CORRESPONDENCE THEOREM IN TROPICAL GEOMETRY

by

Sami Erman Çineli

B.S., Mathematics, Boğaziçi University, 2013

B.S., Mechanical Engineering, Boğaziçi University, 2013

Submitted to the Institute for Graduate Studies in

Science and Engineering in partial fulfillment of

the requirements for the degree of

Master of Science

Graduate Program in Mathematics

Boğaziçi University

2015

CORRESPONDENCE THEOREM IN TROPICAL GEOMETRY

APPROVED BY:

Assoc. Prof. Ferit Öztürk (Thesis Supervisor)	....
Assist. Prof. William D. Gillam	....
Assist. Prof. Özer Öztürk	....

DATE OF APPROVAL: 08.05.2015

ABSTRACT

CORRESPONDENCE THEOREM IN TROPICAL GEOMETRY

Tropical varieties are piecewise linear co-dimension 1 subsets of $\mathbb{R}^n$. For 1-dimensional tropical varieties, we will define an enumerative problem analogous to the classical enumerative problem in $(\mathbb{C}^*)^2$, which counts curves of degree d and genus g, passing through $3d + g - 1$ points in general position. Combinatorial nature of tropical geometry will lead us to an algortihm, which solves the tropical enumerative problem. By the correspondence theorem in tropical geometry, which basically says that the two problems coincide, we will indeed have an algorithm that solves the classical enumerative problem. Throughout this work, we will follow a paper [1] of Grigory Mikhalkin, where he introduces and proves the correspondence theorem in tropical geometry.

ÖZET

TROPİKAL GEOMETRİDE EŞLEME TEOREMİ

Tropikal cebirsel şekiller $\mathbb{R}^n$'nin parçalı lineer, ek boyutu 1 olan altkümeleridir. Klasik sayma problemi, $(\mathbb{C}^*)^2$ içinde genel konumdaki $3d + g - 1$ noktadan geçen, derecesi d ve cinsi g olan eğrileri sayar. Klasik sayma problemine benzer biçimde 1 boyutlu tropikal cebirsel şekiller için bir sayma problemi tanımlayacağız. Tropikal cebirsel şekillerin kombinatorik yapısı, tropik sayma problemini çözen bir algoritma verecek. Tropikal geometride eşleme teoremi sayesinde, iki sayma probleminin çakıştığını söyleyeceğiz ve klasik sayma problemini çözen bir algoritma elde etmiş olacağız. Bu tez boyunca, tropikal geometride eşleme teoremini öneren ve kanıtlayan Grigory Mikhalkin'in makalesini [1] takip edeceğiz.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

$\mathcal{A}$	Amoeba
C	Tropical curve
$(\mathcal{C}, \mathcal{P})$	Marked tropical curve
$\mathbb{C}^*$	Complex numbers without zero.
e	Edge of a tropical curve
E	Edge of a graph
f	Polynomial
F^{trop}	Tropicalization of F
g	Genus
h	Proper map
H_t	Self diffeomorphism of the complex plane
J_t	Holomorphic structure induced by H_t
$\text{mult}(C)$	Multiplicity of C
$N(g, \triangle)$	Number of complex curves of genus g degree $\triangle$
$N_{trop}(g, \triangle)$	Number of tropical curves of genus g degree $\triangle$
$\mathcal{P}$	Configuration of points in $\mathbb{R}^2$
$\mathcal{Q}$	Configuration of points in $(\mathbb{C}^*)^2$
$\mathbb{R}$	Real numbers
$\mathbb{R}_{trop}$	Tropical semifield
s	Number of integer points in $\partial\triangle$
$\mathcal{S}$	Spine
Subdiv_f	Subdivision associated to f
v	Vertex of a tropical curve
V	Vertex of a graph
V_f	Variety of f
V_t	J_t holomorphic curve
V^{trop}	Tropicalization of V
w	Weights of an edge of a graph

$\partial\triangle$	Boundary of $\triangle$
$\triangle$	Newton polygon (degree) of a polynomial
$\triangle_C$	Newton polygon of C
$\triangle_v$	Newton polygon dual to v
γ	Lattice path
Γ	Weighted finite graph
$\mu(\gamma)$	Multiplicity of γ
$\mu(\gamma)_-$	Negative multiplicity of γ
$\mu(\gamma)_+$	Positive multiplicity of γ

1. INTRODUCTION

The tropical semifield $\mathbb{R}_{trop}$ is obtained by changing the usual addition on $\mathbb{R}$ to taking the maximum, and multiplication to the usual addition. Under these operations, the graph of a polynomial f in n variables becomes a piecewise linear codimension 1 subset of $\mathbb{R}^{n+1}$. Variety V_f associated to f is defined as the corner locus of the graph of f, which is a piecewise linear codimension 1 subset of $\mathbb{R}^n$. We will study the tropical varieties in $\mathbb{R}^2$.

1-dimensional tropical varieties are graphs in $\mathbb{R}^2$ with unbounded edges, whose edges have rational slopes, and whose vertices satisfy the "balancing condition". Conversely it can be shown that any graph in $\mathbb{R}^2$ which satisfies these two conditions is a tropical variety. The latter claim makes studying tropical geometry relatively easier than classical geometry. When studying tropical problems we can forget the polynomials giving the varieties, and instead just consider certain subsets of $\mathbb{R}^2$ (e.g. for enumerative problems).

Tropical geometry is a young and active field. There are not many books on it yet. For first exposure we refer to the articles [2] and [3]. For a more detailed study one can see the books [4] and [5].

The image in $\mathbb{R}^2$ of a complex curve $V \subset (\mathbb{C}^*)^2$ under component-wise (absolute value) logarithm is called the amoeba of V. As the base of the logarithm goes to infinity amoebas of complex curves converge to tropical curves (i.e. tropical varieties). Conversely any tropical curve can be seen as the limit of the amoebas of a certain family of complex curves [6]. The latter claim builds a connection between classical and tropical geometry.

This connection is further developed in [1], by also considering imaginary parts of complex curves. The development is a result of Viro's patchworking method, which gives a way of constructing projective plane curves by gluing others. With this method, Viro constructed all nonsingular, degree 7 curves in the real projective plane upto isotopy. This was a breakthrough in the classification problem of real nonsingular plane curves. The proof of the method can be found in [7], for a more approachable treatment one can see [8].

We will demonstrate a special case of Viro's patchworking method, called combinatorial patchworking. Follow from Figure 1.1. We will construct a degree 2 variety, so we start with the polygon $\triangle$ with vertices $(0,0)$, $(2,0)$ and $(0,2)$ ($\triangle$ will be the Newton polygon of the variety). We take a regular triangulation of $\triangle$, i.e., $\triangle$ should admit a concave function, which is affine-linear on each subtriangle, and not affine-linear on the union of any two. Next we put $\pm$ on the vertices of the triangulation without any restriction. (We are describing one fourth of the figure, the positive quadrant.)

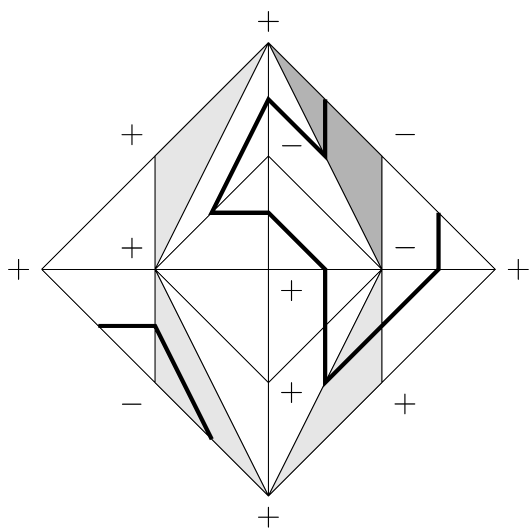

Figure 1.1. Viro's patchworking method.

Each subtriangle represents the Newton polygon of a polynomial whose monomials have coefficients ± 1 (e.g. gray triangle $\triangle'$ corresponds to the polynomial $P_{\triangle'}(x,y) = y^2 - xy - x$). We send the zero locus of $P_{\triangle'}$ in $(\mathbb{R}^+)^2$ diffeomorphically to the interior of $\triangle'$ with the map $X \mapsto \frac{\sum_{i=1}^{3} \alpha_i X^{\alpha_i}}{\sum_{i=1}^{3} X^{\alpha_i}}$; where α_i's are the vertices of $\triangle'$, and $X^{\alpha_i} = x^{\alpha_i^1} y^{\alpha_i^2}$. The black line segment in $\triangle'$ represents the image, and is called the chart of $P_{\triangle'}$. Similarly, we send the zero locus of $P_{\triangle'}$ in the symmetries of $(\mathbb{R}^+)^2$ to the corresponding symmetry of $\triangle'$. See the light gray triangles. We repeat the same process for each subtriangle, and then glue the closure of the charts. Charts in adjacent subtriangles agree on the intersection, since the closure of a chart restricted to an edge e coincides with the chart of P_e (see [7] for details).

By the patchworking theorem, the result represents a variety in $\mathbb{RP}^2$ (identify opposite edges of the diamond to get the projective plane). Furthermore, the method also gives the polynomial of the represented variety. The diffeomorphism we have used is hard to deal with in general. But in this special case, it is reduced to drawing line segments between the edges ending in opposite signs. This is why we carried the signs to the symmetries (e.g. the sign on $(1, 0)$ is changed when carried to $(-1, 0)$, since the corresponding monomial x changes sign under the same symmetry). This was an example of an easy version (but still powerful) of the method, using the full version one can glue more complicated varieties.

Chapters 2 and 3 will be an introduction to tropical geometry. We will discuss the concepts mentioned above. In Chapter 4, we will define the tropical enumerative problem in $\mathbb{R}^2$, which will be analogous to the classical enumerative problem in $(\mathbb{C}^*)^2$. The classical enumerative problem counts complex curves of genus g and degree d, passing through $3d + g - 1$ points in general position. These numbers are denoted by $N(g, d)$, and called the multicomponent Gromov-Witten invariants of $\mathbb{CP}^2$. For a detailed discussion see [9] and [10].

In Chapter 5, we will prove parts of the correspondence theorem in tropical geometry, which basically says that the two enumerative problems coincide. Finally in Chapter 6, we will intoduce the algorithm of Grigory Mikhalkin, which gives a new method (using the tropical approach) for computing the numbers $N(g, d)$.

Throughout this work we will follow “Enumerative tropical algebraic geometry in $\mathbb{R}^2$” by Grigory Mikhalkin.

2. TROPICAL CURVES AS GRAPHS IN $\mathbb{R}^2$

In this chapter we will define tropical curves as images of finite abstract graphs in $\mathbb{R}^2$ under certain maps.

Let $\overline{\Gamma}$ be a weighted finite graph, i.e. a positive integer is associated to each edge of $\overline{\Gamma}$. As an abstract graph, $\overline{\Gamma}$ is compact. By removing the set of all 1-valent vertices ν_1, $\Gamma = \overline{\Gamma} \setminus \nu_1$, we make $\overline{\Gamma}$ non-compact. In the following we extend Definition 2.2 in [1] by adding an extra (third) condition.

