense subset of $\mathfrak{B}$. Hence, it is enough to show that f_{z_0} has no local minimum on the open set $\mathfrak{G}\mathfrak{F}$.

Let us assume that (ξ_0, η_0) is in $\mathfrak{G}\mathfrak{F}$. Then f_{z_0} has a local minimum at (ξ_0, η_0) . Since ξ_0, η_0 have infinite orders in $\langle \xi_0, \eta_0 \rangle$, we have $\xi_0 \cdot z_0 \neq z$ and $\eta_0 \cdot z_0 \neq z$. Then there exists a sequence (z_i) of points on the open hyperbolic line between z_0 and $\xi_0 \cdot z_0 \neq z_0$ which converges to $\xi_0 \cdot z_0$. Also there exists a (w_i) of points on the open hyperbolic line between z_0 and $\eta_0 \cdot z_0 \neq z_0$ which converges to $\eta_0 \cdot z_0$. Let (ξ_i) and (η_i) be two sequences of hyperbolic isometries converging to identity such that $\xi_i \cdot z_0 = z_i$ and

$\eta_i \cdot z_0 = w_i$. Then $(\xi_i \xi, \eta_i \eta)$ converges to (ξ_0, η_0) and $f_{z_0}(\xi_i \xi, \eta_i \eta) < f_{z_0}(\xi_0, \eta_0)$ for every i . Therefore, (ξ_0, η_0) is not a local minimum of f_{z_0} . Hence, (ξ_0, η_0) is in $\mathfrak{B}$. Since, the limit set Λ_Γ of $\Gamma = \langle \xi, \eta \rangle$ is not the sphere at infinity, (ξ, η) is in $\mathfrak{G}\mathfrak{F}$. Therefore, $f_{z_0}(\xi, \eta) = \max(\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0)) \geq \log 3$. This finishes the proof of the $\log 3$ theorem.

1.3 Steps to find a lower bound for hyperbolic displacements

Since the Patterson density $\mathcal{M}_{\Gamma, z_0}$ is the area density $\mathcal{A}$ when $\Lambda_\Gamma = S_\infty$, the calculation of the number $\log 3$ follows from the application of Lemma 1.1.1 with the choices $\gamma = \psi^{-1}$, $\nu = \nu_\psi$, $a = \frac{1}{4}$ and $b = \frac{3}{4}$ where $\psi \in \{\xi, \eta\}$. These choices of a and b follow from the equalities $\mu_\xi = \mu_{\xi^{-1}}$ and $\mu_\eta = \mu_{\eta^{-1}}$ given in Lemma 1.2.1 part (i v) which requires the use of a point z_0 on the common perpendicular $\ell(\xi, \eta)$ of ξ and η . It is possible to calculate the number $\log 3$ without referring to $\ell(\xi, \eta)$ as follows:

Let us apply Lemma 1.1.1 with choices

$$\gamma = \xi^{-1}, \nu = \nu_\xi, a = \nu_{\xi^{-1}}(S_\infty), b = \int \lambda_{\xi^{-1}, z_0}^2 d\nu$$

and

$$\gamma = \eta^{-1}, \nu = \nu_\eta, a = \nu_{\eta^{-1}}(S_\infty), b = \int \lambda_{\eta^{-1}, z_0}^2 d\nu$$

where $\int \lambda_{\psi^{-1}, z_0}^2 d\nu = 1 - \int d\nu_\psi$ for every $\psi \in \{\xi, \eta\}$ by Lemma 1.2.1 (i i i). Then the conclusion of Lemma

1.1.1 gives the inequalities

$$e^{2\text{dist}(z_0, \xi^{-1} \cdot z_0)} \geq \frac{\left(1 - \int d\nu_\xi\right) \left(1 - \int d\nu_{\xi^{-1}}\right)}{\int d\nu_\xi \cdot \int d\nu_{\xi^{-1}}} \quad (1.8)$$

$$e^{2\text{dist}(z_0, \eta^{-1} \cdot z_0)} \geq \frac{\left(1 - \int dv_\eta\right) \left(1 - \int dv_{\eta^{-1}}\right)}{\int dv_\eta \cdot \int dv_{\eta^{-1}}}. \quad (1.9)$$

I propose to consider the constants on the right-hand side of the inequalities given in (Equation 1.8) and (Equation 1.9) as the values of the functions

$$f_1(x_1, x_2, x_3, x_4) = \frac{1-x_1}{x_1} \cdot \frac{1-x_2}{x_2} \quad \text{and} \quad f_2(x_1, x_2, x_3, x_4) = \frac{1-x_3}{x_3} \cdot \frac{1-x_4}{x_4}$$

at the point $(\int dv_\xi, \int dv_{\xi^{-1}}, \int dv_\eta, \int dv_{\eta^{-1}}) \in \mathbf{R}^4$ where $\int dv_\xi + \int dv_{\xi^{-1}} + \int dv_\eta + \int dv_{\eta^{-1}} = 1$. Then the calculation of the number $\log 3$ in the proof of the $\log 3$ theorem becomes a consequence of the statement

$$\inf_{\mathbf{x} \in \Delta^3} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}))\} = 9$$

where $\Delta^3 = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}_+^4 : x_1 + x_2 + x_3 + x_4 = 1\}$. More specifically, the calculation of the number $\log 3$ reduces the calculation of the infimum of maximum of f_1 and f_2 over the simplex Δ^3 . The functions f_1 and f_2 will be called the *displacement functions* for the paradoxical decompositions of Γ .

The arguments developed by Culler and Shalen to prove the $\log 3$ theorem combined with this calculation technique of the number $\log 3$ suggests a step-by-step process to find a lower bound for the displacements under the set of isometries $\{\gamma_i \in \Gamma : i \in I\}$ of $\mathbf{H}^3$ in the case $\Lambda_\Gamma = S_\infty$. Here I denotes an index set. This process can be described as follows: Let z_0 be a fixed point in $\mathbf{H}^3$. Since Γ is a free group on the generators ξ and η , it is possible to define other decompositions of Γ (see §3),

in particular, a decomposition concentrated on the words given in $\{\gamma_i \in \Gamma : i \in I\}$. Therefore, the first step of this process is to

S_1 . Introduce a decomposition of Γ which contains each word in $\{\gamma_i \in \Gamma : i \in I\}$.

Similar to the group theoretical relations $\gamma J_{\gamma^{-1}} = \Gamma - J_\gamma$ for $\gamma \in \{\xi, \xi^{-1}, \eta, \eta^{-1}\}$ originated from the paradoxical decomposition of Γ , the decomposition of Γ introduced in step S_1 for the isometries $\{\gamma_i \in \Gamma : i \in I\}$ provide some group theoretical relations between the sets of words in the decomposition. Applying Proposition 1.1.3 to the orbits of these sets of words in the decomposition for $\{\gamma_i \in \Gamma : i \in I\}$ provides an analog of Lemma 1.2.1. Hence, the second step is to

S_2 . Prove an analog of Lemma 1.2.1 for the decomposition introduced in step S_1 .

An analog of Lemma 1.2.1 for the set of isometries $\{\gamma_i \in \Gamma : i \in I\}$ in S_2 gives a decomposition of the Patterson-Sullivan measure μ_{z_0} and equations similar to the ones introduced in part (iii) of Lemma 1.2.1. Using these equations and summands of μ_{z_0} , it is possible to produce inequalities similar to (Equation 1.8) and (Equation 1.9) by applying Lemma 1.1.1. These inequalities provide lower bounds for the displacements in $\{\gamma_i \in \Gamma : i \in I\}$. The expressions on the right-hand side of these inequalities can be considered as the values of the functions, obtained by replacing each total measure with a variable, at a point in some simplex Δ . The dimension of the simplex is determined by the number of summands in the decomposition of μ_{z_0} . The functions obtained in this way will be called the *displacement functions*. In other words, the next step is to

S_3 . Use analogs of Lemma 1.2.1 and Lemma 1.1.1 to determine the displacement functions.

Remember that Lemma 1.1.1 provides lower bounds for every isometry in $\{\gamma_i \in \Gamma : i \in I\}$. The maximum of these lower bounds is greater than or equal to the infimum of the maximum of the displacement functions over the simplex Δ . Therefore, the fourth step is to

S₄. Calculate the greatest lower bound of the displacement functions over the simplex Δ .

In order to complete the case where $\Lambda_\Gamma = S_\infty$ it is necessary to show that the number obtained in step S_4 is a lower bound for the maximum of the displacements of the isometries in $\{\gamma_i \in \Gamma : i \in I\}$ when every Γ -invariant conformal density is a constant multiple of the area density. Hence, the fifth step is to

S₅. Prove an analog of Proposition 1.2 for the isometries in $\{\gamma_i \in \Gamma : i \in I\}$.

The last step covers the case $\Lambda_\Gamma \neq S_\infty$. Since $\Gamma = \langle \xi, \eta \rangle$ is a free group on the generators ξ and η , every γ_i is of the form $\psi_1^i \cdots \psi_m^i$, where $m \geq 0$, each ψ_j^i is an element of $\{\xi, \xi^{-1}, \eta, \eta^{-1}\}$, and $\psi_{j+1}^i \neq (\psi_j^i)^{-1}$ for $j = 1, \dots, m-1$ and $i \in I$. Let $\Gamma' = \langle \xi', \eta' \rangle$ where $(\xi', \eta') \in \mathfrak{D}$. Let us define the function σ between the set $\{\xi, \xi^{-1}, \eta, \eta^{-1}\}$ and $\{\xi', (\xi')^{-1}, \eta', (\eta')^{-1}\}$ such that $\sigma(\xi) = \xi'$, $\sigma(\xi^{-1}) = (\xi')^{-1}$, $\sigma(\eta) = \eta'$ and $\sigma(\eta^{-1}) = (\eta')^{-1}$. Let γ'_i be the word in Γ' defined as $\sigma(\psi_1^i) \cdots \sigma(\psi_m^i)$ for every $i \in I$. Then the final step is to

S₆. Show that f_{z_0} does not have a local minimum on $\mathfrak{G}\mathfrak{F}$ where

$$\begin{aligned} f_{z_0} &: \mathfrak{D} \longrightarrow \mathbf{R} \\ (\xi', \eta') &\longmapsto \max_{i \in I} (\text{dist}(z_0, \gamma'_i \cdot z_0)). \end{aligned}$$

For the rest of this text, I will follow steps S_1 - S_6 for the set $\{\xi, \eta, \xi\eta\} \subset \Gamma$ of isometries of $\mathbf{H}^3$ to prove

Theorem 5.1.1. The organization of this thesis is as follows:

In Chapter 2, I give the necessary calculations in detail to obtain the number $\log 3$ for the $\log 3$ theorem from the point of view I propose. I show that $\inf_{\mathbf{x} \in \Delta^3} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}))\} = 9$ which follows from the facts that

A. $\inf_{\mathbf{x} \in \Delta^3} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}))\} = \min_{\mathbf{x} \in \Delta^3} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}))\},$

B. There exists a point $\mathbf{x}^* \in \Delta^* \subset \Delta^3$ such that $\min_{\mathbf{x} \in \Delta^3} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}))\} = f_1(\mathbf{x}^*),$

proved in Lemma 2.2.1 (§2.1) where $\Delta^* = \{\mathbf{x} \in \Delta^3 : f_1(\mathbf{x}) = f_2(\mathbf{x})\}$. The statement $\mathbf{x}^* \in \Delta^*$ is deduced from the observation that neither f_1 nor f_2 has a local minimum in Δ^3 . The number $\log 3$ is obtained by evaluating f_1 at the point $\mathbf{x}^*$. The coordinates of $\mathbf{x}^*$ are calculated using the fact that the determinants of the quadratic equations derived from the equality $f_1(\mathbf{x}) = f_2(\mathbf{x})$ under different parametrization of the simplex Δ^3 is 0.

In the rest of the chapters, I concentrate on the set $\{\xi, \eta, \xi\eta\} \subset \Gamma$. In Chapter 3, I explain how to obtain decompositions of Γ other than the paradoxical decompositions $\Gamma_{\mathcal{D}_p}$. I introduce the decomposition $\Gamma_{\mathcal{D}} = \{1\} \amalg \Psi_r \amalg \coprod_{\psi \in \Psi} J_{\psi}$ of Γ which concentrates on isometries ξ , η , and $\xi\eta$ where $\Psi = \{\xi\eta, \xi^2, \xi\eta^{-1}, \eta, \xi^{-1}, \eta^{-1}\xi^{-1}, \eta^{-2}, \eta^{-1}\xi\}$ and $\Psi_r = \{\xi, \eta^{-1}\}$. I prove Lemma 3.2.1 in §3.2 which is the analog of Lemma 1.2.1 for the decomposition $\Gamma_{\mathcal{D}}$. Using Lemma 3.2.1 and Lemma 1.2.1 together, I determine the displacement functions for the displacements $\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0)$ and $\text{dist}(z_0, \xi\eta \cdot z_0)$ in §3. There are fourteen displacement functions $f_1, f_2, \dots, f_{14}$. It is possible to show that only first eight of these functions $f_1, f_2, \dots, f_8$ are significant to find a lower bound for the maximum of the displacements $\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0)$ and $\text{dist}(z_0, \xi\eta \cdot z_0)$. Similar to the calcu-

lation of the $\inf_{\mathbf{x} \in \Delta^3} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}))\}$, the calculation of the infimum of maximum of the functions $f_1, f_2, \dots, f_8$ follows from the statements

$$C. \inf_{\mathbf{x} \in \Delta^7} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_8(\mathbf{x}))\} = \min_{\mathbf{x} \in \Delta^7} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_8(\mathbf{x}))\},$$

$$D. \text{ There exists a point } \mathbf{x}^* \in \Delta^* \subset \Delta^7 \text{ such that } \min_{\mathbf{x} \in \Delta^7} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_8(\mathbf{x}))\} = f_1(\mathbf{x}^*).$$

proved in Lemma 4.1.14 (§4.1) where $\Delta^* = \{\mathbf{x} \in \Delta^7 : f_j(\mathbf{x}) = f_k(\mathbf{x}) \text{ for every } j, k = 1, 2, \dots, 8\}$. Although the proof of the observation that $\mathbf{x}^* \in \Delta^* \subset \Delta^7$ also uses the fact that none of the functions f_j for $j = 1, 2, \dots, 8$ has a local minimum in Δ^7 , it is much more involved. It requires a case by case study of the values of the functions f_j for $j = 1, 2, \dots, 8$ at the point $\mathbf{x}^*$. In the sections §§4.1.1–§§4.1.4, I use the first order partial derivatives of the functions f_j for $j = 1, 2, \dots, 8$ with different parametrization of the simplex Δ^7 to show that there are certain direction vectors in $\mathbf{R}^8$ so that moving along these vectors reduces the number of possible cases for the values of the functions f_j for $j = 1, 2, \dots, 8$ at the point $\mathbf{x}^*$ to only four main situation:

- I. $f_j(\mathbf{x}^*) < \alpha_*$ for $j = 1, 3, 4, 5, 8$ and $f_j(\mathbf{x}^*) = \alpha_*$ for $j = 2, 6, 7$,
- II. $f_j(\mathbf{x}^*) < \alpha_*$ for $j = 1, 2, 5, 6, 7, 8$ and $f_j(\mathbf{x}^*) = \alpha_*$ for $j = 3, 4$,
- III. $f_j(\mathbf{x}^*) = \alpha_*$ for $j = 1, 2, 5, 6, 7, 8$ and $f_j(\mathbf{x}^*) < \alpha_*$ for $j = 3, 4$,
- IV. $f_j(\mathbf{x}^*) = \alpha_*$ for $j = 2, 3, 4, 6, 7$ and $f_j(\mathbf{x}^*) < \alpha_*$ for $j = 1, 5, 8$

where $\alpha_* = \min_{\mathbf{x} \in \Delta^7} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_8(\mathbf{x}))\}$. In each of these cases, I prove in §§4.1.6 that there exists a piece of curve in Δ^7 passing through the point $\mathbf{x}^*$ such that moving along this curve produces a point in Δ^7 at which a smaller minimum for the maximum of the functions f_j for $j = 1, 2, \dots, 8$

is attained. This leaves only the case $\mathbf{x}^* \in \Delta^*$ which suggest a method to find the coordinates of the point $\mathbf{x}^*$ and hence the number α^* (§§4.1.6). In §4.2, I introduce Proposition 4.0.1 the analog of Proposition 1.2.2 for the displacements $\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0)$ and $\text{dist}(z_0, \xi\eta \cdot z_0)$ to complete the case $\Lambda_\Gamma = S_\infty$. In the last chapter, I show that the function defined as

$$\begin{aligned} f_{z_0} : \mathfrak{D} &\longrightarrow \mathbf{R} \\ (\xi', \eta') &\mapsto \max(\text{dist}(z_0, \xi' \cdot z_0), \text{dist}(z_0, \eta' \cdot z_0), \text{dist}(z_0, \xi'\eta' \cdot z_0)). \end{aligned}$$

has no local minimum on $\mathfrak{G}\mathfrak{F}$ to conclude the proof of Theorem 5.1.1.

CHAPTER 2

PROOF OF THE LOG 3 THEOREM REVISITED

2.1 Displacement functions for the paradoxical decompositions of Γ

Suppose that Γ , ξ , and η satisfy the hypotheses of the Proposition 1.2.2. Let z_0 be a constant point in $\mathbf{H}^3$. Since Γ and $\Xi = \{\xi, \eta\}$ satisfy the hypotheses of Lemma 1.2.1, there exists a number $D \in [0, 2]$, a conformal density $\mathcal{M}_{\Gamma, z_0}$ and a family of Borel measures $(\nu_\psi)_{\psi \in \Psi}$ where $\Psi = \{\xi, \xi^{-1}, \eta, \eta^{-1}\}$ for which conclusions (i), (ii) and, (iii) of Lemma 1.2.1 hold. But the hypotheses of Proposition 1.2.2 imply that $\mathcal{M}_{\Gamma, z_0}$ is a constant multiple of the area density $\mathcal{A}$. Therefore, we get $D = 2$. Since μ_{z_0} has total mass 1 by Lemma 1.2.1(i), we have $\mathcal{M}_{\Gamma, z_0} = \mathcal{A}$. We know that

$$\nu_\xi(S_\infty) = \int d\nu_\xi, \nu_{\xi^{-1}}(S_\infty) = \int d\nu_{\xi^{-1}}, \nu_\eta(S_\infty) = \int d\nu_\eta, \nu_{\eta^{-1}}(S_\infty) = \int d\nu_{\eta^{-1}}.$$

Let us call

$$\int d\nu_\xi = m_1, \int d\nu_{\xi^{-1}} = m_2, \int d\nu_\eta = m_3, \int d\nu_{\eta^{-1}} = m_4.$$

By Lemma 1.2.1(i) and (ii), we know that $\nu_\xi(S_\infty) + \nu_{\xi^{-1}}(S_\infty) + \nu_{\eta^{-1}}(S_\infty) + \nu_\eta(S_\infty) = 1$. We claim that $0 < \nu_\psi(S_\infty) < 1$ for every $\psi \in \Psi$. Let us assume that $\nu_\xi(S_\infty) = 0$. Then we get $\int \lambda_{\xi^{-1}, z_0}^2 d\nu_\xi = 0$. By Lemma 1.2.1(iii), we obtain that $1 - \nu_{\xi^{-1}}(S_\infty) = 0$ or $\nu_{\xi^{-1}}(S_\infty) = 1$. Hence, we see that $\nu_\eta(S_\infty) = 0$ and $\nu_{\eta^{-1}}(S_\infty) = 0$. The last equalities imply that $\int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-1}} = 0$ and $\int \lambda_{\eta^{-1}, z_0}^2 d\nu_\eta = 0$. Then we get $1 - \nu_\eta(S_\infty) = 0$ and $1 - \nu_{\eta^{-1}}(S_\infty) = 0$ by Lemma 1.2.1(iii) implying that $\nu_\eta(S_\infty) = 1$ and $\nu_{\eta^{-1}}(S_\infty) =$

1. This is a contradiction. Similar arguments can be given for the measures $\nu_{\xi^{-1}}$, $\nu_{\eta^{-1}}$ and $\nu_{\eta^{-1}}$. Therefore, we have $\nu_\psi(S_\infty) \neq 0$ for every $\psi \in \Psi$. Let us assume that $\nu_\psi(S_\infty) = 1$ for some $\psi \in \Psi$. Then $\nu_{\psi^{-1}}(S_\infty) = 0$. Repeating the argument above for $\nu_{\psi^{-1}}(S_\infty)$ gives a contradiction. In other words, we see that $\nu_\psi(S_\infty) \neq 1$ for every $\psi \in \Psi$. Hence, we conclude that $0 < \nu_\psi(S_\infty) < 1$ for every $\psi \in \Psi$ or equivalently, $0 < m_i < 1$ for every $i = 1, 2, 3, 4$. Note that

$$0 < \int \lambda_{\psi, z_0}^2 d\nu_{\psi^{-1}} = 1 - \int d\nu_\psi < 1$$

for every $\psi \in \Psi$. Therefore, we can apply Lemma 1.1.1 for every $\psi \in \Psi$.

Let $\gamma = \xi$, $\nu = \nu_{\xi^{-1}}$, $a = \nu_{\xi^{-1}}(S_\infty)$, and $b = \int \lambda_{\xi, z_0}^2 d\nu_{\xi^{-1}}$. By the conclusion of Lemma 1.1.1, we obtain that

$$e^{2\text{dist}(z_0, \xi, z_0)} \geq \frac{\int \lambda_{\xi, z_0}^2 d\nu_{\xi^{-1}} \left(1 - \int d\nu_{\xi^{-1}}\right)}{\int d\nu_{\xi^{-1}} \left(1 - \int \lambda_{\xi, z_0}^2 d\nu_{\xi^{-1}}\right)}$$

By Lemma 1.2.1(iii), we see that

$$\begin{aligned} e^{2\text{dist}(z_0, \xi, z_0)} &\geq \frac{\int \lambda_{\xi, z_0}^2 d\nu_{\xi^{-1}} \left(1 - \int d\nu_{\xi^{-1}}\right)}{\int d\nu_{\xi^{-1}} \left(1 - \int \lambda_{\xi, z_0}^2 d\nu_{\xi^{-1}}\right)} \\ &= \frac{\left(1 - \int d\nu_\xi\right) \left(1 - \int d\nu_{\xi^{-1}}\right)}{\int d\nu_\xi \cdot \int d\nu_{\xi^{-1}}} \\ &= \frac{(1 - m_1)(1 - m_2)}{m_1 m_2}. \end{aligned} \tag{2.1}$$

Therefore, the expression in (Equation 2.1) gives a lower bound for the displacement $\text{dist}(z_0, \xi \cdot z_0)$.

Let us replace the constants m_1, m_2, m_3 and m_4 with the variables x_1, x_2, x_3 and x_4 respectively.

Then we see that the displacement function is

$$f_1(x_1, x_2, x_3, x_4) = \frac{1-x_1}{x_1} \cdot \frac{1-x_2}{x_2}.$$

Let $\gamma = \xi^{-1}$, $v = v_\xi$, $a = v_\xi(S_\infty)$, and $b = \int \lambda_{\xi^{-1}, z_0}^2 dv_\xi$. By the conclusion of Lemma 1.1.1, we calculate that

$$e^{2\text{dist}(z_0, \xi^{-1} \cdot z_0)} \geq \frac{\int \lambda_{\xi^{-1}, z_0}^2 dv_\xi \left(1 - \int dv_\xi\right)}{\int dv_\xi \left(1 - \int \lambda_{\xi^{-1}, z_0}^2 dv_\xi\right)}$$

By Lemma 1.2.1(iii), we find that

$$\begin{aligned} e^{2\text{dist}(z_0, \xi^{-1} \cdot z_0)} &\geq \frac{\int \lambda_{\xi^{-1}, z_0}^2 dv_\xi \left(1 - \int dv_\xi\right)}{\int dv_\xi \left(1 - \int \lambda_{\xi^{-1}, z_0}^2 dv_\xi\right)} \\ &= \frac{\left(1 - \int dv_{\xi^{-1}}\right) \left(1 - \int dv_\xi\right)}{\int dv_{\xi^{-1}} \cdot \int dv_\xi} \\ &= \frac{(1-m_2)(1-m_1)}{m_2 m_1}. \end{aligned} \tag{2.2}$$

Therefore, the expression in (Equation 2.2) gives a lower bound for the displacement $\text{dist}(z_0, \xi^{-1} \cdot z_0)$.

In this case, the displacement function is

$$f_3(x_1, x_2, x_3, x_4) = \frac{1-x_2}{x_2} \cdot \frac{1-x_1}{x_1}.$$

Let $\gamma = \eta$, $v = v_{\eta^{-1}}$, $a = v_{\eta^{-1}}(S_\infty)$, and $b = \int \lambda_{\eta, z_0}^2 dv_{\eta^{-1}}$. Let $\gamma = \xi^{-1}$, $v = v_\xi$, $a = v_\xi(S_\infty)$, and $b = \int \lambda_{\xi^{-1}, z_0}^2 dv_\xi$. By the conclusion of Lemma 1.1.1, we have

$$e^{2\text{dist}(z_0, \eta \cdot z_0)} \geq \frac{\int \lambda_{\eta, z_0}^2 dv_{\eta^{-1}} \left(1 - \int dv_{\eta^{-1}}\right)}{\int dv_{\eta^{-1}} \left(1 - \int \lambda_{\eta, z_0}^2 dv_{\eta^{-1}}\right)}.$$

By Lemma 1.2.1(iii), we obtain that

$$\begin{aligned} e^{2\text{dist}(z_0, \eta \cdot z_0)} &\geq \frac{\int \lambda_{\eta, z_0}^2 dv_{\eta^{-1}} \left(1 - \int dv_{\eta^{-1}}\right)}{\int dv_{\eta^{-1}} \left(1 - \int \lambda_{\eta, z_0}^2 dv_{\eta^{-1}}\right)} \\ &= \frac{\left(1 - \int dv_\eta\right) \left(1 - \int dv_{\eta^{-1}}\right)}{\int dv_\eta \cdot \int dv_{\eta^{-1}}} \\ &= \frac{(1-m_3)(1-m_4)}{m_3 m_4}. \end{aligned} \tag{2.3}$$

Therefore, the expression in (Equation 2.3) is a lower bound for the displacement $\text{dist}(z_0, \eta \cdot z_0)$. In

this case, the displacement function is

$$f_2(x_1, x_2, x_3, x_4) = \frac{1-x_3}{x_3} \cdot \frac{1-x_4}{x_4}.$$

Similarly, let $\gamma = \eta^{-1}$, $\nu = \nu_\eta$, $a = \nu_\eta(S_\infty)$, and $b = \int \lambda_{\eta^{-1}, z_0}^2 d\nu_\eta$ and apply Lemma 1.1.1. Then we see that

$$e^{2\text{dist}(z_0, \eta^{-1} \cdot z_0)} \geq \frac{\int \lambda_{\eta^{-1}, z_0}^2 d\nu_\eta \left(1 - \int d\nu_\eta\right)}{\int d\nu_\eta \left(1 - \int \lambda_{\eta^{-1}, z_0}^2 d\nu_\eta\right)}$$

By Lemma 1.2.1(iii), we find that

$$\begin{aligned} e^{2\text{dist}(z_0, \eta^{-1} \cdot z_0)} &\geq \frac{\int \lambda_{\eta^{-1}, z_0}^2 d\nu_\eta \left(1 - \int d\nu_\eta\right)}{\int d\nu_\eta \left(1 - \int \lambda_{\eta^{-1}, z_0}^2 d\nu_\eta\right)} \\ &= \frac{\left(1 - \int d\nu_{\eta^{-1}}\right) \left(1 - \int d\nu_\eta\right)}{\int d\nu_{\eta^{-1}} \cdot \int d\nu_\eta} \\ &= \frac{(1 - m_4)(1 - m_3)}{m_4 m_3}. \end{aligned} \tag{2.4}$$

Therefore, the expression in (Equation 2.4) is a lower bound for the displacement $\text{dist}(z_0, \eta \cdot z_0)$.

In this case, the displacement function is

$$f_4(x_1, x_2, x_3, x_4) = \frac{1 - x_4}{x_4} \cdot \frac{1 - x_3}{x_3}.$$

The expressions on the right-hand side of the equalities (Equation 2.1), (Equation 2.2), (Equation 2.3), and (Equation 2.4) can be considered as the values of the functions f_1 , f_2 , f_3 and f_4 at the point $\mathbf{m} = (m_1, m_2, m_3, m_4) \in \Delta^3$ where Δ^3 is the 3-dimensional unit simplex $\{(x_1, x_2, x_3, x_4) \in \mathbf{R}_+^4 : \sum_{i=1}^4 x_i = 1\}$.

In order to calculate a lower bound for the $\max(\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0))$ we need to calculate the infimum of maximum of the functions f_1, f_2, f_3 and f_4 over the simplex Δ^3 . Let us define the following continuous function

$$\begin{aligned} G_0 &: \Delta^3 \rightarrow \mathbf{R} \\ \mathbf{x} &\mapsto \max(f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}), f_4(\mathbf{x})). \end{aligned}$$

We want to find $\inf_{\mathbf{x} \in \Delta^3} G_0(\mathbf{x})$.

We need different parametrization of Δ^3 . Let $\mathbf{p}_1$ be the parametrization $(1 - x_2 - x_3 - x_4, x_2, x_3, x_4)$ of Δ^3 where $0 < x_2 < 1, 0 < x_3 < 1 - x_2, 0 < x_4 < 1 - x_2 - x_3$ and $\mathbf{p}_2$ be the parametrization $(x_1, 1 - x_1 - x_3 - x_4, x_3, x_4)$ of Δ^3 where $0 < x_1 < 1, 0 < x_3 < 1 - x_1, 0 < x_4 < 1 - x_1 - x_3$. Also let $\mathbf{p}_3$ be the parametrization $(x_1, x_2, 1 - x_1 - x_2 - x_4, x_4)$ of Δ^3 where $0 < x_1 < 1, 0 < x_2 < 1 - x_1, 0 < x_4 < 1 - x_1 - x_2$ and $\mathbf{p}_4$ be the parametrization $(x_1, x_2, x_3, 1 - x_1 - x_2 - x_3)$ of Δ^3 where $0 < x_1 < 1, 0 < x_2 < 1 - x_1, 0 < x_3 < 1 - x_1 - x_2$. Points of Δ^3 will be written in bold fonts.

2.2 Infimum of maximum of f_1, f_2, f_3 and f_4 over the simplex Δ^3

Note that $\inf_{\mathbf{x} \in \Delta^3} G_0(\mathbf{x}) = \inf_{\mathbf{x} \in \Delta^3} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}))\}$ since $f_1(\mathbf{x}) = f_3(\mathbf{x})$ and $f_2(\mathbf{x}) = f_4(\mathbf{x})$ for every $\mathbf{x} \in \Delta^3$. Therefore, it is enough to prove the following:

Lemma 2.2.1. *If $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ are the functions defined as:*

$$f_1(\mathbf{x}) = \frac{1 - x_1}{x_1} \cdot \frac{1 - x_2}{x_2}, \text{ and } f_2(\mathbf{x}) = \frac{1 - x_3}{x_3} \cdot \frac{1 - x_4}{x_4},$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \Delta^3$, then we have $\inf_{\mathbf{x} \in \Delta^3} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}))\} = 9$.

Proof. Let us define the following continuous function:

$$\begin{aligned} F_0 &: \Delta^3 \rightarrow \mathbf{R} \\ \mathbf{x} &\mapsto \max(f_1(\mathbf{x}), f_2(\mathbf{x})). \end{aligned}$$

Note that $f_1(\mathbf{x}) > 1$ and $f_2(\mathbf{x}) > 1$ for every $\mathbf{x} \in \Delta^3$. Because if we have $f_1(\mathbf{x}) = 1$ for some point $\mathbf{x} = (x_1, x_2, x_3, x_4)$ in Δ^3 , then we get $x_1 + x_2 = 1$. This implies that $x_3 = 0$ and $x_4 = 0$. But the point $(x_1, x_2, 0, 0)$ is not in Δ^3 . Similarly, if $f_2(\mathbf{x}) = 1$ for some point $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \Delta^3$, then we get $x_3 + x_4 = 1$. This implies that $x_1 = 0$ and $x_2 = 0$. But the point $(0, 0, x_3, x_4)$ is not in Δ^3 . Therefore, we have $\inf_{\mathbf{x} \in \Delta^3} F_0(\mathbf{x}) \geq 1$. We need to show that $\inf_{\mathbf{x} \in \Delta^3} F_0(\mathbf{x}) = 9$. First, we prove that

$$\inf_{\mathbf{x} \in \Delta^3} F_0(\mathbf{x}) = \min_{\mathbf{x} \in \Delta^3} F_0(\mathbf{x}).$$

If we let $\Delta_k = \{(x_1, x_2, x_3, x_4) \in \Delta^3 : \frac{1}{k} \leq x_i \leq 1 - \frac{1}{2k} \text{ for } i = 1, 2, 3, 4\}$ for $k \geq 2$, then F has an absolute minimum α_k at a point $\mathbf{x}_k$ in Δ_k since Δ_k is compact. (α_k) is a decreasing sequence since we have $\Delta_k \subset \Delta_{k+1}$. Let us consider the sequence $(\mathbf{x}_k)$. We claim that the sequence $(\mathbf{x}_k)$ cannot have a limit point on the boundary of Δ^3 . Let us assume to the contrary that $\mathbf{x}_k \rightarrow \mathbf{x}_0 \in \partial\Delta^3$ as $k \rightarrow \infty$. If b_1, b_2, b_3, b_4 denote the coordinates of the point $\mathbf{x}_0$, then $b_i = 0$ for some $i = 1, 2, 3, 4$. Suppose, for example, that $b_1 = 0$. This implies that $b_2 = 1$ since otherwise f_1 would tend to infinity through the sequence $\mathbf{x}_k$. But it is supposed to be tending to an infimum. Therefore, we must have $b_3 = 0$ and $b_4 = 0$. If we repeat this argument for $b_3 = 0$, we see that $b_4 = 1$ which gives a contradiction.

We claim that there exists a positive integer k_0 so that $\mathbf{x}_k = \mathbf{x}_{k+1}$ for every $k \geq k_0$. Let us assume otherwise. This means that we have a subsequence $(\mathbf{x}_{k_j})$ of the sequence $(\mathbf{x}_k)$ so that $\mathbf{x}_{k_j} \in \Delta_{k_{j+1}} - \Delta_{k_j}$ for every integer $j > 0$. Since we have $\cup_{k=2}^{\infty} \Delta_k = \Delta^3$, we get $F(\mathbf{x}_{k_j}) \rightarrow \infty$ as $j \rightarrow \infty$. But this contradicts with the fact that (α_k) is a decreasing sequence. Therefore, we have $\mathbf{x}_k = \mathbf{x}_{k+1}$ for every $k \geq k_0$. In other words, the absolute minimum $F_0(\mathbf{x}_{k_0})$ of F exists on Δ^3 . Then, we obviously have $\inf_{\mathbf{x} \in \Delta^3} F_0(\mathbf{x}) = \min_{\mathbf{x} \in \Delta^3} F_0(\mathbf{x})$. Thus, we need to calculate the $\min_{\mathbf{x} \in \Delta^3} F_0(\mathbf{x})$. To this purpose, we find a point in Δ^3 where the minimum is attained.

Let $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*)$ be a point where $F_0(\mathbf{x}^*) = \min_{\mathbf{x} \in \Delta^3} F_0(\mathbf{x})$ and $\Delta^* = \{\mathbf{x} \in \Delta^3 : f_1(\mathbf{x}) = f_2(\mathbf{x})\}$. We claim that $\mathbf{x}^* \in \Delta^*$. Before we prove the claim, we show that f_1 and f_2 has no critical point in Δ^3 . If we use the parametrization p_1 of Δ^3 , we see that

$$\frac{\partial f_1}{\partial x_1} = -\frac{1}{x_1^2} \cdot \frac{1-x_2}{x_2}, \quad \frac{\partial f_1}{\partial x_2} = -\frac{1}{x_2^2} \cdot \frac{1-x_1}{x_1}, \quad \frac{\partial f_1}{\partial x_3} = 0.$$

Note that we have $(f_1)_1 < 0$ for every $\mathbf{x} \in \Delta^3$. Therefore, f_1 has no critical point in Δ^3 . Since f_1 is differentiable everywhere in Δ^3 , f_1 has no local minimum. Similarly, we have the partial derivatives

$$\begin{aligned} \frac{\partial f_2}{\partial x_1} &= -\frac{1-x_3}{x_3} \cdot \frac{1}{(1-x_1-x_2-x_3)^2}, \\ \frac{\partial f_2}{\partial x_2} &= -\frac{1-x_3}{x_3} \cdot \frac{1}{(1-x_1-x_2-x_3)^2}, \\ \frac{\partial f_2}{\partial x_3} &= -\frac{x_1+x_2}{x_3^2} \cdot \frac{1-x_1-x_2-2x_3}{(1-x_1-x_2-x_3)^2}. \end{aligned}$$

We see that $(f_2)_1 < 0$ for every $\mathbf{x} \in \Delta^3$. Therefore, f_2 has no critical point in Δ^3 . Since f_2 is also differentiable everywhere in Δ^3 , f_2 has no local minimum. Now we prove the claim:

Let us assume otherwise that $f_1(\mathbf{x}^*) \neq f_2(\mathbf{x}^*)$. Then we have either $f_1(\mathbf{x}^*) > f_2(\mathbf{x}^*)$ or $f_1(\mathbf{x}^*) < f_2(\mathbf{x}^*)$. If we assume that $f_1(\mathbf{x}^*) > f_2(\mathbf{x}^*)$, then there exists a neighborhood U of $\mathbf{x}^*$ in Δ^3 so that $f_1(\mathbf{x}) > f_2(\mathbf{x})$ for every $\mathbf{x} \in U$ since f_1 is continuous on Δ^3 . Hence, we get $F(\mathbf{x}) = f_1(\mathbf{x})$ for every $\mathbf{x} \in U$. Since we have $f_1(\mathbf{x}^*) = F(\mathbf{x}^*)$, f_1 has a local minimum at $\mathbf{x}^* \in U$. This is a contradiction. Let us assume that $f_1(\mathbf{x}^*) < f_2(\mathbf{x}^*)$. Since f_2 is continuous on Δ^3 , we get $f_1(\mathbf{x}) < f_2(\mathbf{x})$ for every $\mathbf{x}$ in some neighborhood V of $\mathbf{x}^*$ where we get $F(\mathbf{x}) = f_2(\mathbf{x})$ for every $\mathbf{x} \in V$. Then, f_2 has a local minimum at $\mathbf{x}^* \in V$, a contradiction. Hence, we get $\mathbf{x}^* \in \Delta^*$.

Using the parametrization $\mathfrak{p}_4$ of Δ^3 , we consider the equation $f_1(\mathbf{x}^*) = f_2(\mathbf{x}^*)$. We obtain the following equality

$$\begin{aligned} \frac{1-x_1^*}{x_1^*} \cdot \frac{1-x_2^*}{x_2^*} &= \frac{1-x_3^*}{x_3^*} \cdot \frac{x_1^*+x_2^*+x_3^*}{1-x_1^*-x_2^*-x_3^*} \\ x_3^*(1-x_1^*)(1-x_2^*)(1-x_1^*-x_2^*-x_3^*) &= x_1^*x_2^*(1-x_3^*)(x_1^*+x_2^*+x_3^*) \\ (1-x_1^*)(1-x_2^*)((1-x_1^*-x_2^*)x_3^*-(x_3^*)^2) &= x_1^*x_2^*((1-x_3^*)(x_1^*+x_2^*)+(1-x_3^*)x_3^*). \end{aligned}$$

We simplify the equality above and obtain that

$$(1-x_1^*-x_2^*)(x_3^*)^2 - (1-x_1^*-x_2^*)^2x_3^* + x_1^*x_2^*(x_1^*+x_2^*) = 0.$$

Therefore, we can consider x_3^* as a solution of the quadratic equation $q_1(x) = 0$ where

$$q_1(x) = (1-x_1^*-x_2^*)x^2 - (1-x_1^*-x_2^*)^2x + x_1^*x_2^*(x_1^*+x_2^*).$$

Let us define the function

$$\begin{aligned} D_1(\mathbf{x}) &= D_1(x_1, x_2) \\ &= (1 - x_1 - x_2)^4 - 4x_1x_2(x_1 + x_2)(1 - x_1 - x_2). \end{aligned}$$

Since $D_1(\mathbf{x}^*)$ corresponds to the discriminant of $q_1(x)$, we have $D_1(\mathbf{x}^*) \geq 0$. Hence, we get

$$x_3^* = \frac{(1 - x_1^* - x_2^*)^2 \pm \sqrt{D_1(\mathbf{x}^*)}}{2(1 - x_1^* - x_2^*)}.$$

We claim that $D_1(\mathbf{x}^*) = 0$. Let us assume otherwise that $D_1(\mathbf{x}^*) > 0$. Since D_1 is continuous, there exists an $\epsilon > 0$ so that $D_1(x'_1, x'_2) > 0$ for

$$x'_1 = x_1^* + \epsilon,$$

$$x'_2 = x_2^*.$$

Therefore, if we say

$$\tilde{q}_1(x) = (1 - x'_1 - x'_2)x^2 - (1 - x'_1 - x'_2)^2x + x'_1x'_2(x'_1 + x'_2),$$

we see that the quadratic equation $\tilde{q}_1(x) = 0$ has two real roots

$$x'_3 = \frac{(1 - x'_1 - x'_2)^2 \pm \sqrt{D_1(x'_1, x'_2)}}{2(1 - x'_1 - x'_2)}.$$

In fact, we can choose $\epsilon > 0$ so that the coefficients of $q_1(x)$ and $\tilde{q}_1(x)$ are sufficiently close to each other to conclude that $0 < x'_i < 1$ for every $i = 1, 2, 3, 4$ where we set

$$x'_4 = 1 - \sum_{i=1}^3 x'_i.$$

Therefore, if we consider the point $\mathbf{x}' = (x'_1, x'_2, x'_3, x'_4) \in \Delta^3$, we see that $f_1(\mathbf{x}') = f_2(\mathbf{x}')$. But we obtain that

$$\begin{aligned} F(\mathbf{x}') &= f_1(\mathbf{x}') \\ &= \left(\frac{1}{x_1^* + \epsilon} - 1 \right) \left(\frac{1}{x_2^*} - 1 \right) \\ &< F(\mathbf{x}^*). \end{aligned}$$

This contradicts with the minimality of $F(\mathbf{x}^*)$. Thus, we get $D_1(x_1^*, x_2^*) = 0$.

Then, we know that

$$x_3^* = \frac{1 - x_1^* - x_2^*}{2}.$$

Since we have $x_4^* = 1 - x_1^* - x_2^* - x_3^*$, we obtain $x_3^* = x_4^*$.

We give the same argument above using the parametrization p_2 . In other words, we consider the equation $f_1(\mathbf{x}^*) = f_2(\mathbf{x}^*)$ and see that

$$\frac{1 - x_1^*}{x_1^*} \cdot \frac{x_1^* + x_3^* + x_4^*}{1 - x_1^* - x_3^* - x_4^*} = \frac{1 - x_3^*}{x_3^*} \cdot \frac{1 - x_4^*}{x_4^*}$$

We simplify the equality above and obtain that

$$(1 - x_3^* - x_4^*)(x_1^*)^2 - (1 - x_3^* - x_4^*)^2 x_1^* + x_3^* x_4^* (x_3^* + x_4^*) = 0.$$

Therefore, we can consider x_1^* as a solution of the quadratic equation $q_2(x) = 0$ where

$$q_2(x) = (1 - x_3^* - x_4^*)x^2 - (1 - x_3^* - x_4^*)^2 x + x_3^* x_4^* (x_3^* + x_4^*).$$

Let us define the following function

$$\begin{aligned} D_2(\mathbf{x}) &= D_2(x_3, x_4) \\ &= (1 - x_3 - x_4)^4 - 4x_3 x_4 (x_3 + x_4)(1 - x_3 - x_4). \end{aligned}$$

Since $D_2(\mathbf{x}^*)$ corresponds to the discriminant of $q_2(x)$, we get $D_2(\mathbf{x}^*) \geq 0$. Thus, we have

$$x_1^* = \frac{(1 - x_3^* - x_4^*)^2 \pm \sqrt{D_2(\mathbf{x}^*)}}{2(1 - x_3^* - x_4^*)}.$$

We claim that $D_2(\mathbf{x}^*) = 0$. Let us assume otherwise that $D_2(\mathbf{x}^*) > 0$. Since D_3 is continuous, there exists an $\epsilon > 0$ so that $D_3(x'_3, x'_4) > 0$ for

$$x'_3 = x_3^* + \epsilon,$$

$$x'_4 = x_4^*.$$

Therefore, if we say

$$\tilde{q}_2(x) = (1 - x'_3 - x'_4)x^2 - (1 - x'_3 - x'_4)^2x + x'_3x'_4(x'_3 + x'_4),$$

we see that the quadratic equation $\tilde{q}_2(x) = 0$ has two real roots

$$x'_1 = \frac{(1 - x'_3 - x'_4)^2 \pm \sqrt{D_2(x'_3, x'_4)}}{2(1 - x'_3 - x'_4)}.$$

In fact, we can choose $\epsilon > 0$ so that the coefficients of $q_2(x)$ and $\tilde{q}_2(x)$ are sufficiently close to each other to conclude that $0 < x'_i < 1$ for every $i = 1, 2, 3, 4$ where we set

$$x'_2 = 1 - \sum_{i=1, i \neq 2}^4 x'_i.$$

Therefore, if we consider the point $\mathbf{x}' = (x'_1, x'_2, x'_3, x'_4) \in \Delta^3$, we see that $f_1(\mathbf{x}') = f_2(\mathbf{x}')$. But we obtain that

$$\begin{aligned} F(\mathbf{x}') &= f_2(\mathbf{x}') \\ &= \left(\frac{1}{x'_3 + \epsilon} - 1 \right) \left(\frac{1}{x'_4} - 1 \right) \\ &< \left(\frac{1}{x'_3} - 1 \right) \left(\frac{1}{x'_4} - 1 \right) \\ &= F(\mathbf{x}^*), \end{aligned}$$

a contradiction. Hence, we have $D_2(\mathbf{x}^*) = 0$.

Then, we get

$$x_1^* = \frac{1 - x_3^* - x_4^*}{2}.$$

Since we have $x_2^* = 1 - x_1^* - x_3^* - x_4^*$, we obtain that $x_1^* = x_2^*$.

We know that the point $\mathbf{x}^* = (x_1^*, x_1^*, x_3^*, x_3^*) \in \Delta^*$. Using the equality $f_1(\mathbf{x}^*) = f_2(\mathbf{x}^*)$ once again, we calculate that

$$\begin{aligned} \left(\frac{1 - x_1^*}{x_1^*} \right)^2 &= \left(\frac{1 - x_3^*}{x_3^*} \right)^2 \\ \frac{1}{x_1^*} - 1 &= \frac{1}{x_3^*} - 1 \end{aligned}$$

which implies that $x_1^* = x_3^*$. Hence, we get $\mathbf{x}^* = (x_1^*, x_1^*, x_1^*, x_1^*)$.

Remember that we have $\mathbf{x}^* \in \Delta^*$. Therefore, x_1^* satisfies the equation $x_1^* + x_1^* + x_1^* + x_1^* = 1$. In other words, we get that $x_1^* = \frac{1}{4}$. Then we obtain that $\mathbf{x}^* = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Finally, we calculate that

$$\begin{aligned} \min_{\mathbf{x} \in \Delta^3} F(\mathbf{x}) &= F_0(\mathbf{x}^*) \\ &= f_1(\mathbf{x}^*) \\ &= f_2(\mathbf{x}^*) \\ &= \left(\frac{1 - \frac{1}{4}}{\frac{1}{4}} \right)^2 \\ &= 9. \end{aligned}$$

This proves that $\inf_{\mathbf{x} \in \Delta^3} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}))\} = 9$.

□

2.3 Proof of Proposition 1.2.2

Lemma 2.2.1 suggests a slightly different proof for Proposition 1.2.2 than the one presented in (5). This proof is given as follows:

Proof of Proposition 1.2.2. We know $\text{dist}(z_0, \xi \cdot z_0) = \text{dist}(z_0, \xi^{-1} \cdot z_0)$, $\text{dist}(z_0, \eta \cdot z_0) = \text{dist}(z_0, \eta^{-1} \cdot z_0)$.

Therefore, it is enough to show that $\max(\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0)) \geq \log 3$. Let $\mathbf{m} = (m_1, m_2, m_3, m_4)$ in Δ^3 where m_1, m_2, m_3 and m_4 are the total measures defined in §2.1. By the discussion given in §2.1, we have

$$\max(\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0)) \geq \frac{1}{2} \log(\max(\frac{1-m_1}{m_1} \cdot \frac{1-m_2}{m_2}, \frac{1-m_3}{m_3} \cdot \frac{1-m_4}{m_4})).$$

Using the fact that $\frac{1}{2} \log t$ is a strictly increasing function, we obtain that

$$\frac{1}{2} \log(\max(\frac{1-m_1}{m_1} \cdot \frac{1-m_2}{m_2}, \frac{1-m_3}{m_3} \cdot \frac{1-m_4}{m_4})) \geq \frac{1}{2} \log(\inf_{\mathbf{x} \in \Delta^3} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}))\}).$$

In Lemma 2.1, we showed that $\inf_{\mathbf{x} \in \Delta^3} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}))\} = 9$. Hence, we conclude that:

$$\begin{aligned} \max(\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0)) &\geq \frac{1}{2} \log(\inf_{\mathbf{x} \in \Delta^3} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}))\}) \\ &= \frac{1}{2} \log 9 \\ &= \log 3. \end{aligned}$$

This finishes the proof of the proposition. □

As I mentioned in Introduction, it is possible to extend the techniques used in the calculation of the number $\log 3$ to find lower bounds for other hyperbolic displacements. In the following chapters, I will carry out similar calculations to determine a lower bound for the displacements $\text{dist}(z_0, \xi \cdot z_0)$, $\text{dist}(z_0, \eta \cdot z_0)$ and $\text{dist}(z_0, \xi\eta \cdot z_0)$.

I would like to emphasize a few key points used in the proof of Lemma 2.2.1. Remember that f_1 and f_2 have no local minimum in the simplex Δ^3 . This fact implied that $\mathbf{x}^* \in \Delta^*$. The proof of Lemma 2.2.1 showed that the main ingredients used to calculate the number $\log 3$ are

- A. $\inf_{\mathbf{x} \in \Delta^3} F_0(\mathbf{x}) = \min_{\mathbf{x} \in \Delta^3} F_0(\mathbf{x})$,
- B. There exists a point $\mathbf{x}^* \in \Delta^* \subset \Delta^3$ such that $\min_{\mathbf{x} \in \Delta^3} F_0(\mathbf{x}) = f_1(\mathbf{x}^*)$.

Note that Lemma 2.2.1 establishes that the point $\mathbf{x}^* \in \Delta^3$ is unique. If we assume the uniqueness of the point $\mathbf{x}^*$ a priori together with (A), it is possible suggest an alternative way of finding the coordinates of the point $\mathbf{x}^*$ which reduces the number of calculations necessary to find the number $\log 3$.

Let $T_1, T_2 : \mathbf{R}^4 \rightarrow \mathbf{R}^4$ be two transformations defined by

$$(x_1, x_2, x_3, x_4) \mapsto (x_2, x_1, x_4, x_3)$$

and

$$(x_1, x_2, x_3, x_4) \mapsto (x_3, x_4, x_1, x_2),$$

respectively. Note that the simplex Δ^3 is preserved by the transformations T_1 and T_2 , i.e., $T_1(\Delta^3) = \Delta^3$, $T_2(\Delta^3) = \Delta^3$. By the definition of T_1 , we have $f_1(T_1(\mathbf{x})) = f_1(\mathbf{x})$ and $f_2(T_1(\mathbf{x})) = f_2(\mathbf{x})$ for every $\mathbf{x} \in \Delta^3$. Let us define the function $H_1 : \Delta^3 \rightarrow \mathbf{R}$ by $H_1(\mathbf{x}) = \max\{(f_i \circ T_1)(\mathbf{x}) : i = 1, 2\}$. Then we see that $F_0(\mathbf{x}) = H_1(\mathbf{x})$ for every $\mathbf{x} \in \Delta^3$ and $\min_{\mathbf{x} \in \Delta^3} F_0 = \min_{\mathbf{x} \in \Delta^3} H_1(\mathbf{x})$. Since F_0 takes its minimum value at the point $\mathbf{x}^*$ and $\{f_i(\mathbf{x}^*) : i = 1, 2\} = \{(f_i \circ T_1)(\mathbf{x}^*) : i = 1, 2\}$, H_1 takes its minimum value at the point $T_1^{-1}(\mathbf{x}^*)$. We assumed that $\mathbf{x}^*$ is unique. In other words, we obtain that $T_1^{-1}(\mathbf{x}^*) = \mathbf{x}^*$. This means that $x_1^* = x_2^*$, and $x_3^* = x_4^*$. A similar argument can be given for the transformation T_2 .

By the definition of T_2 , we have $f_1(T_2(\mathbf{x})) = f_1(\mathbf{x})$ and $f_2(T_2(\mathbf{x})) = f_2(\mathbf{x})$ for every $\mathbf{x} \in \Delta^3$. Let us define the function $H_2 : \Delta^3 \rightarrow \mathbf{R}$ by $H_2(\mathbf{x}) = \max\{(f_i \circ T_2)(\mathbf{x}) : i = 1, 2\}$. Then we see that $F_0(\mathbf{x}) = H_2(\mathbf{x})$ for every $\mathbf{x} \in \Delta^3$ and $\min_{\mathbf{x} \in \Delta^3} F_0 = \min_{\mathbf{x} \in \Delta^3} H_2(\mathbf{x})$. Since F_0 takes its minimum value at the point $\mathbf{x}^*$ and $\{f_i(\mathbf{x}^*) : i = 1, 2\} = \{(f_i \circ T_2)(\mathbf{x}^*) : i = 1, 2\}$, H_2 takes its minimum value at the point $T_2^{-1}(\mathbf{x}^*)$. We assumed that $\mathbf{x}^*$ is unique. Hence, we obtain that $T_2^{-1}(\mathbf{x}^*) = \mathbf{x}^*$. This means that $x_1^* = x_3^*$, and $x_2^* = x_4^*$. Hence, we obtain that $x_i^* = x_j^*$ for every $i, j = 1, 2, 3, 4$. Since $\mathbf{x}^* \in \Delta^3$, we see that $4x_i^* = 1$ for every i . Finally, we find that $\mathbf{x}^* = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Then a simple evaluation shows that $\min_{\mathbf{x} \in \Delta^3} F_0(\mathbf{x}) = 9$. Note that in this calculation method, we don't refer to the statement $\mathbf{x}^* \in \Delta^*$ given in (B).

CHAPTER 3

DECOMPOSITIONS OF 2-GENERATOR FREE GROUPS

Let Γ be a group which is free on a finite generating set Ξ . Every element γ of Γ can be written uniquely as a reduced word $\psi_1 \cdots \psi_m$, where $m \geq 0$, each ψ_i is an element of $\Xi \amalg \Xi^{-1}$, and $\psi_{i+1} \neq \psi_i^{-1}$ for $i = 1, \dots, m-1$. If $\gamma \neq 1$, i.e. $m > 0$, we shall call $\psi_1 \dots \psi_n$ the *initial word of length $n > 0$* of γ . Let Ψ be a set of initial words of length less than or equal to n . For each initial word $\psi \in \Psi$, let J_ψ denote the set of non-trivial elements of Γ that have initial word ψ . Depending on the number of elements in Ξ and lengths of initial words in Ψ there may be a set of words which are not contained in any of J_ψ . Let us call this set the *residue set* of Ψ and denote it by Ψ_r .

Definition 3.0.1. *The pair (Ψ, Ψ_r) is called a decomposition of Γ if*

$$\Gamma = \{1\} \amalg \Psi_r \amalg \coprod_{\psi \in \Psi} J_\psi.$$

The decomposition of Γ corresponding to the pair (Ψ, Ψ_r) will be denoted by $\Gamma_{\mathcal{D}}$ where $\mathcal{D} = (\Psi, \Psi_r)$.

3.1 Decomposition of Γ for the isometries ξ, η and $\xi\eta$

We know that $\Gamma = \langle \xi, \eta \rangle$ is a free group on the generating set $\Xi = \{\xi, \eta\}$ ((5), Proposition 9.2). The decomposition $\Gamma_{\mathcal{D}_p}$ where $\mathcal{D}_p = (\Xi \amalg \Xi^{-1}, \emptyset)$ corresponds to the paradoxical decompositions of Γ used in the proof of the log3 theorem. In this section, I introduce a decomposition of Γ that contains

ξ , η and $\xi\eta$. Let $\Psi = \{\xi\eta, \xi^2, \xi\eta^{-1}, \eta, \xi^{-1}, \eta^{-1}\xi^{-1}, \eta^{-2}, \eta^{-1}\xi\}$. Then the residue set of Ψ is $\Psi_r = \{\xi, \eta^{-1}\}$.

It is easy to see that

$$\begin{aligned}\Gamma &= \{1\} \amalg \Psi_r \amalg J_{\xi\eta} \amalg J_{\xi^2} \amalg J_{\xi\eta^{-1}} \amalg J_{\eta} \amalg J_{\xi^{-1}} \amalg J_{\eta^{-1}\xi^{-1}} \amalg J_{\eta^{-2}} \amalg J_{\eta^{-1}\xi} \\ &= \{1\} \amalg \Psi_r \amalg \coprod_{\psi \in \Psi} J_{\psi}.\end{aligned}$$

Therefore $\Gamma_{\mathcal{D}}$ is a decomposition of Γ where $\mathcal{D} = (\Psi, \Psi_r)$. Note that this decomposition contains the words ξ , η and $\xi\eta$. I will concentrate on the decomposition $\Gamma_{\mathcal{D}}$ to prove Theorem 5.1.1.

3.2 An analog of Lemma 1.2.1

In this section, I prove Lemma 3.2.1 which is an analog of Lemma 1.4 for the decomposition $\Gamma_{\mathcal{D}}$.

Before I proceed, I introduce some notation to shorten some of the expressions I use. Let

$$J_{\Psi} := \coprod_{\psi \in \Psi} J_{\psi}$$

where $\amalg$ denotes the disjoint union. Let us define the following sets

$$\Psi_1 = \{\xi\eta, \xi^2, \xi\eta^{-1}\},$$

$$\Psi_2 = \{\eta^{-1}\xi^{-1}, \eta^{-2}, \eta^{-1}\xi\},$$

$$\Psi_3 = \{\xi\eta, \xi^2, \xi\eta^{-1}, \eta, \xi^{-1}\},$$

$$\Psi_4 = \{\eta^{-1}\xi^{-1}, \eta^{-2}, \eta^{-1}\xi, \eta, \xi^{-1}\}.$$

We also define the following sets

$$\begin{aligned}
\Phi_1 &= \{\eta^{-2}, \eta^{-1}\xi, \eta^{-1}\xi^{-1}, \xi\eta, \xi^2, \xi\eta^{-1}, \eta\}, \\
\Phi_2 &= \{\xi\eta, \xi^2, \xi\eta^{-1}, \xi^{-1}, \eta^{-1}\xi^{-1}, \eta^{-2}, \eta^{-1}\xi\}, \\
\Phi_3 &= \{\xi\eta, \xi^2, \xi\eta^{-1}, \eta, \xi^{-1}, \eta^{-1}\xi^{-1}, \eta^{-1}\xi\}, \\
\Phi_4 &= \{\xi\eta, \xi^2, \eta, \xi^{-1}, \eta^{-1}\xi^{-1}, \eta^{-2}, \eta^{-1}\xi\}, \\
\Phi_5 &= \{\xi\eta, \xi^2, \xi\eta^{-1}, \eta, \xi^{-1}, \eta^{-1}\xi^{-1}, \eta^{-2}\}, \\
\Phi_6 &= \{\xi\eta, \xi\eta^{-1}, \eta, \xi^{-1}, \eta^{-1}\xi^{-1}, \eta^{-2}, \eta^{-1}\xi\}.
\end{aligned}$$

Lemma 3.2.1. *Let Γ be a Kleinian group which is free on a generating set $\{\xi, \eta\}$ and $\Gamma_{\mathcal{D}}$ be the decomposition of Γ described above. Let z_0 be any point of $\mathbf{H}^3$. If Ψ_i and Φ_j for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5, 6$ are the sets defined above, then there exists a number $D \in [0, 2]$, a Γ -invariant D -conformal density $\mathcal{M}_{\Gamma, z_0} = (\mu_z)$ for $\mathbf{H}^3$ and a family $(\nu_\psi)_{\psi \in \Psi}$ of Borel measures on S_∞ such that*

$$(1) \quad \mu_{z_0}(S_\infty) = 1;$$

$$(2) \quad \mu_{z_0} = \sum_{\psi \in \Psi} \nu_\psi; \text{ and}$$

$$(3) \quad (a) \text{ for } \eta^{-1}\xi^{-1}, \xi\eta \in \Psi \text{ we have}$$

$$\begin{aligned}
\int \lambda_{\xi\eta, z_0}^D d\mu_{V_{\eta^{-1}\xi^{-1}}} &= \int \lambda_{\xi, z_0}^D d\mu_{V_{\xi^{-1}}} \\
&= 1 - \sum_{\psi \in \Psi_1} \int d\nu_\psi,
\end{aligned}$$

$$\begin{aligned}
\int \lambda_{\eta^{-1}\xi^{-1}, z_0}^D d\mu_{V_{\xi\eta}} &= \int \lambda_{\eta^{-1}, z_0}^D d\mu_{V_\eta} \\
&= 1 - \sum_{\psi \in \Psi_2} \int d\nu_\psi,
\end{aligned}$$

(b) for $\eta, \xi^{-1} \in \Psi$ we have

$$\begin{aligned}
\int \lambda_{\xi^{-1}, z_0}^D d\mu_{V_{\xi\eta^{-1}}} &= \int \lambda_{\eta, z_0}^D d\mu_{V_{\eta^{-2}}} \\
&= 1 - \sum_{\psi \in \Psi_3} \int d\nu_\psi,
\end{aligned}$$

(c) similarly, we also have

$$\begin{aligned}
\int \lambda_{\xi^{-1}, z_0}^D d\mu_{V_{\xi^2}} &= \int \lambda_{\eta, z_0}^D d\mu_{V_{\eta^{-1}\xi}} \\
&= 1 - \sum_{\psi \in \Psi_4} \int d\nu_\psi,
\end{aligned}$$

(4) Additionally, we have

$$\begin{aligned}
\int \lambda_{\eta, z_0}^D d\mu_{V_{\eta^{-1}\xi^{-1}}} &= \int d\nu_{\xi^{-1}} \\
&= 1 - \sum_{\psi \in \Phi_1} \int d\nu_\psi,
\end{aligned}$$

$$\begin{aligned}
\int \lambda_{\xi^{-1}, z_0}^D d\mu_{V_{\xi\eta}} &= \int d\nu_\eta, \\
&= 1 - \sum_{\psi \in \Phi_2} \int d\nu_\psi,
\end{aligned}$$

$$\begin{aligned}
\int \lambda_{\eta^{-1}\xi^{-1}, z_0}^D d\mu_{V_{\xi\eta^{-1}}} &= \int dv_{\eta^{-2}}, \\
&= 1 - \sum_{\psi \in \Phi_3} \int dv_{\psi},
\end{aligned}$$

$$\begin{aligned}
\int \lambda_{\xi\eta, z_0}^D d\mu_{V_{\eta^{-2}}} &= \int dv_{\xi\eta^{-1}}, \\
&= 1 - \sum_{\psi \in \Phi_4} \int dv_{\psi},
\end{aligned}$$

$$\begin{aligned}
\int \lambda_{\eta^{-1}\xi^{-1}, z_0}^D d\mu_{V_{\xi^2}} &= \int dv_{\eta^{-1}\xi}, \\
&= 1 - \sum_{\psi \in \Phi_5} \int dv_{\psi},
\end{aligned}$$

$$\begin{aligned}
\int \lambda_{\xi\eta, z_0}^D d\mu_{V_{\eta^{-1}\xi}} &= \int dv_{\xi^2}, \\
&= 1 - \sum_{\psi \in \Phi_6} \int dv_{\psi},
\end{aligned}$$

(5) Finally, we have

$$\begin{aligned}
\int \lambda_{\eta^{-2}, z_0}^D d\mu_{V_{\eta}} &= 1 - \int dv_{\eta^{-2}} \\
&= \sum_{\psi \in \Phi_3} \int dv_{\psi},
\end{aligned}$$

$$\begin{aligned}
\int \lambda_{\xi\eta^{-1}, z_0}^D d\mu_{V_\eta} &= 1 - \int dv_{\xi\eta^{-1}}, \\
&= \sum_{\psi \in \Phi_4} \int dv_\psi,
\end{aligned}$$

$$\begin{aligned}
\int \lambda_{\eta^{-1}\xi, z_0}^D d\mu_{V_{\xi^{-1}}} &= 1 - \int dv_{\eta^{-1}\xi}, \\
&= \sum_{\psi \in \Phi_5} \int dv_\psi,
\end{aligned}$$

$$\begin{aligned}
\int \lambda_{\xi^2, z_0}^D d\mu_{V_{\xi^{-1}}} &= 1 - \int dv_{\xi^2}, \\
&= \sum_{\psi \in \Phi_6} \int dv_\psi.
\end{aligned}$$

Proof. We know that $\Gamma_{\mathcal{D}}$ is a decomposition of Γ . Since Γ is discrete and torsion-free, it acts freely on $\mathbf{H}^3$. Hence the orbit $W = \Gamma \cdot z_0$ is a disjoint union

$$W = \{z_0\} \amalg V_0 \amalg \coprod_{\psi \in \Psi} V_\psi$$

where $V_0 = \{\gamma \cdot z_0 : \gamma \in (J_1)_r\}$ and $V_\psi = \{\gamma \cdot z_0 : \gamma \in J_\psi\}$. Note that V_0 is a finite set which is equal to $V_\xi \cup V_{\eta^{-1}}$ where $V_\xi = \{\xi \cdot z_0\}$ and $V_{\eta^{-1}} = \{\eta^{-1} \cdot z_0\}$. Let $\mathcal{V}$ denote the finite collection of all subsets of W which are sub-unions of the above disjoint union; that is, $\mathcal{V}$ consists of all sets of the form $\coprod_{\psi \in \Psi'} V_\psi$ or $V_0 \amalg \coprod_{\psi \in \Psi'} V_\psi$ or $\{z_0\} \amalg \coprod_{\psi \in \Psi'} V_\psi$ or $\{z_0\} \amalg V_0 \amalg \coprod_{\psi \in \Psi'} V_\psi$ for $\Psi' \subset \Psi$. We shall apply Proposition 1.1.3 with these choices of W and $\mathcal{V}$. We take D to be a number, and $(\mathcal{M}_V)_{V \in \mathcal{V}}$ a family of conformal densities, for which conditions (i)-(iv) of Proposition 1.1.3 are satisfied. We write $\mathcal{M}_V = (\mu_{V,z})_{z \in \mathbf{H}^3}$.

