QUALITATIVE AND QUANTITATIVE ANALYSIS OF STOCHASTIC MODELS IN MATHEMATICAL EPIDEMIOLOGY

by

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We introduce random fluctuations on contact and recovery rates in three basic deterministic models in mathematical epidemiology and obtain stochastic counterparts. This paper addresses qualitative and quantitative analysis of stochastic SIS model with disease deaths and demographic effects, and stochastic SIR models with/without disease deaths and demographic effects. We prove the global existence of a unique strong solution and discuss stochastic asymptotic stability of disease free and endemic equilibria. We also investigate numerical properties of these models and prove the convergence of the Balanced Implicit Method approximation to the analytic solution. We simulate the models with fairly realistic parameters to visualize our conclusions.

DEDICATION

To My Family

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TABLE OF CONTENTS

LIST OF FIGURES

INTRODUCTION

SIR models are dynamical systems, as they can vary with respect to time. They are building blocks of compartmental models in mathematical epidemiology which divide the population into three classes; *Susceptible*, *Infected*, and *Removed*. Generally, these models admit two types of equilibria; disease free and endemic equilibrium. If the disease free equilibrium is globally asymptotically stable then the disease dies out. If the endemic equilibrium is globally asymptotically stable then the disease persists in the population at the equilibrium level.

The simplest SIR model, described by Kermack and McKendrick [1], is

$$\begin{aligned} S'(t) &= -\beta S(t)I(t) \\ I'(t) &= \beta S(t)I(t) - \alpha I(t) \\ R'(t) &= \alpha I(t) \end{aligned} \tag{1}$$

with initial value $(S(0), I(0), R(0)) = (S_0, I_0, 0)$, where an infection rate β and removed rate α are positive. βSI denote the number of new infections in unit time, and $1/\alpha$ is the mean of the infective period.

This model is too simple to be an effective model for most real world scenarios. It is desirable to include more assumptions in the model in order to improve its predictive power and its applicability. In this paper we consider models with demographic effects (births and deaths) and disease related deaths (non-constant population).

Establishing global stability of an endemic equilibrium of epidemic model is generally a nontrivial problem. Global stability of many epidemic models (deterministic) has been established by applying Poincaré-Bendixson theorem and Dulacs criterion [3, 4, 5, 6, 8, 9, 10, 11].

Lyapunov's direct method [12] and LaSalle's invariance principle [13] has been used in [14, 15, 16, 17] to study the stability of standard SIR models. A historic function $V(x_1, ..., x_d) = \sum_{i=1}^{d} c_i(x_i - x_i^e - x_i^e \ln \frac{x_i}{x_i^e})$ is considered as a good candidate for the Lyapunov function for epidemic models and used in recent literature [18, 19, 20, 21, 22, 23, 25, 26, 29, 30] to prove the global stability. This function has been used in Lotka-Volterra models [24, 32, 31] and discovered by Volterra himself.

Stochastic modelling is the most natural way to describe an infectious disease, because the process of transmission of infectious diseases is inherently random. Deterministic models describe the spread under the assumption of mass action, relying on the law of large numbers. However, when the population is small or the number of infected is not large, stochasticity can have a major impact [33]. Other properties that are unique to the stochastic epidemic models include the probability of an outbreak and the eradication of an endemic disease. In this paper we present three stochastic models for the spread of an infectious disease. Over the last few years, Lyapunov's direct method for the global stability of stochastic epidemic models has been used in [27, 28, 34].

Below is a brief description of the results highlighted in each chapter. The first chapter is about the preliminary. It provides the basics on mathematical epidemiology and compartmental models. The second part of the chapter mentions some definitions and inequalities in probability theory which will be used throughout the paper. It also contains theorems on existence and uniqueness of solutions to SDE's, stability of equilibrium solutions, and convergence of numerical methods.

Chapter 2 deals with qualitative analysis of stochastic SIR model with demographic effects. First of all we perturb the deterministic system (2.1), given in chapter 2, by a white noise, $\frac{dW(t)}{dt}$ and obtain a stochastic model

$$\begin{aligned} dS &= [-\beta SI + \mu(N-S)]\,dt - SI\ F_1(S,I)\,dW_1 \\ dI &= [\beta SI - (\alpha+\mu)I]\,dt + SI\ F_1(S,I)\,dW_1 - I\ F_2(S,I)\,dW_2 \\ dR &= (\alpha I - \mu R)\,dt + I\ F_2(S,I)\,dW_2 \end{aligned} \tag{2}$$

where an infection rate β, removed rate α, per capita death rate μ, and population size N are positive, the functions F_i's are locally Lipschitz continuous on $\mathbb{D} := \{(S,I) : S \geq 0, I \geq 0, S+I \leq N\}$ for $i = 1,2$ and W_i are Wiener processes defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

Because the total population size is constant, $R\ (= N-S-I)$ is determined once S and I are known, and we can drop the R equation from our model, leaving the two-dimensional system

$$\begin{aligned} dS &= [-\beta SI + \mu(N-S)]\,dt - SI\ F_1(S,I)\,dW_1 \\ dI &= [\beta SI - (\alpha+\mu)I]\,dt + SI\ F_1(S,I)\,dW_1 - I\ F_2(S,I)\,dW_2 \end{aligned} \tag{3}$$

Next the global existence of a unique strong solution on $\mathbb{D}$ is proven. It is then shown that the disease free equilibrium $(S,I) = (N,0)$ and the endemic equilibrium $(S,I) = \left(\frac{\alpha+\mu}{\beta}, \frac{\mu N}{\alpha+\beta} - \frac{\mu}{\beta}\right)$ of the system (3) are stochastically asymptotically stability under some assumptions.

Chapter 3 introduces a stochastic SIS model (Susceptible-Infected-Susceptible) with disease deaths and demographic effects

$$\begin{aligned} dS &= \left[-\beta SI + \mu(K-S) + \alpha I\right] \, dt - SI \; F_1(S,I) \; dW_1 + I \; F_2(S,I) \; dW_2 \\ dI &= [\beta SI - (\alpha+\gamma+\mu)I] \; dt + SI \; F_1(S,I) \; dW_1 - I \; F_2(S,I) \; dW_2 \end{aligned} \quad (4)$$

where K is the carrying capacity (maximum population size) and γ is the per capita disease related death rate. Since $N = S + I$ we obtain $N' = \mu(K - N) - \gamma I$ by adding the above equations. Therefore the population size N is not constant and may vary in time.

In this chapter, it is shown that a unique strong solution to the system (4) globally exists on $\mathbb{D}$. In addition stochastic asymptotic stability of the disease free equilibrium $(S, I) = (K, 0)$ and the endemic equilibrium $(S, I) = \left(\frac{\alpha+\gamma+\mu}{\beta}, \frac{\mu K}{\gamma+\mu} - \frac{\mu(\alpha+\gamma+\mu)}{\beta(\gamma+\mu)}\right)$ of the system (4) are proven.

Chapter 4 deals with global existence of a unique strong solution and stochastic asymptotic stability of the equilibria $(S, I, R) = (K, 0, 0)$ and $(S, I, R) = \left(\frac{\alpha+\gamma+\mu}{\beta}, \frac{\mu K}{\alpha+\gamma+\mu} - \frac{\mu}{\beta}, \frac{\alpha K}{\alpha+\gamma+\mu} - \frac{\alpha}{\beta}\right)$ of a stochastic SIR model with disease deaths and demographic effects

$$\begin{aligned} dS &= [-\beta SI + \mu(K-S)] \; dt - SI \; F_1(S,I,R) \; dW_1 \\ dI &= [\beta SI - (\alpha+\gamma+\mu)I] \; dt + SI \; F_1(S,I,R) \; dW_1 - I \; F_2(S,I,R) \; dW_2 \\ dR &= (\alpha I - \mu R) \; dt + I \; F_2(S,I,R) \; dW_2. \end{aligned} \quad (5)$$

In Chapter 5 the below balanced implicit method approximation is considered,

$$Y_{n+1} = Y_n + f(Y_n, t_n)\Delta_n + \sum_{j=1}^{2} g^j(Y_n, t_n)\Delta W_n^j + c(Y_n, t_n)(Y_n - Y_{n+1}) \quad (6)$$

where $c(Y_n, t_n) = A \; I_{3\times 3}$ for the unit matrix $I_{3\times 3}$ and

$$A = (\alpha+\gamma+\mu+\beta I_n) \; \Delta_n + K \; |F_1(Y_n) \; \Delta W_n^1| + \frac{K}{R_n} \; |F_2(Y_n) \; \Delta W_n^2| \quad (7)$$

for a discretization of SIR model with disease deaths and demographic effect

$$dX = f(X,t)\ dt + g(X,t)\ dW \tag{8}$$

where

$$X = \begin{pmatrix} S \\ I \\ R \end{pmatrix},\ f(X,t) = \begin{pmatrix} -\beta SI + \mu(K-S) \\ \beta SI - (\alpha+\gamma+\mu)I \\ \alpha I - \mu R \end{pmatrix},\ g(X,t) = \begin{pmatrix} -SIF_1(X) & 0 \\ SIF_1(X) & -IF_2(X) \\ 0 & IF_2(X) \end{pmatrix} \text{ and } dW = \begin{pmatrix} dW^1 \\ dW^2 \end{pmatrix}$$

and the W^j's are i.i.d. Wiener processes defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and independent of the initial value $X(0) = x_0 \in \mathbb{R}^3$ with $\mathbb{E}\|x_0\|^2 < \infty$.

Convergence of numerical approximation (6) to analytic solution X is proven by using Invariance property and V-stability of the numerical solution Y_n, Mean Square Hölder Continuity of Martingale Part and Mean Square Contractivity of X, and Local Uniform Boundedness, Local Mean Square Hölder Continuity, Mean and Mean square Consistency of X and Y_n.

We also show that the discretized stochastic SIS model with disease deaths and demographic effect (5.26) is invariant with respect to $\mathbb{D}$.

In chapter 6 we simulate two stochastic models mentioned and discretized in the previous chapter using realistic parameters.

CHAPTER 1

PRELIMINARIES

1.1 MATHEMATICAL EPIDEMIOLOGY

Mathematical epidemiology studies the spread of diseases in populations by using tools from mathematics, statistics, and computer science. Because of ethic concerns and the nature of diseases, it is difficult to do experiments searching an effective strategy for the management of diseases. Mathematical models may be needed.

An outbreak of a disease that spreads rapidly and widely is called epidemic, and a disease that exists permanently in a particular location is called endemic. Every year millions of people die because of infectious diseases such as measles, influenza, tuberculosis, or pneumonia. Diseases such as malaria, typhus, cholera, and sleeping sickness are endemic in many parts of the world. [35].

Modern mathematical epidemiology is based on compartmental models and was initially developed by R.A. Ross, W.H. Hamer, A.G. McKendrick, and W.O. Kermack between 1900 and 1935. SIR models are building blocks of compartmental models which divide the population into three classes: *Susceptible* -the number of individuals who are susceptible to the disease, that is, who are not (yet) infected, *Infected* - the number of infected individuals, and *Removed* - the number of individuals who have been infected and then removed from the possibility of being infected again or of spreading infection [35]. SIR models can be represented by a flow diagram shown in Figure 1.

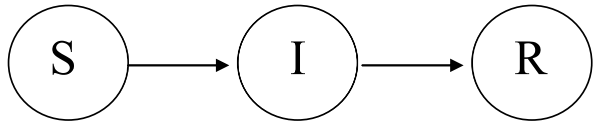


Figure 1.1: Flow chart for the SIR model without demography for constant population size.

The simplest SIR model, described by Kermack and McKendrick [1], is based on the assumptions:

i) Constant population size (no births, deaths, or migration).

ii) The only way an individual can leave the susceptible class is to become infected. The only way an individual can leave the infected class is to recover from the disease. Once an individual has recovered, the individual received immunity.

iii) Homogeneous mixing (all members of the population have identical rates of contacts).

Under these assumptions, Kermack and McKendrick [1] purposed the model

$$\begin{aligned} S'(t) &= -\beta S(t)I(t) \\ I'(t) &= \beta S(t)I(t) - \alpha I(t) \\ R'(t) &= \alpha I(t) \end{aligned} \tag{1.1}$$

where an infection rate β, and removed rate α are positive. βSI denote the number of new infections in unit time, and $1/\alpha$ is the mean of the infective period. This system has the initial condition $S(0) > 0$, $I(0) > 0$, and $R(0) = 0$.

The basic reproduction number, universally represented by the symbol $\mathcal{R}_0$, is the expected number of secondary infections produced when one infected individual entered a fully susceptible population [36, 6]. It is one of the most important quantities in epidemiology. The basic reproduction number is a threshold parameter that tells us how quickly a disease is going to spread a population. It determines whether there is an epidemic or not. If $\mathcal{R}_0 < 1$, the infection dies out (with probability 1), while if $\mathcal{R}_0 > 1$ there is an epidemic (with probability > 0) [37]. $\mathcal{R}_0$ plays an important role in stability analysis of a model. If the disease-free equilibrium is globally asymptotically stable then the disease dies out. If the endemic equilibrium is globally asymptotically stable then the disease persists in the population at the equilibrium level [22]. The basic reproduction number for the model (1.1) is $\mathcal{R}_0 = \frac{\beta}{\alpha}$.

Model (1.1) is too simple to be an effective model for most real world scenarios. It is desirable to include more assumptions in the model in order to improve its predictive power and its applicability. We can consider models with; demographic effects (births and deaths), disease related deaths (non-constant population), more compartments (exposed period, asymptomatic stage, isolation, quarantine, or vaccination), vertical transmissions (mother-to-child transmission), vector transmissions (transmissions by organisms that carry an infectious pathogen), heterogeneous mixing (diverse population), age-saturated populations, or stochastic models [35, 38]

If the disease confers no immunity against reinfection then infectives return to the susceptible class. Such a model is called an SIS model. These models are appropriate for most diseases transmitted by bacterial or helminth agents [35].

All infectious diseases are subject to randomness in terms of the nature of transmission. This thesis addresses qualitative and quantitative analysis of three basic stochastic models in mathematical epidemiology; SIS model with disease deaths and demographic effects, and SIR models with/without disease deaths and demographic effects. To incorporate stochasticity, noise is introduced directly into the deterministic models.

1.2 MATHEMATICAL BACKGROUND

1.2.1 Solutions of SDE

Consider a d-dimensional stochastic differential equation of the form

$$dX(t) = f\big(X(t),t\big)\ dt + g\big(X(t),t\big)\ dW(t) \tag{1.2}$$

$$X(t_0) = X_0, \quad t_0 \le t \le T < \infty,$$

where $f : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^{d\times m}$ are Borel measurable, $W = \{W(t)\}_{t\ge t_0}$ is an $\mathbb{R}^m$-valued Wiener process, and X_0 is an $\mathbb{R}^d$-valued random variable.

Definition. A *strong solution* $X = \{X(t)\}_{t\ge t_0}$ of SDE (1.2) is a $\{\mathcal{F}_t\}_{t\ge t_0}$-adapted stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge t_0}, \mathbb{P})$ such that

i) W is a $\{\mathcal{F}_t\}_{t\ge t_0}$-adapted Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge t_0}, \mathbb{P})$ and the increments $W(t) - W(t_0)$ is independent of a random variable X_0 for $t \ge t_0$.

ii) $\displaystyle\int_{t_0}^{t} \big(\|f\big(X(s),s\big)\| + \|g\big(X(s),s\big)\|^2\big)\ ds < \infty$

iii) X is continuous, and for all $t \ge t_0$,

$$X(t) = X_0 + \int_{t_0}^{t} f\big(X(s),s\big)\ ds + \int_{t_0}^{t} g\big(X(s),s\big)\ dW(s) \qquad \text{a.s.} \tag{1.3}$$

There is another solution concept in SDE's, a *weak solution*. For these solutions the path behavior is not essential, only the distribution of a solution X is of interest. The initial condition X_0 and the coefficients f and g are given, but a Wiener process W is not given. The pair (X, W) that satisfies the equation (1.3) needs to be determined. However we are only interested in a strong solution in this paper.

Definition. The *Frobenius norm* (Hilbert-Schmidt norm) of a matrix $\mathbf{A}_{m\times n}$ is defined by $\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2} = \sqrt{\sum_{i=1}^{m}\|A_{i.}\|_2^2} = \sqrt{\sum_{j=1}^{n}\|A_{.j}\|_2^2} = \sqrt{tr(AA^T)}$. The Frobenius norm $\|.\|_F$ and the Euclidean vector norm $\|.\|_2$ are compatible by

$$\|Ax\|_2^2 = \sum_{i=1}^{m}|A_{i.}x|^2 \le \sum_{i=1}^{m}\|A_{i.}\|_2^2\|x\|_2^2 = \|A\|_F^2\|x\|_2^2 \quad \text{for n-dimensional vector } x.$$

The vector space of $m \times n$ matrices with real entries is an inner product space with the inner product $< A, B >:= tr(AB^T)$, for matrices $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{m\times n}$.

Theorem 1.2.1. ***([39] Existence and uniqueness of Solutions)***

Consider a stochastic differential equation of the form

$$dX(t) = f\big(X(t), t\big)\ dt + g\big(X(t), t\big)\ dW(t) \tag{1.4}$$

$$X(t_0) = X_0, \quad t_0 \le t \le T < \infty,$$

where $f : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^{d\times m}$ are Borel measurable and W is an $\mathbb{R}^m$ valued Wiener process with independent component processes. Suppose X_0 is a random variable independent of $W(t) - W(t_0)$ for $t \ge t_0$, and $\mathbb{E}\|X_0\|^2 < \infty$.

There is a unique $\mathbb{R}^d$-valued strong solution X of (1.4) defined on $[t_0, T]$ if there exist two positive constants L_1 and L_2 for all $t \in [t_0, T]$ and x,$y \in \mathbb{R}^d$ such that *i) (**Lipschitz condition**)*

$$\|f(x,t) - f(y,t)\| + \|g(x,t) - g(y,t)\| \leq L_1 \; \|x - y\|, \tag{1.5}$$

*ii) (**Linear growth condition**)*

$$\|f(x,t)\|^2 + \|g(x,t)\|^2 \leq L_2 \; \big(1 + \|x\|^2\big). \tag{1.6}$$

Furthermore, the solution X is continuous with probability 1 and $\sup_{t\in[t_0,T]} \mathbb{E}\|X(t)\|^2 < \infty$.

Definition. The infinitesimal generator (differential operator) $\mathcal{L}$ associated with the SDE (1.4) is defined as

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{m}\Big(g(x,t)g^T(x,t)\Big)_{ij}\,\frac{\partial^2}{\partial x_i \partial x_j}.$$

Definition. A continuous time stochastic process $\{X(t)\}_{t\geq t_0}$ is *invariant* (a.s.) with respect to fixed simply connected domain $\mathbb{D} \subset \mathbb{R}^d$ if and only if

$$\mathbb{P}\big(X(t) \in \mathbb{D}\big) = 1 \qquad \text{for all } t \geq t_0 \tag{1.7}$$

The next theorem is taken from [40] (Thm. 5, pp. 132-133).

Theorem 1.2.2. *(**$\mathbb{D}$-invariance**) (Khas'minskiĭ [44])*

Let $\mathbb{D}$ and $\mathbb{D}_n$ be open sets in $\mathbb{R}^n$ with

$$\mathbb{D}_n \;\subseteq\; \mathbb{D}_{n+1}, \qquad \overline{\mathbb{D}}_n \;\subseteq\; \mathbb{D}, \quad \textit{and } \mathbb{D} \;= \bigcup_n \; \mathbb{D}_n$$

and suppose f and g satisfy the existence and uniqueness conditions for solutions of (1.4) on each set $t > t_0$, $x \in \mathbb{D}_n$. Suppose, further there is a nonnegative continuous function $V : \mathbb{D} \times [t_0, T] \to \mathbb{R}_+$ with continuous partial derivatives $\partial V/\partial t$, $\partial V/\partial x_i$, and $\partial^2 V/\partial x_i \partial x_j$ and satisfying $\mathcal{L}V \leq c\, V$ for some positive constant c and $t > t_0$, $x \in \mathbb{D}$. If also,

$$\inf_{t>t_0, x\in\mathbb{D}\setminus\mathbb{D}_n} V(x,t) \to \infty \qquad \text{as } n \to \infty$$

then, for any X_0 independent of W such that $\mathbb{P}(X_0 \in \mathbb{D}) = 1$, there is a unique solution X of (1.4) with $X(0) = X_0$, and $X(t) \in \mathbb{D}$ for all $t > 0$, that is $\mathbb{P}(\tau_{\mathbb{D}} = \infty) = 1$.

1.2.2 Stability of Solutions

We use the following assumption to define the stability concepts.

Assumption 1. Consider a d-dimensional stochastic differential equation

$$dX(t) = f\big(X(t), t\big)\, dt + g\big(X(t), t\big)\, dW(t), \quad t \geq t_0, \quad X(t_0) = x_0. \tag{1.8}$$

Assume that f and g satisfy, in addition to the assumptions of the existence and uniqueness Theorem 1.2.1, $f(x^e, t) = 0$ and $g(x^e, t) = 0$ for all equilibrium solutions x^e for $t \geq t_0$, and they have continuous coefficients with respect to t. Furthermore assume that x_0 is a nonrandom constant with probability 1.

Definition. Suppose that the Assumption 1 is satisfied.

i) Then the equilibrium solution x^e is said to be *stochastically stable* (stable in probability) if for every $\epsilon > 0$ and $s \geq t_0$

$$\lim_{x_0 \to x^e} \mathbb{P}\left(\sup_{t_0 \leq s < \infty} \|X(s, x_0, t)\| \geq \epsilon \right) \;=\; 0 \tag{1.9}$$

where $X(s, x_0, t)$ denotes the solution of (1.8) satisfying $X(s) = x_0$.

ii) The equilibrium solution x^e is said to be *stochastically asymptotically stable* if it is stochastically stable and

$$\lim_{x_0 \to x^e} \mathbb{P}\left(\lim_{t \to \infty} X(s, x_0, t) = 0 \right) \;=\; 1, \tag{1.10}$$

iii) and *stochastically asymptotically stable in the large* if, further,

$$\mathbb{P}\left(\lim_{t \to \infty} X(s, x_0, t) = 0 \right) \;=\; 1 \text{ for all } x_0 \in \mathbb{R}^d. \tag{1.11}$$

Theorem 1.2.3. *(Arnold, L. [41]) Suppose that the Assumption 1 is satisfied.*

i) Suppose that there exist a positive definite function $V(x,t) \in C^{2,1}\big(U_h \times [t_0, \infty)\big)$, where $U_h = \{x \in \mathbb{R}^d : \|x\| < h\}$ for $h > 0$, such that

$$\mathcal{L}V(x,t) \leq 0, \quad t \geq t_0, \quad 0 < \|x\| \leq h. \tag{1.12}$$

Then, the equilibrium solution of (1.8) is stochastically stable.

ii) If, in addition, V is decrescent $\big($there exists a positive definite function V_1 such that $V(x,t) \leq V_1(x)$ for all $x \in U_h\big)$ and $\mathcal{L}V(x,t)$ is negative definite, then the equilibrium solution of (1.8) is stochastically asymptotically stable.

iii) If the assumptions of part ii) hold for a radially unbounded function $V(x,t)$ i.e. $V(x,t) \to \infty$ as $\|x\| \to \infty$ uniformly on t, defined everywhere on $\mathbb{R}^d \times [t_0, \infty)$, then the equilibrium solution of (1.8) is stochastically asymptotically stable in the large.

*V is called a **Lyapunov function**.*

1.2.3 Convergence of Numerical Methods

This section stated the main concepts of numerical approximation of stochastic processes as appears in Schurz [46]. Let $X_{Z,s}(t)$, $Y_{Z,s}(t)$ be the one step representations of stochastic processes X, Y evaluated at time $t \geq s$, started from $Z \in H_2([0,T], \mu, H)$. They are supposed to be constructable along any $\mathcal{F}_t$-adapted discretization of the given deterministic finite time interval $[0,T]$ and could depend on a certain mesh size Δ_{max}. Assume that there are deterministic real constants $r_0, r_{SM}, r_2 \geq 0,\ 0 < \delta_0 \leq 1$ such that we have

(A1) **Strong $(\mathbb{D}_t)$-invariance of X,Y**

(A2) **V-Stability of Y**, i.e. $\exists\ V : H_2([0,T], \mu, H) \to \mathbb{R}_+, V(Y(t))$ is $\mathcal{F}_t$-adapted and $\exists$ real constant K_S^Y $\forall t, h : 0 \leq h \leq \delta_0,\ 0 \leq t,\ t+h \leq T$

$$\mathbb{E}[V(Y_{Y(t),t})(t+h)|\mathcal{F}_t] \leq e^{2K_S^Y h} V(Y(t)), \tag{1.13}$$

(A3) **Mean Square Contractivity of X**, i.e. $\exists$ real constant K_C^X such that $\forall X(t), Y(t) \in \mathbb{D}_t$ $\forall t, h : 0 \leq h \leq \delta_0,\ 0 \leq t,\ t+h \leq T$

$$\mathbb{E}[\|X_{X(t),t}(t+h) - X_{Y(t),t}(t+h)\|_H^2\ |X(t), Y(t)] \leq e^{2K_C^X h} \|X(t) - Y(t)\|_H^2 \tag{1.14}$$

(A4) **Mean Consistency of (X,Y)** with rate $r_0 > 0$, i.e. $\exists$ real constant K_0^C such that

$\exists Z(t) \in \mathbb{D}_t \ \forall t, h : 0 \le h \le \delta_0,\ 0 \le t,\ t+h \le T$

$$\left\| \mathbb{E}[X_{Z(t),t}(t+h)|Z(t)] - \mathbb{E}[Y_{Z(t),t}(t+h)|Z(t)] \right\|_H \le K_0^C \sqrt{V(Z(t))}\ h^{r_0} \tag{1.15}$$

(A5) **Mean Square Consistency of (X,Y)** with rate $r_2 > 0$, i.e. $\exists$ real constant K_2^C such that $\forall Z(t) \in \mathbb{D}_t \ \forall t, h : 0 \le h \le \delta_0,\ 0 \le t,\ t+h \le T$

$$\Big(\mathbb{E}[\|X_{Z(t),t}(t+h) - Y_{Z(t),t}(t+h)\|_H^2 \ |Z(t)]\Big)^{1/2} \le K_2^C \sqrt{V(Z(t))}\ h^{r_2} \tag{1.16}$$

(A6) **Mean Square Hölder-type Smoothness of Martingale part of X** with rate $r_{SM} \in [0, 0.5]$, i.e. $\exists$ real constant $K_{SM} \ge 0$ such that $\forall X(t), Y(t) \in \mathbb{D}_t \ \forall t, h : 0 \le h \le \delta_0,\ 0 \le t,\ t+h \le T$

$$\mathbb{E}\|M_{X(t),t}(t+h) - M_{Y(t),t}(t+h)\|_H^2 \le (K_{SM})^2 \ \mathbb{E}\|X(t) - Y(t)\|_H^2 \ h^{2r_{SM}} \tag{1.17}$$

where $M_{z,t}(t+h) = X_{z,t}(t+h) - \mathbb{E}[X_{z,t}(t+h)|\mathcal{F}_t]$ for $z = X(t), Y(t)$.

(A7) **Interplay between consistency rates** given by $r_0 \ge r_2 + r_{SM} \ge 1$.

(A8) **Initial moment V-boundedness** $\mathbb{E}[V(X_0)] + \mathbb{E}[V(Y_0)] < \infty$

Stochastic approximation problems satisfying the assumptions (A1)-(A8) on H_2 are called **well-posed**.

Theorem 1.2.4. *(Schurz [46]) Assume that the conditions (A1)-(A8) are satisfied and* $\mathbb{E}\|X_0 - Y_0\|_H^2 < K_{initial} \Delta_{max}^{r_g}$*. Then stochastic processes* $X, Y \in H_2([0,T], \mu, H)$ *converge to each other with respect to the natural induced metric*

$$m(X,Y) = (< X - Y, X - Y >_{H_2})^{1/2}$$

with worst case convergence rate $r_g = r_2 + r_{SM} - 1$.

1.2.4 Inequalities

1. Hölder's Inequality: If D is a measurable subset of $\mathbb{R}^n$ with the Lebesgue measure μ, and f and g are measurable functions on D, then

$$\left\| \int_D f(u)g(u)d\mu(u) \right\| \leq \left(\int_D \|f(u)\|^p d\mu(u) \right)^{1/p} \left(\int_D \|g(u)\|^q d\mu(u) \right)^{1/q} \tag{1.18}$$

for $1/p + 1/q = 1$ with $1 < p, q < \infty$. Also,

$$\|\mathbb{E}(XY)\| \leq \left(\mathbb{E}\|X\|^p\right)^{1/p} \left(\mathbb{E}\|Y\|^q\right)^{1/q}. \tag{1.19}$$

One of the most used example is $\|\mathbb{E}(X)\| \leq \left(\mathbb{E}\|X\|^2\right)^{1/2}$.

2. Cauchy-Bunyakovsky-Schwarz Inequality:

$$\left\| \int_D f(u)g(u)du \right\|^2 \leq \left(\int_D \|f(u)\|^2 du \right) \left(\int_D \|g(u)\|^2 du \right) \tag{1.20}$$

3. Jensen's Inequality: $f\big(\mathbb{E}(x)\big) \leq \mathbb{E}\big(f(x)\big)$ for the convex function f. One of the most used example is $\|\mathbb{E}x\| \leq \mathbb{E}\|x\|$.

4. It$\hat{o}$ Isometry:

$$\mathbb{E} \left\| \int_s^t f(u)dW(u) \right\|^2 = \mathbb{E} \int_s^t \|f(u)\|^2 du \tag{1.21}$$

5. Algebraic Inequalities:

$$\begin{aligned} (a \pm b)^2 &\leq 2(a^2 + b^2) \\ (a + b + c)^2 &\leq 3(a^2 + b^2 + c^2) \end{aligned} \tag{1.22}$$

CHAPTER 2

STOCHASTIC SIR MODEL

2.1 DETERMINISTIC MODEL

An SIR model with births and deaths, due to Kermack and McKendrick [1], is

$$\begin{aligned} S'(t) &= -\beta S(t)I(t) + \mu\big(N - S(t)\big) \\ I'(t) &= \beta S(t)I(t) - (\alpha + \mu)I(t) \\ R'(t) &= \alpha I(t) - \mu R(t) \end{aligned} \tag{2.1}$$

where an infection rate β, removed rate α, and per capita death rate μ are positive. The population size $N = S(t) + I(t) + R(t)$ is held constant for all time by balancing births and deaths. The transfer diagram for the SIR model is shown in Figure 1.

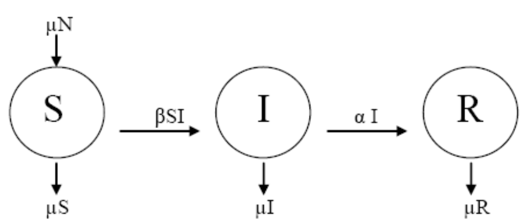


Figure 2.1: Flow chart for the SIR model with constant population size.

In the model, μN is the number of births (immigrants) in unit time, βSI is the number of new infections in unit time, and $1/\alpha$ is the mean of the infective period.

The basic reproduction number for this model is $\mathcal{R}_0 = \dfrac{\beta N}{\alpha + \mu}$.

2.2 STOCHASTIC MODEL

We perturbed the deterministic system (2.1) by a white noise, $\frac{dW(t)}{dt}$, and obtained a stochastic model by replacing the rates β, and α by $\beta + F_1\big(S(t), I(t)\big)\frac{dW_1(t)}{dt}$, and $\alpha + F_2\big(S(t), I(t)\big)\frac{dW_2(t)}{dt}$ respectively, where F_i's are locally Lipschitz-continuous on $\mathbb{D} = \{(S, I) \in \mathbb{R}^2;\ S \geq 0,\ I \geq 0,\ S + I \leq N\}$ and W_i's are i.i.d. Wiener processes defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ for all $t \geq 0$.

Therefore stochastic SIR model is,

$$\begin{aligned} dS(t) &= \Big(-\beta S(t)I(t) + \mu\big(N - S(t)\big)\Big)\ dt - S(t)I(t)\ F_1\big(S(t), I(t)\big)\ dW_1(t) \\ dI(t) &= \Big(\beta S(t)I(t) - (\alpha+\mu)I(t)\Big)\ dt + S(t)I(t)\ F_1\big(S(t), I(t)\big)\ dW_1(t) - I(t)\ F_2\big(S(t), I(t)\big)\ dW_2(t) \\ dR(t) &= \Big(\alpha I(t) - \mu R(t)\Big)\ dt + I(t)\ F_2\big(S(t), I(t)\big)\ dW_2(t) \end{aligned} \tag{2.2}$$

where the parameters α, β and μ are positive and the functions F_i's are locally Lipschitz-continuous on $\mathbb{D}$ for all $t \geq 0$.

Because the total population size is constant, $R\ (= N - S - I)$ is determined once S and I are known, and we can drop the R equation from our model, leaving the two-dimensional system

$$\begin{aligned} dS(t) &= \Big(-\beta S(t)I(t) + \mu\big(N - S(t)\big)\Big)\ dt - S(t)I(t)\ F_1\big(S(t), I(t)\big)\ dW_1(t) \\ dI(t) &= \Big(\beta S(t)I(t) - (\alpha+\mu)I(t)\Big)\ dt + S(t)I(t)\ F_1\big(S(t), I(t)\big)\ dW_1(t) - I(t)\ F_2\big(S(t), I(t)\big)\ dW_2(t) \end{aligned} \tag{2.3}$$

The disease free equilibrium is $(S_1, I_1) = (N, 0)$. There exist a unique endemic equilibrium

$$(S_2, I_2) = \left(\frac{\alpha+\mu}{\beta}, \frac{\mu N}{\alpha+\beta} - \frac{\mu}{\beta}\right) = \left(\frac{N}{\mathcal{R}_0}, \frac{\mu N}{\alpha+\beta}\left(1 - \frac{1}{\mathcal{R}_0}\right)\right) = \left(\frac{N}{\mathcal{R}_0}, \frac{\mu}{\beta}(\mathcal{R}_0 - 1)\right) \tag{2.4}$$

if $\mathcal{R}_0 > 1$ and $F_i(S_2, I_2) = 0$.

