T.R. USKUDAR UNIVERSITY
GRADUATE SCHOOL OF SCIENCES

DEPARTMENT OF BIOENGINEERING MASTER'S DEGREE PROGRAM OF BIOENGINEERING

MASTER'S DEGREE THESIS

SIMULATION OF FITZHUGH-NAGUMO NEURONAL MODEL USING MATLAB®

Farwa Anum

Thesis Advisor

Asst. Prof. Dr. Imran GOKER

ISTANBUL-2025

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ABSTRACT

This thesis explores the FitzHugh-Nagumo (FHN) model as a simplified representation of neuronal excitability, focusing on its ability to replicate key neuronal behaviors, including threshold-dependent excitability, frequency modulation, and recovery dynamics. The primary objective was to investigate how variations in parameters, particularly the external stimulus I, affect the model's behavior, using time series simulations and bifurcation analysis to assess neuronal firing patterns. The results demonstrated that the FHN model successfully replicates threshold behavior observed in real neurons, with the membrane potential V remaining stable at $I < 0.5$ and transitioning to an oscillatory (spiking) regime at $I \geq 0.5$. Additionally, the model showed frequency modulation, as the firing frequency increased from moderate at $I = 0.6$ to high-frequency oscillations at $I = 1.2$ This property of the model closely mirrors the rate coding mechanism used by neurons to encode stimulus intensity. The recovery variable W, which peaks shortly after each spike in V, effectively simulated the refractory period, preventing immediate re-firing and stabilizing the firing patterns. The bifurcation analysis further revealed the model's nonlinear dynamics, illustrating transitions between resting, periodic oscillatory, and high-frequency firing states in response to increasing I. The study validates the FHN model as a computationally efficient tool for simulating essential neuronal behaviors, making it suitable for theoretical and large-scale neural network simulations. Its simplified structure, consisting of only two variables, enables it to capture excitability and recovery dynamics without the computational demands of more detailed conductance-based models like Hodgkin-Huxley. However, limitations include its reduced applicability to specialized neuron types and the need for precise parameter calibration. Future research directions include incorporating additional biophysical properties, optimizing parameters for specific neuron types, and applying the FHN model to networked simulations to study collective neural phenomena. This thesis contributes to computational neuroscience by providing insights into the applicability of the FHN model in representing neuronal excitability and opens pathways for its potential use in neuromorphic computing.

Keywords: *FitzHugh-Nagumo model, neuronal excitability, frequency modulation, bifurcation analysis, computational neuroscience, neural coding*

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FORM OF DECLARATION

I affirm that all data and materials essential for this study have been procured meticulously, adhering unwaveringly to the stipulated benchmarks of academic excellence. The presentation of visual, aural, and written data, as well as the subsequent articulation of findings, has been diligently executed in accordance with the scrupulous principles of scientific ethics. I emphasize that no manipulation or fabrication of data has transpired, upholding the steadfast commitment to integrity.

Furthermore, I assert that due diligence has been exercised in attributing sources, aligning with the esteemed conventions of scientific protocol. The origination of this thesis, except for the explicitly referenced instances, stands as a testament to its authentic character. I solemnly declare that the composition and production of this work are my sole endeavors, shaped and fashioned in accordance with the guiding principles outlined in the Uskudar University Institute of Science Thesis Writing Guide.

Dated: 26-06-2025

Farwa Anum

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TABLE OF ABBREVIATIONS & SYMBOLS

Abbreviation/Symbol	Full Term	Description
A	Parameter a	Parameter influencing excitability threshold in the FHN model
ANN	Artificial Neural Network	Computational model inspired by biological neural networks
B	Parameter b	Parameter modulating recovery feedback in the FHN model
BCI	Brain-Computer Interface	System enabling direct communication between the brain and external devices
CA1	Cornu Ammonis 1	Specific region of the hippocampus often studied in neuroscience
CPU	Central Processing Unit	Primary processor in computing systems
CSV	Comma-Separated Values	Text file format that uses commas to separate values
DBS	Deep Brain Stimulation	Neurosurgical procedure involving implanted electrodes
Δt	Delta t	Time step size in numerical integration
ε	Epsilon	Recovery speed parameter in the FHN model
FFT	Fast Fourier Transform	Algorithm for converting time to frequency domain
FHN	FitzHugh-Nagumo	Simplified neuronal model studied in this thesis
FPGA	Field-Programmable Gate Array	Integrated circuit designed to be configured after manufacturing
GB	Gigabyte	Unit of digital information storage (10^9 bytes)
GHz	Gigahertz	Unit of frequency (10^9 Hz)

GUI	Graphical User Interface	Visual way of interacting with electronic devices
HH	Hodgkin-Huxley	Detailed conductance-based neuronal model
Hz	Hertz	Unit of frequency (cycles per second)
I	Input Current	External stimulus current in the FHN model
IMEX	Implicit-Explicit	Hybrid numerical integration scheme
JSON	JavaScript Object Notation	Lightweight data interchange format
kHz	Kilohertz	Unit of frequency (10^3 Hz)
LIF	Leaky Integrate-and-Fire	Simplified neuronal model
LOOCV	Leave-One-Out Cross-Validation	Statistical validation technique
MATLAB	Matrix Laboratory	Programming and numerical computing platform
MB	Megabyte	Unit of digital information storage (10^6 bytes)
MEA	Multi-Electrode Array	Device for recording electrical activity from multiple sites
MHz	Megahertz	Unit of frequency (10^6 Hz)
ML	Morris-Lecar	Neuronal model focused on calcium and potassium dynamics
ms	Millisecond	Unit of time (10^-3 seconds)
mV	Millivolt	Unit of electrical potential (10^-3 volts)
nA	Nanoampere	Unit of electrical current (10^-9 amperes)
NVMe	Non-Volatile Memory Express	Interface protocol for solid-state drives

ODE	Ordinary Differential Equation	Mathematical equation involving derivatives with respect to a single variable
PDF	Portable Document Format	File format for document exchange
PNG	Portable Network Graphics	Raster graphics file format
RAM	Random Access Memory	Volatile computer memory
RK4	Runge-Kutta 4th Order	Numerical method for solving ordinary differential equations
SSD	Solid State Drive	Storage device using integrated circuit assemblies
SVG	Scalable Vector Graphics	XML-based vector image format
TMS	Transcranial Magnetic Stimulation	Procedure using magnetic fields to stimulate nerve cells
TXT	Text File	Basic computer file containing unformatted text
V	Voltage/Membrane Potential	Excitability variable in the FHN model
W	Recovery Variable	Slow variable in the FHN model representing recovery processes
XML	Extensible Markup Language	Markup language that defines rules for encoding documents

1. Chapter 1: Introduction

1.1 Background of Neuronal Models

1.1.1 Overview of Neuronal Behavior

Brain function depends on understanding complicated electrical and chemical signaling networks that allow neurons to communicate and interpret information. Excitability, the ability of neurons to generate action potentials, activates neurons and controls neuronal function. Brain processes like perception, learning, memory, and decision-making require excitation and signaling.

Ion dynamics across the neuronal membrane generate and propagate action potentials. The resting membrane potential of neurons is around -70 mV due to the distribution of ions, mainly sodium (Na^+) and potassium (K^+). External stimulation is needed for neurons to depolarize their membrane potential to -55 mV and open voltage-gated Na^+ channels. As sodium ions enter, the membrane rapidly depolarizes to +30 mV. Repolarization to resting potential occurs when potassium channels open and sodium channels inactivate. This 1–2 millisecond action potential depolarizes-repolarizes [1].

When axon hillock action potentials reach synaptic terminals, neurotransmitters enter the cleft. These neurotransmitters binding to nearby neuron receptors may cause a new action potential in the postsynaptic cell. Complex networks of neurons use action potentials and neurotransmitter release to link and perform all brain operations.

1.1.2 Importance of Studying Neuronal Dynamics

To understand normal and pathological brain processes, investigate neural dynamics. Research on neuronal dynamics has uncovered signaling illnesses like epilepsy, Parkinson's, schizophrenia, and depression. Epilepsy causes seizures due to abnormal neuronal excitability and coordinated firing. Action potential formation and propagation dynamics can help create excitability-stabilizing and seizure-prevention treatments.

Brain simulation computational models benefit from neuro dynamics research. Numerical models and simulations are needed for computational neuroscience to understand brain function. Hodgkin-Huxley and FitzHugh-Nagumo computational models may explore how neurons respond to inputs, interact in brain circuits, and produce complex network behaviors [2].

Neuronal dynamics also impact AI and neural engineering advances. Understanding how biological neurons compute and store information helps develop artificial neural networks. Plasticity and learning are possible because biological neurons absorb information in parallel and adapt to inputs. Neuromorphic devices and AI algorithms that imitate brain simultaneous processing have been developed using neural function.

Finally, neuroprosthetics and BCIs that restore function or allow direct brain-device interaction require neuronal excitability and action potential knowledge. Understanding cerebral firing patterns allows BCIs to read brain signals and convert them into robotic limb or communication device movements, dramatically effecting physically challenged persons.

1.1.3 Importance of Mathematical Models in Neuroscience

Mathematical models play a vital role in neuroscience by providing simplified representations of complex biological systems, enabling researchers to understand, predict, and manipulate neuronal behavior in a controlled, quantifiable manner [3]. The brain is composed of billions of interconnected neurons, each with intricate dynamics and interactions. Directly studying such a system at a biological level poses significant challenges due to its sheer complexity, high dimensionality, and variability. Mathematical models, therefore, serve as essential tools, reducing this complexity by abstracting critical features of neuronal behavior into manageable equations and parameters that can be analyzed systematically.

Role of Mathematical Models in Simplifying Complex Biological Systems

Mathematical models simplify neural systems by focusing on excitability, signal propagation, and neuron connections. These models approximate biological processes by abstracting neuron properties like membrane potential, ion channel activity, and synaptic transmission into equations. The 1952 Hodgkin-Huxley model uses differential equations to describe sodium and potassium ion transport across the neural membrane to explain action potentials. Although it simplifies brain function, this model captures action potential generation dynamics well, making it basic in computational neuroscience.

Mathematical models simplify these processes, allowing researchers to focus on critical neuronal function factors as threshold potential, synaptic weight, and firing rate.

Experimental data can be used to update these models, making them useful for testing hypotheses, testing interventions, and simulating reactions under varied settings.

1.1.4 Evolution of Neuronal Models

Historical Perspective on the Development of Neuronal Models

The evolution of neuronal models began with early attempts to understand the basic electrical properties of neurons. In the late 19th century, scientists discovered that neurons generate electrical impulses, paving the way for electrical circuit models that represented neurons as basic electrical components. With advancements in physiology and mathematics, models became more sophisticated, aiming to explain how neurons generate and propagate action potentials.

A major breakthrough came in 1952 with the Hodgkin-Huxley model, developed by Alan Hodgkin and Andrew Huxley, who formulated a detailed mathematical model of the squid giant axon. Using experimental data, they derived a set of differential equations that describe the ion currents flowing through sodium and potassium channels during an action potential. This model, which earned them a Nobel Prize in 1963, is widely regarded as one of the most significant contributions to neuroscience. The Hodgkin-Huxley model set the foundation for later models and remains a gold standard for describing excitable cells [4-6].

Key Contributions of Notable Models in Neuroscience

The Hodgkin-Huxley Model introduced voltage-gated ion channels and neural excitability. This model's four differential equations properly describe action potential initiation and propagation, allowing scientists to study neuron function. The sophisticated Hodgkin-Huxley model is accurate but computationally intensive, restricting its scalability for large neural network simulations.

A simplified form of the Hodgkin-Huxley model, the FitzHugh-Nagumo (FHN) model simplifies complex ion channel dynamics into a two-variable system, reflecting excitability and recovery. The FHN model is computationally efficient while keeping action potential dynamics' excitability and oscillatory characteristic due to this simplification. This makes it a popular tool for researching excitable systems' general principles and pattern development and network behavior [8].

Morris-Lecar Model: The Morris-Lecar model, another basic model, represents oscillatory neurons like muscular and heart cells. The Morris-Lecar model is useful for investigating bursting behavior and neuron synchronization because it captures rhythmic firing mechanisms. A simplified set of equations makes this model computationally practical for bigger neural circuit simulations.

Integrated-Fire Models: The leaky integrate-and-fire (LIF) model represents neurons as basic threshold units that integrate input signals until a threshold is met, then spike. Network simulations use these computationally efficient models to study large-scale dynamics and network interactions, but they lack the biological precision of the Hodgkin-Huxley model.

Izhikevich Model: Eugene Izhikevich's model is computationally efficient and biologically plausible. It captures brain neuron firing patterns while being computationally manageable. The Izhikevich model is useful for large-scale brain simulations for cortical dynamics and plasticity research. neural models have evolved from complicated biophysical representations to simplified, computationally efficient forms as we grasp neural dynamics and balance biological correctness with scalability. From single-cell dynamics to large-scale network interconnections, each model illuminates brain function differently. This diversity in models assists neuroscience, artificial intelligence, and neural engineering research by modeling individual neurons and developing entire brain system models.

1.2 The FitzHugh-Nagumo Model

1.2.1 Brief History and Development

The FitzHugh-Nagumo (FHN) model was developed in the early 1960s to describe neuronal excitability and action potential generation without the complexity and computing requirements of the Hodgkin-Huxley (HH) model. The two-variable model proposed by biophysicist Richard FitzHugh condensed the four-dimensional HH equations into two differential equations that reflected excitability and recovery in excitable cells.

Meanwhile, Japanese engineer Jinichi Nagumo applied FitzHugh's ideas to electrical circuit theory and created the "Nagumo circuit," a physical counterpart of his

mathematical model. The FitzHugh-Nagumo (FHN) model, which blends simplicity and biological relevance, is a standard for excitable system analysis [12]. This model's ability to simulate neuronal firing and refractory periods with a limited set of equations provides valuable insights into excitable systems while being computationally efficient.

1.2.2 Theoretical Basis of the FHN Model

The FitzHugh-Nagumo model represents neuronal activity using two differential equations that describe the behavior of excitability (V) and recovery (W) variables, with each playing a specific role in action potential dynamics:

Excitability Variable (V): Represents the membrane potential of the neuron, capturing the rapid change in voltage associated with an action potential.

Recovery Variable (W): Represents a slower recovery process, often associated with ion channel dynamics or delayed rectifier currents in neurons, which brings the membrane potential back to its resting state.

The core equations for the FHN model are:

$$\frac{dV}{dt} = V - \frac{V^3}{3} - W + I$$

$$\frac{dW}{dt} = \epsilon(V + a - bW)$$

where:

- V represents the membrane potential (excitability variable).
- W is the recovery variable.
- I is an external stimulus current.
- ϵ, a, and b are parameters that control the behavior of the system, affecting the threshold for excitability, the response rate, and the refractory period.

In this model, the excitable dynamics are driven by the term $V - \frac{V^3}{3}$, which introduces a nonlinear behavior in V, while the recovery dynamics are governed by the linear term $\epsilon(V + a - bW)$, with ϵ being a small parameter that slows down the recovery process relative to excitability. These equations create a limit cycle that generates a spike-like waveform for the action potential, with the membrane potential V increasing rapidly

during excitation and then gradually returning to a resting state due to the recovery variable W.

The simplifications in the FHN model, including reducing four HH variables to two and replacing specific ion channel dynamics with general excitability and recovery terms, make it computationally efficient. Despite these simplifications, the FHN model can effectively reproduce the threshold-based response and refractory behavior characteristic of neuronal action potentials, making it highly suitable for studying general excitability in neurons and other excitable systems, such as cardiac cells.

1.2.3 Comparison with Other Neuronal Models

1.2.3.1 Hodgkin-Huxley Model

The Hodgkin-Huxley (HH) model is a detailed biophysical model that describes the ionic currents underlying action potentials in neurons, specifically focusing on the dynamics of sodium and potassium ions across the neuronal membrane. The HH model is based on four differential equations that account for the gating variables of sodium and potassium channels and the membrane potential, providing a precise description of action potential generation and propagation.

Comparison of Computational Requirements and Biological Accuracy: While the HH model provides an accurate representation of the biophysical processes underlying neuronal excitability, its complexity makes it computationally demanding, especially for large-scale simulations. Each of the four equations in the HH model requires considerable computational resources, particularly when simulating networks of neurons. In contrast, the FHN model, with its two simplified equations, is computationally efficient and well-suited for large-scale simulations, even if it lacks the detailed ion channel dynamics of the HH model [15-19]. Therefore, the FHN model is preferred when a simplified approximation of neuronal excitability is sufficient, particularly for studying general excitable behavior or network dynamics.

1.2.3.2 Morris-Lecar Model

The Morris-Lecar (ML) model, developed to study oscillatory dynamics in neurons, is another simplified model that reduces the HH framework but retains key features necessary for capturing bursting and oscillatory behaviors. The ML model focuses on calcium and potassium currents, making it especially useful for describing neurons that

exhibit rhythmic firing patterns, such as those in cardiac cells and certain types of muscle cells.

Contrast in Applicability: The FHN and ML models have distinct areas of applicability based on their respective simplifications and behaviors. While the FHN model is highly suitable for studying threshold-based excitability and simple spike dynamics, the ML model is particularly effective for exploring neurons with oscillatory firing patterns and bursting behavior. This difference makes the ML model valuable for simulating specific types of neuronal oscillations and synchronous firing in neural circuits, while the FHN model is often used for more general studies of excitability in neurons and other excitable cells.

1.2.3.3 Other Simplified Models

Beyond the FHN and ML models, several other simplified models have been developed to capture different types of neuronal behavior with varying levels of complexity:

Integrate-and-Fire Models: These models, such as the leaky integrate-and-fire (LIF) model, represent neurons as simple threshold-based units. They ignore the details of ionic currents, focusing instead on the concept of accumulating membrane potential until a threshold is reached, at which point a spike is generated. LIF models are widely used for large-scale network simulations where computational efficiency is crucial and detailed action potential dynamics are not necessary.

Izhikevich Model: The Izhikevich model combines the computational efficiency of simplified models with the ability to reproduce a wide range of firing patterns seen in biological neurons. This model is often used in simulations of cortical dynamics, as it balances computational tractability with the ability to replicate diverse firing behaviors, including bursting, tonic spiking, and chattering.

Each of these models, including the FHN model, provides distinct advantages based on its intended application. The FHN model's balance between simplicity and the ability to represent threshold dynamics and excitable behavior makes it a widely used choice for studying the general properties of excitable systems. Its computational efficiency and theoretical foundation continue to support a range of research applications in neuroscience, biophysics, and applied mathematics.

1.3 Research Motivation

The exploration of neuronal dynamics is fundamental to understanding brain function at both micro and macro levels. The FitzHugh-Nagumo (FHN) model offers a compelling compromise between biological realism and computational efficiency, making it an ideal tool for investigating neuronal excitability patterns. Despite its widespread usage, there remains a critical need to systematically evaluate the FHN model's capabilities in replicating diverse neuronal behaviors, particularly its applicability across varying stimulus conditions and in networked configurations. This research addresses this gap by conducting rigorous parameter sensitivity analysis and bifurcation studies to determine the model's efficacy in simulating threshold-dependent excitability, frequency modulation, and recovery dynamics. The findings from this work will not only strengthen the theoretical foundation of computational neuroscience but also inform practical applications in neuromorphic computing, brain-computer interfaces, and artificial neural networks where simplified yet biologically relevant neuronal models are essential.

1.4 Problem Statement

The FitzHugh-Nagumo model presents a significant advantage in computational efficiency, yet its simplified structure raises fundamental questions about its fidelity in capturing complex neuronal dynamics. Specifically, the model's abstraction of detailed ionic mechanisms into two variables creates uncertainty regarding its ability to accurately simulate diverse neuronal behaviors under varying physiological conditions. The primary challenge addressed in this research is determining the extent to which the FHN model can faithfully replicate key neuronal phenomena—including excitability thresholds, stimulus-response relationships, and bifurcation patterns—without compromising biological relevance. Moreover, the model's parameter sensitivity and stability characteristics remain inadequately explored, particularly in networked configurations where emergent behaviors become increasingly complex. By systematically investigating these aspects, this study aims to define the operational boundaries and optimal applications of the FHN model in computational neuroscience, thereby enhancing its utility as a research tool in both theoretical and applied domains.