We set $\mathcal{M}_{\Gamma, z_0} = \mathcal{M}_W$, and $\nu_\psi = \mu_{V_\psi, z_0}$ for each $\psi \in \Psi$. According to Proposition 1.2, $\mathcal{M}$ is Γ -invariant. It follows from condition (i) of Proposition 1.1.3 and the definition of a conformal density that $\mu_{z_0} = \mu_{W, z_0} \neq 0$; hence after multiplying by a constant we may assume that μ_{z_0} has total mass 1. This is conclusion (i) of Lemma 3.2.1. We shall show that the other conclusions of Lemma 3.2.1 also hold with these definitions of $\mathcal{M}$ and ν_ψ .

By (1) above and condition (ii) of Proposition 1.1.3, we have

$$\begin{aligned} \mu_{z_0} &= \mu_{W, z_0} \\ &= \mu_{\{z_0\}, z_0} + \mu_{V_0, z_0} + \sum_{\psi \in \Psi} \mu_{V_\psi, z_0}. \end{aligned}$$

But we have $\mu_{\{z_0\}, z_0} = 0$ and $\mu_{V_0, z_0} = 0$ by condition (i) of Proposition 1.1.3. Hence we get

$$\begin{aligned} \mu_{z_0} &= \sum_{\psi \in \Psi} \mu_{V_\psi, z_0} \\ &= \sum_{\psi \in \Psi} \nu_\psi. \end{aligned}$$

which is the conclusion (ii) of Lemma 3.2.1.

Let us write $\{\eta^{-1}\} \amalg J_{\Psi'} = J_{\eta^{-1}, \Psi'}$, $\{\xi\} \amalg J_{\Psi'} = J_{\xi, \Psi'}$ and $\{\eta^{-1}\} \amalg \{\xi\} \amalg J_{\Psi'} = J_{\eta^{-1}, \xi, \Psi'}$ for every $\Psi' \in \{\Psi_i : i = 1, 2, 3, 4\}$. Similarly, let us write $\{\eta^{-1}\} \amalg J_\Phi = J_{\eta^{-1}, \Phi}$, $\{\xi\} \amalg J_\Phi = J_{\xi, \Phi}$ and $\{\eta^{-1}\} \amalg \{\xi\} \amalg J_\Phi = J_{\eta^{-1}, \xi, \Phi}$ for every $\Phi \in \{\Phi_j : j = 1, 2, 3, 4, 5, 6\}$. It follows from the definition of the sets J_ψ that for each $\psi \in \{\xi\eta, \xi^2, \xi\eta^{-1}, \eta, \xi^{-1}, \eta^{-1}\xi^{-1}, \eta^{-2}, \eta^{-1}\xi\}$, we have the relations analogous to the relations given in the decomposition $\Gamma_{\mathcal{D}_p}$.

These relations are

$$\xi \eta J_{\eta^{-1} \xi^{-1}} = \xi J_{\xi^{-1}}$$

$$= \Gamma - J_{\xi, \Psi_1},$$

$$\eta^{-1} \xi^{-1} J_{\xi \eta} = \eta^{-1} J_{\eta}$$

$$= \Gamma - J_{\eta^{-1}, \Psi_2},$$

$$\xi^{-1} J_{\xi \eta^{-1}} = \eta J_{\eta^{-2}}$$

$$= \Gamma - J_{\xi, \Psi_3}$$

$$\xi^{-1} J_{\xi^2} = \eta J_{\eta^{-1} \xi}$$

$$= \Gamma - J_{\eta^{-1}, \Psi_4}$$

$$\eta J_{\eta^{-1} \xi^{-1}} = J_{\xi^{-1}}$$

$$= \Gamma - J_{\eta^{-1}, \xi, \Phi_1},$$

$$\eta^{-1} \xi^{-1} J_{\xi^2} = J_{\eta^{-1} \xi}$$

$$= \Gamma - J_{\eta^{-1}, \xi, \Phi_2},$$

$$\eta^{-1} \xi^{-1} J_{\xi \eta^{-1}} = J_{\eta^{-2}}$$

$$= \Gamma - J_{\eta^{-1}, \xi, \Phi_3},$$

$$\xi^{-1} J_{\xi \eta} = J_{\eta}$$

$$= \Gamma - J_{\eta^{-1}, \xi, \Phi_4}.$$

We also have

$$\begin{aligned}
 \xi \eta J_{\eta^{-2}} &= J_{\xi \eta^{-1}} \\
 &= \Gamma - J_{\eta^{-1}, \xi, \Phi_5}, \\
 \xi \eta J_{\eta^{-1} \xi} &= J_{\xi^2} \\
 &= \Gamma - J_{\eta^{-1}, \xi, \Phi_6}.
 \end{aligned}$$

For the set $\eta^{-2} J_\eta$ some additional argument is necessary. We know that $\eta^2 \in J_\eta$. Therefore, we have $1 \in \eta^{-2} J_\eta$. Let w be a word in J_η . Then we have $w = \eta w_1$ for some word $w_1 \in \Gamma$. We get $\eta^{-2} w = \eta^{-1} w_1$. We know that the first letter of w_1 cannot be η^{-1} but it can be either η , ξ or ξ^{-1} . Let us assume that the first letter of w_1 is η and $w_1 \neq \eta$. Then we have $w_1 = \eta w_2$ for some word $w_2 \in \Gamma$. Similar to w_1 , the first letter of w_2 cannot be η^{-1} but it can be either η , ξ or ξ^{-1} . Since we get $\eta^{-2} w = w_2$ in this case, we conclude that $\{1\} \amalg J_\eta \amalg J_{\xi \eta} \amalg J_{\xi \eta^{-1}} \amalg J_{\xi^2} \amalg J_{\xi^{-1}} \subset \eta^{-2} J_\eta$. Let us assume that the first letter of w_1 is ξ . Then we get $w_1 = \xi w_2$ for some $w_2 \in \Gamma$. In this case, we see that $\eta^{-2} w = \eta^{-1} \xi w_2$. This means that $J_{\eta^{-1} \xi} \subset \eta^{-2} J_\eta$. Let us assume that the first letter of w_1 is ξ^{-1} . Then we get $w_1 = \xi^{-1} w_2$ for some $w_2 \in \Gamma$. In this case, we see that $\eta^{-2} w = \eta^{-1} \xi^{-1} w_2$. Therefore, we see that $J_{\eta^{-1} \xi^{-1}} \subset \eta^{-2} J_\eta$. In other words, $\eta^{-2} J_\eta$ contains every word in Γ except the ones start with η^{-2} . Hence, we obtain that

$$\eta^{-2} J_\eta = \Gamma - J_{\eta^{-2}}.$$

We know that $\eta \xi^{-1} \in J_\eta$. Therefore, we have $1 \in \xi \eta^{-1} J_\eta$. Let w be a word in J_η .

Then we have $w = \eta w_1$ for some word $w_1 \in \Gamma$. We get $\xi \eta^{-1} w = \xi w_1$. We know that the first letter of w_1 cannot be η^{-1} but it can be either η , ξ or ξ^{-1} . Let us assume that the first letter of w_1 is ξ^{-1} and $w_1 \neq \xi^{-1}$. Then we have $w_1 = \xi^{-1} w_2$ for some word $w_2 \in \Gamma$. The first letter of w_2 cannot be ξ but it can be either η , η^{-1} or ξ^{-1} . Since we get $\xi \eta^{-1} w = w_2$ in this case, we conclude that $\{1\} \amalg J_\eta \amalg J_{\eta^{-1}\xi} \amalg J_{\eta^{-1}\xi^{-1}} \amalg J_{\eta^{-2}} \amalg J_{\xi^{-1}} \subset \xi \eta^{-1} J_\eta$. Let us assume that the first letter of w_1 is ξ . Then we get $w_1 = \xi w_2$ for some $w_2 \in \Gamma$. In this case, we see that $\xi \eta^{-1} w = \xi^2 w_2$. This means that $J_{\xi^2} \subset \xi \eta^{-1} J_\eta$. Let us assume that the first letter of w_1 is η . Then we get $w_1 = \eta w_2$ for some $w_2 \in \Gamma$. In this case, we see that $\xi \eta^{-1} w = \xi \eta w_2$. Therefore, we see that $J_{\xi \eta} \subset \xi \eta^{-1} J_\eta$. In other words, $\xi \eta^{-1} J_\eta$ contains every word in Γ except the ones start with $\xi \eta^{-1}$. Hence, we find that

$$\xi \eta^{-1} J_\eta = \Gamma - J_{\xi \eta^{-1}}.$$

We know that $\xi^{-1} \eta \in J_{\xi^{-1}}$. Therefore, we have $1 \in \eta^{-1} \xi J_{\xi^{-1}}$. Let w be a word in $J_{\xi^{-1}}$. Then we have $w = \xi^{-1} w_1$ for some word $w_1 \in \Gamma$. We get $\eta^{-1} \xi w = \eta^{-1} w_1$. We know that the first letter of w_1 cannot be ξ but it can be either η , η^{-1} or ξ^{-1} . Let us assume that the first letter of w_1 is η and $w_1 \neq \eta$. Then we have $w_1 = \eta w_2$ for some word $w_2 \in \Gamma$. The first letter of w_2 cannot be η^{-1} but it can be either η , ξ^{-1} or ξ . Since we get $\eta^{-1} \xi w = w_2$ in this case, we conclude that $\{1\} \amalg J_\eta \amalg J_{\xi \eta^{-1}} \amalg J_{\xi \eta} \amalg J_{\xi^2} \amalg J_{\xi^{-1}} \subset \eta^{-1} \xi J_{\xi^{-1}}$. Let us assume that the first letter of w_1 is η^{-1} . Then we get $w_1 = \eta^{-1} w_2$ for some $w_2 \in \Gamma$. In this case, we see that $\eta^{-1} \xi w = \eta^{-2} w_2$. This means that $J_{\eta^{-2}} \subset \eta^{-1} \xi J_{\xi^{-1}}$. Let us assume that the first letter of w_1 is ξ^{-1} . Then we get $w_1 = \xi^{-1} w_2$ for some $w_2 \in \Gamma$. In this case, we see that $\eta^{-1} \xi w = \eta^{-1} \xi^{-1} w_2$. Therefore, we see that $J_{\eta^{-1} \xi^{-1}} \subset \eta^{-1} \xi J_{\xi^{-1}}$.

In other words, $\eta^{-1}\xi J_\eta$ contains every word in Γ except the ones start with $\eta^{-1}\xi$. Hence, we see that

$$\eta^{-1}\xi J_{\xi^{-1}} = \Gamma - J_{\eta^{-1}\xi}.$$

We know that $\xi^{-2} \in J_{\xi^{-1}}$. Therefore, we have $1 \in \xi^2 J_{\xi^{-1}}$. Let w be a word in $J_{\xi^{-1}}$. Then we have $w = \xi^{-1}w_1$ for some word $w_1 \in \Gamma$. We get $\eta^{-1}\xi w = \eta^{-1}w_1$. We know that the first letter of w_1 cannot be ξ but it can be either η , η^{-1} or ξ^{-1} . Let us assume that the first letter of w_1 is ξ^{-1} and $w_1 \neq \xi^{-1}$. Then we have $w_1 = \xi^{-1}w_2$ for some word $w_2 \in \Gamma$. The first letter of w_2 cannot be ξ but it can be either η , ξ^{-1} or η^{-1} . Since we get $\xi^2 w = w_2$ in this case, we conclude that $\{1\} \amalg J_\eta \amalg J_{\eta^{-1}\xi} \amalg J_{\eta^{-1}\xi^{-1}} \amalg J_{\eta^{-2}} \amalg J_{\xi^{-1}} \subset \xi^2 J_{\xi^{-1}}$. Let us assume that the first letter of w_1 is η^{-1} . Then we get $w_1 = \eta^{-1}w_2$ for some $w_2 \in \Gamma$. In this case, we see that $\xi^2 w = \xi \eta^{-1}w_2$. This means that $J_{\xi\eta^{-1}} \subset \xi^2 J_{\xi^{-1}}$. Let us assume that the first letter of w_1 is η . Then we get $w_1 = \eta w_2$ for some $w_2 \in \Gamma$. In this case, we see that $\xi^2 w = \xi \eta w_2$, which implies that $J_{\xi\eta} \subset \xi^2 J_{\xi^{-1}}$. In other words, $\xi^2 J_\eta$ contains every word in Γ except the ones start with ξ^2 . Hence, we see that

$$\xi^2 J_{\xi^{-1}} = \Gamma - J_{\xi^2}.$$

Using the relations we obtained above between the sets J_ψ for $\psi \in \Psi$, we derive the relations between the set V_ψ for $\psi \in \Psi$.

Before we list these relation, we introduce some notation: $V_{\eta^{-1}} \amalg V_{\Psi'} = V_{\eta^{-1}, \Psi'}$, $\{z_0\} \amalg V_{\eta^{-1}} \amalg V_{\Psi'} = V_{z_0, \eta^{-1}, \Psi'}$, $V_\xi \amalg V_{\Psi'} = V_{\xi, \Psi'}$, $\{z_0\} \amalg V_\xi \amalg V_{\Psi'} = V_{z_0, \xi, \Psi'}$, $V_{\eta^{-1}} \amalg V_\xi \amalg V_\Psi = V_{\eta^{-1}, \xi, \Psi}$ and $\{z_0\} \amalg V_{\eta^{-1}} \amalg V_\xi \amalg V_{\Psi'} = V_{z_0, \eta^{-1}, \xi, \Psi'}$ for every $\Psi' \in \{\Psi_i : i = 1, 2, 3, 4\}$. We write $V_{\eta^{-1}} \amalg V_\Phi = V_{\eta^{-1}, \Phi}$, $\{z_0\} \amalg V_{\eta^{-1}} \amalg V_\Phi = V_{z_0, \eta^{-1}, \Phi}$,

$V_\xi \amalg V_\Phi = V_{\xi, \Phi}$, $\{z_0\} \amalg V_\xi \amalg V_\Phi = V_{z_0, \xi, \Phi}$, $V_{\eta^{-1}} \amalg V_\xi \amalg V_\Phi = V_{\eta^{-1}, \xi, \Phi}$ and $\{z_0\} \amalg V_{\eta^{-1}} \amalg V_\xi \amalg V_\Phi = V_{z_0, \eta^{-1}, \xi, \Phi}$ for every $\Phi \in \{\Phi_j : j = 1, 2, 3, 4, 5, 6\}$. Then we obtain that

$$\begin{aligned} \xi \eta V_{\eta^{-1} \xi^{-1}} &= \xi V_{\xi^{-1}} \\ &= W - V_{\xi, \Psi_1}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \eta^{-1} \xi^{-1} V_{\xi \eta} &= \eta^{-1} V_\eta \\ &= W - V_{\eta^{-1}, \Psi_2}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \xi^{-1} V_{\xi \eta^{-1}} &= \eta V_{\eta^{-2}} \\ &= W - V_{z_0, \xi, \Psi_3}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \xi^{-1} V_{\xi^2} &= \eta V_{\eta^{-1} \xi} \\ &= W - V_{z_0, \eta^{-1}, \Psi_4}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \eta V_{\eta^{-1} \xi^{-1}} &= V_{\xi^{-1}} \\ &= W - V_{z_0, \xi, \eta^{-1}, \Phi_1}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \eta^{-1} \xi^{-1} V_{\xi \eta^{-1}} &= V_{\eta^{-2}} \\ &= W - V_{z_0, \xi, \eta^{-1}, \Phi_2}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \eta^{-1} \xi^{-1} V_{\xi^2} &= V_{\eta^{-1} \xi} \\ &= W - V_{z_0, \xi, \eta^{-1}, \Phi_3}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \xi^{-1} V_{\xi \eta} &= V_\eta \\ &= W - V_{z_0, \xi, \eta^{-1}, \Phi_4}. \end{aligned} \tag{3.8}$$

We also obtain the relations

$$\begin{aligned}\xi\eta V_{\eta^{-2}} &= V_{\xi\eta^{-1}} \\ &= W - V_{z_0, \xi, \eta^{-1}, \Phi_5},\end{aligned}\tag{3.9}$$

$$\begin{aligned}\xi\eta V_{\eta^{-1}\xi} &= V_{\xi^2} \\ &= W - V_{z_0, \xi, \eta^{-1}, \Phi_6},\end{aligned}\tag{3.10}$$

$$\eta^{-2}V_\eta = W - V_{\eta^{-2}},\tag{3.11}$$

$$\xi\eta^{-1}V_\eta = W - V_{\xi\eta^{-1}},\tag{3.12}$$

$$\eta^{-1}\xi V_{\xi^{-1}} = W - V_{\eta^{-1}\xi},\tag{3.13}$$

$$\xi^2 V_{\xi^{-1}} = W - V_{\xi^2}.\tag{3.14}$$

We start with the equations in (Equation 3.1). Let $\gamma_1 = \xi\eta$, $\gamma_2 = \xi$, $\gamma^1 = \eta^{-1}\xi^{-1}$ and $\gamma^2 = \xi^{-1}$. We know that $\gamma_i V_{\gamma^i} = W - V_{\xi, \Psi_1}$. Since we have $W - V_{\xi, \Psi_1} = V_{z_0, \eta^{-1}, \Psi_4} \in \mathcal{V}$, condition (iii) of Proposition 1.1.3 gives

$$\mathcal{M}_{V_{\gamma^i}} = (\gamma_i)_\infty^* (\mathcal{M}_{W - V_{\xi, \Psi_1}}).\tag{3.15}$$

On the other hand, by Proposition 1.1.3(ii) we have

$$\mathcal{M}_{\Gamma, z_0} = \mathcal{M}_{W - V_{\xi, \Psi_1}} + \mathcal{M}_{V_{\xi, \Psi_1}}.$$

Thus (Equation 3.15) can be rewritten in the form

$$\mathcal{M}_{V_{\gamma^i}} = (\gamma_i)_\infty^* (\mathcal{M}_{\Gamma, z_0} - \mathcal{M}_{V_{\xi, \Psi_1}}).$$

Since V_ξ is finite, Proposition 1.1.3(*iv*) implies $\mu_{V_\xi, z_0} = 0$ or $\nu_\xi = 0$. Then by Proposition 1.1.3(*ii*), we have

$$\begin{aligned} \mu_{V_{\gamma^i}, \gamma^i \cdot z_0} &= (\gamma_i)_\infty^* (\mu_{z_0} - \sum_{\psi \in \Psi_1} \mu_{V_\psi, z_0}) \\ &= (\gamma_i)_\infty^* (\mu_{z_0} - \sum_{\psi \in \Psi_1} \nu_\psi). \end{aligned} \tag{3.16}$$

Since $\mathcal{M}_{V_{\gamma^i}}$ is a D -conformal density, we have

$$\begin{aligned} d\mu_{V_{\gamma^i}, \gamma^i \cdot z_0} &= \mathcal{P}(z_0, \gamma^i \cdot z_0, \cdot)^D d\mu_{V_{\gamma^i}} \\ &= \lambda_{\gamma^i, z_0}^D d\mu_{V_{\gamma^i}} \end{aligned}$$

where the last equality follows from (Equation 1.1). Hence, equating the total masses of the two sides of (Equation 3.16), we obtain

$$\begin{aligned} \int \lambda_{\gamma^i, z_0}^D d\mu_{V_{\gamma^i}} &= \int d((\gamma_i)_\infty^* (\mu_{z_0} - \sum_{\psi \in \Psi_1} \nu_\psi)) \\ &= \int d(\mu_{z_0} - \sum_{\psi \in \Psi_1} \nu_\psi) \\ &= 1 - \sum_{\psi \in \Psi_1} \int d\nu_\psi. \end{aligned}$$

for $i = 1, 2$. This finishes the proof of part 3(a) of the lemma.

For (Equation 3.2), we carry out the same operations above. Let $\gamma_1 = \eta^{-1}\xi^{-1}$, $\gamma_2 = \eta^{-1}$, $\gamma^1 = \xi\eta$ and $\gamma^2 = \eta$. We know that

$$\gamma_i V_{\gamma^i} = W - V_{\eta^{-1}, \Psi_2}.$$

Since we have $W - V_{\eta^{-1}, \Psi_2} = V_{z_0, \xi, \Psi_3} \in \mathcal{V}$, condition (iii) of Proposition 1.1.3 gives

$$\mathcal{M}_{V_{\gamma^i}} = (\gamma_i)_\infty^* (\mathcal{M}_{W - V_{\eta^{-1}, \Psi_2}}). \quad (3.17)$$

On the other hand, by Proposition 1.1.3(ii) we have

$$\mathcal{M}_{\Gamma, z_0} = \mathcal{M}_{W - V_{\eta^{-1}, \Psi_2}} + \mathcal{M}_{V_{\eta^{-1}, \Psi_2}}.$$

Thus (Equation 3.17) can be rewritten in the form

$$\mathcal{M}_{V_{\gamma^i}} = (\gamma_i)_\infty^* (\mathcal{M}_{\Gamma, z_0} - \mathcal{M}_{V_{\eta^{-1}, \Psi_2}}).$$

Since $V_{\eta^{-1}}$ is finite, Proposition 1.1.3(iv) implies $\mu_{V_{\eta^{-1}}, z_0} = 0$ or $\nu_{\eta^{-1}} = 0$. Then by Proposition 1.1.3(ii) we have

$$\begin{aligned} \mu_{V_{\gamma^i}, \gamma_i \cdot z_0} &= (\gamma_i)_\infty^* (\mu_{z_0} - \sum_{\psi \in \Psi_2} \mu_{V_\psi, z_0}) \\ &= (\gamma_i)_\infty^* (\mu_{z_0} - \sum_{\psi \in \Psi_2} \nu_\psi). \end{aligned} \quad (3.18)$$

Since $\mathcal{M}_{V_{\gamma^i}}$ is a D -conformal density, we have

$$\begin{aligned} d\mu_{V_{\gamma^i}, \gamma^i \cdot z_0} &= \mathcal{P}(z_0, \gamma^i \cdot z_0, \cdot)^D d\mu_{V_{\gamma^i}} \\ &= \lambda_{\gamma^i, z_0}^D d\mu_{V_{\gamma^i}} \end{aligned}$$

where the last equality follows from (Equation 1.1). Hence, equating the total masses of the two sides of (Equation 3.18), we obtain

$$\begin{aligned} \int \lambda_{\gamma^i, z_0}^D d\mu_{V_{\gamma^i}} &= \int d((\gamma^i)_\infty^*(\mu_{z_0} - \sum_{\psi \in \Psi_2} \nu_\psi)) \\ &= \int d(\mu_{z_0} - \sum_{\psi \in \Psi_2} \nu_\psi) \\ &= 1 - \sum_{\psi \in \Psi_2} \int d\nu_\psi. \end{aligned}$$

for $i = 1, 2$.

Let us continue with (Equation 3.3). Let us say $\gamma_1 = \xi^{-1}$, $\gamma_2 = \eta$, $\gamma^1 = \xi\eta^{-1}$ and $\gamma^2 = \eta^{-2}$. We know that $\gamma_i V_{\gamma^i} = W - V_{z_0, \xi, \Psi_3}$. Since we have $W - V_{z_0, \xi, \Psi_3} = V_{\eta^{-1}, \Psi_1} \in \mathcal{V}$, condition (iii) of Proposition 1.1.3 gives

$$\mathcal{M}_{V_{\gamma^i}} = (\gamma^i)_\infty^*(\mathcal{M}_{W - V_{z_0, \xi, \Psi_3}}). \quad (3.19)$$

On the other hand, by Proposition 1.2(ii) we have

$$\mathcal{M}_{\Gamma, z_0} = \mathcal{M}_{W - V_{z_0, \xi, \Psi_3}} + \mathcal{M}_{V_{z_0, \xi, \Psi_3}}.$$

Thus (Equation 3.19) can be rewritten in the form

$$\mathcal{M}_{V_{\gamma^i}} = (\gamma_i)_\infty^* (\mathcal{M}_{\Gamma, z_0} - \mathcal{M}_{V_{z_0, \xi, \Psi_3}}).$$

Since $\{z_0\}$ and V_ξ are finite, Proposition 1.1.3(*iv*) implies $\mu_{\{z_0\}, z_0} = 0$ or $\nu_{\{z_0\}} = 0$ and $\mu_{V_\xi, z_0} = 0$ or $\nu_\xi = 0$.

Then by Proposition 1.1.3(*ii*) we have

$$\begin{aligned} \mu_{V_{\gamma^i}, \gamma^i \cdot z_0} &= (\gamma_i)_\infty^* (\mu_{z_0} - \sum_{\psi \in \Psi_3} \mu_{V_\psi, z_0}) \\ &= (\gamma_i)_\infty^* (\mu_{z_0} - \sum_{\psi \in \Psi_3} \nu_\psi). \end{aligned} \tag{3.20}$$

Since $\mathcal{M}_{V_{\gamma^i}}$ is a D -conformal density for every $i = 1, 2$, we have

$$\begin{aligned} d\mu_{V_{\gamma^i}, \gamma^i \cdot z_0} &= \mathcal{P}(z_0, \gamma^i \cdot z_0, \cdot)^D d\mu_{V_{\gamma^i}} \\ &= \lambda_{\gamma^i, z_0}^D d\mu_{V_{\gamma^i}} \end{aligned}$$

where the last equality follows from (Equation 1.1). Hence, equating the total masses of the two sides

of (17), we obtain

$$\begin{aligned} \int \lambda_{\gamma^i, z_0}^D d\mu_{V_{\gamma^i}} &= \int d((\gamma_i)_\infty^* (\mu_{z_0} - \sum_{\psi \in \Psi_3} \nu_\psi)) \\ &= \int d(\mu_{z_0} - \sum_{\psi \in \Psi_3} \nu_\psi) \\ &= 1 - \sum_{\psi \in \Psi_3} \int d\nu_\psi. \end{aligned}$$

for every $i = 1, 2$. This finished the proof of part 3(c) of the lemma.

For (Equation 3.5), we carry out the same calculations above. Let $\gamma_1 = \xi^{-1}$, $\gamma_2 = \eta$, $\gamma^1 = \xi^2$ and $\gamma^2 = \eta^{-1}\xi$. We know that $\gamma_i V_{\gamma^i} = W - V_{z_0, \eta^{-1}, \Psi_4}$. Since we have $W - V_{z_0, \eta^{-1}, \Psi_4} = V_{\xi, \Psi_{1,1}} \in \mathcal{V}$, condition (iii) of Proposition 1.1.3 gives

$$\mathcal{M}_{V_{\gamma^i}} = (\gamma_i)_\infty^* (\mathcal{M}_{W - V_{z_0, \eta^{-1}, \Psi_4}}). \quad (3.21)$$

On the other hand, by Proposition 1.1.3(ii) we have

$$\mathcal{M}_{\Gamma, z_0} = \mathcal{M}_{V_{z_0, \eta^{-1}, \Psi_4}} + \mathcal{M}_{W - V_{z_0, \eta^{-1}, \Psi_4}}.$$

Thus (Equation 3.21) can be rewritten in the form

$$\mathcal{M}_{V_{\gamma^i}} = (\gamma_i)_\infty^* (\mathcal{M}_{\Gamma, z_0} - \mathcal{M}_{V_{z_0, \eta^{-1}, \Psi_4}}).$$

Since $\{z_0\}$ and $V_{\eta^{-1}}$ are finite, Proposition 1.1.3(iv) implies $\mu_{\{z_0\}, z_0} = 0$ or $\nu_{\{z_0\}} = 0$ and $\mu_{V_{\eta^{-1}}, z_0} = 0$ or $\nu_{\eta^{-1}} = 0$. Then by Proposition 1.1.3(ii) we have

$$\begin{aligned} \mu_{V_{\gamma^i}, \gamma^i \cdot z_0} &= (\gamma_i)_\infty^* (\mu_{z_0} - \sum_{\psi \in \Psi_4} \mu_{V_\psi, z_0}) \\ &= (\gamma_i)_\infty^* (\mu_{z_0} - \sum_{\psi \in \Psi_4} \nu_\psi). \end{aligned} \quad (3.22)$$

Since $\mathcal{M}_{V_{\gamma^i}}$ is a D -conformal density for every $i = 1, 2$, we have

$$\begin{aligned} d\mu_{V_{\gamma^i}, \gamma^i \cdot z_0} &= \mathcal{P}(z_0, \gamma^i \cdot z_0, \cdot)^D d\mu_{V_{\gamma^i}} \\ &= \lambda_{\gamma^i, z_0}^D d\mu_{V_{\gamma^i}} \end{aligned}$$

where the last equality follows from (Equation 1.1). Hence equating the total masses of the two sides of (Equation 3.22), we obtain

$$\begin{aligned} \int \lambda_{\gamma^i, z_0}^D d\mu_{V_{\gamma^i}} &= \int d((\gamma^i)_\infty^*(\mu_{z_0} - \sum_{\psi \in \Psi_4} \nu_\psi)) \\ &= \int d(\mu_{z_0} - \sum_{\psi \in \Psi_4} \nu_\psi) \\ &= 1 - \sum_{\psi \in \Psi_4} \int d\nu_\psi \end{aligned}$$

for every $i = 1, 2$. This completes the proof of part 3(d) of the lemma.

Let us consider the equations in (Equation 3.5), (Equation 3.6), (Equation 3.8) and (Equation 3.9).

We need to carry out the following calculations. Let $\gamma_1 = \eta$, $\gamma_2 = \eta^{-1}\xi^{-1}$, $\gamma_4 = \xi^{-1}$, $\gamma_5 = \xi\eta$, $\gamma^1 = \eta^{-1}\xi^{-1}$, $\gamma^2 = \xi\eta^{-1}$, $\gamma^4 = \xi\eta$ and $\gamma^5 = \eta^{-2}$. Also let $\beta^1 = \xi^{-1}$, $\beta^2 = \eta^{-2}$, $\beta^4 = \eta$ and $\beta^5 = \xi\eta^{-1}$. We know that $\gamma_j V_{\gamma^j} = W - V_{z_0, \xi, \eta^{-1}, \Phi_j}$. Since we have $W - V_{z_0, \xi, \eta^{-1}, \Phi_j} = V_{\beta^j} \in \mathcal{V}$, condition (iii) of Proposition 1.1.3 gives

$$\mathcal{M}_{V_{\gamma^j}} = (\gamma_j)_\infty^*(\mathcal{M}_{W - V_{z_0, \xi, \eta^{-1}, \Phi_j}}). \quad (3.23)$$

On the other hand, by Proposition 1.1.3(ii) we have

$$\mathcal{M}_{\Gamma, z_0} = \mathcal{M}_{V_{z_0, \xi, \eta^{-1}, \Phi_j}} + \mathcal{M}_{W-V_{z_0, \xi, \eta^{-1}, \Phi_j}}.$$

Thus (Equation 3.23) can be rewritten in the form

$$\mathcal{M}_{V_{\gamma^j}} = (\gamma_j)_\infty^* (\mathcal{M}_{\Gamma, z_0} - \mathcal{M}_{V_{z_0, \xi, \eta^{-1}, \Phi_j}}).$$

Since $\{z_0\}$ and $V_{\eta^{-1}}$ are finite, Proposition 1.1.3(iv) implies $\mu_{\{z_0\}, z_0} = 0$ or $\nu_{\{z_0\}} = 0$ and $\mu_{V_{\eta^{-1}}, z_0} = 0$ or $\nu_{\eta^{-1}} = 0$. Then by Proposition 1.1.3(ii) we have

$$\begin{aligned} \mu_{V_{\gamma^j}, \gamma^j \cdot z_0} &= (\gamma_j)_\infty^* (\mu_{z_0} - \sum_{\psi \in \Phi_j} \mu_{V_\psi, z_0}) \\ &= (\gamma_j)_\infty^* (\mu_{z_0} - \sum_{\psi \in \Phi_j} \nu_\psi). \end{aligned} \tag{3.24}$$

Since $\mathcal{M}_{V_{\gamma^j}}$ is a D -conformal density for every $j = 1, 2, 4, 5$, we have

$$\begin{aligned} d\mu_{V_{\gamma^j}, \gamma^j \cdot z_0} &= \mathcal{P}(z_0, \gamma^j \cdot z_0, \cdot)^D d\mu_{V_{\gamma^j}} \\ &= \lambda_{\gamma^j, z_0}^D d\mu_{V_{\gamma^j}} \end{aligned}$$

where the last equality follows from (Equation 1.1). We equate the total masses of the two sides of (Equation 3.24).

Then we obtain

$$\begin{aligned}
 \int \lambda_{\gamma_j, z_0}^D d\mu_{V_{\gamma_j}} &= \int d((\gamma_j)_\infty^*(\mu_{z_0} - \sum_{\psi \in \Phi_j} \nu_\psi)) \\
 &= \int d(\mu_{z_0} - \sum_{\psi \in \Phi_j} \nu_\psi) \\
 &= 1 - \sum_{\psi \in \Phi_j} \int d\nu_\psi \\
 &= \int d\nu_{\beta_j}.
 \end{aligned}$$

for every $j = 1, 2, 4, 5$.

Let us consider the equations (Equation 3.8) and (Equation 3.10). Let us say $\gamma_3 = \eta^{-1}\xi^{-1}$, $\gamma_6 = \xi\eta$, $\gamma^3 = \xi^2$, $\gamma^6 = \eta^{-1}\xi$, $\beta_3 = \eta^{-1}\xi$ and $\beta_6 = \xi^2$. We know that $\gamma_j V_{\gamma_j} = W - V_{z_0, \xi, \eta^{-1}, \Phi_j}$. Since we have $W - V_{z_0, \xi, \eta^{-1}, \Phi_j} = V_{\beta_j} \in \mathcal{V}$, condition (iii) of Proposition 1.1.3 gives

$$\mathcal{M}_{V_{\gamma_j}} = (\gamma_j)_\infty^*(\mathcal{M}_{W - V_{z_0, \xi, \eta^{-1}, \Phi_j}}). \quad (3.25)$$

On the other hand, by Proposition 1.1.3(ii) we have

$$\mathcal{M}_{\Gamma, z_0} = \mathcal{M}_{V_{z_0, \xi, \eta^{-1}, \Phi_j}} + \mathcal{M}_{W - V_{z_0, \xi, \eta^{-1}, \Phi_j}}.$$

Thus (Equation 3.25) can be rewritten in the form

$$\mathcal{M}_{V_{\gamma_j}} = (\gamma_j)_\infty^*(\mathcal{M}_{\Gamma, z_0} - \mathcal{M}_{V_{z_0, \xi, \eta^{-1}, \Phi_j}}).$$

Since $\{z_0\}$ and $V_{\eta^{-1}}$ are finite, Proposition 1.1.3(*iv*) implies $\mu_{\{z_0\}, z_0} = 0$ or $\nu_{\{z_0\}} = 0$ and $\mu_{V_{\eta^{-1}}, z_0} = 0$ or $\nu_{\eta^{-1}} = 0$. Then by Proposition 1.1.3(*ii*) we have

$$\begin{aligned}\mu_{V_{\gamma^j}, \gamma^j \cdot z_0} &= (\gamma_j)_\infty^* (\mu_{z_0} - \sum_{\psi \in \Phi_j} \mu_{V_\psi, z_0}) \\ &= (\gamma_j)_\infty^* (\mu_{z_0} - \sum_{\psi \in \Phi_j} \nu_\psi).\end{aligned}\tag{3.26}$$

Since $\mathcal{M}_{V_{\gamma^j}}$ is a D -conformal density for every $j = 3, 6$, we have

$$\begin{aligned}d\mu_{V_{\gamma^j}, \gamma^j \cdot z_0} &= \mathcal{P}(z_0, \gamma^j \cdot z_0, \cdot)^D d\mu_{V_{\gamma^j}} \\ &= \lambda_{\gamma_j, z_0}^D d\mu_{V_{\gamma^j}}\end{aligned}$$

where the last equality follows from Proposition 2.4 in [1]. Hence equating the total masses of the two sides of (Equation 3.26), we obtain

$$\begin{aligned}\int \lambda_{\gamma_j, z_0}^D d\mu_{V_{\gamma^j}} &= \int d((\gamma_j)_\infty^* (\mu_{z_0} - \sum_{\psi \in \Phi_j} \nu_\psi)) \\ &= \int d(\mu_{z_0} - \sum_{\psi \in \Phi_j} \nu_\psi) \\ &= 1 - \sum_{\psi \in \Phi_j} \int d\nu_\psi \\ &= \int d\nu_{\beta_j}.\end{aligned}$$

for every $j = 3, 6$.

For (Equation 3.11), (Equation 3.12), (Equation 3.13) and (Equation 3.14), we perform the following calculations. Let $\gamma_1 = \eta^{-2}$, $\gamma_2 = \xi\eta^{-1}$, $\beta_1 = \eta^{-1}\xi$ and $\beta_2 = \xi^2$. We know that $\gamma_j V_\eta = W - V_{\gamma_j}$

and $\beta_j V_{\xi^{-1}} = W - V_{\beta_j}$ for $j = 1, 2$. Since we have $W - V_{\gamma_1} = V_{z_0, \eta^{-1}, \xi, \Phi_3} \in \mathcal{V}$, $W - V_{\gamma_2} = V_{z_0, \eta^{-1}, \xi, \Phi_4} \in \mathcal{V}$, $W - V_{\beta_1} = V_{z_0, \eta^{-1}, \xi, \Phi_5} \in \mathcal{V}$ and $W - V_{\beta_2} = V_{z_0, \eta^{-1}, \xi, \Phi_6} \in \mathcal{V}$, condition (iii) of Proposition 1.1.3 gives

$$\mathcal{M}_{V_\eta} = (\gamma_j)_\infty^*(\mathcal{M}_{W-V_{\gamma_j}}) \quad \text{and} \quad \mathcal{M}_{V_{\xi^{-1}}} = (\beta_j)_\infty^*(\mathcal{M}_{W-V_{\beta_j}}). \quad (3.27)$$

On the other hand, by Proposition 1.1.3(ii) we have

$$\mathcal{M}_{\Gamma, z_0} = \mathcal{M}_{V_{\gamma_j}} + \mathcal{M}_{W-V_{\gamma_j}} \quad \text{and} \quad \mathcal{M}_{\Gamma, z_0} = \mathcal{M}_{V_{\beta_j}} + \mathcal{M}_{W-V_{\beta_j}}.$$

Thus (Equation 3.27) and (Equation 3.27) can be rewritten in the form

$$\mathcal{M}_{V_\eta} = (\gamma_j)_\infty^*(\mathcal{M}_{\Gamma, z_0} - \mathcal{M}_{V_{\gamma_j}}) \quad \text{and} \quad \mathcal{M}_{V_{\xi^{-1}}} = (\beta_j)_\infty^*(\mathcal{M}_{\Gamma, z_0} - \mathcal{M}_{V_{\beta_j}}).$$

Then by Proposition 1.1.3(ii) we have

$$\begin{aligned} \mu_{V_\eta, \eta \cdot z_0} &= (\gamma_j)_\infty^*(\mu_{z_0} - \mu_{V_{\gamma_j}, z_0}) \\ &= (\gamma_j)_\infty^*(\mu_{z_0} - \nu_{\gamma_j}) \end{aligned} \quad (3.28)$$

and

$$\begin{aligned} \mu_{V_{\xi^{-1}}, \xi^{-1} \cdot z_0} &= (\beta_j)_\infty^*(\mu_{z_0} - \mu_{V_{\beta_j}, z_0}) \\ &= (\beta_j)_\infty^*(\mu_{z_0} - \nu_{\beta_j}). \end{aligned} \quad (3.29)$$

Since $\mathcal{M}_{V_\eta}$ and $\mathcal{M}_{V_{\xi^{-1}}}$ are D -conformal densities, we obtain that

$$\begin{aligned} d\mu_{V_\eta, \eta \cdot z_0} &= \mathcal{P}(z_0, \gamma_i \cdot z_0, \cdot)^D d\mu_{V_\eta} \\ &= \lambda_{\gamma_i, z_0}^D d\mu_{V_\eta} \end{aligned}$$

and

$$\begin{aligned} d\mu_{V_{\xi^{-1}}, \eta \cdot z_0} &= \mathcal{P}(z_0, \beta_j \cdot z_0, \cdot)^D d\mu_{V_{\xi^{-1}}} \\ &= \lambda_{\beta_j, z_0}^D d\mu_{V_\eta} \end{aligned}$$

where the last equalities follow from (Equation 1.1). Hence equating the total masses of the two sides of (Equation 3.28) and (Equation 3.29), we obtain

$$\begin{aligned} \int \lambda_{\gamma_j, z_0}^D d\mu_{V_\eta} &= \int d((\gamma_j)_\infty^*(\mu_{z_0} - \nu_{\gamma_j})) \\ &= \int d(\mu_{z_0} - \nu_{\gamma_j}) \\ &= 1 - \int d\nu_{\gamma_j} \end{aligned}$$

and

$$\begin{aligned} \int \lambda_{\beta_j, z_0}^D d\mu_{V_{\xi^{-1}}} &= \int d((\beta_j)_\infty^*(\mu_{z_0} - \nu_{\beta_j})) \\ &= \int d(\mu_{z_0} - \nu_{\beta_j}) \\ &= 1 - \int d\nu_{\beta_j} \end{aligned}$$

for every $j = 1, 2$. This finishes the proof. $\square$

3.3 Displacement functions for the decomposition $\Gamma_{\mathcal{D}}$

Suppose that Γ , ξ , and η satisfy the hypotheses of the Proposition 1.1.3. Let z_0 be any point in $\mathbf{H}^3$. Since Γ and $\Xi = \{\xi, \eta\}$ satisfy the hypotheses of Lemma 3.2.1, there exists a number $D \in [0, 2]$, a conformal density $\mathcal{M}_{\Gamma, z_0}$ and a family of Borel measures $(\nu_\psi)_{\psi \in \Psi}$ where

$$\Psi = \{\xi\eta, \xi^2, \xi\eta^{-1}, \eta, \xi^{-1}, \eta^{-1}\xi^{-1}, \eta^{-2}, \eta^{-1}\xi\}$$

for which conclusions (i), (ii), (iii) and, (iv) of Lemma 3.2.1 hold. But the hypotheses of Proposition 1.1.3 imply that $\mathcal{M}_{\Gamma, z_0}$ is a constant multiple of the area density $\mathcal{A}$. Therefore, we get $D = 2$. Since μ_{z_0} has total mass 1 by Lemma 3.2.1(i), we have $\mathcal{M}_{\Gamma, z_0} = \mathcal{A}$. We know that

$$\nu_\psi(S_\infty) = \int d\nu_\psi$$

for every $\psi \in \Psi$. Let us call

$$\int d\nu_{\xi\eta} = m_1, \int d\nu_{\xi^2} = m_2, \int d\nu_{\xi\eta^{-1}} = m_3, \int d\nu_{\eta^{-1}\xi^{-1}} = m_4,$$

$$\int d\nu_{\eta^{-2}} = m_5, \int d\nu_{\eta^{-1}\xi} = m_6, \int d\nu_\eta = m_7, \int d\nu_{\xi^{-1}} = m_8.$$

By Lemma 3.2.1(1), we know that $\sum_{\psi \in \Psi} \nu_\psi(S_\infty) = 1$. We claim that $0 < \nu_\psi(S_\infty) < 1$ for every $\psi \in \Psi$.

Let us assume that $\nu_{\xi^{-1}}(S_\infty) = 0$. Then we get $\int \lambda_{\xi, z_0}^2 d\nu_{\xi^{-1}} = 0$. By Lemma 3.2.1(3)(a), we obtain that

$1 - \sum_{\psi \in \Psi_1} \nu_\psi(S_\infty) = 0$. This implies that $\nu_\psi(S_\infty) = 0$ for every $\psi \in \Psi_4$. In particular, we see that $\nu_{\eta^{-1}\xi}(S_\infty) = 0$ since $\eta^{-1}\xi$ is in Ψ_4 . Since $\int \lambda_{\eta^{-1}\xi, z_0}^2 d\nu_{\xi^{-1}} = 0$ or $\nu_{\eta^{-1}\xi}(S_\infty) = 1$ by Lemma 3.2.1(5), we get a contradiction. Therefore, we have $\nu_{\xi^{-1}}(S_\infty) \neq 0$.

Let us assume that $\nu_\eta(S_\infty) = 0$. Then we get $\int \lambda_{\eta^{-2}, z_0}^2 d\nu_\eta = 0$. We obtain that $\sum_{\psi \in \Phi_3} \nu_\psi(S_\infty) = 0$ by Lemma 3.2.1(5). This implies that $\nu_\psi(S_\infty) = 0$ for every $\psi \in \Phi_3$ or $\nu_{\eta^{-2}}(S_\infty) = 1$. On the other hand, we see that $\int \lambda_{\xi\eta^{-1}, z_0}^2 d\nu_\eta = 0$ implying that $\nu_\psi(S_\infty) = 0$ for every $\psi \in \Phi_4$ by Lemma 3.2.1(5). But this gives a contradiction since η^{-2} is in Φ_4 . Hence, we obtain that $\nu_\eta(S_\infty) \neq 0$.

If $\nu_{\xi\eta}(S_\infty) = 0$, then $\int \lambda_{\eta^{-1}\xi^{-1}, z_0}^2 d\nu_{\xi\eta} = 0$. By Lemma 3.2.1(3)(a), we see that $1 - \sum_{\psi \in \Psi_2} \nu_\psi(S_\infty) = 0$. Thus, we conclude that $\nu_\psi(S_\infty) = 0$ for every $\psi \in \Psi_3$. In particular, we have $\nu_{\xi^{-1}}(S_\infty) = 0$ since ξ^{-1} is in Ψ_3 . This is a contradiction because we know that $\nu_{\xi^{-1}}(S_\infty) \neq 0$. Therefore, we find that $\nu_{\xi\eta}(S_\infty) \neq 0$.

Let us assume that $\nu_{\xi^2}(S_\infty) = 0$. Then we obtain that $\int \lambda_{\xi^{-1}, z_0}^2 d\nu_{\xi^2} = 0$. We conclude that $1 - \sum_{\psi \in \Psi_4} \nu_\psi(S_\infty) = 0$ by Lemma 3.2.1(3)(c). This implies that $\nu_\psi(S_\infty) = 0$ for every $\psi \in \Psi_1$ or $\nu_{\xi\eta}(S_\infty) = 0$ since $\xi\eta$ is in Ψ_1 . But this is a contradiction because we showed above that $\nu_{\xi\eta}(S_\infty) \neq 0$. Therefore, we have $\nu_{\xi^2}(S_\infty) \neq 0$.

If $\nu_{\xi\eta^{-1}}(S_\infty) = 0$, then $\int \lambda_{\xi^{-1}, z_0}^2 d\nu_{\xi\eta^{-1}} = 0$. By Lemma 3.2.1(3)(b), we see that $1 - \sum_{\psi \in \Psi_3} \nu_\psi(S_\infty) = 0$. Therefore, we conclude that $\nu_\psi(S_\infty) = 0$ for every $\psi \in \Psi_2$. In particular, we have $\nu_{\eta^{-1}\xi^{-1}}(S_\infty) = 0$ since $\eta^{-1}\xi^{-1}$ is in Ψ_2 . Then we see that $\int \lambda_{\xi\eta, z_0}^2 d\nu_{\eta^{-1}\xi^{-1}} = 0$ which gives that $1 - \sum_{\psi \in \Psi_1} \nu_\psi = 0$. Therefore, we find that $\nu_\psi(S_\infty) = 0$ for every $\psi \in \Psi_4$. In particular, $\nu_{\xi^{-1}}(S_\infty) = 0$ since $\xi^{-1} \in \Psi_4$. This is a contradiction because we know that $\nu_{\xi^{-1}}(S_\infty) \neq 0$. Therefore, we obtain that $\nu_{\xi\eta^{-1}}(S_\infty) \neq 0$.

Let us assume that $\nu_{\eta^{-1}\xi^{-1}}(S_\infty) = 0$. Then we get $\int \lambda_{\xi\eta, z_0}^2 d\nu_{\eta^{-1}\xi^{-1}} = 0$. By Lemma 3.2.1(3)(a), we obtain that $1 - \sum_{\psi \in \Psi_1} \nu_\psi(S_\infty) = 0$. The last equality shows that $\nu_\psi(S_\infty) = 0$ for every $\psi \in \Psi_4$. Therefore,

we see that $\nu_\eta(S_\infty) = 0$ since η is in Ψ_4 . This gives a contradiction because we know that $\nu_\eta(S_\infty) \neq 0$.

Thus, we find that $\nu_{\eta^{-1}\xi^{-1}}(S_\infty) \neq 0$.

If $\nu_{\eta^{-2}}(S_\infty) = 0$, then $\int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-2}} = 0$. By Lemma 3.2.1(3)(b), we see that $1 - \sum_{\psi \in \Psi_3} \nu_\psi(S_\infty) = 0$. In other words, we have $\nu_\psi(S_\infty) = 0$ for every $\psi \in \Psi_2$. This implies that $\nu_{\eta^{-1}\xi^{-1}}(S_\infty) = 0$ which is a contradiction. Because we already showed that $\nu_{\eta^{-1}\xi^{-1}}(S_\infty) \neq 0$. Therefore, we see that $\nu_{\eta^{-2}}(S_\infty) \neq 0$.

Let us assume that $\nu_{\eta^{-1}\xi}(S_\infty) = 0$. Then we find that $\int \lambda_{\xi\eta, z_0}^2 d\nu_{\eta^{-1}\xi} = 0$ which implies that $1 - \sum_{\psi \in \Phi_6} \nu_\psi = 0$. By Lemma 3.2.1(4), the last equality gives that $\nu_{\xi^2}(S_\infty) = 0$. This is a contradiction. Therefore, we have $\nu_{\eta^{-1}\xi}(S_\infty) \neq 0$.

By Lemma 3.2.1(1), it is clear that $\nu_\psi \neq 1$ for every $\psi \in \Psi$. Because otherwise $\sum_{\psi \in \Psi} \nu_\psi = 1$ would imply that $\nu_{\psi'}(S_\infty) = 0$ for every $\psi' \neq \psi$ in Ψ , a contradiction. Hence, we conclude that $0 < \nu_\psi(S_\infty) < 1$ for every $\psi \in \Psi$, or equivalently, $0 < m_i < 1$ for every $i = 1, 2, \dots, 8$ such that $\sum_1^8 m_i = 1$. Every integral on the right-hand side of the equalities given in Lemma 3.2.1(1), (2), (3), (4), and (5) can be written in the following form

$$\int \lambda_{\psi, z_0}^2 d\nu_{\psi'} = 1 - \sum_{\phi \in \Phi} \nu_\phi$$

for suitably chosen $\psi, \psi' \in \Psi$ and $\Phi \in \{\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6\}$ or $\Phi \subset \Psi$. We know that $0 < \nu_\psi(S_\infty) < 1$ for every $\psi \in \Psi$. Therefore, we obtain that

$$0 < \int \lambda_{\psi, z_0}^2 d\nu_{\psi'} < 1.$$

This means that we can apply Lemma 1.1.1 for every suitably chosen $\psi, \psi' \in \Psi$.

Let $\gamma = \xi\eta$, $\nu = \nu_{\eta^{-1}\xi^{-1}}$, $a = \nu_{\eta^{-1}\xi^{-1}}(S_\infty)$ and $b = \int \lambda_{\xi\eta, z_0}^2 d\nu_{\eta^{-1}\xi^{-1}}$ and apply Lemma 1.1.1. Then, we obtain that

$$e^{2\text{dist}(z_0, \xi\eta \cdot z_0)} \geq \frac{\int \lambda_{\xi\eta, z_0}^2 d\nu_{\eta^{-1}\xi^{-1}} \cdot \left(1 - \int d\nu_{\eta^{-1}\xi^{-1}}\right)}{\int d\nu_{\eta^{-1}\xi^{-1}} \cdot \left(1 - \int \lambda_{\xi\eta, z_0}^2 d\mu_{\eta^{-1}\xi^{-1}}\right)}.$$

By Lemma 3.2.1(3)(a), we obtain that

$$\begin{aligned} e^{2\text{dist}(z_0, \xi\eta \cdot z_0)} &\geq \frac{\int \lambda_{\xi\eta, z_0}^2 d\nu_{\eta^{-1}\xi^{-1}} \cdot \left(1 - \int d\nu_{\eta^{-1}\xi^{-1}}\right)}{\int d\nu_{\eta^{-1}\xi^{-1}} \cdot \left(1 - \int \lambda_{\xi\eta, z_0}^2 d\mu_{\eta^{-1}\xi^{-1}}\right)} \\ &= \frac{\left(1 - \sum_{\psi \in \Psi_1} \int d\nu_\psi\right) \cdot \left(1 - \int d\nu_{\eta^{-1}\xi^{-1}}\right)}{\left(\sum_{\psi \in \Psi_1} \int d\nu_\psi\right) \cdot \int d\nu_{\eta^{-1}\xi^{-1}}} \\ &= \frac{\left(1 - \int d\nu_{\xi\eta} - \int d\nu_{\xi^2} - \int d\nu_{\xi\eta^{-1}}\right) \cdot \left(1 - \int d\nu_{\eta^{-1}\xi^{-1}}\right)}{\left(\int d\nu_{\xi\eta} + \int d\nu_{\xi^2} + \int d\nu_{\xi\eta^{-1}}\right) \cdot \int d\nu_{\eta^{-1}\xi^{-1}}} \\ &= \frac{1 - m_1 - m_2 - m_3}{m_1 + m_2 + m_3} \cdot \frac{1 - m_4}{m_4} \end{aligned} \tag{3.30}$$

by Lemma 3.2.1. Hence, the expression in (Equation 3.30) is a lower bound for the displacement $\text{dist}(z_0, \xi\eta \cdot z_0)$. Replacing each constant m_i with the variable x_i , we obtain the displacement function for this case as

$$f_1(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_1 - x_2 - x_3}{x_1 + x_2 + x_3} \cdot \frac{1 - x_4}{x_4}.$$

Let $\gamma = \eta^{-1}\xi^{-1}$, $\nu = \nu_{\xi\eta}$, $a = \nu_{\xi\eta}(S_\infty)$ and $b = \int \lambda_{\eta^{-1}\xi^{-1}, z_0}^2 d\nu_{\xi\eta}$ and apply Lemma 1.1.1. We get that

$$e^{2\text{dist}(z_0, \eta^{-1}\xi^{-1} \cdot z_0)} \geq \frac{\int \lambda_{\eta^{-1}\xi^{-1}, z_0}^2 d\nu_{\xi\eta} \cdot \left(1 - \int d\nu_{\xi\eta}\right)}{\int d\nu_{\xi\eta} \cdot \left(1 - \int \lambda_{\eta^{-1}\xi^{-1}, z_0}^2 d\nu_{\xi\eta}\right)}.$$

By Lemma 3.2.1(3)(a), we obtain that

$$\begin{aligned} e^{2\text{dist}(z_0, \eta^{-1}\xi^{-1} \cdot z_0)} &\geq \frac{\int \lambda_{\eta^{-1}\xi^{-1}, z_0}^2 d\nu_{\xi\eta} \cdot \left(1 - \int d\nu_{\xi\eta}\right)}{\int d\nu_{\xi\eta} \cdot \left(1 - \int \lambda_{\eta^{-1}\xi^{-1}, z_0}^2 d\nu_{\xi\eta}\right)} \\ &= \frac{\left(1 - \sum_{\psi \in \Psi_2} \int d\nu_\psi\right) \cdot \left(1 - \int d\nu_{\xi\eta}\right)}{\left(\sum_{\psi \in \Psi_2} \int d\nu_\psi\right) \cdot \int d\nu_{\xi\eta}} \\ &= \frac{\left(1 - \int d\nu_{\eta^{-1}\xi^{-1}} - \int d\nu_{\eta^{-2}} - \int d\nu_{\eta^{-1}\xi}\right) \cdot \left(1 - \int d\nu_{\xi\eta}\right)}{\left(\int d\nu_{\eta^{-1}\xi^{-1}} + \int d\nu_{\eta^{-2}} + \int d\nu_{\eta^{-1}\xi}\right) \cdot \int d\nu_{\xi\eta}} \\ &= \frac{1 - m_4 - m_5 - m_6}{m_4 + m_5 + m_6} \cdot \frac{1 - m_1}{m_1} \end{aligned} \tag{3.31}$$

Therefore, the constant in (Equation 3.31) is a lower bound for the displacement $\text{dist}(z_0, \eta^{-1}\xi^{-1} \cdot z_0)$.

In this case, we set the displacement function as

$$f_2(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_4 - x_5 - x_6}{x_4 + x_5 + x_6} \cdot \frac{1 - x_1}{x_1}.$$

Let $\gamma = \xi$, $\nu = \nu_{\xi^{-1}}$, $a = \nu_{\xi^{-1}}(S_\infty)$ and $b = \int \lambda_{\xi, z_0}^2 d\nu_{\xi^{-1}}$ and apply Lemma 1.1.1. Then we obtain that

$$e^{2\text{dist}(z_0, \xi \cdot z_0)} \geq \frac{\int \lambda_{\xi, z_0}^2 d\nu_{\xi^{-1}} \cdot \left(1 - \int d\nu_{\xi^{-1}}\right)}{\int d\nu_{\xi^{-1}} \cdot \left(1 - \int \lambda_{\xi, z_0}^2 d\nu_{\xi^{-1}}\right)}.$$

By Lemma 3.2.1(3)(a), we find that

$$\begin{aligned} e^{2\text{dist}(z_0, \xi \cdot z_0)} &\geq \frac{\int \lambda_{\xi, z_0}^2 d\nu_{\xi^{-1}} \cdot \left(1 - \int d\nu_{\xi^{-1}}\right)}{\int d\nu_{\xi^{-1}} \cdot \left(1 - \int \lambda_{\xi, z_0}^2 d\nu_{\xi^{-1}}\right)} \\ &= \frac{\left(1 - \sum_{\psi \in \Psi_1} \int d\nu_\psi\right) \cdot \left(1 - \int d\nu_{\xi^{-1}}\right)}{\left(\sum_{\psi \in \Psi_1} \int d\nu_\psi\right) \cdot \int d\nu_{\xi^{-1}}} \\ &= \frac{\left(1 - \int d\nu_{\xi\eta} - \int d\nu_{\xi^2} - \int d\nu_{\xi\eta^{-1}}\right) \cdot \left(1 - \int d\nu_{\xi^{-1}}\right)}{\left(\int d\nu_{\xi\eta} + \int d\nu_{\xi^2} + \int d\nu_{\xi\eta^{-1}}\right) \cdot \int d\nu_{\xi^{-1}}} \\ &= \frac{1 - m_1 - m_2 - m_3}{m_1 + m_2 + m_3} \cdot \frac{1 - m_8}{m_8} \end{aligned} \tag{3.32}$$

Therefore, the expression we obtained in (Equation 3.32) is a lower bound for the displacement $\text{dist}(z_0, \xi \cdot z_0)$. In this case, the displacement function is chosen as

$$f_3(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_1 - x_2 - x_3}{x_1 + x_2 + x_3} \cdot \frac{1 - x_8}{x_8}.$$

Let $\gamma = \eta^{-1}$, $\nu = \mu_\eta$, $a = \mu_\eta(S_\infty)$ and $b = \int \lambda_{\eta^{-1}, z_0}^2 d\nu_\eta$ and apply Lemma 1.1.1. Then we obtain that

$$e^{2\text{dist}(z_0, \eta^{-1} \cdot z_0)} \geq \frac{\int \lambda_{\eta^{-1}, z_0}^2 d\nu_\eta \cdot \left(1 - \int d\nu_\eta\right)}{\int d\nu_\eta \cdot \left(1 - \int \lambda_{\eta^{-1}, z_0}^2 d\nu_\eta\right)}.$$

By Lemma 3.2.1(3)(a), we find that

$$\begin{aligned} e^{2\text{dist}(z_0, \eta^{-1} \cdot z_0)} &\geq \frac{\int \lambda_{\eta^{-1}, z_0}^2 d\nu_\eta \cdot \left(1 - \int d\nu_\eta\right)}{\int d\nu_\eta \cdot \left(1 - \int \lambda_{\eta^{-1}, z_0}^2 d\nu_\eta\right)} \\ &= \frac{\left(1 - \sum_{\psi \in \Psi_2} \int d\nu_\psi\right) \cdot \left(1 - \int d\nu_\eta\right)}{\left(\sum_{\psi \in \Psi_2} \int d\nu_\psi\right) \cdot \int d\nu_\eta} \\ &= \frac{\left(1 - \int d\nu_{\eta^{-1}\xi^{-1}} - \int d\nu_{\eta^{-2}} - \int d\nu_{\eta^{-1}\xi}\right) \cdot \left(1 - \int d\nu_\eta\right)}{\left(\int d\nu_{\eta^{-1}\xi^{-1}} + \int d\nu_{\eta^{-2}} + \int d\nu_{\eta^{-1}\xi}\right) \cdot \int d\nu_\eta} \\ &= \frac{1 - m_4 - m_5 - m_6}{m_4 + m_5 + m_6} \cdot \frac{1 - m_7}{m_7} \end{aligned} \tag{3.33}$$

Therefore, the number obtained in the equation (Equation 3.33) is a lower bound for the displacement $\text{dist}(z_0, \eta^{-1} \cdot z_0)$. In this case, the displacement function is

$$f_4(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_1 - x_2 - x_3}{x_1 + x_2 + x_3} \cdot \frac{1 - x_7}{x_7}.$$

Let $\gamma = \eta$, $\nu = \nu_{\eta^{-1}\xi}$, $a = \nu_{\eta^{-1}\xi}(S_\infty)$ and $b = \int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-1}\xi}$ and apply Lemma 1.1.1. Then we obtain that

$$e^{2\text{dist}(z_0, \eta \cdot z_0)} \geq \frac{\int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-1}\xi} \cdot \left(1 - \int d\nu_{\eta^{-1}\xi}\right)}{\int d\nu_{\eta^{-1}\xi} \cdot \left(1 - \int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-1}\xi}\right)}.$$

By Lemma 3.2.1(3)(c), we see that

$$\begin{aligned} e^{2\text{dist}(z_0, \eta \cdot z_0)} &\geq \frac{\int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-1}\xi} \cdot \left(1 - \int d\nu_{\eta^{-1}\xi}\right)}{\int d\nu_{\eta^{-1}\xi} \cdot \left(1 - \int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-1}\xi}\right)} \\ &= \frac{\left(1 - \sum_{\psi \in \Psi_4} \int d\nu_\psi\right) \cdot \left(1 - \int d\nu_{\eta^{-1}\xi}\right)}{\left(\sum_{\psi \in \Psi_4} \int d\nu_\psi\right) \cdot \int d\nu_{\eta^{-1}\xi}} \\ &= \frac{\left(1 - \int d\nu_{\eta^{-1}\xi^{-1}} - \int d\nu_{\eta^{-2}} - \int d\nu_{\eta^{-1}\xi} - \int d\nu_\eta - \int d\nu_{\xi^{-1}}\right) \cdot \left(1 - \int d\nu_{\eta^{-1}\xi}\right)}{\left(\int d\nu_{\eta^{-1}\xi^{-1}} + \int d\nu_{\eta^{-2}} + \int d\nu_{\eta^{-1}\xi} + \int d\nu_\eta + \int d\nu_{\xi^{-1}}\right) \cdot \int d\nu_{\eta^{-1}\xi}} \\ &= \frac{1 - m_4 - m_5 - m_6 - m_7 - m_8}{m_4 + m_5 + m_6 + m_7 + m_8} \cdot \frac{1 - m_6}{m_6} \end{aligned} \quad (3.34)$$

In this case, the constant given in expression (Equation 3.34) is a lower bound for the displacement $\text{dist}(z_0, \eta \cdot z_0)$. Therefore, we set the displacement function as

$$f_5(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_4 - x_5 - x_6 - x_7 - x_8}{x_4 + x_5 + x_6 + x_7 + x_8} \cdot \frac{1 - x_6}{x_6}.$$

Let $\gamma = \xi^{-1}$, $\nu = \nu_{\xi^2}$, $a = \nu_{\xi^2}(S_\infty)$ and $b = \int \lambda_{\xi^{-1}, z_0}^2 d\mu_{\xi^2}$ and apply Lemma 1.1.1. Then we obtain that

$$e^{2\text{dist}(z_0, \xi^{-1} \cdot z_0)} \geq \frac{\int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-1}\xi} \cdot \left(1 - \int d\nu_{\xi^2}\right)}{\int d\nu_{\xi^2} \cdot \left(1 - \int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-1}\xi}\right)}.$$

By Lemma 3.2.1(3)(c), we see that

$$\begin{aligned} e^{2\text{dist}(z_0, \xi^{-1} \cdot z_0)} &\geq \frac{\int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-1}\xi} \cdot \left(1 - \int d\nu_{\xi^2}\right)}{\int d\nu_{\xi^2} \cdot \left(1 - \int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-1}\xi}\right)} \\ &= \frac{\left(1 - \sum_{\psi \in \Psi_4} \int d\nu_\psi\right) \cdot \left(1 - \int d\nu_{\xi^2}\right)}{\left(\sum_{\psi \in \Psi_4} \int d\nu_\psi\right) \cdot \int d\nu_{\xi^2}} \\ &= \frac{\left(1 - \int d\nu_{\eta^{-1}\xi^{-1}} - \int d\nu_{\eta^{-2}} - \int d\nu_{\eta^{-1}\xi} - \int d\nu_\eta - \int d\nu_{\xi^{-1}}\right) \cdot \left(1 - \int d\nu_{\xi^2}\right)}{\left(\int d\nu_{\eta^{-1}\xi^{-1}} + \int d\nu_{\eta^{-2}} + \int d\nu_{\eta^{-1}\xi} + \int d\nu_\eta + \int d\nu_{\xi^{-1}}\right) \cdot \int d\nu_{\xi^2}} \\ &= \frac{1 - m_4 - m_5 - m_6 - m_7 - m_8}{m_4 + m_5 + m_6 + m_7 + m_8} \cdot \frac{1 - m_2}{m_2} \end{aligned} \quad (3.35)$$

The number obtained in (Equation 3.35) is a lower bound for the displacement $\text{dist}(z_0, \xi^{-1} \cdot z_0)$.