2.2.1 Existence and Uniqueness of Solutions

Consider the stochastic SIR model

$$\begin{pmatrix} dS(t) \\ dI(t) \end{pmatrix} = \underbrace{\begin{pmatrix} -\beta S(t)I(t) + \mu\big(N - S(t)\big) \\ \beta S(t)I(t) - (\alpha+\mu)I(t) \end{pmatrix}}_{= f\left(S(t),I(t),t\right)} dt \tag{2.5}$$
$$+ \underbrace{\begin{pmatrix} -S(t)I(t)F_1\big(S(t),I(t)\big) & 0 \\ S(t)I(t)F_1\big(S(t),I(t)\big) & -I(t)F_2\big(S(t),I(t)\big) \end{pmatrix}}_{= g\left(S(t),I(t),t\right)} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}$$

with initial condition $\big(S(t_0), I(t_0)\big) = (S_0, I_0)$, where the parameters α, β, γ, and μ are positive and F_i are locally Lipschitz-continuous on $\mathbb{D} = \{(S,I) \in \mathbb{R}^2;\ S \geq 0,\ I \geq 0,\ S+I \leq N\}$ for all $t \geq t_0$.

Lemma 2.2.1. *The coefficient $f(S,I,t)$ is locally Lipschitz-continuous on $\mathbb{D}$ with Lipschitz constant $L_1 = 8\beta^2N^2 + 2(\alpha+\mu)^2$.*

Proof. For all $t \geq t_0$, $(S,I) \in \mathbb{D}$ and $(S^*, I^*) \in \mathbb{D}$

$$\Big\| f(S,I,t) - f(S^*,I^*,t) \Big\|^2 = \left\| \begin{pmatrix} -\beta(SI - S^*I^*) - \mu(S - S^*) \\ \beta(SI - S^*I^*) - (\alpha+\mu)(I - I^*) \end{pmatrix} \right\|^2$$

$$\begin{aligned}
&= \Big(-\beta(SI - S^*I^*) - \mu(S - S^*) \Big)^2 + \Big(\beta(SI - S^*I^*) - (\alpha+\mu)(I - I^*) \Big)^2 \\
&\overset{(1.22)}{\leq} 2\beta^2(SI - S^*I^*)^2 + 2\mu^2(S - S^*)^2 + 2\beta^2(SI - S^*I^*)^2 + 2(\alpha+\mu)^2(I - I^*)^2 \\
&= 4\beta^2(SI - S^*I^*)^2 + 2\mu^2(S - S^*)^2 + 2(\alpha+\mu)^2(I - I^*)^2 \\
&= 4\beta^2(SI - S^*I + S^*I - S^*I^*)^2 + 2\mu^2(S - S^*)^2 + 2(\alpha+\mu)^2(I - I^*)^2 \\
&\leq 8\beta^2(S - S^*)^2I^2 + 8\beta^2S^{*2}(I - I^*)^2 + 2\mu^2(S - S^*)^2 + 2(\alpha+\mu)^2(I - I^*)^2
\end{aligned}$$

$$\overset{I\leq N \& S^*\leq N}{\leq} (8\beta^2N^2+2\mu^2)(S-S^*)^2+\Big(8\beta^2N^2+2(\alpha+\mu)^2\Big)(I-I^*)^2$$

$$\leq \Big(8\beta^2N^2+2(\alpha+\mu)^2\Big)\Big((S-S^*)^2+(I-I^*)^2\Big)$$

$$= L_1\left\|\begin{pmatrix}S-S^* & I-I^*\end{pmatrix}^T\right\|^2 \text{ where } L_1=8\beta^2N^2+2(\alpha+\mu)^2$$

□

Lemma 2.2.2. *The coefficient $g(S,I,t)$ is locally Lipschitz-continuous on $\mathbb{D}$ with Lipschitz constant* $L_2=\sup_{(S,I)\in\ \mathbb{D}}\Big\{4\widetilde{L}_1N^4+8F_1^2(S,I)N^2+2\widetilde{L}_2N^2+2F_2^2(S,I)\Big\}$.

Proof. For all $t\geq t_0$, $(S,I)\in\mathbb{D}$ and $(S^*,I^*)\in\mathbb{D}$,

$$\Big\|F_i(S,I,t)-F_i(S^*,I^*,t)\Big\|_F^2\leq\widetilde{L}_i\Big((S-S^*)^2+(I-I^*)^2\Big), \tag{2.6}$$

since F_i's are locally Lipschitz-continuous on $\mathbb{D}$.

$$\begin{aligned}
&\Big\|g(S,I,t)-g(S^*,I^*,t)\Big\|_F^2\\
=&\left\|\begin{pmatrix}-SIF_1(S,I)+S^*I^*F_1(S^*,I^*) & 0\\ SIF_1(S,I)-S^*I^*F_1(S^*,I^*) & -IF_2(S,I)+I^*F_2(S^*,I^*)\end{pmatrix}\right\|_F^2\\
=&\,2\Big(SIF_1(S,I)-S^*I^*F_1(S^*,I^*)\Big)^2+\Big(IF_2(S,I)-I^*F_2(S^*,I^*)\Big)^2\\
=&\,2\Big(SIF_1(S,I)-SIF_1(S^*,I^*)+SIF_1(S^*,I^*)-S^*I^*F_1(S^*,I^*)\Big)^2\\
&+\Big(IF_2(S,I)-IF_2(S^*,I^*)+IF_2(S^*,I^*)-I^*F_2(S^*,I^*)\Big)^2\\
=&\,2\Big\{SI\Big(F_1(S,I)-F_1(S^*,I^*)\Big)+\Big(SI-S^*I^*\Big)F_1(S^*,I^*)\Big\}^2\\
&+\Big\{I\Big(F_2(S,I)-F_2(S^*,I^*)\Big)+\Big(I-I^*\Big)F_2(S^*,I^*)\Big\}^2\\
\overset{(1.22)}{\leq}&\,4S^2I^2\Big(F_1(S,I)-F_1(S^*,I^*)\Big)^2+4F_1^2(S^*,I^*)\Big(SI-S^*I+S^*I-S^*I^*\Big)^2\\
&+2I^2\Big(F_2(S,I)-F_2(S^*,I^*)\Big)^2+2F_2^2(S^*,I^*)(I-I^*)^2
\end{aligned}$$

$$
\begin{aligned}
&\overset{(2.6)}{\leq} \quad 4S^2I^2\widetilde{L}_1\Big((S-S^*)^2+(I-I^*)^2\Big)+8F_1^2(S^*,I^*)I^2(S-S^*)^2 \\
&\quad +8F_1^2(S^*,I^*)S^{*2}(I-I^*)^2+2I^2\widetilde{L}_2\Big((S-S^*)^2+(I-I^*)^2\Big)+2F_2^2(S^*,I^*)(I-I^*)^2 \\
&\leq \quad \Big(4\widetilde{L}_1N^4+8F_1^2(S^*,I^*)N^2+2\widetilde{L}_2N^2\Big)(S-S^*)^2 \\
&\quad +\Big(4\widetilde{L}_1N^4+8F_1^2(S^*,I^*)N^2+2\widetilde{L}_2N^2+2F_2^2(S^*,I^*)\Big)(I-I^*)^2 \\
&\leq \quad \sup_{(S,I)\in\ \mathbb{D}}\Big\{4\widetilde{L}_1N^4+8F_1^2(S^*,I^*)N^2+2\widetilde{L}_2N^2+2F_2^2(S^*,I^*)\Big\}\Big((S-S^*)^2+(I-I^*)^2\Big) \\
&= \quad L_2\left\|\begin{pmatrix}S-S^* & I-I^*\end{pmatrix}^T\right\|_F^2
\end{aligned}
$$

where $L_2=\sup_{(S,I)\in\ \mathbb{D}}\Big\{4\widetilde{L}_1N^4+8F_1^2(S,I)N^2+2\widetilde{L}_2N^2+2F_2^2(S,I)\Big\}$ □

Lemma 2.2.3. *$f(S,I,t)$ satisfies linear growth condition on $\mathbb{D}$ with the growth coefficient $L_3=\max\Big\{4\beta^2N^2+2(\alpha+\mu)^2,2\mu^2N^2,2\mu^2\Big\}$.*

Proof. For all $t\geq t_0$ and $(S,I)\in\mathbb{D}$,

$$
\begin{aligned}
\Big\|f(S,I,t)\Big\|^2 \quad &= \quad \left\|\begin{pmatrix}-\beta SI+\mu(N-S)\\ \beta SI-(\alpha+\mu)I\end{pmatrix}\right\|^2 \\
&= \quad \Big(-\beta SI+\mu(N-S)\Big)^2+\Big(\beta SI-(\alpha+\mu)I\Big)^2 \\
&\overset{(1.22)}{\leq} \quad 4\beta^2S^2I^2+2\mu^2(N-S)^2+2(\alpha+\mu)^2I^2 \\
&\leq \quad 4\beta^2S^2I^2+2\mu^2N^2+2\mu^2S^2+2(\alpha+\mu)^2I^2 \\
&\overset{S\leq N}{\leq} \quad 4\beta^2N^2I^2+2\mu^2N^2+2\mu^2S^2+2(\alpha+\mu)^2I^2 \\
&\leq \quad \max\Big\{4\beta^2N^2+2(\alpha+\mu)^2,2\mu^2N^2,2\mu^2\Big\}\Big(1+S^2+I^2\Big) \\
&= \quad L_3(1+S^2+I^2)=L_3\left(1+\left\|\begin{pmatrix}S & I\end{pmatrix}^T\right\|^2\right)
\end{aligned}
$$

where $L_3=\max\Big\{4\beta^2N^2+2(\alpha+\mu)^2,2\mu^2N^2,2\mu^2\Big\}$. □

Lemma 2.2.4. *$g(S,I,t)$ satisfies linear growth condition on $\mathbb{D}$ with the growth coefficient $L_4 = \max\Big\{\sup_{(S,I)\in\ \mathbb{D}} 2N^2F_1^2(S,I), \sup_{(S,I)\in\ \mathbb{D}} F_2^2(S,I)\Big\}$.*

Proof. For all $t \geq t_0$ and $(S,I)\in\mathbb{D}$,

$$\begin{aligned}
\Big\|g(S,I,t)\Big\|_F^2 &= \left\|\begin{pmatrix} -SIF_1(S,I) & 0) \\ SIF_1(S,I) & -IF_2(S,I)\end{pmatrix}\right\|_F^2 \\
&= 2S^2I^2F_1^2(S,I) + I^2F_2^2(S,I) \\
&\leq 2N^2S^2F_1^2(S,I) + I^2F_2^2(S,I) \\
&\leq \max\Big\{\sup_{(S,I)\in\ \mathbb{D}} 2N^2F_1^2(S,I), \sup_{(S,I)\in\ \mathbb{D}} F_2^2(S,I)\Big\}(S^2+I^2) \\
&\leq L_4(1+S^2+I^2) = L_4\left(1+\Big\|\begin{pmatrix}S & I\end{pmatrix}^T\Big\|_F^2\right)
\end{aligned}$$

where $L_4 = \max\Big\{\sup_{(S,I)\in\ \mathbb{D}} 2N^2F_1^2(S,I), \sup_{(S,I)\in\ \mathbb{D}} F_2^2(S,I)\Big\}$.

□

Theorem 2.2.5. *Let $\big(S(t_0), I(t_0)\big) = \big(S_0, I_0\big) \in \mathbb{D} = \big\{(S,I)\in\mathbb{R}^2 : S\geq 0, I\geq 0, S+I\leq N\big\}$, and $\big(S_0,I_0\big)$is independent of $W(t)-W(t_0)$ for $t\geq t_0$. Then the stochastic SIR model (2.3) admits a unique global solution $\big(S(t),I(t)\big)$on $t\geq t_0$ and this solution is invariant with respect to $\mathbb{D}$, where α, β, γ, and μ are positive and F_i's are locally Lipschitz-continuous on $\mathbb{D}$.*

Proof. We use Theorem 1.2.2 and follow ideas of Schurz [43]. Since the coefficients of the system (2.3) are locally Lipschitz-continuous and satisfy linear growth condition on $\mathbb{D}$, for any initial value $(S_0,I_0)\in\mathbb{D}$ there is a unique local solution on $t\in\big[t_0,\tau(\mathbb{D})\big)$, where $\tau(\mathbb{D})$ is the random time of first exit of stochastic process $\big(S(t),I(t)\big)$ from the domain $\mathbb{D}$, started in $\big(S(s),I(s)\big) = \big(S_0,I_0\big)\in\mathbb{D}$ at the initial time $s\in[t_0,\infty)$. To make this solution global, we need to prove that $\mathbb{P}\big(\tau(\mathbb{D})=\infty\big)=1$ a.s.

Let $\mathbb{D}_n=\big\{(S,I): e^{-n}<S<N-e^{-n},\ e^{-n}<I<N-e^{-n},\ S+I\le N\big\}$ for $n\in\mathbb{N}$. The system (2.3) has a unique solution up to stopping time $\tau(\mathbb{D}_n)$.

Let $V(S,I)=I-\ln I+S-\ln S+N-S-\ln(N-S)$ defined on $\overset{\circ}{\mathbb{D}}=\big\{(S,I)\in\mathbb{R}^2: S>0,\ I>0,\ S+I\le N\big\}$ and assume that $\mathbb{E}V(S_0,I_0)<\infty$. Note that $V(S,I)\ge 3$ for $(S,I)\in\overset{\circ}{\mathbb{D}}$.

Let $W(S,I,t)=e^{-c(t-s)}V(S,I)$ defined on $\overset{\circ}{\mathbb{D}}\times[s,\infty)$, where

$$c=\frac{1}{3}\left(N(N+1)\beta+\alpha+2\mu+\sup_{(S,I)\in\overset{\circ}{\mathbb{D}}}\left(\frac{3}{2}S^2F_1^2(S,I)+\frac{1}{2}F_2^2(S,I)\right)\right). \tag{2.7}$$

Then,

$$\mathcal{L}V(S,I)=\big(-\beta SI+\mu(N-S)\big)\frac{\partial V}{\partial S}+\big(\beta SI-(\alpha+\mu)I\big)\frac{\partial V}{\partial I}+A(F_1,F_2)$$

for $(S,I)\in\overset{\circ}{\mathbb{D}}$.

An upper bound of the last term, A in $\mathcal{L}V(S,I)$ can be obtained by

$$\begin{aligned}
& A(F_1,F_2)=\frac{1}{2}\sum_{i,j=1}^{2}\left(gg^T\right)_{ij}\frac{\partial^2 V}{\partial x_i\partial x_j}\\
= \quad & \frac{1}{2}\Big(S^2I^2F_1^2(S,I)\frac{\partial^2 V}{\partial S^2}-2S^2I^2F_1^2(S,I)\frac{\partial^2 V}{\partial SI}+\big(S^2I^2F_1^2(S,I)+I^2F_2^2(S,I)\big)\frac{\partial^2 V}{\partial I^2}\Big)\\
= \quad & \frac{1}{2}\left(S^2I^2F_1^2(S,I)\left(\frac{1}{S^2}+\frac{1}{(N-S)^2}\right)+\big(S^2I^2F_1^2(S,I)+I^2F_2^2(S,I)\big)\frac{1}{I^2}\right)\\
= \quad & \frac{1}{2}\left(\left(I^2+\frac{S^2I^2}{(N-S)^2}+S^2\right)F_1^2(S,I)+F_2^2(S,I)\right)\\
\le \quad & \frac{1}{2}\big((I^2+2S^2)F_1^2(S,I)+F_2^2(S,I)\big) \qquad \text{by } S+I\le N \ \Rightarrow\ \frac{I}{N-S}\le 1\\
\overset{S\le N,\ I\le N}{\le} \quad & \sup_{(S,I)\in\overset{\circ}{\mathbb{D}}}\Big(\frac{3}{2}N^2F_1^2(S,I)+\frac{1}{2}F_2^2(S,I)\Big).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{L}V(S,I) &\leq \big(-\beta SI+\mu(N-S)\big)\left(-\frac{1}{S}+\frac{1}{N-S}\right)+\big(\beta SI-(\alpha+\mu)I\big)\left(1-\frac{1}{I}\right)\\
&\qquad + \sup_{(S,I)\in \overset{\circ}{\mathbb{D}}}\Big(\frac{3}{2}N^2F_1^2(S,I)+\frac{1}{2}F_2^2(S,I)\Big)\\
&\leq \beta I-\frac{\beta SI}{N-S}-\frac{\mu(N-S)}{S}+\mu+\beta SI-(\alpha+\mu)I-\beta S+\alpha+\mu\\
&\qquad + \sup_{(S,I)\in \overset{\circ}{\mathbb{D}}}\Big(\frac{3}{2}N^2F_1^2(S,I)+\frac{1}{2}F_2^2(S,I)\Big)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}V(S,I) &\leq \beta I+\mu+\beta SI+\alpha+\mu+\sup_{(S,I)\in \overset{\circ}{\mathbb{D}}}\Big(\frac{3}{2}N^2F_1^2(S,I)+\frac{1}{2}F_2^2(S,I)\Big)\\
&\leq \beta N+\beta N^2+\alpha+2\mu+\sup_{(S,I)\in \overset{\circ}{\mathbb{D}}}\Big(\frac{3}{2}N^2F_1^2(S,I)+\frac{1}{2}F_2^2(S,I)\Big)\\
&\overset{(3.5)}{=} 3\ c.
\end{aligned}$$

Therefore, $\mathcal{L}V(S,I)\leq c\ V(S,I)$ on $\in \overset{\circ}{\mathbb{D}}$ since $V(S,I)\geq 3$ on $\in \overset{\circ}{\mathbb{D}}$,

Furthermore, $\mathcal{L}W(S,I,t)=e^{-c(t-s)}\Big(-c\ V(S,I)+\mathcal{L}V(S,I)\Big)\leq 0.$

Note that,

$$\begin{aligned}
\inf_{(S,I)\in \overset{\circ}{\mathbb{D}}\backslash\mathbb{D}_n} V(S,I) &\geq \inf_{(S,I)\in \overset{\circ}{\mathbb{D}}\backslash\mathbb{D}_n}\big(I-\ln I\big)+\inf_{(S,I)\in \overset{\circ}{\mathbb{D}}\backslash\mathbb{D}_n}\big(S-\ln S\big)\\
&\qquad +\inf_{(S,I)\in \overset{\circ}{\mathbb{D}}\backslash\mathbb{D}_n}\big(N-S-\ln(N-S)\big)\\
&\geq e^{-n}+n+e^{-n}+n+1\\
&> 2n+1.
\end{aligned}$$

Now, define $\tau_n=\min\{t,\tau(\mathbb{D}_n)\}$ and apply Dynkin's formula

$$\begin{aligned}
\mathbb{E}\ W(S(\tau_n),I(\tau_n),\tau_n) &= \mathbb{E}\ W(S(s),I(s),s)+\mathbb{E}\int_s^{\tau_n}\mathcal{L}W(S(u),I(u),u)\ du\\
&\leq \mathbb{E}\ W(S(s),I(s),s)\\
&= \mathbb{E}V(S(s),I(s))=\mathbb{E}V(S_0,I_0).
\end{aligned}$$

Next, $\mathbb{E}\Big[e^{c(t-\tau_n)}V\big(S(\tau_n),I(\tau_n)\big)\Big] \;=\; \mathbb{E}\Big[e^{c(t-s)}e^{-c(\tau_n-s)}V\big(S(\tau_n),I(\tau_n)\big)\Big]$

$$= \mathbb{E}\Big[e^{c(t-s)}\; W\big(S(\tau_n),I(\tau_n),\tau_n\big)\Big]$$

$$\le e^{c(t-s)}\mathbb{E}V(S_0,I_0).$$

We have $\inf_{(S,I)\in \overset{\circ}{\mathbb{D}}\setminus\mathbb{D}_n} V(S,I) > 2n+1$ and $\mathbb{E}\Big[e^{c(t-\tau_n)}V\big(S(\tau_n),I(\tau_n)\big)\Big] \le e^{c(t-s)}\mathbb{E}V(S_0,I_0)$.

Therefore, $0 \le \mathbb{P}\big(\tau(\overset{\circ}{\mathbb{D}}) < t\big) \overset{\mathbb{D}_n\subseteq \overset{\circ}{\mathbb{D}}}{\le} \mathbb{P}\big(\tau(\mathbb{D}_n) < t\big)$

$$= \mathbb{P}\big(\tau_n < t\big)$$

$$= \mathbb{E}\big(\mathbf{1}_{\tau_n<t}\big) \qquad \text{where } \mathbf{1} \text{ is the indicator function}$$

$$\le \mathbb{E}\left(e^{c(t-\tau_n)}\;\frac{V\Big(S\big(\tau(\mathbb{D}_n)\big),I\big(\tau(\mathbb{D}_n)\big)\Big)}{\inf_{(S,I)\in \overset{\circ}{\mathbb{D}}\setminus\mathbb{D}_n} V(S,I)}\;\mathbf{1}_{\tau_n<t}\right)$$

$$\le e^{c(t-s)}\;\frac{\mathbb{E}V(S_0,I_0)}{\inf_{(S,I)\in \overset{\circ}{\mathbb{D}}\setminus\mathbb{D}_n} V(S,I)}$$

$$\le e^{c(t-s)}\;\frac{\mathbb{E}V(S_0,I_0)}{2n+1} \qquad \to \quad 0 \qquad \text{as } \; n\to\infty$$

for all $(S_0,I_0)\in\; \mathbb{D}_n$ (for large n), and for all fixed $t\in[s,\infty)$.

Thus $\mathbb{P}\big(\tau(\overset{\circ}{\mathbb{D}})<t\big) = \mathbb{P}\big(\tau(\mathbb{D}_n)<t\big) = 0$ for $(S_0,I_0)\in\overset{\circ}{\mathbb{D}}$ and $t\ge 0$, that is, $\mathbb{P}\big(\tau(\overset{\circ}{\mathbb{D}})=\infty\big)=1$. This proves the invariance property and the global existence of the solution $\big(S(t),I(t)\big)$ on $\overset{\circ}{\mathbb{D}}$. Uniqueness and continuity of the solution is obtained by a result from Theorem 1.2.2.

Note that $I=0$ and $S=0$ are not in our domain $\overset{\circ}{\mathbb{D}}$. We study these cases separately.

i) If $I(t)=0$, the system (2.3) becomes the ODE $dS(t)=\mu\big(N-S(t)\big)dt$ with initial condition $S(t_0)\in D_1=[0,N]$. Since the right hand side of the ODE is continuous on D_1 then the solution $S(t)$ globally exists on D_1 for all $t\ge t_0$.

$ii)$ If $S(t) = 0$,

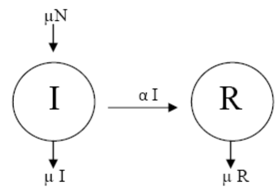


Figure 2.2: Flow chart of SIR model with disease deaths for $S = 0$.

the model turns out, $dI(t) = \Big(\mu N - (\alpha + \mu)I(t)\Big)dt - I(t)F(I(t))dW(t)$. If the initial condition $I(t_0) = I_0 \in D_2 = (0, N]$ then, the above SDE has a unique global solution on D_2. One can prove that by using a function $V(I) = I - \ln I$ defined on D_2 and Theorem 1.2.2.

Hence the proof is complete. The unique solution, $\big(S(t), I(t)\big)$ globally exists and invariant with respect to the whole domain $\mathbb{D} = \Big\{(S, I) \in \mathbb{R}^2 : S \geq 0, I \geq 0, S + I \leq N\Big\}$ for all $\big(S_0, I_0\big) \in \mathbb{D}$ and $t \geq t_0$.

$\square$

2.2.2 Stability of Disease Free Equilibrium

Consider, the stochastic SIR model

$$\begin{aligned} dS &= \Big(- \beta SI + \mu(N - S)\Big)\ dt - SI\ F_1(S, I)\ dW_1 \\ dI &= \Big(\beta SI - (\alpha + \mu)I\Big)\ dt + SI\ F_1(S, I)\ dW_1 - I\ F_2(S, I)\ dW_2 \end{aligned} \tag{2.8}$$

where α, β, γ, and μ are positive and F_i's are locally Lipschitz continuous on $\mathbb{D} = \Big\{(S, I) : S \geq 0, I \geq 0, S + I \leq N\Big\}$ for all $t \geq t_0$.

Theorem 2.2.6. *The disease free equilibrium solution $(S_1, I_1) = (N, 0)$ of (2.8) is stochastically asymptotically stable on $\mathbb{D}$ if the basic reproduction number $\mathcal{R}_0 = \dfrac{\beta N}{\alpha+\mu} \leq 1$ and $\sup\limits_{(S,I)\in\mathbb{D}} F_2^2(S,I) \leq 2(\alpha+\mu)$.*

Proof. Define a Lyapunov function $V(S,I) = \frac{1}{2}(S-N+I)^2 + \frac{\alpha+2\mu}{\alpha+\mu}NI$ on $\mathbb{D}$. Then,

$$\begin{aligned}
\mathcal{L}V(S,I) &= (S-N+I)\Big(-\beta SI + \mu(N-S)\Big) + \left((S-N+I) + \frac{\alpha+2\mu}{\alpha+\mu}N\right)\Big(\beta SI - (\alpha+\mu)I\Big) \\
&\quad + \frac{1}{2}\Big(S^2I^2F_1^2(S,I) - 2S^2I^2F_1^2(S,I) + S^2I^2F_1^2(S,I) + I^2F_2^2(S,I)\Big) \\
&= (S-N+I)\Big(\mu(N-S) - (\alpha+\mu)I\Big) + \frac{\alpha+2\mu}{\alpha+\mu}N\Big(\beta SI - (\alpha+\mu)I\Big) + \frac{1}{2}I^2F_2^2(S,I) \\
&= -\mu(N-S)^2 + (\alpha+\mu)(N-S)I + \mu I(N-S) - (\alpha+\mu)I^2 + \frac{\alpha+2\mu}{\alpha+\mu}\beta NSI \\
&\quad - N(\alpha+2\mu)I + \frac{1}{2}I^2F_2^2(S,I) \\
&= -\mu(N-S)^2 + (\alpha+2\mu)(N-S)I - (\alpha+\mu)I^2 + \frac{\alpha+2\mu}{\alpha+\mu}\beta NSI - N(\alpha+2\mu)I \\
&\quad + \frac{1}{2}I^2F_2^2(S,I) \\
&= -\mu(N-S)^2 + (\alpha+2\mu)NI - (\alpha+2\mu)SI - (\alpha+\mu)I^2 + \frac{\alpha+2\mu}{\alpha+\mu}\beta NSI \\
&\quad - N(\alpha+2\mu)I + \frac{1}{2}I^2F_2^2(S,I)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}V(S,I) &= -\mu(N-S)^2 - (\alpha+2\mu)SI - (\alpha+\mu)I^2 + \frac{\alpha+2\mu}{\alpha+\mu}\beta NSI + \frac{1}{2}I^2F_2^2(S,I) \\
&= -\mu(N-S)^2 - (\alpha+\mu)I^2 - (\alpha+2\mu)\left(1 - \frac{\beta N}{\alpha+\mu}\right)SI + \frac{1}{2}I^2F_2^2(S,I) \\
&\leq -\mu(N-S)^2 - \left(\alpha+\mu - \frac{1}{2}\sup_{(S,I)\in\mathbb{D}} F_2^2(S,I)\right)I^2 - (\alpha+2\mu)(1-\mathcal{R}_0)SI
\end{aligned}$$

Therefore, $\mathcal{L}V(S,I)$ is negative definite if $\mathcal{R}_0 \leq 1$ and $\sup\limits_{(S,I)\in\mathbb{D}} F_2^2(S,I) \leq 2(\alpha+\mu)$. Theorem 1.2.3 completes the proof. □

Remark. Note that stochastic SIR model with constant F_i's

$$\begin{aligned}
dS &= \Big(-\beta SI + \mu(N-S)\Big)\,dt - \sigma_1\ SI\ dW_1 \\
dI &= \Big(\beta SI - (\alpha+\mu)I\Big)\,dt + \sigma_1 SI\ dW_1 - \sigma_2\ I\ \ dW_2
\end{aligned} \tag{2.9}$$

has only one equilibrium solution, which is the disease free equilibrium $(N, 0)$, and it is stochastically asymptotically stable if $\mathcal{R}_0 \leq 1$ and $\sigma_2^2 \leq 2(\alpha + \mu)$.

2.2.3 Stability of Endemic Equilibrium

The model (2.8) has a unique endemic equilibrium solution

$$(S_2, I_2) = \left(\frac{\alpha+\mu}{\beta}, \frac{\mu N}{\alpha+\beta} - \frac{\mu}{\beta}\right) = \left(\frac{N}{\mathcal{R}_0}, \frac{\mu N}{\alpha+\beta}\left(1 - \frac{1}{\mathcal{R}_0}\right)\right) = \left(\frac{N}{\mathcal{R}_0}, \frac{\mu}{\beta}(\mathcal{R}_0 - 1)\right) \qquad (2.10)$$

if $\mathcal{R}_0 > 1$ and $F_i(S_2, I_2) = 0$.

Theorem 2.2.7. *The endemic equilibrium solution, (S_2, I_2), of the system (2.8) is stochastically asymptotically stable on $\mathbb{D} = \{(S, I) : S > 0, I > 0, S + I \leq N\}$ if $\mathcal{R}_0 > 1$ for some $F_i(S, I)$ such that $F_i(S_2, I_2) = 0$ and satisfies*

$$-\mu(S - S_2)^2 - (\alpha+\mu)(I - I_2)^2 + \frac{\alpha + 2\mu}{2\beta} I_2 S^2 F_1^2(S, I) + \frac{1}{2}\left(\frac{\alpha+2\mu}{\beta} I_2 + I^2\right) F_2^2(S, I) < 0. \qquad (2.11)$$

Proof. Note that the conditions $\mathcal{R}_0 > 1$ and $F_i(S_2, I_2) = 0$ are needed for the existence of the endemic equilibrium solution. The following identities are needed in the proof;

i) $$\beta S - (\alpha+\mu) = \beta\left(S - \frac{\alpha+\mu}{\beta}\right) = \beta(S - S_2) \qquad (2.12)$$

ii) $$\begin{aligned} \mu(N - S) - (\alpha+\mu)I &= \mu N - \mu(S - S_2) - \mu S_2 - (\alpha+\mu)(I - I_2) - (\alpha+\mu)I_2 \\ &= -\mu(S - S_2) - (\alpha+\mu)(I - I_2) + \mu N - \mu S_2 - (\alpha+\mu)I_2 \\ &\overset{(2.10)}{=} -\mu(S - S_2) - (\alpha+\mu)(I - I_2) + \mu N\left(1 - \frac{1}{\mathcal{R}_0}\right) \\ &\qquad - (\alpha+\mu)\frac{\mu N}{\alpha+\mu}\left(1 - \frac{1}{\mathcal{R}_0}\right) \\ &= -\mu(S - S_2) - (\alpha+\mu)(I - I_2). \end{aligned} \qquad (2.13)$$

Now, define a Lyapunov function

$$V(S, I) = \frac{1}{2}(S - S_2 + I - I_2)^2 + \frac{\alpha+2\mu}{\beta}\left(I - I_2 - I_2 \ln\frac{I}{I_2}\right)$$

on $\mathbb{D} = \{(S, I) : S > 0, I > 0, S + I \le N\}$. Then,

$$\begin{aligned}
\mathcal{L}V(S,I) &= \big(-\beta SI + \mu(N-S)\big)(S - S_2 + I - I_2) \\
&\quad + \big(\beta SI - (\alpha+\mu)I\big)\left(S - S_2 + I - I_2 + \frac{\alpha+2\mu}{\beta}\left(1 - \frac{I_2}{I}\right)\right) \\
&\quad + \frac{1}{2}S^2I^2F_1^2(S,I) - S^2I^2F_1^2(S,I) + \left(\frac{1}{2} + \frac{\alpha+2\mu}{2\beta}\frac{I_2}{I^2}\right)\big(S^2I^2F_1^2(S,I) + I^2F_2^2(S,I)\big) \\
&= (S - S_2 + I - I_2)\big(-\beta SI + \mu(N-S) + \beta SI - (\alpha+\mu)I\big) \\
&\quad + \frac{\alpha+2\mu}{\beta}\left(1 - \frac{I_2}{I}\right)(\beta SI - (\alpha+\mu)I) \\
&\quad + \frac{\alpha+2\mu}{2\beta}I_2S^2F_1^2(S,I) + \frac{1}{2}\left(\frac{\alpha+2\mu}{\beta}I_2 + I^2\right)F_2^2(S,I) \\
&= (S - S_2 + I - I_2)\big(\mu(N-S) - (\alpha+\mu)I\big) + \frac{\alpha+2\mu}{\beta}(I - I_2)(\beta S - (\alpha+\mu)) \\
&\quad + \frac{\alpha+2\mu}{2\beta}I_2S^2F_1^2(S,I) + \frac{1}{2}\left(\frac{\alpha+2\mu}{\beta}I_2 + I^2\right)F_2^2(S,I)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}V(S,I) &= (S - S_2 + I - I_2)\Big(-\mu(S - S_2) - (\alpha+\mu)(I - I_2)\Big) + \frac{\alpha+2\mu}{\beta}(I - I_2)\,\beta\,(S - S_2) \\
&\quad + \frac{\alpha+2\mu}{2\beta}I_2S^2F_1^2(S,I) + \frac{1}{2}\left(\frac{\alpha+2\mu}{\beta}I_2 + I^2\right)F_2^2(S,I) \\
&= -\mu(S - S_2)^2 - (\alpha+\mu)(S - S_2)(I - I_2) - \mu(I - I_2)(S - S_2) - (\alpha+\mu)(I - I_2)^2 \\
&\quad + (\alpha+2\mu)(I - I_2)(S - S_2) + \frac{\alpha+2\mu}{2\beta}I_2S^2F_1^2(S,I) + \frac{1}{2}\left(\frac{\alpha+2\mu}{\beta}I_2 + I^2\right)F_2^2(S,I) \\
&= -\mu(S - S_2)^2 - (\alpha+\mu)(I - I_2)^2 \\
&\quad + \frac{\alpha+2\mu}{2\beta}I_2S^2F_1^2(S,I) + \frac{1}{2}\left(\frac{\alpha+2\mu}{\beta}I_2 + I^2\right)F_2^2(S,I)
\end{aligned}$$

$\mathcal{L}V(S,I) = 0$ only at (S_2, I_2) and by the choice of suitable functions $F_i(S,I)$ that satisfy (2.11), one can easily obtained $\mathcal{L}V(S,I) < 0$ on $\mathbb{D}\backslash(S_2, I_2)$. Hence $\mathcal{L}V(S,I)$ is negative definite on $\mathbb{D}$ for some suitable $F_i(S,I)$. Therefore, by Theorem 1.2.3, the endemic equilibrium is stochastically asymptotically stable if $\mathcal{R}_0 > 1$ and for some suitable functions $F_i(S,I)$ such that $F_i(S_2, I_2) = 0$, and satisfies the condition (2.11). □

Note that, $F_i(S,I)$'s may have product terms containing such as $S - S_2$ or $I - I_2$. See below examples for illustration.

