Asymmetric Information and Financial Markets

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by

Han Nazmi Ozsoylev

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Andrew McLennan, Adviser   Jan Werner, Adviser

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This is to certify that we have examined this copy of a doctoral dissertation by

Han Nazmi Ozsoylev

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Andrew McLennan	**Jan Werner**
Name of Faculty Advisor	Name of Faculty Advisor
Andrew McLennan	J. Werner
Signature of Faculty Advisor	Signature of Faculty Advisor
11/5/04	Nov. 5, 2004
Date	Date

GRADUATE SCHOOL

- To my parents -

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Abstract

In real-world financial markets, traders constantly communicate and learn from each others' actions. Yet, standard rational expectations models assume away such social interaction and let traders interact only through the price system. Also, rational expectations models are currently far from giving a completely satisfactory explanation regarding the causes and common features of crashes and frenzies. This dissertation has two general objectives: (1) to analyze the impacts of social interaction on asset pricing, trading behavior and aggregation of dispersed private information in financial markets, and (2) to account for some common features of crashes and frenzies in stock prices.

The first essay of the dissertation proposes a generalized noisy rational expectations model which accommodates social interaction in financial markets. On top of the information conveyed through the price system, each trader infers additional information by observing some of the other traders' security demands. Whom a trader observes is determined by a directed graph that represents the social network. The main contribution of this essay is that it shows social interaction can impair the aggregation of dispersed private information in the price system. The essay also analyzes how the presence of disjoint clusters in a social network can affect portfolio decision-making. In a stylized social network, we show that agents located across different clusters of the network make different portfolio decisions while those in the same cluster make similar ones. The latter result is broadly consistent with empirical findings.

The second essay tries to explain two important features of large stock price movements: *amplification* and *asymmetry*. Large price movements are often *amplified reactions* to tangible information and they display an *asymmetry*: the number of crashes is higher than the number of frenzies. Our explanation for these features involves the use of hedging (portfolio insurance) strategies in the stock market. Hedgers, who use these strategies, sell assets as the price declines and buy when the price increases. We show that hedgers amplify the impact of news and liquidity shocks on price movements. Convex hedging strategies cause overreaction to negative news and liquidity shocks, hence they create an asymmetry biased towards crashes. An important class of hedging strategies, namely put-option replication, satisfies the convexity condition when the asset prices are highly volatile. Risk aversion is shown to be essential for the asymmetry of price movements. Also, we show that differential information enhances both amplification and asymmetry delivered by hedging.

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Chapter 1

Introduction

An important feature distinguishing assets from ordinary consumption goods is the uncertainty inherent in their nature: stocks, bonds, currencies, derivatives all entail uncertain future payoffs. In financial markets, different agents typically have different information regarding the uncertain asset payoffs. Some have superior or even conflicting information compared to others. Some have no information. In the presence of such asymmetry in information, agents try to infer each other's information so that they can better assess the uncertain payoffs of assets. Rational expectations models investigate how information is inferred and conveyed in financial markets. In these models, only asset prices convey information across agents, that is, agents infer each other's information only by analyzing asset prices. However, in real-world financial markets, agents also learn from each other's actions (sales and purchase orders). Often there is even direct communication of information among agents. Such social interaction is assumed away in the rational expectations models. One objective of this dissertation is to analyze the impacts of social interaction on asset pricing, trading behavior and aggregation of dispersed private information in financial markets.

Another shortcoming of the rational expectations theory is that it is currently far from giving a completely satisfactory explanation regarding the causes and common features of crashes and frenzies in stock prices. These large price reactions, which are typically disproportionate to tangible information triggering them, continue to puzzle both economists and laymen. The empirical fact that the number of crashes exceeds the number of frenzies also deserves an explanation. This brings us to the second objective of this dissertation: to account for some common features of crashes and frenzies in stock prices.

Before embarking on the analysis, we use this introductory chapter to give a summary account of rational expectations theory and its application to financial markets. Using a basic financial market model, similar to the ones used in Chapters 2 and 3, we demonstrate the role of asset prices in conveying information. First we discuss why ignoring informational content of asset prices leads to conceptual difficulties. Then we explain how asset prices convey information in a rational expectations framework and give a formal definition of rational expectations equilibrium. We also discuss how agents can infer all private information from asset prices in this framework, and explain its problematic consequences. Different ways to resolve these problems are shown. Next we discuss some shortcomings associated with the application of rational expectations concept to financial markets, which motivate our dissertation. Finally, we conclude by providing an overview of the essays in this dissertation.

1.1 Rational Expectations

In the presence of asymmetry in information among agents, asset prices play a dual role: they determine agents' budgets, and convey information. Asset prices convey information in the following sense. Agents' expectations about the uncertain payoffs determine their demands, hence affect asset prices through market clearing. An agent's expectation is conditional on her information, and all agents in the economy are aware of this fact. Therefore, when new information causes revision of asset prices, those, who are unaware of this new information, can infer part of it from the change in prices.

One of the earliest studies, treating prices as conveyers of information, is Hayek's seminal 1945 paper, "The Use of Knowledge in Society". Since then, many economists proposed different ways to model the informational role of asset prices. Among these proposals, *rational expectations approach* has been, arguably, the most prominent. Muth introduced this approach in his 1961 paper and he argued that agents' subjective expectations concerning economic variables will coincide with the true or objective conditional expectations of those variables (Snowdon, Wane and Wynarczyk, 1994). So, in Muth's model, agents have rational expectations about future events and therefore they do not make systematic forecast errors when predicting the future.

In the context of financial markets, rational expectations models assume that agents know the functional relation between asset prices and agents' diverse information. Actually, modelling an economy, in which agents ignore informational content of prices, leads to a conceptual difficulty: if agents believe that prices do not convey information and act on this belief, the outcome of the model will still reveal that market clearing prices are correlated with agents' diverse information and thus convey information. Therefore agents, who ignore informational content of prices, make systematic forecast errors. Systematic errors are unlikely and unsustainable in an economy with rational individuals, and this is why rational expectations models are commonly used by economists to analyze problems involving asymmetric information.

Though we have discussed why ignoring informational content of asset prices leads to a conceptual difficulty, we haven't exactly explained "how". That is, we haven't explained how the outcome of a model can predict that asset prices convey information even if agents believe that they do not. To rigorously explain how this can happen, we will use a specific model. The model we will be using is commonly employed by economists, and the same model will also help us to expose how rational expectations theory is applied to financial markets and what some of its implications are. Here is our model of a financial market economy:

In a two-period economy, n agents, indexed by $i = 1, ..., n$, trade and consume. Trade takes place in the first period and consumption of a single good in the second. Trading in the first period is over one risk-free and one risky asset. The risky asset has a future random payoff $\tilde{X}$ (in units of the single consumption good). Agents observe the realization of $\tilde{X}$ in the second period. The risk-free security pays 1 unit of the consumption good in the second period and its first period price is normalized to 1. Each agent i is endowed with deterministic wealth w_{0i} in units of the consumption good. Therefore, if agent i chooses to hold z_i units of the risky asset, her portfolio yields the random final wealth

$$\tilde{w}_{1i} = z_i\tilde{X} + (w_{0i} - pz_i),$$

where p is the price of the risky asset in the first period. Also, agent i has a CARA utility function, $u_i(\tilde{w}_{1i}) = -exp(-\rho\tilde{w}_{1i})$, where $\rho \in (0, \infty)$ denotes the absolute risk aversion coefficient, which is identical for all agents.

Before trading takes place, agent i receives a private random signal $\tilde{s}_i$, $\tilde{s}_i = \tilde{X} + \tilde{\epsilon}_i$, where $\tilde{X}$ and $\tilde{\epsilon}_i$ are mutually independent and jointly normally distributed. The random vector $(\tilde{X}, \tilde{\epsilon}_1, ..., \tilde{\epsilon}_n)$ has the mean $(\mu_x, 0, ..., 0)$, and the nonsingular variance-covariance matrix $(\sigma_x^2, \sigma_\epsilon^2, ..., \sigma_\epsilon^2)I_{n+1}$, where I_{n+1} denotes the $(n+1)$ dimensional identity matrix.

The net supply of the risky security (i.e., liquidity) is l, and this is known by all agents in the first period.

1.1.1 Ignoring the Informational Content of Price

Following Lintner (1969), we first consider a *competitive equilibrium* in which agents ignore the informational content of price. That is, we consider an economy where agents form their expectations about the risky payoff conditional only on their private signals. Formally, we employ the following equilibrium definition for the economy:

Given signal realizations $(s_1, ..., s_n)$ of $(\tilde{s}_1, ..., \tilde{s}_n)$, a competitive equilibrium consists of a risky security price p and demands $\{z_i(s_i, p)\}_{i=1,...,n}$ such that

(a) $z_i(s_i, p) \in argmax_{z_i} \mathrm{E}[u_i(\tilde{w}_{1i})|s_i], \quad \forall i = 1, ..., n,$

(b) $\sum_{i=1}^{n} z_i(s_i, p) = l$.

To determine the competitive equilibrium, we first compute the mean and variance of $\tilde{X}$ conditional on the private signal realization s_i. Following joint normality of $\tilde{X}$ and $\tilde{s}_i$, we have

$$\mathrm{E}[\tilde{X}|s_i] = \mathrm{E}[\tilde{X}] + \frac{\mathrm{cov}(\tilde{X}, \tilde{s}_i)}{\mathrm{var}(\tilde{s}_i)}(s_i - \mathrm{E}[\tilde{s}_i]) = \mu_x + \frac{\sigma_x^2}{\sigma_x^2 + \sigma_\epsilon^2}(s_i - \mu_x), \quad (1.1.1a)$$

$$\mathrm{var}(\tilde{X}|s_i) = \mathrm{var}(\tilde{X}) - \frac{\left(\mathrm{cov}(\tilde{X}, \tilde{s}_i)\right)^2}{\mathrm{var}(\tilde{s}_i)} = {\sigma_x}^2\left(1 - \frac{\sigma_x^2}{\sigma_x^2 + \sigma_\epsilon^2}\right). \quad (1.1.1b)$$

Solving agent i's problem

$$\max_{z_i} \mathrm{E}[-\exp(-\rho \tilde{w}_{1i})|s_i]$$
$$\text{s. to} \quad \tilde{w}_{1i} = z_i \tilde{X} + (w_{0i} - p z_i),$$

we derive[1]

$$z_i(s_i, p) = \frac{\mathrm{E}[\tilde{X}|s_i] - p}{\rho \mathrm{var}(\tilde{X}|s_i)} \quad (1.1.2)$$

as the asset demand of agent i. Equation (1.1.2) presents a plausible asset demand: agent i holds a positive amount of the risky asset if and only if her expectation of the risky payoff $\tilde{X}$, conditional on her private signal s_i, exceeds the risky asset price p.

Next, the market clearing condition $\sum_{i=1}^{n} z_i(s_i, p) = l$ implies $\sum_{i=1}^{n} \frac{\mathrm{E}[\tilde{X}|s_i] - p}{\rho \mathrm{var}(\tilde{X}|s_i)} = l$. Solving for p gives us a competitive equilibrium price for the risky asset in the following form:

$$p = \left(\sum_{i=1}^{n} \frac{1}{\rho \mathrm{var}(\tilde{X}|s_i)}\right)^{-1} \left(\sum_{i=1}^{n} \frac{\mathrm{E}[\tilde{X}|s_i]}{\rho \mathrm{var}(\tilde{X}|s_i)} - l\right).$$

Substituting for $\mathrm{E}[\tilde{X}|s_i]$ and $\mathrm{var}(\tilde{X}|s_i)$ from (1.1.1a)-(1.1.1b) yields

$$p = \frac{\sigma_\epsilon^2}{\sigma_x^2 + \sigma_\epsilon^2}\mu_x + \frac{\sigma_x^2}{\sigma_x^2 + \sigma_\epsilon^2}\sum_{i=1}^{n} s_i - \frac{\sigma_x^2 \sigma_\epsilon^2}{\sigma_x^2 + \sigma_\epsilon^2}\frac{\rho}{n} l. \quad (1.1.3)$$

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[1] For any given normally distributed random variable $\tilde{y}$, $\mathrm{E}[e^{\tilde{y}}] = \exp\left(\mathrm{E}[\tilde{y}] + \frac{\mathrm{var}(\tilde{y})}{2}\right)$. Therefore,

$$\mathrm{E}[-\exp(-\rho \tilde{w}_{1i})|s_i] = -\exp\left[-\rho\left(\mathrm{E}[\tilde{w}_{1i}|s_i] - \frac{\rho}{2}\mathrm{var}[\tilde{w}_{1i}|s_i]\right)\right],$$

and using this equality, we can easily derive (1.1.2).

Equation (1.1.3) reveals the relationship between the signals $(s_1, ..., s_n)$ and the competitive equilibrium risky asset price p. When agents ignore the informational content of price, as they do in this competitive equilibrium, they essentially ignore this relationship. This means that agents' beliefs about the statistical relationship between their information and eventual outcomes is different from the one predicted by the model and its equilibrium. In other words, agents make systematic forecast errors in the competitive equilibrium. It is implausible to expect rational individuals to make systematic forecast errors, and therefore the competitive equilibrium is conceptually unsatisfactory.

1.1.2 The Rational Expectations Approach

Suppose instead that agents initially know the functional relation between the equilibrium price and signals $(s_1, ..., s_n)$. That is, suppose agents act on the hypothesis that risky asset price is given by

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n),$$

where the function $P : \mathbb{R}^n \to \mathbb{R}$ actually delivers the risky asset price for all realizations $(s_1, ..., s_n)$ of $(\tilde{s}_1, ..., \tilde{s}_n)$. Then each agent i determines her risky asset demand using her expectation of the risky payoff $\tilde{X}$ conditional on the realizations $\tilde{s}_i = s_i$ and $P(\tilde{s}_1, ..., \tilde{s}_n) = p$. An equilibrium, where agents act as described, rules out systematic forecast errors, because agents already know the functional relation between equilibrium price and signals and use this information while determining their demands. Such an equilibrium is called *rational expectations equilibrium* since agents have rational expectations about the functional relation between price and signals. Formally, this equilibrium is defined as follows:

A rational expectations equilibrium (REE) consists of a risky asset price function $P(s_1, ..., s_n)$ *and demands* $\{z_i(s_i, p)\}_{i=1,...,n}$ *such that for all realizations of* $(s_1, ..., s_n)$ *of* $(\tilde{s}_1, ..., \tilde{s}_n)$

(a) $z_i(s_i, p) \in argmax_{z_i} \mathrm{E}\left[u_i(\tilde{w}_{1i}) \middle| s_i, p = P(s_1, ..., s_n)\right], \qquad \forall\, i = 1, ..., n;$

(b) $\sum_{i=1}^{n} z_i(P(s_1, ..., s_n), s_i) = l.$

Now let us derive a rational expectations equilibrium for our financial market model. First we claim that a rational expectations equilibrium price satisfying the following property exists:

$$\tilde{p} \equiv P(\tilde{s}_1, ..., \tilde{s}_n) = \pi_0 + \pi_s \tilde{\mathcal{S}}, \tag{1.1.4}$$

where $\tilde{\mathcal{S}} = \frac{1}{n}\sum_{i=1}^{n} \tilde{s}_i$. Note that $\tilde{\mathcal{S}}$ is a *sufficient statistic for signals* $(\tilde{s}_1, ..., \tilde{s}_n)$ in the sense that conditional distribution of risky payoff $\tilde{X}$ given $(\tilde{s}_1, ..., \tilde{s}_n)$ is the same as the conditional distribution of $\tilde{X}$ given $\tilde{\mathcal{S}}$.[2] In particular, for all realizations $(s_1, ..., s_n, \mathcal{S})$ of $(\tilde{s}_1, ..., \tilde{s}_n, \tilde{\mathcal{S}})$

$$\begin{aligned} \mathrm{E}[\tilde{X}|s_1, ..., s_n] &= \mathrm{E}[\tilde{X}|\mathcal{S}], \\ \mathrm{var}(\tilde{X}|s_1, ..., s_n) &= \mathrm{var}(\tilde{X}|\mathcal{S}). \end{aligned}$$

Therefore, those who have access to the knowledge of $\mathcal{S}$ in our economy, can form their expectations of the risky payoff as if they knew all agents' private signals. Intuitively this is so because the sum of signals of identical precision is more informative than any individual signal.

Using the fact that $\tilde{\mathcal{S}}$ is a sufficient statistic for $(\tilde{s}_1, ..., \tilde{s}_n)$, we compute the mean and variance of $\tilde{X}$ conditional on $\tilde{s}_i = s_i$ and $\tilde{p} = p$:

$$\begin{aligned} \mathrm{E}[\tilde{X}|s_i, p] &= \mathrm{E}[\tilde{X}|s_i, \pi_0 + \pi_s \mathcal{S}] = \mathrm{E}[\tilde{X}|s_i, \mathcal{S}] = \mathrm{E}[\tilde{X}|\mathcal{S}] \\ &= \mathrm{E}[\tilde{X}] + \frac{\mathrm{cov}(\tilde{X}, \tilde{\mathcal{S}})}{\mathrm{var}(\tilde{\mathcal{S}})}(\mathcal{S} - \mathrm{E}[\tilde{\mathcal{S}}]) = \mu_x + \frac{\sigma_x^2}{\sigma_x^2 + \frac{1}{n}\sigma_\epsilon^2}(\frac{1}{n}\sum_{i=1}^{n} s_i - \mu_x), \quad (1.1.5) \\ \mathrm{var}(\tilde{X}|s_i, p) &= \mathrm{var}(\tilde{X}|s_i, \pi_0 + \pi_s \mathcal{S}) = \mathrm{var}(\tilde{X}|s_i, \mathcal{S}) = \mathrm{var}(\tilde{X}|\mathcal{S}) \\ &= \mathrm{var}(\tilde{X}) - \frac{\left(\mathrm{cov}(\tilde{X}, \tilde{\mathcal{S}})\right)^2}{\mathrm{var}(\tilde{\mathcal{S}})} = \sigma_x^2\left(1 - \frac{\sigma_x^2}{\sigma_x^2 + \frac{1}{n}\sigma_\epsilon^2}\right). \quad (1.1.6) \end{aligned}$$

Solving for agent i's maximization problem yields[3]

$$z_i(s_i, p) = \frac{\mathrm{E}[\tilde{X}|s_i, p] - p}{\rho \mathrm{var}(\tilde{X}|s_i, p)} = \frac{\mathrm{E}[\tilde{X}|\mathcal{S}] - p}{\rho \mathrm{var}(\tilde{X}|\mathcal{S})} \tag{1.1.7}$$

[2]For a detailed and more formal treatment of this subject, see §2.7. It is easy to check that definitions of "sufficient statistic" given in §2.7 and above are equivalent for this setup. Also, the reader can refer to Huang and Litzenberger (1988) for a proof showing that $\tilde{\mathcal{S}} = \frac{1}{n}\sum_{i=1}^{n} \tilde{s}_i$ is a sufficient statistic for signals $(\tilde{s}_1, ..., \tilde{s}_n)$.

[3]See footnote 1.

as the risky asset demand of agent i. Imposing market clearing condition $\sum_{i=1}^{n} z_i(s_i, p) = l$ and substituting for $\mathrm{E}[\tilde{X}|S]$, $\mathrm{var}(\tilde{X}|S)$ from (1.1.5)-(1.1.6) leads us to the following rational expectations equilibrium price:

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n) = \frac{\sigma_\epsilon^2}{n\sigma_x^2 + \sigma_\epsilon^2}\left(\mu_x - \frac{\rho\sigma_x^2}{n}\, l\right) + \frac{n\sigma_x^2}{n\sigma_x^2 + \sigma_\epsilon^2}\,\tilde{S}. \tag{1.1.8}$$

As we have claimed in the beginning, the REE price satisfies the functional form given in (1.1.4).

Grossman (1976) derives this REE using a different approach. He first considers an artificial economy identical to ours except that all agents receive a signal equal to S. He solves for the equilibrium of this artificial economy. Then he verifies that this equilibrium is also a REE for the financial market model analyzed above. DeMarzo and Skiadas (1998) establish uniqueness of the REE for this setup.

Equation (1.1.8) reveals an important feature of the REE price: it is a sufficient statistic for all signals in the economy. That is, the REE price reveals all that is worth knowing about private signals. Therefore, the REE price, which is a sufficient statistic for all signals of the economy, is called *fully revealing*.

A fully revealing REE price involves an important conceptual difficulty. If the price reveals all that is worth knowing about private signals $(s_1, ..., s_n)$, agents do not need to act on their private signals and can condition only on price while forming their expectations. If agents do not act on their private signals at all, then it is unclear why the price reveals any information regarding agents' diverse private signals in the first place.

This difficulty can be overcome if the equilibrium price *partially reveals* information in the sense that it is not a sufficient statistic for any of the private signals. In such a case, agents have to act on their own private signals as well as price, and this solves the problem.

1.1.3 Partially Revealing Price

Next we discuss under what conditions an equilibrium price is partially revealing. First notice from (1.1.8) that any change in the fully revealing REE price can be due to either a change in liquidity l or a change in private signals $(s_1, ..., s_n)$. Since agents know the level of liquidity in our financial market model, they can always infer the true reason for price change. For instance, when the risky asset price increases and the liquidity level stays constant, agents infer that price change occurs because the risky asset demand increases, which has to be due to more optimistic private signals, on average.

If there were an additional source of uncertainty in our model on top of risky payoff $\tilde{X}$, then inference of the true reasons for a price change would be problematic. For instance, if the liquidity (net supply of risky security) were random, i.e., if l were taken to be the realization of a random variable $\tilde{l}$, then agents would not be able to infer whether it is the liquidity or the signals triggering the change in price. In such an event, the price would not be fully revealing. To illustrate this better, let us take $\tilde{l}$, $\tilde{X}$, $\tilde{\epsilon}_i$, $i = 1, ..., n$, to be mutually independent and $\tilde{l}$ to be normally distributed with mean 0 and variance σ_l^2. The presence of this new uncertainty requires us to redefine the equilibrium:

A (noisy) rational expectations equilibrium consists of a risky asset price function $P(s_1, ..., s_n, l)$ *and demands* $\{z_i(s_i, p)\}_{i=1,...,n}$ *such that for all realizations of* $(s_1, ..., s_n, l)$ *of* $(\tilde{s}_1, ..., \tilde{s}_n, \tilde{l})$

(a) $z_i(s_i, p) \in argmax_{z_i} \mathrm{E}\left[u_i(\tilde{w}_{1i}) | s_i, p = P(s_1, ..., s_n, l)\right], \qquad \forall i = 1, ..., n;$

(b) $\sum_{i=1}^{n} z_i(P(s_1, ..., s_n), s_i) = l.$

Next we derive the noisy REE for our revised financial market model. We first conjecture that there exists an equilibrium price given by

$$\tilde{p} \equiv P(\tilde{s}_1, ..., \tilde{s}_n, \tilde{l}) = \pi_0 + \pi_s \sum_{i=1}^{n} \tilde{s}_i - \gamma \tilde{l}. \tag{1.1.9}$$

Then the random vector $(\tilde{X}, \tilde{s}_i, \tilde{p})$, $i = 1, ..., n$, is jointly normally distributed with a mean vector

$$(\mu_x, \mu_x, n\,\pi_0 + \pi_s \mu_x),$$

and a variance-covariance matrix

$$\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \equiv \begin{bmatrix} \sigma_x^2 & \sigma_x^2 & n\pi_s\,\sigma_x^2 \\ \sigma_x^2 & \sigma_x^2+\sigma_\epsilon^2 & n\pi_s\sigma_x^2+\pi_s\sigma_\epsilon^2 \\ n\pi_s\,\sigma_x^2 & n\pi_s\sigma_x^2+\pi_s\sigma_\epsilon^2 & n^2\pi_s^2\sigma_x^2+n\pi_s^2\sigma_\epsilon^2+\gamma^2\sigma_l^2 \end{bmatrix},$$

where $V_{11}=\sigma_x^2$, V_{12} is a 1×2 vector, V_{21} is a 2×1 vector, and V_{22} is a 2×2 matrix. Following normal distribution theory, the posterior distribution of $\tilde{X}$ given a realization (s_i, p) is normal with mean and variance of the form

$$\begin{aligned} \mathrm{E}[\tilde{X}|s_i,p] &= \mu_x + V_{12}{V_{22}}^{-1}\begin{pmatrix} s_i-\mu_x \\ p-\pi_0-n\pi_s\mu_x \end{pmatrix} \\ &= a_{0i}+a_{1i}\,s_i+a_{2i}\,p, \qquad (1.1.10) \\ \mathrm{var}(\tilde{X}|s_i,p) &= V_{11}-V_{12}{V_{22}}^{-1}V_{21} \\ &= b_i, \qquad (1.1.11) \end{aligned}$$

where

$$a_{0i} = \frac{\mu_x\sigma_\epsilon^2[(n-1)\pi_s^2{\sigma_\epsilon}^2+\gamma^2\sigma_l^2]-\sigma_x^2(n-1)\pi_s{\sigma_\epsilon}^2\pi_0}{(\sigma_x^2+\sigma_\epsilon^2)[(n-1)\pi_s^2\sigma_\epsilon^2+\gamma^2\sigma_l^2]+\sigma_x^2\sigma_\epsilon^2(n-1)^2\pi_s^2}, \qquad (1.1.12a)$$

$$a_{1i} = \frac{\sigma_x^2\gamma^2\sigma_l^2\,s_i}{(\sigma_x^2+\sigma_\epsilon^2)[(n-1)\pi_s^2\sigma_\epsilon^2+\gamma^2\sigma_l^2]+\sigma_x^2\sigma_\epsilon^2(n-1)^2\pi_s^2}, \qquad (1.1.12b)$$

$$a_{3i} = \frac{\sigma_x^2(n-1)\pi_s{\sigma_\epsilon}^2\,p}{(\sigma_x^2+\sigma_\epsilon^2)[(n-1)\pi_s^2\sigma_\epsilon^2+\gamma^2\sigma_l^2]+\sigma_x^2\sigma_\epsilon^2(n-1)^2\pi_s^2}, \qquad (1.1.12c)$$

$$b_i = \frac{\sigma_x^2\sigma_\epsilon^2[(n-1)\pi_s^2\sigma_\epsilon^2+\gamma^2\sigma_l^2]}{(\sigma_x^2+\sigma_\epsilon^2)[(n-1)\pi_s^2\sigma_\epsilon^2+\gamma^2\sigma_l^2]+\sigma_x^2\sigma_\epsilon^2(n-1)^2\pi_s^2}. \qquad (1.1.12d)$$

Given that realizations of $\tilde{s}_i$ and $\tilde{p}$ are s_i and p, agent i's demand for the risky asset is[4]

$$z_i(s_i,p) = \frac{\mathrm{E}[\tilde{X}|s_i,p]-p}{\rho\mathrm{var}(\tilde{X}|s_i,p)} = \frac{a_{0i}+a_{1i}s_i+(a_{2i}-1)p}{\rho b_i}, \qquad (1.1.13)$$

Imposing market clearing condition $\sum_{i=1}^n z_i(s_i,p)=l$ and substituting for $\mathrm{E}[\tilde{X}|s_i,p]$, $\mathrm{var}(\tilde{X}|s_i,p)$ from (1.1.10)-(1.1.11) yields

$$p = P(s_1,\ldots,s_n,l) = \left(\sum_{i=1}^n \frac{1-a_{2i}}{\rho b_i}\right)^{-1}\left(\sum_{i=1}^n \frac{a_{0i}+a_{1i}s_i}{\rho b_i}-l\right). \qquad (1.1.14)$$

[4]See footnote 1.

Expectations based on the conjectured price function (1.1.9) are rational if and only if the coefficients π_0, π_s, γ are the same as the corresponding coefficients in (1.1.14). Thus,

$$\pi_0 = \gamma \sum_{i=1}^{n} \frac{a_{0i}}{\rho b_i}, \tag{1.1.15a}$$

$$\pi_s = \gamma \frac{a_{1i}}{\rho b_i}, \tag{1.1.15b}$$

$$\gamma = \left(\sum_{i=1}^{n} \frac{1 - a_{2i}}{\rho b_i} \right)^{-1}. \tag{1.1.15c}$$

Substituting for a_{0i}, a_{1i}, a_{2i}, b_i from (1.1.12a)-(1.1.12d) yields

$$\pi_s = \frac{\gamma}{\rho\sigma_\epsilon^2} \frac{\gamma^2 {\sigma_l}^2}{(n-1){\pi_s}^2\sigma_\epsilon^2 + \gamma^2 {\sigma_l}^2}, \tag{1.1.16a}$$

$$\frac{1}{\gamma} = \frac{n(\sigma_x^2 + \sigma_\epsilon^2)}{\rho\sigma_x^2\sigma_\epsilon^2} + \frac{n(n-1)\pi_s\left((n-1)\pi_s - 1\right)}{\rho\left((n-1)\pi_s^2\sigma_\epsilon^2 + \gamma^2{\sigma_l}^2\right)}, \tag{1.1.16b}$$

$$\pi_0 = \frac{n\mu_x}{\rho\sigma_x^2} - \gamma\pi_0 \frac{n(n-1)\pi_s}{\rho\left((n-1)\pi_s^2\sigma_\epsilon^2 + \gamma^2{\sigma_l}^2\right)}. \tag{1.1.16c}$$

The unique solution for the system of equations (1.1.16a)-(1.1.16c) is given by[5]

$$\begin{aligned} \pi_s &= \gamma q, \\ \gamma &= \frac{1 + \frac{1}{\rho}\frac{n(n-1)q}{((n-1)q^2{\sigma_\epsilon}^2 + {\sigma_l}^2)}}{n\frac{{\sigma_x}^2 + {\sigma_\epsilon}^2}{\rho{\sigma_x}^2{\sigma_\epsilon}^2} + \frac{1}{\rho}\frac{n(n-1)q^2}{((n-1)q^2{\sigma_\epsilon}^2 + {\sigma_l}^2)}}, \\ \pi_0 &= \frac{\frac{n\mu_x}{\rho{\sigma_x}^2}}{n\frac{{\sigma_x}^2 + {\sigma_\epsilon}^2}{\rho{\sigma_x}^2{\sigma_\epsilon}^2} + \frac{n(n-1)q^2}{\rho((n-1)q^2{\sigma_\epsilon}^2 + {\sigma_l}^2)}}, \end{aligned} \tag{1.1.17}$$

where

$$q = \sqrt[3]{\frac{{\sigma_l}^2}{2(n-1)\rho{\sigma_\epsilon}^4}} \left(\sqrt[3]{1 + \sqrt{1 + \frac{4}{27}\frac{\rho^2{\sigma_l}^2{\sigma_\epsilon}^2}{n-1}}} - \sqrt[3]{-1 + \sqrt{1 + \frac{4}{27}\frac{\rho^2{\sigma_l}^2{\sigma_\epsilon}^2}{n-1}}} \right).$$

So, as we conjectured in the beginning , there exists a noisy REE price

$$P(\tilde{s}_1, ..., \tilde{s}_n, \tilde{l}) = \pi_0 + \pi_s \sum_{i=1}^{n} s_i - \gamma l,$$

where π_0, π_s, and γ are defined in (1.1.17).

[5]See the proof of Proposition 2.1 for a detailed exposition of this derivation.

Note that in this equilibrium, when the risky asset price increases, an agent is uncertain whether it is because on average agents are getting better signals or because liquidity of the risky asset is lower. Since agents cannot understand the true reasons for price changes, they cannot fully infer the overall informational content of other agents' signals. Therefore, the noisy REE price is *not fully revealing*, i.e., it is *partially revealing*.

The noisy REE setup, presented in this section, is due to Hellwig (1980). Grossman and Stiglitz (1980) introduces the random liquidity $\tilde{l}$ in a simplified setting. There are only two groups of agents in their model: the informed, who receive the same private signal s, and the uninformed. Given the price conjecture $P(s,l) = \pi_0 + \pi_s\, s - \gamma\, l$, the price provides additional information to the uninformed traders. Since informed traders already know s, price does not provide any additional information to them. Also, due to the presence of random liquidity $\tilde{l}$, uninformed traders can only partially infer the private signal of informed traders. Therefore, as in Hellwig (1980), the noisy REE of Grossman-Stiglitz model is also partially revealing. Note one crucial difference between Hellwig's model and Grosmann and Stiglitz's model: the Grossman-Stiglitz model captures the role of asset prices as conveyers of information, however, it cannot account for the information aggregation role of asset prices since the information is not dispersed among multiple agents in their simplified setup. On the other hand, the noisy REE price of Hellwig setup aggregates the private signals dispersed among multiple agents, as depicted in (1.1.9).

Our elaborations above show that both Hellwig (1980) and Grossman and Stiglitz (1980) obtain partially revealing prices by introducing uncertainty through liquidity. Obviously, there are other ways in which a source of uncertainty can be introduced to our financial market model: one might, for instance, consider uncertainty regarding endowments or preferences. However, in such cases, tractability of analysis is generally lost. Thus, following Hellwig (1980) and Grossman and Stiglitz (1980), most noisy rational expectations models obtain partially revealing prices by letting liquidity of risky asset be random. We will also do so in chapters 2 and 3. In particular, the noisy REE setup of Hellwig (1980) will be the backbone for the model of Chapter 2. Our model will generalize Hellwig's model, and the noisy REE of Hellwig setup, derived above, will provide a benchmark to evaluate the implications of our generalization. Also, the model of Chapter 3 will be a modified version of the setup of Grossman and Stiglitz (1980). Further particulars on the modifications carried out in Chapters 2 and 3 as well as the implications of these modifications can be found in the following sections.

1.2 Motivation of the Dissertation

Section 1.1 has briefly explained the rational expectations approach to information transmission in financial markets. From our discussions in that section, we learned that an equilibrium, which ignores the informational content of price, leads to all agents making systematic forecast errors. This conceptual difficulty brought us to the introduction of fully revealing rational expectations equilibria. We learned that such equilibria are not very interesting as they are not likely to arise: recall that if some agent i uses the information contained in the fully revealing REE price, she can afford to disregard her own private information s_i, and if all agents do so, it becomes unclear why the price should reveal anything on the informations s_1, ..., s_n in the first place. That led us to change our original financial market model by letting liquidity of risky security be random. Then we have constructed a noisy rational expectations equilibrium, which is partially revealing and free from the conceptual difficulties associated with the previous types of equilibria. This, of course, does not mean that noisy rational expectations approach to information transmission has no shortcomings of its own.

For instance, questions such as why subgroups of traders tend to buy or sell the same security *en masse* or to what extent the existence of this behavior is associated with asset prices cannot be answered in the models of noisy rational expectations. Is group psychology the driving force behind the highly correlated trading patterns among certain subgroups? Or, is it because traders make their investment decisions by observing the decisions of those around them? Especially, in the past decade, both economists and laymen have become increasingly interested in these questions. According to the recent empirical studies, the group psychology does not appear to be the predominant factor in the highly correlated trading patterns among subgroups of traders. These studies rather suggest that individual traders are influenced by others around them and this drives them into buying or selling the same security en masse. One of the earliest studies, supporting this argument, is due to Shiller and Pound (1989). Shiller and Pound (1989) survey 131 stock market investors and ask them what prompted their initial interest in their most recent stock purchase or sale. The survey reveals that it was the discussions with peers for the majority of the investors. Another study due to Hong, Kubik, and Stein (2002a) shows that mutual-fund managers are heavily influenced by the decisions of other fund managers working in the same city: a fund manager is more likely to hold (or buy, or sell) a particular stock in any quarter if other managers from different fund families located in the same city are holding (or buying, or selling) that same stock. The authors interpret this using an epidemic model where investors spread information about stocks directly to one another by word of mouth. In a different empirical work (2002b), the same authors also argue that stock market participation is influenced by social interaction, i.e., "social investors" find market more attractive when more of their peers participate. In light of these empirical studies, it is obvious why noisy rational expectations models cannot explain the en masse purchases or sales of the same stock among certain subgroups of traders: these models assume away social interaction as conveyer of information and let agents infer information only from the price system. In other words, in the noisy REE models, agents do not communicate or learn from each others' actions. That is exactly what we try to change in the rational expectations approach to information transmission. Our first essay introduces social interaction as a conveyer of information by generalizing Hellwig's (1980) model, given in § 1.1.3. In this essay, we model social interaction in the form of observations of others' actions. In particular, we let agents infer information from risky asset demands of some of the other agents. Whom they observe is determined by a social network. In our generalized model, agents choose their risky asset demands conditional on their private signals, price, and demands of others whom they observe in the social network. Such a generalization allows us to answer the questions posed in the beginning and compare the implications of our theoretical setting with the empirical findings given above.

Noisy rational expectations models are also far from giving a compelling explanation regarding the causes and common features of crashes and frenzies. One common feature of these large price swings is that they are often disproportionate reactions to tangible information. Cutler, Poterba and Summers (1989) document this for the postwar movements in the S&P 500 index. Another feature of crashes and frenzies is the *asymmetry* in the frequency of their appearances: the number of crashes is higher than that of frenzies. For instance, Hong and Stein (2002) report that nine of the ten largest one-day price movements in the S&P 500 since 1947 were decreases. Boldrin and Levine's (2001) analysis of S&P 500 between 1889 and 1984 also confirms the asymmetry. According to their study, there is one annual negative deviation with magnitude larger than 50% but no positive deviation exceeds this value. The number of annual negative deviations with magnitudes larger than 40% is 3 and that of positive deviations is none. 6 annual negative deviations have magnitudes exceeding 30% compared to 4 positive ones exceeding the same value. Magnitudes of 14 annual negative deviations exceed 20% and magnitudes of only 10 positive deviations exceed this value.