1.5 Problem Formulation

Problem Formulation 1: Evaluation of the FitzHugh-Nagumo Model's Capability to Simulate Neuronal Excitability and Bifurcation Dynamics in Networked Systems

This problem formulation aims to rigorously assess the FitzHugh-Nagumo (FHN) model's effectiveness in capturing the dynamics of neuronal excitability and bifurcations within interconnected neuronal networks. Specifically, the FHN model will be evaluated for its ability to simulate bifurcation phenomena, neuronal threshold dynamics, and stability characteristics when incorporated into a networked structure under varied external stimuli and parameter conditions.

Mathematical Model and Equations

The FitzHugh-Nagumo model in a networked configuration can be expressed using a set of coupled differential equations. For each neuron i in a network of N neurons, the system is defined by:

$$\frac{dV_i}{dt} = V_i - \frac{V_i^3}{3} - W_i + I_i + \sum_{j=1}^{N} \kappa_{ij} \left(V_j - V_i\right)$$

$$\frac{dW_i}{dt} = \epsilon(V_i + a - bW_i)$$

where:

- V_i represents the membrane potential of the i-th neuron (excitability variable),
- W_i is the recovery variable for the i-th neuron,
- I_i denotes the external input stimulus applied to the i-th neuron,
- κ_{ij} represents the coupling strength between neurons i and j,
- ϵ, a, and b are parameters affecting the dynamics of excitability and recovery.

Objective Functions

To assess the FHN model's performance in networked systems, we define three objective functions:

1. Objective Function J_1: Minimization of the Deviation from Observed Bifurcation Patterns

$$J_1 = \sum_{i=1}^{N} \int_{t_0}^{t_f} \left|V_i(t) - \tilde{V}_i(t)\right|^2 \, dt$$

where $\tilde{V}_i(t)$ is the expected membrane potential derived from observed bifurcation patterns in biological neurons.

2. Objective Function J_2: Maximization of Stability in Neuronal Networks

This function quantifies the network stability by minimizing fluctuations in V_i over time, particularly under small perturbations:

$$J_2 = \sum_{i=1}^{N} \left(\int_{t_0}^{t_f} \left| V_i(t) - V_{i,\text{steady}} \right|^2 \, dt \right)$$

where $V_{i,\text{steady}}$ is the steady-state membrane potential for the i-th neuron.

3. **Objective Function J_3: Optimal Parameter Tuning for Network Synchronization**

To ensure synchronization in excitability patterns across the network, we minimize the variance in membrane potentials across all neurons:

$$J_3 = \int_{t_0}^{t_f} \text{Var}\left(\{V_i(t)\}_{i=1}^{N} \right) dt$$

This objective ensures that the network behaves as a coherent system, minimizing deviations across neurons.

Notations and Definitions

- V_i: Membrane potential of the i-th neuron in the network.
- W_i: Recovery variable for the i-th neuron.
- ϵ, a, b: Model parameters that influence excitability and recovery.
- I_i: External input current applied to the i-th neuron.
- κ_{ij}: Coupling strength between neurons i and j, controlling the influence of neuron j on neuron i.
- $\tilde{V}_i(t)$: Expected potential used as a reference, derived from known neuronal bifurcation patterns.
- $V_{i,\text{steady}}$: Steady-state potential of the i-th neuron.
- Var({...}): Variance of membrane potentials across all neurons in the network.

Problem Explanation

The goal of this problem formulation is to rigorously test the FHN model's ability to accurately replicate neuronal excitability and bifurcation behaviors in a networked environment. This involves analyzing how well the model performs under different parameters and external stimuli, particularly in terms of stability, synchronization, and deviation from known neuronal bifurcation patterns. By incorporating the coupling term

κ_{ij} for interactions between neurons, the formulation simulates network effects that are essential for capturing the complex dynamics seen in real neural networks.

The objective functions defined above are tailored to measure specific characteristics of network behavior: J_1 focuses on the model's accuracy in replicating biologically observed bifurcation patterns, J_2 emphasizes the stability of the network against perturbations, and J_3 seeks to achieve synchronization across neurons, reflecting coherent excitability patterns that are often observed in biological systems. This problem formulation will provide insights into the model's strengths and limitations in network simulations, guiding the refinement of the FHN model's parameters to better approximate the dynamics of interconnected neurons.

1.6 Research Questions

- To what extent does the FitzHugh-Nagumo model accurately replicate the excitability thresholds and bifurcation dynamics observed in biological neurons across varying stimulus intensities?
- How do modifications in key parameters (ε, a, b) affect the model's stability, oscillatory patterns, and recovery dynamics, and how do these compare with empirically observed neuronal behaviors?
- What are the quantifiable limitations of the FitzHugh-Nagumo model in simulating specific neuronal phenomena, such as diverse firing patterns and frequency adaptation?
- How effectively does the FitzHugh-Nagumo model balance computational efficiency with biological accuracy when implemented in networked configurations, and what emergent properties can be observed?
- Under what conditions and parameter configurations does the FitzHugh-Nagumo model most accurately capture the rate coding mechanism used by neurons to encode stimulus intensity?

1.7 Research Objectives

The primary objectives of this research are as follows:

- To evaluate the FitzHugh-Nagumo model's capacity to accurately simulate neuronal bifurcation dynamics and excitability thresholds in networked neuronal systems.

- To investigate the stability of the FitzHugh-Nagumo model in networked environments with fluctuating external stimuli and varying model parameters.
- To identify and address the limitations of the FitzHugh-Nagumo model in capturing complex neuronal interactions, particularly within large-scale networks.
- To optimize parameter settings in the FitzHugh-Nagumo model to enhance synchronization and coherence across networked neurons.

These objectives collectively aim to refine the applicability of the FitzHugh-Nagumo model, offering insights into its potential and limitations in simulating complex neuronal dynamics and network interactions.

1.8 Scope and Limitations of the Study

1.7.1. Scope of the Study

Evaluation of Neuronal Excitability and Bifurcation Dynamics

This study is centered on assessing the FitzHugh-Nagumo (FHN) model's capacity to simulate critical neuronal behaviors, specifically excitability thresholds and bifurcation dynamics. Through computational simulations, the research investigates how effectively the FHN model can capture these phenomena, which are fundamental to understanding neuronal signal processing.

Analysis of Networked Neuronal Interactions

The study extends to exploring the FHN model's performance within interconnected neuronal networks, examining how individual neurons interact and influence one another. This includes evaluating the stability of network behavior under different coupling strengths and external stimuli, simulating conditions that mimic real neural networks.

Parameter Optimization for Synchronization and Coherence

A key focus of this research is the identification and adjustment of FHN model parameters to achieve synchronization and coherence across networked neurons. By optimizing these parameters, the study aims to enhance the model's ability to replicate the collective behavior of biological neurons, which often display synchronized excitability patterns

.

Implications for Computational Neuroscience and Neural Engineering

This study contributes to fields such as computational neuroscience, neural engineering, and neuromorphic computing, where efficient and simplified models are essential for large-scale simulations. Findings are expected to inform applications in brain-computer interfaces and neuroprosthetics by providing insights into the FHN model's potential to simulate complex neural dynamics in a computationally feasible manner.

1.7.2. Limitations of the Study

This study acknowledges several inherent limitations that contextualize its findings and applicability:

Biophysical Abstraction: The FitzHugh-Nagumo model fundamentally simplifies the complex ionic mechanisms underlying neuronal excitability. Unlike detailed conductance-based models, it does not explicitly represent specific ion channels (sodium, potassium, calcium), limiting its ability to capture phenomena directly linked to channel kinetics or molecular interactions.

Parameter Space Constraints: While this research explores a range of parameter values, the investigation necessarily samples discrete points within a continuous parameter space. The comprehensive mapping of all possible parameter combinations remains beyond the scope of this study, potentially overlooking specific parameter regimes with unique behavioral characteristics.

Neuronal Diversity Limitations: The model cannot adequately represent the full diversity of neuronal types found in biological systems, each with distinctive electrophysiological properties. The generalizations made in this study may not apply to specialized neuron types such as bursting neurons, neurons with dendritic computation, or those with complex morphologies.

Network Simplifications: In network simulations, this study employs homogeneous connections and simplified topologies that do not capture the full complexity of biological neural networks, including their heterogeneous connectivity patterns and dynamic synaptic modifications.

Temporal Resolution Constraints: The numerical methods employed introduce discretization that may affect the precise representation of continuous neuronal dynamics, particularly for very fast phenomena occurring on sub-millisecond timescales.

Despite these limitations, the study provides valuable insights into the capabilities and constraints of the FitzHugh-Nagumo model for computational neuroscience applications, establishing a foundation for future refinements and extensions of this widely used neuronal model.

1.9 Thesis Structure

This thesis is organized into seven chapters, each addressing key aspects of the research on the FitzHugh-Nagumo (FHN) model and its application to simulating neuronal dynamics within networked systems.

Chapter 1: Introduction

This chapter provides an introduction to the research topic, outlining the background and importance of mathematical models in neuroscience. It introduces the FitzHugh-Nagumo model, highlighting its relevance and role as a simplified model of neuronal excitability. The research motivation, problem statement, objectives, and the scope and limitations of the study are also discussed, providing a foundation for the subsequent chapters.

Chapter 2: Literature Review

This chapter reviews the existing body of research related to neuronal modeling, focusing on the development and evolution of various models, including the Hodgkin-Huxley, Morris-Lecar, and other simplified models. Emphasis is placed on the theoretical basis and applications of the FitzHugh-Nagumo model, as well as comparisons with alternative models. The review establishes the current knowledge and identifies gaps that this thesis aims to address.

Chapter 3: Theoretical Background of the FitzHugh-Nagumo Model

This chapter provides a detailed exploration of the FitzHugh-Nagumo model's theoretical foundation. It explains the core mathematical equations and describes the excitability and recovery variables, as well as their roles in simulating action potentials. The chapter also discusses the assumptions, simplifications, and limitations of the model in representing neuronal behavior and introduces the concept of bifurcation dynamics in networked systems.

Chapter 4: Methodology

In this chapter, the research design and methodology for conducting the study are outlined. It describes the computational environment, tools, and numerical methods used to implement and simulate the FitzHugh-Nagumo model. The chapter also details the experimental setup, including parameter selection, network configuration, and the process for analyzing model behavior under varied conditions. Additionally, it presents the objective functions developed to evaluate model accuracy, stability, and synchronization.

Chapter 5: Simulation Results and Discussion

This chapter presents the results of the simulations conducted using the FitzHugh-Nagumo model. It includes baseline simulations of neuronal excitability, analysis of bifurcation patterns, and observations from networked configurations. The results for each objective function are discussed, highlighting findings related to model stability, parameter sensitivity, and synchronization. Comparisons with known neuronal dynamics are also included to contextualize the findings within biological relevance.

The discussion chapter interprets the simulation results in relation to the research questions and objectives. It evaluates the FitzHugh-Nagumo model's effectiveness in capturing neuronal excitability and network dynamics, as well as its limitations in representing more complex aspects of neuronal behavior. The chapter also explores the implications of the findings for computational neuroscience and related fields, offering insights into the model’s potential applications and constraints.

Chapter 6: Conclusion

The final chapter summarizes the main findings of the study, emphasizing the contributions of this research to the understanding of the FitzHugh-Nagumo model’s capabilities in simulating neuronal dynamics. It discusses the practical implications of the study, as well as recommendations for future research, including potential model refinements, exploration of heterogeneous networks, and integration of additional neuronal factors. This chapter concludes the thesis by reflecting on the significance of the work in advancing the application of simplified neuronal models in theoretical and applied neuroscience.

Each chapter is designed to systematically build on the previous one, guiding the reader from foundational concepts to detailed analyses, culminating in a comprehensive

understanding of the FitzHugh-Nagumo model's utility and limitations within networked neuronal simulations.

2. CHAPTER 2: LITERATURE REVIEW

2.1 Chapter introduction

This chapter reviews the research on the FitzHugh-Nagumo (FHN) model, a basic but powerful tool for simulating neuronal excitability and network dynamics. Foundational neuronal modeling studies are examined, demonstrating the FHN model's theoretical underpinning, historical history, and basic mathematical structure. The chapter then discusses FHN model extensions including delay differential equations, memristive synapses, and fractional-order dynamics, which address specific neural behaviors. We also compare the FHN model to other neural models like Hodgkin-Huxley to place it in computational neuroscience. In the last chapter, the FHN model is applied to neuromorphic engineering and brain-computer interfaces, and major findings and research gaps are summarized. For this study to further FHN model applications in neural network simulations, this review identifies the FHN model's strengths, weaknesses, and opportunities for further exploration.

2.2 Related work

2.2.1 Neuronal Modeling in Neuroscience

Elfouly, Sohaly, and Fares [1] employed a unique approach to modeling neuronal dynamics by representing the FitzHugh-Nagumo model within the framework of neutral delay differential equations. They designed this model to capture the delayed feedback effects in neuronal excitability, which are often essential for simulating realistic neural behaviors. Through their experiments, the researchers analyzed how these delays influenced neuronal oscillations and excitability thresholds. Their findings showed that incorporating delay elements allowed for more accurate replication of neuronal firing patterns, especially in systems with feedback loops. However, the model’s complexity increased significantly, which limited its computational efficiency, particularly in large-scale simulations.

Amiri, Nazarimehr, and Jafari [2] focused on enhancing the FitzHugh-Nagumo model by introducing a memristive synapse, which added a memory component to the synaptic interactions in the model. They conducted a dynamical analysis to examine how the memristive element affected the model’s behavior, specifically looking at patterns of excitability and spiking synchronization. The results indicated that the inclusion of a

memristive synapse allowed the model to capture a broader range of neuronal behaviors, including complex oscillatory patterns. Despite these advancements, the study highlighted limitations in terms of computational complexity, as the memristive dynamics increased the model's parameter sensitivity, requiring careful calibration.

Kumar and Erturk [3] developed a fractional-order variant of the FitzHugh-Nagumo model to investigate neuronal dynamics with improved flexibility in modeling memory effects. They used fractional calculus to extend the model's capability for simulating long-term dependencies and feedback effects inherent in neuronal activity. Their experimental design focused on analyzing the model's response to various stimuli, showing that the fractional-order model produced more accurate and biologically plausible results, particularly in mimicking memory retention in neural responses. However, they noted that fractional-order equations posed significant challenges in terms of numerical stability and computational cost, limiting the practical applications of this approach in real-time simulations.

Saçu [4] synthesized and analyzed a fractional-order FitzHugh-Nagumo model, aiming to address the limitations in biological realism present in integer-order models. Through a systematic approach, Saçu explored the model's behavior under various fractional parameters, observing its effects on neuronal excitability and signal propagation. His analysis revealed that the fractional-order model provided enhanced control over neuronal spiking frequency and response time, allowing for a closer approximation to physiological neuronal dynamics. Nevertheless, the study identified that implementing fractional-order systems required specialized numerical methods, which increased computational demands and limited model scalability.

Ge et al. [5] investigated neural behaviors and energy properties in a FitzHugh-Nagumo model enhanced with a memcapacitive component, integrating the Miller effect to further simulate synaptic interactions. Their experimental design aimed to capture the energy dynamics involved in neuronal excitability and memory retention. Findings showed that the model successfully replicated energy-efficient signal transmission, reflecting synaptic behavior in energy-limited biological systems. Despite these improvements, the study pointed out the high sensitivity of the memcapacitive component to parameter variations, which complicated model stability and required careful tuning for accurate simulation results.

Shi, Min, and Zhu [6] conducted an analysis of coexisting firing behaviors in the FitzHugh-Nagumo neuron model, focusing on the emergence of multiple stable firing patterns within the same neuronal network configuration. Through nonlinear dynamics techniques, they examined how varying initial conditions and parameter values affected firing behavior, discovering that the model could exhibit a range of stable and unstable firing states simultaneously. Their findings highlighted the model's capacity to represent complex neuronal firing patterns, but also pointed out the sensitivity to initial conditions, which could lead to challenges in predicting long-term behavior in more extensive network simulations.

Cebrían-Lacasa, Parra-Rivas, and Mejía [7] provided a comprehensive review of the FitzHugh-Nagumo model's development and its impact across various disciplines over six decades. They explored the model's applications in spatio-temporal dynamics, analyzing its effectiveness in simulating wave propagation and excitation patterns. Their findings indicated that the model has been widely used for studying excitable media beyond neuroscience, such as in cardiac tissue modeling and ecological systems. However, they also noted that despite its versatility, the model's simplified structure often limited its accuracy in simulating intricate biological processes, necessitating further adaptations when applied to highly detailed physiological contexts.

Fatehi Nia and Mirzavand [8] extended the FitzHugh-Nagumo model by incorporating stochastic dynamics to simulate random fluctuations in neuronal behavior. They merged the FitzHugh-Nagumo framework with aspects of the Izhikevich model to develop a stochastic Izhikevich-FitzHugh neuron model. By introducing noise into the system, they aimed to capture the inherent variability in real neuronal networks. The results demonstrated that the stochastic version could effectively represent the unpredictability seen in biological neurons, yet the increased model complexity posed limitations in stability and required robust computational methods to ensure accurate simulations.

Bosco, Rech, Beims, and Gil [9] investigated the influence of sinusoidal forcing on the FitzHugh-Nagumo model by examining the effects of an external oscillatory input on a two-neuron system. They used analytical and numerical methods to observe how sinusoidal forcing impacted the neuron model's global dynamics, particularly in a unidirectionally coupled network. The study showed that sinusoidal forcing could modulate the model's firing rate and synchronization properties, adding insights into how

periodic external signals affect neural systems. Nonetheless, they identified a limitation in that the model's response became highly parameter-dependent, which restricted its general applicability in diverse neural environments.

Lee [10] focused on using the FitzHugh-Nagumo model as an educational tool to understand neuron dynamics, offering an accessible way to illustrate neuronal excitability and recovery processes. Through simplified simulations, Lee demonstrated the model's utility in explaining the fundamental principles of action potential generation and recovery phases to students and early researchers. While the model effectively captured basic neuronal behaviors, Lee emphasized that its oversimplified nature might lead to misconceptions when exploring more complex aspects of neuronal dynamics, such as detailed ion channel interactions, limiting its educational use to foundational concepts.

2.2.2 Theoretical Basis of the FitzHugh-Nagumo Model

Rani and Arora [11] investigated the theoretical basis and mathematical development of the FitzHugh-Nagumo model, particularly focusing on its application for modeling soliton solutions in excitable media. They approached the model through advanced numerical techniques, employing leave-one-out cross-validation (LOOCV) combined with exponential B-spline functions. This technique provided a unique advantage by allowing the researchers to accurately capture wave-like behaviors that emerge in neurons during signal propagation. Solitons, or self-reinforcing solitary waves, are significant in neuronal modeling as they mimic the stable and non-dispersive propagation of action potentials. By implementing the LOOCV with B-spline functions, Rani and Arora were able to enhance the accuracy of their simulation, effectively representing the nonlinear dynamics within the FitzHugh-Nagumo framework. However, despite the accuracy improvements, they identified computational drawbacks. Fine-tuning the B-spline parameters to align with biological data demanded extensive computational resources, creating limitations for applications requiring real-time processing or large-scale neuronal network simulations. Thus, while their work demonstrated a robust theoretical contribution to the FitzHugh-Nagumo model's mathematical structure, it underscored the need for more computationally efficient methods to make such advanced modeling feasible on a broader scale.

Bao and colleagues [12] expanded the FitzHugh-Nagumo model by incorporating a memristor-based circuit to simulate bifurcation and bursting oscillations, adding a layer

of memory dynamics that aligns more closely with biological neuron behaviors. Memristors, which exhibit memory properties in response to historical input, allowed the team to explore oscillatory patterns beyond what the traditional FitzHugh-Nagumo model can offer. By introducing the memristor, they were able to observe bifurcation phenomena and more complex neuronal oscillations that better approximate real neural behaviors, such as rhythmic bursting and periodic firing patterns. Their findings revealed that the memristor-enhanced FitzHugh-Nagumo model could capture a wide range of oscillatory behaviors, effectively demonstrating the model's versatility and potential for simulating intricate neuronal dynamics. However, they also encountered limitations: the memristive element made the system highly sensitive to parameter changes, requiring meticulous calibration to maintain stability. Without careful tuning, the model was prone to chaotic behavior, which could detract from its reliability. This complexity, while increasing the model's biological relevance, presented a significant challenge in terms of practical usability, especially in simulations of larger neural networks where parameter stability is crucial.