Example 2.2.1. The endemic equilibrium solution, (S_2, I_2), of the system (2.8) is stochastically asymptotically stable on $\mathbb{D} = \{(S,I) : S > 0, I > 0, S + I \le N\}$ if

$$1 < \mathcal{R}_0 < 1 + \frac{\beta^2}{\alpha + 2\mu} \frac{1}{\sup_{(S,I)\in\ \mathbb{D}} S^2\phi_1^2(S,I)} \tag{2.14}$$

for $F_1(S,I) = (S - S_2)\phi_1(S,I)$ and $F_2(S,I) = 0$, where a Lipschitz-continuous function $\phi_1(S,I)$ defined on $\mathbb{D}$ such that $SF_1(S,I)$ is Lipschitz-continuous on $\mathbb{D}$.

$$\begin{aligned}\mathcal{L}V(S,I) &= -\mu(S-S_2)^2 - (\alpha+\mu)(I-I_2)^2 + \frac{\alpha+2\mu}{2\beta}\frac{\mu}{\beta}(\mathcal{R}_0 - 1)S^2(S-S_2)^2\phi_1^2(S,I) \\ &\le -\left(\mu - (\mathcal{R}_0-1)\frac{\mu(\alpha+2\mu)}{2\beta^2}\sup_{(S,I)\in\ \mathbb{D}} S^2\phi_1^2(S,I)\right)(S-S_2)^2 - (\alpha+\mu)(I-I_2)^2\end{aligned}$$

$\mathcal{L}V(S,I)$ is negative definite on $\mathbb{D}$ if $\mu > (\mathcal{R}_0 - 1)\dfrac{\mu(\alpha+2\mu)}{2\beta^2}\sup\limits_{(S,I)\in\ \mathbb{D}} S^2\phi_1^2(S,I)$. Therefore, $\mathcal{R}_0 < 1 + \dfrac{2\beta^2}{(\alpha+2\mu)}\dfrac{1}{\sup_{(S,I)\in\ \mathbb{D}} S^2\phi_1^2(S,I)}$. By using the assumption on the existence of the endemic equilibrium solution,$\mathcal{R}_0 > 1$, we obtain (2.14).

Example 2.2.2. The endemic equilibrium solution, (S_2, I_2), of the system (2.8) is stochastically asymptotically stable on $\mathbb{D} = \{(S,I) : S > 0, I > 0, S + I \le N\}$ if

$$1 < \mathcal{R}_0 < 1 + \frac{2\beta^2}{\mu(\alpha+2\mu)\sup_{(S,I)\in\ \mathbb{D}}\phi_2^2(S,I)}\left(\alpha + \mu - \sup_{(S,I)\in\ \mathbb{D}} I^2\phi_2^2(S,I)\right) \tag{2.15}$$

for $F_1(S,I) = 0$ and $F_2(S,I) = (I - I_2)\phi_2(S,I)$, where a Lipschitz-continuous function $\phi_2(S,I)$ defined on $\mathbb{D}$ such that $\sup\limits_{(S,I)\in\ \mathbb{D}} I^2\phi_2^2(S,I) < \alpha + \mu$.

$$\mathcal{L}V(S,I) \le -\mu(S-S_2)^2 - \left(\alpha + \mu - \sup_{(S,I)\in\ \mathbb{D}}\left\{\left(\frac{\mu(\alpha+2\mu)}{2\beta^2}(\mathcal{R}_0 - 1) + I^2\right)\phi_2^2(S,I)\right\}\right)(I-I_2)^2 \tag{2.16}$$

$\mathcal{L}V(S,I)$ is negative definite on $\mathbb{D}$ if $\alpha+\mu \ > \sup\limits_{(S,I)\in\ \mathbb{D}} \Big\{\Big(\dfrac{\mu(\alpha+2\mu)}{2\beta^2}\ (\mathcal{R}_0-1)+I^2\Big)\ \phi_2^2(S,I)\Big\}$.

Therefore, $\mathcal{R}_0 < 1 + \dfrac{2\beta^2}{\mu(\alpha+2\mu)\sup_{(S,I)\in\ \mathbb{D}}\phi_2^2(S,I)}\ \Big(\alpha+\mu-\sup\limits_{(S,I)\in\ \mathbb{D}} I^2\phi_2^2(S,I)\Big)$.

Example 2.2.3. The endemic equilibrium solution, (S_2, I_2), of the system (2.8) is stochastically asymptotically stable on $\mathbb{D} = \Big\{(S,I) : S > 0, I > 0, S + I \leq N\Big\}$ if

$$1 < \mathcal{R}_0 < 1 + \min\left\{\frac{2\beta^2\Big(\mu - \sup_{(S,I)\in\ \mathbb{D}} I^2\phi_4^2(S,I)\Big)}{\mu(\alpha+2\mu)\ \sup_{(S,I)\in\ \mathbb{D}}\phi_4^2(S,I)}\ ,\ \frac{2\beta^2(\alpha+\mu)}{\mu(\alpha+2\mu)\ \sup_{(S,I)\in\ \mathbb{D}} S^2\phi_3^2(S,I)}\right\} \tag{2.17}$$

for $F_1(S,I) = (I-I_2)\phi_3(S,I)$ and $F_2(S,I) = (S-S_2)\phi_4(S,I)$, where Lipschitz-continuous functions $\phi_3(S,I)$, and $\phi_4(S,I)$ defined on $\mathbb{D}$ such that $\sup\limits_{(S,I)\in\ \mathbb{D}} I^2\phi_4^2(S,I)\ <\ \mu$.

$$\begin{aligned}\mathcal{L}V(S,I) \ \leq\ & -\left(\mu - \sup_{(S,I)\in\ \mathbb{D}}\Big\{\Big(\frac{\mu(\alpha+2\mu)}{2\beta^2}\ (\mathcal{R}_0-1)+I^2\Big)\ \phi_4^2(S,I)\Big\}\right)(S-S_2)^2 \\ & -\left(\mu - \frac{\mu(\alpha+2\mu)}{2\beta^2}\ (\mathcal{R}_0-1)\ \sup_{(S,I)\in\ \mathbb{D}} S^2\phi_2^3(S,I)\}\right)(I-I_2)^2\end{aligned}$$

$\mathcal{L}V(S,I)$ is negative definite on $\mathbb{D}$ if

$$\begin{aligned}\mu \quad &> \quad (\mathcal{R}_0-1)\frac{\mu(\alpha+2\mu)}{2\beta^2}\sup_{(S,I)\in\ \mathbb{D}}\phi_4^2(S,I) + \sup_{(S,I)\in\ \mathbb{D}} I^2\phi_4^2(S,I) \\ \Rightarrow \mathcal{R}_0 \quad &< \quad 1 + \frac{2\beta^2\Big(\mu - \sup_{(S,I)\in\ \mathbb{D}} I^2\phi_4^2(S,I)\Big)}{\mu(\alpha+2\mu)\ \sup_{(S,I)\in\ \mathbb{D}}\phi_4^2(S,I)}\end{aligned}$$

and

$$\begin{aligned}\alpha+\mu \quad &> \quad (\mathcal{R}_0-1)\frac{\mu(\alpha+2\mu)}{2\beta^2}\sup_{(S,I)\in\ \mathbb{D}} S^2\phi_3^2(S,I) \\ \Rightarrow \mathcal{R}_0 \quad &< \quad 1 + \frac{2\beta^2\Big(\mu - \sup_{(S,I)\in\ \mathbb{D}} I^2\phi_4^2(S,I)\Big)}{\mu(\alpha+2\mu)\ \sup_{(S,I)\in\ \mathbb{D}}\phi_4^2(S,I)}.\end{aligned}$$

CHAPTER 3

STOCHASTIC SIS MODEL WITH DISEASE DEATHS

3.1 DETERMINISTIC MODEL

The SIS model with disease deaths has the form

$$\begin{aligned} S'(t) &= -\beta S(t)I(t) + \mu\big(K - S(t)\big) + \alpha I(t) \\ I'(t) &= \beta S(t)I(t) - (\alpha + \gamma + \mu)I(t) \end{aligned} \tag{3.1}$$

where the parameters α, β, γ, and μ are positive.

Since $N(t) = S(t) + I(t) \Rightarrow N'(t) = \mu\big(K - N(t)\big) - \gamma I(t)$ the total population size N may vary in time.

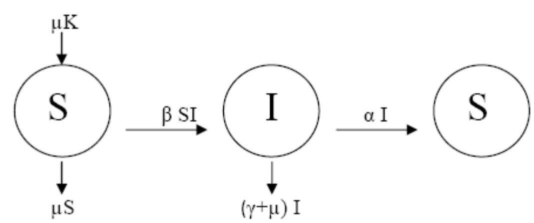


Figure 3.1: Flow chart for SIS model with disease deaths.

In the model, μK is the number of births in unit time, K is the carrying capacity (i.e. maximum population size), μ is the per capita death rate, γ is the per capita disease related death rate, βSI is the number of new infections in unit time, α is the per capita recovery rate with no immunity.

The basic reproduction number for this model is $\mathcal{R}_0 = \dfrac{\beta K}{\alpha + \gamma + \mu}$.

3.2 STOCHASTIC MODEL

We perturbed the deterministic system (3.1) by a white noise, $\frac{dW(t)}{dt}$, and obtained the stochastic version of SIS model by replacing the rates β, and α by $\beta + F_1(S(t), I(t))\dfrac{dW_1(t)}{dt}$, and $\alpha + F_2(S(t), I(t))\dfrac{dW_2(t)}{dt}$respectively, where F_i's are locally Lipschitz-continuous on $\mathbb{D} = \{(S, I) \in \mathbb{R}^2; S \geq 0, I \geq 0, S + I \leq K\}$and W_i's are i.i.d. Wiener processes defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

$$\begin{aligned} dS(t) &= \Big(-\beta S(t)I(t) + \mu(K - S(t)) + \alpha I(t)\Big)\ dt - S(t)I(t)\ F_1(S(t), I(t))\ dW_1(t) \\ &\qquad + I(t)\ F_2(S(t), I(t))\ dW_2(t) \\ dI(t) &= \Big(\beta S(t)I(t) - (\alpha + \gamma + \mu)I(t)\Big)\ dt + S(t)I(t)\ F_1(S(t), I(t))\ dW_1(t) \\ &\qquad - I(t)\ F_2(S(t), I(t))\ dW_2(t) \end{aligned} \tag{3.2}$$

The disease free equilibrium is $(S_1, I_1) = (K, 0)$. There exist a unique endemic equilibrium

$$\begin{aligned} (S_2, I_2) &= \left(\frac{\alpha + \gamma + \mu}{\beta}, \frac{\mu K}{\gamma + \mu} - \frac{\mu(\alpha + \gamma + \mu)}{\beta(\gamma + \mu)}\right) \\ &= \left(\frac{K}{\mathcal{R}_0}, \frac{\mu K}{\gamma + \mu}\left(1 - \frac{1}{\mathcal{R}_0}\right)\right) \\ &= \left(\frac{K}{\mathcal{R}_0}, \frac{\mu(\alpha + \gamma + \mu)}{\beta(\gamma + \mu)}(\mathcal{R}_0 - 1)\right) \end{aligned} \tag{3.3}$$

if $\mathcal{R}_0 > 1$ and $F_i(S_2, I_2) = 0$.

3.2.1 Existence and Uniqueness of Solutions

Consider the stochastic SIS model with disease deaths

$$\begin{pmatrix} dS \\ dI \end{pmatrix} = \underbrace{\begin{pmatrix} -\beta SI + \mu(K-S) + \alpha I \\ \beta SI - (\alpha+\gamma+\mu)I \end{pmatrix}}_{= f(S,I,t)} dt + \underbrace{\begin{pmatrix} -SIF_1(S,I) & IF_2(S,I) \\ SIF_1(S,I) & -IF_2(S,I) \end{pmatrix}}_{= g(S,I,t)} \begin{pmatrix} dW_1 \\ dW_2 \end{pmatrix}$$

with initial condition $(S(t_0), I(t_0)) = (S_0, I_0)$, where the parameters α, β, γ, and μ are positive and $F_i(S,I)$ are locally Lipschitz-continuous on $\mathbb{D} := \{(S,I) \in \mathbb{R}^2 : S \geq 0, I \geq 0, S+I \leq K\}$.

Lemma 3.2.1. *The coefficient $f(S,I,t)$ is locally Lipschitz-continuous on $\mathbb{D}$ with Lipschitz constant L_1 where $L_1^2 = \max\Big\{10\beta^2K^2 + 3\mu^2, 10\beta^2K^2 + 3\alpha^2 + 2(\alpha+\gamma+\mu)^2\Big\}$.*

Proof. For all $t \geq t_0$, $(S,I) \in \mathbb{D}$ and $(S^*, I^*) \in \mathbb{D}$,

$$\Big\| f(S,I,t) - f(S^*,I^*,t) \Big\|^2 = \left\| \begin{pmatrix} -\beta(SI - S^*I^*) - \mu(S-S^*) + \alpha(I-I^*) \\ \beta(SI - S^*I^*) - (\alpha+\gamma+\mu)(I-I^*) \end{pmatrix} \right\|^2$$

$$\begin{aligned}
&= \Big(-\beta(SI - S^*I^*) - \mu(S-S^*) + \alpha(I-I^*) \Big)^2 \\
&\quad + \Big(\beta(SI - S^*I^*) - (\alpha+\gamma+\mu)(I-I^*) \Big)^2 \\
&\overset{(1.22)}{\leq} 3\beta^2(SI - S^*I^*)^2 + 3\mu^2(S-S^*)^2 + 3\alpha^2(I-I^*)^2 \\
&\quad + 2\beta^2(SI - S^*I^*)^2 + 2(\alpha+\gamma+\mu)^2(I-I^*)^2 \\
&= 5\beta^2(SI - S^*I^*)^2 + 3\mu^2(S-S^*)^2 + \big(3\alpha^2 + 2(\alpha+\gamma+\mu)^2\big)(I-I^*)^2 \\
&= 5\beta^2\big((S-S^*)I + S^*(I-I^*)\big)^2 + 3\mu^2(S-S^*)^2 + \big(3\alpha^2 + 2(\alpha+\gamma+\mu)^2\big)(I-I^*)^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad 10\beta^2(S-S^*)^2 I^2 + 10\beta^2 S^{*2}(I-I^*)^2 + 3\mu^2(S-S^*)^2 \\
&\quad\quad + \Big(3\alpha^2 + 2(\alpha+\gamma+\mu)^2\Big)(I-I^*)^2 \\
&\overset{I\leq K}{\leq} \quad (10\beta^2K^2 + 3\mu^2)(S-S^*)^2 + \Big(10\beta^2K^2 + 3\alpha^2 + 2(\alpha+\gamma+\mu)^2\Big)(I-I^*)^2 \\
&\leq \quad \max\Big\{10\beta^2K^2 + 3\mu^2, 10\beta^2K^2 + 3\alpha^2 + 2(\alpha+\gamma+\mu)^2\Big\}\Big((S-S^*)^2 + (I-I^*)^2\Big) \\
&= \quad L_1^2 \Big\| \begin{pmatrix} S-S^* & I-I^* \end{pmatrix}^T \Big\|^2
\end{aligned}
$$

where $L_1^2 = \max\Big\{10\beta^2K^2 + 3\mu^2, 10\beta^2K^2 + 3\alpha^2 + 2(\alpha+\gamma+\mu)^2\Big\}$. □

Lemma 3.2.2. *The coefficient $g(S, I, t)$ is locally Lipschitz-continuous on $\mathbb{D}$ with Lipschitz constant L_2 where* $L_2^2 = \sup_{(S,I)\in\ \mathbb{D}} \Big\{4\widetilde{L}_1K^4 + 8F_1^2(S,I)K^2 + 4\widetilde{L}_2K^2 + 4F_2^2(S,I)\Big\}$.

Proof. For all $t \geq t_0$, $(S, I) \in \mathbb{D}$ and $(S^*, I^*) \in \mathbb{D}$,

$$
\Big\| F_i(S,I,t) - F_i(S^*,I^*,t) \Big\|_F^2 \leq \widetilde{L}_i\Big((S-S^*)^2 + (I-I^*)^2\Big). \tag{3.4}
$$

since F_i's are locally Lipschitz-continuous on $\mathbb{D}$.

$$
\begin{aligned}
& \Big\| g(S,I,t) - g(S^*,I^*,t) \Big\|_F^2 \\
= \quad & \left\| \begin{pmatrix} -SIF_1(S,I) + S^*I^*F_1(S^*,I^*) & IF_2(S,I) - I^*F_2(S^*,I^*) \\ SIF_1(S,I) - S^*I^*F_1(S^*,I^*) & -IF_2(S,I) + I^*F_2(S^*,I^*) \end{pmatrix} \right\|_F^2 \\
= \quad & 2\Big(SIF_1(S,I) - S^*I^*F_1(S^*,I^*)\Big)^2 + 2\Big(IF_2(S,I) - I^*F_2(S^*,I^*)\Big)^2 \\
= \quad & 2\Big(SIF_1(S,I) - SIF_1(S^*,I^*) + SIF_1(S^*,I^*) - S^*I^*F_1(S^*,I^*)\Big)^2 \\
& +2\Big(IF_2(S,I) - IF_2(S^*,I^*) + IF_2(S^*,I^*) - I^*F_2(S^*,I^*)\Big)^2 \\
= \quad & 2\Big\{SI\Big(F_1(S,I) - F_1(S^*,I^*)\Big) + \Big(SI - S^*I^*\Big)F_1(S^*,I^*)\Big\}^2 \\
& +2\Big\{I\Big(F_2(S,I) - F_2(S^*,I^*)\Big) + \Big(I - I^*\Big)F_2(S^*,I^*)\Big\}^2
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(1.22)}{\leq} \quad 4S^2I^2\Big(F_1(S,I)-F_1(S^*,I^*)\Big)^2+4F_1^2(S^*,I^*)\Big(SI-S^*I+S^*I-S^*I^*\Big)^2\\
&\qquad +4I^2\Big(F_2(S,I)-F_2(S^*,I^*)\Big)^2+4F_2^2(S^*,I^*)(I-I^*)^2\\
&\overset{(3.4)}{\leq} \quad 4S^2I^2\widetilde{L}_1\Big((S-S^*)^2+(I-I^*)^2\Big)+8F_1^2(S^*,I^*)I^2(S-S^*)^2\\
&\qquad +8F_1^2(S^*,I^*)S^{*2}(I-I^*)^2+4I^2\widetilde{L}_2\Big((S-S^*)^2+(I-I^*)^2\Big)+4F_2^2(S^*,I^*)(I-I^*)^2\\
&\leq \quad \Big(4\widetilde{L}_1K^4+8F_1^2(S^*,I^*)K^2+4\widetilde{L}_2K^2\Big)(S-S^*)^2\\
&\qquad +\Big(4\widetilde{L}_1K^4+8F_1^2(S^*,I^*)K^2+4\widetilde{L}_2K^2+4F_2^2(S^*,I^*)\Big)(I-I^*)^2\\
&\leq \quad \sup_{(S,I)\in\,\mathbb{D}}\Big\{4\widetilde{L}_1K^4+8F_1^2(S^*,I^*)K^2+4\widetilde{L}_2K^2+4F_2^2(S^*,I^*)\Big\}\Big((S-S^*)^2+(I-I^*)^2\Big)\\
&= \quad L_2^2\left\|\begin{pmatrix}S-S^* & I-I^*\end{pmatrix}^T\right\|_F^2
\end{aligned}
$$

where $L_2^2=\sup_{(S,I)\in\,\mathbb{D}}\Big\{4\widetilde{L}_1K^4+8F_1^2(S,I)K^2+4\widetilde{L}_2K^2+4F_2^2(S,I)\Big\}.$ □

Lemma 3.2.3. *$f(S,I,t)$ satisfies linear growth condition on $\mathbb{D}$ with the growth coefficient $L_3^2=\max\Big\{5\beta^2K^2+3\alpha^2+2(\alpha+\gamma+\mu)^2, 3\mu^2K^2, 3\mu^2\Big\}$.*

Proof. For all $t\geq t_0$ and $(S,I)\in\mathbb{D}$,

$$
\begin{aligned}
\Big\|f(S,I,t)\Big\|^2 \quad &= \quad \left\|\begin{pmatrix}-\beta SI+\mu(K-S)+\alpha I\\ \beta SI-(\alpha+\gamma+\mu)I\end{pmatrix}\right\|^2\\
&= \quad \Big(-\beta SI+\mu(K-S)+\alpha I\Big)^2+\Big(\beta SI-(\alpha+\gamma+\mu)I\Big)^2\\
&\overset{(1.22)}{\leq} \quad 5\beta^2S^2I^2+3\mu^2(K-S)^2+(3\alpha^2+2(\alpha+\gamma+\mu)^2)I^2\\
&\leq \quad 5\beta^2S^2I^2+3\mu^2K^2+3\mu^2S^2+(3\alpha^2+2(\alpha+\gamma+\mu)^2)I^2\\
&\overset{S\leq K}{\leq} \quad 5\beta^2K^2I^2+3\mu^2K^2+3\mu^2S^2+(3\alpha^2+2(\alpha+\gamma+\mu)^2)I^2\\
&\leq \quad \max\Big\{5\beta^2K^2+3\alpha^2+2(\alpha+\gamma+\mu)^2, 3\mu^2K^2, 3\mu^2\Big\}\Big(1+S^2+I^2\Big)\\
&= \quad L_3^2(1+S^2+I^2)=L_3^2\left(1+\left\|\begin{pmatrix}S & I\end{pmatrix}^T\right\|^2\right)
\end{aligned}
$$

where $L_3^2 = \max\Big\{5\beta^2K^2 + 3\alpha^2 + 2(\alpha+\gamma+\mu)^2, 3\mu^2K^2, 3\mu^2\Big\}$. □

Lemma 3.2.4. *$g(S,I,t)$ satisfies linear growth condition on $\mathbb{D}$ with the growth coefficient $L_4^2 = \max\Big\{\sup_{(S,I)\in\mathbb{D}} 2K^2F_1^2(S,I), \sup_{(S,I)\in\mathbb{D}} 2F_2^2(S,I)\Big\}$.*

Proof. For all $t \geq t_0$ and $(S,I) \in \mathbb{D}$,

$$\begin{aligned}
\Big\|g(S,I,t)\Big\|_F^2 &= \left\|\begin{pmatrix} -SIF_1(S,I) & IF_2(S,I) \\ SIF_1(S,I) & -IF_2(S,I)\end{pmatrix}\right\|_F^2 \\
&= 2S^2I^2F_1^2(S,I) + 2I^2F_2^2(S,I) \\
&\leq 2K^2S^2F_1^2(S,I) + 2I^2F_2^2(S,I) \\
&\leq \max\Big\{\sup_{(S,I)\in\mathbb{D}} 2K^2F_1^2(S,I), \sup_{(S,I)\in\mathbb{D}} 2F_2^2(S,I)\Big\}(S^2+I^2) \\
&\leq L_4^2(1+S^2+I^2) = L_4^2\left(1 + \Big\|\begin{pmatrix}S & I\end{pmatrix}^T\Big\|_F^2\right)
\end{aligned}$$

where $L_4^2 = \max\Big\{\sup_{(S,I)\in\mathbb{D}} 2K^2F_1^2(S,I), \sup_{(S,I)\in\mathbb{D}} 2F_2^2(S,I)\Big\}$. □

Theorem 3.2.5. *Let $\big(S(t_0), I(t_0)\big) = \big(S_0, I_0\big) \in \mathbb{D} = \big\{(S,I) \in \mathbb{R}^2; S \geq 0, I \geq 0, S+I \leq K\big\}$, and $\big(S_0, I_0\big)$ is independent of $W(t) - W(t_0)$ for $t \geq t_0$. Then the stochastic SIR model with disease deaths (3.2) admits a unique global solution $\big(S(t), I(t)\big)$ on $t \geq t_0$ and this solution is invariant with respect to $\mathbb{D}$, where α, β, γ, μ, and K are positive and F_i's are locally Lipschitz-continuous on $\mathbb{D}$.*

Proof. We use Theorem 1.2.2 and adapt ideas of [43]. Since the coefficients of the system (3.2) are locally Lipschitz-continuous and satisfy linear growth condition on $\mathbb{D}$, for any initial value $(S_0, I_0) \in \mathbb{D}$ there is a unique local solution on $t \in \big[t_0, \tau(\mathbb{D})\big)$, where $\tau(\mathbb{D})$ is the random time of first exit of stochastic process $\big(S(t), I(t)\big)$ from the domain $\mathbb{D}$, started in $\big(S(s), I(s)\big) = \big(S_0, I_0\big) \in \mathbb{D}$ at the initial time $s \in [t_0, \infty)$. To make this solution global, we need to prove that $\mathbb{P}\big(\tau(\mathbb{D}) = \infty\big) = 1$ a.s.

Let $\mathbb{D}_n := \big\{(S, I) : e^{-n} < S < K - e^{-n}, e^{-n} < I < K - e^{-n}, S + I \le K\big\}$ for $n \in \mathbb{N}$. The system (3.2) has a unique solution up to stopping time $\tau(\mathbb{D}_n)$.

Let $V(S, I) = I - \ln I + K - S - \ln(K - S)$ defined on $\overset{\circ}{\mathbb{D}} := \big\{(S, I) \in \mathbb{R}^2 : S > 0, I > 0, S + I \le K\big\}$ and assume that $\mathbb{E}V(S_0, I_0) < \infty$. Note that $V(S, I) \ge 2$ for $(S, I) \in \overset{\circ}{\mathbb{D}}$.

Let $W(S, I, t) = e^{-c(t-s)}V(S, I)$ defined on $\overset{\circ}{\mathbb{D}} \times [s, \infty)$, where

$$c = \frac{1}{2}\Big(2\beta K^2 + 2\alpha + \gamma + 2\mu + \sup_{(S,I)\in \overset{\circ}{\mathbb{D}}} \big(S^2 F_1^2(S, I) + F_2^2(S, I)\big)\Big). \tag{3.5}$$

Then,

$$\mathcal{L}V(S, I) = \big(-\beta SI + \mu(K - S) + \alpha I\big)\frac{\partial V}{\partial S} + \big(\beta SI - (\alpha + \gamma + \mu)I\big)\frac{\partial V}{\partial I} + A(F_1, F_2)$$

for $(S, I) \in \overset{\circ}{\mathbb{D}}$.

An upper bound of the last term, $A(F_1, F_2)$, in $\mathcal{L}V(S, I)$ can be obtained by

$$\begin{aligned}
A(F_1, F_2) &= \frac{1}{2}\sum_{i,j=1}^{2}\Big(gg^T\Big)_{ij}\frac{\partial^2 V}{\partial x_i \partial x_j} \\
&= \frac{1}{2}\big(S^2 I^2 F_1^2(S, I) + I^2 F_2^2(S, I)\big)\big(\frac{\partial^2 V}{\partial S^2} - 2\frac{\partial^2 V}{\partial SI} + \frac{\partial^2 V}{\partial I^2}\big) \\
&= \frac{1}{2}\big(S^2 I^2 F_1^2(S, I) + I^2 F_2^2(S, I)\big)\left(\frac{1}{(K - S)^2} + \frac{1}{I^2}\right) \\
&= \frac{1}{2}\frac{S^2 I^2}{(K - S)^2}F_1^2(S, I) + \frac{1}{2}S^2 F_1^2(S, I) + \frac{1}{2}\frac{I^2}{(K - S)^2}F_2^2(S, I) + \frac{1}{2}F_2^2(S, I) \\
&\le \sup_{(S,I)\in \overset{\circ}{\mathbb{D}}}\Big(S^2 F_1^2(S, I) + F_2^2(S, I)\Big) \qquad \text{by } \frac{I}{K - S} \le 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{L}V(S, I) \le{}& \big(-\beta SI + \mu(K - S) + \alpha I\big)\left(-1 + \frac{1}{K - S}\right) \\
&+ \big(\beta SI - (\alpha + \gamma + \mu)I\big)\left(1 - \frac{1}{I}\right) + \sup_{(S,I)\in \mathbb{D}}\Big(S^2 F_1^2(S, I) + F_2^2(S, I)\Big)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}V(S,I) &\leq \beta SI - \mu(K-S) - \alpha I - \beta\frac{SI}{K-S} + \mu + \alpha\frac{I}{K-S} + \beta SI \\
&\quad -(\alpha+\gamma+\mu)I - \beta S + \alpha + \gamma + \mu + \sup_{(S,I)\in\ \mathbb{D}}\Big(S^2F_1^2(S,I) + F_2^2(S,I)\Big) \\
&\leq 2\beta SI + \mu + \alpha\frac{I}{K-S} + \alpha + \gamma + \mu + \sup_{(S,I)\in\ \mathbb{D}}\Big(S^2F_1^2(S,I) + F_2^2(S,I)\Big) \\
&\overset{\frac{I}{K-S}\leq 1}{\leq} 2\beta K^2 + 2\alpha + \gamma + 2\mu + \sup_{(S,I)\in\ \overset{\circ}{\mathbb{D}}}\Big(S^2F_1^2(S,I) + F_2^2(S,I)\Big) \\
&\overset{(3.5)}{=} 2c
\end{aligned}
$$

Therefore $\mathcal{L}V(S,I) \leq c\ V(S,I)$ on $\overset{\circ}{\mathbb{D}}$ since $V(S,I) \geq 2$ for $(S,I) \in \overset{\circ}{\mathbb{D}}$.

Hence $\mathcal{L}W(S,I,t) = e^{-c(t-s)}\Big(-c\ V(S,I) + \mathcal{L}V(S,I)\Big) \leq 0$.

$$
\begin{aligned}
\text{Note that } \inf_{(S,I)\in\ \overset{\circ}{\mathbb{D}}\setminus\mathbb{D}_n} V(S,I) &\geq \inf_{(S,I)\in\ \overset{\circ}{\mathbb{D}}\setminus\mathbb{D}_n}\Big(I - \ln I\Big) + \inf_{(S,I)\in\ \overset{\circ}{\mathbb{D}}\setminus\mathbb{D}_n}\Big(K - S - \ln(K-S)\Big) \\
&\geq e^{-n} + n + 1 \\
&> n + 1.
\end{aligned}
$$

Now, define $\tau_n := \min\{t, \tau(\mathbb{D}_n)\}$ and apply Dynkin's formula

$$
\begin{aligned}
\mathbb{E}\ W(S(\tau_n), I(\tau_n), \tau_n) &= \mathbb{E}\ W(S(s), I(s), s) + \mathbb{E}\int_s^{\tau_n} \mathcal{L}W(S(u), I(u), u)\ du \\
&\leq \mathbb{E}\ W(S(s), I(s), s) \\
&= \mathbb{E}V(S(s), I(s)) = \mathbb{E}V(S_0, I_0).
\end{aligned}
$$

$$
\begin{aligned}
\text{Next, } \mathbb{E}\Big[e^{c(t-\tau_n)}V\big(S(\tau_n), I(\tau_n)\big)\Big] &= \mathbb{E}\Big[e^{c(t-s)}e^{-c(\tau_n - s)}V\big(S(\tau_n), I(\tau_n)\big)\Big] \\
&= \mathbb{E}\Big[e^{c(t-s)}\ W\big(S(\tau_n), I(\tau_n), \tau_n\big)\Big] \\
&\leq e^{c(t-s)}\mathbb{E}V(S_0, I_0).
\end{aligned}
$$

We have $\inf_{(S,I)\in \mathring{\mathbb{D}}\setminus\mathbb{D}_n} V(S,I) > n+1$ and $\mathbb{E}\Big[e^{c(t-\tau_n)}V\big(S(\tau_n),I(\tau_n)\big)\Big] \le e^{c(t-s)}\mathbb{E}V(S_0,I_0)$.

Therefore

$$
\begin{aligned}
0 \le \mathbb{P}\big(\tau(\mathring{\mathbb{D}}) < t\big) &\overset{\mathbb{D}_n\subseteq \mathring{\mathbb{D}}}{\le} \mathbb{P}\big(\tau(\mathbb{D}_n) < t\big) \\
&= \mathbb{P}\big(\tau_n < t\big) \\
&= \mathbb{E}\big(\mathbf{1}_{\tau_n<t}\big) \qquad \text{where } \mathbf{1} \text{ is the indicator function} \\
&\le \mathbb{E}\left(e^{c(t-\tau_n)}\ \frac{V\Big(S\big(\tau(\mathbb{D}_n)\big), I\big(\tau(\mathbb{D}_n)\big)\Big)}{\inf_{(S,I)\in \mathring{\mathbb{D}}\setminus\mathbb{D}_n} V(S,I)}\ \mathbf{1}_{\tau_n<t}\right) \\
&\le e^{c(t-s)}\ \frac{\mathbb{E}V(S_0,I_0)}{\inf_{(S,I)\in \mathring{\mathbb{D}}\setminus\mathbb{D}_n} V(S,I)} \\
&\le e^{c(t-s)}\ \frac{\mathbb{E}V(S_0,I_0)}{n+1} \qquad \to \quad 0 \qquad \text{as } n\to\infty
\end{aligned}
$$

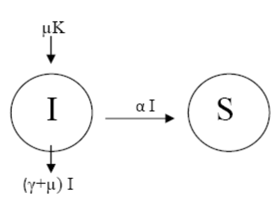
for all $(S_0, I_0) \in \mathbb{D}_n$ (for large n), and for all fixed $t \in [s,\infty)$.

Thus $\mathbb{P}\big(\tau(\mathring{\mathbb{D}}) < t\big) = \mathbb{P}\big(\tau(\mathbb{D}_n) < t\big) = 0$ for $(S_0, I_0) \in \mathring{\mathbb{D}}$ and $t \ge t_0$, that is, $\mathbb{P}\big(\tau(\mathring{\mathbb{D}}) = \infty\big) = 1$. This proves the invariance property and the existence of the solution $\big(S(t), I(t)\big)$ on $\mathring{\mathbb{D}}$.