Our second essay tries to account for the common features of large price swings, discussed above. To do so, we modify the model of Grosmann and Stiglitz (1980), given in § 1.1.3: in addition to informed and uninformed agents, we introduce hedgers to our model. Hedgers use portfolio insurance strategies, which means they sell after the market declines and buy after the market rises. Obviously, the strategies followed by hedgers are in contrast with the conventional supply schedules, which are increasing functions of price. Several empirical studies provide evidence regarding the existence of hedgers in the stock market and their role in large price swings. One of them is Brady Commission Report (1988), which blames portfolio insurance strategies for deepening the decline hence perhaps causing the crash during 1987. The studies of Chicago Mercantile Exchange, Miller, Hawke, Malkiel, and Scholes (1987), Commodity Futures Trading Commission (1987), Securities and Exchange Commission (1987) also emphasize the role of hedgers in the 1987 crash. The stop-loss orders, which are primitive portfolio insurance strategies, are also seen as a possible contributing factor to the 1929 crash (Gennotte and Leland (1990)). All these studies validate the need for addition of hedgers to the standard Grossman-Stiglitz model while investigating large price swings. Gennotte and Leland (1990) are the first ones making this modification, and following them, we use a similar modified model for our analysis. We will discuss the implications of this modification in detail in the next section, but before doing so, let us provide an intuitive explanation regarding why presence of hedgers can be the key factor to account for the common features of crashes and frenzies. First, let us consider the stop-loss orders. Hedgers, employing stop-loss orders, sell the asset after the asset price falls under some exercise value. The aim of this is to protect one's portfolio against future potential losses. However, by employing stop-loss orders, hedgers put an additional downward pressure on the asset price once the price begins to fall. This means that even if tangible information initially triggers a small decline in price, the use of stop-loss orders will insure a further fall in price due to the sales coming from hedgers. Hence, we have the desired disproportionate price reaction to tangible information. On the other hand, when asset price increases due to new information, stop-loss orders do not bring any further purchases, hence, they do not cause additional increase in asset price. So, we are likely to observe an asymmetry biased towards crashes in a stock market where stop-loss orders prevail. The mechanics of modern hedging strategies, such as put-option replication, is quite similar to that of stop-loss orders except that we see purchases from hedgers in a bull market and sales in a bearish one. So, both price declines and increases become disproportionate reactions to tangible information triggering the initial reaction. Moreover, if hedgers, on average, make larger sales compared to their purchases, the number of crashes will be likely to be higher than that of frenzies. In our second essay, we will argue that hedgers do make larger sales than purchases, on average, when they predominantly employ put-option replication strategies in a volatile stock market.

To briefly summarize our discussions in this section, our dissertation focuses on two main shortcomings of noisy rational expectations models. First, these models cannot explain the highly correlated trading patterns observed among subgroups of traders and how this relates to the asset prices. In light of the recent empirical studies, our proposal to overcome this shortcoming is to introduce social interaction as a medium of information transmission, on top of the price system. Second, the noisy REE models cannot provide a compelling explanation regarding certain common features of crashes and frenzies. We argue that introducing hedgers into the noisy REE models will help us to account for the mentioned features. Several empirical studies are in line with our argument and they provide evidence regarding the pivotal role of hedgers in large price swings.

1.3 Overview of the Dissertation

This dissertation consists of two self-contained essays. These essays propose new models to overcome the shortcomings of (noisy) rational expectations approach listed above.

Our first essay proposes a generalized rational expectations model which accommodates social interaction in financial markets. On top of the information conveyed through the price system, each agent infers additional information by observing some of the other agents' security demands. Whom an agent observes is determined by a directed graph that represents the social network. We define a notion of equilibrium for this generalized framework and investigate the implications of our model in several natural settings, including cyclic interaction schemes (where the social network is a cycle) and hierarchic interaction schemes (where the network is a tree). Three results from this essay are especially worth mentioning. First, an agent's signal affects price less than her uphill neighbor's signal in a hierarchic interaction scheme when social interaction is the only conveyer of information. In other words, hierarchy in observation leads to hierarchy in influence regarding security pricing. Second, the essay strongly suggests that agents located across different clusters of a social network make different portfolio decisions while those in the same cluster make similar ones. Third, social interaction can lead to inefficient information aggregation in the price system. That is, in the presence of social interaction, some agents' signals can affect price more than other agents' signals regardless of the signals' precisions. This means that inefficient information aggregation allows for an imprecise signal to have a disproportionately large impact on price. Therefore, the inefficiency brought by social interaction may help us to account for crashes and frenzies.

Our second essay tries to explain why large price movements are often disproportionate reactions to tangible information and why they exhibit asymmetry (i.e., why number of crashes exceed number of frenzies). We attribute these phenomena to an asymmetric amplification mechanism which is illustrated in a rational expectations framework. The formal setup is a simplified version of Gennotte and Leland (1990) where the simplification allows for a unique and easily tractable equilibrium. There are four types of traders in our model. Insider receives a private random signal on the payoff structure of assets. In addition, we have risk averse uninformed outsiders, liquidity traders whose demands are price inelastic, and hedgers. The hedgers employ portfolio insurance strategies and sell assets as the price declines and buy when the price increases. The presence of such traders causes a destabilizing effect. A small shift in other traders' demand, possibly due to incoming news or a liquidity shock, now causes a larger price impact since hedgers are simply following the price trend in their supply schedules. In the case when hedging supply is a convex function of price, hedgers overreact to price declines in their orders. Hence, convexity of hedging strategies deliver the desired asymmetry. We prove that an important class of hedging strategies (namely, put-option replication) indeed satisfies the convexity condition in a highly volatile market. We also analyze the roles of risk aversion and differential information in our analysis: risk aversion is essential for the asymmetry of price movements and differential information enhances both amplification and asymmetry delivered by hedging.

Remarks. Section 1.1 relies on Admati (1989), Brunnermeier (2001), Hellwig (1980), and Huang and Litzenberger (1988). For a detailed account of rational expectations equilibrium in a broader context, reader may refer to Jordan and Radner (1982).

Chapter 2

Rational Expectations and Social Interaction in Financial Markets

> Investing in speculative assets is a social activity. Investors spend a substantial part of their leisure time discussing investments, reading about investments, or gossiping about others' successes or failures in investing.
>
> Robert J. Shiller[1]

2.1 Introduction

Interaction among heterogenous agents is an important and pervasive feature of any economic environment. In standard noisy rational expectations models,[2] agents interact only through the price system: they make their decisions in *isolation*, using only their private information and the information con-

[1]Shiller (1984)

[2]For a critical survey, see Admati (1989).

veyed by security prices.[3] Yet, in the real world, agents communicate, and learn from each others' actions. That is, the *economic agent* is also a *social agent.*

The general objective of this essay is to introduce social interaction into the modelling of financial markets. In particular, we propose a rational expectations model in which each agent observes security demands of some of the other agents, whom we refer to as "*uphill neighbors*",[4] on top of security price and her private signal on risky security payoff. Our model assumes that social interaction takes place through observations of uphill neighbors' demands. The *uphill neighborhood* of an agent is determined by a given directed graph, which represents the social network. Such a model obviously encompasses the conventional rational expectations model: once the underlying social network is taken to be a trivial graph without any edges, all agents are isolated, and we return to the standard rational expectations economy where only price conveys information.

Our generalized framework presents a new conveyer and aggregator of information on top of price: uphill neighbors' demands. The information inferred from an uphill neighbor's demand not only refers to that uphill neighbor's private signal but also aggregates signals of agents observed by that uphill neighbor, and signals of those who are observed by agents observed by that uphill neighbor, and so on. We also have a byproduct due to this generalized framework: in addition to risk aversion and information accuracy, asymmetric interaction patterns in the given social network are also a source of heterogeneity for the economy.

The main novelty of this essay is that it accommodates both social interaction and price as informational conveyers in a *tractable* model which exploits parametric assumptions of constant absolute

[3] For a critical assessment of interaction in economic systems, see Kirman (1996).

[4] We use the term "uphill neighbor" rather than "neighbor" to emphasize that observations are not necessarily bilateral. That is, for agents A and B of the economy, all of the following interaction patterns are possible:

- A observes B's demand while B does not observe A's, in which case B is an uphill neighbor of A;
- B observes A's demand while A does not observe B's, in which case A is an uphill neighbor of B;
- Both A and B observe each other's demands, in which case both A and B are uphill neighbors of each other.

risk aversion and normal distribution. These assumptions considerably simplify the analysis and they facilitate closed-form solutions.

This essay provides a thorough investigation of social interaction's effect on security pricing and portfolio decision-making. In this investigation, we consider various natural settings, including cyclic interaction schemes (where the social network is a cycle) and hierarchic interaction schemes (where the social network is a tree). Three implications of our model are especially worth mentioning. First, it will be seen that an agent's signal affects price less than her uphill neighbor's signal in a hierarchic interaction scheme when social interaction is the only conveyer of information. In other words, hierarchy in observation will lead to hierarchy in influence regarding security pricing. Second, in a stylized social network, we will see that agents located across different clusters of the network make different portfolio decisions while those in the same cluster make similar ones. Third, we will show that social interaction can lead to inefficient information aggregation in the price system. That is, in the presence of social interaction, some agents' signals can affect price more than other agents' signals regardless of the signals' precisions. The inefficiency brought by social interaction may help us to account for crashes and frenzies. Cutler, Poterba and Summers (1989) document that there were virtually no significant events prior to many large price swings in the stock market. Since inefficient information aggregation allows for an imprecise signal to have a disproportionate impact on price, a significant event is not necessary to trigger a price swing in the presence of social interaction.

There have been several empirical studies highlighting the role of social interaction as a conveyer of information in financial markets. One of the earliest studies is due to Shiller and Pound (1989): their survey questions 131 investors in the stock market. Majority of these investors asserted that their initial interest in their most recent stock purchase was prompted by discussions with their peers. Of course, the evidence here is only suggestive, but the idea of information transmission via social interaction seems very reasonable in light of this survey. Recent empirical studies provide further evidence of information transmission through social channels in financial markets. Hong, Kubik, and Stein (2002a) show that mutual-fund managers are heavily influenced by the decisions of other fund managers working in the same city. In particular, authors observe the following pattern in fund managers' decisions: a fund manager is likely to hold (or buy, or sell) a particular stock in any quarter if other managers from different fund families located in the same city are holding (or buying, or selling) that same stock. The authors interpret this using an epidemic model where investors spread information about stocks directly to one another by word of mouth. In a different empirical work (2002b), the same authors also argue that stock market participation is influenced by social interaction, i.e., "social investors" find market more attractive when more of their peers participate.

It is also worth mentioning the theoretical literatures, to which our essay is related. In the models of social learning theory[5], social interaction takes place through sequential observations of others' actions over time. The memory of past actions may reduce (or completely prevent) social learning in these models. This causes inefficient information aggregation, as is the case in our essay. However, unlike social learning theory, we let agents interact according to a social network in a static financial market economy. Another closely related paper is due to DeMarzo, Vayanos and Zwiebel (2003). Their paper proposes a boundedly-rational model of opinion formation in social networks.[6] The pivotal assumption of their model is *persuasion bias*, which allows double counting of repeated information. This means if an agent receives information from different agents, who happen to share information with each other, the agent acts as if the pieces of information received from these agents are mutually independent. This assumption leads to *social influence*: agents, who are "well-connected" in the social network, may have more influence in the overall formation of opinions in the economy regardless of their information accuracies. We obtain a similar result in our essay, however, we do not dispense with rationality.

[5]The seminal papers in this literature are Banerjee (1992), Bikchandani, Hirshleifer and Welch (1992), and Ellison and Fudenberg (1995). Avery and Zemsky (1998) investigate the implications of social learning theory in the financial markets. Bikchandani and Sharma (2001), and Devenow and Welch (1996) provide extensive literature reviews on social learning in financial markets.

[6]In a companion paper (2001), the authors explore the implications of their original model in financial markets.

Our essay is organized as follows. In Section 2.2, we recall the standard rational expectations model (à la Hellwig), and obtain the unique rational expectations equilibrium (REE) for the economy consisting of agents homogenous in risk aversion and signal precision. The obtained REE serves as a benchmark for us to assess the implications of the presence of social interaction. Section 2.3 exhibits our generalized REE model which accommodates social interaction in the financial market. We explain how we model social interaction and introduce the equilibrium concept for the generalized framework. Sections 2.4 and 2.5 investigate the implications of our model in several natural settings, including cyclic interaction schemes (cycles) and hierarchic interaction schemes (trees). In Section 2.4, we show the non-existence of linear equilibrium price when the social network is a cycle. In Section 2.5, we prove existence of linear equilibrium price for hierarchic interaction schemes (trees) provided there is sufficiently volatile liquidity in the economy. Later we characterize the obtained equilibrium under certain restrictions regarding the pattern of interaction and level of liquidity variance. Section 2.6 investigates how presence of disjoint clusters in a social network affects portfolio decision-making. In particular, the section analyzes social networks consisting of multiple disjoint stars. Section 2.7 exhibits that social interaction can impair aggregation of dispersed private information through the price system. Section 2.8 concludes. The proofs are provided in the appendix.

2.2 Rational Expectations when Only Price Conveys Information

We begin by recalling the standard rational expectations approach to information transmission in financial markets: when only price conveys information. The following description is due to Hellwig (1980).

There are $n \geq 2$ agents, indexed by $i = 1, ..., n$. In a two-period economy, trade takes place in the first period and consumption of a single good in the second. A risk-free security and a risky security are traded. The risky security has a future stochastic payoff $\tilde{X}$, which realizes in the second period.

The price and the payoff of the risk-free security are normalized to 1. We let p be the price of the risky security.[7] Each agent i is endowed with deterministic wealth w_{0i} (in units of consumption good). If agent i purchases z_i units of the risky security, her portfolio yields the random final wealth

$$\tilde{w}_{1i} = z_i \tilde{X} + (w_{0i} - p z_i).$$

We specify agents' preferences by the following assumption:

A1. *All agents have CARA preferences: for $i = 1, ..., n$, agent i's expected utility of final wealth is $\mathrm{E}_i[u_i(\tilde{w}_{1i})] = \mathrm{E}_i[-exp(-\rho_i \tilde{w}_{1i})]$, where $\rho_i \in (0, \infty)$ denotes (absolute) risk aversion coefficient. The expectation operator, E_i, is conditional on agent i's information $\mathcal{I}_i$.*

Under this assumption, agent i's demand is independent of her initial wealth w_{0i}. It only depends on price p and information $\mathcal{I}_i$.

Prior to the first period, each agent i receives a private random signal $\tilde{s}_i$, which communicates the true stochastic payoff $\tilde{X}$ perturbed by some additive noise $\tilde{\epsilon}_i$, i.e., $\tilde{s}_i = \tilde{X} + \tilde{\epsilon}_i$. Also, the net supply (liquidity) of the risky security L is taken to be the realization of a random variable $\tilde{L}$. We impose normal distributions for the random parameters of the economy:

A2. *The random[8] vector $(\tilde{X}, \tilde{L}, \tilde{\epsilon}_1, ..., \tilde{\epsilon}_n)$ is normally distributed with mean $(\mu_x, 0, 0, ..., 0)$, and nonsingular variance-covariance matrix $({\sigma_x}^2, {\sigma_L}^2, {\sigma_{\epsilon_1}}^2, ..., {\sigma_{\epsilon_n}}^2)\, I_{n+2}$, where I_{n+2} denotes the $(n+2)$ dimensional identity matrix.*[9]

Price p that prevails in the market depends on the realized liquidity L and the private signals $s_1, ..., s_n$. Considering the whole range of realizations of the random variables $\tilde{L}$ and $\tilde{s}_1, ..., \tilde{s}_n$, the

[7] From now on, the terms *price* and *demand* will be exclusively used for the risky security price and demand, respectively, unless otherwise stated.

[8] Throughout the text, we use the following convention: random variables are denoted with *tilde* (such as $\tilde{y}$), and the realizations of random variables are denoted without *tilde* (such as y).

[9] Nonsingularity guarantees ${\sigma_x}^2 \neq 0$, ${\sigma_L}^2 \neq 0$, ${\sigma_{\epsilon_i}}^2 \neq 0$, $i = 1, ..., n$.

realized market prices generate a random variable $\tilde{p}$. The following assumption imposes the hypothesis that expectations, determined through the observations of private signals and price, are rational.

A3. *For $i = 1, ..., n$, agent i knows the joint distribution of the random vector $(\tilde{X}, \tilde{s}_i, \tilde{p}, \tilde{L})$ and the realizations s_i, p. In particular, agent i's expectation operator, E_i, is conditional on the information $\mathcal{I}_i = (s_i, p)$.*

For the described economy, the following definition is standard:

DEFINITION OF EQUILIBRIUM. *A rational expectations equilibrium (REE) consists of a risky security price function $P(s_1, ..., s_n; L)$ and demands $\{z_i(s_i, p)\}_{i=1,...,n}$ such that for all realizations of $(s_1, ..., s_n; L)$ of $(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L})$*

(a) $z_i(s_i, p) \in argmax_{z_i} \mathrm{E}\left[u_i(\tilde{w}_{1i}) \middle| s_i, p = P(s_1, ..., s_n; L)\right], \qquad \forall i = 1, ..., n;$

(b) $\sum_{i=1}^{n} z_i(P(s_1, ..., s_n; L), s_i) = L.$

Hellwig (1980) shows the existence of a *linear* REE price under assumptions A1-A3, and provides a partial characterization of the equilibrium. A closed-form solution for linear equilibrium is not available in the full generality of the model. Neither is the uniqueness of linear equilibrium guaranteed. Both of these properties have been established for the linear REE in a large economy: by Hellwig (1980), in the limit of linear REE where the number of agents tends to infinity, and by Admati (1985), in a model with a continuum of agents.[10]

To facilitate the analysis of social interaction later in this essay, we retain the finite-agent economy. We impose the following assumption in order to obtain a unique linear REE in closed-form:

A4. *For all $i = 1, ..., n$, $\rho_i = \rho$ and ${\sigma_{\epsilon_i}}^2 = {\sigma_\epsilon}^2$.*

[10] Admati's (1985) model is a generalization of Hellwig's (1980) to the multi-security case.

Assumption A4 simplifies the analysis by ruling out heterogeneity in risk aversion and signal precision. Since we will be imposing A4 in the presence of social interaction as well, the following result is a benchmark for us:

Proposition 2.1 *Assume A1, A2, A3, and A4. There exists a unique linear REE price*

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \pi_0 + \sum_{i=1}^{n} \pi_i \, \tilde{s}_i - \gamma \, \tilde{L} \tag{2.2.1}$$

with

$$\begin{aligned} \pi_i &= \gamma \, q, \qquad i = 1, ..., n, \\ \gamma &= \frac{1 + \frac{1}{\rho} \frac{n(n-1)q}{((n-1)q^2\sigma_\epsilon{}^2 + \sigma_L{}^2)}}{n \frac{\sigma_x{}^2 + \sigma_\epsilon{}^2}{\rho\sigma_x{}^2\sigma_\epsilon{}^2} + \frac{1}{\rho} \frac{n(n-1)q^2}{((n-1)q^2\sigma_\epsilon{}^2 + \sigma_L{}^2)}}, \\ \pi_0 &= \frac{\frac{n\mu_x}{\rho\sigma_x{}^2}}{n \frac{\sigma_x{}^2 + \sigma_\epsilon{}^2}{\rho\sigma_x{}^2\sigma_\epsilon{}^2} + \frac{n(n-1)q^2}{\rho((n-1)q^2\sigma_\epsilon{}^2 + \sigma_L{}^2)}}, \end{aligned}$$

where

$$q = \sqrt[3]{\frac{\sigma_L{}^2}{2(n-1)\rho\sigma_\epsilon{}^4}} \left(\sqrt[3]{1 + \sqrt{1 + \frac{4}{27} \frac{\rho^2\sigma_L{}^2\sigma_\epsilon{}^2}{n-1}}} - \sqrt[3]{-1 + \sqrt{1 + \frac{4}{27} \frac{\rho^2\sigma_L{}^2\sigma_\epsilon{}^2}{n-1}}} \right).$$

The weights of agents' signals in the REE price (2.2.1) are equal to each other. That is, each agents' signal has the same effect on the equilibrium price as long as agents are homogenous in their signal precisions and risk aversions. The forthcoming sections will show that the equilibrium properties established here do not necessarily hold in the presence of social interaction. In particular, we will see that existence and uniqueness of a linear equilibrium price fail for certain social interaction schemes. Also, each agent's signal will no longer have the same effect on equilibrium price.

2.3 Rational Expectations with Social Interaction

In this section, we propose a framework which allows formation of rational expectations in the presence of social interaction. Our model retains basic features of the model considered in §2.2 with the notable exception of assumption A3. In particular, we carry over the description of the physical environment, and assumptions A1, A2, A4. Assumption A3 is replaced by S3.

We consider financial markets in which information is transmitted not only through prices but also through direct interaction among agents. The latter allows agents to infer information from the actions of those whom they interact with. Each agent interacts with a few of the other agents, and the pattern of interaction among all agents defines a social network. The social network is modelled by a simple directed graph,[11] with vertices representing the agents, and directed edges representing the directions of information transmission. In particular, our model assumes that information transmission takes place through demand observations. That is, agents observe the demands of those to whom they are linked by the edges of the network, where the direction of each edge indicates the direction of observation. Agents also observe their private signals and price.

The agents whose demands are observed by agent i are called i's *uphill neighbors*, and the set of i's uphill neighbors is denoted by $\mathcal{N}_i$. So, agent i observes risky security demand z_j of agent j for all $j \in \mathcal{N}_i$. The set $\mathcal{N}_i$ may be empty. Conditional on the information inferred from the private signal s_i, price p, and demand observations $\{z_j\}_{j\in\mathcal{N}_i}$, agent i derives an expectation for the risky security payoff $\tilde{X}$, and then determines her own demand based on this expectation.

Price p and demands $\{z_i\}_{i=1,\ldots,n}$ that prevail in the market depend on the realized signals s_1, ..., s_n and liquidity L. Therefore, the random vector $(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L})$ generates random variables $\tilde{p}$ and $\{\tilde{z}_i\}_{i=1,\ldots,n}$. The following assumption imposes the hypothesis that expectations, derived from private signals, price, and demand observations, are rational.

S3. *For $i = 1, ..., n$, agent i knows the joint distribution of $\left(\tilde{X}, \tilde{s}_i, \tilde{p}, \{\tilde{z}_j\}_{j\in\mathcal{N}_i}, \tilde{L}\right)$, and the realizations s_i, p, $\{z_j\}_{j\in\mathcal{N}_i}$. In particular, agent i's expectation operator, E_i, is conditional on the*

[11] A graph is called *simple* if multiple edges between the same pair of vertices or edges connecting a vertex to itself are forbidden. A graph is called *directed* if edges exhibit inherent direction, implying every relationship so represented is asymmetric.

information $\mathcal{I}_i = (s_i, p, \{z_j\}_{j\in\mathcal{N}_i})$.

We employ the following equilibrium definition for the economy described above:

DEFINITION OF EQUILIBRIUM. *A rational expectations equilibrium with social interaction (REESI) consists of a risky security price function* $P(s_1, ..., s_n; L)$ *and demands* $\{z_i(s_i, p, \{z_j\}_{j\in\mathcal{N}_i})\}_{i=1,...,n}$ *such that for all realizations* $(s_1, ..., s_n; L)$ *of* $(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L})$

(a) $z_i(s_i, p, \{z_j\}_{j\in\mathcal{N}_i}) \in argmax_{z_i}\mathrm{E}\left[u_i(\tilde{w}_{1i})\middle|s_i, p = P(s_1, ..., s_n; L), \{z_j = z_j(s_j, p, \{z_k\}_{k\in\mathcal{N}_j})\}_{j\in\mathcal{N}_i}\right]$, $\forall\, i = 1, ..., n$,

(b) $\sum_{i=1}^{n} z_i(s_i, P(s_1, ..., s_n; L), \{z_j\}_{j\in\mathcal{N}_i}) = L$.

The first equilibrium condition implies that each agent maximizes her expected utility of final wealth based on her private signal as well as information inferred from price and uphill neighbors' demands. The second condition states that market clears. Together the two conditions are consistent in the sense that the price and demands, which clear the market, correctly reflect the decision procedure of agents, and agents' optimal decisions correctly rely on information inferred from the price and uphill neighbors' demands.

Note that the basic REE model of §2.2 is a special case of the model introduced here: if the social network is a graph without any edges, meaning that no agent observes another agent's demand, then S3 reduces to A3 and REESI reduces to REE. As it is common in the literature on rational expectations, we will focus on a REESI price that is linear in private signals and liquidity, that is, REESI of the form

$$P(s_1, ..., s_n; L) = \pi_0 + \sum_{i=1}^{n} \pi_i s_i - \gamma L.$$

Also, from now on we will use the terms *equilibrium* and *linear equilibrium price* to refer to *REESI* and *linear REESI price*, respectively.

In the remainder of this essay, the analysis is restricted to the social networks where each agent i has at most one uphill neighbor. Such restriction reflects the *limited interaction capability* of agents, but it is imposed primarily for the sake of tractability. The unique agent (if exists), who is the uphill neighbor of i, will be denoted by i^-.

2.4 Non-Existence of Linear Equilibrium in Cycles

In this section we explore the information transmission in a financial market economy in which the social network is a cycle. The cycle represents a symmetric interaction scheme in the sense that each agent observes demands of the same number of agents. We find that social interaction in a cycle leads to non-existence of a linear equilibrium price.

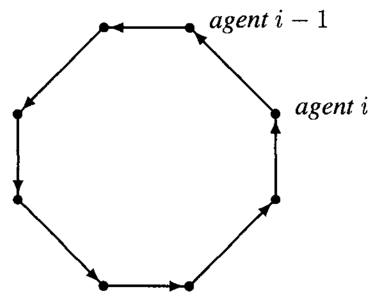

Figure 2.1: A CYCLE REPRESENTING SOCIAL INTERACTION

Arrows represent the direction of demand observations. For all $i = 1, .., n$, *agent* i *has information that comes from demand of agent* $i - 1 \pmod{n}$ *on top of* s_i *and* p.

Proposition 2.2 *Assume A1, A2, S3, A4. If the social network is a cycle, there does not exist any linear REESI price.*

The key to this non-existence result lies in the interaction pattern created by the cycle: each agent's inference from social interaction contains information from all other agents' inferences from social interaction, including her own. Thus, the agent forms expectation on the payoff of the risky security by referring to her very own expectation. However, this *self-referral* leads to *infinite regress*,[12] that is, the agent cannot disentangle her own expectation from others expectations' while she tries to infer information from social interaction, and hence cannot form an expectation at all. Therefore an equilibrium cannot exist.

One can also approach this infinite regress problem by considering price and demand formation in a Walrasian tâtonnement. For clarity and simplicity, let us consider a 2-agent economy with agents 1 and 2. Obviously, 1 and 2 are uphill neighbors of each other in a cycle. The Walrasian auctioneer is unaware of agents' signals and begins the auction by announcing a price p_0 *au hasard.* Agents know their own signals and reveal their asset demands to the auctioneer knowing that trade at p_0 will take place if and only if p_0 clears the market. Thus each agent's revealed demand takes account of the information carried by p_0, meaning that agents' demands are of the form $z_{i,0}(p_0; s_i),\ i = 1, 2$. No actual trading occurs until each agent has a chance to revise her demand, so the auctioneer calls a new price p_1 conditional on the information inferred from revealed demands at p_0. Simultaneously, agents learn their uphill neighbors' revealed demands at p_0. Thus, each agent's revealed demand takes account of the information carried by prices p_0, p_1, and her uphill neighbor's demand at p_0, i.e., revealed demands at p_1 take the form $z_{i,1}\left(p_0, p_1; z_{j,0}(p_0; s_j); s_i\right),\ i \neq j$. So, by the time p_1 is announced, agents begin to forecast the forecasts of their uphill neighbors on the asset's payoff. Next, auctioneer announces price p_2 conditional on the revealed demands at p_0 and p_1 while agents learn their uphill neighbors' revealed demands at p_1. Given this new information, agents' revealed demands are given by $z_{i,2}\left(p_0, p_1, p_2; z_{j,0}(p_0; s_j), z_{j,1}\left(p_0, p_1; z_{i,0}(p_0; s_i); s_j\right); s_i\right),\ i \neq j$. That is, by the time p_2 is announced, agents not only forecast their uphill neighbors' forecasts, but also forecast the forecasts that their uphill neighbors make of their forecasts. As new prices are announced by the auctioneer in order to converge to a market clearing price, this regress on forecasts continue ad infinitum. Since the auctioneer announces prices conditional on agents' revealed demands, he faces the same infinite regress

[12]See Townsend (1983a, 1983b), or Singleton (1987) for more on the issues pertaining to infinite regress.

problem that agents do, and cannot reach a market clearing price for given s_1, s_2.

Recall from Proposition 2.1 that a linear equilibrium always exists in an economy in which only price conveys information. Proposition 2.2 shows that additional information transmission through social interaction can cause the dissolution of linear equilibrium. Therefore, this section gives a good indication of the large impact of social interaction on price formation in the market.

2.5 Existence and Characterization in Hierarchies

We now turn our focus to social networks in the form of acyclic graphs, in particular, trees. Trees have a natural appeal, because they represent *hierarchic schemes* of social interaction. They also bring asymmetry into the interaction pattern, hence heterogeneity into the financial market economy.

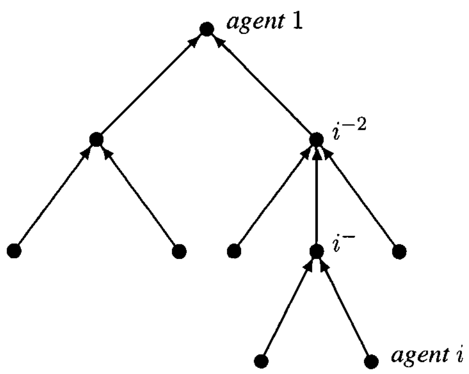

Figure 2.2: A TREE REPRESENTING SOCIAL INTERACTION

Arrows indicate the direction of demand observations. For all $i > 1$, agent i has information that comes from observing demand of agent i^- (i.e., her predecessor in the tree) on top of her private signal s_i and price p. Agent 1 only observes s_1 and p.

Before going into the formal analysis of trees, we introduce some convention and notation for this interaction scheme. Without loss of generality we take agent 1 to be the root of the tree. For any agent $i > 1$, i^- (or equivalently i^{-1}) denotes the uphill neighbor of i in the tree. i^{-2} denotes the uphill neighbor of i^-, and i^{-k} denotes the uphill neighbor of i^{-k+1}, $k > 2$. We refer to i^{-k} as the k^{th} *predecessor of agent* i, $k \geq 1$. Also, as a notational convenience, i^{-0} denotes agent i herself. Given j as a predecessor of i, we let $l_{i,j}$ denote the integer that satisfies $i^{-l_{i,j}} = j$. That is, $l_{i,j}$ gives the order of precedence of agent j according to agent i in the tree. Thus $l_{i,i} = 0$ and $l_{i,i^-} = 1$. For any agent i, $\mathcal{H}(i)$ denotes the set of predecessors of i, i.e., $\mathcal{H}(i) = \left\{i^{-k} \in \{1, ..., n\} : 1 \leq k \leq l_{i,1}\right\}$. Note that $1 \in \mathcal{H}(i)$ for all $i > 1$. On the other hand, $\mathcal{H}^{-1}(i)$ denotes the set of agents that are preceded by agent i, i.e., $\mathcal{H}^{-1}(i) = \left\{j \in \{1, ..., n\} : i = j^{-k},\ \ k \geq 1\right\}$. Therefore $\mathcal{H}^{-1}(1) = \{2, 3, ..., n\}$, and $i \in \mathcal{H}^{-1}(i^-)$ for all $i > 1$. The elements of $\mathcal{H}^{-1}(i)$ are referred to as the *successors of agent* i.

2.5.1 Existence with Sufficiently Volatile Liquidity

Section 2.4 has already established that a linear equilibrium does not necessarily exist for any given social network. The particular problem with social interaction in cycle was *self-referral* in the formation of expectations. Obviously, self-referral is no more a problem in hierarchic interaction schemes (i.e., trees) since no agent is a predecessor of herself and therefore no agent refers back to her own expectation through social interaction. In fact, if the social network is a tree, we can prove the existence of a linear equilibrium provided there is sufficiently volatile liquidity in the economy:

Proposition 2.3 *Assume A1, A2, S3, A4. Suppose the social network is a tree. There is a level of liquidity variance* $\underline{\sigma_L}^2 < \infty$ *such that for all* $\sigma_L{}^2 \geq \underline{\sigma_L}^2$*, a linear REESI price exists.*

To see the role of liquidity variance in existence, we need to explain a complication that comes along with information transmission through social interaction.

First, recall the *dual role* of price in the standard REE models: price allocates assets through market clearing and it also conveys information. Therefore, the determination of equilibrium (REE) can be considered as a fixed-point problem in the space of functions relating the asset price to signals and liquidity. Formally speaking, given any function $\phi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, we can initially assume that all agents act on the hypothesis that price function is given by $\tilde{p} = \phi(\tilde{s}_1, ..., \tilde{s}_n, \tilde{L})$. For all $i = 1, ..., n$, the function ϕ determines the joint distribution of $(\tilde{X}, \tilde{s}_i, \tilde{p}, \tilde{L})$ and therefore the conditional distribution of $\tilde{X}$ given the realized information $\mathcal{I}_i = (s_i, p)$. This implies that each agent i's demand z_i depends on the price p, the signal s_i, and the function ϕ. Since market clearing dictates $\sum_{i=1}^{n} z_i(p, s_i; f) = L$, we can derive a new function $T\phi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ such that $T\phi(s_1, ..., s_n, L)$ is the market clearing price given that agents form their expectations conditional on the hypothesis that $\tilde{p} = \phi(\tilde{s}_1, ..., \tilde{s}_n, \tilde{L})$. Expectations being rational, ϕ has to be a fixed point of T, meaning that

$$T\phi(s_1, ..., s_n, L) = \phi(s_1, ..., s_n, L), \quad \forall (s_1, ..., s_n, L) \in \mathbb{R}^{n+1}.$$

Such a fixed point ϕ of T delivers us a REE.

On the other hand, when social interaction is present, both equilibrium demands and price play the so-called dual roles: they both clear the market and transmit information. Therefore, in the presence of social interaction, the determination of equilibrium (REESI) can be formulated as a fixed-point problem in the space of functions relating both asset price and demands to signals and liquidity. Given arbitrary functions $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ and $g_i : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, $i = 1, ..., n$, suppose that each agent i acts on the hypothesis that price function and her uphill neighbors' demands are given by $\tilde{p} = f(\tilde{s}_1, ..., \tilde{s}_n, \tilde{L})$ and $\tilde{z}_j = g_j(\tilde{s}_1, ..., \tilde{s}_n, \tilde{L})$, $j \in \mathcal{N}_i$, respectively. Then agent i's demand z_i depends on the price p, the signal s_i, uphill neighbors' demands $\{z_j\}_{j\in\mathcal{N}_i}$ as well as the functions f and $\{g_j\}_{j\in\mathcal{N}_i}$, which determine the joint distribution of the random vector $(\tilde{X}, \tilde{s}_i, \tilde{p}, \{\tilde{z}_j\}_{j\in\mathcal{N}_i}, \tilde{L})$. Let $h = (f, g_1, ..., g_n)$. From the market clearing condition

$$\sum_{i=1}^{n} z_i(p, s_i, \{z_j\}_{j\in\mathcal{N}_i}; f, \{g_j\}_{j\in\mathcal{N}_i})) = L,$$

a new function $\mathcal{T}h : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ can be derived such that $(\mathcal{T}h(s_1, ..., s_n, L))_1$ is the market clearing price and $(\mathcal{T}h(s_1, ..., s_n, L))_{i+1}$, $i = 1, ..., n$, are the market clearing demands[13] given that each agent i

[13]For any given vector $\zeta \in \mathbb{R}^n$, ζ_k denotes the kth component of ζ, i.e., $\zeta = (\zeta_1, ..., \zeta_k, ..., \zeta_n)$.

forms her expectation conditional on the hypothesis that $\tilde{p} = f(\tilde{s}_1, ..., \tilde{s}_n, \tilde{L})$ and $\tilde{z}_j = g_j(\tilde{s}_1, ..., \tilde{s}_n, \tilde{L})$, $j \in \mathcal{N}_i$. The rationality of expectations requires that the vector $h = (f, g_1, ..., g_n)$ is a fixed point of $\mathcal{T}$, meaning that for all $(s_1, ..., s_n, L) \in \mathbb{R}^{n+1}$,

$$\begin{aligned} (\mathcal{T}h(s_1, ..., s_n, L))_1 &= f(s_1, ..., s_n, L), \quad \text{and} \\ (\mathcal{T}h(s_1, ..., s_n, L))_{i+1} &= g_i(s_1, ..., s_n, L), \quad i = 1, ..., n. \end{aligned}$$

In this formulation, a fixed point $(f, g_1, ..., g_n)$ of $\mathcal{T}$ delivers us a REESI.

Obviously, in the determination of REESI we face a more demanding and complicated fixed point problem compared to the case of REE. All said and done, the main complication related to social interaction goes back to the fact that there are two conveyers of information, namely price and social interaction, in REESI compared to the unique conveyer of information, price, in REE.

The variance of liquidity helps us to reduce this complication. First, consider the limit case where the variance of liquidity, ${\sigma_L}^2$, tends to infinity. In this limit case variations in price reflect variations in liquidity rather than variations in signals, thus agents cannot infer any information from price about others' signals on the risky payoff. This means only social interaction conveys information in the economy. With a unique conveyer we face a less complicated fixed point problem, and proving existence of a linear REESI in the limit ${\sigma_L}^2 \to \infty$ is quite similar to proving existence of a linear REE since both employ unique conveyers of information. Once we establish existence in the limit, we can extend the existence of linear REESI from limit ${\sigma_L}^2 \to \infty$ to an interval $(\underline{\sigma_L}^2, \infty)$ using a continuity argument pertaining to the variance of liquidity. This gives us Proposition 2.3.

