OPTIMAL DECISIONS UNDER RECURSIVE UTILITY

by

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Dedication

To my wife Asiye, and my parents.

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Abstract

Recursive utility functions control the investors relative risk aversion (RRA) and elasticity of intertemporal substitution (EIS) by different parameters. They are generalization of expected utility functions in which the RRA and the EIS are controlled by the same parameter. This is widely discussed in the empirical literature. Also, the timing of the resolution of uncertainty matters in recursive setting. Recursive utility functions are widely used in the literature in order to explain many macroeconomic issues like the equity premium puzzle, risk free rate puzzle, and stock market participation. We want to have a deep understanding about the effects and relations of the model parameters. We use the Epstein-Zin preferences on a binomial tree and find the analytical closed form solution for the optimal allocations in consumption, risk free and risky assets. We give numerical results for the effects of model parameters. Numerical results show that the dependence of consumption on RRA parameter is insignificant. Then, we extend our model by adding lifetime uncertainty. We check the interchangeability of EIS parameter, and the subjective time discount factor under different cases like incomplete markets and stochastic lifetime. Analytically, these two parameters are not interchangeable, but numerically they are under certain lifetime, complete and incomplete markets. However, accuracy is much smaller for the uncertain lifetime model. Then, we analyze the welfare loss of suboptimal allocations. We find that effect of suboptimal allocation in bond holdings is insignificant. The welfare loss is larger when the suboptimal allocation is in stock holdings and consumption, but it is still modest. Finally, we numerically show that a representative agent exists when the heterogeneity is in RRA or EIS parameter. However, this is not true when the heterogeneity is in subjective time discount factor or survival probability.

Chapter 1

Introduction

Recursive utility functions (or Epstein and Zin preferences) control the investors relative risk aversion (RRA) and elasticity of intertemporal substitution (EIS) by different parameters. Recursive utility functions are the generalization of expected utility functions in which the relative risk aversion and the elasticity of intertemporal substitution are controlled by the same parameter. However, in recursive setting one has the freedom of assigning separate parameters on these two attitudes towards risk. This is widely discussed in the empirical literature such as, [13], [12], [15], [11], [19], and [3].

Another interesting property of recursive utility functions is that, the timing of the resolution of uncertainty matters. Because, a recursive utility function is indifferent to the timing of resolution of uncertainty if and only if it is an expected utility functional (see [7]). In a recursive setting, the representative agent prefers the early (late) resolution of uncertainty if the RRA is larger (smaller) than the EIS.

Recursive utility functions are widely used in the literature in order to explain many macroeconomic issues like the equity premium puzzle, risk free rate puzzle, and stock market participation.

[9] prove the existence of recursive intertemporal utility functions, and the existence of optima to corresponding optimization problems. In their continuation paper, [10], they investigate the testable restrictions on the time-series behavior of consumption and asset returns with recursive preferences. They find that the elasticity of

intertemporal substitution is less than one, relative risk aversion is close to one, and consumers prefer the late resolution of uncertainty.

[18] studies the implications for general equilibrium asset pricing of the recursive utility preferences and finds evidence that does not solve the equity premium puzzle introduced by [15]. In this case, an additional puzzle emerges which is called the the risk-free rate puzzle.

[2] argue that considering the consumption growth of stockholders and two asset returns partially explains the equity premium puzzle.

[4] model consumption and dividend growth rates and give empirical support by using Epstein and Zin preferences which justifies the observed equity premium, the volatility of the market return, dividend-yield, and the risk-free rate.

[17] provides axiomatic foundation for recursive preferences without assuming specific exogenous objective beliefs.

[5] obtain an analytical solution for the optimal policies for a three-date model with a stochastic interest rate.

These are only a very brief list of papers that used the recursive preferences. Our work is motivated by this huge literature using recursive utility functions in which the risk aversion and elasticity of intertemporal substitution are separated clearly.

We want to have a deep understanding about the effects and relations of the model parameters. We use the Epstein-Zin preferences on a binomial tree and find the analytical closed form solution for the optimal allocations in consumption, risk free and risky assets. We give numerical results for the effects of model parameters. For example, [5] find that the consumption depends on the risk aversion parameter alpha analytically. Our results also support this conclusion, however, the numerical results show that the dependence is insignificant. Then, we extend our model such that it includes lifetime uncertainty of the representative agent. We check the interchangeability of elasticity of intertemporal substitution, and the subjective time discount factor under different cases like incomplete markets, stochastic lifetime. Numerically, we show that these two parameters are interchangeable under the certain life time, complete and incomplete markets. However, the accuracy is much smaller when the stochastic lifetime is introduced to the model.

We analyze the effects of suboptimal allocation of decision variables on initial utility and we calculate the level of compensation needed to reach the level of initial utility with optimal allocations. We find that the effect of suboptimal allocation in bond and stock holdings is insignificant as long as the agent consumes and saves at the optimal level. In other words, suboptimal allocation in the composition of savings does not affect welfare significantly. The welfare loss is larger when stock holdings and consumption are chosen suboptimally. However, this loss is modest for reasonable levels of model parameters. [6] showed that in a power utility setting without intertemporal consumption, welfare costs of suboptimal decisions in risky and risk-free assets are relatively small. Our results support their argument with recursive utility.

Finally, we analyze the existence of the representative agent in a society whose agents have heterogeneous utility functions. We look at heterogeneity in RRA, EIS, subjective time discount factor and survival probability. We numerically show that a representative agent exists when the heterogeneity is in RRA or EIS parameter. However, it is not true when the heterogeneity is in subjective time discount factor or survival probability.

The remainder of the paper is organized as follows. In Chapter 2, we give the results for complete markets with and without uncertainty of lifetime. The results under incomplete markets results are presented in Chapter 3. In Chapter 4, we give the results for suboptimal allocations. In Chapter 5, the existence of representative agent is discussed. Chapter 6 concludes.

Chapter 2

The Model in Complete Markets

In this chapter, the recursive utility function and the budget constraint, in a general form will be given. Moreover, the specific form will be given and used in the analysis. Below, the model and its solution in deterministic life span will be given.

2.1 Certain Horizon

Analytical Solution

We use the Epstein-Zin type preferences (see [9]) and the notation of [5]. We consider a single representative agent who has a certain lifetime utility function at time t as, $U_t = f(c_t, \mu_t(U_{t+1}))$ for $t < T$ and at the terminal date T as $U_T = B(W_T)$, where f is an aggregator function, $\mu_t(U_{t+1})$ is the certainty equivalent of the distribution of time $t+1$ utility, U_{t+1}, conditional upon time-t information. $B(W_T)$ is the bequest function which denotes the joy of giving.

In our model the aggregator is:

$$f(c, v) = [c^{\phi} + \beta v^{\phi}]^{1/\phi};\ 0 \neq \phi \leq 1,\ \beta > 0 \tag{2.1}$$

The certainty equivalent of the future utility is:

$$\mu_t = (E_t U_{t+1}^{\alpha})^{1/\alpha};\ 0 \neq \alpha \leq 1 \tag{2.2}$$

The bequest function is:

$$B(W_T) = f(W_T, 0), \quad \phi \neq 0 \tag{2.3}$$

Therefore, combining the aggregator, certainty equivalent and the bequest function yields the recursive utility, U_t for any time $t \in \{1, 2, ..., T\}$ as:

$$U_t = [C_t^\phi + \beta(E_t U_{t+1}^\alpha)^{\phi/\alpha}]^{1/\phi}; \quad 0 \neq \phi \leq 1, \; 0 \neq \alpha \leq 1 \tag{2.4}$$

α denotes relative risk aversion parameter, ϕ denotes elasticity of intertemporal substitution parameter, and β is the subjective time discount factor.

Early (late) resolution of uncertainty is preferred by the agent whose relative risk aversion parameter α is less (greater) than the elasticity of intertemporal substitution parameter ϕ. This is one of the differences between recursive utility functions and standard additive expected utility functions. In the standard setting, the agent is indifferent between early and late resolution of uncertainty. There are cases where early resolution of uncertainty is is preferred to the late resolution or vice versa.

The individual decides how much to consume, how much to invest in risk free assets and risky stocks, given the initial endowment W_0, on a binomial tree. Hence the budget constraint at an arbitrary time t is:

$$B_t + C_t + \Pi_t S_t = W_t \tag{2.5}$$

where W_t is the wealth at time t, which is the sum of returns from investment in bonds and stocks. This can also be written as:

$$W_t = B_{t-1}(1 + r) + \Pi_{t-1} S_t = B_{t-1} R + \Pi_{t-1} S_t \tag{2.6}$$

We are assuming that everything is known at time t except the choice variables, which are B_t, C_t and Π_t. There are no transaction costs. It is assumed that lending and borrowing rates are equal. The variables used in the model at time t can be defined as:

$$
\begin{aligned}
B_t &= dollars\ invested\ in\ bonds \\
C_t &= dollars\ consumed \\
\Pi_t &= number\ of\ stock\ holdings \\
S_t &= stock\ price \\
\Pi_t S_t &= dollars\ invested\ in\ stocks \\
R &= (1+r)\ risk\ free\ gross\ return
\end{aligned}
$$

There are two states for the stock price which are S_t^u and S_t^d.

$\nearrow S_t^u = uS_{t-1}$ with probability p

S_{t-1}

$\searrow S_t^d = dS_{t-1}$ with probability $q = 1 - p$

Markets are complete if and only if the number of independent securities and the number of states are equal. Since there are two states, and two independent securities, then it can be said that the markets are complete.

In this model, ϕ measures the elasticity of intertemporal substitution, and α measures the relative risk aversion. More precisely, $\frac{1}{1-\phi}$ is the elasticity of intertemporal substitution and $1 - \alpha$ is the coefficient of relative risk aversion. This is another main advantage of using recursive utility because it disentangles the relative risk aversion and elasticity of intertemporal substitution. When $\alpha = \phi \neq 0$, one obtains the power specification of the standard expected utility U_t as:

$$U_t = [E_t(\sum_{j=0}^{T-t-1} \beta^j C_{t+j}^{\alpha} + \beta^{T-t} W_T^{\alpha})]^{1/\alpha} \tag{2.7}$$

in which the relative risk aversion is $1-\alpha$ and the elasticity of intertemporal substitution is $1/(1-\alpha)$. So both are determined by a single parameter, α. Having different parameters for EIS and RRA, provides flexibility and freedom to the model. In other words, the agent can have a large RRA and EIS at the same time. However, in the standard setting, one of them has to be large if the other one is small.

Since it is a finite time model, in order to find optimal choice variables, we maximize the recursive utility backwards, starting from the last period T. We find the variables as a percentage of wealth in backward solution and by going forward in time, we can find the dollar amounts allocated to decision variables.

There are no more periods after time T, so we assume that bond and stock holdings at time T are zero, i.e. $B_T = 0;\ \Pi_T = 0$. Since there is no utility expected in periods after T, the equation (2.4) at time T becomes:

$$U_T = C_T = W_T = B_{T-1}R + \Pi_{T-1}S_T \tag{2.8}$$

which means that the agent consumes everything finally.

More specifically, utility in the up state is:

$$U_T^u = B_{T-1}R + \Pi_{T-1}S_T^u = B_{T-1}R + \Pi_{T-1}S_{T-1}u \tag{2.9}$$

and utility in the down state is:

$$U_T^d = B_{T-1}R + \Pi_{T-1}S_T^d = B_{T-1}R + \Pi_{T-1}S_{T-1}d \tag{2.10}$$

After the explicit substitution of inner expectation, the problem at time $T-1$ becomes:

$$\begin{aligned} \max_{\{B_{T-1},C_{T-1},\Pi_{T-1}\}}\{ \quad & C_{T-1}^{\phi} + \beta[p(B_{T-1}R + \Pi_{T-1}S_{T-1}u)^{\alpha} \\ & + q(B_{T-1}R + \Pi_{T-1}S_{T-1}d)^{\alpha}]^{\phi/\alpha}\}^{1/\phi} \end{aligned}$$

subject to:

$$B_{T-1} + C_{T-1} + \Pi_{T-1}S_{T-1} = W_{T-1} \tag{2.11}$$

The substitution of the budget constraint into the equation gives:

$$\begin{aligned} & \max_{\{B_{T-1},\Pi_{T-1}\}}\{(w_{T-1} - B_{T-1} - \Pi_{T-1}S_{T-1})^{\phi} + \\ & \beta[p(B_{T-1}R + \Pi_{T-1}S_{T-1}u)^{\alpha} + q(B_{T-1}R + \Pi_{T-1}S_{T-1}d)^{\alpha}]^{\phi/\alpha}\}^{1/\phi} \end{aligned} \tag{2.12}$$

In order to maximize U_{T-1}, it is sufficient to solve the following system simultaneously.

$$\begin{aligned} \partial U_{T-1}/\partial B_{T-1} &= 0 \\ \partial U_{T-1}/\partial \Pi_{T-1} &= 0 \end{aligned} \tag{2.13}$$

The following constants k, l, and m are used in the solution of the system above.

$$\begin{aligned} k &= [\frac{q(R-d)}{p(u-R)}]^{\frac{1}{\alpha-1}} \\ l &= \frac{R(k-1)}{u-kd} \end{aligned} \tag{2.14}$$

$$m_{T-1} = \{\beta[p(lu+R)^{\alpha} + q(ld+R)^{\alpha}]^{\frac{\phi-\alpha}{\alpha}}(pu(lu+R)^{\alpha-1} + qd(ld+R)^{\alpha-1})\}^{\frac{1}{\phi-1}}$$

As a result, the dollar amounts allocated to bond holdings, and stock holdings at time $T-1$ can be found as following:

$$\begin{aligned} B_{T-1} &= \frac{1}{m_{T-1}+l+1}W_{T-1} \\ \Pi_{T-1}S_{T-1} &= \frac{l}{m_{T-1}+l+1}W_{T-1} \end{aligned}$$

The allocation in consumption can be obtained from the budget constraint, which is given below:

$$C_{T-1} = \frac{m_{T-1}}{m_{T-1}+l+1}W_{T-1}$$

So, if we summarize[1], we get the allocations as the following:

$$\begin{aligned} B_{T-1} &= \frac{1}{m_{T-1}+l+1}W_{T-1}, \text{let } b_{T-1} = \frac{1}{m_{T-1}+l+1} \\ C_{T-1} &= \frac{m_{T-1}}{m_{T-1}+l+1}W_{T-1}, \text{let } c_{T-1} = \frac{m_{T-1}}{m_{T-1}+l+1} \\ \Pi_{T-1}S_{T-1} &= \frac{l}{m_{T-1}+l+1}W_{T-1}, \text{let } \pi_{T-1} = \frac{l}{m_{T-1}+l+1} \end{aligned}$$

[1]See appendix for details.

then,

$$\begin{aligned} B_{T-1} &= b_{T-1}W_{T-1} \\ C_{T-1} &= c_{T-1}W_{T-1} \\ \Pi_{T-1}S_{T-1} &= \pi_{T-1}W_{T-1} \end{aligned} \tag{2.15}$$

The lower case letters in the above equations (2.15), $b_{T-1},\ c_{T-1},\ \pi_{T-1}$ (ratio coefficients) do not depend on the state. For example, c_{T-1} is the ratio of consumption to wealth at time $T-1$ no matter what the state is. However, the upper case letters $B_{T-1},\ C_{T-1}$ and $\Pi_{T-1}S_{T-1}$ are dollar amounts allocated to bond holdings, consumption, and stock holdings, respectively. These allocations depend on the state through wealth W_{T-1}.