Note that $I = 0$ and $S = 0$ are not in the domain $\mathring{\mathbb{D}}$. We study these cases separately.

i) If $I(t) = 0$, the system (3.2) becomes the ODE $dS(t) = \mu\big(N - S(t)\big)dt$ with initial condition $S(t_0) \in D_1 = [0, K]$. Since the right hand side of the ODE is continuous on D_1 then the solution $S(t)$ globally exists on D_1 for all $t \ge t_0$.

ii) If $S(t) = 0$ then we have $dI(t) = \Big(\mu K - (\alpha+\gamma+\mu)I(t)\Big)dt - I(t)F(I(t))dW(t)$.



If the initial condition $I(t_0) = I_0 \in D_2 = (0, K]$ then, the above SDE has a unique global solution on D_2. One can prove that by using a function $V(I) = I - \ln I$ defined on D_2 and similar calculations.

Hence the proof is complete. The unique solution $\big(S(t), I(t)\big)$ exists globally and invariant with respect to the whole domain $\mathbb{D} = \big\{(S, I) \in \mathbb{R}^2; S \geq 0, I \geq 0, S + I \leq K\big\}$ for all $(S_0, I_0) \in \mathbb{D}$ and $t \geq t_0$ by Theorem 1.2.2. $\square$

3.2.2 Stability of Disease Free Equilibrium

Consider, the stochastic SIS model with disease deaths

$$\begin{aligned} dS &= \Big(- \beta SI + \mu(K - S) + \alpha I\Big)\, dt - SI\ F_1(S, I)\, dW_1 + I\ F_2(S, I)\, dW_2 \\ dI &= \Big(\beta SI - (\alpha + \gamma + \mu)I\Big)\, dt + SI\ F_1(S, I)\, dW_1 - I\ F_2(S, I)\, dW_2 \end{aligned} \tag{3.6}$$

where α, β, γ, μ, and K are positive and F_i's are locally Lipschitz-continuous on $\mathbb{D} = \big\{(S, I) : S \geq 0, I \geq 0, S + I \leq K\big\}$.

Theorem 3.2.6. *The disease free equilibrium solution $(S_1, I_1) = (K, 0)$ of (3.6) is stochastically asymptotically stable on $\mathbb{D}$ if the basic reproduction number $\mathcal{R}_0 = \dfrac{\beta K}{\alpha + \gamma + \mu} < 1$, more precisely if $\beta K \leq \alpha$.*

Proof. Define a Lyapunov function $V(S, I) = \dfrac{1}{2}(S - K + I)^2 + \dfrac{\gamma}{\beta}(K - S)$ on $\mathbb{D}$. Then,

$$\begin{aligned} \mathcal{L}V(S, I) &= (S - K + I)\Big(- \beta SI + \mu(K - S) + \alpha I + \beta SI - (\alpha + \gamma + \mu)I\Big) \\ &\qquad - \frac{\gamma}{\beta}\Big(- \beta SI + \mu(K - S) + \alpha I\Big) \\ &= (S - K + I)\Big(\mu(K - S - I) - \gamma I\Big) - \frac{\gamma}{\beta}\Big(- \beta SI + \mu K - \mu S + \alpha I\Big) \\ &= -\mu(K - S - I)^2 - \gamma(S - K)I - \gamma I^2 + \gamma SI - \frac{\gamma\mu}{\beta}K + \frac{\gamma\mu}{\beta}S - \frac{\alpha\gamma}{\beta}I \end{aligned}$$

$$\begin{aligned}
\mathcal{L}V(S,I) &= -\mu(K-S-I)^2 - \gamma I^2 - \gamma SI + \gamma KI + \gamma SI - \frac{\gamma\mu}{\beta}K + \frac{\gamma\mu}{\beta}S - \frac{\alpha\gamma}{\beta}I \\
&= -\mu(K-S-I)^2 - \gamma I^2 - \gamma\left(\frac{\alpha}{\beta} - K\right)I - \frac{\gamma\mu}{\beta}(K-S) \\
&\leq 0 \qquad\qquad \text{if } \frac{\alpha}{\beta} - K \geq 0. \qquad (3.7)
\end{aligned}$$

Rewritten the stability condition, $\frac{\alpha}{\beta} - K \geq 0$, in terms of the basic reproduction number, we have

$$\frac{\alpha}{\beta} - K \geq 0 \quad \Rightarrow \quad \beta K \leq \alpha < \alpha + \gamma + \mu \quad \Rightarrow \quad \frac{\beta K}{\alpha + \gamma + \mu} < 1.$$

Therefore $\mathcal{L}V(S,I)$ is negative definite on $\mathbb{D}$ if $\mathcal{R}_0 \leq 1$. Theorem 1.2.3 completes the proof. □

Remark. If the F_i's in the stochastic SIS model with disease deaths are constants

$$\begin{aligned}
dS &= \Big(-\beta SI + \mu(K-S) + \alpha I\Big)\,dt - \sigma_1\, SI\, dW_1 + \sigma_2 I\, dW_2 \\
dI &= \Big(\beta SI - (\alpha + \gamma + \mu)I\Big)\,dt + \sigma_1 SI\, dW_1 - \sigma_2 I\, dW_2 \qquad (3.8)
\end{aligned}$$

then the above system has only one equilibrium solution, which is disease free equilibrium $(K, 0)$, and it is stochastically asymptotically stable if $\mathcal{R}_0 \leq 1$.

3.2.3 Stability of Endemic Equilibrium

The model (3.6) has a unique endemic equilibrium solution

$$\begin{aligned}
(S_2, I_2) &= \left(\frac{\alpha+\gamma+\mu}{\beta}, \frac{\mu K}{\gamma+\mu} - \frac{\mu(\alpha+\gamma+\mu)}{\beta(\gamma+\mu)}\right) \\
&= \left(\frac{K}{\mathcal{R}_0}, \frac{\mu K}{\gamma+\mu}\left(1 - \frac{1}{\mathcal{R}_0}\right)\right) \qquad (3.9) \\
&= \left(\frac{K}{\mathcal{R}_0}, \frac{\mu(\alpha+\gamma+\mu)}{\beta(\gamma+\mu)}(\mathcal{R}_0 - 1)\right)
\end{aligned}$$

if $\mathcal{R}_0 > 1$ and $F_i(S_2, I_2) = 0$.

Theorem 3.2.7. *The endemic equilibrium solution, (S_2, I_2), of the system (3.6) is stochastically asymptotically stable on $\mathbb{D} = \big\{(S,I) : S > 0, I > 0, S + I \leq K\big\}$ if $\mathcal{R}_0 > 1$ for some $F_i(S,I)$ such that $F_i(S_2, I_2) = 0$ and satisfies*

$$-\mu(S-S_2)^2 - (\gamma+\mu)(I-I_2)^2 + \frac{(2\mu+\gamma)I_2}{2\beta}\Big(S^2F_1^2(S,I) + F_2^2(S,I)\Big) < 0. \tag{3.10}$$

Proof. Note that the conditions $\mathcal{R}_0 > 1$ and $F_i(S_2, I_2) = 0$ are needed for the existence of the endemic equilibrium solution. The following identities are needed in the proof;

$$\text{i)} \qquad \beta S - (\alpha+\gamma+\mu) = \beta\left(S - \frac{\alpha+\gamma+\mu}{\beta}\right) = \beta(S - S_2) \tag{3.11}$$

$$\begin{aligned}
\text{ii)} \qquad \mu(K-S) - (\gamma+\mu)I &= \mu K - \mu(S-S_2) - \mu S_2 - (\gamma+\mu)(I-I_2) - (\gamma+\mu)I_2 \\
&= -\mu(S-S_2) - (\gamma+\mu)(I-I_2) + \mu K - \mu S_2 - (\gamma+\mu)I_2 \\
&\overset{(3.9)}{=} -\mu(S-S_2) - (\gamma+\mu)(I-I_2) + \mu K\left(1 - \frac{1}{\mathcal{R}_0}\right) \\
&\qquad - (\gamma+\mu)\frac{\mu K}{\gamma+\mu}\left(1 - \frac{1}{\mathcal{R}_0}\right) \\
&= -\mu(S-S_2) - (\gamma+\mu)(I-I_2)
\end{aligned} \tag{3.12}$$

Now, define a Lyapunov function

$$V(S,I) = S - S_2 + I - I_2 - (S_2+I_2)\ln\left(\frac{S+I}{S_2+I_2}\right) + \frac{2\mu+\gamma}{\beta K}\left(I - I_2 - I_2\ln\frac{I}{I_2}\right)$$

on $\mathbb{D} = \big\{(S,I) : S > 0, I > 0, S + I \leq K\big\}$. Then,

$$\begin{aligned}
\mathcal{L}V(S,I) &= \big(-\beta SI + \mu(K-S) + \alpha I\big)\left(1 - \frac{S_2+I_2}{S+I}\right) \\
&+ \big(\beta SI - (\alpha+\gamma+\mu)I\big)\left(1 - \frac{S_2+I_2}{S+I} + \frac{2\mu+\gamma}{\beta K}\left(1 - \frac{I_2}{I}\right)\right) \\
&+ \frac{I^2}{2}\Big(S^2F_1^2(S,I) + F_2^2(S,I)\Big)\left(\frac{S_2+I_2}{(S+I)^2} - 2\frac{S_2+I_2}{(S+I)^2} + \frac{S_2+I_2}{(S+I)^2} + \frac{2\mu+\gamma}{\beta K}\frac{I_2}{I^2}\right)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}V(S,I) \quad &= \quad \frac{S-S_2+I-I_2}{S+I}\Big(-\beta SI+\mu(K-S)+\alpha I+\beta SI-(\alpha+\gamma+\mu)I\Big) \\
&\quad +\frac{2\mu+\gamma}{\beta K}\ \frac{I-I_2}{I}\Big(\beta S-(\alpha+\gamma+\mu)\Big)I \\
&\quad +\frac{2\mu+\gamma}{\beta K}\frac{I^2}{2}\frac{I_2}{I^2}\Big(S^2F_1^2(S,I)+F_2^2(S,I)\Big) \\
&= \quad \frac{1}{S+I}\Big(S-S_2+I-I_2\Big)\Big(\mu(K-S)-(\gamma+\mu)I\Big) \\
&\quad +\frac{2\mu+\gamma}{\beta K}(I-I_2)\Big(\beta S-(\alpha+\gamma+\mu)\Big) \\
&\quad +\frac{(2\mu+\gamma)I_2}{2\beta K}\Big(S^2F_1^2(S,I)+F_2^2(S,I)\Big) \\
&\overset{(3.11)\&(3.12)}{=} \quad \frac{1}{S+I}\Big(S-S_2+I-I_2\Big)\Big(-\mu(S-S_2)-(\gamma+\mu)(I-I_2)\Big) \\
&\quad +\frac{2\mu+\gamma}{\beta K}(I-I_2)\Big(\beta(S-S_2)\Big)+\frac{(2\mu+\gamma)I_2}{2\beta K}\Big(S^2F_1^2(S,I)+F_2^2(S,I)\Big) \\
&= \quad -\frac{\mu}{S+I}(S-S_2)^2-\frac{\gamma+\mu}{S+I}(I-I_2)^2-\frac{2\mu+\gamma}{S+I}(S-S_2)(I-I_2) \\
&\quad +\frac{2\mu+\gamma}{K}(S-S_2)(I-I_2)+\frac{(2\mu+\gamma)I_2}{2\beta K}\Big(S^2F_1^2(S,I)+F_2^2(S,I)\Big) \\
&\overset{S+I\le K}{\le} \quad -\frac{\mu}{K}(S-S_2)^2-\frac{\gamma+\mu}{K}(I-I_2)^2-\frac{2\mu+\gamma}{K}(S-S_2)(I-I_2) \\
&\quad +\frac{2\mu+\gamma}{K}(S-S_2)(I-I_2)+\frac{(2\mu+\gamma)I_2}{2\beta K}\Big(S^2F_1^2(S,I)+F_2^2(S,I)\Big) \\
&= \quad -\frac{\mu}{K}(S-S_2)^2-\frac{\gamma+\mu}{K}(I-I_2)^2+\frac{(2\mu+\gamma)I_2}{2\beta K}\Big(S^2F_1^2(S,I)+F_2^2(S,I)\Big).
\end{aligned}
$$

$\mathcal{L}V(S,I) = 0$ only at (S_2, I_2) and by the choice of suitable functions $F_i(S,I)$ that satisfy (3.10), one can easily obtained $\mathcal{L}V(S,I) < 0$ on $\mathbb{D}\backslash(S_2,I_2)$. Hence $\mathcal{L}V(S,I)$ is negative definite on $\mathbb{D}$ for some suitable $F_i(S,I)$.

Therefore, by Theorem 1.2.3, the endemic equilibrium is stochastically asymptotically stable on $\mathbb{D}$ if $\mathcal{R}_0 > 1$ and for some suitable functions $F_i(S,I)$ such that $F_i(S_2,I_2) = 0$, and satisfies the condition (3.10). □

CHAPTER 4

STOCHASTIC SIR MODEL WITH DISEASE DEATHS

4.1 DETERMINISTIC MODEL

We were able to reduce the system of three differential equations (2.2) to the system of two differential equations (2.3) because of the assumption that the total population N is constant. If there are deaths due to the disease the total population size N may vary in time and it would be necessary to use a three-dimensional system as a model. The SIR model with disease deaths has the form

$$\begin{aligned} S'(t) &= \beta S(t)I(t) + \mu(K - S(t)) \\ I'(t) &= \beta S(t)I(t) - \big(\alpha + \gamma + \mu\big)I(t) \\ R'(t) &= \alpha I(t) - \mu R(t) \end{aligned} \tag{4.1}$$

A transfer diagram for the model is

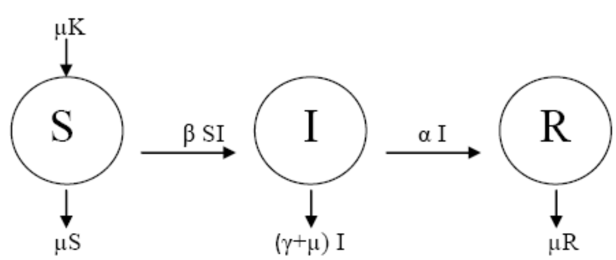


Figure 4.1: Flow chart for the SIR model with births and deaths.

The basic reproduction number for this model is $\mathcal{R}_0 = \dfrac{\beta K}{\alpha + \gamma + \mu}$.

4.2 STOCHASTIC MODEL

We perturbed the deterministic system (4.1) by a white noise, $\frac{dW(t)}{dt}$, and obtained the stochastic version of SIR model with disease deaths by replacing the rates β, and α by $\beta + F_1(S(t), I(t), R(t))\dfrac{dW_1(t)}{dt}$, and $\alpha + F_2(S(t), I(t), R(t))\dfrac{dW_2(t)}{dt}$ respectively, where F_i's are locally Lipschitz-continuous on $\mathbb{D} = \{(S, I, R) \in \mathbb{R}^3; S \geq 0, I \geq 0, R \geq 0, S + I + R \leq K\}$ for all $t \geq 0$ and W_i are i.i.d. Wiener processes defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Stochastic SIR model with disease death has a form

$$
\begin{aligned}
dS(t) &= \Big(-\beta S(t)I(t) + \mu(K - S(t))\Big)\, dt - S(t)I(t)\ F_1\big(S(t), I(t), R(t)\big)\ dW_1(t) \\
dI(t) &= \Big(\beta S(t)I(t) - \big(\alpha + \gamma + \mu\big)I(t)\Big)\, dt + S(t)I(t)\ F_1\big(S(t), I(t), R(t)\big)\ dW_1(t) \\
&\qquad - I(t)\ F_2\big(S(t), I(t), R(t)\big)\ dW_2(t) \\
dR(t) &= \Big(\alpha I(t) - \mu R(t)\Big)\, dt + I(t)\ F_2\big(S(t), I(t), R(t)\big)\ dW_2(t)
\end{aligned}
\tag{4.2}
$$

where an infection rate β, removed rate α, disease-related death rate γ, and per capita death rate μ are positive and locally Lipschitz-continuous functions F_i's are defined on $\mathbb{D}$ for all $t \geq 0$. By adding the above equations we obtain $N'(t) = \mu(K - N(t)) - \gamma I(t)$. Thus the total population size is not constant and K represents a carrying capacity (maximum possible population size).

The disease free equilibrium is $(S_1, I_1, R_1) = (K, 0, 0)$. There exist a unique endemic equilibrium

$$
\begin{aligned}
(S_2, I_2, R_2) &= \left(\frac{\alpha + \gamma + \mu}{\beta},\ \frac{\mu K}{\alpha + \gamma + \mu} - \frac{\mu}{\beta},\ \frac{\alpha K}{\alpha + \gamma + \mu} - \frac{\alpha}{\beta}\right) \\
&= \left(\frac{K}{\mathcal{R}_0},\ \frac{\mu}{\beta}\left(\mathcal{R}_0 - 1\right),\ \frac{\alpha}{\beta}\left(\mathcal{R}_0 - 1\right)\right)
\end{aligned}
\tag{4.3}
$$

if $\mathcal{R}_0 > 1$ and $F_i(S_2, I_2, R_2) = 0$.

4.2.1 Existence and Uniqueness of Solutions

Consider the stochastic SIR model with disease deaths

$$\begin{pmatrix} dS(t) \\ dI(t) \\ dR(t) \end{pmatrix} = \underbrace{\begin{pmatrix} -\beta S(t)I(t) + \mu(K - S(t)) \\ \beta S(t)I(t) - (\alpha+\gamma+\mu)I(t) \\ \alpha I(t) - \mu R(t) \end{pmatrix}}_{= f(S(t),I(t),R(t),t)} dt \tag{4.4}$$

$$+ \underbrace{\begin{pmatrix} -S(t)I(t)F_1(S(t),I(t),R(t)) & 0 \\ S(t)I(t)F_1(S(t),I(t),R(t)) & -I(t)F_2(S(t),I(t),R(t)) \\ 0 & I(t)F_2(S(t),I(t),R(t)) \end{pmatrix}}_{= g(S(t),I(t),R(t),t)} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}$$

with initial condition $S\big((t_0), I(t_0), R(t_0)\big) = (S_0, I_0, R_0)$, where the parameters α, β, γ, and μ are positive and F_i are locally Lipschitz-continuous on $\mathbb{D} = \{(S, I, R) \in \mathbb{R}^3 : S \geq 0, I \geq 0, R \geq 0, S + I + R \leq K\}$ for all $t \geq t_0$.

Lemma 4.2.1. *The coefficient $f(S, I, R, t)$ is locally Lipschitz-continuous on $\mathbb{D}$ with the Lipschitz constant $L_1^2 = 8\beta^2K^2 + 2(\alpha + \gamma + \mu)^2 + 2\alpha^2$.*

Proof. For all $t \geq t_0$, $(S, I, R) \in \mathbb{D}$ and $(S^*, I^*, R^*) \in \mathbb{D}$

$$\Big\| f(S,I,R,t) - f(S^*,I^*,R^*,t) \Big\|^2 = \left\| \begin{pmatrix} -\beta(SI - S^*I^*) - \mu(S - S^*) \\ \beta(SI - S^*I^*) - (\alpha+\gamma+\mu)(I - I^*) \\ \alpha(I - I^*) - \mu(R - R^*) \end{pmatrix} \right\|^2$$

$$= \Big(-\beta(SI - S^*I^*) - \mu(S - S^*)\Big)^2 + \Big(\beta(SI - S^*I^*) - (\alpha+\gamma+\mu)(I - I^*)\Big)^2 + \Big(\alpha(I - I^*) - \mu(R - R^*)\Big)^2$$

$$
\begin{aligned}
&\overset{(1.22)}{\leq} \quad 2\beta^2(SI - S^*I^*)^2 + 2\mu^2(S-S^*)^2 + 2\beta^2(SI - S^*I^*)^2 \\
&\quad +2(\alpha+\gamma+\mu)^2(I-I^*)^2 + 2\alpha^2(I-I^*)^2 + 2\mu^2(R-R^*)^2 \\
&= \quad 4\beta^2\Big((S-S^*)I + S^*(I-I^*)\Big)^2 + 2\mu^2(S-S^*)^2 \\
&\quad +\Big(2(\alpha+\gamma+\mu)^2 + 2\alpha^2\Big)(I-I^*)^2 + 2\mu^2(R-R^*)^2 \\
&\leq \quad 8\beta^2 I^2(S-S^*)^2 + 8\beta^2 S^{*2}(I-I^*) + 2\mu^2(S-S^*)^2 \\
&\quad +\Big(2(\alpha+\gamma+\mu)^2 + 2\alpha^2\Big)(I-I^*)^2 + 2\mu^2(R-R^*)^2 \\
&\leq \quad (8\beta^2K^2 + 2\mu^2)(S-S^*)^2 + \Big(8\beta^2K^2 + 2(\alpha+\gamma+\mu)^2 + 2\alpha^2\Big)(I-I^*)^2 \\
&\quad +2\mu^2(R-R^*)^2 \\
&\leq \quad \Big(8\beta^2K^2 + 2(\alpha+\gamma+\mu)^2 + 2\alpha^2\Big)\Big((S-S^*)^2 + (I-I^*)^2 + (R-R^*)^2\Big) \\
&= \quad L_1^2 \left\|\begin{pmatrix} S-S^* & I-I^* & R-R^* \end{pmatrix}^T\right\|^2 \text{ where } L_1^2 = 8\beta^2K^2 + 2(\alpha+\gamma+\mu)^2 + 2\alpha^2.
\end{aligned}
$$

□

Lemma 4.2.2. *The coefficient $g(S, I, R, t)$ is locally Lipschitz-continuous on $\mathbb{D}$ with the Lipschitz constant $L_2^2 = 4\widetilde{L}_1K^4 + 4\widetilde{L}_2K^2 + \sup_{(S,I,R)\in\,\mathbb{D}} \Big\{8K^2F_1^2(S,I,R) + 4F_2^2(S,I,R)\Big\}$.*

Proof. For all $t \geq t_0$, $(S, I, R) \in \mathbb{D}$ and $(S^*, I^*, R^*) \in \mathbb{D}$,

$$
\Big\|F_i(S,I,R,t) - F_i(S^*,I^*,R^*,t)\Big\|_F^2 \leq \widetilde{L}_i\Big((S-S^*)^2 + (I-I^*)^2 + (R-R^*)^2\Big), \qquad (4.5)
$$

since F_i's are locally Lipschitz-continuous on $\mathbb{D}$.

For a simplicity in reading, let $F_i = F_i(S, I, R)$ and $F_i^* = F_i(S^*, I^*, R^*)$.

$$
\begin{aligned}
\Big\|g(S,I,R,t)-g(S^*,I^*,R^*,t)\Big\|_F^2 &= \left\|\begin{pmatrix} -SIF_1+S^*I^*F_1^* & 0 \\ SIF_1-S^*I^*F_1^* & -IF_2+I^*F_2^* \\ 0 & IF_2-I^*F_2^* \end{pmatrix}\right\|_F^2 \\
&= 2\Big(SIF_1-S^*I^*F_1^*\Big)^2+2\Big(IF_2-I^*F_2^*\Big)^2 \\
&= 2\Big(SIF_1-SIF_1^*+SIF_1^*-S^*I^*F_1^*\Big)^2+2\Big(IF_2-IF_2^*+IF_2^*-I^*F_2^*\Big)^2 \\
&= 2\Big\{SI\Big(F_1-F_1^*\Big)+\Big(SI-S^*I^*\Big)F_1^*\Big\}^2+2\Big\{I\Big(F_2-F_2^*\Big)+\Big(I-I^*\Big)F_2^*\Big\}^2 \\
&\overset{(1.22)}{\leq} 4S^2I^2\Big(F_1-F_1^*\Big)^2+4F_1^{*2}\Big(SI-S^*I+S^*I-S^*I^*\Big)^2 \\
&\quad +4I^2\Big(F_2-F_2^*\Big)^2+4F_2^{*2}(I-I^*)^2 \\
&\overset{(4.5)}{\leq} 4S^2I^2\widetilde{L}_1\Big((S-S^*)^2+(I-I^*)^2+(R-R^*)^2\Big)+8F_1^{*2}I^2(S-S^*)^2 \\
&\quad +8F_1^{*2}S^{*2}(I-I^*)^2+4I^2\widetilde{L}_2\Big((S-S^*)^2+(I-I^*)^2+(R-R^*)^2\Big) \\
&\quad +4F_2^{*2}(I-I^*)^2 \\
&\leq \Big(4\widetilde{L}_1K^4+8F_1^{*2}K^2+4\widetilde{L}_2K^2\Big)(S-S^*)^2 \\
&\quad +\Big(4\widetilde{L}_1K^4+8F_1^{*2}K^2+4\widetilde{L}_2K^2+4F_2^{*2}\Big)(I-I^*)^2 \\
&\quad +\Big(4\widetilde{L}_1K^4+4\widetilde{L}_2K^2\Big)(R-R^*)^2 \\
&\leq \sup_{(S,I,R)\in\ \mathbb{D}}\Big\{4\widetilde{L}_1K^4+8F_1^{*2}K^2+4\widetilde{L}_2K^2+4F_2^{*2}\Big\}\Big((S-S^*)^2+(I-I^*)^2+(R-R^*)^2\Big) \\
&= L_2^2\Big\|\Big(S-S^* \quad I-I^* \quad R-R^*\Big)^T\Big\|_F^2
\end{aligned}
$$

where $L_2^2=4\widetilde{L}_1K^4+4\widetilde{L}_2K^2+\sup_{(S,I,R)\in\ \mathbb{D}}\Big\{8K^2F_1^2(S,I,R)+4F_2^2(S,I,R)\Big\}$. □

Lemma 4.2.3. *$f(S,I,R,t)$ satisfies linear growth condition on $\mathbb{D}$ with the growth coefficient $L_3^2=\max\Big\{4\beta^2K^4+4\mu^2K^2,\ 4\mu^2,\ 2(\alpha+\gamma+\mu)^2+2\alpha^2\Big\}$.*

Proof. For all $t \geq t_0$ and $(S, I, R) \in \mathbb{D}$,

$$\begin{aligned}
\Big\|f(S,I,R,t)\Big\|^2 &= \left\|\begin{pmatrix} -\beta SI + \mu(K-S) \\ \beta SI - (\alpha+\gamma+\mu)I \\ \alpha I - \mu R \end{pmatrix}\right\|^2 \\
&= \Big(-\beta SI + \mu(K-S)\Big)^2 + \Big(\beta SI - (\alpha+\gamma+\mu)I\Big)^2 + \Big(\alpha I - \mu R\Big)^2 \\
&\overset{(1.22)}{\leq} 4\beta^2 S^2 I^2 + 4\mu^2(K-S)^2 + \Big(2(\alpha+\gamma+\mu)^2 + 2\alpha^2\Big)I^2 + 2\mu^2 R^2 \\
&\leq 4\beta^2 S^2 I^2 + 4\mu^2 K^2 + 4\mu^2 S^2 + \Big(2(\alpha+\gamma+\mu)^2 + 2\alpha^2\Big)I^2 + 2\mu^2 R^2 \\
&\leq 4\beta^2 K^4 + 4\mu^2 K^2 + 4\mu^2 S^2 + \Big(2(\alpha+\gamma+\mu)^2 + 2\alpha^2\Big)I^2 + 2\mu^2 R^2 \\
&\leq L_3^2(1 + S^2 + I^2 + R^2) = L_3\left(1 + \Big\|\begin{pmatrix} S & I & R\end{pmatrix}^T\Big\|^2\right)
\end{aligned}$$

where $L_3^2 = \max\Big\{4\beta^2 K^4 + 4\mu^2 K^2,\ 4\mu^2,\ 2(\alpha+\gamma+\mu)^2 + 2\alpha^2, 2\mu^2\Big\}$. □

Lemma 4.2.4. *$g(S,I,R,t)$ satisfies linear growth condition on $\mathbb{D}$ with the growth coefficient $L_4^2 = \max\Big\{\sup_{(S,I,R)\in\ \mathbb{D}} 2K^2F_1^2(S,I,R), \sup_{(S,I,R)\in\ \mathbb{D}} 2F_2^2(S,I,R)\Big\}$.*

Proof. For all $t \geq t_0$ and $(S, I, R) \in \mathbb{D}$,

$$\begin{aligned}
\Big\|g(S,I,R,t)\Big\|_F^2 &= \left\|\begin{pmatrix} -SIF_1(S,I,R) & 0 \\ SIF_1(S,I,R) & -IF_2(S,I,R) \\ 0 & IF_2(S,I,R) \end{pmatrix}\right\|_F^2 \\
&= 2S^2I^2F_1^2(S,I,R) + 2I^2F_2^2(S,I,R) \\
&\leq 2K^2S^2F_1^2(S,I,R) + 2I^2F_2^2(S,I,R) \\
&\leq \max\Big\{\sup_{(S,I,R)\in\ \mathbb{D}} 2K^2F_1^2(S,I,R), \sup_{(S,I,R)\in\ \mathbb{D}} 2F_2^2(S,I,R)\Big\}(S^2 + I^2 + R^2) \\
&\leq L_4^2(1 + S^2 + I^2 + R^2) = L_4\left(1 + \Big\|\begin{pmatrix} S & I & R\end{pmatrix}^T\Big\|_F^2\right)
\end{aligned}$$

where $L_4^2 = \max\Big\{ \sup\limits_{(S,I,R)\in\ \mathbb{D}} 2K^2F_1^2(S,I,R), \sup\limits_{(S,I,R)\in\ \mathbb{D}} 2F_2^2(S,I,R)\Big\}$. □

Theorem 4.2.5. *Let $\big(S(t_0), I(t_0), R(t_0)\big) = \big(S_0, I_0, R_0\big) \in \mathbb{D} = \big\{(S,I,R) \in \mathbb{R}^3, t \geq t_0 : S \geq 0, I \geq 0, R \geq 0, S+I+R \leq K\big\}$, and $\big(S_0, I_0, R_0\big)$ is independent of $W(t) - W(t_0)$ for $t \geq t_0$. Then the stochastic SIR model with disease deaths (4.2) admits a unique global solution $\big(S(t), I(t), R(t)\big)$ on $t \geq t_0$ and this solution is invariant with respect to $\mathbb{D}$.*

Proof. We use Theorem 1.2.2 and follow ideas of [43]. Since the coefficients of the system (4.2) are locally Lipschitz-continuous and satisfy linear growth condition on $\mathbb{D}$, for any initial value $(S_0, I_0, R_0) \in \mathbb{D}$ there is a unique local solution on $t \in \big[t_0, \tau(\mathbb{D})\big)$, where $\tau(\mathbb{D})$ is the random time of first exit of stochastic process $\big(S(t), I(t), R(t)\big)$ from the domain $\mathbb{D}$, started in $\big(S(s), I(s), R(t)\big) = \big(S_0, I_0, R_0\big) \in \mathbb{D}$ at the initial time $s \in [t_0, \infty)$. To make this solution global, we need to prove that $\mathbb{P}\big(\tau(\mathbb{D}) = \infty\big) = 1$ a.s.

Let $\mathbb{D}_n := \big\{(S,I) : e^{-n} < S < K - e^{-n}, e^{-n} < I < K - e^{-n}, e^{-n} < R < K - e^{-n}, S+I+R \leq K\big\}$ for $n \in \mathbb{N}$. The system (4.2) has a unique solution up to stopping time $\tau(\mathbb{D}_n)$.

Let $V(S,I,R) = I - \ln I + S - \ln S + K - S - \ln(K-S) + K - R - \ln(K-R)$ defined on $\overset{\circ}{\mathbb{D}} = \big\{(S,I,R) \in \mathbb{R}^3, t \geq t_0 : S > 0, I > 0, R > 0, S+I+R \leq K\big\}$ and assume that $\mathbb{E}V(S,I,R) < \infty$.

Note that $V(S,I,R) \geq 4$ for $(S,I,R) \in \overset{\circ}{\mathbb{D}}$.

Let $W(S,I,R,t) = e^{-c(t-s)}V(S,I,R)$ defined on $\overset{\circ}{\mathbb{D}} \times [s, \infty)$,

$$\text{where } c = \frac{1}{4}\left(\beta K(1+K) + 2\alpha + \gamma + 3\mu + \mu K + \sup_{(S,I,R)\in\ \overset{\circ}{\mathbb{D}}} \left\{\frac{3K^2}{2}F_1^2(S,I,R) + F_2^2(S,I,R)\right\}\right). \tag{4.6}$$

Then,

$$\mathcal{L}V(S,I,R) = \big(-\beta SI + \mu(K-S)\big)\frac{\partial V}{\partial S} + \big(\beta SI - (\alpha+\gamma+\mu)I\big)\frac{\partial V}{\partial I} + \big(\alpha I - \mu R\big)\frac{\partial V}{\partial R} + A$$

for $(S,I,R) \in \overset{\circ}{\mathbb{D}}$.