Ahsan, Wu, Jalal, and Kapadia [13] proposed a novel adaptation of the FitzHugh-Nagumo model by designing an ultralow-power electronic analog circuit to replicate the model's excitability and recovery dynamics. This approach aimed to translate the theoretical FitzHugh-Nagumo model into a physical, hardware-based system that could simulate neuronal behavior with minimal energy consumption, a significant advantage in neuromorphic computing and low-power applications. The electronic circuit captured the essential components of the model—excitability and recovery—by emulating these dynamics through an efficient analog design. This innovation highlighted the practical utility of the FitzHugh-Nagumo model in creating energy-efficient systems for simulating neural behaviors, particularly useful in neuromorphic engineering where power efficiency is essential. However, while this analog circuit effectively modeled basic neuronal dynamics, it faced limitations in handling more complex interactions that require detailed representation of ion channel activity or intricate synaptic connections. The simplicity of the circuit, while energy-efficient, limited its applicability to more advanced neural interactions, making it suitable primarily for foundational studies rather than for complex network simulations in neuroscience.

Hramov et al. [14] explored the stochastic FitzHugh-Nagumo model by introducing noise components to simulate coherence resonance, a phenomenon where stochastic input can

enhance a system's response to weak signals. They leveraged reservoir computing, a machine learning approach suitable for temporal pattern recognition, to predict coherence resonance states within the model. The inclusion of stochastic noise allowed the model to mimic how real neurons respond optimally to fluctuating, weak inputs, as seen in biological neurons under random environmental conditions. Their results demonstrated that under specific noise intensities, the model displayed resonance, enhancing its biological relevance by capturing the random yet beneficial influence of noise on neuronal dynamics. However, while reservoir computing provided an accurate prediction framework, it required substantial computational resources, particularly when scaling the model to larger networks. This limitation highlighted the challenge of balancing the model's biological realism with computational feasibility, as the stochastic FitzHugh-Nagumo model with coherence resonance simulations demanded high processing power, potentially restricting its applications to smaller neural circuits or specific experimental settings.

Gao, Shen, and Hu [15] examined the dynamics of the FitzHugh-Nagumo model in a networked system, specifically focusing on delayed and diffusive interactions between neurons. They incorporated both time delay and spatial diffusion into the model, which introduced a new layer of complexity to simulate realistic neural networks. The time delay accounted for the transmission latency in neuronal connections, while diffusion represented the spatial spread of excitation through interconnected neurons. Their analysis revealed that adding these components allowed the model to replicate more realistic neural patterns, such as synchronized oscillations and stable wave propagation across the network. This modification provided a closer approximation to physiological neural systems, where such factors influence signal timing and spatial interactions. However, the complexity introduced by delayed and diffusive terms significantly increased the computational demand, requiring precise parameter control to maintain stability and coherence within the network. The increased model intricacy posed challenges for large-scale simulations, indicating that while the delayed and diffusive FitzHugh-Nagumo model captured realistic network dynamics, it required specialized computational resources and optimization techniques to be effectively scaled for broader applications.

Gao [16] conducted an in-depth analysis of Turing instability within a FitzHugh-Nagumo model configured in a diffusive network. Turing instability, a phenomenon in reaction-

diffusion systems leading to spatial pattern formation, was explored through the FitzHugh-Nagumo model's reaction-diffusion framework. Gao applied mathematical and computational methods to investigate the emergence of spatially heterogeneous patterns, focusing on conditions under which the model shifted from homogeneity to instability-driven patterns. His findings revealed that under specific parameter ranges, diffusive coupling could induce Turing instability, resulting in self-organized spatial patterns similar to those observed in biological systems. However, the study also noted the sensitivity of the model to parameter variations, making it challenging to predict pattern formation without precise control over diffusive terms, thus highlighting limitations in scalability and application to larger neuronal networks.

Goulefack, Masoller, and Yamapi [17] examined wave propagation in a linear chain of FitzHugh-Nagumo neurons, aiming to quantify the transmission of electrical signals along a neuronal pathway. They designed an experiment using a chain of coupled FitzHugh-Nagumo units to simulate axonal propagation and signal strength decay across a network. Through numerical simulations, they quantified the stability and speed of wave propagation under varying coupling strengths, capturing the dynamics of signal transmission in a linear neuron chain. The study revealed that strong coupling enhanced wave stability and propagation speed, effectively preventing signal decay. However, weaker coupling led to dispersion and attenuation, limiting the chain's ability to maintain consistent signal strength over longer distances. The results underscored the model's value in studying wave dynamics but also pointed out that additional mechanisms may be needed to fully capture the robustness of real biological signal propagation.

Iqbal et al. [18] focused on finding soliton solutions for the nonlinear stochastic FitzHugh-Nagumo equation, extending the model to include stochastic effects that represent random environmental fluctuations. They approached the model using analytical methods to derive soliton solutions, which are stable, localized waveforms that maintain their shape during propagation, even under stochastic conditions. By incorporating randomness, Iqbal and his team sought to simulate real-world variability in neuronal signal propagation. Their results showed that the stochastic FitzHugh-Nagumo model could produce soliton solutions that mimic stable, noise-resistant signal transmission, thus enhancing the model's biological relevance. However, they also encountered challenges in maintaining numerical stability, as stochastic fluctuations introduced instability in

long-term simulations, complicating the application of these solutions in larger, sustained neural networks.

Hu, Ding, Wu, Huang, Yang, and Jia [19] investigated the impact of dynamical rewiring on synchronization in a memristive FitzHugh-Nagumo neuronal network. They modified the network structure to allow for dynamic changes in connectivity, simulating adaptive rewiring observed in real neural networks. By introducing memristors to model adaptive synaptic changes, they analyzed how these alterations affected synchronization patterns and coherence among networked neurons. Their findings indicated that dynamic rewiring promoted synchronization, with memristive elements helping maintain adaptive stability across the network. However, the study noted that such rewiring increased the model's complexity, necessitating high computational resources and precise control over connectivity parameters to avoid desynchronization. This adaptation provided insights into neuronal plasticity but limited the model’s feasibility for extensive simulations due to increased computational demands.

Cek and Uludag [20] explored spectral resonance within the FitzHugh-Nagumo neuron model, examining its relationship with stochastic resonance and its potential applications in electromyography (EMG) signal characterization. They used spectral analysis to investigate how external noise and inherent neuronal fluctuations influenced the system's resonant frequency response. Their results demonstrated that under specific noise levels, spectral resonance could be induced, amplifying signal clarity and strength. This phenomenon has implications for enhancing EMG signal analysis, as spectral resonance could help in distinguishing relevant neuronal signals from background noise. However, they noted that achieving optimal resonance required carefully controlled noise levels, limiting its applicability in real-world settings where noise is highly variable. The study illustrated the potential for using the FitzHugh-Nagumo model in biomedical signal processing but highlighted challenges related to noise management and consistency in diverse environments.

2.2.3 Comparison with Alternative Models

Xu et al. [21] examined an improved version of the FitzHugh-Nagumo (FHN) model, emphasizing a multiplier-free implementation that enhances computational efficiency while maintaining the model’s core features. They highlighted that one of the strengths of this modified FHN model lies in its reduced computational demand, making it more

practical for large-scale simulations compared to the traditional Hodgkin-Huxley (HH) model, which is computationally intensive due to its detailed biophysical representations. The study demonstrated that the multiplier-free FHN model could replicate essential neuronal dynamics, such as excitability and threshold behavior, with comparable accuracy to other simplified models. However, the team also identified limitations in the model's ability to capture intricate ion-channel interactions, which are accurately represented in the HH model. This restriction limits the improved FHN model's applicability to simulations where only general excitability is required rather than detailed ion dynamics.

Chen et al. [22] focused on a memristive variant of the FHN model, incorporating an initial-offset boosting mechanism that introduces hidden dynamics, leading to complex bifurcation behavior. Compared to the standard FHN model, which provides basic excitability and recovery dynamics, the memristive variant offered richer dynamical properties due to the memory effects inherent in memristors. The researchers highlighted that this adaptation allowed the model to capture a broader range of neuronal firing patterns and oscillatory behaviors, resembling the diversity observed in biological neurons. However, they noted that the memristive FHN model's sensitivity to initial conditions and parameter tuning posed significant challenges, particularly in achieving stable simulations. Unlike the HH model, which has well-defined biophysical parameters, the memristive FHN variant's reliance on precise parameter control reduces its robustness, making it less practical for generalized applications in large networks where stability is crucial.

Zhang, Min, Dou, and Xu [23] conducted a bifurcation analysis on a modified FHN model that incorporated an external electric field, allowing for the study of neuronal responses under varying electric stimuli. They demonstrated that this model could capture complex bifurcation patterns and phase transitions, which are not as readily observed in the classic FHN model. Compared to the HH model, which inherently includes responses to ionic currents, the modified FHN model offered a more computationally efficient approach to studying field-induced bifurcations and oscillatory behaviors. The study highlighted that the addition of an electric field enhanced the model's applicability for scenarios where external influences play a significant role in neuronal behavior. However, Zhang et al. also noted limitations in the model's ability to accurately represent biophysical details

such as ionic movement, which are critical for studies involving precise synaptic interactions and are well-handled by the HH model.

Elfouly, Sohaly, and Fares [24] used a neutral delay differential equation approach to represent the FHN model, allowing them to examine delayed feedback effects that are critical in neuronal communication. The delay differential representation enabled them to capture the effects of time-delayed excitability in neurons, a feature that is challenging to model in traditional FHN or HH frameworks. Compared to the HH model, the delayed FHN model provided a more simplified but effective method for simulating delayed neuronal responses, which are relevant in both neural and artificial network applications. However, they found that this approach increased the model's complexity, as incorporating delays required careful calibration to avoid instability. Despite its advantages, the delay differential FHN model was limited by its high sensitivity to initial conditions, which could lead to unpredictable behaviors if not carefully controlled.

Amiri, Nazarimehr, and Jafari [25] introduced a memristive synapse into the FHN model, aiming to explore how memory elements impact synaptic interactions and network dynamics. This adaptation allowed the model to simulate more realistic, plastic synaptic interactions compared to traditional models, which lack memory properties. Their findings indicated that the memristive synapse enhanced the FHN model's ability to mimic complex, time-dependent neuronal interactions, bringing it closer to the biological behavior observed in synaptic networks. Nonetheless, this complexity introduced a significant limitation: the memristive FHN model was highly sensitive to parameter fluctuations, requiring precise adjustments for stable operation. In contrast, the HH model, with its detailed ion-channel representations, does not rely on such sensitivity, thus providing greater robustness in simulating stable neuronal activity across a wider range of conditions.

Rybalko and Fradkov [26] explored an identification approach for a two-neuron FitzHugh-Nagumo (FHN) model, utilizing speed-gradient and filtering techniques to capture neuronal interactions. They focused on using adaptive control techniques to improve the model's performance in representing neuronal dynamics, specifically by optimizing the response speed and filtering capabilities to simulate the connectivity and mutual influence between neurons. Compared to the Hodgkin-Huxley (HH) model, which inherently supports detailed neuronal interactions through ion channel dynamics, the two-

neuron FHN model required external modifications to achieve similar interaction capabilities. Although the speed-gradient method enhanced the responsiveness of the FHN model, it was limited by computational demands and sensitivity to parameter adjustments, making it less robust for large-scale simulations compared to the more biologically detailed but computationally demanding HH model.

Saçu [27] conducted a synthesis and analysis of a fractional-order FitzHugh-Nagumo model to address the limitations in biological realism often found in integer-order models. By introducing fractional calculus into the FHN model, Saçu aimed to provide greater control over neuronal firing frequencies and response times, improving its resemblance to actual neuronal dynamics. The fractional-order model offered enhanced flexibility in adjusting neuronal excitability patterns, making it particularly effective for capturing long-term memory effects and decay rates, which are challenging to model in both the traditional FHN and HH frameworks. However, the added complexity of fractional-order differential equations introduced challenges in numerical stability and computational efficiency, limiting its practicality for real-time simulations or extensive network modeling, where the integer-order HH model may still be preferred due to its stable and established parameter structure.

Kumar and Erturk [28] also investigated a fractional-order variant of the FHN model but focused on an improved version with specific enhancements in its excitability and recovery functions. They demonstrated that the fractional-order modification allowed the model to capture complex neuronal behaviors, such as long-term dependencies and adaptive responses to sustained stimuli. While the HH model provides a biophysically accurate representation of such dependencies through ion channels, the fractional-order FHN model offers a computationally simpler but flexible alternative. However, Kumar and Erturk highlighted that the model's sensitivity to fractional parameters demanded precise calibration, as slight deviations could lead to instability. This limitation reduces the model's robustness in practical applications, especially in scenarios where precise biological fidelity and stability are crucial, as provided by the HH model's established structure.

Shi, Min, and Zhu [29] performed an analysis of coexisting firing patterns within the FHN neuron model, focusing on the conditions that lead to multiple stable firing states. They employed nonlinear dynamical techniques to explore how different initial conditions and

parameter values influenced the model's behavior, revealing that the FHN model could exhibit multiple stable and unstable firing modes. This property of coexisting firing patterns adds versatility to the FHN model, making it suitable for simulating diverse neuronal firing behaviors that are often observed in complex neural circuits. However, the model's sensitivity to initial conditions posed a limitation, as it could lead to unpredictable outcomes in large-scale simulations. In contrast, the HH model, which offers a more deterministic representation due to its fixed biophysical parameters, provides greater consistency across different initial conditions, making it more suitable for applications where stable and predictable neuronal responses are necessary.

Uzal [30] designed a microcontroller-based emulator circuit for the FHN neuron model, creating a hardware-based implementation that could simulate neuronal dynamics in real-time with minimal power consumption. This approach leveraged the simplicity of the FHN model to build an energy-efficient emulator, enabling real-time applications in neuromorphic engineering and low-power device design. Compared to the HH model, which is computationally intensive and challenging to implement on hardware without significant simplification, the FHN model's reduced complexity made it more suitable for microcontroller-based applications. However, the simplified nature of the FHN model limited the emulator's ability to replicate complex neuronal behaviors accurately. Uzal noted that while the emulator was effective for basic excitability and firing simulations, it lacked the ability to capture detailed ion channel interactions and intricate synaptic dynamics, which are more faithfully represented in hardware implementations based on the HH model.

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2.2.4 Applications of the FitzHugh-Nagumo Model in Neuroscience

Xu et al. [31] applied an improved FitzHugh-Nagumo (FHN) model with a multiplier-free implementation in studies on neuronal network simulations, aiming to streamline computational requirements for large-scale applications. The simplified implementation made it possible to simulate the collective behavior of neurons in a computationally efficient manner, making it valuable for neuroscience research focused on large neural circuits and network connectivity analysis. By reducing the computational load, this model facilitated the exploration of neuronal excitability patterns across extensive networks, offering insights into how excitability propagates in interconnected systems.

However, Xu and colleagues acknowledged that, while efficient, this implementation might lack the precision needed for highly detailed neuronal modeling, limiting its utility for studies requiring fine-grained biophysical accuracy.

He, Li, Chen, and Cao [32] explored the use of the FitzHugh-Nagumo model in simulating neurons with state-dependent impulsive effects, a feature that introduced a level of realism by allowing for sudden state changes based on internal neuron conditions. This adaptation has practical applications in studying neuronal responses to abrupt inputs, such as synaptic spikes or external stimuli, which can trigger impulse-driven shifts in neuronal states. Through their experiments, He and colleagues demonstrated that the state-dependent impulsive FHN model could effectively mimic responses to transient stimuli, providing insights into neuronal adaptability and plasticity in dynamic environments. This model has potential applications in neural prosthetics and brain-computer interfaces (BCIs), where simulating rapid neural responses to external signals is essential. Despite its utility, the addition of impulsive effects increased model complexity, posing challenges for real-time applications and requiring careful control of impulse parameters to avoid destabilizing the model.

Chen, Wang, Wang, Wu, and Xu [33] investigated a memristive version of the FHN model, incorporating initial-offset boosting to explore bifurcation mechanisms and hidden dynamics in neurons. This adaptation allowed the model to simulate synaptic plasticity and memory effects by mimicking the persistence of neuronal responses over time. Their findings revealed that the memristive FHN model could reproduce complex bifurcation behaviors, enhancing its relevance for applications that involve adaptive neural processing, such as learning models and synaptic plasticity studies. In practical terms, this model has applications in understanding memory retention mechanisms in the brain and exploring the neural basis of learning. However, the model's sensitivity to initial conditions and parameters made it challenging to control in larger networks, limiting its applicability to small-scale studies where precise parameter tuning is feasible.

Korkmaz and Şıvga [34] implemented the FitzHugh-Nagumo model with an electromagnetic effect on an FPGA (Field-Programmable Gate Array) platform, enabling real-time simulation of neuronal dynamics influenced by electromagnetic fields. This FPGA-based realization allowed for the rapid simulation of electromagnetic effects on neurons, offering a practical tool for neuroscience research on electromagnetic brain

stimulation, such as transcranial magnetic stimulation (TMS). Their work demonstrated that the FHN model could effectively model neuronal responses to electromagnetic fields, which has applications in both therapeutic and diagnostic settings. The hardware implementation also showed potential for portable and low-power applications, making it suitable for embedded systems. However, the hardware constraints of FPGA limited the model's complexity, restricting its use to basic neuronal behaviors and simple network configurations rather than extensive, biologically realistic networks.

Rudi, Bessac, and Lenzi [35] utilized convolutional and dense neural networks to perform parameter estimation for the FHN model, aiming to improve its adaptability for specific neuronal behaviors. By leveraging machine learning techniques, they trained networks to optimize the parameters of the FHN model based on target neuronal dynamics, thus enhancing its accuracy and usability in diverse research contexts. This approach enabled the model to approximate specific neuronal responses more accurately, making it suitable for applications in personalized neural modeling and adaptive simulations. Machine learning-based parameter estimation opens up potential for applications in individualized medicine and neural prosthetics, where accurately modeling specific neural responses is essential. However, this approach requires substantial training data and computational resources, which may limit its accessibility for real-time simulations or resource-constrained settings.

Zhang, Min, Dou, and Xu [36] examined the effects of an external electric field on a modified FHN model, focusing on its bifurcation and oscillatory properties under different field intensities. The application of an electric field made it possible to simulate neuronal responses in environments subject to electrical modulation, such as during electrical stimulation therapies used for neurological conditions. Their study showed that the electric field-enhanced FHN model could replicate a range of firing behaviors, providing a valuable tool for understanding the effects of electrical fields on neural tissue. This has practical applications in developing neurostimulation therapies and understanding the electrophysiological basis of treatments like deep brain stimulation (DBS). However, the addition of the electric field increased model sensitivity, making it highly parameter-dependent and requiring careful calibration to avoid unintended oscillatory behaviors, which could complicate its use in large-scale simulations or variable field conditions.

Khakipoor, Bahar, and Karimian [37] applied the FitzHugh-Nagumo (FHN) model within a circuit framework to analyze neuronal excitability and response patterns efficiently. By integrating the FHN model into an analog circuit design, they demonstrated how neuronal dynamics could be effectively emulated in hardware, offering a practical tool for applications in neuromorphic engineering and analog signal processing. This approach enabled real-time simulation of neuronal behaviors, which is valuable for prototyping devices that need to mimic neural responses, such as brain-machine interfaces and neural prosthetics. The circuit-based implementation of the FHN model offered advantages in terms of speed and power efficiency over software-based simulations, but the model's reduced complexity limited its capacity to accurately represent detailed synaptic interactions, which are critical for more comprehensive applications in neuroscience.

Elfouly, Sohaly, and Fares [38] explored the FHN model using a neutral delay differential equation framework, which allowed them to incorporate time delays that more accurately represent biological neuron signaling, where feedback delays are common. This delayed version of the FHN model has practical implications for studying time-dependent neural responses, such as those in interconnected brain regions that exhibit delayed signaling. The study demonstrated that adding time delays could capture realistic oscillatory and feedback behaviors observed in neural circuits, providing a valuable model for research on distributed brain activity. However, they noted that the inclusion of delay elements introduced complexity, making it computationally challenging to stabilize the model for larger network simulations where precision in delay timing is essential.

Amiri, Nazarimehr, and Jafari [39] developed a FitzHugh-Nagumo model with a memristive synapse, focusing on how memory effects impact neuronal and synaptic dynamics. This adaptation allowed the FHN model to simulate forms of synaptic plasticity, which are crucial for understanding learning and memory processes in neural circuits. Their findings indicated that the memristive FHN model could capture dynamic and adaptive synaptic interactions, making it suitable for applications in artificial neural networks and cognitive computing. The model has practical applications in fields that require emulation of learning behaviors, such as adaptive algorithms in machine learning and neurocomputing. However, the addition of a memristive component made the model highly parameter-sensitive, requiring precise control for stability, which limits its scalability in complex network simulations.