2.5.2 Linear Equilibrium when Only Social Interaction Conveys Information

Although Proposition 2.3 provides an equilibrium existence result, we have not been able to derive a REESI in closed form. Therefore, characterization of the equilibrium in its full generality proves to be quite a challenging task. Given this difficulty, we rather focus on the polar cases. There are two polar cases pertaining to information transmission. One is when price is the unique conveyer of information, and this case is already analyzed in standard REE models. The other is when only social interaction conveys information. As we discussed in §2.5.1, the latter is obtained in the limit when variance of liquidity tends to infinity.

Proposition 2.4 *Assume A1, A2, S3, A4. Suppose the social network is a tree. If* ${\sigma_L}^2 \to \infty$, *then*

(a) there is a sequence of linear REESI prices that converges to $\tilde{p}^s = \pi_0^s + \sum_{i=1}^n \pi_i^s \tilde{s}_i - \gamma^s \tilde{L}$, *where*

$$\begin{aligned}
\pi_0^s &= \gamma^s \frac{n\mu_x}{\rho {\sigma_x}^2}, \\
\pi_i^s &= \gamma^s \left(1 + \sum_{m \in \mathcal{H}^{-1}(i)} 1\right) \frac{1}{\rho {\sigma_\epsilon}^2}, \qquad i = 1, ..., n, \\
\gamma^s &= \frac{1}{\frac{n}{\rho {\sigma_x}^2} + \frac{n}{\rho {\sigma_\epsilon}^2} + \frac{1}{\rho {\sigma_\epsilon}^2} \sum_{i=2}^n l_{i,1}};
\end{aligned}$$

(b) the corresponding sequence of REESI demands of agent i*, for* $i = 1, ..., n$*, converges to*

$$\tilde{z}_i^s = \frac{\mu_x}{\rho {\sigma_x}^2} + \frac{1}{\rho {\sigma_\epsilon}^2} \sum_{k=0}^{l_{i,1}} \tilde{s}_{i-k} - \left(\frac{1}{\rho {\sigma_x}^2} + (l_{i,1} + 1) \frac{1}{\rho {\sigma_\epsilon}^2}\right) \tilde{p}^s.$$

From part (a) of Proposition 2.4, it is straightforward to see that $\pi_i^s < \pi_{i-}^s$, $i = 2, ..., n$. This means that an agent's signal affects the price less than her uphill neighbor's signal, and therefore, by induction, an agent's signal affects the price less than all her predecessors' signals. Actually, following part (a), we can say that the weight of an agent's signal in price is proportional to the number of successors of that agent. To put it differently, the higher the agent is in the hierarchy (i.e., tree), the larger the effect of her signal in price. This is so because, for a given agent, social interaction conveys information only about her predecessors' signals. Therefore, all agents' demands are more sensitive to their predecessors' signals compared to their successors' signals, and consequently, price becomes more sensitive to signals of agents with larger number of successors. Note that our last comment on demand sensitivity is justified by part (b) of Proposition 2.4.

We can find results similar to Proposition 2.4 in the social learning theory literature. In particular, DeMarzo, Vayanos and Zwiebel (2003) show that the position and connections of an agent in the social network can make that agent more influential in the formation of other agents' opinions. However, they employ a boundedly rational model of opinion formation in social networks in order to obtain this result whereas we do not need to dispense with rationality.

2.5.3 Social Interaction in Stars

In this section we consider social networks in the form of stars. Star is a special tree in which all vertices except the root are terminal nodes (see Figure 2.3). So, in terms of information transmission, we have a central agent who gets to be observed by every other agent in the economy.

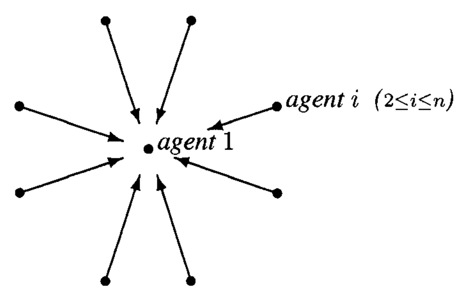

Figure 2.3: A STAR REPRESENTING SOCIAL INTERACTION

Arrows indicate the direction of demand observation. So, for $2 \leq i \leq n$, agent i has additional information that comes from observation of agent 1's demand on top of her private signal s_i and price p.

In the analysis of star, we are motivated by the Security Exchange Commission (SEC) regulations enforcing information disclosure. Empowered by the Securities Exchange Act of 1934 and disclosure rules following this act, SEC can require any publicly traded company to offer full disclosure in events related to company's stocks such as repurchase plans, splits, defaults as well as any major portfolio position taken in company's stocks by directors, officers, and principal stockholders. In the latter case, SEC effectively imposes a star as the social network: the CEO (or the principal stockholder) of the company can be considered as the root of the star, and the minor stockholders can be considered as the terminal nodes.

When the social network is a star, we can prove the existence of a linear REESI without relying on highly volatile liquidity. Moreover, the obtained equilibrium price is unique and has closed form.

Proposition 2.5 *Assume A1, A2, S3, A4. Suppose the social network is a star in which agent* 1 *is the root. There exists a unique linear REESI price of the form* $\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \pi_0^* + \sum_{i=1}^n \pi_i^* \tilde{s}_i - \gamma^* \tilde{L}$ *with non-zero* γ^**. The linear REESI price has*

$$\begin{aligned}
\pi_i^* &= \gamma^* q^*, \qquad i{=}2,...,n; \\
\pi_1^* &= \gamma^* \frac{n}{\rho{\sigma_\epsilon}^2 + \frac{(n-1)q^*{\sigma_\epsilon}^2}{(n-1)(q^*)^2{\sigma_\epsilon}^2+{\sigma_L}^2} + (n-1)\frac{(n-2)q^*{\sigma_\epsilon}^2}{(n-2)(q^*)^2{\sigma_\epsilon}^2+{\sigma_L}^2}}, \\
\gamma^* &= \frac{1+\frac{1}{\rho}\frac{(n-1)q}{(n-1)(q^*)^2{\sigma_\epsilon}^2+{\sigma_L}^2}+\frac{1}{\rho}\frac{(n-1)(n-2)q^*}{(n-2)(q^*)^2{\sigma_\epsilon}^2+{\sigma_L}^2}}{\frac{n}{\rho{\sigma_x}^2}+\frac{2n-1}{\rho{\sigma_\epsilon}^2}+\frac{1}{\rho}\frac{(n-1)^2(q^*)^2}{(n-1)(q^*)^2{\sigma_\epsilon}^2+{\sigma_L}^2}+\frac{1}{\rho}\frac{(n-1)(n-2)^2(q^*)^2}{(n-2)(q^*)^2{\sigma_\epsilon}^2+{\sigma_L}^2}}, \\
\pi_0^* &= \frac{\gamma^* n \frac{\mu_x}{\rho{\sigma_x}^2}}{1+\frac{1}{\rho}\frac{(n-1)q^*}{(n-1)(q^*)^2{\sigma_\epsilon}^2+{\sigma_L}^2}+\frac{1}{\rho}\frac{(n-1)(n-2)q^*}{(n-2)(q^*)^2{\sigma_\epsilon}^2+{\sigma_L}^2}},
\end{aligned}$$

where $q^* = \sqrt[3]{\frac{{\sigma_L}^2}{2(n-2)\rho{\sigma_\epsilon}^4}}\left(\sqrt[3]{1+\sqrt{1+\frac{4}{27}\frac{\rho^2{\sigma_L}^2{\sigma_\epsilon}^2}{n-2}}}-\sqrt[3]{-1+\sqrt{1+\frac{4}{27}\frac{\rho^2{\sigma_L}^2{\sigma_\epsilon}^2}{n-2}}}\right)$.

Proposition 2.5 shows that all agents' signals but agent 1's equally affect the price, i.e, $\pi_i^* = \pi_j^*, \forall i, j \in \{2, ...n\}$. This is so because agents $2, ..., n$ are completely homogenous with respect to their signal precisions, risk aversions, and locations in the social network.

Next we investigate the case where social interaction is the only conveyer of information (i.e., when ${\sigma_L}^2 \to \infty$). Going back to the analogy between star and the information disclosure imposed on the CEO by SEC, it is plausible to expect the effect of disclosure to surpass any information revelation through price in the event of a major change in the CEO's portfolio holdings. Therefore we believe that the limit case, where social interaction is the only conveyer of information, does not fall too short of reality. Formally, we have the following result:

Corollary 2.1 *Assume A1, A2, S3, A4. Suppose the social network is a star in which agent* 1 *is the root. As* ${\sigma_L}^2 \to \infty$, *the linear REESI price converges to* $\tilde{p}^{*s} = \pi_0^{*s} + \sum_{i=1}^n \pi_i^{*s}\tilde{s}_i - \gamma^{*s}\tilde{L}$, *where*

$$\pi_0^{*s} = \gamma^{*s}\frac{n\mu_x}{\rho{\sigma_x}^2}, \qquad \pi_1^{*s} = \gamma^{*s}\frac{n}{\rho{\sigma_\epsilon}^2}, \qquad \gamma^{*s} = \frac{1}{\frac{n}{\rho{\sigma_x}^2} + \frac{2n-1}{\rho{\sigma_\epsilon}^2}}, \qquad \pi_i^{*s} = \gamma^{*s}\frac{1}{\rho{\sigma_\epsilon}^2}, \quad i = 2, ..., n.$$

From Corollary 2.1, we observe that the weight of agent 1's signal in price is greater than the sum of the weights of all other agents' signals, i.e., $\pi_1^{*s} > \sum_{i=2}^n \pi_i^{*s}$. This discrepancy between the weight of agent 1's signal and others is only due to agent 1's location in the social network, because all agents are otherwise homogenous in their signal precisions and risk aversions.

2.6 Social Interaction in Multiple Stars

A recent paper by Hong, Kubik and Stein (2002) finds that a mutual-fund manager is more likely to hold (or buy, or sell) a particular stock in any quarter if other managers from different fund families located in the same city are also holding (or buying, or selling) that same stock. Another study by Feng and Seasholes (2002) shows that the correlation of buys (as well as sells) within a geographic region is highly significant at a weekly frequency whereas no such significant correlation can be seen across different geographic regions.

Although it is hard for us to fully address the issues raised by these studies in our framework, we take a small step by looking at how presence of disjoint clusters in a social network can affect portfolio decision-making. In particular, we analyze a social network consisting of multiple disjoint stars and examine the correlation between agents' demands in that network.

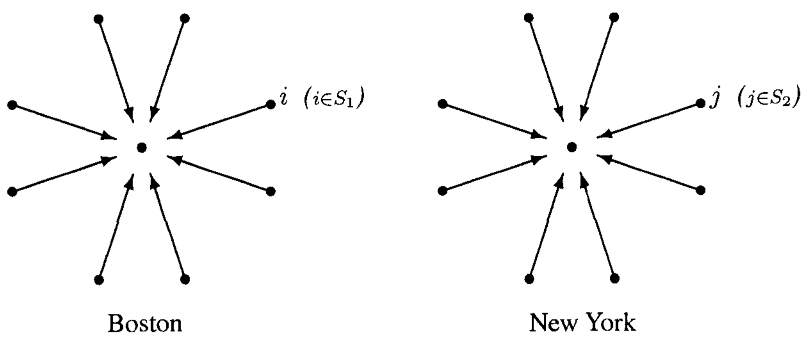

Figure 2.4: MULTIPLE STARS REPRESENTING SOCIAL INTERACTION

Proposition 2.6 *Assume A1, A2, S3, A4. Suppose the social network consists of multiple disjoint identical stars,* S_1, S_2, ..., S_m, *with* $n > m \geq 2$ *and* $\frac{n}{m} \in \mathbb{Z}$. *If* ${\sigma_L}^2$ *is sufficiently large, then linear REESI demands,* $\{\tilde{z}_i\}_{i=1,\ldots,n}$, *satisfy the following property*

$$\text{corr}(\tilde{z}_i, \tilde{z}_j) > \text{corr}(\tilde{z}_k, \tilde{z}_l), \qquad i,j \in S_r,\ k \in S_{r'},\ l \in S_{r''},\quad i^-, j^-, k^-, l^- \neq \emptyset,\quad r' \neq r''.$$

Recall that variations in price reflect variations in liquidity rather than variations in signals if liquidity is highly volatile. Therefore, when liquidity is highly volatile, social interaction becomes a more reliable predictor of the risky payoff compared to price. Proposition 2.6 expresses the simple fact that the correlation of demands of agents within the same star is larger than the correlation of demands of agents located across different stars when social interaction is the more reliable predictor of risky payoff.[14] If we treat each disjoint star as a social network representation of a city (or a geographical region), then Proposition 2.6 provides a theoretical justification for the findings above. Surely, modelling social network across different cities as multiple disjoint stars is a naive and primitive approach, but the consistency between our result and empirical findings is encouraging.

[14]However, there is a condition for this statement to hold: all agents have to be at the terminal nodes of the stars. The correlations become highly complicated when one of the agents in question is at the root of a star, and, in that case, the inequality in Proposition 2.6 does not necessarily hold.

2.7 Information Aggregation and Social Interaction

We now explore the effect of social interaction on the aggregation of information through the price system. This is especially important in economies where information is dispersed among many agents, because, by aggregating private signals of the agents, price also reveals some or all of the information carried by these signals.

To facilitate the formal analysis, we first define *sufficient statistic* following Huang and Litzenberger (1988):

DEFINITION. *Let $\tilde{\tau}$, $\tilde{\theta}$ be random vectors, and τ, θ be their corresponding realizations. Let $\tilde{z}$ be a random variable with realization z. Also, let $f(\tau, z|\theta)$ be the joint density function of $\tilde{\tau}$ and $\tilde{z}$ conditional on $\tilde{\theta}$. The random variable $\tilde{z}$ is a* ***sufficient statistic*** *for the joint density f if there exist functions g_1 and g_2 such that for all realizations τ, z and θ*

$$f(\tau, z|\theta) = g_1(\tau, z)\, g_2(z, \theta).$$

Using Bayes' rule, one can verify that if $\tilde{z}$ is a sufficient statistic for $f(\tau, z|\theta)$, the joint density of $\tilde{\theta}$ conditional on $\tilde{\tau}$ and $\tilde{z}$ is independent of $\tilde{\tau}$. Applying this to our model, if $\tilde{z}$ is a sufficient statistic for the joint density of $(\tilde{s}_1, ..., \tilde{s}_n, \tilde{z})$ conditional on $\tilde{X}$, then the conditional density of $\tilde{X}$ given $(\tilde{s}_1, ..., \tilde{s}_n)$ and $\tilde{z}$ is independent of $(\tilde{s}_1, ..., \tilde{s}_n)$. In particular,

$$\mathrm{E}[\tilde{X}|\tilde{s}_1, ...\tilde{s}_n, \tilde{z}] = \mathrm{E}[\tilde{X}|\tilde{z}], \tag{2.7.1}$$

$$\mathrm{var}(\tilde{X}|\tilde{s}_1, ...\tilde{s}_n, \tilde{z}) = \mathrm{var}(\tilde{X}|\tilde{z}). \tag{2.7.2}$$

This means that agents, who have access to the knowledge of $\tilde{z}$, know all the relevant information in the economy. In order to analyze the performance of price $\tilde{p}$ as an aggregator of diverse information, we first check whether price is a sufficient statistic for the conditional density of $(\tilde{s}_1, ..., \tilde{s}_n, \tilde{p})$ given $\tilde{X}$. If it is, then (2.7.1) and (2.7.2) hold for $\tilde{z} = \tilde{p}$. However, price can not be a sufficient statistic in our framework due to the existence of random liquidity (noise). In other words, price $\tilde{p}$ can not be *fully revealing*. This is a crucial feature of price in our model; because if price were fully revealing, social interaction would not convey any *additional information*. Therefore we rather check whether the informational content of price relevant to risky payoff $\tilde{X}$ can be a sufficient statistic. If it is, then (2.7.1) and (2.7.2) hold with $\tilde{z}$ being the informational content of $\tilde{p}$ relevant to $\tilde{X}$. With a linear price function, we can easily identify the informational content of price relevant to the risky payoff: let price $\tilde{p}$ be a linear function of the information vector $(\tilde{s}_1, ..., \tilde{s}_n)$ and liquidity (noise) $\tilde{L}$ such that

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \pi_0 + \sum_{i=1}^{n} \pi_i \tilde{s}_i - \gamma \tilde{L}. \tag{2.7.3}$$

Then the *informational content of* $\tilde{p}$ *relevant to* $\tilde{X}$ is given by $\sum_{i=1}^{n} \pi_i \tilde{s}_i$.

Accordingly, we define efficiency of price pertaining to information aggregation as follows:

DEFINITION. *Let price $\tilde{p}$ be of the form (2.7.3). We say that price efficiently aggregates information $(\tilde{s}_1, ..., \tilde{s}_n)$ if the informational content of $\tilde{p}$ relevant to $\tilde{X}$, $\sum_{i=1}^{n} \pi_i \tilde{s}_i$, is a sufficient statistic for the joint density of $(\tilde{s}_1, ..., \tilde{s}_n, \sum_{i=1}^{n} \pi_i \tilde{s}_i)$ conditional on $\tilde{X}$, so that*

$$\begin{aligned} \mathrm{E}\left[\tilde{X}\middle|\tilde{s}_1, ...\tilde{s}_n, \sum_{i=1}^{n} \pi_i \tilde{s}_i\right] &= \mathrm{E}\left[\tilde{X}\middle|\sum_{i=1}^{n} \pi_i \tilde{s}_i\right], \\ \mathrm{var}\left(\tilde{X}\middle|\tilde{s}_1, ...\tilde{s}_n, \sum_{i=1}^{n} \pi_i \tilde{s}_i\right) &= \mathrm{var}\left(\tilde{X}\middle|\sum_{i=1}^{n} \pi_i \tilde{s}_i\right). \end{aligned}$$

Under assumptions A2 and A4 (i.e., joint normality of signals and i.i.d. error terms), we can verify that

$$\tilde{\mathcal{S}} \equiv \frac{1}{{\sigma_x}^2 + {\sigma_\epsilon}^2} \sum_{i=1}^{n} \tilde{s}_i$$

is a sufficient statistic for the joint density of $(\tilde{s}_1, ..., \tilde{s}_n, \tilde{\mathcal{S}})$ conditional on $\tilde{X}$. Intuitively, this follows, because the precision weighted sum of signals is more informative than any individual $\tilde{s}_i$. Taking a glance back at Proposition 2.1, we notice that the informational content of the REE price relevant to

$\tilde{X}$ carries equal weights for all signals, i.e., it is equal to $\tilde{S}$ multiplied by a constant. This means the informational content of the REE price relevant to $\tilde{X}$ is informationally equivalent to $\tilde{S}$. So we have the following result:

Proposition 2.7 *Assume A1, A2, A3, A4. Then the linear REE price efficiently aggregates information* $(\tilde{s}_1, ..., \tilde{s}_n)$.

The presence of social interaction significantly affects the performance of price as an aggregator of information. In an economy with interacting agents, one of the determinants for the weights of signals in the linear equilibrium price is the interaction pattern itself. In particular, asymmetry in the interaction pattern can cause some agents' signals be overweighed in the equilibrium price even though all signals have the same precision. As a consequence, the information aggregation through the price system can be impaired by social interaction. We see this happening in social networks in the form of stars.

Proposition 2.8 *Assume A1, A2, S3, A4. Suppose the social network is a star in which agent* 1 *is the root. Then the linear REESI price does not efficiently aggregate information* $(\tilde{s}_1, ..., \tilde{s}_n)$ *for the generic exogenous parameters* ${\sigma_\epsilon}^2$, ${\sigma_L}^2$ *and* n.

Propositions 2.7 and 2.8 suggest that using REE price to assess real world markets may lead to false implications. One problem attributed to the informationally efficient REE is its inability to explain large price swings in the stock market. Cutler, Poterba and Summers (1989) document that there were no significant events prior to many large price swings for the postwar S&P 500 index. Therefore, if information is aggregated efficiently in price so that each signal is weighted proportional to its precision (as is the case in REE), it is hard to account for large price swings. Proposition 2.8 shows a new way out of this dilemma: the swings might be due to inefficient information aggregation brought by social interaction. Since inefficient information aggregation allows for an imprecise signal to have a disproportionately large impact on price, a significant event is no more necessary for a large swing in the presence of social interaction.

DeMarzo, Vayanos and Zwiebel (2003) provide an actual example from the stock market, which nicely illustrates inefficient information aggregation brought by social interaction and its implications: "[...] consider the individuals participating in internet chat rooms on financial investments. Many of these individuals have inaccurate (and sometimes false) information. However, the fact that they have a large audience [...] can give them enough influence to affect market prices. For example, the Wall Street Journal (November 6, 2000) reports: 'Preliminary figures show that market manipulation accounted for 8% of the roughly 500 cases the SEC brought in fiscal 2000, ended September 30, up from 3% in fiscal 1999. Manipulation on the Internet is where the action is, and appears to be replacing brokerage boiler rooms of the past, said SEC enforcement-division director Richard Walker'."

2.8 Concluding Remarks

Three significant results emerge from our analysis of social interaction in financial markets. First, when social interaction is the only conveyer of information in a hierarchic interaction scheme, an agent's signal affects the price less than her predecessor's signal. That is, the hierarchy in observation leads to a hierarchy in influence regarding security pricing. Second, we show that presence of disjoint clusters in a social network can cause different portfolio decisions to be taken across different clusters. In particular, when the social network consists of multiple disjoint stars, correlation of demands of agents within the same star is larger than correlation of demands of agents located across different stars provided that social interaction is a more reliable predictor of risky payoff than price is. Third, and most tentatively, in the presence of social interaction signals of some of the agents can affect price more than others' signals regardless of the signals' precisions. This allows for an imprecise signal to have a disproportionate effect on price. We refer to this situation as inefficient information aggregation.

All these results are broadly consistent with the empirical findings. In particular, the second result is consistent with the finding that fund managers located in the same city display more similar trading patterns than those in different cities (Hong, Kubik and Stein (2002)). Also, inefficient information aggregation brought by social interaction may possibly account for crashes and frenzies. Cutler, Poterba and Summers (1989) document that there were virtually no significant events prior to many large price swings in the stock market. Since inefficient information aggregation allows for an imprecise signal to have a large impact on price, a significant event is not necessary to trigger a price swing in the presence of social interaction.

This essay is just a step toward an understanding of the role of social interaction in the functioning of financial markets. We shall mention several ways to extend our analysis. One is allowing for heterogeneity in risk aversion and signal precision among the traders. Naturally, one expects to see traders with less risk aversion and more precise signals to have more influence in the valuation of securities. A richer class of interaction patterns may also bring out new results. Another extension pertains to the competitive behavior in our model. How imperfectly competitive markets would operate in the presence of social interaction remains to be seen. Above all, this essay considers social interaction only in the form of observations of other agents's security demands. Modelling social interaction as direct observation of other agents's signals, cheap talk or a strategic exchange of signals are the natural routes to be taken to extend our analysis.

2.9 Appendix: Proofs

THE REE Á LA HELLWIG

For the proof of Proposition 2.1, we use the following lemmas from Hellwig (1980):

Lemma 2.1 *Assume A1, A2, and A3. Then there exists a linear REE price*

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \pi_0 + \sum_{i=1}^{n} \pi_i \, \tilde{s}_i - \gamma \, \tilde{L} \quad \text{with non-zero } \gamma, \tag{2.9.1}$$

and REE price coefficients satisfy the following:

$$\pi_i = \frac{\gamma}{\rho_i {\sigma_i}^2} \frac{\sum_{k=1}^{n} {\pi_k}^2 {\sigma_k}^2 + \gamma^2 {\sigma_L}^2 - \pi_i \pi {\sigma_i}^2}{\sum_{k=1}^{n} {\pi_k}^2 {\sigma_k}^2 + \gamma^2 {\sigma_L}^2 - {\pi_i}^2 {\sigma_i}^2}, \quad i=1,\dots,n, \tag{2.9.2a}$$

$$\frac{1}{\gamma} = \sum_{i=1}^{n} \frac{{\sigma_x}^2 + \sigma_i^2}{\rho_i {\sigma_x}^2 {\sigma_i}^2} + \sum_{i=1}^{n} \frac{(\pi - \pi_i)^2 - (\pi - \pi_i)}{\rho_i \left(\sum_{k=1}^{n} {\pi_k}^2 {\sigma_k}^2 + \gamma^2 {\sigma_L}^2 - {\pi_i}^2 {\sigma_i}^2 \right)}, \tag{2.9.2b}$$

$$\pi_0 = \frac{\mu_x}{{\sigma_x}^2} \sum_{i=1}^{n} \frac{1}{\rho_i} - \gamma \pi_0 \sum_{i=1}^{n} \frac{\pi - \pi_i}{\rho_i \left(\sum_{k=1}^{n} {\pi_k}^2 {\sigma_k}^2 + \gamma^2 {\sigma_L}^2 - {\pi_i}^2 {\sigma_i}^2 \right)}, \tag{2.9.2c}$$

where $\pi \equiv \sum_{k=1}^{n} \pi_k$.

Proof. See Proposition 3.3 and Eqs. (7a)-(7c) in Hellwig (1980).

Lemma 2.2 *Assume A1, A2, and A3. Let* $\pi_0, \{\pi_i\}_{i=1,\dots,n}, \gamma$ *be the coefficients of an REE price function.*

(a) If $\rho_i \geq \rho_j$ *and* ${\sigma_i}^2 \geq {\sigma_j}^2$, *then* $\pi_i \leq \pi_j$.

(b) If $\rho_i \geq \rho_j$ *and* ${\sigma_i}^2 = {\sigma_j}^2$, *then* $\rho_i \pi_i \geq \rho_j \pi_j$.

(c) If $\rho_i = \rho_j$ *and* ${\sigma_i}^2 \geq {\sigma_j}^2$, *then* ${\sigma_i}^2 \pi_i \leq {\sigma_j}^2 \pi_j$.

If one of the inequalities in the premises of statements (a)-(c) is strict, the inequality in the corresponding conclusion is also strict.

Proof. See Proposition 4.1 in Hellwig (1980).

Proof of Proposition 2.1. The existence of a linear REE price follows from Lemma 2.1. We also have $\pi_i = \pi_j, \ \forall i, j$ from Lemma 2.2. Define a new variable $q \equiv \frac{\pi_i}{\gamma}$. Rewriting (2.9.2a), one obtains

$$(n-1){\sigma_\epsilon}^2 q^3 + {\sigma_L}^2 q - \frac{{\sigma_L}^2}{\rho {\sigma_\epsilon}^2} = 0. \tag{2.9.3a}$$

Let $f(x) \equiv (n-1){\sigma_\epsilon}^2 x^3 + {\sigma_L}^2 x - \frac{{\sigma_L}^2}{\rho {\sigma_\epsilon}^2}$. f is a strictly increasing continuous function, and it takes both positive and negative values. Therefore f must take value 0 for a unique $x \in \mathbb{R}$. This implies the existence of a *unique solution* q for equation (2.9.3a).

Rewriting (2.9.2b) and (2.9.2c), one gets

$$\gamma = \frac{1 + \frac{n(n-1)q}{\rho((n-1)q^2{\sigma_\epsilon}^2 + {\sigma_L}^2)}}{n\frac{{\sigma_x}^2+{\sigma_\epsilon}^2}{\rho{\sigma_x}^2{\sigma_\epsilon}^2} + \frac{n(n-1)q^2}{\rho((n-1)q^2{\sigma_\epsilon}^2+{\sigma_L}^2)}}, \tag{2.9.3b}$$

$$\pi_0 = \frac{\frac{n\mu_x}{\rho{\sigma_x}^2}}{n\frac{{\sigma_x}^2+{\sigma_\epsilon}^2}{\rho{\sigma_x}^2{\sigma_\epsilon}^2} + \frac{n(n-1)q^2}{\rho((n-1)q^2{\sigma_\epsilon}^2+{\sigma_L}^2)}}. \tag{2.9.3c}$$

Since there exists a unique q satisfying equation (2.9.3a) (which is equivalent to (2.9.2a)), γ and π_0 are also uniquely given by the equations above. Hence uniqueness of price coefficients $\pi_i = \gamma\, q$, $i = 1, \ldots, n$, follow.

To derive the closed-form solution, notice that (2.9.3a) is a cubic polynomial in q without a quadratic component. Thus by *Cardano's Formula* (see, e.g., Artin (1991)), the unique (real) solution of (2.9.3a) is given by

$$q = \sqrt[3]{\frac{{\sigma_L}^2}{2(n-1)\rho{\sigma_\epsilon}^4} + \sqrt{\frac{{\sigma_L}^4}{4(n-1)^2\rho^2{\sigma_\epsilon}^8} + \frac{{\sigma_L}^6}{27(n-1)^3{\sigma_\epsilon}^6}}} - \sqrt[3]{-\frac{{\sigma_L}^2}{2(n-1)\rho{\sigma_\epsilon}^4} + \sqrt{\frac{{\sigma_L}^4}{4(n-1)^2\rho^2{\sigma_\epsilon}^8} + \frac{{\sigma_L}^6}{27(n-1)^3{\sigma_\epsilon}^6}}}.$$

Rewriting q, using equations (2.9.3b), (2.9.3c), and the fact that $\pi_i = \gamma\, q$, one obtains the desired result.

□

NON-EXISTENCE IN CYCLES

Lemma 2.3 *Assume A1, A2, S3, and A4. Suppose that the social network is a cycle and there exists a linear REESI price. Then, for all $i = 1, \ldots, n$, the corresponding REESI demand of agent i's uphill neighbor is given by*

$$\tilde{z}_{i^-} = \zeta_i\, \tilde{p} + \tilde{\delta}_i$$

such that $\tilde{\delta}_i$ is a linear function of signals $(\tilde{s}_1, ..., \tilde{s}_n)$. Moreover,

(a) the random vector $(\tilde{\delta}_1, ..., \tilde{\delta}_n)$ satisfying the property stated above is unique;

(b) for all $i = 1, ..., n$, agent i knows the joint distribution of $(\tilde{X}, \tilde{s}_i, \tilde{p}, \tilde{\delta}_i, \tilde{L})$ and the realization δ_i;

(c) the conditional distribution of $\tilde{X}$ given (s_i, p, δ_i) is the same as the conditional distribution of $\tilde{X}$ given (s_i, p, z_{i^-}), i.e., the informational contents of (s_i, p, δ_i) and (s_i, p, z_{i^-}) related to the risky payoff $\tilde{X}$ are same.

Proof. Without loss of generality we relabel the agents in the cycle so that for $i = 1, ..., n$, agent i observes demand of agent $i-1$, i.e., $i^- = i - 1 \pmod n$. Let the linear REESI price be given by

$$\tilde{p} \;=\; P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \pi_0 + \sum_{i=1}^{n} \pi_i\, \tilde{s}_i - \gamma\, \tilde{L}. \tag{2.9.4}$$

Assume, initially, that the corresponding REESI demands $\{\tilde{z}_j\}_{j=1,...,n}$ are normal random variables. Then the conditional distribution of risky payoff $\tilde{X}$ as assessed by agent j has the mean

$$\mathrm{E}[\tilde{X}|\mathcal{I}_j] \equiv \mathrm{E}[\tilde{X}|s_j, p, z_{j-1}] = a_{0j} + a_{1j}\, s_j + a_{2j}\, p + a_{3j}\, z_{j-1},$$

and the variance

$$\mathrm{var}(\tilde{X}|\mathcal{I}_j) \equiv \mathrm{var}(\tilde{X}|s_j, p, z_{j-1}) = b_j,$$

where the values of coefficients a_{0j}, a_{1j}, a_{2j}, a_{3j}, b_j depend on the variance-covariance matrix of $(\tilde{X}, \tilde{s}_j, \tilde{p}, \tilde{z}_{j-1})$. However, note that these coefficients are independent of the realizations (X, s_j, p, z_{j-1}). The CARA-Gaussian setup dictates the following (risky security) demand for agent j:

$$z_j = \frac{\mathrm{E}[\tilde{X}|\mathcal{I}_j] - p}{\rho_j \mathrm{var}(\tilde{X}|\mathcal{I}_j)} = \frac{a_{0j} + a_{1j}\, s_j + a_{2j}\, p + a_{3j}\, z_{j-1}}{\rho_j\, b_j}. \tag{2.9.5}$$

Since (2.9.5) holds for all $j = 1, ..., n$, recursive substitution for $\{z_{j-k}\}_{k=1,...,n}$ yields

$$\begin{aligned} z_j &= \frac{a_{0j}}{\rho\, b_j} + \frac{a_{1j}}{\rho\, b_j}\, s_j + \frac{a_{2j}}{\rho\, b_j}\, p + \sum_{k=1}^{n-1}\left(\prod_{m=0}^{k-1} \frac{a_{3,j-m}}{\rho\, b_{j-m}}\right)\left(\frac{a_{0,j-k}}{\rho\, b_{j-k}} + \frac{a_{1,j-k}}{\rho\, b_{j-k}}\, s_{j-k} + \frac{a_{2,j-k}}{\rho\, b_{j-k}}\, p\right) \\ &+ \prod_{m=0}^{n-1} \frac{a_{3,j-m}}{\rho\, b_{j-m}} z_{j-n}. \end{aligned}$$

Using the simple fact that $j - n \equiv j \pmod n$ and rearranging the terms, we get

$$z_j = \sum_{k=1}^{n} \frac{\prod_{m=0}^{k-1} \frac{a_{3,j-m}}{\rho\, b_{j-m}}}{1 - \prod_{m=0}^{n-1} \frac{a_{3,j-m}}{\rho\, b_{j-m}}} \left(\frac{a_{0,j-k}}{\rho\, b_{j-k}} + \frac{a_{1,j-k}}{\rho\, b_{j-k}}\, s_{j-k} + \frac{a_{2,j-k}}{\rho\, b_{j-k}}\, p\right).$$

If we take $j = i^- \equiv i - 1$ and let

$$\begin{aligned} \zeta_i &= \sum_{k=1}^{n} \frac{\prod_{m=0}^{k-1} \frac{a_{3,i-1-m}}{\rho\, b_{i-1-m}}}{1 - \prod_{m=0}^{n-1} \frac{a_{3,i-1-m}}{\rho\, b_{i-1-m}}}, \\ \delta_i &= \sum_{k=1}^{n} \frac{\prod_{m=0}^{k-1} \frac{a_{3,i-1-m}}{\rho\, b_{i-1-m}}}{1 - \prod_{m=0}^{n-1} \frac{a_{3,i-1-m}}{\rho\, b_{i-1-m}}} \frac{a_{0,i-1-k}}{\rho\, b_{i-1-k}} + \sum_{k=1}^{n} \frac{\prod_{m=0}^{k-1} \frac{a_{3,i-1-m}}{\rho\, b_{i-1-m}}}{1 - \prod_{m=0}^{n-1} \frac{a_{3,i-1-m}}{\rho\, b_{i-1-m}}} \frac{a_{1,i-1-k}}{\rho\, b_{i-1-k}}\, s_{i-1-k}, \end{aligned}$$

then $z_{i^-} = \zeta_i\, p + \delta_i$. Since this equality holds for all realizations $(s_1, ..., s_n, L)$ of $(\tilde{s}_1, ..., \tilde{s}_n, \tilde{L})$,

$$\tilde{z}_{i^-} = \zeta_i\, \tilde{p} + \tilde{\delta}_i \tag{2.9.6}$$

with

$$\tilde{\delta}_i = \sum_{k=1}^{n} \frac{\prod_{m=0}^{k-1} \frac{a_{3,i-1-m}}{\rho\, b_{i-1-m}}}{1 - \prod_{m=0}^{n-1} \frac{a_{3,i-1-m}}{\rho\, b_{i-1-m}}} \frac{a_{0,i-1-k}}{\rho\, b_{i-1-k}} + \sum_{k=1}^{n} \frac{\prod_{m=0}^{k-1} \frac{a_{3,i-1-m}}{\rho\, b_{i-1-m}}}{1 - \prod_{m=0}^{n-1} \frac{a_{3,i-1-m}}{\rho\, b_{i-1-m}}} \frac{a_{1,i-1-k}}{\rho\, b_{i-1-k}}\, \tilde{s}_{i-1-k}.$$

Since $\tilde{p}$ is dependent on $\tilde{L}$, $\tilde{\delta}_i$ is the unique linear function of signals $(\tilde{s}_1, ..., \tilde{s}_n)$ satisfying (2.9.6).

Also, note that $\mathrm{cov}(\tilde{z}_{i^-}, \tilde{L}) = \gamma\, \zeta_i = \mathrm{cov}(\tilde{p}, \tilde{L})\, \zeta_i$. Following assumption S3,

$$\zeta_i = \frac{\mathrm{cov}(\tilde{z}_{i^-}, \tilde{L})}{\mathrm{cov}(\tilde{p}, \tilde{L})}$$

is a known constant for agent i. Then, again due to S3, agent i knows the joint distribution of $(\tilde{X}, \tilde{s}_i, \tilde{p}, \tilde{\delta}_i, \tilde{L})$ and the realization δ_i.

Part (c) of the lemma simply follows from the fact that $\tilde{\delta}_i$ is a linear function of $\tilde{p}$ and $\tilde{z}_{i^-}$.