An upper bound of the last term, A, in $\mathcal{L}V$ can be obtained by

$$\begin{aligned}
A &= \frac{1}{2}\sum_{i,j=1}^{2}\left(gg^T\right)_{ij}\frac{\partial^2 V}{\partial x_i \partial x_j} \\
&= \frac{1}{2}S^2I^2F_1^2(S,I,R)\left(\frac{\partial^2 V}{\partial S^2} - 2\frac{\partial^2 V}{\partial SI} + \frac{\partial^2 V}{\partial I^2}\right) + \frac{1}{2}I^2F_2^2(S,I,R)\left(\frac{\partial^2 V}{\partial I^2} - 2\frac{\partial^2 V}{\partial IR} + \frac{\partial^2 V}{\partial R^2}\right) \\
&= \frac{1}{2}S^2I^2F_1^2(S,I,R)\left(\frac{1}{(K-S)^2} + \frac{1}{S^2} + \frac{1}{I^2}\right) + \frac{1}{2}I^2F_2^2(S,I,R)\left(\frac{1}{I^2} + \frac{1}{(K-R)^2}\right) \\
&= \frac{1}{2}\left(S^2F_1^2(S,I,R) + I^2F_1^2(S,I,R) + S^2F_1^2(S,I,R) + F_2^2(S,I,R) + F_2^2(S,I,R)\right) \\
&\qquad\qquad\qquad\qquad \text{by } \frac{I}{K-S} \le 1 \text{ and } \frac{I}{K-R} \le 1 \\
&= \left(S^2 + \frac{I^2}{2}\right)F_1^2(S,I,R) + F_2^2(S,I,R) \\
&= \sup_{(S,I,R)\in \overset{\circ}{\mathbb{D}}}\left\{\frac{3K^2}{2}F_1^2(S,I,R) + F_2^2(S,I,R)\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{L}V(S,I,R) &\le \big(-\beta SI + \mu(K-S)\big)\left(\frac{1}{K-S} - \frac{1}{S}\right) + \big(\beta SI - (\alpha+\gamma+\mu)I\big)\left(1 - \frac{1}{I}\right) \\
&\quad + \big(\alpha I - \mu R\big)\left(\frac{1}{K-R} - 1\right) + \sup_{(S,I,R)\in \overset{\circ}{\mathbb{D}}}\left\{\frac{3K^2}{2}F_1^2(S,I,R) + F_2^2(S,I,R)\right\} \\
&= -\frac{\beta SI}{K-S} + \beta I + \mu - \frac{\mu K}{S} + \mu + \beta SI - \beta S - (\alpha+\gamma+\mu)I \\
&\quad + \alpha + \gamma + \mu + \alpha\frac{I}{K-R} - \alpha I - \frac{\mu R}{K-R} + \mu R \\
&\quad + \sup_{(S,I,R)\in \overset{\circ}{\mathbb{D}}}\left\{\frac{3K^2}{2}F_1^2(S,I,R) + F_2^2(S,I,R)\right\} \\
&= \beta I + \mu + \mu + \beta SI + \alpha + \gamma + \mu + \alpha\frac{I}{K-R} + \mu R \\
&\quad + \sup_{(S,I,R)\in \overset{\circ}{\mathbb{D}}}\left\{\frac{3K^2}{2}F_1^2(S,I,R) + F_2^2(S,I,R)\right\}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}V(S,I,R) &\overset{\frac{I}{K-R}\le 1}{\le} \beta I + \beta SI + 2\alpha + \gamma + 3\mu + \mu R \\
&\quad + \sup_{(S,I,R)\in \overset{\circ}{\mathbb{D}}} \left\{ \frac{3K^2}{2} F_1^2(S,I,R) + F_2^2(S,I,R) \right\} \\
&= \beta K(1+K) + 2\alpha + \gamma + 3\mu + \mu K \\
&\quad + \sup_{(S,I,R)\in \overset{\circ}{\mathbb{D}}} \left\{ \frac{3K^2}{2} F_1^2(S,I,R) + F_2^2(S,I,R) \right\} \\
&\overset{(4.6)}{=} 4c.
\end{aligned}
$$

Therefore, $\mathcal{L}V(S,I,R) \le c\, V(S,I,R)$ since $V(S,I,R) \ge 4$ for $(S,I,R) \in \overset{\circ}{\mathbb{D}}$.

Hence $\mathcal{L}W(S,I,R,t) = e^{-c(t-s)}\Big(-c\, V(S,I,R) + \mathcal{L}V(S,I,R)\Big) \le 0.$

Note that, $\inf_{(S,I,R)\in \overset{\circ}{\mathbb{D}}\setminus \mathbb{D}_n} V(S,I,R) > 2n+2$ for $n \in \mathbb{N}$.

Now define $\tau_n := \min\{t, \tau(\mathbb{D}_n)\}$ and apply Dynkin's formula

$$
\begin{aligned}
\mathbb{E}\, W(S(\tau_n), I(\tau_n), R(\tau_n), \tau_n) &= \mathbb{E}\, W(S(s), I(s), R(s), s) + \mathbb{E}\int_s^{\tau_n} \mathcal{L}W(S(u), I(u), R(u), u)\, du \\
&\le \mathbb{E}\, W(S(s), I(s), R(s), s) \\
&= \mathbb{E}V(S(s), I(s), R(s)) = \mathbb{E}V(S_0, I_0, R_0).
\end{aligned}
$$

$$
\begin{aligned}
\text{Next, } \mathbb{E}\Big[e^{c(t-\tau_n)} V\big(S(\tau_n), I(\tau_n), R(\tau_n)\big)\Big] &= \mathbb{E}\Big[e^{c(t-s)} e^{-c(\tau_n - s)} V\big(S(\tau_n), I(\tau_n), R(\tau_n)\big)\Big] \\
&= \mathbb{E}\Big[e^{c(t-s)}\ W\big(S(\tau_n), I(\tau_n), R(\tau_n), \tau_n\big)\Big] \\
&\le e^{c(t-s)} \mathbb{E}V(S_0, I_0, R_0).
\end{aligned}
$$

We have $\inf_{(S,I,R)\in \overset{\circ}{\mathbb{D}}\setminus \mathbb{D}_n} V(S,I,R) > 2n+2$ and $\mathbb{E}\Big[e^{c(t-\tau_n)} V\big(S(\tau_n), I(\tau_n), R(\tau_n)\big)\Big] \le e^{c(t-s)}\mathbb{E}V(S_0, I_0, R_0)$. Therefore

$$
\begin{aligned}
0 \le \mathbb{P}\big(\tau(\overset{\circ}{\mathbb{D}}) < t\big) &\overset{\mathbb{D}_n \subseteq \overset{\circ}{\mathbb{D}}}{\le} \mathbb{P}\big(\tau(\mathbb{D}_n) < t\big) \\
&= \mathbb{P}\big(\tau_n < t\big) \\
&= \mathbb{E}\big(\mathbf{1}_{\tau_n < t}\big) \qquad \text{where } \mathbf{1} \text{ is the indicator function} \\
&\le \mathbb{E}\left(e^{c(t-\tau_n)} \ \frac{V\Big(S\big(\tau(\mathbb{D}_n)\big), I\big(\tau(\mathbb{D}_n)\big), R\big(\tau(\mathbb{D}_n)\big)\Big)}{\inf_{(S,I,R)\in \overset{\circ}{\mathbb{D}}\setminus\mathbb{D}_n} V(S,I,R)} \ \mathbf{1}_{\tau_n<t} \right) \\
&\le e^{c(t-s)} \ \frac{\mathbb{E}V(S_0,I_0,R_0)}{\inf_{(S,I,R)\in \overset{\circ}{\mathbb{D}}\setminus\mathbb{D}_n} V(S,I,R)} \\
&\le e^{c(t-s)} \ \frac{\mathbb{E}V(S_0,I_0,R_0)}{2n+2} \qquad \to \ 0 \qquad \text{as } \ n \to \infty
\end{aligned}
$$

for all $(S_0, I_0, R_0) \in \mathbb{D}_n$ (for large n), and for all fixed $t \in [s, \infty)$.

Thus $\mathbb{P}\big(\tau(\overset{\circ}{\mathbb{D}}) < t\big) = \mathbb{P}\big(\tau(\mathbb{D}_n) < t\big) = 0$ for $(S_0, I_0, R_0) \in \overset{\circ}{\mathbb{D}}$ and $t \ge t_0$, that is, $\mathbb{P}\big(\tau(\overset{\circ}{\mathbb{D}}) = \infty\big) = 1$.

This proves the invariance property and the global existence of the solution $\big(S(t), I(t), R(t)\big)$ on $\overset{\circ}{\mathbb{D}}$. Uniqueness and continuity of the solution is obtained by Theorem 1.2.2.

Note that $I = 0$, $S = 0$, and $R = 0$ are not in the domain $\overset{\circ}{\mathbb{D}}$. We study these cases separately.

i) If $I(t) = 0$, the system (4.2) becomes an ODE

$$
\begin{aligned}
dS(t) &= \mu\big(K - S(t)\big)dt \\
dR(t) &= -\mu R(t)dt
\end{aligned} \tag{4.7}
$$

with initial condition $(S_0, R_0) \in D_1 = \{(S,R); S > 0, R > 0, S + R \le K\}$. Since the right hand side of the ODE is continuous on D_1, the solution $(S(t), R(t))$ globally exists on D_1 for all $t \ge t_0$.

ii) If $S(t) = 0$,

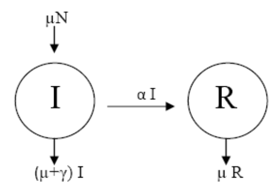


Figure 4.2: Flow chart of SIR model with disease deaths for $S = 0$.

the model becomes,

$$\begin{aligned} dI(t) &= \Big(\mu K - (\alpha+\gamma+\mu)I(t)\Big)dt - I(t)F(I(t),R(t))dW(t) \\ dR(t) &= \big(\alpha I(t) - \mu R(t)\big)dt + I(t)F(I(t),R(t))dW(t). \end{aligned} \tag{4.8}$$

If the initial condition $(S_0, R_0) \in D_2 = \{(I,R); I > 0, R \geq 0, S + R \leq K\}$ then, the above SDE has a unique global solution on D_2. One can prove it using a function $V(I,R) = I - \ln I + K - R - \ln(K-R)$ defined on D_2 in similar calculations.

iii) If $R(t) = 0$ (no recovery from the disease),

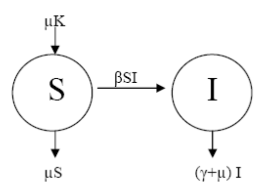


Figure 4.3: Flow chart of SIR model with disease deaths for $R = 0$.

the model becomes

$$\begin{aligned} dS(t) &= \big(-\beta S(t)I(t) + \mu(K - S(t))\big)dt - S(t)I(t)F(I(t),S(t))dW(t) \\ dI(t) &= \big(\beta S(t)I(t) - (\alpha+\mu)I(t)\big)dt + S(t)I(t)F(I(t),R(t))dW(t) \end{aligned} \tag{4.9}$$

If the initial condition $(S_0, I_0) \in D_3 = \{(I,S), t \geq t_0 : S \leq 0, I > 0, S + I \leq K\}$ then, the

SDE (4.9) has a unique global solution on D_3. It can be proven with using the function $V(I,S) = I - \ln I + K - S - \ln(K-S)$ defined on D_3.

Hence the proof is complete. The unique solution $\big(S(t), I(t), R(t)\big)$ exists globally and invariant with respect to the whole domain $\mathbb{D} = \big\{(S,I,R) \in \mathbb{R}^3; S \geq 0, I \geq 0, R \geq 0, S+I+R \leq K\big\}$ for all $(S_0, I_0, R_0) \in \mathbb{D}$ and $t \geq t_0$. □

4.2.2 Stability of Disease Free Equilibrium

Consider the stochastic SIR model with disease deaths

$$\begin{aligned} dS &= \Big(-\beta SI + \mu(K-S)\Big)\, dt - SI\ F_1\big(S,I,R\big)\ dW_1 \\ dI &= \Big(\beta SI - (\alpha+\gamma+\mu)I\Big)\, dt + SI\ F_1\big(S,I,R\big)\ dW_1 - I\ F_2\big(S,I,R\big)\ dW_2 \\ dR &= \Big(\alpha I - \mu R\Big)\, dt + I\ F_2\big(S,I,R\big)\ dW_2 \end{aligned} \tag{4.10}$$

where α, β, γ, μ, and K are positive and F_i's are locally Lipschitz-continuous on $\mathbb{D} = \big\{(S,I,R) : S \geq 0, I \geq 0, R \geq 0, S+I+R \leq K\big\}$.

Theorem 4.2.6. *The disease free equilibrium solution $(S_1, I_1, R_1) = (K, 0, 0)$ of (4.10) is stochastically asymptotically stable on $\mathbb{D}$ if the basic reproduction number $\mathcal{R}_0 = \dfrac{\beta K}{\alpha+\gamma+\mu} < 1$, more precisely if $\beta K \leq \gamma$.*

Proof. Define a Lyapunov function $V(S,I,R) = \dfrac{1}{2}(S-K+I+R)^2 + KI + KR$ on $\mathbb{D}$. Then,

$$\begin{aligned} \mathcal{L}V(S,I,R) &= (S-K+I+R)\Big(-\beta SI + \mu(K-S) + \beta SI - (\alpha+\gamma+\mu)I + \alpha I - \mu R\Big) \\ &\quad + K\Big(\beta SI - (\alpha+\gamma+\mu)I + \alpha I - \mu R\Big) \\ &\quad + \frac{I^2}{2}\Big(S^2F_1^2(S,I,R) - 2S^2F_1^2(S,I,R) + S^2F_1^2(S,I,R) \\ &\qquad + F_2^2(S,I,R) - 2F_2^2(S,I,R) + F_2^2(S,I,R)\Big) \end{aligned}$$

$$
\begin{aligned}
\mathcal{L}V(S,I,R) &= (S-K+I+R)\Big(\mu(K-S-I-R)-\gamma I\Big)+K\Big(\beta SI-(\gamma+\mu)I-\mu R\Big) \\
&= -\mu(S-K+I+R)^2+\gamma(K-S-I-R)I-\gamma I\Big) \\
&\quad +K\Big(\beta SI-(\gamma+\mu)I-\mu R\Big) \\
&= -\mu(S-K+I+R)^2+\gamma KI-\gamma SI-\gamma I^2-\gamma IR \\
&\quad +\beta KSI-\gamma KI-\mu KI-\mu KR \\
&= -\mu(S-K+I+R)^2-\gamma I^2-\gamma IR-\mu KI-\mu KR-(\gamma-\beta K)SI.
\end{aligned}
$$

Hence $\mathcal{L}V(S,I,R)$ is negative definite if $\gamma-\beta K\geq 0$.

By rewriting the stability condition, $\gamma-\beta K\geq 0$, in terms of the basic reproduction number, we have

$$
\beta K\leq\gamma<\alpha+\gamma+\mu \quad\Rightarrow\quad \frac{\beta K}{\alpha+\gamma+\mu}<1.
$$

Therefore $\mathcal{L}V(S,I,R)$ is negative definite if $\mathcal{R}_0\leq 1$. Theorem 1.2.3 completes the proof. □

4.2.3 Stability of Endemic Equilibrium

The model (4.10) has a unique endemic equilibrium solution

$$
\begin{aligned}
(S_2,I_2,R_2) &= \left(\frac{\alpha+\gamma+\mu}{\beta},\ \frac{\mu K}{\alpha+\gamma+\mu}-\frac{\mu}{\beta},\ \frac{\alpha K}{\alpha+\gamma+\mu}-\frac{\alpha}{\beta}\right) \\
&= \left(\frac{K}{\mathcal{R}_0},\ \frac{\mu}{\beta}\left(\mathcal{R}_0-1\right),\ \frac{\alpha}{\beta}\left(\mathcal{R}_0-1\right)\right)
\end{aligned}
\tag{4.11}
$$

if $\mathcal{R}_0>1$ and $F_i(S_2,I_2,R_2)=0$.

Theorem 4.2.7. *The endemic equilibrium solution, (S_2,I_2,R_2) of the system (4.10) is stochastically asymptotically stable on $\mathbb{D}=\Big\{(S,I,R): S>0, I>0, R>0, S+I+R\leq K\Big\}$*

if $\mathcal{R}_0 > 1$ *for some* $F_i(S,I,R)$ *such that* $F_i(S_2,I_2,R_2) = 0$ *and satisfies* $\mathcal{L}V \leq 0$ *where*

$$\begin{aligned} \mathcal{L}V &= -\frac{\mu}{K}(S - S_2 + I - I_2 + R - R_2)^2 - \frac{\gamma}{K}(I - I_2)^2 - b\ \mu(S - S_2)^2 \\ &\quad -c\ \mu(R - R_2)^2 + \frac{1}{2}\left(aI_2 + bK^2\right) S^2 F_1^2(S,I,R) + \frac{1}{2}\left(cI_2 + bK^2\right) F_2^2(S,I,R) \end{aligned} \tag{4.12}$$

for $a = \dfrac{\gamma}{\beta K} + bS_2$, $b > 0$ *and* $c = \dfrac{\gamma}{\alpha K}$.

Proof. Note that the conditions $\mathcal{R}_0 > 1$ and $F_i(S_2,I_2,R_2) = 0$ are needed for the existence of the endemic equilibrium solution. The following identities help to prove the theorem.

i) $$S_2 + I_2 + R_2 = \frac{\alpha+\gamma+\mu}{\beta} + \frac{\mu K}{\alpha+\gamma+\mu} - \frac{\mu}{\beta} + \frac{\alpha K}{\alpha+\gamma+\mu} - \frac{\alpha}{\beta} = \frac{\gamma}{\beta} + \frac{\alpha+\mu}{\alpha+\gamma+\mu}K \tag{4.13}$$

ii) $$\begin{aligned} \mu K - \mu(S+I+R) - \gamma I &= \mu K - \mu(S - S_2 + I - I_2 + R - R_2) - \gamma(I - I_2) \\ &\quad -\mu(S_2 + I_2 + R_2) - \gamma I_2 \\ &= -\mu(S - S_2 + I - I_2 + R - R_2) - \gamma(I - I_2) + \mu K \\ &\quad -\mu\left(\frac{\gamma}{\beta} + \frac{\alpha+\mu}{\alpha+\gamma+\mu}K\right) - \gamma\frac{\mu}{\beta}\left(\mathcal{R}_0 - 1\right) \\ &= -\mu(S - S_2 + I - I_2 + R - R_2) - \gamma(I - I_2) + \mu K - \frac{\gamma\mu}{\beta} \\ &\quad -\mu K\frac{\alpha+\mu}{\alpha+\gamma+\mu} - \gamma\frac{\mu}{\beta}\mathcal{R}_0 + \gamma\frac{\mu}{\beta} \\ &= -\mu(S - S_2 + I - I_2 + R - R_2) - \gamma(I - I_2) \\ &\quad +\mu K\left(1 - \frac{\alpha+\mu}{\alpha+\gamma+\mu}\right) - \gamma\frac{\mu}{\beta}\mathcal{R}_0 \\ &= -\mu(S - S_2 + I - I_2 + R - R_2) - \gamma(I - I_2) \\ &\quad +\mu K\frac{\gamma}{\alpha+\gamma+\mu} - \gamma\frac{\mu}{\beta}\mathcal{R}_0 \\ &= -\mu(S - S_2 + I - I_2 + R - R_2) - \gamma(I - I_2) \end{aligned} \tag{4.14}$$

$$\begin{aligned}
\text{iii)}\quad -\beta SI+\mu(K-S) &= -\beta(S-S_2)I-\beta S_2 I+\mu K-\mu(S-S_2)-\mu S_2\\
&= -\beta(S-S_2)I-\beta S_2(I-I_2)-\mu(S-S_2)-\beta S_2 I_2+\mu K-\mu S_2\\
&= -\beta(S-S_2)I-\beta S_2(I-I_2)-\mu(S-S_2)-\beta\frac{K}{\mathcal{R}_0}\frac{\mu}{\beta}(\mathcal{R}_0-1)\\
&\quad +\mu K-\mu\frac{K}{\mathcal{R}_0}\\
&= -\beta(S-S_2)I-\beta S_2(I-I_2)-\mu(S-S_2) \qquad (4.15)
\end{aligned}$$

$$\text{iv)}\quad \beta S-(\alpha+\gamma+\mu)=\beta\left(S-\frac{\alpha+\gamma+\mu}{\beta}\right)=\beta(S-S_2) \qquad (4.16)$$

$$\begin{aligned}
\text{v)}\quad \alpha I-\mu R &= \alpha(I-I_2)-\mu(R-R_2)+\alpha I_2-\mu R_2\\
&= \alpha(I-I_2)-\mu(R-R_2)+\alpha\frac{\mu}{\beta}(\mathcal{R}_0-1)-\mu\frac{\alpha}{\beta}(\mathcal{R}_0-1)\\
&= \alpha(I-I_2)-\mu(R-R_2). \qquad (4.17)
\end{aligned}$$

Now we are ready to prove. Define a Lyapunov function

$$\begin{aligned}
V(S,I,R) &= S-S_2+I-I_2+R-R_2-(S_2+I_2+R_2)\ln\frac{S+I+R}{S_2+I_2+R_2}\\
&\quad +a\left(I-I_2-I_2\ln\frac{I}{I_2}\right)+\frac{b}{2}(S-S_2)^2+\frac{c}{2}(R-R_2)^2 \qquad (4.18)
\end{aligned}$$

on $\mathbb{D}$, where the positive constants $a=\frac{\gamma}{\beta K}+bS_2$, $b>0$, and $c=\frac{\gamma}{\alpha K}$.

Then, $\mathcal{L}V(S,I,R)=\big(-\beta SI+\mu(K-S)\big)\frac{\partial V}{\partial S}+\big(\beta SI-(\alpha+\gamma+\mu)I\big)\frac{\partial V}{\partial I}+\big(\alpha I-\mu R\big)\frac{\partial V}{\partial R}+A(F_1,F_2)$ on $(S,I,R)\in\mathbb{D}$, where the last term $A(F_1,F_2)$ is

$$\begin{aligned}
A(F_1,F_2) &= \frac{1}{2}S^2I^2F_1^2(S,I,R)\left(\frac{\partial^2 V}{\partial S^2}-2\frac{\partial^2 V}{\partial SI}+\frac{\partial^2 V}{\partial I^2}\right)\\
&\quad +\frac{1}{2}I^2F_2^2(S,I,R)\left(\frac{\partial^2 V}{\partial I^2}-2\frac{\partial^2 V}{\partial IR}+\frac{\partial^2 V}{\partial R^2}\right)\\
&= \frac{S_2+I_2+R_2}{2(S+I+R)^2}\left\{S^2I^2F_1^2(S,I,R)\Big(1-2+1\Big)+I^2F_2^2(S,I,R)\Big(1-2+1\Big)\right\}\\
&\quad +\frac{1}{2}\left\{S^2I^2F_1^2(S,I,R)\left(b+a\frac{I_2}{I^2}\right)+I^2F_2^2(S,I,R)\left(c+a\frac{I_2}{I^2}\right)\right\}\\
&= \frac{1}{2}\left(aI_2+I^2b\right)S^2F_1^2(S,I,R)+\frac{1}{2}\left(cI_2+I^2b\right)F_2^2(S,I,R)
\end{aligned}$$

Then $\mathcal{L}V(S,I,R)$ becomes

$$\begin{aligned}
\mathcal{L}V(S,I,R) &= \left(1-\frac{S_2+I_2+R_2}{S+I+R}\right)\Big(-\beta SI+\mu(K-S)+\beta SI-(\alpha+\gamma+\mu)I+\alpha I-\mu R\Big) \\
&\quad +b(S-S_2)\big(-\beta SI+\mu(K-S)\big)+a\frac{I-I_2}{I}\big(\beta SI-(\alpha+\gamma+\mu)I\big) \\
&\quad +c(R-R_2)\big(\alpha I-\mu R\big)+A(F_1,F_2) \\
&= \frac{S-S_2+I-I_2+R-R_2}{S+I+R}\Big(-\mu(S-S_2+I-I_2+R-R_2)-\gamma(I-I_2)\Big) \\
&\quad +b(S-S_2)\Big(-\beta(S-S_2)I-\beta S_2(I-I_2)-\mu(S-S_2)\Big)+a\beta(I-I_2)(S-S_2) \\
&\quad +c(R-R_2)\Big(\alpha(I-I_2)-\mu(R-R_2)\Big)+A(F_1,F_2) \qquad \text{by the above identities} \\
&= -\frac{\mu}{S+I+R}(S-S_2+I-I_2+R-R_2)^2-\frac{\gamma}{S+I+R}(I-I_2)^2 \\
&\quad -\frac{\gamma}{S+I+R}(S-S_2)(I-I_2)-\frac{\gamma}{S+I+R}(I-I_2)(R-R_2) \\
&\quad -b\ \beta I(S-S_2)^2-b\ \beta S_2(S-S_2)(I-I_2)-b\ \mu(S-S_2)^2 \\
&\quad +a\ \beta(S-S_2)(I-I_2)+c\ \alpha(I-I_2)(R-R_2)-c\ \mu(R-R_2)^2+A(F_1,F_2) \\
&\leq -\frac{\mu}{K}(S-S_2+I-I_2+R-R_2)^2-\frac{\gamma}{K}(I-I_2)^2-b\ \Big(\beta I+\mu\Big)(S-S_2)^2 \\
&\quad -c\ \mu(R-R_2)^2-\left(\frac{\gamma}{K}+b\ \beta S_2-a\ \beta\right)(S-S_2)(I-I_2) \\
&\quad -\left(\frac{\gamma}{K}-c\ \alpha\right)(I-I_2)(I-R_2)+A(F_1,F_2).
\end{aligned}$$

Since $A(F_1,F_2)=\frac{1}{2}\left(aI_2+I^2b\right)S^2F_1^2(S,I,R)+\frac{1}{2}\left(cI_2+I^2b\right)F_2^2(S,I,R)$, $a=\dfrac{\gamma}{\beta K}+bS_2$ and $c=\dfrac{\gamma}{\alpha K}$

$$\begin{aligned}
\mathcal{L}V(S,I,R) &\leq -\frac{\mu}{K}(S-S_2+I-I_2+R-R_2)^2-\frac{\gamma}{K}(I-I_2)^2-b\ \Big(\beta I+\mu\Big)(S-S_2)^2 \\
&\quad -c\ \mu(R-R_2)^2+\frac{1}{2}\left(aI_2+I^2b\right)S^2F_1^2(S,I,R)+\frac{1}{2}\left(cI_2+I^2b\right)F_2^2(S,I,R) \\
&\leq -\frac{\mu}{K}(S-S_2+I-I_2+R-R_2)^2-\frac{\gamma}{K}(I-I_2)^2-b\ \mu(S-S_2)^2 \\
&\quad -c\ \mu(R-R_2)^2+\frac{1}{2}\left(aI_2+bK^2\right)S^2F_1^2(S,I,R)+\frac{1}{2}\left(cI_2+bK^2\right)F_2^2(S,I,R).
\end{aligned}$$

$\mathcal{L}V(S,I,R)=0$ only at (S_2,I_2,R_2) and by the choice of suitable functions $F_i(S,I,R)$, one can easily obtained $\mathcal{L}V(S,I,R) < 0$ on $\mathbb{D}\backslash(S_2,I_2,R_2)$. Hence $\mathcal{L}V(S,I,R)$ is negative definite on $\mathbb{D}$ for some suitable $F_i(S,I,R)$.

Therefore, by Theorem 1.2.3, the endemic equilibrium is stochastically asymptotically stable on $\mathbb{D}$ if $\mathcal{R}_0 > 1$ and for some suitable functions $F_i(S,I,R)$ such that $F_i(S_2,I_2,R_2) = 0$ and satisfies the condition (4.12).

□

Remark. If the F_i's in the stochastic SIR model with disease deaths are constant

$$\begin{aligned} dS &= \Big(-\beta SI + \mu(K-S)\Big)\,dt - \sigma_1\,SI\,dW_1 \\ dI &= \Big(\beta SI - \big(\alpha+\gamma+\mu\big)I\Big)\,dt + \sigma_1\,SI\,dW_1 - \sigma_2\,I\,dW_2 \\ dR &= \Big(\alpha I - \mu R\Big)\,dt + \sigma_2\,I\,dW_2 \end{aligned} \tag{4.19}$$

then the above system has only one equilibrium solution, which is disease free equilibrium $(K,0,0)$ and it is stochastically asymptotically stable if $\mathcal{R}_0 \leq 1$.

CHAPTER 5

NUMERICAL ANALYSIS OF STOCHASTIC MODELS IN EPIDEMIOLOGY

5.1 BALANCED IMPLICIT METHOD

Consider d-dimensional stochastic differential equation

$$dX(t) = a(X(t), t)\ dt\ +\ \sum_{j=1}^{m} b^j(X(t), t)\ dW^j(t) \tag{5.1}$$

with adapted initial values $X(0) = x_0 \in \mathbb{R}^d$, where W^j's are independent Wiener processes on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and a, and b are continuous vector fields.

The balanced implicit outer theta method (Schurz [45]) is defined by

$$\begin{aligned} X_{n+1} \ &= \ X_n + \Big[\Theta_n\ a(X_{n+1}, t_{n+1}) + (I - \Theta_n)\ a(X_n, t_n)\Big]\ h_n \\ &+ \ \sum_{j=1}^{m} b^j(X_n, t_n)\Delta W_n^j + \sum_{j=0}^{m} c^j(X_n, t_n)(X_n - X_{n+1})\ |\Delta W_n^j| \end{aligned} \tag{5.2}$$

where I is the unit matrix in $\mathbb{R}^{d\times d}$, $\Delta W_n^0 = h_n$, and $\Delta W_n^j = W^j(t_{n+1}) - W^j(t_n)$ along partitions $0 = t_0 < t_1 < t_2 < ..... < t_n < t_{n+1} < .... < t_{n_T} = T < \infty$ of finite time intervals $[0, T]$, with appropriate (bounded) matrices c^j with continuous entries.

The parameter matrices $\{\Theta_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{d\times d}$ determine the degree of implicitness. The most used are with scalar choices $\Theta_n = \theta_n I$, where $\theta_n \in \mathbb{R}^1$ and I is unit matrix in $R^{d\times d}$.

Throughout this chapter, the initial time is fixed $[0, T]$ with non-random and finite terminal time T. Let $\|.\| = \|.\|_d$ be Euclidean vector norm on $\mathbb{R}^d$, and $\|.\|_F$ be Frobenius norm on $\mathbb{R}^{d\times d}$.

5.2 NUMERICAL ANALYSIS OF STOCHASTIC SIR MODEL WITH DISEASE DEATHS

Consider the below balanced implicit method

$$Y_{n+1} = Y_n + f(Y_n, t_n)\Delta_n + \sum_{j=1}^{2} g^j(Y_n, t_n)\Delta W_n^j + c(Y_n, t_n)(Y_n - Y_{n+1}) \tag{5.3}$$

where $c(Y_n, t_n) = A\ I_{3\times 3}$ for the unit matrix $I_{3\times 3}$ and

$$A = (\alpha + \gamma + \mu + \beta I_n)\ \Delta_n + K\ |F_1(Y_n)\ \Delta W_n^1| + \frac{K}{R_n}\ |F_2(Y_n)\ \Delta W_n^2|$$

for a discretization of SIR model with disease deaths

$$dX(t) = f(X(t), t)\ dt + g(X(t), t)\ dW(t) \tag{5.4}$$

where

$$X(t) \ = \ \begin{pmatrix} S(t) \\ I(t) \\ R(t) \end{pmatrix}, \quad f(X(t), t) = \begin{pmatrix} -\beta S(t)I(t) + \mu(K - S(t)) \\ \beta S(t)I(t) - (\alpha + \gamma + \mu)I(t) \\ \alpha I(t) - \mu R(t) \end{pmatrix}$$

$$g(X(t), t) \ = \ \begin{pmatrix} -S(t)I(t)F_1(X(t)) & 0 \\ S(t)I(t)F_1(X(t)) & -I(t)F_2(X(t)) \\ 0 & I(t)F_2(X(t)) \end{pmatrix}, \text{ and } \quad dW(t) = \begin{pmatrix} dW^1(t) \\ dW^2(t) \end{pmatrix}.$$

Recall that, the parameters α, β, γ, μ, and K are positive. F_i's locally Lipschitz-continuous functions defined on $\mathbb{D} = \{(S, I, R) : S > 0,\ I \geq 0,\ R > 0,\ S + I + R \leq K\}$ for all $t \geq 0$ with coefficients $\widetilde{L}_i$ respectively, i=1,2. The W^j's are i.i.d. Wiener processes defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and independent of the initial value $X(0) = x_0 \in \mathbb{R}^3$ with $\mathbb{E}\|x_0\|^2 < \infty$. Coefficients f and g are Lipschitz-continuous and satisfy linear growth condition on $\mathbb{D}$. Furthermore, let's assume that f, g are Hölder continuous with order 1/2 in time and 1 in space on $\mathbb{D}$.

$$\begin{aligned}
\|f(X(t),t) - f(Y(t),s)\| &\leq L_1\Big(\|X(t) - Y(t)\| + |t-s|^{1/2}\Big) \\
\|g(X(t),t) - g(Y(t),s)\|_F^2 &\leq L_2^2\Big(\|X(t) - Y(t)\|^2 + |t-s|\Big) \\
\|f(X(t),t)\|^2 &\leq L_3^2\Big(1 + \|X(t)\|^2\Big) \\
\|g(X(t),t)\|_F^2 &\leq L_4^2\Big(1 + \|X(t)\|^2\Big)
\end{aligned}$$

where

$$\begin{aligned}
L_1^2 &= 8\beta^2K^2 + 2(\alpha+\gamma+\mu)^2 + 2\alpha^2 \\
L_2^2 &= 4\widetilde{L}_1K^4 + 4\widetilde{L}_2K^2 + \sup_{(S,I,R)\in\ \mathbb{D}}\Big\{8K^2F_1^2(S,I,R) + 4F_2^2(S,I,R)\Big\} \\
L_3^2 &= \max\Big\{4\beta^2K^4 + 4\mu^2K^2,\ 4\mu^2,\ 2(\alpha+\gamma+\mu)^2 + 2\alpha^2\Big\} \\
L_4^2 &= \max\Big\{\sup_{(S,I,R)\in\ \mathbb{D}} 2K^2F_1^2(S,I,R), \sup_{(S,I,R)\in\ \mathbb{D}} 2F_2^2(S,I,R)\Big\}. \qquad (5.5)
\end{aligned}$$

5.2.1 Invariance property of the numerical solution Y_n with respect to $\mathbb{D}$

Definition. A random sequence $\{Y_i\}_{i\in\mathbb{N}}$ is called **invariant** with respect to given domain $\mathbb{D} \subset \mathbb{R}^d$ if and only if $\mathbb{P}(Y_i \in \mathbb{D}) = 1$ for all $i \in \mathbb{N}$.

Theorem 5.2.1. *Assume that the initial condition* $Y_0 = (S_0, I_0, R_0) \in \mathbb{D} = \{(S_n, I_n, R_n) : S_n > 0,\ I_n \geq 0,\ R_n > 0,\ S_n + I_n + R_n \leq K\}$ *is independent of* $W(t)$ *for* $t \geq 0$*. The numerical solution* $\{Y_i\}_{i\in\mathbb{N}}$ *governed by* (5.4) *is invariant with respect to* $\mathbb{D}$.