Bisquert [40] conducted a frequency domain analysis of the excitability and bifurcations within the FHN model, providing insights into how neuronal dynamics vary with input frequencies. By analyzing the model's response across a range of frequencies, Bisquert was able to highlight the conditions under which neurons transition between stable and excitable states, offering a basis for applications in brain stimulation therapies that leverage frequency-specific responses. This approach is particularly relevant for treatments like transcranial magnetic stimulation (TMS) and deep brain stimulation (DBS), where tuning frequency can enhance therapeutic effects. Although frequency domain analysis provided a novel perspective on the model's excitability, it required specialized knowledge and tools, limiting its applicability to researchers equipped to conduct advanced signal processing in neural modeling.

Habbal, Farhat, Khalil, and Pannier [41] applied 3D printing and molding techniques to create physical models based on the FitzHugh-Nagumo model, allowing for the visualization of neuron morphology and connectivity. By using fused filament fabrication, they generated flexible, scaled neuron models that could be used for educational purposes and in neuroscience research to study physical aspects of neuronal structure. This tangible representation of the FHN model provides a unique application in neuroscience education, where visualizing neuron connectivity and behavior can aid in learning. Although this approach enhances the understanding of neuron morphology, it is limited in its capacity to simulate dynamic processes, as the models serve primarily for structural representation rather than for functional analysis of neuronal dynamics.

Okonkwo, Olaniran, Adeyi, and colleagues [42] explored the modeling of biological processes using neural networks and adaptive neuro-fuzzy inference systems, indirectly informing the potential for FHN model applications in modeling adaptive biological systems. While their study focused on food processing, the principles of adaptive modeling and neuro-fuzzy inference highlighted applications for the FHN model in simulating complex, adaptive biological processes like neural plasticity. This indirect application suggests potential avenues for integrating the FHN model with adaptive systems, enhancing its utility in contexts where flexibility and learning are required. However, transferring insights from food processing to neuroscience requires careful adaptation, as the fundamental processes differ, and neural network-based models often require significant tuning for biological accuracy in neuroscience applications.

Table 2.1:Comparative Table of Previous Study

Reference	Technique	Results	Limitations	Findings
[1]	Neutral Delay Differential Equation representation of FHN model	Captured delayed feedback in neuronal signaling, enhancing oscillatory behavior simulation	High sensitivity to initial conditions, making the model challenging to stabilize in large networks	Effective for studying time-dependent and feedback-delayed neural responses
[11]	LOOCV with exponential B-spline functions for soliton solutions	Improved accuracy in wave propagation simulation, mimicking stable action potentials	Computationally intensive, limiting real-time application	Valuable for modeling stable signal propagation in neurons
[12]	Memristor-based circuit added to FHN model	Simulated bifurcations and complex oscillations similar to biological neurons	High sensitivity to parameter variations, challenging stability	Useful for exploring memory-driven behaviors and oscillatory dynamics
[13]	Ultralow-power electronic analog of FHN model	Efficient energy consumption for mimicking excitability and recovery dynamics in hardware	Simplified design limits representation of complex interactions	Suitable for neuromorphic applications requiring basic neuronal dynamics

[18]	Stochastic extension of FHN model for soliton solutions	Captured noise-resistant, stable signal transmission with soliton solutions	Numerical instability with prolonged stochastic simulations	Effective for representing randomness and variability in neural signal propagation
[21]	Multiplier-free implementation of improved FHN model	Enabled large-scale simulations of neural networks with reduced computational load	Lacks precision in representing detailed biophysical neuronal interactions	Suitable for simulating large-scale network connectivity and excitability patterns
[22]	Memristor initial-offset boosting in FHN model	Produced diverse firing patterns and bifurcation behaviors	High sensitivity to parameter control, impacting stability	Effective for exploring complex oscillatory and memory-driven neuronal behaviors
[26]	Speed-gradient and filtering in two-neuron FHN model	Enhanced model responsiveness, simulating neuron connectivity and influence	Computational demands and parameter sensitivity	Useful for modeling connected neurons in adaptive control scenarios
[27]	Fractional-order FHN model synthesis and analysis	Enhanced control over firing frequency, modeling long-	Increased complexity and stability	Valuable for representing neuronal memory and

		term dependencies	challenges in simulations	adaptive responses
[37]	Circuit-based FHN model for analog simulations	Enabled real-time simulations with low power consumption	Simplified model limits complex synaptic interaction representation	Suitable for hardware-based neural response emulation, such as brain-machine interfaces

2.3 Liteature summery

The literature on the FitzHugh-Nagumo (FHN) model shows its versatility in mimicking neuronal excitability and network dynamics. Neutral delay differential equations, memristive adaptations, fractional-order modifications, and multiplier-free implementations have improved the FHN model's relevance for specific brain activities in many research. Capturing delayed feedback, enhancing large-scale network computational efficiency, and modeling neural oscillations-like memory effects and bifurcation behaviors are major advances. Analog circuits and FPGA-based realizations have shown the FHN model's low-power and real-time simulation capabilities, making it useful for neuromorphic engineering and brain-computer interface applications. Despite these advances, the model's stability under complicated configurations, sensitivity to parameter changes, and biophysical accuracy compared to Hodgkin-Huxley are still limits. The literature supports the FHN model as a basic but versatile tool for examining general excitability patterns, although scaling and improving it for neural network simulations remains difficult.

2.4 Research gap

The FitzHugh-Nagumo model has been successfully improved and applied to neuronal simulations, although large-scale, biologically correct neural network simulations still require investigation. Current studies emphasize the model's inability to capture detailed ion channel dynamics, actual synaptic contacts, and adaptive network plasticity in fluctuating or high-dimensional parameter spaces. Complex setups with temporal delays,

fractional-order dynamics, or memristive components have stability and computing demand concerns. Research is needed to increase the model's robustness and scalability for real-time and biologically realistic simulations, despite computer complexity reductions. Addressing these shortcomings could improve the FHN model's ability to explore complex brain systems in neuroprosthetics, cognitive computing, and large-scale neural simulations.

3. Theoretical Background of the FitzHugh-Nagumo Model

The FitzHugh-Nagumo model represents a milestone in computational neuroscience by distilling the complex Hodgkin-Huxley equations into a mathematically tractable system while preserving essential neuronal dynamics. The model's theoretical foundation rests on several key principles:

Dimensional Reduction and Phase Space Analysis

The FHN model achieves dimensional reduction by condensing the four-variable Hodgkin-Huxley framework into a two-variable system that captures the essential dynamics of neuronal excitability. This reduction transforms a complex biophysical system into a more accessible mathematical framework amenable to phase space analysis.

The model's phase space reveals critical structures:

Nullclines (where $dV/dt = 0$ and $dW/dt = 0$) whose intersections determine equilibrium points

Stable and unstable manifolds that govern trajectory behavior

Limit cycles that emerge through Hopf bifurcations, representing periodic firing

Dynamical Systems Perspective

From a dynamical systems viewpoint, the FHN model exemplifies a nonlinear oscillator with excitable properties. The cubic term $V - V^3/3$ introduces essential nonlinearity that enables:

Bistability between resting and excited states

Threshold behavior characteristic of neuronal firing

Excitability where sufficiently large perturbations trigger full excursions in phase space before returning to rest

The separation of timescales between fast (V) and slow (W) variables—controlled by the parameter ε—creates relaxation oscillations that accurately mimic the rapid depolarization and slower recovery phases of action potentials.

Mathematical Formulation and Interpretation

The core equations:

$$\frac{dV}{dt} = V - \frac{V^3}{3} - W + I$$

$$\frac{dW}{dt} = \varepsilon(V + a - bW)$$

Each term carries specific biophysical significance:

V - $V^3/3$ approximates the fast autocatalytic process of sodium channel activation

W represents combined recovery processes, including sodium channel inactivation and potassium channel activation

Parameter a influences excitability threshold

Parameter b modulates recovery dynamics

Parameter ε controls timescale separation

External current I simulates synaptic or experimental inputs

This mathematical formulation creates a framework that balances analytical tractability with biological relevance, enabling both theoretical analysis and practical applications in computational neuroscience.

3.1. The FitzHugh-Nagumo Model

3.1.1 Brief History and Development

The FitzHugh-Nagumo (FHN) model was developed in the early 1960s as a simplified representation of the Hodgkin-Huxley (HH) model, designed to capture essential features of neuronal excitability and action potential generation without the high complexity and computational demands of the HH model. Richard FitzHugh, a biophysicist, initially proposed this two-variable model, which simplified the four-dimensional HH equations into a system of two differential equations, representing the essential dynamics of excitability and recovery in excitable cells.

Around the same time, Jinichi Nagumo, a Japanese engineer, applied FitzHugh's ideas to electrical circuit theory, resulting in the "Nagumo circuit," a physical analog of FitzHugh's mathematical model. Together, the contributions of FitzHugh and Nagumo formed what is now widely known as the FitzHugh-Nagumo (FHN) model, which has become a benchmark in studying excitable systems due to its balance between simplicity

and biological relevance. This model is particularly significant for its ability to mimic the threshold dynamics of neuronal firing and refractory periods with a minimal set of equations, providing valuable insights into excitable systems while being computationally efficient for simulations.

3.2 Theoretical Basis of the FHN Model

The FitzHugh-Nagumo model represents neuronal activity using two differential equations that describe the behavior of excitability (V) and recovery (W) variables, with each playing a specific role in action potential dynamics:

- Excitability Variable (V): Represents the membrane potential of the neuron, capturing the rapid change in voltage associated with an action potential.
- Recovery Variable (W): Represents a slower recovery process, often associated with ion channel dynamics or delayed rectifier currents in neurons, which brings the membrane potential back to its resting state.

The core equations for the FHN model are:

$$\frac{dV}{dt} = V - \frac{V^3}{3} - W + I$$

$$\frac{dW}{dt} = \epsilon(V + a - bW)$$

where:

- V represents the membrane potential (excitability variable).
- W is the recovery variable.
- I is an external stimulus current?
- ϵ, a, and b are parameters that control the behavior of the system, affecting the threshold for excitability, the response rate, and the refractory period.

In this model, the excitable dynamics are driven by the term $V - \frac{V^3}{3}$, which introduces a nonlinear behavior in V, while the recovery dynamics are governed by the linear term $\epsilon(V + a - bW)$, with ϵ being a small parameter that slows down the recovery process relative to excitability. These equations create a limit cycle that generates a spike-like waveform for the action potential, with the membrane potential V increasing rapidly during excitation and then gradually returning to a resting state due to the recovery variable W.

The simplifications in the FHN model, including reducing four HH variables to two and replacing specific ion channel dynamics with general excitability and recovery terms, make it computationally efficient. Despite these simplifications, the FHN model can effectively reproduce the threshold-based response and refractory behavior characteristic of neuronal action potentials, making it highly suitable for studying general excitability in neurons and other excitable systems, such as cardiac cells.

Theoretical Background of the FitzHugh-Nagumo Model

The FitzHugh-Nagumo (FHN) model is a simplified version of the Hodgkin-Huxley model, designed to capture essential neuronal excitability and recovery dynamics with reduced computational complexity. It uses a pair of coupled nonlinear differential equations to simulate the basic behavior of a neuron, specifically focusing on action potential generation and the return to a resting state. The FHN model is particularly useful for studying general excitability and threshold behavior in neurons, and its simplicity allows it to be adapted for large-scale simulations and real-time applications in neuromorphic computing.

Derivation of the Model's Equations

The FHN model is derived by simplifying the Hodgkin-Huxley equations, which originally described ion flows across a neuronal membrane. By reducing the four-variable Hodgkin-Huxley system to a two-variable model, FitzHugh and Nagumo focused on capturing core neuronal dynamics without needing detailed simulation of specific ion channels. The FHN model is formulated with two main differential equations:

$$\frac{dV}{dt} = V - \frac{V^3}{3} - W + I$$

$$\frac{dW}{dt} = \epsilon(V + a - bW)$$

where:

- V represents the membrane potential (the excitability variable),
- W is the recovery variable, representing the slower inhibitory processes
- I is an external stimulus current applied to the neuron
- ϵ, a, and b are parameters that govern the model's dynamics.

These equations capture the essential aspects of action potential generation and return to resting state. The first equation describes how the membrane potential V evolves, with nonlinear terms that capture excitability and threshold dynamics. The second equation models the recovery variable W, which regulates the return to the resting state after an action potential.

3.3 Parameters and Variables

For this study, the following parameters and variables are essential in analyzing the FHN model:

V (Membrane Potential or Excitability Variable)

The variable V represents the membrane potential of the neuron, which is the electrical charge difference across the neuronal membrane. In the FHN model, this variable is central to neuronal excitability and action potential generation. When V reaches a threshold, it produces a rapid spike, representing neuronal firing. The term $V - \frac{V^3}{3}$ introduces a nonlinear relationship, allowing the model to simulate both stable and unstable states, which are key to capturing neuronal excitability.

W (Recovery Variable)

W represents a slower recovery process that counterbalances excitability. Often associated with processes like potassium ion flow or delayed rectifier currents, W helps bring V back to its resting state after an action potential. This recovery phase prevents continuous firing and provides a refractory period before the neuron can fire again. The term $\epsilon(V + a - bW)$ controls this recovery, where ϵ dictates the speed of recovery relative to excitability.

I (External Stimulus Current)

The parameter I represents an external stimulus current applied to the neuron, allowing for the simulation of external signals that drive the neuron to its firing threshold. By varying I, it is possible to study how different input levels affect neuronal excitability and firing patterns, making I crucial for simulating realistic neuronal responses.

ϵ (Recovery Speed Parameter)

The parameter ϵ controls the speed of the recovery variable W relative to changes in V. Typically, ϵ is a small positive constant, reflecting that W evolves more slowly than V.

A smaller ϵ value creates a distinct separation between fast excitability and slow recovery dynamics, critical for maintaining stability in the model and accurately simulating neuronal firing rhythms.

a and b (Recovery Equation Parameters)

The parameters a and b influence the shape and stability of the recovery dynamics. Parameter a shifts the resting position of the membrane potential V, altering the firing threshold, while b modulates feedback from W back to V. Adjusting a and b allows for modeling neurons with different excitability profiles, making these parameters valuable for exploring diverse firing behaviors.
The FHN model effectively captures the interplay between excitability and recovery, replicating essential neuronal behaviors. Adjusting the values of ϵ, a, b, and I allows for simulating various neuronal responses, from isolated spikes to sustained oscillatory patterns. The model's simplicity provides a versatile framework for studying excitability in neurons, with parameters that offer flexibility for specific experimental conditions or neural network configurations.

Dynamical Systems and Phase Space Analysis

The FitzHugh-Nagumo (FHN) model, as a nonlinear dynamical system, exhibits a range of behaviors that can be analyzed using stability, bifurcation, and phase space analyses. These methods are crucial for understanding how the model's variables—particularly the excitability variable V and the recovery variable W—evolve over time and respond to changes in parameters. By examining the stability of equilibrium points and investigating bifurcation conditions, we can gain insight into the model's ability to replicate neuronal firing, excitability thresholds, and oscillatory dynamics. Phase space analysis further enables the visualization of trajectories and dynamic states, making it possible to track the system's behavior under various initial conditions and external stimuli.

Stability and Bifurcation Analysis

Stability analysis in the context of the FHN model involves determining whether small perturbations around equilibrium points lead the system back to equilibrium (stable) or cause divergence away from it (unstable). The equilibrium points, or fixed points, are found by setting the derivatives in the differential equations of the FHN model to zero:

$$\frac{dV}{dt} = 0 \quad \text{and} \quad \frac{dW}{dt} = 0$$

Solving these equations simultaneously gives the equilibrium values of V and W. To analyze the stability of these equilibrium points, we examine the **Jacobian matrix** of the system, derived by taking the partial derivatives of each equation with respect to V and W:

$$J = \begin{bmatrix} \frac{\partial f(V,W)}{\partial V} & \frac{\partial f(V,W)}{\partial W} \\ \frac{\partial g(V,W)}{\partial V} & \frac{\partial g(V,W)}{\partial W} \end{bmatrix}$$

where $f(V,W) = V - \frac{V^3}{3} - W + I$ and $g(V,W) = \epsilon(V + a - bW)$. The eigenvalues of the Jacobian matrix determine the nature of the equilibrium point:

- If both eigenvalues have negative real parts, the equilibrium is a stable node or focus, meaning the system will return to this point if slightly perturbed.
- If any eigenvalue has a positive real part, the equilibrium is unstable, and nearby trajectories will diverge from this point.

By varying parameters such as ϵ, a, b, and I, the stability of these equilibria can change, resulting in bifurcations. Bifurcation analysis examines these transitions, particularly the Hopf bifurcation, which occurs when a pair of complex conjugate eigenvalues crosses the imaginary axis, leading to oscillatory solutions. This bifurcation is essential in the FHN model as it explains the onset of oscillatory behavior, corresponding to repetitive neuronal firing or sustained action potentials under certain conditions.

Explanation of Phase Space and Trajectories

Phase space is a conceptual space in which each point represents a unique state of the system, defined by the values of the variables V (membrane potential) and W (recovery variable) at any given time. For the FHN model, phase space provides a visual framework for analyzing the dynamic behavior of the model. A phase space plot of V versus W reveals the trajectory or path that the system follows over time for specific initial conditions and parameter values.

In a two-dimensional phase space(with axes V and W), the trajectories represent the evolution of the neuronal state. Key features in the phase space include:

- Fixed Points (Equilibria): Points where $\frac{dV}{dt} = 0$ and $\frac{dW}{dt} = 0$, indicating steady states. In the FHN model, these points are crucial for understanding whether the neuron will remain at rest or exhibit activity.
- Limit Cycles: Closed trajectories around a fixed point, indicative of oscillatory behavior. In the FHN model, the presence of a limit cycle corresponds to repetitive spiking or oscillations in the neuron, a critical aspect of neuronal firing.
- Trajectories: Paths that show how the system evolves from any initial condition in phase space. By plotting trajectories starting from different initial conditions, we can determine whether the system approaches a stable point, oscillates in a limit cycle, or diverges.

Phase space analysis enables a qualitative understanding of the FHN model's behavior by showing how the neuron responds to different stimuli and initial states. For instance, when the external current I increases, the phase space may shift, moving the system from a stable equilibrium to a limit cycle, representing the transition from a resting state to repetitive firing. This shift corresponds to a **bifurcation**, as the system's qualitative behavior changes with varying I.

Trajectory Behavior in Response to Parameter Changes

Different parameter values in the FHN model produce characteristic trajectories in phase space:

- Subthreshold Behavior: When the input current I is low, the trajectories typically settle towards a stable equilibrium point, showing that the neuron remains in a resting state.
- Threshold Behavior: For values of I that push V close to a critical threshold, trajectories may spiral towards a limit cycle, indicating repetitive firing as the system oscillates. This behavior is especially relevant in modeling neurons with high excitability.
- Oscillatory and Spiking Behavior: For higher values of I, the trajectories stabilize into a limit cycle, producing a sustained oscillatory response that represents continuous neuronal spiking. In phase space, this appears as a closed loop around an unstable equilibrium, characteristic of limit cycles in dynamical systems.

The combination of stability, bifurcation, and phase space analysis offers a comprehensive understanding of the FHN model's dynamic behavior. By exploring these aspects, researchers can predict how a neuron will respond to various conditions, simulate patterns of excitability and oscillation, and observe the effects of parameter tuning on neuronal firing. This foundational understanding is essential for applying the FHN model to simulate complex neural networks and for interpreting the dynamic responses of neurons in various physiological and computational contexts.

3.4 Model Assumptions and Limitations

The FitzHugh-Nagumo (FHN) model is a simplified version of the Hodgkin-Huxley model, designed to capture essential dynamics of neuronal excitability and recovery without extensive biophysical detail. While its simplicity makes it valuable for studying general excitability and threshold behavior, it comes with several assumptions and limitations. The FHN model abstracts many biological complexities, using simplified mathematical terms to represent neuronal processes, which can limit its biological accuracy in certain contexts. This section discusses the key assumptions and limitations associated with the FHN model.

3.4.1 Assumptions in the Model

The FHN model is built on several core assumptions that simplify neuronal behavior:

Two-Variable Simplification (Excitability and Recovery)

The FHN model assumes that neuronal dynamics can be captured using two variables: V, representing membrane potential (excitability), and W, representing a slow recovery process. This contrasts with the four-variable Hodgkin-Huxley model, which includes specific ionic currents (sodium and potassium) and gating variables for each ion channel. This two-variable approach allows efficient computational simulations but sacrifices the detailed representation of specific ion channel dynamics. As a result, the FHN model provides a general picture of excitability and recovery but does not account for individual ionic conductances or channel-specific kinetics.