Therefore, in this case, the displacement function is

$$f_6(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_4 - x_5 - x_6 - x_7 - x_8}{x_4 + x_5 + x_6 + x_7 + x_8} \cdot \frac{1 - x_2}{x_2}.$$

Let $\gamma = \xi^{-1}$, $\nu = \nu_{\xi\eta^{-1}}$, $a = \mu_{\xi\eta^{-1}}(S_\infty)$ and $b = \int \lambda_{\xi^{-1}, z_0}^2 d\nu_{\xi\eta^{-1}}$ and apply Lemma 1.1.1. Then we obtain that

$$e^{2\text{dist}(z_0, \xi^{-1} \cdot z_0)} \geq \frac{\int \lambda_{\xi^{-1}, z_0}^2 d\nu_{\xi\eta^{-1}} \cdot \left(1 - \int d\nu_{\xi\eta^{-1}}\right)}{\int d\nu_{\xi\eta^{-1}} \cdot \left(1 - \int \lambda_{\eta, z_0}^2 d\nu_{\xi\eta^{-1}}\right)}.$$

By Lemma 3.2.1(3)(b), we find that

$$\begin{aligned} e^{2\text{dist}(z_0, \xi^{-1} \cdot z_0)} &\geq \frac{\int \lambda_{\xi^{-1}, z_0}^2 d\nu_{\xi\eta^{-1}} \cdot \left(1 - \int d\nu_{\xi\eta^{-1}}\right)}{\int d\nu_{\xi\eta^{-1}} \cdot \left(1 - \int \lambda_{\eta, z_0}^2 d\nu_{\xi\eta^{-1}}\right)} \\ &= \frac{\left(1 - \sum_{\psi \in \Psi_3} \int d\nu_\psi\right) \cdot \left(1 - \int d\nu_{\xi\eta^{-1}}\right)}{\left(\sum_{\psi \in \Psi_3} \int d\nu_\psi\right) \cdot \int d\nu_{\xi\eta^{-1}}} \\ &= \frac{\left(1 - \int d\nu_{\xi\eta} - \int d\nu_{\xi^2} - \int d\nu_{\xi\eta^{-1}} - \int d\nu_\eta - \int d\nu_{\xi^{-1}}\right) \cdot \left(1 - \int d\nu_{\xi\eta^{-1}}\right)}{\left(\int d\nu_{\xi\eta} + \int d\nu_{\xi^2} + \int d\nu_{\xi\eta^{-1}} + \int d\nu_\eta + \int d\nu_{\xi^{-1}}\right) \cdot \int d\nu_{\xi\eta^{-1}}} \\ &= \frac{1 - m_1 - m_2 - m_3 - m_7 - m_8}{m_1 + m_2 + m_3 + m_7 + m_8} \cdot \frac{1 - m_3}{m_3} \end{aligned} \tag{3.36}$$

In this case, the constant given in the expression in (Equation 3.36) is a lower bound for the displacement $\text{dist}(z_0, \xi^{-1} \cdot z_0)$ again. We choose the displacement function as

$$f_7(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_1 - x_2 - x_3 - x_7 - x_8}{x_1 + x_2 + x_3 + x_7 + x_8} \cdot \frac{1 - x_3}{x_3}.$$

Let $\gamma = \eta$, $\nu = \nu_{\eta^{-2}}$, $a = \nu_{\eta^{-2}}(S_\infty)$ and $b = \int \lambda_{\xi^{-1}, z_0}^2 d\nu_{\eta^{-2}}$ and apply Lemma 1.1.1. Then we obtain that

$$e^{2\text{dist}(z_0, \eta \cdot z_0)} \geq \frac{\int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-2}} \cdot \left(1 - \int d\nu_{\eta^{-2}}\right)}{\int d\nu_{\eta^{-2}} \cdot \left(1 - \int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-2}}\right)}$$

By Lemma 3.2.1(3)(b), we find that

$$\begin{aligned} e^{2\text{dist}(z_0, \eta \cdot z_0)} &\geq \frac{\int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-2}} \cdot \left(1 - \int d\nu_{\eta^{-2}}\right)}{\int d\nu_{\eta^{-2}} \cdot \left(1 - \int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-2}}\right)} \\ &= \frac{\left(1 - \sum_{\psi \in \Psi_3} \int d\nu_\psi\right) \cdot \left(1 - \int d\nu_{\eta^{-2}}\right)}{\left(\sum_{\psi \in \Psi_3} \int d\nu_\psi\right) \cdot \int d\nu_{\eta^{-2}}} \\ &= \frac{\left(1 - \int d\nu_{\xi\eta} - \int d\nu_{\xi^2} - \int d\nu_{\xi\eta^{-1}} - \int d\nu_\eta - \int d\nu_{\xi^{-1}}\right) \cdot \left(1 - \int d\nu_{\eta^{-2}}\right)}{\left(\int d\mu_{\xi\eta} + \int d\mu_{\xi^2} + \int d\mu_{\xi\eta^{-1}} + \int d\mu_\eta + \int d\nu_{\xi^{-1}}\right) \cdot \int d\nu_{\eta^{-2}}} \\ &= \frac{1 - m_1 - m_2 - m_3 - m_7 - m_8}{m_1 + m_2 + m_3 + m_7 + m_8} \cdot \frac{1 - m_5}{m_5} \end{aligned} \tag{3.37}$$

Since the constant calculated in (Equation 3.37) is a lower bound for the displacement $\text{dist}(z_0, \eta \cdot z_0)$,

we set the displacement function as

$$f_8(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_1 - x_2 - x_3 - x_7 - x_8}{x_1 + x_2 + x_3 + x_7 + x_8} \cdot \frac{1 - x_5}{x_5}.$$

Let $\gamma = \xi\eta$, $\nu = \nu_{\eta^{-2}}$, $a = \nu_{\eta^{-2}}(S_\infty)$ and $b = \int \lambda_{\xi\eta, z_0}^2 d\nu_{\eta^{-2}}$ and apply Lemma 1.1.1. Then we obtain that

$$e^{2\text{dist}(z_0, \xi\eta \cdot z_0)} \geq \frac{\int \lambda_{\xi\eta, z_0}^2 d\nu_{\eta^{-2}} \cdot \left(1 - \int d\nu_{\eta^{-2}}\right)}{\int d\nu_{\eta^{-2}} \cdot \left(1 - \int \lambda_{\xi\eta, z_0}^2 d\nu_{\eta^{-2}}\right)}$$

By Lemma 3.2.1(4), we find that

$$\begin{aligned} e^{2\text{dist}(z_0, \xi\eta \cdot z_0)} &\geq \frac{\int \lambda_{\xi\eta, z_0}^2 d\nu_{\eta^{-2}} \cdot \left(1 - \int d\nu_{\eta^{-2}}\right)}{\int d\nu_{\eta^{-2}} \cdot \left(1 - \int \lambda_{\xi\eta, z_0}^2 d\nu_{\eta^{-2}}\right)} \\ &= \frac{\left(1 - \sum_{\psi \in \Psi_6} \int d\nu_\psi\right) \cdot \left(1 - \int d\nu_{\eta^{-2}}\right)}{\left(\sum_{\psi \in \Psi_6} \int d\nu_\psi\right) \cdot \int d\nu_{\eta^{-2}}} \\ &= \frac{m_3}{1 - mc_3} \cdot \frac{1 - m_5}{m_5} \\ &= \frac{1 - \sum_{i=1, i \neq 3}^8 m_i}{\sum_{i=1, i \neq 3}^8 m_i} \cdot \frac{1 - m_5}{m_5}. \end{aligned} \tag{3.38}$$

In (Equation 3.38), we obtain a lower bound for the displacement $\text{dist}(z_0, \xi\eta \cdot z_0)$. In this case, we need to assign the following function as the displacement function

$$f_9(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_1 - x_2 - x_4 - x_5 - x_6 - x_7 - x_8}{x_1 + x_2 + x_4 + x_5 + x_6 + x_7 + x_8} \cdot \frac{1 - x_5}{x_5}.$$

Let $\gamma = \xi\eta$, $\nu = \nu_{\eta^{-1}\xi}$, $a = \nu_{\xi^2}(S_\infty)$ and $b = \int \lambda_{\xi\eta, z_0}^2 d\nu_{\xi^2}$ and apply Lemma 1.1.1. Then we obtain that

$$e^{2\text{dist}(z_0, \xi\eta \cdot z_0)} \geq \frac{\int \lambda_{\xi\eta, z_0}^2 d\nu_{\eta^{-1}\xi} \cdot \left(1 - \int d\nu_{\eta^{-1}\xi}\right)}{\int d\nu_{\eta^{-1}\xi} \cdot \left(1 - \int \lambda_{\xi\eta, z_0}^2 d\nu_{\eta^{-1}\xi}\right)}.$$

By Lemma 3.2.1(4), we see that

$$\begin{aligned} e^{2\text{dist}(z_0, \xi\eta \cdot z_0)} &\geq \frac{\int \lambda_{\xi\eta, z_0}^2 d\nu_{\eta^{-1}\xi} \cdot \left(1 - \int d\nu_{\eta^{-1}\xi}\right)}{\int d\nu_{\eta^{-1}\xi} \cdot \left(1 - \int \lambda_{\xi\eta, z_0}^2 d\nu_{\eta^{-1}\xi}\right)} \\ &= \frac{\left(1 - \sum_{\psi \in \Psi_9} \int d\nu_\psi\right) \cdot \left(1 - \int d\nu_{\eta^{-1}\xi}\right)}{\left(\sum_{\psi \in \Psi_9} \int d\nu_\psi\right) \cdot \int d\nu_{\eta^{-1}\xi}} \\ &= \frac{m_2}{1 - m_2} \cdot \frac{1 - m_6}{m_6} \\ &= \frac{1 - \sum_{i=1, i \neq 2}^8 m_i}{\sum_{i=1, i \neq 2}^8 m_i} \cdot \frac{1 - m_6}{m_6} \end{aligned} \tag{3.39}$$

The constant obtained in (Equation 3.39) is a lower bound for the displacement $\text{dist}(z_0, \xi\eta \cdot z_0)$. In this case, the displacement function is

$$f_{10}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_1 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8}{x_1 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8} \cdot \frac{1 - x_6}{x_6}.$$

Let $\gamma = \eta^{-1}\xi^{-1}$, $\nu = \nu_{\xi^2}$, $a = \nu_{\xi^2}(S_\infty)$ and $b = \int \lambda_{\eta^{-1}\xi^{-1}, z_0}^2 d\nu_{\xi^2}$ and apply Lemma 1.1.1. Then we obtain that

$$e^{2\text{dist}(z_0, \eta^{-1}\xi^{-1} \cdot z_0)} \geq \frac{\int \lambda_{\eta^{-1}\xi^{-1}, z_0}^2 d\nu_{\xi^2} \cdot \left(1 - \int d\nu_{\xi^2}\right)}{\int d\nu_{\xi^2} \cdot \left(1 - \int \lambda_{\eta^{-1}\xi^{-1}, z_0}^2 d\nu_{\xi^2}\right)}$$

By Lemma 3.2.1(4), we see that

$$\begin{aligned} e^{2\text{dist}(z_0, \eta^{-1}\xi^{-1} \cdot z_0)} &\geq \frac{\int \lambda_{\eta^{-1}\xi^{-1}, z_0}^2 d\nu_{\xi^2} \cdot \left(1 - \int d\nu_{\xi^2}\right)}{\int d\nu_{\xi^2} \cdot \left(1 - \int \lambda_{\eta^{-1}\xi^{-1}, z_0}^2 d\nu_{\xi^2}\right)} \\ &= \frac{\left(1 - \sum_{\psi \in \Psi_5} \int d\nu_\psi\right) \cdot \left(1 - \int d\nu_{\xi^2}\right)}{\left(\sum_{\psi \in \Psi_5} \int d\nu_\psi\right) \cdot \int d\nu_{\xi^2}} \\ &= \frac{m_6}{1 - m_6} \cdot \frac{1 - m_2}{m_2} \\ &= \frac{1 - \sum_{i=1, i \neq 6}^8 m_i}{\sum_{i=1, i \neq 6}^8 m_i} \cdot \frac{1 - m_2}{m_2} \end{aligned} \tag{3.40}$$

In this case, we obtain a lower bound for the displacement $\text{dist}(z_0, \eta^{-1}\xi^{-1} \cdot z_0)$. We set the displacement function as

$$f_{11}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_1 - x_2 - x_3 - x_4 - x_5 - x_7 - x_8}{x_1 + x_2 + x_3 + x_4 + x_5 + x_7 + x_8} \cdot \frac{1 - x_2}{x_2}.$$

Let $\gamma = \eta^{-1}\xi^{-1}$, $\nu = \nu_{\xi\eta^{-1}}$, $a = \nu_{\xi\eta^{-1}}(S_\infty)$ and $b = \int \lambda_{\eta^{-1}\xi^{-1}, z_0}^2 d\nu_{\xi^2}$ and apply Lemma 1.1.1. Then we obtain that

$$e^{2\text{dist}(z_0, \eta^{-1}\xi^{-1} \cdot z_0)} \geq \frac{\int \lambda_{\eta^{-1}\xi^{-1}, z_0}^2 d\nu_{\xi\eta^{-1}} \cdot \left(1 - \int d\nu_{\xi\eta^{-1}}\right)}{\int d\nu_{\xi\eta^{-1}} \cdot \left(1 - \int \lambda_{\eta^{-1}\xi^{-1}, z_0}^2 d\nu_{\xi\eta^{-1}}\right)}.$$

By Lemma 3.2.1(4), we see that

$$\begin{aligned} e^{2\text{dist}(z_0, \eta^{-1}\xi^{-1} \cdot z_0)} &\geq \frac{\int \lambda_{\eta^{-1}\xi^{-1}, z_0}^2 d\nu_{\xi\eta^{-1}} \cdot \left(1 - \int d\nu_{\xi\eta^{-1}}\right)}{\int d\nu_{\xi\eta^{-1}} \cdot \left(1 - \int \lambda_{\eta^{-1}\xi^{-1}, z_0}^2 d\nu_{\xi\eta^{-1}}\right)} \\ &= \frac{\left(1 - \sum_{\psi \in \Psi_7} \int d\nu_\psi\right) \cdot \left(1 - \int d\nu_{\xi\eta^{-1}}\right)}{\left(\sum_{\psi \in \Psi_7} \int d\nu_\psi\right) \cdot \int d\nu_{\xi\eta^{-1}}} \\ &= \frac{m_5}{1 - m_5} \cdot \frac{1 - m_3}{m_3} \\ &= \frac{1 - \sum_{i=1, i \neq 5}^8 m_i}{\sum_{i=1, i \neq 5}^8 m_i} \cdot \frac{1 - m_3}{m_3} \end{aligned} \tag{3.41}$$

The number obtained in (Equation 3.41) provides another lower bound for the displacement $\text{dist}(z_0, \eta^{-1}\xi^{-1} \cdot z_0)$. In this case, the displacement function is

$$f_{12}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_1 - x_2 - x_3 - x_4 - x_6 - x_7 - x_8}{x_1 + x_2 + x_3 + x_4 + x_6 + x_7 + x_8} \cdot \frac{1 - x_3}{x_3}.$$

Let $\gamma = \xi^{-1}$, $\nu = \nu_{\xi\eta}$, $a = \nu_{\xi\eta}(S_\infty)$ and $b = \int \lambda_{\xi^{-1}, z_0}^2 d\nu_{\xi\eta}$ and apply Lemma 1.1.1. Then we obtain that

$$e^{2\text{dist}(z_0, \xi^{-1} \cdot z_0)} \geq \frac{\int \lambda_{\xi^{-1}, z_0}^2 d\nu_{\xi\eta} \cdot \left(1 - \int d\nu_{\xi\eta}\right)}{\int d\nu_{\xi\eta} \cdot \left(1 - \int \lambda_{\xi^{-1}, z_0}^2 d\nu_{\xi\eta}\right)}.$$

By Lemma 3.2.1(4), we see that

$$\begin{aligned} e^{2\text{dist}(z_0, \xi^{-1} \cdot z_0)} &\geq \frac{\int \lambda_{\xi^{-1}, z_0}^2 d\nu_{\xi\eta} \cdot \left(1 - \int d\nu_{\xi\eta}\right)}{\int d\nu_{\xi\eta} \cdot \left(1 - \int \lambda_{\xi^{-1}, z_0}^2 d\nu_{\xi\eta}\right)} \\ &= \frac{\left(1 - \sum_{\psi \in \Psi_{10}} \int d\nu_\psi\right) \cdot \left(1 - \int d\nu_{\xi\eta}\right)}{\left(\sum_{\psi \in \Psi_{10}} \int d\nu_\psi\right) \cdot \int d\nu_{\xi\eta}} \\ &= \frac{m_7}{1 - m_7} \cdot \frac{1 - m_1}{m_1} \\ &= \frac{1 - \sum_{i=1, i \neq 7}^8 m_i}{\sum_{i=1, i \neq 7}^8 m_i} \cdot \frac{1 - m_1}{m_1} \end{aligned} \tag{3.42}$$

We have another lower bound for the displacement $\text{dist}(z_0, \xi^{-1} \cdot z_0)$ given in (Equation 3.42). The corresponding displacement function is

$$f_{13}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_8}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_8} \cdot \frac{1 - x_1}{x_1}.$$

Let $\gamma = \eta$, $\nu = \nu_{\eta^{-1}\xi^{-1}}$, $a = \nu_{\eta^{-1}\xi^{-1}}(S_\infty)$ and $b = \int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-1}\xi^{-1}}$ and apply Lemma 1.1.1. Then we obtain that

$$e^{2\text{dist}(z_0, \eta \cdot z_0)} \geq \frac{\int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-1}\xi^{-1}} \cdot \left(1 - \int d\nu_{\eta^{-1}\xi^{-1}}\right)}{\int d\nu_{\eta^{-1}\xi^{-1}} \cdot \left(1 - \int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-1}\xi^{-1}}\right)}.$$

By Lemma 3.2.1(4), we see that

$$\begin{aligned} e^{2\text{dist}(z_0, \eta \cdot z_0)} &\geq \frac{\int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-1}\xi^{-1}} \cdot \left(1 - \int d\nu_{\eta^{-1}\xi^{-1}}\right)}{\int d\nu_{\eta^{-1}\xi^{-1}} \cdot \left(1 - \int \lambda_{\eta, z_0}^2 d\nu_{\eta^{-1}\xi^{-1}}\right)} \\ &= \frac{\left(1 - \sum_{\psi \in \Psi_8} \int d\nu_\psi\right) \cdot \left(1 - \int d\nu_{\eta^{-1}\xi^{-1}}\right)}{\left(\sum_{\psi \in \Psi_8} \int d\nu_\psi\right) \cdot \int d\nu_{\eta^{-1}\xi^{-1}}} \\ &= \frac{m_8}{1 - m_8} \cdot \frac{1 - m_4}{m_4} \\ &= \frac{1 - \sum_{i=1}^7 m_i}{\sum_{i=1}^7 m_i} \cdot \frac{1 - m_4}{m_4} \end{aligned} \tag{3.43}$$

In this case, we obtain a lower bound for the displacement $\text{dist}(z_0, \eta \cdot z_0)$ in (Equation 3.43). We set the displacement the displacement function as

$$f_{14}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7} \cdot \frac{1 - x_4}{x_4}.$$

Let $\gamma = \eta^{-2}$, $\nu = \nu_\eta$, $a = \nu_\eta(S_\infty)$ and $b = \int \lambda_{\eta^{-2}, z_0}^2 d\nu_\eta$ and apply Lemma 1.1.1. Then we obtain that

$$e^{2\text{dist}(z_0, \eta^{-2} \cdot z_0)} \geq \frac{\int \lambda_{\eta^{-2}, z_0}^2 d\nu_\eta \cdot \left(1 - \int d\nu_\eta\right)}{\int d\nu_\eta \cdot \left(1 - \int \lambda_{\eta^{-2}, z_0}^2 d\nu_\eta\right)}.$$

By Lemma 3.2.1(5), we find that

$$\begin{aligned} e^{2\text{dist}(z_0, \eta^{-2} \cdot z_0)} &\geq \frac{\int \lambda_{\eta^{-2}, z_0}^2 d\nu_\eta \cdot \left(1 - \int d\nu_\eta\right)}{\int d\nu_\eta \cdot \left(1 - \int \lambda_{\eta^{-2}, z_0}^2 d\nu_\eta\right)} \\ &= \frac{\left(1 - \int d\nu_{\eta^{-2}}\right) \cdot \left(1 - \int d\nu_\eta\right)}{\left(\int d\nu_{\eta^{-2}}\right) \cdot \int d\nu_\eta} \\ &= \frac{1 - m_5}{m_5} \cdot \frac{1 - m_7}{m_7}. \end{aligned} \tag{3.44}$$

The expression (Equation 3.44) gives a lower bound for the displacement $\text{dist}(z_0, \eta^{-2} \cdot z_0)$ in this case.

The displacement function is

$$f_{15}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_5}{x_5} \cdot \frac{1 - x_7}{x_7}.$$

Let $\gamma = \xi\eta^{-1}$, $\nu = \nu_\eta$, $a = \nu_\eta(S_\infty)$ and $b = \int \lambda_{\xi\eta^{-1}, z_0}^2 d\nu_\eta$ and apply Lemma 1.1.1. Then we obtain that

$$e^{2\text{dist}(z_0, \xi\eta^{-1} \cdot z_0)} \geq \frac{\int \lambda_{\xi\eta^{-1}, z_0}^2 d\nu_\eta \cdot \left(1 - \int d\nu_\eta\right)}{\int d\nu_\eta \cdot \left(1 - \int \lambda_{\xi\eta^{-1}, z_0}^2 d\nu_\eta\right)}.$$

By Lemma 3.2.1(5), we see that

$$\begin{aligned}
 e^{2\text{dist}(z_0, \xi\eta^{-1}\cdot z_0)} &\geq \frac{\int \lambda_{\xi\eta^{-1}, z_0}^2 dv_\eta \cdot \left(1 - \int dv_\eta\right)}{\int dv_\eta \cdot \left(1 - \int \lambda_{\xi\eta^{-1}, z_0}^2 dv_\eta\right)} \\
 &= \frac{\left(1 - \int dv_{\xi\eta^{-1}}\right) \cdot \left(1 - \int dv_\eta\right)}{\left(\int dv_{\xi\eta^{-1}}\right) \cdot \int dv_\eta} \\
 &= \frac{1 - m_3}{m_3} \cdot \frac{1 - m_7}{m_7}.
 \end{aligned} \tag{3.45}$$

The constant obtained in (Equation 3.45) is a lower bound for the displacement $\text{dist}(z_0, \xi\eta^{-1}\cdot z_0)$. In this case, we set the displacement function as

$$f_{16}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_3}{x_3} \cdot \frac{1 - x_7}{x_7}.$$

Let $\gamma = \eta^{-1}\xi$, $v = v_{\xi^{-1}}$, $a = v_\xi(S_\infty)$ and $b = \int \lambda_{\eta^{-1}\xi, z_0}^2 dv_{\xi^{-1}}$ and apply Lemma 1.1.1. Then by Lemma 3.2.1(5), we obtain that

$$\begin{aligned}
 e^{2\text{dist}(z_0, \eta^{-1}\xi\cdot z)} &\geq \frac{\int \lambda_{\eta^{-1}\xi, z_0}^2 dv_{\xi^{-1}} \cdot \left(1 - \int dv_{\xi^{-1}}\right)}{\int dv_{\xi^{-1}} \cdot \left(1 - \int \lambda_{\eta^{-1}\xi, z_0}^2 dv_{\xi^{-1}}\right)} \\
 &= \frac{\left(1 - \int dv_{\eta^{-1}\xi}\right) \cdot \left(1 - \int dv_{\xi^{-1}}\right)}{\left(\int dv_{\eta^{-1}\xi}\right) \cdot \int dv_{\xi^{-1}}} \\
 &= \frac{1 - m_6}{m_6} \cdot \frac{1 - m_8}{m_8}.
 \end{aligned} \tag{3.46}$$

In this case, we find a lower bound for the displacement $\text{dist}(z_0, \eta^{-1}\xi\cdot z)$ given in (Equation 3.46).

The displacement function is

$$f_{17}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_6}{x_6} \cdot \frac{1 - x_8}{x_8}.$$

Let $\gamma = \xi^2$, $v = v_{\xi^{-1}}$, $a = v_{\xi^{-1}}(S_\infty)$ and $b = \int \lambda_{\xi^2, z_0}^2 dv_{\xi^{-1}}$ and apply Lemma 1.1.1. Then by Lemma 3.2.1(5), we obtain that

$$\begin{aligned} e^{2\text{dist}(z_0, \xi^2 \cdot z_0)} &\geq \frac{\int \lambda_{\xi^2, z_0}^2 dv_{\xi^{-1}} \cdot \left(1 - \int dv_{\xi^{-1}}\right)}{\int dv_{\xi^{-1}} \cdot \left(1 - \int \lambda_{\xi^2, z_0}^2 dv_{\xi^{-1}}\right)} \\ &= \frac{\left(1 - \int dv_{\xi^2}\right) \cdot \left(1 - \int dv_{\xi^{-1}}\right)}{\left(\int dv_{\xi^2}\right) \cdot \int dv_{\xi^{-1}}} \\ &= \frac{1 - m_2}{m_2} \cdot \frac{1 - m_8}{m_8}. \end{aligned} \tag{3.47}$$

The constant obtained in (Equation 3.47) provides a lower bound for the displacement $\text{dist}(z_0, \xi^2 \cdot z_0)$.

We set the displacement function as

$$f_{18}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \frac{1 - x_2}{x_2} \cdot \frac{1 - x_8}{x_8}.$$

We have eighteen lower bounds given in the expressions (Equation 3.30)-(Equation 3.47) for the displacements $\text{dist}(z_0, \xi \cdot z_0)$, $\text{dist}(z_0, \eta \cdot z_0)$, $\text{dist}(z_0, \xi^{-1} \cdot z_0)$, $\text{dist}(z_0, \eta^{-2} \cdot z_0)$, $\text{dist}(z_0, \eta^{-1} \xi \cdot z)$, and $\text{dist}(z_0, \xi \eta^{-1} \cdot z_0)$. Therefore, there are eighteen displacement functions. Remember that we have $\text{dist}(z_0, \xi \cdot z_0) = \text{dist}(z_0, \xi^{-1} \cdot z_0)$, $\text{dist}(z_0, \eta \cdot z_0) = \text{dist}(z_0, \eta^{-1} \cdot z_0)$, $\text{dist}(z_0, \xi \eta \cdot z_0) = \text{dist}(z_0, \eta^{-1} \xi^{-1} \cdot z_0)$

hold for every $z_0 \in \mathbf{H}^3$. Since we are interested in only $\text{dist}(z_0, \xi \cdot z_0)$, $\text{dist}(z_0, \eta \cdot z_0)$ and $\text{dist}(z_0, \xi\eta \cdot z_0)$, we concentrate on the first fourteen displacement functions.

The constants obtained in (Equation 3.30)-(Equation 3.47) can be considered as the values of these functions $f_1, f_2, \dots, f_{14}$ at the point $\mathbf{m} = (m_1, m_2, \dots, m_8) \in \Delta^7$ where Δ^7 denotes the 7-dimensional simplex $\{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbf{R}_+^8 : \sum_{i=1}^8 x_i = 1\}$. In order to calculate a lower bound for the maximum of the displacement $\text{dist}(z_0, \xi \cdot z_0)$, $\text{dist}(z_0, \eta \cdot z_0)$, and $\text{dist}(z_0, \xi\eta \cdot z_0)$, it is required to find the greatest lower bound for the maximum of the functions $f_1, f_2, \dots, f_{14}$ over the simplex Δ^7 . In other words, if G is the continuous function defined as

$$\begin{aligned} G_1 & : \Delta^7 \rightarrow \mathbf{R} \\ \mathbf{x} & \mapsto \max(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_{14}(\mathbf{x})), \end{aligned}$$

we need to calculate $\inf_{\mathbf{x} \in \Delta^7} G_1(\mathbf{x})$. We will need different parametrization of Δ^7 : Let $\mathfrak{P}_i$ denotes the parametrization $(x_{j_1}, x_{j_2}, \dots, x_i = 1 - \sum_{k=j_1}^{j_7} x_k, \dots, x_{j_7})$ of Δ^7 where $0 < x_{j_1} < 1$, $0 < x_{j_2} < 1 - x_{j_1}, \dots$, $0 < x_{j_7} < 1 - \sum_{k=j_1}^{j_6} x_k$ such that $\{j_1, j_2, \dots, j_7\} = \{1, 2, \dots, i-1, i+1, \dots, 8\}$ and $j_1 < j_2 < \dots < j_7$. Points of Δ^7 will be written in bold fonts.

CHAPTER 4

DISPLACEMENTS WHEN THE LIMIT SET OF Γ IS S_∞

In this chapter, I aim to prove the following proposition which is the analog of Proposition 1.2.2 for the set of isometries $\{\xi, \eta, \xi\eta\}$.

Proposition 4.0.1. *Let ξ and η be two orientation-preserving isometries of $\mathbf{H}^3$. Suppose that the group Γ generated by ξ and η is discrete and is free on the generators ξ and η . Suppose in addition that every Γ -invariant conformal density for $\mathbf{H}^3$ is a constant multiple of the area density. Then for any $z \in \mathbf{H}^3$ we have*

$$\max(\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0), \text{dist}(z_0, \xi\eta \cdot z_0)) \geq \frac{1}{2} \log(5 + 3\sqrt{2}),$$

Proposition 4.0.1 is required to complete the proof of Theorem 5.1.1 in the case $\Lambda_\Gamma = S_\infty$. The number given in the proposition above is obtained by calculating $\inf_{\mathbf{x} \in \Delta^7} G_1(\mathbf{x})$. To this purpose, I gather some facts that are necessary to complete this task in the next section.

4.1 Infimum of maximum of $f_1, f_2, \dots, f_{14}$ over the simplex Δ^7

We consider the displacement functions $f_i : \Delta^7 \rightarrow \mathbf{R}_+$ for $i = 1, \dots, 14$ where

$$\begin{aligned} f_1(\mathbf{x}) &= \frac{1 - x_1 - x_2 - x_3}{x_1 + x_2 + x_3} \cdot \frac{1 - x_4}{x_4}, \\ f_2(\mathbf{x}) &= \frac{1 - x_4 - x_5 - x_6}{x_4 + x_5 + x_6} \cdot \frac{1 - x_1}{x_1}, \\ f_3(\mathbf{x}) &= \frac{1 - x_1 - x_2 - x_3}{x_1 + x_2 + x_3} \cdot \frac{1 - x_8}{x_8}, \end{aligned}$$

$$\begin{aligned}
f_4(\mathbf{x}) &= \frac{1-x_4-x_5-x_6}{x_4+x_5+x_6} \cdot \frac{1-x_7}{x_7}, \\
f_5(\mathbf{x}) &= \frac{1-x_4-x_5-x_6-x_7-x_8}{x_4+x_5+x_6+x_7+x_8} \cdot \frac{1-x_6}{x_6}, \\
f_6(\mathbf{x}) &= \frac{1-x_4-x_5-x_6-x_7-x_8}{x_4+x_5+x_6+x_7+x_8} \cdot \frac{1-x_2}{x_2}, \\
f_7(\mathbf{x}) &= \frac{1-x_1-x_2-x_3-x_7-x_8}{x_1+x_2+x_3+x_7+x_8} \cdot \frac{1-x_3}{x_3}, \\
f_8(\mathbf{x}) &= \frac{1-x_1-x_2-x_3-x_7-x_8}{x_1+x_2+x_3+x_7+x_8} \cdot \frac{1-x_5}{x_5}, \\
f_9(\mathbf{x}) &= \frac{1-x_1-x_2-x_4-x_5-x_6-x_7-x_8}{x_1+x_2+x_4+x_5+x_6+x_7+x_8} \cdot \frac{1-x_5}{x_5}, \\
f_{10}(\mathbf{x}) &= \frac{1-x_1-x_3-x_4-x_5-x_6-x_7-x_8}{x_1+x_3+x_4+x_5+x_6+x_7+x_8} \cdot \frac{1-x_6}{x_6}, \\
f_{11}(\mathbf{x}) &= \frac{1-x_1-x_2-x_3-x_4-x_5-x_7-x_8}{x_1+x_2+x_3+x_4+x_5+x_7+x_8} \cdot \frac{1-x_2}{x_2}, \\
f_{12}(\mathbf{x}) &= \frac{1-x_1-x_2-x_3-x_4-x_6-x_7-x_8}{x_1+x_2+x_3+x_4+x_6+x_7+x_8} \cdot \frac{1-x_3}{x_3}, \\
f_{13}(\mathbf{x}) &= \frac{1-x_1-x_2-x_3-x_4-x_5-x_6-x_8}{x_1+x_2+x_3+x_4+x_5+x_6+x_8} \cdot \frac{1-x_1}{x_1}, \\
f_{14}(\mathbf{x}) &= \frac{1-x_1-x_2-x_3-x_4-x_5-x_6-x_7}{x_1+x_2+x_3+x_4+x_5+x_6+x_7} \cdot \frac{1-x_4}{x_4}.
\end{aligned}$$

Let us define the following continuous function

$$\begin{aligned}
F_1 &: \Delta^7 \rightarrow \mathbf{R} \\
\mathbf{x} &\mapsto \max(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_8(\mathbf{x})).
\end{aligned}$$

We will show that $\inf_{\mathbf{x} \in \Delta^7} G_1(\mathbf{x}) = \inf_{\mathbf{x} \in \Delta^7} F_1(\mathbf{x})$ (see Lemma 4.1.14). Therefore, it is enough to calculate $\inf_{\mathbf{x} \in \Delta^7} F_1(\mathbf{x})$. Since $f_i(\mathbf{x}) > 1$ for every $\mathbf{x} \in \Delta^7$ for $i = 1, 2, 3, 4, 5, 6, 7, 8$, we have $\inf_{\mathbf{x} \in \Delta^7} F_1(\mathbf{x}) \geq 1$.

Let us define the following subsets $\Delta_k = \{\mathbf{x} \in \Delta^7 : \frac{1}{k} \leq x_i \leq 1 - \frac{1}{2k} \text{ for } i = 1, \dots, 8\}$ of Δ^7 for every $k \geq 2$. Then F_1 has an absolute minimum α_k at some point $\mathbf{x}_k$ in each Δ_k since Δ_k is compact.

Let us consider the sequences (α_k) and $(\mathbf{x}_k)$. It is clear that (α_k) is a decreasing sequence because $\Delta_k < \Delta_{k+1}$. Since it is bounded below by 1, it converges to a finite number greater than or equal to 1. This means that $F_1(\mathbf{x}_k)$ approaches to $\inf_{\mathbf{x} \in \Delta^7} F_1(\mathbf{x})$.

We claim that the sequence $(\mathbf{x}_k)$ cannot have a limit point on the boundary of Δ^7 . Let us assume contrary that $\mathbf{x}_k \rightarrow \mathbf{x}_0 \in \Delta^7$ as $k \rightarrow \infty$. If $b_1, b_2, \dots, b_8$ denote the coordinates of the point $\mathbf{x}_0$, then $b_i = 0$ for some $i \in \{1, 2, \dots, 8\}$. Let us assume that $b_1 = 0$. Then using the function f_2 , we conclude that $b_4 + b_5 + b_6 = 1$. Because otherwise f_2 would tend to infinity through the sequence $(\mathbf{x}_k)$. But it is supposed to be tending to an infimum. Therefore, we must have $b_i = 0$ for every $i \in \{2, 3, 7, 8\}$. In particular, we get $b_8 = 0$ which implies that $b_1 + b_2 + b_3 = 1$. Because otherwise f_3 would tend to infinity through the sequence $(\mathbf{x}_k)$. But it is supposed to be tending to an infimum. Since $b_2 = 0$ and $b_3 = 0$, we find that $b_1 = 1$. This is a contradiction.

If $b_2 = 0$, then we get $b_4 + b_5 + b_6 + b_7 + b_8 = 1$ using the function f_6 . Because otherwise f_6 would tend to infinity through the sequence $(\mathbf{x}_k)$. But it is supposed to be tending to an infimum. Therefore, we must have $b_i = 0$ for every $i \in \{1, 2, 3\}$. We already showed that $b_1 \neq 0$, a contradiction.

Let us assume that $b_3 = 0$. Since the function f_7 cannot tend to infinity through the sequence $(\mathbf{x}_k)$, we conclude that $b_1 + b_2 + b_7 + b_8 = 1$. This means that $b_i = 0$ for every $i \in \{4, 5, 6\}$. Since we have $b_4 + b_5 + b_6 = 0$, we find that $b_8 = 1$. This implies that $b_1 = 0$ which is a contradiction.

Let us assume that $b_4 = 0$. Using the function f_1 , we conclude that $b_1 + b_2 + b_3 = 1$. Otherwise, f_1 would tend to infinity through the sequence $(\mathbf{x}_k)$. Therefore, we get that $b_i = 0$ for every $i \in \{4, 5, 6, 7, 8\}$. Since the function f_4 cannot approach to infinity through the sequence $(\mathbf{x}_k)$, we obtain that $b_4 + b_5 + b_6 = 1$. This gives a contradiction.

If $b_5 = 0$, then we find that $b_1 + b_2 + b_3 + b_7 + b_8 = 1$ using the function f_8 . Because otherwise f_8 tends to infinity through the sequence $(\mathbf{x}_k)$. This implies that $b_4 = 0$. Since we already show that $b_4 \neq 0$, this gives a contradiction. Therefore, we get $b_5 \neq 0$.

Let us assume that $b_6 = 0$. Using the function f_5 , we conclude that $b_4 + b_5 + b_6 + b_7 + b_8 = 1$. f_5 cannot approach to infinity through the sequence $(\mathbf{x}_k)$ because it is supposed to tend to an infimum. Consequently, we derive that $b_i = 0$ for every $i \in \{1, 2, 3\}$ which is a contradiction.

If $b_7 = 0$, then we get $b_4 + b_5 + b_6 = 1$. This implies that $b_i = 0$ for every $i \in \{1, 2, 3, 7, 8\}$. This is a contradiction because we know that $b_1 \neq 0$. Let us assume that $b_8 = 0$. Then we get $x_1 + x_2 + x_3 = 1$ or equivalently $b_i = 0$ for every $i \in \{4, 5, 6, 7, 8\}$. Since we showed that $b_4 \neq 0$, this is again a contradiction. Hence, the sequence $(\mathbf{x})_k$ cannot have a limit on the boundary of Δ^7 .

We claim that there exists a positive integer k_0 so that $\mathbf{x}_k = \mathbf{x}_{k+1}$ for every $k \geq k_0$. Otherwise, we would have a subsequence $(\mathbf{x}_{k_j})$ of the sequence $(\mathbf{x}_k)$ so that $\mathbf{x}_{k_j} \in \Delta_{k_{j+1}} - \Delta_{k_j}$ for every integer $j > 0$. Since we have $\cup_{k=2}^{\infty} \Delta_k = \Delta^7$, we get $F_1(\mathbf{x}_{k_j}) = \alpha_{k_j} \rightarrow \infty$ as $j \rightarrow \infty$. This is a contradiction. Therefore, we have $\mathbf{x}_k = \mathbf{x}_{k+1}$ for every $k \geq k_0$. In other words, the absolute minimum $F_1(\mathbf{x}_{k_0})$ of F_1 exists on Δ^7 . Then, we obviously have $\inf_{\mathbf{x} \in \Delta^7} F_1(\mathbf{x}) = \min_{\mathbf{x} \in \Delta^7} F_1(\mathbf{x})$. In the following sections, we calculate $\alpha_* = \min_{\mathbf{x} \in \Delta^7} F_1(\mathbf{x})$.

It is possible to find an interval that contains α_* . Note that

$$\mathbf{y}^* = \left(\frac{\sqrt{2}-1}{2}, \frac{3-2\sqrt{2}}{4}, \frac{3-2\sqrt{2}}{4}, \frac{\sqrt{2}-1}{2}, \frac{3-2\sqrt{2}}{4}, \frac{3-2\sqrt{2}}{4}, \frac{\sqrt{2}-1}{2}, \frac{\sqrt{2}-1}{2} \right)$$

is a point in Δ^7 such that $f_i(\mathbf{y}^*) = 5 + 3\sqrt{2}$ for every $i = 1, 2, 3, 4, 5, 6, 7, 8$. Therefore, we get $F_1(\mathbf{y}^*) = 5 + 3\sqrt{2}$ which implies that $\alpha_* \leq 5 + 3\sqrt{2}$. To find a lower bound for α_* we consider the functions f_3 and f_4 . It is clear that $\min_{\mathbf{x} \in \Delta^7} F_1(\mathbf{x}) \geq \inf_{\mathbf{x} \in \Delta^7} \{\max(f_3(\mathbf{x}), f_4(\mathbf{x}))\}$. If we apply the following substitution

$$\begin{aligned} X_1 &= x_1 + x_2 + x_3 \\ X_2 &= x_8 \\ X_3 &= x_4 + x_5 + x_6 \\ X_4 &= x_7 \end{aligned}$$

we see that

$$\begin{aligned} f_3(\mathbf{x}) &= \frac{(1 - X_1)(1 - X_2)}{X_1 X_2}, \\ f_4(\mathbf{x}) &= \frac{(1 - X_3)(1 - X_4)}{X_3 X_4} \end{aligned}$$

where $X_1 + X_2 + X_3 + X_4 = 1$. By Lemma 2.2.1, we know that $\inf_{\mathbf{x} \in \Delta^3} \{\max(f_3(\mathbf{x}), f_4(\mathbf{x}))\} = 9$. Hence, we obtain that $9 \leq \alpha_*$. This means that the minimum value of F_1 lies in the interval $I_\alpha = [9, 5 + 3\sqrt{2}]$.

In order to calculate α_* , we need to consider the extreme values of the functions $\{f_1, f_2, \dots, f_8\}$ on the simplex Δ^7 . Some of the first order partial derivatives of the functions $\{f_1, f_2, \dots, f_8\}$ using the parametrization $\mathfrak{P}_8$ are given in the Appendix A.1. If $(f_j)_i$ denotes the first order partial derivative of the function f_j according to the variable x_i , we see that $(f_1)_1(\mathbf{x})$, $(f_2)_1(\mathbf{x})$, $(f_4)_4(\mathbf{x})$, $(f_7)_3(\mathbf{x})$ are all negative and $(f_3)_4(\mathbf{x})$, $(f_5)_1(\mathbf{x})$, $(f_6)_1(\mathbf{x})$, $(f_8)_4(\mathbf{x})$ are all positive for every $\mathbf{x} \in \Delta^7$. Therefore, the following

equations $(f_j)_i = 0$ for $i = 1, 2, \dots, 8$ has no solution in Δ^7 for any $j = 1, 2, \dots, 8$. Consequently, none of the functions $f_1, f_2, \dots, f_8$ has a local extremum in Δ^7 .

Let us consider the set $\{f_1(\mathbf{x}^*), f_2(\mathbf{x}^*), \dots, f_8(\mathbf{x}^*)\}$. If we have $f_j(\mathbf{x}^*) \neq f_k(\mathbf{x}^*)$ for every $j \neq k$, then the set of real numbers $\{f_1(\mathbf{x}^*), f_2(\mathbf{x}^*), \dots, f_8(\mathbf{x}^*)\}$ has a unique largest element. By renumbering the functions, we can assume that $f_1(\mathbf{x}^*)$ is the largest value. Therefore, we get that $f_1(\mathbf{x}^*) = \alpha_*$. By continuity, there exists a neighborhood U of $\mathbf{x}^*$ contained in Δ^7 so that $F(\mathbf{x}) = f_1(\mathbf{x})$ for every $\mathbf{x} \in U$. Since F has a minimum at $\mathbf{x}^*$, then f_1 has a local minimum at $\mathbf{x}^*$. But this contradicts with the fact that f_1 has no local extremum in Δ^7 . Therefore, there exists two functions f_{j_1}, f_{j_2} such that $f_{j_1}(\mathbf{x}^*) = f_{j_2}(\mathbf{x}^*)$ where $j_1, j_2 \in \{1, 2, \dots, 8\}$ and $j_1 \neq j_2$. Hence, at least two of the functions attain the value α_* at the point $\mathbf{x}^*$.

In the following sections, we give a list of all possible cases for the values of the functions $f_1, f_2, f_3, \dots, f_8$ at the point $\mathbf{x}^*$.

4.1.1 The relationship between the functions f_2, f_6 and f_7

Let us assume that $f_2(\mathbf{x}^*) < \alpha_*$, $f_6(\mathbf{x}^*) = \alpha_*$, $f_7(\mathbf{x}^*) = \alpha_*$ and $f_j(\mathbf{x}^*) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$. If we say $I_1 = \{2, 6, 7\}$, then it is possible to obtain a point $\mathbf{z} \in \Delta^7$ so that $f_k(\mathbf{z}) < \alpha_*$ for every $k \in I_1$ and $f_j(\mathbf{z}) = f_j(\mathbf{x}^*)$ for every $j \in \{1, 3, 4, 5, 8\}$ as follows:

If we use the parametrization $\mathfrak{P}_2$ and calculate the directional derivatives of all the functions $f_1, f_2, \dots, f_8$ in the direction of the vector

$$\vec{u}_1 = \langle -1, 0, 0, 0, 0, 0, 0 \rangle,$$

we get

$$\begin{aligned} D_{\vec{u}_1} f_2(\mathbf{x}) &> 0 \\ D_{\vec{u}_1} f_j(\mathbf{x}) &= 0, \quad 1 \leq j \leq 8, j \neq 2, 6 \\ D_{\vec{u}_1} f_6(\mathbf{x}) &< 0 \end{aligned}$$

for every $\mathbf{x} \in \Delta^7$ (see Appendix A.4). Then we conclude that f_2 is increasing, f_6 is decreasing and f_j is constant for every $j \in \{1, 3, 4, 5, 6, 8\}$ on the line segment $\ell_1 \cap \Delta^7$ where $\ell_1 = \{\vec{\mathbf{x}}^* + t\vec{u}_1 : t \in \mathbf{R}\}$. Since f_2 is continuous on $\ell_1 \cap \Delta$ and $f_2(\mathbf{x}^*) < \alpha_*$, there exists an $\epsilon_1 > 0$ so that $f_2(\mathbf{x}) < \alpha_*$ for every $\mathbf{x} \in \ell_1^{\epsilon_1} = \{\vec{\mathbf{x}}^* + t\vec{u}_1 : t \in (0, \epsilon_1)\}$. Let $\mathbf{y}$ be a point in $\ell_1^{\epsilon_1}$. Then we get $f_2(\mathbf{y}) < \alpha_*$. Since f_6 is decreasing on the line segment $\ell_1^{\epsilon_1} \subset \ell_1$ and $f_6(\mathbf{x}^*) = \alpha_*$, we see that $f_6(\mathbf{y}) < \alpha_*$. In other words, if we move along the line segment starting at $\mathbf{x}^*$ in the direction of the vector $\vec{u}_1$ for a sufficiently small amount, we get a point $\mathbf{y} \in \Delta^7$ so that the values of f_2 and f_6 at $\mathbf{y}$ are less than α_* and the values of the functions $f_1, f_3, f_4, f_5, f_7, f_8$ at the point $\mathbf{y}$ are the same as at the point $\mathbf{x}^*$, i.e. $f_j(\mathbf{y}) = f_j(\mathbf{x}^*)$ for every $j \in \{1, 3, 4, 5, 7, 8\}$. Let us call the perturbation performed in the way described above when $f_2(\mathbf{x}) < \alpha_*$, $f_7(\mathbf{x}) = \alpha_*$, $f_6(\mathbf{x}) = \alpha_*$ and $f_j(\mathbf{x}) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$ for some $\mathbf{x} \in \Delta^7$ $pert_{\vec{u}_1}^2$ and the point obtained by applying the perturbation $pert_{\vec{u}_1}^2$ at the point $\mathbf{x}$ $pert_{\vec{u}_1}^2(\mathbf{x})$. We will use the following notation to denote this process:

$$\begin{aligned} &[\mathbf{x} : f_2 < \alpha_*, f_6 = \alpha_*, f_7 = \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*] \\ &\quad \downarrow pert_{\vec{u}_1}^2 \\ &[pert_{\vec{u}_1}^2(\mathbf{x}) : f_2 < \alpha_*, f_6 < \alpha_*, f_7 = \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*]. \end{aligned}$$

In the case above, we have $\mathbf{y} = pert_{\vec{u}_1}^2(\mathbf{x}^*)$.

The expression $[\mathbf{x} : f_2 < \alpha_*, f_6 = \alpha_*, f_7 = \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*]$ means that $f_2(\mathbf{x}) < \alpha_*$, $f_6(\mathbf{x}) = \alpha_*$, $f_7(\mathbf{x}) = \alpha_*$, $f_j(\mathbf{x}) \leq \alpha_*$ for every $j \in I_2 \cup I_3$ and $F(\mathbf{x}) = \alpha_*$.

If we calculate the directional derivatives of all the functions $f_1, f_2, \dots, f_8$ in the direction of the vector

$$\vec{u}_2 = \langle -1, 1, 0, 0, 0, 0, 0 \rangle,$$

we obtain that

$$\begin{aligned} D_{\vec{u}_2} f_2(\mathbf{x}) &> 0 \\ D_{\vec{u}_2} f_j(\mathbf{x}) &= 0, \quad 1 \leq j \leq 8, \quad j \neq 2, 7 \\ D_{\vec{u}_2} f_7(\mathbf{x}) &< 0 \end{aligned}$$

for every $\mathbf{x} \in \Delta^7$, particularly for $\mathbf{y}$ (see Appendix A.4). Then we conclude that f_2 is increasing, f_7 is decreasing and f_j is constant for every $j \in \{1, 3, 4, 5, 6, 8\}$ on the line segment $\ell_2 \cap \Delta$ where $\ell_2 = \{\vec{\mathbf{y}} + t\vec{u}_2 : t \in \mathbf{R}\}$. Since f_2 is continuous on $\ell_2 \cap \Delta^7$ and $f_2(\mathbf{y}) < \alpha_*$, there exists an $\epsilon_2 > 0$ so that $f_2(\mathbf{x}) < \alpha_*$ for every $\mathbf{x} \in \ell_2^{\epsilon_2} = \{\vec{\mathbf{y}} + t\vec{u}_2 : t \in (0, \epsilon_2)\}$. Let $\mathbf{z}$ be a point in $\ell_2^{\epsilon_2}$. Then we get $f_2(\mathbf{z}) < \alpha_*$. Since f_7 is decreasing on the line segment $\ell_2^{\epsilon_2} \subset \ell_2$ and $f_7(\mathbf{y}) = \alpha_*$, we see that $f_7(\mathbf{z}) < \alpha_*$. In other words, if we move along the line segment starting at $\mathbf{y}$ in the direction of the vector $\vec{u}_2$ for a sufficiently small amount, we get a point $\mathbf{z} \in \Delta^7$ so that the values of f_2 , f_6 , and f_7 at $\mathbf{z}$ are all less than α_* , and the values of f_1, f_3, f_4, f_5, f_8 at the point $\mathbf{z}$ are the same as at the point $\mathbf{y}$, i.e. $f_j(\mathbf{z}) = f_j(\mathbf{y})$ for every $j \in \{1, 3, 4, 5, 8\}$. Thus, we obtain a point $\mathbf{z}$ such that $f_j(\mathbf{z}) < \alpha_*$ for every $j \in I_1$ and $f_j(\mathbf{z}) = f_j(\mathbf{x}^*)$ for every $j \in \{1, 3, 4, 5, 8\}$. Let us call the perturbation performed in the way described above when $f_2(\mathbf{x}) < \alpha_*$, $f_7(\mathbf{x}) = \alpha_*$, $f_6(\mathbf{x}) < \alpha_*$ and $f_j(\mathbf{x}) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$ for some $\mathbf{x} \in \Delta^7$ *per* $t_{\vec{u}_2}^2$ and the point obtained using the point $\mathbf{x}$ *per* $t_{\vec{u}_2}^2(\mathbf{x})$.

We will use the following notation to denote this process:

$$\begin{aligned} & [\mathbf{x} : f_2 < \alpha_*, f_6 < \alpha_*, f_7 = \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*] \\ & \quad \downarrow \text{pert}_{\vec{u}_2}^2 \\ & [\text{pert}_{\vec{u}_2}^2(\mathbf{x}) : f_2 < \alpha_*, f_6 < \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*]. \end{aligned}$$

Therefore, we get $\mathbf{z} = \text{pert}_{\vec{u}_2}^2(\mathbf{y})$ or $\mathbf{z} = \text{pert}_{\vec{u}_2}^2(\text{pert}_{\vec{u}_1}^2(\mathbf{x}^*))$.

Let us assume that $f_6(\mathbf{x}^*) < \alpha_*$, $f_7(\mathbf{x}^*) = \alpha_*$, $f_2(\mathbf{x}^*) = \alpha_*$ and $f_j(\mathbf{x}^*) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$. If this is the case, then it is possible to obtain a point $\mathbf{z} \in \Delta^7$ so that $f_k(\mathbf{z}) < \alpha_*$ for every $k \in I_1$ and $f_j(\mathbf{z}) = f_j(\mathbf{x}^*)$ for every $j \in \{1, 3, 4, 5, 8\}$ as follows:

We calculate the directional derivatives of all the functions $f_1, f_2, \dots, f_8$ in the direction of the vector

$$\vec{u}_3 = \langle 0, 1, 0, 0, 0, 0, 0 \rangle.$$

Then we see that

$$\begin{aligned} D_{\vec{u}_3} f_6(\mathbf{x}) &> 0 \\ D_{\vec{u}_3} f_j(\mathbf{x}) &= 0, \quad 1 \leq j \leq 8, \quad j \neq 6, 7 \\ D_{\vec{u}_3} f_7(\mathbf{x}) &< 0 \end{aligned}$$

for every $\mathbf{x} \in \Delta^7$ (see Appendix A.4). Hence, we conclude that f_6 is increasing, f_7 is decreasing and f_j is constant for every $j \in \{1, 2, 3, 4, 5, 8\}$ on the line segment $\ell_3 \cap \Delta^7$ where $\ell_3 = \{\vec{\mathbf{x}}^* + t\vec{u}_3 : t \in \mathbf{R}\}$. Since f_6 is continuous on $\ell_3 \cap \Delta^7$ and $f_6(\mathbf{x}^*) < \alpha_*$, there exists an $\epsilon_3 > 0$ so that $f_6(\mathbf{x}) < \alpha_*$ for every $\mathbf{x} \in \ell_3^{\epsilon_3} = \{\vec{\mathbf{x}}^* + t\vec{u}_3 : t \in (0, \epsilon_3)\}$. Let $\mathbf{y}$ be a point in $\ell_3^{\epsilon_3}$. Then we get $f_6(\mathbf{y}) < \alpha_*$. Since f_7 is decreasing on the line segment $\ell_3^{\epsilon_3} \subset \ell_3$ and $f_7(\mathbf{x}^*) = \alpha_*$, we see that $f_7(\mathbf{y}) < \alpha_*$. If we move along the line segment

starting at $\mathbf{x}^*$ in the direction of the vector $\vec{u}_3$ for a sufficiently small amount, we obtain a point $\mathbf{y} \in \Delta^7$ such that the values of f_6 and f_7 at $\mathbf{y}$ are less than α_* and the values of $f_1, f_2, f_3, f_4, f_5, f_8$ at the point $\mathbf{y}$ are the same as at $\mathbf{x}^*$, i.e. $f_j(\mathbf{y}) = f_j(\mathbf{x}^*)$ for every $j \in \{1, 2, 3, 4, 5, 8\}$. Let us call the perturbation performed in the way described above when $f_6(\mathbf{x}) < \alpha_*$, $f_7(\mathbf{x}) = \alpha_*$, $f_2(\mathbf{x}) = \alpha_*$ and $f_j(\mathbf{x}) \leq \alpha_*$ for every $j \in \{1, 3, 4, 5, 8\}$ for some $\mathbf{x} \in \Delta^7$ $pert_{\vec{u}_3}^6(\mathbf{x})$ and the point obtained starting at $\mathbf{x}$ $pert_{\vec{u}_3}^6(\mathbf{x})$.

We will use the following notation to denote this process:

$$\begin{array}{c} [\mathbf{x} : f_2 = \alpha_*, f_6 < \alpha_*, f_7 = \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*] \\ \downarrow pert_{\vec{u}_3}^6 \\ [pert_{\vec{u}_3}^6(\mathbf{x}) : f_2 = \alpha_*, f_6 < \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*]. \end{array}$$

Therefore, we have $\mathbf{y} = pert_{\vec{u}_3}^6(\mathbf{x}^*)$ in the perturbation above.

If we calculate the directional derivative of all the functions $f_1, f_2, \dots, f_8$ in the direction of the vector

$$\vec{u}_4 = -\vec{u}_2,$$

we observe that

$$\begin{aligned} D_{\vec{u}_4} f_2(\mathbf{x}) &< 0 \\ D_{\vec{u}_4} f_j(\mathbf{x}) &= 0, \quad 1 \leq j \leq 8, \quad j \neq 2, 6 \\ D_{\vec{u}_4} f_6(\mathbf{x}) &> 0 \end{aligned}$$

for every $\mathbf{x} \in \Delta^7$, particularly for $\mathbf{y}$ (see Appendix A.4). Therefore, we conclude that f_6 is increasing, f_2 is decreasing and f_j is constant for every $j \in \{1, 3, 4, 5, 7, 8\}$ on the line segment $\ell_4 \cap \Delta^7$ where $\ell_4 = \{\vec{\mathbf{y}} + t\vec{u}_4 : t \in \mathbf{R}\}$. Since f_6 is continuous on $\ell_4 \cap \Delta^7$ and $f_6(\mathbf{y}) < \alpha_*$, there exists an $\epsilon_4 > 0$ so that

$f_6(\mathbf{x}) < \alpha_*$ for every $\mathbf{x} \in \ell_4^{\epsilon_4} = \{\vec{\mathbf{y}} + t\vec{u}_4 : t \in (0, \epsilon_4)\}$. Let $\mathbf{z}$ be a point in $\ell_4^{\epsilon_4}$. Then we get $f_6(\mathbf{z}) < \alpha_*$. Since f_2 is decreasing on the line segment $\ell_4^{\epsilon_4} \subset \ell_4$ and $f_2(\mathbf{z}) = \alpha_*$, we see that $f_2(\mathbf{z}) < \alpha_*$. Thus if we move along the line segment starting at $\mathbf{y}$ in the direction of the vector $\vec{u}_4$ for a sufficiently small amount, we get a point $\mathbf{z}$ in Δ^7 so that the values of f_2 and f_6 at the point $\mathbf{z}$ are less than α_* and the values of $f_1, f_3, f_4, f_5, f_7, f_8$ at $\mathbf{z}$ are the same as at $\mathbf{y}$, i.e. $f_j(\mathbf{z}) = f_j(\mathbf{y})$ for every $j \in \{1, 3, 4, 5, 7, 8\}$. Hence we obtain a point $\mathbf{z}$ such that $f_j(\mathbf{z}) < \alpha_*$ for every $j \in I_1$ and $f_j(\mathbf{z}) = f_j(\mathbf{x}^*)$ for every $j \in \{1, 3, 4, 5, 8\}$. Let us call the perturbation performed in the way described above when $f_6(\mathbf{x}) < \alpha_*$, $f_7(\mathbf{x}) < \alpha_*$, $f_2(\mathbf{x}) = \alpha_*$ and $f_j(\mathbf{x}) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$ for some $\mathbf{x} \in \Delta^7$ $\text{pert}_{\vec{u}_4}^6$ and the point obtained applying the perturbation at $\mathbf{x}$ $\text{pert}_{\vec{u}_4}^6(\mathbf{x})$.

We will use the following notation to denote this process:

$$\begin{array}{c} [\mathbf{x} : f_2 = \alpha_*, f_6 < \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*] \\ \downarrow \text{pert}_{\vec{u}_4}^6 \\ [\text{pert}_{\vec{u}_4}^6(\mathbf{x}) : f_2 < \alpha_*, f_6 < \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*]. \end{array}$$

Therefore, we obtain $\mathbf{z} = \text{pert}_{\vec{u}_4}^6(\mathbf{y})$ or $\mathbf{z} = \text{pert}_{\vec{u}_4}^6(\text{pert}_{\vec{u}_3}^6(\mathbf{x}^*))$.

Let us assume that $f_7(\mathbf{x}^*) < \alpha_*$, $f_2(\mathbf{x}^*) = \alpha_*$, $f_6(\mathbf{x}^*) = \alpha_*$ and $f_j(\mathbf{x}^*) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$. Then it is possible to obtain a point $\mathbf{z} \in \Delta^7$ so that $f_k(\mathbf{z}) < \alpha_*$ for every $k \in I_1$ and $f_j(\mathbf{z}) = f_j(\mathbf{x}^*)$ for every $j \in \{1, 3, 4, 5, 8\}$ as follows:

If we calculate the directional derivatives of all the functions $f_1, f_2, \dots, f_8$ in the direction of the vector

$$\vec{u}_5 = -\vec{u}_1,$$

we see that

$$\begin{aligned} D_{\vec{u}_5} f_2(\mathbf{x}) &< 0 \\ D_{\vec{u}_5} f_j(\mathbf{x}) &= 0, \quad 1 \leq j \leq 8, \quad j \neq 2, 7 \\ D_{\vec{u}_5} f_7(\mathbf{x}) &> 0 \end{aligned}$$

for every $\mathbf{x} \in \Delta^7$ (see Appendix A.4). Then we conclude that f_7 is increasing, f_2 is decreasing and f_j is constant for every $j \in \{1, 2, 3, 4, 5, 8\}$ on the line segment $\ell_5 \cap \Delta^7$ where $\ell_5 = \{\vec{\mathbf{x}}^* + t\vec{u}_5 : t \in \mathbf{R}\}$. Since f_7 is continuous on $\ell_5 \cap \Delta^7$ and $f_7(\mathbf{x}^*) < \alpha_*$, there exists an $\epsilon_5 > 0$ so that $f_7(\mathbf{x}) < \alpha_*$ for every $\mathbf{x} \in \ell_5^{\epsilon_5} = \{\vec{\mathbf{x}}^* + t\vec{u}_5 : t \in (0, \epsilon_5)\}$. Let $\mathbf{y}$ be a point in $\ell_5^{\epsilon_5}$. Then we get $f_7(\mathbf{y}) < \alpha_*$. Since f_2 is decreasing on the line segment $\ell_5^{\epsilon_5} \subset \ell_5$ and $f_2(\mathbf{x}^*) = \alpha_*$, we see that $f_2(\mathbf{y}) < \alpha_*$. In other words, if we move along the line segment starting at $\mathbf{x}^*$ in the direction of the vector $\vec{u}_5$ for a sufficiently small amount, we get a point $\mathbf{y}$ in Δ^7 so that the values of the functions f_7 and f_2 at $\mathbf{y}$ are less than α_* and the values of the functions $f_1, f_3, f_4, f_5, f_6, f_8$ at $\mathbf{y}$ are the same as at $\mathbf{x}^*$, i.e., $f_j(\mathbf{y}) = f_j(\mathbf{x}^*)$ for every $j \in \{1, 3, 4, 5, 6, 8\}$. Let us call the perturbation performed in the way described above when $f_7(\mathbf{x}) < \alpha_*$, $f_2(\mathbf{x}) = \alpha_*$, $f_6(\mathbf{x}) = \alpha_*$ and $f_j(\mathbf{x}) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$ for some $\mathbf{x} \in \Delta^7$ $pert_{\vec{u}_5}^7$ and the point obtained using the point $\mathbf{x}$ $pert_{\vec{u}_5}^7(\mathbf{x})$.

We will use the following notation to denote this process:

$$\begin{aligned} &[\mathbf{x} : f_2 = \alpha_*, f_6 = \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*] \\ &\quad \downarrow pert_{\vec{u}_5}^7 \\ &[pert_{\vec{u}_5}^7(\mathbf{x}) : f_2 < \alpha_*, f_6 = \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*]. \end{aligned}$$

Hence, we have $\mathbf{y} = pert_{\vec{u}_5}^7(\mathbf{x}^*)$ in the perturbation above.

If we calculate the directional derivatives of all the functions $f_1, f_2, \dots, f_8$ in the direction of the vector

$$\vec{u}_6 = -\vec{u}_3,$$

we get

$$\begin{aligned} D_{\vec{u}_6} f_6(\mathbf{x}) &< 0 \\ D_{\vec{u}_6} f_j(\mathbf{x}) &= 0, \quad 1 \leq j \leq 8, \quad j \neq 6, 7 \\ D_{\vec{u}_6} f_7(\mathbf{x}) &> 0 \end{aligned}$$

for every $\mathbf{x} \in \Delta^7$, particularly for $\mathbf{y}$ (see Appendix A.4). Therefore, we conclude that f_7 is increasing, f_6 is decreasing and f_j is constant for every $j \in \{1, 2, 3, 4, 5, 8\}$ on the line segment $\ell_6 \cap \Delta^7$ where $\ell_6 = \{\vec{\mathbf{y}} + t\vec{u}_6 : t \in \mathbf{R}\}$. Since f_7 is continuous on $\ell_6 \cap \Delta^7$ and $f_7(\mathbf{y}) < \alpha_*$, there exists an $\epsilon_6 > 0$ so that $f_7(\mathbf{x}) < \alpha_*$ for every $\mathbf{x} \in \ell_6^{\epsilon_6} = \{\vec{\mathbf{y}} + t\vec{u}_6 : t \in (0, \epsilon_6)\}$. Let $\mathbf{z}$ be a point in $\ell_6^{\epsilon_6}$. Then we get $f_7(\mathbf{z}) < \alpha_*$. Since f_6 is decreasing on the line segment $\ell_6^{\epsilon_6} \subset \ell_6$ and $f_6(\mathbf{y}) = \alpha_*$, we see that $f_6(\mathbf{z}) < \alpha_*$. This means that if we move along the line segment starting at $\mathbf{y}$ in the direction of the vector $\vec{u}_6$ for a sufficiently small amount, we obtain a point $\mathbf{z}$ such that the values of f_6 and f_7 are less than α_* and the values of all the other functions $f_1, f_2, f_3, f_4, f_5, f_8$ are the same as at $\mathbf{y}$, i.e., $f_j(\mathbf{z}) = f_j(\mathbf{y})$ for every $j \in \{1, 2, 3, 4, 5, 8\}$. In other words, we obtain a point $\mathbf{z} \in \Delta^7$ such that $f_j(\mathbf{z}) < \alpha_*$ for every $j \in I_1$ and $f_j(\mathbf{z}) = f_j(\mathbf{x}^*)$ for every $j \in \{1, 3, 4, 5, 8\}$. Let us call the perturbation performed in the way described above when $f_7(\mathbf{x}) < \alpha_*$, $f_6(\mathbf{x}) = \alpha_*$, $f_2(\mathbf{x}) < \alpha_*$ and $f_j(\mathbf{x}) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$ for some $\mathbf{x} \in \Delta^7$ *per* $t_{\vec{u}_6}^7$ and the point obtained using $\mathbf{x}$ *per* $t_{\vec{u}_6}^7(\mathbf{x})$.