Proof. By rewriting the numerical method (5.3) we obtained,

$$
\begin{aligned}
S_{n+1} &= S_n + \Big[-\beta S_n I_n + \mu(K - S_n)\Big]\Delta_n - S_n I_n F_1(Y_n)\Delta W_n^1 + A(S_n - S_{n+1}) \\
I_{n+1} &= I_n + \Big[\beta S_n I_n - (\alpha+\gamma+\mu) I_n\Big]\Delta_n + S_n I_n F_1(Y_n)\Delta W_n^1 - I_n F_2(Y_n)\Delta W_n^2 + A(I_n - I_{n+1}) \\
R_{n+1} &= R_n + \Big[\alpha I_n - \mu R_n\Big]\Delta_n + I_n F_2(Y_n)\Delta W_n^2 + A(R_n - R_{n+1})
\end{aligned}
\tag{5.6}
$$

$$
\begin{aligned}
S_{n+1} &= S_n + \frac{1}{1+A}\Big\{[-\beta S_n I_n + \mu(K - S_n)]\Delta_n - S_n I_n F_1(Y_n)\Delta W_n^1\Big\} \\
I_{n+1} &= I_n + \frac{1}{1+A}\Big\{[\beta S_n I_n - (\alpha+\gamma+\mu) I_n]\Delta_n + S_n I_n F_1(Y_n)\Delta W_n^1 - I_n F_2(Y_n)\Delta W_n^2\Big\} \\
R_{n+1} &= R_n + \frac{1}{1+A}\Big\{[\alpha I_n - \mu R_n]\Delta_n + I_n F_2(Y_n)\Delta W_n^2\Big\}.
\end{aligned}
\tag{5.7}
$$

Let's use an induction on $n \in \mathbb{N}$. Assume that $S_0 + I_0 + R_0 \leq K$, and $S_n + I_n + R_n \leq K$ with $S_n, R_n > 0$ and $I_n \geq 0$. Then,

$$
\begin{aligned}
S_{n+1} + R_{n+1} + I_{n+1} &= S_n + I_n + R_n + \frac{\mu\Delta_n}{1+A}\Big(K - S_n - I_n - R_n\Big) - \frac{\gamma I_n \Delta_n}{1+A} \\
&\leq S_n + I_n + R_n + \frac{\mu\Delta_n}{1+A}\Big(K - S_n - I_n - R_n\Big) \\
&= \left(1 - \frac{\mu\Delta_n}{1+A}\right)(S_n + I_n + R_n) + \frac{\mu\Delta_n}{1+A}K \\
&\leq \left(1 - \frac{\mu\Delta_n}{1+A}\right)K + \frac{\mu\Delta_n}{1+A}K \qquad \text{since} \quad \frac{\mu\Delta_n}{1+A} \leq 1 \\
&= K.
\end{aligned}
$$

Positivity of numerical solution can be proven by substituting $A = (\alpha + \gamma + \mu + \beta I_n)\Delta_n + K\ |F_1(Y_n)\Delta W_n^1| + \frac{K}{R_n}|F_2(Y_n)\Delta W_n^2|$ into (5.7),

$$\begin{aligned}
S_{n+1} &= \frac{1}{1+A}\Big\{S_n + AS_n + [-\beta S_n I_n + \mu(K-S_n)]\Delta_n - S_n I_n F_1(Y_n)\Delta W_n^1\Big\} \\
&= \frac{1}{1+A}\Big\{S_n + (\alpha+\gamma+\mu+\beta I_n)S_n\Delta_n + KS_n|F_1(Y_n)\Delta W_n^1| \\
&\quad + \frac{KS_n}{R_n}|F_2(Y_n)\Delta W_n^2| + [-\beta S_n I_n + \mu(K-S_n)]\Delta_n - S_n I_n F_1(Y_n)\Delta W_n^1\Big\} \\
&= \frac{\mu K\Delta_n}{1+A} + \frac{S_n}{1+A}\Big\{1 + (\alpha+\gamma)\Delta_n + K\ |F_1(Y_n)\Delta W_n^1| - I_n F_1(Y_n)\Delta W_n^1 \\
&\quad + \frac{K}{R_n}|F_2(Y_n)\Delta W_n^2|\Big\} \\
&> 0 \qquad (5.8)
\end{aligned}$$

$$\begin{aligned}
I_{n+1} &= \frac{I_n}{1+A}\Big\{1 + A + [\beta S_n - (\alpha+\gamma+\mu)]\Delta_n + S_n F_1(Y_n)\Delta W_n^1 - F_2(Y_n)\Delta W_n^2\Big\} \\
&= \frac{I_n}{1+A}\Big\{1 + (\alpha+\gamma+\mu+\beta I_n)\ \Delta_n + K|F_1(Y_n)\ \Delta W_n^1| + \frac{K}{R_n}|F_2(Y_n)\ \Delta W_n^2| \\
&\quad + [\beta S_n - (\alpha+\gamma+\mu)]\Delta_n + S_n F_1(Y_n)\Delta W_n^1 - F_2(Y_n)\Delta W_n^2\Big\} \\
&= \frac{I_n}{1+A}\Big\{1 + \beta(I_n + S_n)\ \Delta_n + K\ |F_1(Y_n)\Delta W_n^1| + S_n F_1(Y_n)\Delta W_n^1 \\
&\quad - F_2(Y_n)\Delta W_n^2 + \frac{K}{R_n}|F_2(Y_n)\Delta W_n^2|\Big\} \\
&\geq 0 \qquad \text{since} \quad \frac{K}{R_n} \geq 1 \qquad (5.9)
\end{aligned}$$

$$\begin{aligned}
R_{n+1} &= \frac{1}{1+A}\Big\{R_n + AR_n + [\alpha I_n - \mu R_n]\Delta_n + I_n F_2(Y_n)\Delta W_n^2\Big\} \\
&= \frac{1}{1+A}\Big\{R_n + (\alpha+\gamma+\mu+\beta I_n)R_n\Delta_n + K|F_1(Y_n)\Delta W_n^1|R_n \\
&\quad + \frac{K}{R_n}|F_2(Y_n)\Delta W_n^2|R_n + [\alpha I_n - \mu R_n]\Delta_n + I_n F_2(Y_n)\Delta W_n^2\Big\} \\
&= \frac{1}{1+A}\Big\{R_n + (\alpha+\gamma+\beta I_n)R_n\Delta_n + \alpha I_n\Delta_n \\
&\quad + K|F_1(Y_n)\Delta W_n^1|R_n + K|F_2(Y_n)\Delta W_n^2| + I_n F_2(Y_n)\Delta W_n^2\Big\} \\
&> 0. \qquad (5.10)
\end{aligned}$$

Therefore, for all $n \in \mathbb{N} : \mathbb{P}(Y_n = (S_n, I_n, R_n) \in \mathbb{D}) = 1.$ □

5.2.2 V-stability of the Numerical Solution Y_n

We have

$$\begin{aligned} dX(t) &= f(X(t),t)\ dt + g(X(t),t)\ dW(t), \quad t \geq 0, \quad X(0) = x_0 \\ Y_{n+1} &= Y_n + \frac{1}{1+A}\ f(Y_n)\Delta_n + \frac{1}{1+A}\ g(Y_n)\Delta W_n, \quad Y_0 = x_0. \end{aligned}$$

The one step representations of the solution X and the numerical solution Y are

$$\begin{aligned} X_{x_0,s}(t) &= x_0 + \int_s^t f(X(u),u)\ du + \int_s^t g(X(u),u)\ dW(u) \\ Y_{x_0,s}(t) &= x_0 + \frac{1}{1+A}\ f(x_0,s)\int_s^t du + \frac{1}{1+A}\ g(x_0,s)\int_s^t dW(u) \end{aligned} \tag{5.11}$$

for all $t > s \geq 0$, $|t-s| \leq 1$, and $X(s) = x_0 \in \mathbb{D}$.

Now,

$$\begin{aligned} 1 + \|Y_{x_0,s}(t)\|^2 &= 1 + \left\| x_0 + \frac{1}{1+A}\ f(x_0,s)\ (t-s) + \frac{1}{1+A}\ g(x_0,s)\big(W(t) - W(s)\big) \right\|^2 \\ &= 1 + \|x_0\|^2 + \left\| \frac{1}{1+A} \left[f(x_0,s)\ (t-s) + g(x_0,s)\big(W(t)-W(s)\big)\right] \right\|^2 \\ &\quad +2\left\langle x_0, \frac{1}{1+A}\ \left[f(x_0,s)\ (t-s) + g(x_0,s)\Big(W(t)-W(s)\Big)\right] \right\rangle \\ &\leq 1 + \|x_0\|^2 + \left\| \frac{1}{1+A} \right\|^2 \big\| f(x_0,s)\ (t-s) + g(x_0,s)\big(W(t)-W(s)\big)\big\|^2 \\ &\quad +2\left\langle x_0, \frac{1}{1+A}\ \left[f(x_0,s)\ (t-s) + g(x_0,s)\big(W(t)-W(s)\big)\right] \right\rangle \\ &\overset{\|\frac{1}{1+A}\|^2<1}{<} 1 + \|x_0\|^2 + \|f(x_0,s)\ (t-s)\|^2 + \big\|g(x_0,s)\big(W(t)-W(s)\big)\big\|^2 \\ &\quad +2\Big\langle f(x_0,s)\ (t-s),\ g(x_0,s)\big(W(t)-W(s)\big)\Big\rangle \\ &\quad +2\left\langle x_0,\ \frac{1}{1+A}\ \left[f(x_0,s)\ (t-s) + g(x_0,s)\big(W(t)-W(s)\big)\right] \right\rangle \end{aligned}$$

Since $W(t) - W(s) \sim \mathcal{N}(0, t-s)$

$$
\begin{aligned}
1+\mathbb{E}\|Y_{x_0,s}(t)\|^2 \quad &\leq \quad 1+\|x_0\|^2+\|f(x_0,s)\|^2\ (t-s)^2+\|g(x_0,s)\|_F^2\,\mathbb{E}\big(W(t)-W(s)\big)^2 \\
&\quad +2\Big\langle x_0,\ \frac{1}{1+A}\ f(x_0,s)\ (t-s)\Big\rangle \\
&\leq \quad 1+\|x_0\|^2+L_3^2(1+\|x_0\|^2)(t-s)^2+L_4^2(1+\|x_0\|^2)(t-s) \\
&\quad +2\|x_0\|\ \left\|\frac{1}{1+A}\right\|\ \|f(x_0,s)\|\ (t-s) \\
&\overset{(t-s)^2\leq t-s}{\leq} \quad \Big(1+(L_3^2+L_4^2)(t-s)\Big)(1+\|x_0\|^2)+2L_3\|x_0\|\ \sqrt{1+\|x_0\|^2}\ (t-s) \\
&\overset{\|x_0\|\leq\sqrt{1+\|x_0\|^2}}{\leq} \quad \Big(1+(2L_3+L_3^2+L_4^2)(t-s)\Big)(1+\|x_0\|^2) \\
&\leq \quad e^{(2L_3+L_3^2+L_4^2)(t-s)}(1+\|x_0\|^2) \qquad (5.12)
\end{aligned}
$$

the last step obtained by the Taylor series expansion around s, i.e. $e^{a(t-s)} \geq 1+a(t-s)$.

Hence, the approximation Y is V-stable with $V(x_0)=1+\|x_0\|^2$.

5.2.3 Mean Square Contractivity of X

$$
\begin{aligned}
\mathbb{E}\,\|X_{x_0,s}(t)-X_{y_0,s}(t)\|^2 \;&=\; \mathbb{E}\Big\|x_0-y_0+\int_s^t \Big[f(X_{x_0,s}(u),u)-f(X_{y_0,s}(u),u)\Big]\ du \\
&\qquad +\int_s^t \Big[g(X_{x_0,s}(u),u)-g(X_{y_0,s}(u),u)\Big]\ dW(u)\Big\|^2 \\
&\leq\; 3\|x_0-y_0\|^2+3\mathbb{E}\Big\|\int_s^t \Big[f(X_{x_0,s}(u),u)-f(X_{y_0,s}(u),u)\Big]\ du\Big\|^2 \\
&\qquad +3\mathbb{E}\Big\|\int_s^t \Big[g(X_{x_0,s}(u),u)-g(X_{y_0,s}(u),u)\Big]\ dW(u)\Big\|^2
\end{aligned}
$$

By the Hölder's inequality, $\left\|\int_s^t a(u)\ du\right\|^2 \leq \left(\int_s^t \|1\|^2\ du\right)\left(\int_s^t \|a(u)\|^2\ du\right)$, and Itô isometry, $\mathbb{E}\ \left\|\int_s^t a(u)\ dW(u)\right\|^2 = \mathbb{E}\int_s^t\ \|a(u)\|^2\ du$,

$$
\begin{aligned}
\mathbb{E}\,\|X_{x_0,s}(t) - X_{y_0,s}(t)\|^2 \quad & \leq \quad 3\|x_0 - y_0\|^2 + 3(t-s)\ \mathbb{E}\int_s^t \|f(X_{x_0,s}(u),u) - f(X_{y_0,s}(u),u)\|^2\ du \\
& \qquad + 3\ \mathbb{E}\int_s^t \|g(X_{x_0,s}(u),u) - g(X_{y_0,s}(u),u)\|_F^2\ du \\
& \overset{t-s\leq 1}{\leq} \quad 3\|x_0 - y_0\|^2 + 3(L_1^2 + L_2^2)\ \int_s^t\ \mathbb{E}\|X_{x_0,s}(u) - X_{y_0,s}(u)\|^2\ du
\end{aligned}
$$

After applying the Gronwall inequality, we obtained

$$
\mathbb{E}\,\|X_{x_0,s}(t) - X_{y_0,s}(t)\|^2 \leq 3\|x_0 - y_0\|^2\ e^{3(L_1^2+L_2^2)(t-s)}. \tag{5.13}
$$

We may choose K_C^X such that $e^{2K_C^X\ (t-s)} \leq 3e^{3(L_1^2+L_2^2)(t-s)}$, for instance $K_C^X = \frac{3}{2}(L_1^2 + L_2^2) + \frac{\ln 3}{2}$.

Hence $\mathbb{E}\,\|X_{x_0,s}(t) - X_{y_0,s}(t)\|^2 \leq \|x_0 - y_0\|^2\ e^{2K_C^X(t-s)}$

5.2.4 Local Uniform Boundedness of X and Y_n

$$
\begin{aligned}
1 + \mathbb{E}\|X_{x_0,s}(t)\|^2 \quad & = \quad 1 + \mathbb{E}\|x_0 + \int_s^t f(X(u),u)\ du + \int_s^t g(X(u),u)\ dW(u)\|^2 \\
& \leq \quad 1 + 3\|x_0\|^2 + 3\mathbb{E}\left\|\int_s^t f(X(u),u)\ du\right\|^2 + 3\mathbb{E}\left\|\int_s^t g(X(u),u)\ dW(u)\right\|^2 \\
& \leq \quad 1 + 3\|x_0\|^2 + 3(t-s)\ \mathbb{E}\int_s^t \|f(X(u),u)\|^2 du + 3\mathbb{E}\int_s^t \|g(X(u),u)\|_F^2 du \\
& \leq \quad 1 + 3\|x_0\|^2 + 3L_3^2\ \int_s^t \big(1 + \mathbb{E}\|x(u)\|^2\big) du + 3L_4^2\ \int_s^t \big(1 + \mathbb{E}\|x(u)\|^2\big) du \\
& \leq \quad 3 + 3\|x_0\|^2 + 3(L_3^2 + L_4^2)\ \int_s^t \big(1 + \mathbb{E}\|x(u)\|^2\big) du \\
& = \quad 3(1 + \|x_0\|^2)\ e^{3(L_3^2+L_4^2)(t-s)} \qquad (5.14) \\
& = \quad c_1^2(1 + \|x_0\|^2)
\end{aligned}
$$

where $c_1^2 = 3e^{3(L_3^2+L_4^2)(t-s)}$.

$$
\begin{aligned}
1+\|Y_{x_0,s}(t)\|^2 &= 1+\left\|x_0+\frac{1}{1+A}f(x_0,s)\int_s^t du+\frac{1}{1+A}g(x_0,s)\int_s^t dW(u)\right\|^2 \\
&\le 1+3\|x_0\|^2+3\left\|\frac{1}{1+A}\right\|^2\|f(x_0,s)\|^2\left\|\int_s^t du\right\|^2 \\
&\quad +3\left\|\frac{1}{1+A}\right\|^2\|g(x_0,s)\|_F^2\left\|\int_s^t dW(u)\right\|^2 \\
&\le 1+3\|x_0\|^2+3L_3^2(1+\|x_0\|^2)(t-s)^2+3L_4^2(1+\|x_0\|^2)\left\|\int_s^t dW(u)\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
1+\mathbb{E}\|Y_{x_0,s}(t)\|^2 &\le 1+3\|x_0\|^2+3L_3^2(1+\|x_0\|^2)(t-s)^2+3L_4^2(1+\|x_0\|^2)\ \mathbb{E}\ \left\|\int_s^t dW(u)\right\|^2 \\
&= 1+3\|x_0\|^2+3L_3^2(1+\|x_0\|^2)(t-s)^2+3L_4^2(1+\|x_0\|^2)\ \mathbb{E}\ \int_s^t \|1\|^2 du \\
&\le 3+3\|x_0\|^2+3L_3^2(1+\|x_0\|^2)(t-s)^2+3L_4^2(1+\|x_0\|^2)(t-s) \\
&\le 3\Big[1+(L_3^2+L_4^2)(t-s)\Big](1+\|x_0\|^2) \\
&= c_2^2(1+\|x_0\|^2) \qquad (5.15)
\end{aligned}
$$

where $c_2^2=3\Big[1+(L_3^2+L_4^2)(t-s)\Big]$.

5.2.5 Local Mean Square Hölder Continuity of X and Y_n

$$
\begin{aligned}
\mathbb{E}\|X_{x_0,s}-x_0\|^2 &= \mathbb{E}\left\|\int_s^t f(X(u),u)\ du+\int_s^t g(X(u),u)\ dW(u)\right\|^2 \\
&\le 2\mathbb{E}\left\|\int_s^t f(X(u),u)\ du\right\|^2+2\mathbb{E}\left\|\int_s^t g(X(u),u)\ dW(u)\right\|^2 \\
&\le 2(t-s)\mathbb{E}\int_s^t \|f(X(u),u)\|^2\ du+2\mathbb{E}\int_s^t \|g(X(u),u)\|_F^2\ du \\
&\le 2(L_3^2+L_4^2)\int_s^t (1+\mathbb{E}\|X(u)\|^2)\ du \qquad (5.16)
\end{aligned}
$$

By the local uniform boundedness of X (5.14)

$$\begin{aligned}
\mathbb{E}\|X_{x_0,s}(t)-x_0\|^2 &\leq 2(L_3^2+L_4^2)(1+\|x_0\|^2)\int_s^t 3e^{3(L_3^2+L_4^2)(u-s)}\,du \\
&= 6(L_3^2+L_4^2)(1+\|x_0\|^2)\frac{e^{3(L_3^2+L_4^2)(t-s)}-1}{3(L_3^2+L_4^2)} \\
&= 2\Big[e^{3(L_3^2+L_4^2)(t-s)}-1\Big](1+\|x_0\|^2) \\
&= 2\frac{e^{3(L_3^2+L_4^2)(t-s)}-1}{t-s}(1+\|x_0\|^2)(t-s). \qquad (5.17)
\end{aligned}$$

We may choose $c_3^2 \leq 2\max_{|z|\leq 1}\dfrac{e^{3(L_3^2+L_4^2)z}-1}{z} < \infty$.

Hence,

$$\mathbb{E}\|X_{x_0,s}-x_0\|^2 \leq c_3^2(1+\|x_0\|^2)(t-s) \qquad (5.18)$$

. That proves the local Hölder continuity of X in the mean square sense (with exponent 1/2 in time).

$$\begin{aligned}
\|Y_{x_0,s}-x_0\|^2 &= \left\|\frac{1}{1+A}f(x_0,s)\int_s^t du + \frac{1}{1+A}g(x_0,s)\int_s^t dW(u)\right\|^2 \\
&\leq \left\|\frac{1}{1+A}\right\|^2\|f(x_0,s)\|^2\left\|\int_s^t du\right\|^2 + \left\|\frac{1}{1+A}\right\|^2\|g(x_0,s)\|_F^2\left\|\int_s^t dW(u)\right\|^2 \\
&\quad +2\Big\langle\frac{1}{1+A}f(x_0,s)\int_s^t du,\ \ \frac{1}{1+A}g(x_0,s)\int_s^t dW(u)\Big\rangle \\
&\leq L_3^2(1+\|x_0\|^2)(t-s)^2 + L_4^2(1+\|x_0\|^2)\left\|\int_s^t dW(u)\right\|^2 \\
&\quad +2\Big\langle\frac{1}{1+A}f(x_0,s)\int_s^t du,\ \ \frac{1}{1+A}g(x_0,s)\int_s^t dW(u)\Big\rangle
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\|Y_{x_0,s}-x_0\|^2 &\leq L_3^2(1+\|x_0\|^2)(t-s)^2+L_4^2(1+\|x_0\|^2)\mathbb{E}\left\|\int_s^t dW(u)\right\|^2\\
&\quad +2\mathbb{E}\Big\langle\frac{1}{1+A}f(x_0,s)\int_s^t du,\ \ \frac{1}{1+A}g(x_0,s)\int_s^t dW(u)\Big\rangle\\
&\leq L_3^2(1+\|x_0\|^2)(t-s)^2+L_4^2(1+\|x_0\|^2)\mathbb{E}\int_s^t \|1\|^2\, du\\
&\leq (L_3^2+L_4^2)(t-s)(1+\|x_0\|^2)\\
&= c_4^2(t-s)(1+\|x_0\|^2) \qquad (5.19)
\end{aligned}$$

where $c_4^2 = L_3^2+L_4^2$. That completes the proof of the local Hölder continuity of the numerical solution Y_n in mean square sense.

5.2.6 Mean Consistency of X, Y_n

First, let's estimate $\mathbb{E}\left(\frac{A}{1+A}\right)$, and $\mathbb{E}\left(\frac{A}{1+A}\int_s^t dW(u)\right)$ for

$$\begin{aligned}
A &= (\alpha+\gamma+\mu+\beta I_n)(t-s)+K\ \left|F_1(X_{x_0,s}(t))\big(W^1(t)-W^1(s)\big)\right|\\
&\quad +\frac{K}{R_n}\ \left|F_2(X_{x_0,s}(t))\big(W^2(t)-W^2(s)\big)\right|\\
&\leq a(t-s)+b|W(t)-W(s)| \qquad (5.20)
\end{aligned}$$

for finite constants a, and b. Since $Y_{x_0,s}(t)$ is invariant with respect to the bounded domain $\mathbb{D}$.

Since $0<\frac{1}{1+A}<1$, then $0<\frac{A}{1+A}<A$.

i) $\mathbb{E}\left(\frac{A}{1+A}\right)<\mathbb{E}(A)\leq a\ (t-s)+b\ \mathbb{E}|W(t)-W(s)|$

We know that $|W(t)-W(s)| \sim$ Half-Normal $\left(\sqrt{\frac{2}{\pi}}\ \sqrt{t-s},\ \ \left(1-\frac{2}{\pi}\right)(t-s)\right)$ by $W(t)-W(s)\sim\mathcal{N}(0,t-s)$.

Therefore

$$\mathbb{E}\left(\frac{A}{1+A}\right)\leq a\ (t-s)+b\ \sqrt{\frac{2}{\pi}}\ (t-s)^{1/2}<a\ (t-s)+b\ (t-s)^{1/2} \qquad (5.21)$$

$$\textbf{ii)}\mathbb{E}\left(\frac{A}{1+A}\int_s^t dW(u)\right) \leq a\ (t-s)\mathbb{E}\int_s^t dW(u) + b\ \mathbb{E}\left\{|W(t)-W(s)|\int_s^t dW(u)\right\}$$
$$= 0 \qquad (5.22)$$

Now,

$$\begin{aligned} X_{x_0,s(t)} - Y_{x_0,s}(t) &= \int_s^t f(X(u),u)\ du + \int_s^t g(X(u),u)\ dW(u) \\ &\quad -\frac{1}{1+A}f(x_0,s)\int_s^t\ du - \frac{1}{1+A}g(x_0,s)\int_s^t\ dW(u) \\ &= \int_s^t [f(X(u),u)-f(x_0,s)]\ du + \int_s^t [g(X(u),u)-g(x_0,s)]\ dW(u) \\ &\quad +\left(1-\frac{1}{1+A}\right)f(x_0,s)\int_s^t\ du + \left(1-\frac{1}{1+A}\right)g(x_0,s)\int_s^t\ dW(u) \end{aligned}$$

$$\begin{aligned} \mathbb{E}\left(X_{x_0,s}(t) - Y_{x_0,s}(t)\right) &= \mathbb{E}\int_s^t [f(X(u),u)-f(x_0,s)]\ du \\ &\quad +f(x_0,s)(t-s)\ \mathbb{E}\left(\frac{A}{1+A}\right) \\ &\quad +g(x_0,s)\ \mathbb{E}\left(\frac{A}{1+A}\int_s^t\ dW(u)\right) \\ &\overset{(5.21),(5.22)}{\leq} \mathbb{E}\int_s^t [f(X(u),u)-f(x_0,s)]\ du + a\ f(x_0,s)(t-s)^2 \\ &\quad +b\ f(x_0,s)(t-s)^{3/2} \end{aligned}$$

$$\begin{aligned} \|\mathbb{E}\left(X_{x_0,s}(t) - Y_{x_0,s}(t)\right)\| &\leq \left\|\mathbb{E}\int_s^t [f(X(u),u)-f(x_0,s)]\ du\right\| + a\ \|f(x_0,s)\|(t-s)^2 \\ &\quad +b\ \|f(x_0,s)\|(t-s)^{3/2} \\ &\leq \mathbb{E}\left\|\int_s^t [f(X(u),u)-f(x_0,s)]\ du\right\| + a\ L_3\ \sqrt{1+\|x_0\|^2}(t-s)^2 \\ &\quad +b\ L_3\ \sqrt{1+\|x_0\|^2}(t-s)^{3/2} \\ &\leq \int_s^t \mathbb{E}\,\|[f(X(u),u)-f(x_0,s)]\|\ du + a\ L_3\ \sqrt{1+\|x_0\|^2}(t-s)^2 \\ &\quad +b\ L_3\ \sqrt{1+\|x_0\|^2}(t-s)^{3/2} \end{aligned}$$

$$
\begin{aligned}
\|\mathbb{E}\left(X_{x_0,s}(t)-Y_{x_0,s}(t)\right)\| &\leq L_1 \int_s^t \left\{|u-s|^{1/2}+\mathbb{E}\|X(u)-x_0\|\right\}\, du \\
&\quad +a\ L_3\ \sqrt{1+\|x_0\|^2}(t-s)^2+b\ L_3\ \sqrt{1+\|x_0\|^2}(t-s)^{3/2} \\
&\leq \frac{2}{3}L_1\ |t-s|^{3/2}+L_1\ \int_s^t \left(\mathbb{E}\|X(u)-x_0\|^2\right)^{1/2}\, du \\
&\quad +a\ L_3\ \sqrt{1+\|x_0\|^2}(t-s)^2+b\ L_3\ \sqrt{1+\|x_0\|^2}(t-s)^{3/2} \\
&\overset{(5.18)}{\leq} \frac{2}{3}L_1\ |t-s|^{3/2}+L_1\ c_3\ \int_s^t \sqrt{1+\|x_0\|^2}u^{1/2}\, du \\
&\quad +a\ L_3\ \sqrt{1+\|x_0\|^2}(t-s)^2+b\ L_3\ \sqrt{1+\|x_0\|^2}(t-s)^{3/2} \\
&\leq \left[\frac{2}{3}L_1+\left(\frac{2}{3}L_1\ c_3+a\ L_3+b\ L_3\right)\sqrt{1+\|x_0\|^2}\right](t-s)^{3/2} \\
&\leq \left(\frac{2}{3}L_1+\frac{2}{3}L_1\ c_3+a\ L_3+b\ L_3\right)\sqrt{1+\|x_0\|^2}(t-s)^{3/2}
\end{aligned}
$$

by $1 \leq \sqrt{1+\|x_0\|^2}$. Therefore, the numerical method is mean consistent with rate $r_0 = 1.5$.

5.2.7 Mean Square Consistency of X, Y_n

First, we estimate $\mathbb{E}\left\|\frac{A}{1+A}\int_s^t dW(u)\right\|^2$, using Cauchy-Bunyakovsky-Schwarz inequality, Itô isometry and properties of Wiener processes.

$$
\begin{aligned}
\frac{A}{1+A}\int_s^t dW(u) &\leq \frac{A}{1+A}\left|\int_s^t dW(u)\right| \leq A\left|\int_s^t dW(u)\right| \\
\left\|\frac{A}{1+A}\int_s^t dW(u)\right\|^2 &\leq \left\|A\left|\int_s^t dW(u)\right|\right\|^2 \\
&\leq \|A\|^2\left\|\int_s^t dW(u)\right\|^2 \\
&\overset{(5.20)}{\leq} 2\Big(a^2(t-s)^2+b^2\big(W(t)-W(s)\big)^2\Big)\big(W(t)-W(s)\big)^2
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left\|\frac{A}{1+A}\int_s^t dW(u)\right\|^2 &\leq 2a^2(t-s)^2\mathbb{E}\big(W(t)-W(s)\big)^2+2b^2\mathbb{E}\big(W(t)-W(s)\big)^4 \\
&\leq 2a^2(t-s)^2(t-s)+2b^2 3(t-s)^2 \\
&= 2a^2(t-s)^3+6b^2(t-s)^2. \qquad (5.23)
\end{aligned}
$$

Then,

$$\begin{aligned}
\mathbb{E}\left\|X_{x_0,s}(t)-Y_{x_0,s}(t)\right\|^2 &= \mathbb{E}\| \int_s^t \left[f(X(u),u)-f(x_0,s)\right]\ du \\
&\quad + \int_s^t \left[g(X(u),u)-g(x_0,s)\right]\ dW(u) \\
&\quad + \left(1-\frac{1}{1+A}\right) f(x_0,s)\int_s^t\ du \\
&\quad + \left(1-\frac{1}{1+A}\right) g(x_0,s)\int_s^t\ dW(u)\|^2 \\
&\leq 4\mathbb{E}\left\|\int_s^t \left[f(X(u),u)-f(x_0,s)\right]\ du\right\|^2 \\
&\quad +4\mathbb{E}\left\|\int_s^t \left[g(X(u),u)-g(x_0,s)\right]\ dW(u)\right\|^2 \\
&\quad +4\mathbb{E}\left\|\frac{A}{1+A} f(x_0,s)\int_s^t\ du\right\|^2 \\
&\quad +4\mathbb{E}\left\|\frac{A}{1+A} g(x_0,s)\int_s^t\ dW(u)\right\|^2 \\
&\leq 4(t-s)\mathbb{E}\int_s^t \left\|f(X(u),u)-f(x_0,s)\right\|^2\ du \\
&\quad +4\mathbb{E}\int_s^t \left\|g(X(u),u)-g(x_0,s)\right\|^2\ du + 4(t-s)^2\mathbb{E}\left\|f(x_0,s)\right\|^2 \\
&\quad +4\mathbb{E}\left\{\left\|g(x_0,s)\right\|_F^2 \left\|\frac{A}{1+A}\int_s^t\ dW(u)\right\|^2\right\} \\
&\leq 4(t-s)L_1^2\int_s^t \left\{(u-s)+\mathbb{E}\left\|X_{x_0,s}(u)-x_0\right\|^2\right\}\ du \\
&\quad +4L_2^2\int_s^t \left\{(u-s)+\mathbb{E}\left\|X_{x_0,s}(u)-x_0\right\|^2\right\}\ du \\
&\quad +4(t-s)^2L_3^2\Big(1+\|x_0\|^2\Big) \\
&\quad +4L_4^2\Big(2a^2(t-s)^3+6b^2(t-s)^2\Big)\Big(1+\|x_0\|^2\Big) \\
&\leq 4\Big(L_1^2(t-s)+L_2^2\Big)\left[\frac{1}{2}(t-s)^2+\frac{c_3^2}{2}(t-s)^2\Big(1+\|x_0\|^2\Big)\right] \\
&\quad +4\Big[L_3^2(t-s)^2+L_4^2\big(2a^2(t-s)^3+6b^2(t-s)^2\big)\Big]\Big(1+\|x_0\|^2\Big)
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left\|X_{x_0,s}(t)-Y_{x_0,s}(t)\right\|^2 &\le 4\Big(L_1^2(t-s)+L_2^2\Big)\left[\frac{1}{2}(t-s)^2\Big(1+\|x_0\|^2\Big)\right] \\
&\quad +4\Big(L_1^2(t-s)+L_2^2\Big)\left[\frac{c_3^2}{2}(t-s)^2\Big(1+\|x_0\|^2\Big)\right] \\
&\quad +4\Big[L_3^2(t-s)^2+L_4^2\big(2a^2(t-s)^3+6b^2(t-s)^2\big)\Big]\Big(1+\|x_0\|^2\Big) \\
&= 2(1+c_3^2)\big[L_1^2(t-s)+L_2^2\big](t-s)^2\Big(1+\|x_0\|^2\Big) \\
&\quad +4\big[L_3^2(t-s)^2+L_4^2\big(2a^2(t-s)^3+6b^2(t-s)^2\big)\big]\Big(1+\|x_0\|^2\Big) \\
&\le c_5^2(t-s)^2\Big(1+\|x_0\|^2\Big) \qquad (5.24)
\end{aligned}
$$

where $c_5^2 = 2(1+c_3^2)(L_1^2+L_2^2)+4L_3^2+8(a^2+3b^2)L_4^2$. Hence, the numerical method is mean square consistent with rate $r_2 = 1$.

5.2.8 Mean Square Hölder Continuity of Martingale Part of X

$$
\begin{aligned}
&\mathbb{E}\left\|\int_s^t \left[g(X_{x_0,s}(u),u)-g(X_{y_0,s}(u),u)\right]\ dW(u)\right\|^2 \\
&= \int_s^t \mathbb{E}\left\|g(X_{x_0,s}(u),u)-g(X_{y_0,s}(u),u)\right\|_F^2 du \\
&\le L_2^2\int_s^t \mathbb{E}\left\|X_{x_0,s}(u)-X_{y_0,s}(u)\right\|^2 du \\
&\overset{(5.13)}{\le} 3\ L_2^2\ \|x_0-y_0\|^2 \int_s^t e^{3(L_1^2+L_2^2)(u-s)}du \\
&= 3\ L_2^2\ \frac{e^{3(L_1^2+L_2^2)(t-s)}-1}{3(L_1^2+L_2^2)}\ \|x_0-y_0\|^2 \\
&= \frac{L_2^2}{L_1^2+L_2^2}\ \frac{e^{3(L_1^2+L_2^2)(t-s)}-1}{t-s}\ (t-s)\ \|x_0-y_0\|^2.
\end{aligned}
$$

We may choose $r_{SM} = 0.5$ and $K_{SM} \le \dfrac{L_2^2}{L_1^2+L_2^2}\ \max\limits_{|z|\le 1} \dfrac{e^{3(L_1^2+L_2^2)z}-1}{z} < \infty$.