Definition 2.1. *A proper map $h : \Gamma \to \mathbb{R}^2$ is called a* parametrized tropical curve *if it satisfies the following three conditions.*

- *For every edge $E \subset \Gamma$ the restriction $h|_E$ is either an embedding or a constant map. The image $h(E)$ is contained in a line $l \subset \mathbb{R}^2$ such that the slope of l is rational.*
- *For every vertex $V \in \Gamma$ we have the following property. Let $E_1, ...E_m \subset \Gamma$ be the edges adjacent to V, let $w_1, ..., w_m \in \mathbb{N}$ be their weights and let $v_1, ..., v_m \in \mathbb{Z}^2$ be the primitive integer vectors at the point $h(V)$ in the direction of the edges $h(E_j)$ (We take $v_j = 0$ if $h(E_j)$ is a point.) We have*

$$\sum_{j=1}^{m} w_j v_j = 0.$$

- *For every vertex V of Γ we require $h(V)$ to be a vertex of $h(\Gamma)$, where vertices of $h(\Gamma)$ are identified in the canonical way.*

We will refer to the second condition as the balancing condition. We require non-zero weights on the edges of Γ because otherwise balancing condition would be less restrictive than desired (e.g. the direction of an edge with zero weight would be free in the image).

Properness of h tells us that the non-compact edges of Γ cannot be contracted to a point, more precisely they should be mapped to unbounded line segments in $\mathbb{R}^2$. Note that the image of the two distinct edges of Γ may intersect at a point or on a segment, or they may coincide.

We say two parametrized tropical curves $h : \Gamma \to \mathbb{R}^2$, $h' : \Gamma' \to \mathbb{R}^2$ are equivalent if there exits a homeomorphism $\Phi : \Gamma \to \Gamma'$, which respects the weights of the edges and satisfies $h = h' \circ \Phi$. We will consider parametrized tropical curves upto this equivalence.

The image $h(\Gamma) = C$ is called an unparametrized tropical curve, or a tropical 1-cycle if no connected component of Γ is contracted to a point. Note that two distinct parametrized tropical curves may have the same image in $\mathbb{R}^2$.

A tropical 1-cycle C is a piecewise linear graph in $\mathbb{R}^2$. We define the edges and vertices of C in the canonical way. We want to associate weights to the edges, which makes C a balanced graph. We do this in the following way; if e is an edge of C, then $w(e)$ is the sum of the weights of the edges, or possibly segments of the edges, in $h^{-1}(e)$. Without the third condition in Definition 2.1, which does not exist in the original definition ([1], Definition 2.2), we would have the following ambiguity:

The figure below shows a parametrized tropical curve. (If we do not see the weight number on an edge, we understand that the edge has weight 1.) If we want the tropical 1-cycle C to be a balanced graph, we need to see the middle line segment as 3 edges. Otherwise it would have weight 8, which is against the balancing condition.

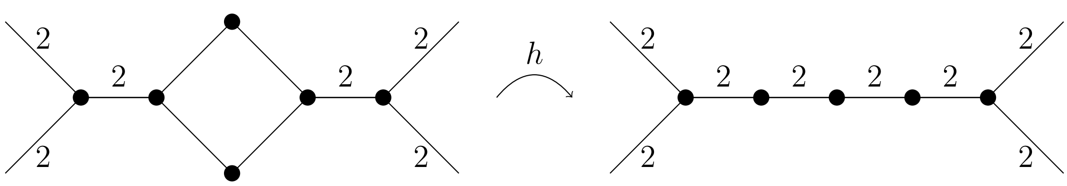

Figure 2.1. A parametrized tropical curve.

This identification of an edge in C is not canonical. However, the third condition in Definition 2.1 solves this problem, and guarantees that each tropical 1-cycle satisfies the balancing condition.

Note that we do not have a one-to-one correspondence between the vertices of Γ and C. Two distinct vertices in Γ may have the same image under h; or the intersection of two edges in C may create an extra vertex, which is not the image of any vertex in Γ.

2.1. Degree of a tropical 1-cycle

Let $C \subset \mathbb{R}^2$ be a tropical 1-cycle, and $\tau_1, \tau_1, ..., \tau_n \subset \mathbb{Z}^2$ be distinct primitive integer vectors in the direction of the unbounded edges of C. We multiply each τ_i by the sum m_i of the weights of the edges going to infinity in the direction of τ_i. We define the degree of C as the set $\{m_1\tau_1,\ m_2\tau_2, ..., m_n\tau_n\}$. By the balancing condition we have

$$\sum_{i=1}^{n} m_i\tau_i = 0.$$

To see this, apply the balancing condition to each vertex of C. In the sum over all vertices, bounded edges will appear twice with primitive vectors pointing in opposite directions. So they will cancel each other out. What remains is the desired equality above.

2.2. Genus of a tropical curve

We say that a parametrized tropical curve $h : \Gamma \to \mathbb{R}^2$ is reducible, in case Γ is disconnected. We say a tropical 1-cyle C is reducible, if C can be seen as a union of two distinct tropical 1-cycles. (Distinct as subsets of $\mathbb{R}^2$, we disregard weights on the edges.) Clearly, a reducible tropical 1-cycle can be represented by a reducible parametrized tropical curve.

Definition 2.2 ([1], Definition 2.9). *The* genus *of a parametrized tropical curve* $\Gamma \to$

$\mathbb{R}^2$ *is dim* $H_1(\Gamma) - $ *dim* $H_0(\Gamma) + 1$. *In particular, for irreducible parameterized curves the genus is the first Betti number of* Γ. *The genus of a tropical 1-cycle* $C \subset \mathbb{R}^2$ *is the minimum genus among all parametrizations of* C.

Note that if C is a tropical 1-cycle which admits a parametrization $h : \Gamma \to \mathbb{R}^2$ with a non-empty vertex set, then C can be parametrized by a graph of arbitrarily large genus. Definition 2.1 allows us to add loops to a vertex V of Γ, and then contract them back to V while satisfying the balancing condition. Cleary this can be done even if C is an embedded 3-valent graph, although being 3-valent puts serious restrictions on the parametrizing graph.

Figure 2.2 gives a better understanding of the minimum genus requirement in Definition 2.2. We can resolve the 4-valent vertex and parametrize the tropical 1-cycle C by a graph of genus zero.

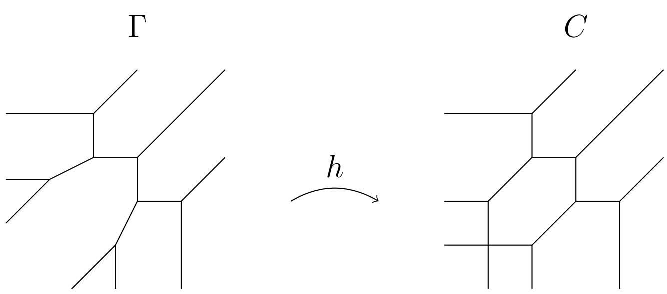

Figure 2.2. C has genus zero.

3. TROPICAL ALGEBRA

In this chapter we will first introduce the tropical semifield $\mathbb{R}_{trop}$, and then define varieties of tropical polynomials over $\mathbb{R}_{trop}$. Later we will see that tropical varieties (of dimension 1) and tropical 1-cycles coincide.

3.1. The tropical semifield $\mathbb{R}_{trop}$

$(\mathbb{R}_{>0}, +, \cdot)$ is a semifield with the usual addition and multiplication of real numbers. For each $t \in (1, \infty)$ we carry the semifield structure to $\mathbb{R}$ with the map $\log_t : \mathbb{R}_{>0} \to \mathbb{R}$, so that the resulting semifield will be isomorphic to $(\mathbb{R}_{>0}, +, \cdot)$. Denote the resulting semifield by $(\mathbb{R}, \oplus_t, \odot_t)$, where

$$x \oplus_t y = \log_t(t^x + t^y) \text{ and } x \odot_t y = \log_t(t^x.t^y) = x + y.$$

Now define $\oplus_\infty$ and $\odot_\infty$ as the limits of $\oplus_t$ and $\odot_t$ as t goes to infinity. Equip $\mathbb{R}$ with these operations. The result $(\mathbb{R}, \oplus_\infty, \odot_\infty)$ is called the tropical semifield, denoted by $\mathbb{R}_{trop}$.

From now on we will denote $x \oplus_\infty y$ by "$x + y$" and $x \odot_\infty y$ by "$x \cdot y$". (Or by "xy".) Note that "$x + y$" $= \max\{x, y\}$ and "$x \cdot y$" $= x + y$. We took the limit of an isomorphic family of semifields and we obtained a "new", non-isomorphic, semifield $\mathbb{R}_{trop}$. To see this latter claim, observe the following; $(\mathbb{R}_{>0}, +, \cdot)$ can be extended to a field by adding additive inverses of its elements, however this is not the case for $\mathbb{R}_{trop}$ since "$x + x$" $= x$ for all $x \in \mathbb{R}_{trop}$.

We could have defined the tropical semifield $\mathbb{R}_{trop}$ directly, but the limiting process is crucial. It is the connection between tropical and classical geometry.

3.2. Varieties of tropical polynomials over $\mathbb{R}_{trop}$

Let A be a finite subset of $(\mathbb{Z}_{\geq 0})^2$ and f be a tropical polynomial in two variables, $f(x,y) = \text{“}\sum_{(i,j)\in A} a_{ij}x^iy^j\text{”} = \max_{(i,j)\in A}\{a_{ij} + ix + jy\}$. We define the variety $V_f \subset \mathbb{R}^2$ associated to f as the set of points where $f : \mathbb{R}^2 \to \mathbb{R}$ is not smooth. Since f is a piecewise linear function it is equivalent to saying that: V_f is the corner locus of f.

In classical geometry, we equate polynomials to the additive identity of the ground field to compute the varieties associated to them. But $\mathbb{R}_{trop}$ does not contain an additive identity, its only option is $-\infty$. Even if we extended $\mathbb{R}_{trop}$ so that it contained $-\infty$, the solution set of $f(x,y) = -\infty$ would not be meaningful. On the other hand, we can still consider the two definitions analogous in the sense of the following paragraph.

Extending the definition to three variables, the variety associated to “$z + f$” is the set $\{(x,y,z) : (x,y) \in V_f \text{ and } z \leq f(x,y)\}$. Observe that $V(\text{“}z + f\text{”}) \cap \{z = t\}$ looks like V_f for a sufficiently small t.

Definition 3.1 ([1], Definition 3.2)**.** *The polygon $\triangle = ConvexHull(A)$ is called the* Newton polygon *of f or alternatively the* degree *of f.*

Proposition 3.2 ([1], Proposition 3.3)**.** *V_f is the set of points in $\mathbb{R}^2$ where more than one monomial of f reaches the maximal value.*

Proof. If only one monomial reaches the maximal value at $x \in \mathbb{R}^2$, then that monomial will dominate in a neighborhood $\mathcal{N}$ of x. f would be linear on $\mathcal{N}$, and hence smooth at x. Since two distinct monomials cannot agree on an open set, the converse is also true. □

Example 3.3. *Let $f(x,y) = \text{“}x^2 + y^2 + 1\text{”}$ and $g(x,y) = \text{“}x^2 + \frac{1}{2}x + \frac{1}{2}y + xy + y^2 + 1\text{”}$. We will find the varieties associated to f, g, “xf” and “$2f$”.*

We need to understand first what we mean by $f(x,y) = \text{“}x^2 + y^2 + 1\text{”}$. In the classical sense, x and $1x$ are equal. But in tropical algebra, “x” $= x$ and “$1x$” $= 1 + x$.