We will use the following notation to denote this process:

$$\begin{aligned} & [\mathbf{x} : f_2 < \alpha_*, f_6 = \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*] \\ & \quad \downarrow \text{pert}_{\bar{u}_6}^7 \\ & [\text{pert}_{\bar{u}_6}^7(\mathbf{x}) : f_2 < \alpha_*, f_6 < \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*]. \end{aligned}$$

Thus, we obtain $\mathbf{z} = \text{pert}_{\bar{u}_6}^7(\mathbf{y})$ or $\mathbf{z} = \text{pert}_{\bar{u}_6}^7(\text{pert}_{\bar{u}_5}^7(\mathbf{x}^*))$.

Let us assume that $f_2(\mathbf{x}^*) < \alpha_*$, $f_6(\mathbf{x}^*) < \alpha_*$, $f_7(\mathbf{x}^*) = \alpha_*$ and $f_j(\mathbf{x}^*) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$. Then it is possible to find a point $\mathbf{y} \in \Delta^7$ so that $f_k(\mathbf{y}) < \alpha_*$ for every $k \in I_1$ and $f_j(\mathbf{y}) = f_j(\mathbf{x}^*)$ for every $j \in \{1, 3, 4, 5, 8\}$ by applying the following perturbation

$$\begin{aligned} & [\mathbf{x}^* : f_2 < \alpha_*, f_6 < \alpha_*, f_7 = \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*] \\ & \quad \downarrow \text{pert}_{\bar{u}_2}^2 \\ & [\mathbf{y} : f_2 < \alpha_*, f_6 < \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*]. \end{aligned}$$

If we assume that $f_2(\mathbf{x}^*) < \alpha_*$, $f_6(\mathbf{x}^*) = \alpha_*$, $f_7(\mathbf{x}^*) < \alpha_*$ and $f_j(\mathbf{x}^*) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$, it is possible to find a point $\mathbf{y} \in \Delta^7$ so that $f_k(\mathbf{y}) < \alpha_*$ for every $k \in I_1$ and $f_j(\mathbf{y}) = f_j(\mathbf{x}^*)$ for every $j \in \{1, 3, 4, 5, 8\}$ by applying the following perturbation

$$\begin{aligned} & [\mathbf{x}^* : f_2 < \alpha_*, f_6 = \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*] \\ & \quad \downarrow \text{pert}_{\bar{u}_6}^7 \\ & [\mathbf{y} : f_2 < \alpha_*, f_6 < \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*]. \end{aligned}$$

Let us assume that $f_2(\mathbf{x}^*) = \alpha_*$, $f_6(\mathbf{x}^*) < \alpha_*$, $f_7(\mathbf{x}^*) < \alpha_*$ and $f_j(\mathbf{x}^*) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$. Then it is possible to a point $\mathbf{y} \in \Delta^7$ so that $f_k(\mathbf{y}) < \alpha_*$ for every $k \in I_1$ and $f_j(\mathbf{y}) = f_j(\mathbf{x}^*)$ for every $j \in \{1, 3, 4, 5, 8\}$ by applying the following perturbation

$$\begin{array}{c} [\mathbf{x}^* : f_2 = \alpha_*, f_6 < \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*] \\ \downarrow \text{pert}_{\bar{u}_4}^6 \\ [\mathbf{y} : f_2 < \alpha_*, f_6 < \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*]. \end{array}$$

Finally, let us assume that $f_2(\mathbf{x}^*) < \alpha_*$, $f_6(\mathbf{x}^*) < \alpha_*$, $f_7(\mathbf{x}^*) < \alpha_*$ and $f_j(\mathbf{x}^*) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$. If this is the case, we know that there exists two distinct indices $j_1, j_2 \in \{1, 3, 4, 5, 8\}$ so that f_{j_1} and f_{j_2} take the value α_* at the point $\mathbf{x}^*$.

Hence, we conclude that if one or two of the functions f_2, f_6, f_7 take a value less than α_* at the point $\mathbf{x}^*$, it is possible to find a point ($\mathbf{y}$ or $\mathbf{z}$) obtained by applying one or two perturbations chosen from the list

$$L_1 = \{\text{pert}_{\bar{u}_1}^2, \text{pert}_{\bar{u}_2}^2, \text{pert}_{\bar{u}_3}^6, \text{pert}_{\bar{u}_4}^6, \text{pert}_{\bar{u}_5}^7, \text{pert}_{\bar{u}_6}^7\}$$

so that all of the functions f_2, f_6, f_7 take a value less than α_* at that point ($\mathbf{y}$ or $\mathbf{z}$) and the values of the functions f_1, f_3, f_4, f_5, f_8 at that point are the same as at the point $\mathbf{x}^*$.

The relationship between the functions f_2, f_6 and f_7 described in this section implies the following: if we assume $f_j(\mathbf{x}^*) < \alpha_*$ for every $j \in \{1, 3, 4, 5, 8\}$, we see that $f_i(\mathbf{x}^*) = \alpha_*$ for every $i \in I_1$. Because, if we have $f_7(\mathbf{x}^*) = \alpha_*$, $f_6(\mathbf{x}^*) = \alpha_*$ but $f_j(\mathbf{x}^*) < \alpha_*$ for every $j \in \{1, 2, 3, 4, 5, 8\}$, then there exists a point $\mathbf{z} = \text{pert}_{\bar{u}_2}^2(\text{pert}_{\bar{u}_1}^2(\mathbf{x}^*)) \in \Delta^7$ so that $f_j(\mathbf{z}) < \alpha_*$ for every $j \in I_1$ and $f_j(\mathbf{z}) = f_j(\mathbf{x}^*) < \alpha_*$ for every $j \in \{1, 3, 4, 5, 8\}$. This contradicts with the minimality of α_* . Similarly, if we have the case

that $f_7(\mathbf{x}^*) = \alpha_*$, $f_2(\mathbf{x}^*) = \alpha_*$ but $f_j(\mathbf{x}^*) < \alpha_*$ for every $j \in \{1, 3, 4, 5, 6, 8\}$, then there exists a point $\mathbf{z} = \text{pert}_{\tilde{u}_4}^6(\text{pert}_{\tilde{u}_3}^6(\mathbf{x}^*)) \in \Delta^7$ so that $f_j(\mathbf{z}) < \alpha_*$ for every $j \in I_1$ and $f_j(\mathbf{z}) = f_j(\mathbf{x}^*) < \alpha_*$ for every $j \in \{1, 3, 4, 5, 8\}$. This is a contradiction. If we have $f_6(\mathbf{x}^*) = \alpha_*$, $f_2(\mathbf{x}^*) = \alpha_*$ but $f_j(\mathbf{x}^*) < \alpha_*$ for every $j \in \{1, 3, 4, 5, 7, 8\}$, then there exists a point $\mathbf{z} = \text{pert}_{\tilde{u}_6}^7(\text{pert}_{\tilde{u}_5}^7(\mathbf{x}^*)) \in \Delta^7$ so that $f_j(\mathbf{z}) < \alpha_*$ for every $j \in I_1$ and $f_j(\mathbf{z}) = f_j(\mathbf{x}^*) < \alpha_*$ for every $j \in \{1, 3, 4, 5, 8\}$, a contradiction. Therefore, the following case

- I. $f_j(\mathbf{x}^*) < \alpha_*$ for $j = 1, 3, 4, 5, 8$ and $f_j(\mathbf{x}^*) = \alpha_*$ for $j = 2, 6, 7$.

requires further investigation. We will denote the set $\{f_2, f_6, f_7\}$ with $\{f_{I_1}\}$.

4.1.2 The relationship between the functions f_3 and f_4

We have the following relationship between f_3 and f_4 . We use the parametrization $\mathfrak{P}_7$ to calculate the partial derivatives of $f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8$ (see Appendix A.2).

Let us assume that $f_3(\mathbf{x}^*) < \alpha_*$, $f_4(\mathbf{x}^*) = \alpha_*$ and $f_j(\mathbf{x}^*) \leq \alpha_*$ for $j = 1, 2, 5, 6, 7, 8$. If we say $I_2 = \{3, 4\}$, then we show that there exists a point $\mathbf{y} \in \Delta^7$ so that $f_k(\mathbf{y}) < \alpha_*$ for every $k \in I_2$ and $f_j(\mathbf{y}) = f_j(\mathbf{x}^*)$ for every $j \in \{1, 2, 5, 6, 7, 8\}$ as follows: If we calculate the directional derivatives of all the functions $f_1, f_2, \dots, f_8$ in the direction of the vector

$$\tilde{u}_7 = \langle 0, 0, 0, 0, 0, 0, -1 \rangle,$$

we get

$$D_{\tilde{u}_7} f_3(\mathbf{x}) > 0$$

$$D_{\tilde{u}_7} f_j(\mathbf{x}) = 0, \quad 1 \leq j \leq 8, \quad j \neq 3, 4$$

$$D_{\tilde{u}_7} f_4(\mathbf{x}) < 0$$

for every $\mathbf{x} \in \Delta^7$. Then we conclude that f_3 is increasing, f_4 is decreasing and f_j is constant for every $j \in \{1, 3, 4, 5, 6, 8\}$ on the line segment $\ell_7 \cap \Delta^7$ where $\ell_7 = \{\bar{\mathbf{x}}^* + t\vec{u}_7 : t \in \mathbf{R}\}$. Since f_3 is continuous on $\ell_7 \cap \Delta^7$ and $f_3(\mathbf{x}^*) < \alpha_*$, there exists an $\epsilon_7 > 0$ so that $f_3(\mathbf{x}) < \alpha_*$ for every $\mathbf{x} \in \ell_7^{\epsilon_7} = \{\bar{\mathbf{x}}^* + t\vec{u}_7 : t \in (0, \epsilon_7)\}$. Let $\mathbf{y}$ be a point in $\ell_7^{\epsilon_7}$. Then we get $f_3(\mathbf{y}) < \alpha_*$. Since f_4 is decreasing on the line segment $\ell_7^{\epsilon_7} \subset \ell_7$ and $f_4(\mathbf{x}^*) = \alpha_*$, we see that $f_4(\mathbf{y}) < \alpha_*$. Therefore if we move along the line segment $\ell_7^{\epsilon_7}$ starting at $\mathbf{x}^*$ in the direction of the vector $\vec{u}_7$ for a sufficiently small amount, we obtain a point $\mathbf{y}$ in Δ^7 so that the value of f_3 and f_4 at $\mathbf{y}$ are less than α_* and the values of all the other functions $f_1, f_2, f_5, f_6, f_7, f_8$ at $\mathbf{y}$ are the same as at $\mathbf{x}^*$. Hence, we get a point $\mathbf{y}$ such that $f_3(\mathbf{x}_1) < \alpha_*$, $f_4(\mathbf{x}_1) < \alpha_*$ and $f_j(\mathbf{y}) = f_j(\mathbf{x}^*)$ for $i = 1, 2, 5, 6, 7, 8$. Let us call the perturbation performed in the way described above when $f_3(\mathbf{x}) < \alpha_*$, $f_4(\mathbf{x}) = \alpha_*$ and $f_j(\mathbf{x}) \leq \alpha_*$ for $j \in \{1, 2, 5, 6, 7, 8\}$ for some $\mathbf{x} \in \Delta^7$ $per t_{\vec{u}_7}^3$ and the point obtained using $\mathbf{x}$ $per t_{\vec{u}_7}^3(\mathbf{x})$.

We will use the following notation to denote this process:

$$\begin{array}{c} [\mathbf{x} : f_3 < \alpha_*, f_4 = \alpha_*, f_{j \notin I_2} \leq \alpha_*; F = \alpha_*] \\ \downarrow per t_{\vec{u}_7}^3 \\ [per t_{\vec{u}_7}^3(\mathbf{x}) : f_2 < \alpha_*, f_4 < \alpha_*, f_{j \notin I_2} \leq \alpha_*; F = \alpha_*]. \end{array}$$

Thus, we obtain $\mathbf{y} = per t_{\vec{u}_7}^3(\mathbf{x}^*)$ in the process above.

If we start with the assumption that $f_4(\mathbf{x}^*) < \alpha_*$, $f_3(\mathbf{x}^*) = \alpha_*$ and $f_j(\mathbf{x}^*) \leq \alpha_*$ for $j \in \{1, 2, 5, 6, 7, 8\}$ and calculate the directional derivatives of all the functions $f_1, f_2, \dots, f_8$ in the direction of the vector

$$\vec{u}_8 = -\vec{u}_7,$$

we see that

$$\begin{aligned} D_{\vec{u}_8} f_3(\mathbf{x}) &< 0 \\ D_{\vec{u}_8} f_j(\mathbf{x}) &= 0, \quad 1 \leq j \leq 8, \quad j \neq 3, 4 \\ D_{\vec{u}_8} f_4(\mathbf{x}) &> 0 \end{aligned}$$

for every $\mathbf{x} \in \Delta^7$. We conclude that f_4 is increasing, f_3 is decreasing and f_j is constant for every $j \in \{1, 3, 4, 5, 6, 8\}$ on the line segment $\ell_8 \cap \Delta^7$ where $\ell_8 = \{\vec{\mathbf{x}}^* + t\vec{u}_8 : t \in \mathbf{R}\}$. Since f_4 is continuous on $\ell_7 \cap \Delta^7$ and $f_4(\mathbf{x}^*) < \alpha_*$, there exists an $\epsilon_8 > 0$ so that $f_4(\mathbf{x}) < \alpha_*$ for every $\mathbf{x} \in \ell_8^{\epsilon_8} = \{\vec{\mathbf{x}}^* + t\vec{u}_8 : t \in (0, \epsilon_8)\}$. Let $\mathbf{y}$ be a point in $\ell_8^{\epsilon_8}$. Then we get $f_4(\mathbf{y}) < \alpha_*$. Since f_3 is decreasing on the line segment $\ell_8^{\epsilon_8} \subset \ell_8$ and $f_3(\mathbf{x}^*) = \alpha_*$, we see that $f_3(\mathbf{y}) < \alpha_*$. Hence, if we move along the line segment $\ell_8^{\epsilon_8}$ starting at $\mathbf{x}^*$ in the direction of the vector $\vec{u}_8$ for a sufficiently small amount, we obtain a point $\mathbf{y}$ in Δ^7 so that the value of f_3 and f_4 at $\mathbf{y}$ are less than α_* and the values of all the other functions $f_1, f_2, f_5, f_6, f_7, f_8$ at $\mathbf{y}$ are the same as at $\mathbf{x}^*$. Hence, we get a point $\mathbf{y}$ such that $f_3(\mathbf{x}) < \alpha_*$, $f_4(\mathbf{x}) < \alpha_*$ and $f_j(\mathbf{y}) = f_j(\mathbf{x}^*)$ for $i = 1, 2, 5, 6, 7, 8$. Let us call the perturbation performed in the way described above when $f_3(\mathbf{x}) = \alpha_*$, $f_4(\mathbf{x}) < \alpha_*$ and $f_j(\mathbf{x}) \leq \alpha_*$ for every $j \in \{1, 2, 5, 6, 7, 8\}$ for some $\mathbf{x} \in \Delta^7$ $pert_{\vec{u}_8}^4$ and the point obtained using $\mathbf{x}$ $pert_{\vec{u}_8}^4(\mathbf{x})$.

We will use the following notation to denote this process:

$$\begin{aligned} &[\mathbf{x} : f_3 = \alpha_*, f_4 < \alpha_*, f_{j \notin I_2} \leq \alpha_*; F = \alpha_*] \\ &\quad \downarrow pert_{\vec{u}_8}^4 \\ &[pert_{\vec{u}_8}^4(\mathbf{x}) : f_3 < \alpha_*, f_4 < \alpha_*, f_{j \notin I_2} \leq \alpha_*; F = \alpha_*]. \end{aligned}$$

Therefore, we obtain $\mathbf{y} = pert_{\vec{u}_8}^4(\mathbf{x}^*)$ in the process above.

Finally, let us assume that $f_3(\mathbf{x}^*) < \alpha_*$, $f_4(\mathbf{x}^*) < \alpha_*$ and $f_j(\mathbf{x}^*) \leq \alpha_*$ for $j \in \{1, 2, 5, 6, 7, 8\}$. If this is the case, we know that there exists two distinct indices $j_1, j_2 \in \{1, 2, 5, 6, 7, 8\}$ so that f_{j_1} and f_{j_2} take the value α_* at the point $\mathbf{x}^*$.

Hence we conclude that if one of the functions f_3 or f_4 take a value less than α_* at the point $\mathbf{x}^*$, it is possible to find a point ($\mathbf{y}$) obtained by applying a perturbations chosen from the list

$$L_2 = \{pert_{u_7}^3, pert_{u_8}^4\}$$

so that both of the functions f_3 and f_4 take a value less than α_* at that point ($\mathbf{y}$) and the values of the functions $f_1, f_2, f_5, f_6, f_7, f_8$ at that point are the same as at the point $\mathbf{x}^*$.

The relationship between the functions f_2 , f_6 and f_7 described in this section implies the following: if we assume that $f_j(\mathbf{x}^*) < \alpha_*$ for $j = 1, 2, 5, 6, 7, 8$, we see that $f_3(\mathbf{x}^*) = \alpha_*$ and $f_4(\mathbf{x}^*) = \alpha_*$. Because if we have $f_4(\mathbf{x}^*) = \alpha_*$ and $f_j(\mathbf{x}^*) < \alpha_*$ for every $j \in \{1, 2, 3, 5, 6, 7, 8\}$, then there exists a point $\mathbf{y} = pert_{u_7}^3(\mathbf{x}^*) \in \Delta^7$ so that $f_3(\mathbf{y}) < \alpha_*$, $f_4(\mathbf{y}) < \alpha_*$ and $f_j(\mathbf{y}) = f_j(\mathbf{x}^*) < \alpha_*$ for every $j \in \{1, 2, 5, 6, 7, 8\}$. This contradicts with the minimality of α_* . Similarly, if we have $f_3(\mathbf{x}^*) = \alpha_*$ but $f_j(\mathbf{x}^*) < \alpha_*$ for every $j \in \{1, 2, 4, 5, 6, 7, 8\}$, then there exists a point $\mathbf{y} = pert_{u_8}^4(\mathbf{x}^*) \in \Delta^7$ so that $f_3(\mathbf{y}) < \alpha_*$, $f_4(\mathbf{y}) < \alpha_*$ and $f_j(\mathbf{y}) = f_j(\mathbf{x}^*) < \alpha_*$ for every $j \in \{1, 2, 5, 6, 7, 8\}$, a contradiction. Therefore, the following case

II. $f_j(\mathbf{x}^*) < \alpha_*$ for $j = 1, 2, 5, 6, 7, 8$ and $f_j(\mathbf{x}^*) = \alpha_*$ for $j = 3, 4$.

needs further investigation. We will denote the set $\{f_3, f_4\}$ with $\{f_{I_2}\}$. Note that case II can be obtained directly from the fact that there exists two functions f_{j_1}, f_{j_2} such that $f_{j_1}(\mathbf{x}^*) = f_{j_2}(\mathbf{x}^*) = \alpha_*$ where $j_1, j_2 \in$

$\{1, 2, \dots, 8\}$ and $j_1 \neq j_2$. If we assume that $f_j(\mathbf{x}^*) < \alpha_*$ for $j = 1, 2, 5, 6, 7, 8$, then it is clear that $j_1 = 3$ and $j_2 = 4$.

4.1.3 The relationship between the functions f_1, f_5 and f_8

The functions f_1, f_5 and f_8 are also related in a similar manner to f_2, f_6 and f_7 . Let us assume that we have $f_1(\mathbf{x}^*) < \alpha_*$, $f_5(\mathbf{x}^*) = \alpha_*$ and $f_8(\mathbf{x}^*) = \alpha_*$ and $f_j(\mathbf{x}^*) \leq \alpha_*$ for $j = 2, 3, 4, 6, 7$. If we say $I_3 = \{1, 5, 8\}$, then we show that there exists a point $\mathbf{z} \in \Delta^7$ so that $f_k(\mathbf{z}) < \alpha_*$ for every $k \in I_3$ and $f_j(\mathbf{z}) = f_j(\mathbf{x}^*)$ for every $j = 2, 3, 4, 6, 7$.

If we use the parametrization $\mathfrak{P}_4$ and calculate the directional derivatives of the functions $f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8$ in the direction of the vector

$$\vec{u}_9 = \langle 0, 0, 0, -1, 0, 0, 0 \rangle,$$

we see that

$$D_{\vec{u}_9} f_1(\mathbf{x}) > 0$$

$$D_{\vec{u}_9} f_j(\mathbf{x}) = 0, \quad 2 \leq j \leq 7$$

$$D_{\vec{u}_9} f_8(\mathbf{x}) < 0$$

for every $\mathbf{x} \in \Delta^7$ (see Appendix A.3). Then we conclude that f_1 is increasing, f_8 is decreasing and f_j is constant for every $j \in \{2, 3, 4, 5, 6, 7\}$ on the line segment $\ell_9 \cap \Delta^7$ where $\ell_9 = \{\vec{\mathbf{x}}^* + t\vec{u}_9 : t \in \mathbf{R}\}$. Since f_1 is continuous on $\ell_9 \cap \Delta^7$ and $f_1(\mathbf{x}^*) < \alpha_*$, there exists an $\epsilon_9 > 0$ so that $f_1(\mathbf{x}) < \alpha_*$ for every $\mathbf{x} \in \ell_9^{\epsilon_9} = \{\vec{\mathbf{x}}^* + t\vec{u}_9 : t \in (0, \epsilon_9)\}$. Let $\mathbf{y}$ be a point in $\ell_9^{\epsilon_9}$. Then we get $f_1(\mathbf{y}) < \alpha_*$. Since f_8 is decreasing on the line segment $\ell_9^{\epsilon_9} \subset \ell_9$ and $f_8(\mathbf{x}^*) = \alpha_*$, we see that $f_8(\mathbf{y}) < \alpha_*$. Therefore, if we move in the direction

of the vector $\vec{u}_9$ for a sufficiently small amount, we obtain a point $\mathbf{y} \in \Delta^7$ so that the value of f_1 at $\mathbf{y}$ is still less than α_* , the values of $f_2, f_3, f_4, f_5, f_6, f_7$ at $\mathbf{y}$ are the same as at $\mathbf{x}^*$, i.e., $f_j(\mathbf{y}) = f_j(\mathbf{x}^*)$ for every $j \in \{2, 3, 4, 5, 6, 7\}$, and the value of f_8 at $\mathbf{y}$ is less than α_* . Let us call the perturbation performed in the way described above when $f_1(\mathbf{x}) < \alpha_*$, $f_5(\mathbf{x}) = \alpha_*$, $f_8(\mathbf{x}) = \alpha_*$ and $f_j(\mathbf{x}) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$ for some $\mathbf{x} \in \Delta^7$ $pert_{\vec{u}_9}^1$ and the point obtained by applying the perturbation $pert_{\vec{u}_9}^1$ using the point $\mathbf{x}$ $pert_{\vec{u}_9}^1(\mathbf{x})$. We will use the following notation to denote this process:

$$\begin{array}{c} [\mathbf{x} : f_1 < \alpha_*, f_5 = \alpha_*, f_8 = \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*] \\ \downarrow pert_{\vec{u}_9}^1 \\ [pert_{\vec{u}_9}^1(\mathbf{x}) : f_1 < \alpha_*, f_5 = \alpha_*, f_8 < \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*]. \end{array}$$

In the case above, we have $\mathbf{y} = pert_{\vec{u}_9}^1(\mathbf{x}^*)$.

If we take the directional derivative in the direction of the vector

$$\vec{u}_{10} = \langle 0, 0, 0, 0, 1, 0, 0 \rangle,$$

we get that

$$D_{\vec{u}_{10}} f_1(\mathbf{x}) > 0$$

$$D_{\vec{u}_{10}} f_j(\mathbf{x}) = 0, \quad 2 \leq j \leq 8, j \neq 5$$

$$D_{\vec{u}_{10}} f_5(\mathbf{x}) < 0$$

for every $\mathbf{x} \in \Delta^7$, particularly for $\mathbf{y}$. Then we conclude that f_1 is increasing, f_5 is decreasing and f_j is constant for every $j \in \{2, 3, 4, 6, 7, 8\}$ on the line segment $\ell_{10} \cap \Delta^7$ where $\ell_{10} = \{\vec{\mathbf{y}} + t\vec{u}_{10} : t \in \mathbf{R}\}$. Since f_1 is continuous on $\ell_{10} \cap \Delta^7$ and $f_1(\mathbf{y}) < \alpha_*$, there exists an $\epsilon_{10} > 0$ so that $f_1(\mathbf{x}) < \alpha_*$ for every

$\mathbf{x} \in \ell_{10}^{\epsilon_{10}} = \{\tilde{\mathbf{y}} + t\vec{u}_{10} : t \in (0, \epsilon_{10})\}$. Let $\mathbf{z}$ be a point in $\ell_{10}^{\epsilon_{10}}$. Then we get $f_1(\mathbf{z}) < \alpha_*$. Since f_5 is decreasing on the line segment $\ell_{10}^{\epsilon_{10}} \subset \ell_{10}$ and $f_5(\mathbf{y}) = \alpha_*$, we see that $f_5(\mathbf{z}) < \alpha_*$. Thus, if we move in the direction of the vector $\vec{u}_{10}$ for a sufficiently small amount, we obtain a point $\mathbf{z} \in \Delta^7$ so that the value of f_5 at $\mathbf{z}$ is less than α_* , the values of $f_2, f_3, f_4, f_6, f_7, f_8$ at $\mathbf{z}$ are the same as at $\mathbf{y}$, i.e., $f_j(\mathbf{z}) = f_j(\mathbf{y})$ for every $j \in \{2, 3, 4, 6, 7, 8\}$, and the value of f_1 at $\mathbf{z}$ stays less than α_* . Hence, we obtain a point $\mathbf{z}$ so that $f_1(\mathbf{z}) < \alpha_*$, $f_5(\mathbf{z}) < \alpha_*$ and $f_8(\mathbf{z}) < \alpha_*$ and $f_j(\mathbf{z}) = f_j(\mathbf{x}^*)$ for every $j \in \{2, 3, 4, 6, 7\}$. Let us call the perturbation performed in the way described above when $f_1(\mathbf{x}) < \alpha_*$, $f_5(\mathbf{x}) = \alpha_*$, $f_8(\mathbf{x}) = \alpha_*$ and $f_j(\mathbf{x}) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$ for some $\mathbf{x} \in \Delta^7 \text{ pert}_{\vec{u}_{10}}^1$ and the point obtained by applying the perturbation $\text{pert}_{\vec{u}_{10}}^1$ using the point $\mathbf{x} \text{ pert}_{\vec{u}_{10}}^1(\mathbf{x})$. We will use the following notation to denote this process:

$$\begin{aligned} & [\mathbf{x} : f_1 < \alpha_*, f_5 = \alpha_*, f_8 < \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*] \\ & \quad \downarrow \text{pert}_{\vec{u}_{10}}^1 \\ & [\text{pert}_{\vec{u}_{10}}^1(\mathbf{x}) : f_1 < \alpha_*, f_5 < \alpha_*, f_8 < \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*]. \end{aligned}$$

In the case above, we have $\mathbf{z} = \text{pert}_{\vec{u}_{10}}^1(\mathbf{y})$ or $\mathbf{z} = \text{pert}_{\vec{u}_{10}}^1(\text{pert}_{\vec{u}_9}^1(\mathbf{x}^*))$.

We can follow similar steps if we start with the assumption that $f_5(\mathbf{x}^*) < \alpha_*$, $f_1(\mathbf{x}^*) = \alpha_*$ and $f_8(\mathbf{x}^*) = \alpha_*$ and $f_i(\mathbf{x}^*) \leq \alpha_*$ for $i = 2, 3, 4, 6, 7$. Specifically, we show that there exists a point $\mathbf{z} \in \Delta^7$ so that $f_k(\mathbf{z}) < \alpha_*$ for every $k \in I_3$ and $f_j(\mathbf{z}) = f_j(\mathbf{x}^*)$ for every $j = 2, 3, 4, 6, 7$.

If we calculate the directional derivative of the functions $f_1, f_2, \dots, f_8$ in the direction of the vector

$$\vec{u}_{11} = \langle 0, 0, 0, 1, -1, 0, 0 \rangle,$$

we obtain that

$$D_{\vec{u}_{11}} f_5(\mathbf{x}) > 0$$

$$D_{\vec{u}_{11}} f_j(\mathbf{x}) = 0, \quad 1 \leq j \leq 7, j \neq 5$$

$$D_{\vec{u}_{11}} f_8(\mathbf{x}) < 0$$

for every $\mathbf{x} \in \Delta^7$, particularly for $\mathbf{y}$. Therefore, we see that f_5 is increasing, f_8 is decreasing and f_j is constant for every $j \in \{1, 2, 3, 4, 6, 7\}$ on the line segment $\ell_{11} \cap \Delta^7$ where $\ell_{11} = \{\vec{\mathbf{y}} + t\vec{u}_{11} : t \in \mathbf{R}\}$. Since f_5 is continuous on $\ell_{11} \cap \Delta^7$ and $f_5(\mathbf{y}) < \alpha_*$, there exists an $\epsilon_{11} > 0$ so that $f_5(\mathbf{x}) < \alpha_*$ for every $\mathbf{x} \in \ell_{11}^{\epsilon_{11}} = \{\vec{\mathbf{y}} + t\vec{u}_{11} : t \in (0, \epsilon_{11})\}$. Let $\mathbf{z}$ be a point in $\ell_{11}^{\epsilon_{11}}$. Then we get $f_5(\mathbf{z}) < \alpha_*$. Since f_8 is decreasing on the line segment $\ell_{11}^{\epsilon_{11}} \subset \ell_{11}$ and $f_8(\mathbf{y}) = \alpha_*$, we see that $f_8(\mathbf{z}) < \alpha_*$. This means that if we move sufficiently small amount in the direction of the vector $\vec{u}_{12}$, we obtain a point $\mathbf{z} \in \Delta^7$ so that the value of f_5 at $\mathbf{z}$ stays less than α_* and the values of $f_1, f_2, f_3, f_4, f_6, f_7$ at $\mathbf{z}$ are the same as at $\mathbf{y}$, i.e., $f_j(\mathbf{y}) = f_j(\mathbf{x}^*)$ for every $j \in \{1, 2, 3, 4, 6, 7\}$. But the value of f_8 at $\mathbf{z}$ becomes less than α_* . Therefore we obtain a point $\mathbf{z}$ such that $f_8(\mathbf{z}) < \alpha_*$, $f_1(\mathbf{z}) < \alpha_*$, $f_5(\mathbf{z}) < \alpha_*$ and $f_j(\mathbf{z}) = f_j(\mathbf{x}^*)$ for every $j \in \{2, 3, 4, 6, 7\}$. Let us call the perturbation performed in the way described above when $f_5(\mathbf{x}) < \alpha_*$, $f_1(\mathbf{x}) < \alpha_*$, $f_8(\mathbf{x}) = \alpha_*$ and $f_j(\mathbf{x}) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$ for some $\mathbf{x} \in \Delta^7$ $per t_{\vec{u}_{11}}^5$ and the point obtained by applying the perturbation $per t_{\vec{u}_{11}}^5$ using the point $\mathbf{x}$ $per t_{\vec{u}_{11}}^5(\mathbf{x})$. We will use the following notation to denote this process

$$\begin{array}{c} [\mathbf{x} : f_1 = \alpha_*, f_5 < \alpha_*, f_8 = \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*] \\ \downarrow per t_{\vec{u}_{11}}^5 \\ [per t_{\vec{u}_{11}}^5(\mathbf{x}) : f_1 = \alpha_*, f_5 < \alpha_*, f_8 < \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*]. \end{array}$$

In the case above, we have $\mathbf{y} = per t_{\vec{u}_{11}}^5(\mathbf{x}^*)$.

We calculate the directional derivatives of the functions $f_1, f_2, \dots, f_8$ in the direction of the vector

$$\vec{u}_{12} = -\vec{u}_{10}.$$

We see that

$$D_{\vec{u}_{12}} f_1(\mathbf{x}) < 0$$

$$D_{\vec{u}_{12}} f_j(\mathbf{x}) = 0, \quad 2 \leq j \leq 8, j \neq 5$$

$$D_{\vec{u}_{12}} f_5(\mathbf{x}) > 0$$

for every $\mathbf{x} \in \Delta^7$. Then we conclude that f_5 is increasing, f_1 is decreasing and f_j is constant for every $j \in \{2, 3, 4, 6, 7, 8\}$ on the line segment $\ell_{12} \cap \Delta^7$ where $\ell_{12} = \{\vec{\mathbf{x}}^* + t\vec{u}_{12} : t \in \mathbf{R}\}$. Since f_5 is continuous on $\ell_{12} \cap \Delta^7$ and $f_5(\mathbf{x}^*) < \alpha_*$, there exists an $\epsilon_{12} > 0$ so that $f_5(\mathbf{x}) < \alpha_*$ for every $\mathbf{x} \in \ell_{12}^{\epsilon_{12}} = \{\vec{\mathbf{x}}^* + t\vec{u}_{12} : t \in (0, \epsilon_{12})\}$. Let $\mathbf{y}$ be a point in $\ell_{12}^{\epsilon_{12}}$. Then we get $f_5(\mathbf{y}) < \alpha_*$. Since f_1 is decreasing on the line segment $\ell_{12}^{\epsilon_{12}} \subset \ell_{12}$ and $f_1(\mathbf{x}^*) = \alpha_*$, we see that $f_1(\mathbf{y}) < \alpha_*$. Therefore, if we move sufficiently small amount in the direction of the vector $\vec{u}_{11}$, we obtain a point $\mathbf{y} \in \Delta^7$ so that the value of f_5 at $\mathbf{y}$ is less than α_* , the values of $f_2, f_3, f_4, f_6, f_7, f_8$ at $\mathbf{y}$ are the same as at $\mathbf{x}^*$, i.e., $f_j(\mathbf{y}) = f_j(\mathbf{x}^*)$ for every $j \in \{2, 3, 4, 6, 7, 8\}$, and the value of f_1 at $\mathbf{y}$ is less than α_* .

Let us call the perturbation performed in the way described above when $f_5(\mathbf{x}) < \alpha_*$, $f_1(\mathbf{x}) = \alpha_*$, $f_8(\mathbf{x}) = \alpha_*$ and $f_j(\mathbf{x}) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$ for some $\mathbf{x} \in \Delta^7$ $pert_{\vec{u}_{12}}^5$ and the point obtained by applying the perturbation $pert_{\vec{u}_{12}}^5$ using the point $\mathbf{x}$ $pert_{\vec{u}_{12}}^5(\mathbf{x})$. In the case above, we have $\mathbf{z} = pert_{\vec{u}_{12}}^5(\mathbf{y})$ or $\mathbf{z} = pert_{\vec{u}_{12}}^5(pert_{\vec{u}_{11}}^5(\mathbf{x}^*))$.

We will use the following notation to denote this process

$$\begin{aligned} & [\mathbf{x} : f_1 = \alpha_*, f_5 < \alpha_*, f_8 < \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*] \\ & \quad \downarrow \text{pert}_{\vec{u}_{12}}^5 \\ & [\text{pert}_{\vec{u}_{12}}^5(\mathbf{x}) : f_1 < \alpha_*, f_5 < \alpha_*, f_8 < \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*]. \end{aligned}$$

Let us assume that $f_8(\mathbf{x}^*) < \alpha_*$, $f_1(\mathbf{x}^*) = \alpha_*$, $f_5(\mathbf{x}^*) = \alpha_*$ and $f_i(\mathbf{x}^*) \leq \alpha_*$ for $i = 2, 3, 4, 6, 7$. Then we calculate the directional derivatives of all functions $f_1, f_2, \dots, f_8$ in the direction of the vector

$$\vec{u}_{13} = -\vec{u}_9,$$

we see that

$$D_{\vec{u}_{13}} f_1(\mathbf{x}) < 0$$

$$D_{\vec{u}_{13}} f_j(\mathbf{x}) = 0, \quad 2 \leq j \leq 7$$

$$D_{\vec{u}_{13}} f_8(\mathbf{x}) > 0$$

for every $\mathbf{x} \in \Delta^7$. Hence, we know that f_8 is increasing, f_1 is decreasing and f_j is constant for every $j \in \{2, 3, 4, 5, 6, 7\}$ on the line segment $\ell_{13} \cap \Delta^7$ where $\ell_{13} = \{\vec{\mathbf{x}}^* + t\vec{u}_{13} : t \in \mathbf{R}\}$. Since f_8 is continuous on $\ell_{13} \cap \Delta^7$ and $f_8(\mathbf{x}^*) < \alpha_*$, there exists an $\epsilon_{13} > 0$ so that $f_8(\mathbf{x}) < \alpha_*$ for every $\mathbf{x} \in \ell_{13}^{\epsilon_{13}} = \{\vec{\mathbf{x}}^* + t\vec{u}_{13} : t \in (0, \epsilon_{13})\}$. Let $\mathbf{y}$ be a point in $\ell_{13}^{\epsilon_{13}}$. Then we get $f_8(\mathbf{y}) < \alpha_*$. Since f_1 is decreasing on the line segment $\ell_{13}^{\epsilon_{13}} \subset \ell_{13}$ and $f_1(\mathbf{x}^*) = \alpha_*$, we see that $f_1(\mathbf{y}) < \alpha_*$. This means that if we move sufficiently small amount in the direction of the vector $\vec{u}_{13}$, we obtain a point $\mathbf{y} \in \Delta^7$ so that the value of f_1 at $\mathbf{y}$ is less than α_* , the values of $f_2, f_3, f_4, f_5, f_6, f_7$ at $\mathbf{y}$ are the same as at $\mathbf{x}^*$, i.e., $f_j(\mathbf{y}) = f_j(\mathbf{x}^*)$ for every $j \in \{2, 3, 4, 5, 6, 7\}$ and the value of f_8 at $\mathbf{y}$ stays less than α_* . Let us call the perturbation performed in

the way described above when $f_1(\mathbf{x}) = \alpha_*$, $f_5(\mathbf{x}) = \alpha_*$, $f_8(\mathbf{x}) < \alpha_*$ and $f_j(\mathbf{x}) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$ for some $\mathbf{x} \in \Delta^7$ $per t_{\vec{u}_{13}}^8$ and the point obtained by applying the perturbation $per t_{\vec{u}_{13}}^8$ using the point $\mathbf{x}$ $per t_{\vec{u}_{13}}^8(\mathbf{x})$.

We will use the following notation to denote this process

$$\begin{array}{c} [\mathbf{x} : f_1 = \alpha_*, f_5 = \alpha_*, f_8 < \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*] \\ \downarrow per t_{\vec{u}_{13}}^8 \\ [per t_{\vec{u}_{13}}^8(\mathbf{x}) : f_1 < \alpha_*, f_5 = \alpha_*, f_8 = \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*]. \end{array}$$

In the case above, we have $\mathbf{y} = per t_{\vec{u}_{13}}^8(\mathbf{x}^*)$.

If we calculate the directional derivatives of all the functions in the direction of the vector

$$\vec{u}_{14} = -\vec{u}_{11}.$$

We see that

$$\begin{aligned} D_{\vec{u}_{14}} f_5(\mathbf{x}) &< 0 \\ D_{\vec{u}_{14}} f_j(\mathbf{x}) &= 0, \quad 1 \leq j \leq 7, j \neq 5 \\ D_{\vec{u}_{14}} f_8(\mathbf{x}) &> 0 \end{aligned}$$

for every $\mathbf{x} \in \Delta^7$. Then we conclude that f_8 is increasing, f_5 is decreasing and f_j is constant for every $j \in \{1, 2, 3, 4, 6, 7\}$ on the line segment $\ell_{14} \cap \Delta^7$ where $\ell_{14} = \{\vec{\mathbf{y}} + t\vec{u}_{14} : t \in \mathbf{R}\}$. Since f_8 is continuous on $\ell_{14} \cap \Delta^7$ and $f_8(\mathbf{y}) < \alpha_*$, there exists an $\epsilon_{14} > 0$ so that $f_8(\mathbf{x}) < \alpha_*$ for every $\mathbf{x} \in \ell_{14}^{\epsilon_{14}} = \{\vec{\mathbf{y}} + t\vec{u}_{14} : t \in (0, \epsilon_{14})\}$. Let $\mathbf{z}$ be a point in $\ell_{14}^{\epsilon_{14}}$. Then we get $f_8(\mathbf{z}) < \alpha_*$. Since f_5 is decreasing on the line segment $\ell_{14}^{\epsilon_{14}} \subset \ell_{14}$ and $f_5(\mathbf{y}) = \alpha_*$, we see that $f_5(\mathbf{z}) < \alpha_*$. In other words, if we move in the direction of the

vector $\vec{u}_{14}$ for sufficiently small amount, we obtain a point $\mathbf{z} \in \Delta^7$ so that the value of f_5 at $\mathbf{z}$ is less than α_* , the values of $f_1, f_2, f_3, f_4, f_6, f_7$ at $\mathbf{z}$ are the same as at $\mathbf{y}$, i.e., $f_j(\mathbf{z}) = f_j(\mathbf{y})$ for every $j \in \{1, 2, 3, 4, 6, 7\}$, and the value of f_8 at $\mathbf{z}$ stays less than α_* . Therefore we obtain a point $\mathbf{z}$ such that $f_8(\mathbf{z}) < \alpha_*$, $f_1(\mathbf{z}) < \alpha_*$, $f_5(\mathbf{z}) < \alpha_*$ and $f_j(\mathbf{z}) = f_j(\mathbf{y})$ for every $j \in \{2, 3, 4, 6, 7\}$. Let us call the perturbation performed in the way described above when $f_5(\mathbf{x}) < \alpha_*$, $f_1(\mathbf{x}) < \alpha_*$, $f_8(\mathbf{x}) = \alpha_*$ and $f_j(\mathbf{x}) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$ for some $\mathbf{x} \in \Delta^7$ $pert_{\vec{u}_{14}}^8$ and the point obtained by applying the perturbation $pert_{\vec{u}_{14}}^8$ using the point $\mathbf{x}$ $pert_{\vec{u}_{14}}^8(\mathbf{x})$. We will use the following notation to denote this process

$$\begin{array}{c} [\mathbf{x} : f_1 < \alpha_*, f_5 < \alpha_*, f_8 = \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*] \\ \downarrow pert_{\vec{u}_{14}}^8 \\ [pert_{\vec{u}_{14}}^8(\mathbf{x}) : f_1 < \alpha_*, f_5 < \alpha_*, f_8 < \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*]. \end{array}$$

In the case above, we have $\mathbf{z} = pert_{\vec{u}_{13}}^8(\mathbf{y})$ or $\mathbf{z} = pert_{\vec{u}_{14}}^8(pert_{\vec{u}_{13}}^8(\mathbf{x}^*))$.

Let us assume that $f_1(\mathbf{x}^*) < \alpha_*$, $f_5(\mathbf{x}^*) < \alpha_*$, $f_8(\mathbf{x}^*) = \alpha_*$ and $f_j(\mathbf{x}^*) \leq \alpha_*$ for $j \in I_1 \cup I_2$. Then it is possible to find a point $\mathbf{y} \in \Delta^7$ so that $f_k(\mathbf{y}) < \alpha_*$ for every $k \in I_3$ and $f_j(\mathbf{y}) = f_j(\mathbf{x}^*)$ for every $j \in I_1 \cup I_2$ by applying the following perturbation

$$\begin{array}{c} [\mathbf{x}^* : f_1 < \alpha_*, f_5 < \alpha_*, f_8 = \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*] \\ \downarrow pert_{\vec{u}_{14}}^8 \\ [\mathbf{y} : f_1 < \alpha_*, f_5 < \alpha_*, f_8 < \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*]. \end{array}$$

If we assume that $f_1(\mathbf{x}^*) < \alpha_*$, $f_5(\mathbf{x}^*) = \alpha_*$, $f_8(\mathbf{x}^*) < \alpha_*$ and $f_j(\mathbf{x}^*) \leq \alpha_*$ for $j \in I_1 \cup I_2$, it is possible to find a point $\mathbf{y} \in \Delta^7$ so that $f_k(\mathbf{y}) < \alpha_*$ for every $k \in I_1$ and $f_j(\mathbf{y}) = f_j(\mathbf{x}^*)$ for every $j \in I_1 \cup I_2$.

We apply the following perturbation

$$\begin{aligned} & [\mathbf{x}^* : f_1 < \alpha_*, f_5 = \alpha_*, f_8 < \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*] \\ & \quad \downarrow \text{pert}_{\tilde{u}_9}^1 \\ & [\mathbf{y} : f_1 < \alpha_*, f_5 < \alpha_*, f_8 < \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*]. \end{aligned}$$

Let us assume that $f_1(\mathbf{x}^*) = \alpha_*$, $f_5(\mathbf{x}^*) < \alpha_*$, $f_8(\mathbf{x}^*) < \alpha_*$ and $f_j(\mathbf{x}^*) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$. Then it is possible to a point $\mathbf{y} \in \Delta^7$ so that $f_k(\mathbf{y}) < \alpha_*$ for every $k \in I_1$ and $f_j(\mathbf{y}) = f_j(\mathbf{x}^*)$ for every $j \in \{1, 3, 4, 5, 8\}$ by applying the following perturbation

$$\begin{aligned} & [\mathbf{x}^* : f_1 = \alpha_*, f_5 < \alpha_*, f_8 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*] \\ & \quad \downarrow \text{pert}_{\tilde{u}_{11}}^5 \\ & [\mathbf{y} : f_1 < \alpha_*, f_5 < \alpha_*, f_8 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*]. \end{aligned}$$

Finally, let us assume that $f_1(\mathbf{x}^*) < \alpha_*$, $f_5(\mathbf{x}^*) < \alpha_*$, $f_8(\mathbf{x}^*) < \alpha_*$ and $f_j(\mathbf{x}^*) \leq \alpha_*$ for $j \in \{1, 3, 4, 5, 8\}$. If this is the case, we know that there exists two distinct indices $j_1, j_2 \in \{2, 3, 4, 6, 7\}$ so that f_{j_1} and f_{j_2} take the value α_* at the point $\mathbf{x}^*$.

Hence, we conclude that if one or two of the functions f_1, f_5, f_8 take a value less than α_* at the point $\mathbf{x}^*$, it is possible to find a point ($\mathbf{y}$ or $\mathbf{z}$) obtained by applying one or two perturbations chosen from the list

$$L_3 = \{\text{pert}_{\tilde{u}_9}^1, \text{pert}_{\tilde{u}_{10}}^1, \text{pert}_{\tilde{u}_{11}}^5, \text{pert}_{\tilde{u}_{12}}^5, \text{pert}_{\tilde{u}_{13}}^8, \text{pert}_{\tilde{u}_{14}}^8\}$$

so that all of the functions f_1, f_5, f_8 take a value less than α_* at that point ($\mathbf{y}$ or $\mathbf{z}$) and the values of the functions f_1, f_3, f_4, f_5, f_8 at that point are the same as at the point $\mathbf{x}^*$.

The relationship between the functions f_1, f_5 and f_8 described in this section suggests the following: the assumption $f_i(\mathbf{x}^*) < \alpha_*$ for every $i \in I_1 \cup I_2$ implies that $f_k(\mathbf{x}^*) = \alpha_*$ for every $k \in I_3$. Because if we have $f_1(\mathbf{x}^*) < \alpha_*$, $f_5(\mathbf{x}^*) = \alpha_*$ and $f_8(\mathbf{x}^*) = \alpha_*$, then there is a point $\mathbf{z} = \text{pert}_{\bar{u}_{10}}^1(\text{pert}_{\bar{u}_9}^1(\mathbf{x}^*)) \in \Delta^7$ such that $f_k(\mathbf{z}) < \alpha_*$ for every $k \in I_3$ and $f_j(\mathbf{z}) = f_j(\mathbf{x}^*)$ for every $j \in I_1 \cup I_2$. This contradicts with the minimality of α_* . If we have $f_1(\mathbf{x}^*) = \alpha_*$, $f_5(\mathbf{x}^*) < \alpha_*$ and $f_8(\mathbf{x}^*) = \alpha_*$, then there is a point $\mathbf{z} = \text{pert}_{\bar{u}_{12}}^5(\text{pert}_{\bar{u}_{11}}^5(\mathbf{x}^*)) \in \Delta^7$ such that $f_k(\mathbf{z}) < \alpha_*$ for every $k \in I_3$ and $f_j(\mathbf{z}) = f_j(\mathbf{x}^*) < \alpha_*$ for every $j \in I_1 \cup I_2$. This gives a contradiction. If we have $f_1(\mathbf{x}^*) = \alpha_*$, $f_5(\mathbf{x}^*) = \alpha_*$ and $f_8(\mathbf{x}^*) < \alpha_*$, then there is a point $\mathbf{z} = \text{pert}_{\bar{u}_{14}}^8(\text{pert}_{\bar{u}_{13}}^8(\mathbf{x}^*)) \in \Delta^7$ such that $f_k(\mathbf{z}) < \alpha_*$ for every $k \in I_3$ and $f_j(\mathbf{z}) = f_j(\mathbf{x}^*) < \alpha_*$ for every $j \in I_1 \cup I_2$, a contradiction. Therefore, the following case

$\bar{\text{I.}} \quad f_j(\mathbf{x}^*) = \alpha_*$ for $j = 1, 5, 8$ and $f_j(\mathbf{x}^*) < \alpha_*$ for $j = 2, 3, 4, 6, 7$.

is one of the cases that requires further consideration. We will denote the set $\{f_1, f_5, f_8\}$ with $\{f_{I_3}\}$.

4.1.4 The relationship between the sets $\{f_{I_1}\}, \{f_{I_2}\}$ and $\{f_{I_3}\}$

We numerate the case above as $\hat{\text{I}}$ since the existence of a point in Δ^7 which satisfies the case I implies the existence of a point in Δ^7 which satisfies the case $\hat{\text{I}}$. Similarly, the existence of a point in Δ^7 which satisfies the case $\hat{\text{I}}$ implies the existence of a point in Δ^7 which satisfies the case I. Because if we consider the transformation $L: \Delta^7 \rightarrow \Delta^7$ defined by

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mapsto (x_4, x_5, x_6, x_1, x_2, x_3, x_8, x_7),$$

we see that all the functions $f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8$ are preserved and

$$f_2(\mathbf{x}^*) = f_1(L(\mathbf{x}^*))$$

$$f_6(\mathbf{x}^*) = f_8(L(\mathbf{x}^*))$$

$$f_7(\mathbf{x}^*) = f_5(L(\mathbf{x}^*))$$

$$f_3(\mathbf{x}^*) = f_4(L(\mathbf{x}^*)).$$

Hence, if we have $f_j(\mathbf{x}^*) < \alpha_*$ for $j = 1, 3, 4, 5, 8$ and $f_j(\mathbf{x}^*) = \alpha_*$ for $j = 2, 6, 7$, then we have $f_j(L(\mathbf{x}^*)) = \alpha_*$ for $j = 1, 5, 8$ and $f_j(L(\mathbf{x}^*)) < \alpha_*$ for $j = 2, 3, 4, 6, 7$. Similarly, if we have $f_j(\mathbf{x}^*) < \alpha_*$ for $j = 1, 5, 8$ and $f_j(\mathbf{x}^*) = \alpha_*$ for $j = 2, 3, 4, 6, 7$, then we have $f_j(L(\mathbf{x}^*)) = \alpha_*$ for $j = 1, 3, 4, 5, 8$ and $f_j(L(\mathbf{x}^*)) < \alpha_*$ for $j = 2, 6, 7$. Therefore, we consider the case I and $\bar{I}$ as one case.

The relationships between the functions $\{f_i : i \in I_1 \cup I_2 \cup I_3\}$ we presented above show that if we have $f_k(\mathbf{x}^*) < \alpha_*$ for some $k \in I_1$, we can find a point $\mathbf{y}$ or $\mathbf{z}$ in Δ^7 using perturbations chosen from the list L_1 so that $f_k(\mathbf{y}) < \alpha_*$ for every $k \in I_1$ and $F(\mathbf{y}) = \alpha_*$ or $f_k(\mathbf{z}) < \alpha_*$ for every $k \in I_1$ and $F(\mathbf{z}) = \alpha_*$. Similarly, if we have $f_k(\mathbf{x}^*) < \alpha_*$ for some $k \in I_2$, then we can find a point $\mathbf{y} \in \Delta^7$ using a perturbation from the list L_2 so that $f_k(\mathbf{y}) < \alpha_*$ for every $k \in I_2$ and $F(\mathbf{y}) = \alpha_*$. If there exists a $k \in I_3$ so that $f_k(\mathbf{x}^*) < \alpha_*$, then we can find a point $\mathbf{z} \in \Delta^7$ using one or two perturbations from the list L_3 so that $f_k(\mathbf{z}) < \alpha_*$ for every $k \in I_3$ and $F(\mathbf{z}) = \alpha_*$.

Let assume that $f_k(\mathbf{x}^*) < \alpha_*$ for every $k \in I_2$. If there exists a $k_1 \in I_1$ such that $f_{k_1}(\mathbf{x}^*) < \alpha_*$, then we have $f_{k_2}(\mathbf{x}^*) = \alpha_*$ for every $k_2 \in I_3$. Otherwise, we apply perturbations chosen from the list L_3 and obtain a point $\mathbf{y} \in \Delta^7$ so that every function $f_k(\mathbf{y}) < \alpha_*$ for every $k \in I_2 \cup I_3$. Then we apply one or

two perturbations chosen from the list L_1 and we obtain a point $\mathbf{z} \in \Delta^7$ so that $f_k(\mathbf{z}) < \alpha_*$ for every $k = I_1 \cup I_2 \cup I_3$. This contradicts with the minimality of α_* . Therefore, we see that $f_{k_2}(\mathbf{x}^*) = \alpha_*$ for every $k_2 \in I_3$. But then after applying permutations from the list L_1 , we obtain case $\hat{\text{I}}$, or equivalently case I.

If there exists a $k_2 \in I_3$ such that $f_{k_2}(\mathbf{x}^*) < \alpha_*$, then we have $f_{k_1}(\mathbf{x}^*) = \alpha_*$ for every $k_1 \in I_1$. Otherwise, we apply perturbations chosen from the list L_1 and obtain a point $\mathbf{y} \in \Delta^7$ so that $f_k(\mathbf{y}) < \alpha_*$ for every $k = I_1 \cup I_3$. Then we apply one or two perturbations chosen from the list L_1 and we obtain a point $\mathbf{z} \in \Delta^7$ so that $f_k(\mathbf{z}) < \alpha_*$ for every $k = I_1 \cup I_2 \cup I_3$. This contradicts with the minimality of α_* . Thus, we see that $f_{k_2}(\mathbf{x}^*) = \alpha_*$ for every $k_2 \in I_3$. But then this is case I. Therefore, a new case arises if we assume that $f_k(\mathbf{x}^*) = \alpha_*$ for every $k \in I_1 \cup I_3$. Let us numerate this case as

III. $f_j(\mathbf{x}^*) = \alpha_*$ for $j = 1, 2, 5, 6, 7, 8$ and $f_j(\mathbf{x}^*) < \alpha_*$ for $j = 3, 4$.

Let us assume that $f_j(\mathbf{x}^*) < \alpha_*$ for every $j \in I_3$. If there exists $k_1 \in I_1$ so that $f_{k_1}(\mathbf{x}^*) < \alpha_*$, then we claim that $f_{k_2}(\mathbf{x}^*) = \alpha_*$ for every $k_2 \in I_2$. Because if there exists $k_2 \in I_2$ such that $f_{k_2}(\mathbf{x}^*) < \alpha_*$, we apply a perturbation from the list L_2 and obtain a point $\mathbf{y} \in \Delta^7$ so that $f_k(\mathbf{y}) < \alpha_*$ for every $k \in I_2 \cup I_3$. Then we apply perturbations from the list L_1 and obtain a point $\mathbf{z} \in \Delta^7$ so that $f_k(\mathbf{z}) < \alpha_*$ for every $k \in I_1 \cup I_2 \cup I_3$. This contradicts with the minimality of α_* . Therefore, we see that $f_{k_2}(\mathbf{x}^*) = \alpha_*$ for every $k_2 \in I_2$. But then this gives case II.

If there exists a $k_2 \in I_2$ such that $f_{k_2}(\mathbf{x}^*) < \alpha_*$, then we have $f_{k_1}(\mathbf{x}^*) = \alpha_*$ for every $k_1 \in I_1$. Otherwise, we apply perturbations from the list L_3 and obtain a point $\mathbf{y} \in \Delta^7$ so that $f_k(\mathbf{y}) < \alpha_*$ for every $k = I_1 \cup I_3$. Then we apply perturbations from the list L_2 and obtain a point $\mathbf{z} \in \Delta^7$ so that $f_k(\mathbf{z}) < \alpha_*$ for every $k = 1, 2, \dots, 8$. This contradicts with the minimality of α_* . Hence, we conclude that $f_{k_2}(\mathbf{x}^*) = \alpha_*$

for every $k_2 \in I_2$. But then this is case II again. Therefore, another new case arises if we assume that $f_k(\mathbf{x}^*) = \alpha_*$ for every $k \in I_1 \cup I_2$. Specifically, we obtain the following case

$$\text{IV. } f_j(\mathbf{x}^*) = \alpha_* \text{ for } j = 2, 3, 4, 6, 7 \text{ and } f_j(\mathbf{x}^*) < \alpha_* \text{ for } j = 1, 5, 8$$

which needs further investigation.

Let us assume that $f_j(\mathbf{x}^*) < \alpha_*$ for every $j \in I_1$. If there exists $k_1 \in I_3$ so that $f_{k_1}(\mathbf{x}^*) < \alpha_*$, then we claim that $f_{k_2}(\mathbf{x}^*) = \alpha_*$ for every $k_2 \in I_2$. Because if there exists $k_2 \in I_2$ such that $f_{k_2}(\mathbf{x}^*) < \alpha_*$, we apply a perturbation from the list L_2 and obtain a point $\mathbf{y} \in \Delta^7$ so that $f_k(\mathbf{y}) < \alpha_*$ for every $k \in I_1 \cup I_2$. Then we apply perturbations from the list L_3 and obtain a point $\mathbf{z} \in \Delta^7$ so that $f_k(\mathbf{z}) < \alpha_*$ for every $k \in I_1 \cup I_2 \cup I_3$. This contradicts with the minimality of α_* . Thus, we see that $f_{k_2}(\mathbf{x}^*) = \alpha_*$ for every $k_2 \in I_2$. But then this gives case II.

If there exists a $k_2 \in I_2$ such that $f_{k_2}(\mathbf{x}^*) < \alpha_*$, then we have $f_{k_1}(\mathbf{x}^*) = \alpha_*$ for every $k_1 \in I_3$. Otherwise, we apply perturbations from the list L_3 and obtain a point $\mathbf{y} \in \Delta^7$ so that $f_k(\mathbf{y}) < \alpha_*$ for every $k \in I_1 \cup I_3$. Then we apply perturbations from the list L_2 and obtain a point $\mathbf{z} \in \Delta^7$ so that $f_k(\mathbf{z}) < \alpha_*$ for every $k = 1, 2, \dots, 8$. This contradicts with the minimality of α_* . Therefore, we conclude that $f_{k_2}(\mathbf{x}^*) = \alpha_*$ for every $k_2 \in I_3$. But then this is case I. Therefore, another new case arises if we assume that $f_k(\mathbf{x}^*) = \alpha_*$ for every $k \in I_2 \cup I_3$. Finally, we obtain the following case

$$\overline{\text{IV.}} \quad f_j(\mathbf{x}^*) = \alpha_* \text{ for } j = 1, 3, 4, 5, 8 \text{ and } f_j(\mathbf{x}^*) < \alpha_* \text{ for } j = 2, 6, 7.$$

In fact, case $\overline{\text{IV}}$ is not a separate case. Because if we have $f_j(\mathbf{x}^*) = \alpha_*$ for $j = 1, 3, 4, 5, 8$ and $f_j(\mathbf{x}^*) < \alpha_*$ for $j = 2, 6, 7$, then we have $f_j(L(\mathbf{x}^*)) < \alpha_*$ for $j = 1, 5, 8$ and $f_j(L(\mathbf{x}^*)) = \alpha_*$ for $j = 2, 3, 4, 6, 7$. Similarly,

if we have $f_j(\mathbf{x}^*) < \alpha_*$ for $j = 1, 5, 8$ and $f_j(\mathbf{x}^*) = \alpha_*$ for $j = 2, 3, 4, 6, 7$ then we have $f_j(L(\mathbf{x}^*)) = \alpha_*$ for $j = 1, 3, 4, 5, 8$ and $f_j(L(\mathbf{x}^*)) < \alpha_*$ for $j = 2, 6, 7$. Therefore, we consider the case IV and $\overline{\text{IV}}$ as one case.

4.1.5 Cases I, II, III and IV are not possible at the point $\mathbf{x}^*$

Let us define the following subset $\Delta^* = \{\mathbf{x} \in \Delta^7 : f_j(\mathbf{x}) = f_k(\mathbf{x}) \text{ for every } j, k = 1, 2, \dots, 8\}$ of Δ^7 . Note that Δ^* may be empty. If it is not empty, $\mathbf{x}^*$ may not be an element of Δ^* . We aim to prove the following lemma

Lemma 4.1.1. Δ^* is nonempty and $\mathbf{x}^* \in \Delta^*$.

To this purpose, we need to show that cases I, II, III and IV are not possible at the point $\mathbf{x}^*$. We start with case I. Let us define the continuous functions $A : \Delta^7 \rightarrow \mathbf{R}$ and $B : \Delta^7 \rightarrow \mathbf{R}$ where

$$A(\mathbf{x}) = \frac{1 - x_1 - x_2 - x_3}{x_1 + x_2 + x_3} \quad \text{and} \quad B(\mathbf{x}) = \frac{1 - x_4 - x_5 - x_6}{x_4 + x_5 + x_6}.$$

Lemma 4.1.2. Case I is not possible at the point $\mathbf{x}^*$.

Proof. If we use the parametrization $\mathfrak{P}_8$, case I gives us

$$\begin{aligned} A(\mathbf{x}^*) \cdot \frac{1 - x_4^*}{x_4^*} &< \alpha_*, \\ B(\mathbf{x}^*) \cdot \frac{1 - x_1^*}{x_1^*} &= \alpha_*, \\ A(\mathbf{x}^*) \cdot \frac{\sum_{i=1}^7 x_i^*}{1 - \sum_{i=1}^7 x_i^*} &< \alpha_*, \end{aligned}$$

$$\begin{aligned}
B(\mathbf{x}^*) \cdot \frac{1-x_7^*}{x_7^*} &< \alpha_*, \\
\frac{1}{A(\mathbf{x}^*)} \cdot \frac{1-x_6^*}{x_6^*} &< \alpha_*,
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
\frac{1}{A(\mathbf{x}^*)} \cdot \frac{1-x_2^*}{x_2^*} &= \alpha_*, \\
\frac{1}{B(\mathbf{x}^*)} \cdot \frac{1-x_3^*}{x_3^*} &= \alpha_*, \\
\frac{1}{B(\mathbf{x}^*)} \cdot \frac{1-x_5^*}{x_5^*} &< \alpha_*.
\end{aligned} \tag{4.2}$$

Equality in (Equation 4.2) shows that

$$\begin{aligned}
\frac{1}{A(\mathbf{x}^*)} \cdot \frac{1-x_2^*}{x_2^*} &= \alpha_* \\
\frac{x_1^* + x_2^* + x_3^*}{1-x_1^* - x_2^* - x_3^*} \cdot \frac{1-x_2^*}{x_2^*} &= \alpha_* \\
(x_1^* + x_2^* + x_3^*)(1-x_2^*) &= \alpha_*(1-x_1^* - x_2^* - x_3^*)x_2^* \\
x_1^* + x_2^* + x_3^* - x_1^*x_2^* - (x_2^*)^2 - x_2^*x_3^* &= \alpha_*x_2^* - \alpha_*x_1^*x_2^* - \alpha_*(x_2^*)^2 - \alpha_*x_2^*x_3^* \\
(1-\alpha_*)(x_2^*)^2 - (1-\alpha_*)(1-x_1^* - x_3^*)x_2^* - (x_1^* + x_3^*) &= 0.
\end{aligned}$$

In other words, x_2^* is a solution of the quadratic equation $q_3(x) = 0$ where

$$q_3(x) = (1-\alpha_*)x^2 - (1-\alpha_*)(1-x_1^* - x_3^*)x - (x_1^* + x_3^*).$$

Therefore, we get that

$$x_2^* = \frac{(1-\alpha_*)(1-x_1^* - x_3^*) \pm \sqrt{(1-\alpha_*)^2(1-x_1^* - x_3^*)^2 + 4(1-\alpha_*)(x_1^* + x_3^*)}}{2(1-\alpha_*)}.$$

Let us define the function

$$\begin{aligned} D_3(\mathbf{x}) &= D_3(x_1, x_3) \\ &= (1 - \alpha_*)^2(1 - x_1 - x_3)^2 + 4(1 - \alpha_*)(x_1 + x_3). \end{aligned}$$

$D_3(\mathbf{x}^*)$ corresponds to the discriminant of $q_3(x)$. Therefore, we have $D_3(\mathbf{x}^*) \geq 0$. Let us assume that $D_3(\mathbf{x}^*) > 0$. Since $D_3(\mathbf{x})$ is continuous, there exists an $\epsilon > 0$ so that $D_3(x'_1, x'_3) > 0$ for

$$\begin{aligned} x'_1 &= x_1^* + \epsilon, \\ x'_3 &= x_3^* + \epsilon. \end{aligned}$$

Therefore, if we define

$$\tilde{q}_3(x) = (1 - \alpha_*)x^2 - (1 - \alpha_*)(1 - x'_1 - x'_3)x - (x'_1 + x'_3),$$

we see that the quadratic equation $\tilde{q}_3(x) = 0$ has two real roots

$$x'_2 = \frac{(1 - \alpha_*)(1 - x'_1 - x'_3) + m\sqrt{D_3(x'_1, x'_3)}}{2(1 - \alpha_*)},$$

and

$$x'_2 = \frac{(1 - \alpha_*)(1 - x'_1 - x'_3) - \sqrt{D_3(x'_1, x'_3)}}{2(1 - \alpha_*)}.$$

In fact, we can choose $\epsilon > 0$ so that the coefficients of $q_3(x)$ and $\tilde{q}(x)$ are sufficiently close to each other to conclude that $0 < x'_i < 1$ for $i = 1, 2, 3, 8$ where we set $x'_4 = x_4^*$, $x'_5 = x_5^*$, $x'_6 = x_6^*$, $x'_7 = x_7^*$ and

$$x'_8 = 1 - \sum_{i=1}^7 x'_i.$$

Moreover, using the continuity of $A(\mathbf{x})$, $\epsilon > 0$ can be chosen so that the following two inequalities are also satisfied:

$$\begin{aligned} A(x'_1, x'_2, x'_3) \cdot \frac{1 - x'_8}{x'_8} &< \alpha_*, \\ \frac{1}{A(x'_1, x'_2, x'_3)} \cdot \frac{1 - x'_6}{x'_6} &< \alpha_*. \end{aligned}$$

Since $\tilde{q}_3(x'_2) = 0$, we see that $f_6(\mathbf{x}') = \alpha_*$. We also have $A(x_1^*, x_2^*, x_3^*) = A(x'_1, x'_2, x'_3)$, $B(x_1^*, x_2^*, x_3^*) = B(x'_1, x'_2, x'_3)$ and

$$\begin{aligned} \frac{1 - x'_1}{x'_1} &< \frac{1 - x_1^*}{x_1^*}, \\ \frac{1 - x'_3}{x'_3} &< \frac{1 - x_3^*}{x_3^*}. \end{aligned}$$

Therefore, if we consider the point $\mathbf{x}' = (x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, x'_7, x'_8) \in \Delta^7$, we see that $f_6(\mathbf{x}') = \alpha_*$ and $f_j(\mathbf{x}') < \alpha_*$ for every $j \in \{1, 2, 3, 4, 5, 7, 8\}$. Then we apply the following perturbation

$$\begin{aligned} &[\mathbf{x}' : f_2 < \alpha_*, f_6 = \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} < \alpha_*; F = \alpha_*] \\ &\quad \downarrow \text{pert}_{\bar{u}_6}^7 \\ &[\mathbf{y} : f_2 < \alpha_*, f_6 < \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} < \alpha_*; F < \alpha_*]. \end{aligned}$$

In this way, we obtain a point $\mathbf{y} \in \Delta^7$ so that $F(\mathbf{y}) < \alpha_*$. This contradicts with the minimality of α_* .