Finally, note that both $\tilde{p}$ and $\tilde{\delta}_j$ are normally distributed since they are linear functions of the normally distributed random vector $(\tilde{s}_1, ..., \tilde{s}_n, \tilde{L})$. Thus our initial assumption of $\{\tilde{z}_j\}_{j=1,...,n}$ being normal random variables is verified by equation (2.9.6). □

Proof of Proposition 2.2: Without loss of generality, relabel the agents in the cycle so that $i^- = i - 1$ (mod n) for all $i = 1, ..., n$. Suppose there exists a linear REESI price, which is given by

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \mathring{\pi}_0 + \sum_{i=1}^{n} \mathring{\pi}_i \, \tilde{s}_i - \mathring{\gamma} \, \tilde{L}. \tag{2.9.7}$$

We have already established that there exists a random variable $\tilde{\delta}_i$ with

$$\tilde{z}_{i-1} = \zeta_i \, \tilde{p} + \tilde{\delta}_i,$$

satisfying the properties listed in Lemma 2.3. In particular, following part (c) of the lemma,

$$\mathrm{E}[\tilde{X}|s_i, p, z_{i-1}] = \mathrm{E}[\tilde{X}|s_i, p, \delta_i] \quad \text{and} \quad \mathrm{var}(\tilde{X}|s_i, p, z_{i-1}) = \mathrm{var}(s_i, p, \delta_i).$$

Let $\mathring{V}_i$ denote the variance-covariance matrix of $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$, and $\mathring{W}_i$ be the covariance matrix of $\tilde{X}$ and $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$. Since $(\tilde{X}, \tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$ is jointly normally distributed, the conditional distribution of risky payoff $\tilde{X}$ as assessed by agent i has the moments

$$\begin{aligned} \mathrm{E}[\tilde{X}|s_i, p, z_{i-1}] &= \mathrm{E}[\tilde{X}|s_i, p, \delta_i] = \mathring{a}_{0i} + \mathring{a}_{1i} s_i + \mathring{a}_{2i} p + \mathring{a}_{3i} \delta_i, \\ \mathrm{var}(\tilde{X}|s_i, p, z_{i-1}) &= \mathrm{var}(\tilde{X}|s_i, p, \delta_i) = \mathring{b}_i, \end{aligned}$$

where the values of the coefficients $\mathring{a}_{0i}, \mathring{a}_{1i}, \mathring{a}_{2i}, \mathring{a}_{3i}, \mathring{b}_i$ depend on $\mathring{V}_i$ and $\mathring{W}_i$. Due to the homogeneity of agents, identical signals and total symmetry in the interaction pattern, we necessarily have $\mathring{a}_0, \mathring{a}_1, \mathring{a}_2, \mathring{a}_3, \mathring{b}$ such that

$$\mathring{a}_{0i} = \mathring{a}_0, \quad \mathring{a}_{1i} = \mathring{a}_1, \quad \mathring{a}_{2i} = \mathring{a}_2, \quad \mathring{a}_{3i} = \mathring{a}_3, \quad \mathring{b}_i = \mathring{b}; \quad \forall i = 1, ..., n.$$

So given the CARA-Gaussian setup, demands of agents will be given by

$$z_i = \frac{\mathrm{E}[\tilde{X}|s_i, p, z_{i-1}] - p}{\rho \operatorname{var}(\tilde{X}|s_i, p, z_{i-1})} = \frac{\mathring{a}_0 + \mathring{a}_1 s_i + (\mathring{a}_2 - 1)p + \mathring{a}_3 \delta_i}{\rho \mathring{b}}, \qquad i = 1, ..., n. \quad (2.9.8)$$

Now we can derive $\{\tilde{\delta}_i\}_{i=1,...,n}$ explicitly:

Claim 2.1 *For each* $i = 1, ..., n$,

$$\tilde{\delta}_i = \frac{1}{1 - \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^n} \sum_{k=1}^{n} \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^{k-1} \left(\frac{\mathring{a}_0}{\rho \mathring{b}} + \frac{\mathring{a}_1}{\rho \mathring{b}} \tilde{s}_{i-k}\right). \quad (2.9.9)$$

Proof. Following (2.9.8) and the fact that $\tilde{\delta}_j = \tilde{z}_{j-1} - \zeta_j \tilde{p}$, $j = 1, ..., n$,

$$\tilde{z}_{i-1} = \frac{\mathring{a}_0 + \mathring{a}_1 \tilde{s}_{i-1} + (\mathring{a}_2 - 1)p + \mathring{a}_3 (\tilde{z}_{i-2} - \zeta_{i-1} \tilde{p})}{\rho \mathring{b}}. \quad (2.9.10)$$

Recursively using (2.9.10) to substitute for $\{z_{i-k}\}_{k=2,...,n}$, we derive

$$\begin{aligned}\tilde{z}_{i-1} = & \sum_{k=1}^{n} \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^{k-1} \left(\frac{\mathring{a}_0}{\rho \mathring{b}} + \frac{\mathring{a}_1}{\rho \mathring{b}} \tilde{s}_{i-k} + \frac{\mathring{a}_2 - 1}{\rho \mathring{b}} \tilde{p}\right) - \sum_{k=1}^{n-1} \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^{k} \zeta_{i-k} p \\ & + \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^{n} (\tilde{z}_{i-n-1} - \zeta_{i-n} \tilde{p}).\end{aligned}$$

Since $i - n \equiv i \pmod n$ and $\tilde{\delta}_i = \tilde{z}_{i-1} - \zeta_i \tilde{p}$, this equation yields

$$\tilde{z}_{i-1} = \sum_{k=1}^{n} \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^{k-1} \left(\frac{\mathring{a}_0}{\rho \mathring{b}} + \frac{\mathring{a}_1}{\rho \mathring{b}} \tilde{s}_{i-k} + \frac{\mathring{a}_2 - 1}{\rho \mathring{b}} \tilde{p}\right) - \sum_{k=1}^{n-1} \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^{k} \zeta_{i-k} p + \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^{n} \tilde{\delta}_i.$$

Then due to Lemma 2.3,

$$\tilde{\delta}_i = \sum_{k=1}^{n} \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^{k-1} \left(\frac{\mathring{a}_0}{\rho \mathring{b}} + \frac{\mathring{a}_1}{\rho \mathring{b}} \tilde{s}_{i-k}\right) + \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^{n} \tilde{\delta}_i,$$

which, in turn, implies

$$\tilde{\delta}_i = \frac{1}{1 - \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^n} \sum_{k=1}^{n} \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^{k-1} \left(\frac{\mathring{a}_0}{\rho \mathring{b}} + \frac{\mathring{a}_1}{\rho \mathring{b}} \tilde{s}_{i-k}\right). \quad \square \text{ [end of claim]}$$

Note that the weights of agents' signals in price are equal to each other due to the strict homogeneity in the market, i.e., for all $i = 1, ..., n$ there exists some $\mathring{\pi}$ dependent on demand coefficients

$\mathring{a}_0, \mathring{a}_1, \mathring{a}_2, \mathring{a}_3, \mathring{b}$ such that $\mathring{\pi}_i = \mathring{\pi}$. Actually one can verify that

$$\mathring{\pi} = \mathring{\gamma}\frac{\mathring{a}_1}{\rho\mathring{b}}\left(1+\frac{\frac{\mathring{a}_3}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\right), \tag{2.9.11a}$$

$$\mathring{\gamma} = \frac{\rho\mathring{b}}{n(1-\mathring{a}_2)}, \tag{2.9.11b}$$

$$\mathring{\pi}_0 = \mathring{\gamma}\frac{n\mathring{a}_0}{\rho\mathring{b}}\left(1+\frac{\frac{\mathring{a}_3}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\right), \tag{2.9.11c}$$

using market clearing condition

$$\sum_{i=1}^{n} z_i(s_i, p, \delta_i) = L,$$

and solving for p.

Now we can write down the variance-covariance matrix $\mathring{V}_i$ of $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$:

$$\begin{bmatrix} {\sigma_x}^2+{\sigma_\epsilon}^2 & n\mathring{\pi}\,{\sigma_x}^2+\mathring{\pi}\,{\sigma_\epsilon}^2 & \frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_x}^2+\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n-1}}{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^n}\,{\sigma_\epsilon}^2 \\ n\mathring{\pi}\,{\sigma_x}^2+\mathring{\pi}\,{\sigma_\epsilon}^2 & n^2\mathring{\pi}^2\,{\sigma_x}^2+n\mathring{\pi}^2\,{\sigma_\epsilon}^2+\mathring{\gamma}^2\,{\sigma_L}^2 & n\mathring{\pi}\,\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_x}^2+\mathring{\pi}\,\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_\epsilon}^2 \\ \frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_x}^2+\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n-1}}{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^n}\,{\sigma_\epsilon}^2 & n\mathring{\pi}\,\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_x}^2+\mathring{\pi}\,\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_\epsilon}^2 & \frac{\left(\frac{\mathring{a}_1}{\rho\mathring{b}}\right)^2}{\left(1-\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^2}\,{\sigma_x}^2+\frac{\left(\frac{\mathring{a}_1}{\rho\mathring{b}}\right)^2}{\left(1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^n\right)^2}\,\frac{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{2n}}{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^2}\,{\sigma_\epsilon}^2 \end{bmatrix}.$$

It is a straightforward observation that $\mathring{V}_i = \mathring{V}_j$ for all i, j. Let $\mathring{V}$ denote the variance-covariance matrix of $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$ for all i. The covariance matrix of $\tilde{X}$ and $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$ is of the form

$$\mathring{W}_i = \mathring{W} = {\sigma_x}^2\begin{bmatrix} 1 \\ n\mathring{\pi} \\ \frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}} \end{bmatrix}.$$

Normal distribution theory dictates that $[\mathring{a}_1 \;\; \mathring{a}_2 \;\; \mathring{a}_3]\;\mathring{V} = \mathring{W}'$, which further implies

$$\mathring{a}_1\,({\sigma_x}^2+{\sigma_\epsilon}^2)+\mathring{a}_2\,(n\mathring{\pi}{\sigma_x}^2+\mathring{\pi}{\sigma_\epsilon}^2)+\mathring{a}_3\left(\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_x}^2+\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n-1}}{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^n}\,{\sigma_\epsilon}^2\right) = {\sigma_x}^2, \tag{2.9.12a}$$

$$\mathring{a}_1\,(n\mathring{\pi}{\sigma_x}^2+\mathring{\pi}{\sigma_\epsilon}^2)+\mathring{a}_2\,(n^2\mathring{\pi}^2{\sigma_x}^2+n\mathring{\pi}^2{\sigma_\epsilon}^2+\mathring{\gamma}^2{\sigma_L}^2)+\mathring{a}_3\left(n\mathring{\pi}\,\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_x}^2+\mathring{\pi}\,\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_\epsilon}^2\right) = n\mathring{\pi}{\sigma_x}^2, \tag{2.9.12b}$$

$$\begin{aligned} &\mathring{a}_1\left(\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_x}^2+\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n-1}}{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^n}\,{\sigma_\epsilon}^2\right)+\mathring{a}_2\left(n\mathring{\pi}\,\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_x}^2+\mathring{\pi}\,\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_\epsilon}^2\right)+ \\ &\quad\mathring{a}_3\left(\frac{\left(\frac{\mathring{a}_1}{\rho\mathring{b}}\right)^2}{\left(1-\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^2}\,{\sigma_x}^2+\frac{\left(\frac{\mathring{a}_1}{\rho\mathring{b}}\right)^2}{\left(1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^n\right)^2}\,\frac{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{2n}}{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^2}\,{\sigma_\epsilon}^2\right) = \frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}{\sigma_x}^2. \end{aligned} \tag{2.9.12c}$$

Multiplying (2.9.12a) with $-\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}$, adding the result to (2.9.12c), and dividing the derived equation by $\rho\mathring{b}$, one acquires

$$-\left(\frac{\mathring{a}_1}{\rho\mathring{b}}\right)^2 \frac{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n-1}}{\left(1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^n\right)\left(1-\frac{\mathring{a}_3}{\rho\mathring{b}}\right)}\sigma_\epsilon{}^2+\frac{\mathring{a}_3}{\rho\mathring{b}}\left(\frac{\mathring{a}_1}{\rho\mathring{b}}\right)^2\left(\frac{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n-1}}{\left(1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^n\right)\left(1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^2\right)}\right)\sigma_\epsilon{}^2=0.$$

Simplifying the equation above, we get

$$\left(\frac{\mathring{a}_1}{\rho\mathring{b}}\right)^2 \frac{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n-1}}{\left(1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^n\right)\left(1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^2\right)}=0.$$

The only solution to this equation is $\frac{\mathring{a}_1}{\rho\mathring{b}}=0$, which in turn implies $\mathring{a}_1=0$. Then (2.9.12a) reduces to

$$\mathring{a}_2(n\mathring{\pi}\sigma_x{}^2+\mathring{\pi}\sigma_\epsilon{}^2)=\sigma_x{}^2, \tag{2.9.13}$$

while plugging $\mathring{a}_1=0$ in (2.9.11a) results with

$$\mathring{\pi}=0. \tag{2.9.14}$$

Eqs. (2.9.13) and (2.9.14) together imply $\sigma_x{}^2=0$, which violates A2. Thus there cannot exist any linear REESI price. □

HIERARCHIC INTERACTION SCHEMES (TREES)

Lemma 2.4 *Assume A1, A2, S3, and A4. Suppose that the social network is a tree and there exists a linear REESI price. Then, for all i such that $i^- \neq \emptyset$, the corresponding REESI demand of agent i's uphill neighbor is given by*

$$\tilde{z}_{i^-}=\zeta_i\tilde{p}+\tilde{\delta}_i$$

such that $\tilde{\delta}_i$ is a linear function of signals $(\tilde{s}_1,\dots,\tilde{s}_n)$. Moreover,

(a) the random vector $(\tilde{\delta}_i)_{i^-\neq\emptyset}$ satisfying the property stated above is unique;

(b) for all $i = 1$ *such that* $i^- \neq \emptyset$, *agent* i *knows the joint distribution of* $(\tilde{X}, \tilde{s}_i, \tilde{p}, \tilde{\delta}_i, \tilde{L})$ *and the realization* δ_i*;*

(c) the conditional distribution of $\tilde{X}$ *given* (s_i, p, δ_i) *is the same as the conditional distribution of* $\tilde{X}$ *given* (s_i, p, z_{i^-})*, i.e., the informational contents of* (s_i, p, δ_i) *and* (s_i, p, z_{i^-}) *related to the risky payoff* $\tilde{X}$ *are same.*

Proof. The proof is similar to the proof of Lemma 2.3.

Lemma 2.5 *Assume A1, A2, S3, A4. Let the social network be a tree. Then there exists a linear REESI price within the class of functions of the form*

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \pi_0^h + \sum_{i=1}^{n} \pi_i^h \, \tilde{s}_i - \gamma^h \, \tilde{L} \quad \text{with non-zero } \gamma^h \tag{2.9.15}$$

if and only if the following system of equations has a solution in $\left(\{\pi_i^h\}_{i=1,...,n}, \gamma^h, \pi_0^h\right)$*:*

$$\begin{aligned} \pi_i^h &= \gamma^h \left(1 + \sum_{m \in \mathcal{H}^{-1}(i)} \prod_{j=0}^{l_{m,i}-1} \theta_{3,m^{-j}}^h\right) \theta_{1i}^h, \qquad i = 1, ..., n \\ \gamma^h &= \left(\sum_{i=1}^{n} \frac{1}{\rho b_i^h} - \theta_{2i}^h\right)^{-1} \neq 0, \\ \pi_0^h &= \gamma^h \sum_{i=1}^{n} \theta_{0i}^h, \end{aligned}$$

where $\left(\{\theta_{0i}^h, \theta_{1i}^h, \theta_{2i}^h, b_i^h\}_{i=1,...,n}; \{\theta_{3i}^h\}_{i=2,...,n}\right)$ *constitute the equilibrium demand coefficients of*

$$\begin{aligned} z_1(s_1, p) &= \theta_{01}^h + \theta_{11}^h + \left(\theta_{21}^h - \frac{1}{\rho b_1^h}\right) p, \\ z_i(s_i, p, \delta_{i^-}^i) &= \theta_{0i}^h + \theta_{1i}^h + \left(\theta_{2i}^h - \frac{1}{\rho b_i^h}\right) p + \theta_{3i}^h \delta_i^i \;, \qquad i > 1 \end{aligned}$$

such that

$$
\begin{aligned}
\theta^h_{01} &= \mu_x\left(\frac{1}{\rho b^h_1} - \theta^h_{11} - \theta^h_{21}\pi^h\right) - \theta^h_{21}\pi^h_0, \\
\theta^h_{11} &= \frac{1}{\rho{\sigma_\epsilon}^2}\frac{\left(\sum_{k=1}^n(\pi^h_k)^2 - \pi^h\pi^h_1\right){\sigma_\epsilon}^2 + (\gamma^h)^2{\sigma_L}^2}{\sum_{k\neq 1}(\pi^h_k)^2{\sigma_\epsilon}^2 + (\gamma^h)^2{\sigma_L}^2}, \\
\theta^h_{21} &= \frac{\pi^h - \pi^h_1}{\rho\left(\sum_{k\neq 1}(\pi^h_k)^2{\sigma_\epsilon}^2 + (\gamma^h)^2{\sigma_L}^2\right)}, \\
b^h_1 &= \frac{1}{\rho}\left(\frac{1}{\rho{\sigma_x}^2} + \theta^h_{11} + \theta^h_{21}\pi^h\right)^{-1};
\end{aligned}
$$

and for $i > 1$

$$
\begin{aligned}
\theta^h_{0i} &= \mu_x\left(\tfrac{1}{\rho b^h_i} - \theta^h_{1i} - \theta^h_{2i}\pi^h - \theta^h_{3i}\left[\theta^h_{1,i-} + \sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta^h_{3,i-j}\right)\theta^h_{1,i-k}\right]\right) - \theta^h_{2i}\pi^h_0, \\
\theta^h_{1i} &= \tfrac{1}{\rho\beta_i}\Bigg[\left(\left(\sum_{k=1}^n(\pi^h_k)^2 - \pi^h\pi^h_i\right){\sigma_\epsilon}^2 + (\gamma^h)^2{\sigma_L}^2\right)\left(\left(\theta^h_{1,i-}\right)^2 + \sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta^h_{3,i-j}\right)^2\left(\theta_{1,i-k}\right)^2\right) + \\
&\quad \left(\theta^h_{1,i-}(\pi^h_i - \pi^h_{i-}) + \sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta^h_{3,i-j}\right)\theta^h_{1,i-k}(\pi^h_i - \pi^h_{i-k})\right)\left(\theta^h_{1,i-}\pi^h_{i-} + \sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta^h_{3,i-j}\right)\theta^h_{1,i-k}\pi^h_{i-k}\right){\sigma_\epsilon}^2\Bigg], \\
\theta^h_{2i} &= \tfrac{1}{\rho\beta_i}\Bigg[(\pi^h - \pi^h_i)\left(\left(\theta^h_{1,i-}\right)^2 + \sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta^h_{3,i-j}\right)^2\left(\theta^h_{1,i-k}\right)^2\right){\sigma_\epsilon}^2 - \\
&\quad \left(\theta^h_{1,i-} + \sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta^h_{3,i-j}\right)\theta^h_{1,i-k}\right)\left(\theta^h_{1,i-}\pi^h_{i-} + \sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta^h_{3,i-j}\right)\theta^h_{1,i-k}\pi^h_{i-k}\right){\sigma_\epsilon}^2\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\theta^h_{3i} &= \tfrac{1}{\rho\beta_i}\Bigg[\left(\sum_{k\neq i}(\pi^h_k)^2{\sigma_\epsilon}^2 + (\gamma^h)^2{\sigma_L}^2\right)\left(\theta^h_{1,i-} + \sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta^h_{3,i-j}\right)\theta^h_{1,i-k}\right) - \\
&\quad (\pi^h - \pi^h_i)\left(\theta^h_{1,i-}\pi^h_{i-} + \sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta^h_{3,i-j}\right)\theta^h_{1,i-k}\pi^h_{i-k}\right){\sigma_\epsilon}^2\Bigg], \\
b^h_i &= \tfrac{1}{\rho}\left(\tfrac{1}{\rho{\sigma_x}^2} + \theta^h_{1i} + \theta^h_{2i}\pi^h + \theta^h_{3i}\left[\theta^h_{1,i-} + \sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta^h_{3,i-j}\right)\theta^h_{1,i-k}\right]\right)^{-1}, \\
\beta_i &= {\sigma_\epsilon}^2\Bigg[\left(\sum_{k\neq i}(\pi^h_k)^2{\sigma_\epsilon}^2 + (\gamma^h)^2{\sigma_L}^2\right)\left(\left(\theta^h_{1,i-}\right)^2 + \sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta^h_{3,i-j}\right)^2\left(\theta^h_{1,i-k}\right)^2\right) - \\
&\quad \left(\theta^h_{1,i-}\pi^h_{i-} + \sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta^h_{3,i-j}\right)\theta^h_{1,i-k}\pi^h_{i-k}\right)^2{\sigma_\epsilon}^2\Bigg],
\end{aligned}
$$

with $\pi^h = \sum_{i=1}^n \pi^h_i$.

Proof: Recall that there exists a random variable $\tilde{\delta}_i$ with $\tilde{z}_{i-1} = \zeta_i \tilde{p} + \tilde{\delta}_i$, satisfying the properties listed in Lemma 2.4. Assuming that risky security price $\tilde{p}$ is given by (2.9.15), the random vectors $(\tilde{X}, \tilde{s}_1, \tilde{p})$ and $(\tilde{X}, \tilde{s}_i, \tilde{p}, \tilde{\delta}_i)_{i=2,..,n}$ are jointly normally distributed. Let V_1^h be the variance-covariance matrix of $(\tilde{s}_1, \tilde{p})$, and W_1^h be the covariance matrix of $\tilde{X}$ and $(\tilde{s}_1, \tilde{p})$. Also, for $i = 2, ..., n$, let V_i^h denote the variance-covariance matrix of $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$, and W_i^h denote the covariance matrix of $\tilde{X}$ and $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$. Since the social network is a tree, agent 1's information is $\mathcal{I}_1 = (s_1, p)$, and for $i = 2, ..., n$, agent i's information is $\mathcal{I}_i = (s_1, p, z_{i-})$. Therefore, the conditional distribution of risky payoff $\tilde{X}$ as assessed by agent 1 has the moments

$$\begin{aligned} \mathrm{E}[\tilde{X}|s_1, p] &= \mathrm{E}[\tilde{X}|s_1, p] = a_{01}^h + a_{11}^h s_1 + a_{21}^h p, \\ \mathrm{var}(\tilde{X}|s_1, p) &= \mathrm{var}(\tilde{X}|s_1, p) = b_1^h, \end{aligned}$$

and the conditional distribution of risky payoff $\tilde{X}$ as assessed by agent i, $i = 2, ..., n$, has the moments

$$\begin{aligned} \mathrm{E}[\tilde{X}|s_i, p, z_{i-}] &= \mathrm{E}[\tilde{X}|s_i, p, \delta_i] = a_{0i}^h + a_{1i}^h s_i + a_{2i}^h p + a_{3i}^h \delta_i, \\ \mathrm{var}(\tilde{X}|s_i, p, z_{i-}) &= \mathrm{var}(\tilde{X}|s_i, p, \delta_i) = b_i^h, \end{aligned}$$

where a_{0i}^h, a_{1i}^h, a_{2i}^h, a_{3i}^h, and b_i^h depend on the variance-covariance matrix V_i^h and W_i^h for all $i = 1, ..., n$.

Given the CARA-Gaussian setup, the demands of agents are of the form

$$z_1 = \frac{\mathrm{E}[\tilde{X}|s_1, p] - p}{\rho \mathrm{var}(\tilde{X}|s_1, p)} = \frac{a_{01}^h + a_{11}^h s_1 + (a_{21}^h - 1)p}{\rho b_1^h}, \tag{2.9.16a}$$

and for $i = 2, ..., n$

$$z_i = \frac{\mathrm{E}[\tilde{X}|s_i, p, z_{i-}] - p}{\rho \mathrm{var}(\tilde{X}|s_i, p, z_{i-})} = \frac{a_{0i}^h + a_{1i}^h s_i + (a_{2i}^h - 1)p + a_{3i}\delta_i}{\rho b_i^h}. \tag{2.9.16b}$$

Now we can derive the random variables $\{\tilde{\delta}_i\}_{i=1,...,n}$ explicitly:

Claim 2.2 *For each* $i = 2, ..., n,$[15]

$$\tilde{\delta}_i = \frac{a^h_{0,i^-}}{\rho b^h_{i^-}} + \frac{a^h_{1,i^-}}{\rho b^h_{i^-}} \tilde{s}_{i^-} + \sum_{k=2}^{l_{i,1}} \left(\prod_{m=1}^{k-1} \frac{a^h_{3,i^-m}}{\rho b^h_{i^-m}} \right) \left(\frac{a^h_{0,i^-k}}{\rho b^h_{0,i^-k}} + \frac{a^h_{1,i^-k}}{\rho b^h_{i^-k}} \tilde{s}_{i^-k} \right). \tag{2.9.17}$$

Proof. First recall that $\tilde{\delta}_j = \tilde{z}_{j-1} - \zeta_j \tilde{p}$, $j = 1, ..., n$. Then following (2.9.16a), for all i such that $i^- = 1$,

$$\tilde{\delta}_i = \frac{a^h_{01} + a^h_{11}\tilde{s}_1 + (a^h_{21} - 1)\tilde{p}}{\rho b^h_1} - \zeta_j \tilde{p}.$$

Due to Lemma 2.4, ζ_j must equal $\frac{a^h_{21}-1}{\rho b^h_1}$ so that

$$\tilde{\delta}_i = \frac{a^h_{01} + a^h_{11}\tilde{s}_1}{\rho b^h_1}, \quad \forall i \text{ s.t. } i^- = 1. \tag{2.9.18}$$

On the other hand, following (2.9.16a), for all i with $i^- > 1$,

$$\tilde{z}_{i^-} = \frac{a^h_{0,i^-} + a^h_{1,i^-}\tilde{s}_{i^-} + (a^h_{2,i^-} - 1)p + a_{3,i^-}(\tilde{z}_{i^-2} - \zeta_{i^-}\tilde{p})}{\rho b^h_{i^-}}. \tag{2.9.19}$$

Recursively using (2.9.18) and (2.9.19) to substitute for $\{z_{i^-k}\}_{k=2,...,l_{i,1}}$, we derive

$$\begin{aligned} \tilde{z}_{i^-} = & \frac{a^h_{0,i^-}}{\rho b^h_{i^-}} + \frac{a^h_{1,i^-}}{\rho b^h_{i^-}}\tilde{s}_{i^-} + \frac{a^h_{2,i^-} - 1}{\rho b^h_{i^-}}\tilde{p} \\ & + \sum_{k=2}^{l_{i,1}-1} \left(\prod_{m=1}^{k-1} \frac{a^h_{3,i^-m}}{\rho b^h_{i^-m}} \right) \left(\frac{a^h_{0,i^-k}}{\rho b^h_{0,i^-k}} + \frac{a^h_{1,i^-k}}{\rho b^h_{i^-k}} \tilde{s}_{i^-k} + \frac{a^h_{2i^-k} - 1}{\rho b^h_{i^-k}}\tilde{p} \right) \\ & - \sum_{k=1}^{l_{i,1}-1} \left(\prod_{m=1}^{k} \frac{a^h_{3,i^-m}}{\rho b^h_{i^-m}} \right) \zeta_{i^-k}\tilde{p} + \left(\prod_{m=1}^{l_{i,1}} \frac{a^h_{3,i^-m}}{\rho b^h_{i^-m}} \right) (\tilde{z}_{i^-l_{i,1}} - \zeta_{i^-l_{i,1}+1}\tilde{p}). \end{aligned}$$

[15] If $l_{i,1} = 1$, i.e., $i^- = 1$, then (2.9.17) simply reduces to

$$\tilde{\delta}_i = \frac{a^h_{0,i^-} + a^h_{1,i^-}\tilde{s}_{i^-}}{\rho b^h_{i^-}} = \frac{a^h_{01} + a^h_{11}\tilde{s}_1}{\rho b^h_1},$$

which is consistent with (2.9.18).

Since $i^{-l_{i,1}} = 1$ by definition, using (2.9.18), the equation above reduces to

$$\begin{aligned}\tilde{z}_{i^-} &= \frac{a^h_{0,i^-}}{\rho b^h_{i^-}} + \frac{a^h_{1,i^-}}{\rho b^h_{i^-}}\tilde{s}_{i^-} + \frac{a^h_{2,i^-} - 1}{\rho b^h_{i^-}}\tilde{p} \\ &+ \sum_{k=2}^{l_{i,1}-1}\left(\prod_{m=1}^{k-1}\frac{a^h_{3,i^{-m}}}{\rho b^h_{i^{-m}}}\right)\left(\frac{a^h_{0,i^{-k}}}{\rho b^h_{0,i^{-k}}} + \frac{a^h_{1,i^{-k}}}{\rho b^h_{i^{-k}}}\tilde{s}_{i^{-k}} + \frac{a^h_{2i^{-k}} - 1}{\rho b^h_{i^{-k}}}\tilde{p}\right) \\ &- \sum_{k=1}^{l_{i,1}-1}\left(\prod_{m=1}^{k}\frac{a^h_{3,i^{-m}}}{\rho b^h_{i^{-m}}}\right)\zeta_{i^{-k}}\tilde{p} + \left(\prod_{m=1}^{l_{i,1}}\frac{a^h_{3,i^{-m}}}{\rho b^h_{i^{-m}}}\right)\frac{a^h_{01} + a^h_{11}\tilde{s}_1}{\rho b^h_1}.\end{aligned}$$

Then due to Lemma 2.4,

$$\begin{aligned}\tilde{\delta}_i &= \frac{a^h_{0,i^-}}{\rho b^h_{i^-}} + \frac{a^h_{1,i^-}}{\rho b^h_{i^-}}\tilde{s}_{i^-} + \sum_{k=2}^{l_{i,1}-1}\left(\prod_{m=1}^{k-1}\frac{a^h_{3,i^{-m}}}{\rho b^h_{i^{-m}}}\right)\left(\frac{a^h_{0,i^{-k}}}{\rho b^h_{0,i^{-k}}} + \frac{a^h_{1,i^{-k}}}{\rho b^h_{i^{-k}}}\tilde{s}_{i^{-k}}\right) \\ &+ \left(\prod_{m=1}^{l_{i,1}}\frac{a^h_{3,i^{-m}}}{\rho b^h_{i^{-m}}}\right)\frac{a^h_{01} + a^h_{11}\tilde{s}_1}{\rho b^h_1}.\end{aligned}$$

and using the relation $i^{-l_{i,1}} = 1$ gives us

$$\tilde{\delta}_i = \frac{a^h_{0,i^-}}{\rho b^h_{i^-}} + \frac{a^h_{1,i^-}}{\rho b^h_{i^-}}\tilde{s}_{i^-} + \sum_{k=2}^{l_{i,1}}\left(\prod_{m=1}^{k-1}\frac{a^h_{3,i^{-m}}}{\rho b^h_{i^{-m}}}\right)\left(\frac{a^h_{0,i^{-k}}}{\rho b^h_{0,i^{-k}}} + \frac{a^h_{1,i^{-k}}}{\rho b^h_{i^{-k}}}\tilde{s}_{i^{-k}}\right).$$ □ *[end of claim]*

Having this explicit expression for $\tilde{\delta}_i$, we can write down the distribution matrices. For notational convenience, we define $\pi^h = \sum_{k=1}^n \pi^h_k$. The variance-covariance matrix of $(\tilde{s}_1, \tilde{p})$ is

$$V^h_1 = \begin{bmatrix} {\sigma_x}^2 + {\sigma_\epsilon}^2 & \pi^h{\sigma_x}^2 + \pi^h_1{\sigma_\epsilon}^2 \\ \pi^h{\sigma_x}^2 + \pi^h_1{\sigma_\epsilon}^2 & (\pi^h)^2{\sigma_x}^2 + \sum_{k=1}^n(\pi^h_k)^2{\sigma_k}^2 + (\gamma^h)^2{\sigma_L}^2 \end{bmatrix},$$

and the covariance matrix of $\tilde{X}$ and $(\tilde{s}_1, \tilde{p})$ is

$$W^h_1 = {\sigma_x}^2\begin{bmatrix} 1 \\ \pi^h \end{bmatrix}.$$

For $i > 1$, the variance-covariance matrix of $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$ is

$$V^h_i = \begin{bmatrix} {\sigma_x}^2 + {\sigma_\epsilon}^2 & \pi^h{\sigma_x}^2 + \pi^h_i{\sigma_\epsilon}^2 & \mathrm{cov}(\tilde{s}_i, \tilde{\delta}^i_{i^-}) \\ \pi^h{\sigma_x}^2 + \pi^h_i{\sigma_\epsilon}^2 & (\pi^h)^2{\sigma_x}^2 + \sum_{k=1}^n(\pi^h_k)^2{\sigma_k}^2 + (\gamma^h)^2{\sigma_L}^2 & \mathrm{cov}(\tilde{p}, \tilde{\delta}^i_{i^-}) \\ \mathrm{cov}(\tilde{s}_i, \tilde{\delta}^i_{i^-}) & \mathrm{cov}(\tilde{p}, \tilde{\delta}^i_{i^-}) & \mathrm{var}(\tilde{\delta}^i_{i^-}) \end{bmatrix},$$

where

$$\begin{aligned}
\operatorname{cov}(\tilde{s}_i,\tilde{\delta}_i) &= \left(\frac{a^h_{1,i-}}{\rho b^h_{i-}}+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a^h_{3,i-j}}{\rho b^h_{i-j}}\right)\frac{a^h_{1,i-k}}{\rho b^h_{i-k}}\right){\sigma_x}^2,\\
\operatorname{cov}(\tilde{p},\tilde{\delta}_i) &= \left(\frac{a^h_{1,i-}}{\rho b^h_{i-}}+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a^h_{3,i-j}}{\rho b^h_{i-j}}\right)\frac{a^h_{1,i-k}}{\rho b^h_{i-k}}\right)\pi^h{\sigma_x}^2 + \left(\frac{a^h_{1,i-}}{\rho b^h_{i-}}\pi^h_{i-}+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a^h_{3,i-j}}{\rho b^h_{i-j}}\right)\frac{a^h_{1,i-k}}{\rho b^h_{i-k}}\pi^h_{i-k}\right){\sigma_\epsilon}^2,\\
\operatorname{var}(\tilde{\delta}_i) &= \left(\frac{a^h_{1,i-}}{\rho b^h_{i-}}+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a^h_{3,i-j}}{\rho b^h_{i-j}}\right)\frac{a^h_{1,i-k}}{\rho b^h_{i-k}}\right)^2{\sigma_x}^2 + \left(\left(\frac{a^h_{1,i-}}{\rho b^h_{i-}}\right)^2+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a^h_{3,i-j}}{\rho b^h_{i-j}}\right)^2\left(\frac{a^h_{1,i-k}}{\rho b^h_{i-k}}\right)^2\right){\sigma_\epsilon}^2,
\end{aligned}$$

and the covariance matrix of $\tilde{X}$ and $(\tilde{s}_i,\tilde{p},\tilde{\delta}_i)$ is

$$W^h_i = {\sigma_x}^2\begin{bmatrix} 1 \\ \pi^h \\ \frac{a^h_{1,i-}}{\rho b^h_{i-}}+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a^h_{3,i-j}}{\rho b^h_{i-j}}\right)\frac{a^h_{1,i-k}}{\rho b^h_{i-k}}\end{bmatrix}.$$

Normal distribution theory dictates

$$\begin{bmatrix}a^h_{11} & a^h_{21}\end{bmatrix} = W^{h\prime}_1(V^h_1)^{-1}, \tag{2.9.20a}$$

$$b^h_1 = {\sigma_x}^2 - W^{h\prime}_1(V^h_1)^{-1}W^h_1, \tag{2.9.20b}$$

$$a^h_{01} = \mu_x - W^{h\prime}_1(V^h_1)^{-1}\begin{bmatrix}\mu_x \\ \pi^h_0+\pi^h\mu_x\end{bmatrix}, \tag{2.9.20c}$$

and for all $i > 1$

$$\begin{bmatrix}a^h_{1i} & a^h_{2i} & a^h_{3i}\end{bmatrix} = W^{h\prime}_i(V^h_i)^{-1}, \tag{2.9.20d}$$

$$b^h_i = {\sigma_x}^2 - W^{h\prime}_i(V^h_i)^{-1}W^h_i, \tag{2.9.20e}$$

$$a^h_{0i} = \mu_x - W^{h\prime}_i(V^h_i)^{-1}\begin{bmatrix}\mu_x \\ \pi^h_0+\pi^h\mu_x \\ \left(\frac{a^h_{1,i-}}{\rho b^h_{i-}}+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a^h_{3,i-j}}{\rho b^h_{i-j}}\right)\frac{a^h_{1,i-k}}{\rho b^h_{i-k}}\right)\mu_x\end{bmatrix} \tag{2.9.20f}$$