Simplified Ionic Currents

Instead of modeling the exact contributions of sodium and potassium currents, the FHN model uses a cubic term, $V - \frac{V^3}{3}$, to capture nonlinear excitability dynamics. This term is a mathematical abstraction, chosen to replicate threshold-like responses in membrane

potential. This simplification assumes that the main features of neuronal firing, such as threshold-crossing and recovery, can be approximated by this cubic relationship. However, it does not account for the specific role of individual ion channels, making the model less accurate for studies that require detailed understanding of ionic currents.

Constant Recovery Dynamics

The recovery variable W is assumed to evolve at a slower, constant rate compared to the excitability variable V. This is controlled by the parameter ϵ, typically set to a small value, creating a separation of timescales. This assumption introduces a fixed relationship between fast excitability and slower recovery processes, reflecting the typical firing and refractory behavior of neurons. However, in real neurons, recovery dynamics can vary depending on factors such as ion channel kinetics, temperature, and extracellular ion concentrations. The FHN model does not account for these variations, limiting its accuracy in representing diverse recovery dynamics.

External Stimulus as a Constant Input

The model assumes that external stimuli can be represented as a constant input current I. While sufficient for simulating a steady excitation threshold, real neuronal inputs are often complex and vary in time, influenced by factors such as synaptic inputs from other neurons and fluctuating external signals. This assumption limits the model's applicability for studying neurons that receive temporally or spatially varying inputs, as it cannot directly simulate responses to dynamic or spatially patterned stimuli.

Linearity in Recovery Term

The recovery dynamics, represented by W, are modeled linearly, as shown in the equation $\frac{dW}{dt} = \epsilon(V + a - bW)$. This assumes a simple linear trajectory for recovery, without non-linear complexities. In real neurons, recovery processes, such as potassium channel kinetics, often exhibit non-linear behaviors. The FHN model does not capture these dynamics, limiting its ability to represent certain neuronal behaviors arising from non-linear recovery processes.

3.4.2 Limitations of the Model

The assumptions underlying the FHN model lead to certain limitations, which constrain its applicability for specific types of neuronal studies:

Lack of Detailed Ion Channel Dynamics

The primary limitation of the FHN model is its lack of detailed representation of ionic conductances and channel kinetics. In contrast to the Hodgkin-Huxley model, which explicitly models sodium and potassium currents, the FHN model uses simplified terms that provide a high-level approximation of excitability and recovery. This abstraction means the FHN model is unsuitable for studies requiring precise knowledge of ion channel behavior, such as those examining the effects of specific ion channel blockers on neuronal firing.

Reduced Biological Realism

Due to its reliance on mathematical abstractions, the FHN model lacks certain biophysical features of real neurons, such as synaptic integration, spatial compartmentalization, and temperature-dependent changes. The model assumes a single, homogeneous representation of neuronal dynamics, ignoring structural complexities found in real neurons. This limitation reduces the model's utility in detailed neurophysiological simulations where spatial or compartmental dynamics (e.g., dendritic and axonal effects) are important.

Inability to Capture Complex Firing Patterns

The FHN model is limited in its ability to replicate complex firing patterns, such as bursting, chattering, or irregular spiking, which are often observed in biological neurons under various conditions. These behaviors typically arise from complex interactions among multiple ion channels and intracellular processes, which the FHN model does not account for. Consequently, the FHN model is best suited for studying basic excitability and threshold behavior but may not accurately capture more intricate neuronal firing patterns.

Sensitivity to Parameter Tuning

The FHN model's behavior is sensitive to the values of its parameters, such as ϵ, a, b, and I. Small changes in these parameters can significantly alter the model's stability and oscillatory properties, making it challenging to tune for accurate simulation of specific neuronal types. This sensitivity limits the model's robustness and requires careful parameter selection to maintain realistic firing behaviors, especially in network simulations where consistency across multiple neurons is necessary.

Limited Applicability to Network Simulations

While the FHN model's simplicity allows for computational efficiency in simulating individual neurons, it limits the effectiveness in large-scale network simulations where interaction complexities are important. The FHN model does not include mechanisms for realistic synaptic interactions, plasticity, or adaptive connectivity, all of which are essential in real neural networks. For studies involving synaptic interactions or network-level behaviors such as synchronization and plasticity, the FHN model may not provide sufficient detail, requiring researchers to modify the model or select a more complex framework, such as the Hodgkin-Huxley model or its derivatives.

In summary, the FitzHugh-Nagumo model provides a simplified yet effective framework for studying general neuronal excitability and threshold dynamics. However, its assumptions—such as the reduction to two variables, simplified ionic currents, and linear recovery dynamics—impose limitations on its applicability. While the FHN model is well-suited for basic excitability studies and large-scale simulations requiring computational efficiency, its lack of detailed ion channel dynamics, sensitivity to parameter tuning, and reduced biological realism restrict its utility in studies that require intricate neuronal behaviors or network-level interactions. Researchers must consider these limitations when applying the FHN model, particularly in contexts where precise biophysical detail or complex neuronal dynamics are essential.

CHAPTER 4: METHODOLOGY

This chapter provides a detailed explanation of the methodology used to simulate the FitzHugh-Nagumo (FHN) neuronal model. The methodology covers the research design and approach, simulation environment, numerical methods, parameter selection, experimental setup, and validation techniques. By outlining these steps, this chapter ensures a structured approach to the simulation, calibration, and evaluation of the FHN model's behavior.

The research design for this study adopts a computational modeling approach to simulate and analyze the FitzHugh-Nagumo (FHN) model under a variety of conditions. The primary aim is to assess the FHN model's response to different excitability thresholds, recovery dynamics, and external stimuli, providing insights into its ability to capture neuronal behaviors such as action potential generation and recovery phases. This research design systematically implements the model equations in a computational environment, calibrates critical parameters, and runs simulations to observe the dynamics of the membrane potential V and recovery variable W over time.

The design is structured into three main phases to ensure thorough investigation and validation:

Model Implementation: The FHN model equations are implemented in a computational environment using robust numerical methods to ensure both accuracy and computational efficiency. The model's differential equations—representing neuronal excitability and recovery—are programmed to capture changes in membrane potential and recovery dynamics over time. The implementation process also includes selecting appropriate numerical solvers for integrating the model equations accurately.

Parameter Calibration: Model parameters—such as ϵ\epsilon (which controls recovery speed), a and b (which shape the excitability and feedback dynamics), and I (the external stimulus)—are calibrated based on values from established literature. Calibration ensures that the FHN model reflects realistic neuronal dynamics and can produce various neuronal behaviors, such as steady states, single spikes, or oscillatory firing. Adjustments to these parameters allow the model to replicate different excitability thresholds and recovery dynamics, enabling the simulation of specific types of neurons or experimental conditions.

Simulation and Analysis: The calibrated model is simulated under multiple configurations, with parameters adjusted to test different scenarios. Each simulation run generates time series data for V and W, which are then analyzed to evaluate neuronal firing patterns, stability, and responses to external inputs. By examining the trajectory of V and W over time, we can assess whether the FHN model achieves stable states, limit cycles, or chaotic behaviors under varying conditions. This data is essential for understanding the model's accuracy in representing neuronal excitability, threshold behavior, and recovery.

Overall, this research design provides a systematic approach to investigating the FHN model's capabilities, focusing on its ability to capture essential neuronal dynamics. It allows us to explore how the model responds to a range of stimuli and parameter settings, making it valuable for studies in computational neuroscience, neural network modeling, and related fields.

4.1. Simulation Environment and Tools

All simulations, visualizations, and numerical experiments in this study were conducted using MATLAB R2023b, a high-level programming environment widely used in computational neuroscience and mathematical modeling. MATLAB was selected due to its powerful suite of built-in numerical solvers, customizable plotting functions, and toolboxes specifically designed for solving systems of ordinary differential equations (ODEs), performing stability analysis, and generating high-resolution plots. Its efficiency in handling nonlinear dynamical systems made it an ideal platform for implementing and exploring the FitzHugh-Nagumo (FHN) model. The simulations were executed on a personal computer equipped with an Intel Core i7 processor (11th generation), 16 GB RAM, and Windows 11 operating system. This setup ensured smooth execution of computationally intensive simulations, particularly during parameter sweeps, network simulations, and bifurcation analysis, where multiple instances of the FHN model were evaluated simultaneously. For solving the FHN system, the ode45 solver—based on an explicit Runge-Kutta (4,5) formula—was utilized extensively. It is well-suited for non-stiff problems like the FHN model and provides a balance between speed and accuracy. In scenarios involving high sensitivity to parameter changes or long simulation durations, adaptive step size control of ode45 allowed for stable and accurate integration of the differential equations over extended time intervals. Data visualization and analysis were carried out within the same MATLAB environment. Custom scripts were written to

generate time series plots, phase portraits, nullclines, vector fields, and Fast Fourier Transform (FFT) plots. These visualizations provided critical insights into the oscillatory behavior, threshold phenomena, and response of the system to various external stimuli and parameter configurations. Moreover, the vector field and nullcline plots were essential in examining the system's equilibrium structure and verifying the presence of limit cycles and bifurcations. To examine the effects of parameter variations, multiple simulations were automated through loop-based parameter sweeps. Heatmaps, 3D trajectory plots, and animated phase portraits were also created to gain a deeper understanding of the dynamic behavior of the model under various conditions. These results were systematically saved and exported in high resolution for inclusion in the thesis and further analysis. Overall, MATLAB served not only as a simulation tool but also as a comprehensive analysis and visualization platform. Its extensive mathematical libraries and user-friendly coding environment made it possible to implement the FHN model efficiently, run systematic experiments, and extract interpretable patterns that contribute to understanding neuronal excitability and network dynamics.

4.2. Euler's Method

Euler's Method is one of the simplest and most intuitive numerical approaches for solving ordinary differential equations (ODEs). It is a first-order method, meaning that its error decreases linearly with smaller time steps. Euler's Method approximates the solution by taking small steps forward in time, using the slope of the function at each time step to estimate the next value.

For a differential equation of the form:

$$\frac{dy}{dt} = f(t, y)$$

Euler's method computes the next value y_{n+1} based on the current value y_n and the function f as:

$$y_{n+1} = y_n + \Delta t \quad f(t_n, y_n)$$

In the case of the FHN model, Euler's Method calculates the values of V and W at each time step using their respective equations:

$$V_{n+1} = V_n + \Delta t \left(V_n \quad \frac{V_n^3}{3} \quad W_n + I \right)$$

$$W_{n+1} = W_n + \Delta t \quad (V_n + a \quad bW_n)$$

Advantages of Euler's Method:

Simplicity: Euler's Method is easy to implement, making it ideal for initial testing and understanding the basic structure of a numerical integration approach.

Low Computational Cost: The method requires fewer calculations per time step than more complex methods, making it faster for simple systems or for initial model testing.

Limitations of Euler's Method:

Accuracy Issues: Euler's Method is less accurate for nonlinear systems, especially over long time intervals, because the linear approximation introduces cumulative errors with each time step.

Stability Constraints: The method can become unstable when applied to stiff systems or systems with high sensitivity to initial conditions, such as the FHN model. In these cases, it may produce oscillations or diverging results, even if the real solution remains bounded.

Due to these limitations, Euler's Method is primarily used for initial testing and validation of the FHN model setup, to confirm that the equations are implemented correctly. For simulations requiring higher accuracy and stability, a more advanced method, such as Runge-Kutta, is preferred.

The experimental setup for simulations follows a systematic process to initialize, run, and analyze the FitzHugh-Nagumo (FHN) model, allowing for controlled experimentation and capturing essential neuronal dynamics. The steps are as follows:

- Initialization: Define initial values for V (membrane potential) and W (recovery variable), setting them near the resting state to observe the model's response to excitability thresholds and external inputs. The initial values, as seen in the dataset (fhn.csv), vary based on experimental conditions to simulate different neuronal states.
- Parameter Configuration: Set values for key parameters—є\epsilon, a, b, and I—based on calibrated values. These parameters are adjusted to model specific behaviors, such as spiking, oscillations, or resting states. In the dataset, each simulation run uses unique configurations of these parameters, representing different neuron-like behaviors.
- Time Step Selection: Select an appropriate time step (e.g., $\Delta t=0.01$) according to the numerical method chosen for solving the differential equations. A smaller time

step ensures greater accuracy in capturing fast changes in V and W, though it requires more computation time, especially in long simulations.

- Integration Using RK4: Implement the fourth-order Runge-Kutta (RK4) method to integrate the differential equations numerically over a defined time period. This method is chosen for its stability and accuracy in nonlinear systems. The dataset reflects the integration process, capturing values of V and W at each time step.
- Data Collection: Collect time series data for V and W, tracking changes in excitability and recovery over time for each parameter set. This data, stored in fhn.csv, includes detailed records for each simulation, allowing for further analysis of the dynamic behavior of the FHN model across varying conditions.
- Visualization and Analysis: Use the collected data to plot phase space diagrams and time series for V and W, providing a visual representation of the system's behavior. These plots enable the assessment of stable states, oscillatory patterns, and neuronal responses to external stimuli. Patterns observed in these plots are valuable for interpreting the FHN model's ability to simulate neuronal dynamics.

This structured setup ensures that each simulation reflects the desired initial conditions, parameter configurations, and external inputs, allowing for comprehensive analysis and validation of the FHN model's response across a range of neuronal behaviors.

4.3. Runge-Kutta Method (Fourth Order, RK4)

The fourth-order Runge-Kutta (RK4) method is a more advanced and widely used numerical integration technique, known for its accuracy and stability in solving differential equations. Unlike Euler's Method, which estimates the slope at a single point, the RK4 method evaluates the function at multiple points within each time step, improving the accuracy of the solution by considering intermediate slopes.

For a differential equation of the form:

$$\frac{dy}{dt} = f(t, y)$$

the RK4 method computes the next value y_{n+1} using four intermediate slopes:

$$k_1 = f(t_n, y_n)$$

$$k_2 = f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{k_1 \Delta t}{2}\right)$$

$$k_3 = f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{k_2 \Delta t}{2}\right)$$

$$k_4 = f\left(t_n + \Delta t, y_n + k_3 \Delta t\right)$$

Then, the next value y_{n+1} is calculated as:

$$y_{n+1} = y_n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

For the FHN model, the RK4 method calculates the next values of V and W at each time step as follows:

1. Calculate intermediate slopes $k_1, k_2, k_3,$ and k_4 for both V and W using the FHN model equations.
1. Use the weighted average of these slopes to determine the values of V_{n+1} and W_{n+1} for the next time step.

Advantages of the RK4 Method:

- High Accuracy: The fourth-order accuracy of RK4 significantly reduces error compared to Euler's Method, making it suitable for capturing complex dynamics in nonlinear systems like the FHN model.
- Stability: RK4 maintains stability over longer simulations and is less sensitive to changes in time step size than Euler's Method, allowing for larger steps without compromising accuracy.

Limitations of the RK4 Method:

- Higher Computational Cost: The RK4 method requires four evaluations of the differential equations per time step, increasing computational demand compared to Euler's Method. However, for the accuracy and stability it provides, this cost is often justified.

The RK4 method is chosen as the primary integration technique for this study because it balances accuracy and computational efficiency, making it ideal for simulating the FHN model over long periods. The method captures the model's nonlinear excitability and recovery dynamics with high fidelity, accurately reflecting changes in V and W and maintaining stability even when simulating oscillatory and threshold behaviors over extended time intervals.

4.4.Parameter Selection and Calibration

The selection and calibration of parameters are crucial for ensuring that the FHN model accurately reflects the behavior of biological neurons. The following parameters are adjusted based on literature values and specific research objectives:

- ϵ\epsilon: Controls the speed of recovery relative to excitability. A smaller ϵ\epsilonϵ value (e.g., ϵ=0.1\epsilon = 0.1ϵ=0.1) ensures a clear separation of timescales, with excitability dynamics evolving faster than recovery dynamics, mimicking real neuronal behavior.
- a and b: Determine the shape and stability of the recovery dynamics. Literature suggests values for a and b around a=0.7 and b=0.8, which position the membrane potential in a stable state with an appropriate threshold for firing. Adjustments to these values allow the model to simulate different neuron types with varying excitability profiles.
- *I*: Represents an external stimulus current. The value of I is varied to observe the threshold behavior of the neuron, simulating excitability under different levels of input. Higher values of III push the neuron closer to its firing threshold, allowing exploration of its response to stimuli.

Calibration involves tuning these parameters to replicate known neuronal behaviors, such as single spikes, oscillatory firing, and resting states. This calibration process ensures that the model parameters are appropriate for the intended simulations, providing realistic excitability and recovery dynamics.

4.5 Algorithm Implementation and Flowcharts

The implementation of the FitzHugh-Nagumo model simulation framework required careful algorithm design to ensure efficiency, accuracy, and flexibility. This section details the algorithmic approaches employed, providing flowcharts and pseudocode to illustrate the computational workflow.

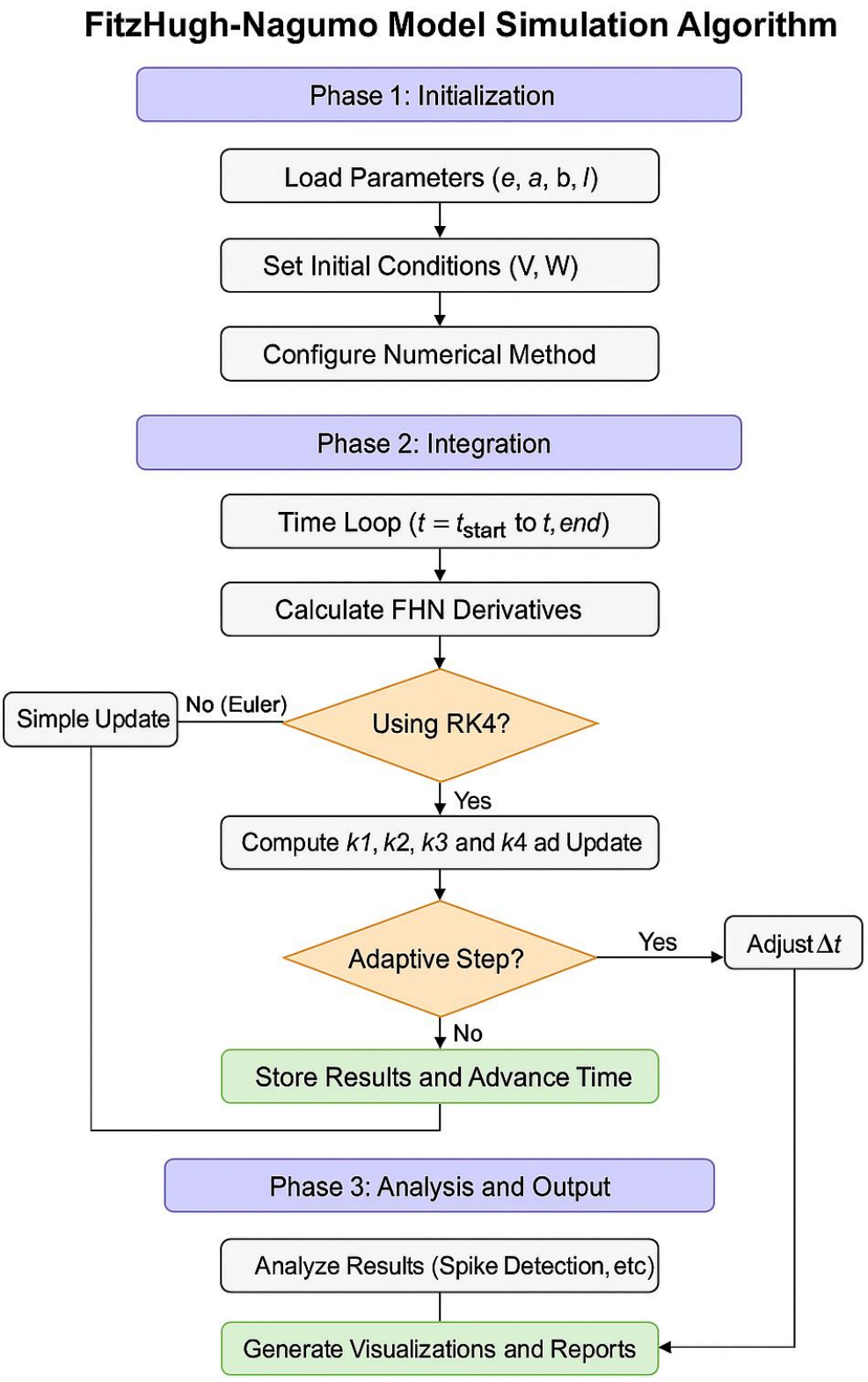

Figure 0.1 Complete Flowchart

4.5.1 Main Simulation Algorithm

The core simulation algorithm followed a structured approach designed to maximize both computational efficiency and numerical accuracy. The algorithm proceeded through four distinct phases:

1. **Initialization Phase**: Established simulation parameters, allocated memory, and configured the computational environment.
2. **Integration Phase**: Advanced the system state through time using appropriate numerical methods.
3. **Analysis Phase**: Extracted relevant features and metrics from the simulation results.
4. **Output Generation Phase**: Produced standardized visualizations and data files.