After substituting the results to U_{T-1}, U_{T-1} is:

$$\begin{aligned} &= \left\{C_{T-1}^{\phi} + \beta[p(B_{T-1}R + \Pi_{T-1}S_{T-1}u)^{\alpha} + q(B_{T-1}R + \Pi_{T-1}S_{T-1}d)^{\alpha}]^{\phi/\alpha}\right\}^{1/\phi} \\ &= \left\{\begin{array}{c} (c_{T-1}W_{T-1})^{\phi} + \beta[p(W_{T-1}(b_{T-1}R + u\pi_{T-1}))^{\alpha} \\ +q(W_{T-1}(b_{T-1}R + d\pi_{T-1}))^{\alpha}]^{\phi/\alpha} \end{array}\right\}^{1/\phi} \\ &= W_{T-1}\left\{c_{T-1}^{\phi} + \beta[p(b_{T-1}R + u\pi_{T-1})^{\alpha} + q(b_{T-1}R + d\pi_{T-1})^{\alpha}]^{\phi/\alpha}\right\}^{1/\phi} \end{aligned}$$

Let's define A_{T-1} as:

$$A_{T-1} = \left\{c_{T-1}^{\phi} + \beta[p(b_{T-1}R + u\pi_{T-1})^{\alpha} + q(b_{T-1}R + d\pi_{T-1})^{\alpha}]^{\phi/\alpha}\right\}^{1/\phi}$$

A_{T-1} , also does not depend on the state. Rewriting the utility as $U_{T-1} = W_{T-1}A_{T-1}$, shows that the recursive utility is linear in wealth. It is equal to:

$$\begin{aligned} W^u_{T-1} &= B_{T-2}R + \Pi_{T-2}S^u_{T-1} = B_{T-2}R + \Pi_{T-2}S_{T-2}u \\ W^d_{T-1} &= B_{T-2}R + \Pi_{T-2}S^d_{T-1} = B_{T-2}R + \Pi_{T-2}S_{T-2}d \end{aligned}$$

in the up and down states, respectively.

By (2.8), $U_T = W_T$, so we can set $A_T = 1$. At the end of the backward maximization, A_0 will give the initial utility for the normalized level of wealth ($W_0 = 1$), because $U_0 = A_0W_0$ implies that $U_0 = A_0$ when $W_0 = 1$.

In order to see the backward recursive solution for the maximization problem, it is crucial to look at time $T-2$ setup. Now, if we write the maximization problem for $t = T-2$:

$$\max_{\{B_{T-2}, C_{T-2}, \Pi_{T-2}\}} \{C^{\phi}_{T-2} + \beta[p(U^u_{T-1})^{\alpha} + q(U^d_{T-1})^{\alpha}]^{\phi/\alpha}\}^{1/\phi}$$

subject to :

$$B_{T-2} + C_{T-2} + \Pi_{T-2}S_T = W_{T-2} \tag{2.16}$$

If we substitute U^u_{T-1} and U^d_{T-1} to our maximization problem, we get:

$$\max_{\{B_{T-2}, C_{T-2}, \Pi_{T-2}\}} \{C^{\phi}_{T-2} + \beta[p(W^u_{T-1}A_{T-1})^{\alpha} + q(W^d_{T-1}A_{T-1})^{\alpha}]^{\phi/\alpha}\}^{1/\phi}$$

subject to:

$$B_{T-2} + C_{T-2} + \Pi_{T-2}S_T = W_{T-2} \tag{2.17}$$

After the substitution of W^u_{T-1}, $W^d_{T-1}$and the budget constraint, the problem becomes:

$$\max_{\{B_{T-2},\Pi_{T-2}\}}\{(w_{T-2}-B_{T-2}-\Pi_{T-2}S_{T-2})^{\phi}+ \tag{2.18}$$
$$\beta A^{\phi}_{T-1}[p(B_{T-2}R+\Pi_{T-2}S_{T-2}u)^{\alpha}+q(B_{T-2}R+\Pi_{T-2}S_{T-2}d)^{\alpha}]^{\phi/\alpha}\}^{1/\phi}$$

Let's define $\beta_{T-1}=\beta A^{\phi}_{T-1}$ ($\beta_T=\beta$ by definition of A_T), then the problem becomes:

$$\max_{\{B_{T-2},\Pi_{T-2}\}}\{(w_{T-2}-B_{T-2}-\Pi_{T-2}S_{T-2})^{\phi}+ \tag{2.19}$$
$$\beta_{T-1}[p(B_{T-2}R+\Pi_{T-2}S_{T-2}u)^{\alpha}+q(B_{T-2}R+\Pi_{T-2}S_{T-2}d)^{\alpha}]^{\phi/\alpha}\}^{1/\phi}$$

The maximization problem (2.19) at time $T-2$, is the same as (2.12) except the time dependence and β in equation (2.12) is replaced by β_{T-1}. So, we can write a general recursive solution for time t as following:

The problem :

$$\max U_{t-1}=[C^{\phi}_{t-1}+\beta(E_{t-1}U^{\alpha}_t)^{\phi/\alpha}]^{1/\phi}$$

subject to:

$$B_{t-1}+C_{t-1}+\Pi_{t-1}S_{t-1}=W_{t-1} \tag{2.20}$$

Step 1.

Find k, l, and X, which are time and state independent.

$$\begin{aligned} k &= [\frac{q(R-d)}{p(u-R)}]^{\frac{1}{\alpha-1}} \\ l &= \frac{R(k-1)}{u-kd} \\ X &= \{[p(lu+R)^{\alpha}+q(ld+R)^{\alpha}]^{\frac{\phi-\alpha}{\alpha}}(pu(lu+R)^{\alpha-1}+qd(ld+R)^{\alpha-1})\}^{\frac{1}{\phi-1}} \end{aligned}$$

Step 2.

For $t \leq T$, find $m_{t-1},\ b_{t-1},\ c_{t-1}$, and π_{t-1}.

$$
\begin{aligned}
m_{t-1} &= \beta_t^{\frac{1}{\phi-1}} X \\
b_{t-1} &= \frac{1}{m_{t-1}+l+1} \\
c_{t-1} &= \frac{m_{t-1}}{m_{t-1}+l+1} \\
\pi_{t-1} &= \frac{l}{m_{t-1}+l+1}
\end{aligned}
$$

where $\beta_T = \beta$, $b_T = 0$, $c_T = 1$, and $\pi_T = 0$ by assumption.

Step 3.

Find

$$A_{t-1} = \left\{ c_{t-1}^{\phi} + \beta_t [p(b_{t-1}R + u\pi_{t-1})^{\alpha} + q(b_{t-1}R + d\pi_{t-1})^{\alpha}]^{\phi/\alpha} \right\}^{1/\phi}$$

where it is assumed that $A_T = 1$.

Step 4.

Update beta, i.e. find $\beta_{t-1} = \beta A_{t-1}^{\phi}$. Then, decrease t by 1, i.e. $t = t-1$ and go to Step 2, until $t = 1$. If $t = 1$, move to Step 5. Hence, we have the ratio coefficients.

In order to find the dollar amounts allocated to consumption and investment for any path starting from $t = 0$, we continue with Step 5.

Step 5.

$$
\begin{aligned}
B_t &= b_t W_t \\
C_t &= c_t W_t \\
\Pi_t S_t &= \pi_t W_t
\end{aligned}
$$

where W_0 is the initial wealth and $W_t = B_{t-1}R + \Pi_{t-1}S_t$ for $t \geq 1$.

Definition 1. *$\omega_{0,t}$ is the weight of bond holdings, and $\omega_{1,t}$ is the weight of stock holdings in total savings. So, $\omega_{0,t} + \omega_{1,t} = 1$.*

Proposition 1. *$\omega_{0,t}$ and $\omega_{1,t}$ do not depend on the elasticity of intertemporal substitution parameter, ϕ, and subjective time discount parameter, β. Also, they are constant in time.*

Proof.

$$
\begin{aligned}
\omega_{1,t} &= \frac{\Pi_{T-1}S_{T-1}}{B_{T-1} + \Pi_{T-1}S_{T-1}} \\
&= \frac{l}{l+1} \\
&= \frac{R(k-1)}{k(R-d) + (u-R)} \\
&= \frac{R[(\frac{q(R-d)}{p(u-R)})^{\frac{1}{\alpha-1}} - 1]}{(R-d)[\frac{q(R-d)}{p(u-R)}]^{\frac{1}{\alpha-1}} + (u-R)}
\end{aligned}
$$

We clearly see that it does not depend on ϕ, β, and time t. Similarly, the same result is true for $\omega_{0,t}$, because, $\omega_{0,t} = 1 - \omega_{1,t}$. □

Now, in the following subsections of 2.1, the effects of the parameters will be studied numerically together with the notion of interchangeability of beta and phi.

2.1.1 The Effects of Alpha

Alpha measures the relative risk aversion of the agent, i.e. $RRA = 1-\alpha$. So, decreasing alpha increases the RRA. Among the three parameters in the model, alpha, phi and beta; the only one that affects the composition of savings, between stocks and bonds, is alpha. However, alpha does not affect how much the investor consumes, significantly.

We look at the percentage changes in bond holdings, stock holdings and consumption stream for three different values of alpha (-2,-3,-4), for different values of phi and beta.

· The decrease in the initial utility Uzero is insignificant , which is less than 0.2 percent. So the individuals risk aversion level does not affect the initial utility. If we consider the initial utility as a level of satisfaction, there is no significant difference between the satisfaction of risk averse agents and risk lover agents according to recursive type of preferences.

Alpha = -2 to -3 : Uzero decreases by at most 0.2 percent

Alpha = -3 to -4 : Uzero decreases by at most 0.1 percent

· There is significant increase in the bond holdings;

Alpha = -2 to -3 : bond holdings increase around 13 percent

Alpha = -3 to -4 : bond holdings increase around 7 percent

· There is significant decrease in the stock holdings;

Alpha = -2 to -3 : stock holdings decrease around 25 percent

Alpha = -3 to -4 : stock holdings decrease around 20 percent

· According to the analytical results, consumption depends on risk aversion parameter alpha, which is consistent with the results of [5]. However, this dependence is insignificant. The increase in the consumption level is less than 0.1 percent for an increase in risk aversion by one.

Alpha = -2 to -3 : consumption increases around 0.09 percent

Alpha = -3 to -4 : consumption increases around 0.06 percent

As a result; as alpha changes there is no significant change in the amount of investment as a percentage of wealth, but there is significant change in the composition of investment (bond or stock) no matter what the level of phi and beta is. Also, the percentage of stock holdings in total savings decreases as alpha increases.

34 % of total saving is stock holdings when alpha is -2

26 % of total saving is stock holdings when alpha is -3

21 % of total saving is stock holdings when alpha is -4

These numbers are almost the same for different values of phi and beta.

2.1.2 The Effects of Beta

Beta is the subjective time discount factor. As beta increases, the agent becomes more patient. Beta has a significant effect on savings and consumption. However, it does not have an effect on the composition of the savings which can be seen from the previous proposition. We looked at the percentage change in bond holdings, stock holdings and consumption for three different values of beta; 0.90, 0.95, and 0.99 when phi and alpha takes different values. The effects of change in beta from 0.90 to 0.95 and from 0.95 to 0.99 are summarized as below:

· There is significant increase in the initial utility,Uzero.

This effect is same for different values of alpha and fixed phi, but the effect of beta decreases as phi increases.

0.90 to 0.95 : Uzero increases by 104 % when phi = 0.3

0.95 to 0.99 : Uzero increases by 88 % when phi = 0.3

0.90 to 0.95 : Uzero increases by 48 % when phi = 0.5

0.95 to 0.99 : Uzero increases by 45 % when phi = 0.5

· There is small increase in the bond holdings when beta changes.

This effect is same for different values of alpha and fixed phi, but the effect of beta is higher for higher values of phi.

0.90 to 0.95 : bond holdings increase on average 4.8 % when phi = 0.3

0.95 to 0.99 : bond holdings increase on average 3.2 % when phi = 0.3

0.90 to 0.95 : bond holdings increase on average 7.1 % when phi = 0.5

0.95 to 0.99 : bond holdings increase on average 4.7 % when phi = 0.5

· The effect of beta on the stock holdings is same as the effect on bond holdings given above. By the previous proposition and this result, we conclude that the composition of savings does not depend on beta. In other words, the percentage of stock holdings and bond holdings in total savings do not change with beta.

· There is significant decrease in consumption when beta increases. This is intuitive, since as beta increases, the agent becomes more patient. Thus the agent saves more and consumes less to be able to consume more in the future. This effect is same for different values of alpha and fixed phi, but the effect of beta is higher for higher values of phi.