Let us assume that $D_3(\mathbf{x}^*) = 0$. Then we get

$$x_2^* = \frac{1 - x_1^* - x_3^*}{2},$$

which implies that

$$x_2^* = x_4^* + x_5^* + x_6^* + x_7^* + x_8^*.$$

Therefore, we can say that $x_6^* < x_2^*$. Then using the equality in (Equation 4.2), we obtain that

$$\begin{aligned} \alpha_* &= \frac{x_1^* + x_2^* + x_3^*}{1 - x_1^* - x_2^* - x_3^*} \cdot \left(\frac{1}{x_2^*} - 1 \right) \\ &= \frac{1 - x_4^* - x_5^* - x_6^* - x_7^* - x_8^*}{x_4^* + x_5^* + x_6^* + x_7^* + x_8^*} \cdot \left(\frac{1}{x_2^*} - 1 \right) \\ &= \left(\frac{1 - x_2^*}{x_2^*} \right) \cdot \left(\frac{1}{x_2^*} - 1 \right) \\ &= \left(\frac{1}{x_2^*} - 1 \right) \left(\frac{1}{x_2^*} - 1 \right) \\ &< \left(\frac{1}{x_2^*} - 1 \right) \left(\frac{1}{x_6^*} - 1 \right) \end{aligned}$$

But by inequality in (Equation 4.1) we also have

$$\left(\frac{1}{x_2^*} - 1 \right) \left(\frac{1}{x_6^*} - 1 \right) < \alpha_*.$$

This is a contradiction. Thus, case I is not possible at the point $\mathbf{x}^*$. □

Lemma 4.1.3. *Case II is not possible at the point $\mathbf{x}^*$.*

Proof. We need to use two parametrization simultaneously in this case.

If we start with the parametrization $\mathfrak{P}_8$ first, we see that

$$A(\mathbf{x}^*) \cdot \frac{1 - x_4^*}{x_4^*} < \alpha_* \quad (4.3)$$

$$B(\mathbf{x}^*) \cdot \frac{1 - x_1^*}{x_1^*} < \alpha_*$$

$$A(\mathbf{x}^*) \cdot \frac{\sum_{i=1}^7 x_i^*}{1 - \sum_{i=1}^7 x_i^*} = \alpha_* \quad (4.4)$$

$$B(\mathbf{x}^*) \cdot \frac{1 - x_7^*}{x_7^*} = \alpha_*$$

$$\frac{1}{A(\mathbf{x}^*)} \cdot \frac{1 - x_6^*}{x_6^*} < \alpha_*$$

$$\frac{1}{A(\mathbf{x}^*)} \cdot \frac{1 - x_2^*}{x_2^*} < \alpha_*$$

$$\frac{1}{B(\mathbf{x}^*)} \cdot \frac{1 - x_3^*}{x_3^*} < \alpha_*$$

$$\frac{1}{B(\mathbf{x}^*)} \cdot \frac{1 - x_5^*}{x_5^*} < \alpha_*$$

By (Equation 4.4), we find that $x_1^* + x_2^* + x_3^*$ is a solution for the quadratic equation $q_4(x) = 0$ where

$$q_4(x) = (\alpha_* - 1)x^2 + (1 - \alpha_*)\left(1 - \sum_{k=4}^7 x_k^*\right)x + \sum_{k=4}^7 x_k^*.$$

which follows from the calculations below:

We have

$$A(\mathbf{x}^*) \cdot \frac{\sum_{i=1}^7 x_i^*}{1 - \sum_{i=1}^7 x_i^*} = \alpha_*.$$

Then we obtain that

$$\begin{aligned} \frac{1 - x_1^* - x_2^* - x_3^*}{x_1^* + x_2^* + x_3^*} \cdot \frac{\sum_{i=1}^7 x_i^*}{1 - \sum_{i=1}^7 x_i^*} &= \alpha_* \\ (1 - x_1^* - x_2^* - x_3^*) \left(\sum_{i=1}^7 x_i^* \right) &= \alpha_* (x_1^* + x_2^* + x_3^*) \left(1 - \sum_{i=1}^7 x_i^* \right) \\ (\alpha_* - 1)(x_1^* + x_2^* + x_3^*)^2 - (1 - \alpha_*) \left(1 - \sum_{i=4}^7 x_i^* \right) (x_1^* + x_2^* + x_3^*) + \sum_{i=4}^7 x_i^* &= 0. \end{aligned}$$

Therefore, we find that

$$x_1^* + x_2^* + x_3^* = \frac{-(1 - \alpha_*) \left(1 - \sum_{k=4}^7 x_k^* \right) \pm \sqrt{(1 - \alpha_*)^2 \left(1 - \sum_{k=4}^7 x_k^* \right)^2 - 4(\alpha_* - 1) \left(\sum_{k=4}^7 x_k^* \right)}}{2(\alpha_* - 1)}.$$

Let us define the function

$$\begin{aligned} D_4(\mathbf{x}) &= D_4(x_4, x_5, x_6, x_7) \\ &= (1 - \alpha_*)^2 \left(1 - \sum_{k=4}^7 x_k \right)^2 - 4(\alpha_* - 1) \left(\sum_{k=4}^7 x_k \right). \end{aligned}$$

Since $D_4(\mathbf{x}^*)$ corresponds to the discriminant of $q_4(x)$, we get $D_4(\mathbf{x}^*) \geq 0$.

If we express the case IV using the parametrization $\mathfrak{P}_7$, we get

$$\begin{aligned}
 A(\mathbf{x}^*) \cdot \frac{1-x_4^*}{x_4^*} &< \alpha_* \\
 B(\mathbf{x}^*) \cdot \frac{1-x_1^*}{x_1^*} &< \alpha_* \\
 A(\mathbf{x}^*) \cdot \frac{1-x_8^*}{x_8^*} &= \alpha_* \\
 B(\mathbf{x}^*) \cdot \frac{\sum_{i=1, i \neq 7}^8 x_i^*}{1 - \sum_{i=1, i \neq 7}^8 x_i^*} &= \alpha_* \\
 \frac{1}{A(\mathbf{x}^*)} \cdot \frac{1-x_6^*}{x_6^*} &< \alpha_* \\
 \frac{1}{A(\mathbf{x}^*)} \cdot \frac{1-x_2^*}{x_2^*} &< \alpha_* \\
 \frac{1}{B(\mathbf{x}^*)} \cdot \frac{1-x_3^*}{x_3^*} &< \alpha_* \\
 \frac{1}{B(\mathbf{x}^*)} \cdot \frac{1-x_5^*}{x_5^*} &< \alpha_*.
 \end{aligned} \tag{4.5}$$

Using equality in (Equation 4.5), we see that $x_4^* + x_5^* + x_6^*$ is a solution for the quadratic equation

$q_5(x) = 0$ where

$$q_5(x) = (\alpha_* - 1)x^2 + (1 - \alpha_*)(1 - \sum_{k=1}^3 x_k^* - x_8^*)x + \sum_{k=1}^3 x_k^* + x_8^*.$$

We have

$$B(\mathbf{x}^*) \cdot \frac{\sum_{i=1, i \neq 7}^8 x_i^*}{1 - \sum_{i=1, i \neq 7}^8 x_i^*} = \alpha_*.$$

We carry out the calculations:

$$\begin{aligned}
 & \frac{1 - \sum_{i=4}^6 x_i^*}{\sum_{i=4}^6 x_i^*} \cdot \frac{\sum_{i=1, i \neq 7}^8 x_i^*}{1 - \sum_{i=1, i \neq 7}^8 x_i^*} = \alpha_* \\
 & \alpha_* \left(\sum_{i=4}^6 x_i^* \right) \left(1 - \sum_{i=1, i \neq 7}^8 x_i^* \right) = \left(1 - \sum_{i=4}^6 x_i^* \right) \left(\sum_{i=1, i \neq 7}^8 x_i^* \right) \\
 & (\alpha_* - 1) \left(\sum_{i=4}^6 x_i^* \right)^2 - (1 - \alpha_*) \left(1 - \sum_{k=1,2,3,8} x_k^* \right) \left(\sum_{i=1}^3 x_i^* \right) + \sum_{k=1,2,3,8} x_k^* = 0.
 \end{aligned}$$

Therefore, we get that

$$x_4^* + x_5^* + x_6^* = \frac{(\alpha_* - 1) \left(1 - \sum_{k=1,2,3,8} x_k^* \right) \pm \sqrt{(1 - \alpha_*)^2 \left(1 - \sum_{k=1,2,3,8} x_k^* \right)^2 - 4(\alpha_* - 1) \sum_{k=1,2,3,8} x_k^*}}{2(\alpha_* - 1)}.$$

Let us define the function

$$\begin{aligned}
 D_5(\mathbf{x}) &= D_5(x_1, x_2, x_3, x_8) \\
 &= (1 - \alpha_*)^2 \left(1 - \sum_{k=1,2,3,8} x_k^* \right)^2 - 4(\alpha_* - 1) \left(\sum_{k=1,2,3,8} x_k^* \right).
 \end{aligned}$$

Note that $D_5(\mathbf{x}^*)$ corresponds to the discriminant of $q_5(x)$. Therefore, we have $D_5(\mathbf{x}^*) \geq 0$. Thus, we obtain either:

i) $D_4(\mathbf{x}^*) > 0$ or,

ii) $D_5(\mathbf{x}^*) > 0$ or,

iii) $D_4(\mathbf{x}^*) = 0$ and $D_5(\mathbf{x}^*) = 0$.

We examine each of these cases:

(i) Since we have $D_4(\mathbf{x}^*) > 0$ and $D_4(\mathbf{x})$ is continuous, there exists an $\epsilon > 0$ so that $D_4(x'_4, x'_5, x'_6, x'_7) > 0$

for $x'_4 = x_4^*$, $x'_5 = x_5^*$, $x'_6 = x_6^*$, $x'_7 = x_7^* + \epsilon$. Hence, if we define

$$\tilde{q}_4(x) = (\alpha_* - 1)x^2 + (1 - \alpha_*) \left(1 - \sum_{k=4}^7 x'_k \right) x + \sum_{k=4}^7 x'_k,$$

we see that the quadratic equation $\tilde{q}_4(x) = 0$ has two real roots. If we set

$$x'_1 = x_1^*,$$

$$x'_3 = x_3^*,$$

and

$$x'_1 + x'_2 + x'_3 = \frac{-(1 - \alpha_*)(1 - \sum_{k=4}^7 x'_k) \pm \sqrt{D_4(x'_4, x'_5, x'_6, x'_7)}}{2(\alpha_* - 1)},$$

we get $\tilde{q}_4(x'_1 + x'_2 + x'_3) = 0$. Moreover, we can choose $\epsilon > 0$ so that the coefficients of $q_4(x)$ and $\tilde{q}_4(x)$ are

sufficiently close to each other to conclude that x'_2 is close enough to x_2^* and $0 < x'_i < 1$ for $i = 2, 7, 8$

where $x'_8 = 1 - \sum_{i=1}^7 x'_i$. In fact, since $A(\mathbf{x})$ is continuous, $\epsilon > 0$ can be chosen so that the following

inequalities are also satisfied:

$$\begin{aligned} A(x'_1, x'_2, x'_3) \cdot \frac{1 - x'_4}{x'_4} &< \alpha_*, \\ \frac{1}{A(x'_1, x'_2, x'_3)} \cdot \frac{1 - x'_6}{x'_6} &< \alpha_*, \\ \frac{1}{A(x'_1, x'_2, x'_3)} \cdot \frac{1 - x'_2}{x'_2} &< \alpha_*. \end{aligned}$$

Since $\tilde{q}_4(x'_1 + x'_2 + x'_3) = 0$, we see that $f_3(x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, x'_7, x'_8) = \alpha_*$. Note that we have the equality

$$B(x_1^*, x_2^*, x_3^*) = B(x'_1, x'_2, x'_3) \text{ and}$$

$$\frac{1 - x'_7}{x'_7} < \frac{1 - x_7^*}{x_7^*}.$$

Therefore, if we consider the point $\mathbf{x}' = (x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, x'_7, x'_8) \in \Delta^7$, we see that $f_3(\mathbf{x}') = \alpha_*$ and $f_j(\mathbf{x}') < \alpha_*$ for every $j \in \{1, 2, 4, 5, 6, 7, 8\}$. Then we apply the following perturbation

$$\begin{array}{c} [\mathbf{x}' : f_3 = \alpha_*, f_4 < \alpha_*, f_{j \notin I_2} < \alpha_*; F = \alpha_*] \\ \downarrow \text{pert}_{\bar{u}_8}^A \\ [\mathbf{y} : f_3 < \alpha_*, f_4 < \alpha_*, f_{j \notin I_2} < \alpha_*; F < \alpha_*]. \end{array}$$

In this way, we obtain a point $\mathbf{y} \in \Delta^7$ so that $F(\mathbf{y}) < \alpha_*$. This contradicts with the minimality of α_* .

(ii) Since we have $D_5(\mathbf{x}^*) > 0$ and $D_5(\mathbf{x})$ is continuous, there exists an $\epsilon > 0$ so that $D_5(x'_1, x'_2, x'_3, x'_8) > 0$

for $x'_1 = x_1^*, x'_2 = x_2^*, x'_3 = x_3^*, x'_8 = x_8^* + \epsilon$. Hence, if we define

$$\tilde{q}_5(x) = (\alpha_* - 1)x^2 + (1 - \alpha_*) \left(1 - \sum_{k=1,2,3,8} x'_k \right) x + \left(\sum_{k=1,2,3,8} x'_k \right),$$

we see that the quadratic equation $\tilde{q}_5(x) = 0$ has two real roots. If we set $x'_4 = x_4^*, x'_6 = x_6^*$ and

$$x'_4 + x'_5 + x'_6 = \frac{(\alpha_* - 1)(1 - \sum_{k=1,2,3,8} x'_k) \pm \sqrt{D_5(x'_1, x'_2, x'_3, x'_8)}}{2(\alpha_* - 1)},$$

we get $\tilde{q}_5(x'_4 + x'_5 + x'_6) = 0$.

Moreover, we can choose $\epsilon > 0$ so that the coefficients of $q_5(x)$ and $\tilde{q}_5(x)$ are sufficiently close to each other to conclude that $x'_4 + x'_5 + x'_6$ is close enough to $x_4^* + x_5^* + x_6^*$ and $0 < x'_i < 1$ for $i = 5, 7, 8$ where $x'_7 = 1 - \sum_{i=1, i \neq 7}^8 x'_i$. In fact, since $B(\mathbf{x})$ is continuous, $\epsilon > 0$ can be chosen so that the following inequalities are also satisfied:

$$\begin{aligned} B(x'_4, x'_5, x'_6) \cdot \frac{1 - x'_1}{x'_1} &< \alpha_*, \\ \frac{1}{B(x'_4, x'_5, x'_6)} \cdot \frac{1 - x'_3}{x'_3} &< \alpha_*, \\ \frac{1}{B(x'_4, x'_5, x'_6)} \cdot \frac{1 - x'_5}{x'_5} &< \alpha_*. \end{aligned}$$

Note that $\tilde{q}_5(x'_3 + x'_4 + x'_5) = 0$ implies that $f_4(x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, x'_7, x'_8) = \alpha_*$. We have the equation

$$A(x_1^*, x_2^*, x_3^*) = A(x'_1, x'_2, x'_3) \text{ and}$$

$$\frac{1 - x'_8}{x'_8} < \frac{1 - x_8^*}{x_8^*},$$

If we consider the point $\mathbf{x}' = (x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, x'_7, x'_8) \in \Delta^7$, we see that $f_4(\mathbf{x}') = \alpha_*$ and $f_j(\mathbf{x}') < \alpha_*$ for every $j \in \{1, 2, 3, 5, 6, 7, 8\}$.

Then we apply the following perturbation

$$\begin{aligned} [\mathbf{x}' : f_3 = \alpha_*, f_4 < \alpha_*, f_{j \notin I_2} < \alpha_*; F = \alpha_*] \\ \downarrow \text{pert}_{\bar{u}_8}^A \\ [\mathbf{y} : f_3 < \alpha_*, f_4 < \alpha_*, f_{j \notin I_2} < \alpha_*; F < \alpha_*]. \end{aligned}$$

In this way, we obtain a point $\mathbf{y} \in \Delta^7$ so that $F(\mathbf{y}) < \alpha_*$. This contradicts with the minimality of α_* .

(iii) Since we have $D_4(\mathbf{x}^*) = 0$ and $D_5(\mathbf{x}^*) = 0$, we get

$$\begin{aligned} x_1^* + x_2^* + x_3^* &= \frac{1 - \sum_{k=4}^7 x_k^*}{2} \\ &= x_8^*, \end{aligned}$$

and

$$\begin{aligned} x_4^* + x_5^* + x_6^* &= \frac{1 - \sum_{k=1,2,3,8} x_k^*}{2} \\ &= x_7^*. \end{aligned}$$

Using (Equation 4.4) and (Equation 4.5), we get

$$\left(\frac{1 - x_8^*}{x_8^*} \right)^2 = \left(\frac{1 - x_7^*}{x_7^*} \right)^2$$

which implies that $x_7^* = x_8^*$. We know that $\sum_{i=1}^8 x_i^* = 1$. Therefore we have $x_8^* = 1/4$ and $x_7^* = 1/4$.

Then we obtain

$$\begin{aligned} \alpha_* &= \left(\frac{1 - x_8^*}{x_8^*} \right)^2 \\ &= \left(\frac{1 - \frac{1}{4}}{\frac{1}{4}} \right)^2 \\ &= 9. \end{aligned}$$

Since we have $x_4^* < x_7^*$, we get $x_4^* < 1/4$. We calculate that

$$\begin{aligned} A(\mathbf{x}^*) \cdot \left(\frac{1}{x_4^*} - 1 \right) &= \left(\frac{1}{x_8^*} - 1 \right) \left(\frac{1}{x_4^*} - 1 \right) \\ &> 9 \\ &= \alpha_*. \end{aligned}$$

This contradicts with the assumption in (Equation 4.3) of the case II. Since we obtain a contradiction in any of the cases *i*, *ii*, and *iii*, case II is not possible at the point $\mathbf{x}^*$. $\square$

Lemma 4.1.4. *Case III is not possible at the point $\mathbf{x}^*$.*

Proof. In this case, we have

$$A(\mathbf{x}^*) \cdot \frac{1 - x_4^*}{x_4^*} = \alpha_* \quad (4.6)$$

$$B(\mathbf{x}^*) \cdot \frac{1 - x_1^*}{x_1^*} = \alpha_* \quad (4.7)$$

$$A(\mathbf{x}^*) \cdot \frac{1 - x_8^*}{x_8^*} < \alpha_* \quad (4.8)$$

$$B(\mathbf{x}^*) \cdot \frac{1 - x_7^*}{x_7^*} < \alpha_* \quad (4.9)$$

$$\frac{1}{A(\mathbf{x}^*)} \cdot \frac{1 - x_6^*}{x_6^*} = \alpha_* \quad (4.10)$$

$$\frac{1}{A(\mathbf{x}^*)} \cdot \frac{1 - x_2^*}{x_2^*} = \alpha_* \quad (4.11)$$

$$\frac{1}{B(\mathbf{x}^*)} \cdot \frac{1 - x_3^*}{x_3^*} = \alpha_* \quad (4.12)$$

$$\frac{1}{B(\mathbf{x}^*)} \cdot \frac{1 - x_5^*}{x_5^*} = \alpha_*. \quad (4.13)$$

Using (Equation 4.10) and (Equation 4.11), we obtain that $x_2^* = x_6^*$. Similarly, we get $x_3^* = x_5^*$ by using (Equation 4.12) and (Equation 4.13). Now if we consider equalities in (Equation 4.6) and (Equation 4.7), we see that

$$\begin{aligned}
 \frac{1 - x_1^* - x_6^* - x_5^*}{x_1^* + x_6^* + x_5^*} \cdot \frac{1 - x_4^*}{x_4^*} &= \frac{1 - x_1^* - x_2^* - x_3^*}{x_1^* + x_2^* + x_3^*} \cdot \frac{1 - x_4^*}{x_4^*} \\
 &= A(\mathbf{x}^*) \cdot \frac{1 - x_4^*}{x_4^*} \\
 &= B(\mathbf{x}^*) \cdot \frac{1 - x_1^*}{x_1^*} \\
 &= \frac{1 - x_4^* - x_5^* - x_6^*}{x_4^* + x_5^* + x_6^*} \cdot \frac{1 - x_1^*}{x_1^*}
 \end{aligned}$$

If we simplify the equality above, we get

$$(x_1^* - x_4^*)(x_5^* + x_6^*)(1 - x_1^* - x_4^* - x_5^* - x_6^*) = 0.$$

Since we know that $x_5^* + x_6^* > 0$ and $1 - x_1^* - x_4^* - x_5^* - x_6^* > 0$ from the discussion given in §4.1, we obtain that $x_1^* = x_4^*$. Then we notice that $A(\mathbf{x}^*) = B(\mathbf{x}^*)$. Hence, if we use equalities in (Equation 4.11) and (Equation 4.12), we find that $x_2^* = x_3^*$. Hence, we have

$$x_1^* = x_4^*,$$

$$x_2^* = x_3^*$$

$$= x_5^*$$

$$= x_6^*.$$

Remember that $\sum_{i=1}^8 x_i^* = 1$. Therefore, we see that $x_1^* + 2x_2^* < \frac{1}{2}$. We know that

$$\frac{1 - x_1^* - 2x_2^*}{x_1^* + 2x_2^*} \cdot \frac{1 - x_1^*}{x_1^*} = \frac{x_1^* + 2x_2^*}{1 - x_1^* - 2x_2^*} \cdot \frac{1 - x_2^*}{x_2^*}$$

from the equalities in (Equation 4.6) and (Equation 4.10). Let $t = x_1^* + 2x_2^*$. If we solve the equation above for t , we see that

$$\begin{aligned} \frac{1-t}{t} \cdot \frac{1-x_1^*}{x_1^*} &= \frac{t}{1-t} \cdot \frac{1-x_2^*}{x_2^*} \\ \left(\frac{1-t}{t}\right)^2 &= \frac{x_1^*(1-x_2^*)}{x_2^*(1-x_1^*)} \\ \frac{1}{t} - 1 &= \sqrt{\frac{x_1^*(1-x_2^*)}{x_2^*(1-x_1^*)}} \end{aligned}$$

or

$$x_1^* + 2x_2^* = \frac{1}{\sqrt{\frac{x_1^*(1-x_2^*)}{x_2^*(1-x_1^*)}}} < \frac{1}{2}.$$

Therefore, we get

$$\sqrt{\frac{x_1^*(1-x_2^*)}{x_2^*(1-x_1^*)}} = \frac{\sqrt{\frac{1}{x_2^*} - 1}}{\sqrt{\frac{1}{x_1^*} - 1}} > 1,$$

which implies that $x_2^* < x_1^*$. Thus, we obtain that $x_2^* < \frac{1}{6}$.

If we consider the equality in (Equation 4.6), we see that

$$\begin{aligned}
 A(\mathbf{x}^*) \cdot \frac{1-x_1^*}{x_1^*} &= \alpha_* \\
 \frac{1-x_1^*-2x_2^*}{x_1^*+2x_2^*} \cdot \frac{1-x_1^*}{x_1^*} &= \alpha_* \\
 (1-x_1^*-2x_2^*)(1-x_1^*) &= \alpha_*(x_1^*+2x_2^*)x_1^* \\
 (1-\alpha_*)(x_1^*)^2 + (-2+2x_2^*-2\alpha_*x_2^*)x_1^* + 1-2x_2^* &= 0.
 \end{aligned}$$

Hence, x_1^* is a solution of the quadratic equation $q_6(x) = 0$ where

$$q_6(x) = (1-\alpha_*)x^2 + (-2+2x_2^*-2\alpha_*x_2^*)x + 1-2x_2^*.$$

Then we calculate that

$$x_1^* = \frac{-(-2+2x_2^*-2\alpha_*x_2^*) \pm \sqrt{(-2+2x_2^*-2\alpha_*x_2^*)^2 - 4(1-\alpha_*)(1-2x_2^*)}}{2(1-\alpha_*)}.$$

Let us define the continuous function

$$\begin{aligned}
 D_6(\mathbf{x}) &= D_6(x_2) \\
 &= (-2+2x_2-2\alpha_*x_2)^2 - 4(1-\alpha_*)(1-2x_2).
 \end{aligned}$$

Since $D_6(\mathbf{x}^*)$ corresponds to the discriminant of $q_6(x)$, we have $D_6(\mathbf{x}^*) \geq 0$.

Let us assume that $D_6(\mathbf{x}^*) = 0$. This assumption gives us the following quadratic equation in x_2^* :

$$(4 - 8\alpha_* + 4\alpha_*^2)(x_2^*)^2 + 4\alpha_* = 0.$$

Therefore, we get

$$x_2^* = \frac{\sqrt{-\alpha_*}}{\alpha_* - 1} \text{ or } x_2^* = -\frac{\sqrt{-\alpha_*}}{\alpha_* - 1}.$$

Since x_2^* is a positive real number, we obtain a contradiction. Therefore, we have $D_6(\mathbf{x}^*) > 0$. If we assume that

$$x_1^* = \frac{-(-2 + 2x_2^* - 2\alpha_* x_2^*) + \sqrt{D_6(\mathbf{x}^*)}}{2(1 - \alpha_*)},$$

we see that

$$2(1 - \alpha_*)x_1^* + (-2 + 2x_2^* - 2\alpha_* x_2^*) = \sqrt{D_6(\mathbf{x}^*)}.$$

Because the expression $2(1 - \alpha_*)x_1^* - 2(1 + (\alpha_* - 1)x_2^*)$ is strictly less than 0, we get a contradiction.

Therefore, we have

$$x_1^* = \frac{-(-2 + 2x_2^* - 2\alpha_* x_2^*) - \sqrt{D_6(\mathbf{x}^*)}}{2(1 - \alpha_*)},$$

or

$$x_1^* = \frac{(-1 + (1 - \alpha_*)x_2^*) + \sqrt{(-1 + (1 - \alpha_*)x_2^*)^2 - (1 - \alpha_*)(1 - 2x_2^*)}}{\alpha_* - 1}.$$

Since $D_6(\mathbf{x})$ is continuous and $D_6(\mathbf{x}^*) > 0$, there exists an $\epsilon > 0$ so that $D_6(x'_2) > 0$ and $x'_2 \in (0, \frac{1}{6})$ for $x'_2 = x_2^* + \epsilon$. Hence, if we define

$$\tilde{q}_6(x) = (1 - \alpha_*)x^2 + (-2 + 2x'_2 - 2\alpha_*x'_2)x + 1 - 2x'_2,$$

we see that the quadratic equation $\tilde{q}_6(x) = 0$ has two real solutions. Thus, if we set

$$x'_1 = \frac{(-1 + (1 - \alpha_*)x'_2) + \sqrt{(-1 + (1 - \alpha_*)x'_2)^2 - (1 - \alpha_*)(1 - 2x'_2)}}{\alpha_* - 1},$$

we get $\tilde{q}_6(x'_1) = 0$. In fact, we can choose $\epsilon > 0$ so that the coefficients of $q_6(x)$ and $\tilde{q}_6(x)$ are sufficiently close to each other to conclude that $0 < x'_i < 1$ for $i = 1, 3, 4, 5, 6, 7$ where we have $x'_i = x'_2$ for $i = 3, 5, 6$, $x'_4 = x'_1$, $x'_7 = x_7^*$ and $x'_8 = 1 - \sum_{i=1}^7 x'_i$. Moreover, since $A(\mathbf{x})$ is continuous, $\epsilon > 0$ can be chosen so that the following inequalities are also satisfied:

$$\begin{aligned} A(x'_1, x'_2, x'_3) \cdot \frac{1 - x'_8}{x'_8} &< \alpha_*, \\ B(x'_1, x'_2, x'_3) \cdot \frac{1 - x'_7}{x'_7} &< \alpha_*. \end{aligned}$$

Since we have $\tilde{q}_6(x'_1) = 0$, we obtain that $f_1(x'_1, x'_2, x'_3, x'_4) = \alpha_*$ and $f_2(x'_4, x'_5, x'_6, x'_1) = \alpha_*$. At this stage of the proof, we define the function:

$$\begin{aligned} h_\alpha(x) &= \frac{(-1 + (1 - \alpha)x) + \sqrt{(-1 + (1 - \alpha)x)^2 - (1 - \alpha)(1 - 2x)}}{(\alpha - 1)} \\ &= \frac{(-1 + (1 - \alpha)x) + \sqrt{(1 - \alpha)^2 x^2 + \alpha}}{(\alpha - 1)}, \end{aligned}$$

and we define the function

$$H_\alpha(x) = \frac{h_\alpha(x) + 2x}{1 - h_\alpha(x) - 2x} \cdot \frac{1-x}{x}$$

Then we prove the following:

Lemma 4.1.5. $H_\alpha(x)$ is a strictly decreasing function around a neighborhood I of x_2^* for every $\alpha \in I_\alpha$.

Proof. We know that $x_2^* < \frac{1}{6}$. Let $I = (0, \frac{1}{6})$. Let us call $g_\alpha(x) = (1 - \alpha)^2 x^2 + \alpha$. The equation $g_\alpha(x) = 0$ has no solution since the discriminant of $g_\alpha(x)$ is $-4(1 - \alpha)^2 \alpha < 0$. We have $g_\alpha(0) = \alpha$ and

$$\begin{aligned} g_\alpha\left(\frac{1}{6}\right) &= (1 - 2\alpha + \alpha^2)\left(\frac{1}{6}\right)^2 + \alpha \\ &= \frac{1}{36} + \frac{17}{18}\alpha + \frac{1}{36}\alpha^2 \\ &> 0. \end{aligned}$$

Therefore, we get $\frac{97}{9} < g_\alpha\left(\frac{1}{6}\right) < \frac{107}{18} + \frac{11}{3}\sqrt{2}$. Since we have $g'_\alpha(x) = 2(1 - \alpha)^2 x > 0$ for every $x \in I$ and $\alpha \in I_\alpha$, we see that $9 < g_\alpha(x) < \frac{107}{18} + \frac{11}{3}\sqrt{2}$ for every x and α . If we calculate the derivative of h_α , we obtain that

$$h'_\alpha(x) = -1 + \frac{x(-1 + \alpha)}{\sqrt{g_\alpha(x)}}.$$

Note that $h'_\alpha(x)$ is defined and continuous for every $x \in (0, \frac{1}{6})$.

We also calculate $h''_{\alpha}(x)$ as

$$h''_{\alpha}(x) = \frac{(-1+\alpha)\alpha}{g_{\alpha}(x)\sqrt{g_{\alpha}(x)}}.$$

We see that $h''_{\alpha}(x) > 0$ for every x . Since we have $\lim_{x \rightarrow 0^+} h'_{\alpha}(x) = -1$ and

$$\lim_{x \rightarrow \frac{1}{6}^-} h'_{\alpha}(x) = -1 + \frac{-1+\alpha}{6\sqrt{g_{\alpha}(\frac{1}{6})}},$$

we obtain that

$$-1 < h'_{\alpha}(x) < -1 + \frac{-1+\alpha}{6\sqrt{g_{\alpha}(\frac{1}{6})}}.$$

From the inequality

$$\left(\frac{856 - 528\sqrt{2}}{2737} \right) \sqrt{\frac{107}{2} + 33\sqrt{2}} < \frac{-1+\alpha}{6\sqrt{g_{\alpha}(\frac{1}{6})}} < \left(\frac{2}{97} + \frac{3}{194}\sqrt{2} \right) \sqrt{97},$$

we get that

$$-1 < h'_{\alpha}(x) < -1 + \left(\frac{2}{97} + \frac{3}{194}\sqrt{2} \right) \sqrt{97}.$$

In particular, we find that

$$1 < h'_{\alpha}(x) + 2 < 1 + \left(\frac{2}{97} + \frac{3}{194}\sqrt{2} \right) \sqrt{97}.$$

This means that the function $h_{\alpha}(x) + 2x$ is increasing. Then we calculate that

$$\frac{-1+\sqrt{\alpha}}{\alpha-1} < h_{\alpha}(x) + 2x < \frac{-\frac{5}{6} - \frac{1}{6}\alpha + \sqrt{g_{\alpha}(\frac{1}{6})}}{\alpha-1} + \frac{1}{3}.$$

More specifically, we obtain

$$0 < \left(1 - \sqrt{5 + 3\sqrt{2}}\right) \left(2 - \frac{3}{2}\sqrt{2}\right) < h_\alpha(x) + 2x < \frac{1}{24} + \frac{\sqrt{97}}{24}.$$

This means $H_\alpha(x) > 0$ for every x and α .

Our aim is to show that $H'_\alpha(x) < 0$ for every x and α . Let us calculate the derivative of H_α using logarithmic differentiation. We obtain that following

$$\begin{aligned} \frac{H'_\alpha}{H_\alpha} &= \frac{h'_\alpha + 2}{h_\alpha + 2x} - \frac{-h'_\alpha - 2}{1 - h_\alpha - 2x} + \frac{-1}{1-x} - \frac{1}{x} \\ &= \frac{h'_\alpha + 2}{(1 - h_\alpha - 2x)(h_\alpha + 2x)} - \frac{1}{(1-x)x} \\ &= \frac{(h'_\alpha + 2)(1-x)x - (1 - h_\alpha - 2x)(h_\alpha + 2x)}{(1 - h_\alpha - 2x)(h_\alpha + 2x)(1-x)x}. \end{aligned}$$

Note that we have

$$(1 - h_\alpha(x) - 2x)(h_\alpha(x) + 2x)(1-x)x > 0$$

for every x and α . We also have

$$\lim_{x \rightarrow 0^+} \left((h'_\alpha(x) + 2)(1-x)x - (1 - h_\alpha(x) - 2x)(h_\alpha(x) + 2x) \right) < 0.$$

Since the expression inside the limit is continuous, it is enough to show that the equation

$$(h'_\alpha(x) + 2)(1-x)x - (1 - h_\alpha(x) - 2x)(h_\alpha(x) + 2x) = 0$$

has no solution for any x and α . Let us assume otherwise. Then we obtain

$$(1 + \frac{x(-1 + \alpha)}{\sqrt{g_\alpha}})(1 - x)x = (1 - \frac{(-1 + (1 - \alpha)x) + \sqrt{g_\alpha}}{(\alpha - 1)} - 2x)(\frac{(-1 + (1 - \alpha)x) + \sqrt{g_\alpha}}{(\alpha - 1)} + 2x)$$

We simplify the equality above as follows:

$$(\alpha - 1)^2(\sqrt{g_\alpha} + (\alpha - 1)x)(1 - x)x = \sqrt{g_\alpha}(\alpha - \alpha x - \sqrt{g_\alpha} + x)(-1 + \alpha x + \sqrt{g_\alpha} - x)$$

We expand the equation above and obtain that

$$(\alpha - 1)^2(x - x^2)\sqrt{g_\alpha} + (\alpha - 1)^3(x^2 - x^3) = (-\alpha + (\alpha^2 - 1)x - (1 - \alpha)^2x^2 - g_\alpha)\sqrt{g_\alpha} + (\alpha + 1 + 2x(1 - \alpha))g_\alpha.$$

Let us collect all the terms containing $\sqrt{g_\alpha}$ on the left hand-side of the equation and the rest on the right hand-side:

$$((\alpha - 1)^2(x - x^2) - (-\alpha + (\alpha^2 - 1)x - (1 - \alpha)^2x^2 - g_\alpha))\sqrt{g_\alpha} = -(\alpha - 1)^3(x^2 - x^3) + (\alpha + 1 + 2x(1 - \alpha))g_\alpha.$$

Then we substitute $g_\alpha(x) = (1 - \alpha)^2x^2 + \alpha$ in the first expression of the term on the left hand-side and find that

$$((\alpha - 1)^2(x - x^2) - (-2\alpha + (\alpha^2 - 1)x - 2(1 - \alpha)^2x^2))\sqrt{g_\alpha} = -(\alpha - 1)^3(x^2 - x^3) + (\alpha + 1 + 2x(1 - \alpha))g_\alpha.$$

Next we simplify the first term on the left hand-side and obtain that

$$((\alpha - 1)^2 x^2 - 2(\alpha - 1)x + 2\alpha)\sqrt{g_\alpha} = -(\alpha - 1)^3(x^2 - x^3) + (\alpha + 1 + 2x(1 - \alpha))g_\alpha.$$

Then we substitute $g_\alpha(x) = (1 - \alpha)^2 x^2 + \alpha$ on the right hand-side and simplify. We get

$$((\alpha - 1)^2 x^2 - 2(\alpha - 1)x + 2\alpha)\sqrt{g_\alpha} = -(\alpha - 1)^3 x^3 + 2(\alpha - 1)^2 x^2 + 2\alpha(1 - \alpha)x + \alpha^2 + \alpha.$$

We divide both sides by $(\alpha - 1)^2 x^2 - 2(\alpha - 1)x + 2\alpha$ to obtain

$$\sqrt{g_\alpha(x)} = \frac{-(\alpha - 1)^3 x^3 + 2(\alpha - 1)^2 x^2 + 2\alpha(1 - \alpha)x + \alpha^2 + \alpha}{(\alpha - 1)^2 x^2 - 2(\alpha - 1)x + 2\alpha}$$

Let us say $f_\alpha(x) = \frac{r_\alpha(x)}{s_\alpha(x)}$ where we define the functions $r_\alpha(x) = -(\alpha - 1)^3 x^3 + 2(\alpha - 1)^2 x^2 + 2\alpha(1 - \alpha)x + \alpha^2 + \alpha$ and $s_\alpha(x) = (\alpha - 1)^2 x^2 - 2(\alpha - 1)x + 2\alpha$. Then we calculate that

$$r'_\alpha s_\alpha - r_\alpha s'_\alpha = -(\alpha - 1)^2((\alpha - 1)^3 x^4 - 4(\alpha - 1)^2 x^3 + 4(\alpha^2 - 1)x^2 + 2\alpha(\alpha - 3)x + 2\alpha).$$

Since we have $x \in (0, \frac{1}{6})$, we get $x^3 < x^2$. Then we obtain that $4(\alpha^2 - 1)x^3 < 4(\alpha^2 - 1)x^2$. If we consider the expressions $4(\alpha^2 - 1)$ and $4(\alpha - 1)^2$, we see that $4(\alpha^2 - 1)x^3 > 4(\alpha - 1)^2 x^3$ since $4(\alpha^2 - 1) - 4(\alpha - 1)^2 = 8(\alpha - 1) > 0$. In other words, we obtain that $4(\alpha^2 - 1)x^2 - 4(\alpha - 1)^2 x^3 > 0$. Since we have the expression

$$(\alpha - 1)^3 x^4 + 2\alpha(\alpha - 3)x + 2\alpha > 0,$$

we obtain that $f'_\alpha < 0$. This means f_α is strictly decreasing on $(0, \frac{1}{6})$ for every $\alpha \in I_\alpha$. Note that we have $\lim_{x \rightarrow 0^+} f_\alpha(x) = \frac{\alpha+1}{2}$ and

$$\lim_{x \rightarrow \frac{1}{6}^-} f_\alpha(x) = -\frac{\alpha^3 - 159\alpha^2 - 261\alpha - 13}{6(\alpha^2 + 58\alpha + 13)}.$$

Thus, we get

$$\frac{907}{231} < f_\alpha(x) < \frac{6+3\sqrt{2}}{2},$$

which implies that

$$f_\alpha(x) > \sqrt{\frac{107}{18} + \frac{11}{3}\sqrt{2}}.$$

This gives a contradiction because we know that

$$3 < \sqrt{g_\alpha(x)} < \sqrt{\frac{107}{18} + \frac{11}{3}\sqrt{2}}.$$

Hence, we obtain that $H'_\alpha(x) < 0$ for every α . This completes the proof of the lemma. $\square$

By the lemma above, we know that $H_\alpha(x)$ is a strictly decreasing function for every $\alpha \in I_\alpha$ over the interval $(0, \frac{1}{6})$. Note that we have $h_{\alpha_*}(x'_2) = x'_1$ and $h_{\alpha_*}(x_2^*) = x_1^*$. Since we set $x'_2 > x_2^*$, we obtain that

$$\begin{aligned} f_5(x'_1, x'_2) &= \frac{x'_1 + 2x'_2}{1 - x'_1 - 2x'_2} \cdot \frac{1 - x'_2}{x'_2} \\ &= \frac{h_{\alpha_*}(x'_2) + 2x'_2}{1 - h_{\alpha_*}(x'_2) - 2x'_2} \cdot \frac{1 - x'_2}{x'_2} \\ &= H_{\alpha_*}(x'_2). \end{aligned}$$

Then we get

$$\begin{aligned}
 H_{\alpha_*}(x'_2) &< H_{\alpha_*}(x_2^*) \\
 &= \frac{h_{\alpha_*}(x_2^*) + 2x_2^*}{1 - h_{\alpha_*}(x_2^*) - 2x_2^*} \cdot \frac{1 - x_2^*}{x_2^*} \\
 &= \frac{x_1^* + 2x_2^*}{1 - x_1^* - 2x_2^*} \cdot \frac{1 - x_2^*}{x_2^*} \\
 &= \alpha_*.
 \end{aligned}$$

Remember that we have $f_i(x'_1, x'_2) = f_j(x'_1, x'_2)$ for every $i, j = 5, 6, 7, 8$.

If we consider the point $\mathbf{x}' = (x'_1, x'_2, x'_3, x'_4, x'_6, x'_7, x'_8) \in \Delta^7$, we see that $f_1(\mathbf{x}') = \alpha_*$, $f_2(\mathbf{x}') = \alpha_*$, and $f_j(\mathbf{x}') < \alpha_*$ for $j = 3, 4, 5, 6, 7, 8$.

Then we apply the following perturbations

$$\begin{aligned}
 &[\mathbf{x}' : f_1 = \alpha_*, f_5 < \alpha_*, f_8 < \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*] \\
 &\quad \downarrow \text{pert}_{\vec{u}_3}^6 \\
 &[\mathbf{y} : f_1 < \alpha_*, f_5 < \alpha_*, f_8 < \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*]
 \end{aligned}$$

and

$$\begin{aligned}
 &[\mathbf{y} : f_2 = \alpha_*, f_6 < \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} < \alpha_*; F = \alpha_*] \\
 &\quad \downarrow \text{pert}_{\vec{u}_4}^6 \\
 &[\mathbf{z} : f_2 < \alpha_*, f_6 < \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} < \alpha_*; F < \alpha_*].
 \end{aligned}$$

In this way, we obtain a point $\mathbf{z} \in \Delta^7$ so that $F(\mathbf{z}) < \alpha_*$. This contradicts with the minimality of α_* .

Therefore, case III is not possible at the point $\mathbf{x}^*$. □

Before proceeding to case IV, it is necessary to state two preliminary lemmas:

Lemma 4.1.6. *Let us define the following continuous functions*

$$g(\alpha) = \frac{(1 - \alpha^2)\sqrt{\alpha} + \alpha^3 + \alpha + \sqrt{d(\alpha)}}{2(-1 + (\alpha^3 - \alpha)\sqrt{\alpha} + \alpha^2)}$$

and

$$h(\alpha) = \frac{(1 - \alpha^2)\sqrt{\alpha} + \alpha^3 + \alpha - \sqrt{d(\alpha)}}{2(-1 + (\alpha^3 - \alpha)\sqrt{\alpha} + \alpha^2)}$$

over the interval I_α where

$$d(\alpha) = \alpha^6 + 6\alpha^3 + \alpha^2 + 4 + (4\alpha^4 + 6\alpha + 4)\sqrt{\alpha} - (3\alpha^5 + 6\alpha^4 + 3\alpha + (2\alpha^5 + 4\alpha^3 + 8\alpha^2)\sqrt{\alpha}).$$

Then neither $G^+(\alpha)$ nor $G^-(\alpha)$ takes the value 1 over the interval I_α if we have

$$G^+(\alpha) = \frac{(\sqrt{\alpha} + 1)(\alpha - 1)(g(\alpha))^2 + 2(\alpha - 1)g(\alpha) + \sqrt{\alpha} + 3}{(\sqrt{\alpha} + 1)((\alpha - 1)g(\alpha) + 1)}$$

and

$$G^-(\alpha) = \frac{(\sqrt{\alpha} + 1)(\alpha - 1)(h(\alpha))^2 + 2(\alpha - 1)h(\alpha) + \sqrt{\alpha} + 3}{(\sqrt{\alpha} + 1)((\alpha - 1)h(\alpha) + 1)}.$$

Proof. See Appendix B.1. □

Lemma 4.1.7. *Let us define the following continuous functions*

$$g(\alpha) = \frac{(\alpha - 1)^2 + \sqrt{(\alpha - 1)(\alpha^3 - 3\alpha^2 - 5\alpha - 9)}}{2(\alpha^2 - 1)},$$

$$h(\alpha) = \frac{(\alpha - 1)^2 - \sqrt{(\alpha - 1)(\alpha^3 - 3\alpha^2 - 5\alpha - 9)}}{2(\alpha^2 - 1)}$$

over the interval I_α . Then neither $G^+(\alpha)$ nor $G^-(\alpha)$ takes the value $\frac{1}{2}$ over the interval I_α where

$$G^+(\alpha) = \frac{(\alpha + 1)(\alpha^2 - 1)(g(\alpha))^2 + 5(\alpha^2 - 1)g(\alpha) + \alpha^2 + 9\alpha - 4}{(\alpha + 1)(\alpha^2 - 1)g(\alpha) + (\alpha + 1)(3\alpha - 1)}$$

and

$$G^-(\alpha) = \frac{(\alpha + 1)(\alpha^2 - 1)(h(\alpha))^2 + 5(\alpha^2 - 1)h(\alpha) + \alpha^2 + 9\alpha - 4}{(\alpha + 1)(\alpha^2 - 1)h(\alpha) + (\alpha + 1)(3\alpha - 1)}.$$

Proof. See Appendix B.2. □

Lemma 4.1.8. *Case IV is not possible at the point $\mathbf{x}^*$.*

Proof. Let us assume that case IV holds at the point $\mathbf{x}^*$. If we use the parametrization $\mathfrak{P}_8$, we see that

$$A(\mathbf{x}^*) \cdot \frac{1 - x_4^*}{x_4^*} < \alpha_* \tag{4.14}$$

$$B(\mathbf{x}^*) \cdot \frac{1 - x_1^*}{x_1^*} = \alpha_* \tag{4.15}$$

$$A(\mathbf{x}^*) \cdot \frac{\sum_{i=1}^7 x_i^*}{1 - \sum_{i=1}^7 x_i^*} = \alpha_* \tag{4.16}$$

$$B(\mathbf{x}^*) \cdot \frac{1 - x_7^*}{x_7^*} = \alpha_* \tag{4.17}$$

$$\frac{1}{A(\mathbf{x}^*)} \cdot \frac{1 - x_6^*}{x_6^*} < \alpha_* \tag{4.18}$$

$$\frac{1}{A(\mathbf{x}^*)} \cdot \frac{1 - x_2^*}{x_2^*} = \alpha_* \tag{4.19}$$

$$\frac{1}{B(\mathbf{x}^*)} \cdot \frac{1-x_3^*}{x_3^*} = \alpha_* \quad (4.20)$$

$$\frac{1}{B(\mathbf{x}^*)} \cdot \frac{1-x_5^*}{x_5^*} < \alpha_*. \quad (4.21)$$

From the equality in (Equation 4.19), we notice that x_2^* is a solution of the equation $q_3(x) = 0$. Therefore, we get

$$x_2^* = \frac{(1-\alpha_*)(1-x_1^*-x_3^*) \pm \sqrt{D_3(x_1^*, x_3^*)}}{2(1-\alpha_*)}.$$

Since we also have the inequality in (Equation 4.18), we know that $D_3(\mathbf{x}^*) > 0$ from case I. There is two choices for x_2^* . Let us assume that

$$\begin{aligned} x_2^* &= \frac{(1-\alpha_*)(1-x_1^*-x_3^*) - \sqrt{D_3(x_1^*, x_3^*)}}{2(1-\alpha_*)} \\ 2(1-\alpha_*)x_2^* &= (1-\alpha_*)(1-x_1^*-x_3^*) - \sqrt{D_3(x_1^*, x_3^*)} \\ 2(1-\alpha_*)x_2^* - (1-\alpha_*)(1-x_1^*-x_3^*) &= -\sqrt{D_3(x_1^*, x_3^*)}. \end{aligned}$$

Then we get

$$(1-\alpha_*)(x_1^* + 2x_2^* + x_3^* - 1) = -\sqrt{D_3(x_1^*, x_3^*)}.$$

Since $1-\alpha_*$ is a negative number, we obtain that $x_1^* + 2x_2^* + x_3^* > 1$. This implies that $x_2^* > x_4^* + x_5^* + x_6^* + x_7^* + x_8^*$. Therefore, we get $x_2^* > x_6^*$. Then, we obtain that

$$\frac{1-x_2^*}{x_2^*} < \frac{1-x_6^*}{x_6^*},$$

and consequently we find that

$$\begin{aligned}\alpha_* &= \frac{1}{A(\mathbf{x}^*)} \cdot \frac{1-x_2^*}{x_2^*} \\ &< \frac{1}{A(\mathbf{x}^*)} \cdot \frac{1-x_6^*}{x_6^*} \\ &< \alpha_*.\end{aligned}$$

using expressions in (Equation 4.19) and (Equation 4.18). This is a contradiction. Therefore, we get

$x_1^* + 2x_2^* + x_3^* < 1$. In other words, we obtain that

$$x_2^* = \frac{(1-\alpha_*)(1-x_1^*-x_3^*) + \sqrt{D_3(x_1^*, x_3^*)}}{2(1-\alpha_*)}. \quad (4.22)$$

We know that $x_1^* + x_2^* + x_3^*$ is a solution of the equation $q_4(x)$ from case II which we also have in case IV. Hence, we get $D_4(\mathbf{x}^*) \geq 0$. Let us assume that $D_4(\mathbf{x}^*) = 0$. Then we get $x_1^* + x_2^* + x_3^* = x_8^*$. Using the equality in (Equation 4.16), we see that

$$\left(\frac{1-x_8^*}{x_8^*} \right)^2 = \alpha_*.$$

In other words, we get

$$x_8^* = \frac{1}{\sqrt{\alpha_*} + 1}.$$

Since we know x_8^* in terms of α_* , we can calculate every entry of $\mathbf{x}^*$ in terms of α_* .

From the equality in (Equation 4.19), we know that

$$\begin{aligned} \frac{x_1^* + x_2^* + x_3^*}{1 - x_1^* - x_2^* - x_3^*} \cdot \frac{1 - x_2^*}{x_2^*} &= \frac{x_8^*}{1 - x_8^*} \cdot \frac{1 - x_2^*}{x_2^*} \\ &= \alpha_*. \end{aligned}$$

We can solve x_2^* in terms of α_* :

$$x_2^* = \frac{1}{\alpha_* \sqrt{\alpha_*} + 1}.$$

If we multiply the equalities (Equation 4.15) and (Equation 4.20) side by side, we obtain that

$$\frac{1 - x_1^*}{x_1^*} \cdot \frac{1 - x_3^*}{x_3^*} = \alpha_*^2.$$

In other words we obtain that

$$1 - (x_1^* + x_3^*) = (\alpha_*^2 - 1)x_1^* x_3^*. \quad (4.23)$$

Since we know $x_1^* + x_2^* + x_3^*$ and x_2^* in terms of α_* , we can find $x_1^* + x_3^*$ in terms of α_* :

$$\begin{aligned} x_1^* + x_3^* &= x_1^* + x_2^* + x_3^* - x_2^* \\ &= \frac{1}{\sqrt{\alpha_*} + 1} - \frac{1}{\alpha_* \sqrt{\alpha_*} + 1} \\ &= \frac{\alpha_* - \sqrt{\alpha_*}}{\alpha_* \sqrt{\alpha_*} + 1}. \end{aligned}$$

In other words, we get

$$x_3^* = \frac{\alpha_* - \sqrt{\alpha_*}}{\alpha_* \sqrt{\alpha_*} + 1} - x_1^*$$

If we replace x_3^* in the (Equation 4.23) with the expression above, we obtain the following quadratic equation in x_1^* :

$$(\alpha_* \sqrt{\alpha_*} + 1)(\alpha_*^2 - 1)(x_1^*)^2 - \sqrt{\alpha_*}(\alpha_*^2 - 1)(\sqrt{\alpha_*} - 1)x_1^* + \alpha_* \sqrt{\alpha_*} + 1 - \alpha_* + \sqrt{\alpha_*} = 0.$$

Then we find that

$$x_1^* = \frac{(1 - \alpha_*^2)\sqrt{\alpha_*} + \alpha_*^3 - \alpha_* \pm \sqrt{d(\alpha_*)}}{2(-1 + (\alpha_*^3 - \alpha_*)\sqrt{\alpha_*} + \alpha_*^2)}.$$

where

$$d(\alpha) = \alpha^6 - 3\alpha^5 - 6\alpha^4 + 6\alpha^3 + \alpha^2 - 3\alpha + 4 - (2\alpha^5 - 4\alpha^4 + 4\alpha^3 + 8\alpha^2 - 6\alpha - 4)\sqrt{\alpha}.$$

Let us define the functions

$$g(\alpha) = \frac{(1 - \alpha^2)\sqrt{\alpha} + \alpha^3 - \alpha + \sqrt{d(\alpha)}}{2(-1 + (\alpha^3 - \alpha)\sqrt{\alpha} + \alpha^2)}$$

and

$$h(\alpha) = \frac{(1 - \alpha^2)\sqrt{\alpha} + \alpha^3 - \alpha - \sqrt{d(\alpha)}}{2(-1 + (\alpha^3 - \alpha)\sqrt{\alpha} + \alpha^2)}.$$

Note that $x_1^* = g(\alpha_*)$ or $x_1^* = h(\alpha_*)$. Using the equality in (Equation 4.15), we calculate that

$$x_4^* + x_5^* + x_6^* = \frac{1 - x_1^*}{(\alpha_* - 1)x_1^* + 1}.$$

Remember that we have $x_1^* = x_7^*$. Since we know that

$$1 = \sum_{i=1}^8 x_i^*$$

we see that

$$\begin{aligned}
1 &= \sum_{i=1}^8 x_i^* \\
&= x_1^* + x_4^* + x_5^* + x_6^* + 2x_8^* \\
&= x_1^* + \frac{1 - x_1^*}{(\alpha_* - 1)x_1^* + 1} + 2x_8^* \\
&= \frac{(\alpha_* - 1)(x_1^*)^2 + 1}{(\alpha_* - 1)x_1^* + 1} + \frac{2}{\sqrt{\alpha_*} + 1} \\
&= \frac{(\sqrt{\alpha_*} + 1)((\alpha_* - 1)(x_1^*)^2 + 1) + 2((\alpha_* - 1)x_1^* + 1)}{(\sqrt{\alpha_*} + 1)((\alpha_* - 1)x_1^* + 1)} \\
&= \frac{(\sqrt{\alpha_*} + 1)(\alpha_* - 1)(x_1^*)^2 + 2(\alpha_* - 1)x_1^* + \sqrt{\alpha_*} + 3}{(\sqrt{\alpha_*} + 1)((\alpha_* - 1)x_1^* + 1)}.
\end{aligned}$$

Then We obtain that

$$1 = \frac{(\sqrt{\alpha_*} + 1)(\alpha_* - 1)(g(\alpha_*))^2 + 2(\alpha_* - 1)g(\alpha_*) + \sqrt{\alpha_*} + 3}{(\sqrt{\alpha_*} + 1)((\alpha_* - 1)g(\alpha_*) + 1)}$$

if $x_1^* = g(\alpha_*)$ or

$$1 = \frac{(\sqrt{\alpha_*} + 1)(\alpha_* - 1)(h(\alpha_*))^2 + 2(\alpha_* - 1)h(\alpha_*) + \sqrt{\alpha_*} + 3}{(\sqrt{\alpha_*} + 1)((\alpha_* - 1)h(\alpha_*) + 1)}$$

if $x_1^* = h(\alpha_*)$. Let us define the following continuous functions

$$G^+(\alpha) = \frac{(\sqrt{\alpha} + 1)(\alpha - 1)(g(\alpha))^2 + 2(\alpha - 1)g(\alpha) + \sqrt{\alpha} + 3}{(\sqrt{\alpha} + 1)((\alpha - 1)g(\alpha) + 1)}$$

and

$$G^-(\alpha) = \frac{(\sqrt{\alpha} + 1)(\alpha - 1)(h(\alpha))^2 + 2(\alpha - 1)h(\alpha) + \sqrt{\alpha} + 3}{(\sqrt{\alpha} + 1)((\alpha - 1)h(\alpha) + 1)}.$$

Then we must have $G^\pm(\alpha_*) = 1$ where $\alpha_* \in I_\alpha$. In Lemma 4.1.6, we show that this is not the case because $G^\pm(\alpha) \neq 1$ for every $\alpha \in I_\alpha$. Therefore, we have $D_4(\mathbf{x}^*) > 0$. We need to consider two cases:

(i) $x_1^* + x_2^* + x_3^* < x_8^*$ or,

(ii) $x_1^* + x_2^* + x_3^* > x_8^*$.

(i) Let us assume that $x_1^* + x_2^* + x_3^* < x_8^*$. We know that $x_1^* + 2x_2^* + x_3^* < 1$. Therefore, we have

$$x_2^* = \frac{(1 - \alpha_*)(1 - x_1^* - x_3^*) + \sqrt{D_3(x_1^*, x_3^*)}}{2(1 - \alpha_*)}.$$

Note that D_3 can be considered as a function of $x_1 + x_3$. Let us say $x_1 + x_3 = x$. Since we know that $D_3(x)$ is continuous and $D_3(x_1^* + x_3^*) > 0$, there exists an $\epsilon > 0$ so that $D_3(x_1' + x_3') > 0$ where

$$\begin{aligned} x_1' &= x_1^* + \frac{\epsilon}{2} \\ x_3' &= x_3^* + \frac{\epsilon}{2}. \end{aligned}$$

Hence, if we define

$$\tilde{q}_3(x) = (1 - \alpha)x^2 - (1 - \alpha)(1 - x_1' - x_3')x - (x_1' + x_3'),$$

we see that one of roots of the quadratic equation $\tilde{q}_3(x) = 0$ is

$$x_2' = \frac{(1 - \alpha_*)(1 - x_1' - x_3') + \sqrt{D_3(x_1' + x_3')}}{2(1 - \alpha_*)}.$$

We can choose $\epsilon > 0$ so that the coefficients of $q_3(x)$ and $\tilde{q}_3(x)$ are sufficiently close each other to conclude that $0 < x'_i < 1$ for $i = 1, 2, \dots, 8$ where we set $x'_4 = x_4^*$, $x'_5 = x_5^*$, $x'_6 = x_6^*$, $x'_7 = x_7^*$ and

$$x'_8 = 1 - \sum_{i=1}^7 x'_i.$$

Using the continuity of $A(x_1, x_2, x_3)$, $\epsilon > 0$ can be chosen so that the following two inequalities also hold

$$\begin{aligned} A(x'_1, x'_2, x'_3) \cdot \frac{1 - x'_4}{x'_4} &< \alpha_*, \\ \frac{1}{A(x'_1, x'_2, x'_3)} \cdot \frac{1 - x'_6}{x'_6} &< \alpha_*. \end{aligned}$$

Since we have $\tilde{q}_3(x'_2) = 0$, we get $f_6(x'_1, x'_2, x'_3) = \alpha_*$. Note that we also have $B(x_1^*, x_2^*, x_3^*) = B(x'_1, x'_2, x'_3)$ and

$$\begin{aligned} \frac{1 - x'_1}{x'_1} &< \frac{1 - x_1^*}{x_1^*}, \\ \frac{1 - x'_3}{x'_3} &< \frac{1 - x_3^*}{x_3^*}. \end{aligned}$$

Therefore, we obtain that $f_2(x'_1, x'_4, x'_5, x'_6) < \alpha_*$, $f_7(x'_3, x'_4, x'_5, x'_6) < \alpha_*$ and $f_8(x'_4, x'_5, x'_6) < \alpha_*$. At this stage of the proof, we define the continuous function

$$h_\alpha(x) = \frac{(1 - \alpha)(1 - x) + \sqrt{(1 - \alpha)^2(1 - x)^2 + 4(1 - \alpha)x}}{2(1 - \alpha)},$$

and the continuous function

$$H_\alpha(x) = \frac{1-x-h_\alpha(x)}{x+h_\alpha(x)} \cdot \frac{x+h_\alpha(x)+K}{1-x-h_\alpha(x)-K}$$

where $K = x_4^* + x_5^* + x_6^* + x_7^*$. Then we prove that

Lemma 4.1.9. $H_\alpha(x)$ is a strictly decreasing function around a neighborhood I of $x_1^* + x_3^*$ for every $\alpha \in I_\alpha$.

Proof. Remember that we have $x_1^* = x_7^*$ from the equalities in (Equation 4.15) and (Equation 4.17).

From the expressions in (Equation 4.20) and (Equation 4.21), we derive that $x_3^* < x_5^*$. By the assumption of (i), we know that $x_1^* + x_3^* < x_8^*$. Therefore, we get that $0 < x_1^* + x_3^* < \frac{1}{3}$. We need to show that

$H'_\alpha(x_1^* + x_3^*) < 0$ for every $\alpha \in I_\alpha$.