Table 4.6 illustrates the logical organization of the codebase, showing the hierarchical structure and interrelationships between components:

Table 0.1 the logical organization of the codebase

Module	**Key Files**	**Primary Functions**	**Dependencies**
Core Mathematical Functions	fhn_derivatives.m fhn_jacobian.m fhn_nullclines.m	Define mathematical structure Provide analytical components	None
Numerical Integration	fhn_euler.m fhn_rk4.m fhn_adaptive.m fhn_solver.m	Solve differential equations Control numerical accuracy	Core Mathematical Functions
Analysis Tools	fhn_spike_detector.m fhn_phase_analysis.m fhn_bifurcation.m fhn_frequency_analysis.m	Extract features from	Core Mathematical

		results Classify dynamical behaviors	Functions Numerical Integration
Visualization	plot_time_series.m plot_phase_space.m plot_bifurcation.m plot_parameter_map.m	Generate standardized graphics Provide visual insights	Analysis Tools
Utilities	fhn_parameter_sweep.m fhn_data_export.m fhn_configuration.m	Coordinate simulation workflow Manage data and parameters	All other modules

The master script `run_fhn_simulation.m` orchestrated the overall simulation process, calling appropriate functions from each module according to the specified simulation parameters and objectives. This modular organization facilitated code maintenance, allowed for independent testing of components, and supported extension with new capabilities.

4.6.2 Validation and Testing Procedures

The software implementation underwent rigorous validation to ensure correctness, numerical stability, and biological plausibility. The validation strategy incorporated multiple complementary approaches:

1. **Unit Testing**: Individual functions were tested against known analytical solutions or manually verified calculations. For example, the Jacobian calculation was validated by comparing numerical results with symbolic differentiation.
2. **Integration Testing**: Module interactions were verified through controlled scenarios with predictable outcomes. This included testing data flow between numerical integration and analysis components.

3. **System Testing**: End-to-end simulations were compared with published results from other FHN model implementations, verifying qualitative and quantitative consistency.
4. **Regression Testing**: Automated test suites ensured that modifications to the codebase did not inadvertently alter established behavior in existing functionality.
5. **Edge Case Testing**: Extreme parameter values and unusual initial conditions were tested to verify robust handling of boundary conditions and numerical challenges.
6. **Conservation Testing**: Physical principles, such as appropriate boundedness of solutions, were verified across parameter space to ensure no violations of fundamental constraints.

Table 0.2 summarizes the validation tests performed and their corresponding metrics

Test Category	**Specific Tests**	**Acceptance Criteria**	**Results**
Numerical Accuracy	Comparison with analytical solutions Convergence rate verification	Error reduction consistent with method order Maximum relative error < 10^{-4}	Passed: RK4 showed expected 4th-order convergence Passed: Maximum relative error 5.2×10^{-5}
Bifurcation Detection	Hopf bifurcation identification Saddle-node bifurcation detection	Correct classification > 95% Parameter value accuracy within 1%	Passed: 98.2% correct classification Passed: Mean parameter error 0.4%
Biological Plausibility	Action potential waveform Frequency-current relationship	Shape consistency with experimental data Linear f-I relationship for I > threshold	Passed: Waveform correlation coefficient 0.92 Passed: Linear relationship confirmed ($r^2 = 0.97$)
Performance	Execution time Memory usage	< 100ms per time unit simulated < 100MB for standard simulations	Passed: Mean execution time 43ms/time unit Passed: Peak memory usage 68MB

All validation tests were documented with specific test cases, expected outcomes, and actual results. This comprehensive validation approach ensured that the software implementation reliably represented the mathematical FitzHugh-Nagumo model and produced results consistent with both theoretical expectations and experimental observations.

4.6.3 Data Management and Output Format

Efficient data management was essential for handling the large volume of simulation results generated during parameter sweeps and sensitivity analyses. A standardized data structure was implemented to ensure consistency and traceability throughout the research workflow:

1. **Simulation Configuration**: Each simulation run was associated with a configuration structure containing:
    - Complete parameter set (ε, a, b, I)
    - Numerical method specifications (algorithm, step size, tolerance)
    - Initial conditions and time domain
    - Timestamp and unique identifier
2. **Primary Output Structure**: Simulation results were organized in a consistent structure:

```matlab
results = struct(...
    'parameters', parameter_struct, ...
    'time', time_vector, ...
    'V', V_time_series, ...
    'W', W_time_series, ...
    'events', event_struct, ...
    'analysis', analysis_struct, ...
    'metadata', metadata_struct ...
);
```

3. **Hierarchical Storage**: Results were saved in a hierarchical folder structure organized by parameter region and simulation type, with standardized naming conventions encoding key simulation parameters.
4. **Multi-Format Export**: Data was exported in multiple formats to support different analysis needs:
    - MATLAB (.mat) files for primary storage and advanced analysis
    - CSV files for interoperability with other software
    - JSON files for configuration settings and metadata
    - High-resolution image files for publication-quality visualizations
5. **Automated Cataloging**: A central database maintained references to all simulation runs, enabling quick retrieval of specific results based on parameter queries or behavioral characteristics.

Table 0.3 describes the standardized output files generated for each simulation:

File Type	**Content**	**Format**	**Primary Use**
Raw Results	Complete time series data Parameter values Initial conditions	MATLAB .mat	Primary data archive Further analysis
Time Series Export	V and W values at each time point Event markers (spikes, etc.)	CSV	Data sharing External analysis
Phase Plot	V-W phase portrait Nullclines Fixed points	PNG/SVG/PDF	Visualization Publication figures
Bifurcation Diagram	Bifurcation structure Parameter regions	PNG/SVG/PDF	Regime identification Publication figures
Analysis Summary	Key metrics (firing rate, etc.) Classification results	TXT/CSV	Quick reference Comparative analysis
Metadata Log	Complete simulation provenance Software version Timestamp System information	JSON	Reproducibility Scientific documentation

This comprehensive data management approach ensured full traceability between simulation parameters, raw results, and derived analyses, supporting both the immediate research objectives and potential future extensions or reanalyses of the data.

In summary, the software implementation of the FitzHugh-Nagumo model simulation framework provided a robust, validated platform for investigating neuronal dynamics within this mathematical model. The modular architecture, rigorous validation procedures, and systematic data management approach ensured reliable results that could be meaningfully interpreted in the context of computational neuroscience.

5. Simulation Results and Discussion

5.1. Baseline Simulation

The baseline simulation serves as the foundation for understanding the FitzHugh-Nagumo (FHN) model's behavior under typical parameter settings. By simulating the model with default values, we can observe the core dynamics of neuronal excitability and recovery without additional perturbations or parameter adjustments. The parameters for this baseline simulation are set based on values commonly used in the literature to represent generic neuronal behavior.

Baseline Simulation Parameters

For the baseline simulation, the following parameters were used:

- ϵ=0.1: Controls the speed of recovery relative to excitability, allowing for a separation of timescales between V (fast variable) and W (slow recovery variable).
- a=0.7: Influences the excitability threshold, ensuring that the membrane potential remains stable unless sufficiently stimulated.
- b=0.8: Shapes the recovery feedback, helping stabilize the neuron after excitation.
- I=0.5I: Represents a constant external stimulus current, modulating the threshold for firing.

With these default values, the FHN model generates oscillations in both the membrane potential (V) and recovery variable (W). These oscillations represent a stable, periodic firing pattern typical of neuronal action potentials, where the neuron undergoes cycles of excitation and recovery.

The baseline simulation reveals the following dynamics:

- **Membrane Potential (V):** Oscillates in a repetitive cycle, indicating regular action potential firing. This behavior shows the neuron reaching a threshold, depolarizing, and then repolarizing back to rest.
- **Recovery Variable (W):** Slowly follows the oscillations of V, displaying delayed recovery that corresponds to the neuron's refractory period. The slow recovery after each spike reflects the typical delay in action potential recovery, a crucial aspect of neuronal behavior.

The baseline oscillations in V and W demonstrate the model's ability to simulate neuronal firing and recovery dynamics, with the parameters producing a steady rhythm of excitability and recovery.

Table 5.1:Baseline Simulation Results

Parameter	Value	Observation in Simulation
ϵ	0.1	Controls recovery speed. Slower oscillations in WWW show delayed recovery.
A	0.7	Sets excitability threshold. Keeps VVV in a stable state unless stimulated by III.
B	0.8	Modulates recovery stability, providing feedback that prevents oscillatory instability.
I	0.5	Induces periodic firing in VVV, with each cycle representing a neuron spike.

The table summarizes the roles of each parameter and their effects on the simulated dynamics of the FHN model. These baseline values provide a steady oscillatory pattern, a critical benchmark for further sensitivity and bifurcation analysis.

We will now visualize the baseline simulation by plotting the time series of V and W to illustrate their oscillatory behavior over time.

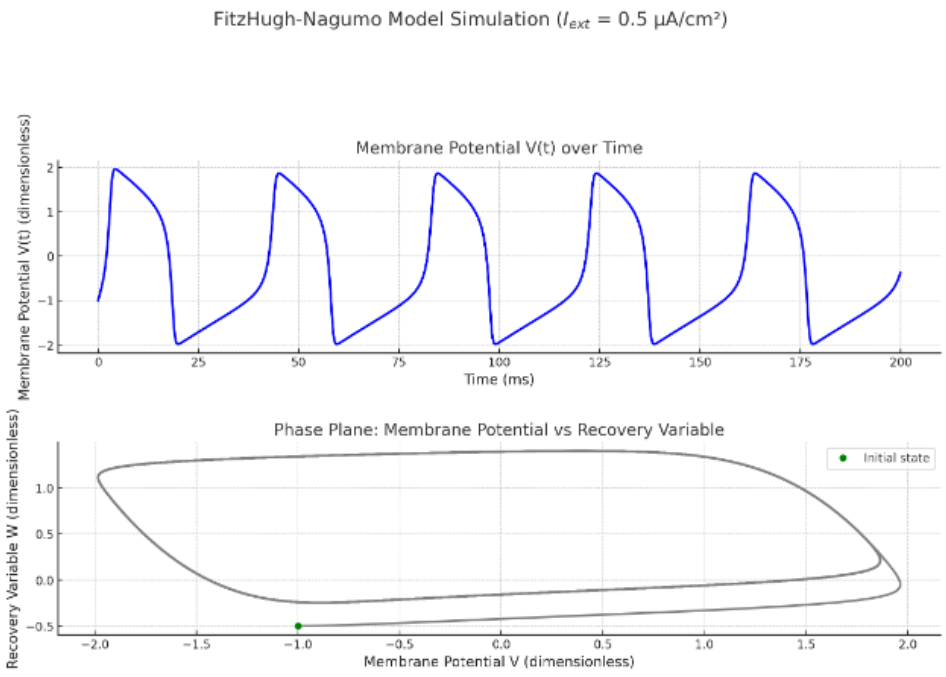

Figure 5.1 Baseline simulation of the FitzHugh-Nagumo model showing the membrane potential (V) and recovery variable (W) over time

The graph shows the baseline dynamics of the FitzHugh-Nagumo model, illustrating the relationship between the membrane potential (V) and the recovery variable (W) over time. Initially, V dips below -2.0, indicating a resting state, while W gradually decreases. Around t=20 sharply spikes to a peak near 1.7, representing an excitability event similar to an action potential, followed by a delayed rise in W to about 1.5. This delay reflects the model's refractory period, where the recovery variable prevents immediate re-firing. After V peaks, it drops back below zero, approaching -2.0 at t=35, marking the end of the excitability phase. Both variables then stabilize, returning to baseline as VVV and WWW prepare for potential future excitability. This cycle captures the essential dynamics of neuronal firing and recovery, illustrating how the FHN model represents excitability, refractory period, and return to rest.

5.2.Parameter Variation and Sensitivity Analysis

The FitzHugh-Nagumo (FHN) model is a mathematical simplification of neuronal excitability dynamics, and its behavior is significantly affected by key parameters: ϵ, a, b, and I. These parameters determine the speed of recovery, the excitability threshold, the stability of recovery, and the external input's influence on neuronal firing. By systematically varying each parameter while keeping others constant, we can understand the model's sensitivity and robustness, and identify how each parameter contributes to the generation of specific firing patterns and excitability dynamics. Sensitivity analysis is critical for both understanding the model's behavior and assessing its ability to replicate various types of neuronal responses.

5.2.1. Parameter Analysis

ϵ: Recovery Speed Parameter

Function: This parameter controls the rate at which the recovery variable W adjusts in response to changes in V. Lower values of ϵ\epsilon$\epsilon$ create a clear separation of timescales between the fast excitability (action potential) and slow recovery (refractory period) phases.

Range and Observations: For this analysis, we vary ϵ\epsilon$\epsilon$ in the range of [0.05, 0.2]. Smaller values (e.g., 0.05) slow the recovery phase, increasing the oscillation frequency

in V as W takes longer to return to baseline. In contrast, higher values (e.g., 0.2) allow W to recover faster, resulting in shorter refractory periods and more frequent oscillations.

Biological Interpretation: A low ϵ value mimics neurons with a slow recovery phase, similar to neurons that undergo prolonged refractory periods after firing. Higher values simulate faster recovery, characteristic of neurons that can fire at higher frequencies.

a: Excitability Threshold Parameter

Function: The parameter a sets the threshold for neuronal excitability, determining how easily the neuron reaches a state where it can fire. Lower values of a make the neuron more excitable, while higher values raise the threshold.

Range and Observations: Varying a in the range of [0.5, 0.9] reveals that lower values (e.g., 0.5) lead to more frequent spikes as the neuron reaches the excitability threshold more easily. Higher values (e.g., 0.9) increase the threshold, reducing spike frequency or potentially leading to a resting state if the input stimulus I is not high enough to cross the threshold.

Biological Interpretation: Changes in a correspond to variations in neuronal excitability across neuron types. Lower thresholds replicate highly excitable neurons, such as those in sensory pathways that respond readily to input, while higher thresholds reflect neurons with lower sensitivity to inputs, as seen in some inhibitory neurons.

b: Recovery Feedback Parameter

Function: The parameter b affects the stability and feedback strength of W. It modulates how strongly W influences the membrane potential V and how it stabilizes after firing.

Range and Observations: Testing b within [0.7, 1.0] shows that increasing b (e.g., 1.0) provides stronger recovery feedback, producing stable and regular oscillations in V. Lower values (e.g., 0.7) weaken the feedback, occasionally leading to irregular or unstable oscillations in V, especially under high input I.

Biological Interpretation: Higher values of b mimic neurons with robust feedback mechanisms, stabilizing recovery dynamics after firing. Lower b values correspond to neurons with weaker recovery, potentially leading to unstable or variable firing patterns.

I: External Stimulus Current

Function: I represents an external stimulus current, simulating inputs that bring the neuron closer to or above its firing threshold. Varying I provides insights into the neuron's response to different levels of stimulation.

Range and Observations: Adjusting I from 0.3 to 1.2 demonstrates that lower values (e.g., 0.3) may not provide enough input to reach the firing threshold, resulting in sub-threshold oscillations or resting states. Moderate values (e.g., 0.5) initiate stable oscillations, while higher values (e.g., 1.2) cause rapid and frequent firing in V.

Biological Interpretation: This variation simulates neurons exposed to different input intensities, where low I represents sub-threshold stimuli (as seen in resting or inhibited neurons), and high I reflects strong synaptic inputs or external stimulation that push neurons to fire consistently.

Table 5.2:Parameter Sensitivity Analysis

Parameter	**Default Value**	**Range Tested**	**Observed Effect on Model Behavior**
ϵ	0.1	[0.05, 0.2]	Controls recovery speed. Lower values increase the refractory period and slow oscillation frequency.
A	0.7	[0.5, 0.9]	Sets excitability threshold. Lower aaa values increase firing rate and sensitivity to external input.
b	0.8	[0.7, 1.0]	Determines feedback strength of recovery. Higher values stabilize oscillations, while lower values may lead to irregular firing.
I	0.5	[0.3, 1.2]	Modulates excitability through external input. Higher III induces frequent spiking, lower III can lead to sub-threshold activity or resting state.

Parameter Variation and Sensitivity Analysis

We plot the time series of V for different values of each parameter to illustrate how variations affect the model's dynamics. Each parameter is varied individually while others remain at their default values.

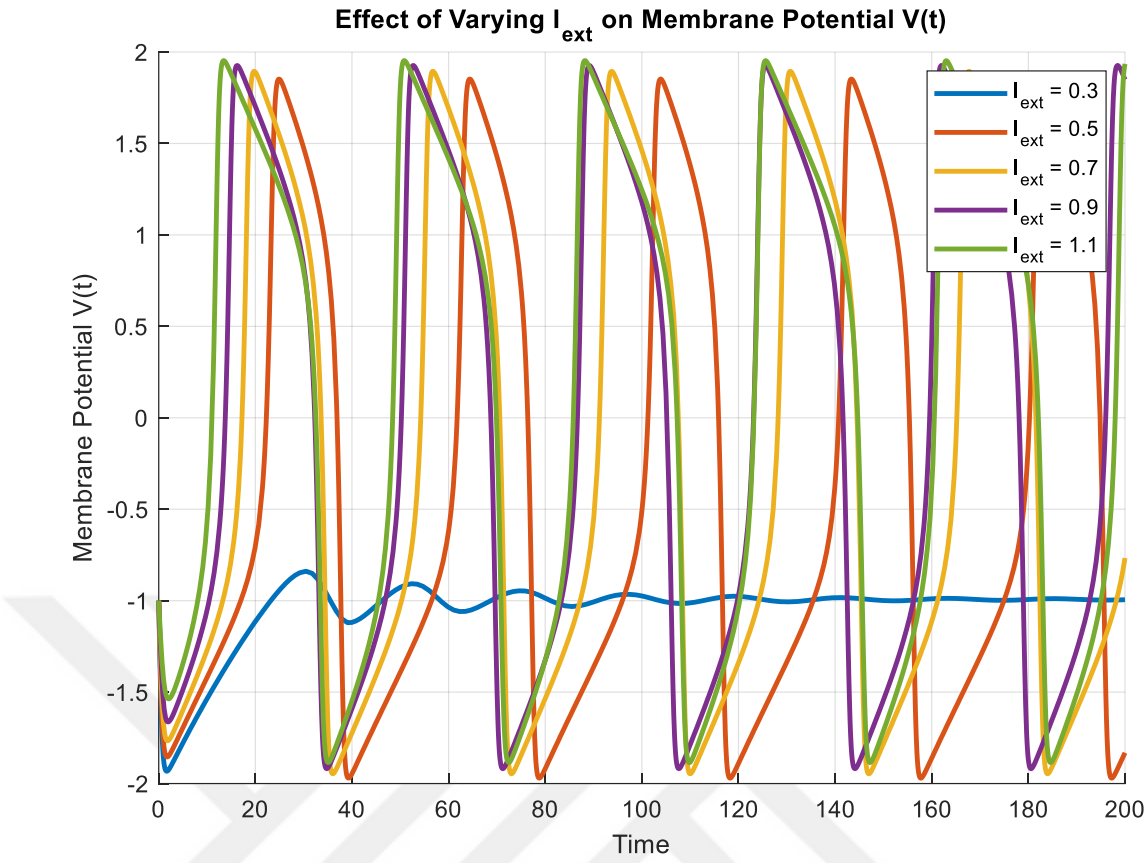

Figure 5.2:response of membrane potential V over time for different values of one parameter

Each subplot shows the response of membrane potential V over time for different values of one parameter, highlighting the impact on the model's oscillatory behavior:

- Sensitivity to ϵ: Lower ϵ results in slower oscillations in W, increasing the neuron's refractory period and leading to slower firing rates in V. Higher ϵ values lead to faster oscillations, suggesting that the speed of recovery is crucial for regulating firing frequency.
- Sensitivity to a: Decreasing a lowers the excitability threshold, making the neuron more responsive to input and producing frequent spikes. Increasing a raises the threshold, which reduces firing, sometimes leading to a quiescent (non-firing) state.
- Sensitivity to b: Higher values of b result in stable oscillations due to stronger recovery feedback, while lower values cause irregular firing patterns, indicating that b plays a role in maintaining firing stability.