So what should we understand from $f(x,y) =$ "$x^2 + y^2 + 1$"*? The coefficient of* x^2 *in* f *is not 1, it is 0. Multiplying with 0 does not kill a monomial in the tropical world. Writing* f *in the form* "$0 \cdot x^2 + 0 \cdot y^2 + 1$" *may be more convenient, but as a convention we omit writing the 0 coefficients.*

Then $f(x,y) = \max\{2x, 2y, 1\}$*, so the variety associated to* f *is the set* $\{1 = 2x : y \leq \frac{1}{2}\} \cup \{1 = 2y : x \leq \frac{1}{2}\} \cup \{2y = 2x : x, y \geq \frac{1}{2}\}$*. See Figure 3.1.*

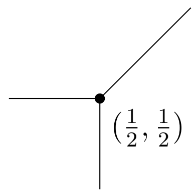

Figure 3.1. Tropical variety V_f of f.

A short computation shows that the varieties of all the tropical polynomials in the example coincide. Even the degrees $\triangle$ *of* f, g *and* "$2f$" *coincide. So given a tropical variety, there are ambiguities in selecting the polynomial representing it. We have actually seen all three types of ambiguities that may happen. We will first list these three ambiguities, then describe a way to select a unique polynomial from among all the representatives.*

Let $A \subset \mathbb{R}^2$ be a finite set. We say $\phi : A \to \mathbb{R}$ is concave if for all $a_1, a_2, ..., a_n \in A$ and for all $t_1, t_2, ..., t_n \geq 0$ such that $\sum_{i=1}^{n} t_i = 1$ and $\sum_{i=1}^{n} t_i a_i \in A$ we have

$$\phi(\sum_{i=1}^{n} t_i a_i) \geq \sum_{i=1}^{n} t_i \phi(a_i).$$

Let $f(x,y) =$ "$\sum_{(i,j)\in A} a_{ij} x^i y^j$". Here are the three ambiguities:

(i) "$x^m y^n f$", where $(m,n) \in \mathbb{Z}^2$ in the positive quadrant. The Newton polygon of "$x^m y^n f$" is a translate of the Newton polygon of f. Since this ambiguity is subject to change in the Newton polygon, when we fix the degree there is no ambiguity.

(ii) "cf", where $c \in \mathbb{R}_{trop}$. We solve this ambiguity by setting $\max_{(i,j)\in\triangle\cap\mathbb{Z}^2}\{a_{ij}\} = 0$.

(iii) Let $(i,j) \in \triangle \cap \mathbb{Z}^2$ and suppose that the function $(i,j) \mapsto a_{ij}$ is not concave, set $a_{ij} = -\infty$ if $a_{ij} \notin A$. Let g be the smallest function that satisfies the concavity condition and $g \geq f$. Then the varieties of f and g coincide. (Observe that the g given in the example is as described here.)

The first two ambiguities are easy to understand, but the third one is little mysterious. Why these are all the possible ambiguities, and what is really described in the third ambiguity will be clear after we introduce lattice subdivisions in Section 3.2.2.

So there is no loss of generality in just considering polynomials, which satisfy the concavity condition. After we fix the degree and set $\max_{(i,j)\in\triangle\cap\mathbb{Z}^2}\{a_{ij}\} = 0$ (i.e. we solve the first two ambiguities), we can uniquely select a polynomial from among all the representatives of a variety, as desired.

3.2.1. Compactness of the space of tropical varieties

This is an independent section of this chapter. We will need it in the proof of the main theorem in Chapter 5.

The space $\mathcal{M}_\triangle$ of all tropical varieties with Newton polygon $\triangle$ is not compact. We compactify $\mathcal{M}_\triangle$ by adding $\mathcal{M}_{\triangle'}$ for all $\triangle' \subset \triangle$.

Proposition 3.4 ([1], Proposition 3.9)**.** *Let $C_k \subset \mathbb{R}^2$, $k \in \mathbb{N}$, be a sequence of tropical varieties with the same Newton polygon $\triangle$. Then there exists a subsequence which converges to a tropical variety C whose Newton polygon $\triangle_C$ is contained in $\triangle$ (note that C is empty if $\triangle_C$ is a point). The convergence is in the Hausdorff metric when restricted to any compact subset in $\mathbb{R}^2$. Furthermore, if the Newton polygon of $\triangle_C$ coincides with $\triangle$, then the convergence is in the Hausdorff metric in the whole $\mathbb{R}^2$.*

Proof. Let $f^{C_k} =$ "$\sum_{(i,j)\in\triangle\cap\mathbb{Z}^2} a_{ij}^{C_k} x^i y^j$" be the polynomial representing C_k chosen as before, i.e., f^{C_k} satisfies the concavity condition and $\max_{(i,j)\in\triangle\cap\mathbb{Z}^2}\{a_{ij}^{C_k}\} = 0$.

We will choose a subsequence (C_{k_n}) such that for at least one pair $(i,j) \in \triangle\cap\mathbb{Z}^2$ the sequence $(a_{ij}^{C_{k_n}})$ converges, and for all other $(i^{'},j^{'}) \neq (i,j)$ the sequences $(a_{i^{'}j^{'}}^{C_{k_n}})$ either converge or go to $-\infty$.

We will choose (C_{k_n}) inductively. We choose first a subsequence which guarantees the convergence criterion. The condition $\max_{(i,j)\in\triangle\cap\mathbb{Z}^2}\{a_{ij}^{C_k}\} = 0$ allows us to do this. Then for a pair $(i^{'},j^{'}) \neq (i,j)$ we choose a subsequence of the chosen subsequence, which satisfies the second condition above. Since $\triangle\cap\mathbb{Z}^2$ is finite, this process has an end. Now let C be the variety of the tropical polynomial with coefficients coming from the limits, "$\sum a_{ij}^{\infty} x^i y^j$". (If $(a_{ij}^{C_{k_n}})$ diverges, we omit the corresponding monomial.)

Obviously $\triangle_C$ is contained in $\triangle$. Suppose that they are equal. Then by the concavity condition $(a_{ij}^{C_{k_n}})$ converges for all $(i,j) \in \triangle\cap\mathbb{Z}^2$. Hence, C_{k_n} converges to C in the whole $\mathbb{R}^2$ with respect to the Hausdorff metric. Now suppose $\triangle_C$ is a proper subset of $\triangle$. Since $(a_{ij}^{C_{k_n}})$ goes to $-\infty$ for all $(i,j) \notin \triangle_C$, for large k_n the corresponding points in C_{k_n} disappear from any bounded subset of $\mathbb{R}^2$. (Section 3.2.2 makes this argument more explicitly.) □

3.2.2. Lattice subdivision of $\triangle$ associated to a tropical variety

Let $f(x,y) =$ "$\sum_{(i,j)\in A} a_{ij} x^i y^j$". Recall that the Newton polygon of f is $\triangle =$ Convexhull(A). The lattice subdivision of $\triangle$ associated to f is given in the following way.

Let $\tilde{\triangle} = \text{Convexhull}\{(i,j,t) : (i,j) \in A,\ t \leq a_{ij}\} \subset \mathbb{R}^3$. Using the projection $\mathbb{R}^3 \to \mathbb{R}^2$ to the first two coordinates, we write $\triangle$ as the union of the bounded faces in $\tilde{\triangle}$. The resulting subdivision of $\triangle$ is denoted by Subdiv$_f$. We will see that the tropical variety V_f is dual to Subdiv$_f$.

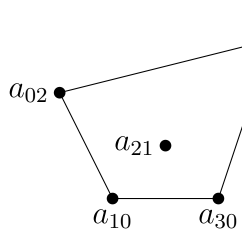

Figure 3.2. A two dimensional polygon $\triangle'$ in Subdiv_f.

Let $\triangle' \in \text{Subdiv}_f$ be a two dimensional polygon, and $f^{\triangle'} = $ "$\sum_{(i,j)\in\triangle'\cap A} a_{ij}x^iy^j$" be the corresponding truncated polynomial. Let I be the minimal set of indices in $\triangle'$ such that $\text{Convexhull}(I) = \triangle'$. See Figure 3.2, where $I = \{(0,2),(4,3),(3,0),(1,0)\}$.

Since $\triangle'$ is a face of $\tilde{\triangle}$, the varieties of $f^{\triangle'}$ and $f^{\triangle'\cap I}$ coincide. For a contradiction suppose that the maximal value is uniquely attained by the monomial $a_{21}+2x+y$ at the point (x,y). Since $(2,1)$ is in the convex hull of the set I, there exists $t_1,t_2,t_3,t_4 \geq 0$, where $\sum_{i=1}^{4} t_i = 1$, such that

$$(2,1) = t_1(0,2) + t_2(4,3) + t_3(3,0) + t_4(1,0).$$

Since $\triangle'$ is a face of $\tilde{\triangle}$, we also have

$$a_{21} \leq t_1a_{02} + t_2a_{43} + t_3a_{30} + t_4a_{10}.$$

By adding the two, we get a contradiction.

So the monomials that are not vertices of Subdiv_f do not make any contribution to V_f. For simplicity we drop them. But all the remaining monomials uniquely attain a maximal value. In other words, we cannot drop any other monomial without changing V_f.

Let v be a vertex of Subdiv_f, as in Figure 3.3. Select a plane which intersects $\tilde{\triangle}$ only at v. Such a plane exists, since $\tilde{\triangle}$ is convex. Write the normal of the plane of the form $n =< x, y, 1 >$. Then the corresponding monomial uniquely attains the maximal value at the point (x, y). For the proof observe that evaluating a monomial at the point (x, y) is algebraically equivalent to taking the inner product of the vectors $< i, j, a_{ij} >$ and $< x, y, 1 >$. So vertices of Subdiv_f are dual to connected components in $\mathbb{R}^2 \setminus V_f$. (Remember that V_f is the corner locus of the graph of f.)

Let $\triangle' \in \text{Subdiv}_f$ again be a two dimensional polygon, and $f^{\triangle'}$ be the truncated polynomial. In the figure below, we see the corresponding face of $\tilde{\triangle}$. Again thinking as taking the inner product, we see that all the monomials in $f^{\triangle'}$ agree when multiplied with the normal vector $n =< x, y, 1 >$ of the face. Moreover, no other monomial of f dominates at the point (x, y), since otherwise $\triangle'$ would not be in the subdivision. This is obvious in Figure 3.3.

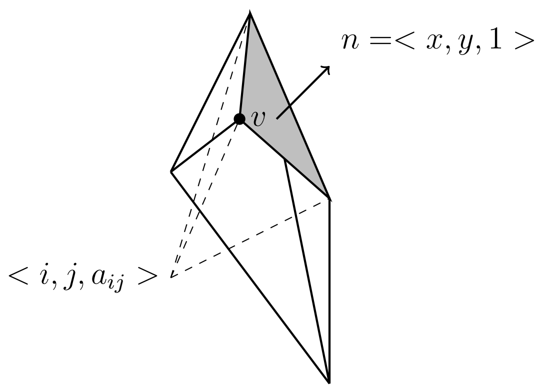

Figure 3.3. Convex polygon $\tilde{\triangle} \subset \mathbb{R}^3$.

Similarly, we see that for each edge in $\triangle'$ there is a line segment in V_f, which emanates from (x, y) in the direction perpendicular to the edge. Moreover, this is the complete set of edges adjacent to (x, y). So for each two dimensional polygon in Subdiv_f, there is a vertex in V_f with a neighborhood dual to $\triangle'$. It remains to show that these are all the vertices of V_f. This would also finish the duality of edges, since an extra edge would cause an extra vertex.