After elementary but somewhat tedious manipulations, one acquires the following result:

$$\begin{aligned}
\frac{a^h_{11}}{\rho b^h_1} &= \frac{1}{\rho{\sigma_\epsilon}^2}\frac{\left(\sum_{k=1}^{n}(\pi^h_k)^2-\pi^h\pi^h_1\right){\sigma_\epsilon}^2+(\gamma^h)^2{\sigma_L}^2}{\sum_{k\neq 1}(\pi^h_k)^2{\sigma_\epsilon}^2+(\gamma^h)^2{\sigma_L}^2},\\
\frac{a^h_{21}}{\rho b^h_1} &= \frac{\pi^h-\pi^h_1}{\rho\left(\sum_{k\neq 1}(\pi^h_k)^2{\sigma_\epsilon}^2+(\gamma^h)^2{\sigma_L}^2\right)},
\end{aligned}$$

and for $i > 1$

$$\begin{aligned}
\frac{a_{1i}^h}{\rho b_i^h} &= \frac{1}{\rho\beta_i}\Bigg[\Big(\big(\textstyle\sum_{k=1}^{n}(\pi_k^h)^2-\pi^h\pi_i^h\big)\sigma_\epsilon{}^2+(\gamma^h)^2\sigma_L{}^2\Big)\Bigg(\left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}\right)^2+\textstyle\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)^2\left(\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\right)^2\Bigg)+ \\
&\quad \left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}(\pi_i^h-\pi_{i-}^h)+\textstyle\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}(\pi_i^h-\pi_{i-k}^h)\right)\times \\
&\quad \left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}\pi_{i-}^h+\textstyle\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\pi_{i-k}^h\right)\sigma_\epsilon{}^2\Bigg], \\
\frac{a_{2i}^h}{\rho b_i^h} &= \frac{1}{\rho\beta_i}\Bigg[(\pi^h-\pi_i^h)\Bigg(\left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}\right)^2+\textstyle\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)^2\left(\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\right)^2\Bigg)\sigma_\epsilon{}^2 - \\
&\quad \left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}+\textstyle\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\right)\left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}\pi_{i-}^h+\textstyle\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\pi_{i-k}^h\right)\sigma_\epsilon{}^2\Bigg], \\
\frac{a_{3i}^h}{\rho b_i^h} &= \frac{1}{\rho\beta_i}\Bigg[\Big(\textstyle\sum_{k\neq i}(\pi_k^h)^2\sigma_\epsilon{}^2+(\gamma^h)^2\sigma_L{}^2\Big)\left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}+\textstyle\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\right)- \\
&\quad (\pi^h-\pi_i^h)\left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}\pi_{i-}^h+\textstyle\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\pi_{i-k}^h\right)\sigma_\epsilon{}^2\Bigg]; \\
\beta_i &= \sigma_\epsilon{}^2\Bigg[\Big(\textstyle\sum_{k\neq i}(\pi_k^h)^2\sigma_\epsilon{}^2+(\gamma^h)^2\sigma_L{}^2\Big)\Bigg(\left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}\right)^2+\textstyle\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)^2\left(\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\right)^2\Bigg)- \\
&\quad \left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}\pi_{i-}^h+\textstyle\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\pi_{i-k}^h\right)^2\sigma_\epsilon{}^2\Bigg].
\end{aligned}$$

Now define

$$\theta_{01}^h \equiv \frac{a_{01}^h}{\rho b_1^h} \qquad \theta_{11}^h \equiv \frac{a_{11}^h}{\rho b_1^h}, \qquad \theta_{21}^h \equiv \frac{a_{21}^h}{\rho b_1^h},$$

and for $i > 1$

$$\theta_{0i}^h \equiv \frac{a_{0i}^h}{\rho b_i^h} \qquad \theta_{1i}^h \equiv \frac{a_{1i}^h}{\rho b_i^h}, \qquad \theta_{2i}^h \equiv \frac{a_{2i}^h}{\rho b_i^h}, \qquad \theta_{3i}^h \equiv \frac{a_{3i}^h}{\rho b_i^h}.$$

We can rewrite the equations above as follows

$$\theta_{11}^h = \frac{1}{\rho\sigma_\epsilon{}^2}\frac{\left(\sum_{k=1}^n(\pi_k^h)^2-\pi^h\pi_1^h\right)\sigma_\epsilon{}^2+(\gamma^h)^2\sigma_L{}^2}{\sum_{k\neq 1}(\pi_k^h)^2\sigma_\epsilon{}^2+(\gamma^h)^2\sigma_L{}^2}, \tag{2.9.22a}$$

$$\theta_{21}^h = \frac{\pi^h-\pi_1^h}{\rho\left(\sum_{k\neq 1}(\pi_k^h)^2\sigma_\epsilon{}^2+(\gamma^h)^2\sigma_L{}^2\right)}, \tag{2.9.22b}$$

$$\begin{aligned}\theta_{1i}^h = \frac{1}{\rho\beta_i}\Bigg[&\left(\left(\sum_{k=1}^n(\pi_k^h)^2-\pi^h\pi_i^h\right)\sigma_\epsilon{}^2+(\gamma^h)^2\sigma_L{}^2\right)\left(\left(\theta_{1,i-}^h\right)^2+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta_{3,i-j}^h\right)^2\left(\theta_{1,i-k}\right)^2\right)+\\&\left(\theta_{1,i-}^h(\pi_i^h-\pi_{i-}^h)+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta_{3,i-j}^h\right)\theta_{1,i-k}^h(\pi_i^h-\pi_{i-k}^h)\right)\times\\&\left(\theta_{1,i-}^h\pi_{i-}^h+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta_{3,i-j}^h\right)\theta_{1,i-k}^h\pi_{i-k}^h\right)\sigma_\epsilon{}^2\Bigg],\end{aligned} \tag{2.9.22c}$$

$$\begin{aligned}\theta_{2i}^h = \frac{1}{\rho\beta_i}\Bigg[&(\pi^h-\pi_i^h)\left(\left(\theta_{1,i-}^h\right)^2+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta_{3,i-j}^h\right)^2\left(\theta_{1,i-k}^h\right)^2\right)\sigma_\epsilon{}^2-\\&\left(\theta_{1,i-}^h+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta_{3,i-j}^h\right)\theta_{1,i-k}^h\right)\left(\theta_{1,i-}^h\pi_{i-}^h+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta_{3,i-j}^h\right)\theta_{1,i-k}^h\pi_{i-k}^h\right)\sigma_\epsilon{}^2\Bigg],\end{aligned} \tag{2.9.22d}$$

$$\begin{aligned}\theta_{3,i}^h = \frac{1}{\rho\beta_i}\Bigg[&\left(\sum_{k\neq i}(\pi_k^h)^2\sigma_\epsilon{}^2+(\gamma^h)^2\sigma_L{}^2\right)\left(\theta_{1,i-}^h+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta_{3,i-j}^h\right)\theta_{1,i-k}^h\right)-\\&(\pi^h-\pi_i^h)\left(\theta_{1,i-}^h\pi_{i-}^h+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta_{3,i-j}^h\right)\theta_{1,i-k}^h\pi_{i-k}^h\right)\sigma_\epsilon{}^2\Bigg];\end{aligned} \tag{2.9.22e}$$

$$\begin{aligned}\beta_i = \sigma_\epsilon{}^2\Bigg[&\left(\sum_{k\neq i}(\pi_k^h)^2\sigma_\epsilon{}^2+(\gamma^h)^2\sigma_L{}^2\right)\left(\left(\theta_{1,i-}^h\right)^2+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta_{3,i-j}^h\right)^2\left(\theta_{1,i-k}^h\right)^2\right)-\\&\left(\theta_{1,i-}^h\pi_{i-}^h+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta_{3,i-j}^h\right)\theta_{1,i-k}^h\pi_{i-k}^h\right)^2\sigma_\epsilon{}^2\Bigg].\end{aligned}$$

Moreover, from (2.9.20b) and (2.9.20e) we have

$$b_1^h = \sigma_x{}^2\left(1-a_{11}^h-a_{21}^h\pi^h\right),$$

$$b_i^h = \sigma_x{}^2\left(1-a_{1i}^h-a_{2i}^h\pi^h-a_{3i}^h\left[\frac{a_{1,i-}^h}{\rho b_{i-}^h}+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\right]\right), \quad i>1.$$

Following these equations, one can easily acquire

$$b_1^h = \frac{1}{\rho}\left(\frac{1}{\rho\sigma_x{}^2}+\theta_{11}^h+\theta_{21}^h\pi^h\right)^{-1}, \tag{2.9.23a}$$

$$b_i^h = \frac{1}{\rho}\left(\frac{1}{\rho\sigma_x{}^2}+\theta_{1i}^h+\theta_{2i}^h\pi^h+\theta_{3i}^h\left[\theta_{1,i-}^h+\sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta_{3,i-j}^h\right)\theta_{1,i-k}^h\right]\right)^{-1}, \quad i>1. \tag{2.9.23b}$$

We also have from (2.9.20c) and (2.9.20f)

$$\begin{aligned} a_{01}^h &= \mu_x - a_{11}^h\mu_x - a_{21}^h\left(\pi_0^h + \pi^h\mu_x\right), \\ a_{0i}^h &= \mu_x - a_{1i}^h\mu_x - a_{2i}^h\left(\pi_0^h + \pi^h\mu_x\right) - a_{3i}^h\left(\frac{a_{1,i-}^h}{\rho b_{i-}^h} + \sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\right)\mu_x, \qquad i > 1. \end{aligned}$$

Hence the following holds

$$\theta_{01}^h = \mu_x\left(\frac{1}{\rho b_1^h} - \theta_{11}^h - \theta_{21}^h\pi^h\right) - \theta_{21}^h\pi_0^h, \tag{2.9.24a}$$

$$\theta_{0i}^h = \mu_x\left(\frac{1}{\rho b_i^h} - \theta_{1i}^h - \theta_{2i}^h\pi^h - \theta_{3i}^h\left[\theta_{1,i-}^h + \sum_{k=2}^{l_{i,1}}\left(\prod_{j=1}^{k-1}\theta_{3,i-j}^h\right)\theta_{1,i-k}^h\right]\right) - \theta_{2i}^h\pi_0^h. \tag{2.9.24b}$$

On the other hand, we have the market clearing condition given by

$$z_1(s_1, p) + \sum_{i>1} z_i(s_i, p, \delta_{i-}^i) = L.$$

Substituting from (2.9.16a)-(2.9.16b) and solving for p, we get

$$p = \left(\sum_{i=1}^n \frac{1 - a_{2i}^h}{\rho b_i^h}\right)^{-1}\left(\sum_{i=1}^n \frac{a_{0i}^h}{\rho b_i^h} + \sum_{i=1}^n \frac{a_{1i}^h}{\rho b_i^h} s_i + \sum_{i>1}\frac{a_{3i}^h}{\rho b_i^h}\delta_{i-}^i - L\right).$$

A further substitution from (2.9.17) results with

$$p = \left(\sum_{i=1}^n \frac{1 - a_{2i}^h}{\rho b_i^h}\right)^{-1}\left(\sum_{i=1}^n \frac{a_{0i}^h}{\rho b_i^h} + \sum_{i=1}^n \frac{a_{1i}^h}{\rho b_i^h} s_i + \sum_{i>1}\sum_{k=1}^{l_{i,1}}\left(\prod_{j=0}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h} s_{i-k} - L\right).$$

Rearranging terms and substituting $\{\theta_{j,i}^h\}_{j=1,2,3}$ for $\{\frac{a_{j,i}^h}{\rho b_i^h}\}_{j=1,2,3}$ yields

$$p = \left(\sum_{i=1}^n \frac{1}{\rho b_i^h} - \theta_{2i}^h\right)^{-1}\left(\sum_{i=1}^n \theta_{0i}^h + \sum_{i=1}^n\left(1 + \sum_{m\in H^{-1}(i)}\prod_{j=0}^{l_{m,i}-1}\theta_{3,m-j}^h\right)\theta_{1i}^h s_i - L\right). \tag{2.9.25}$$

We observe that the market clearing price induced by hypothesis (2.9.15) is again linear in private signals and liquidity supply. Expectations formed relying on (2.9.15) would be rational if and only if the coefficients π_0^h, $\{\pi_i^h\}_{i=1,\ldots,n}$, γ^h in (2.9.15) are the same as the corresponding coefficients in

(2.9.25). Therefore the following conditions hold:

$$\pi_i^h = \gamma^h \left(1 + \sum_{m\in H^{-1}(i)} \prod_{j=0}^{l_{m,i}-1} \theta_{3,m^{-j}}^h\right) \theta_{1i}^h, \qquad i = 1, ..., n \tag{2.9.26a}$$

$$\gamma^h = \left(\sum_{i=1}^{n} \frac{1}{\rho b_i^h} - \theta_{2i}^h\right)^{-1}, \tag{2.9.26b}$$

$$\pi_0^h = \gamma^h \sum_{i=1}^{n} \theta_{0i}^h. \tag{2.9.26c}$$

Recall that θ_{0i}^h, θ_{1i}^h, θ_{2i}^h, θ_{3i}^h, and b_i^h depend on the price coefficients $\pi_0^h, \pi_1^h, ..., \pi_n^h, \gamma^h$ (see Eqns. (2.9.22a)-(2.9.24b)). The existence of a linear REESI price as given by (2.9.15) is equivalent to the existence of a solution to the system of equations (2.9.26a)-(2.9.26c) and (2.9.22a)-(2.9.24b) in arguments $(\pi_0^h, \pi_1^h, ..., \pi_n^h, \gamma^h)$ with $\gamma^h \neq 0$, which essentially presents a *fixed point problem* as given in the statement of this lemma. □

Proof of Proposition 2.3: Suppose price $\tilde{p}$ is given by (2.9.15). Let $\pi_0^h, \pi_1^h, ..., \pi_n^h, \gamma^h$ be the coefficients of our hypothetical price $\tilde{p}$. To establish that there exists a linear REESI price in a hierarchic interaction scheme, it suffices to show that there exists a solution to the fixed point problem given in Lemma 2.5.

Define

$$Q_i^h \equiv \frac{\pi_i^h}{\gamma^h}, \qquad i = 1, 2, ..., n,$$

$$Q^h \equiv \frac{\pi^h}{\gamma^h},$$

where $\pi^h = \sum_{i=1}^n \pi_i^h$. Also let

$$\Theta_{11}^h = \frac{1}{\rho\sigma_\epsilon{}^2} \frac{\left(\sum_{k=1}^n (Q_k^h)^2 - Q^h Q_1^h\right)\sigma_\epsilon{}^2 + \sigma_L{}^2}{\sum_{k\neq 1}(Q_k^h)^2\sigma_\epsilon{}^2 + \sigma_L{}^2}, \tag{2.9.27a}$$

$$\Theta_{21}^h = \frac{Q^h - Q_1^h}{\rho\left(\sum_{k\neq 1}(Q_k^h)^2\sigma_\epsilon{}^2 + \sigma_L{}^2\right)}, \tag{2.9.27b}$$

and for $i = 2, ..., n$

$$\begin{aligned}
\Theta_{1i}^h &= \tfrac{1}{\rho B_i}\Bigg[\Big(\big(\textstyle\sum_{k=1}^n (Q_k^h)^2 - Q^h Q_i^h\big)\sigma_\epsilon{}^2 + \sigma_L{}^2\Big)\Big(\big(\Theta_{1,i-}^h\big)^2 + \sum_{k=2}^{l_{i,1}}\big(\prod_{j=1}^{k-1}\Theta_{3,i-j}^h\big)^2\big(\Theta_{1,i-k}\big)^2\Big) + \\
&\Big(\Theta_{1,i-}^h (Q_i^h - Q_{i-}^h) + \textstyle\sum_{k=2}^{l_{i,1}}\big(\prod_{j=1}^{k-1}\Theta_{3,i-j}^h\big)\Theta_{1,i-k}^h (Q_i^h - Q_{i-k}^h)\Big)\times \\
&\Big(\Theta_{1,i-}^h Q_{i-}^h + \textstyle\sum_{k=2}^{l_{i,1}}\big(\prod_{j=1}^{k-1}\Theta_{3,i-j}^h\big)\Theta_{1,i-k}^h Q_{i-k}^h\Big)\sigma_\epsilon{}^2\Bigg], \qquad (2.9.27c)
\end{aligned}$$

$$\begin{aligned}
\Theta_{2i}^h &= \tfrac{1}{\rho B_i}\Bigg[(Q^h - Q_i^h)\Big(\big(\Theta_{1,i-}^h\big)^2 + \textstyle\sum_{k=2}^{l_{i,1}}\big(\prod_{j=1}^{k-1}\Theta_{3,i-j}^h\big)^2\big(\Theta_{1,i-k}^h\big)^2\Big)\sigma_\epsilon{}^2 - \\
&\Big(\Theta_{1,i-}^h + \textstyle\sum_{k=2}^{l_{i,1}}\big(\prod_{j=1}^{k-1}\Theta_{3,i-j}^h\big)\Theta_{1,i-k}^h\Big)\Big(\Theta_{1,i-}^h Q_{i-}^h + \sum_{k=2}^{l_{i,1}}\big(\prod_{j=1}^{k-1}\Theta_{3,i-j}^h\big)\Theta_{1,i-k}^h Q_{i-k}^h\Big)\sigma_\epsilon{}^2\Bigg], \\
&\qquad (2.9.27d)
\end{aligned}$$

$$\begin{aligned}
\Theta_{3i}^h &= \tfrac{1}{\rho B_i}\Bigg[\Big(\textstyle\sum_{k\neq i}(Q_k^h)^2\sigma_\epsilon{}^2 + \sigma_L{}^2\Big)\Big(\Theta_{1,i-}^h + \sum_{k=2}^{l_{i,1}}\big(\prod_{j=1}^{k-1}\Theta_{3,i-j}^h\big)\Theta_{1,i-k}^h\Big) - \\
&(Q^h - Q_i^h)\Big(\Theta_{1,i-}^h Q_{i-}^h + \textstyle\sum_{k=2}^{l_{i,1}}\big(\prod_{j=1}^{k-1}\Theta_{3,i-j}^h\big)\Theta_{1,i-k}^h Q_{i-k}^h\Big)\sigma_\epsilon{}^2\Bigg]; \qquad (2.9.27e)
\end{aligned}$$

$$\begin{aligned}
B_i &= \sigma_\epsilon{}^2\Bigg[\Big(\textstyle\sum_{k\neq i}(Q_k^h)^2\sigma_\epsilon{}^2 + \sigma_L{}^2\Big)\Big(\big(\Theta_{1,i-}^h\big)^2 + \sum_{k=2}^{l_{i,1}}\big(\prod_{j=1}^{k-1}\Theta_{3,i-j}^h\big)^2\big(\Theta_{1,i-k}^h\big)^2\Big) - \\
&\Big(\Theta_{1,i-}^h Q_{i-}^h + \textstyle\sum_{k=2}^{l_{i,1}}\big(\prod_{j=1}^{k-1}\Theta_{3,i-j}^h\big)\Theta_{1,i-k}^h Q_{i-k}^h\Big)^2\sigma_\epsilon{}^2\Bigg]. \qquad (2.9.27f)
\end{aligned}$$

Notice that

$$\begin{aligned}
\Theta_{1i}^h &= \theta_{1i}^h, \\
\Theta_{3i}^h &= \theta_{3i}^h; \\
\Theta_{2i}^h &= \theta_{2i}^h \gamma^h, \qquad i = 1, ..., n,
\end{aligned}$$

where θ_{ji}, $j = 1, 2, 3$, $i = 1, 2, ..., n$, are given as in Lemma 2.5.

From Lemma 2.5, one further has

$$\frac{1}{\rho b_1^h} = \frac{1}{\rho {\sigma_x}^2} + \Theta_{11}^h + \Theta_{21}^h Q^h, \tag{2.9.28a}$$

$$\frac{1}{\rho b_i^h} = \frac{1}{\rho {\sigma_x}^2} + \Theta_{1i}^h + \Theta_{2i}^h Q^h + \Theta_{3i}^h \left(\Theta_{1,i^-}^h + \sum_{k=2}^{l_{i,1}} (\prod_{j=1}^{k-1} \Theta_{3,i^{-j}}^h) \Theta_{1,i^{-k}}^h \right), \quad i = 2, ..., n \tag{2.9.28b}$$

$$\theta_{01}^h = \frac{\mu_x}{\rho {\sigma_x}^2} - \Theta_{21}^h \frac{\pi_0^h}{\gamma^h}, \tag{2.9.28c}$$

$$\theta_{0i}^h = \frac{\mu_x}{\rho {\sigma_x}^2} - \Theta_{2i}^h \frac{\pi_0^h}{\gamma^h}, \qquad i = 2, ..., n. \tag{2.9.28d}$$

And also:

$$Q_i^h = \left(1 + \sum_{m \in \mathcal{H}^{-1}(i)} \prod_{j=0}^{l_{m,i}-1} \Theta_{3,m^{-j}}^h \right) \Theta_{1i}^h, \qquad i = 1, ..., n \tag{2.9.29a}$$

$$\gamma^h = \left(\sum_{i=1}^{n} \frac{1}{\rho b_i^h} - \frac{\Theta_{2i}^h}{\gamma^h} \right)^{-1}, \tag{2.9.29b}$$

$$\pi_0^h = \gamma^h \sum_{i=1}^{n} \theta_{0i}^h, \tag{2.9.29c}$$

Since Θ_{ji}^h, $j = 1, 3$, $i = 1, ..., n$, only depend on $\{Q_i^h\}_{i=1,...,n}$, equation (2.9.29a) can be analyzed independently from equations (2.9.29b)-(2.9.29c). Essentially (2.9.29a) is a fixed point problem in the arguments $(Q_1^h, ..., Q_n^h)$. To show the existence of a solution (i.e., a fixed point) for this problem, we proceed as follows: let

$$\Omega^h \equiv \left[0, \frac{2}{\rho {\sigma_\epsilon}^2} (1 + n 2^n) \right]^n \in \mathbb{R}_+^n$$

and define $\mathcal{F} : \Omega^h \to \mathbb{R}^n$, $\mathcal{G} : \Omega^h \to \Omega^h$ by the conditions

$$\left(\mathcal{F}(Q_1^h, ..., Q_n^h) \right)_i = \left(1 + \sum_{m \in \mathcal{H}^{-1}(i)} \prod_{j=0}^{l_{m,i}-1} \Theta_{3,m^{-j}}^h \right) \Theta_{1i}^h, \qquad i = 1, ..., n;$$

and

$$\begin{aligned}
&\left(\mathcal{G}(Q_1^h, ..., Q_n^h) \right)_i = 0 && \text{if } \left(\mathcal{F}(Q_1^h, ..., Q_n^h) \right)_i < 0, \\
&\left(\mathcal{G}(Q_1^h, ..., Q_n^h) \right)_i = \left(\mathcal{F}(Q_1^h, ..., Q_n^h) \right)_i && \text{if } 0 \le \left(\mathcal{F}(Q_1^h, ..., Q_n^h) \right)_i \le \tfrac{2}{\rho {\sigma_\epsilon}^2} (1 + n 2^n), \\
&\left(\mathcal{G}(Q_1^h, ..., Q_n^h) \right)_i = \tfrac{2}{\rho {\sigma_\epsilon}^2} (1 + n 2^n) && \text{if } \left(\mathcal{F}(Q_1^h, ..., Q_n^h) \right)_i > \tfrac{2}{\rho {\sigma_\epsilon}^2} (1 + n 2^n), \quad i = 1, ..., n.
\end{aligned}$$

Due to assumptions $S1$ and $S2$, Ω^h is compact and $\mathcal{G}$ is continuous. Thus we can employ *Brouwer's Theorem*, which implies $\mathcal{G}$ has a fixed point

$$(\hat{Q}_1^h, ..., \hat{Q}_n^h) \in \Omega^h.$$

Now we would like to show

$$0 < \hat{Q}_i^h < \frac{2}{\rho{\sigma_\epsilon}^2}(1 + n2^n), \qquad i = 1, ..., n$$

so that we end up with

$$(\hat{Q}_1^h, ..., \hat{Q}_n^h) = \mathcal{F}(\hat{Q}_1^h, ..., \hat{Q}_n^h).$$

We will be able to show this *given sufficiently high level of liquidity variance* ${\sigma_L}^2$ *in price*. However, before proceeding with our arguments, let us introduce the following notation:

$$\hat{\Theta}_{ji} = \Theta_{ji}\Big|_{(Q_1^h,...,Q_n^h)=(\hat{Q}_1^h,...,\hat{Q}_n^h)}, \qquad j = 1, 2, 3, \quad i = 1, ..., n.$$

Claim 2.3 *There exists* $\underline{\sigma_L}^2 < \infty$ *such that* $\forall\, {\sigma_L}^2 > \underline{\sigma_L}^2$ *and* $\forall\, (\hat{Q}_1^h, ..., \hat{Q}_n^h) \in \Omega^h$

$$\begin{aligned} 0 &< \hat{\Theta}_{1i} < \frac{2}{\rho{\sigma_\epsilon}^2}, \qquad & i = 1, 2, ..., n, \\ 0 &< \hat{\Theta}_{3i} < 2, \qquad & i = 2, ..., n. \end{aligned}$$

Proof. Since $0 \leq \hat{Q}_i^h \leq \frac{2}{\rho{\sigma_\epsilon}^2}(1 + n2^n)$, $i = 1, ..., n$, following (2.9.27a)

$$\lim_{{\sigma_L}^2 \to \infty} \Theta_{11}^h = \frac{1}{\rho{\sigma_\epsilon}^2}.$$

Furthermore, from (2.9.27e) and the equality above one derives

$$\lim_{{\sigma_L}^2 \to \infty} \Theta_{32}^h = 1.$$

Now proceed by induction: suppose for all $i \leq k$,

$$\begin{aligned} \lim_{{\sigma_L}^2 \to \infty} \Theta_{1,i-1}^h &= \frac{1}{\rho{\sigma_\epsilon}^2}, \\ \lim_{{\sigma_L}^2 \to \infty} \Theta_{3,i}^h &= 1. \end{aligned}$$

Then from (2.9.27c)

$$\lim_{\sigma_L{}^2 \to \infty} \Theta_{1,k}^h = \frac{1}{\rho\sigma_\epsilon{}^2},$$

and from (2.9.27e)

$$\begin{aligned}\lim_{\sigma_L{}^2 \to \infty} \Theta_{3,k+1}^h &= \frac{1}{\rho\sigma_\epsilon{}^2} \frac{\frac{1}{\rho\sigma_\epsilon{}^2} + \frac{1}{\rho\sigma_\epsilon{}^2}\sum_{k=2}^{l_{i,1}} 1}{\left(\frac{1}{\rho\sigma_\epsilon{}^2}\right)^2 + \left(\frac{1}{\rho\sigma_\epsilon{}^2}\right)^2 \sum_{k=2}^{l_{i,1}} 1} \\ &= 1.\end{aligned}$$

Thus, by the (strong) induction argument above, we have

$$\begin{aligned}\lim_{\sigma_L{}^2 \to \infty} \Theta_{1i}^h &= \frac{1}{\rho\sigma_\epsilon{}^2}, \qquad i = 1, 2, ..., n, \\ \lim_{\sigma_L{}^2 \to \infty} \Theta_{3i}^h &= 1, \qquad i = 2, ..., n.\end{aligned}$$

By continuity of Θ_{1i}^h and Θ_{3i}^h as functions of $\sigma_L{}^2$, there exists $\underline{\sigma_L}{}^2 < \infty$ such that $\forall \sigma_L{}^2 \geq \underline{\sigma_L}{}^2$

$$\begin{aligned}0 < \hat{\Theta}_{1i} &< \frac{2}{\rho\sigma_\epsilon{}^2}, \qquad i = 1, 2, ..., n, \\ 0 < \hat{\Theta}_{3i} &< 2, \qquad i = 2, ..., n.\end{aligned}$$

□ [*end of claim*]

Claim 2.4 *For all* $\sigma_L{}^2 > \underline{\sigma_L}{}^2$

$$\hat{Q}_i^h > 0, \qquad i = 1, ..., n.$$

Proof. Let us fix liquidity variance $\sigma_L{}^2$ so that $\sigma_L{}^2 > \underline{\sigma_L}{}^2$. Suppose there exists $i_{1 \leq i \leq n}$ such that $\hat{Q}_i^h = 0$. Then due to the way we defined $\mathcal{F}$ and $\mathcal{G}$, the following must hold:

$$\left(\left(1 + \sum_{m \in \mathcal{H}^{-1}(i)} \prod_{j=0}^{l_{m,i}-1} \hat{\Theta}_{3,m-j}^h\right) \hat{\Theta}_{1i}^h\right)\Bigg|_{\hat{Q}_i^h = 0} \leq 0.$$

However Claim 2.3 shows that $\forall\, (\hat{Q}_1^h, ..., \hat{Q}_n^h) \in \Omega^h = \left[0, \frac{2}{\rho\sigma_\epsilon{}^2}\left(1 + n2^n\right)\right]^n$

$$\begin{aligned}\hat{\Theta}_{1i}^h &> 0, \qquad i = 1, 2, ..., n, \\ \hat{\Theta}_{3i}^h &> 0, \qquad i = 2, ..., n.\end{aligned}$$

Then we have a clear violation of the inequality above since

$$\left.\left(\left(1+\sum_{m\in\mathcal{H}^{-1}(i)}\prod_{j=0}^{l_{m,i}-1}\hat{\Theta}^{h}_{3,m^{-j}}\right)\hat{\Theta}^{h}_{1i}\right)\right|_{\hat{Q}^{h}_{i}=0} > 0.$$

Thus it must be true that $\hat{Q}^{h}_{i} > 0, i = 1, ..., n.$ □ *[end of claim]*

Claim 2.5 *For all* $\sigma_L{}^2 > \underline{\sigma_L}{}^2$

$$\hat{Q}^{h}_{i} < \frac{2}{\rho\sigma_\epsilon{}^2}\left(1+n2^n\right), \qquad i = 1, ..., n.$$

Proof. Fix liquidity variance $\sigma_L{}^2$ so that $\sigma_L{}^2 > \underline{\sigma_L}{}^2$, and suppose there exists $i_{1\leq i\leq n}$ such that $\hat{Q}^{h}_{i} = \frac{2}{\rho\sigma_\epsilon{}^2}\left(1+n2^n\right)$. Once again, due to the way we defined $\mathcal{F}$ and $\mathcal{G}$, this implies

$$\left.\left(\left(1+\sum_{m\in\mathcal{H}^{-1}(i)}\prod_{j=0}^{l_{m,i}-1}\hat{\Theta}^{h}_{3,m^{-j}}\right)\hat{\Theta}^{h}_{1i}\right)\right|_{\hat{Q}^{h}_{i}=\frac{2}{\rho\sigma_\epsilon{}^2}(1+n2^n)} \geq \frac{2}{\rho\sigma_\epsilon{}^2}\left(1+n2^n\right).$$

But following Claim 2.3, $\forall\left(\hat{Q}^{h}_{1}, ..., \hat{Q}^{h}_{n}\right) \in \Omega^h = \left[0, \frac{2}{\rho\sigma_\epsilon{}^2}\left(1+n2^n\right)\right]^n$

$$\begin{aligned}\hat{\Theta}^{h}_{1i} &< \frac{2}{\rho\sigma_\epsilon{}^2}, & i &= 1, 2, ..., n,\\ \hat{\Theta}^{h}_{3i} &< 2, & i &= 2, ..., n.\end{aligned}$$

Then one necessarily has

$$\begin{aligned}\left.\left(\left(1+\sum_{m\in\mathcal{H}^{-1}(i)}\prod_{j=0}^{l_{m,i}-1}\hat{\Theta}^{h}_{3,m^{-j}}\right)\hat{\Theta}^{h}_{1i}\right)\right|_{\hat{Q}^{h}_{i}=\frac{2}{\rho\sigma_\epsilon{}^2}(1+n2^n)} &< \frac{2}{\rho\sigma_\epsilon{}^2}\left(1+\sum_{m\in\mathcal{H}^{-1}(i)}2^{l_{m,i}}\right)\\ &< \frac{2}{\rho\sigma_\epsilon{}^2}\left(1+n2^n\right),\end{aligned}$$

which violates the inequality stated in the beginning of our proof. Thus $\hat{Q}^{h}_{i} < \frac{2}{\rho\sigma_\epsilon{}^2}\left(1+n2^n\right), \ i = 1, ..., n.$ □ *[end of claim]*

Claims 2.3, 2.4 and 2.5 prove that $\left(\hat{Q}^{h}_{1}, ..., \hat{Q}^{h}_{n}\right)$ is in fact a fixed point of $\mathcal{F}$ for $\sigma_L{}^2 > \underline{\sigma_L}{}^2$, which directly implies that $\left(\hat{Q}^{h}_{1}, ..., \hat{Q}^{h}_{n}\right)$ is a solution to equation (2.9.29a) for $\sigma_L{}^2 > \underline{\sigma_L}{}^2$.

Now if we can show that there exists a solution to the original fixed point problem, namely the one given in Lemma 2.5, then we will have established the existence of a linear equilibrium price. Rewriting (2.9.29b) and (2.9.29c) to solve for γ^h, π_0^h, and substituting $(\hat{Q}_1^h, ..., \hat{Q}_n^h)$ for $(Q_1^h, ..., Q_n^h)$, one gets

$$\begin{aligned}
\hat{\gamma}^h &= \frac{1 + \sum_{i=1}^n \hat{\Theta}_{2i}^h}{\frac{n}{\rho {\sigma_x}^2} + \sum_{i=1}^n \left(\hat{\Theta}_{1i}^h + \hat{\Theta}_{2i}^h \sum_{k=1}^n \hat{Q}_k^h \right) + \sum_{i=2}^n \hat{\Theta}_{3i}^h \left(\hat{\Theta}_{1,i^-}^h + \sum_{k=2}^{l_{i,1}} \left(\prod_{j=1}^{k-1} \hat{\Theta}_{3,i^{-j}}^h \right) \hat{\Theta}_{1,i^{-k}}^h \right)}, \\
\hat{\pi}_0^h &= \hat{\gamma}^h \frac{\frac{n\mu_x}{\rho {\sigma_x}^2}}{1 + \sum_{i=1}^n \hat{\Theta}_{2i}^h},
\end{aligned}$$

where $\{\hat{\Theta}_{ji}^h\}_{j=1,2,3;i=1,...,n}$ are functions of $(\hat{Q}_1^h, ..., \hat{Q}_n^h)$. Also substituting for $\hat{\pi}_i^h = \hat{\gamma}^h \hat{Q}_i^h$, $i = 1, ..., n$, one derives $(\hat{\pi}_1^h, ..., \hat{\pi}_n^h, \hat{\gamma}^h, \hat{\pi}_0^h)$ as a solution to the fixed point problem given in Lemma 2.5 for all ${\sigma_L}^2 > \underline{\sigma_L}^2$. Hence we have the desired existence result for the linear equilibrium price.

However, before ending this proof, we need to verify one last thing: whether γ^h is nonzero (since it is required by 2.9.15). First, note that $\hat{\Theta}_{2i}^h \to 0 \; \forall \, i$ as ${\sigma_L}^2 \to \infty$, and consecutively γ^h converges to

$$\frac{1}{\frac{n}{\rho {\sigma_x}^2} + \frac{n}{\rho {\sigma_\epsilon}^2} + \frac{1}{\rho {\sigma_\epsilon}^2} \sum_{i=1}^n l_{i,1}}.$$

So for sufficiently large ${\sigma_L}^2$, γ^h will be nonzero, and (if need be) by redefining $\underline{\sigma_L}^2$, we can always guarantee this. □

Proof of Proposition 2.4: The proof of Proposition 2.3 guarantees that for sufficiently large ${\sigma_L}^2$ there exists a linear REESI price in the hierarchic scheme with

$$0 \le \frac{\pi_i^h}{\gamma^h} < \frac{2}{\rho {\sigma_\epsilon}^2} \left(1 + n 2^n\right), \qquad \forall i = 1, ..., n.$$

Then for this linear equilibrium, one can show that

$$\lim_{{\sigma_L}^2 \to \infty} \theta_{1i}^h = \frac{1}{\rho {\sigma_\epsilon}^2}, \qquad i = 1, ..., n,$$

and

$$\lim_{\sigma_L{}^2 \to \infty} \theta_{3i}^h = 1, \qquad i = 2, ..., n$$

with arguments similar to those used in Claim 2.3 within the proof of Proposition 2.3. One also has

$$\gamma^h \longrightarrow \frac{1}{\frac{n}{\rho\sigma_x{}^2} + \frac{n}{\rho\sigma_\epsilon{}^2} + \frac{1}{\rho\sigma_\epsilon{}^2}\sum_{i=1}^n l_{i,1}}$$

as $\sigma_L{}^2 \to \infty$. Using Lemma 2.5, it can be further shown

$$\begin{aligned} \lim_{\sigma_L{}^2 \to \infty} \theta_{2i}^h &= 0, \qquad i = 1, ..., n, \\ \lim_{\sigma_L{}^2 \to \infty} b_i^h &= \frac{1}{\rho\left(\frac{1}{\rho\sigma_x{}^2} + (l_{i,1}+1)\frac{1}{\rho\sigma_\epsilon{}^2}\right)}, \qquad i = 1, ..., n. \end{aligned}$$

for the same linear equilibrium. Remainder of the proof will directly follow from the equations that yield price coefficients in Lemma 2.5. □

Proof of Proposition 2.5: We assume risky security price $\tilde{p}$ is given by a linear function of the form

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \pi_0^* + \sum_{i=1}^n \pi_i^* \, \tilde{s}_i - \gamma^* \, \tilde{L} \quad \text{with non-zero } \gamma^*.$$

Since star is a special case of hierarchic interaction, we will be able to employ Lemma 2.5 in our proof. In particular, star has the characteristic that $l_{i,1} = 1, \forall i \in \{2, ..., n\}$.