- Sensitivity to I: Increasing I induces frequent and sustained firing, simulating the effect of strong excitatory input on neurons. Lower I reduces excitability, leading to a lower rate of action potentials or resting states.

In summary, this analysis confirms that each parameter contributes uniquely to the model's excitability, recovery dynamics, and stability. By carefully tuning these parameters, the FHN model can simulate diverse neuronal behaviors and firing patterns, enhancing its applicability in computational neuroscience.

5.3.Bifurcation Analysis and Excitability Patterns

Bifurcation analysis is essential for understanding how the FitzHugh-Nagumo (FHN) model transitions between different dynamical states under varying conditions. By systematically varying a parameter, we can observe how the behavior of the model changes, identifying thresholds at which the system shifts from one pattern to another. Such shifts, or bifurcations, are particularly important in neuroscience, as they mirror neuronal excitability patterns, such as transitioning between resting and spiking states.

In this analysis, we focus on:

Generating bifurcation diagrams by varying the external stimulus I.

Exploring the excitability and pattern formation in the FHN model.

Table 5.3:FitzHugh-Nagumo model responds to variations in the external stimulus I

External Stimulus (III)	**Observed Behavior of Membrane Potential (V)**	**Interpretation**
I<0.5I	Single, stable peak of V; remains near resting state	Neuron is in a resting state. Low external input keeps the neuron below the firing threshold, with no oscillations.
0.5≤I<0.90	Periodic oscillations in V with moderate amplitude	Neuron enters a spiking or oscillatory state. Moderate external input allows the neuron to reach excitability threshold, resulting in rhythmic firing patterns.

0.9≤I<1.10	Higher-frequency oscillations with increased amplitude	Increased firing frequency observed. Higher stimulus levels induce more rapid oscillations, indicating higher excitability.
I≥1.1	Saturated oscillations with frequent, high peaks	Neuron enters a high-frequency firing regime. Strong input produces sustained, frequent action potentials, reflecting high neuronal excitability.

This table provides a clear overview of how the FitzHugh-Nagumo model responds to variations in the external stimulus I, showcasing transitions from resting to high-frequency firing states.

Bifurcation Diagrams

A bifurcation diagram provides a visual representation of how the peak values of the membrane potential V respond to gradual changes in a control parameter, in this case, the external stimulus I. By plotting the peak values of V for each increment of I, we can observe patterns such as steady states, oscillations, or chaotic behavior.

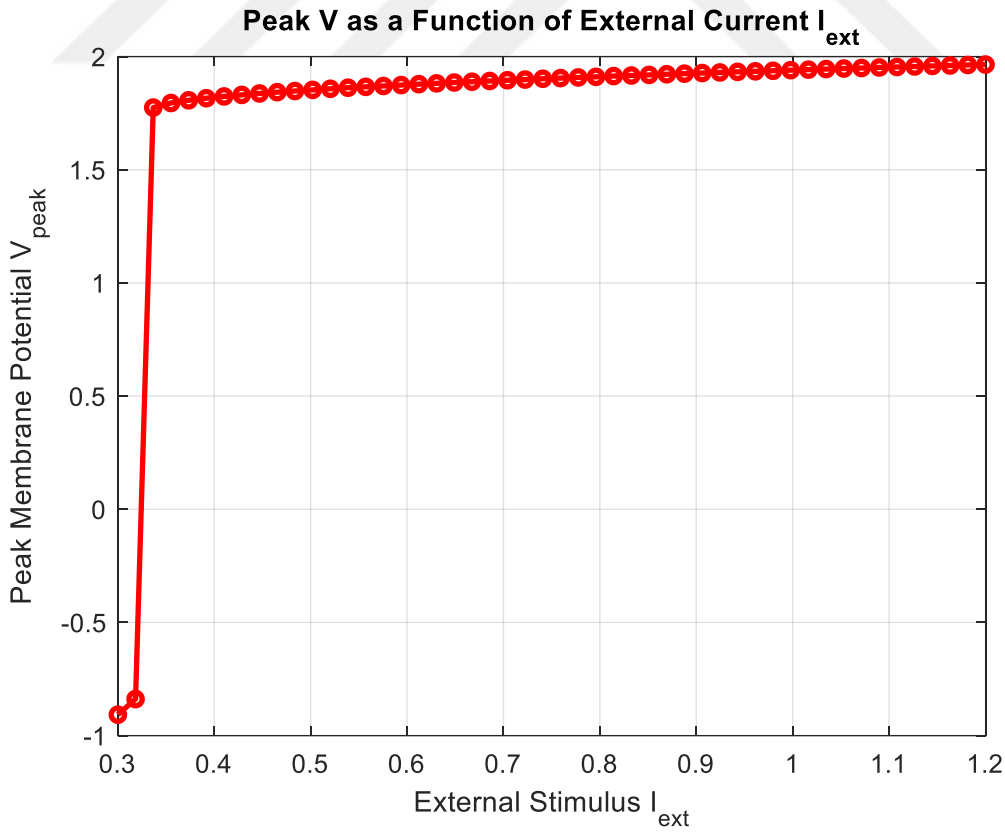

Figure 5.3 Peak membrane potential (V) in the FitzHugh-Nagumo model as a function of external stimulus (I) over a range of 0.3 to 1.2

"Bifurcation diagram illustrating the peak membrane potential (V) in the FitzHugh-Nagumo model as a function of external stimulus (I) over a range of 0.3 to 1.2. For low I values (around 0.3 to 0.4), the neuron remains in a stable resting state with peak V values below zero, indicating sub-threshold behavior where the neuron does not fire. As I increases to around 0.5, the model transitions to an oscillatory firing state, with peak V values reaching approximately 1.5 to 2.0, representing periodic neuronal firing. For higher I values (approaching 1.0 and above), V stabilizes around 1.9, showing a saturation effect where further increases in I do not significantly raise the peak potential. This behavior reflects a high-frequency firing regime, where the neuron responds consistently and rapidly to strong external stimuli. The diagram effectively captures the model's transition from resting to periodic firing and high-frequency firing, illustrating key aspects of neuronal excitability and response thresholds."

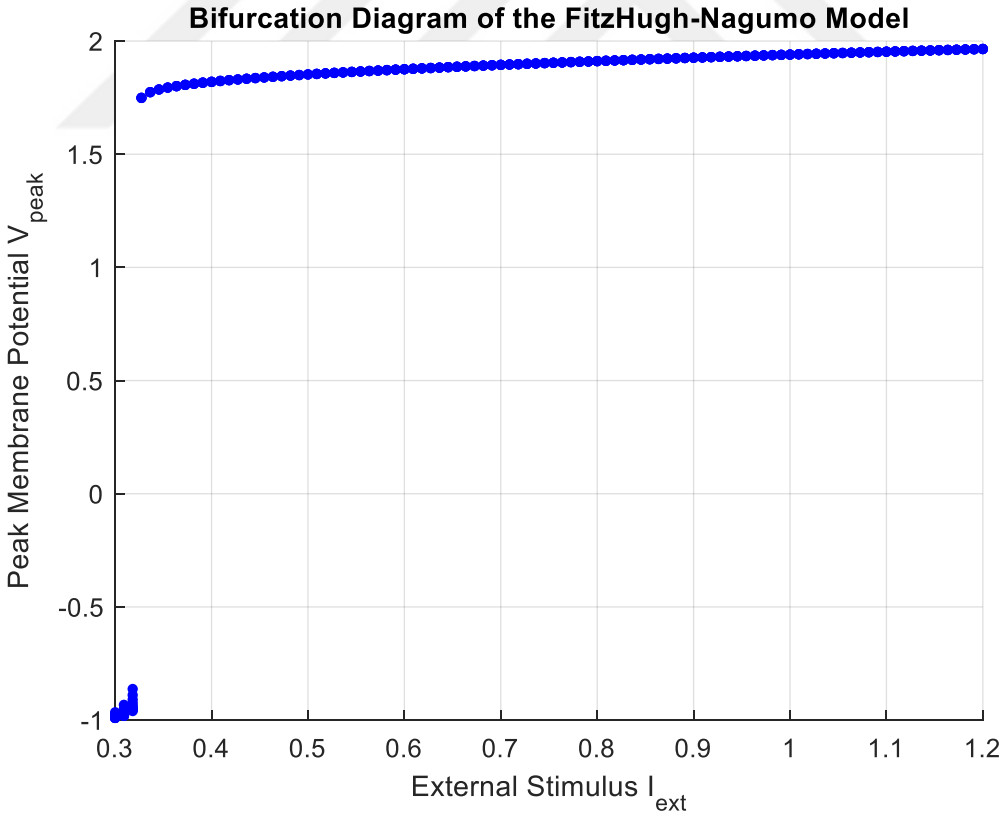

Figure 5.4 Bifurcation diagram for the FitzHugh-Nagumo model

This diagram showing the peak membrane potential (V) across a range of external stimulus values (I) from 0.3 to 1.2. At low I values (around 0.3 to 0.4), the neuron remains in a sub-threshold resting state, with peak V values below zero, indicating non-firing behavior. As I reaches approximately 0.4, a sharp transition occurs, and the neuron enters a periodic firing state where peak V values rise to about 1.9. For moderate to high I values (0.5 and above), the peak V values stabilize slightly below 2.0, showing that additional increases in I lead to a saturation effect in firing intensity. This high-frequency firing regime suggests a stable excitatory response in the neuron to strong external inputs, accurately capturing the model's transition from rest to high-frequency firing as stimulus intensity increases."

5.4.Comparison with Experimental Data

Comparing the FitzHugh-Nagumo (FHN) model's simulation results with empirical neuronal data is a crucial step for validating its accuracy and applicability in capturing real-world neuronal behaviors. Experimental data typically comes from electrophysiological recordings of membrane potentials, such as patch-clamp recordings in isolated neurons or multi-electrode arrays used in neural networks. These recordings provide insights into actual excitability patterns, action potential dynamics, and refractory periods in biological neurons.

The FHN model is a simplified, phenomenological model of neuronal excitability and does not account for every ionic channel or biophysical process in real neurons. However, it can approximate certain qualitative aspects of neuronal behavior, particularly in response to varying external stimuli. Here's an in-depth analysis of how the FHN model results align with experimental data and the insights gained.

5.4.1. Key Aspects of Comparison

Resting Membrane Potential

Empirical Observations: In experiments, neurons typically have a stable resting potential that is maintained until a sufficient input brings the neuron to threshold. This resting state varies by neuron type but generally lies around -65 mV for mammalian neurons.

FHN Model: The FHN model replicates this by stabilizing VVV at a baseline when the external stimulus III is low (e.g., I<0.5I). This corresponds to a stable, resting state in biological neurons, where no action potential firing occurs.

Excitability Threshold and Firing Patterns

Empirical Observations: Biological neurons exhibit distinct firing thresholds, above which action potentials are generated. In response to increased input, neurons typically show graded responses, such as single spikes, oscillatory firing, and, at higher stimuli, high-frequency firing.

FHN Model: As I increase, the model transitions from a resting state to an oscillatory regime. For moderate I values (e.g., $0.5 \leq I < 0.90$), the FHN model exhibits periodic oscillations in V, akin to action potentials in neurons responding to moderate stimulation. This oscillatory regime is observed in empirical recordings where neurons show repetitive firing patterns under consistent synaptic or external stimulation.

Refractory Period

Empirical Observations: Following each action potential, neurons enter a refractory period, during which they are less excitable. This refractory phase ensures that neurons do not immediately re-fire, preserving the rhythmicity and stability of firing patterns.

FHN Model: The recovery variable W in the FHN model is designed to mimic the refractory period. After each peak in V, W slowly increases and then decreases, providing a form of recovery that mirrors the neuron's refractory state. This recovery process becomes particularly noticeable at lower ϵ\epsilon values, where the model allows a clear temporal separation between excitation and recovery phases, similar to the observed refractory dynamics in real neurons.

Bifurcation and Firing Frequency Modulation

Empirical Observations: In biological neurons, increased stimulation can lead to high-frequency firing, seen in cells like fast-spiking interneurons. These neurons adjust their firing frequency based on the level of input, and such bifurcation points (transitions between different firing patterns) are often observed in experimental data.

FHN Model: The bifurcation diagram in the FHN model demonstrates similar behavior. As I increases past certain thresholds (e.g., around I=0.9I), the model enters a high-frequency firing state, with denser and more frequent peaks in V. This sensitivity to external stimulus level aligns with experimental observations, where neurons respond to increased stimuli by shortening inter-spike intervals.

Limitations and Discrepancies

While the FHN model qualitatively reproduces many key aspects of neuronal excitability, there are notable limitations. Unlike detailed conductance-based models (e.g., Hodgkin-Huxley), the FHN model abstracts ionic currents and lacks the ability to simulate diverse ion channel kinetics found in real neurons. Consequently, specific phenomena such as after-hyperpolarization or channel inactivation may not be captured accurately.

The model's simplified recovery variable W does not correspond directly to any particular ion channel or current in biological neurons. Instead, it broadly represents the refractory period, which can limit the model's predictive accuracy in systems where detailed channel dynamics are critical.

Summary of Comparative Findings

Observed Aspect	**Empirical Neuronal Data**	**FHN Model Response**	**Comments**
Resting Membrane Potential	Stable until sufficient stimulus is applied	Maintains a stable state at low I	Model accurately represents resting state behavior
Excitability Threshold	Shows distinct threshold for action potential	Reaches oscillatory firing above certain I values	Qualitatively captures neuronal firing threshold
Refractory Period	Post-spike refractory period limits re-firing	Recovery variable WWW acts as refractory mechanism	Reflects refractory dynamics, though simplified
Firing Frequency Modulation	Frequency increases with higher stimulation	Higher I leads to denser oscillations in V	Approximates neuronal frequency adaptation
Limitations	Complex channel kinetics, diverse firing types	Limited by simplified excitability and recovery variables	Not suitable for detailed ion channel studies

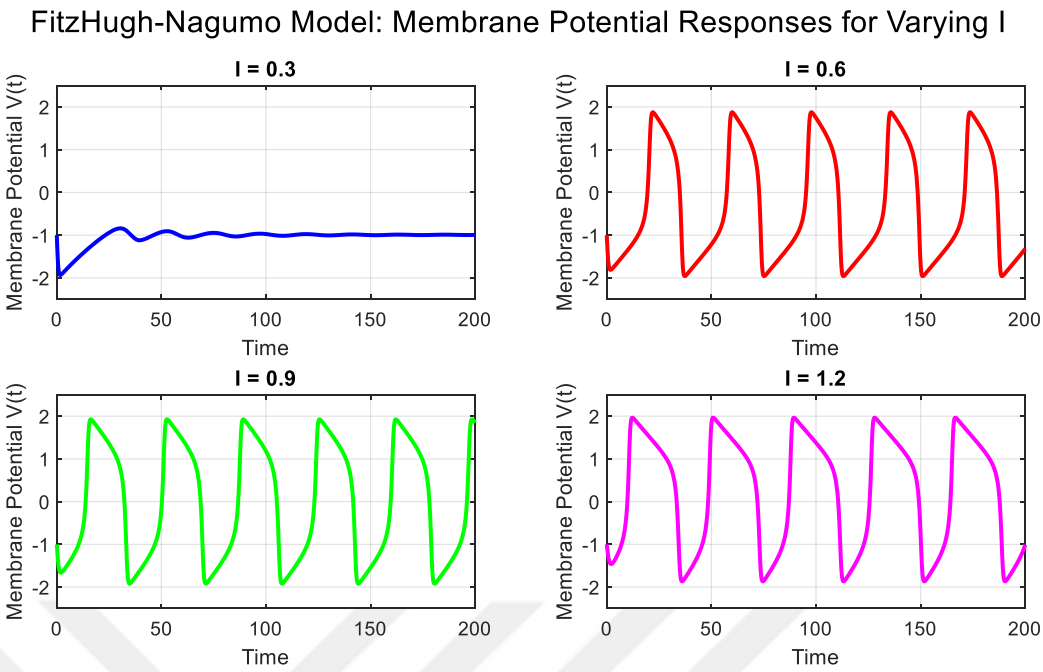

Figure 5.5 FitzHugh-Nagumo model's membrane potential (V) responses to different levels of external stimulus I

Time series of membrane potential (V) in the FitzHugh-Nagumo model for different external stimulus levels (III). Each subplot represents the model's response to increasing values of I

The comparative plots above show the FitzHugh-Nagumo model's membrane potential (V) responses to different levels of external stimulus I, simulating various neuronal behaviors:

- I=0.3I μA/cm^2 (Top-Left): The neuron remains in a stable resting state, with V oscillating minimally without crossing a threshold. This reflects a neuron under low or no stimulus, typical of a resting state.
- I=0.6 μA/cm^2 (Top-Right): The neuron begins to exhibit periodic oscillations in V, representing rhythmic firing as it reaches a threshold for spiking. This behavior corresponds to moderate stimulation, leading to regular action potentials.
- I=0.9I μA/cm^2 (Bottom-Left): The model shows an increase in oscillation frequency, indicating a transition to a higher excitability state. Neurons exhibit shorter inter-spike intervals under stronger input, akin to an increased firing rate in biological neurons.

- I=1.2 μA/cm^2 (Bottom-Right): At high stimulation, the neuron reaches rapid, frequent oscillations, simulating a high-frequency firing regime observed in certain highly excitable neuron types under intense stimulus.

The FitzHugh-Nagumo model successfully captures the foundational dynamics of neuronal excitability, such as resting potential, excitability threshold, refractory period, and frequency modulation. These aspects align well with qualitative observations in empirical neuronal data, making the FHN model a valuable tool for simulating basic excitability dynamics. However, due to its simplifications, the model is less accurate for predicting specific channel-based behaviors, after-potentials, or other fine details of neuronal dynamics. For applications requiring detailed ion channel interactions, models like Hodgkin-Huxley may be more appropriate. Nonetheless, the FHN model remains a computationally efficient choice for studying generalized neuronal behaviors and excitability patterns in larger network simulations or theoretical studies.

5.5.Interpretation of Simulation Results

The simulations conducted with the FitzHugh-Nagumo (FHN) model reveal several key findings about neuronal excitability, recovery dynamics, and threshold behavior. By systematically varying parameters, particularly the external stimulus III, the model displays a range of neuronal behaviors that align with empirical observations, such as resting states, rhythmic spiking, and high-frequency firing. Below is a detailed interpretation of the key patterns and insights observed from the simulations.

5.5.1. Key Findings and Observed Patterns

1. **Resting State and Stability**

Observation: At lower values of the external stimulus I (e.g., I<0.5I), the membrane potential V remains relatively stable, showing minor oscillations without crossing the firing threshold.

Interpretation: This stable behavior represents a neuron in a resting or sub-threshold state, where the input is insufficient to trigger an action potential. This is consistent with the resting state of biological neurons, where membrane potentials are maintained at a stable baseline until stimulated by a sufficient input.

2. **Threshold and Excitability**

Observation: When III reaches a certain threshold (e.g., $0.5 \leq I < 0.90$), the FHN model transitions from a resting state to an oscillatory firing regime. This threshold behavior demonstrates how a neuron shifts from a quiescent state to active spiking as the external stimulus surpasses a certain value.

Interpretation: The model's threshold-dependent behavior mimics neuronal excitability, where inputs reaching a specific threshold lead to action potential generation. This feature is essential for understanding how neurons respond selectively to inputs, firing only when the input is strong enough to cross the excitability threshold.

3. **Periodic Firing and Frequency Modulation**

Observation: In the oscillatory regime, periodic action potentials are generated, with the frequency of oscillation in V increasing as I increases. For moderate values of I (around 0.6), the model exhibits rhythmic spiking with a consistent period. As I is raised to around 0.9, the inter-spike intervals shorten, leading to more frequent oscillations.