Any other vertex in V_f would give rise to a two dimensional polygon in the interior of $\tilde{\triangle}$. But then when multiplied with the normal of the polygon, corresponding monomials would not take the maximal value. Hence, such a point could not appear in V_f.

Definition 3.5 ([1], Definition 3.13)**.** *The* combinatorial type *of a tropical variety $V_f \subset \mathbb{R}^2$ is the equivalence class of all V_g such that $Subdiv_g = Subdiv_f$.*

Example 3.6. *Let $f(x,y) =$ "$x^2 + y^2 + 1$" and $g(x,y) =$ "$x^2 + \frac{1}{2}x + \frac{1}{2}y + xy + y^2 + 1$". We will show that $Subdiv_g = Subdiv_f$, and $V_f = V_g$.*

Note that being in the same combinatorial type does not imply coincidence of varieties in general. For example "$x^2 + y^2 + 1$" and "$x^2 + y^2 + 5$" are in the same combinatorial type but their varieties do not coincide.

Both f and g have the same Newton polygon $\triangle$. Since f has only three monomials, there is no further subdivision of $\triangle$. In the figure below, we write the coefficients of g near the integer points of $\triangle$. Since the function $(i,j) \mapsto a_{ij}$ is affine-linear on $\triangle$, $Subdiv_g$ is also trivial. Hence V_f and V_g are in the same combinatorial type. Note also that we have, $V_f = V_g$. (Remember that we could drop a monomial, if it is not a vertex of $\tilde{\triangle}$.)

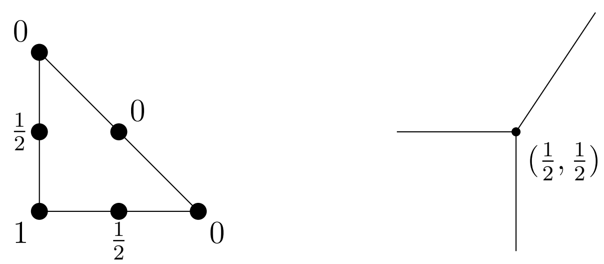

Figure 3.4. Subdiv_f, Subdiv_g; V_f and V_g.

In Figure 3.5, there is a more complicated example of the duality. With a little work one can find a tropical polynomial which gives the variety.

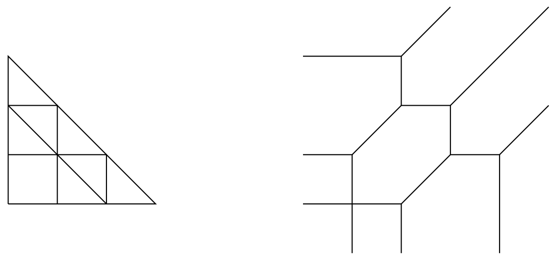

Figure 3.5. An example of the duality.

Given a Newton polygon $\triangle$; thanks to duality, we can determine shapes of all tropical varieties just working on the possible subdivisions of $\triangle$. But we have to be careful because there are subdivisions which do not have a dual tropical variety [3]. (See Figure 3.6) This subdivision of the square cannot come from a tropical polynomial. In fact, there is no concave function on it, which is affine-linear on each two dimensional subpolygon and not affine-linear on union of any two dimensional subpolygon. Moreover, we cannot draw the dual of it either. (When we try, we see that there is no choice for the lengths of the dual edges corresponding to the middle square.)

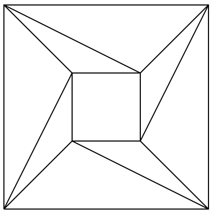

Figure 3.6. A non-realizable subdivision.

Now take any Newton polygon and a subdivision of it, which comes from a concave function as described above. Obviously there exists a tropical polynomial which gives that subdivision and hence a tropical variety dual to it. (Just use the concave function to determine the coefficients of the monomials.) We will see in Section 3.2.3 that somehow the converse of this statement is also true.

3.2.3. Tropical varieties and tropical 1-cycles coincide

We will first associate weights to the edges of a tropical variety, which make it a balanced graph. Since such a graph can be parametrized by an abstract graph (e.g. considering it as an embedded abstract graph in $\mathbb{R}^2$ itself), we will be finished with one side of the coincidence.

Let $f(x, y) =$ " $\sum_{(i,j)\in A} a_{ij}x^i y^j$ ", and $\triangle' \in \text{Subdiv}_f$. We have seen that at the vertex dual to $\triangle'$, the local picture looks like that given in the figure below. To the edge e, we associate the weight $w(e) = \gcd\,(|i' - i|, |j' - j|)$. Since each edge is perpendicular to its dual, with this choice of weights V_f satisfies the balancing condition.

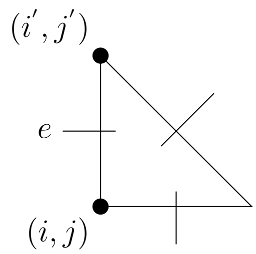

Figure 3.7. Tropical varieties are balanced graphs.

Theorem 3.7 ([1], Corollary 3.16)**.** *Any tropical 1-cycle $C \subset \mathbb{R}^2$ is a tropical variety for some polynomial f. Conversely, any tropical variety in $\mathbb{R}^2$ can be parametrized by a tropical curve.*

We have seen that the converse of the statement is true. For the other direction; we give the idea of a proof, a complete proof can be found in [6].

What we will be basically doing is tracing back the arguments we gave in the previous section. Let $v = (x, y)$ be a vertex of the tropical 1-cycle C. For each edge e adjacent to v draw a perpendicular line segment with integer length $w(e)$. By the balancing condition, when we glue these line segments (which we call the edges of $\triangle_v$) the result will be a two dimensional polygon $\triangle_v$. See Figure 3.8.

Now glue each $\triangle_v$ by identifying edges perpendicular to the same edge in C. Denote the resulting polygon by $\triangle$. Translate $\triangle$ in $\mathbb{R}^2$ so that it has non-negative integer vertices. $\triangle$ is convex, since we have defined the vertices of C in the canonical way (being non-convex creates an extra vertex). Now we have a candidate for a Newton polygon and a subdivision of it, which is dual to C by construction. It remains to show two things; the subdivision is a valid one, and there exists a tropical polynomial f such that $V_f = C$ as sets.

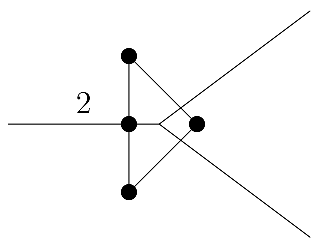

Figure 3.8. From a tropical 1-cycle to a tropical variety.

Orient $\triangle_v$ in $\mathbb{R}^3$ so that the vector $< x, y, 1 >$ will be positively normal to it. We want to glue the oriented $\triangle_v$'s to obtain the upper face of a convex object $\tilde{\triangle}$ in $\mathbb{R}^3$, so that projecting back will give $\triangle$. But observe that when you just orient and then project back to $\mathbb{R}^2$, you do not get the same subpolygon $\triangle_v$. So orient and dilate if necessary to solve this problem. Now we start gluing oriented $\triangle_v$'s.

Take any two adjacent subpolygons in $\triangle$. (In the remainder of this section; when we say a subpolygon, we mean a two dimensional subpolygon.) Since their intersection is exactly a common edge, there is no obstruction in gluing their oriented versions. Moreover, our choice of normal vectors guarantees that the glued object $\triangle^g$ (g stands for being glued) is an upper face of a convex set in $\mathbb{R}^3$. If there were only two subpolygons in $\triangle$, we would be finished.

We will extend $\triangle^g$ inductively, by adding an adjacent subpolygon at each step. Take an adjacent subpolygon $\triangle_{v=(x,y)}$; if $\triangle^g \cap \triangle_{v=(x,y)}$ is exactly an edge, there is again no obstruction in gluing them. (There is an ambiguity in our naming of objects. We denote oriented and non-oriented subpolygons by the same name. But what we mean must be clear from the context.) Suppose that the intersection contains more than one edge. We will make a reduction, we assume that the intersection $\triangle^g \cap \triangle_{v=(x,y)}$ is connected. (Since $\triangle$ itself is a convex object, we can do this. If $\triangle_{v=(x,y)}$ does not satisfy this condition, glue another adjacent subpolygon first.) Now take any two adjacent edges from the intersection. Since the integer lengths of the common edges of adjacent subpolygons coincide, there is an orientation of $\triangle_{v=(x,y)}$ which glues to these two edges without any obstruction. We will see that the orientation described above coincides with the orientation given by the vector $< x, y, 1 >$. Then we would be finished, since the argument would be valid for any adjacent pair from the intersection.

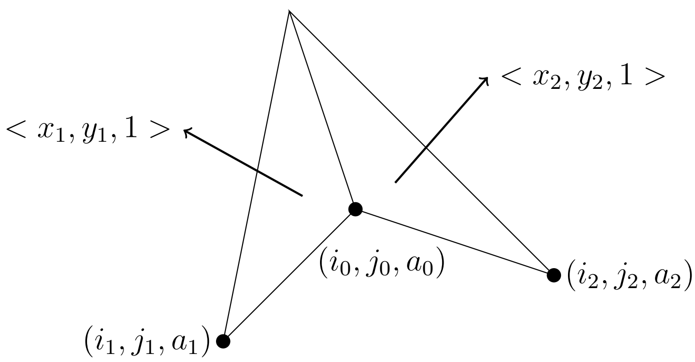

Figure 3.9. Gluing subpolygons.

The picture looks like that in Figure 3.9. Assume $a_0 = 0$. We will compute the coordinates a_1 and a_2. The equation for the plane on the left hand side is

$$x_1 x + y_1 y + z - (x_1 i_0 + y_1 j_0) = 0.$$

We have

$$a_1 = x_1(i_0 - i_1) + y_1(j_0 - j_1), \text{ and similarly } a_2 = x_2(i_0 - i_2) + y_2(j_0 - j_2).$$

We compute the cross product n of the vectors $< i_1 - i_0,\ j_1 - j_0,\ a_1 >$ and $< i_2 -$

$i_0,\ j_2 - j_0,\ a_2 >$,

$$n =< \frac{a_2(j_1 - j_0) - a_1(j_2 - j_0)}{(i_1 - i_0)(j_2 - j_0) - (i_2 - i_0)(j_1 - j_0)}, \frac{a_1(i_2 - i_0) - a_2(i_1 - i_0)}{(i_1 - i_0)(j_2 - j_0) - (i_2 - i_0)(j_1 - j_0)}, 1 > .$$

To finish, compute the point (x, y) using the data;

$$(x_1, y_1),\ < j_0 - j_1,\ i_1 - i_0 > \text{ and } (x_2, y_2),\ < j_0 - j_2,\ i_2 - i_0 >,$$

to see that n coincides with the vector $< x, y, 1 >$.

So we have glued all subpolygons while respecting their orientation coming from the coordinates of the corresponding vertex in C. We have done each step in the gluing process at an edge in a convex manner. So we see that the subdivision dual to C is a valid one. It remains to show that there exists a tropical polynomial f such that $V_f = C$. But we have already have it, just use the heights of the vertices in the glued polygon for the coefficients of the polynomial.