0.90 to 0.95 : consumption decreases on average 15 % when phi = 0.3

0.95 to 0.99 : consumption decreases on average 13 % when phi = 0.3

0.90 to 0.95 : consumption decreases on average 18 % when phi = 0.5

0.95 to 0.99 : consumption decreases on average 17 % when phi = 0.5

As a result; as beta changes there is significant decrease in consumption as a percentage of wealth, and there is increase in risky and risk free investment no matter what the level of alpha is. But the percentage of bond holdings and stock holdings in total savings do not change.

2.1.3 The Effects of Phi

Phi measure the elasticity of intertemporal substitution (EIS). More precisely, EIS = $1/(1-\phi)$. So, EIS increases with ϕ. The effects of change in phi from 0.1 to 0.3 and from 0.3 to 0.5 are summarized below.

· There is significant decrease in the initial utility,Uzero. This effect is same for different values of alpha and phi

0.1 to 0.3 : Uzero decreases by 99 %

0.3 to 0.5 : Uzero decreases by 94 %

· There is small change in the bond holdings when phi increases. This effect is same for different values of alpha and fixed beta, but the effect of phi is slightly higher for low values of beta.

0.1 to 0.3 : bond holdings decrease on average 1.9 % when beta = 0.90

0.3 to 0.5 : bond holdings decrease on average 3.6 % when beta = 0.90

0.1 to 0.3 : bond holdings decrease on average 0.7 % when beta = 0.95

0.3 to 0.5 : bond holdings decrease on average 1.3 % when beta = 0.95

However, the bond holdings increase around 0.03 % as phi increases when beta is 0.99. This increase can be a decrease for very small changes in the interest rate. So, we can say that the effect of phi is negligible on bond holding when beta is high.

· The effects of phi on stock holdings is the same as the effects on bond holdings given above. From this result, we conclude that the composition of savings does not depend on phi, i.e., the percentage of stock holdings in total savings does not change with phi.

· There is increase in consumption when phi increases. This effect is same for different values of alpha and fixed beta, but the effect of phi is higher for smaller values of beta.

0.1 to 0.3 : consumption increases on average 6.5 % when beta = 0.90

0.3 to 0.5 : consumption increases on average 11 % when beta = 0.90

0.1 to 0.3 : consumption increases on average 3.2 % when beta = 0.95

0.3 to 0.5 : consumption increases on average 5.5 % when beta = 0.95

However, the consumption decreases around 0.1 % as phi increases when beta is 0.99. But this decrease can be an increase for very small changes in the interest rate. So we can say that the effect of phi is negligible on consumption when beta is high.

The effect of phi on total savings is the opposite of the effect on consumption, but much less in percentage. Also, the composition of total savings does not change at all as phi changes. For example, the percentage of stock holdings in total savings is 34% when alpha=-2 and 26% when alpha =-3, for any value of phi and beta.

As a result; as phi changes there is change in consumption as a percentage of wealth, and there is small change in investment (bond or stock) no matter what the level of alpha is. However, the percentage of wealth in stock holdings (and bond holdings) in total savings does not change.

The initial utility is greater than W_0, when phi is positive and it is less than W_0 when phi is negative. Since the agent gets a non-optimal initial utility of W_0 if he consumes all the initial endowment at time $t = 0$, we only considered positive values of phi.

2.1.4 Interchangeability of Beta and Phi

Definition 2. *(Interchangeability) What we mean with the interchangeability of beta and phi is the following: Fix beta (at say 0.95) and phi (at say 0.5). Then we find the optimal allocation between consumption and savings for other parameter values. Next, we change beta to say 0.97 and find the optimal allocation again for all the other* *parameter values. We know that optimal consumption will now be lower (more savings, since the investor is more patient). Say optimal consumption ratio for the first beta (0.95) is c_1 and for the second beta (0.97) is c_2, with $c_2 < c_1$. With interchangeability, we mean that there will be another value of phi different from 0.5 such that for the original beta, (beta = 0.95) and all the other parameter values, the optimal consumption ratio is c_2.*

That is, if we can achieve the same effects on the optimal consumption decision of the investor by changing beta or by changing phi, we say that beta and phi are interchangeable. There are two cases to be considered; obtaining the effect of change in phi by a change in beta and vice versa.

Proposition 2. *The subjective time discount factor β and the elasticity of intertemporal substitution parameter ϕ are not interchangeable analytically.*

Proof. Done by using backward induction.

The definition above can be written in the following way using our notation.

Let $c_t^1(\beta, \phi)$ be the optimal consumption ratio for a fixed beta and phi at time t.

Let $c_t^2(\beta_1, \phi)$ be the optimal consumption ratio when only the beta is changed.

We say that beta and phi are interchangeable if there is a ϕ_1 such that:

$$c_t^2(\beta_1, \phi) = c_t^3(\beta, \phi_1) \;\; \forall t \in \{1, 2, ..., T\}$$

The optimal consumption ratio depends on m_t and l, but l does not depend on beta and phi. So,

$$c_t^2(\beta_1, \phi) = c_t^3(\beta, \phi_1)$$

is true if:

$$m_t^2(\beta_1, \phi) = m_t^3(\beta, \phi_1) \;\; \forall t \in \{1, 2, ..., T\}.$$

Here, ϕ, β, and β_1 are given constants, and ϕ_1 is the only unknown in the equation. By definition,

$$m_t^2(\beta_1, \phi) = m_t^3(\beta, \phi_1)$$

is true for $t = T$.

When $t = T - 1$, we can find ϕ_1 in terms of ϕ, β, and β_1 and the other constants. Similarly, we can find ϕ_1 in terms of ϕ, β, and β_1 and the other constants at time $t = T - 2$. In order to say that beta and phi are interchangeable, the solutions of ϕ_1 at time $t = T - 1$, and $t = T - 2$ must be the same. However, these two results are different. The rest of the details are given in the appendix. □

On the other hand, if we look at some numerical results, we see that the absolute difference between consumptions, $\mid c_t^2(\beta_1, \phi) - c_t^3(\beta, \phi_1) \mid$ is sufficiently small $\forall\ t \in \{1, 2, ..., T\}$ for reasonable levels of beta and phi. This result implies that beta and phi are interchangeable numerically.

Under the certain lifetime horizon, beta can be interchanged by phi for big changes in beta when beta is small and for small changes in beta when beta is large up to a high level of accuracy. Error[2] in the following tables is defined as the minimized maximum absolute difference between the consumptions $\mathbf{c_2}$ and $\bar{c_2}$, i.e. error $= \max \mid \mathbf{c_2} - \bar{c_2} \mid$. The following tables exhibit selected results.

Also, phi can be interchanged by beta for a much larger range than beta.

[2]Error can be made much smaller, but beta does not chance much

Table 2.1: Interchangeability of Beta by Phi

	$\mathbf{c}1$	$\mathbf{c}2$	$\bar{c}2$	$Error$
(ϕ, β)	$(0.4, 0.95)$	$(0.4, 0.97)$	$(0.0029, 0.95)$	0.0043
(ϕ, β)	$(0.3, 0.95)$	$(0.3, 0.97)$	$(0.0029, 0.95)$	0.0066
(ϕ, β)	$(0.4, 0.80)$	$(0.4, 0.85)$	$(0.1589, 0.80)$	$1.46e-7$
(ϕ, β)	$(0.3, 0.80)$	$(0.3, 0.80)$	$(0.0187, 0.80)$	$2.98e-7$

Table 2.2: Interchangeability of Phi by Beta

	$\mathbf{c}1$	$\mathbf{c}2$	$\bar{c}2$	$Error$
(ϕ, β)	$(0.4, 0.95)$	$(0.5, 0.95)$	$(0.4, 0.94231)$	$2.64e-7$
(ϕ, β)	$(0.4, 0.85)$	$(0.5, 0.85)$	$(0.4, 0.82472)$	$8.54e-7$
(ϕ, β)	$(0.1, 0.95)$	$(0.4, 0.95)$	$(0.1, 0.93147)$	$9.53e-7$
(ϕ, β)	$(0.1, 0.85)$	$(0.4, 0.85)$	$(0.4, 0.78823)$	$9.69e-7$

2.2 Uncertain Horizon

The recursive utility for an uncertain lifetime for any $t \in \{1, 2, ..., T\}$ is defined as:

$$U_t = [C_t^{\phi} + \beta(\rho_{t+1}(EU_{t+1}^{\alpha})^{\phi/\alpha} + (1-\rho_{t+1})B(W_{t+1}))]^{1/\phi}$$

subject to:

$$B_t + C_t + \Pi_t S_t = W_t \tag{2.21}$$

Similarly, for simplicity we take the bequest as zero, so the above equation (2.21) with the budget constraint becomes:

$$U_t = [C_t^{\phi} + \beta\rho_{t+1}(EU_{t+1}^{\alpha})^{\phi/\alpha}]^{1/\phi}; \quad 0 \neq \phi \leq 1,\ 0 \neq \alpha \leq 1$$

subject to:

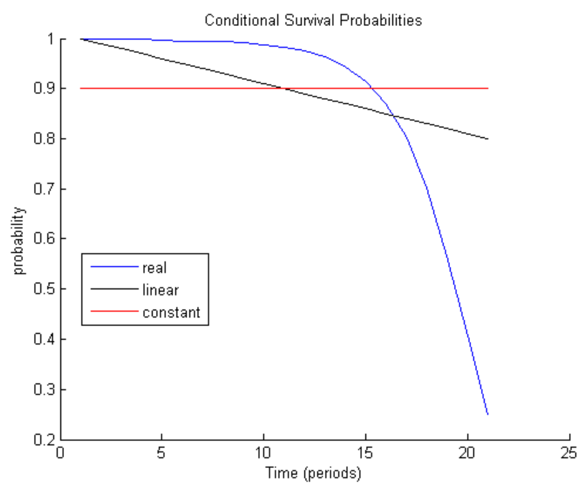
$$B_t + C_t + \Pi_t S_t = W_t = B_{t-1}(1+r) + \Pi_{t-1} S_t \tag{2.22}$$

where ρ_t is the conditional survival probability. More precisely:

$\rho_t =$ Prob(the agent is alive at time t$|$the agent is alive at time t-1$)$.

When $\rho_t = 1$ for all t, we get the results of Section 2.1 and by the definition of the model, $\rho_t = 0$ for $t > T$. The following Figure 2.1 displays three conditional lifetime probability scenarios; constant conditional probability, $\rho_t = 0.9$ for all t, linearly decaying conditional probability and the conditional lifetime probability, taken from US data (see [1]).

Figure 2.1: Conditional Survival Probabilities



2.2.1 Analytical Solution

By using similar steps in Section 2.1, the analytical solution for the uncertain lifetime horizon model can be obtained by the following recursive steps.

The problem:

$$\max_{\{B_{t-1},C_{t-1},\Pi_{t-1}\}} U_{t-1} = [C_{t-1}^{\phi} + \beta\rho_t(EU_t^{\alpha})^{\phi/\alpha}]^{1/\phi}$$

subject to:

$$B_{t-1} + C_{t-1} + \Pi_{t-1}S_{t-1} = W_{t-1}$$

Step 1. Find k, l, and X, which are time and state independent.

$$\begin{aligned} k &= [\frac{q(R-d)}{p(u-R)}]^{\frac{1}{\alpha-1}} \\ l &= \frac{R(k-1)}{u-kd} \\ X &= \{[p(lu+R)^{\alpha} + q(ld+R)^{\alpha}]^{\frac{\phi-\alpha}{\alpha}}(pu(lu+R)^{\alpha-1} + qd(ld+R)^{\alpha-1})\}^{\frac{1}{\phi-1}} \end{aligned}$$

Step 2. For $t \leq T$:

Find $m_{t-1},\ b_{t-1},\ c_{t-1}$, and π_{t-1}.

$$\begin{aligned} m_{t-1} &= \beta_t^{\frac{1}{\phi-1}} X \\ b_{t-1} &= \frac{1}{m_{t-1}+l+1} \\ c_{t-1} &= \frac{m_{t-1}}{m_{t-1}+l+1} \\ \pi_{t-1} &= \frac{l}{m_{t-1}+l+1} \end{aligned}$$

where $\beta_T = \rho_T\beta$, $b_T = 0$, $c_T = 1$, and $\pi_T = 0$ by assumption.

Step 3. Find

$$A_{t-1} = \left\{c_{t-1}^{\phi} + \beta_t[p(b_{t-1}R + u\pi_{t-1})^{\alpha} + q(b_{t-1}R + d\pi_{t-1})^{\alpha}]^{\phi/\alpha}\right\}^{1/\phi}$$

By assumption $A_T = 1$.

Step 4.

Update beta, i.e. find $\beta_{t-1} = \beta\rho_{t-1}A_{t-1}^{\phi}$. Then, decrease t by 1, i.e. $t = t-1$ and go to Step 2, until $t = 1$. If $t = 1$, move to Step 5. Hence, we have the ratio coefficients.

Similarly, in order to find the dollar amounts allocated to consumption and investment for any path starting from $t = 0$, we continue with Step 5.

Step 5.

$$\begin{aligned} B_t &= b_t W_t \\ C_t &= c_t W_t \\ \Pi_t S_t &= \pi_t W_t \end{aligned}$$

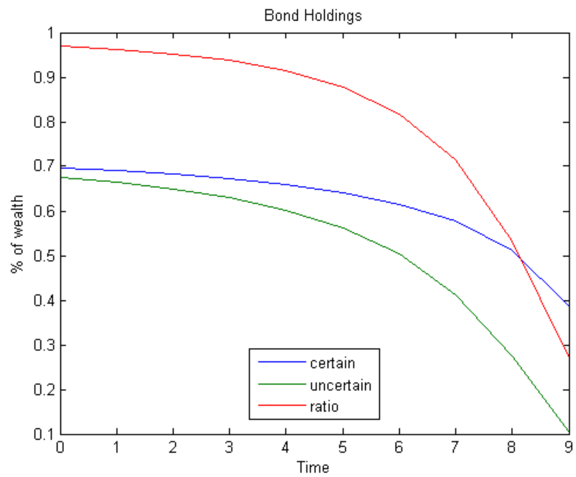
where W_0 is the initial wealth and $W_t = B_{t-1}R + \Pi_{t-1}S_t$ for $t \geq 1$.