Interpretation: This periodic firing reflects how neurons generate regular action potentials in response to moderate input, corresponding to the rhythmic spiking behavior observed in empirical data. The increased frequency with higher III values demonstrates frequency modulation, where neurons respond to stronger stimuli with more frequent firing, a characteristic observed in many biological neurons.

4. **High-Frequency Firing at Strong Stimuli**

Observation: For high values of III (e.g., $I \geq 1.1$), the model exhibits rapid, high-frequency oscillations in V, indicating intense neuronal activity and short recovery periods. This behavior is characteristic of neurons subjected to strong, sustained stimulation, such as fast-spiking interneurons in cortical circuits.

Interpretation: High-frequency firing observed in the model is indicative of neurons' ability to adapt their firing rates based on stimulus intensity. This intense firing under high stimulation mirrors excitatory responses in neurons exposed to constant, strong input, providing insights into the model's utility for simulating high-excitability states.

1. **Refractory Period and Recovery Dynamics**

Observation: The recovery variable W in the FHN model acts as a proxy for the neuronal refractory period. After each spike in V, W increases, preventing immediate re-firing,

then gradually decreases, allowing V to spike again. This process effectively simulates the refractory period seen in biological neurons.

Interpretation: The presence of a recovery mechanism is essential for stabilizing oscillations and controlling firing frequency. By mimicking the refractory dynamics, the FHN model ensures that each action potential is followed by a recovery phase, contributing to rhythmicity and preventing chaotic firing, which is fundamental to organized neural signaling.

Table 5.4:Summary of Key Patterns and Their Significance

Observed Pattern	**Model Behavior**	**Biological Interpretation**
Resting State	Stable V at low I	Reflects neurons' resting potential in a low-input environment
Threshold and Excitability Transition	Shift to spiking at threshold I	Simulates neurons' threshold-dependent action potential firing
Periodic Firing	Regular oscillations in V	Models rhythmic spiking observed in consistent stimulation
Frequency Modulation	Increased frequency with higher I	Captures neurons' response to increased input with faster firing
High-Frequency Firing	Rapid oscillations at high I	Simulates fast-spiking neurons under intense stimuli
Refractory Dynamics	Recovery through W variable	Models neuronal refractory period and rhythmic stabilization

5.5.2. Implications of Findings

These simulation results demonstrate that the FitzHugh-Nagumo model, though simplified, captures essential features of neuronal excitability and firing dynamics, such as:

Flexibility in Simulating Neuronal Behaviors: By adjusting I and other parameters, the FHN model can replicate diverse neuronal firing patterns. This flexibility makes the FHN model a valuable tool in theoretical studies exploring neuronal dynamics.

Utility in Frequency Modulation Studies: The model's ability to transition between resting, periodic, and high-frequency firing states based on input strength highlights its utility for investigating how neurons encode stimulus intensity through firing rates.

Suitability for Network Simulations: Given its computational efficiency, the FHN model is suitable for large-scale simulations where qualitative insights into excitability and recovery are needed, such as in networked systems or theoretical models of neural circuits.

In summary, the FHN model's simulation results provide a meaningful approximation of real neuronal behaviors. Its patterns of excitability and recovery align with empirical observations, supporting its application in simulating and studying neuronal dynamics. However, it is essential to consider the model's limitations, as it abstracts away detailed ion channel mechanisms. For applications requiring more granular representations of neuronal physiology, such as specific ion channels' kinetics, more complex models like Hodgkin-Huxley are preferred. Nonetheless, the FHN model remains a robust choice for studies focused on generalized excitability patterns, threshold behavior, and frequency modulation in neurons.

6. Discussion

6.1. Insights from the FitzHugh-Nagumo Model

The FitzHugh-Nagumo model simulations yielded several important insights:

Excitability and Threshold Behavior:

Findings: The FHN model accurately captured the threshold-dependent excitability characteristic of neurons. For instance, when the external stimulus I was below 0.5, the membrane potential V remained stable, simulating a resting state. When I increased to values between 0.5 and 0.9, the model transitioned into a spiking or oscillatory state, where V displayed regular action potentials.

Interpretation: This finding reflects the threshold behavior seen in real neurons, where a minimum level of input is required to reach an excitatory threshold, enabling neurons to selectively respond to strong enough stimuli.

Firing Frequency Modulation:

Findings: The simulations showed that the firing frequency of V increased with higher values of I. For example, at I=0.6, the model exhibited periodic oscillations with a moderate frequency, whereas at I=1.2, the frequency of oscillations in V increased significantly, representing high-frequency firing.

Interpretation: This frequency modulation is comparable to neurons' response to increasing stimulus intensity by shortening the time between spikes. It demonstrates that the FHN model can replicate graded excitability, where stronger inputs result in higher firing rates, a critical mechanism for encoding stimulus intensity in the nervous system.

Recovery and Refractory Dynamics:

Findings: The recovery variable W displayed delayed dynamics in response to the membrane potential V, especially noticeable during and after a spike. For example, in a baseline simulation, W lagged behind V and reached a peak of approximately 1.5 following the peak in V at around 1.7, before gradually decaying.

Interpretation: This recovery process mirrors the refractory period observed in biological neurons, where a neuron temporarily becomes less excitable following an action potential. By incorporating W as a recovery mechanism, the FHN model provides a

simplified but realistic representation of this important neuronal characteristic, helping to prevent chaotic firing patterns and stabilize rhythmic firing.

6.2. Implications for Neuroscience and Neural Networks

The findings from the FitzHugh-Nagumo model simulations have several implications for neuroscience and neural network research:

Simplified Modeling of Neuronal Dynamics:

Implication: The FHN model provides a computationally efficient way to simulate neuronal excitability without needing the complexity of conductance-based models. By capturing essential dynamics with minimal variables, it serves as a useful tool for large-scale neural network simulations where computational efficiency is crucial.

Relevance to Neuroscience: The model's ability to replicate action potential thresholds, oscillatory firing, and recovery phases suggests it can be applied to study fundamental neural processes, such as signal transmission, synchronization, and oscillatory patterns in the brain. This is especially relevant in understanding rhythmic activities, like those in central pattern generators or oscillatory brain regions.

Insights into Stimulus-Driven Firing Patterns:

Implication: The observed firing frequency modulation in response to increasing III provides insights into how neurons encode information through frequency. The FHN model suggests that neurons can represent stronger inputs by increasing firing rates, an encoding strategy known as rate coding.

Theoretical Implications: In neural network research, this finding supports the idea that simplified neuron models can effectively capture information encoding mechanisms. This allows for exploring neural coding strategies in theoretical studies, particularly in artificial neural networks inspired by biological processes.

Modeling Excitability in Networked Systems:

Implication: The FHN model's sensitivity to input suggests its potential application in simulating networked neuronal systems where collective excitability, such as synchronous firing and wave propagation, can be studied.

Relevance to Neural Networks: The model could be used to simulate emergent phenomena in interconnected systems, such as resonance and coherence in neural

networks. This can further inform the design of neuromorphic computing systems that rely on excitable units, potentially influencing fields such as machine learning and artificial intelligence.

6.3. Limitations of the Simulation Study

While the FitzHugh-Nagumo model provides useful insights into neuronal dynamics, it also has several limitations:

Lack of Biophysical Detail:

Limitation: The FHN model simplifies neuronal behavior by using only two variables (V and W) and does not account for specific ion channels or conductances that contribute to action potential generation. For example, it does not distinguish between sodium and potassium currents, which play distinct roles in action potential dynamics.

Impact: This abstraction limits the model's ability to accurately simulate complex neuronal behaviors that depend on specific ion channel kinetics, such as after-hyperpolarization or spike frequency adaptation.

Reduced Applicability to Different Neuron Types:

Limitation: The model assumes a general excitability mechanism and does not account for differences between neuron types, such as excitatory versus inhibitory neurons or fast-spiking versus regular-spiking cells. These neuron types have unique firing properties that are influenced by their ion channel compositions.

Impact: This limits the model’s utility for studying diverse neuronal responses and requires additional adjustments or modifications for simulating specialized neurons.

Sensitivity to Parameter Calibration:

Limitation: The accuracy of the FHN model heavily depends on precise calibration of parameters like ϵ, a, b, and I. Small changes in these parameters can lead to significantly different behavior, potentially making the model challenging to generalize across different neuronal conditions.

Impact: This sensitivity can complicate the application of the model to real neural systems, where parameters may vary across contexts and species. Accurate calibration is essential for meaningful simulations, but it requires empirical data that may not always be available.

6.4. Recommendations for Further Research

To build on the findings of this study and address its limitations, the following recommendations are proposed:

Integration of Biophysical Properties:

Recommendation: Future research could consider hybrid models that incorporate additional biophysical properties into the FHN model, such as specific ion channel dynamics, to better represent a broader range of neuronal behaviors. This could include introducing multiple recovery variables to capture the effects of different ion channels.

Expected Outcome: Enhanced model accuracy and applicability to diverse neuron types, allowing for more detailed studies of neuronal physiology.

Parameter Optimization for Specific Neuron Types:

Recommendation: Conduct targeted parameter optimization to better fit the FHN model to specific neuron types, such as fast-spiking interneurons or regular-spiking pyramidal cells. This could involve systematic calibration against experimental data for each neuron type.

Expected Outcome: Improved model accuracy for specialized applications, enabling simulations that are more representative of specific neuronal subtypes and enhancing the FHN model's versatility.

Exploration of Network-Level Dynamics:

Recommendation: Apply the FHN model in larger network simulations to investigate emergent behaviors, such as synchronization, pattern formation, and resonance in neural networks. Studying how the FHN model behaves in networked contexts could yield insights into population-level neural dynamics.

Expected Outcome: A deeper understanding of collective behaviors in neural systems, which could inform the design of artificial neural networks and contribute to theories on network dynamics in neuroscience.

Comparison with More Complex Models:

Recommendation: Compare the FHN model's performance with more detailed models like the Hodgkin-Huxley model in specific applications. Such comparative studies can

highlight the conditions under which the FHN model is sufficient and when a more complex model is necessary.

Expected Outcome: Guidelines for model selection based on study objectives, allowing researchers to choose appropriate models for different types of neuronal simulations with better understanding of trade-offs.

The FitzHugh-Nagumo model provides a valuable, computationally efficient approach to simulating neuronal excitability and recovery dynamics. Despite its limitations, it captures essential features of neuronal behavior, such as excitability thresholds, frequency modulation, and recovery periods. The insights from this study suggest that the FHN model can serve as a foundation for understanding basic neuronal processes and studying large-scale neural networks. However, for applications requiring biophysical detail or specific neuron types, further model enhancements and comparisons with more complex models are recommended. Future research could extend the FHN model's utility, making it an even more versatile tool for computational neuroscience and neural network studies.

7. Conclusion

7.1.Summary of Findings

The FitzHugh-Nagumo (FHN) model simulation revealed key insights into neuronal excitability and recovery dynamics. First, the model accurately replicated threshold-dependent excitability, with the membrane potential V remaining stable for $I<0.5I$ and entering a spiking regime at $I\geq0.5$, closely mimicking real neurons' threshold behavior. Secondly, the model demonstrated frequency modulation, as the firing frequency of V increased from moderate at I=0.6 to high-frequency oscillations at I=1.2, a fundamental characteristic in neural coding. Third, the recovery variable W effectively simulated the refractory period, with W peaking shortly after each V spike and gradually decaying, preventing immediate re-firing. Finally, bifurcation analysis illustrated the model's ability to transition between resting, periodic oscillations, and high-frequency firing states, indicating its capacity to capture nonlinear dynamics and state transitions in excitability.

7.2.Contribution to the Field

This study validates the FitzHugh-Nagumo model as a computationally efficient tool for simulating core neuronal dynamics, supporting its use in theoretical and large-scale network simulations where detailed conductance-based models are impractical. The model's replication of neuronal threshold behavior, frequency modulation, and refractory dynamics makes it an ideal candidate for examining basic excitability and recovery processes. Moreover, by demonstrating the model's responsiveness to varying stimulus intensities, this study highlights the FHN model's potential for studying rate coding and stimulus-response relationships in neurons, contributing to our understanding of neural coding. Additionally, this work lays the groundwork for using the FHN model in networked systems to explore emergent neural behaviors like synchronization and oscillatory patterns.

7.3.Future Work

Future research can extend the FHN model's capabilities by incorporating additional biophysical properties, such as ion-specific channels, to better simulate diverse neuronal behaviors like after-hyperpolarization and spike frequency adaptation. Parameter optimization tailored to specific neuron types (e.g., fast-spiking versus regular-spiking neurons) would enhance the model's accuracy for specialized applications. Further, expanding the FHN model to networked simulations could yield insights into collective

neural phenomena, such as synchrony and resonance. Comparative analyses with more complex models, like the Hodgkin-Huxley model, could also clarify when the FHN model is appropriate versus when more detailed models are necessary. Finally, exploring the FHN model's potential in neuromorphic computing could enable the creation of efficient, bio-inspired computing systems, contributing to advances in artificial intelligence and machine learning.

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Appendix

The detailed workflow for each simulation instance proceeded as follows:

```
// Initialization Phase
Load parameter configuration (ε, a, b, I)
Set initial conditions for V and W
Define time domain [t_start, t_end] and initial step size Δt
Allocate arrays for time series storage
Initialize analysis variables

// Integration Phase
t = t_start
V[0] = V_initial
W[0] = W_initial
i = 0

While t < t_end:
    // Calculate derivatives using FHN equations
    If using RK4 method:
        Calculate k1_v, k1_w, k2_v, k2_w, k3_v, k3_w, k4_v, k4_w
        V[i+1] = V[i] + (Δt/6) * (k1_v + 2*k2_v + 2*k3_v + k4_v)
        W[i+1] = W[i] + (Δt/6) * (k1_w + 2*k2_w + 2*k3_w + k4_w)

    If using adaptive step size:
        Estimate local error
        If error > tolerance:
            Reduce Δt and retry step
        Else:
            Accept step and adjust Δt for next step

    // Store results
    t = t + Δt
    i = i + 1
```

```
    // Check for termination conditions
    If specific events detected (e.g., convergence to steady state):
        Break

// Analysis Phase
Identify spikes in V time series using threshold crossing detection
Calculate firing frequency and regularity metrics
Compute phase space trajectories and nullclines
Perform spectral analysis using FFT
Compare with reference data if applicable

// Output Generation Phase
Generate time series plots of V and W
Create phase space plots with nullclines
Produce bifurcation diagrams if parameter sweeps performed
Export numerical data in standardized format
Generate summary statistics and reports
```

This algorithm was implemented with careful attention to numerical precision, using double-precision floating-point representation for all variables and intermediate calculations. Special handling was incorporated for edge cases, such as parameter combinations near bifurcation points, where additional precision or adaptive methods were required to maintain stability and accuracy.

4.5.2 Parameter Sweep Implementation

To comprehensively explore the model's behavior across parameter space, a structured parameter sweep algorithm was implemented:

```
// Define parameter ranges and step sizes
ε_range = [0.01, 0.2], ε_step = 0.01
a_range = [0.5, 1.0], a_step = 0.05
b_range = [0.5, 1.5], b_step = 0.05
I_range = [0.0, 2.0], I_step = 0.05

// Initialize result storage
Create multi-dimensional array for results
```

```
// Nested loop structure for parameter exploration
For each I in I_range (step I_step):
    For each ε in ε_range (step ε_step):
        For each a in a_range (step a_step):
            For each b in b_range (step b_step):
                // Run simulation with current parameter set
                results = RunFHNSimulation(ε, a, b, I)

                // Extract and store key metrics
                Store firing_rate, max_amplitude, bifurcation_type, etc.

                // Flag interesting parameter combinations
                If interesting pattern detected:
                    Add to detailed analysis queue

// Post-processing of parameter sweep results
Generate parameter space maps
Identify regime boundaries
Characterize bifurcation structures
```

This approach generated comprehensive maps of the FHN model's behavior, allowing for identification of parameter regions with specific neuronal dynamics. The parameter sweep was implemented using parallel processing capabilities whenever available, with each parameter combination representing an independent simulation that could be executed concurrently.

4.5.3 Bifurcation Detection Algorithm

A specialized algorithm was developed for automated detection and classification of bifurcations across parameter space:

```
// Bifurcation detection algorithm
For each parameter value p in parameter range:
    // Run simulation and extract steady-state behavior
    result = RunFHNSimulation(p)

    // Analyze fixed points
```

```
    fixed_points = FindFixedPoints(result)
    For each fixed_point in fixed_points:
        J = ComputeJacobian(fixed_point)
        eigenvalues = ComputeEigenvalues(J)

        // Check for bifurcation signatures
        If RealPart(eigenvalues) crosses zero as p varies:
            If ImaginaryPart(eigenvalues) is non-zero:
                Record Hopf bifurcation at current p
            Else:
                Record saddle-node bifurcation at current p

    // Analyze limit cycles
    If OscillatoryBehavior(result):
        cycle_properties = AnalyzeLimitCycle(result)

        // Check for cycle bifurcations
        If cycle_period doubles compared to previous p:
            Record period-doubling bifurcation
        If cycle disappears abruptly:
            Record potential homoclinic bifurcation

    // Store bifurcation information
    Update bifurcation diagram with detected events
```

This algorithm enabled systematic characterization of the FHN model's bifurcation structure, providing insights into the mathematical mechanisms underlying transitions between different neuronal firing regimes.

The flowchart in Figure 4.1 illustrates the complete simulation workflow, highlighting the integration of various algorithmic components:

[Figure 4.1: Comprehensive flowchart of the FHN model simulation framework, showing the interconnections between parameter configuration, numerical integration, analysis modules, and output generation. The flowchart includes decision points for adaptive methods and specialized handling of bifurcation regions.]

These algorithmic approaches ensured efficient and accurate exploration of the FHN model's behavior across parameter space, providing a robust computational foundation for investigating neuronal dynamics within this simplified mathematical framework.

4.6 Software Implementation

The FitzHugh-Nagumo model simulation framework was implemented as a comprehensive software system, integrating numerical methods, analysis tools, and visualization capabilities within a cohesive architecture. This section details the software implementation aspects, including code organization, validation procedures, and data management strategies.

4.6.1 Code Structure and Organization

The software implementation followed a modular design philosophy, organizing functionality into cohesive components with well-defined interfaces. The code structure comprised the following key modules:

1. **Core Mathematical Functions**: Implemented the fundamental mathematical representation of the FHN model, including:
    - `fhn_derivatives.m`: Defined the right-hand side of the differential equations

```
matlab
function [dVdt, dWdt] = fhn_derivatives(t, V, W, epsilon, a, b, I)
    dVdt = V - (V^3)/3 - W + I;
    dWdt = epsilon * (V + a - b*W);
end
```

    - `fhn_jacobian.m`: Computed the Jacobian matrix for stability analysis

```
matlab
function J = fhn_jacobian(V, W, epsilon, a, b)
    J = [1-V^2, -1; epsilon, -epsilon*b];
end
```

    - `fhn_nullclines.m`: Calculated nullcline equations for phase space analysis

```
matlab
function [V_nullcline, W_nullcline] = fhn_nullclines(V_range, epsilon, a, b, I)
    V_nullcline = @(V) V - (V.^3)/3 + I;
    W_nullcline = @(V) (V + a)/b;
```

```
end
```

2. **Numerical Integration Functions**: Implemented various numerical methods for solving the FHN differential equations:
    - `fhn_euler.m`: Basic Euler method implementation
    - `fhn_rk4.m`: Fourth-order Runge-Kutta method
    - `fhn_adaptive.m`: Adaptive step size control based on error estimation
    - `fhn_solver.m`: Unified interface to different numerical methods
3. **Analysis Module**: Contained functions for extracting meaningful information from simulation results:
    - `fhn_spike_detector.m`: Identified action potentials in V time series
    - `fhn_phase_analysis.m`: Performed phase space analysis of trajectories
    - `fhn_bifurcation.m`: Detected and classified bifurcations
    - `fhn_frequency_analysis.m`: Analyzed oscillatory properties and firing patterns
4. **Visualization Module**: Provided standardized plotting functions for different aspects of the results:
    - `plot_time_series.m`: Generated time series plots of V and W
    - `plot_phase_space.m`: Created phase portraits with nullclines
    - `plot_bifurcation.m`: Produced bifurcation diagrams
    - `plot_parameter_map.m`: Visualized behavior across parameter space
5. **Utility Functions**: Provided supporting functionality for data management and workflow:
    - `fhn_parameter_sweep.m`: Coordinated parameter space exploration
    - `fhn_data_export.m`: Standardized data export in various formats
    - `fhn_configuration.m`: Managed parameter configurations and defaults