Note again that, there is no one to one correspondence between subdivisions and dual tropical varieties. Indeed, given a valid subdivision, there are infinitely many choices for varieties dual to it. Just find one and translate it in $\mathbb{R}^2$, or change the lengths of the bounded edges without destroying the balancing condition. How do we know that such movements still give a tropical variety? We have just seen that any balanced graph in $\mathbb{R}^2$ is a tropical variety.

4. TWO ENUMERATIVE PROBLEMS

In this chapter we will define two enumerative problems. Namely, the tropical enumerative problem in $\mathbb{R}^2$ and a classical enumerative problem in $(\mathbb{C}^*)^2$. But first, we need some definitions and facts regarding tropical curves, which is the subject of the following section.

4.1. Enumeration of tropical curves in $\mathbb{R}^2$

In the previous chapter, we have seen that tropical 1-cycles in $\mathbb{R}^2$ and tropical varieties coincide. We call them both tropical curves.

Note that we gave different definitions for degrees of tropical 1-cycles and tropical varieties, but by duality, we see that both carry the same information. Moreover, subdivision of the Newton polygon tells us the shape of the variety associated to it, including the information regarding the weights. (The weight of an edge is the number of integer points in the dual edge of the subdivision.)

First we will see a special class of tropical curves, which will be the object of the tropical enumerative problem.

Definition 4.1 ([1], Definition 4.2)**.** *A parametrized tropical curve $h : \Gamma \to \mathbb{R}^2$ is called* simple *if it satisfies all of the following conditions.*

- *The graph Γ is 3-valent.*
- *The map h is an immersion.*
- *For any $y \in \mathbb{R}^2$ the inverse image $h^{-1}(y)$ consists of at most two points.*
- *For any vertex V of Γ the inverse image $h^{-1}(h(V))$ contains a unique point.*

A tropical 1-cycle $C \subset \mathbb{R}^2$ is called simple if it admits a simple parametrization.

Here are two facts about simple tropical curves:

(i) ([1], Proposition 4.3) A simple tropical 1-cycle $C \subset \mathbb{R}^2$ admits a unique simple tropical parametrization. Furthermore, any of its non-simple parametrizations has a strictly larger genus.

This allows us to use simple tropical 1-cycles and their simple parametrizations interchangeably in our arguments.

(ii) ([1], Lemma 4.5) A tropical curve $C \subset \mathbb{R}^2$ is simple if and only if it is the variety of a tropical polynomial such that $\mathrm{Subdiv_f}$ is a subdivision into triangles and parallelograms.

In Chapter 6, we will give a combinatorial solution to the tropical enumerative problem. Basically we will be counting simple tropical curves with Newton polygon $\triangle$ over all possible subdivisions of $\triangle$. Fact (ii) is our first restriction for subdividing.

We need to define the notion of being in general position in the tropical setting, which will be closely related to being simple defined above.

Definition 4.2 ([1], Definition 4.7)**.** *Points $p_1, ..., p_k \in \mathbb{R}^2$ are said to be in* tropically general position *if for any tropical curve $h : \Gamma \to \mathbb{R}^2$ of genus g and with x ends such that $k \geq g + x - 1$ and $p_1, ..., p_k \in h(\Gamma)$ we have the following conditions.*

- *The curve $h : \Gamma \to \mathbb{R}^2$ is simple.*
- *Inverse images $h^{-1}(p_1), ..., h^{-1}(p_k)$ are disjoint from the vertices of C.*
- *$k = g + x - 1$.*

We are ready to define the tropical enumerative problem.

4.1.1. Tropical enumerative problem in $\mathbb{R}^2$

Fix a Newton polygon $\triangle$, where $s = \#(\partial\triangle \cap \mathbb{Z}^2)$, and a genus $g \in \mathbb{Z}$. Consider a configuration $\mathcal{P} = \{p_1, p_2, ..., p_{s+g-1}\} \subset \mathbb{R}^2$ of $s + g - 1$ points in tropically general position.

Definition 4.3 ([1], Definition 2.6)**.** *Let V be a 3-valent vertex of a tropical curve $C \subset \mathbb{R}^2$. Let v_1, v_2, v_3 be the primitive integer vectors adjacent to V and w_1, w_2, w_3 be their weights. We define the* multiplicity *of V as the area of the parallelogram spanned by the vectors w_1v_1 and w_2v_2 (well-defined by the balancing condition).*

Definition 4.4 ([1], Definition 4.15)**.** *The* multiplicity *$mult(C)$ of a tropical curve $C \subset \mathbb{R}^2$ of degree $\triangle$ and genus g passing through $\mathcal{P}$ equals the product of the multiplicities of all the 3-valent vertices of C.*

We define $N_{trop}(g, \triangle)$ to be the number of all tropical curves of genus g degree $\triangle$ passing through $\mathcal{P}$, counted with multiplicity. Note that multiplicity of a tropical curve C becomes the product of two times the areas of triangles in Subdiv_C.

4.2. A classical enumerative problem in $(\mathbb{C}^*)^2$

Again fix a Newton polygon $\triangle$, where $s = \#(\partial\triangle \cap \mathbb{Z}^2)$, and a genus $g \in \mathbb{Z}$. Consider a configuration $\mathcal{Q} = \{q_1, q_2, ..., q_{s+g-1}\} \subset (\mathbb{C}^*)^2$ of $s + g - 1$ points in general position. A complex algebraic curve $C \in (\mathbb{C}^*)^2$ is given as the zero locus of a polynomial with complex coefficients. We define $N(g, \triangle)$ to be the number of all complex curves of genus g and degree $\triangle$, passing through $\mathcal{Q}$. We will not examine the number $N(g, \triangle)$, without any further comment we take it to be invariant under $\mathcal{Q}$. See [10] for details.

At this step we do not know whether the number $N_{trop}(g, \triangle)$ is independent from the choice $\mathcal{P}$, or not. We just know that $N_{trop}(g, \triangle)$ is finite ([1], Lemma 4.22). In the next chapter our main theorem will give us the independence, which basically says $N_{trop}(g, \triangle) = N(g, \triangle)$.

Note that in the tropical problem, we count curves with multiplicity. This is crucial, since otherwise even the independence of $N_{trop}(g, \triangle)$ from $\mathcal{P}$ is not true. For an example, see Example 4.14 in [1].

5. THE CORRESPONDENCE THEOREM

In this chapter we will see the correspondence between the tropical enumerative problem, and the classical enumerative problem defined in Section 4.2. We have defined the numbers $N_{trop}(g, \triangle)$ and $N(g, \triangle)$ in Chapter 4. Let $\mathcal{P}$ be a configuration of $s+g-1$ points in $\mathbb{R}^2$, where $s = \#(\partial\triangle \cap \mathbb{Z}^2)$.

Theorem 5.1 ([1], Theorem 1)**.** *For any tropically generic choice of $\mathcal{P}$ we have $N_{trop}(g, \triangle) = N(g, \triangle)$.*

Furthermore, there exists a configuration $\mathcal{Q} \subset (\mathbb{C}^*)^2$ of $s+g-1$ points in general position such that for every tropical curve C of genus g and degree $\triangle$ passing through $\mathcal{P}$ we have mult(C) distinct complex curves of genus g and degree $\triangle$ passing through $\mathcal{Q}$. These curves are distinct for distinct C.

We will give a partial proof of Theorem 5.1. Before that, we need to introduce some new concepts.

5.1. Degeneration of complex structure on $(\mathbb{C}^*)^2$

Let $t > 1$ be a real number. We define the self-diffeomorphism

$$\begin{aligned} H_t : (\mathbb{C}^*)^2 &\to (\mathbb{C}^*)^2 \\ (z, w) &\mapsto (|z|^{\frac{1}{log t}} \frac{z}{|z|}, |w|^{\frac{1}{log t}} \frac{w}{|w|}). \end{aligned}$$

For each t, H_t induces a new complex structure J_t on $(\mathbb{C}^*)^2$. We will compute J_t in logarithmic polar coordinates.

$$\begin{aligned} \mathcal{L}og \circ H_t : (\mathbb{C}^*)^2 &\to \mathbb{R}^2 \times iT^2 \\ (x_1 + iy_1,\ x_2 + iy_2) &\mapsto (\frac{1}{\log t} \ln r_1, \frac{1}{\log t} \ln r_2, \theta_1, \theta_2) \end{aligned}$$

where $r_i^2 = x_i^2 + y_i^2$ and $\theta_i = \tan^{-1}\frac{y_i}{x_i}$ for $i = 1, 2$.

$$D(\mathcal{L}og \circ H_t) = \begin{bmatrix} \frac{1}{\log t}\frac{x_1}{r_1^2} & 0 & \frac{1}{\log t}\frac{y_1}{r_1^2} & 0 \\ 0 & \frac{1}{\log t}\frac{x_2}{r_2^2} & 0 & \frac{1}{\log t}\frac{y_2}{r_2^2} \\ \frac{-y_1}{r_1^2} & 0 & \frac{x_1}{r_1^2} & 0 \\ 0 & \frac{-y_2}{r_2^2} & 0 & \frac{x_2}{r_2^2} \end{bmatrix}$$

J_t in logarithmic polar coordinates is

$$J_t = D(\mathcal{L}og \circ H_t) \circ i \circ D(\mathcal{L}og \circ H_t)^{-1} = \begin{bmatrix} 0 & 0 & \frac{-1}{\log t} & 0 \\ 0 & 0 & 0 & \frac{-1}{\log t} \\ \log t & 0 & 0 & 0 \\ 0 & \log t & 0 & 0 \end{bmatrix}.$$

We have $J_t^2 = -I$.

A curve V_t is holomorphic with respect to J_t if and only if $V_t = H_t(V)$ for some holomorphic curve V with respect to the standard complex structure i. By definition, $V \subset (\mathbb{C}^*)^2$ is the zero locus of a polynomial with complex coefficients. Moreover, we say V_t is of genus g and degree $\triangle$ if and only if V is of genus g and degree $\triangle$.

Proposition 5.2 ([1], Proposition 8.1)**.** *For generic t we have $N(g, \triangle)$ J_t-holomorphic curves passing through $\mathcal{Q}$.*

Proof. For a generic t, the points $H_t^{-1}(\mathcal{Q})$ are also generic. So the proposition follows easily since genus and degree of a J_t-holomorphic curve V_t are defined to be as the genus and the degree of $H_t^{-1}(V_t)$. □

Proposition 5.2 allows us to work with J_t holomorphic curves in the proof of Theorem 5.1.

5.2. Amoebas of J_t holomorphic curves

Let V_t be a J_t holomorphic curve. We define the amoeba $\mathcal{A}$ of V_t as $\mathrm{Log}(V_t)$, where

$$\begin{aligned} \mathrm{Log} : (\mathbb{C}^*)^2 &\to \mathbb{R}^2 \\ (z_1, z_2) &\mapsto (\log|z_1|, \log|z_2|). \end{aligned}$$

Example 5.3. *We will find the amoeba $\mathcal{A}$ of the curve $z_1 + z_2 + 1 = 0$.*

We write the set $Z(z_1 + z_2 + 1 = 0) \cap (\mathbb{C}^*)^2$ explicitly as

$$\{(t^x e^{i\theta}, -1 - t^x e^{i\theta}) : x \in \mathbb{R}, \theta \in [0, 2\pi) \text{ and } (x, \theta) \neq (0, 0)\}.$$

So $\mathrm{Log}(z_1, z_2) = (x, \log_t |1 + t^x e^{i\theta}|)$. Boundary of the image has three components corresponding to the following sets: $\{x, \theta = 0\}, \{x < 0, \theta = \pi\}, \{x > 0, \theta = \pi\}$. In Figure 5.1 we see the graph of the boundary for, $t = e$ and $t = 10$. For a fixed x, zero locus is mapped to a vertical line inside the boundary. (There are two points in the preimage for each value inside the boundary.)