2.2.2 The Effects of Conditional Survival Probability

In the following sections, the effects of uncertainty on the choice variables are analyzed. The figures, given below, contain three graphs: one for the certain lifetime model, one for the uncertain lifetime model, and one for their ratio (Uncertain/Certain). The conditional probability we used for the lifetime is the one from real US data[3].

Risk aversion parameter is taken as 3 (alpha = -2), beta = 0.95, and phi = 0.4.

Figure 2.2: Bond Holdings



Bond Holdings

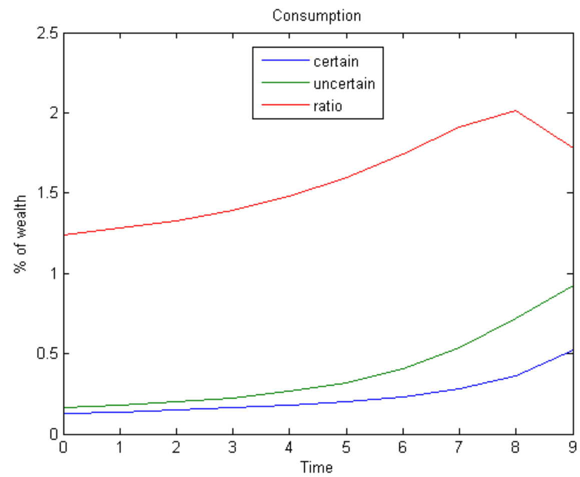
Adding uncertainty to the model affects the bond holdings significantly. As we see in the Figure 2.2, there is big difference towards the end of the lifetime. On average, bond holdings decrease by 25 percent with uncertainty. This is intuitive, because if the agent is not sure whether he will live or die tomorrow, he has incentive to consume more and save less today.

Consumption

Uncertainty affects the lifetime consumption significantly. As displayed in the Figure 2.3, the ratio of uncertain lifetime consumption to the certain lifetime consumption is more than 1.50 after the middle of the lifetime horizon. On average, consumption increases by 55 percent with the uncertainty.

[3]Constant conditional survival probability has the effects same as when beta is multiplied by a factor equal to the constant value of the survival probability.

Figure 2.3: Consumption



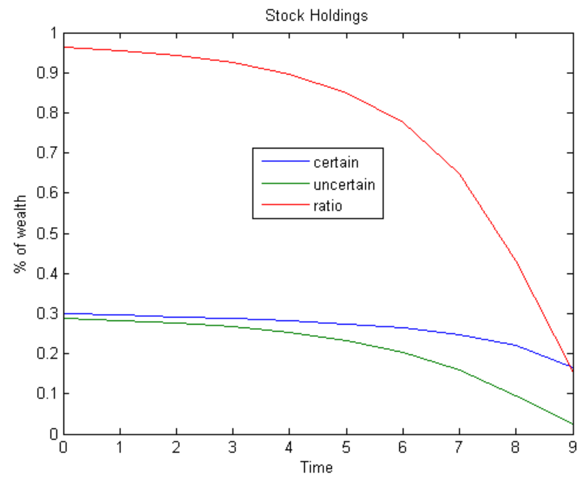
Stock Holdings

The effect of uncertainty on stock holdings is the same as it is on the bond holdings. As we see in the Figure 2.4, there is big difference towards the end of lifetime. Like bond holdings, on average, stock holdings decrease by 25 percent with the uncertainty. Hence, on average the total savings in bonds and stocks decrease by 25 percent which also shows that adding uncertainty to the model does not change the composition of savings between risky and risk free securities.

Adding uncertainty to the lifetime, partially explains low level of savings and high levels of consumption.

Now, the analysis of the effects of parameters on the choice variables are given in the following sections.

Figure 2.4: Stock Holdings



2.2.3 The Effects of Alpha

The effects of alpha in an uncertain lifetime are almost same as the effects in certain lifetime given in Section 2.1.2 for different choices of survival probabilities. The only difference is in consumption. The effect of alpha becomes even smaller with the uncertainty.

2.2.4 The Effects of Beta

Similar to certain lifetime horizon, beta has a significant effect on how much saved and how much consumed, and it does not have an effect on the composition of the savings when the lifetime is uncertain which can be seen from the ratio of savings in stocks to total savings.

$$\frac{\Pi_{T-1}S_{T-1}}{B_{T-1}+\Pi_{T-1}S_{T-1}} = \frac{l}{l+1}$$

Moreover, the above ratio is equal to the ratio in the certain life time model, i.e., adding uncertainty to lifetime does not change the effect of beta on the composition of savings.

However, initial utility Uzero becomes less sensitive to the changes in beta with the addition of uncertainty to the lifetime.

For example, if we want to compare some numerical values for the average effect of change in beta from 0.90 to 0.95 for the two models, one with certain lifetime and one with uncertain lifetime, (in percentage) a summary can be seen below.

Table 2.3: Effects of Beta

	$Certain\ Life$		$Uncertain\ Life$	
	$\phi = 0.3$	$\phi = 0.5$	$\phi = 0.3$	$\phi = 0.5$
$\Delta Uzero$	104	48	73	34
$\Delta Bond, \Delta Stock$	4.8	7.1	6.5	9.5
$\Delta Consumption$	-15	-18	-10.5	-12.8

As a result, as beta changes there is significant decrease in consumption as a percentage of wealth, and this decrease is smaller for the uncertain lifetime model. There is increase in the investment (bond or stock) and this increase is higher for the uncertain lifetime model no matter what the level of alpha is. It is also interesting to note that the effect of change in beta on bond and stock holdings is decreasing towards the end of the lifetime for the certain lifetime model; however, it is increasing for the uncertain lifetime model.

2.2.5 The Effects of Phi

Adding uncertainty to the model changes the effects of phi on savings and consumption significantly. For example, if we compare some numerical values for the average effect of change (in percentage), in phi from 0.1 to 0.3 for the two models, one with certain lifetime and one with uncertain lifetime, we obtain the following results.

Table 2.4: Effects of Phi

	$Certain\ Life$		$Uncertain\ Life$	
	$\beta = 0.90$	$\beta = 0.95$	$\beta = 0.90$	$\beta = 0.95$
$\Delta Bond, \Delta Stock$	-1.9	-0.7	-10	-8.9
$\Delta Consumption$	6.5	3.2	9.6	7.8

There is significant difference between the two models in the effect of phi on bond and stock holdings. Adding uncertainty to the model increases the effect of phi in absolute value towards the end of lifetime, however, the same effect is almost uniform for the certainty case throughout the lifetime.

2.2.6 Interchangeability of Beta and Phi

Corollary 1. *Time discount factor β and elasticity of intertemporal substitution parameter ϕ are not interchangeable analytically when the agent has stochastic lifetime.*

Proof. If we consider the subjective discount factor as $\beta'_t = \beta\rho_{t+1}$, the result follows from the previous proposition. □

Now, for the interchangeability of beta and phi, we check he numerical results. We also look at the percentage error which can be written as:

$$\%error = \max\{\mid \mathbf{c}2 - \bar{\mathbf{c}}2 \mid ./\mathbf{c}2\}$$

Beta cannot be interchanged by phi or vice versa for a small or big range. Numerical results are presented in Table 2.5 and 2.6.

By comparing the tables of Section 2.1.5 (Table 2.1 and 2.2) and Section 2.2.6 (Table 2.5 and 2.6), adding lifetime uncertainty to the model decreases the accuracy

Table 2.5: Interchangeability of Beta by Phi, Uncertain Horizon

	$\mathbf{c}1$	$\mathbf{c}2$	$\bar{c2}$	$Error$	$\%\ Error$
(ϕ, β)	$(0.4, 0.95)$	$(0.4, 0.97)$	$(0.3458, 0.95)$	0.0133	7.10
(ϕ, β)	$(0.3, 0.95)$	$(0.3, 0.97)$	$(0.2479, 0.95)$	0.0118	5.37
(ϕ, β)	$(0.4, 0.80)$	$(0.4, 0.85)$	$(0.2870, 0.80)$	0.0248	6.11
(ϕ, β)	$(0.3, 0.80)$	$(0.3, 0.80)$	$(0.1811, 0.80)$	0.0237	7.56

Table 2.6: Interchangeability of Phi by Beta, Uncertain Horizon

	$\mathbf{c}1$	$\mathbf{c}2$	$\bar{c2}$	$Error$	$\%\ Error$
(ϕ, β)	$(0.4, 0.95)$	$(0.5, 0.95)$	$(0.4, 0.9096)$	0.0291	10.20
(ϕ, β)	$(0.4, 0.85)$	$(0.5, 0.85)$	$(0.4, 0.8008)$	0.0267	04.43
(ϕ, β)	$(0.1, 0.95)$	$(0.4, 0.95)$	$(0.1, 0.8327)$	0.0636	38.94
(ϕ, β)	$(0.1, 0.85)$	$(0.4, 0.85)$	$(0.4, 0.7187)$	0.0571	12.01

in the interchangeability of beta and phi. We cannot clearly state that beta and phi are interchangeable, anymore.

It is also interesting to note that in the uncertain lifetime model, phi do not decrease as much as before to get the same effect, and beta decreases more to compensate the effect in consumption.

Chapter 3

The Model in Incomplete Markets

In this chapter, we generalize the number of states that the stock can take.

3.1 Certain Horizon

3.1.1 Solution

As in Section 2.1, the utility function for any $t \in \{1, 2, ..., T\}$ is:

$$U_{t-1} = [C_{t-1}^{\phi} + \beta(E_{t-1}U_t^{\alpha})^{\phi/\alpha}]^{1/\phi};\ \ 0 \neq \phi \leq 1,\ 0 \neq \alpha \leq 1$$

subject to:

$$B_{t-1} + C_{t-1} + \Pi_{t-1}S_{t-1} = W_{t-1} \tag{3.1}$$

Now, instead of having only two states, there are n ≥ 3 states for the stock price.

$$\begin{array}{ll} \nearrow\ S_t^{u_1} = S_{t-1}u_1 & \text{with probability } p_1 \\ S_{t-1} \rightarrow S_t^{u_2} = S_{t-1}u_2 & \text{with probability } p_2 \\ \quad . & \\ \quad . & \\ \searrow\ S_t^{u_n} = S_{t-1}u_n & \text{with probability } p_n \end{array}$$

where $\sum_{i=1}^{n} p_i = 1$, and u_j's are the returns of the stock in different states.

We can now say that the markets are incomplete, because the number of states is larger than the number of independent securities.

The objective becomes solving the following maximization problem (3.2) after writing the inner expectation explicitly:

$$\max_{\{B_{T-1},C_{T-1},\Pi_{T-1}\}} U_{T-1} = \{C_{T-1}^{\phi} + \beta[\sum_{i=1}^{n} p_i (U_T^{u_i})^{\alpha}]^{\phi/\alpha}\}^{1/\phi}$$

subject to:

$$B_{T-1} + C_{T-1} + \Pi_{T-1} S_{T-1} = W_{T-1} \tag{3.2}$$

Similarly, we will solve the problem starting from the period $t = T - 1$.

$$\max_{\{B_{T-1},C_{T-1},\Pi_{T-1}\}} U_{T-1} = \{C_{T-1}^{\phi} + \beta[\sum_{i=1}^{n} p_i (B_{T-1}R + \Pi_{T-1} S_{T-1} u_i)^{\alpha}]^{\phi/\alpha}\}^{1/\phi}$$

subject to:

$$B_{T-1} + C_{T-1} + \Pi_{T-1} S_{T-1} = W_{T-1} \tag{3.3}$$

Now, if we solve the budget constraint for C_{T-1} and substitute it into the above maximization problem, we get:

$$\begin{aligned} \max_{\{B_{T-1},\Pi_{T-1}\}} U_{T-1} \{ \quad & (w_{T-1} - B_{T-1} - \Pi_{T-1} S_{T-1})^{\phi} \qquad (3.4) \\ & + \beta[\sum_{i=1}^{n} p_i (B_{T-1}R + \Pi_{T-1} S_{T-1} u_i)^{\alpha}]^{\phi/\alpha}\}^{1/\phi} \end{aligned}$$

In order to maximize U_{T-1}, again, we need to solve the following system of equations simultaneously.

$$\begin{aligned} \partial U_{T-1} / \partial B_{T-1} &= 0 \\ \partial U_{T-1} / \partial \Pi_{T-1} &= 0 \qquad (3.5) \end{aligned}$$

Equivalently:

$$\begin{aligned}\beta[\sum_{i=1}^{n} p_i(BR + \Pi S u_i)^{\alpha}]^{\frac{\phi-\alpha}{\alpha}} \sum_{i=1}^{n} p_i u_i (BR + \Pi S u_i)^{\alpha-1} &= (W - B - \Pi S)^{\phi-1} \\ \sum_{i=1}^{n} p_i(u_i - R)(BR + \Pi S u_i)^{\alpha-1} &= 0 \end{aligned} \quad (3.6)$$

where the variables B, Π, S and W have time $T-1$ dependence.

The above system, which is a nonlinear system of two equations and two unknowns, can be solved numerically in general and explicitly when $n = 2$.