Hence, $\mathbb{E}\left\|\int_s^t \left[g(X_{x_0,s}(u),u)-g(X_{y_0,s}(u),u)\right]\ dW(u)\right\|^2 = K_{SM}\|x_0-y_0\|^2(t-s)$. The martingale part of X is Hölder continuous in mean square sense with rate $r_{SM} = 0.5$.

5.2.9 Local Moment V-Boundedness and Convergence Rate

The local moment V-boundedness is satisfied with $V(x_0) = 1 + \|x_0\|^2$

$$\begin{aligned}\mathbb{E}[V(x_0)] + \mathbb{E}[V(y_0)] &= \mathbb{E}[1 + \|x_0\|^2] + \mathbb{E}[1 + \|y_0\|^2] \\ &= 2 + \mathbb{E}\|x_0\|^2 + \mathbb{E}\|y_0\|^2 < \infty \end{aligned} \tag{5.25}$$

by the view of assumptions on initial condition $X(0) = x_0 \in \mathbb{D}$, and $\mathbb{E}\|x_0\|^2 < \infty$.

We obtained, $r_0 = 1.5$, $r_2 = 1$, and $r_{SM} = 0.5$ and they satisfy the condition $r_0 \geq r_2 + r_{SM} \geq 1$. Hence the convergence rate is 0.5 by $r_g = r_2 + r_{SM} - 1 = 1 + 0.5 - 1 = 0.5$.

Theorem 5.2.2. *Numerical approximation Y_n (5.3) converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ to an analytic solution X with the rate* 0.5 *for the stochastic SIR model (5.4).*

Proof. The problem is well posed, i.e. it satisfied the conditions (A1)-(A8) in the section (1.2.3). Hence, numerical approximation Y_n converge to analytic solution X with the rate $r_g = 0.5$ by the convergence Theorem 1.2.4. □

5.3 NUMERICAL ANALYSIS OF STOCHASTIC SIS MODEL WITH DISEASE DEATHS

Consider the below balanced implicit method

$$Y_{n+1} = f(Y_n, t_n)\Delta_n + \sum_{j=1}^{2} g^j(Y_n, t_n)\Delta W_n^j + c(Y_n, t_n)(Y_n - Y_{n+1}) \tag{5.26}$$

where $c(Y_n, t_n) = A\ I_{2\times 2}$ for the unit matrix $I_{2\times 2}$ and

$$A = (\alpha + \gamma + \mu + \beta I_n)\ \Delta_n + K\ |F_1(Y_n)\ \Delta W_n^1| + \frac{K}{S_n}\ |F_2(Y_n)\ \Delta W_n^2|$$

for a discretization of SIS model with disease deaths

$$dX(t) = f(X(t), t))\ dt + g(X(t), t))\ dW(t) \tag{5.27}$$

where

$$X(t) = \begin{pmatrix} S(t) \\ \\ I(t) \end{pmatrix}, \quad f(X(t),t) = \begin{pmatrix} -\beta S(t)I(t) + \mu(K - S(t)) + \alpha I \\ \\ \beta S(t)I(t) - (\alpha+\gamma+\mu)I(t) \end{pmatrix}$$

$$g(X(t),t) = \begin{pmatrix} -S(t)I(t)F_1(X(t)) & I(t)F_2(X(t)) \\ \\ S(t)I(t)F_1(X(t)) & -I(t)F_2(X(t)) \end{pmatrix}, \text{ and } \quad dW(t) = \begin{pmatrix} dW^1(t) \\ \\ dW^2(t) \end{pmatrix}$$

Recall that, the parameters α, β, γ, μ, and K are positive. F_i's locally Lipschitz-continuous functions defined on $\mathbb{D} = \{(S(t), I(t)) : S(t) > 0,\ I(t) \geq 0,\ S(t) + I(t) \leq K\}$ for all $t \geq 0$ with coefficients $\widetilde{L}_i$ respectively, i=1,2. The W^j's are i.i.d. Wiener processes defined on a complete probability basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and independent of the initial value $X(0) = x_0 \in \mathbb{R}^2$ with $\mathbb{E}\|x_0\|^2 < \infty$. Coefficients f and g are Lipschitz-continuous and satisfy linear growth condition on $\mathbb{D}$. Furthermore, let's assume that f, g are Hölder continuous with order 1/2, 1 in time respectively, and with order 1, 2 in space respectively on $\mathbb{D}$.

$$\begin{aligned}
\|f(X(t),t) - f(Y(t),s)\| &\leq L_1\Big(\|X(t) - Y(t)\| + |t-s|^{1/2}\Big) \\
\|g(X(t),t) - g(Y(t),s)\|_F^2 &\leq L_2^2\Big(\|X(t) - Y(t)\|^2 + |t-s|\Big) \\
\|f(X(t),t)\|^2 &\leq L_3^2\Big(1 + \|X(t)\|^2\Big) \\
\|g(X(t),t)\|_F^2 &\leq L_4^2\Big(1 + \|X(t)\|^2\Big)
\end{aligned}$$

where

$$\begin{aligned}
L_1^2 &= \max\Big\{10\beta^2K^2 + 3\mu^2, 10\beta^2K^2 + 3\alpha^2 + 2(\alpha+\gamma+\mu)^2\Big\} \\
L_2^2 &= \sup_{(S,I)\in\ \mathbb{D}}\Big\{4\widetilde{L}_1K^4 + 8F_1^2(S,I)K^2 + 4\widetilde{L}_2K^2 + 4F_2^2(S,I)\Big\} \\
L_3^2 &= \max\Big\{5\beta^2K^2 + 3\alpha^2 + 2(\alpha+\gamma+\mu)^2, 3\mu^2K^2, 3\mu^2\Big\} \\
L_4^2 &= \max\Big\{\sup_{(S,I)\in\ \mathbb{D}} 2K^2F_1^2(S,I), \sup_{(S,I)\in\ \mathbb{D}} 2F_2^2(S,I)\Big\}. \qquad (5.28)
\end{aligned}$$

5.3.1 Invariance property of the numerical solution Y_n with respect to $\mathbb{D}$

Theorem 5.3.1. *Assume that the initial condition* $Y_0 = (S_0, I_0) \in \mathbb{D} = \{(S_n, I_n) : S_n > 0,\ I_n \geq 0,\ S_n + I_n \leq K\}$ *is independent of* $W(t)$ *for* $t \geq 0$*. The numerical solution* $\{Y_i\}_{i\in\mathbb{N}}$ *governed by* (5.27) *is invariant with respect to* $\mathbb{D}$*.*

Proof. By rewriting the numerical method (5.26) we obtained,

$$\begin{aligned} S_{n+1} &= S_n + \Big[-\beta S_n I_n + \mu(K - S_n) + \alpha I_n\Big]\Delta_n \\ &\quad -S_n I_n F_1(Y_n)\Delta W_n^1 + I_n F_2(Y_n)\Delta W_n^2 + A(S_n - S_{n+1}) \\ I_{n+1} &= I_n + \Big[\beta S_n I_n - (\alpha + \gamma + \mu) I_n\Big]\Delta_n \\ &\quad +S_n I_n F_1(Y_n)\Delta W_n^1 - I_n F_2(Y_n)\Delta W_n^2 + A(I_n - I_{n+1}) \end{aligned} \tag{5.29}$$

$$\begin{aligned} S_{n+1} &= S_n + \frac{1}{1+A}\Big\{[-\beta S_n I_n + \mu(K - S_n) + \alpha I_n]\Delta_n - S_n I_n F_1(Y_n)\Delta W_n^1 + I_n F_2(Y_n)\Delta W_n^2\Big\} \\ I_{n+1} &= I_n + \frac{1}{1+A}\Big\{[\beta S_n I_n - (\alpha + \gamma + \mu) I_n]\Delta_n + S_n I_n F_1(Y_n)\Delta W_n^1 - I_n F_2(Y_n)\Delta W_n^2\Big\}. \end{aligned} \tag{5.30}$$

Let's use an induction on $n \in \mathbb{N}$. Assume that $S_0 + I_0 \leq K$, and $S_n + I_n \leq K$ with $S_n > 0$ and $I_n \geq 0$. Then,

$$\begin{aligned} S_{n+1} + I_{n+1} &= S_n + I_n + \frac{\mu\Delta_n}{1+A}\Big(K - S_n - I_n\Big) - \frac{\gamma I_n \Delta_n}{1+A} \\ &\leq S_n + I_n + \frac{\mu\Delta_n}{1+A}\Big(K - S_n - I_n\Big) \\ &= \left(1 - \frac{\mu\Delta_n}{1+A}\right)(S_n + I_n) + \frac{\mu\Delta_n}{1+A}K \\ &\leq \left(1 - \frac{\mu\Delta_n}{1+A}\right)K + \frac{\mu\Delta_n}{1+A}K \qquad \text{since} \quad \frac{\mu\Delta_n}{1+A} \leq 1 \\ &= K. \end{aligned}$$

Positivity of numerical solution can be proven by substituting $A = (\alpha+\gamma+\mu+\beta I_n)\Delta_n + K\ |F_1(Y_n)\Delta W_n^1| + \frac{K}{S_n}|F_2(Y_n)\Delta W_n^2|$ into (5.30),

$$\begin{aligned}
S_{n+1} &= \frac{1}{1+A}\Big\{S_n + AS_n + [-\beta S_n I_n + \mu(K-S_n) + \alpha I_n]\Delta_n \\
&\qquad - S_n I_n F_1(Y_n)\Delta W_n^1 + I_n F_2(Y_n)\Delta W_n^2\Big\} \\
&= \frac{1}{1+A}\Big\{S_n + (\alpha+\gamma+\mu+\beta I_n)S_n\Delta_n + KS_n|F_1(Y_n)\Delta W_n^1| + \frac{KS_n}{S_n}|F_2(Y_n)\Delta W_n^2| \\
&\qquad + [-\beta S_n I_n + \mu(K-S_n) + \alpha I_n]\Delta_n - S_n I_n F_1(Y_n)\Delta W_n^1 + I_n F_2(Y_n)\Delta W_n^2\Big\} \\
&= \frac{1}{1+A}\Big\{S_n + (\alpha+\gamma)S_n\Delta_n + KS_n|F_1(Y_n)\Delta W_n^1| + K|F_2(Y_n)\Delta W_n^2| \\
&\qquad + \mu KS_n\Delta_n + \alpha I_n\Delta_n - S_n I_n F_1(Y_n)\Delta W_n^1 + I_n F_2(Y_n)\Delta W_n^2\Big\} \\
&> 0 \qquad (5.31)
\end{aligned}$$

$$\begin{aligned}
I_{n+1} &= \frac{I_n}{1+A}\Big\{1 + A + [\beta S_n - (\alpha+\gamma+\mu)]\Delta_n + S_n F_1(Y_n)\Delta W_n^1 - F_2(Y_n)\Delta W_n^2\Big\} \\
&= \frac{I_n}{1+A}\Big\{1 + (\alpha+\gamma+\mu+\beta I_n)\ \Delta_n + K|F_1(Y_n)\ \Delta W_n^1| + \frac{K}{S_n}|F_2(Y_n)\ \Delta W_n^2| \\
&\qquad + [\beta S_n - (\alpha+\gamma+\mu)]\Delta_n + S_n F_1(Y_n)\Delta W_n^1 - F_2(Y_n)\Delta W_n^2\Big\} \\
&= \frac{I_n}{1+A}\Big\{1 + \beta(I_n+S_n)\ \Delta_n + K\ |F_1(Y_n)\Delta W_n^1| + S_n F_1(Y_n)\Delta W_n^1 \\
&\qquad - F_2(Y_n)\Delta W_n^2 + \frac{K}{S_n}|F_2(Y_n)\Delta W_n^2|\Big\} \\
&\geq 0 \qquad \text{since} \quad \frac{K}{S_n} \geq 1. \qquad (5.32)
\end{aligned}$$

Therefore, $\mathbb{P}(Y_n = (S_n, I_n) \in \mathbb{D}) = 1$. □

5.3.2 Convergence of the numerical solution Y_n

Theorem 5.3.2. *Numerical approximation Y_n (5.26) converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ to analytic solution X with the rate 0.5 for the stochastic SIS model (5.27).*

Proof. The numerical solution Y_n is invariant with respect to $\mathbb{D}$, and the coefficients f and g are uniformly Lipschitz continuous and satisfy linear growth condition on the domain $\mathbb{D}$. Furthermore, the problem is well-posed. Hence the numerical approximation Y_n converges to an analytic solution X with the rate 0.5 by the convergence Theorem 1.2.4. We omit the details due to similar calculations as in the proof of convergence of the numerical solution for the stochastic SIR model with disease deaths. The only differences appear in the coefficients. □

CHAPTER 6

SIMULATIONS WITH USING BALANCED IMPLICIT METHOD

We simulate two stochastic models mentioned and discretized on the previous chapter (SIR and SIS model with disease deaths). From a mathematical point of view we are only interested in models with non-negative parameters α, β, γ, and μ. But all these parameters have meaning in epidemiology.

(i) $K > 0$ is a maximum population size, i.e. carrying capacity.

(ii) μ is a per capita death rate (birth rate) per unit time. Hence μK is a number natural births and μS a the number of natural deaths in the susceptible population. In our simulations, we used $\mu = 1/75$ corresponding to a human life expectancy of 75 years.

(iii) $1/\alpha$ is a mean of the infective period. We used $\alpha = 73, 52, 26$, and 13 corresponding to infectives recover after a mean infective period of $1/73, 1/52, 1/26$, and $1/13$ year (5 days, 1 week, 2, and 4 weeks) respectively.

(iv) $1/\gamma$ is a mean of the disease related death period. We used $\gamma = 73, 52, 26$, and 13 describing a disease from which infectives die because of the disease after a mean period of $1/73, 1/52, 1/26$, and $1/13$ year respectively.

(v) β is an infection rate (contact rate), βSI is the number of new infectives in unit time. We used $\beta = 0.1$ and 0.05 explaining that average infectives makes contact sufficiently to transmit infection with $0.1K$ and $0.05K$ others per year.

6.1 STOCHASTIC SIR MODEL WITH DISEASE DEATHS

Consider the model

$$\begin{aligned} dS &= \Big(-\beta SI + \mu(K-S) \Big)\, dt - \frac{1}{K^3} SI(S-S_2)\, dW_1 \\ dI &= \Big(\beta SI - \big(\alpha+\gamma+\mu\big) I \Big)\, dt + \frac{1}{K^3} SI(S-S_2)\, dW_1 - \frac{1}{K^2} I(I-I_2)\, dW_2 \\ dR &= \Big(\alpha I - \mu R \Big)\, dt + \frac{1}{K^2} I(I-I_2)\, dW_2 \end{aligned} \tag{6.1}$$

where $\alpha, \beta, \gamma, \mu$ and K are positive constants and $(S_2, I_2, R_2) = \left(\frac{K}{\mathcal{R}_0},\ \frac{\mu}{\beta}(\mathcal{R}_0 - 1),\ \frac{\alpha}{\beta}(\mathcal{R}_0 - 1) \right)$ for $\mathcal{R}_0 = \frac{\beta K}{\alpha+\gamma+\mu}$.

Theorem 4.2.5 proved the global existence of a unique solution of the system (6.1) in $\mathbb{D} = \big\{ (S,I,R) \in \mathbb{R}^3 : S \geq 0, I \geq 0, R \geq 0, S+I+R \leq K \big\}$ for all $t \geq t_0$ if the initial value $(S_0, I_0, R_0) \in \mathbb{D}$.

Theorem 4.2.6 proved stochastic asymptotic stability of disease free equilibrium solution $(S_1, I_1, R_1) = (K, 0, 0)$ to the SDE (6.1) by the help of the Lyapunov function $V(S,I,R) = \frac{1}{2}(S-K+I+R)^2 + KI + KR$. The following simulations verify the theorem.

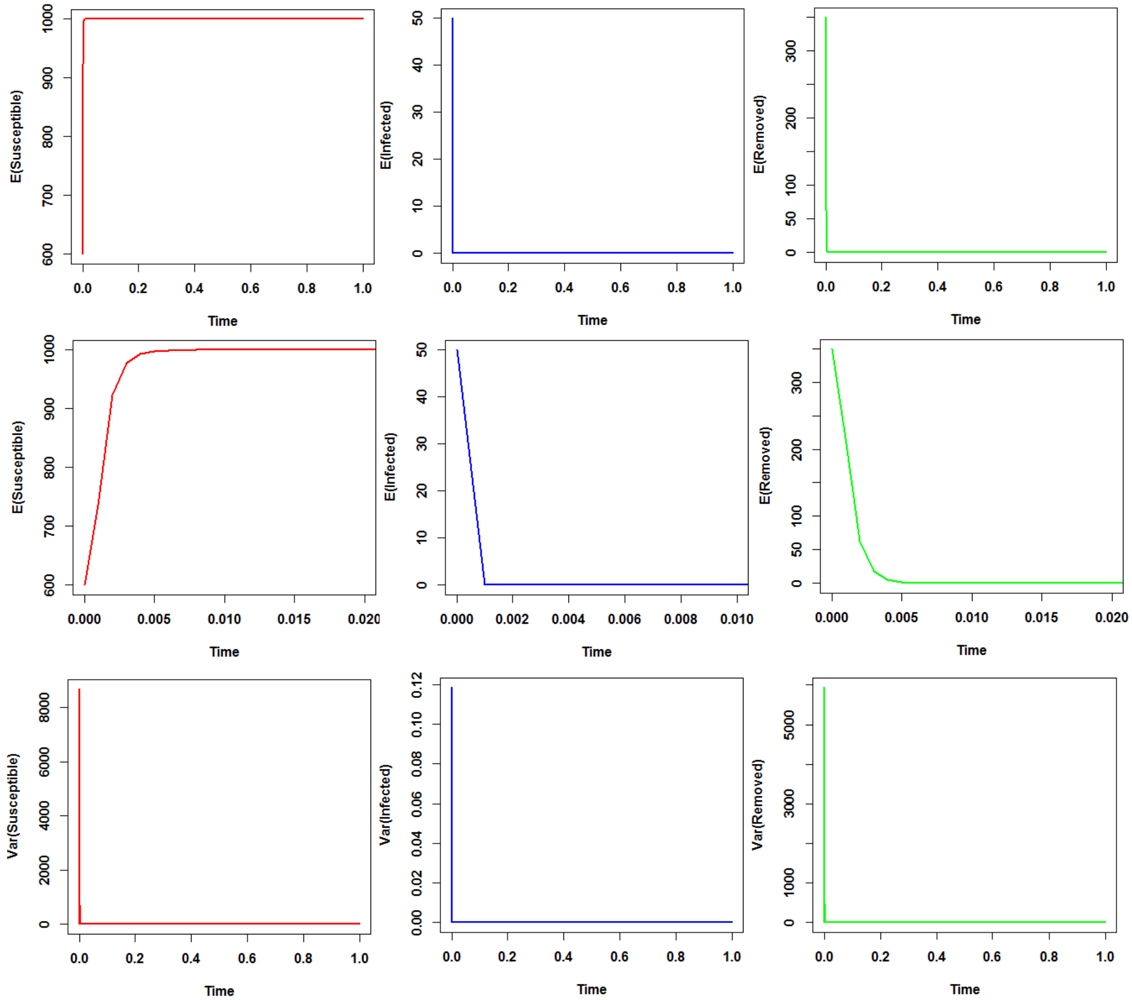


Figure 6.1: The disease free equilibrium $(S, I, R) = (1000, 0, 0)$ is stochastically asymptotically stable since $\mathcal{R}_0 = 0.96 < 1$ for $\alpha = 52$, $\beta = 0.1$, $\gamma = 52$, $\mu = 0.013$, $K = 1000$. Here we use initial value $(S_0, I_0, R_0) = (600, 50, 350)$ and step size $\Delta = 10^{-3}$. Expectations and variances are taken for 10000 trajectories.

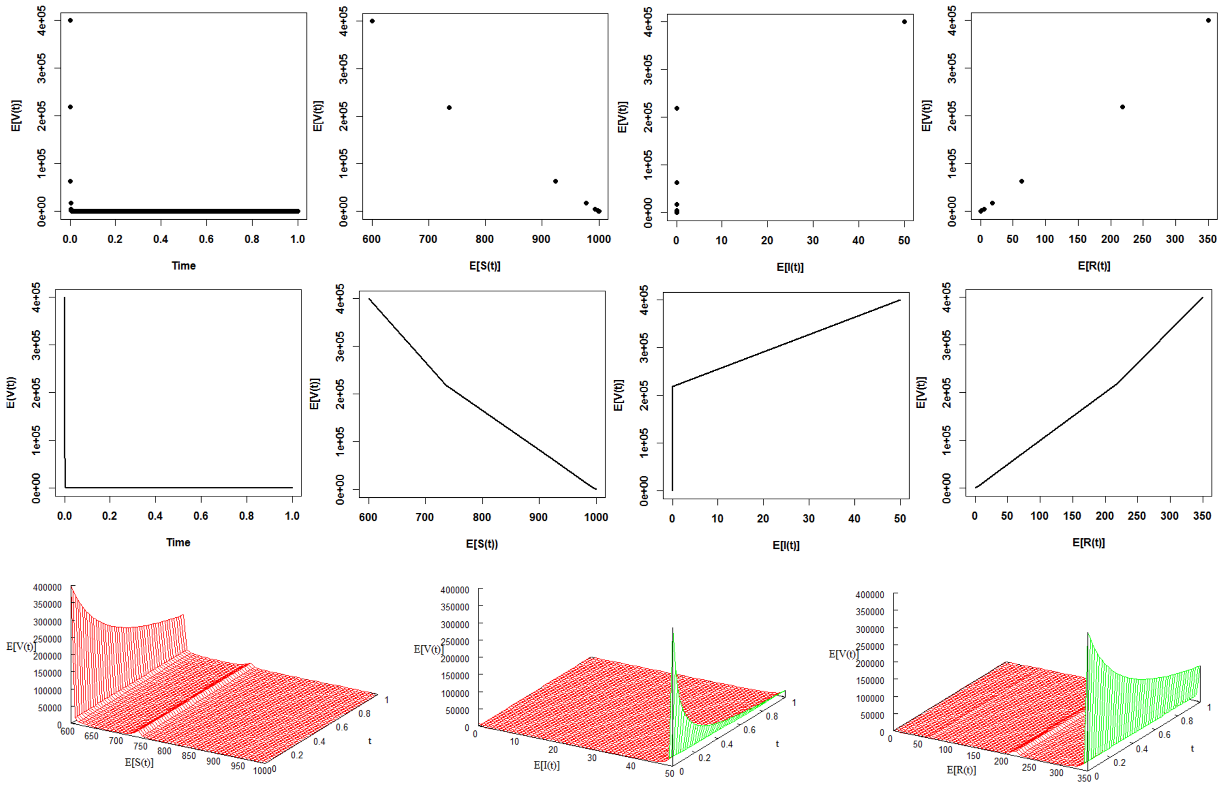


Figure 6.2: Expected value of the Lyapunov function $V(S,I,R) = \frac{1}{2}(S-K+I+R)^2 + KI + KR$, which is used in the proof of stochastic asymptotic stability of the disease free equilibrium solution $(S,I,R) = (1000,0,0)$ to the system (6.1). Here we use the same parameters $\alpha = 52$, $\beta = 0.1$, $\gamma = 52$, $\mu = 0.013$, $K = 1000$ i.e. $\mathcal{R}_0 = 0.96 < 1$ and the initial value $(S_0, I_0, R_0) = (600, 50, 350)$ with step size $\Delta = 10^{-3}$. Expectations are taken for 10000 trajectories.

Theorem 4.2.7 proved stochastic asymptotic stability of the endemic equilibrium solution $(S_2, I_2, R_2) = \left(\frac{K}{\mathcal{R}_0},\ \frac{\mu}{\beta}(\mathcal{R}_0 - 1),\ \frac{\alpha}{\beta}(\mathcal{R}_0 - 1)\right)$ to the system (6.1) on $\{(S, I, R) : S > 0, I > 0, R > 0, S + I + R \le K\}$ under the assumptions $\mathcal{R}_0 = \frac{\beta K}{\alpha+\gamma+\mu} > 1$ and $\mathcal{L}V$ is negative definite, which requires nonnegativity of the constants $\phi := b\ \mu - \frac{1}{2K^4}(a\ I_2 + b\ K^2)$ and $\psi : \frac{\gamma}{K} - \frac{1}{2K^4}(c\ I_2 + b\ K^2)$, where

$$\mathcal{L}V = -\frac{\mu}{K}(S - S_2 + I - I_2 + R - R_2)^2 - \psi\ (I - I_2)^2 - \phi\ (S - S_2)^2 - c\ \mu(R - R_2)^2$$

and $a = \frac{\gamma}{\beta K} + bS_2$, $b > 0$ and $c = \frac{\gamma}{\alpha K}$.

Recall that we used

$$\begin{aligned} V(S, I, R) &= S - S_2 + I - I_2 + R - R_2 - (S_2 + I_2 + R_2)\ln\frac{S + I + R}{S_2 + I_2 + R_2} \\ &\quad + a\left(I - I_2 - I_2 \ln\frac{I}{I_2}\right) + \frac{b}{2}(S - S_2)^2 + \frac{c}{2}(R - R_2)^2 \end{aligned}$$

as a Lyapunov function in the proof of the Theorem 4.2.7.

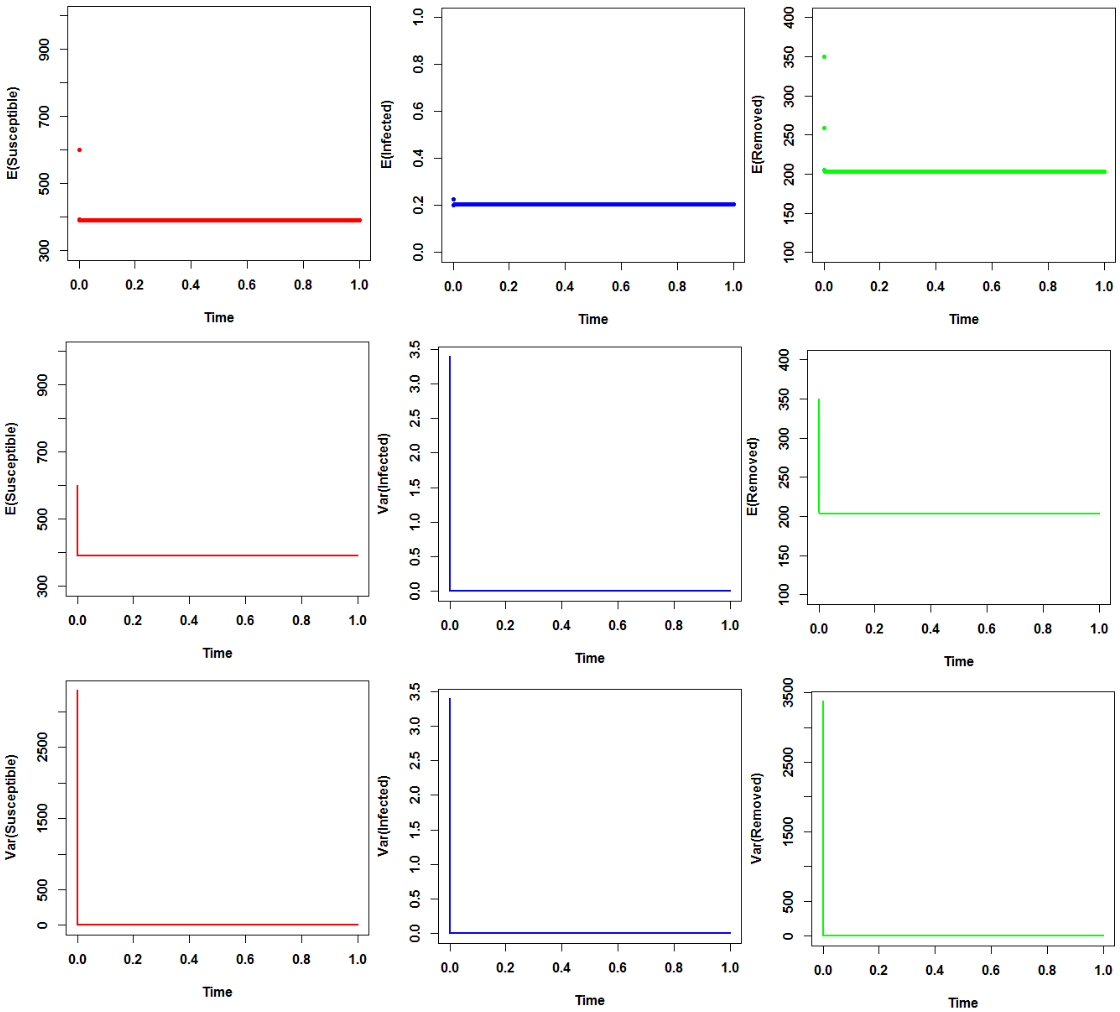


Figure 6.3: The endemic equilibrium $(S_2, I_2, R_2) = (390.13, 0.203, 203.222)$ is stochastically asymptotically stable since $\mathcal{R}_0 = 2.56 > 1$ and $\mathcal{L}V \leq 0$ $(\phi = 0.013, \psi = 0.026)$ for $\alpha = 13$, $\beta = 0.1$, $\gamma = 26$, $\mu = 0.013$, $K = 1000$, $b = 1$. We use initial value $(S_0, I_0, R_0) = (600, 50, 350)$ and step size $\Delta = 10^{-3}$. Expectations and variances are taken for 10000 trajectories.

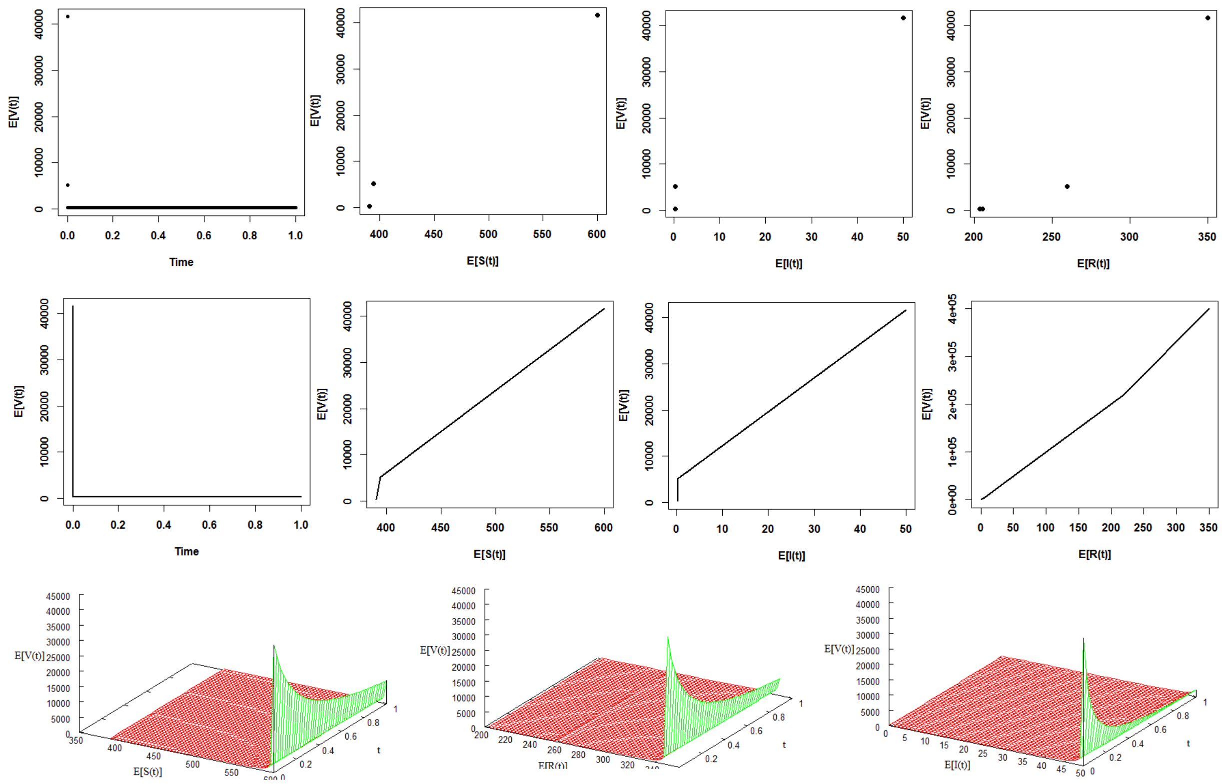


Figure 6.4: Expected value of the Lyapunov function $V(S,I,R) = S - S_2 + I - I_2 + R - R_2 - (S_2 + I_2 + R_2)\ln\frac{S+I+R}{S_2+I_2+R_2} + a\left(I - I_2 - I_2 \ln\frac{I}{I_2}\right) + \frac{b}{2}(S-S_2)^2 + \frac{c}{2}(R-R_2)^2$ where $a = \frac{\gamma}{\beta K} + bS_2$, $b > 0$, and $c = \frac{\gamma}{\alpha K}$, which is used in the proof of stochastic asymptotic stability of the endemic equilibrium solution $(S_2, I_2, R_2) = (390.13, 0.203, 203.222)$ to the SDE (6.1). Here we use the same parameters $\alpha = 13$, $\beta = 0.1$, $\gamma = 26$, $\mu = 0.013$, $K = 1000$, $b = 1$ i.e. $\mathcal{R}_0 = 2.56 > 1$, $\mathcal{L}V \le 0$ $(\phi = 0.013, \psi = 0.026)$ and the initial value $(S_0, I_0, R_0) = (600, 50, 350)$ with step size $\Delta = 10^{-3}$. Expectations are taken for 10000 trajectories.