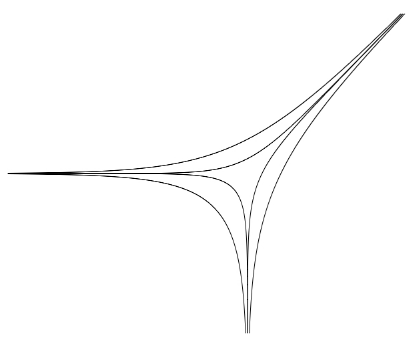

Figure 5.1. Amoeba $\mathcal{A}$ of the curve $z_1 + z_2 + 1 = 0$.

If V_t is a J_t-holomorphic curve with the Newton polygon $\triangle$ then its amoeba $\mathcal{A}_t = \mathrm{Log}(V_t)$ contains a tropical curve $\mathcal{S}_t$ with the same Newton polygon $\triangle$ [11]. $\mathcal{S}_t$ is called the spine of the amoeba $\mathcal{A}_t$.

5.3. Proof of Theorem 5.1

Fix a degree $\triangle$, where $s = \#(\partial\triangle \cap \mathbb{Z}^2)$, and a genus $g \in \mathbb{Z}$. Let $\mathcal{Q} = \{q_1, q_2, ..., q_{s+g-1}\} \subset (\mathbb{C}^*)^2$ be a configuration of $s + g - 1$ points in general position, such that $\mathcal{P} = \{\log|q_1|, \log|q_2|, ..., \log|q_{s+g-1}|\} \subset \mathbb{R}^2$ is in tropically general position.

By ([1], Lemma 4.22) there are finitely many tropical curves $C_1, C_2, ..., C_m$ of genus g and degree $\triangle$, passing through $\mathcal{P}$. Recall that for the tropical enumerative problem we count the curves with multiplicity, hence $N_{trop}(g, \triangle) \geq m$.

Proposition 5.2 and the following two lemmas finishes the proof of Theorem 5.1. We will discuss the proof of Lemma 5.4 in Section 5.3.1. Later we will give a sketch of the proof of Lemma 5.5. We only discuss it for the sake of completeness. We refer to [1] for the actual proof.

Lemma 5.4 ([1], Lemma 8.3)**.** *For any $\epsilon > 0$ there exists $T > 1$ such that for generic $t > T$ if V is a J_t-holomorphic curve of genus g, degree $\triangle$ and passing through $\mathcal{Q}$, then its amoeba* $\mathrm{Log}(V)$ *is contained in the ϵ-neighborhood $\mathcal{N}_\epsilon(C_j)$ of C_j for some $j = 1, ..., m$.*

Lemma 5.5 ([1], Lemma 8.4)**.** *For a sufficiently small $\epsilon > 0$ and a sufficiently large generic $t > 0$ the multiplicity mult(C_j) of each C_j is equal to the number of the J_t-holomorphic curves V of genus g and degree $\triangle$ passing through $\mathcal{Q}$ and such that* $\mathrm{Log}(V)$ *is contained in $\mathcal{N}_\epsilon(C_j)$.*

5.3.1. Proof of Lemma 5.4

A holomorphic curve $V \subset (\mathbb{C}^*)^2$ is given by a polynomial

$$F(z_1, z_2) = \sum_{(i,j)\in\triangle} a_{i,j} z_1^i z_2^j.$$

To a curve $V \subset (\mathbb{C}^*)^2$, we associate its tropicalization $V^{trop} \subset \mathbb{R}^2$, which is given by the tropical polynomial

$$F^{trop}(y_1, y_2) = \max_{(i,j)\in\triangle} \{iy_1 + jy_2 + \log|a_{i,j}|\}.$$

Lemma 5.6 ([1], Lemma 8.5)**.** *The amoeba* $\mathrm{Log}(V)$ *is contained in the* δ*-neighborhood of* V^{trop} *where* $\delta = \log(\#(\triangle \cap \mathbb{Z}^2) - 1)$.

Proof. Suppose $(y_1, y_2) \in \mathrm{Log}(V)$ is not contained in the δ-neighborhood of V^{trop}. Then there exists $(i^{'}, j^{'})$ such that

$$i^{'}y_1 + j^{'}y_2 + \log|a_{i^{'},j^{'}}| > iy_1 + jy_2 + \log|a_{i,j}| + \delta$$

for all $(i, j) \neq (i^{'}, j^{'})$. To see this consider the graph of F^{trop}. There exists unique $(i^{'}, j^{'})$ such that $F^{trop}(y_1, y_2) = i^{'}y_1 + j^{'}y_2 + \log|a_{i^{'},j^{'}}|$. Existence is obvious, uniqueness comes from the fact that $(y_1, y_2) \notin V^{trop}$. Now for any (i, j) such that the line $iy_1 + jy_2 + \log|a_{i,j}| = i^{'}y_1 + j^{'}y_2 + \log|a_{i^{'},j^{'}}|$ is a corner locus of the graph of F^{trop} we have the desired inequality, since the gradient of the line is greater than or equal to 1. For the remaining (i, j)'s, $iy_1 + jy_2 + \log|a_{i,j}|$ is even smaller.

Let $(z_1, z_2) \in V$ such that $\mathrm{Log}(z_1, z_2) = (y_1, y_2)$. Since $\sum_{(i,j)} a_{i,j} z_1^i z_2^j = 0$, we have

$$|a_{i^{'},j^{'}} z_1^{i^{'}} z_2^{j^{'}}| = |\sum_{(i,j)\neq(i^{'},j^{'})} a_{i,j} z_1^i z_2^j|.$$

Apply log to both sides of the equality

$$\begin{aligned} i^{'}y_1 + j^{'}y_2 + \log|a_{i^{'},j^{'}}| &= \log|\sum_{(i,j)\neq(i^{'},j^{'})} a_{i,j} z_1^i z_2^j| \\ &\leq \log((\#(\triangle \cap \mathbb{Z}^2) - 1) \max_{(i,j)\neq(i^{'},j^{'})} |a_{i,j} z_1^i z_2^j|) \\ &= \delta + \log(\max_{(i,j)\neq(i^{'},j^{'})} |a_{i,j} z_1^i z_2^j|). \end{aligned}$$

Contradiction. □

Corollary 5.7 ([1], Corollary 8.6)**.** *The amoeba* $\mathrm{Log}(V_t)$ *of a* J_t*-holomorphic curve* $V_t = H_t(V)$ *is contained in the* δ*-neighborhood of some tropical curve in* $\mathbb{R}^2$*, where* $\delta = \log_t(\#(\triangle \cap \mathbb{Z}^2))$*.*

Proof. Observe that $\mathrm{Log}(V_t) = \mathrm{Log}_t(V) = \frac{\mathrm{Log}(V)}{\log t}$. Since $\frac{V^{trop}}{\log t}$ is still a balanced graph, we have the desired tropical curve. □

Let us denote the tropical curve $\frac{V^{trop}}{\log t}$ by V_t^{trop}. Observe that V_t^{trop} can be given by the variety of the tropical polynomial

$$F_t^{trop}(y_1, y_2) = \max_{(i,j)\in\triangle} \{iy_1 + jy_2 + \log_t |a_{i,j}|\}.$$

Take a sequence of J_k-holomorphic curves V_k of genus g and degree $\triangle$, passing through $\mathcal{Q}$, where $k \to \infty$ (k's are generic as in Proposition 5.2).

Proposition 5.8 ([1], Proposition 8.7)**.** *There is a subsequence* V_{k_n}*,* $n \in \mathbb{N}$*, such that the sets* $\mathcal{A}_{k_n} \subset \mathbb{R}^2$ *converge in the Hausdorff metric in the whole* $\mathbb{R}^2$ *to some tropical curve* C_j*, where* $j \in \{1, ..., m\}$*.*

Proof. For each k, let V_k^{trop} be the tropical curve given by the Corollary 5.7. V_k^{trop} has the same degree $\triangle$ for all k. Recall how the corresponding tropical polynomials F_k^{trop} are given.

We may choose the coefficients for the polynomials $F_k(z_1, z_2) = \sum_{(i,j)\in\triangle} a_{i,j}^k z_1^i z_2^j$ giving the varieties $H_k^{-1}(V_k)$ so that upto selecting a subsequence (V_{k_n}), the sequence $(\log_{k_n} |a_{i,j}^{k_n}|)$ converges for all $(i, j) \in \triangle$. (For the proof see [12]. We thank Erwan Brugallé, who is a co-author of [12], for pointing out this fact for us.) As in the proof of Proposition 3.4, this choice guarantees the convergence of $(V_{k_n}^{trop})$ to a tropical curve C with degree $\triangle$, with respect to the Hausdorff metric in the whole $\mathbb{R}^2$.

Corollary 5.7 ensures that $(\mathcal{A}_{k_n})$ also converges to C in the same mode of convergence. Then C passes through $\mathcal{P}$, since $\mathcal{A}_{k_n}$ contains $\mathcal{P}$ for all k_n. It remains to show that C is of genus g.

Since $\mathcal{P}$ is a configuration of $s+g-1$ points in tropically general position, genus of C cannot be strictly less than g. Moreover, we will prove next (Lemma 5.9) that we can extract a convergent (in every compact subset of $(\mathbb{C}^*)^2$, with respect to the Hausdorff metric) subsequence from (V_{k_n}). This finishes the proof, since it guarantees that the genus of C also cannot be strictly greater than g ([1], Proposition 6.7). Note that C is a tropical curve of genus g and degree $\triangle$, passing through $\mathcal{P}$. Hence, $C = C_j$ for some $j \in \{1, ..., m\}$.

□

Lemma 5.9. *Let (V_k) be a sequence of J_k-holomorphic curves. If the sequence of amoebas $(\mathcal{A}_k)$ converges in the Hausdorff metric in the whole $\mathbb{R}^2$, then there is a subsequence (V_{k_n}) that converges in every compact subset of $(\mathbb{C}^*)^2$, with respect to the Hausdorff metric.*

Proof. We identify $(\mathbb{C}^*)^2$ with $(\mathbb{R}_{>0})^2 \times T^2$. Note that the metric induced on T^2 is not flat. We will show that we may work as it is flat, since we only require convergence in compact subsets of $(\mathbb{C}^*)^2$.

$\pi_1 \circ Log^{-1} : \mathbb{R}^2 \to (\mathbb{R}_{>0})^2$ is a well-defined continuous function. So by assumption the projection of V_k to the first coordinate converges (in every compact subset of $(\mathbb{R}_{>0})^2$, with respect to the Hausdorff metric).

The topology on T^2 induced from $(\mathbb{C}^*)^2$ under the projection $\pi_2 : (\mathbb{C}^*)^2 \to T^2$ agrees with the standard topology on T^2. So this guarantees that the sets $\pi_2(V_k)$ are closed in T^2 with respect to the Euclidean metric and hence, compact. Moreover, again since T^2 is compact, the Hausdorff metric defined on the compact subsets of T^2 gives us a compact metric space. So there is a subsequence (V_{k_n}) which converges under π_2

in T^2, with respect to the Hausdorff metric coming from the Euclidean metric on T^2.