Proposition 3. *The solution* (B_{T-1}, Π_{T-1}) *to the system (3.6) has the following form*[1].

$$\begin{aligned} B_{T-1} &= b_{T-1} W_{T-1} \\ \Pi_{T-1} S_{T-1} &= \pi_{T-1} W_{T-1} \end{aligned} \quad (3.7)$$

Proof. Substituting the solution 3.7 the the system 3.6 gives the desired result. □

Now, by using the above proposition, let's assume that the solution to the system (3.6) is as in (3.7), and set

$$\begin{aligned} b_{T-1} &= b \\ \pi_{T-1} &= \pi \end{aligned}$$

[1]This statement can be generalized for any time t after replacing β by β_t, where $\beta_t = \beta A_t^{\phi}$

which will be the numerical solutions of the following system (3.8).

$$
\begin{aligned}
\beta[\sum_{i=1}^{n} p_i(bR+\pi u_i)^{\alpha}]^{(\phi-\alpha)/\alpha}[\sum_{i=1}^{n} p_i u_i(bR+\pi u_i)^{\alpha-1}] &= (1-b-\pi)^{\phi-1} \\
\sum_{i=1}^{n} p_i(u_i-R)(bR+\pi u_i)^{\alpha-1} &= 0
\end{aligned}
\tag{3.8}
$$

By using the budget constraint, we get:

$$
C_{T-1} = (1-b_{T-1}-\pi_{T-1})W_{T-1} := c_{T-1}W_{T-1} \tag{3.9}
$$

If we summarize, the solution, in terms of wealth is:

$$
\begin{aligned}
B_{T-1} &= b_{T-1}W_{T-1} \\
C_{T-1} &= c_{T-1}W_{T-1} \\
\Pi_{T-1}S_{T-1} &= \pi_{T-1}W_{T-1}
\end{aligned}
\tag{3.10}
$$

Here, again the lower case letters, b_{T-1}, c_{T-1}, and π_{T-1} which are called as the ratio coefficients, do not depend on the state. However, B_{T-1}, C_{T-1}, and $\Pi_{T-1}S_{T-1}$ depend on the state through wealth W_{T-1} as before.

Now, let's define

$$
A_{T-1} = \{c_{T-1}^{\phi} + \beta_T[\sum_{i=1}^{n} p_i(b_{T-1}R+\pi_{T-1}u_i)^{\alpha}]^{\phi/\alpha}\}^{1/\phi} \tag{3.11}
$$

which does not depend on the state. Rewriting the maximized utility as:

$$
U_{T-1} = W_{T-1}A_{T-1}
$$

shows that, the recursive utility is linear in wealth. More precisely:

$$\begin{aligned} U_{T-1}^{u_j} &= W_{T-1}^{u_j} A_{T-1} \\ W_{T-1}^{u_j} &= B_{T-2}R + \Pi_{T-2} S_{T-1}^{u_j} = B_{T-2}R + \Pi_{T-2} S_{T-2} u_j \end{aligned}$$

for $j \in \{1, 2, ..., n\}$

As in the binomial model, $U_T = W_T$, so we can set $A_T = 1$, and at the end of the backward maximization, A_0 will give the initial utility for the normalized level of wealth. Hence, $U_0 = A_0$ when $W_0 = 1$.

In order to see the backward recursive solution to the maximization problem, again, we look at the time $T - 2$ setup. Now, if we write the maximization problem for $t = T - 2$,

$$\max_{\{B_{T-2}, C_{T-2}, \Pi_{T-2}\}} \{C_{T-2}^{\phi} + \beta[\sum_{i=1}^{n} p_i (U_{T-1}^{u_i})^{\alpha}]^{\phi/\alpha}\}^{1/\phi}$$

subject to:

$$B_{T-2} + C_{T-2} + \Pi_{T-2} S_T = W_{T-2} \tag{3.12}$$

If we substitute $U_{T-1}^{u_j}$, the maximization problem becomes:

$$\max_{\{B_{T-2}, C_{T-2}, \Pi_{T-2}\}} \{C_{T-2}^{\phi} + \beta[\sum_{i=1}^{n} p_i (W_{T-1}^{u_i} A_{T-1})^{\alpha}]^{\phi/\alpha}\}^{1/\phi}$$

subject to:

$$B_{T-2} + C_{T-2} + \Pi_{T-2} S_T = W_{T-2} \tag{3.13}$$

After the substitution of $W_{T-1}^{u_j}$, we get:

$$\max_{\{B_{T-2}, C_{T-2}, \Pi_{T-2}\}} \{C_{T-2}^{\phi} + \beta[\sum_{i=1}^{n} p_i (B_{T-2}R + \Pi_{T-2} S_{T-2} u_i)^{\alpha} A_{T-1}^{\alpha}]^{\phi/\alpha}\}^{1/\phi}$$

subject to:

$$B_{T-2} + C_{T-2} + \Pi_{T-2} S_T = W_{T-2} \tag{3.14}$$

If we solve the budget constraint for C_{T-2} and substitute into the above maximization problem, we get:

$$\max_{\{B_{T-2},\Pi_{T-2}\}} \{(w_{T-2} - B_{T-2} - \Pi_{T-2} S_{T-2})^{\phi} \tag{3.15}$$
$$+ \beta A_{T-1}^{\phi} [\sum_{i=1}^{n} p_i (B_{T-2} R + \Pi_{T-2} S_{T-2} u_i)^{\alpha}]^{\phi/\alpha}\}^{1/\phi}$$

Let's define $\beta_{T-1} = \beta A_{T-1}^{\phi}$, then the problem becomes:

$$\max_{\{B_{T-2},\Pi_{T-2}\}} \{(w_{T-2} - B_{T-2} - \Pi_{T-2} S_{T-2})^{\phi} \tag{3.16}$$
$$+ \beta_{T-1} [\sum_{i=1}^{n} p_i (B_{T-2} R + \Pi_{T-2} S_{T-2} u_i)^{\alpha}]^{\phi/\alpha}\}^{1/\phi}$$

The maximization problem above at time $T-2$, is the same as (3.4) except the time dependence and β in equation (3.4) is replaced by β_{T-1}. So, we can write a general recursive solution for any time t as following:

General recursive solution:

The problem :

$$\max U_{t-1} = [C_{t-1}^{\phi} + \beta(E_{t-1} U_t^{\alpha})^{\phi/\alpha}]^{1/\phi}$$

subject to:

$$B_{t-1} + C_{t-1} + \Pi_{t-1} S_{t-1} = W_{t-1} \tag{3.17}$$

Step 0.

Set $t = T$, and define $\beta_T = \beta$ which is given as a parameter.

Step 1.

Solve the following system for b, and π numerically and get c from the constraint $c = 1 - b - \pi$.

$$\begin{aligned}
\sum_{i=1}^{n} p_i(u_i - R)(bR + \pi u_i)^{\alpha-1} &= 0 \\
\beta_t[\sum_{i=1}^{n} p_i(bR + \pi u_i)^{\alpha}]^{(\phi-\alpha)/\alpha}[\sum_{i=1}^{n} p_i u_i(bR + \pi u_i)^{\alpha-1}] &= (1 - b - \pi)^{\phi-1}
\end{aligned}$$

Then, define $(b_{t-1}, c_{t-1}, \pi_{t-1}) = (b, c, \pi)$.

Step 2. Find

$$A_{t-1} = \{c_{t-1}^{\phi} + \beta_t[\sum_{i=1}^{n} p_i(b_{t-1}R + \pi_{t-1}u_i)^{\alpha}]^{\phi/\alpha}\}^{1/\phi}$$

where it is assumed that $A_T = 1$.

Step 3.

Update beta, i.e. find $\beta_{t-1} = \beta A_{t-1}^{\phi}$. Then, decrease t by 1, i.e. $t = t - 1$ and go to Step 1, until $t = 1$. If $t = 1$, move to Step 4. Hence, we have the ratio coefficients.

In order to find the dollar amounts allocated to consumption and investment for any path starting from t=0,we continue with Step 4.

Step 4.

$$\begin{aligned}
B_t &= b_t W_t \\
C_t &= c_t W_t \\
\Pi_t S_t &= \pi_t W_t
\end{aligned}$$

where W_0 is the initial wealth and $W_t = B_{t-1}R + \Pi_{t-1}S_t$ for $t \geq 1$.

Hence, we get the partially closed form recursive solution to the maximization problem when the number of states is 3 or more.

Now, in the following subsections of 3.1, the effects of the parameters are be discussed together with the market incompleteness.

3.1.2 The Effects of Market Incompleteness

In order to see the effects of market incompleteness on the effects of parameters discussed in Section 2.1.2 to Section 2.1.4, we compare the two models with almost the same parameters except the number of states and the stock returns in each state. Also, we keep the expected stock return equal in both models.

In general, alpha, beta and phi have almost same effects on choice variables. However, the effect of alpha on bond holdings is significantly different. The effect of alpha on bond holdings in incomplete markets is larger. For example, if we change alpha from -2 to -3 (RRA increases from 3 to 4), bond holdings increase around 20 percent, on average, in incomplete markets, whereas the change in bond holdings is only about 13 percent in complete markets. Moreover, the difference increases as the number of states increases.

Market incompleteness has no effect on the optimal level of consumption. Agent consumes the same amount in complete and incomplete markets, independent of the number of states and the return of the stock, as long as the parameter phi is the same in both cases. However, when the markets are incomplete, for the same stock return, the composition of savings changes and the agent holds around 15 percent less bonds and 25 percent more stocks whereas the percentage of total savings in wealth stays constant. Change in the composition of savings is due to the decrease in the variance of stock return thanks to the higher number of states, but total savings do not change.

To sum up, consumption and savings do not depend on the number of states of the stock.

3.1.3 Interchangeability of Beta and Phi

As in complete markets, beta and phi are interchangeable; beta can be interchanged by phi and phi can be interchanged by beta. The results are almost same as in Table 2.1 and Table 2.2. Errors are slightly larger, which could be due to the numerical solution of the two by two system. However, the difference is insignificant. Hence, interchangeability of beta and phi does not depend on the market incompleteness.

3.2 Uncertain Horizon

3.2.1 Solution

$$\max_{\{B_{t-1},C_{t-1},\Pi_{t-1}\}} U_{t-1} = \{C_{t-1}^{\phi} + \beta\rho_t[\sum_{i=1}^{n} p_i(U_t^{u_i})^{\alpha}]^{\phi/\alpha}\}^{1/\phi}$$

subject to:

$$B_{t-1} + C_{t-1} + \Pi_{t-1}S_{t-1} = W_{t-1} \tag{3.18}$$

where ρ_t is the conditional survival probability. When $\rho_t = 1$ for all t, we get the results of Section 3.1

Similarly, we can solve the model recursively and get the recursive solution starting from time T.

3.2.2 The Effects of Uncertainty in Incomplete Markets

1. On decision variables

Adding the uncertainty increases the consumption level and decreases the savings. However, the composition of savings remain unchanged. Bond and stock holdings decrease by 25 percent and consumption increases by 55 percent on average with the addition of uncertainty to the model. These numbers are exactly the same as the ones for the complete market case. Also, if we compare the following Figure 3.1, 3.2 and 3.3 with the Figure 2.2, 2.3 and 2.4, there is almost no difference between the effects in the two markets. This can easily be seen from the ratio graphs in the following figures. So, the effects of uncertainty are same in both complete and incomplete markets. Hence the effects of uncertainty in the lifetime are independent of market incompleteness.

Figure 3.1: Bond Holdings in Incomplete Markets

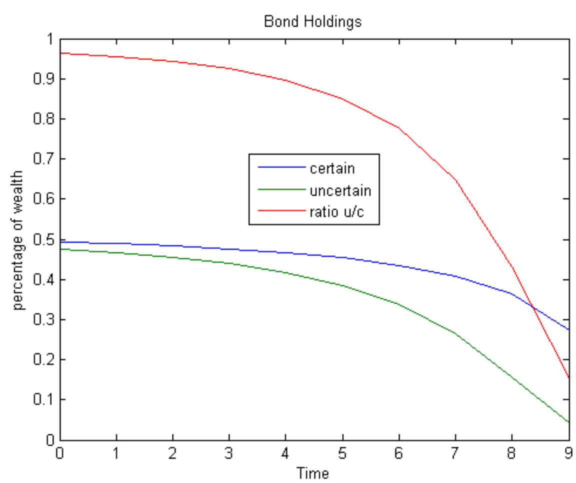
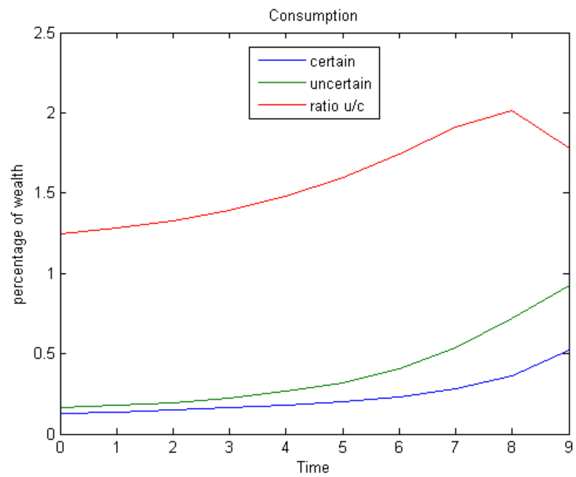


Figure 3.2: Consumption in Incomplete Markets



2. On the Effects of Parameters

In order to see the effects of utility parameters, we compare the certain lifetime model and the uncertain lifetime model in incomplete markets. The effects of alpha is almost the same, which shows that the effects of alpha does not depend on the lifetime uncertainty. However, some of the effects of beta and phi change after adding lifetime uncertainty to the model. For example, the effect of beta (0.90 to 0.95) on initial utility and consumption decreases significantly. The numerical comparison can be seen in Table 3.1. Also, the effect of phi (0.1 to 0.3) on bond holdings, stock holdings and consumption increases with the uncertainty. Results are compared in Table 3.2.