Let us calculate the derivative of $H_\alpha(x)$ using logarithmic differentiation:

$$\begin{aligned} \frac{H'_\alpha}{H_\alpha} &= \frac{-1-h'_\alpha}{1-x-h_\alpha} - \frac{1+h'_\alpha}{x+h_\alpha} + \frac{1+h'_\alpha}{x+h_\alpha+K} - \frac{-1-h'_\alpha}{1-x-h_\alpha-K} \\ &= (1+h'_\alpha) \left[\frac{-1}{1-x-h_\alpha} - \frac{1}{x+h_\alpha} + \frac{1}{x+h_\alpha+K} - \frac{-1}{1-x-h_\alpha-K} \right] \\ &= (1+h'_\alpha) \left[\frac{-x-h_\alpha-1+x+h_\alpha}{(1-x-h_\alpha)(x+h_\alpha)} + \frac{x+h_\alpha+K+1-x-h_\alpha-K}{(x+h_\alpha+K)(1-x-h_\alpha-K)} \right] \\ &= (1+h'_\alpha) \left[\frac{-1}{(1-x-h_\alpha)(x+h_\alpha)} + \frac{1}{(x+h_\alpha+K)(1-x-h_\alpha-K)} \right] \\ &= (1+h'_\alpha) \left[\frac{-(x+h_\alpha+K)(1-x-h_\alpha-K) + (1-x-h_\alpha)(x+h_\alpha)}{(1-x-h_\alpha)(x+h_\alpha)(x+h_\alpha+K)(1-x-h_\alpha-K)} \right] \\ &= (1+h'_\alpha) \left[\frac{K(K-1+2x+2h_\alpha)}{(1-x-h_\alpha)(x+h_\alpha)(x+h_\alpha+K)(1-x-h_\alpha-K)} \right]. \end{aligned}$$

Note that we have

$$\begin{aligned}
 K(K-1+2(x_1^*+x_3^*)+2h_\alpha(x_1^*+x_3^*)) &= \left(\sum_{k=4}^7 x_k^* - 1 + 2x_1^* + 2x_3^* + 2x_2^*\right) \sum_{k=4}^7 x_k^* \\
 &= (x_1^* + x_2^* + x_3^* - x_8^*) \sum_{k=4}^7 x_k^* \\
 &< 0
 \end{aligned}$$

by the assumption of (i). Thus, it is enough to show that $1 + h'_\alpha(x_1^* + x_3^*) > 0$ for every $\alpha \in I_\alpha$. To this purpose, we will show that $1 + h'_\alpha(x) > 0$ on the interval $(0, \frac{1}{3})$ for every $\alpha \in I_\alpha$. We calculate that

$$\begin{aligned}
 1 + h'_\alpha(x) &= 1 - \frac{1}{2} - \frac{1}{2(\alpha-1)} \cdot \frac{-2(1-\alpha)^2(1-x) + 4(1-\alpha)}{2\sqrt{(1-\alpha)^2(1-x)^2 + 4(1-\alpha)x}} \\
 &= \frac{1}{2} + \frac{1}{2} \cdot \frac{(\alpha-1)(1-x) + 2}{\sqrt{(1-\alpha)^2(1-x)^2 + 4(1-\alpha)x}}.
 \end{aligned}$$

Note that we have $(\alpha-1)(1-x)+2 > 0$ on the interval $(0, \frac{1}{3})$ for every $\alpha \in I_\alpha$. This proves that $1 + h'_\alpha(x) > 0$ on the interval $(0, \frac{1}{3})$ for every $\alpha \in I_\alpha$. In particular, we get $1 + h'_\alpha(x_1^* + x_3^*) > 0$ for every $\alpha \in I_\alpha$. Since $H_\alpha(x)$ is continuous and $H_\alpha(x_1^* + x_3^*) > 0$, there exists a neighborhood I_0 of $x_1^* + x_3^*$ so that $H_\alpha(x) > 0$ for every $x \in I_0$. Similarly, there exists a neighborhood I_1 of $x_1^* + x_3^*$ so that $(1-x-h_\alpha)(x+h_\alpha)(x+h_\alpha+K)(1-x-h_\alpha-K) > 0$ because the factors are continuous and $1-(x_1^*+x_3^*)-h_\alpha(x_1^*+x_3^*) = 1-(x_1^*+x_3^*)-x_2^*$, $x_1^*+x_3^*+h_\alpha(x_1^*+x_3^*) = x_1^*+x_2^*+x_3^*$, $x_1^*+x_3^*+h_\alpha(x_1^*+x_3^*)+K = \sum_{k=1}^6 x_k^*$, $1-(x_1^*+x_3^*)-h_\alpha(x_1^*+x_3^*)-K = x_7^*+x_8^*$ are all positive. Hence, the calculations given above show that $H'_\alpha(x_1^* + x_3^*) < 0$. Therefore, there exists a neighborhood $I = (0, \frac{1}{3}) \cap I_0 \cap I_1$ of $x_1^* + x_3^*$ so that $H'_\alpha(x) < 0$ for every $x \in I$. This finishes the proof. $\square$

Note that we have $h_{\alpha_*}(x_1^* + x_3^*) = x_2^*$ and $h_{\alpha_*}(x_1' + x_3') = x_2'$. Since we set $x_1' + x_3' > x_1^* + x_3^*$, we obtain that

$$\begin{aligned}
 f_3(x_1', x_2', x_3', x_4', x_5', x_6', x_7') &= \frac{1 - x_1' - x_2' - x_3'}{x_1' + x_2' + x_3'} \cdot \frac{\sum_{i=1}^7 x_i'}{1 - \sum_{i=1}^7 x_i'} \\
 &= \frac{1 - (x_1' + x_3') - h_{\alpha_*}(x_1' + x_3')}{x_1' + x_3' + h_{\alpha_*}(x_1' + x_3')} \cdot \frac{x_1' + x_3' + h_{\alpha_*}(x_1' + x_3') + \sum_{i=4}^7 x_i'}{1 - x_1' - x_3' - h_{\alpha_*}(x_1' + x_3') - \sum_{i=4}^7 x_i'} \\
 &= H_{\alpha_*}(x_1' + x_3')
 \end{aligned}$$

Since we know that $H_{\alpha_*}(x)$ is a strictly decreasing function on an interval that contains $x_1^* + x_3^*$ by Lemma 4.1.9, we get that

$$\begin{aligned}
 H_{\alpha_*}(x_1' + x_3') &< H_{\alpha_*}(x_1^* + x_3^*) \\
 &= \frac{1 - (x_1^* + x_3^*) - h_{\alpha_*}(x_1^* + x_3^*)}{x_1^* + x_3^* + h_{\alpha_*}(x_1^* + x_3^*)} \cdot \frac{x_1^* + x_3^* + h_{\alpha_*}(x_1^* + x_3^*) + \sum_{i=4}^7 x_i^*}{1 - x_1^* - x_3^* - h_{\alpha_*}(x_1^* + x_3^*) - \sum_{i=4}^7 x_i^*} \\
 &= \frac{1 - x_1^* - x_2^* - x_3^*}{x_1^* + x_2^* + x_3^*} \cdot \frac{\sum_{i=1}^7 x_i^*}{1 - \sum_{i=1}^7 x_i^*} \\
 &= \alpha_*.
 \end{aligned}$$

If we consider the point $\mathbf{x}' = (x_1', x_2', x_3', x_4', x_5', x_6', x_7', x_8') \in \Delta^7$, we see that $f_j(\mathbf{x}') < \alpha_*$ for every $j = 1, 2, 3, 5, 7, 8$ and $f_j(\mathbf{x}') = \alpha_*$ for $j = 4, 6$. Then we apply the following perturbations

$$\begin{array}{c}
 [\mathbf{x}' : f_3 < \alpha_*, f_4 = \alpha_*, f_{j \notin I_2} \leq \alpha_*; F = \alpha_*] \\
 \downarrow \text{pert}_{\bar{u}_7}^3 \\
 [\mathbf{y} : f_3 < \alpha_*, f_4 < \alpha_*, f_{j \notin I_3} \leq \alpha_*; F = \alpha_*]
 \end{array}$$

and

$$\begin{array}{c} [\mathbf{y}: f_2 < \alpha_*, f_6 = \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} < \alpha_*; F = \alpha_*] \\ \downarrow \text{pert}_{\bar{u}_4}^6 \\ [\mathbf{z}: f_2 < \alpha_*, f_6 < \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} < \alpha_*; F < \alpha_*]. \end{array}$$

Hence, we obtain a point $\mathbf{z} \in \Delta^7$ so that $F(\mathbf{z}) < \alpha_*$. This contradicts with the minimality of α_* . Therefore, we cannot have $x_1^* + x_2^* + x_3^* < x_8^*$.

(ii) Let us assume that $x_1^* + x_2^* + x_3^* > x_8^*$. We need to divide the case (ii) into three parts:

(iii) $x_1^* > x_8^*$,

(iv) $x_1^* = x_8^*$,

(v) $x_1^* < x_8^*$.

(iii) Let us assume that $x_8^* < x_1^*$. Remember that we have $x_1^* = x_7^*$. Therefore, we obtain that $x_8^* < x_1^*$. We know that $q_4(x_1^* + x_2^* + x_3^*) = 0$ by the equality in (Equation 4.4) where

$$q_4(x) = (\alpha_* - 1)x^2 + (1 - \alpha_*) \left(1 - \sum_{k=4}^7 x_k^* \right) x + \sum_{k=4}^7 x_k^*.$$

The assumption $x_1^* + x_2^* + x_3^* > x_8^*$ implies that $D_4(\mathbf{x}^*) > 0$. Let us denote the second root of $q_4(x) = 0$ with t_1 . Then we get

$$t_1 = \frac{(\alpha_* - 1) \left(1 - \sum_{k=4}^7 x_k^* \right) - \sqrt{D_4(\mathbf{x}^*)}}{2(\alpha_* - 1)}.$$

We also know that $q_5(x_4^* + x_5^* + x_6^*) = 0$ from the equality in (Equation 4.5). Therefore, we get $D_5(\mathbf{x}^*) \geq$

0. Let us denote the second root of $q_5(x) = 0$ with t_2 . In other words, we obtain that

$$t_2 = \frac{(\alpha_* - 1) \left(1 - \sum_{k=1,2,3,8} x_k^* \right) - \sqrt{D_5(\mathbf{x}^*)}}{2(\alpha_* - 1)}.$$

The formula for the addition of the roots of a quadratic equation for $q_4(x) = 0$ gives that

$$x_1^* + x_2^* + x_3^* + t_1 = 1 - \sum_{k=4}^7 x_k^*.$$

Therefore, we obtain that $t_1 = x_8^*$. Similarly, if we consider the quadratic equation $q_5(x) = 0$ where

$$q_5(x) = (\alpha_* - 1)x^2 + (1 - \alpha_*) \left(1 - \sum_{k=1}^3 x_k^* - x_8^* \right) x + \sum_{k=1}^3 x_k^* + x_8^*,$$

we know that $q_5(x_4^* + x_5^* + x_6^*) = 0$. Let us denote the second root of $q_5(x)$ with t_2 . The formula for the

addition of the roots of a quadratic equation for $q_5(x) = 0$ gives that

$$x_4^* + x_5^* + x_6^* + t_2 = 1 - \sum_{k=1,2,3,8} x_k^*.$$

In other words, we get that $t_2 = x_7^*$. Let us define the following linear function

$$\begin{aligned} L(x) &= q_4(x) - q_5(x) \\ &= \left(\sum_{k=4}^7 x_k^* - \sum_{k=1,2,3,8} x_k^* \right) ((\alpha_* - 1)x + 1). \end{aligned}$$

Using the expressions in (Equation 4.14) and (Equation 4.16), we conclude that $x_8^* < x_4^*$. We have $x_2^* < x_6^*$ and $x_3^* < x_5^*$. Then, we obtain that $x_2^* + x_3^* + x_8^* < x_4^* + x_5^* + x_6^*$. The last inequality implies that $x_1^* + x_2^* + x_3^* + x_8^* < x_4^* + x_5^* + x_6^* + x_7^*$. Therefore, L has only one root x_0 where

$$x_0 = \frac{-1}{\alpha_* + 1} < 0.$$

Note that we have

$$\begin{aligned} L(0) &= q_4(0) - q_5(0) \\ &= \sum_{k=4}^7 x_k^* - \sum_{k=1,2,3,8} x_k^* \\ &> 0. \end{aligned}$$

It is clear that $x_1^* < x_1^* + x_2^* + x_3^*$. Since we have $x_8^* < x_1^* < x_1^* + x_2^* + x_3^*$, we get $q_4(x_1^*) < 0$. Thus, if we calculate $L(x)$ at the point x_1^* , we see that

$$\begin{aligned} L(x_1^*) &= q_4(x_1^*) - q_5(x_1^*) \\ &= q_4(x_1^*) \\ &< 0. \end{aligned}$$

By the Intermediate Value Theorem, $L(x)$ must have a root in the interval $(0, 1)$. This is a contradiction. Hence, we cannot have $x_8^* < x_1^*$.

(iv) Let us assume that $x_1^* = x_8^*$. If we compare the equalities in (Equation 4.15) and (Equation 4.16), we see that $A(\mathbf{x}^*) = B(\mathbf{x}^*)$. This means that $x_1^* + x_2^* + x_3^* = x_4^* + x_6^* + x_7^*$. Using the expressions in (Equation 4.14) and (Equation 4.15), we conclude that $x_1^* < x_4^*$. We know that $x_2^* < x_6^*$ and $x_3^* < x_5^*$ from the expressions given in (Equation 4.18)-(Equation 4.21).

This is a contradiction because we obtain that

$$\begin{aligned} x_1^* + x_2^* + x_3^* &= x_4^* + x_5^* + x_6^* \\ &> x_1^* + x_2^* + x_3^*. \end{aligned}$$

Therefore, we cannot have $x_1^* = x_8^*$.

(v) Let us assume that $x_1^* < x_8^*$. We need to divide the case (v) into three parts:

$$(vi) \quad x_4^* + x_5^* + x_6^* < \frac{1}{2},$$

$$(vii) \quad x_4^* + x_5^* + x_6^* = \frac{1}{2},$$

$$(viii) \quad x_4^* + x_5^* + x_6^* > \frac{1}{2}.$$

(vi) Let us assume that $x_4^* + x_5^* + x_6^* < \frac{1}{2}$. If we consider the equalities in (Equation 4.15) and (Equation 4.20), we see that

$$\begin{aligned} \frac{B(\mathbf{x}^*) \cdot \frac{1-x_1^*}{x_1^*}}{\frac{1}{B(\mathbf{x}^*)} \cdot \frac{1-x_3^*}{x_3^*}} &= 1 \\ B^2(\mathbf{x}^*) &= \frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)}. \end{aligned}$$

We solve the equality above for $B(\mathbf{x}^*)$. We obtain that

$$B(\mathbf{x}^*) = \sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)}}.$$

The equality above implies that

$$x_4^* + x_5^* + x_6^* = \frac{1}{\sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)}} + 1} < \frac{1}{2}.$$

Therefore, we conclude that $x_3^* < x_1^*$. Since we know that $x_8^* < x_4^*$ from the expressions in (Equation 4.14) and (Equation 4.17), we see that $x_1^* < x_4^* + x_5^* + x_6^*$. We know that $x_1^* + x_2^* + x_3^*$ is a root of the quadratic equation $q_4(x) = 0$. The condition of (ii) implies that

$$x_1^* + x_2^* + x_3^* = \frac{(\alpha_* - 1) \left(1 - \sum_{k=4}^7 x_k^* \right) + \sqrt{D_4(x_4^*, x_5^*, x_6^*, x_7^*)}}{2(\alpha_* - 1)}.$$

or

$$x_2^* = \frac{(\alpha_* - 1) \left(1 - 3x_1^* - 2x_3^* - \sum_{k=4}^6 x_k^* \right) + \sqrt{D_4(x_4^*, x_5^*, x_6^*, x_7^*)}}{2(\alpha_* - 1)}. \quad (4.24)$$

At the beginning of the proof, we showed that

$$x_2^* = \frac{(1 - \alpha_*)(1 - x_1^* - x_3^*) + \sqrt{D_3(x_1^*, x_3^*)}}{2(1 - \alpha_*)}.$$

In other words, the equations $q_3(x) = 0$ and $q_4(x) = 0$ has x_2^* as a common solution. Let us define the following two variable function

$$U(x_1, x_3) = \frac{1}{\sqrt{\frac{x_1(1-x_3)}{x_3(1-x_1)} + 1}}.$$

Note that we have $x_4^* + x_5^* + x_6^* = U(x_1^*, x_3^*)$. Since we also know that $x_1^* = x_7^*$ from (Equation 4.15) and (Equation 4.17), we obtain that

$$D_4(x_4^*, x_5^*, x_6^*, x_7^*) = (1 - \alpha_*)^2(1 - x_1^* - U(x_1^*, x_3^*))^2 - 4(\alpha_* - 1)(x_1^* + U(x_1^*, x_3^*)).$$

In other words, we see that

$$x_2^* = \frac{(\alpha_* - 1)(1 - 3x_1^* - 2x_3^* - U(x_1^*, x_3^*)) + \sqrt{D_4(x_4^*, x_5^*, x_6^*, x_7^*)}}{2(\alpha_* - 1)}.$$

Let us define the following two variable functions:

$$F_\alpha^1(x_1, x_3) = \frac{(\alpha - 1)(1 - 3x_1 - 2x_3 - U(x_1, x_3)) + \sqrt{d_4(x_1, x_3)}}{2(\alpha - 1)}.$$

and

$$F_\alpha^2(x_1, x_3) = \frac{(1 - \alpha)(1 - x_1 - x_3) + \sqrt{D_3(x_1, x_3)}}{2(1 - \alpha)}$$

where we define the function

$$d_4(x_1, x_3) = (1 - \alpha)^2(1 - x_1 - U(x_1, x_3))^2 - 4(\alpha - 1)(x_1 + U(x_1, x_3))$$

and the function

$$D_3(x_1, x_3) = (1 - \alpha)^2(1 - x_1 - x_3)^2 - 4(\alpha - 1)(x_1 + x_3).$$

Note that we have $x_2^* = F_{\alpha^*}^1(x_1^*, x_3^*)$ by (Equation 4.24) and $x_2^* = F_{\alpha^*}^2(x_1^*, x_3^*)$ by (Equation 4.22) and

$d_4(x_1^*, x_3^*) = D_4(x_4^*, x_5^*, x_6^*, x_7^*)$. Next, we prove the following

Lemma 4.1.10. *For any given $\alpha \in I_\alpha$, let F_α^1 and F_α^2 be the functions defined above. Then there exists a neighborhood of (x_1^*, x_3^*) so that the equation $F^1(x_1, x_3) - F^2(x_1, x_3) = 0$ defines x_3 as a function of x_1 over that neighborhood.*

Proof. We need to show that

$$\frac{\partial F_\alpha^1}{\partial x_3}(x_1^*, x_3^*) - \frac{\partial F_\alpha^2}{\partial x_3}(x_1^*, x_3^*) \neq 0$$

for any given $\alpha \in I_\alpha$. Let us say $R = \{(x_1, x_3) \in \mathbf{R}^2 : 0 < x_1 < \frac{1}{2}, 0 < x_3 < \frac{1}{4}, x_3 < x_1\}$. We know that

$(x_1^*, x_3^*) \in R$. We will show that for any given $\alpha \in I_\alpha$, we have

$$\frac{\partial F_\alpha^1}{\partial x_3}(x_1, x_3) - \frac{\partial F_\alpha^2}{\partial x_3}(x_1, x_3) \neq 0$$

for every $(x_1, x_3) \in R$. If we calculate $\frac{\partial F_\alpha^1}{\partial x_3}$ and $\frac{\partial F_\alpha^2}{\partial x_3}$, we see that

$$\begin{aligned} \frac{\partial F_\alpha^1}{\partial x_3} &= -1 - \frac{1}{2} \cdot \frac{\partial U}{\partial x_3} - \frac{(\alpha - 1) \frac{\partial U}{\partial x_3} (1 - x_1 - U) + 2 \frac{\partial U}{\partial x_3}}{\sqrt{d_4(x_1, x_3)}} \\ &= -1 - \left(\frac{1}{2} + \frac{(\alpha - 1)(1 - x_1 - U) + 2}{\sqrt{d_4(x_1, x_3)}} \right) \frac{\partial U}{\partial x_3}. \end{aligned}$$

Therefore, we obtain that

$$\frac{\partial F_\alpha^2}{\partial x_3} = -\frac{1}{2} + \frac{(\alpha - 1)(1 - x_1 - x_3) + 2}{\sqrt{D_3(x_1, x_3)}}.$$

It is clear that

$$\frac{\partial F_\alpha^2}{\partial x_3}(x_1^*, x_3^*) > -\frac{1}{2}$$

since $(\alpha - 1)(1 - x_1 - x_3) + 2 > 0$ on the region R for every $\alpha \in I_\alpha$. Note that

$$\frac{\partial U}{\partial x_3} = \frac{\frac{x_1}{x_3^2(1-x_1)}}{2 \left(\sqrt{\frac{x_1(1-x_3)}{x_3(1-x_1)}} + 1 \right)^2 \sqrt{\frac{x_1(1-x_3)}{x_3(1-x_1)}}} > 0$$

for every $(x_1, x_3) \in R$. Since we have

$$U(x_1^*, x_3^*) = x_4^* + x_5^* + x_6^* < \frac{1}{2}$$

by the condition (ν) , we conclude that

$$\frac{\partial F_\alpha^1}{\partial x_3}(x_1^*, x_3^*) < -1$$

for every $\alpha \in I_\alpha$. By the Implicit Function Theorem, there exists a neighborhood of (x_1^*, x_3^*) such that

x_3 can be written as a function of x_1 . □

Note that we have

$$\begin{aligned}
 f_2(\mathbf{x}^*) &= f_4(\mathbf{x}^*) \\
 &= B(\mathbf{x}^*) \cdot \frac{1-x_1^*}{x_1^*} \\
 &= \frac{1-U(x_1^*, x_3^*)}{U(x_1^*, x_3^*)} \cdot \frac{1-x_1^*}{x_1^*} \\
 &= \frac{1 - \frac{1}{\sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)} + 1}}}{\sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)} + 1}} \cdot \frac{1-x_1^*}{x_1^*} \\
 &= \sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)}} \cdot \frac{1-x_1^*}{x_1^*} \\
 &= \sqrt{\frac{(1-x_1^*)(1-x_3^*)}{x_1^* x_3^*}}.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 f_7(\mathbf{x}^*) &= \frac{1}{B(\mathbf{x}^*)} \cdot \frac{1-x_3^*}{x_3^*} \\
 &= \frac{U(x_1^*, x_3^*)}{1-U(x_1^*, x_3^*)} \cdot \frac{1-x_3^*}{x_3^*} \\
 &= \frac{1}{\sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)} + 1}} \cdot \frac{1-x_3^*}{x_3^*} \\
 &= \sqrt{\frac{x_3^*(1-x_1^*)}{x_1^*(1-x_3^*)}} \cdot \frac{1-x_3^*}{x_3^*} \\
 &= \sqrt{\frac{(1-x_1^*)(1-x_3^*)}{x_1^* x_3^*}}.
 \end{aligned}$$

In other words, we observe that $f_2(\mathbf{x}^*) = f_4(\mathbf{x}^*)$ and $f_4(\mathbf{x}^*) = f_7(\mathbf{x}^*)$. At this point, we state the following lemma:

Lemma 4.1.11. *Assume that together with Case IV, the inequalities $x_1^* + x_2^* + x_3^* > x_8^*$, $x_1^* < x_8^*$, and $x_4^* + x_5^* + x_6^* < \frac{1}{2}$ hold at the point $\mathbf{x}^*$. Then we have*

$$\frac{x_2^*(1-x_1^*-2x_2^*-x_3^*)(1-x_1^*-x_2^*-x_3^*)}{x_8^*(x_1^*+x_2^*+x_3^*-x_8^*)(x_1^*+x_2^*+x_3^*)} \neq \frac{x_1^*(1-x_1^*)-x_3^*(1-x_3^*)}{(x_4^*+x_5^*+x_6^*)(1-x_4^*-x_5^*-x_6^*)-x_1^*(1-x_1^*)}.$$

Proof. See Appendix C. □

From Lemma 4.1.10, we know that the equation $F_\alpha^1(x, x_3) = F_\alpha^2(x_1, x_3)$ defines x_3 as a function of x_1 around a neighborhood of (x_1^*, x_3^*) , i.e., $x_3 = x_3(x_1)$. Let us replace the variable x_1 with x . Then we define the following functions

$$h_\alpha(x) = x_3(x)$$

and

$$H_\alpha(x) = \sqrt{\frac{(1-x)(1-h_\alpha(x))}{xh_\alpha(x)}}.$$

It is possible to prove the following lemma:

Lemma 4.1.12. *$H_{\alpha_*}(x)$ is a strictly monotone function around a neighborhood of x_1^* .*

Proof. Let us use the logarithmic differentiation to calculate the following: Since we have

$$\log H_{\alpha_*}(x) = \frac{1}{2} \log \left(\frac{(1-x)(1-h_{\alpha_*}(x))}{xh_{\alpha_*}(x)} \right)$$

we get that

$$\begin{aligned}
 \frac{H'_{\alpha_*}}{H_{\alpha_*}} &= \frac{1}{2} \left[\frac{-1}{1-x} - \frac{h'_{\alpha_*}}{1-h_{\alpha_*}} - \frac{1}{x} - \frac{h'_{\alpha_*}}{h_{\alpha_*}} \right] \\
 &= -\frac{1}{2} \left[\frac{1}{1-x} + \frac{h'_{\alpha_*}}{1-h_{\alpha_*}} + \frac{1}{x} + \frac{h'_{\alpha_*}}{h_{\alpha_*}} \right] \\
 &= -\frac{1}{2} \left[\frac{h_{\alpha_*}(1-h_{\alpha_*}) + x(1-x)h'_{\alpha_*}}{xh_{\alpha_*}(1-x)(1-h_{\alpha_*})} \right].
 \end{aligned}$$

We need to show that $H_{\alpha_*}(x_1^*) \neq 0$. Let us assume otherwise. Then we get

$$h_{\alpha_*}(x_1^*)(1-h_{\alpha_*}(x_1^*)) + x_1^*(1-x_1^*)h'_{\alpha_*}(x_1^*) = 0.$$

Solving the previous equality for $h'_{\alpha_*}(x_1^*)$, we find that

$$\begin{aligned}
 h'_{\alpha_*}(x_1^*) &= -\frac{h_{\alpha_*}(x_1^*)(1-h_{\alpha_*}(x_1^*))}{x_1^*(1-x_1^*)} \\
 &= -\frac{x_3^*(1-x_3^*)}{x_1^*(1-x_1^*)}.
 \end{aligned}$$

Remember that $h_{\alpha_*}(x) = x_3(x)$ and $x_3(x_1^*) = x_3^*$. If we use the equality

$$F_{\alpha_*}^1(x, h_{\alpha_*}(x)) = F_{\alpha_*}^2(x, h_{\alpha_*}(x))$$

to calculate $h'_{\alpha_*}(x)$, we obtain that

$$\frac{d'_4}{2\sqrt{d_4}} + \frac{D'_3}{2\sqrt{D_3}} = (\alpha_* - 1)(2 + h'_{\alpha_*} + U'), \quad (4.25)$$

$$\frac{dU}{dx} = - \frac{\frac{1-h_{\alpha_*}}{h_{\alpha_*}(1-x)} - \frac{xh'_{\alpha_*}}{h_{\alpha_*}(1-x)} - \frac{x(1-h_{\alpha_*})h'_{\alpha_*}}{h_{\alpha_*}^2(1-x)} + \frac{x(1-h_{\alpha_*})}{h_{\alpha_*}(1-x)^2}}{2 \left(\sqrt{\frac{x(1-h_{\alpha_*})}{h_{\alpha_*}(1-x)}} + 1 \right)^2 \sqrt{\frac{x(1-h_{\alpha_*})}{h_{\alpha_*}(1-x)}}}.$$

We evaluate the derivative above at the point x_1^* . We obtain that

$$\begin{aligned} U'(x_1^*) &= - \frac{\frac{1-h_{\alpha_*}(x_1^*)}{h_{\alpha_*}(x_1^*)(1-x_1^*)} - \frac{x_1^*h'_{\alpha_*}(x_1^*)}{h_{\alpha_*}(x_1^*)(1-x_1^*)} - \frac{x_1^*(1-h_{\alpha_*}(x_1^*))h'_{\alpha_*}(x_1^*)}{h_{\alpha_*}^2(x_1^*)(1-x_1^*)} + \frac{x_1^*(1-h_{\alpha_*}(x_1^*))}{h_{\alpha_*}(x_1^*)(1-x_1^*)^2}}{2 \left(\sqrt{\frac{x_1^*(1-h_{\alpha_*}(x_1^*))}{h_{\alpha_*}(x_1^*)(1-x_1^*)}} + 1 \right)^2 \sqrt{\frac{x_1^*(1-h_{\alpha_*}(x_1^*))}{h_{\alpha_*}(x_1^*)(1-x_1^*)}}} \\ &= - \frac{\frac{1-x_3^*}{x_3^*(1-x_1^*)} - \frac{x_1^*h'_{\alpha_*}(x_1^*)}{x_3^*(1-x_1^*)} - \frac{x_1^*(1-x_3^*)h'_{\alpha_*}(x_1^*)}{(x_3^*)^2(1-x_1^*)} + \frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)^2}}{2 \left(\sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)}} + 1 \right)^2 \sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)}}} \\ &= - \frac{\frac{1-x_3^*}{x_3^*(1-x_1^*)} - \frac{x_1^* \left(-\frac{x_3^*(1-x_3^*)}{x_1^*(1-x_1^*)} \right)}{x_3^*(1-x_1^*)} - \frac{x_1^*(1-x_3^*) \left(-\frac{x_3^*(1-x_3^*)}{x_1^*(1-x_1^*)} \right)}{(x_3^*)^2(1-x_1^*)} + \frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)^2}}{2 \left(\sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)}} + 1 \right)^2 \sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)}}} \\ &= - \frac{\frac{1-x_3^*}{x_3^*(1-x_1^*)} + \frac{1-x_3^*}{(1-x_1^*)^2} + \frac{(1-x_3^*)^2}{x_3^*(1-x_1^*)^2} + \frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)^2}}{2 \left(\sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)}} + 1 \right)^2 \sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)}}} \\ &= - \frac{(1-x_3^*)(1-x_1^*) + x_3^*(1-x_3^*) + (1-x_3^*)^2 + x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)^2}}{2 \left(\sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)}} + 1 \right)^2 \sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)}}} \\ &= - \frac{\frac{2(1-x_3^*)}{x_3^*(1-x_1^*)^2}}{2 \left(\sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)}} + 1 \right)^2 \sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)}}} \end{aligned}$$

It is possible to simplify $U'(x_1^*)$ using the equalities given in (Equation 4.20) and (Equation 4.15) as follows:

$$\begin{aligned}
 U'(x_1^*) &= -\frac{1-x_3^*}{x_3^*(1-x_1^*)^2 \left(\sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)}} + 1 \right)^2 \sqrt{\frac{x_1^*(1-x_3^*)}{x_3^*(1-x_1^*)}}} \\
 &= -\frac{1}{B(\mathbf{x}^*)} \cdot \frac{1-x_3^*}{x_3^*} \cdot \frac{U^2(x_1^*, x_3^*)}{(1-x_1^*)^2} \\
 &= -\alpha_* \cdot \frac{(x_4^* + x_5^* + x_6^*)^2}{(1-x_1^*)^2} \\
 &= -B(\mathbf{x}^*) \cdot \frac{1-x_1^*}{x_1^*} \cdot \frac{(x_4^* + x_5^* + x_6^*)^2}{(1-x_1^*)^2} \\
 &= -\frac{1-x_4^*-x_5^*-x_6^*}{x_4^*+x_5^*+x_6^*} \cdot \frac{1-x_1^*}{x_1^*} \cdot \frac{(x_4^* + x_5^* + x_6^*)^2}{(1-x_1^*)^2} \\
 &= -\frac{(x_4^* + x_5^* + x_6^*)(1-x_4^*-x_5^*-x_6^*)}{x_1^*(1-x_1^*)}.
 \end{aligned}$$

Note that we have

$$\begin{aligned}
 d'_4(x_1^*) &= 2(\alpha_* - 1)^2(-1 - U')(1 - x_1^* - U(x_1^*, x_3^*)) - 4(\alpha_* - 1)(1 + U'(x_1^*)) \\
 &= -2(\alpha_* - 1)(1 + U'(x_1^*))((\alpha_* - 1)(1 - x_1^* - U(x_1^*, x_3^*)) + 2)
 \end{aligned}$$

and

$$\begin{aligned}
 D'_3(x_1^*) &= 2(\alpha_* - 1)^2(-1 - h'_{\alpha_*}(x_1^*))(1 - x_1^* - h_{\alpha_*}(x_1^*)) - 4(\alpha_* - 1)(1 + h'_{\alpha_*}(x_1^*)) \\
 &= -2(\alpha_* - 1)(1 + h'_{\alpha_*}(x_1^*))((\alpha_* - 1)(1 - x_1^* - h_{\alpha_*}(x_1^*)) + 2)
 \end{aligned}$$

If we simplify the equality in (Equation 4.25) by multiplying both sides with $4\sqrt{d_4}\sqrt{D_3}$, we find that

$$d'_4\sqrt{D_3} + D'_3\sqrt{d_4} = 2\sqrt{d_4}\sqrt{D_3}(\alpha_* - 1)(2 + h'_{\alpha_*} + U') \quad (4.26)$$

Note that we have

$$\sqrt{D_3(x_1^*, x_3^*)} = (\alpha_* - 1)(1 - x_1^* - 2x_2^* - x_3^*)$$

and

$$\sqrt{d_4(x_1^*, x_3^*)} = (\alpha_* - 1)(x_1^* + x_2^* + x_3^* - x_8^*).$$

If we plug the two expressions above into the terms $d'_4\sqrt{D_3}$ and $D'_3\sqrt{d_4}$, we get

$$d'_4\sqrt{D_3} = -2(\alpha_* - 1)^2(1 + U'(x_1^*))((\alpha_* - 1)(1 - x_1^* - U(x_1^*, x_3^*)) + 2)(1 - x_1^* - 2x_2^* - x_3^*),$$

$$D'_3\sqrt{d_4} = -2(\alpha_* - 1)^2(1 + h'_{\alpha_*}(x_1^*))((\alpha_* - 1)(1 - x_1^* - h_{\alpha_*}(x_1^*)) + 2)(x_1^* + x_2^* + x_3^* - x_8^*),$$

and

$$2\sqrt{d_4}\sqrt{D_3}(\alpha_* - 1)(2 + h'_{\alpha_*} + U') = 2(\alpha_* - 1)^3 \left(1 - \sum_{i=1}^3 x_i^* - x_2^*\right) \left(\sum_{i=1}^3 x_i^*\right) (2 + h'_{\alpha_*} + U'). \quad (4.27)$$

Let us say

$$A_1 = ((\alpha_* - 1)(1 - x_1^* - U(x_1^*, x_3^*)) + 2)(1 - x_1^* - 2x_2^* - x_3^*),$$

$$A_2 = ((\alpha_* - 1)(1 - x_1^* - h_{\alpha_*}(x_1^*)) + 2)(x_1^* + x_2^* + x_3^* - x_8^*),$$

$$A_3 = (1 - x_1^* - 2x_2^* - x_3^*)(x_1^* + x_2^* + x_3^* - x_8^*).$$

The term $2(\alpha_* - 1)^2$ is a common factor for left-hand side of the equality in (Equation 4.26) and the right-hand sides of the equality in (Equation 4.27). If we divide both sides with $2(\alpha_* - 1)^2$, we obtain that

$$\begin{aligned} -(1 + U')A_1 - (1 + h'_{\alpha_*})A_2 &= (\alpha_* - 1)A_3(2 + h'_{\alpha_*} + U') \\ -(1 + U')A_1 - (1 + h'_{\alpha_*})A_2 &= (\alpha_* - 1)A_3(1 + h'_{\alpha_*} + 1 + U') \\ -(1 + U')((\alpha_* - 1)A_3 + A_1) &= (1 + h'_{\alpha_*})((\alpha_* - 1)A_3 + A_2). \end{aligned} \quad (4.28)$$

We simplify $(\alpha_* - 1)A_3 + A_1$ as follows:

$$\begin{aligned} (\alpha_* - 1)A_3 + A_1 &= ((\alpha_* - 1)(1 - x_1^* - U(x_1^*, x_3^*) + x_1^* + x_2^* + x_3^* - x_8^*) + 2)(1 - x_1^* - 2x_2^* - x_3^*) \\ &= ((\alpha_* - 1)(1 - x_1^* - x_4^* - x_5^* - x_6^* + x_1^* + x_2^* + x_3^* - x_8^*) + 2)(1 - x_1^* - 2x_2^* - x_3^*) \\ &= (\alpha_* - 1)(2x_1^* + 2x_2^* + 2x_3^*)(1 - x_1^* - 2x_2^* - x_3^*) + 2(1 - x_1^* - 2x_2^* - x_3^*) \\ &= 2(1 - x_1^* - 2x_2^* - x_3^*)((\alpha_* - 1)(x_1^* + x_2^* + x_3^*) + 1) \\ &= 2(1 - x_1^* - 2x_2^* - x_3^*) \left(\left(\frac{1 - x_1^* - x_2^* - x_3^*}{x_1^* + x_2^* + x_3^*} \cdot \frac{1 - x_8^*}{x_8^*} - 1 \right) (x_1^* + x_2^* + x_3^*) + 1 \right) \\ &= 2(1 - x_1^* - 2x_2^* - x_3^*) \left(\frac{1 - x_1^* - x_2^* - x_3^* - x_8^*}{(x_1^* + x_2^* + x_3^*)x_8^*} \cdot (x_1^* + x_2^* + x_3^*) + 1 \right) \\ &= 2(1 - x_1^* - 2x_2^* - x_3^*) \left(\frac{1 - x_1^* - x_2^* - x_3^* - x_8^*}{x_8^*} \cdot + 1 \right) \\ &= 2(1 - x_1^* - 2x_2^* - x_3^*) \cdot \frac{1 - x_1^* - x_2^* - x_3^*}{x_8^*}. \end{aligned} \quad (4.29)$$

Similarly, we simplify $(\alpha_* - 1)A_3 + A_2$ as follows:

$$\begin{aligned}
(\alpha_* - 1)A_3 + A_2 &= ((\alpha_* - 1)(1 - x_1^* - h_{\alpha_*}(x_1^*) + 1 - x_1^* - 2x_2^* - x_3^*) + 2)(x_1^* + x_2^* + x_3^* - x_8^*) \\
&= ((\alpha_* - 1)(1 - x_1^* - x_3^* + 1 - x_1^* - 2x_2^* - x_3^*) + 2)(x_1^* + x_2^* + x_3^* - x_8^*) \\
&= ((\alpha_* - 1)(2 - 2x_1^* - 2x_2^* - 2x_3^*) + 2)(x_1^* + x_2^* + x_3^* - x_8^*) \\
&= 2(x_1^* + x_2^* + x_3^* - x_8^*) \left(\left(\frac{x_1^* + x_2^* + x_3^*}{1 - x_1^* - x_2^* - x_3^*} \cdot \frac{1 - x_2^*}{x_2^*} - 1 \right) (1 - x_1^* - x_2^* - x_3^*) + 1 \right) \\
&= 2(x_1^* + x_2^* + x_3^* - x_8^*) \left(\frac{x_1^* + x_2^* + x_3^* - x_2^*}{(1 - x_1^* - x_2^* - x_3^*)x_2^*} (1 - x_1^* - x_2^* - x_3^*) + 1 \right) \\
&= 2(x_1^* + x_2^* + x_3^* - x_8^*) \cdot \frac{x_1^* + x_2^* + x_3^*}{x_2^*}. \tag{4.30}
\end{aligned}$$

We know that

$$\begin{aligned}
-(1 + U'(x_1^*)) &= -1 + \frac{(x_4^* + x_5^* + x_6^*)(1 - x_4^* - x_5^* - x_6^*)}{x_1^*(1 - x_1^*)} \\
&= \frac{(x_4^* + x_5^* + x_6^*)(1 - x_4^* - x_5^* - x_6^*) - x_1^*(1 - x_1^*)}{x_1^*(1 - x_1^*)} \tag{4.31}
\end{aligned}$$

and

$$\begin{aligned}
1 + h'_{\alpha_*}(x_1^*) &= 1 - \frac{x_3^*(1 - x_3^*)}{x_1^*(1 - x_1^*)} \\
&= \frac{x_1^*(1 - x_1^*) - x_3^*(1 - x_3^*)}{x_1^*(1 - x_1^*)}. \tag{4.32}
\end{aligned}$$

Let $u^* = x_4^* + x_5^* + x_6^*$ and $t^* = x_1^* + x_2^* + x_3^*$. Then if we plug the expressions in (Equation 4.29),

(Equation 4.30), (Equation 4.31) and (Equation 4.32) into the equality in (Equation 4.28),

we see that

$$\begin{aligned} 2(1-x_1^*-2x_2^*-x_3^*) \cdot \frac{1-t^*}{x_8^*} \cdot \frac{u^*(1-u^*)-x_1^*(1-x_1^*)}{x_1^*(1-x_1^*)} &= 2(t^*-x_8^*) \cdot \frac{t^*}{x_2^*} \cdot \frac{x_1^*(1-x_1^*)-x_3^*(1-x_3^*)}{x_1^*(1-x_1^*)} \\ &= \frac{(1-t^*)(1-x_1^*-2x_2^*-x_3^*)x_2^*}{x_8^* t^* (t^*-x_8^*)} = \frac{x_1^*(1-x_1^*)-x_3^*(1-x_3^*)}{u^*(1-u^*)-x_1^*(1-x_1^*)} \end{aligned}$$

or

$$\frac{x_2^*(1-x_1^*-2x_2^*-x_3^*)(1-x_1^*-x_2^*-x_3^*)}{x_8^*(x_1^*+x_2^*+x_3^*-x_8^*)(x_1^*+x_2^*+x_3^*)} = \frac{x_1^*(1-x_1^*)-x_3^*(1-x_3^*)}{(x_4^*+x_5^*+x_6^*)(1-x_4^*-x_5^*-x_6^*)-x_1^*(1-x_1^*)}.$$

Lemma 4.12 shows that this is not the case. In other words, we obtain a contradiction. This means that we have $H'_{\alpha_*}(x_1^*) \neq 0$, which finishes the proof. $\square$

By the lemma above, we know that $H_{\alpha_*}(x)$ is a strictly monotone function around a neighborhood of the point x_1^* . If $H_{\alpha_*}(x)$ is a strictly decreasing function around a neighborhood of x_1^* , for a sufficiently small $\epsilon > 0$, we introduce the point $\mathbf{x}' = (x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, x'_7, x'_8) \in \Delta^7$ where $x'_4 = x_4^*$, $x'_5 = x_5^*$, $x'_7 = x'_1$ and

$$x'_1 = x_1^* + \epsilon,$$

$$x'_3 = h_{\alpha_*}(x'_1),$$

$$x'_2 = F^1(x'_1, x'_3)$$

$$x'_6 = U(x'_1, x'_3) - x'_4 - x'_5,$$

$$x'_8 = 1 - \sum_{k=1}^7 x'_k.$$

such that $0 < x'_k < 1$ for every $k = 1, 2, \dots, 8$.

If $H_{\alpha_*}(x)$ is a strictly increasing function around a neighborhood of x_1^* , for a sufficiently small $\epsilon > 0$, we introduce the point $\mathbf{x}' = (x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, x'_7, x'_8) \in \Delta^7$ where $x'_4 = x_4^*$, $x'_5 = x_5^*$, $x'_7 = x_1'$ and

$$\begin{aligned} x'_1 &= x_1^* - \epsilon, \\ x'_3 &= h_{\alpha_*}(x'_1), \\ x'_2 &= F^1(x'_1, x'_3) \\ x'_6 &= U(x'_1, x'_3) - x'_4 - x'_5, \\ x'_8 &= 1 - \sum_{k=1}^7 x'_k. \end{aligned}$$

But note that in either case, we obtain that

$$\begin{aligned} f_2(\mathbf{x}') &= f_4(\mathbf{x}') \\ &= f_7(\mathbf{x}') \\ &= \sqrt{\frac{(1-x'_1)(1-x'_3)}{x'_1 x'_3}} \\ &= \sqrt{\frac{(1-x'_1)(1-h_{\alpha_*}(x'_1))}{x'_1 h_{\alpha_*}(x'_1)}} \\ &= H_{\alpha_*}(x'_1) \\ &< H_{\alpha_*}(x_1^*) \\ &= \sqrt{\frac{(1-x_1^*)(1-h_{\alpha_*}(x_1^*))}{x_1^* h_{\alpha_*}(x_1^*)}} \\ &= \sqrt{\frac{(1-x_1^*)(1-x_3^*)}{x_1^* x_3^*}} \\ &= \alpha_*. \end{aligned}$$

Since x'_2 is on the curve defined by the equality $F_{\alpha_*}^1(x, h_{\alpha_*}(x)) = F_{\alpha_*}^2(x, h_{\alpha_*}(x))$ on a neighborhood of x_1^* such that $x'_2 = F_{\alpha_*}^1(x'_1, h_{\alpha_*}(x'_1))$ and $x'_2 = F_{\alpha_*}^2(x'_1, h_{\alpha_*}(x'_1))$, we get $f_6(\mathbf{x}') = \alpha_*$ and $f_3(\mathbf{x}') = \alpha_*$. Then we apply the following perturbations

$$\begin{aligned}
 & [\mathbf{x}' : f_2 < \alpha_*, f_6 = \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*] \\
 & \quad \downarrow \text{pert}_{\bar{u}_6}^7 \\
 & [\mathbf{y} : f_2 < \alpha_*, f_6 < \alpha_*, f_7 < \alpha_*, f_{j \notin I_1} \leq \alpha_*; F = \alpha_*], \\
 & \quad \downarrow \text{pert}_{\bar{u}_8}^A \\
 & [\mathbf{z} : f_3 < \alpha_*, f_4 < \alpha_*, f_{j \notin I_2} < \alpha_*; F < \alpha_*].
 \end{aligned}$$

and obtain a point $\mathbf{z} \in \Delta^7$ so that $F(\mathbf{z}) < \alpha_*$. This contradicts with the minimality of α_* . Hence, we cannot have $x_3^* + x_4^* + x_5^* < \frac{1}{2}$.

(vi) Let us assume that $x_4^* + x_5^* + x_6^* = \frac{1}{2}$. Then we get $B(\mathbf{x}^*) = 1$ and $x_1^* = x_3^*$. If we solve the equality in (Equation 4.15) for x_1^* , we see that

$$x_1^* = \frac{1}{\alpha_* + 1}.$$

Using the expression in (Equation 4.19) and the equality above, we get the following

$$\begin{aligned}
 \frac{2x_1^* + x_2^*}{1 - 2x_1^* - x_2^*} \cdot \frac{1 - x_2^*}{x_2^*} &= \alpha_* \\
 \frac{\frac{2}{\alpha_* + 1} + x_2^*}{1 - \frac{2}{\alpha_* + 1} - x_2^*} \cdot \frac{1 - x_2^*}{x_2^*} &= \alpha_*
 \end{aligned}$$

If we simplify the last equality above, we obtain that

$$\begin{aligned}\frac{2 + (\alpha_* + 1)x_2^*}{\alpha_* - 1 - (\alpha_* + 1)x_2^*} \cdot \frac{1 - x_2^*}{x_2^*} &= \alpha_* \\ (2 + (\alpha_* + 1)x_2^*)(1 - x_2^*) &= \alpha_*(\alpha_* - 1 - (\alpha_* + 1)x_2^*)x_2^* \\ 2 + (\alpha_* - 1)x_2^* - (\alpha_* + 1)(x_2^*)^2 &= \alpha_*(\alpha_* - 1)x_2^* - \alpha_*(\alpha_* + 1)(x_2^*)^2 \\ (\alpha_*^2 - 1)(x_2^*)^2 - (\alpha_* - 1)^2 x_2^* + 2 &= 0.\end{aligned}$$

Therefore, we see that

$$x_2^* = \frac{(\alpha_* - 1)^2 \pm \sqrt{(\alpha_* - 1)(\alpha_*^3 - 3\alpha_*^2 - 5\alpha_* - 9)}}{2(\alpha_*^2 - 1)}.$$

Let us define the following functions

$$g(\alpha) = \frac{(\alpha - 1)^2 + \sqrt{(\alpha - 1)(\alpha^3 - 3\alpha^2 - 5\alpha - 9)}}{2(\alpha^2 - 1)}$$

and

$$h(\alpha) = \frac{(\alpha - 1)^2 - \sqrt{(\alpha - 1)(\alpha^3 - 3\alpha^2 - 5\alpha - 9)}}{2(\alpha^2 - 1)}.$$

Then we see that $x_2^* = g(\alpha_*)$ or $x_2^* = h(\alpha_*)$. If we plug x_1^* in the (Equation 4.16) and solve for x_8^* , we find that

$$\frac{1 - 2x_1^* - x_2^*}{2x_1^* + x_2^*} \cdot \frac{1 - x_8^*}{x_8^*} = \alpha_*.$$

Then we calculate that

$$\begin{aligned}
 \frac{1 - \frac{2}{\alpha_* + 1} - x_2^*}{\frac{2}{\alpha_* + 1} + x_2^*} \cdot \frac{1 - x_8^*}{x_8^*} &= \alpha_* \\
 \frac{\alpha_* - 1 - (\alpha_* + 1)x_2^*}{2 + (\alpha_* + 1)x_2^*} \cdot \frac{1 - x_8^*}{x_8^*} &= \alpha_* \\
 \frac{1}{x_8^*} - 1 &= \frac{\alpha_* (2 + (\alpha_* + 1)x_2^*)}{(\alpha_* - 1 - (\alpha_* + 1)x_2^*)} \\
 \frac{1}{x_8^*} &= \frac{\alpha_* (2 + (\alpha_* + 1)x_2^*)}{(\alpha_* - 1 - (\alpha_* + 1)x_2^*)} + 1 \\
 x_8^* &= -\frac{1 - \alpha_* + (1 + \alpha_*)x_2^*}{3\alpha_* - 1 + (\alpha_*^2 - 1)x_2^*}.
 \end{aligned}$$

Remember that we have $x_1^* = x_7^*$, $x_1^* = x_3^*$ and $x_4^* + x_5^* + x_6^* = \frac{1}{2}$. Since we have $\sum_{i=1}^8 x_i^* = 1$, we find that

$$\begin{aligned}
 1 &= 3x_1^* + x_2^* + \frac{1}{2} + x_8^* \\
 &= \frac{3}{\alpha_* + 1} + x_2^* + \frac{1}{2} - \frac{1 - \alpha_* + (1 + \alpha_*)x_2^*}{3\alpha_* - 1 + (\alpha_*^2 - 1)x_2^*}.
 \end{aligned}$$

Equivalently, we obtain that

$$\begin{aligned}
 \frac{1}{2} &= \frac{3}{\alpha_* + 1} + \frac{(\alpha_*^2 - 1)(x_2^*)^2 + 2(\alpha_* - 1)x_2^* + \alpha_* - 1}{3\alpha_* - 1 + (\alpha_*^2 - 1)x_2^*} \\
 &= \frac{3(3\alpha_* - 1 + (\alpha_*^2 - 1)x_2^*) + ((\alpha_*^2 - 1)(x_2^*)^2 + 2(\alpha_* - 1)x_2^* + \alpha_* - 1)(\alpha_* + 1)}{(3\alpha_* - 1 + (\alpha_*^2 - 1)x_2^*)(\alpha_* + 1)} \\
 &= \frac{(\alpha_* + 1)(\alpha_*^2 - 1)(x_2^*)^2 + (3(\alpha_*^2 - 1) + 2(\alpha_*^2 - 1))x_2^* + \alpha_*^2 + 9\alpha_* - 4}{(\alpha_* + 1)(\alpha_*^2 - 1)x_2^* + (\alpha_* + 1)(3\alpha_* - 1)} \\
 &= \frac{(\alpha_* + 1)(\alpha_*^2 - 1)(x_2^*)^2 + 5(\alpha_*^2 - 1)x_2^* + \alpha_*^2 + 9\alpha_* - 4}{(\alpha_* + 1)(\alpha_*^2 - 1)x_2^* + (\alpha_* + 1)(3\alpha_* - 1)}.
 \end{aligned}$$

where $\alpha_* \in I_\alpha$. In other words, we get

$$\frac{1}{2} = \frac{(\alpha_* + 1)(\alpha_*^2 - 1)(g(\alpha_*))^2 + 5(\alpha_*^2 - 1)g(\alpha_*) + \alpha_*^2 + 9\alpha_* - 4}{(\alpha_* + 1)(\alpha_*^2 - 1)g(\alpha_*) + (\alpha_* + 1)(3\alpha_* - 1)}$$

if $x_2^* = g(\alpha_*)$ or

$$\frac{1}{2} = \frac{(\alpha_* + 1)(\alpha_*^2 - 1)(h(\alpha_*))^2 + 5(\alpha_*^2 - 1)h(\alpha_*) + \alpha_*^2 + 9\alpha_* - 4}{(\alpha_* + 1)(\alpha_*^2 - 1)h(\alpha_*) + (\alpha_* + 1)(3\alpha_* - 1)}$$

if $x_2^* = h(\alpha_*)$. Let us define the following continuous functions

$$G^+(\alpha) = \frac{(\alpha + 1)(\alpha^2 - 1)(g(\alpha))^2 + 5(\alpha^2 - 1)g(\alpha) + \alpha^2 + 9\alpha - 4}{(\alpha + 1)(\alpha^2 - 1)g(\alpha) + (\alpha + 1)(3\alpha - 1)}$$

and

$$G^-(\alpha) = \frac{(\alpha + 1)(\alpha^2 - 1)(h(\alpha))^2 + 5(\alpha^2 - 1)h(\alpha) + \alpha^2 + 9\alpha - 4}{(\alpha + 1)(\alpha^2 - 1)h(\alpha) + (\alpha + 1)(3\alpha - 1)}.$$

Then we must have $G^\pm(\alpha_*) = \frac{1}{2}$ for $\alpha_* \in I_\alpha$. In the Lemma 4.1.7, we showed that $G^\pm(\alpha) \neq \frac{1}{2}$ for every

$\alpha \in I_\alpha$. Therefore, we get a contradiction. Hence, we get $x_4^* + x_5^* + x_6^* \neq \frac{1}{2}$.

(vii) Let us assume that $x_4^* + x_5^* + x_6^* > \frac{1}{2}$. We will show that $x_8^* < \frac{1}{4} < x_1^* + x_2^* + x_3^*$. Let assume that

$\frac{1}{4} \leq x_8^*$. Note that if we have $x_8^* = \frac{1}{4}$, then we get $x_1^* + x_2^* + x_3^* > \frac{1}{4}$. Thus, we see that

$$\begin{aligned} x_1^* + x_2^* + x_3^* + x_4^* + x_5^* + x_6^* + x_8^* &> \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \\ &= 1. \end{aligned}$$

This is a contradiction.

If we have $\frac{1}{4} < x_8^* < x_1^* + x_2^* + x_3^*$, then we calculate that

$$\begin{aligned} \frac{1-x_8^*}{x_8^*} &< 3, \\ \frac{1-x_1^*-x_2^*-x_3^*}{x_1^*+x_2^*+x_3^*} &< 3. \end{aligned}$$

In other words, we get $\alpha_* = f_3(\mathbf{x}^*) < 9$. We know that $\alpha_* \in I_\alpha$. Therefore, we get a contradiction. Let us assume that $x_1^* + x_2^* + x_3^* \leq \frac{1}{4}$. Then we see that $A(\mathbf{x}^*) \geq 3$. Then we get

$$\begin{aligned} 3 \cdot \frac{1-x_8^*}{x_8^*} &\leq A(\mathbf{x}^*) \cdot \frac{1-x_8^*}{x_8^*} \\ &= \alpha_*. \end{aligned}$$

We perform the following operations

$$\begin{aligned} 3 \cdot \frac{1-x_8^*}{x_8^*} &\leq \alpha_* \\ \frac{1}{x_8^*} - 1 &\leq \frac{\alpha_*}{3} \\ \frac{1}{x_8^*} &\leq \frac{\alpha_*}{3} + 1 \\ \frac{3}{\alpha_* + 3} &\leq x_8^*. \end{aligned}$$

The last inequality implies that

$$\frac{12}{23} - \frac{9}{46}\sqrt{2} \leq x_8^* < x_1^* + x_2^* + x_3^*.$$

Therefore, we have

$$\frac{24}{23} - \frac{9}{23}\sqrt{2} \leq x_1^* + x_2^* + x_3^* + x_8^*.$$

Since we assumed that $\frac{1}{2} < x_4^* + x_5^* + x_6^*$, we obtain $\frac{71}{46} - \frac{9}{23}\sqrt{2} < x_1^* + x_2^* + x_3^* + x_4^* + x_5^* + x_6^* + x_8^*$. Thus, we get that $x_7^* < -\frac{25}{46} + \frac{9}{23}\sqrt{2}$. Then we follow the the operations below

$$\begin{aligned} x_7^* &< -\frac{25}{46} + \frac{9}{23}\sqrt{2} \\ 50 + 36\sqrt{2} &< \frac{1}{x_7^*} \\ 49 + 36\sqrt{2} &< \frac{1}{x_7^*} - 1 \\ B(\mathbf{x}^*) (49 + 36\sqrt{2}) &< B(\mathbf{x}^*) \cdot \left(\frac{1}{x_7^*} - 1 \right) \\ &= \alpha_*. \end{aligned}$$

We get that

$$\begin{aligned} B(\mathbf{x}^*) (49 + 36\sqrt{2}) &< \alpha_* \\ B(\mathbf{x}^*) &\leq \frac{\alpha_*}{49 + 36\sqrt{2}} \\ \frac{1}{x_4^* + x_5^* + x_6^*} - 1 &< \frac{\alpha_*}{49 + 36\sqrt{2}} \\ \frac{1}{x_4^* + x_5^* + x_6^*} &< \frac{\alpha_* + 49 + 36\sqrt{2}}{49 + 36\sqrt{2}} \\ \frac{49 + 36\sqrt{2}}{\alpha_* + 49 + 36\sqrt{2}} &< x_4^* + x_5^* + x_6^* \\ \frac{9}{7} - \frac{11}{42}\sqrt{2} &< x_4^* + x_5^* + x_6^*. \end{aligned}$$

Finally, we conclude that

$$x_1^* + x_2^* + x_3^* + x_4^* + x_5^* + x_6^* = \frac{12}{23} - \frac{9}{46}\sqrt{2} + \frac{9}{7} - \frac{11}{42}\sqrt{2} > 1$$

which is a contradiction. Therefore, we have $x_8^* < \frac{1}{4} < x_1^* + x_2^* + x_3^*$. This implies that $q_4(\frac{1}{4}) < 0$. We follow the calculations below

$$\begin{aligned} (\alpha_* - 1)\frac{1}{16} + (1 - \alpha_*)\frac{1}{4}(1 - \sum_{k=4}^7 x_k^*) + \sum_{k=4}^7 x_k^* &< 0 \\ \left(\frac{\alpha_* - 1}{4} + 1\right) \sum_{k=4}^7 x_k^* &< \frac{\alpha_* - 1}{4} - \frac{\alpha_* - 1}{16} \\ \sum_{k=4}^7 x_k^* &< \frac{3(\alpha_* - 1)}{4(\alpha_* + 3)} \\ \sum_{k=4}^7 x_k^* &< \frac{21}{92} + \frac{9}{46}\sqrt{2}. \end{aligned}$$

We assume that $\frac{1}{2} < x_4^* + x_5^* + x_6^*$. Therefore, we see that $q_5(\frac{1}{2}) < 0$. Then we follow the calculations below

$$\begin{aligned} (\alpha_* - 1)\frac{1}{4} + (1 - \alpha_*)\frac{1}{2}(1 - \sum_{k=1,2,3,8} x_k^*) + \sum_{k=1,2,3,8} x_k^* &< 0 \\ \left(\frac{\alpha_* - 1}{2} + 1\right) \sum_{k=1,2,3,8} x_k^* &< \frac{\alpha_* - 1}{2} - \frac{\alpha_* - 1}{4} \\ \sum_{k=1,2,3,8} x_k^* &< \frac{\alpha_* - 1}{2(\alpha_* + 1)} \\ \sum_{k=1,2,3,8} x_k^* &< \frac{1}{6} + \frac{1}{6}\sqrt{2}. \end{aligned}$$

In other words, we obtain that

$$\begin{aligned}
 1 &= \sum_{k=1}^8 x_k^* \\
 &= \sum_{k=1,2,3,8} x_k^* + \sum_{k=4}^7 x_k^* \\
 &< \frac{21}{92} + \frac{9}{46}\sqrt{2} + \frac{1}{6} + \frac{1}{6}\sqrt{2} \\
 &= \frac{109}{276} + \frac{25}{69}\sqrt{2} \\
 &< 1,
 \end{aligned}$$

a contradiction. Since we have a contradiction in any of the cases (v) , (vi) , and (vii) , case IV is not possible at the point $\mathbf{x}^*$.

□

Proof of Lemma 4.2. Let us assume that $\mathbf{x}^* \notin \Delta^*$. Then there exists a $j_0 \in I_1 \cup I_2 \cup I_3$ such that $f_{j_0}(\mathbf{x}^*) < \alpha_*$. If we have $j_0 \in I_1$, then we apply perturbations from the list L_1 and find a point $\mathbf{z}_1 \in \Delta^7$ such that $f_i(\mathbf{z}_1) < \alpha_*$ for every $i \in I_1$. If there does not exist any $j_1 \in I_2 \cup I_3$ such that $f_{j_1}(\mathbf{z}_1) < \alpha_*$, then we get the case $\overline{\text{IV}}$. We know that case $\overline{\text{IV}}$ is equivalent to the case IV.

If there exists $j_1 \in I_2 \cup I_3$ such that $f_{j_1}(\mathbf{z}_1) < \alpha_*$, we have two possibilities to consider. If $j_1 \in I_2$, then we apply perturbations from the list L_2 and obtain a point $\mathbf{z}_2 \in \Delta^7$ such that $f_j(\mathbf{z}_2) < \alpha_*$ for every $j \in I_1 \cup I_2$. Note that we must have $f_j(\mathbf{z}_1) = \alpha_*$ for every $j \in I_3$. Otherwise application of perturbations from the list L_3 gives a point in Δ^7 at which a smaller maximum value is attained, a contradiction. Therefore, we obtain the case I. If $j_1 \in I_3$, then we apply perturbations from the list L_3 and obtain a

point $\mathbf{z}_2 \in \Delta^7$ such that $f_j(\mathbf{z}_2) < \alpha_*$ for every $j \in I_1 \cup I_3$. Note that we must have $f_j(\mathbf{z}_1) = \alpha_*$ for every $j \in I_2$. Then we get the case II.

If we have $j_0 \in I_2$, then we apply perturbations from the list L_2 and find a point $\mathbf{z}_1 \in \Delta^7$ such that $f_i(\mathbf{z}_1) < \alpha_*$ for every $i \in I_2$. If there does not exist any $j_1 \in I_1 \cup I_3$ such that $f_{j_1}(\mathbf{z}_1) < \alpha_*$, then we have the case III.

If there exists $j_1 \in I_1 \cup I_3$ such that $f_{j_1}(\mathbf{z}_1) < \alpha_*$, we have two possibilities to consider. If $j_1 \in I_1$, then we apply perturbations from the list L_1 and obtain a point $\mathbf{z}_2 \in \Delta^7$ such that $f_j(\mathbf{z}_2) < \alpha_*$ for every $j \in I_1 \cup I_2$. We must have $f_j(\mathbf{z}_1) = \alpha_*$ for every $j \in I_3$. This means that we get the case I. If $j_1 \in I_3$, then we apply perturbations from the list L_3 and obtain a point $\mathbf{z}_2 \in \Delta^7$ such that $f_j(\mathbf{z}_2) < \alpha_*$ for every $j \in I_2 \cup I_3$. We must have $f_j(\mathbf{z}_1) = \alpha_*$ for every $j \in I_1$. This means that we get the case $\bar{\text{I}}$ which is equivalent to case I.

If we have $j_0 \in I_3$, then we apply perturbations from the list L_3 and find a point $\mathbf{z}_1 \in \Delta^7$ such that $f_i(\mathbf{z}_1) < \alpha_*$ for every $i \in I_3$. If there does not exist any $j_1 \in I_1 \cup I_2$ such that $f_{j_1}(\mathbf{z}_1) < \alpha_*$, then we have the case IV.

If there exists $j_1 \in I_1 \cup I_2$ such that $f_{j_1}(\mathbf{z}_1) < \alpha_*$, we have two possibilities to consider. If $j_1 \in I_1$, then we apply perturbations from the list L_1 and obtain a point $\mathbf{z}_2 \in \Delta^7$ such that $f_j(\mathbf{z}_2) < \alpha_*$ for every $j \in I_1 \cup I_3$. We must have $f_j(\mathbf{z}_1) = \alpha_*$ for every $j \in I_2$. This means that we get the case II. If $j_1 \in I_2$, then we apply perturbations from the list L_2 and obtain a point $\mathbf{z}_2 \in \Delta^7$ such that $f_j(\mathbf{z}_2) < \alpha_*$ for every $j \in I_2 \cup I_3$. We must have $f_j(\mathbf{z}_1) = \alpha_*$ for every $j \in I_1$. This means that we get the case $\bar{\text{I}}$ which is equivalent to case I.

In Lemmas 4.1.2, 4.1.3, 4.1.4 and 4.1.8, we showed that none of the cases I, II, III, or IV can happen at a point where F_1 takes its minimum value. Therefore, we must have $\mathbf{x}^* \in \Delta^*$.

□

4.1.6 Calculation of the infimum

Lemma 4.1.13. *If $\{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$ is the set of functions defined in §4.1, then we have*

$$\inf_{\mathbf{x} \in \Delta^7} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_8(\mathbf{x}))\} = 5 + 3\sqrt{2}.$$

Proof. By Lemma 4.1.1, we know that $\mathbf{x}^* \in \Delta^*$. This means that we have

$$A(\mathbf{x}^*) \cdot \frac{1 - x_4^*}{x_4^*} = \alpha_* \quad (4.33)$$

$$B(\mathbf{x}^*) \cdot \frac{1 - x_1^*}{x_1^*} = \alpha_* \quad (4.34)$$

$$A(\mathbf{x}^*) \cdot \frac{1 - x_8^*}{x_8^*} = \alpha_* \quad (4.35)$$

$$B(\mathbf{x}^*) \cdot \frac{1 - x_7^*}{x_7^*} = \alpha_* \quad (4.36)$$

$$\frac{1}{A(\mathbf{x}^*)} \cdot \frac{1 - x_6^*}{x_6^*} = \alpha_* \quad (4.37)$$

$$\frac{1}{A(\mathbf{x}^*)} \cdot \frac{1 - x_2^*}{x_2^*} = \alpha_* \quad (4.38)$$

$$\frac{1}{B(\mathbf{x}^*)} \cdot \frac{1 - x_3^*}{x_3^*} = \alpha_* \quad (4.39)$$

$$\frac{1}{B(\mathbf{x}^*)} \cdot \frac{1 - x_5^*}{x_5^*} = \alpha_*. \quad (4.40)$$

Using (Equation 4.33) and (Equation 4.35), (Equation 4.34) and (Equation 4.36), (Equation 4.37) and (Equation 4.39), (Equation 4.38) and (Equation 4.40) together, we get $x_4^* = x_8^*$, $x_1^* = x_7^*$, $x_6^* = x_2^*$, and $x_3^* = x_5^*$. Using (Equation 4.33) and (Equation 4.34), we see that

$$\begin{aligned} \frac{1 - x_1^* - x_2^* - x_3^*}{x_1^* + x_2^* + x_3^*} \cdot \frac{1 - x_4^*}{x_1^*} &= \frac{1 - x_4^* - x_5^* - x_6^*}{x_4^* + x_5^* + x_6^*} \cdot \frac{1 - x_1^*}{x_1^*} \\ &= \frac{1 - x_4^* - x_2^* - x_3^*}{x_4^* + x_2^* + x_3^*} \cdot \frac{1 - x_1^*}{x_1^*} \end{aligned}$$

If we simplify the equality above, we obtain $(x_1^* - x_4^*)(x_2^* + x_3^*)(1 - x_1^* - x_2^* - x_3^* - x_4^*) = 0$ which implies that $x_1^* = x_4^*$. Therefore, we get that $A(\mathbf{x}^*) = B(\mathbf{x}^*)$. Then using (Equation 4.38) and (Equation 4.39), we see that $x_2^* = x_3^*$. In other words, we obtain $x_i^* = x_j^*$ for every $i, j \in \{1, 4, 7, 8\}$ and $x_i^* = x_j^*$ for every $i, j \in \{2, 3, 5, 6\}$. Since we know that $\sum_{i=1}^8 x_i^* = 1$, we have $x_1^* + x_2^* = \frac{1}{4}$ or $x_2^* = \frac{1}{4} - x_1^*$. Using (Equation 4.34) and (Equation 4.38) we have

$$\frac{1 - x_1^* - 2x_2^*}{x_1^* + 2x_2^*} \cdot \frac{1 - x_1^*}{x_1^*} = \frac{x_1^* + 2x_2^*}{1 - x_1^* - 2x_2^*} \cdot \frac{1 - x_2^*}{x_2^*}.$$

We simplify the equation above and obtain that

$$x_2^* - 3x_2^*x_1^* - (x_1^*)^2x_2^* - 4(x_2^*)^2 + 4(x_2^*)^2x_1^* + 4(x_2^*)^3 - (x_1^*)^3 = 0.$$

Then we substitute $x_2^* = \frac{1}{4} - x_1^*$ and get

$$\frac{1}{16} - \frac{1}{4}x_1^* - \frac{1}{4}(x_1^*)^2 = 0.$$

We solve the quadratic equation above for x_1^* . We find that $x_1^* = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$. We need to choose the positive solution for x_1^* . Then we calculate that

$$x_2^* = \frac{3}{4} - \frac{\sqrt{2}}{2}.$$

In particular, we obtain

$$\mathbf{x}^* = \left(\frac{\sqrt{2}-1}{2}, \frac{3-2\sqrt{2}}{4}, \frac{3-2\sqrt{2}}{4}, \frac{\sqrt{2}-1}{2}, \frac{3-2\sqrt{2}}{4}, \frac{3-2\sqrt{2}}{4}, \frac{\sqrt{2}-1}{2}, \frac{\sqrt{2}-1}{2} \right).$$

Finally, we calculate that

$$\begin{aligned} \inf_{\mathbf{x} \in \Delta^7} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_{14}(\mathbf{x}))\} &= \inf_{\mathbf{x} \in \Delta^7} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_8(\mathbf{x}))\} \\ &= f_1(\mathbf{x}^*) \\ &= \frac{1 - \frac{\sqrt{2}-1}{2} - 2 \cdot \frac{3-2\sqrt{2}}{4}}{\frac{\sqrt{2}-1}{2} + 2 \cdot \frac{3-2\sqrt{2}}{4}} \cdot \frac{1 - \frac{\sqrt{2}-1}{2}}{\frac{\sqrt{2}-1}{2}} \\ &= \frac{2 - \sqrt{2} + 1 - 3 + 2\sqrt{2}}{\sqrt{2} - 1 + 3 - 2\sqrt{2}} \cdot \frac{2 - \sqrt{2} + 1}{\sqrt{2} - 1} \\ &= \frac{\sqrt{2}}{2 - \sqrt{2}} \cdot \frac{3 - \sqrt{2}}{\sqrt{2} - 1} \\ &= \frac{3\sqrt{2} - 2}{3\sqrt{2} - 4} \cdot \frac{3\sqrt{2} + 4}{3\sqrt{2} + 4} \\ &= \frac{10 + 6\sqrt{2}}{2} \\ &= 5 + 3\sqrt{2}. \end{aligned}$$

□

We are ready to state the result we need to prove Proposition 4.0.1:

Lemma 4.1.14. *Let $\{f_i\}_{i=1}^{14}$ is the set of functions given in §4.1. We have $\min_{\mathbf{x} \in \Delta^7} G_1(\mathbf{x}) = \min_{\mathbf{x} \in \Delta^7} F_1(\mathbf{x})$.*

Proof. Let $\mathbf{z}^* = (z_1^*, z_2^*, z_3^*, z_4^*, z_5^*, z_6^*, z_7^*, z_8^*) \in \Delta^7$ be a point at which G_1 takes its minimum value β .