We will construct a candidate V for the limit of (V_{k_n}) in $(\mathbb{C}^*)^2$. For every compact subset of $(\mathbb{C}^*)^2$, we can extract a convergent subsequence from (V_{k_n}). We will define V on compact subsets of $(\mathbb{C}^*)^2$ as the limit of the corresponding subsequences. Observe that the projection of V to both coordinates should agree with the limits of the projected sequences.

We will show that for every compact subset K of $(\mathbb{C}^*)^2$ we can keep the Hausdorff distance between the sets $V_{k_n} \cap K$ and $V \cap K$ arbitrarily small for large k_n. For the sake of simplicity, we will treat curves as subsets of $\mathbb{C}^*$ (i.e. we will argue in one coordinate). Fix $z_2 \in V \cap K$.

$$\begin{aligned}\inf_{z_1 \in V_{k_n}} |z_1 - z_2| &= \inf_{z_1 \in V_{k_n}} |r_1 e^{i\theta_1} - r_2 e^{i\theta_2}| \\ &\leq \inf_{z_1 \in V_{k_n}} \{|r_1 e^{i\theta_1} - r_2 e^{i\theta_1}| + r_2 |e^{i\theta_1} - e^{i\theta_2}|\} \\ &\leq \inf_{z_1 \in V_{k_n}} \{|r_1 - r_2| + r_2 |e^{i\theta_1} - e^{i\theta_2}|\}\end{aligned}$$

where $z_j = r_j e^{i\theta_j}$, $j = 1, 2$. Since r_2 is fixed, we can keep the right hand side of the inequality arbitrarily small for large k_n. But note that since $V \cap K$ is bounded, for sufficiently large k_n we can keep the right hand side small for all $z_2 \in V \cap K$. □

Proposition 5.8 finishes the proof of Lemma 5.4. To see this; consider the negation of Lemma 5.4, and then choose an appropriate sequence, which contradicts the assertion of Proposition 5.8.

5.3.2. Sketch of the proof of Lemma 5.5.

Lemma 5.4 tells us that for large $k > 0$, amoebas of J_k holomorphic curves (of genus g and degree $\triangle$, passing through $\mathcal{Q}$) are contained in $\mathcal{N}_\epsilon(C_j)$ for some $j \in \{1, ..., m\}$. We need to show that for each $j \in \{1, ..., m\}$, there are exactly $\text{mult}(C_j)$

distinct J_k holomorphic curves contained in $\mathcal{N}_\epsilon(C_j)$. [1] shows this in two steps. For the first step, we need the notion of complex tropical curves.

Let (V_k) be a convergent sequence (in every compact subset of $(\mathbb{C}^*)^2$, with respect to the Hausdorff metric) of J_k holomorphic curves, where $k \to \infty$. The limit is called a complex tropical curve, and denoted by V_∞. Complex tropical curves project onto tropical curves (in $\mathbb{R}^2$) under Log ([1], Proposition 6.1). So we may think of V_∞ both as the limit of J_k holomorphic curves, and as a tropical curve equipped with a phase. The latter is the approach of the first step.

Let V_∞ be a complex tropical curve and $C = \text{Log}(V_\infty)$. The degree $\triangle$ of V_∞ is defined as the degree of C, and the genus of V_∞ is defined as the minimum genus among the sequences of J_k holomorphic curves (of the same genus) that converge to V_∞.

First step (Proposition 6.18 in [1]): There are $\frac{\text{mult}(C_j)}{\mu_{\text{edge}}(C_j)}$ simple complex tropical curves V_∞ (see Definition 6.12 in [1]) of genus g and degree $\triangle$, passing through $\mathcal{Q}$ that project onto C_j; where $\mu_{\text{edge}}(C_j)$ is the product of the weights of the edges, of the (simply) parametrizing graph $h : \Gamma \to C$, that are disjoint from $h^{-1}(\mathcal{P})$ times the product of the squares of the weights of the edges that are not disjoint.

Second step (we will give a sketch of the proof in [1]): For large $k > 0$ there are exactly $\mu_{\text{edge}}(C_j)$ distinct J_k holomorphic curves (of genus g and degree $\triangle$, passing through $\mathcal{Q}$) contained in $\mathcal{N}_\epsilon(V_\infty)$, where V_∞ is a simple complex tropical curve (of genus g and degree $\triangle$, passing through $\mathcal{Q}$) that projects onto C_j. These two steps prove Lemma 5.5.

In the remainder of this section: V_∞ will denote a simple complex tropical curve that projects to C_j and V_k a J_k holomorphic curve, where both are of genus g and degree $\triangle$, passing through $\mathcal{Q}$.

Suppose that for large k, there exists V_k with $\text{Log}(V_k) \subset \mathcal{N}_\epsilon(C_j)$. Under this assumption, Proposition 8.11 in [1] shows that $V_k = H_k(\{f_k^\zeta = 0\})$, where

$$f_k^\zeta = \sum_{j \in \triangle \cap \mathbb{Z}^2} \arg(\zeta_j) t^{\log|\zeta_j|} z^j$$

for some $\zeta \in \mathcal{D} \subset (\mathbb{C}^*)^n$ and $n = \#(\triangle \cap \mathbb{Z}^2)$. Here $\mathcal{D}$ is a small neighborhood, which only depends on C_j and $\mathcal{Q}$ ([1], Proposition 8.10). Furthermore (still under the assumption), for any $\triangle' \in \mathrm{Subdiv}_{C_j}$ and for $\zeta \in \mathcal{D}$, changing ζ_j with $j \notin \triangle'$ has little effect on $H_k(\{f_k^\zeta = 0\}) \cap \mathrm{Log}^{-1}(U(\triangle'))$ for large k, where $U(\triangle')$ is a small neighborhood of the dual of $\triangle'$ in C_j ([1], Proposition 8.12). The latter claim is analogous to the approach used in Viro's patchworking method, and it provides a way to solve the problem locally in each $\triangle' \in \mathrm{Subdiv}_{C_j}$. (Note that the latter claim is not exactly the same as the assertion of Proposition 8.12 in [1]. We use Proposition 8.12 in the last paragraph of this section without the assumption.)

Now, [1] reverses the problem and counts J_k holomorphic curves of genus g and degree $\triangle$, passing through $\mathcal{Q}$, and whose coefficients come from $\mathcal{D}$. Using Proposition 8.14 in [1], a compatibility condition is given on the elements of $\mathcal{D}$, such that $\zeta \in \mathcal{D}$ is called compatible if and only if $H_k(\{f_k^\zeta = 0\})$ is of genus g, passing through $\mathcal{Q}$.

The compatibility condition applies locally, i.e., it is checked on each $\triangle' \in \mathrm{Subdiv}_{C_j}$ separately. Proposition 8.16, Proposition 8.17 and Proposition 8.21 in [1] count the number of compatible coefficients; respectively for an edge, a parallelogram and a triangle in Subdiv_{C_j}. But although the compatibility condition applies locally, it is not guaranteed that the coefficients found for each $\triangle' \in \mathrm{Subdiv}_{C_j}$ are still compatible when combined. Proposition 8.23 in [1] solves this problem using Proposition 8.12 in [1], and proves the second step when $\mu_{\mathrm{edge}}(C_j) = 1$. (Note that [1] does not argue using a fixed V_∞ until the end.) Finally, Lemma 8.24 in [1] finishes the proof of Lemma 5.5 in the general case.

6. COUNTING COMPLEX CURVES BY LATTICE PATHS

In this chapter we will give an algorithm to find the number $N_{top}(g, \triangle)$.

Definition 6.1 ([1], Definition 7.1)**.** *A path* $\gamma : [0, n] \to \mathbb{R}^2$*,* $n \in \mathbb{N}$*, is called a* lattice path *if* $\gamma|_{[j-1,j]}$*,* $j = 1, ..., n$*, is an affine-linear map and* $\gamma(j) \in \mathbb{Z}^2$*,* $j \in 0, ..., n$*.*

Choose a linear map $\lambda : \mathbb{R}^2 \to \mathbb{R}$, which is injective on $\mathbb{Z}^2$. Let p and q be points in $\triangle$ where λ takes its minimum and maximum, respectively. It is clear that p, $q \in \partial\triangle$. Moreover, since $\triangle$ is the convex hull of its boundary vertices and λ is linear, we ensure that p and q are vertices of $\partial\triangle$.

We will say that a lattice path γ is λ-increasing if $\lambda \circ \gamma$ is increasing. Let $n \in \mathbb{N}$ and $s = \#(\partial\triangle \cap \mathbb{Z}^2)$. We define two λ-increasing lattice paths, which divides $\partial\triangle$ into two pieces (see Figure 6.1);

$$\alpha^+ : [0, n_+] \to \partial\triangle \quad \text{and} \quad \alpha^- : [0, n_-] \to \partial\triangle,$$

where $\alpha^+(0) = \alpha^-(0) = p$, $\alpha^+(n_+) = \alpha^-(n_-) = q$ and $n_+ + n_- = s$. To differentiate the two, we assume α^+ makes a clockwise and α^- a counterclockwise rotation around $\partial\triangle$. Let $\gamma : [0, n] \to \triangle$ be a λ-increasing lattice path such that $\gamma(0) = p$ and $\gamma(n) = q$ (we do not claim the existence of γ for all $n \in \mathbb{N}$). For an example see Figure 6.1.

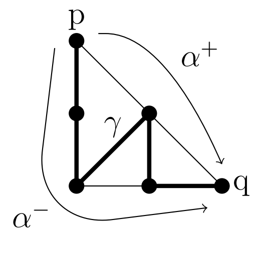

Figure 6.1. γ, where $\lambda(x, y) = x - \epsilon y$ for small $\epsilon > 0$, and $n = 5$.

We will first define the positive multiplicity $\mu_+(\gamma)$ of the curve γ inductively.

(i) If $\gamma = \alpha^+$, set $\mu_+(\gamma) = 1$. If $\gamma \neq \alpha^+$, take the smallest integer $k \in [1, n-1]$ where γ makes a left turn (i.e. γ makes a counter clockwise rotation at k).

(ii) If there is no such k, set $\mu_+(\gamma) = 0$. If k exists, define two λ-increasing lattice paths γ'_+, γ''_+ connecting p and q.

- $\gamma'_+ : [0, n-1] \to \triangle$; $\gamma'_+(j) = \gamma(j)$ for $j < k$ and $\gamma'_+(j) = \gamma(j+1)$ for $j > k$.
- $\gamma''_+ : [0, n] \to \triangle$; $\gamma''_+(j) = \gamma(j)$ for $j \neq k$ and $\gamma''_+(k) = \gamma(k-1)+\gamma(k+1)-\gamma(k)$. If $\gamma''_+(k) \notin \triangle$, set $\mu_+(\gamma''_+) = 0$. See Figure 6.2.

Set $\mu_+(\gamma) = 2 \cdot \text{Area}(T)\mu_+(\gamma'_+) + \mu_+(\gamma''_+)$ where T is the triangle with vertices $\gamma(k-1)$, $\gamma(k)$ and $\gamma(k+1)$.

(iii) Go back to (i), to compute $\mu_+(\gamma'_+)$ and $\mu_+(\gamma''_+)$.

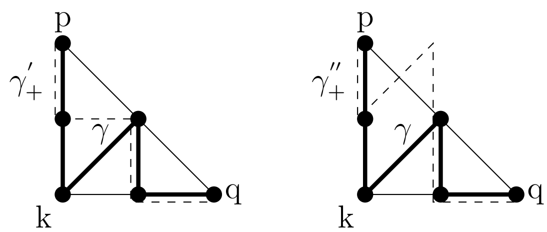

Figure 6.2. γ'_+ and γ''_+.