Table 3.1: Effects of Beta in Incomplete Markets

	$Certain\ Life$		$Uncertain\ Life$	
	$\phi = 0.3$	$\phi = 0.5$	$\phi = 0.3$	$\phi = 0.5$
$\Delta Uzero$	105	48	75	34
$\Delta Consumption$	-15	-18	-12	-13.8

Figure 3.3: Stock Holdings in Incomplete Markets

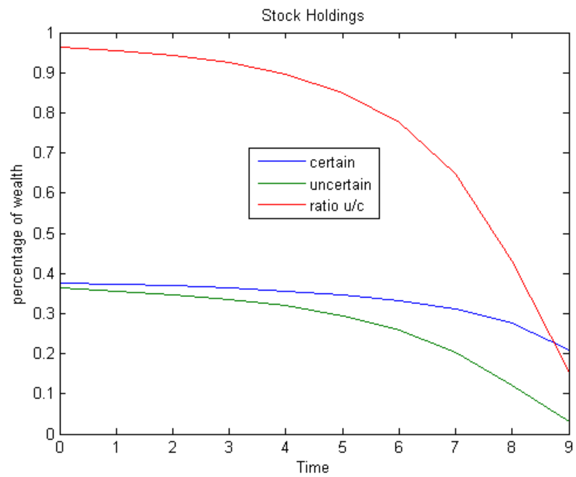


Table 3.2: Effects of Phi in Incomplete Markets

	Certain Life		*Uncertain Life*	
	$\beta = 0.90$	$\beta = 0.95$	$\beta = 0.90$	$\beta = 0.95$
$\Delta Bond, \Delta Stock$	-1.9	-0.7	-4.1	-3
$\Delta Consumption$	6.5	3.2	9.3	7.3

Also, as in Section 3.1.2, adding market incompleteness to the model when there is lifetime uncertainty, does not have any different significant effect on the results. In other words, the results in uncertain lifetime model in incomplete markets are not significantly different from the results in uncertain lifetime model in complete markets.

3.2.3 Interchangeability of Beta and Phi

The results for the interchangeability of beta and phi, in incomplete markets when the lifetime is uncertain, are similar to the results in Table 2.5 and Table 2.6. The results show that, same as in complete markets, beta and phi are not interchangeable when markets incomplete and the lifetime is uncertain. By the results of Section 3.1.3, it can be concluded that, due to the lifetime uncertainty, beta and phi are not interchangeable.

If we make an over all conclusion by using the results of sections 2.1.5, 2.2.5, 3.1.3, and 3.2.4, we can say that the lifetime uncertainty is the only factor affecting the interchangeability of beta and phi in our analysis.

Chapter 4

Compensation for Suboptimal Allocations

In this chapter[1], the effects of suboptimal allocation on initial utility U_0 are checked. We also present the level of compensation need to be given to an agent who chooses suboptimal allocations, in order to reach the utility level when all choice variables are set to their optimal levels.

By following the definition of [6], define $EU_0(W_0, \mathbf{x})$ as the expected initial utility. Also, define $W_0(EU_0, \mathbf{x})$ as the level of initial wealth required to achieve the given level of expected initial utility, EU_0 under the life time portfolio allocation $\mathbf{x} = (\overrightarrow{b}, \overrightarrow{c}, \overrightarrow{\pi})$.

The quantity $\overline{W_0}[EU_0(W_0, \mathbf{x}^*), \mathbf{x}]$ denotes the initial wealth required to achieve, under policy $\mathbf{x}$, the same level of expected initial utility as is achieved under the optimal policy, $\mathbf{x}^*$ starting with the initial wealth W_0. By definition, it follows that $\overline{W_0}[EU_0(W_0, \mathbf{x}^*), \mathbf{x}] > W_0$ for a suboptimal allocation $\mathbf{x}$.

Compensation is defined as $\frac{\overline{W_0}}{W_0} - 1$.

Our strategy is to pick a suboptimal level for one of the choice variables, pick an optimal level for another one, and determine the third choice variable from the budget constraint. Under this case, we look at the welfare loss and the compensation need to be given to the agent.

[1] alpha = -2, beta = 0.95, phi = 0.4, and T = 11 periods

4.1 Bonds

We are looking at a scenario where the agent's bond holdings are different from the optimal level and the consumption is at the optimal level. Stock holdings are found from the budget constraint. As we see in Figure 4.1 and Table 4.1, deviations of bond holdings around the optimal level do not change Uzero significantly both for the certain lifetime model and the uncertain lifetime model. For example, an agent who holds the bonds at a level 30 percent more than the optimum and keeps the consumption at the optimal level, needs a compensation of 0.23 percent of initial wealth in certain lifetime model, and 0.19 percent in uncertain lifetime model. Hence, the welfare loss due to suboptimal allocation in bonds is negligible.

Figure 4.1: Welfare Loss of Suboptimal Allocations in Bonds Holdings

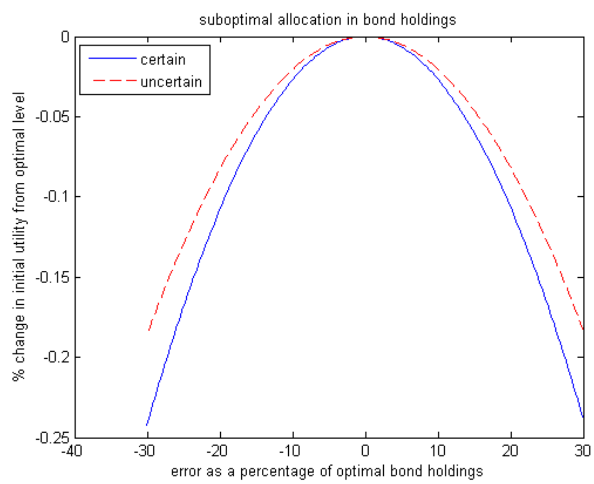


Table 4.1: Compensation for Suboptimal Bond Holdings

$Model$	$Certain\ Life$		$Uncertain\ Life$	
$\%\ Error$	0.30	-0.30	0.30	-0.30
$\%\ Change\ in\ U_0$	-0.23	-0.24	-0.19	-0.18
$Compensation(\%W_0)$	0.23	0.24	0.19	0.18

4.2 Consumption

Suppose that the agent consumes different from the optimal level and keeps the bond holdings at the optimal level. Stock holdings satisfy the budget constraint. As we see in Figure 4.2, deviations around the optimal level of consumption result in small change in U_0. For example, in certain lifetime model, if the agent consumes around 30 percent more than the optimal level, and keeps the bond holdings at the optimal level, the initial utility decreases 2.71 percent. The effect of a 30 percent less consumption results in a decrease of 4.14 percent in initial utility. Also, level of compensation is 2.79 percent and 4.31 percent increase in the initial wealth W_0, respectively. In other words, an agent who has 2.79 (4.31) percent more initial wealth and consumes 30 percent more (less) than the optimal level gets an initial utility same as an a agent who sets the choice variables at the optimal level. This result also shows that, the effect of an error in optimal consumption level is not symmetric.

Table 4.2 summarizes the above result and in addition, it presents the results under uncertain lifetime horizon.

4.3 Stocks

Similarly, now the agent holds stocks different from the optimal level and keeps the bond holdings at the optimal level. Consumption is set to a level which satisfies the

Figure 4.2: Welfare Loss of Suboptimal Allocations in Consumption

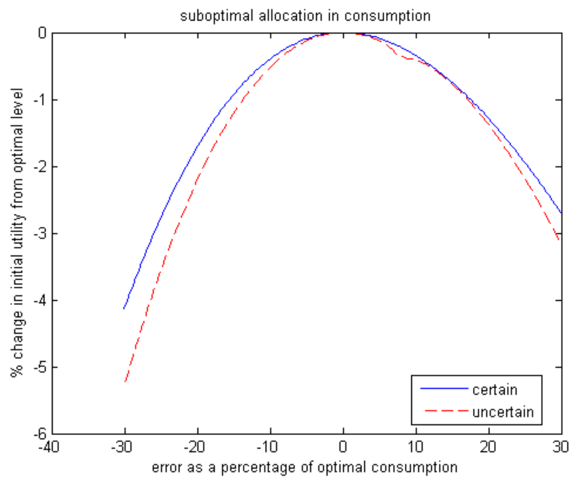


Table 4.2: Compensation for Suboptimal Consumption

Model	*Certain Life*		*Uncertain Life*	
% Error	0.30	-0.30	0.30	-0.30
% Change in U_0	-2.71	-4.14	-3.19	-5.29
Compensation$(\%W_0)$	2.79	4.31	3.30	5.60

budget constraint. As we see in Figure 4.3, a 10 percent deviation around the optimal level of stock holding results in a change of less than 1 percent in U_0. However, if the deviation from the optimal level increases, the decrease in the initial utility increases more compared to the other two sections 4.1 and 4.2. For example, in certain lifetime model, if the agent holds around 30 percent more stocks than the optimal level, and keeps the bond holdings at an optimal level, the initial utility decreases 12.57 percent and the effect of a 30 percent less stock holding results in a decrease of 6.15 percent in initial utility. Also, level of compensation is 14.37 percent and 6.55 percent increase in the initial wealth W_0, respectively. In other words, an agent who has 14.37 (6.55) percent more wealth initially and holds 30 percent more (less) stocks than the optimal level, derives an initial utility same as an a agent who sets the choice variables at the optimal level. The effect of an error in optimal stock holding is not symmetric, holding less than the optimal level in stocks has a smaller negative effect on initial utility compared to holding more at the same percent. Also, in the uncertain lifetime model, the effect of suboptimal allocation in stock holdings is not as significant as the one in the certain lifetime model.

Figure 4.3: Welfare Loss of Suboptimal Allocations in Stock Holdings

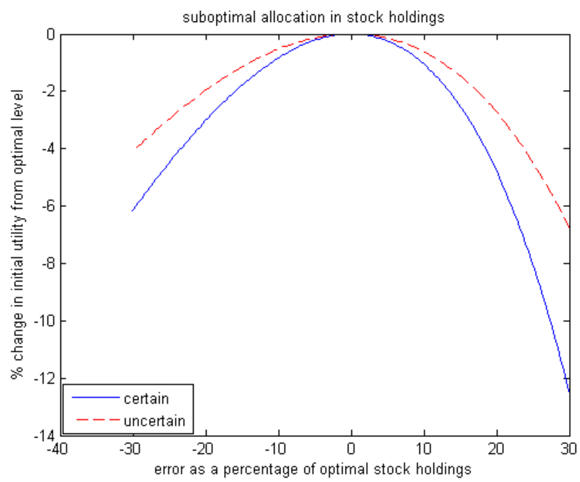


Table 4.3 summarizes the above result and in addition, it presents the results when the lifetime is uncertain.

Table 4.3: Compensation for Suboptimal Stock Holdings

$Model$	$Certain\ Life$		$Uncertain\ Life$	
$\%\ Error$	0.30	-0.30	0.30	-0.30
$\%\ Change\ in\ U_0$	-12.57	-6.15	-6.78	-4.10
$Compensation(\%W_0)$	14.37	6.55	7.27	4.28

If the agent decides to consume at the optimal level, and adjusts the suboptimal allocation in stock holdings by bond holdings, the effects are smaller than those in Section 4.1.

Chapter 5

Representative Agent in Recursive Settings

In this chapter, mainly two questions are addressed. First one is that, if we have a society of heterogeneous agents, can we find a representative agent whose income is the aggregate income of the society and whose allocations are the aggregate allocations of the society? In general, finding the representative agent analytically is not easy. [8] state that, in general, the representative agent does not have recursive utility, even if the agents in the economy have recursive utility. Second question is about the effects of state probabilities on representative agent's parameters.

Heterogeneity in alpha and/or phi, beta and survival probability, are also discussed.

In the following sections, to minimize the error between the actual allocations of the representative agent and that of aggregate of heterogeneous agents, the ratio coefficients cannot be used anymore. It is because, minimizing the difference between the ratio coefficients is not equivalent to minimizing the difference between the variables. Simply, consider an equally weighted society with two types of heterogeneous agents with initial wealths W_0^1 and W_0^2. Suppose that the optimal lifetime consumption for these agents are C_t^1 and C_t^2. We are looking for a representative agent whose consumption and wealth at time t are $C_t = C_t^1 + C_t^2$, $W_t = W_t^1 + W_t^2$ respectively. In general,

$\frac{C_t}{W_t}(=\frac{C_t^1+C_t^2}{W_t^1+W_t^2})$ is not equal to the sum $\frac{C_t^1}{W_t^1}+\frac{C_t^2}{W_t^2}$[1]. Thus, we cannot look for a representative agent by minimizing the ratio coefficients. However, the binomial tree can still be used for the wealth process because, the binomial tree for the wealth process is recombining.

Proposition 4. *The growth rates of the ratio coefficient of bonds and stocks are equal.i.e.*

$$\frac{b_t}{b_{t-1}} = \frac{\pi_t}{\pi_{t-1}} \quad \forall t \in \{1, 2, ..., T\}$$

Proof. By using the results for b_t and π_t,

$$\frac{b_t}{b_{t-1}} = \frac{\frac{1}{m_t+l+1}}{\frac{1}{m_{t-1}+l+1}} = \frac{m_{t-1}+l+1}{m_t+l+1}$$

and

$$\frac{\pi_t}{\pi_{t-1}} = \frac{\frac{l}{m_t+l+1}}{\frac{l}{m_{t-1}+l+1}} = \frac{m_{t-1}+l+1}{m_t+l+1}$$

which gives the desired result. At the optimal level, the growth rates of bond holdings and stock holdings, as a percentage of wealth, are equal. □

Corollary 2. *The wealth process is recombining on the binomial tree.*

Proof. W_{t-1}, the wealth at time $t-1$, after one up and one down state is W_{t+1}^{ud}, and it is equal to W_{t+1}^{du} after one down and one up state. The wealth process is recombining if:

$$W_{t+1}^{ud} = W_{t+1}^{du}$$

By using the result of the above proposition we get:

$$(b_{t-1}R + \pi_{t-1}u)(b_tR + \pi_t d) = (b_{t-1}R + \pi_{t-1}d)(b_tR + \pi_t u)$$

[1]Same argument is true for allocations in bonds and stocks

Also,

$$W_{t+1}^{ud} = (b_{t-1}R + \pi_{t-1}u)(b_tR + \pi_t d)W_{t-1}$$

and

$$W_{t+1}^{du} = (b_{t-1}R + \pi_{t-1}d)(b_tR + \pi_t u)W_{t-1}$$

Hence we get the desired result. □

5.1 Heterogeneity in Alpha and Phi

The heterogeneity in the risk aversion parameter, alpha and elasticity of intertemporal substitution parameter, phi is considered in this section.