Then we get

$$G_1(\mathbf{x}^*) \geq G_1(\mathbf{z}^*).$$

But we have $f_i(\mathbf{x}^*) = 5 + 3\sqrt{2}$ for every $i \in \{1, 2, \dots, 8\}$ and $f_i(\mathbf{x}^*) = 1$ for every $i \in \{9, \dots, 14\}$ because $x_k^* = x_j^*$ for every $k, j \in \{1, 4, 7, 8\}$ and $x_r^* = x_s^*$ for every $r, s \in \{2, 3, 5, 6\}$. Then we get

$$5 + 3\sqrt{2} = G_1(\mathbf{x}^*) \geq \beta.$$

We know that $\beta \geq 5 + 3\sqrt{2}$ because $\{f_i\}_{i=1}^8 \subset \{f_i\}_{i=1}^{14}$. Hence, we obtain that $\beta = 5 + 3\sqrt{2}$. $\square$

4.2 Proof of Proposition 4.0.1

In this section, I give the proof of Proposition 4.0.1 which requires the inequalities similar to the ones used in the proof of Proposition 1.2.2.

Proof of Proposition 4.0.1. We know $\text{dist}(z_0, \xi \cdot z_0) = \text{dist}(z_0, \xi^{-1} \cdot z_0)$, $\text{dist}(z_0, \eta \cdot z_0) = \text{dist}(z_0, \eta^{-1} \cdot z_0)$ and $\text{dist}(z_0, \xi\eta \cdot z_0) = \text{dist}(z_0, \eta^{-1}\xi^{-1} \cdot z_0)$ hold. Let $\mathbf{m} = (m_1, m_2, \dots, m_8)$ where m_i is the total measure defined in §3.3. By the discussion given in §3.3, we have

$$\max(\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0), \text{dist}(z_0, \xi\eta \cdot z_0)) \geq \frac{1}{2} \log(\max(f_1(\mathbf{m}), f_2(\mathbf{m}), \dots, f_{14}(\mathbf{m}))).$$

Since the function $\frac{1}{2} \log t$ is strictly increasing, we get

$$\begin{aligned} \max(\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0), \text{dist}(z_0, \xi\eta \cdot z_0)) &\geq \frac{1}{2} \log(\max(f_1(\mathbf{m}), f_2(\mathbf{m}), \dots, f_{14}(\mathbf{m}))) \\ &\geq \frac{1}{2} \log(\inf_{\mathbf{x} \in \Delta^7} \{\max(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_{14}(\mathbf{x}))\}) \\ &= \frac{1}{2} \log(5 + 3\sqrt{2}) \end{aligned}$$

by Lemma 4.1.14. □

I would like to emphasize the similarities between the following statements

C. $\inf_{\mathbf{x} \in \Delta^7} F_1(\mathbf{x}) = \min_{\mathbf{x} \in \Delta^7} F_1(\mathbf{x}),$

D. There exists a point $\mathbf{x}^* \in \Delta^* \subset \Delta^7$ such that $\min_{\mathbf{x} \in \Delta^7} F_1(\mathbf{x}) = f_i(\mathbf{x}^*)$ for any $i \in \{1, 2, \dots, 8\},$

used in the proof of Lemma 4.1.13 to calculate the number $\frac{1}{2} \log(5 + 3\sqrt{2})$ and the statements (A) and (B) listed in §2.3. Although it is straight forward to observe the fact in (B), it takes considerable amount of calculations to prove the statement given in (D). Analogous to Lemma 2.2.1, Lemma 4.1.13 shows that the point $\mathbf{x}^*$ is unique. Assuming the uniqueness of the point $\mathbf{x}^*$ a priori together with (C) suggests an alternative way of finding the coordinates of the point $\mathbf{x}^* \in \Delta^7$.

Let $T_1, T_2 : \mathbf{R}^8 \rightarrow \mathbf{R}^8$ be two transformations defined by

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mapsto (x_4, x_5, x_6, x_1, x_2, x_3, x_8, x_7)$$

and

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mapsto (x_1, x_3, x_2, x_4, x_5, x_6, x_7, x_8)$$

respectively. Note that the simplex Δ^7 is preserved by the transformations T_1 and T_2 , i.e., $T_1(\Delta^7) = \Delta^7$, $T_2(\Delta^7) = \Delta^7$. By the definitions of T_1 , we have $f_1(T_1(\mathbf{x})) = f_2(\mathbf{x})$, $f_2(T_1(\mathbf{x})) = f_1(\mathbf{x})$, $f_3(T_1(\mathbf{x})) = f_4(\mathbf{x})$, $f_4(T_1(\mathbf{x})) = f_3(\mathbf{x})$, $f_5(T_1(\mathbf{x})) = f_7(\mathbf{x})$ and $f_7(T_1(\mathbf{x})) = f_5(\mathbf{x})$, $f_6(T_1(\mathbf{x})) = f_8(\mathbf{x})$, $f_8(T_1(\mathbf{x})) = f_6(\mathbf{x})$ for every $\mathbf{x} \in \Delta^7$. In other words, we may conclude that $\{f_i : i = 1, 2, \dots, 8\} = \{f_i \circ T_1 : i = 1, 2, \dots, 8\}$. Let us define the following function $H_1 : \Delta^7 \rightarrow \mathbf{R}$ where

$$H_1(\mathbf{x}) = \max\{(f_i \circ T_1)(\mathbf{x}) : i = 1, 2, \dots, 8\}.$$

Then, we see that $F_1(\mathbf{x}) = H_1(\mathbf{x})$ for every $\mathbf{x} \in \Delta^7$ and $\min_{\mathbf{x} \in \Delta^7} F_1(\mathbf{x}) = \min_{\mathbf{x} \in \Delta^7} H_1(\mathbf{x})$. Since F_1 takes its minimum value at the point $\mathbf{x}^*$ and $\{f_i(\mathbf{x}^*) : i = 1, 2, \dots, 8\} = \{(f_i \circ T_1)(\mathbf{x}^*) : i = 1, 2, \dots, 8\}$, H_1 takes its minimum value at the point $T_1^{-1}(\mathbf{x}^*)$. We assumed that $\mathbf{x}^*$ is unique. Hence, we obtain that $T_1^{-1}(\mathbf{x}^*) = \mathbf{x}^*$. This means that $x_1^* = x_4^*$, $x_2^* = x_5^*$, $x_3^* = x_6^*$ and $x_7^* = x_8^*$. A similar argument can be given for the transformation T_2 .

By definition of T_2 , we have $f_i(T_2(\mathbf{x})) = f_i(\mathbf{x})$ for every $i = 1, 2, 3, 4, 5, 8$ and $f_6(T_2(\mathbf{x})) = f_7(\mathbf{x})$ and $f_7(T_2(\mathbf{x})) = f_6(\mathbf{x})$ for every $\mathbf{x} \in \Delta^7$. Let us define the following function $H_2 : \Delta^7 \rightarrow \mathbf{R}$ where

$$H_2(\mathbf{x}) = \max\{(f_i \circ T_2)(\mathbf{x}) : i = 1, 2, \dots, 8\}.$$

Then, we see that $F_1(\mathbf{x}) = H_2(\mathbf{x})$ for every $\mathbf{x} \in \Delta^7$ and $\min_{\mathbf{x} \in \Delta^7} F_1(\mathbf{x}) = \min_{\mathbf{x} \in \Delta^7} H_2(\mathbf{x})$. Since F_1 takes its minimum value at the point $\mathbf{x}^*$ and $\{f_i(\mathbf{x}^*) : i = 1, 2, \dots, 8\} = \{(f_i \circ T_2)(\mathbf{x}^*) : i = 1, 2, \dots, 8\}$, H_2 takes its minimum value at the point $T_2^{-1}(\mathbf{x}^*)$. We assumed that $\mathbf{x}^*$ is unique. Hence, we obtain that $T_2^{-1}(\mathbf{x}^*) = \mathbf{x}^*$. This means that $x_2^* = x_3^*$. In other words, we find that $\mathbf{x}^* = (x_1^*, x_2^*, x_2^*, x_1^*, x_2^*, x_2^*, x_7^*, x_7^*)$ where

$2x_1^* + 4x_2^* + 2x_7^* = 1$. This form of the point $\mathbf{x}^*$ reduces the calculation of the infimum of maximum of the functions $f_1, f_2, \dots, f_8$ to a similar problem with a smaller set of functions with a smaller set of variables as follows: we define the following functions $g_i : \Delta \rightarrow \mathbf{R}$ for $i = 1, 2, 3$ where

$$\begin{aligned} g_1(y_1, y_2, y_3) &= \frac{1 - y_1 - 2y_2}{y_1 + 2y_2} \cdot \frac{1 - y_1}{y_1} \\ g_2(y_1, y_2, y_3) &= \frac{1 - y_1 - 2y_2}{y_1 + 2y_2} \cdot \frac{1 - y_3}{y_3} \\ g_3(y_1, y_2, y_3) &= \frac{y_1 + 2y_2}{1 - y_2 - 2y_3} \cdot \frac{1 - y_2}{y_2} \end{aligned}$$

and $\Delta = \{(y_1, y_2, y_3) \in \mathbf{R}_+^3 : 2y_1 + 4y_2 + 2y_3 = 1\}$. Let us introduce the following continuous function

$$\begin{aligned} F_2 &: \Delta \rightarrow \mathbf{R} \\ \mathbf{y} &\mapsto \max(g_1(\mathbf{y}), g_2(\mathbf{y}), g_3(\mathbf{y})). \end{aligned}$$

Then it is possible to show that $\inf_{\mathbf{y} \in \Delta} F_2(\mathbf{y}) = 5 + 3\sqrt{2}$ which is attained at the point

$$\mathbf{y}^* = \left(\frac{\sqrt{2}-1}{2}, \frac{3-\sqrt{2}}{4}, \frac{\sqrt{2}-1}{2} \right).$$

Then the argument I presented above shows that $\inf_{\mathbf{x} \in \Delta^*} F_1(\mathbf{x}) = \inf_{\mathbf{y} \in \Delta} F_2(\mathbf{y})$ and

$$\mathbf{x}^* = \left(\frac{\sqrt{2}-1}{2}, \frac{3-2\sqrt{2}}{4}, \frac{3-2\sqrt{2}}{4}, \frac{\sqrt{2}-1}{2}, \frac{3-2\sqrt{2}}{4}, \frac{3-2\sqrt{2}}{4}, \frac{\sqrt{2}-1}{2}, \frac{\sqrt{2}-1}{2} \right)$$

since $y_1^* = x_1^*$, $y_2^* = x_2^*$ and $y_3^* = x_7^*$. Note that in the discussion above, we don't refer to the statement

$\mathbf{x}^* \in \Delta^*$ given in (D).

I will leave the details of the calculations that show that $\inf_{\mathbf{y} \in \Delta} F_2(\mathbf{y}) = 5 + 3\sqrt{2}$ and the investigation of the likely conditions such as the convexity properties of the functions F_0, F_1 and F_2 that might lead to a proof of the uniqueness of the point $\mathbf{x}^*$ to future studies.



CHAPTER 5

THE $\frac{1}{2}\log(5+3\sqrt{2})$ THEOREM

In this chapter, I present the details of the proof of Theorem 5.1.1 to complete this thesis.

5.1 Proof of Theorem 5.1.1

Theorem 5.1.1. *Let ξ and η be non-commuting isometries of $\mathbf{H}^3$. Suppose that ξ and η generate a torsion-free discrete group which is topologically tame, is not co-compact and contains no parabolics. Then for any $z \in \mathbf{H}^3$ we have*

$$\max(\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0), \text{dist}(z_0, \xi\eta \cdot z_0)) \geq \frac{1}{2}\log(5+3\sqrt{2}).$$

Proof. Let $\Gamma = \langle \xi, \eta \rangle$. Since Γ is a torsion-free discrete group which is not co-compact and contains no parabolics, it is a free group on the generators ξ and η (5), Proposition 9.2). Then there are two cases $\Lambda_\Gamma = S_\infty$ or $\Lambda_\Gamma \neq S_\infty$.

Let us assume that $\Lambda_\Gamma = S_\infty$. Since Γ is topologically tame and $\Lambda_\Gamma = S_\infty$, every superharmonic function on the manifold $M = \mathbf{H}^3/\Gamma$ is constant (5) Proposition 6.8 and (3) Theorem 7.2) which implies that every Γ -invariant superharmonic function is constant on $\mathbf{H}^3$. Then every Γ -invariant D -conformal density $\mathcal{M}$ is a constant multiple of the area density $\mathcal{A}$ (5), Proposition 3.9). Hence, Proposition 4.0.1 implies that

$$\max(\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0), \text{dist}(z_0, \xi\eta \cdot z_0)) \geq \frac{1}{2}\log(5+3\sqrt{2}),$$

which gives the conclusion of Theorem 5.1.1 in the case $\Lambda_\Gamma = S_\infty$.

Let us assume that $\Lambda_\Gamma \neq S_\infty$. Let V denote the variety $PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$ with the complex topology. Remember that $\mathfrak{D}$ is the set of all points $(\xi', \eta') \in V$ such that the group $\langle \xi', \eta' \rangle$ generated by ξ' and η' is a free Kleinian group of rank 2. We know that $\mathfrak{D}$ is a closed subset of $V((4))$. The set of all points $(\xi', \eta') \in \mathfrak{D}$ such that $\Lambda_{\Gamma'} \neq S_\infty$ where $\Gamma' = \langle \xi', \eta' \rangle$ is denoted by $\mathfrak{G}\mathfrak{F}$ which is an open subset of V ((7), Theorem 8.1). The frontier $\mathfrak{B}$ of $\mathfrak{G}\mathfrak{F}$ contains a dense subset $\mathfrak{C}$ of elements $(\xi', \eta') \in \mathfrak{B}$ so that $\Lambda_{\Gamma'} = S_\infty$ ((5), Theorem 8.2). Let z_0 be a given point in $\mathbf{H}^3$. Let us consider the function

$$\begin{aligned} f_{z_0} : \mathfrak{D} &\longrightarrow \mathbf{R} \\ (\xi', \eta') &\longmapsto \max(\text{dist}(z_0, \xi' \cdot z_0), \text{dist}(z_0, \eta' \cdot z_0), \text{dist}(z_0, \xi'\eta' \cdot z_0)). \end{aligned}$$

Since the set $f_{z_0}^{-1}([0, B]) = \{(\xi', \eta') \in V : \max(\text{dist}(z_0, \xi' \cdot z_0), \text{dist}(z_0, \eta' \cdot z_0)) \leq B\}$ is compact for every $B \geq 0$, f_{z_0} is a continuous, proper, non-negative valued function on V . Therefore, f_{z_0} must take a minimum value on $\overline{\mathfrak{G}\mathfrak{F}}$. Let $(\xi_0, \eta_0) \in \overline{\mathfrak{G}\mathfrak{F}}$ be a point at which f_{z_0} takes a minimum value. The fact that every superharmonic function on $M = \mathbf{H}^3/\Gamma'$ is constant for every $(\xi', \eta') \in \mathfrak{C}$ implies that $\mathcal{M}_{\Gamma', z_0} = \mathcal{A}$ for every $(\xi', \eta') \in \mathfrak{C}$ where $\Gamma' = \langle \xi', \eta' \rangle$. Then Proposition 4.0.1 gives that $f_{z_0}(\xi', \eta') \geq \frac{1}{2} \log(5 + 3\sqrt{2})$ for every $(\xi', \eta') \in \mathfrak{C}$. If (ξ_0, η_0) is in $\mathfrak{B}$, then

$$f_{z_0}(\xi_0, \eta_0) \geq \frac{1}{2} \log(5 + 3\sqrt{2})$$

since $\mathfrak{C}$ is a dense subset of $\mathfrak{B}$. Hence, it is enough to show that f_{z_0} has no local minimum on the open set $\mathfrak{G}\mathfrak{F}$.

Let us assume that (ξ_0, η_0) is in $\mathfrak{GF}$. Then f_{z_0} has a local minimum at (ξ_0, η_0) . Since ξ_0, η_0 and $\xi_0\eta_0$ have infinite orders in $\langle \xi_0, \eta_0 \rangle$, we have $\xi_0 \cdot z \neq z$, $\eta_0 \cdot z \neq z$ and $\xi_0\eta_0 \cdot z \neq z$ for every $z \in \mathbf{H}^3$. In particular, we get that $\xi_0 \cdot z_0 \neq z_0$, $\eta_0 \cdot z_0 \neq z_0$ and $\xi_0\eta_0 \cdot z_0 \neq z_0$. Therefore, there exists hyperbolic geodesic segments joining z_0 to $\xi_0 \cdot z_0$, z_0 to $\eta_0 \cdot z_0$ and z_0 to $\xi_0\eta_0 \cdot z_0$. Since we have $\text{dist}(z_0, \xi_0\eta_0 \cdot z_0) = \text{dist}(\xi_0^{-1} \cdot z_0, \eta_0 \cdot z_0)$ and $\text{dist}(z_0, \xi \cdot z_0) = \text{dist}(z_0, \xi^{-1} \cdot z_0)$, we consider the geodesic triangle $\triangle_{PQR}$ where $P = \xi_0^{-1} \cdot z_0$, $Q = z_0$ and $R = \eta_0 \cdot z_0$. We will show that there exist a point P' on the line segment QP arbitrarily close to P and a point R' on the line segment QR arbitrarily close to R such that $\text{dist}(Q, P') < \text{dist}(Q, P)$, $\text{dist}(Q, R') < \text{dist}(Q, R)$ and $\text{dist}(P', R') < \text{dist}(P, R)$.

Let us say $p = \text{dist}(R, Q)$, $q = \text{dist}(P, R)$, $r = \text{dist}(P, Q)$ and θ is the angle between the line segments PQ and QR . It follows from the hyperbolic law of cosines that

$$\cosh q = \cosh r \cosh p - \sinh r \sinh p \cos \theta.$$

Assume that $p = r$. Then we have $0 < \theta < \pi$ since $\xi_0^{-1} \cdot z_0 \neq \eta_0 \cdot z_0$ and $\xi_0 \cdot z_0 \neq \eta_0 \cdot z_0$. Let us say P' be a given point on the line segment PQ and R' be the point on the line segment QR such that $\text{dist}(P', Q) = \text{dist}(R', Q)$. If we keep the angle θ constant, then the distance between P' and R' can be considered as a function of the distance between P' and Q . In other words, we may define the following function

$$\begin{aligned} f(x) &= \cosh^2 x - \sinh^2 x \cos \theta \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \cdot \cos \theta \end{aligned}$$

where $0 < x < \text{dist}(Q, P)$. Let us calculate the derivative of f :

$$\begin{aligned} f'(x) &= \sinh 2x - \sinh 2x \cos \theta \\ &= \frac{e^{2x} - e^{-2x}}{2} (1 - \cos \theta) \end{aligned}$$

Since we have $\sinh 2x > 0$ and $1 - \cos \theta > 0$, we see that $f' > 0$ for every $0 < x < \text{dist}(Q, P)$ which means that f is a strictly increasing function. Therefore, for $\epsilon > 0$, however small, there exists a point P' on the line segment QP and a corresponding point R' on the line segment QR such that $\text{dist}(Q, P') = \text{dist}(Q, R')$ where $\text{dist}(P, P') < \epsilon$, $\text{dist}(R', R) < \epsilon$ and $\text{dist}(P', R') < \text{dist}(P, R)$.

Let us consider the case $p \neq r$. Without loss of generality, let us assume that $p > r$. If we keep r and θ constant, the distance between P and a point on the line segment QR can be given as a function of the distance between Q and that point. In other words, we may define the following function

$$\begin{aligned} g(x) &= \cosh r \cosh x - \sinh r \sinh x \cos \theta \\ &= \frac{e^r + e^{-r}}{2} \cdot \frac{e^x + e^{-x}}{2} - \frac{e^r - e^{-r}}{2} \cdot \frac{e^x - e^{-x}}{2} \cdot \cos \theta \end{aligned}$$

where $r < x < \text{dist}(Q, R)$. Let us calculate the derivative of g :

$$\begin{aligned} g'(x) &= \cosh r \sinh x - \sinh r \cosh x \cos \theta \\ &= \frac{e^r + e^{-r}}{2} \cdot \frac{e^x - e^{-x}}{2} - \frac{e^r - e^{-r}}{2} \cdot \frac{e^x + e^{-x}}{2} \cdot \cos \theta \end{aligned}$$

Let us assume that $0 < \theta < \pi$. Then we see that

$$\frac{e^r + e^{-r}}{2} \cdot \frac{e^x - e^{-x}}{2} - \frac{e^r - e^{-r}}{2} \cdot \frac{e^x + e^{-x}}{2} < \frac{e^r + e^{-r}}{2} \cdot \frac{e^x - e^{-x}}{2} - \frac{e^r - e^{-r}}{2} \cdot \frac{e^x + e^{-x}}{2} \cdot \cos \theta. \quad (5.1)$$

Let us follow the operations below:

$$\begin{aligned} \frac{e^r + e^{-r}}{2} \cdot \frac{e^x - e^{-x}}{2} - \frac{e^r - e^{-r}}{2} &= \frac{e^{r+x} - e^{r-x} + e^{-r+x} - e^{-r-x} - (e^{r+x} + e^{r-x} - e^{-r+x} - e^{-r-x})}{4} \\ &= \frac{-2e^{r-x} + 2e^{-r+x}}{4} \\ &= -\frac{e^{r-x} - e^{-(r-x)}}{2} \\ &= -\sinh(r-x). \end{aligned}$$

Since we have $x > r$, we see that $-\sinh(r-x) > 0$. Using the inequality in (Equation 5.1), we obtain that $g'(x) > 0$ for every $r < x < \text{dist}(Q, R)$ which means that g is a strictly increasing function. Hence, for $\epsilon > 0$, however small, there exists a point R' on the line segment QR such that $\text{dist}(P, R') < \text{dist}(P, R)$ and $\text{dist}(R, R') < \epsilon$. Let us say $q' = \text{dist}(P, R')$ and $r' = \text{dist}(Q, R')$. Then it follows from the hyperbolic law of cosines that

$$\cosh q' = \cosh r' \cosh p - \sinh r' \sinh p \cos \theta.$$

If we keep p and θ constant, the distance between R' and a point on the line segment QP can be given as a function of the distance between Q and that point. In other words, we may define

$$h(x) = \cosh x \cosh p - \sinh x \sinh p \cos \theta$$

where $0 < x < \text{dist}(P, Q)$. Note that h is a continuous function and $g(\text{dist}(P, R')) < \text{dist}(P, R)$. Therefore, for $\epsilon > 0$, however small, there exists a point P' on the line segment QP such that $\text{dist}(P', R') < \text{dist}(P, R)$ and $\text{dist}(P', P) < \epsilon$. It is straight forward to obtain such points P' and R' in the cases when $\theta = 0$ or $\theta = \pi$ since the points P , Q and R are on the same geodesic.

Let ξ_1 and η_1 be two isometries of $\mathbf{H}^3$ so that $\xi_1^{-1} \cdot z_0 = P'$ and $\eta_1^{-1} \cdot z_0 = R'$. Since the points P' and R' are chosen arbitrarily close to the points P and R , (ξ_1, η_1) is arbitrarily close the point $(\xi_0, \eta_0) \in \mathfrak{G}\mathfrak{F}$. We know that $\mathfrak{G}\mathfrak{F}$ is an open subset of V . Hence, we get (ξ_1, η_1) is in $\mathfrak{G}\mathfrak{F}$. But note that we have

$$\begin{aligned} \text{dist}(z_0, \xi_1 \cdot z_0) &= \text{dist}(z_0, \xi_1^{-1} \cdot z_0) \\ &= \text{dist}(Q, P') \\ &< \text{dist}(Q, P) \\ &= \text{dist}(z_0, \xi_0 \cdot z_0), \end{aligned}$$

Similarly, we obtain that

$$\begin{aligned} \text{dist}(z_0, \eta_1 \cdot z_0) &= \text{dist}(Q, R') \\ &< \text{dist}(Q, R) \\ &= \text{dist}(z_0, \eta_0 \cdot z_0). \end{aligned}$$

We also have

$$\begin{aligned}
 \text{dist}(z_0, \xi_1 \eta_1 \cdot z_0) &= \text{dist}(\xi_1^{-1} \cdot z_0, \eta_1 \cdot z_0) \\
 &= \text{dist}(P', R') \\
 &< \text{dist}(P, R) \\
 &= \text{dist}(z_0, \eta_0 \cdot z_0).
 \end{aligned}$$

This means that we obtain a point $(\xi_1, \eta_1) \in \mathfrak{G}\mathfrak{F}$ so that $f_{z_0}(\xi_1, \eta_1) < f_{z_0}(\xi_0, \eta_0)$. This contradicts with the fact that f_{z_0} has a local minimum at the point (ξ_0, η_0) . Hence, we conclude that (ξ_0, η_0) is not in $\mathfrak{G}\mathfrak{F}$. This completes the proof.

□



APPENDICES

Appendix A

PARTIAL DERIVATIVES

In this section, I list some of the first order partial derivatives of the functions $\{f_i : i \in I_1 \cup I_2 \cup I_3\}$ with a specific parametrization of Δ^7 .

A.1 Partial derivatives with the parametrization $\mathfrak{P}_8$

If we use the parametrization $\mathfrak{P}_8$, each point in Δ^7 has the form

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 = 1 - \sum_{k=1}^7 x_k)$$

where $0 < x_1 < 1, 0 < x_2 < 1 - x_1, 0 < x_3 < 1 - x_1 - x_2, \dots, 0 < x_7 < 1 - \sum_{k=1}^6 x_k$. Then we see that

$$\begin{aligned} f_1(\mathbf{x}) &= \frac{1 - x_1 - x_2 - x_3}{x_1 + x_2 + x_3} \cdot \frac{1 - x_4}{x_4}, \\ f_2(\mathbf{x}) &= \frac{1 - x_4 - x_5 - x_6}{x_4 + x_5 + x_6} \cdot \frac{1 - x_1}{x_1}, \\ f_3(\mathbf{x}) &= \frac{1 - x_1 - x_2 - x_3}{x_1 + x_2 + x_3} \cdot \frac{\sum_{i=1}^7 x_i}{1 - \sum_{i=1}^7 x_i}, \\ f_4(\mathbf{x}) &= \frac{1 - x_4 - x_5 - x_6}{x_4 + x_5 + x_6} \cdot \frac{1 - x_7}{x_7}, \\ f_5(\mathbf{x}) &= \frac{x_1 + x_2 + x_3}{1 - x_1 - x_2 - x_3} \cdot \frac{1 - x_6}{x_6}, \\ f_6(\mathbf{x}) &= \frac{x_1 + x_2 + x_3}{1 - x_1 - x_2 - x_3} \cdot \frac{1 - x_2}{x_2}, \end{aligned}$$

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$$f_7(\mathbf{x}) = \frac{x_4 + x_5 + x_6}{1 - x_4 - x_5 - x_6} \cdot \frac{1 - x_3}{x_3},$$

$$f_8(\mathbf{x}) = \frac{x_4 + x_5 + x_6}{1 - x_4 - x_5 - x_6} \cdot \frac{1 - x_5}{x_5}.$$

All the relevant first order partial derivatives are:

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} &= -\frac{1}{(x_1 + x_2 + x_3)^2} \cdot \frac{1 - x_4}{x_4}, \\ \frac{\partial f_2}{\partial x_1} &= -\frac{1}{x_1^2} \cdot \frac{1 - x_4 - x_5 - x_6}{x_4 + x_5 + x_6}, \\ \frac{\partial f_3}{\partial x_4} &= \frac{1}{\left(1 - \sum_{k=1}^7 x_k\right)^2} \cdot \frac{1 - x_1 - x_2 - x_3}{x_1 + x_2 + x_3}, \\ \frac{\partial f_4}{\partial x_4} &= -\frac{1}{(x_4 + x_5 + x_6)^2} \cdot \frac{1 - x_7}{x_7}, \\ \frac{\partial f_5}{\partial x_1} &= \frac{1}{(1 - x_1 - x_2 - x_3)^2} \cdot \frac{1 - x_6}{x_6}, \\ \frac{\partial f_6}{\partial x_1} &= \frac{1}{(1 - x_1 - x_2 - x_3)^2} \cdot \frac{1 - x_2}{x_2}, \\ \frac{\partial f_7}{\partial x_3} &= -\frac{1}{x_3^2} \cdot \frac{x_4 + x_5 + x_6}{1 - x_4 - x_5 - x_6}, \\ \frac{\partial f_8}{\partial x_4} &= \frac{1}{(1 - x_4 - x_5 - x_6)^2} \cdot \frac{1 - x_5}{x_5}. \end{aligned}$$

A.2 Partial derivatives with the parametrization $\mathfrak{P}_7$

If we use the parametrization $\mathfrak{P}_7$, each point in Δ^7 has the form

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7 = 1 - \sum_{k=1, k \neq 7}^8 x_k, x_8)$$

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where $0 < x_1 < 1$, $0 < x_2 < 1 - x_1$, $0 < x_3 < 1 - x_1 - x_2, \dots, 0 < x_6 < 1 - \sum_{k=1}^5 x_k$, $0 < x_8 < 1 - \sum_{k=1}^6 x_k$. Then

we see that

$$\begin{aligned}
 f_1(\mathbf{x}) &= \frac{1 - x_1 - x_2 - x_3}{x_1 + x_2 + x_3} \cdot \frac{1 - x_4}{x_4}, \\
 f_2(\mathbf{x}) &= \frac{1 - x_4 - x_5 - x_6}{x_4 + x_5 + x_6} \cdot \frac{1 - x_1}{x_1}, \\
 f_3(\mathbf{x}) &= \frac{1 - x_1 - x_2 - x_3}{x_1 + x_2 + x_3} \cdot \frac{1 - x_8}{x_8}, \\
 f_4(\mathbf{x}) &= \frac{1 - x_4 - x_5 - x_6}{x_4 + x_5 + x_6} \cdot \frac{\sum_{i=1, i \neq 7}^8 x_i}{1 - \sum_{i=1, i \neq 7}^8 x_i}, \\
 f_5(\mathbf{x}) &= \frac{x_1 + x_2 + x_3}{1 - x_1 - x_2 - x_3} \cdot \frac{1 - x_6}{x_6}, \\
 f_6(\mathbf{x}) &= \frac{x_1 + x_2 + x_3}{1 - x_1 - x_2 - x_3} \cdot \frac{1 - x_2}{x_2}, \\
 f_7(\mathbf{x}) &= \frac{x_4 + x_5 + x_6}{1 - x_4 - x_5 - x_6} \cdot \frac{1 - x_3}{x_3}, \\
 f_8(\mathbf{x}) &= \frac{x_4 + x_5 + x_6}{1 - x_4 - x_5 - x_6} \cdot \frac{1 - x_5}{x_5}.
 \end{aligned}$$

All relevant first order partial derivatives are:

$$\frac{\partial f_j}{\partial x_8} = 0$$

where $j = 1, 2, 5, 6, 7, 8$ and

$$\frac{\partial f_3}{\partial x_8} = -\frac{1}{x_8^2} \cdot \frac{1 - \sum_{k=1}^3 x_k}{\sum_{k=1}^3 x_k},$$

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$$\frac{\partial f_4}{\partial x_8} = \frac{1}{\left(1 - \sum_{k=1, k \neq 7}^8 x_k\right)^2} \cdot \frac{1 - \sum_{k=4}^6 x_k}{\sum_{k=4}^6 x_k}.$$

A.3 Partial derivatives with the parametrization $\mathfrak{P}_4$

If we use the parametrization $\mathfrak{P}_4$, each point in Δ^7 has the form

$$(x_1, x_2, x_3, x_4 = 1 - \sum_{k=1, k \neq 7}^8 x_k, x_8, x_5, x_6, x_7, x_8)$$

where $0 < x_1 < 1, 0 < x_2 < 1 - x_1, 0 < x_3 < 1 - x_1 - x_2, 0 < x_5 < 1 - x_1 - x_2 - x_3, \dots, 0 < x_8 < 1 - \sum_{k=1, k \neq 4}^7 x_k$.

Then we obtain that

$$\begin{aligned} f_1(\mathbf{x}) &= \frac{1 - x_1 - x_2 - x_3}{x_1 + x_2 + x_3} \cdot \frac{\sum_{i=1, i \neq 4}^8 x_i}{1 - \sum_{i=1, i \neq 4}^8 x_i}, \\ f_2(\mathbf{x}) &= \frac{x_1 + x_2 + x_3 + x_7 + x_8}{1 - x_1 - x_2 - x_3 - x_7 - x_8} \cdot \frac{1 - x_1}{x_1}, \\ f_3(\mathbf{x}) &= \frac{1 - x_1 - x_2 - x_3}{x_1 + x_2 + x_3} \cdot \frac{1 - x_8}{x_8}, \\ f_4(\mathbf{x}) &= \frac{x_1 + x_2 + x_3 + x_7 + x_8}{1 - x_1 - x_2 - x_3 - x_7 - x_8} \cdot \frac{1 - x_7}{x_7}, \\ f_5(\mathbf{x}) &= \frac{x_1 + x_2 + x_3}{1 - x_1 - x_2 - x_3} \cdot \frac{1 - x_6}{x_6}, \\ f_6(\mathbf{x}) &= \frac{x_1 + x_2 + x_3}{1 - x_1 - x_2 - x_3} \cdot \frac{1 - x_2}{x_2}, \\ f_7(\mathbf{x}) &= \frac{1 - x_1 - x_2 - x_3 - x_7 - x_8}{x_1 + x_2 + x_3 + x_7 + x_8} \cdot \frac{1 - x_3}{x_3}, \end{aligned}$$

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$$f_8(\mathbf{x}) = \frac{1 - x_1 - x_2 - x_3 - x_7 - x_8}{x_1 + x_2 + x_3 + x_7 + x_8} \cdot \frac{1 - x_5}{x_5}.$$

Hence, all the relevant first order partial derivatives are:

$$\frac{\partial f_j}{\partial x_i} = 0$$

for every $j = 2, 3, 4, 6, 7$ and for $i = 5, 6$ and

$$\begin{aligned} \frac{\partial f_1}{\partial x_5} &= \frac{\partial f_1}{\partial x_6} \\ &= \frac{1}{\left(\sum_{k=1, k \neq 4}^8 x_k \right)^2} \cdot \frac{1 - x_1 - x_2 - x_3}{x_1 + x_2 + x_3}, \\ \frac{\partial f_5}{\partial x_5} &= 0, \\ \frac{\partial f_5}{\partial x_6} &= -\frac{1}{x_6^2} \cdot \frac{x_1 + x_2 + x_3}{1 - x_1 - x_2 - x_3}, \\ \frac{\partial f_8}{\partial x_5} &= -\frac{1}{x_5^2} \cdot \frac{1 - x_1 - x_2 - x_3 - x_7 - x_8}{x_1 + x_2 + x_3 + x_7 + x_8}, \\ \frac{\partial f_8}{\partial x_6} &= 0. \end{aligned}$$

A.4 Partial derivatives with the parametrization $\mathfrak{P}_2$

If we use the substitution $\mathfrak{P}_2$, each point in Δ^7 has the form

$$(x_1, x_2 = 1 - \sum_{k=1, k \neq 2}^8 x_k, x_3, x_4, x_8, x_5, x_6, x_7, x_8)$$

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where $0 < x_1 < 1$, $0 < x_3 < 1 - x_1$, $0 < x_4 < 1 - x_1 - x_3$, ..., $0 < x_8 < 1 - \sum_{k=1, k \neq 4}^7 x_k$. Then we see that

$$\begin{aligned}
 f_1(\mathbf{x}) &= \frac{x_4 + x_5 + x_6 + x_7 + x_8}{1 - x_4 - x_5 - x_6 - x_7 - x_8} \cdot \frac{1 - x_4}{x_4}, \\
 f_2(\mathbf{x}) &= \frac{1 - x_4 - x_5 - x_6}{x_4 + x_5 + x_6} \cdot \frac{1 - x_1}{x_1}, \\
 f_3(\mathbf{x}) &= \frac{x_4 + x_5 + x_6 + x_7 + x_8}{1 - x_4 - x_5 - x_6 - x_7 - x_8} \cdot \frac{1 - x_8}{x_8}, \\
 f_4(\mathbf{x}) &= \frac{1 - x_4 - x_5 - x_6}{x_4 + x_5 + x_6} \cdot \frac{1 - x_7}{x_7}, \\
 f_5(\mathbf{x}) &= \frac{1 - x_4 - x_5 - x_6 - x_7 - x_8}{x_4 + x_5 + x_6 + x_7 + x_8} \cdot \frac{1 - x_6}{x_6}, \\
 f_6(\mathbf{x}) &= \frac{1 - x_4 - x_5 - x_6 - x_7 - x_8}{x_4 + x_5 + x_6 + x_7 + x_8} \cdot \frac{\sum_{i=1, i \neq 2}^8 x_i}{1 - \sum_{i=1, i \neq 2}^8 x_i}, \\
 f_7(\mathbf{x}) &= \frac{x_4 + x_5 + x_6}{1 - x_4 - x_5 - x_6} \cdot \frac{1 - x_3}{x_3}, \\
 f_8(\mathbf{x}) &= \frac{x_4 + x_5 + x_6}{1 - x_4 - x_5 - x_6} \cdot \frac{1 - x_5}{x_5}.
 \end{aligned}$$

The relevant first order partial derivatives are: $\frac{\partial f_j}{\partial x_1} = 0$ for every $j = 1, 3, 4, 5, 7, 8$, $\frac{\partial f_j}{\partial x_3} = 0$ for every $j = 1, 2, 3, 4, 5, 8$ and

$$\begin{aligned}
 \frac{\partial f_2}{\partial x_1} &= -\frac{1}{x_1^2} \cdot \frac{1 - x_4 - x_5 - x_6}{x_4 + x_5 + x_6}, \\
 \frac{\partial f_6}{\partial x_1} &= \frac{\partial f_6}{\partial x_3} \\
 &= \frac{1}{\left(1 - \sum_{k=1, k \neq 2}^8 x_k\right)^2} \cdot \frac{1 - \sum_{k=4}^8 x_k}{\sum_{k=4}^8 x_k}, \\
 \frac{\partial f_7}{\partial x_3} &= -\frac{1}{x_3^2} \cdot \frac{x_4 + x_5 + x_6}{1 - x_4 - x_5 - x_6}.
 \end{aligned}$$

Appendix B

PROOFS OF LEMMA 4.7 AND LEMMA 4.8

B.1 Proof of Lemma 4.7

Proof. Let us calculate the derivative of $G^+(\alpha)$:

$$(G^+)'(\alpha) = \frac{(g^2 R_1 + g R_2 + R_3)g' + (g^2 R_4 + g R_5 + R_6)}{\sqrt{\alpha}(\sqrt{\alpha} + 1)^2(\alpha g - g + 1)^2}$$

where

$$R_1(\alpha) = \alpha^{\frac{7}{2}} - \alpha^{\frac{5}{2}} + 2\alpha^3 - 4\alpha^2 - \alpha^{\frac{3}{2}} + 2\alpha + \sqrt{\alpha},$$

$$R_2(\alpha) = 2\alpha^{\frac{5}{2}} + 4\alpha^2 - 4\alpha - 2\sqrt{\alpha},$$

$$R_3(\alpha) = -\alpha^{\frac{5}{2}} - 2\alpha^2 + 2\alpha + \sqrt{\alpha},$$

$$R_4(\alpha) = -\alpha^2 + \alpha^{\frac{3}{2}} + 4\alpha + \sqrt{\alpha} - 1,$$

$$R_5(\alpha) = -\alpha^{\frac{3}{2}} - 4\alpha - \sqrt{\alpha} + 2,$$

$$R_6(\alpha) = -1.$$

Let us also calculate the derivative of $g(\alpha)$:

$$g' = -\frac{R_7\sqrt{d} + R_8\sqrt{\alpha} + R_9}{4(\alpha^{\frac{7}{2}} + \alpha^2 - \alpha^{\frac{3}{2}} - 1)^2\sqrt{d}\sqrt{\alpha}}$$

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where

$$R_7(\alpha) = \alpha^6 - 2\alpha^{\frac{11}{2}} - 2\alpha^{\frac{9}{2}} + 9\alpha^4 + 4\alpha^{\frac{7}{2}} + 8\alpha^{\frac{5}{2}} - 3\alpha^2 - 2\alpha^{\frac{3}{2}} + 2\alpha^{\frac{1}{2}} + 1,$$

$$R_8(\alpha) = -3\alpha^8 + 8\alpha^7 - 16\alpha^6 - 24\alpha^5 + 22\alpha^4 + 8\alpha^3 + 8\alpha - 3,$$

$$R_9(\alpha) = \alpha^9 - 6\alpha^8 - 12\alpha^7 + 16\alpha^6 - 14\alpha^5 - 12\alpha^4 + 28\alpha^3 - 3\alpha + 2.$$

Let us write $d(\alpha) = R_{10}(\alpha) - R_{11}(\alpha)$ where

$$R_{10}(\alpha) = \alpha^6 + 6\alpha^3 + \alpha^2 + 4 + (4\alpha^4 + 6\alpha + 4)\sqrt{\alpha},$$

$$R_{11}(\alpha) = 3\alpha^5 + 6\alpha^4 + 3\alpha + (2\alpha^5 + 4\alpha^3 + 8\alpha^2)\sqrt{\alpha}.$$

Note that both R_{10} and R_{11} are increasing functions over the interval I_α because

$$R'_{10}(\alpha) = \frac{1}{2\sqrt{\alpha}} \left(12\alpha^{\frac{11}{2}} + 36\alpha^4 + 36\alpha^{\frac{5}{2}} + 4\alpha^{\frac{3}{2}} + 18\alpha + 4 \right) > 0,$$

$$R'_{11}(\alpha) = \frac{1}{2\sqrt{\alpha}} \left(22\alpha^5 + 30\alpha^{\frac{9}{2}} + 48\alpha^{\frac{7}{2}} + 28\alpha^3 + 40\alpha^2 + 6\alpha^{\frac{1}{2}} \right) > 0$$

for every $\alpha \in I_\alpha$. Let us define the following functions

$$P_1(\alpha) = 8\alpha^{\frac{11}{2}} - 22\alpha^5,$$

$$P_2(\alpha) = 3\alpha^{\frac{11}{2}} + 36\alpha^4 + 36\alpha^{\frac{5}{2}} - 30\alpha^{\frac{9}{2}} - 48\alpha^{\frac{7}{2}}$$

$$P_3(\alpha) = \alpha^{\frac{11}{2}} + 4\alpha^{\frac{3}{2}} + 18\alpha + 4 - 28\alpha^3 - 40\alpha^2 - 6\alpha^{\frac{1}{2}}.$$

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Note that we have

$$d'(\alpha) = \frac{1}{2\sqrt{\alpha}} (P_1(\alpha) + P_2(\alpha) + P_3(\alpha)).$$

We show that $P_i(\alpha) > 0$ for every $\alpha \in I_\alpha$ for $i = 1, 2, 3$. It is clear that $P_1(\alpha) = \alpha^5(8\alpha^{\frac{1}{2}} - 22) > 0$ for every $\alpha \in I_\alpha$ since $\alpha \geq 9$.

Let us calculate the derivative of $P_2(\alpha)$:

$$P'_2(\alpha) = \frac{33}{2}\alpha^{\frac{9}{2}} + 144\alpha^3 + 90\alpha^{\frac{3}{2}} - 135\alpha^{\frac{7}{2}} - 168\alpha^{\frac{5}{2}}.$$

Note that we get

$$\begin{aligned} P'_2(\alpha) &\geq \min_{\alpha \in I_\alpha} \frac{33}{2}\alpha^{\frac{9}{2}} + 144\alpha^3 + 90\alpha^{\frac{3}{2}} - \max_{\alpha \in I_\alpha} 135\alpha^{\frac{7}{2}} + 168\alpha^{\frac{5}{2}} \\ &= \frac{33}{2} \cdot 9^{\frac{9}{2}} + 144 \cdot 9^3 + 90 \cdot 9^{\frac{3}{2}} - 135 \cdot (5 + 3\sqrt{2})^{\frac{7}{2}} - 168 \cdot (5 + 3\sqrt{2})^{\frac{5}{2}} \\ &> 0 \end{aligned}$$

since both of the functions $\frac{33}{2}\alpha^{\frac{9}{2}} + 144\alpha^3 + 90\alpha^{\frac{3}{2}}$ and $135\alpha^{\frac{7}{2}} + 168\alpha^{\frac{5}{2}}$ are clearly increasing over the interval I_α . This means $P_2(\alpha)$ is increasing. Therefore, we obtain that

$$\begin{aligned} P_2(\alpha) &\geq P_2(9) \\ &= 3 \cdot 9^{\frac{11}{2}} + 36 \cdot 9^4 + 36 \cdot 9^{\frac{5}{2}} - 30 \cdot 9^{\frac{9}{2}} - 48 \cdot 9^{\frac{7}{2}} \\ &> 0 \end{aligned}$$

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for every $\alpha \in I_\alpha$. We also carry out a similar calculation for $P_3(\alpha)$. Note that we have

$$\begin{aligned}
 P_3(\alpha) &\geq \min_{\alpha \in I_\alpha} \alpha^{\frac{11}{2}} + 4\alpha^{\frac{3}{2}} + 18\alpha + 4 - \max_{\alpha \in I_\alpha} 28\alpha^3 + 40\alpha^2 + 6\alpha^{\frac{1}{2}} \\
 &= 9^{\frac{11}{2}} + 4 \cdot 9^{\frac{3}{2}} + 18 \cdot 9 + 4 - 28 \cdot (5 + 3\sqrt{2})^3 - 40 \cdot (5 + 3\sqrt{2})^2 - 6 \cdot (5 + 3\sqrt{2})^{\frac{1}{2}} \\
 &> 0
 \end{aligned}$$

since both of the functions $\alpha^{\frac{11}{2}} + 4\alpha^{\frac{3}{2}} + 18\alpha + 4$ and $28\alpha^3 + 40\alpha^2 + 6\alpha^{\frac{1}{2}}$ are clearly increasing over the interval I_α . This means $P_3(\alpha)$ is a positive function for every α . Hence, we conclude that $d'(\alpha) > 0$ for every α . Consequently, $d(\alpha)$ is an increasing function.

We need to find if R_7 is a positive function. Let us write $R_7(\alpha) = r_1(\alpha) - r_2(\alpha)$ where

$$r_1(\alpha) = \alpha^6 + 9\alpha^4 + 4\alpha^{\frac{7}{2}} + 8\alpha^{\frac{5}{2}} + 2\alpha^{\frac{1}{2}} + 1,$$

and

$$r_2(\alpha) = 2\alpha^{\frac{11}{2}} + 2\alpha^{\frac{9}{2}} + 3\alpha^2 + 2\alpha^{\frac{3}{2}}.$$

Then we obtain that

$$\begin{aligned}
 R_7(\alpha) &\geq \min_{\alpha \in I_\alpha} r_1(\alpha) - \max_{\alpha \in I_\alpha} r_2(\alpha) \\
 &= r_1(9) - r_2(5 + 3\sqrt{2}) \\
 &> 0
 \end{aligned}$$

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for every α . Next we examine the expression $R_7\sqrt{d} + R_8\sqrt{\alpha} + R_9$. Since we know that $d(\alpha)$ is increasing, we obtain

$$\begin{aligned} R_7\sqrt{d} + R_8\sqrt{\alpha} + R_9 &\leq R_7 \cdot \max_{\alpha \in I_\alpha} \sqrt{d} + R_8\sqrt{\alpha} + R_9 \\ &= R_7\sqrt{d(5+3\sqrt{2})} + R_8\sqrt{\alpha} + R_9 \\ &= KR_7 + R_8\sqrt{\alpha} + R_9. \end{aligned}$$

where $K = \sqrt{d(5+3\sqrt{2})}$ and

$$\sqrt{d(5+3\sqrt{2})} = 191040 + 135084\sqrt{2} - 54744\sqrt{5+3\sqrt{2}} - 38712\sqrt{5+3\sqrt{2}}\sqrt{2}.$$

We will show that $KR_7 + R_8\sqrt{\alpha} + R_9 < 0$ for every $\alpha \in I_\alpha$. First, we consider the equality

$$KR_7 + R_8\sqrt{\alpha} + R_9 = p_1 - p_2$$

where

$$p_1 = K\left(\alpha^6 + 9\alpha^4 + 4\alpha^{\frac{7}{2}} + 8\alpha^{\frac{5}{2}} + 2\alpha^{\frac{1}{2}} + 1\right) + 8\alpha^{\frac{15}{2}} + 22\alpha^{\frac{9}{2}} + 8\alpha^{\frac{7}{2}} + 8\alpha^{\frac{3}{2}} + \alpha^9 + 16\alpha^6 + 28\alpha^3 + 2$$

and

$$p_2 = K\left(2\alpha^{\frac{11}{2}} + 2\alpha^{\frac{9}{2}} + 3\alpha^2 + 2\alpha^{\frac{3}{2}}\right) + 3\alpha^{\frac{17}{2}} + 16\alpha^{\frac{13}{2}} + 24\alpha^{\frac{11}{2}} + 3\alpha^{\frac{1}{2}} + 6\alpha^8 + 12\alpha^7 + 12\alpha^4 + 14\alpha^5 + 3\alpha.$$

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Note that we have

$$\begin{aligned}
 KR_7 + R_8\sqrt{\alpha} + R_9 &= p_1(\alpha) - p_2(\alpha) \\
 &\leq \max_{\alpha \in I_\alpha} p_1(\alpha) - \min_{\alpha \in I_\alpha} p_2(\alpha) \\
 &= p_1(5 + 3\sqrt{2}) - p_2(9) \\
 &< 0
 \end{aligned}$$

because both p_1 and p_2 are clearly increasing functions over the interval I_α . This implies that $g'(\alpha) > 0$ for every $\alpha \in I_\alpha$.

We know that $g(\alpha)$ is an increasing function. $g(\alpha)$ is also a positive function since $g(9) > 0$. Thus, $g^2(\alpha)$ is an increasing function. Then we obtain

$$\begin{aligned}
 g^2 R_1 + g R_2 + R_3 &\leq \left(\max_{\alpha \in I_\alpha} g(\alpha) \right)^2 R_1 + \left(\max_{\alpha \in I_\alpha} g(\alpha) \right) R_2 + R_3 \\
 &= \left(g(5 + 3\sqrt{2}) \right)^2 R_1 + g(5 + 3\sqrt{2}) R_2 + R_3.
 \end{aligned}$$

To simplify the calculations, let us say $K = g(5 + 3\sqrt{2}) > 0$. We will show that $K^2 R_1 + K R_2 + R_3 < 0$ for every $\alpha \in I_\alpha$. Let us say $K^2 R_1 + K R_2 + R_3 = q_1 - q_2$ where

$$q_1 = K^2 \left(\alpha^{\frac{7}{2}} + 2\alpha^3 + 2\alpha + \alpha^{\frac{1}{2}} \right) + K \left(2\alpha^{\frac{5}{2}} + 4\alpha^2 \right) + 2\alpha + \alpha^{\frac{1}{2}}$$

and

$$q_2 = K^2 \left(\alpha^{\frac{5}{2}} + 4\alpha^2 + \alpha^{\frac{3}{2}} \right) + K \left(4\alpha + 2\alpha^{\frac{1}{2}} \right) + \alpha^{\frac{5}{2}} + 2\alpha^2.$$

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Note that we have

$$\begin{aligned}
 K^2 R_1 + K R_2 + R_3 &= q_1(\alpha) - q_2(\alpha) \\
 &\leq \max_{\alpha \in I_\alpha} q_1(\alpha) - \min_{\alpha \in I_\alpha} q_2(\alpha) \\
 &= q_1(5 + 3\sqrt{2}) - q_2(9) \\
 &< 0
 \end{aligned}$$

for every $\alpha \in I_\alpha$ since both of the functions $q_1(\alpha)$ and $q_2(\alpha)$ are clearly increasing functions. Therefore, we showed that the expression $(g^2 R_1 + g R_2 + R_3)g' < 0$ for every α . We need to show that the expression $g^2 R_4 + g R_5 + R_6$ is also negative for every α .

It is clear that the functions R_5 and R_6 are negative functions. We show that R_4 is also a negative function. Note that we have

$$\begin{aligned}
 R_4(\alpha) &= \alpha^{\frac{3}{2}} + 4\alpha - (\alpha^2 + 1) \\
 &\leq \max_{\alpha \in I_\alpha} \alpha^{\frac{3}{2}} + 4\alpha - \min_{\alpha \in I_\alpha} \alpha^2 + 1 \\
 &= (5 + 3\sqrt{2})^{\frac{3}{2}} + 4 \cdot 5 + 3\sqrt{2} - (9^2 + 1) \\
 &< 0
 \end{aligned}$$

for every $\alpha \in I_\alpha$ since both of the functions $\alpha^{\frac{3}{2}} + 4\alpha$ and $\alpha^2 + 1$ are increasing functions. The last inequality implies that $g^2 R_4 + g R_5 + R_6$ is also negative for every α because we have $g^2 > 0$ and $g > 0$ for every α . Thus we get $(g^2 R_1 + g R_2 + R_3)g' + (g^2 R_4 + g R_5 + R_6) < 0$ for every α . In particular, we

Appendix B (Continued)

obtain that $(G^+)' < 0$ for every α . Thus, the following calculations gives the result of the first part of the lemma:

$$\begin{aligned} G^+(\alpha) &\geq G^+(5+3\sqrt{2}) \\ &= \frac{(\sqrt{5+3\sqrt{2}}+1)(5+3\sqrt{2}-1)(g(5+3\sqrt{2}))^2 + 2(5+3\sqrt{2}-1)g(5+3\sqrt{2}) + \sqrt{5+3\sqrt{2}}+3}{(\sqrt{5+3\sqrt{2}}+1)((5+3\sqrt{2}-1)g(5+3\sqrt{2})+1)} \\ &> 1 \end{aligned}$$

for every $\alpha \in I_\alpha$.

For the second part of the lemma, we calculate the derivative of $G^-(\alpha)$:

$$(G^-)'(\alpha) = \frac{(h^2 R_1 + h R_2 + R_3)h' + (h^2 R_4 + h R_5 + R_6)}{\sqrt{\alpha}(\sqrt{\alpha}+1)^2(\alpha h - h + 1)^2}$$

Note that $R_1, R_2, R_3, R_4, R_5, R_6$ are the same functions we used for the first part of the proof. Let us calculate the derivative of $h(\alpha)$:

$$h' = \frac{\hat{R}_7 \sqrt{d} + \hat{R}_8 \sqrt{\alpha} + \hat{R}_9}{4(\alpha^{\frac{7}{2}} + \alpha^2 - \alpha^{\frac{3}{2}} - 1)^2 \sqrt{d} \sqrt{\alpha}}$$

where

$$\hat{R}_7(\alpha) = -R_7(\alpha),$$

$$\hat{R}_8(\alpha) = R_8(\alpha),$$

$$\hat{R}_9(\alpha) = R_9(\alpha).$$

Appendix B (Continued)

We already know that $\sqrt{d(\alpha)} > 0$ is an increasing function from the first part. Let us examine the expression $-R_7\sqrt{d} + R_8\sqrt{\alpha} + R_9$. Since we know that $d(\alpha)$ is increasing, we obtain

$$\begin{aligned} -R_7\sqrt{d} + R_8\sqrt{\alpha} + R_9 &\leq -R_7 \cdot \min_{\alpha \in I_\alpha} \sqrt{d} + R_8\sqrt{\alpha} + R_9 \\ &= -R_7\sqrt{d(9)} + R_8\sqrt{\alpha} + R_9 \\ &= -33280R_7 + R_8\sqrt{\alpha} + R_9. \end{aligned}$$

To simplify the calculations, let us say $K = -33280$. We will show that $KR_7 + R_8\sqrt{\alpha} + R_9 > 0$ for every $\alpha \in I_\alpha$. First, we consider the equality $KR_7 + R_8\sqrt{\alpha} + R_9 = s_1 - s_2$ where

$$s_1 = -K \left(2\alpha^{\frac{11}{2}} + 2\alpha^{\frac{9}{2}} + 3\alpha^2 + 2\alpha^{\frac{3}{2}} \right) + 8\alpha^{\frac{15}{2}} + 22\alpha^{\frac{9}{2}} + 8\alpha^{\frac{7}{2}} + 8\alpha^{\frac{3}{2}} + \alpha^9 + 16\alpha^6 + 28\alpha^3 + 2$$

and

$$s_2 = -K \left(\alpha^6 + 9\alpha^4 + 4\alpha^{\frac{7}{2}} + 8\alpha^{\frac{5}{2}} + 2\alpha^{\frac{1}{2}} + 1 \right) + 3\alpha^{\frac{17}{2}} + 16\alpha^{\frac{13}{2}} + 24\alpha^{\frac{11}{2}} + 3\alpha^{\frac{1}{2}} + 6\alpha^8 + 12\alpha^7 + 12\alpha^4 + 14\alpha^5 + 3\alpha.$$

Note that we have

$$\begin{aligned} KR_7 + R_8\sqrt{\alpha} + R_9 &= s_1(\alpha) - s_2(\alpha) \\ &\leq \max_{\alpha \in I_\alpha} s_1(\alpha) - \min_{\alpha \in I_\alpha} s_2(\alpha) \\ &= s_1(5 + 3\sqrt{2}) - s_2(9) \\ &< 0. \end{aligned}$$

Appendix B (Continued)

because both s_1 and s_2 are clearly increasing functions over the interval I_α . This implies that $h'(\alpha) < 0$ for every $\alpha \in I_\alpha$.

We know that $h(\alpha)$ is a decreasing function. $h(\alpha)$ is also a positive function since $h(5 + 3\sqrt{2}) > 0$. Therefore, $h^2(\alpha)$ is a decreasing function. Then we obtain

$$\begin{aligned} h^2 R_1 + h R_2 + R_3 &\geq \left(\min_{\alpha \in I_\alpha} h(\alpha) \right)^2 R_1 + \left(\min_{\alpha \in I_\alpha} h(\alpha) \right) R_2 + R_3 \\ &= \left(h(5 + 3\sqrt{2}) \right)^2 R_1 + h(5 + 3\sqrt{2}) R_2 + R_3. \end{aligned}$$

To simplify the calculations, let us say $K = h(5 + 3\sqrt{2}) > 0$. We will show that $K^2 R_1 + K R_2 + R_3 < 0$ for every $\alpha \in I_\alpha$. Let us say $K^2 R_1 + K R_2 + R_3 = t_1 - t_2$ where

$$t_1 = K^2 \left(\alpha^{\frac{7}{2}} + 2\alpha^3 + 2\alpha + \alpha^{\frac{1}{2}} \right) + K \left(2\alpha^{\frac{5}{2}} + 4\alpha^2 \right) + 2\alpha + \alpha^{\frac{1}{2}}$$

and

$$t_2 = K^2 \left(\alpha^{\frac{5}{2}} + 4\alpha^2 + \alpha^{\frac{3}{2}} \right) + K \left(4\alpha + 2\alpha^{\frac{1}{2}} \right) + \alpha^{\frac{5}{2}} + 2\alpha^2.$$

Note that we have

$$\begin{aligned} K^2 R_1 + K R_2 + R_3 &= t_1(\alpha) - t_2(\alpha) \\ &\geq \min_{\alpha \in I_\alpha} t_1(\alpha) - \max_{\alpha \in I_\alpha} t_2(\alpha) \\ &= t_1(9) - t_2(5 + 3\sqrt{2}) \\ &> -381 \end{aligned}$$

Appendix B (Continued)

for every $\alpha \in I_\alpha$ since both of the functions $t_1(\alpha)$ and $t_2(\alpha)$ are clearly increasing functions.

We need to show that $h'(\alpha)$ is an increasing function. To this purpose, we calculate $h''(\alpha)$:

$$h'' = \frac{d^{\frac{3}{2}}(Q_1 - Q_2) + Q_3 - Q_4}{8(\alpha^{\frac{7}{2}} + \alpha^2 - \alpha^{\frac{3}{2}} - 1)^3(d)^{\frac{3}{2}}(\alpha)^{\frac{3}{2}}}$$

where

$$Q_1(\alpha) = 3\alpha^{\frac{19}{2}} + 72\alpha^{\frac{15}{2}} + 24\alpha^7 + 96\alpha^6 + 16\alpha^{\frac{9}{2}} + 60\alpha^4 + 36\alpha^{\frac{7}{2}} + 8\alpha^3 + 48\alpha^{\frac{5}{2}},$$

$$Q_2(\alpha) = 8\alpha^9 + 15\alpha^8 + 36\alpha^{\frac{11}{2}} + 24\alpha^5 + 12\alpha^2 + 11\alpha^{\frac{3}{2}} + 1,$$

$$Q_3(\alpha) = 17\alpha^{18} + 15\alpha^{\frac{35}{2}} + 118\alpha^{\frac{33}{2}} + 162\alpha^{16} + 414\alpha^{15} + 438\alpha^{\frac{29}{2}} + 950\alpha^{\frac{27}{2}} + 810\alpha^{13} +$$

$$2194\alpha^{12} + 2034\alpha^{\frac{23}{2}} + 3336\alpha^{\frac{21}{2}} + 3624\alpha^{10} + 3090\alpha^9 + 3072\alpha^{\frac{17}{2}} + 2166\alpha^{\frac{15}{2}} +$$

$$2442\alpha^7 + 2026\alpha^6 + 1290\alpha^{\frac{11}{2}} + 974\alpha^{\frac{9}{2}} + 37\alpha^{\frac{3}{2}} + 12\alpha^{\frac{1}{2}} + 8,$$

$$Q_4(\alpha) = 3\alpha^{\frac{37}{2}} + 132\alpha^{17} + 798\alpha^{\frac{31}{2}} + 2442\alpha^{14} + 4038\alpha^{\frac{25}{2}} + 4386\alpha^{11} + 4596\alpha^{\frac{19}{2}} + 5346\alpha^8 +$$

$$4326\alpha^{\frac{13}{2}} + 2226\alpha^5 + 90\alpha^4 + 1098\alpha^{\frac{7}{2}} + 6\alpha^3 + 213\alpha^{\frac{5}{2}} + 153\alpha^2 + 6\alpha.$$

We show that $Q_1(\alpha) - Q_2(\alpha) > 0$ for every $\alpha \in I_\alpha$. Let us say $Q_1 - Q_2 = q_1 + q_2 + q_3 + q_4$ where

$$q_1(\alpha) = 96\alpha^6 + 16\alpha^{\frac{9}{2}} + 60\alpha^4 + 36\alpha^{\frac{7}{2}} + 8\alpha^3 + 48\alpha^{\frac{5}{2}} - (24\alpha^5 + 12\alpha^2 + 11\alpha^{\frac{3}{2}} + 1),$$

$$q_2(\alpha) = 3\alpha^{\frac{19}{2}} - 8\alpha^9,$$

$$q_3(\alpha) = 72\alpha^{\frac{15}{2}} - 15\alpha^8,$$

$$q_4(\alpha) = 24\alpha^7 - (36\alpha^{\frac{11}{2}} + 24\alpha^5).$$

Appendix B (Continued)

It is clear that

$$\begin{aligned} 96\alpha^6 + 16\alpha^{\frac{9}{2}} + 60\alpha^4 + 36\alpha^{\frac{7}{2}} + 8\alpha^3 + 48\alpha^{\frac{5}{2}} &> 24\alpha^6 + 12\alpha^{\frac{9}{2}} + 11\alpha^{\frac{7}{2}} + 1 \\ &> 24\alpha^5 + 12\alpha^2 + 11\alpha^{\frac{3}{2}} + 1, \end{aligned}$$

and

$$\begin{aligned} 3\alpha^{\frac{19}{2}} - 8\alpha^9 &= \alpha^9(3\alpha^{\frac{1}{2}} - 8) \\ &> \alpha^9, \end{aligned}$$

and

$$72\alpha^{\frac{15}{2}} - 15\alpha^8 = \alpha^{\frac{15}{2}}(72 - 15\alpha^{\frac{1}{2}}).$$

Hence, we get $q_1(\alpha) > 0$, $q_2(\alpha) > 0$ and $q_3(\alpha) > 0$ for every $\alpha \in I_\alpha$. Therefore, it is enough to show that

for every $\alpha \in I_\alpha$, $q_4(\alpha) > 0$. We see that

$$\begin{aligned} q_4(\alpha) &= \alpha^5 \left(24\alpha^2 - (36\alpha^{\frac{1}{2}} + 24) \right) \\ &\geq \alpha^5 \left(\min_{\alpha \in I_\alpha} 24\alpha^2 - \max_{\alpha \in I_\alpha} (36\alpha^{\frac{1}{2}} + 24) \right) \\ &\geq \alpha^5 \\ &> 0 \end{aligned}$$

Appendix B (Continued)

for every $\alpha \in I_\alpha$. This means that $q_4(\alpha)$ is also a positive function over the interval I_α . This implies that $Q_1(\alpha) - Q_2(\alpha) = q_1(\alpha) + q_2(\alpha) + q_3(\alpha) + q_4(\alpha) > 0$ for every $\alpha \in I_\alpha$.