Throughout the proof, we will specify the variables pertaining to the star with the super-index $*$. So in the case of star, $(\{\theta_{0i}^*, \theta_{1i}^*, \theta_{2i}^*, b_i^*\}_{i=1,...,n}, \{\theta_{3i}^*\}_{i=2,...,n})$ will represent the demand coefficients such that

$$\begin{aligned} z_1(s_1, p) &= \theta_{01}^* + \theta_{11}^* s_1 + \left(\theta_{21}^* - \frac{1}{\rho b_1^*}\right) p, \\ z_i(s_i, p, \delta_1^i) &= \theta_{0i}^* + \theta_{1i}^* s_i + \left(\theta_{2i}^* - \frac{1}{\rho b_i^*}\right) p + \theta_{3i}^* \delta_1^i, \qquad i > 1. \end{aligned}$$

Once again we will use the convention (as in Lemma 2.5) that $\pi^* = \sum_{i=1}^{n} \pi_i^*$. From Lemma 2.5, one observes that following hold for the star:

$$\pi_1^* = \gamma^* \left(1 + \sum_{i=2}^{n} \theta_{3i}^*\right) \theta_{11}^*, \quad (2.9.30a)$$

$$\pi_i^* = \gamma^* \theta_{1i}^*, \quad (2.9.30b)$$

$$\gamma^* = \left(\sum_{i=1}^{n} \frac{1}{\rho b_i^*} - \theta_{2i}^*\right)^{-1}, \quad (2.9.30c)$$

$$\pi_0^* = \gamma^* \sum_{i=1}^{n} \theta_{0i}^*, \quad (2.9.30d)$$

where

$$\theta_{11}^* = \frac{1}{\rho \sigma_\epsilon{}^2} \frac{\left(\sum_{k=1}^{n} (\pi_k^*)^2 - \pi^* \pi_1^*\right) \sigma_\epsilon{}^2 + (\gamma^*)^2 \sigma_L{}^2}{\sum_{k \neq 1} (\pi_k^*)^2 \sigma_\epsilon{}^2 + (\gamma^*)^2 \sigma_L{}^2}, \quad (2.9.31a)$$

$$\theta_{21}^* = \frac{1}{\rho} \frac{\pi^* - \pi_1^*}{\sum_{k \neq 1} (\pi_k^*)^2 \sigma_\epsilon{}^2 + (\gamma^*)^2 \sigma_L{}^2}, \quad (2.9.31b)$$

$$b_1^* = \frac{1}{\rho} \left(\frac{1}{\rho \sigma_x{}^2} + \theta_{11}^* + \theta_{21}^* \pi^*\right)^{-1}, \quad (2.9.31c)$$

$$\theta_{01}^* = \mu_x \left(\frac{1}{\rho b_1^*} - \theta_{11}^* - \theta_{21}^* \pi^*\right) - \theta_{21}^* \pi_0^*; \quad (2.9.31d)$$

and for $i = 2, ..., n$

$$\theta_{1i}^* = \frac{1}{\rho \sigma_\epsilon{}^2} \frac{\left(\sum_{k=1}^{n} (\pi_k^*)^2 - \pi^* \pi_i^* + \pi_1^* (\pi_i^* - \pi_1^*)\right) \sigma_\epsilon{}^2 + (\gamma^*)^2 \sigma_L{}^2}{\sum_{k \neq i} (\pi_k^*)^2 \sigma_\epsilon{}^2 - (\pi_1^*)^2 \sigma_\epsilon{}^2 + (\gamma^*)^2 \sigma_L{}^2}, \quad (2.9.31e)$$

$$\theta_{2i}^* = \frac{1}{\rho} \frac{\pi^* - \pi_i^* - \pi_1^*}{\sum_{k \neq i} (\pi_k^*)^2 \sigma_\epsilon{}^2 - (\pi_1^*)^2 \sigma_\epsilon{}^2 + (\gamma^*)^2 \sigma_L{}^2}, \quad (2.9.31f)$$

$$\theta_{3i}^* = \frac{1}{\rho \sigma_\epsilon{}^2} \frac{\left(\sum_{k \neq i} (\pi_k^*)^2 - \pi_1^* (\pi^* - \pi_1^*)\right) \sigma_\epsilon{}^2 + (\gamma^*)^2 \sigma_L{}^2}{\sum_{k \neq i} (\pi_k^*)^2 \sigma_\epsilon{}^2 - (\pi_1^*)^2 \sigma_\epsilon{}^2 + (\gamma^*)^2 \sigma_L{}^2} \frac{1}{\theta_{11}^*}, \quad (2.9.31g)$$

$$b_i^h = \frac{1}{\rho} \left(\frac{1}{\rho \sigma_x{}^2} + \theta_{1i}^* + \theta_{2i}^* \pi^* + \theta_{3i}^* \theta_{11}^*\right)^{-1}, \quad (2.9.31h)$$

$$\theta_{0i}^* = \mu_x \left(\frac{1}{\rho b_i^*} - \theta_{1i}^* - \theta_{2i}^* \pi^* - \theta_{3i}^* \theta_{11}^*\right) - \theta_{2i}^* \pi_0^*. \quad (2.9.31i)$$

Equations (2.9.30a)-(2.9.30d) and (2.9.31a)-(2.9.31i) together determine the linear equilibrium price through the following equations:

$$\pi_1^* = \gamma^* \left(\frac{1}{\rho\sigma_\epsilon{}^2} \frac{\left(\sum_{k=1}^n (\pi_k^*)^2 - \pi^*\pi_1^*\right)\sigma_\epsilon{}^2 + (\gamma^*)^2 \sigma_L{}^2}{\sum_{k\neq 1}(\pi_k^*)^2\sigma_\epsilon{}^2 + (\gamma^*)^2\sigma_L{}^2} + \frac{1}{\rho\sigma_\epsilon{}^2}\sum_{i=2}^n \frac{\left(\sum_{k\neq i}(\pi_k^*)^2 - \pi_1^*(\pi^* - \pi_i^*)\right)\sigma_\epsilon{}^2 + (\gamma^*)^2\sigma_L{}^2}{\sum_{k\neq i}(\pi_k^*)^2\sigma_\epsilon{}^2 - (\pi_1^*)^2\sigma_\epsilon{}^2 + (\gamma^*)^2\sigma_L{}^2} \right), \tag{2.9.32a}$$

$$\pi_i^* = \gamma^* \frac{1}{\rho\sigma_\epsilon{}^2} \frac{\left(\sum_{k=1}^n (\pi_k^*)^2 - \pi^*\pi_i^* + \pi_1^*(\pi_i^* - \pi_1^*)\right)\sigma_\epsilon{}^2 + (\gamma^*)^2\sigma_L{}^2}{\sum_{k\neq i}(\pi_k^*)^2\sigma_\epsilon{}^2 - (\pi_1^*)^2\sigma_\epsilon{}^2 + (\gamma^*)^2\sigma_L{}^2}, \tag{2.9.32b}$$

$$\frac{1}{\gamma^*} = \frac{n}{\rho\sigma_x{}^2} + \frac{1}{\rho\sigma_\epsilon{}^2} \frac{\left(\sum_{k=1}^n (\pi_k^*)^2 + \pi^*(\pi^* - 2\pi_1^*) - \pi^* + \pi_1^*\right)\sigma_\epsilon{}^2 + (\gamma^*)^2\sigma_L{}^2}{\sum_{k\neq 1}(\pi_k^*)^2\sigma_\epsilon{}^2 + (\gamma^*)^2\sigma_L{}^2} +$$

$$\sum_{k=2}^n \frac{1}{\rho\sigma_\epsilon{}^2} \frac{\left(2\sum_{k\neq 1,i}(\pi_k^*)^2 + (\pi_1^* + \pi_i^*)^2 + \pi^*(\pi^* - 2\pi_1^* - 2\pi_i^*) - \pi^* + \pi_1^* + \pi_i^*\right)\sigma_\epsilon{}^2 + 2(\gamma^*)^2\sigma_L{}^2}{\sum_{k\neq i}(\pi_k^*)^2\sigma_\epsilon{}^2 - (\pi_1^*)^2\sigma_\epsilon{}^2 + (\gamma^*)^2\sigma_L{}^2}, \tag{2.9.32c}$$

$$\pi_0^* = \gamma^* \left(n\frac{\mu_x}{\rho\sigma_x{}^2} - \frac{\pi_0^*}{\rho} \left(\frac{\pi^* - \pi_1^*}{\sum_{k\neq 1}(\pi_k^*)^2\sigma_\epsilon{}^2 + (\gamma^*)^2\sigma_L{}^2} + \sum_{k=2}^n \frac{\pi^* - \pi_i^* - \pi_1^*}{\sum_{k\neq i}(\pi_k^*)^2\sigma_\epsilon{}^2 - (\pi_1^*)^2\sigma_\epsilon{}^2 + (\gamma^*)^2\sigma_L{}^2} \right) \right). \tag{2.9.32d}$$

It turns out that Eqs. (2.9.32a) and (2.9.32b) can be analyzed independently of (2.9.32c) and (2.9.32d). Define

$$Q_i^* = \frac{\pi_i^*}{\gamma^*}, \qquad i = 1, ..., n,$$
$$Q^* = \frac{\pi^*}{\gamma^*}.$$

Then Eqs (2.9.32a)-(2.9.32b) yield the following:

$$Q_1^* = \frac{1}{\rho\sigma_\epsilon{}^2} \frac{\left(\sum_{k=1}^n (Q_k^*)^2 - Q^*Q_1^*\right)\sigma_\epsilon{}^2 + \sigma_L{}^2}{\sum_{k\neq 1}(Q_k^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2} + \frac{1}{\rho\sigma_\epsilon{}^2}\sum_{i=2}^n \frac{\left(\sum_{k\neq i}(Q_k^*)^2 - Q_1^*(Q^* - Q_i^*)\right)\sigma_\epsilon{}^2 + \sigma_L{}^2}{\sum_{k\neq i}(Q_k^*)^2\sigma_\epsilon{}^2 - (Q_1^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2}, \tag{2.9.33a}$$

$$Q_i^* = \frac{1}{\rho\sigma_\epsilon{}^2} \frac{\left(\sum_{k=1}^n (Q_k^*)^2 - Q^*Q_i^* + Q_1^*(Q_i^* - Q_1^*)\right)\sigma_\epsilon{}^2 + \sigma_L{}^2}{\sum_{k\neq i}(Q_k^*)^2\sigma_\epsilon{}^2 - (Q_1^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2}, \qquad i=2,...,n. \tag{2.9.33b}$$

The following result states that agents $i = 2, ..., n$ do not differ with respect to their proportional weights in the price, namely Q_i^*. This is actually straightforward given the complete homogeneity between them in terms of signal precision, risk aversion, and location in the interaction pattern. The rigorous proof goes as follows:

Claim 2.6 *For all* $i, j \in \{2, ..., n\}$,

$$Q_i^* = Q_j^*.$$

Proof. Suppose not, i.e, suppose $\exists\, i, j \in \{2, ..., n\}$ such that $Q_i^* \neq Q_j^*$. Without loss of generality, we can assume $Q_i^* > Q_j^*$. Following (2.9.33b), we have

$$\frac{\left(\sum_{k=1}^n (Q_k^*)^2 - Q^*Q_i^* + Q_1^*(Q_i^* - Q_1^*)\right)\sigma_\epsilon{}^2 + \sigma_L{}^2}{\sum_{k\neq i}(Q_k^*)^2\sigma_\epsilon{}^2 - (Q_1^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2} > \frac{\left(\sum_{k=1}^n (Q_k^*)^2 - Q^*Q_j^* + Q_1^*(Q_j^* - Q_1^*)\right)\sigma_\epsilon{}^2 + \sigma_L{}^2}{\sum_{k\neq j}(Q_k^*)^2\sigma_\epsilon{}^2 - (Q_1^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2}.$$

Since denominators on LHS and RHS of the inequality above are equal and positive, dropping them does not change the direction of inequality. Further simplifications yield

$$Q_i^*(Q_1^* - Q^*) > Q_j^*(Q_1^* - Q^*).$$

Then it must be true that $Q_i^* < Q_j^*$ since $Q^* \geq Q_1^*$. This violates our very first assumption that $Q_i^* > Q_j^*$. Thus we are done. $\square$ [*end of claim*]

Following this claim, we know that $\exists$ some q^* such that

$$Q_i^* = q^*, \qquad \forall i \in \{2, ..., n\}.$$

Now one can rewrite (2.9.33a)-(2.9.33b) as

$$Q_1^* = \frac{1}{\rho\sigma_\epsilon{}^2}\frac{(n-1)q^*(q^*-Q_1^*)\sigma_\epsilon{}^2+\sigma_L{}^2}{(n-1)(q^*)^2\sigma_\epsilon{}^2+\sigma_L{}^2}+\frac{n-1}{\rho\sigma_\epsilon{}^2}\frac{(n-2)q^*(q^*-Q_1^*)\sigma_\epsilon{}^2+\sigma_L{}^2}{(n-2)(q^*)^2\sigma_\epsilon{}^2+\sigma_L{}^2}, \tag{2.9.34a}$$

$$q^* = \frac{\sigma_L{}^2}{\rho\sigma_\epsilon{}^2}\frac{1}{(n-2)(q^*)^2\sigma_\epsilon{}^2+\sigma_L{}^2}, \qquad i=2,\ldots,n. \tag{2.9.34b}$$

Notice that (2.9.34b) is a cubic polynomial in q^* without a quadratic component. From *Cardano's formula*, the unique (real) solution to (2.9.34b) is given by

$$q^*=\sqrt[3]{\frac{\sigma_L{}^2}{2(n-2)\rho\sigma_\epsilon{}^4}}\left(\sqrt[3]{1+\sqrt{1+\frac{4}{27}\frac{\rho^2\sigma_L{}^2\sigma_\epsilon{}^2}{n-2}}}-\sqrt[3]{-1+\sqrt{1+\frac{4}{27}\frac{\rho^2\sigma_L{}^2\sigma_\epsilon{}^2}{n-2}}}\right). \tag{2.9.35}$$

Then Q_1^* is derived uniquely as follows:

$$Q_1^* = n\left(\rho\sigma_\epsilon{}^2+\frac{(n-1)q^*\sigma_\epsilon{}^2}{(n-1)(q^*)^2\sigma_\epsilon{}^2+\sigma_L{}^2}+(n-1)\frac{(n-2)q^*\sigma_\epsilon{}^2}{(n-2)(q^*)^2\sigma_\epsilon{}^2+\sigma_L{}^2}\right)^{-1} \tag{2.9.36}$$

Solving for γ^* from (2.9.32c), we get

$$\gamma^* = \frac{1+\frac{1}{\rho}\frac{(n-1)q}{(n-1)(q^*)^2\sigma_\epsilon{}^2+\sigma_L{}^2}+\frac{1}{\rho}\frac{(n-1)(n-2)q^*}{(n-2)(q^*)^2\sigma_\epsilon{}^2+\sigma_L{}^2}}{\frac{n}{\rho\sigma_x{}^2}+\frac{2n-1}{\rho\sigma_\epsilon{}^2}+\frac{1}{\rho}\frac{(n-1)^2(q^*)^2}{(n-1)(q^*)^2\sigma_\epsilon{}^2+\sigma_L{}^2}+\frac{1}{\rho}\frac{(n-1)(n-2)^2(q^*)^2}{(n-2)(q^*)^2\sigma_\epsilon{}^2+\sigma_L{}^2}}. \tag{2.9.37}$$

Also solving for π_0^* from (2.9.32d) gives us

$$\pi_0^* = \frac{\gamma^* n\frac{\mu_x}{\rho\sigma_x{}^2}}{1+\frac{1}{\rho}\frac{(n-1)q^*}{(n-1)(q^*)^2\sigma_\epsilon{}^2+\sigma_L{}^2}+\frac{1}{\rho}\frac{(n-1)(n-2)q^*}{(n-2)(q^*)^2\sigma_\epsilon{}^2+\sigma_L{}^2}}. \tag{2.9.38}$$

Now substituting for $\pi_i^* = \gamma^* q^*$, $i = 2, ..., n$, and $\pi_1^* = \gamma^* Q_1^*$, we have the desired result through the equations (2.9.35)-(2.9.38). $\square$

Proof of Corollary 2.1: Keeping in mind that $1^- = \emptyset$ and $i^- = 1$, $\forall i = 2, ..., n$, in the star, the limit linear REESI price given in the statement of the Corollary 2.1 immediately follows from Proposition 2.4. Since there exists a unique linear REESI price in the star, the derived limit equilibrium is also the unique limit equilibrium.[16] $\square$

INTERACTION IN MULTIPLE STARS

Lemma 2.6 *Assume A1, A2, S3, and A4. Suppose that the social network consists of multiple disjoint stars and that there exists a linear REESI price. Then, for all i such that $i^- \neq \emptyset$, the corresponding REESI demand of agent i's uphill neighbor is given by*

$$\tilde{z}_{i^-} = \zeta_i \tilde{p} + \tilde{\delta}_i$$

such that $\tilde{\delta}_i$ is a linear function of signals $(\tilde{s}_1, ..., \tilde{s}_n)$. Moreover,

(a) the random vector $(\tilde{\delta}_i)_{i^- \neq \emptyset}$ satisfying the property stated above is unique;

(b) for all i such that $i^- \neq \emptyset$, agent i knows the joint distribution of $(\tilde{X}, \tilde{s}_i, \tilde{p}, \tilde{\delta}_i, \tilde{L})$ and the realization δ_i;

(c) the conditional distribution of $\tilde{X}$ given (s_i, p, δ_i) is the same as the conditional distribution of $\tilde{X}$ given (s_i, p, z_{i^-}), i.e., the informational contents of (s_i, p, δ_i) and (s_i, p, z_{i^-}) related to the risky payoff $\tilde{X}$ are same.

[16] One can get the same result using Proposition 2.5, however, it will require slightly more work.

Proof. Similar to the proof of Lemma 2.3.

Proof of Corollary 2.6: Let linear REESI price $\tilde{p}$ be given by

$$\tilde{p} = \pi_0 + \sum_{i=1}^{n} \pi_i \, \tilde{s}_i - \gamma \, \tilde{L}.$$

Without loss of generality, let us relabel the agents so that k is the root of the star S_k, $k = 1, ..., m$. Having established that there exists a random variable $\tilde{\delta}_i$ with $\tilde{z}_{i-1} = \zeta_i \, \tilde{p} + \tilde{\delta}_i$, satisfying the properties listed in Lemma 2.6; let V_i denote the variance-covariance matrix of $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$ and W_i denote the covariance matrix of $\tilde{X}$ and $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$ for all $i \in \{m+1, ..., n\}$. Also, let V_i be the variance-covariance matrix of $(\tilde{s}_i, \tilde{p})$ and W_i be the covariance matrix of $\tilde{X}$ and $(\tilde{s}_i, \tilde{p})$ for all $i \in \{1, ..., m\}$. Since $(\tilde{X}, \tilde{s}_i, \tilde{p})$ is jointly normally distributed, the conditional distribution of risky payoff $\tilde{X}$ as assessed by agent i, $i \in \{1, ..., m\}$, has the moments

$$\begin{aligned} \mathrm{E}[\tilde{X}|s_i, p] &= a_{0i} + a_{1i}s_i + a_{2i}p, \\ \mathrm{var}(\tilde{X}|s_i, p) &= b_i, \end{aligned}$$

and since $(\tilde{X}, \tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$ is jointly normally distributed, the conditional distribution of risky payoff $\tilde{X}$ as assessed by agent i, $i \in \{m+1, ..., n\}$, has the moments

$$\begin{aligned} \mathrm{E}[\tilde{X}|s_i, p, z_{i-1}] &= \mathrm{E}[\tilde{X}|s_i, p, \delta_i] = a_{0i} + a_{1i}s_i + a_{2i}p + a_{3i}\delta_i, \\ \mathrm{var}(\tilde{X}|s_i, p, z_{i-1}) &= \mathrm{var}(\tilde{X}|s_i, p, \delta_i) = b_i, \end{aligned}$$

where the values of the coefficients $a_{0i}, a_{1i}, a_{2i}, a_{3i}, b_i$ depend on V_i and W_i for all i. Due to the homogeneity of agents in signal precision and risk aversion as well as the symmetry between the disjoint stars, we necessarily have b_1, b_2, $\{a_{j1}\}_{j=0,1,2}$, and $\{a_{j2}\}_{j=0,1,2,3}$, such that

$$\begin{aligned} a_{ji} &= a_{j1}, \ j = 0, 1, 2, \quad b_i = b_1, \ \forall i \in \{1, ..., m\}, \\ a_{ji} &= a_{j2}, \ j = 0, 1, 2, 3, \quad b_i = b_2, \ \forall i \in \{m+1, ..., n\}. \end{aligned}$$

So given the CARA-Gaussian setup, demands of agents will be given by

$$z_i = \frac{\mathrm{E}[\tilde{X}|s_i, p] - p}{\rho\,\mathrm{var}(\tilde{X}|s_i, p)} = \frac{a_{01} + a_{11}s_i + (a_{21} - 1)p}{\rho\, b_1}, \quad i \in \{1, ..., m\}. \tag{2.9.39}$$

$$z_i = \frac{\mathrm{E}[\tilde{X}|s_i, p, z_{i-1}] - p}{\rho\,\mathrm{var}(\tilde{X}|s_i, p, z_{i-1})} = \frac{a_{02} + a_{12}s_i + (a_{22} - 1)p + a_{32}\delta_i}{\rho\, b_2}, \quad i \in \{m+1, ..., n\}. \tag{2.9.40}$$

Following Lemma 2.6, $\forall k \in \{1, ..., m\}$ and $\forall i$ such that $i^- = k$,

$$\delta_i = \frac{a_{01} + a_{11}s_k}{\rho\, b_1}. \tag{2.9.41}$$

On the other hand, due to the homogeneity of agents in signal precision and risk aversion as well as the symmetry between the disjoint stars, we can easily show that there exist π_1 and π_2 such that

$$\begin{aligned} \pi_i &= \pi_1, \quad i \in \{1, ..., m\}, \\ \pi_i &= \pi_2, \quad i \in \{m+1, ..., n\}. \end{aligned}$$

Now using market clearing condition

$$\sum_{i=1}^{m} z_i(s_i, p) + \sum_{i=m+1}^{n} z_i(s_i, p, \delta_i) = L,$$

and solving for p, we derive

$$\pi_1 = \gamma\left(1 + \frac{n-m}{m}\frac{a_{32}}{\rho b_2}\right)\frac{a_{11}}{\rho b_1}, \tag{2.9.42a}$$

$$\pi_2 = \gamma\frac{a_{12}}{\rho b_2}, \tag{2.9.42b}$$

$$\gamma = \left(\frac{m(1-a_{21})}{\rho b_1} + \frac{(n-m)(1-a_{22})}{\rho b_2}\right)^{-1}, \tag{2.9.42c}$$

$$\pi_0 = \gamma\left(n\frac{a_{01}}{\rho b_1} + (n-m)\frac{a_{02}}{\rho b_2}\right), \tag{2.9.42d}$$

The variance-covariance matrix V_i of $(\tilde{s}_i, \tilde{p})$, $i = 1, ..., m$, is

$$\begin{bmatrix} {\sigma_x}^2 + {\sigma_\epsilon}^2 & (m\pi_1 + (n-m)\pi_2)\,{\sigma_x}^2 + \pi_1\,{\sigma_\epsilon}^2 \\ (m\pi_1 + (n-m)\pi_2)\,{\sigma_x}^2 + \pi_1\,{\sigma_\epsilon}^2 & (m\pi_1 + (n-m)\pi_2)^2\,{\sigma_x}^2 + \left(m{\pi_1}^2 + (n-m)\pi_2^2\right){\sigma_\epsilon}^2 + \gamma^2\,{\sigma_L}^2 \end{bmatrix}.$$

The variance-covariance matrix V_i of $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$, $i = m+1, ..., n$, is

$$\begin{bmatrix} {\sigma_x}^2 + {\sigma_\epsilon}^2 & \operatorname{cov}(\tilde{s}_i, \tilde{p}) & \frac{a_{11}}{\rho b_1} {\sigma_x}^2 \\ \operatorname{cov}(\tilde{s}_i, \tilde{p}) & \operatorname{var}(\tilde{p}) & \operatorname{cov}(\tilde{p}, \tilde{\delta}_i) \\ \frac{a_{11}}{\rho b_1} {\sigma_x}^2 & \operatorname{cov}(\tilde{p}, \tilde{\delta}_i) & \left(\frac{a_{11}}{\rho b_1}\right)^2 ({\sigma_x}^2 + {\sigma_\epsilon}^2) \end{bmatrix},$$

where

$$\begin{aligned} \operatorname{cov}(\tilde{s}_i, \tilde{p}) &= (m\pi_1 + (n-m)\pi_2)\, {\sigma_x}^2 + \pi_2\, {\sigma_\epsilon}^2, \\ \operatorname{var}(\tilde{p}) &= (m\pi_1 + (n-m)\pi_2)^2\, {\sigma_x}^2 + \left(m{\pi_1}^2 + (n-m)\pi_2^2\right)\, {\sigma_\epsilon}^2 + \gamma^2\, {\sigma_L}^2, \\ \operatorname{cov}(\tilde{p}, \tilde{\delta}_i) &= \frac{a_{11}}{\rho b_1} (m\pi_1 + (n-m)\pi_2)\, {\sigma_x}^2 + \frac{a_{11}}{\rho b_1} \pi_1\, {\sigma_\epsilon}^2. \end{aligned}$$

On the other hand, the covariance matrix W_i, $i = 1, ..., m$, of $\tilde{X}$ and $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$ is of the form

$$W_i = {\sigma_x}^2 \begin{bmatrix} 1 \\ m\pi_1 + (n-m)\pi_2 \end{bmatrix}.$$

and the covariance matrix W_i, $i = m+1, ..., n$, of $\tilde{X}$ and $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$ is of the form

$$W_i = {\sigma_x}^2 \begin{bmatrix} 1 \\ m\pi_1 + (n-m)\pi_2 \\ \frac{a_{11}}{\rho b_1} \end{bmatrix}.$$

Normal distribution theory dictates that for $i \in \{1, ..., m\}$

$$[a_{11} \;\; a_{21}] = W_i' V_i^{-1}, \tag{2.9.43a}$$

$$b_1 = {\sigma_x}^2 - W_i'(V_i)^{-1} W_i^h, \tag{2.9.43b}$$

$$a_{01} = \mu_x - W_i'(V_i)^{-1} \begin{bmatrix} \mu_x \\ \pi_0 + (m\pi_1 + (n-m)\pi_2)\mu_x \end{bmatrix}, \tag{2.9.43c}$$

and for $i \in \{m+1, ..., n\}$

$$[a_{12} \;\; a_{22} \;\; a_{32}] = W_i'(V_i)^{-1}, \tag{2.9.43d}$$

$$b_2 = {\sigma_x}^2 - W_i'(V_i)^{-1} W_i, \tag{2.9.43e}$$

$$a_{02} = \mu_x - W_i'(V_i)^{-1} \begin{bmatrix} \mu_x \\ \pi_0 + +(m\pi_1 + (n-m)\pi_2)\mu_x \\ \frac{a_{01}}{\rho b_1} + \frac{a_{11}}{\rho b_1} \mu_x \end{bmatrix}. \tag{2.9.43f}$$

Therefore,

$$\frac{a_{11}}{\rho b_1} = \frac{1}{\rho {\sigma_\epsilon}^2} \frac{(n-m)(\pi_2-\pi_1)\pi_2 \, {\sigma_\epsilon}^2 + \gamma^2 \, {\sigma_L}^2}{\big((m-1)\pi_1^2 + (n-m)\pi_2^2\big)\, {\sigma_\epsilon}^2 + \gamma^2 {\sigma_L}^2}, \tag{2.9.44a}$$

$$\frac{a_{21}}{\rho b_1} = \frac{1}{\rho} \frac{(m-1)\pi_1 + (n-m)\pi_2}{\big((m-1)\pi_1^2 + (n-m)\pi_2^2\big)\, {\sigma_\epsilon}^2 + \gamma^2 {\sigma_L}^2}, \tag{2.9.44b}$$

$$b_1 = \frac{\Big(\big((m-1)\pi_1^2+(n-m)\pi_2^2\big){\sigma_\epsilon}^2+\gamma^2{\sigma_L}^2\Big){\sigma_\epsilon}^2{\sigma_x}^2}{\big((m-1)\pi_1^2+(n-m)\pi_2^2\big){\sigma_\epsilon}^4+\gamma^2{\sigma_\epsilon}^2{\sigma_L}^2+\Big(\big((m-1)m\pi_1^2-2(m-1)(m-n)\pi_1\pi_2+(n-m+1)(n-m)\pi_2^2\big){\sigma_\epsilon}^2+\gamma^2{\sigma_L}^2\Big){\sigma_x}^2}, \tag{2.9.44c}$$

and

$$\frac{a_{12}}{\rho b_2} = \frac{1}{\rho {\sigma_\epsilon}^2} \frac{(m-1)\pi_1\,(\pi_1-\pi_2)\,{\sigma_\epsilon}^2 + \gamma^2\,{\sigma_L}^2}{\big((m-1)\pi_1^2 + (n-m-1)\pi_2^2\big)\,{\sigma_\epsilon}^2 + \gamma^2\,{\sigma_L}^2}, \tag{2.9.44d}$$

$$\frac{a_{22}}{\rho b_2} = \frac{1}{\rho} \frac{(m-1)\pi_1 + (n-m-1)\pi_2}{\big((m-1)\pi_1^2 + (n-m-1)\pi_2^2\big)\,{\sigma_\epsilon}^2 + \gamma^2\,{\sigma_L}^2}, \tag{2.9.44e}$$

$$\frac{a_{32}}{\rho b_2} = \frac{1}{\rho\,{\sigma_\epsilon}^2\,\frac{a_{11}}{\rho b_1}} \frac{(n-m-1)(\pi_2-\pi_1)\pi_2{\sigma_\epsilon}^2 + \gamma^2{\sigma_L}^2}{\Big(\big((m-1)\pi_1^2 + (n-m-1)\pi_2^2\big)\,{\sigma_\epsilon}^2 + \gamma^2\,{\sigma_L}^2\Big)} \tag{2.9.44f}$$

$$b_2 = \frac{\Big(\big((m-1)\pi_1^2 + (n-m-1)\pi_2^2\big)\,{\sigma_\epsilon}^2 + \gamma^2{\sigma_L}^2\Big)\,{\sigma_\epsilon}^2{\sigma_x}^2}{D}, \tag{2.9.44g}$$

$$\begin{aligned} D &= \big((m-1)\pi_1^2 + (n-m-1)\pi_2^2\big)\,{\sigma_\epsilon}^4 + \gamma^2{\sigma_\epsilon}^2{\sigma_L}^2 \\ &\quad + \Big(\big((m^2-1)\pi_1^2 - 2(m-1)(1+m-n)\pi_1\pi_2 + (-1+(n-m)^2)\pi_2^2\big)\,{\sigma_\epsilon}^2 + 2\gamma^2{\sigma_L}^2\Big)\,{\sigma_x}^2. \end{aligned}$$

Equations (2.9.42a)-(2.9.42d) and (2.9.44a)-(2.9.44g) together determine the linear equilibrium price through the following equations:

$$\pi_1 = \gamma\left(\frac{1}{\rho{\sigma_\epsilon}^2}\frac{(n-m)(\pi_2-\pi_1)\pi_2\,{\sigma_\epsilon}^2+\gamma^2\,{\sigma_L}^2}{\big((m-1)\pi_1^2+(n-m)\pi_2^2\big){\sigma_\epsilon}^2+\gamma^2{\sigma_L}^2} + \frac{n-m}{m}\frac{1}{\rho{\sigma_\epsilon}^2}\frac{(n-m-1)(\pi_2-\pi_1)\pi_2{\sigma_\epsilon}^2+\gamma^2{\sigma_L}^2}{\Big(\big((m-1)\pi_1^2+(n-m-1)\pi_2^2\big){\sigma_\epsilon}^2+\gamma^2\,{\sigma_L}^2\Big)}\right), \tag{2.9.45a}$$

$$\pi_2 = \gamma\,\frac{1}{\rho{\sigma_\epsilon}^2}\frac{(m-1)\pi_1\,(\pi_1-\pi_2)\,{\sigma_\epsilon}^2+\gamma^2\,{\sigma_L}^2}{\big((m-1)\pi_1^2+(n-m-1)\pi_2^2\big)\,{\sigma_\epsilon}^2+\gamma^2\,{\sigma_L}^2}, \tag{2.9.45b}$$

$$\begin{aligned} \frac{1}{\gamma} &= \frac{m}{\rho}\Bigg(\frac{\big((m-1)\pi_1^2+(n-m)\pi_2^2\big){\sigma_\epsilon}^4+\gamma^2{\sigma_\epsilon}^2{\sigma_L}^2+\Big(\big((m-1)m\pi_1^2-2(m-1)(m-n)\pi_1\pi_2+(n-m+1)(n-m)\pi_2^2\big){\sigma_\epsilon}^2+\gamma^2{\sigma_L}^2\Big){\sigma_x}^2}{\Big(\big((m-1)\pi_1^2+(n-m)\pi_2^2\big){\sigma_\epsilon}^2+\gamma^2{\sigma_L}^2\Big){\sigma_\epsilon}^2{\sigma_x}^2} \\ &\qquad - \frac{(m-1)\pi_1+(n-m)\pi_2}{\big((m-1)\pi_1^2+(n-m)\pi_2^2\big)\,{\sigma_\epsilon}^2+\gamma^2{\sigma_L}^2}\Bigg) + \frac{n-m}{\rho} \\ &\quad \Bigg(\frac{\big((m-1)\pi_1^2+(n-m-1)\pi_2^2\big){\sigma_\epsilon}^4+\gamma^2{\sigma_\epsilon}^2{\sigma_L}^2+\Big(\big((m^2-1)\pi_1^2-2(m-1)(1+m-n)\pi_1\pi_2+(-1+(n-m)^2)\pi_2^2\big){\sigma_\epsilon}^2+2\gamma^2{\sigma_L}^2\Big){\sigma_x}^2}{\Big(\big((m-1)\pi_1^2+(n-m-1)\pi_2^2\big){\sigma_\epsilon}^2+\gamma^2{\sigma_L}^2\Big){\sigma_\epsilon}^2{\sigma_x}^2} \\ &\qquad - \frac{(m-1)\pi_1+(n-m-1)\pi_2}{\big((m-1)\pi_1^2+(n-m-1)\pi_2^2\big)\,{\sigma_\epsilon}^2+\gamma^2\,{\sigma_L}^2}\Bigg). \end{aligned} \tag{2.9.45c}$$

Letting $Q_1 = \frac{\pi_1}{\gamma}$, $Q_2 = \frac{\pi_2}{\gamma}$ yields

$$\begin{aligned}
Q_1 &= \frac{1}{\rho\sigma_\epsilon{}^2}\frac{(n-m)(Q_2-Q_1)Q_2\,\sigma_\epsilon{}^2+\sigma_L{}^2}{\left((m-1)Q_1^2+(n-m)Q_2^2\right)\sigma_\epsilon{}^2+\sigma_L{}^2} \\
&\quad +\frac{n-m}{m}\frac{1}{\rho\sigma_\epsilon{}^2}\frac{(n-m-1)(Q_2-Q_1)Q_2\sigma_\epsilon{}^2+\sigma_L{}^2}{\left(\left((m-1)Q_1^2+(n-m-1)Q_2^2\right)\sigma_\epsilon{}^2+\sigma_L{}^2\right)}, \\
Q_2 &= \frac{1}{\rho\sigma_\epsilon{}^2}\frac{(m-1)Q_1\,(Q_1-Q_2)\,\sigma_\epsilon{}^2+\sigma_L{}^2}{\left((m-1)Q_1^2+(n-m-1)Q_2^2\right)\sigma_\epsilon{}^2+\sigma_L{}^2}.
\end{aligned}$$

One can now verify that

$$\begin{aligned}
\lim_{\sigma_L{}^2\to\infty} Q_1 &= \frac{n}{m}\frac{1}{\rho\,\sigma_\epsilon{}^2}, \\
\lim_{\sigma_L{}^2\to\infty} Q_2 &= \frac{1}{\rho\,\sigma_\epsilon{}^2}.
\end{aligned}$$

Consequently,

$$\lim_{\sigma_L{}^2\to\infty} \pi_1 = \gamma^s\,\frac{n}{m}\frac{1}{\rho\,\sigma_\epsilon{}^2}, \tag{2.9.46a}$$

$$\lim_{\sigma_L{}^2\to\infty} \pi_2 = \gamma^s\,\frac{1}{\rho\,\sigma_\epsilon{}^2}, \tag{2.9.46b}$$

$$\lim_{\sigma_L{}^2\to\infty} \gamma = \frac{1}{\frac{2n-m}{\rho\sigma_\epsilon{}^2}+\frac{n}{\rho\sigma_x{}^2}}, \tag{2.9.46c}$$

and

$$\lim_{\sigma_L{}^2\to\infty} \frac{a_{11}}{\rho b_1} = \frac{1}{\rho\sigma_\epsilon{}^2}, \tag{2.9.46d}$$

$$\lim_{\sigma_L{}^2\to\infty} \frac{a_{21}}{\rho b_1} = 0, \tag{2.9.46e}$$

$$\lim_{\sigma_L{}^2\to\infty} b_1 = \frac{\sigma_\epsilon{}^2\sigma_x{}^2}{\sigma_\epsilon{}^2+\sigma_x{}^2}, \tag{2.9.46f}$$

$$\lim_{\sigma_L{}^2\to\infty} \frac{a_{12}}{\rho b_2} = \frac{1}{\rho\sigma_\epsilon{}^2}, \tag{2.9.46g}$$

$$\lim_{\sigma_L{}^2\to\infty} \frac{a_{22}}{\rho b_2} = 0, \tag{2.9.46h}$$

$$\lim_{\sigma_L{}^2\to\infty} \frac{a_{32}}{\rho b_2} = 1, \tag{2.9.46i}$$

$$\lim_{\sigma_L{}^2\to\infty} b_2 = \frac{\sigma_\epsilon{}^2\sigma_x{}^2}{\sigma_\epsilon{}^2+2\sigma_x{}^2}. \tag{2.9.46j}$$

Now fix $i, j \in S_r$, $k \in S_{r'}$ and $l \in S_{r''}$ such that i^-, j^-, k^-, $l^- \neq \emptyset$, $r' \neq r''$. Following (2.9.39)-(2.9.41), we derive that $\text{cov}(\tilde{z}_i, \tilde{z}_j) - \text{cov}(\tilde{z}_k, \tilde{z}_l)$ equals

$$\left(\left(\frac{a_{22}}{\rho b_2} - \frac{1}{\rho b_2}\right)^2 \pi_1^2 + \left[\frac{a_{11}}{\rho b_1} + \left(\frac{a_{22}}{\rho b_2} - \frac{1}{\rho b_2}\right)\pi_1\right]\left[\frac{a_{11}}{\rho b_1} - \left(\frac{a_{22}}{\rho b_2} - \frac{1}{\rho b_2}\right)\pi_1\right]\right)\sigma_\epsilon{}^2. \quad (2.9.47)$$

Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(\sigma_L{}^2) = \left[\frac{a_{11}}{\rho b_1} + \left(\frac{a_{22}}{\rho b_2} - \frac{1}{\rho b_2}\right)\pi_1\right]\left[\frac{a_{11}}{\rho b_1} - \left(\frac{a_{22}}{\rho b_2} - \frac{1}{\rho b_2}\right)\pi_1\right].$$

From equations (2.9.44a)-(2.9.45c), we verify that f is continuous in $\sigma_L{}^2$. Moreover, following (2.9.46a)-(2.9.46j),

$$\lim_{\sigma_L{}^2 \to \infty} f(\sigma_L{}^2) = \left(\frac{1}{\rho\sigma_\epsilon{}^2}\right)^2 \left(1 - \frac{2n\,\sigma_x{}^2 + n\,\sigma_\epsilon{}^2}{(2n-m)m\,\sigma_x{}^2 + nm\,\sigma_\epsilon{}^2}\right)\left(1 + \frac{2n\,\sigma_x{}^2 + n\,\sigma_\epsilon{}^2}{(2n-m)m\,\sigma_x{}^2 + nm\,\sigma_\epsilon{}^2}\right).$$

Since $n > m \geq 2$, $(2n - m)m \geq 2n$, and consequently $\lim_{\sigma_L{}^2 \to \infty} f(\sigma_L{}^2) > 0$. Then following continuity of f in $\sigma_L{}^2$ and (2.9.47),

$$\text{cov}(\tilde{z}_i, \tilde{z}_j) - \text{cov}(\tilde{z}_k, \tilde{z}_l) > 0$$

for sufficiently large $\sigma_L{}^2$. Moreover, it can be easily verified that $\text{var}(\tilde{z}_i) = \text{var}(\tilde{z}_j) = \text{var}(\tilde{z}_k) = \text{var}(\tilde{z}_l)$. Hence, the result given in the statement of the proposition follows. □

INFORMATION AGGREGATION

Proof of Proposition 2.8: Following Proposition 2.5, the following inequality holds for generic exogenous parameters:

$$q^* \neq \frac{n}{\rho\sigma_\epsilon{}^2 + \frac{(n-1)q^*\sigma_\epsilon{}^2}{(n-1)(q^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2} + (n-1)\frac{(n-2)q^*\sigma_\epsilon{}^2}{(n-2)(q^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2}},$$

where

$$q^* = \sqrt[3]{\frac{\sigma_L{}^2}{2(n-2)\rho\sigma_\epsilon{}^4}}\left(\sqrt[3]{1+\sqrt{1+\frac{4}{27}\frac{\rho^2\sigma_L{}^2\sigma_\epsilon{}^2}{n-2}}} - \sqrt[3]{-1+\sqrt{1+\frac{4}{27}\frac{\rho^2\sigma_L{}^2\sigma_\epsilon{}^2}{n-2}}}\right).$$

Then the price coefficients π_i^* generically satisfy

$$\pi_1^* \neq \pi_i, \qquad \pi_i = \pi_j \qquad \text{for } i,j \in \{2,...,n\}.$$

Suppose to the contrary, the informational content of price $\tilde{p}$, namely $\sum_{i=1}^n \pi_i^* \tilde{s}_i$, is (generically) a sufficient statistic for the joint distribution $(\tilde{s}_1,...,\tilde{s}_n, \sum_{i=1}^n \pi_i^* \tilde{s}_i)$ conditional on $\tilde{X}$. Following (2.7.1) and the fact that $\sum_{i=1}^n \pi_i^* \tilde{s}_i$ adds no information on top of $(\tilde{s}_1,...,\tilde{s}_n)$,

$$\begin{aligned} \mathrm{E}[\tilde{X}|\tilde{s}_1,...,\tilde{s}_n] &= \mathrm{E}\left[\tilde{X}\middle|\tilde{s}_1,...,\tilde{s}_n, \sum_{i=1}^n \pi_i^* \tilde{s}_i\right] \\ &= \mathrm{E}\left[\tilde{X}\middle|\sum_{i=1}^n \pi_i^* \tilde{s}_i\right]. \end{aligned} \tag{2.9.48}$$

On the other hand, since $\tilde{S} = \frac{1}{\sigma_x{}^2+\sigma_\epsilon{}^2}\sum_{i=1}^n \tilde{s}_i$ is also a sufficient statistic for the joint distribution $(\tilde{s}_1,...,\tilde{s}_n,\tilde{S})$ conditional on $\tilde{X}$, by a similar argument

$$\begin{aligned} \mathrm{E}[\tilde{X}|\tilde{s}_1,...,\tilde{s}_n] &= \mathrm{E}[\tilde{X}|\tilde{s}_1,...,\tilde{s}_n,\tilde{S}] \\ &= \mathrm{E}[\tilde{X}|\tilde{S}]. \end{aligned} \tag{2.9.49}$$

(2.9.48) and (2.9.49) imply

$$\begin{aligned} \mathrm{E}\left[\tilde{X}\middle|\sum_{i=1}^n \pi_i^* \tilde{s}_i\right] &= \mathrm{E}[\tilde{X}|\tilde{S}] \\ \Longrightarrow \quad \mu_x + \frac{\mathrm{cov}\left(\tilde{X}, \sum_{i=1}^n \pi_i^* \tilde{s}_i\right)}{\mathrm{var}\left(\sum_{i=1}^n \pi_i^* \tilde{s}_i\right)}\left(\sum_{i=1}^n \pi_i^* \tilde{s}_i - \mathrm{E}\left[\sum_{i=1}^n \pi_i^* \tilde{s}_i\right]\right) &= \mu_x + \frac{\mathrm{cov}(\tilde{X},\tilde{S})}{\mathrm{var}(\tilde{S})}(\tilde{S} - \mathrm{E}[\tilde{S}]) \\ \Longrightarrow \quad \frac{\sum_{i=1}^n \pi_i^* \sigma_x{}^2}{\sum_{i=1}^n (\pi_i^*)^2(\sigma_x{}^2+\sigma_\epsilon{}^2)}\left(\sum_{i=1}^n \pi_i^* \tilde{s}_i - \sum_{i=1}^n \pi_i^* \mu_x\right) &= \sigma_x{}^2\left(\tfrac{1}{\sigma_x{}^2+\sigma_\epsilon{}^2}\sum_{i=1}^n \tilde{s}_i - \tfrac{n}{\sigma_x{}^2+\sigma_\epsilon{}^2}\mu_x\right) \\ \Longrightarrow \quad \frac{\sum_{i=1}^n \pi_i^*}{\sum_{i=1}^n (\pi_i^*)^2}\sum_{i=1}^n \pi_i^*(\tilde{s}_i - \mu_x) &= \sum_{i=1}^n (\tilde{s}_i - \mu_x) \end{aligned}$$

The last equality should hold for all realizations $(s_1,...,s_n)$ of $(\tilde{s}_1,...,\tilde{s}_n)$. Consider the case where $s_1 > 0$ and $s_i = 0$, $i = 2,...,n$. Then we must have

$$\pi_1^* \sum_{i=1}^n \pi_i^* = \sum_{i=1}^n (\pi_i^*)^2.$$

Given that $\pi_1^* \neq \pi_i$ and $\pi_i = \pi_j$ for $i, j \in \{2, ..., n\}$ for generic exogenous parameters, the last equality will (generically) fail to hold. □

Chapter 3

Amplification and Asymmetry in Crashes and Frenzies

Large stock price movements within short periods of time have always drawn economists' attention. We see them in the form of *frenzies*, when the price movement is in the positive direction, and *crashes*, when the direction is negative. This essay focuses on two characteristics of crashes and frenzies: *amplification* and *asymmetry*.

In many cases there seems to be no significant events prior to large price movements. Cutler, Poterba and Summers (1989) document that for the postwar movements in the S&P 500 index. This empirical fact suggests that large price movements are most often *amplified price reactions* to comparatively insignificant information or liquidity shocks.

In addition, there is a substantial difference between the number of crashes and frenzies, and this is what we mean by the *asymmetry*. Hong and Stein (2002) report that nine of the ten largest one-day price movements in the S&P 500 since 1947 were decreases. A broader look at the data also confirms the asymmetry. Boldrin and Levine's (2001) analysis of S&P 500 between 1889 and 1984 reveals that annual negative deviations[1] are, on average, larger than positive ones. We observe the following in the Boldrin and Levine data: There is one annual negative deviation with magnitude larger than 50% but no positive deviation exceeds this value. The number of annual negative deviations with magnitudes larger than 40% is 3 and that of positive deviations is none. There are 6 annual negative deviations of size exceeding 30% compared to 4 positive ones. Finally, 14 annual negative deviations have magnitudes larger than 20% and only 10 positive ones exceed this value. Thus the Boldrin and Levine data also implicates the asymmetry between crashes and frenzies.

This essay offers an explanation for the two characteristics of large price movements, depicted above. Our explanation involves the use of *hedging (portfolio insurance) strategies* in the stock market. Hedgers using these strategies sell after the market has declined and buy after the market rises. Therefore portfolio insurance is *negatively price sensitive* since conventional supply schedules are increasing functions of price. Brady Commission Report (1988) provides evidence for the use of portfolio insurance strategies during the crash of 1987 and furthermore blames these negatively price sensitive strategies for deepening the decline hence perhaps causing the crash. The studies of Chicago Mercantile Exchange, Miller, Hawke, Malkiel, and Scholes (1987), Commodity Futures Trading Commission (1987), Securities and Exchange Commission (1987) also highlight the important role of these strategies in the 1987 crash.[2] As a possible contributing factor to the crash of 1929, we also see arguments focusing on the use of stop-loss orders which are primitive portfolio insurance strategies (Gennotte and Leland (1990)). Gennotte and Leland (1990) explain the '87 crash in concordance with the findings of Brady Report by incorporating hedging (portfolio insurance) into a conventional noisy rational expectations model.

Following Gennotte and Leland (1990) we develop a static noisy rational expectations equilibrium

[1]For the years 1889-1984, Boldrin and Levine (2001) report the real S&P 500 index, and the "deviation" from the difference between the log of the index value of a year and that of a subsequent year.

[2]Shiller (1989)

(REE) model with hedgers using negatively price sensitive strategies in a CARA-Gaussian environment. Our results show that hedging strategies amplify the effect of news and liquidity shocks on price deviations. Convex hedging strategies cause overreaction to negative news and liquidity shocks, hence they create an asymmetry biased towards crashes. An important class of hedging functions (put-option replication strategies[3]) satisfies the convexity condition in a highly volatile market. We also examine the roles of risk aversion and asymmetric information in our analysis. In particular, we show that risk aversion is necessary for asymmetry of price deviations and asymmetric information enhances the amplification and the asymmetry delivered by hedging. Finally we analyze trading behavior of rational agents in the presence of hedgers, and question the emergence of hedging in financial markets.

The focus of our essay is characteristics of certain *dynamic phenomena*, namely crashes and frenzies. This might seem puzzling since we employ a static model for the analysis. However, in our static framework we can interpret comparative statics results on price as dynamic changes over time. In particular, the equilibrium price reactions to changes in the information or liquidity parameters are viewed as fluctuations over time. In the same fashion, crashes and frenzies are interpreted as high sensitivity to changes in information or liquidity parameters. That is, if we see a substantial fall in equilibrium price as a reaction to comparatively insignificant news, we call it a crash (or a frenzy in the case of a price increase) in our setup. Note that, by this interpretation, we also incorporate an observed characteristic, namely amplification, into our definition of crashes and frenzies.

As mentioned above, in our setup hedging (portfolio insurance) is the cause of amplification and asymmetry in large price movements. Hedging strategies are naturally dynamic strategies dependent on the price trend. Before explaining how hedging strategies fit into our static environment, let us discuss why they would cause amplified and asymmetric deviations. For intuition, we can first look at stop-loss orders. With stop-loss orders we see sales after the market has fallen under some exercise value. The

[3]Put-option replication is formally defined in Section 4. See Rubinstein and Leland (1981) for a detailed exposition of the subject.

aim is to protect one's portfolio against future potential losses. Here it is easy to see how a crash can be the result of an amplified price reaction, because stop-loss itself puts a downward pressure on the price once the price begins to fall. Moreover since there is no accompanying upward pressure, we are likely to observe an asymmetry biased towards crashes in an environment where stop-loss orders prevail. In modern hedging strategies, such as put-option replication, the idea is the same, but now we have both upward and downward pressures on the price. That is, we see a buying spree from hedgers in a bull market, and sales in a bearish one; hence comes the amplified price reactions. If the downward pressure of the strategy were to be stronger than the upward one, we would observe asymmetry biased towards crashes. This summarizes most of what we are trying to formalize in §3.2.

Now we can return to the interpretation of hedging in our static environment. All hedging activity is aggregated into a deterministic supply function of price p, say $h(p)$. As we have only one trading period in our model, let us take p^* as our (hypothetical) initial price, and let $h(p^*) = 0$. A fall in the security price leads to positive hedging supply, thus for $p < p^*$, $h(p) > 0$. Similarly, we have positive hedging demand (or negative supply) with increasing price, thus $h(p) < 0$ for $p > p^*$. The more the price increases, the higher the hedging demand (and vice versa); thus we want h to be a *decreasing function* of p. In summary, we will view hedging as the change of a deterministic supply with respect to the change in price p compared to a hypothetical initial price p^*. The supply is deterministic, because with stop-loss there is a specific exercise value to strike on, and with others there are specific formulas to follow, such as Black-Scholes in the case of put-option replication.

Having made an informal introduction to the functioning of our model, we can now discuss how our results compare with others in the literature. Though there is an extensive literature on the amplification observed in crashes and frenzies, the asymmetric feature of these large movements has not been addressed until recent years. Boldrin and Levine (2001), Chalkley and Lee (1998), Hong and Stein (2002), Veldkamp (2002), and Veronesi (1999) address asymmetry of crashes and frenzies. In Boldrin and Levine (2001) the asymmetry in large price movements is driven by the asymmetry in the underlying technology shocks that drive fundamentals. Chalkley and Lee (1998) propose a model with noise traders where risk aversion prevents agents from acting promptly on receiving good news and encourages them to act quickly on receiving bad news. Veronesi's (1999) work is similar to Chalkley and Lee (1998) in spirit as risk aversion makes asset price a convex function of beliefs and leads to underreaction to good news in bad times and overreaction to bad news in good times. Hong and Stein (2002) achieve asymmetry via short-sales constraints, which cause revelation of bad information in bad times and hidden bad information in good times. Veldkamp (2002) explains the asymmetric feature by asymmetric endogenous speed of learning: faster learning in good times causes quick reaction to bad news and hence sudden crashes.

This essay is similar to studies of Chalkley and Lee (1998) and Veronesi (1999) in that we also have risk aversion convexifying price reactions to changes in the underlying parameters, which leads to asymmetry. The difference is that our explanation stems from the use of hedging strategies, which also amplifies the price fluctuations. As mentioned above, hedging is introduced to REE models by Gennotte and Leland (1990) for the first time. However their paper only focuses on the cause of '87 crash, and they define crash as a discontinuity in the price function. Here in this essay, following the REE model proposed by them, we offer an explanation for the asymmetry between crashes and frenzies, and we are not seeking any discontinuities in price.[4] Also, Gennotte and Leland (1990) solve for a single equilibrium, hence discard other possible equilibria from their analysis. In our proposed framework, we are able to derive a unique equilibrium, thus our analysis on amplification and asymmetry does not call for a specific choice of equilibrium or a pricing rule. There is another paper, Jacklin, Kleidon, and Pfleiderer (1992), which also attributes the '87 crash to hedging strategies. Following Glosten and Milgrom (1985), they model a market with bid-ask prices and sequential trading. What delivers crash is the underestimation of the extent of hedging activities. This might cause a rise in the security price due to imperfect information aggregation, and ultimately learning leads to a price correction, in this

[4]Actually we rule out discontinuities to ease the comparative statics exercises.

case to a fall in price. However, as in Gennotte and Leland (1990), the asymmetry is not sought in Jacklin, Kleidon, and Pfleiderer (1992) either.

Our essay is organized as follows. In Section 3.1 we develop a noisy REE model with hedgers and derive the unique equilibrium. Section 3.2 provides the results on amplification and asymmetry in price deviations. Section 3.3 checks whether the conditions for asymmetry derived in §3.2 are satisfied in practice, then we provide a numerical example demonstrating amplification and asymmetry in Section 3.4. Section 3.5 focuses on the roles of risk aversion and asymmetric information pertaining to amplification and asymmetry in crashes and frenzies. Section 3.6 deals with the effect of hedging on rational agents' trading behavior. Finally, Section 3.7 questions the emergence of hedging in the stock market.

3.1 CARA-Gaussian Economy

We employ a static REE model, which is a simplified version of Gennotte and Leland (1990) with one informed trader instead of many informed traders with different Gaussian information sets. We used the method introduced in Demange and Laroque (1995) for the computation of equilibrium price as a simple function of parameters.

3.1.1 The model

First let us give the basic characteristics of the model and then describe how the model functions.

Basic characteristics. We assume two periods of time in our model. Economic agents, whom we will specify later, competitively trade in the first period and consume in the second. There is only one good in the economy, and there are two securities (i.e. two claims on the good): a risk-free security and a risky security with a future stochastic payoff R, which realizes in the second period. The price and the payoff of the risk-free security are normalized to 1.

The four types of agents in our economy are as follows:

(1) *insider (marketmaker)*[5], who observes price p of the risky security, and also observes private random signal S on payoff R of the risky security;

(2) *rational outsiders*, who observe only price p of the risky security;

(3) *liquidity traders*, whose function is to add noise to the economy, that is, they create an exogenously determined random net supply of the risky security;

(4) *hedgers*, who create a deterministic net supply of the risky security. This net supply, h, is a decreasing function of the price p of the risky security.

The informational structure in our model is as follows. The distribution of signal S is common knowledge whereas the realization of the signal is only known to the insider. Similarly, distribution of liquidity supply L is common knowledge, however neither the insider nor outsiders know the realization of L. The hedging supply function h is known to both insider and outsiders.

All random variables in our model are Gaussian. The future payoff of the risky security, R, is a normal random variable with non-zero variance. Insider's signal on R is of the form $S = R + \Omega$, where Ω is distributed with $N(0, \sigma_\Omega)$, $\sigma_\Omega \neq 0$. The liquidity supply, L, is also normal with distribution $N(0, \sigma_L)$. The random variables R, Ω, and L are jointly normally distributed and independent from each other. Note that, throughout the essay, the random variables are denoted by capital letters, and realizations of them are denoted by the corresponding small letters.

Utilities of rational agents, namely the insider and outsiders, exhibit constant absolute risk aversion (CARA). The CARA-Gaussian setup allows us to aggregate outsiders into a single agent, as all outsiders share the same information. From now on we denote the insider by i, and the outsider by o. The constant Arrow-Pratt measure of absolute risk aversion of insider is a_i, and that of outsider is a_o. To be more precise, $\frac{1}{a_o}$ is the sum of all rational outsiders' measures of risk tolerance (as we are

[5]We can justify the price-taking behavior of the single insider by assuming that she represents a continuum of mass one of insiders who act competitively.

aggregating all outsiders into a single agent). We define the *aggregate Arrow-Pratt measure of absolute risk aversion* A by setting $\frac{1}{A} = \frac{1}{a_i} + \frac{1}{a_o}$. Utility functions of insider and outsider are of the form

$$u^j(W_j) = -\mathrm{e}^{-a_j W_j},$$

where W_j is the random final wealth (which realizes in the second period) of agent j, for $j = i, o$. Both agents maximize expected utility of final wealth over the first period and their expectations depend on their Gaussian information. As liquidity traders and hedgers are irrational, their preferences are not specified. However their role and actions were specified before.

As a final note, insider and outsider are endowed with deterministic wealth (holdings of risk-free claim on the good) e_i and e_o, respectively.

The functioning of the model. For $j = i, o$, let e_j, W_j, and I_j denote agent j's initial (deterministic) endowment, final (random) wealth, and Gaussian information, respectively.

In the first period, the risky security is traded on the market against the risk-free security. If agent j purchases D_j units of the risky security at price p, j's random final wealth would be

$$W_j = D_j R + (e_j - pD_j).$$

As the rational agent j maximizes his expected utility of consumption in the second period, the following maximization problems are solved in the first period by $j = i, o$:

$$\max_{D_j} \mathrm{E}[-\mathrm{e}^{-a_j W_j} | I_j] \tag{3.1.1}$$

$$\text{s. to} \quad D_j R + (e_j - pD_j) = W_j,$$

where D_j is the net excess demand of risky security by agent j (negative in case of sales).

Liquidity traders and hedgers determine the total net supply of the risky security in the first period. Thus, in the first period, total supply of the risky security at price p is

$$l + h(p),$$

where l is the realization of random liquidity supply L.

In the second period, all uncertainty is resolved, and consumption takes place without any further trade.

3.1.2 Equilibrium

Next we define the equilibrium price in the fashion of rational expectations equilibrium.

Definition 1. *A rational expectations equilibrium price of the risky security is a function $P(s,l)$ such that, for any realization of signal and liquidity supply (s,l),*

$$D_i(p|s) + D_o(p|P(s,l) = p) = l + h(p), \quad \textit{where}$$

$D_i(p|s)$ solves insider's maximization problem given in (3.1.1), conditional on the observation of the price p and the signal s, [6]

$D_o(p|P(s,l) = p)$ solves outsider's maximization problem given in (3.1.1), conditional on the observation of p and the knowledge about the price function $P(s,l)$ to update the beliefs on s.

Note that as insider is the only informed trader in the economy, observation of risky security's price does not add any information on top of what he already has. We let Σ denote outsider's Gaussian information. From the definition above we already know Σ coincides to the knowledge of $P(s,l) = p$; however we would like to express outsider's information explicitly as a function of s and l in the equilibrium, hence we introduce this new notation. The excess demand functions of insider and outsider

[6] The random variables are denoted by capital letters and realizations of them are denoted by the corresponding small letters.

are given by[7]

$$D_i(p|S=s) = \frac{\mathrm{E}[R|s]-p}{a_i \mathrm{var}(R|S)}, \quad D_o(p|\Sigma=\sigma) = \frac{\mathrm{E}[R|\sigma]-p}{a_o \mathrm{var}(R|\Sigma)}. \tag{3.1.2}$$

The following notation is introduced:[8]

$$a_i^* = a_i \mathrm{var}(R|S), \quad a_o^* = a_o \mathrm{var}(R|\Sigma), \quad \frac{1}{A^*} = \frac{1}{a_i^*} + \frac{1}{a_o^*}.$$

Given joint distributions of R, S, and L, A^* is only a function of insider's risk aversion a_i, and outsider's risk aversion a_o. That is, the value of A^* does not depend on the realization of insider's signal and liquidity supply (since normal conditional variances are independent of realizations). We introduce the following assumption:

S1. $I + A^*h$ is strictly monotone (i.e. either strictly increasing or strictly decreasing). [9]

This assumption guarantees a continuous equilibrium price function that can be used for comparative statics. Without assuming S1, the proof of the existence of an equilibrium still holds, but it leads to price correspondence which may not be single-valued. One now has the following:

Proposition 3.1 (Equilibrium) *Assume S1. Then the unique rational expectations equilibrium price is given by*

$$\begin{aligned} P(s,l) &= f^{-1}\Big(\frac{A^*}{a_i^*}\mathrm{E}[R|s] + \frac{A^*}{a_o^*}\mathrm{E}[R|\sigma] - A^*l\Big) \\ &= f^{-1}\Big(\mathrm{E}[R|\sigma] + \frac{A^*}{a_i^*}(\sigma - \mathrm{E}[R|\sigma])\Big), \end{aligned}$$

[7] Expressions of excess demand functions in CARA-Gaussian environments are well-known, however we still provide the derivations in (B1) of Appendix B.

[8] Note that we abuse the notation here by writing $\mathrm{var}(R|S)$ instead of $\mathrm{var}(R|s)$, i.e. we condition the variance of R on the distribution of signal rather than its realization. However normal conditional variances do not depend on realizations, thus our notation for the variance fits to this characteristic of the Gaussian environment.

[9] I denotes the identity function, i.e. $I(x) = x \ \forall x \in \mathbf{R}$. .

where f^{-1} is the inverse of $f \equiv I + A^*h$, and

$$\sigma = \mathrm{E}[R|s] - a_i \mathrm{var}(R|S)l$$

is the (realization of) outsider's information.

Proof. S1 guarantees that f^{-1} is a well-defined continuous function. Excess demand functions of insider and outsider are also well-defined since varΩ and var$R$ are non-zero.[10] Hence market clearing yields

$$\left(\frac{1}{a_i \mathrm{var}(R|S)} + \frac{1}{a_o \mathrm{var}(R|\Sigma)}\right) p + h(p) = \frac{\mathrm{E}[R|s]}{a_i \mathrm{var}(R|S)} + \frac{\mathrm{E}[R|\sigma]}{a_o \mathrm{var}(R|\Sigma)} - l.$$

Outsider's information σ is revealed by the observation of price and the knowledge of price function. The price function is essentially derived from the market clearing condition above, thus outsider's information coincides with the knowledge of market clearing condition. Since the hedging function h and distributions of S and L are common knowledge, and values of conditional normal variances are independent from realizations[11], outsider can induce the following information from market clearing:

$$\frac{\mathrm{E}[R|s]}{a_i \mathrm{var}(R|S)} - l.$$

Multiplying this argument by a known constant (namely $a_i \mathrm{var}(R|S)$) would not matter for the informational content, therefore outsider's information is equivalent to the knowledge of the realization

$$\sigma = \mathrm{E}[R|s] - a_i \mathrm{var}(R|S)l.$$

Recall that S and L are jointly normally distributed. So Σ (the random distribution σ belongs to) is also normally distributed, and outsider's demand as given in (3.1.2) holds. Rewriting market clearing condition we have

$$p + A^*h(p) = \frac{A^*}{a_i^*}\mathrm{E}[R|s] + \frac{A^*}{a_o^*}\mathrm{E}[R|\sigma] - A^*l,$$

[10] See Lemma A (C1) in Appendix C.

[11] See (A1) in Appendix A.

where A^*, a_i^*, and a_o^* are as defined above. Writing $\frac{A^*}{a_o^*} = 1 - \frac{A^*}{a_i^*}$, and using definition of f; the result follows. □

Note that the equilibrium price of risky security given by Proposition 3.1 is a function of insider's private signal s and liquidity supply l. In the Gaussian framework $\mathrm{E}[R|s]$ is a linear increasing function of s, and given s the assessment of conditional expectation does not put a burden on the agents from the informational perspective since all the parameters necessary to extract its functional form are common knowledge. Therefore the comparative statics results in this essay do not change qualitatively if the equilibrium price is taken as a function of the vector $(\mathrm{E}[R|s], l)$ rather than (s, l). For this purpose we introduce the following notation: let N stand for the random variable $\mathrm{E}[R|S]$, and let ν be the realized value, i.e. $\mathrm{E}[R|s]$. Then the equilibrium price function takes the form[12]

$$P(\nu, l) = f^{-1}\Big(Q(\nu, l)\Big), \quad \text{where} \tag{3.1.3a}$$

$$\begin{aligned} Q(\nu, l) = & -\frac{A^*}{a_o^*}\Big\{1 - \frac{\mathrm{cov}(R, \Sigma)}{\mathrm{var}\Sigma}\Big\}\mathrm{E}R \\ & + \Big\{\frac{\mathrm{cov}(R, \Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}\Big(1 - \frac{\mathrm{cov}(R, \Sigma)}{\mathrm{var}\Sigma}\Big)\Big\}\nu \\ & - \Big\{a_i^*\frac{\mathrm{cov}(R, \Sigma)}{\mathrm{var}\Sigma} + A^*\Big(1 - \frac{\mathrm{cov}(R, \Sigma)}{\mathrm{var}\Sigma}\Big)\Big\}l. \end{aligned} \tag{3.1.3b}$$

3.1.3 Asymmetric price deviations and non-linear prices

As it follows from (3.1.3a), equilibrium price[13] is a function of insider's expectation of risky payoff ν, and liquidity supply l. Here we would like to discuss how the asymmetry between crashes and frenzies emerges in our setup. If $P(\nu, l)$ were linear in (ν, l), negative and positive shocks of same magnitudes would create price deviations of the same size. Then we could only attribute the asymmetry in favor of crashes to more frequent and significant negative shocks. As there is no evidence of more frequent

[12]See (B2) in Appendix B for the derivation.

[13]Now on the term "price" stands for the risky security price unless otherwise stated.

negative news or liquidity shocks in the history of S&P 500, we are interested in asymmetric price deviations triggered by symmetric shocks. Formally, we have the following:

Given (ν_0, l_0) *we say that there is an* **asymmetry in deviations** *at the equilibrium price* $P(\nu_0, l_0)$ *if for some* $(\Delta\nu, \Delta l) > 0$[14]

$$\begin{aligned} P(\nu_0, l_0) - P(\nu_0 - \Delta\nu, l_0) &\neq P(\nu_0 + \Delta\nu, l_0) - P(\nu_0, l_0), \quad \textit{or} \\ P(\nu_0, l_0) - P(\nu_0, l_0 - \Delta l) &\neq P(\nu_0, l_0 + \Delta l) - P(\nu_0, l_0). \end{aligned}$$

Clearly, *non-linearity* of the equilibrium price function in ν or l is necessary and sufficient for asymmetry in price deviations. Recall that $f \equiv I + A^*h$. When there is no hedging supply, $f = I$ and $P(\nu, l)$ is linear in (ν, l) by (3.1.3a)-(3.1.3b). So asymmetric information by itself can not create asymmetric deviations in price. With non-zero A^*, non-linearity of hedging supply h becomes a necessary and sufficient condition for a non-linear equilibrium price function, and consequently for asymmetric deviations in price.

Given (ν_0, l_0) *we say information and liquidity shocks cause a* **bias towards negative price deviations** *within the set* $U_{v_0} \times U_{l_0}$ *if for all* $(\Delta\nu, \Delta l) > 0$ *s.t.* $v_0 - \Delta v,\ v_0 + \Delta v \in U_{v_0}$ *and* $l_0 - \Delta l,\ l_0 + \Delta l \in U_{l_0}$ *the following holds:*

$$\begin{aligned} P(\nu_0, l_0) - P(\nu_0 - \Delta\nu, l_0) &> P(\nu_0 + \Delta\nu, l_0) - P(\nu_0, l_0), \\ P(\nu_0, l_0) - P(\nu_0, l_0 - \Delta l) &> P(\nu_0, l_0 + \Delta l) - P(\nu_0, l_0). \end{aligned}$$

If the price deviations above are significantly large, then we say there exists a **bias towards crashes** *within* $U_{v_0} \times U_{l_0}$.

[14] $(\Delta\nu, \Delta l) > 0$ if and only if both $\Delta\nu$ and Δl are strictly positive.

Suppose equilibrium price function P is continuously differentiable. Then there exists a bias towards negative price deviations within $U_{v0} \times U_{l_0}$ if and only if $P(\nu, l)$ is *strictly concave* in ν and l within $U_{v0} \times U_{l_0}$. This is due to the fact that for a strictly concave and continuously differentiable function g

$$g(x_1) < g(x_0) + g'(x_0)(x_1 - x_0),$$

and letting x_1 equal $x_0 - \Delta x$ and then $x_0 + \Delta x$ one gets

$$g(x_0) - g(x_0 - \Delta x) > g(x_0 + \Delta x) - g(x_0).$$

Note the following obvious that whenever $P(\nu, l)$ is globally concave in ν and l, all shocks will cause a bias towards negative price deviations in the economy. One can also interpret the *strict concavity* of equilibrium price P as *overreaction to negative shocks* compared to the price reaction to positive shocks.

3.2 Amplification and Asymmetry

In this section we present comparative statics of the equilibrium price $P(\nu, l)$. The first-order partial derivatives of $P(\nu, l)$ with respect to ν and l determine the sensitivity of price to changes in the information parameter and liquidity supply, respectively, and the second-order partial derivatives determine the concavity of price function, hence it reveals the nature of bias in the asymmetric price deviations. Our purpose is to see how hedging activity effects the first and second-order partial derivatives of equilibrium price. In particular, we would like to observe higher price sensitivity to changes in the underlying parameters in the presence of hedging activity. This will reveal that hedging amplifies price reactions. We also would like to see equilibrium price as a concave function of the underlying parameters in the presence of hedging (possibly under some condition(s) imposed on hedging function), which will imply the asymmetry biased towards crashes. On top of these, we would like to observe more amplification and more asymmetry as the extent of hedging activity increases.

First we examine the sensitivity of price. Taking partial derivatives of $P(\nu, l)$ with respect to ν and l yield

$$\frac{\partial P(\nu,l)}{\partial \nu} = (f^{-1})'(Q(\nu,l))\left[\tfrac{\text{cov}(R,\Sigma)}{\text{var}\Sigma} + \tfrac{A^*}{a_i^*}\left(1 - \tfrac{\text{cov}(R,\Sigma)}{\text{var}\Sigma}\right)\right], \tag{3.2.1a}$$

$$\frac{\partial P(\nu,l)}{\partial l} = -(f^{-1})'(Q(\nu,l))\left[a_i^*\tfrac{\text{cov}(R,\Sigma)}{\text{var}\Sigma} + A^*\left(1 - \tfrac{\text{cov}(R,\Sigma)}{\text{var}\Sigma}\right)\right]. \tag{3.2.1b}$$

Lemma B (C2) in Appendix C shows that $\frac{\text{cov}(R,\Sigma)}{\text{var}\Sigma} \leq 1$. Therefore[15]

$$sgn\Big(\frac{\partial P(\nu,l)}{\partial \nu}\Big) = sgn\Big((f^{-1})'\Big(Q(\nu,l)\Big),$$
$$sgn\Big(\frac{\partial P(\nu,l)}{\partial l}\Big) = -sgn\Big((f^{-1})'\Big(Q(\nu,l)\Big)\Big).$$

Consider the case with no hedgers in the market, i.e. $h \equiv 0$. Then $(f^{-1})' \equiv 1$ and

$$sgn\Big(\frac{\partial P(\nu,l)}{\partial l}\Big|_{h\equiv 0}\Big) < 0 < sgn\Big(\frac{\partial P(\nu,l)}{\partial \nu}\Big|_{h\equiv 0}\Big).$$

This means when there are no hedgers in the market the equilibrium price is a strictly increasing function of ν and a strictly decreasing function of l. This is in concordance with reality since security prices tend to increase in the presence of good news about their payoffs whereas they tend to fall when liquidity supply of the security increases. Theoretically, presence of hedgers may pervert this observed characteristic of security prices, that is, prices may fall with good news and increase with liquidity supply. Naturally we want to know the condition(s) on hedging activity that would lead to price reactions in accord with reality. So we have the following:

Lemma 3.1 *Let f^{-1} be differentiable. Then $P(\nu, l)$ is strictly increasing in ν and strictly decreasing in l if and only if*

$$I + A^*h \textit{ is strictly increasing.} \tag{S1'}$$

[15] $sgn(.)$ stands for the sign function.

The proof simply follows from (3.2.1a)-(3.2.1b) and the fact that $(f^{-1})'(y) = \frac{1}{1+A^* h'(x)}$, given $y = f(x)$. Also note that S1 necessarily holds whenever one assumes S1$'$.

For the results presented below, we need to incorporate the extent of hedging activity as a parameter into the hedging supply function. So we introduce

$$h(p) = \alpha \Pi(p), \quad \forall p,$$

where α denotes the fraction of assets protected by hedging (portfolio insurance) and Π is a decreasing function of p. One now has the following proposition:

Proposition 3.2 (Amplification) *Let f^{-1} be differentiable, and assume S1$'$ holds. Then as the fraction α of assets protected by hedging (portfolio insurance) increases, the equilibrium price function becomes more sensitive to changes in the information parameter ν and the liquidity parameter l. That is, $|\frac{\partial P(\nu,l)}{\partial \nu}|$ and $|\frac{\partial P(\nu,l)}{\partial l}|$ are increasing functions of α.*

Proposition 3.2 reveals the *amplifying effect* of hedging activity on price movements. One can easily see the intuition behind this result: once the price begins to fall (due to bad news or increasing liquidity supply), there will be more hedging supply of the security which will further push the prices to much lower levels. So in the presence of hedgers one will see amplified price reactions to the triggering events (such as bad news or higher liquidity). Naturally, the more the hedgers, the larger the price reactions. Of course, a similar argument works for the price hikes as well.

Now we would like to analyze the second characteristic of large price movements, namely the *asymmetry* in favor of crashes. For this purpose, we need to check the concavity of equilibrium price with respect to the parameters ν and l (see §3.1.3). In the case of twice-differentiable price functions, concavity is determined by the second-order partial derivatives:

$$\frac{\partial^2 P(\nu,l)}{\partial \nu^2} = (f^{-1})''(Q(\nu,l))\Big\{\frac{\text{cov}(R,\Sigma)}{\text{var}\Sigma} + \frac{A^*}{a_i^*}(1-\frac{\text{cov}(R,\Sigma)}{\text{var}\Sigma})\Big\}^2, \qquad (3.2.2a)$$

$$\frac{\partial^2 P(\nu,l)}{\partial l^2} = (f^{-1})''(Q(\nu,l))\Big\{a_i^*\frac{\text{cov}(R,\Sigma)}{\text{var}\Sigma} + A^*(1-\frac{\text{cov}(R,\Sigma)}{\text{var}\Sigma})\Big\}^2. \qquad (3.2.2b)$$

They lead us to the following result:

Proposition 3.3 (Asymmetry) *Assume $S1'$ holds and that f^{-1} is twice-differentiable. If hedging supply h is a strictly convex function within the set*

$$P(U_{\nu_0}, U_{l_0}) = \{p : p = P(\nu,l) \ \ s.t. \ \ (\nu,l) \in U_{\nu_0} \times U_{l_0}\},$$

then:

(a) information and liquidity shocks cause a bias towards negative price deviations within $U_{\nu_0} \times U_{l_0}$; i.e. $P(\nu,l)$ is strictly concave in ν and strictly concave in l for $(\nu,l) \in U_{\nu_0} \times U_{l_0}$;

(b) the bias becomes more significant within $U_{\nu_0} \times U_{l_0}$ as the fraction α of assets protected by hedging increases; that is, $\frac{\partial^2 P(\nu,l)}{\partial \nu^2}$ and $\frac{\partial^2 P(\nu,l)}{\partial l^2}$ are decreasing functions of α within $U_{\nu_0} \times U_{l_0}$.