5.1.1 Case 1 : $\alpha_1 \neq \alpha_2$

We assume that the agents are heterogeneous only in the risk aversion parameter and there are two agents with risk aversion parameters α_1 and α_2 such that $\alpha_1 \neq \alpha_2$. Without loss of generality, it is assumed that $\alpha_1 < \alpha_2$.

Q1 : Is there a representative agent ?

Yes, with a small error margin, there is $\alpha \in (\alpha_1, \alpha_2)$ such that the optimal decisions for the representative agent are the aggregate of that of the two heterogeneous agents.

Here, it is assumed that the agents have initial wealth of 1, hence the representative agent has initial wealth of two.

To find the error minimizing α, we need the dollar amounts allocated to bond holdings, consumption and stock holdings. It is because, our previous results give the ratio coefficients which are the ratios to wealth. Since the allocations of these two agents are different, the dollar amounts at every node on the binomial tree are needed.

Suppose that B_t^1, C_t^1, ΠS_t^1, are the life time dollar allocations of agent 1 to bond holdings, consumption, and stock holdings respectively. Similarly B_t^2, C_t^2, ΠS_t^2 are those for agent 2. We say that, there is a representative agent with $\alpha \in (\alpha_1, \alpha_2)$, if the following errors are zero at every node for all $t \in \{1, .., T\}$.

Errorbond = $|B_t - (B_t^1 + B_t^2)|$

Errorcons = $|C_t - (C_t^1 + C_t^2)|$

Errorstock = $|\Pi S_t - (\Pi S_t^1 + \Pi S_t^2)|$ and

Maxerror = max(Errorbond, Errorcons, Errorstock)

B_t, C_t, $\Pi_t S_t$ are the allocations of the representative agent. The error defined here is minimized for all kinds of heterogeneity.

The following Table 5.1 and 5.2 present some selected numerical results for reasonable levels of alpha and phi.

Table 5.1: Representative Agent Alpha Small

$\phi = 0.4$		$\phi = 0.6$	
$\alpha_1 = -3$	$\alpha_2 = -2$	$\alpha_1 = -3$	$\alpha_2 = -2$
$\alpha = -2.429$		$\alpha = -2.428$	
$Maxerror = 6.6e - 4$		$Maxerror = 5.9e - 4$	

Table 5.2: Representative Agent Alpha Big

$\phi = 0.4$		$\phi = 0.6$	
$\alpha_1 = -5$	$\alpha_2 = -2$	$\alpha_1 = -5$	$\alpha_2 = -2$
$\alpha = -2.998$		$\alpha = -2.997$	
$Maxerror = 2.7e - 3$		$Maxerror = 2.4e - 3$	

For example, when $\alpha_1 = -3$ and $\alpha_2 = -2$, there is a representative agent with $\alpha = -2.43$, whose wealth and allocations are the aggregate of those of the heterogeneous agents in the society. The risk aversion parameter, alpha, of the representative agent does not depend on the level of EIS parameter, phi. Errors are relatively larger if the difference between α_1 and α_2 is larger. However, the maximum error is still small and negligible.

Q2 : What is the dependence of α (representative agent's risk aversion parameter) on p (up probability of stock return)

As we can also see from the following Table 5.3, α does not have significant dependence on p. The same results are true for different levels of phi and alpha.

Table 5.3: Effect of p on Alpha

$p = 0.60$		$p = 0.65$	
$\alpha_1 = -3$	$\alpha_2 = -2$	$\alpha_1 = -3$	$\alpha_2 = -2$
$\alpha = -2.429$		$\alpha = -2.425$	
$Maxerror = 6.6e - 4$		$Maxerror = 2.8e - 3$	

5.1.2 Case 2 : $\phi_1 \neq \phi_2$

Suppose now that the agents are heterogeneous only in the elasticity of intertemporal substitution parameter and there are two agents with elasticity of intertemporal substitution parameters ϕ_1 and ϕ_2 such that $\phi_1 \neq \phi_2$. Without loss of generality, it is assumed that $\phi_1 < \phi_2$.

Q1 : Is there a representative agent ?

Yes, with a small error margin, there is a representative agent whose optimal allocations are almost equal to the aggregate of the allocations of heterogeneous agents.

Some selected numerical results are presented in Table 5.4 and Table 5.5.

Q2 : What is the dependence of ϕ on p?

Table 5.4: Representative Agent Phi Small

$\alpha = -3$		$\alpha = -2$	
$\phi_1 = 0.4$	$\phi_2 = 0.6$	$\phi_1 = 0.4$	$\phi_2 = 0.6$
$\phi = 0.5192$		$\phi = 0.5191$	
$Maxerror = 4.4e - 4$		$Maxerror = 4.3e - 4$	

Table 5.5: Representative Agent Phi Big

$\alpha = -3$		$\alpha = -2$	
$\phi_1 = 0.2$	$\phi_2 = 0.7$	$\phi_1 = 0.2$	$\phi_2 = 0.7$
$\phi = 0.5589$		$\phi = 0.5589$	
$Maxerror = 2.7e - 3$		$Maxerror = 2.6e - 3$	

Similarly, ϕ does not have significant dependence on p.

Some selected numerical results for the dependence of ϕ on p are presented in Table 5.6.

Table 5.6: Effect of p on Phi

$p = 0.60$		$p = 0.65$	
$\phi_1 = 0.4$	$\phi_2 = 0.6$	$\phi_1 = 0.4$	$\phi_2 = 0.6$
$\phi = 0.5191$		$\phi = 0.5190$	
$Maxerror = 4.3e - 4$		$Maxerror = 3.8e - 4$	

5.1.3 Case 3 : $\alpha_1 \neq \alpha_2$ and $\phi_1 \neq \phi_2$

As a third case, suppose now that the agents are heterogeneous both in the risk aversion parameter and the elasticity of intertemporal substitution parameter. Again, it is assumed that there are two agents with risk aversion parameters α_1 and α_2, and elasticity of intertemporal substitution parameters ϕ_1 and ϕ_2 such that $\alpha_1 \neq \alpha_2$ and $\phi_1 \neq \phi_2$.

Q1 : Is there a representative agent?

Yes, this result follows from the results of Case 1 and Case 2. Because, when only $\alpha_1 \neq \alpha_2$, the representative agent's α does not have a significant dependence on the level of ϕ. Similarly, the representative agent's ϕ does not depend on the level of α significantly. So, when there is heterogeneity in both parameters, (i.e. $\alpha_1 \neq \alpha_2$ and $\phi_1 \neq \phi_2$) we can find the parameters for the representative agent by getting one parameter at a time.

Errors are very close to the errors given in Case 1 and Case 2.

Q2 : What is the dependence of the pair (α, ϕ) on p?

Similarly, the pair (α, ϕ) does not depend on p significantly.

5.2 Heterogeneity in Beta

The heterogeneity in subjective time discount factor, β is checked in this section. It is assumed that there are two agents with $\beta_1 \neq \beta_2$. Without loss of generality, assume that $\beta_1 < \beta_2$, which implies that agent 2 is more patient than agent 1. Table 5.7 and Table 5.8 present selected results for different levels of alpha when $\phi = 0.4$. Also, Table 5.9 and Table 5.10 present selected results for different levels of phi when $\alpha = -3$. We can say that, there is a representative agent if phi is not large (less than 0.5) or the difference between β_1 and β_2 is small (less than 0.05). It is because, in all these cases, the error is less than 0.01[2].

However, when phi is close to one, there is significant difference between the allocations of the representative agent and the aggregate of the allocations of the heterogeneous agents if the agents subjective time discount factors are not close. For example,

[2]The percentage error is also less than 0.01

Table 5.7: Representative Agent Beta Small, α

$\alpha = -4$	$\alpha = -2$
$(\beta_1, \beta_2) = (0.90, 0.96)$	
$\beta = 0.9304$	$\beta = 0.9305$
$Maxerror = 0.0033$	$Maxerror = 0.0035$

Table 5.8: Representative Agent Beta Big, α

$\alpha = -4$	$\alpha = -2$
$(\beta_1, \beta_2) = (0.88, 0.98)$	
$\beta = 0.9312$	$\beta = 0.9314$
$Maxerror = 0.0091$	$Maxerror = 0.0097$

when $\alpha = -2$ and $\phi = 0.8$, the minimized error is around 7 percent for the heterogeneous agents with betas $(\beta_1, \beta_2) = (0.88, 0.98)$. This error is monotonic in phi, and it increases to 10 percent when $\phi = 0.9$. Also, as RRA increases the error decreases, but the decrease in error and error itself are still small.

β does not depend on p significantly. β increases as p increases, however, the change in β is insignificant. For example, when p is increased by 10 percent, β increases by 0.10 percent. So we can say that the dependence of representative agent's subjective discount factor on the state probabilities is insignificant.

5.3 Heterogeneity in Survival Probability

Now, we look at the existence of representative agent when there are heterogeneous groups (or individuals) in the society with different survival probabilities.

Table 5.9: Representative Agent Beta Small, ϕ

$\phi = 0.6$	$\phi = 0.2$
$(\beta_1, \beta_2) = (0.90, 0.96)$	
$\beta = 0.9313$	$\beta = 0.9301$
$Maxerror = 0.0070$	$Maxerror = 0.0020$

Table 5.10: Representative Agent Beta Big, ϕ

$\phi = 0.6$	$\phi = 0.2$
$(\beta_1, \beta_2) = (0.88, 0.98)$	
$\beta = 0.9335$	$\beta = 0.9303$
$Maxerror = 0.0189$	$Maxerror = 0.0054$

Consider that the population consists of k groups with survival probabilities ρ_t^1, ρ_t^2,..,ρ_t^k. Assume that the weight of these groups in the population are $\omega_1, \omega_2 ... \omega_k$ such that $\sum_{j=1}^{k} \omega_j = 1$. Hence, the survival probability of the society is $\rho_t = \sum_{j=1}^{k} \omega_j \rho_t^j$.

We look at two different cases for the representative agent when there is heterogeneity in survival probability numerically. In the first case, there are two heterogeneous groups in the society. Both have linearly decaying survival probabilities which are very close. In the second case, there are two close survival probabilities which are similar to the Figure 2.1.

In the first case it, is hard to say that there is a representative agent who has the pooled survival probability. Because errors are between 3 to 17 percent depending on the level of beta and phi. However, alpha does not have any significant effect on the errors.

In the second case, errors are smaller compared to first case, but still they are not very close to zero. When beta is small and phi is large, errors are at the largest level.

These results are almost independent of small changes in the state probabilities.

Figure 5.1: Linearly Decaying Survival Probabilities

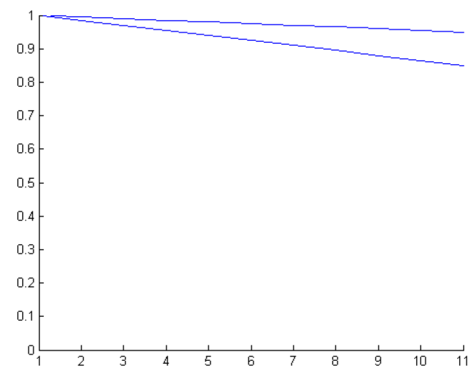


Table 5.11: Error in Aggregate Allocation, Linear, $\beta = 0.90$

$\alpha = -2, -4$	
$\beta = 0.90$	$\beta = 0.90$
$\phi = 0.4$	$\phi = 0.6$
$Errorbond = 2.58\%$	$Errorbond = 4.62\%$
$Errorcons = 6.81\%$	$Errorcons = 16.89\%$
$Errorstock = 2.58\%$	$Errorstock = 4.62\%$

Table 5.12: Error in Aggregate Allocation, Linear, $\beta = 0.95$

$\alpha = -2, -4$	
$\beta = 0.95$	$\beta = 0.95$
$\phi = 0.4$	$\phi = 0.6$
$Errorbond = 2.93\%$	$Errorbond = 5.59\%$
$Errorcons = 5.66\%$	$Errorcons = 14.37\%$
$Errorstock = 2.93\%$	$Errorstock = 5.59\%$

Figure 5.2: Real Survival Probabilities

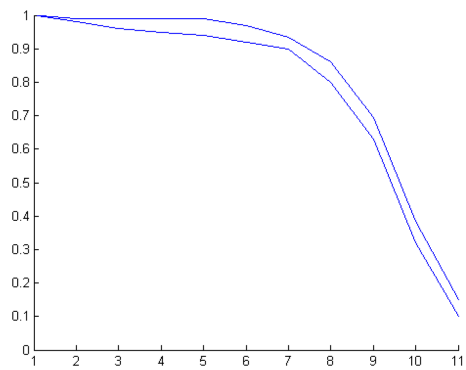


Table 5.13: Error in Aggregate Allocation, Real, $\beta = 0.90$

$\alpha = -2, -4$	
$\beta = 0.90$	$\beta = 0.90$
$\phi = 0.4$	$\phi = 0.6$
$Errorbond = 1.06\%$	$Errorbond = 1.73\%$
$Errorcons = 2.68\%$	$Errorcons = 3.61\%$
$Errorstock = 1.06\%$	$Errorstock = 1.73\%$

Table 5.14: Error in Aggregate Allocation, Real, $\beta = 0.95$

$\alpha = -2, -4$	
$\beta = 0.95$	$\beta = 0.95$
$\phi = 0.4$	$\phi = 0.6$
$Errorbond = 0.83\%$	$Errorbond = 1.25\%$
$Errorcons = 2.23\%$	$Errorcons = 2.73\%$
$Errorstock = 0.83\%$	$Errorstock = 1.25\%$

Chapter 6

Conclusion

We derived the closed form recursive solution for optimal allocation in consumption and investment. Hence we analyzed the effects of model parameters which are RRA, EIS, β and mortality on the decision variables, consumption, bond holdings, and stock holdings.

Life uncertainty increases the consumption and decreases the investment and savings significantly. Adding uncertainty partially explains the low level of participation in risky investments and high level of consumption.