We carry out similar calculations to show that the function $Q_3(\alpha) - Q_4(\alpha) > 0$ over the interval I_α .

By comparing term by term, we see that $q_1(\alpha) > q_2(\alpha)$ for every $\alpha \in I_\alpha$ where

$$\begin{aligned} q_1(\alpha) = & 950\alpha^{\frac{27}{2}} + 810\alpha^{13} + \\ & 2194\alpha^{12} + 2034\alpha^{\frac{23}{2}} + 3336\alpha^{\frac{21}{2}} + 3624\alpha^{10} + 3090\alpha^9 + 3072\alpha^{\frac{17}{2}} + 2166\alpha^{\frac{15}{2}} + \\ & 2442\alpha^7 + 2026\alpha^6 + 1290\alpha^{\frac{11}{2}} + 974\alpha^{\frac{9}{2}} + 37\alpha^{\frac{3}{2}} + 12\alpha^{\frac{1}{2}} + 8 \end{aligned}$$

and

$$\begin{aligned} q_2(\alpha) = & 4386\alpha^{11} + 4596\alpha^{\frac{19}{2}} + 5346\alpha^8 + \\ & 4326\alpha^{\frac{13}{2}} + 2226\alpha^5 + 90\alpha^4 + 1098\alpha^{\frac{7}{2}} + 6\alpha^3 + 213\alpha^{\frac{5}{2}} + 153\alpha^2 + 6\alpha. \end{aligned}$$

Let $Q_3(\alpha) - Q_4(\alpha) = q_1 - q_2 + q_3 + q_4 + q_5$ where

$$q_3(\alpha) = \alpha^{17}(10 - 3\alpha^{\frac{1}{2}}),$$

$$q_4(\alpha) = \alpha^{16}(7\alpha^2 + 15\alpha^{\frac{3}{2}} + 118\alpha^{\frac{1}{2}} - 132\alpha),$$

$$q_5(\alpha) = \alpha^{\frac{31}{2}}(162\alpha^{\frac{1}{2}} - 798) + \alpha^{14}(414\alpha - 2442) + \alpha^{\frac{25}{2}}(438\alpha^2 - 4038).$$

Appendix B (Continued)

Since we have $\alpha \geq 3$, we get $q_3(\alpha) > 0$ for every α . We see that

$$\begin{aligned}
 q_4(\alpha) &= \alpha^{16}(7\alpha^2 + 15\alpha^{\frac{3}{2}} + 118\alpha^{\frac{1}{2}} - 132\alpha) \\
 &\geq \alpha^{16} \left(\min_{\alpha \in I_\alpha} (7\alpha^2 + 15\alpha^{\frac{3}{2}} + 118\alpha^{\frac{1}{2}}) - \max_{\alpha \in I_\alpha} 132\alpha \right) \\
 &\geq \alpha^{16} \\
 &> 0.
 \end{aligned}$$

All the expressions $162\alpha^{\frac{1}{2}} - 798$, $414\alpha - 2442$ and $438\alpha^2 - 4038$ are positive functions since $\alpha \geq 3$.

This implies that $Q_3(\alpha) - Q_4(\alpha) = q_1(\alpha) - q_2(\alpha) + q_3(\alpha) + q_4(\alpha) + q_5(\alpha) > 0$ for every $\alpha \in I_\alpha$. Thus, we conclude that $h''(\alpha) > 0$ over the interval I_α which shows that $h'(\alpha)$ is an increasing function over the interval I_α . Then we calculate that

$$\begin{aligned}
 h'(\alpha) &\geq h'(9) \\
 &= \frac{\hat{R}_7(9)\sqrt{d(9)} + \hat{R}_8(9)\sqrt{9} + \hat{R}_9(9)}{4(9^{\frac{7}{2}} + 9^2 - 9^{\frac{3}{2}} - 1)^2 \sqrt{d(9)}\sqrt{9}} \\
 &\geq -\frac{1}{40}.
 \end{aligned}$$

Let us consider the following inequalities:

$$\begin{aligned}
 h^2 R_4 + h R_5 + R_6 &\geq \left(\min_{\alpha \in I_\alpha} h(\alpha) \right)^2 R_4 + \left(\min_{\alpha \in I_\alpha} h(\alpha) \right) R_5 + R_6 \\
 &= \left(h(5 + 3\sqrt{2}) \right)^2 R_4 + h(5 + 3\sqrt{2}) R_5 + R_6.
 \end{aligned}$$

Appendix B (Continued)

To simplify the calculations, let us say $K = h(5 + 3\sqrt{2}) > 0$. We will show that $K^2 R_4 + K R_5 + R_6 > 0$ for every $\alpha \in I_\alpha$. Let us say $K^2 R_4 + K R_5 + R_6 = u_1 - u_2$ where

$$u_1 = K^2 \left(\alpha^{\frac{3}{2}} + 4\alpha + \alpha^{\frac{1}{2}} \right) + 2K$$

and

$$u_2 = K^2(\alpha^2 + 1) + K(\alpha^{\frac{3}{2}} + 4\alpha + \alpha^{\frac{1}{2}}) + 1.$$

Note that we have

$$\begin{aligned} K^2 R_4 + K R_5 + R_6 &= u_1(\alpha) - u_2(\alpha) \\ &\geq \min_{\alpha \in I_\alpha} u_1(\alpha) - \max_{\alpha \in I_\alpha} u_2(\alpha) \\ &= u_1(9) - u_2(5 + 3\sqrt{2}) \\ &> -6 \end{aligned}$$

for every $\alpha \in I_\alpha$ since both of the functions $u_1(\alpha)$ and $u_2(\alpha)$ are clearly increasing functions. For every $\alpha \in I_\alpha$, we calculate that

$$\begin{aligned} (h^2 R_1 + h R_2 + R_3)h' + (h^2 R_4 + h R_5 + R_6) &> (-381) \cdot \left(\frac{1}{40} \right) + (-6) \\ &> 0. \end{aligned}$$

Appendix B (Continued)

Since we have $\sqrt{\alpha}(\sqrt{\alpha}+1)^2(\alpha h - h + 1)^2 > 0$ for every $\alpha \in I_\alpha$, we obtain $(G^-)'(\alpha) > 0$ for every $\alpha \in I_\alpha$.

Thus, the following calculations gives the result of the second part of the lemma:

$$\begin{aligned} G^-(\alpha) &\geq G^-(9) \\ &= \frac{(\sqrt{9}+1)(9-1)(h(9))^2 + 2(9-1)h(9) + \sqrt{9}+3}{(\sqrt{9}+1)((9-1)h(9)+1)} \\ &> 1 \end{aligned}$$

for every $\alpha \in I_\alpha$. □

B.2 Proof of Lemma 4.8

Proof. Let us calculate the derivative of $G^+(\alpha)$:

$$(G^+)'(\alpha) = -\frac{R_1\sqrt{d} + R_2}{\sqrt{d}(\alpha^2 + 4\alpha - 1 + \sqrt{d})^2(\alpha + 1)^2(\alpha - 1)^2}$$

where

$$d(\alpha) = (\alpha - 1)(\alpha^3 - 3\alpha^2 - 5\alpha - 9),$$

$$R_1(\alpha) = 2(\alpha - 1)^2(\alpha^4 - 2\alpha^3 - 2\alpha^2 - 42\alpha + 5),$$

$$R_2(\alpha) = 2\alpha^8 - 12\alpha^7 - 76\alpha^6 + 180\alpha^5 - 56\alpha^4 + 252\alpha^3 - 628\alpha^2 + 348\alpha - 10$$

$$= 2(\alpha - 1)^2(\alpha^6 - \alpha^5 - 39\alpha^4 - 38\alpha^3 + 36\alpha^2 - 27\alpha + 1).$$

Appendix B (Continued)

We first show that $d(\alpha)$ is an increasing function. To this purpose, we calculate the derivative of $d(\alpha)$;

$$d'(\alpha) = 4\alpha^3 - 12\alpha^2 - 4\alpha - 4.$$

Note that we have

$$\begin{aligned} d'(\alpha) &\geq \lim_{\alpha \rightarrow 9^+} 4\alpha^3 - \lim_{\alpha \rightarrow 5+3\sqrt{2}^-} (12\alpha^2 + 4\alpha + 4) \\ &> 0 \end{aligned}$$

since both of the functions $4\alpha^3$ and $12\alpha^2 + 4\alpha + 4$ are increasing over the interval I_α . Therefore, $d(\alpha)$ is an increasing function and $d(\alpha) \geq d(9) > 0$ for every $\alpha \in I_\alpha$. Let us show that $R_1(\alpha)$ is a positive function over the interval I_α . It is enough to show that $\alpha^4 - 2\alpha^3 - 2\alpha^2 - 42\alpha + 5 > 0$ for every $\alpha \in I_\alpha$.

Let us consider the following calculations

$$\begin{aligned} \alpha^4 - 2\alpha^3 - 2\alpha^2 - 42\alpha + 5 &\geq \left(\min_{\alpha \in I_\alpha} (\alpha^4 + 5) - \max_{\alpha \in I_\alpha} (2\alpha^3 + 2\alpha^2 + 42\alpha) \right) \\ &> 0. \end{aligned}$$

Therefore, we know that $R_1(\alpha)$ is a positive function over the interval I_α . Then we obtain that

$$\begin{aligned} R_1\sqrt{d} + R_2 &\geq \left(\min_{\alpha \in I_\alpha} \sqrt{d} \right) R_1 + R_2 \\ &= \left(\sqrt{d(9)} \right) R_1 + R_2 \\ &= \sqrt{3456} R_1 + R_2. \end{aligned}$$

Appendix B (Continued)

To simplify the calculations, let us say $K = \sqrt{3456} > 0$. We will show that $KR_1 + R_2 > 0$ for every $\alpha \in I_\alpha$.

Note that we have $KR_1 + R_2 = 2(\alpha - 1)^2(u_1 - u_2)$ where

$$u_1(\alpha) = \alpha^6 + K\alpha^4 + 19\alpha^2 + 164\alpha + 5K$$

and

$$u_2(\alpha) = 4\alpha^5 + 47\alpha^4 + 2K\alpha^3 + 2K\alpha^2 + 5.$$

We show that $u_1(\alpha) - u_2(\alpha) > 0$ for every $\alpha \in I_\alpha$. We follow the calculations below:

$$\begin{aligned} u_1(\alpha) - u_2(\alpha) &\geq \min_{\alpha \in I_\alpha} u_1(\alpha) - \max_{\alpha \in I_\alpha} u_2(\alpha) \\ &= u_1(9) - u_2(5 + 3\sqrt{2}) \\ &> 0 \end{aligned}$$

for every $\alpha \in I_\alpha$ since both of the functions $u_1(\alpha)$ and $u_2(\alpha)$ are clearly increasing functions. Therefore, we find that $KR_1 + R_2 > 0$ for every $\alpha \in I_\alpha$. We already have that $\sqrt{d}(\alpha^2 + 4\alpha - 1 + \sqrt{d})^2(\alpha + 1)^2(\alpha - 1)^2 > 0$ for every α . Thus, we get that $(G^+)' < 0$ for every $\alpha \in I_\alpha$. In other words, $G^+(\alpha)$ is a decreasing function over the interval I_α . The following calculation completes the first part of the lemma:

$$\begin{aligned} G^+(\alpha) &\geq G^-(5 + 3\sqrt{2}) \\ &= \frac{(\sqrt{5 + 3\sqrt{2}} + 1)(5 + 3\sqrt{2} - 1)(g(5 + 3\sqrt{2}))^2 + 2(5 + 3\sqrt{2} - 1)g(5 + 3\sqrt{2}) + \sqrt{5 + 3\sqrt{2}} + 3}{(\sqrt{5 + 3\sqrt{2}} + 1)((5 + 3\sqrt{2} - 1)g(5 + 3\sqrt{2}) + 1)} \\ &> \frac{1}{2}. \end{aligned}$$

Appendix B (Continued)

For the second part of the lemma, let us calculate the derivative of $G^-(\alpha)$:

$$(G^-)'(\alpha) = \frac{R_1\sqrt{d} + R_2}{\sqrt{d}(\alpha^2 + 4\alpha - 1 + \sqrt{d})^2(\alpha + 1)^2(\alpha - 1)^2}$$

where

$$d(\alpha) = (\alpha - 1)(\alpha^3 - 3\alpha^2 - 5\alpha - 9),$$

$$R_1(\alpha) = -2(\alpha - 1)^2(\alpha^4 - 2\alpha^3 - 2\alpha^2 - 42\alpha + 5),$$

$$\begin{aligned} R_2(\alpha) &= 2\alpha^8 - 12\alpha^7 - 76\alpha^6 + 180\alpha^5 - 56\alpha^4 - 628\alpha^2 + 252\alpha^3 + 348\alpha - 10 \\ &= 2(\alpha - 1)^2(\alpha^6 - \alpha^5 - 39\alpha^4 - 38\alpha^3 + 36\alpha^2 - 27\alpha + 1). \end{aligned}$$

We already showed that $d(\alpha)$ is an increasing function and $R_1(\alpha)$ is a positive function over the interval I_α . Then we obtain that

$$\begin{aligned} R_1\sqrt{d} + R_2 &\leq \left(\max_{\alpha \in I_\alpha} \sqrt{d} \right) R_1 + R_2 \\ &= \left(\sqrt{d(5 + 3\sqrt{2})} \right) R_1 + R_2 \\ &= R_1\sqrt{1972 + 1392\sqrt{2}} + R_2. \end{aligned}$$

To simplify the calculations, let us say $K = \sqrt{1972 + 1392\sqrt{2}} > 0$. We will show that $KR_1 + R_2 < 0$ for every $\alpha \in I_\alpha$. Note that we have $KR_1 + R_2 = -2(\alpha - 1)^2(t_1 - t_2)$ where

$$t_1(\alpha) = 4\alpha^5 + (47 + K)\alpha^4 + 5K + 5$$

Appendix B (Continued)

and

$$t_2(\alpha) = \alpha^6 + 2K\alpha^3 + (19 + 2K)\alpha^2 + (42K + 164)\alpha.$$

We show that $t_1(\alpha) - t_2(\alpha) > 0$ for every $\alpha \in I_\alpha$. We follow the calculations below:

$$\begin{aligned} t_1(\alpha) - t_2(\alpha) &\geq \min_{\alpha \in I_\alpha} t_1(\alpha) - \max_{\alpha \in I_\alpha} t_2(\alpha) \\ &= t_1(9) - t_2(5 + 3\sqrt{2}) \\ &> 0 \end{aligned}$$

for every $\alpha \in I_\alpha$ since both of the functions $t_1(\alpha)$ and $t_2(\alpha)$ are clearly increasing functions. Therefore, we find that $KR_1 + R_2 < 0$ for every $\alpha \in I_\alpha$. We already have that $\sqrt{d}(\alpha^2 + 4\alpha - 1 + \sqrt{d})^2(\alpha + 1)^2(\alpha - 1)^2 > 0$ for every $\alpha \in I_\alpha$. Therefore, we get that $(G^-)' < 0$ for every $\alpha \in I_\alpha$. In other words, $G^-(\alpha)$ is a decreasing function over the interval I_α . The following calculation completes the first part of the lemma:

$$\begin{aligned} G^-(\alpha) &\geq G^-(5 + 3\sqrt{2}) \\ &= \frac{(\sqrt{5 + 3\sqrt{2}} + 1)(5 + 3\sqrt{2} - 1)(h(5 + 3\sqrt{2}))^2 + 2(5 + 3\sqrt{2} - 1)h(5 + 3\sqrt{2}) + \sqrt{5 + 3\sqrt{2}} + 3}{(\sqrt{5 + 3\sqrt{2}} + 1)((5 + 3\sqrt{2} - 1)h(5 + 3\sqrt{2}) + 1)} \\ &> \frac{1}{2}. \end{aligned}$$

□

Appendix C

PROOF OF LEMMA 4.12

Proof. Let us consider the equalities (58), (61) and (62). We see that

$$\begin{aligned} (1 - x_1^* - x_2^* - x_3^*)(1 - x_8^*) &= \alpha_*(x_1^* + x_2^* + x_3^*)x_8^* \\ (x_1^* + x_2^* + x_3^*)(1 - x_2^*) &= \alpha_*(1 - x_1^* - x_2^* - x_3^*)x_2^* \\ (x_4^* + x_5^* + x_6^*)(1 - x_3^*) &= \alpha_*(1 - x_4^* - x_5^* - x_6^*)x_3^*. \end{aligned}$$

Since we have $\sum_{i=1}^8 x_i^* = 1$ and $x_1^* = x_7^*$, we get that

$$\begin{aligned} (1 - x_1^* - x_2^* - x_3^*)(1 - x_8^*) &= \alpha_*(x_1^* + x_2^* + x_3^*)x_8^* \\ (x_1^* + x_2^* + x_3^*)(1 - x_2^*) &= \alpha_*(1 - x_1^* - x_2^* - x_3^*)x_2^* \\ (1 - 2x_1^* - x_2^* - x_3^* - x_8^*)(1 - x_3^*) &= \alpha_*(2x_1^* + x_2^* + x_3^* + x_8^*)x_3^*. \end{aligned}$$

We expand each sides of the equalities above and obtain that

$$1 + x_1^*x_8^* + x_2^*x_8^* + x_3^*x_8^* - x_1^* - x_2^* - x_3^* - x_8^* = \alpha_*(x_1^*x_8^* + x_2^*x_8^* + x_3^*x_8^*) \quad (\text{C.1})$$

$$x_1^* + x_2^* + x_3^* - x_1^*x_2^* - (x_2^*)^2 - x_2^*x_3^* = \alpha_*(x_2^* - x_1^*x_2^* - (x_2^*)^2 - x_3^*x_2^*) \quad (\text{C.2})$$

$$1 + 2x_1^*x_3^* + x_2^*x_3^* + x_8^*x_3^* + (x_3^*)^2 - 2x_1^* - 2x_3^* - x_2^* - x_8^* = \alpha_*(2x_1^*x_3^* + x_2^*x_3^* + (x_3^*)^2 + x_8^*x_3^*) \quad (\text{C.3})$$

Appendix C (Continued)

We multiply the expression in (Equation C.1) by 1, in (Equation C.2) by -1 and in (Equation C.3) by -1 and add them side-by-side. We see that

$$x_1^* x_8^* - x_2^* + x_2^* x_8^* + x_1^* x_2^* + (x_2^*)^2 - 2x_1^* x_3^* - (x_3^*)^2 = \alpha_* (x_1^* x_8^* - x_2^* + x_2^* x_8^* + x_1^* x_2^* + (x_2^*)^2 - 2x_1^* x_3^* - (x_3^*)^2).$$

Since we have $\alpha_* \in I_\alpha$, we conclude that

$$x_1^* x_8^* - x_2^* + x_2^* x_8^* + x_1^* x_2^* + (x_2^*)^2 - 2x_1^* x_3^* - (x_3^*)^2 = 0. \quad (\text{C.4})$$

We will use the equality above in this proof. Let $I_4 = \{1, 2, 3, 8\}$ and $I_5 = \{4, 5, 6, 7\}$. Remember that $x_1^* = x_7^*$. Let us consider the following quadratic polynomial

$$\begin{aligned} q_4(x) + q_5(x) &= 2(\alpha_* - 1)x^2 + (1 - \alpha_*)(1 - \sum_{k \in I_5} x_k^*)x + \sum_{k \in I_5} x_k^* + (1 - \alpha_*)(1 - \sum_{k \in I_4} x_k^*)x + \sum_{k \in I_4} x_k^* \\ &= 2(\alpha_* - 1)x^2 + (1 - \alpha_*)(2 - \sum_{k=1}^8 x_k^*)x + \sum_{k=1}^8 x_k^* \\ &= 2(\alpha_* - 1)x^2 + (1 - \alpha_*)x + 1. \end{aligned}$$

If we solve the equation $q_4(x) + q_5(x) = 0$, we see that

$$\begin{aligned} x_{+,-} &= \frac{-1 + \alpha_* \pm \sqrt{(\alpha_*)^2 - 10\alpha_* + 9}}{4\alpha_* - 4} \\ &= \frac{1}{4} \pm \frac{1}{4} \sqrt{1 - \frac{8}{\alpha_* - 1}}. \end{aligned}$$

Appendix C (Continued)

Let us define the following two functions over the interval I_α ;

$$x_+(\alpha) = \frac{1}{4} + \frac{1}{4} \sqrt{1 - \frac{8}{\alpha - 1}}$$

and

$$x_-(\alpha) = \frac{1}{4} - \frac{1}{4} \sqrt{1 - \frac{8}{\alpha - 1}}.$$

Note that we have $x_+(\alpha_*) = x_+$ and $x_-(\alpha_*) = x_-$. We know that

$$q_4(x_1^* + x_2^* + x_3^*) + q_5(x_1^* + x_2^* + x_3^*) = q_5(x_1^* + x_2^* + x_3^*) < 0,$$

$$q_4(x_4^* + x_5^* + x_6^*) + q_5(x_4^* + x_5^* + x_6^*) = q_4(x_4^* + x_5^* + x_6^*) > 0,$$

$$q_4(x_1^*) + q_5(x_1^*) = q_4(x_1^*) > 0,$$

$$q_4(x_8^*) + q_5(x_8^*) = q_5(x_8^*) < 0$$

since we have $x_1^* < x_8^*$ and $x_8^* < x_1^* + x_2^* + x_3^*$ and $x_1^* + x_2^* + x_3^* < x_4^* + x_5^* + x_6^*$. Therefore we conclude

that $x_- < x_8^* < x_1^* + x_2^* + x_3^* < x_+$ for any given $\alpha \in I_\alpha$. We divide the interval I_α into two intervals U

and V so that $U = [9, \alpha_0]$ and $V = [\alpha_0, 5 + 3\sqrt{2}]$ where $\alpha_0 = 9.118994613$. First, we concentrate on the

interval U .

Since $x_-(\alpha)$ is a decreasing function over the interval U , the smallest value of $x_-(\alpha)$ is obtained at $\alpha = \alpha_0$. We have

$$\begin{aligned} x_- &\geq \frac{1}{4} - \frac{1}{4} \sqrt{1 - \frac{8}{\alpha_0 - 1}} \\ &= a_1. \end{aligned}$$

Appendix C (Continued)

Therefore, we get that $a_1 < x_8^*$ for every $\alpha \in U$. It is clear that $x_8^* < \frac{1}{4}$. Because if we have $\frac{1}{4} < x_8^* < x_1^* + x_2^* + x_3^*$, then we get

$$\begin{aligned} \alpha_* &= \left(\frac{1}{x_8^*} - 1 \right) \left(\frac{1}{x_1^* + x_2^* + x_3^*} - 1 \right) \\ &< 9, \end{aligned}$$

a contradiction. If we have $x_8^* = \frac{1}{4}$, then we get $x_1^* + x_2^* + x_3^* > \frac{1}{4}$. Thus, we see that

$$\begin{aligned} \alpha_* &= \left(\frac{1}{x_8^*} - 1 \right) \left(\frac{1}{x_1^* + x_2^* + x_3^*} - 1 \right) \\ &\leq 9. \end{aligned}$$

We know that $\alpha_* \geq 9$. Therefore, we find that $\alpha_* = 9$. This implies that $x_1^* + x_2^* + x_3^* = \frac{1}{4}$. This contradicts with hypothesis of Lemma 4.1.11. Hence, we see that $a_1 < x_8^* < \frac{1}{4}$. Let us call $a_2 = \frac{1}{4}$.

It is possible to find a lower and an upper bound for $x_1^* + x_2^* + x_3^*$ using (Equation 4.16). Let us call $t = x_1^* + x_2^* + x_3^*$. We see that

$$\begin{aligned} a_1 &< x_8^* < a_2 \\ \frac{1}{a_1} &> \frac{1}{x_8^*} > \frac{1}{a_2} \\ \frac{1}{a_1} - 1 &> \frac{1}{x_8^*} - 1 > \frac{1}{a_2} - 1 \\ \left(\frac{1}{t} - 1 \right) \left(\frac{1}{a_1} - 1 \right) &> \left(\frac{1}{t} - 1 \right) \cdot \left(\frac{1}{x_8^*} - 1 \right) > \left(\frac{1}{t} - 1 \right) \cdot \left(\frac{1}{a_2} - 1 \right) \\ \left(\frac{1}{t} - 1 \right) \cdot \left(\frac{1}{a_1} - 1 \right) &> \alpha_* > \left(\frac{1}{t} - 1 \right) \cdot \left(\frac{1}{a_2} - 1 \right) \end{aligned}$$

Appendix C (Continued)

We know that $\alpha_* \geq 9$. Then we obtain that

$$\begin{aligned} \left(\frac{1}{t} - 1\right) \cdot \left(\frac{1}{a_1} - 1\right) &> \alpha_* \\ \left(\frac{1}{t} - 1\right) \cdot \left(\frac{1}{a_1} - 1\right) &> 9 \\ \frac{1}{t} - 1 &> \frac{9a_1}{1 - a_1} \\ \frac{1}{t} &> \frac{8a_1 + 1}{1 - a_1} \\ t &< \frac{1 - a_1}{8a_1 + 1} \\ x_1^* + x_2^* + x_3^* &< \frac{1 - a_1}{8a_1 + 1}. \end{aligned}$$

Let us call $a_3 = \frac{1 - a_1}{8a_1 + 1}$.

Since we know that $\alpha_* \leq \alpha_0$, we find that

$$\begin{aligned} \left(\frac{1}{t} - 1\right) \cdot \left(\frac{1}{a_2} - 1\right) &< \alpha_* \\ \left(\frac{1}{t} - 1\right) \cdot \left(\frac{1}{a_2} - 1\right) &< \alpha_0 \\ \frac{1}{t} - 1 &< \frac{\alpha_0 a_2}{1 - a_2} \\ \frac{1}{t} &< \frac{\alpha_0 a_2 - a_2 + 1}{1 - a_2} \\ t &> \frac{1 - a_2}{\alpha_0 a_2 - a_2 + 1} \\ x_1^* + x_2^* + x_3^* &> \frac{1 - a_2}{\alpha_0 a_2 - a_2 + 1}. \end{aligned}$$

Let us call $a_4 = \frac{1 - a_2}{\alpha_0 a_2 - a_2 + 1}$. We have $a_4 < x_1^* + x_2^* + x_3^* < a_3$.

Appendix C (Continued)

Using the equality in (Equation 4.19), it is possible to find a lower and an upper bound for x_2^* .

$$\begin{array}{ccccc}
 a_4 & < & t & < & a_3 \\
 \frac{1}{a_4} & > & \frac{1}{t} & > & \frac{1}{a_3} \\
 \frac{1}{a_4} - 1 & > & \frac{1}{t} - 1 & > & \frac{1}{a_3} - 1 \\
 \frac{a_4}{1 - a_4} & < & \frac{t}{1 - t} & < & \frac{a_3}{1 - a_3} \\
 \left(\frac{1}{x_2^*} - 1 \right) \cdot \left(\frac{a_4}{1 - a_4} \right) & < & \left(\frac{t}{1 - t} \right) \cdot \left(\frac{1}{x_2^*} - 1 \right) & < & \left(\frac{1}{x_2^*} - 1 \right) \cdot \left(\frac{a_3}{1 - a_3} \right) \\
 \left(\frac{1}{x_2^*} - 1 \right) \cdot \left(\frac{a_4}{1 - a_4} \right) & < & \alpha_* & < & \left(\frac{1}{x_2^*} - 1 \right) \cdot \left(\frac{a_3}{1 - a_3} \right)
 \end{array}$$

Since we know that $\alpha_* \leq \alpha_0$, we find that

$$\begin{array}{ccc}
 \left(\frac{1}{x_2^*} - 1 \right) \cdot \left(\frac{a_4}{1 - a_4} \right) & < & \alpha_* \\
 \left(\frac{1}{x_2^*} - 1 \right) \cdot \left(\frac{a_4}{1 - a_4} \right) & < & \alpha_0 \\
 \frac{1}{x_2^*} - 1 & < & \frac{\alpha_0(1 - a_4)}{a_4} \\
 \frac{1}{x_2^*} & < & \frac{\alpha_0(1 - a_4) + a_4}{a_4} \\
 x_2^* & > & \frac{a_4}{\alpha_0(1 - a_4) + a_4}.
 \end{array}$$

Let us call $a_5 = \frac{a_4}{\alpha_0(1 - a_4) + a_4}$.

Using the following inequality, it is possible to find an upper bound for x_2^* .

$$\left(\frac{1}{x_2^*} - 1 \right) \cdot \left(\frac{a_3}{1 - a_3} \right) > \alpha_*$$

Appendix C (Continued)

We know that $\alpha_* \geq 9$. Then we obtain that

$$\begin{aligned} \left(\frac{1}{x_2^*} - 1\right) \cdot \left(\frac{a_3}{1-a_3}\right) &> \alpha_* \\ \left(\frac{1}{x_2^*} - 1\right) \cdot \left(\frac{a_3}{1-a_3}\right) &> 9 \\ \frac{1}{x_2^*} - 1 &> \frac{9(1-a_3)}{a_3} \\ \frac{1}{x_2^*} &> \frac{9-8a_3}{a_3} \\ x_2^* &< \frac{a_3}{9-8a_3}. \end{aligned}$$

Let us call $a_6 = \frac{1-a_4}{8a_4+1}$. We have $a_5 < x_2^* < a_6$.

We follow the operations below to find a lower and an upper bound for x_1^* . We add the inequalities term by term

$$a_4 < x_1^* + x_2^* + x_3^* < a_3$$

and

$$-a_6 < -x_2^* < -a_5$$

and simplify as follows

$$\begin{aligned} a_4 - a_6 &< x_1^* + x_3^* < a_3 - a_5 \\ (a_4 - a_6)^2 &< (x_1^* + x_3^*)^2 < (a_3 - a_5)^2 \\ (a_4 - a_6)^2 &< (x_1^*)^2 + 2x_1^* + (x_3^*)^2 < (a_3 - a_5)^2. \end{aligned}$$

Appendix C (Continued)

The equality in (Equation C.4) implies that $x_1^* x_8^* - x_2^* + x_2^* x_8^* + x_1^* x_2^* + (x_2^*)^2 = 2x_1^* x_3^* + (x_3^*)^2$. Then we obtain that

$$\begin{aligned}
 (a_4 - a_6)^2 &< (x_1^*)^2 + x_1^* x_8^* + x_2^* x_8^* + x_1^* x_2^* + (x_2^*)^2 - x_2^* \\
 (a_4 - a_6)^2 + x_2^* - x_2^* x_8^* - (x_2^*)^2 &< (x_1^*)^2 + x_1^* x_8^* + x_1^* x_2^* \\
 (a_4 - a_6)^2 + x_2^* - x_2^* x_8^* - (x_2^*)^2 + \left(\frac{x_2^* + x_8^*}{2}\right)^2 &< (x_1^*)^2 + x_1^* x_8^* + x_1^* x_2^* + \left(\frac{x_2^* + x_8^*}{2}\right)^2 \\
 \sqrt{(a_4 - a_6)^2 + x_2^* - \frac{3}{4}(x_2^*)^2 + \frac{1}{4}(x_8^*)^2} &< x_1^* + \frac{x_2^* + x_8^*}{2} \\
 \sqrt{(a_4 - a_6)^2 + x_2^* - \frac{3}{4}(x_2^*)^2 + \frac{1}{4}(x_8^*)^2} - \frac{x_2^* + x_8^*}{2} &< x_1^*.
 \end{aligned}$$

Since $x - \frac{3}{4}x^2$ is an increasing function over the interval (a_5, a_6) , we have

$$\sqrt{(a_4 - a_6)^2 + a_5 - \frac{3}{4}a_5^2 + \frac{1}{4}a_1^2} - \frac{a_6 + a_2}{2} < \sqrt{(a_4 - a_6)^2 + x_2^* - \frac{3}{4}(x_2^*)^2 + \frac{1}{4}(x_8^*)^2} - \frac{x_2^* + x_8^*}{2}.$$

Let us call $a_7 = \sqrt{(a_4 - a_6)^2 + a_5 - \frac{3}{4}a_5^2 + \frac{1}{4}a_1^2} - \frac{a_6 + a_2}{2}$. Similarly, we find that

$$\begin{aligned}
 (a_3 - a_5)^2 &> (x_1^*)^2 + x_1^* x_8^* + x_2^* x_8^* + x_1^* x_2^* + (x_2^*)^2 - x_2^* \\
 (a_3 - a_5)^2 + x_2^* - x_2^* x_8^* - (x_2^*)^2 &> (x_1^*)^2 + x_1^* x_8^* + x_1^* x_2^* \\
 (a_3 - a_5)^2 + x_2^* - x_2^* x_8^* - (x_2^*)^2 + \left(\frac{x_2^* + x_8^*}{2}\right)^2 &> (x_1^*)^2 + x_1^* x_8^* + x_1^* x_2^* + \left(\frac{x_2^* + x_8^*}{2}\right)^2 \\
 \sqrt{(a_3 - a_5)^2 + x_2^* - \frac{3}{4}(x_2^*)^2 + \frac{1}{4}(x_8^*)^2} &> x_1^* + \frac{x_2^* + x_8^*}{2} \\
 \sqrt{(a_3 - a_5)^2 + x_2^* - \frac{3}{4}(x_2^*)^2 + \frac{1}{4}(x_8^*)^2} - \frac{x_2^* + x_8^*}{2} &> x_1^*.
 \end{aligned}$$

Appendix C (Continued)

Since $x - \frac{3}{4}x^2$ is an increasing function over the interval (a_5, a_6) , we have

$$\sqrt{(a_3 - a_5)^2 + a_6 - \frac{3}{4}a_6^2 + \frac{1}{4}a_2^2 - \frac{a_1 + a_5}{2}} > \sqrt{(a_3 - a_5)^2 + x_2^* - \frac{3}{4}(x_2^*)^2 + \frac{1}{4}(x_8^*)^2 - \frac{x_2^* + x_8^*}{2}}.$$

Let us call $a_8 = \sqrt{(a_3 - a_5)^2 + a_6 - \frac{3}{4}a_6^2 + \frac{1}{4}a_2^2 - \frac{a_1 + a_5}{2}}$. This means that we get $a_7 < x_1^* < a_8$.

It is possible to find an upper and a lower bound for $x_4^* + x_5^* + x_6^*$ using (Equation 4.15). Let us call

$u = x_4^* + x_5^* + x_6^*$. We see that

$$\begin{aligned} a_7 &< x_1^* < a_8 \\ \frac{1}{a_7} &> \frac{1}{x_1^*} > \frac{1}{a_8} \\ \frac{1}{a_7} - 1 &> \frac{1}{x_1^*} - 1 > \frac{1}{a_8} - 1 \\ \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{a_7} - 1\right) &> \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{x_1^*} - 1\right) > \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{a_8} - 1\right) \\ \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{a_7} - 1\right) &> \alpha_* > \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{a_8} - 1\right) \end{aligned}$$

We know that $\alpha_* \geq 9$. Then we obtain that

$$\begin{aligned} \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{a_7} - 1\right) &> \alpha_* \\ \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{a_7} - 1\right) &> 9 \\ \frac{1}{u} - 1 &> \frac{9a_7}{1 - a_7} \\ u &< \frac{1 - a_7}{8a_7 + 1} \\ x_4^* + x_5^* + x_6^* &< \frac{1 - a_7}{8a_7 + 1}. \end{aligned}$$

Appendix C (Continued)

Let us call $a_9 = \frac{1-a_7}{8a_7+1}$. Since we know that $\alpha_* \leq \alpha_0$, we find that

$$\begin{aligned}
 \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{a_8} - 1\right) &< \alpha_* \\
 \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{a_8} - 1\right) &< \alpha_0 \\
 \frac{1}{u} - 1 &< \frac{\alpha_0 a_8}{1 - a_8} \\
 \frac{1}{u} &< \frac{\alpha_0 a_8}{1 - a_8} + 1 \\
 \frac{1}{u} &< \frac{\alpha_0 a_8 - a_8 + 1}{1 - a_8} \\
 u &> \frac{1 - a_8}{\alpha_0 a_8 - a_8 + 1} \\
 x_4^* + x_5^* + x_6^* &> \frac{1 - a_8}{\alpha_0 a_8 - a_8 + 1}.
 \end{aligned}$$

Let us call $a_{10} = \frac{1-a_8}{\alpha_0 a_8 - a_8 + 1}$. We have $a_{10} < x_4^* + x_5^* + x_6^* < a_9$.

Using the equality in (Equation 4.20), it is possible to find a lower bound and an upper bound for x_3^* . We see that

$$\begin{aligned}
 a_{10} &< u < a_9 \\
 \frac{1}{a_{10}} &> \frac{1}{u} > \frac{1}{a_9} \\
 \frac{1}{a_{10}} - 1 &> \frac{1}{u} - 1 > \frac{1}{a_9} - 1 \\
 \frac{a_{10}}{1 - a_{10}} &< \frac{u}{1 - u} < \frac{a_9}{1 - a_9} \\
 \left(\frac{1}{x_3^*} - 1\right) \cdot \left(\frac{a_{10}}{1 - a_{10}}\right) &< \left(\frac{u}{1 - u}\right) \cdot \left(\frac{1}{x_3^*} - 1\right) < \left(\frac{1}{x_3^*} - 1\right) \cdot \left(\frac{a_9}{1 - a_9}\right) \\
 \left(\frac{1}{x_3^*} - 1\right) \cdot \left(\frac{a_{10}}{1 - a_{10}}\right) &< \alpha_* < \left(\frac{1}{x_3^*} - 1\right) \cdot \left(\frac{a_9}{1 - a_9}\right)
 \end{aligned}$$

Appendix C (Continued)

We know that $\alpha_* \geq 9$. Then we obtain that

$$\begin{aligned}
 \left(\frac{1}{x_3^*} - 1 \right) \cdot \left(\frac{a_9}{1 - a_9} \right) &> \alpha_* \\
 \left(\frac{1}{x_3^*} - 1 \right) \cdot \left(\frac{a_9}{1 - a_9} \right) &> 9 \\
 \frac{1}{x_3^*} - 1 &> \frac{9(1 - a_9)}{a_9} \\
 \frac{1}{x_3^*} &> \frac{9(1 - a_9)}{a_9} + 1 \\
 \frac{1}{x_3^*} &> \frac{9 - 8a_9}{a_9} \\
 x_3^* &< \frac{a_9}{9 - 8a_9}.
 \end{aligned}$$

Let us call $a_{11} = \frac{a_9}{9 - 8a_9}$.

Since we know that $\alpha_* \leq \alpha_0$, we find that

$$\begin{aligned}
 \left(\frac{1}{x_3^*} - 1 \right) \cdot \left(\frac{a_{10}}{1 - a_{10}} \right) &< \alpha_* \\
 \left(\frac{1}{x_3^*} - 1 \right) \cdot \left(\frac{a_{10}}{1 - a_{10}} \right) &< \alpha_0 \\
 \frac{1}{x_3^*} - 1 &< \frac{\alpha_0(1 - a_{10})}{a_{10}} \\
 \frac{1}{x_3^*} &< \frac{\alpha_0(1 - a_{10})}{a_{10}} + 1 \\
 \frac{1}{x_3^*} &< \frac{\alpha_0(1 - a_{10}) + a_{10}}{a_{10}} \\
 x_3^* &> \frac{a_{10}}{\alpha_0(1 - a_{10}) + a_{10}}.
 \end{aligned}$$

Appendix C (Continued)

Let us call $a_{12} = \frac{a_{10}}{\alpha_0(1 - a_{10}) + a_{10}}$. We have $a_{12} < x_3^* < a_{11}$. Therefore, we get the followings

$$a_1 < x_8^* < a_2,$$

$$a_4 < x_1^* + x_2^* + x_3^* < a_3,$$

$$a_5 < x_2^* < a_6,$$

$$a_7 < x_1^* < a_8,$$

$$a_{10} < x_4^* + x_5^* + x_6^* < a_9,$$

$$a_{12} < x_3^* < a_{11}.$$

where

$$a_1 = \frac{1}{4} - \frac{1}{4} \sqrt{1 - \frac{8}{\alpha_0 - 1}} \approx 0.2197341744,$$

$$a_2 = \frac{1}{4} = 0.2500000000,$$

$$a_3 = \frac{1 - a_1}{8a_1 + 1} \approx 0.2829230040,$$

$$a_4 = \frac{1 - a_2}{\alpha_0 a_2 - a_2 + 1} \approx 0.2475452871,$$

$$a_5 = \frac{a_4}{\alpha_0(1 - a_4) + a_4} \approx 0.03482053272,$$

$$a_6 = \frac{1 - a_4}{8a_4 + 1} \approx 0.04199779316,$$

$$a_7 = \sqrt{(a_4 - a_6)^2 + a_5 - \frac{3}{4}a_5^2 + \frac{1}{4}a_1^2} - \frac{a_6 + a_2}{2} \approx 0.1510393664,$$

$$a_8 = \sqrt{(a_3 - a_5)^2 + a_6 - \frac{3}{4}a_6^2 + \frac{1}{4}a_2^2} - \frac{a_1 + a_5}{2} \approx 0.2160224693,$$

$$a_9 = \frac{1 - a_7}{8a_7 + 1} \approx 0.3844382075,$$

$$a_{10} = \frac{1 - a_8}{\alpha_0 a_8 - a_8 + 1} \approx 0.2846805352,$$

Appendix C (Continued)

$$\begin{aligned} a_{11} &= \frac{a_9}{9-8a_9} \approx 0.06488962356, \\ a_{12} &= \frac{a_{10}}{\alpha_0(1-a_{10})+a_{10}} \approx 0.04181758681. \end{aligned}$$

We are ready to conclude the lemma for the interval U . We have

$$\begin{aligned} x_2^*(1-x_1^*-2x_2^*-x_3^*) &= x_2^*(1-x_1^*-x_2^*-x_3^*-x_2^*) \\ &> a_5(1-a_6-a_3). \end{aligned}$$

Similarly, we also have

$$x_8^*(x_1^*+x_2^*+x_3^*-x_8^*) < a_2(a_3-a_1),$$

since $x_8^* < a_2$, $a_4 < x_1^*+x_2^*+x_3^* < a_3$ and $-x_8^* < -a_1$. Then we obtain that

$$\begin{aligned} \frac{x_2^*(1-x_1^*-2x_2^*-x_3^*)}{x_8^*(x_1^*+x_2^*+x_3^*-x_8^*)} \cdot \left(\frac{1}{x_1^*+x_2^*+x_3^*} - 1 \right) &> \frac{a_5(1-a_6-a_3)(1-a_3)}{a_2(a_3-a_1)a_3} \\ &\approx 3.771441275 \\ &> 3.771441270. \end{aligned}$$

We know that $x_1^* < \frac{1}{2}$, $x_3^* < \frac{1}{2}$ and $x_4^*+x_5^*+x_6^* < \frac{1}{2}$. The function $x-x^2$ is an increasing function over the interval $(0, \frac{1}{2})$.

Appendix C (Continued)

Therefore, we obtain the following:

$$\begin{aligned} x_1^*(1-x_1^*)-x_3^*(1-x_3^*) &< \max_{a_7 < x_1^* < a_8} x_1^*(1-x_1^*) - \min_{a_{12} < x_3^* < a_{11}} x_3^*(1-x_3^*) \\ &= a_8(1-a_8) - a_{12}(1-a_{12}) \end{aligned}$$

and

$$\begin{aligned} (x_4^* + x_5^* + x_6^*)(1-x_4^*-x_5^*-x_6^*)-x_1^*(1-x_1^*) &= u(1-u) - x_1^*(1-x_1^*) \\ &> \min_{a_{10} < u < a_9} u(1-u) - \max_{a_7 < x_1^* < a_8} x_1^*(1-x_1^*) \\ &= a_{10}(1-a_{10}) - a_8(1-a_8). \end{aligned}$$

We calculate that

$$\begin{aligned} \frac{x_1^*(1-x_1^*)-x_3^*(1-x_3^*)}{(x_4^*+x_5^*+x_6^*)(1-x_4^*-x_5^*-x_6^*)-x_1^*(1-x_1^*)} &< \frac{a_8(1-a_8)-a_{12}(1-a_{12})}{a_{10}(1-a_{10})-a_8(1-a_8)} \\ &\approx 3.771440985 \\ &< 3.771441270. \end{aligned}$$

Therefore, we see that

$$\frac{x_2^*(1-x_1^*-2x_2^*-x_3^*)(1-x_1^*-x_2^*-x_3^*)}{x_8^*(x_1^*+x_2^*+x_3^*-x_8^*)(x_1^*+x_2^*+x_3^*)} \neq \frac{x_1^*(1-x_1^*)-x_3^*(1-x_3^*)}{(x_4^*+x_5^*+x_6^*)(1-x_4^*-x_5^*-x_6^*)-x_1^*(1-x_1^*)}$$

if $\alpha_* \in U$.

Appendix C (Continued)

We follow the same calculations carried above for the interval V . Since $x_-(\alpha)$ is a decreasing function over the interval V , the smallest value of $x_-(\alpha)$ is obtained at $\alpha_1 = 5 + 3\sqrt{2}$. We have

$$x_- \geq \frac{1}{4} - \frac{1}{4} \sqrt{1 - \frac{8}{5 + 3\sqrt{2} - 1}}.$$

Let us call $a_1 = \frac{1}{4} - \frac{1}{4} \sqrt{1 - \frac{8}{5 + 3\sqrt{2} - 1}}$. Therefore, we get that $a_1 < x_8^*$ for every $\alpha \in V$. We showed in the previous case that $x_8^* < \frac{1}{4}$. Let us call $a_2 = \frac{1}{4}$. It is possible to find a lower and an upper bound for $x_1^* + x_2^* + x_3^*$. Let us call $t = x_1^* + x_2^* + x_3^*$. We see that

$$\begin{aligned} a_1 &< x_8^* < a_2 \\ \frac{1}{a_1} &> \frac{1}{x_8^*} > \frac{1}{a_2} \\ \frac{1}{a_1} - 1 &> \frac{1}{x_8^*} - 1 > \frac{1}{a_2} - 1 \\ \left(\frac{1}{t} - 1\right) \left(\frac{1}{a_1} - 1\right) &> \left(\frac{1}{t} - 1\right) \left(\frac{1}{x_8^*} - 1\right) > \left(\frac{1}{t} - 1\right) \left(\frac{1}{a_2} - 1\right) \\ \left(\frac{1}{t} - 1\right) \left(\frac{1}{a_1} - 1\right) &> \alpha_* > \left(\frac{1}{t} - 1\right) \left(\frac{1}{a_2} - 1\right) \end{aligned}$$

We know that $\alpha_* \geq \alpha_0$. Then we obtain that

$$\begin{aligned} \left(\frac{1}{t} - 1\right) \left(\frac{1}{a_1} - 1\right) &> \alpha_* \\ \left(\frac{1}{t} - 1\right) \left(\frac{1}{a_1} - 1\right) &> \alpha_0 \\ \frac{1}{t} - 1 &> \frac{\alpha_0 a_1}{1 - a_1} \\ t &< \frac{1 - a_1}{\alpha_0 a_1 + 1 - a_1} \\ x_1^* + x_2^* + x_3^* &< \frac{1 - a_1}{\alpha_0 a_1 + 1 - a_1}. \end{aligned}$$

Appendix C (Continued)

Let us call $a_3 = \frac{1 - a_1}{\alpha_0 a_1 + 1 - a_1}$.

Since we know that $\alpha_* \leq \alpha_1$, we find that

$$\begin{aligned} \left(\frac{1}{t} - 1\right) \cdot \left(\frac{1}{a_2} - 1\right) &< \alpha_* \\ \left(\frac{1}{t} - 1\right) \cdot \left(\frac{1}{a_2} - 1\right) &< \alpha_1 \\ \frac{1}{t} - 1 &< \frac{\alpha_1 a_2}{1 - a_2} \\ \frac{1}{t} &< \frac{\alpha_1 a_2 - a_2 + 1}{1 - a_2} \\ t &> \frac{1 - a_2}{\alpha_1 a_2 - a_2 + 1} \\ x_1^* + x_2^* + x_3^* &> \frac{1 - a_2}{\alpha_1 a_2 - a_2 + 1}. \end{aligned}$$

Let us call $a_4 = \frac{1 - a_2}{\alpha_1 a_2 - a_2 + 1}$. We have $a_4 < x_1^* + x_2^* + x_3^* < a_3$.

Using the equality in (Equation 4.19), it is possible to find a lower and an upper bound for x_2^* .

$$\begin{aligned} a_4 &< t < a_3 \\ \frac{1}{a_4} &> \frac{1}{t} > \frac{1}{a_3} \\ \frac{1}{a_4} - 1 &> \frac{1}{t} - 1 > \frac{1}{a_3} - 1 \\ \frac{a_4}{1 - a_4} &< \frac{t}{1 - t} < \frac{a_3}{1 - a_3} \\ \left(\frac{1}{x_2^*} - 1\right) \cdot \left(\frac{a_4}{1 - a_4}\right) &< \left(\frac{t}{1 - t}\right) \cdot \left(\frac{1}{x_2^*} - 1\right) < \left(\frac{1}{x_2^*} - 1\right) \cdot \left(\frac{a_3}{1 - a_3}\right) \\ \left(\frac{1}{x_2^*} - 1\right) \cdot \left(\frac{a_4}{1 - a_4}\right) &< \alpha_* < \left(\frac{1}{x_2^*} - 1\right) \cdot \left(\frac{a_3}{1 - a_3}\right) \end{aligned}$$

Appendix C (Continued)

Since we know that $\alpha_* \leq \alpha_1$, we find that

$$\begin{aligned} \left(\frac{1}{x_2^*} - 1\right) \cdot \left(\frac{a_4}{1-a_4}\right) &< \alpha_* \\ \left(\frac{1}{x_2^*} - 1\right) \cdot \left(\frac{a_4}{1-a_4}\right) &< \alpha_1 \\ \frac{1}{x_2^*} - 1 &< \frac{\alpha_1(1-a_4)}{a_4} \\ \frac{1}{x_2^*} &< \frac{\alpha_1(1-a_4)}{a_4} + 1 \\ \frac{1}{x_2^*} &< \frac{\alpha_1(1-a_4) + a_4}{a_4} \\ x_2^* &> \frac{a_4}{\alpha_1(1-a_4) + a_4}. \end{aligned}$$

Let us call $a_5 = \frac{a_4}{\alpha_1(1-a_4) + a_4}$.

We know that $\alpha_* \geq \alpha_0$. Then we obtain that

$$\begin{aligned} \left(\frac{1}{x_2^*} - 1\right) \cdot \left(\frac{a_3}{1-a_3}\right) &> \alpha_* \\ \left(\frac{1}{x_2^*} - 1\right) \cdot \left(\frac{a_3}{1-a_3}\right) &> \alpha_0 \\ \frac{1}{x_2^*} - 1 &> \frac{\alpha_0(1-a_3)}{a_3} \\ \frac{1}{x_2^*} &> \frac{\alpha_0(1-a_3)}{a_3} + 1 \\ \frac{1}{x_2^*} &> \frac{\alpha_0(1-a_3) + a_3}{a_3} \\ x_2^* &< \frac{a_3}{\alpha_0(1-a_3) + a_3}. \end{aligned}$$

Let us call $a_6 = \frac{a_3}{\alpha_0(1-a_3) + a_3}$. We have $a_5 < x_2^* < a_6$.

Appendix C (Continued)

We follow the operations below to find a lower and an upper bound for x_1^* .

$$\begin{aligned}
 a_4 &< x_1^* + x_2^* + x_3^* < a_3 \\
 -a_6 &< -x_2^* < -a_5 \\
 a_4 - a_6 &< x_1^* + x_3^* < a_3 - a_5 \\
 (a_4 - a_6)^2 &< (x_1^* + x_3^*)^2 < (a_3 - a_5)^2 \\
 (a_4 - a_6)^2 &< (x_1^*)^2 + 2x_1^* + (x_3^*)^2 < (a_3 - a_5)^2.
 \end{aligned}$$

The equality (86) implies that $x_1^* x_8^* - x_2^* + x_2^* x_8^* + x_1^* x_2^* + (x_2^*)^2 = 2x_1^* x_3^* + (x_3^*)^2$. Then we obtain that

$$\begin{aligned}
 (a_4 - a_6)^2 &< (x_1^*)^2 + x_1^* x_8^* + x_2^* x_8^* + x_1^* x_2^* + (x_2^*)^2 - x_2^* \\
 (a_4 - a_6)^2 + x_2^* - x_2^* x_8^* - (x_2^*)^2 &< (x_1^*)^2 + x_1^* x_8^* + x_1^* x_2^* \\
 (a_4 - a_6)^2 + x_2^* - x_2^* x_8^* - (x_2^*)^2 + \left(\frac{x_2^* + x_8^*}{2}\right)^2 &< (x_1^*)^2 + x_1^* x_8^* + x_1^* x_2^* + \left(\frac{x_2^* + x_8^*}{2}\right)^2 \\
 \sqrt{(a_4 - a_6)^2 + x_2^* - \frac{3}{4}(x_2^*)^2 + \frac{1}{4}(x_8^*)^2} &< x_1^* + \frac{x_2^* + x_8^*}{2} \\
 \sqrt{(a_4 - a_6)^2 + x_2^* - \frac{3}{4}(x_2^*)^2 + \frac{1}{4}(x_8^*)^2} - \frac{x_2^* + x_8^*}{2} &< x_1^*.
 \end{aligned}$$

Since $x - \frac{3}{4}x^2$ is an increasing function over the interval (a_5, a_6) , we have

$$\sqrt{(a_4 - a_6)^2 + a_5 - \frac{3}{4}a_5^2 + \frac{1}{4}a_1^2} - \frac{a_6 + a_2}{2} < \sqrt{(a_4 - a_6)^2 + x_2^* - \frac{3}{4}(x_2^*)^2 + \frac{1}{4}(x_8^*)^2} - \frac{x_2^* + x_8^*}{2}.$$

Let us call $a_7 = \sqrt{(a_4 - a_6)^2 + a_5 - \frac{3}{4}a_5^2 + \frac{1}{4}a_1^2} - \frac{a_6 + a_2}{2}$.

Appendix C (Continued)

Similarly, we find that

$$\begin{aligned}
 (a_3 - a_5)^2 &> (x_1^*)^2 + x_1^* x_8^* + x_2^* x_8^* + x_1^* x_2^* + (x_2^*)^2 - x_2^* \\
 (a_3 - a_5)^2 + x_2^* - x_2^* x_8^* - (x_2^*)^2 &> (x_1^*)^2 + x_1^* x_8^* + x_1^* x_2^* \\
 (a_3 - a_5)^2 + x_2^* - x_2^* x_8^* - (x_2^*)^2 + \left(\frac{x_2^* + x_8^*}{2}\right)^2 &> (x_1^*)^2 + x_1^* x_8^* + x_1^* x_2^* + \left(\frac{x_2^* + x_8^*}{2}\right)^2 \\
 \sqrt{(a_3 - a_5)^2 + x_2^* - \frac{3}{4}(x_2^*)^2 + \frac{1}{4}(x_8^*)^2} &> x_1^* + \frac{x_2^* + x_8^*}{2} \\
 \sqrt{(a_3 - a_5)^2 + x_2^* - \frac{3}{4}(x_2^*)^2 + \frac{1}{4}(x_8^*)^2} - \frac{x_2^* + x_8^*}{2} &> x_1^*.
 \end{aligned}$$

Since $x - \frac{3}{4}x^2$ is an increasing function over the interval (a_5, a_6) , we have

$$\sqrt{(a_3 - a_5)^2 + a_6 - \frac{3}{4}a_6^2 + \frac{1}{4}a_2^2} - \frac{a_1 + a_5}{2} > \sqrt{(a_3 - a_5)^2 + x_2^* - \frac{3}{4}(x_2^*)^2 + \frac{1}{4}(x_8^*)^2} - \frac{x_2^* + x_8^*}{2}.$$

Let us call $a_8 = \sqrt{(a_3 - a_5)^2 + a_6 - \frac{3}{4}a_6^2 + \frac{1}{4}a_2^2} - \frac{a_1 + a_5}{2}$. This means that we get $a_7 < x_1^* < a_8$. It is possible to find an upper and a lower bound for $x_4^* + x_5^* + x_6^*$ using (57). Let us call $u = x_4^* + x_5^* + x_6^*$.

We see that

$$\begin{aligned}
 a_7 &< x_1^* < a_8 \\
 \frac{1}{a_7} &> \frac{1}{x_1^*} > \frac{1}{a_8} \\
 \frac{1}{a_7} - 1 &> \frac{1}{x_1^*} - 1 > \frac{1}{a_8} - 1 \\
 \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{a_7} - 1\right) &> \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{x_1^*} - 1\right) > \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{a_8} - 1\right) \\
 \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{a_7} - 1\right) &> \alpha_* > \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{a_8} - 1\right)
 \end{aligned}$$

Appendix C (Continued)

We know that $\alpha_* \geq \alpha_0$. Then we obtain that

$$\begin{aligned}
 \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{a_7} - 1\right) &> \alpha_* \\
 \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{a_7} - 1\right) &> \alpha_0 \\
 \frac{1}{u} - 1 &> \frac{\alpha_0 a_7}{1 - a_7} \\
 \frac{1}{u} &> \frac{\alpha_0 a_7 + 1 - a_7}{1 - a_7} \\
 u &< \frac{1 - a_7}{\alpha_0 a_7 + 1 - a_7} \\
 x_4^* + x_5^* + x_6^* &< \frac{1 - a_7}{\alpha_0 a_7 + 1 - a_7}.
 \end{aligned}$$

Let us call $a_9 = \frac{1 - a_7}{\alpha_0 a_7 + 1 - a_7}$.

Since we know that $\alpha_* \leq \alpha_1$, we find that

$$\begin{aligned}
 \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{a_8} - 1\right) &< \alpha_* \\
 \left(\frac{1}{u} - 1\right) \cdot \left(\frac{1}{a_8} - 1\right) &< \alpha_1 \\
 \frac{1}{u} - 1 &< \frac{\alpha_1 a_8}{1 - a_8} \\
 \frac{1}{u} &< \frac{\alpha_1 a_8 - a_8 + 1}{1 - a_8} \\
 u &> \frac{1 - a_8}{\alpha_1 a_8 - a_8 + 1} \\
 x_4^* + x_5^* + x_6^* &> \frac{1 - a_8}{\alpha_1 a_8 - a_8 + 1}.
 \end{aligned}$$

Let us call $a_{10} = \frac{1 - a_8}{\alpha_1 a_8 - a_8 + 1}$. We have $a_{10} < x_4^* + x_5^* + x_6^* < a_9$.

Appendix C (Continued)

Using the equality (62), it is possible to find a lower bound and an upper bound for x_3^* . We see that

$$\begin{aligned}
 a_{10} &< u < a_9 \\
 \frac{1}{a_{10}} &> \frac{1}{u} > \frac{1}{a_9} \\
 \frac{1}{a_{10}} - 1 &> \frac{1}{u} - 1 > \frac{1}{a_9} - 1 \\
 \frac{a_{10}}{1 - a_{10}} &< \frac{u}{1 - u} < \frac{a_9}{1 - a_9} \\
 \left(\frac{1}{x_3^*} - 1\right) \cdot \left(\frac{a_{10}}{1 - a_{10}}\right) &< \left(\frac{u}{1 - u}\right) \cdot \left(\frac{1}{x_3^*} - 1\right) < \left(\frac{1}{x_3^*} - 1\right) \cdot \left(\frac{a_9}{1 - a_9}\right) \\
 \left(\frac{1}{x_3^*} - 1\right) \cdot \left(\frac{a_{10}}{1 - a_{10}}\right) &< \alpha_* < \left(\frac{1}{x_3^*} - 1\right) \cdot \left(\frac{a_9}{1 - a_9}\right)
 \end{aligned}$$

We know that $\alpha_* \geq \alpha_0$. Then we obtain that

$$\begin{aligned}
 \left(\frac{1}{x_3^*} - 1\right) \cdot \left(\frac{a_9}{1 - a_9}\right) &> \alpha_* \\
 \left(\frac{1}{x_3^*} - 1\right) \cdot \left(\frac{a_9}{1 - a_9}\right) &> \alpha_0 \\
 \frac{1}{x_3^*} - 1 &> \frac{\alpha_0(1 - a_9)}{a_9} \\
 x_3^* &< \frac{a_9}{\alpha_0(1 - a_9) + a_9}.
 \end{aligned}$$

Let us call $a_{11} = \frac{a_9}{\alpha_0(1 - a_9) + a_9}$. Since we know that $\alpha_* \leq \alpha_1$, we find that

$$\begin{aligned}
 \left(\frac{1}{x_3^*} - 1\right) \cdot \left(\frac{a_{10}}{1 - a_{10}}\right) &< \alpha_* \\
 \left(\frac{1}{x_3^*} - 1\right) \cdot \left(\frac{a_{10}}{1 - a_{10}}\right) &< \alpha_1 \\
 \frac{1}{x_3^*} - 1 &< \frac{\alpha_1(1 - a_{10})}{a_{10}} \\
 x_3^* &> \frac{a_{10}}{\alpha_1(1 - a_{10}) + a_{10}}.
 \end{aligned}$$

Appendix C (Continued)

Let us call $a_{12} = \frac{a_{10}}{\alpha_1(1 - a_{10}) + a_{10}}$. We have $a_{12} < x_3^* < a_{11}$. Hence, we get the followings

$$a_1 < x_8^* < a_2,$$

$$a_4 < x_1^* + x_2^* + x_3^* < a_3,$$

$$a_5 < x_2^* < a_6,$$

$$a_7 < x_1^* < a_8,$$

$$a_{10} < x_4^* + x_5^* + x_6^* < a_9,$$

$$a_{12} < x_3^* < a_{11}.$$

where

$$a_1 = \frac{1}{4} - \frac{1}{4} \sqrt{1 - \frac{8}{\alpha_1 - 1}} \approx 0.2197341744,$$

$$a_2 = \frac{1}{4} = 0.2500000000,$$

$$a_3 = \frac{1 - a_1}{\alpha_0 a_1 + 1 - a_1} \approx 0.2802658255,$$

$$a_4 = \frac{1 - a_2}{\alpha_1 a_2 - a_2 + 1} \approx 0.2450451726,$$

$$a_5 = \frac{a_4}{\alpha_1(1 - a_4) + a_4} \approx 0.03392651705,$$

$$a_6 = \frac{a_3}{\alpha_0(1 - a_3) + a_3} \approx 0.04095347309,$$

$$a_7 = \sqrt{(a_4 - a_6)^2 + a_5 - \frac{3}{4}a_5^2 + \frac{1}{4}a_1^2} - \frac{a_6 + a_2}{2} \approx 0.1491203770,$$

$$a_8 = \sqrt{(a_3 - a_5)^2 + a_6 - \frac{3}{4}a_6^2 + \frac{1}{4}a_2^2} - \frac{a_1 + a_5}{2} \approx 0.2137627221,$$

$$a_9 = \frac{1 - a_7}{\alpha_0 a_7 + 1 - a_7} \approx 0.3817066009,$$

$$a_{10} = \frac{1 - a_8}{\alpha_1 a_8 - a_8 + 1} \approx 0.2846654650,$$

Appendix C (Continued)

$$\begin{aligned} a_{11} &= \frac{a_9}{\alpha_0(1-a_9) + a_9} \approx 0.06257148633, \\ a_{12} &= \frac{a_{10}}{\alpha_1(1-a_{10}) + a_{10}} \approx 0.04127832503. \end{aligned}$$

We are ready to conclude the lemma for the interval V . We have

$$\begin{aligned} x_2^*(1 - x_1^* - 2x_2^* - x_3^*) &= x_2^*(1 - x_1^* - x_2^* - x_3^* - x_2^*) \\ &> a_5(1 - a_6 - a_3). \end{aligned}$$

Similarly, we also have

$$x_8^*(x_1^* + x_2^* + x_3^* - x_8^*) < a_2(a_3 - a_1),$$

since $x_8^* < a_2$, $a_4 < x_1^* + x_2^* + x_3^* < a_3$ and $-x_8^* < -a_1$. Then we obtain that

$$\begin{aligned} \frac{x_2^*(1 - x_1^* - 2x_2^* - x_3^*)}{x_8^*(x_1^* + x_2^* + x_3^* - x_8^*)} \cdot \left(\frac{1}{x_1^* + x_2^* + x_3^*} - 1 \right) &> \frac{a_5(1 - a_6 - a_3)(1 - a_3)}{a_2(a_3 - a_1)a_3} \\ &\approx 3.907942936 \\ &> 3.9. \end{aligned}$$

We know that $x_1^* < \frac{1}{2}$, $x_3^* < \frac{1}{2}$ and $x_4^* + x_5^* + x_6^* < \frac{1}{2}$. The function $x - x^2$ is an increasing function over the interval $(0, \frac{1}{2})$.

Appendix C (Continued)

Therefore, we obtain the following:

$$\begin{aligned} x_1^*(1-x_1^*)-x_3^*(1-x_3^*) &< \max_{a_7 < x_1^* < a_8} x_1^*(1-x_1^*) - \min_{a_{12} < x_3^* < a_{11}} x_3^*(1-x_3^*) \\ &= a_8(1-a_8) - a_{12}(1-a_{12}) \end{aligned}$$

and

$$\begin{aligned} (x_4^* + x_5^* + x_6^*)(1-x_4^*-x_5^*-x_6^*)-x_1^*(1-x_1^*) &= u(1-u) - x_1^*(1-x_1^*) \\ &> \min_{a_{10} < u < a_9} u(1-u) - \max_{a_7 < x_1^* < a_8} x_1^*(1-x_1^*) \\ &= a_{10}(1-a_{10}) - a_8(1-a_8). \end{aligned}$$

We calculate that

$$\begin{aligned} \frac{x_1^*(1-x_1^*)-x_3^*(1-x_3^*)}{(x_4^*+x_5^*+x_6^*)(1-x_4^*-x_5^*-x_6^*)-x_1^*(1-x_1^*)} &< \frac{a_8(1-a_8)-a_{12}(1-a_{12})}{a_{10}(1-a_{10})-a_8(1-a_8)} \\ &\approx 3.613150070 \\ &< 3.9. \end{aligned}$$

Therefore, we see that

$$\frac{x_2^*(1-x_1^*-2x_2^*-x_3^*)(1-x_1^*-x_2^*-x_3^*)}{x_8^*(x_1^*+x_2^*+x_3^*-x_8^*)(x_1^*+x_2^*+x_3^*)} \neq \frac{x_1^*(1-x_1^*)-x_3^*(1-x_3^*)}{(x_4^*+x_5^*+x_6^*)(1-x_4^*-x_5^*-x_6^*)-x_1^*(1-x_1^*)}$$

if $\alpha_* \in V$ which finishes the proof of the lemma. □

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