It is easy to check that even if S1′ does not hold, the results above (on amplification and asymmetry) will hold within the domain

$$\{(\nu,l) : (I + A^*h)'(P(\nu,l)) > 0\}.$$

To sum up, under plausible conditions, whenever a shock (either of informational nature or liquidity based) occurs in the economy, the deviation in price is amplified due to hedging. Hence with hedging, the deviations are more likely to be significant, that is they are more likely to be a crash or a frenzy. Moreover if the hedging function is (globally) strictly convex, then a bias towards negative deviations is observed. We can summarize these results as follows:

Corollary 3.1 (Main Result) *Let f^{-1} be twice-differentiable. Assume $S1'$ holds. If hedging supply h is a strictly convex function, then there exists a bias towards crashes in the economy.*

One criticism towards the results of this section might be the extent of their dependence on hedging. After all, having lots of irrational agents, programmed to behave in ways to create amplification and asymmetry, would not be much of an explanation for the characteristics we are examining. Therefore we would like to show that our results do not stem from an imposed environment with a lot of irrational hedgers accompanied by just enough rational traders to equate supply and demand. The main difference between rational traders (insider, outsider) and hedgers is that their demands react differently to price deviations. That is, demand of rational traders is a decreasing function of price whereas demand of hedgers is increasing in price. So we can determine the dominance of a group (namely rational traders or hedgers) in the market by checking the sensitivity of market excess demand with respect to price.

Proposition 3.4 (Market Demand) *Let f^{-1} be differentiable. Then $S1'$ holds if and only if excess market demand of the risky security Z is strictly decreasing in p, where*

$$Z(p) = D_i(p|\nu) + D_o(p|\sigma) - h(p) - l.$$

This proposition shows that demand of rational traders prevail over that of hedgers if and only if $S1'$ holds. Since we get the results of this section with practically one assumption, namely $S1'$, we can say that our results hold within an environment where rationality prevails.

3.3 Put-Option Replication

We now further our analysis by examining a specific hedging (portfolio insurance) strategy: the put-option replication. Put-option replication was the most popular hedging strategy during 1980's, in particular, during the October '87 crash. The formula for the put-option replication is taken from

Gennotte and Leland (1990).[16] We follow their definition of the strategy using notation of our model: Put-option replication is assumed to be applied to a fraction α of risky securities. The incremental hedging supply when new price is p, relative to the supply at the hypothetical initial price ($p^* = 1$), is given by

$$\hat{h}(p) = \alpha\Big(\Phi(d(1)) - \Phi(d(p))\Big),$$

where $\Phi(.)$ is the standard cumulative normal distribution function, and $d(.)$ is derived from the Black-Scholes formula, that is

$$d(p) = \frac{\ln\left(\frac{p}{K}\right) + \frac{1}{2}\text{var}(R|\Sigma)}{\sqrt{\text{var}(R|\Sigma)}}.$$

with K as the striking price of the option (or the protection level in the replication case).[17]

Unfortunately possibility of negative security prices is a caveat of the CARA-Gaussian framework. Naturally we focus on strictly positive prices for the analysis of put-option replication. Note that

$$\hat{h}'(p) = -\frac{\alpha\phi(d(p))}{p\sqrt{\text{var}(R|\Sigma)}},$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is the standard normal density function. Clearly, $\hat{h}$ is decreasing in the domain of strictly positive prices, namely $(0, \infty)$. Extracting $\hat{h}'(p)$, we get

$$\hat{h}'(p) = -\frac{\alpha}{p}\frac{\exp\left(-\frac{1}{2}\left(\frac{\ln\frac{p}{K} + \frac{1}{2}\text{var}(R|\Sigma)}{\sqrt{\text{var}(R|\Sigma)}}\right)^2\right)}{\sqrt{2\pi\text{var}(R|\Sigma)}}.$$

Now it is easy to see the following:

[16] Gennotte and Leland (1990) point out the differences in their formula compared to Black and Scholes (1973). They assume that interest rate has been normalized to zero, and assume a one-year time horizon. Moreover in theirs payoff is normally distributed (as in our model), whereas in Black and Scholes (1973) payoff follows a lognormal process.

[17] In the actual Black-Scholes formula, we would have

$$d(p) = \Big(\ln\left(\frac{p}{K}\right) + \frac{1}{2}\text{var}(R|P)\Big)\left(\sqrt{\text{var}(R|P)}\right)^{-1}.$$

However as we elaborated before in §3.1.2, observing P is equivalent to observing Σ due to the common knowledge assumptions about the distribution S of insider's signal and the distribution L of liquidity supply (see Proposition 3.1).

Lemma 3.2 *Given $p_0 > 0$, there is a sufficiently low value of α which guarantees $\hat{h}'(p) > -\frac{1}{A^*}$ for all $p \in [p_0, \infty)$. In particular, if*

$$\alpha \leq \frac{p_0\sqrt{2\pi \text{var}(R|\Sigma)}}{A^*},$$

then $\hat{h}'(p) > -\frac{1}{A^*}$ for all $p \in [p_0, \infty)$. Moreover as α tends to 0, the set

$$\{p : \hat{h}'(p) > -\frac{1}{A^*}\}$$

will converge to the domain of strictly positive prices $(0, \infty)$.

So by choosing sufficiently low α, we can make S1$'$ hold for $\hat{h}$ over a strict subset of positive prices. It is easy to check that all our proofs will work over this strict subset. To be more precise, our results on amplification and asymmetry will still hold over the domain

$$\{(\nu, l) : \hat{h}'(P(\nu, l)) > -\frac{1}{A^*}\},$$

and this domain will converge to $\{(\nu, l) : P(\nu, l) > 0\}$ as α tends to 0. Recall that S1$'$ essentially concurs with reality since we observe prices rising after good news and falling after liquidity sales, so it is plausible to consider that α is quite low, that is the size of hedging in the stock market is small. We will concretely see our results on amplification and asymmetry working in a numerical example introduced in the next section, where α and various other parameters coincide with real world values.

For the convexity of $\hat{h}$, we need to check the second-order partial derivative:

$$\begin{aligned}
\hat{h}''(p) &= -\frac{\alpha}{\sqrt{\text{var}(R|\Sigma)}} \frac{\phi'(d(p))d'(p)p - \phi(d(p))}{p^2} \\
&= \frac{\alpha\phi(d(p))}{p^2\sqrt{\text{var}(R|\Sigma)}}\Big(\frac{d(p)}{\sqrt{\text{var}(R|\Sigma)}} + 1\Big) \\
&= \frac{\alpha \exp(-d(p)^2)}{p^2\sqrt{2\pi \text{var}(R|\Sigma)}}\Big(\frac{\ln\left(\frac{p}{K}\right)}{\text{var}(R|\Sigma)} + 2\Big).
\end{aligned}$$

The following result can be easily proved using this equation:

Lemma 3.3 *$\hat{h}$ is strictly convex over the domain $\{p : p > \frac{K}{e^{2\text{var}(R|\Sigma)}}\}$. As $\text{var}(R|\Sigma)$ tends to ∞, the domain where $\hat{h}$ is strictly convex will converge to the set of strictly positive prices.*

Using Lemma 3.2 and 3.3, one gets:

Proposition 3.5 (Put-option Replication) *Whenever the vector of parameters (ν, l) are in the domain*

$$\{(\nu, l) : P(\nu, l) > \frac{K}{e^{2\text{var}(R|\Sigma)}}\},$$

α is less than $\frac{K\sqrt{2\pi \text{var}(R|\Sigma)}}{A^* e^{2\text{var}(R|\Sigma)}}$, and hedgers employ put-option replication (i.e., $\hat{h}$) as the hedging strategy, there exists a bias towards crashes in the economy.

Moreover as $\text{var}(R|\Sigma)$ tends to ∞, the domain where the bias towards crashes is observed will converge to $\{(\nu, l) : P(\nu, l) > 0\}$ (i.e. in the limit the bias towards crashes will be observed in the whole range of strictly positive prices).

From the extraction of $\text{var}(R|\Sigma)$,[18] one can observe that high $\text{var}(R|\Sigma)$ is equivalent to high values of $\text{var}R$, $\text{var}S$, or $\text{var}L$. So high $\text{var}(R|\Sigma)$ is essentially equivalent to high market volatility, which is an observed characteristic of stock markets. Therefore we would expect to see a bias towards crashes for a large domain of positive security prices with put-option replication as the hedging strategy.

Remark 3.1 *Of course, one might still be concerned with the sufficiency level of market's high volatility level to observe the bias over a large range of prices. For instance, take the protection level K to be 85 percent of the initial equilibrium price. Recall that initial equilibrium price was fixed to be 1. Then even if one assumes $var(R|\Sigma)$ to be 1, the bias towards crashes will be observed for the range of positive prices within $(0.115, \infty)$. This is a strong implication that put-option replication creates amplification and asymmetry in favor of crashes in stock market.*

[18] See the extraction in the proof of Lemma A (C1) in Appendix C.

Though put-option replication and other portfolio insurance strategies played an important role in modern times, it is hard to use the same argument for the first half of the century. The sophisticated portfolio insurance strategies did not even exist then. However there is a hedging strategy which has been in use arguably as long as stock markets existed: stop-loss. In its most primitive form, hedgers sell their risky securities when the price falls below a predetermined level, say K. Use of this primitive hedging form clearly creates the asymmetry we want: there is an additional downward pressure on sales once price falls below K whereas there is no pressure when market goes up. Hence we get an asymmetry biased towards crashes.

3.4 A Numerical Example: Back to the 80's

The levels of risk aversion, hedging and market volatility necessary for substantial sizes of amplification and asymmetry are, of course, matters of concern. In other words, we do not want to generate amplification and asymmetry through absurdly high values of these parameters. So we examine the following numerical example:

Let us take put-option replication, the most popular portfolio insurance strategy of 80's, as the hedging function. We assume α to be 0.05, which is not far from the hedging size in the '87 crash. The protection level K is assumed to be 85 percent of initial price. Let us fix the initial equilibrium price to be 1 so that K becomes 0.85. Assuming an expected 6 percent return on the risky security compared to a risk-free asset is reasonable for U.S. markets, thus we let $\mathrm{E}[R] = 1.06$. Outsider is assumed to be more risk averse than insider by letting $a_i = 0.70$ and $a_o = 1.40$. Take $\mathrm{var}(R|S)$, $\mathrm{var}(R|\Sigma)$ and $\frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma}$ to be 200, 400 and 0.5, respectively.[19] Note that these values illustrate the informational advantage of insider through $\frac{\mathrm{var}(R|\Sigma)}{\mathrm{var}(R|S)} = 2$. We assume l to be 0 as liquidity supply is not biased.

Then to create a 20 percent price deviation in the negative direction it takes a 2.9 percent fall in

[19] $\frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma}$ always takes values between 0 and 1. See Lemma B (C2) in Appendix C.

the insider's expectation on risky payoff (ν) whereas a positive price deviation of the same magnitude requires a 8.5 percent increase in ν. This example clearly depicts the asymmetry.

Moreover if there were no hedging in the market, a 20 percent price movement in any direction would require a 18.1 percent change in the information parameter ν. Clearly in the case with put-option replication, price is more sensitive to the parameter changes, which illustrates the amplification brought by hedging.

3.5 Roles of Risk Aversion and Asymmetric Information

Lee (1998) makes the following conjecture in the conclusion of his paper: "Under risk aversion it is more difficult to trigger a frenzy than a crash because a surprise of the same degree in the direction of the good state induces a smaller response than the one in the direction of the bad state." Granted Lee's model exploits a totally different mechanism, his conjecture actually pinpoints the role of risk aversion in our analysis. To see that, we first give the following lemma, which is a straightforward consequence of (3) and function f's definition:

Lemma 3.4 *Assume $S1'$. As insider or outsider tends to be risk neutral (i.e. when one of their risk aversion parameters converges to 0), f^{-1} converges to the identity function, and thus $P(\nu, l)$ converges to a function linear in ν and l.*

Note that $P(\nu, l)$ is a continuous function of a_i and a_o. If the equilibrium price converges to a linear function, it simply means that asymmetry in price deviations vanishes (see §3.1.4). So we can explicitly state the following obvious:

Corollary 3.2 (Risk aversion) *Assume $S1'$. As insider or outsider tends to be risk neutral, asymmetry vanishes in the equilibrium price deviations.*

This result also concurs with Chalkley and Lee (1998) and Veronesi (1999). Both papers emphasize convexifying effect of risk aversion on price reactions to changes in underlying parameters. Though risk aversion is not the central reason for asymmetry in our version of the story, it still plays a significant role. Actually, risk aversion allows hedging to be incorporated to the price function. If traders are risk neutral, hedging does not affect price function at all; hence price becomes a linear function, which does not allow for asymmetric deviations.

Having elaborated on the vital role played by risk aversion in our analysis, next we would like to discuss the role of asymmetric information. For convenience, we first define a measure for the level of asymmetry regarding information. Notice that the ratio $\frac{\text{var}(R|\Sigma)}{\text{var}(R|S)}$ gives the imprecision of the information of outsider relative to that of the insider, i.e., given the gaussian nature of our framework this ratio delivers insider's informational advantage over outsider. So we let

$$\mu \equiv \frac{\text{var}(R|\Sigma)}{\text{var}(R|S)},$$

and call the ratio μ, $\mu > 1$, the *measure of asymmetric information*.[20] The bigger the measure μ gets, the larger the asymmetry between insider and outsider is. Now we can easily see how asymmetric information affects our analysis:

Proposition 3.6 (Asymmetric Information) *Assume $S1'$ and that $h'(.) < -\frac{1}{a_i^*}$. Also suppose that f^{-1} is continuously twice-differentiable and hedging supply h is strictly convex. There exits $\bar{\mu} > 1$ such that within the domain $(\bar{\mu}, \infty)$ of the asymmetric information measure μ*

(a) the equilibrium price function becomes more sensitive to changes in the information parameter ν and the liquidity parameter l as μ increases; i.e., $|\frac{\partial P(\nu,l)}{\partial \nu}|$ and $|\frac{\partial P(\nu,l)}{\partial l}|$ are increasing functions of μ,

(b) the bias towards negative deviations becomes more significant as μ increases; that is, $\frac{\partial^2 P(\nu,l)}{\partial \nu^2}$ and $\frac{\partial^2 P(\nu,l)}{\partial l^2}$ are decreasing functions of μ.

[20]Since outsider's information is more imprecise compared to that of insider's, the measure of asymmetric information $\mu \equiv \frac{\text{var}(R|\Sigma)}{\text{var}(R|S)}$ is always strictly greater than 1.

The only new assumption in this proposition, which has not been employed before, is

$$h'(.) < -\frac{1}{a_i^*} \equiv -\frac{1}{a_i \mathrm{var}(R|S)},$$

and this may be justified if the information of insider is sufficiently imprecise (i.e., if $\mathrm{var}(R|S)$ is sufficiently large). The proposition states that, with large enough asymmetry between insider and outsider in terms of information owned, both *amplification* and *asymmetry (of price deviations)* will be more significant as the measure of asymmetric information μ increases. Hence asymmetric information certainly helps our cause.

However, one can still question the necessity of asymmetric information in our analysis. After all, risk aversion and hedging strategies are sufficient ingredients to create asymmetry in price deviations. That is, our analysis will go through without making use of asymmetric information at all. Though, such analysis will be hard to justify when it comes to numerical computations. For instance, in §3.4 we are able to generate significant amplification and asymmetry with risk aversion coefficients $a_i = 0.7$ and $a_o = 1.4$. Without asymmetric information, the same effect would necessitate implausibly high risk aversion coefficients for CARA utility traders.

3.6 Trading Behavior in The Presence of Hedgers

All previous sections have dealt with the effect of hedging on equilibrium price. Now we would like to analyze the effect of hedging on rational agents' trading behavior. Recall that equilibrium demand function of a rational trader is of the form

$$D_j(P(\nu,l)|I_j) = \frac{\mathrm{E}[R|I_j] - P(\nu,l)}{a_j \mathrm{var}(R|I_j)}, \tag{3.6.1}$$

where I_j stands for the Gaussian information of agent $j = i, o$. We can partition the rational demand into the *information effect* $\frac{\mathrm{E}[R|I_j]}{a_j \mathrm{var}(R|I_j)}$, and the *substitution effect* $-\frac{P(\nu,l)}{a_j \mathrm{var}(R|I_j)}$. The overcoming effect among these two determines *the direction of the rational demand reaction* whenever price deviates.

Clearly portfolio allocation of a rational trader would be different depending on whether there are hedgers in the market or not, because the price is affected by the presence of hedgers. However we would like to analyze a more significant impact of hedging on the trading behavior. In particular, we want to see whether a rational trader would change the direction of her reaction to the information and liquidity shocks. We will elaborate on this after the following proposition.

Proposition 3.7 (Trading Behavior) *Assume $S1'$ and that f^{-1} is differentiable. We have the following:*

(a) $D_o(P(\nu,l)|\sigma)$ is decreasing in ν and increasing in l.

(b) $D_i(P(\nu,l)|\nu)$ is increasing in l.

(c) If the fraction α of assets protected by hedging is sufficiently small, then $D_i(P(\nu,l)|\nu)$ is increasing in ν. If α is sufficiently large, then $D_i(P(\nu,l)|\nu)$ is decreasing in ν.

Part (c) of Proposition 3.7 depicts the significant impact of hedging that we are looking for. It is easy to see from the proof that substitution and information effects move in different directions with respect to the changes in the information parameter. To be more specific, information effect is an increasing function of ν, and substitution effect is decreasing in ν. We see that without hedging activity in the market insider would demand more of the risky security when good news arrive; that is, information effect overcomes the substitution effect. In the presence of hedgers this may not be true. If the size of hedging is large enough, the price (hence the substitution effect) might be amplified excessively by hedgers, cancelling the information effect. Then insider's demand will decrease when good news arrive. This is certainly a significant change for insider's trading behavior since he changes the direction of his demand reaction to information shocks.

On the other hand, we do not see hedging affecting outsider's trading behavior to the same extent it affects insider's. In particular, the direction of outsider's demand reaction to information and liquidity shocks does not differ with or without hedgers in the market. However the way outsider reacts to information shocks is interesting. Part (a) of Proposition 3.7 shows that whenever good news come (i.e. when insider's expectation about the risky security increases) outsider decreases her demand of the risky security regardless of the size of hedging activity. This might seem puzzling at first, because conventionally we would expect increasing demand following good news. The reason is actually the noise created by liquidity traders. When good news come the price increases, but outsider is not sure whether it is the good news or liquidity demand that increases the price. Therefore although her expectation on the risky security return increases, the price increase overcomes this effect due to the risk premium associated with the liquidity trading. This translates into substitution effect overcoming information effect in outsider's demand.

Finally, from part (b) of Proposition 3.7 we see that hedging does not change the direction of insider's demand reaction to liquidity shocks.

3.7 Discussion on The Emergence of Hedging

In this section we investigate the emergence of hedging strategies in the stock market. First let us verify that hedging strategy is not optimal for an outsider to employ. We know that hedging demand (i.e, negative hedging supply) is an increasing function of price. If the trigger for the price hike is an increase in the information parameter ν, following Proposition 6.1, outsider's demand decreases. If the trigger for the hike is a decrease in the liquidity supply l, outsider's demand again decreases. So whatever the origin of shock is, we always see hedging and rational demand dictating opposite directions in portfolio allocation. Therefore, hedging strategy is clearly sub-optimal for the outsider, that is it will create a significant ex-ante utility cost (given the observation of price) compared to employing the rational demand schedule. So why do people employ hedging strategies after all?

Ex-post, hedgers might be better off compared to outsiders in the case of information shocks. The reason is that both hedging demand and insider's rational demand are in the same direction (and both are opposite to outsider's direction of demand) when the size of hedging is sufficiently small. Since the insider has the privileged information, it is quite likely that insider is better off compared to outsider (however we cannot say this with certainty as insider's signal is noisy). Hence the hedger is also quite likely to be better off compared to outsider after an information shock. In the case of liquidity shocks, hedger's demand is opposite in direction to both insider and outsider. So if overwhelmingly information shocks trigger price deviations, employing hedging strategies might prove to be ex-post profitable due to the argument above. Of course, this explanation is far from a rigorous treatment of the matter; however we were not able to conduct this task due to the analytical complexity associated with the particular choice of hedging strategy (put-option replication) and normal distributions.

Another interesting point is that whenever the size of hedging is sufficiently large, the hedger's demand is in the opposite direction of both insider and outsider. So a possibly winning strategy for one person will be an almost certainly losing strategy when many people employ it.

Appendix A: Mathematical Preliminaries

A1 Projection theorem. For jointly normally distributed random variables X and Θ, we have the following formulas:

$$\begin{aligned} \mathrm{E}[X|\Theta=\theta] &= \mathrm{E}[X] + \frac{\mathrm{cov}(X,\Theta)}{\mathrm{var}\Theta}(\theta - \mathrm{E}\Theta), \\ \mathrm{var}(X|\Theta) &= \mathrm{var}(X) - \frac{(\mathrm{cov}(X,\Theta))^2}{\mathrm{var}\Theta}. \end{aligned}$$

A2 Rao's formula. For a normal random variable X, the following formula holds:

$$\mathrm{E}[\mathrm{e}^X] = \mathrm{e}^{\left(\mathrm{E}X + \frac{\mathrm{var}X}{2}\right)}.$$

Appendix B: Derivations

B1 Derivation of excess demand functions. Since R is normal, W_j is also normal for $j = i, o$. By Rao's formula (A2) we have

$$\mathrm{E}[u^j(W_j)|I_j] = -\mathrm{e}^{\left(-a_j D_j \mathrm{E}[R|I_j] - a_j(e_j - pD_j) + a_j^2 D_j^2 \frac{\mathrm{var}(R|I_j)}{2}\right)}.$$

Agent $j \in \{i, o\}$ solves the maximization problem, given in (3.1.1). The solution to this problem is

$$D_j(p) = \frac{\mathrm{E}[R|I_j] - p}{a_j \mathrm{var}(R|I_j)}.$$

B2 Derivation of (3.1.3b). Recall that $\sigma = \nu - a_i^* l$. Projection theorem (A1) implies

$$\mathrm{E}[R|\sigma] = \mathrm{E}R + \frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma}(\nu - a_i^* l - \mathrm{E}R).$$

Then

$$\begin{aligned}\mathrm{E}[R|\sigma] + \frac{A^*}{a_i^*}(\sigma - \mathrm{E}[R|\sigma]) &= \left\{\frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}(1 - \frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma})\right\}\nu \\ &\quad -\left\{\frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma} + A^*(1 - \frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma})\right\}l \\ &\quad -\frac{A^*}{a_o^*}\left\{1 - \frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma}\right\}\mathrm{E}R.\end{aligned}$$

B3 Derivatives of the function $\mathbf{f^{-1}}$. Let $y = f(x)$. Then as $f \equiv I + A^* h$, we have

$$(f^{-1})'(y) = \frac{1}{1 + A^* h'(x)}, \quad (f^{-1})''(y) = -\frac{A^* h''(x)}{\left(1 + A^* h'(x)\right)^3}.$$

Appendix C: Proofs

C1 Lemma A. Excess demand functions of insider and outsider are well-defined.

Proof: We only need to show that $\text{var}(R|S)$ and $\text{var}(R|\Sigma)$ are non-zero (see (3.1.2)). As $\text{var}\Omega$ and $\text{var}R$ are non-zero, using (A1) we get

$$\text{var}(R|S) = \text{var}R - \frac{(\text{cov}(R,S))^2}{\text{var}S} = \text{var}R - \frac{(\text{var}R)^2}{\text{var}R + \text{var}\Omega} > 0,$$

$$\begin{aligned}
\text{var}(R|\Sigma) &= \text{var}R - \frac{(\text{cov}(R,\Sigma))^2}{\text{var}\Sigma} \\
&= \text{var}R - \frac{\Big(\text{cov}(R, \text{E}[R|S] - a_i\text{var}(R|S)L)\Big)^2}{\text{var}(\text{E}[R|S] - a_i\text{var}(R|S)L)} \\
&= \text{var}R - \frac{\left(\frac{(\text{cov}(R,S))^2}{\text{var}S}\right)^2}{\left(\frac{(\text{cov}(R,S))^2}{\text{var}S} + a_i{}^2(\text{var}(R|S))^2\text{var}L\right)} \\
&\geq \text{var}R - \frac{(\text{cov}(R,S))^2}{\text{var}S} = \text{var}(R|S) > 0. \ \square
\end{aligned}$$

C2 Lemma B. $\frac{\text{cov}(R,\Sigma)}{\text{var}\Sigma} \leq 1.$

Proof: To prove this inequality, we extract the terms in LHS.

$$\begin{aligned}
\text{cov}(R,\Sigma) &= \text{cov}\Big(R, \text{E}[R|S] - a_i\text{var}(R|S)L\Big) = \text{cov}(R, \text{E}[R,S]) \\
&= \text{cov}\Big(R, \text{E}R + \frac{\text{cov}(R,S)}{\text{var}S}(S - \text{E}R)\Big) = \frac{(\text{cov}(R,S))^2}{\text{var}S},
\end{aligned}$$

$$\begin{aligned}
\text{var}\Sigma &= \text{var}(\text{E}[R|S]) + a_i{}^2(\text{var}(R|S))^2\text{var}L \\
&= \text{var}\Big(\text{E}R + \frac{\text{cov}(R,S)}{\text{var}S}(S - \text{E}R)\Big) + a_i{}^2(\text{var}(R|S))^2\text{var}L \\
&= \frac{(\text{cov}(R,S))^2}{\text{var}S} + a_i{}^2(\text{var}(R|S))^2\text{var}L.
\end{aligned}$$

Hence the result follows. □

C3 Proof of Proposition 3.2. Note that $h' = \alpha\Pi' > -\frac{1}{A^*}$ due to S1$'$. We also know from (C2) that

$\frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma} \leq 1$. Thus from (3.2.1a)-(3.2.1b) and (B3), given $p = P(\nu, l)$ one has

$$\begin{aligned}\left|\frac{\partial P(\nu,l)}{\partial \nu}\right| &= \frac{1}{1+\alpha A^*\Pi'(p)}\left\{\frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}\left(1 - \frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma}\right)\right\},\\ \left|\frac{\partial P(\nu,l)}{\partial l}\right| &= \frac{1}{1+\alpha A^*\Pi'(p)}\left\{a_i^*\frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma} + A^*\left(1 - \frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma}\right)\right\}.\end{aligned}$$

Since Π is a decreasing function, it is straightforward to see that $\left|\frac{\partial P(\nu,l)}{\partial \nu}\right|$ and $\left|\frac{\partial P(\nu,l)}{\partial l}\right|$ are increasing functions of α. □

C4 Proof of Proposition 3.3 Due to S1′, $h' > -\frac{1}{A^*}$. Hence from (B3) it follows that if h is strictly convex within the set $P(U_{\nu_0}, U_{l_0})$, f^{-1} is strictly concave within $P(U_{\nu_0}, U_{l_0})$, consequently $P(\nu, l)$ is strictly concave in ν and strictly concave in l within $U_{\nu_0} \times U_{l_0}$. This proves (a).

Now from (3.2.2a)-(3.2.2b) and (B3), given $p = P(\nu, l)$ we have

$$\begin{aligned}\frac{\partial^2 P(\nu,l)}{\partial \nu^2} &= -\frac{\alpha A^*\Pi''(p)}{\left(1+\alpha A^*\Pi'(p)\right)^3}\left\{\frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}\left(1 - \frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma}\right)\right\}^2,\\ \frac{\partial^2 P(\nu,l)}{\partial l^2} &= -\frac{\alpha A^*\Pi''(p)}{\left(1+\alpha A^*\Pi'(p)\right)^3}\left\{a_i^*\frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma} + A^*\left(1 - \frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma}\right)\right\}^2.\end{aligned}$$

Recall that Π is a strictly decreasing function. If h (and hence Π) is strictly convex in $P\big(U_{\nu_0}, U_{l_0}\big)$, one has $\Pi''(p) > 0$ for $p \in P\big(U_{\nu_0}, U_{l_0}\big)$, thus $\frac{\partial^2 P(\nu,l)}{\partial \nu^2}$ and $\frac{\partial^2 P(\nu,l)}{\partial l^2}$ are decreasing functions of α for $(\nu, l) \in U_{\nu_0} \times U_{l_0}$. Hence (b) is proved. □

C5 Proof of Proposition 3.4. We have

$$\begin{aligned}Z(p) &= -\frac{p}{A^*} - h(p) + \frac{\mathrm{E}[R|s]}{a_i\mathrm{var}(R|S)} + \frac{\mathrm{E}[R|\sigma]}{a_o\mathrm{var}(R|\Sigma)} - l, \quad \text{and thus}\\ Z'(p) &= -\frac{1}{A^*} - h'(p).\end{aligned}$$

Now it is straightforward to see that S1′ holds if and only if Z is strictly decreasing in p. □

C6 Proof of Proposition 3.6. First note that the assumptions employed in the proposition impose

$$-\frac{1}{A^*} < h'(.) < -\frac{1}{a_i^*}.$$

Now take a look at the following equations for given $p = P(\nu, l)$ (which were already derived in C3 and C4):

$$\begin{aligned}
\left|\frac{\partial P(\nu,l)}{\partial \nu}\right| &= \frac{1}{1+A^*h'(p)}\left\{\frac{\text{cov}(R,\Sigma)}{\text{var}\Sigma} + \frac{A^*}{a_i^*}\Big(1-\frac{\text{cov}(R,\Sigma)}{\text{var}\Sigma}\Big)\right\},\\
\left|\frac{\partial P(\nu,l)}{\partial l}\right| &= \frac{1}{1+A^*h'(p)}\left\{a_i^*\frac{\text{cov}(R,\Sigma)}{\text{var}\Sigma} + A^*\Big(1-\frac{\text{cov}(R,\Sigma)}{\text{var}\Sigma}\Big)\right\};\\
\frac{\partial^2 P(\nu,l)}{\partial \nu^2} &= -\frac{A^*h''(p)}{\big(1+A^*h'(p)\big)^3}\left\{\frac{\text{cov}(R,\Sigma)}{\text{var}\Sigma} + \frac{A^*}{a_i^*}\Big(1-\frac{\text{cov}(R,\Sigma)}{\text{var}\Sigma}\Big)\right\}^2,\\
\frac{\partial^2 P(\nu,l)}{\partial l^2} &= -\frac{A^*h''(p)}{\big(1+A^*h'(p)\big)^3}\left\{a_i^*\frac{\text{cov}(R,\Sigma)}{\text{var}\Sigma} + A^*\Big(1-\frac{\text{cov}(R,\Sigma)}{\text{var}\Sigma}\Big)\right\}^2.
\end{aligned}$$

It is easy to see that if we can find a limit value, say $\hat{\mu} > 1$, such that

$$1 + A^*h'(p) \to 0 \quad \text{as} \quad \mu \equiv \frac{\text{var}(R|\Sigma)}{\text{var}(R|S)} \to \hat{\mu},$$

then $|\frac{\partial P(\nu,l)}{\partial \nu}|$ and $|\frac{\partial P(\nu,l)}{\partial l}|$ will tend to ∞ whereas $\frac{\partial^2 P(\nu,l)}{\partial \nu^2}$ and $\frac{\partial^2 P(\nu,l)}{\partial l^2}$ will converge to $-\infty$. Given $p = P(\nu,l)$, $\hat{\mu} = -\frac{1}{a_o\text{var}(R|S)\left(\frac{1}{a_i^*}+h'(p)\right)}$ does the job. That is, as $\mu \to -\frac{1}{a_o\text{var}(R|S)\left(\frac{1}{a_i^*}+h'(p)\right)}$

$$\left|\frac{\partial P(\nu,l)}{\partial \nu}\right|,\ \left|\frac{\partial P(\nu,l)}{\partial l}\right| \to \infty \quad \text{and} \quad \frac{\partial^2 P(\nu,l)}{\partial \nu^2},\ \frac{\partial^2 P(\nu,l)}{\partial l^2} \to -\infty.$$

However, there is one thing we need to check: whether $\hat{\mu} = -\frac{1}{a_o\text{var}(R|S)\left(\frac{1}{a_i^*}+h'(p)\right)} > 1$ or not. Suppose not: using our assumption that $h'(.) < -\frac{1}{a_i^*}$ we get

$$-a_o\text{var}(R|S)\left(\frac{1}{a_i^*}+h'(p)\right) > 1 \implies h'(p) < -\frac{1}{a_o\text{var}(R|S)} - \frac{1}{a_i^*} < -\frac{1}{a_o\text{var}(R|\Sigma)} - \frac{1}{a_i^*} = -\frac{1}{A^*},$$

which violates another assumption of ours, $h'(.) > -\frac{1}{A}$. So it is also true that $\hat{\mu} > 1$.

Now following the limit results derived above and the fact that $P(\nu,l)$ is continuously twice-differentiable, there exits $\bar{\mu} > 1$ such that within the domain $(\bar{\mu}, \infty)$ of the asymmetric information measure μ; $|\frac{\partial P(\nu,l)}{\partial \nu}|$, $|\frac{\partial P(\nu,l)}{\partial l}|$ are increasing in μ and $\frac{\partial^2 P(\nu,l)}{\partial \nu^2}$, $\frac{\partial^2 P(\nu,l)}{\partial l^2}$ are decreasing in μ. □

C7 Proof of Proposition 3.7.

(a) From the extraction of $\mathrm{E}[R|\sigma]$ it follows

$$\frac{\partial \mathrm{E}[R|\sigma]}{\partial \nu} = \frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma}, \quad \frac{\partial \mathrm{E}[R|\sigma]}{\partial l} = -{a_i}^* \frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma}.$$

Recall from (3.2.1a)-(3.2.1b) that

$$\begin{aligned} \frac{\partial P(\nu,l)}{\partial \nu} &= (f^{-1})'(Q(\nu,l))\Big\{\frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}\Big(1 - \frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma}\Big)\Big\}, \\ \frac{\partial P(\nu,l)}{\partial l} &= -(f^{-1})'(Q(\nu,l))\Big\{a_i^* \frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma} + A^*\Big(1 - \frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma}\Big)\Big\}. \end{aligned}$$

Following Lemma B (C2),

$$\frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}\Big(1 - \frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma}\Big) \geq \frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma}.$$

Moreover h is a strictly decreasing function, hence $h'(.) < 1$. Then it follows from (B3) that

$$(f^{-1})'(Q(\nu,l)) = \frac{1}{1 + A^* h'(P(\nu,l))} \geq 1$$

under $S1'$. So

$$\frac{\partial P(\nu,l)}{\partial \nu} \geq \frac{\partial \mathrm{E}[R|\sigma]}{\partial \nu}, \quad \frac{\partial P(\nu,l)}{\partial l} \leq \frac{\partial \mathrm{E}[R|\sigma]}{\partial l};$$

and therefore $D_o(P(\nu,l)|\sigma)$ is decreasing in ν and increasing in l from (3.6.1).

(b) We have $\frac{\partial \mathrm{E}[R|s]}{\partial l} = \frac{\partial \nu}{\partial l} = 0$. On the other hand, $\frac{\partial P(\nu,l)}{\partial l} < 0$ due to $S1'$ (see Lemma 3.1). Thus from (6) one observes that $D_i(P(\nu,l)|\nu)$ is increasing in l.

(c) We have $\frac{\partial \mathrm{E}[R|s]}{\partial \nu} = \frac{\partial \nu}{\partial \nu} = 1$. Recall from (3.2.1a) that

$$\frac{\partial P}{\partial \nu} = (f^{-1})'(Q(\nu, l))\Big\{\frac{\mathrm{cov}(R, \Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}\Big(1 - \frac{\mathrm{cov}(R, \Sigma)}{\mathrm{var}\Sigma}\Big)\Big\}.$$

First of all; $\frac{A^*}{a_i^*} = \frac{a_o^*}{a_i^* + a_o^*} \le 1$, and $\frac{\mathrm{cov}(R,\Sigma)}{\mathrm{var}\Sigma} \le 1$ from Lemma B (C2). So

$$\frac{\mathrm{cov}(R, \Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}\Big(1 - \frac{\mathrm{cov}(R, \Sigma)}{\mathrm{var}\Sigma}\Big) \le 1.$$

On the other hand we have shown in part (a) that $(f^{-1})'\big(Q(\nu, l)\big) \ge 1$. Therefore $\frac{\partial P(\nu,l)}{\partial \nu}$ can be greater or less than 1 depending on the exact value of $(f^{-1})'\big(Q(\nu, l)\big)$. In particular, following from (B3),

$$(f^{-1})'\big(Q(\nu, l)\big) \rightarrow \begin{cases} 1 & \text{as} \quad \alpha \rightarrow 0 \\ \infty & \text{as} \quad \alpha \rightarrow -\frac{1}{A^*\Pi'\left(f^{-1}\left(Q(\nu,l)\right)\right)}. \end{cases}$$

Note that under S1′, α cannot take values larger than $-\frac{1}{A^*\Pi'\left(f^{-1}(.)\right)}$, and also note that $-\frac{1}{A^*\Pi'\left(f^{-1}(.)\right)} \ge 0$ as Π is a decreasing function. Therefore $D_i(P(\nu, l)|\nu)$ is increasing in ν with sufficiently small α, and it is decreasing in ν with sufficiently large α. □

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