In recursive settings, consumption depends on relative risk aversion parameter, alpha, analytically. However, numerical results show that consumption does not depend on RRA significantly. This is an interesting result, since in standard setting where EIS and RRA are determined by the same parameter, consumption depends on alpha. When we have two separate parameters to control EIS and RRA, consumption and savings become independent of RRA numerically. Moreover, the composition of savings does not depend on EIS analytically. In short, the attitude towards risk determines the composition of savings and EIS determines the allocation of income between consumption and savings, in other words, allocation between today's and future consumption.

The interchangeability of subjective time discount parameter, beta and elasticity of intertemporal substitution parameter,phi is introduced. Analytically, beta and phi are not interchangeable. However, under the certain lifetime model, beta and phi are interchangeable numerically. This is not true when the lifetime is uncertain.

In reality, agents may not know their optimal level of allocations. So, the effects of suboptimal allocations in consumption and savings are explored. The welfare loss of suboptimal allocation in consumption and investment is modest, in general. If the agent consumes at the optimal level, but allocates savings between stocks and bonds suboptimally, then the welfare loss is found to be small. This is an interesting result, since it implies that as long as the agent consumes optimally, the composition of savings does not affect welfare significantly. Also, the welfare loss due to suboptimal allocation in stock holdings is asymmetric. Holding less stocks than the optimal level has smaller welfare loss when compared to holding more stocks than the optimal level. This result is important, because it can explain the low level of stock market participation. If the agent does not know his optimal level of stock holdings, his welfare loss is lower with a low level of investment in stocks when compared with a high level of investment in stocks. This might create a tendency on the agent to invest small amount in stocks.

When there is heterogeneity in the risk aversion parameter and/or elasticity of intertemporal substitution parameter, we can find a representative agent whose risk aversion parameter does not depend on the state probability significantly. And, there is representative agent if the heterogeneity is in subjective time discount parameter. This result is not true when the level of elasticity of intertemporal substitution parameter is close to one or the difference between the agents' subjective time discount factors is small. Also, under the recursive utility, the representative agent's allocations are not equal to the aggregate of those of the society when the heterogeneity is in survival probability. From the data, we know that there is heterogeneity among people in survival probability. For instance, women tend to live longer than men, rich people tend to live longer than poor people. So, people may have different survival probability patterns. According to our results, then, it may be better to use heterogeneous agents models than the representative agent models, under the recursive utility.

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Appendix

Proposition 5.

$$\max_{\{B_{T-1}, C_{T-1}, \Pi_{T-1}\}} \{C_{T-1}^{\phi} + \beta[p(B_{T-1}R + \Pi_{T-1}S_{T-1}u)^{\alpha}$$
$$+ q(B_{T-1}R + \Pi_{T-1}S_{T-1}d)^{\alpha}]^{\phi/\alpha}\}^{1/\phi}$$

subject to:

$$B_{T-1} + C_{T-1} + \Pi_{T-1}S_{T-1} = W_{T-1}$$

The solution of the maximization problem above is:

$$\begin{aligned} B_{T-1} &= b_{T-1}W_{T-1} \\ C_{T-1} &= c_{T-1}W_{T-1} \\ \Pi_{T-1}S_{T-1} &= \pi_{T-1}W_{T-1} \end{aligned}$$

where

$$\begin{aligned} b_{T-1} &= \frac{1}{m_{T-1} + l + 1} \\ c_{T-1} &= \frac{m_{T-1}}{m_{T-1} + l + 1} \\ \pi_{T-1} &= \frac{l}{m_{T-1} + l + 1} \end{aligned}$$

and

$$m_{T-1} = \{\beta[p(lu + R)^{\alpha} + q(ld + R)^{\alpha}]^{\frac{\phi-\alpha}{\alpha}}(pu(lu + R)^{\alpha-1} + qd(ld + R)^{\alpha-1})\}^{\frac{1}{\phi-1}}$$

$$
\begin{aligned}
l &= \frac{R(k-1)}{u-kd} \\
k &= [\frac{q(R-d)}{p(u-R)}]^{\frac{1}{\alpha-1}}
\end{aligned}
$$

Proof. After substituting the budget constraint, it is sufficient to solve the following system simultaneously.

$$
\begin{aligned}
\partial U_{T-1}/\partial B_{T-1} &= 0 \\
\partial U_{T-1}/\partial \Pi_{T-1} &= 0
\end{aligned}
$$

If we substitute the budget constraint, the utility is:

$$
\begin{aligned}
U_{T-1} &= \{(w_{T-1} - B_{T-1} - \Pi_{T-1}S_{T-1})^{\phi} + \beta[p(B_{T-1}R + \Pi_{T-1}S_{T-1}u)^{\alpha} + \\
&\quad q(B_{T-1}R + \Pi_{T-1}S_{T-1}d)^{\alpha}]^{\phi/\alpha}\}^{1/\phi}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial U_{T-1}}{\partial \Pi_{T-1}} &= \frac{1}{\phi}(U_{T-1})^{\frac{1}{\phi}-1}\{\phi(w_{T-1} - B_{T-1} - \Pi_{T-1}S_{T-1})^{\phi-1}(-S_{T-1}) + \\
&\quad \beta\frac{\phi}{\alpha}[p(B_{T-1}R + \Pi_{T-1}S_{T-1}u)^{\alpha} + q(B_{T-1}R + \Pi_{T-1}S_{T-1}d)^{\alpha}]^{\frac{\phi}{\alpha}-1} \\
&\quad [\alpha p(B_{T-1}R + \Pi_{T-1}S_{T-1}u)^{\alpha-1}S_{T-1}u + \\
&\quad \alpha q(B_{T-1}R + \Pi_{T-1}S_{T-1}d)^{\alpha-1}S_{T-1}d]\}
\end{aligned}
$$

Now, let

$$
\begin{aligned}
f &= w_{T-1} - B_{T-1} - \Pi_{T-1}S_{T-1} \\
g_1 &= B_{T-1}R + \Pi_{T-1}S_{T-1}u \\
g_2 &= B_{T-1}R + \Pi_{T-1}S_{T-1}d
\end{aligned}
$$

After some simplifications,

$$\frac{\partial U_{T-1}}{\partial \Pi_{T-1}} = S_{T-1}(U_{T-1})^{\frac{1}{\phi}-1}\{-f^{\phi-1} + \beta[pg_1^\alpha + qg_2^\alpha]^{\frac{\phi}{\alpha}-1}[upg_1^{\alpha-1} + dqg_2^{\alpha-1}]\}$$

and similarly,

$$\frac{\partial U_{T-1}}{\partial B_{T-1}} = (U_{T-1})^{\frac{1}{\phi}-1}\{-f^{\phi-1} + \beta R[pg_1^\alpha + qg_2^\alpha]^{\frac{\phi}{\alpha}-1}[pg_1^{\alpha-1} + qg_2^{\alpha-1}]\}$$

Now, solving

$$\begin{aligned} \partial U_{T-1}/\partial B_{T-1} &= 0 \\ \partial U_{T-1}/\partial \Pi_{T-1} &= 0 \end{aligned}$$

is equivalent to solving:

$$\begin{aligned} -f^{\phi-1} + \beta[pg_1^\alpha + qg_2^\alpha]^{\frac{\phi}{\alpha}-1}[upg_1^{\alpha-1} + dqg_2^{\alpha-1}] &= 0 \\ -f^{\phi-1} + \beta R[pg_1^\alpha + qg_2^\alpha]^{\frac{\phi}{\alpha}-1}[pg_1^{\alpha-1} + qg_2^{\alpha-1}] &= 0 \end{aligned}$$

By using these two equations, we get:

$$\begin{aligned} upg_1^{\alpha-1} + dqg_2^{\alpha-1} &= R(pg_1^{\alpha-1} + qg_2^{\alpha-1}) \\ (\frac{g_1}{g_2})^{\alpha-1} &= \frac{q(R-d)}{p(u-R)} \\ \frac{g_1}{g_2} &= [\frac{q(R-d)}{p(u-R)}]^{\frac{1}{\alpha-1}} \end{aligned}$$

Now, let

$$k = [\frac{q(R-d)}{p(u-R)}]^{\frac{1}{\alpha-1}}$$

then,

$$
\begin{aligned}
k &= \frac{g_1}{g_2} \\
&= \frac{B_{T-1}R + \Pi_{T-1}S_{T-1}u}{B_{T-1}R + \Pi_{T-1}S_{T-1}d}
\end{aligned}
$$

Solving for $\Pi_{T-1}S_{T-1}$ gives:

$$\Pi_{T-1}S_{T-1} = B_{T-1}\frac{R(k-1)}{u-kd}$$

Now, define:

$$l = \frac{R(k-1)}{u-kd}$$

then,

$$\Pi_{T-1}S_{T-1} = B_{T-1}l$$

Now, $\partial U_{T-1}/\partial \Pi_{T-1} = 0$ implies that

$$-f^{\phi-1} + \beta[pg_1^{\alpha} + qg_2^{\alpha}]^{\frac{\phi}{\alpha}-1}[upg_1^{\alpha-1} + dqg_2^{\alpha-1}] = 0$$

After substituting the above result, and defining

$$m_{T-1} = \{\beta[p(lu+R)^{\alpha} + q(ld+R)^{\alpha}]^{\frac{\phi-\alpha}{\alpha}}(pu(lu+R)^{\alpha-1} + qd(ld+R)^{\alpha-1})\}^{\frac{1}{\phi-1}}$$

we get

$$-f^{\phi-1} + B_{T-1}^{\phi-1}m_{T-1}^{\phi-1} = 0$$

Now, by using the definition of f, and the budget constraint, we get

$$B_{T-1} = \frac{1}{m_{T-1} + l + 1} W_{T-1}$$

The rest follows from the relation $\Pi_{T-1} S_{T-1} = B_{T-1} l$ and the budget constraint. $\square$

Proof. (Interchangeability)

By contradiction, suppose that there is β_2 such that $m_t^1(\beta_1, \phi_2) = m_t^2(\beta_2, \phi_1)$ for any given t, β_1, $\phi_{1,}$ and ϕ_2 within the domain of the model. By assumption, $m_T^1 = m_T^2$. At time $T - 1$, if we assume that beta and phi are interchangeable, it is equivalent to $m_T^1(\beta_1, \phi_2) = m_T^2(\beta_2, \phi_1)$. By using the definition of m_T, this implies that

$$(\beta_1)^{1/(\phi_2 - 1)} X(\phi_2) = (\beta_2)^{1/(\phi_1 - 1)} X(\phi_1)$$

If we solve for β_2 we get:

$$\beta_2 = (\beta_1)^{\frac{\phi_1 - 1}{\phi_2 - 1}} \left[\frac{X(\phi_2)}{X(\phi_1)} \right]^{\phi_1 - 1}$$

Now, by assumption, this result must satisfy the equality:

$$m_{T-2}^1(\beta_1, \phi_2) = m_{T-2}^2(\beta_2, \phi_1)$$

However, this equality implies that:

$$(\beta_1 A_{T-1}^{\phi_2}(\beta_1, \phi_2))^{1/(\phi_2 - 1)} X(\phi_2) = (\beta_2 A_{T-1}^{\phi_1}(\beta_2, \phi_1))^{1/(\phi_1 - 1)} X(\phi_1)$$

where

$$A_{T-1} = \left\{ c_{T-1}^{\phi} + \beta[p(b_{T-1}R + u\pi_{T-1})^{\alpha} + q(b_{T-1}R + d\pi_{T-1})^{\alpha}]^{\phi/\alpha} \right\}^{1/\phi}$$

and

$$X = \{[p(lu+R)^{\alpha} + q(ld+R)^{\alpha}]^{\frac{\phi-\alpha}{\alpha}} (pu(lu+R)^{\alpha-1} + qd(ld+R)^{\alpha-1})\}^{\frac{1}{\phi-1}}$$

Since X does not depend on β we can write it as $X(\phi)$. Also, we can write A_{T-1} as $A_{T-1}(\beta, \phi)$. After solving for β_2, we get

$$\beta_2 = \frac{[\beta_1 A_{T-1}^{\phi_2}(\beta_1, \phi_2)]^{\frac{\phi_1-1}{\phi_2-1}} \left[\frac{X(\phi_2)}{X(\phi_1)}\right]^{\phi_1-1}}{A_{T-1}^{\phi_1}(\beta_2, \phi_1)}$$

β_2 at time $T-1$ is equal to β_2 at time $T-2$ if and only if $A_t(.,.)$ is equal to 1 for all t. However, $A_t(.,.)$ is monotonically decreasing if $\phi > 0$, and monotonically increasing if $\phi < 0$. Hence, β and ϕ are not interchangeable analytically. □

Remark 1. *: The dependence of the percentage in stock holdings on alpha is given below.*

$$\begin{aligned} \frac{d\omega_{1,t}}{d\alpha} &= \frac{d\omega_{1,t}}{dk}\frac{dk}{d\alpha} \\ \frac{dk}{d\alpha} &= -(\alpha-1)^{-2}\left(\frac{(R-d)q}{p(u-R)}\right)^{\frac{1}{\alpha-1}} \ln\frac{(R-d)q}{p(u-R)} \\ \frac{d\omega_{1,t}}{dk} &= \frac{R(u-d)}{[k(R-d)+(u-R)]^2} \end{aligned}$$

Now, if we multiply the two partial derivatives, $\frac{dk}{d\alpha}$ and $\frac{d\omega_{1,t}}{dk}$, we get the following result for $\frac{d\omega_{1,t}}{d\alpha}$.

$$\begin{aligned}\frac{d\omega_{1,t}}{d\alpha} &= -\frac{R(u-d)}{[k(R-d)+(u-R)]^2}(\alpha-1)^{-2}\left(\frac{(R-d)q}{p(u-R)}\right)^{\frac{1}{\alpha-1}}\ln\frac{(R-d)q}{p(u-R)} \\ &= \frac{-R(u-d)(\alpha-1)^{-2}}{\left[\left[\frac{q(R-d)}{p(u-R)}\right]^{\frac{1}{\alpha-1}}(R-d)+(u-R)\right]^2}\left(\frac{(R-d)q}{p(u-R)}\right)^{\frac{1}{\alpha-1}}\ln\frac{(R-d)q}{p(u-R)}\end{aligned}$$