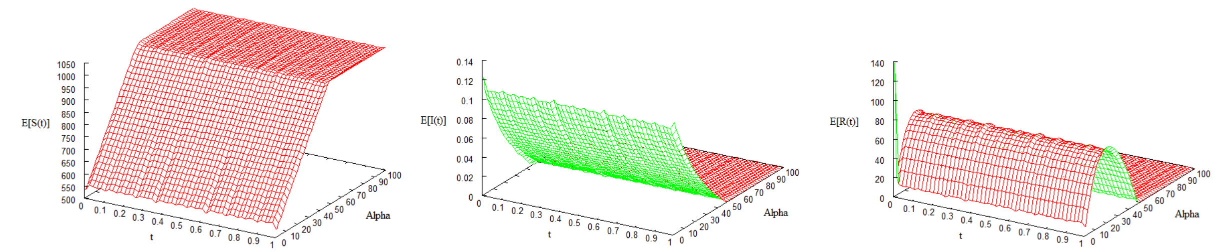


Figure 6.5: Expected values of Susceptible, Infective, and Removed for variable α and fixed $\beta = 0.1$, $\gamma = 52$, $\mu = 0.013$, $K = 1000$. If $\alpha \geq 48$ then $\mathcal{R}_0 = \frac{100}{52.013+\alpha} < 1$, and there exists only one equilibrium $(S, I, R) = (1000, 0, 0)$, which is stochastically asymptotically stable. If $\alpha \leq 47$ then $\mathcal{R}_0 > 1$ and an endemic equilibrium $(S, I, R) = \left(520.13 + 10\alpha, \dfrac{13}{52.013+\alpha} - 0.13, \dfrac{1000\alpha}{52.013+\alpha} - 10\alpha\right)$ is stochastically asymptotically stable. We use initial value $(S_0, I_0, R_0) = (650, 50, 350)$ and step size $\Delta = 10^{-3}$. Expectations are taken for 10000 trajectories.

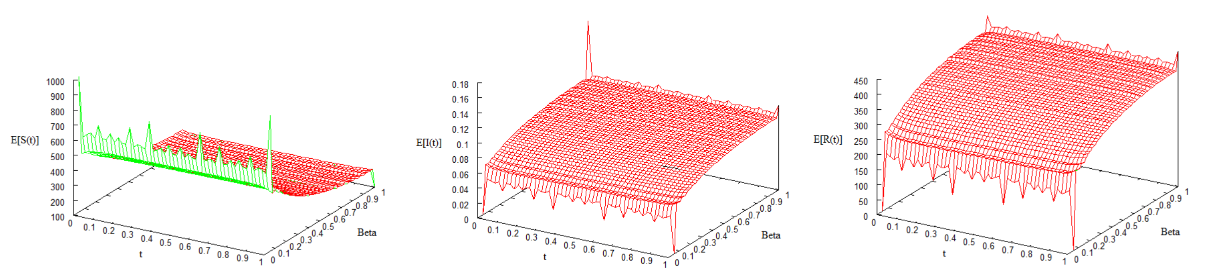


Figure 6.6: Expected values of Susceptible, Infective, and Removed for variable β and fixed $\alpha = 52$, $\gamma = 52$, $\mu = 0.013$, $K = 1000$. If $\beta \leq 0.1$ then $\mathcal{R}_0 = 9.61\beta < 1$, and there exists only one equilibrium $(S, I, R) = (1000, 0, 0)$, which is stochastically asymptotically stable. If $\beta > 0.1$ then $\mathcal{R}_0 > 1$ and an endemic equilibrium $(S, I, R) = \left(\dfrac{104.013}{\beta}, 0.125 - \dfrac{0.013}{\beta}, 499.938 - \dfrac{52}{\beta}\right)$ is stochastically asymptotically stable. We use initial value $(S_0, I_0, R_0) = (650, 50, 350)$ and step size $\Delta = 10^{-3}$. Expectations are taken for 10000 trajectories.

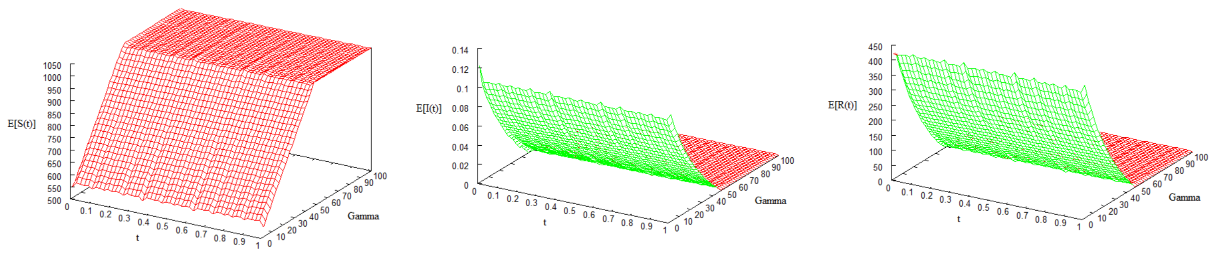


Figure 6.7: Expected values of Susceptible, Infective, and Removed for variable γ and fixed $\beta = 0.1$, $\alpha = 52$, $\mu = 0.013$, $K = 1000$. If $\gamma \geq 48$ then $\mathcal{R}_0 = \frac{100}{52.013+\gamma} < 1$, and there exists only one equilibrium $(S, I, R) = (1000, 0, 0)$, which is stochastically asymptotically stable. If $\gamma \leq 47$ then $\mathcal{R}_0 > 1$ and an endemic equilibrium $(S, I, R) = \left(520.13 + 10\gamma, \dfrac{13}{52.013+\gamma} - 0.13, \dfrac{52000}{52.013+\gamma} - 520\right)$ is stochastically asymptotically stable. We use initial value $(S_0, I_0, R_0) = (650, 50, 350)$ and step size $\Delta = 10^{-3}$. Expectations are taken for 10000 trajectories.

6.2 STOCHASTIC SIS MODEL WITH DISEASE DEATHS

Consider the model

$$\begin{aligned} dS &= \Big(-\beta SI + \mu(K - S) + \alpha I \Big)\, dt - \frac{1}{K^2} SI(S - S_2)\, dW_1 + \frac{1}{K^2} I(I - I_2)\, dW_2 \\ dI &= \Big(\beta SI - (\alpha + \gamma + \mu) I \Big)\, dt + \frac{1}{K^2} SI(S - S_2)\, dW_1 - \frac{1}{K^2} I(I - I_2)\, dW_2 \end{aligned} \tag{6.2}$$

where $\alpha, \beta, \gamma, \mu$ and K are positive constants and $(S_2, I_2) = \left(\frac{K}{\mathcal{R}_0}, \frac{\mu K}{\gamma+\mu} \left(1 - \frac{1}{\mathcal{R}_0}\right)\right)$ for $\mathcal{R}_0 = \frac{\beta K}{\alpha+\gamma+\mu}$. Theorem 3.2.5 proved the existence of a unique global solution of the system (6.2) in $\mathbb{D} = \{(S, I) \in \mathbb{R}^2 : S \geq 0, I \geq 0, S + I \leq K\}$ for all $t \geq t_0$ if the initial value $(S_0, I_0) \in \mathbb{D}$.

If the basic reproduction number $\mathcal{R}_0 < 1$ then the disease free equilibrium solution $(S, I) = (K, 0)$ to the system (6.2) is stochastically asymptotically stable on $\mathbb{D}$ by Theorem 3.2.6 with the help of the Lyapunov function $V(S, I) = \dfrac{1}{2}(S - K + I)^2 + \dfrac{\gamma}{\beta}(K - S)$.

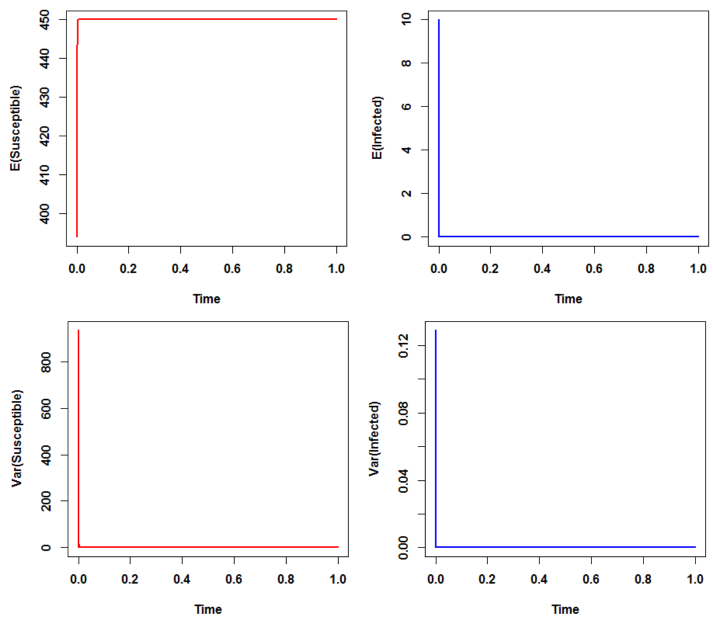


Figure 6.8: The disease free equilibrium $(S, I) = (450, 0)$ is stochastically asymptotically stable since $\mathcal{R}_0 = 0.87 < 1$ for $\alpha = 26$, $\beta = 0.1$, $\gamma = 26$, $\mu = 0.013$, $K = 450$. Here we use initial value $(S_0, I_0) = (400, 10)$ and step size $\Delta = 10^{-3}$. Expectations and variances are taken for 10000 trajectories.

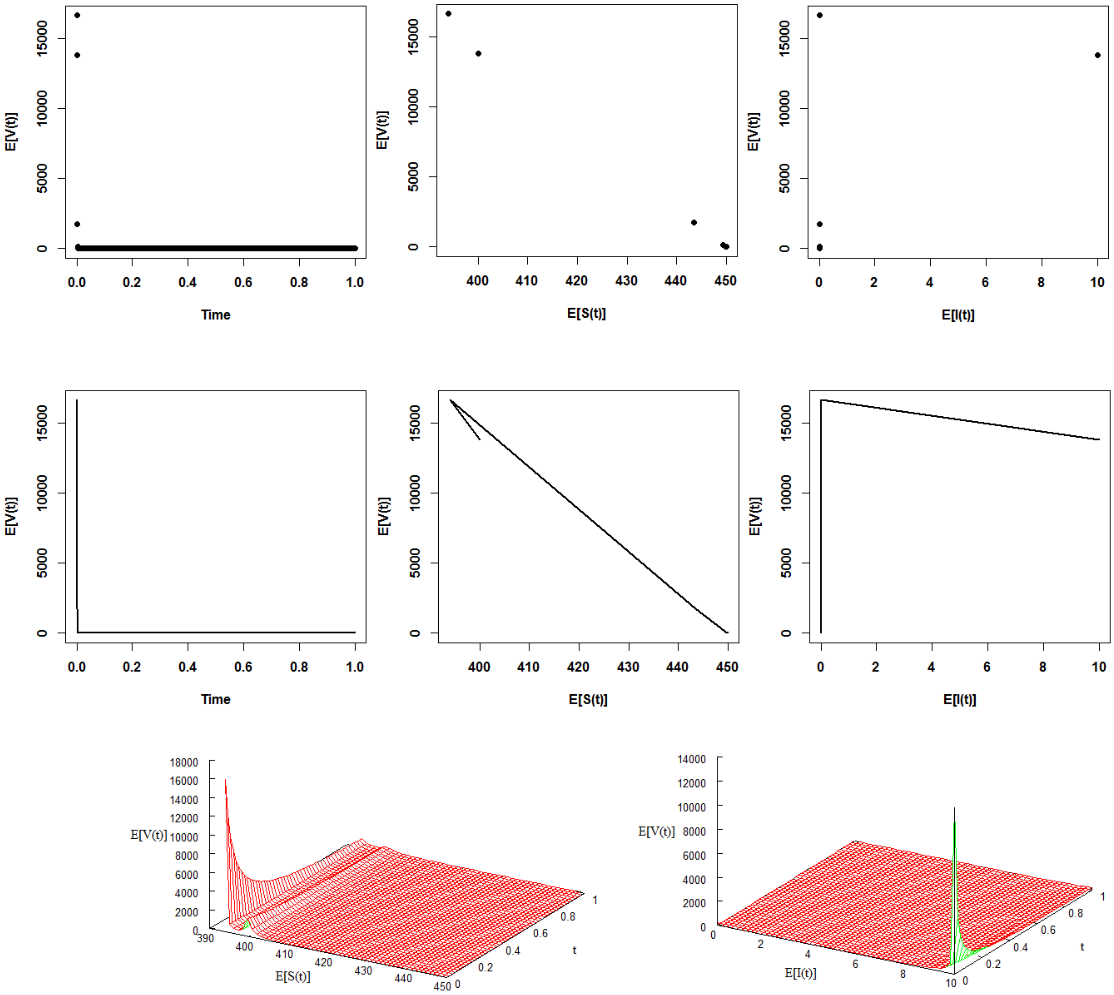


Figure 6.9: Expected value of the Lyapunov function $V(S,I) = \dfrac{1}{2}(S-K+I)^2 + \dfrac{\gamma}{\beta}(K-S)$, which is used in the proof of stochastic asymptotic stability of the disease free equilibrium solution $(S,I) = (450,0)$ to the system (6.2). Here we use the same parameters $\alpha = 26,\ \beta = 0.1,\ \gamma = 26,\ \mu = 0.013,\ K = 450$ i.e. $\mathcal{R}_0 = 0.87 < 1$ and the initial value $(S_0, I_0) = (400, 10)$ with step size $\Delta = 10^{-3}$. Expectations are taken for 10000 trajectories.

Theorem 3.2.7 proved stochastic asymptotic stability of the endemic equilibrium solution $(S_2, I_2) = \left(\frac{K}{\mathcal{R}_0}, \frac{\mu K}{\gamma+\mu}\left(1 - \frac{1}{\mathcal{R}_0}\right)\right)$ to the system (6.2) on $\{(S, I) : S > 0, I > 0, S + I \leq K\}$ under the assumptions $\mathcal{R}_0 = \frac{\beta K}{\alpha+\gamma+\mu} > 1$ and $\mathcal{L}V$ is negative definite, which requires nonnegativity of the constants $\phi := \mu - \frac{2\mu+\gamma}{2\beta K^2}\, I_2$ and $\psi : \mu + \gamma - \frac{2\mu+\gamma}{2\beta K^4}\, I_2$, where $\mathcal{L}V = -\phi\,(S - S_2)^2 - \psi\,(I - I_2)^2$.

Recall that we used $V(S, I) = S - S_2 + I - I_2 - (S_2 + I_2)\ln\left(\frac{S+I}{S_2+I_2}\right) + \frac{2\mu+\gamma}{\beta K}\left(I - I_2 - I_2 \ln\frac{I}{I_2}\right)$ as a Lyapunov function in the proof of the Theorem 3.2.7.

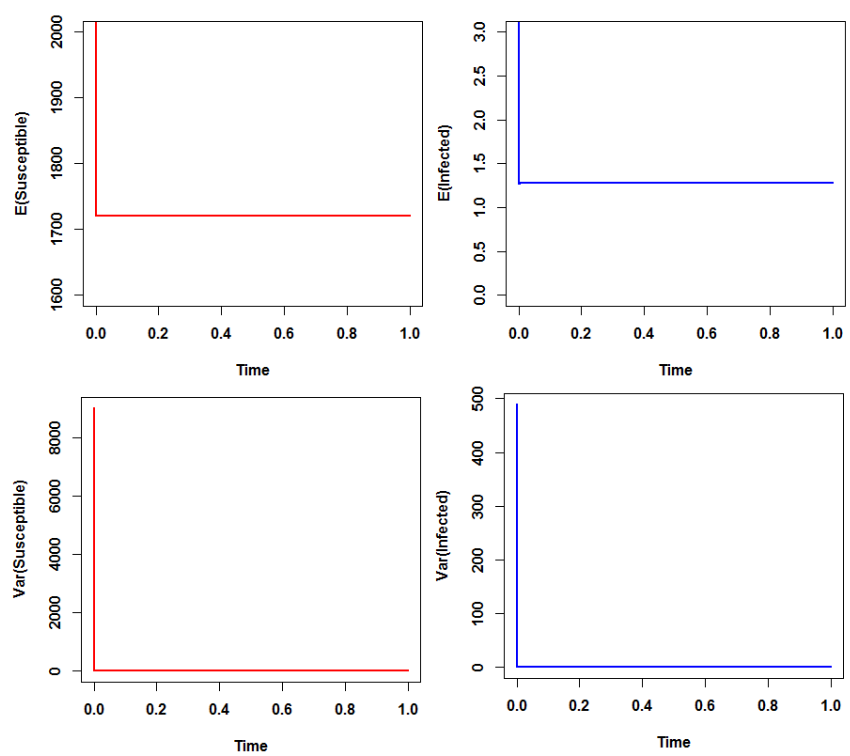


Figure 6.10: The endemic equilibrium $(S_2, I_2) = (1720.26, 1.278)$ is stochastically asymptotically stable since $\mathcal{R}_0 = 1.74 > 1$ and $\mathcal{L}V \leq 0$ ($\phi = 0.013, \psi = 13.013$) for $\alpha = 73$, $\beta = 0.05$, $\gamma = 13$, $\mu = 0.013$, $K = 3000$. We use initial value $(S_0, I_0) = (2990, 10)$ and step size $\Delta = 10^{-3}$. Expectations and variances are taken for 10000 trajectories.

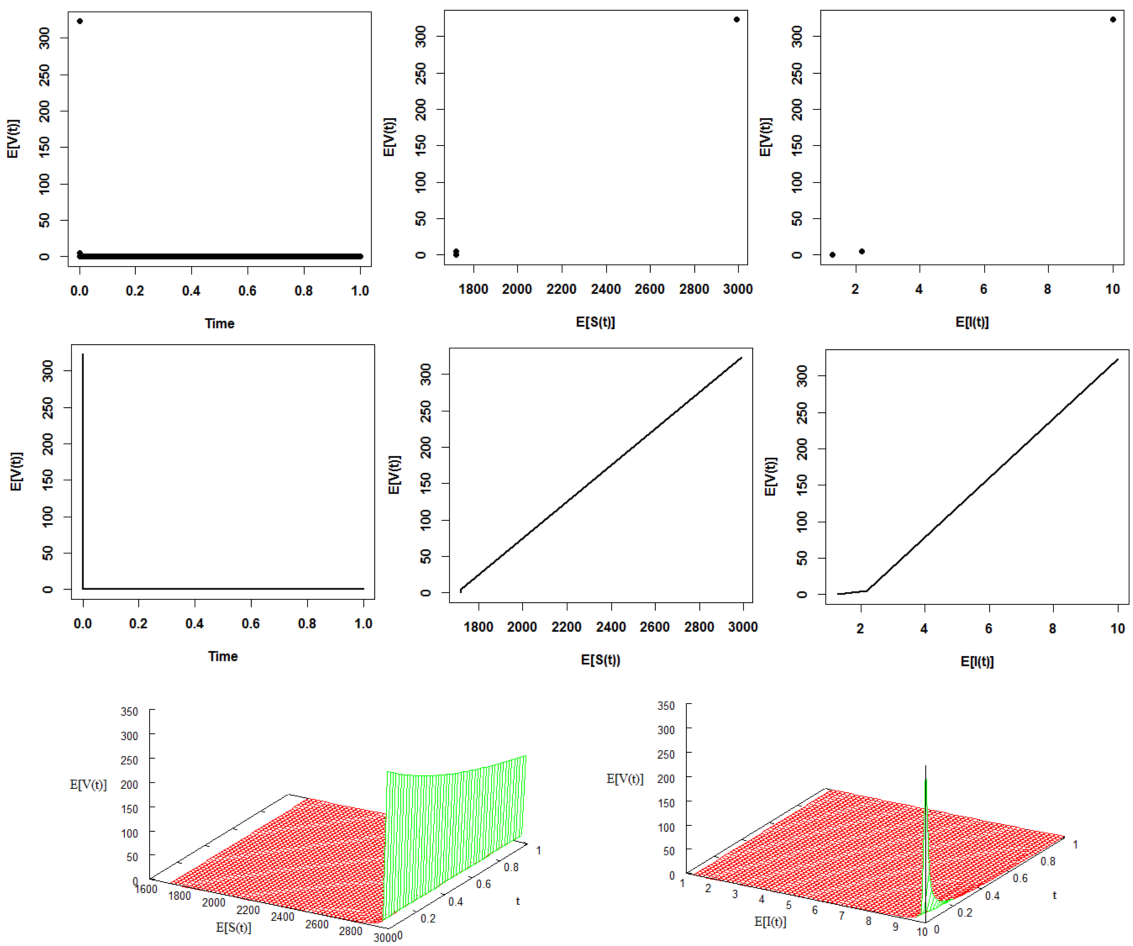


Figure 6.11: Expected value of the Lyapunov function $V(S,I) = S - S_2 + I - I_2 - (S_2 + I_2)\ln\left(\frac{S+I}{S_2+I_2}\right) + \frac{2\mu+\gamma}{\beta K}\left(I - I_2 - I_2 \ln \frac{I}{I_2}\right)$, which is used in the proof of stochastic asymptotic stability of the endemic equilibrium solution $(S_2, I_2) = (1720.26, 1.278)$ to the system (6.2). Here we use the same parameters $\alpha = 73$, $\beta = 0.05$, $\gamma = 13$, $\mu = 0.013$, $K = 3000$ i.e. $\mathcal{R}_0 = 1.74 > 1$, $\mathcal{L}V \leq 0$ $(\phi = 0.013, \psi = 13.013)$ and the initial value $(S_0, I_0) = (2990, 10)$ with step size $\Delta = 10^{-3}$. Expectations are taken for 10000 trajectories.

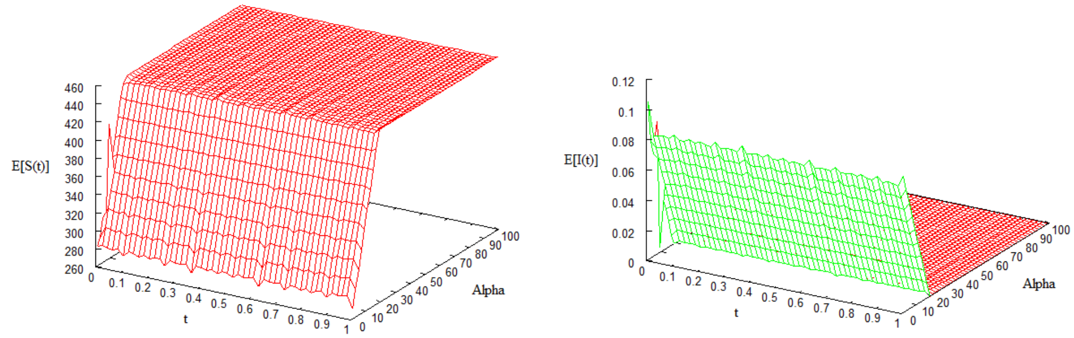


Figure 6.12: Expected values of Susceptible and Infective for variable α and fixed $\beta = 0.1$, $\gamma = 26$, $\mu = 0.013$, $K = 450$. If $\alpha \geq 19$ then $\mathcal{R}_0 = \frac{45}{26.013+\alpha} < 1$, and there exists only one equilibrium $(S, I) = (450, 0)$, which is stochastically asymptotically stable. If $\alpha \leq 18$ then $\mathcal{R}_0 > 1$ and an endemic equilibrium $(S, I) = (260.13 + 10\alpha, 0.095 - 0.005\alpha)$ is stochastically asymptotically stable. We use initial value $(S_0, I_0) = (400, 10)$ and step size $\Delta = 10^{-3}$. Expectations are taken for 10000 trajectories.

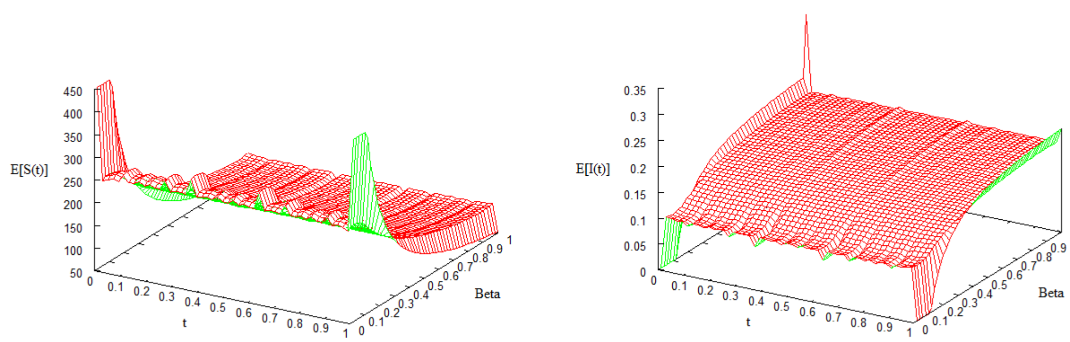


Figure 6.13: Expected values of Susceptible and Infective for variable β and fixed $\alpha = 26$, $\gamma = 26$, $\mu = 0.013$, $K = 450$. If $\beta \leq 0.115$ then $\mathcal{R}_0 = 8.652\beta < 1$, and there exists only one equilibrium $(S, I) = (450, 0)$, which is stochastically asymptotically stable. If $\beta \geq 0.116$ then $\mathcal{R}_0 > 1$ and an endemic equilibrium $(S, I) = \left(\dfrac{52.013}{\beta}, 0.225 - \dfrac{0.0260}{\beta}\right)$ is stochastically asymptotically stable. We use initial value $(S_0, I_0) = (400, 10)$ and step size $\Delta = 10^{-3}$. Expectations are taken for 10000 trajectories.

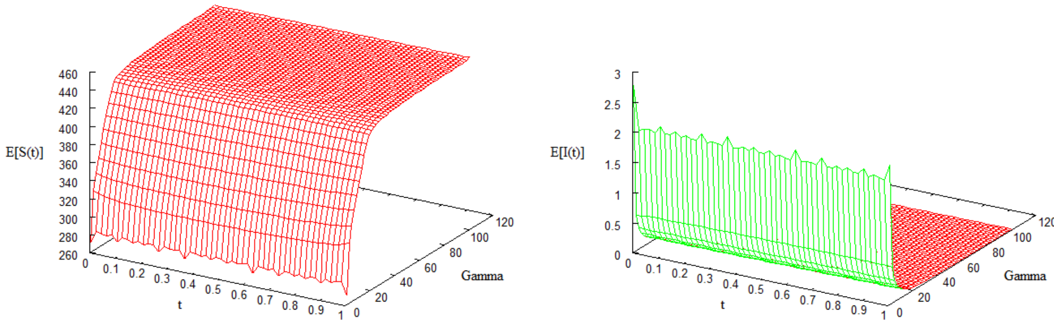


Figure 6.14: Expected values of Susceptible and Infective for variable γ and fixed $\beta = 0.1$, $\alpha = 26$, $\mu = 0.013$, $K = 450$. If $\gamma \geq 19$ then $\mathcal{R}_0 = \frac{45}{26.013+\gamma} < 1$, and there exists only one equilibrium $(S, I) = (450, 0)$, which is stochastically asymptotically stable. If $\gamma \leq 18$ then $\mathcal{R}_0 > 1$ and an endemic equilibrium $(S, I) = \left(260.13 + 10\gamma, \dfrac{2.468 - 0.13\gamma}{0.013 + \gamma}\right)$ is stochastically asymptotically stable. We use initial value $(S_0, I_0) = (400, 10)$ and step size $\Delta = 10^{-3}$. Expectations are taken for 10000 trajectories.

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APPENDIX

Below is a C++ code that we used to simulate the stochastic SIR model with disease deaths (6.1). The random number generating routine is provided by Dr. Schurz.

```
/* Plotting Balanced Implicit Method approximations for Ito SDE*/
/* dS(t)=[-BETA*S*I+MU*(K-S)+ALPHA*I]* dt -F_1*S*I* dW_1(t), S(0)=S0 */
/* dI(t)=[BETA*S*I-(ALPHA+GAMMA+MU)*I]* dt +F1*S*I* dW_1(t)-F_2*I*dW_2(t), I(0)=I0*/
/* dR(t)=[ALPHA*I-\MU*R]* dt +F_2*I*dW_2(t), R(0)=R0*/
/*  THE BASIC REPRODUCTIVE NUMBER R_0=BETA*K/(ALPHA+GAMMA+MU)  */
/*  F_1=(S-S_2)/K^3,     F_2=(I-I_2)/K^2 */

#include <math.h>
#include <stdlib.h>
#include <stdio.h>
#include <time.h>

#define pi 3.141592653
#define T0 0.0  /* Time Interval lower bound */
#define T1 1.0 /* Time Interval upper bound */
#define DELTA 0.001 /* Step size for BIM approximation */
#define BETA 0.1 /* CONTACT RATE average infective makes 0.1*1000=100 contacts per year */
#define ALPHA 13 /* REMOVED RATE infectives recover after a mean infective period of 1/13
    year, i.e. four weeks*/
#define MU 0.013 /* DEATH RATE 1/75=0.013 corresponding to a human life expectancy of 75
    years*/
#define GAMMA 26 /* DISEASE related death RATE infectives dead after a mean death period
    of 1/26 year, i.e. two weeks*/
#define K 1000.0 /* MAXIMUM POPULATION SIZE*/
#define S_2 (ALPHA+GAMMA+MU)/BETA
#define I_2 ((MU*K)/(ALPHA+GAMMA+MU))-(MU/BETA)
#define R_2 ((MU*K)/(ALPHA+GAMMA+MU))-(ALPHA/BETA)
#define S0 600.0 /* Initial value S0 of S at T0 */
#define I0 50.0
#define R0 350.0
#define V0 (S0)-(S_2)+(I0)-(I_2)+(R0)-(R_2)-((S_2)+(I_2)+(R_2))*log(((S0)+(I0)+(R0))/((S_2
```

```
    )+(I_2)+(R_2)))+ a*((I0)-(I_2)-(I_2)*log((I0)/(I_2)))+(0.5)*b*((S0)-(S_2))*((S0)-(S_2)
    )+(0.5)*c*((R0)-(R_2))*((R0)-(R_2));
#define j 10000  /*number of trajectories*/
#define a (GAMMA/(BETA*K))+b*(S_2) /*constant in the Lyapunov function*/
#define b 1  /*constant in the Lyapunov function*/
#define c (GAMMA/(ALPHA*K))  /*constant in the Lyapunov function*/

double dwt1,dwt2;
const double scale_factor=RAND_MAX+1.0;
double uniform();

main()
{
void pol_mas(double t);
void variance( double *arr, int no, double *var);
double tn=0.0,Sn,In,Rn,Vn;
int n=0;
int counter;
double arrSn[j], arrIn[j], arrRn[j];
double varSn, varIn, varRn;
double Sn_tot,In_tot,Rn_tot,Vn_tot;
FILE *pFile=NULL;

srand((unsigned int)time(NULL));
pFile=fopen("SIRdd-mean-variance2.dat","w+");
fprintf(pFile,"tn\tESn\tEIn\tERn\tEVn\tVSn\tVIn\tVRn\n");

tn=0.0;
Sn=S0;
In=I0;
Rn=R0;
Vn=V0;

fprintf(pFile,"%f\t%f\t%f\t%f\t%f\t%f\t%f\t%f\n",tn,Sn,In,Rn,Vn,varSn,varIn,varRn);
```

```
n=1;
do
{
    pol_mas(DELTA);
    tn=T0+n*DELTA;
    Sn_tot=0;
    In_tot=0;
    Rn_tot=0;
    Vn_tot=0;

    for(counter=0; counter<j; counter++)
   {
     Sn=Sn+((-BETA*(Sn)*(In)+MU*(K-(Sn)))*DELTA-((Sn)*(In)*((Sn)-(S_2))/(K*K*K))*dwt1)
         /((1+ALPHA+GAMMA+MU+BETA*(In))*DELTA+K*fabs((((Sn)-(S_2))/(K*K*K))*dwt1)+(K/(Rn))
         *fabs((((In)-(I_2))/(K*K))*dwt2));
     In=In+((BETA*(Sn)*(In)-(ALPHA+GAMMA+MU)*(In))*DELTA+((Sn)*(In)*((Sn)-(S_2))/(K*K*K))*
         dwt1-((In)/(K*K))*((In)-(I_2))*dwt2)/((1+ALPHA+GAMMA+MU+BETA*(In))*DELTA+K*fabs
         ((((Sn)-(S_2))/(K*K*K))*dwt1)+(K/(Rn))*fabs((((In)-(I_2))/(K*K))*dwt2));
     Rn=Rn+((ALPHA*(In)-MU*(Rn))*DELTA+((In)/(K*K))*((In)-(I_2))*dwt2)/((1+ALPHA+GAMMA+MU+
         BETA*(In))*DELTA+K*fabs((((Sn)-(S_2))/(K*K*K))*dwt1)+(K/(Rn))*fabs((((In)-(I_2))
         /(K*K))*dwt2));
     Vn=(Sn)-(S_2)+(In)-(I_2)+(Rn)-(R_2)-((S_2)+(I_2)+(R_2))*log(((Sn)+(In)+(Rn))/((S_2)+(
         I_2)+(R_2)))+ a*((In)-(I_2)-(I_2)*log((In)/(I_2)))+(0.5)*b*((Sn)-(S_2))*((Sn)-(
         S_2))+(0.5)*c*((Rn)-(R_2))*((Rn)-(R_2));

      Sn_tot=(Sn_tot)+(Sn);
      In_tot=(In_tot)+(In);
      Rn_tot=(Rn_tot)+(Rn);
      Vn_tot=Vn_tot+Vn;

      arrSn[counter]=Sn;
      arrIn[counter]=In;
      arrRn[counter]=Rn;
    }
```

```
      variance(arrSn, j, &varSn);
      variance(arrIn, j, &varIn);
      variance(arrRn, j, &varRn);

     fprintf(pFile,"%f\t%f\t%f\t%f\t%f\t%f\t%f\t%f\n",tn,Sn_tot/j,In_tot/j,Rn_tot/j,Vn_tot
         /j,varSn,varIn,varRn);
     n=n+1;
}while(tn<T1);
fclose(pFile);

return 0;
}
void pol_mas(double t)
{
double u1,u2,v1,v2,w,wn;
u1=uniform();
v1=2*u1-1;
u2=uniform();
v2=2*u2-1;
w=v1*v1+v2*v2;
if ((w <= 1.0)&&(w >0.0))
{
wn=sqrt(-2*log(w)/w);
dwt1=sqrt(t)*v1*wn;
dwt2=sqrt(t)*v2*wn;
}
else
pol_mas(t);
return ;
}
double uniform()
{
  double u,U;
  U=rand();
```

```
  u=U/scale_factor;
  return(u);
}
void variance( double *arr, int no, double *var)
{
    int i;
    double sum = 0.0, sum2 = 0.0, tavg;
    for (i = 0; i < no; i++)
  sum += arr[i];
    tavg = sum / (double) no;
    for (i = 0; i < no; i++)
  sum2 += (tavg - arr[i]) * (tavg - arr[i]);
    *var = sum2 / (double) (no - 1);
    }
```

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