Similarly, by changing +'s with −'s and choosing k for a right turn (i.e. for a clockwise rotation) in (i), we compute the negative multiplicity $\mu_-(\gamma)$. Finally, we define the multiplicity $\mu(\gamma)$ of the curve γ to be the product $\mu_-(\gamma) \cdot \mu_+(\gamma)$ of positive and negative multiplicities of γ.

Note that we have only used λ to point out p and q, and to decide whether a path is λ-increasing or not. Multiplicity of a λ-increasing path does not depend on λ.

Now we are ready to state our second main theorem. Let λ, p and q be as above.

Theorem 6.2 ([1], Theorem 2)**.** *The number $N_{trop}(g, \triangle)$ is equal to the number* *(counted with multiplicities) of λ-increasing lattice paths $[0, s+g-1] \to \triangle$ connecting p and q.*

Furthermore, there exists a configuration $\mathcal{P} \subset \mathbb{R}^2$ of $s+g-1$ points in tropically general position such that each λ-increasing lattice path encodes a number of tropical curves of genus g and degree $\triangle$ passing through $\mathcal{P}$ of total multiplicity $\mu(\gamma)$. These curves are distinct for distinct paths.

At this point, we do not know whether the number we obtain from the algorithm depends on λ or not. The proof of Theorem 6.2 will also prove independence.

Example 6.3. *We will calculate $N_{trop}(0, \triangle_2) = 1$, using the algorithm.*

Let $\lambda(x, y) = x - \epsilon y$ for sufficiently small $\epsilon > 0$. Then $p = (0, 2)$ and $q = (2, 0)$. Let us first find λ-increasing lattice paths $\gamma : [0, 5] \to \triangle$ connecting $(0, 2)$ and $(2, 0)$. Since ϵ is positive and very small, the possible moves for a λ-increasing lattice path are;

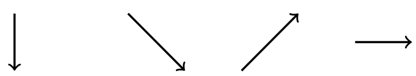

Figure 6.3. Possible moves.

The only possible path γ of length 5 subject to the moves above is

We heve seen in our earlier example that $\mu_+(\gamma_+^{''})$ is zero. So $\mu_+(\gamma) = 2 \cdot Area(T)\mu_+(\gamma_+^{'})$, where $Area(T) = \frac{1}{2}$. Let us calculate $\mu_+(\gamma_+^{'})$.

Since $(\gamma_+^{'})^{''}(k) \notin \triangle$, the number $\mu_+((\gamma_+^{'})^{''})$ is zero (See Figure 6.5). So we have $\mu_+(\gamma_+^{'}) = 2 \cdot Area(T^{'})\mu_+((\gamma_+^{'})^{'})$, where $Area(T^{'}) = \frac{1}{2}$. Similarly we obtain the equality $\mu_+((\gamma_+^{'})^{'}) = \mu_+(((\gamma_+^{'})^{'})^{'})$, where $((\gamma_+^{'})^{'})^{'} = \alpha^+$. So $\mu_+(\gamma) = 1$.

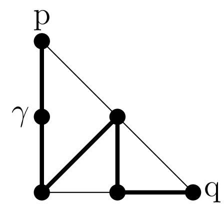

Figure 6.4. Lattice path subject to $\lambda(x, y) = x - \epsilon y$, for small $\epsilon > 0$.

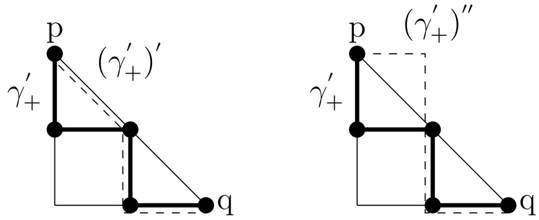

Figure 6.5. $(\gamma_+^{'})^{'}$ and $(\gamma_+^{'})^{''}$.

Similarly $\mu_-(\gamma) = 1$ *since* $\gamma_-^{'} = \alpha^-$ *and the corresponding triangle has area* $\frac{1}{2}$, *and* $\gamma_-^{''}(k) \notin \triangle$. *Hence* $\mu(\gamma) = 1$.

6.1. Proof of Theorem 6.2

Theorem 5.1 tells us that $N_{trop}(g, \triangle)$ does not depend on the choice of $\mathcal{P} = \{p_1, p_2, ..., p_{s+g-1}\}$ as long as $\mathcal{P}$ is in tropically general position. We will explicity show that the algorithm counts the curves passing through some special configuration $\mathcal{P}$, which depends on λ.

Let a and b be the coefficients of λ, i.e., the entries of the matrix representing λ. Since λ is injective on $\mathbb{Z}^2$, $\frac{a}{b}$ is irrational. So we can regard λ as a projection onto an affine-line L with the irrational slope $\frac{a}{b}$.

Choose a configuration $\mathcal{P}$ of $s + g - 1$ points on L so that the distance from p_k to p_{k-1} is much larger than the distance from p_{k-1} to p_{k-2}. We can choose L and the p_k's so that $\mathcal{P}$ is in tropically general position ([1], Page 374). Let C be a tropical curve of genus g and degree $\triangle$, passing through $\mathcal{P}$.

Lemma 6.4 ([1], Lemma 8.26)**.** *C and L intersect exactly at $\mathcal{P}$.*

Proof. Since C is of the right genus and $\mathcal{P}$ is in tropically general position, C is simple. Let K be a component of $\Gamma \setminus h^{-1}(\mathcal{P})$ passing through L under h. K is a tree with one end at infinity ([1], Lemma 4.20). Since the slope of L is irrational, K yields a bounded tree on one side of L. (Otherwise a part of C with positive length could be on L.) But a bounded tree cannot satisfy the balancing condition. □

The pair $(C, \mathcal{P})$ is called a marked tropical curve, and the subdivision of $\triangle$ dual to C together with marking of edges $\mathcal{F}$ is called the combinatorial type of the pair $(C, \mathcal{P})$. See Figure 6.6.

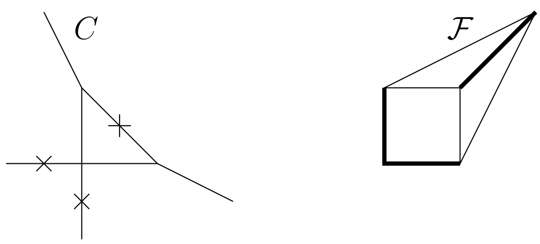

Figure 6.6. $(C, \mathcal{P})$ and its combinatorial type.

As in Figure 6.6 $\mathcal{F}$ can be disconnected. But our choice of $\mathcal{P}$ ensures that $\mathcal{F}$ is a λ-increasing lattice path that connects the vertices p and q (respectively the minimum and the maximum of λ on $\triangle$) as in Theorem 6.2 ([1], Corollary 8.27). Moreover, in each combinatorial type of a marked tropical curve $(C, \mathcal{P})$ there is either one or no tropical curve of genus g with s ends ([1], Lemma 4.22). We will see that the algorithm counts (with multiplicity) exactly the realized combinatorial types for $(C, \mathcal{P})$, g and s.

In a realized combinatorial type, $\mathcal{F}$ must be a λ-increasing lattice path. So we start with drawing a λ-increasing lattice path γ in $\triangle$. Recall that to compute $\mu_+(\gamma)$, we first consider the smallest integer k where γ makes a left turn. A turn would produce a vertex v in the dual picture. But v is required to be the intersection of the edges passing through $\mathcal{P}$ since otherwise we can create an infinite family of tropical curves having the same combinatorial type and passing through $\mathcal{P}$. But then a tropical curve having this combinatorial type cannot have both genus g and s ends ([1], Lemma 4.22).

Since we are interested in simple tropical curves, v should either be 3-valent or 4-valent. $\gamma_+^{'}$ corresponds to a 3-valent vertex and $\gamma_+^{''}$ to a 4-valent one. Remember that the multiplicity of a tropical curve C is the product of two times the areas of triangles in $\mathrm{Subdiv_C}$. So we include a $2 \cdot \mathrm{Area}(T)$ factor in the second step of the algorithm. As we proceed inductively, we count all possible subdivisions (above γ) of $\triangle$ with multiplicity. Similarly we compute $\mu_-(\gamma)$. Note that possible subdivisions below and above γ are independent. Hence to count them all, we multiply $\mu_+(\gamma)$ and $\mu_-(\gamma)$.

Recall that α^+ and α^- are all the non-zero end paths in the algorithm, which guarantees that the corresponding simple tropical curve C has s ends. (Remember that a tropical curve C is simple if and only if $\mathrm{Subdiv_C}$ consists of triangles and parallelograms.) By construction C can be parametrized by a graph Γ where each component of $\Gamma \setminus h^{-1}(\mathcal{P})$ has one end at infinity, then C has genus g ([1], Lemma 4.20). Note that the only non-trivial dead end of the algorithm occurs when γ makes no turn and not equal to α^+ or α^-. But a tropical curve corresponding to that combinatorial type has either less ends than s, or greater genus than g (former is easy to see, for the latter we use again Lemma 4.22 in [1]).

We have showed that the algorithm counts with multiplicity all the possible combinatorial types for $(C, \mathcal{P})$, g and s. It remains to show that we do not count more, i.e. each counted combinatorial type is realizable.

Let $\gamma : [0, s+g-1] \to \triangle$ be a λ-increasing lattice path that connects p and q. For each k, draw a line passing through p_k in the direction perpendicular to $\gamma([k-1, k])$. Take the first intersecting consecutive lines, say $\mathrm{Line}(p_{j-1})$ and $\mathrm{Line}(p_j)$. If the intersection happens to be above (respectively below) L, then it corresponds to the first left turn (respectively right turn) in the algorithm. Now draw a second line $L^{'}$ parallel to L such that the strip in between contains the intersection point $p_{j-1,j}$, but does not contain any other intersection point of the lines Line(p_k) (see Figure 6.7). Such an $L^{'}$ exists, since $\mathcal{P}$ is chosen so that the distance from p_k to p_{k-1} is much larger than the distance from p_{k-1} to p_{k-2}. (It does not matter how large the distance between the points is, one can find a case in which such an $L^{'}$ does not exist. But the key point is that there are finitely many choices for the slopes of the lines. We may find these choices by considering all the possible subdivisions of $\triangle$. After being given all the choices for the slopes of the lines, now we can choose p_k's so that existence of $L^{'}$ is ensured.)

If we want $p_{j-1,j}$ to be a 3-valent vertex (see Figure 6.7), we take $k-1$ points on $L^{'}$ in the following way: $p^{'}_k = \text{Line}(p_k) \cap L^{'}$ for $k < j-1$, $p^{'}_{k-1} = \text{Line}(p_k) \cap L^{'}$ for $k > j$, and $p^{'}_{j-1}$ is the intersection point of the line emanating from $p_{j-1,j}$ with $L^{'}$. Similarly, if we want $p_{j-1,j}$ to be a 4-valent vertex, we take k points on $L^{'}$ in the following way: $p^{'}_k = \text{Line}(p_k) \cap L^{'}$ for $k \neq j, j-1$, $p^{'}_{j-1} = \text{Line}(p_j) \cap L^{'}$, and $p^{'}_j = \text{Line}(p_{j-1}) \cap L^{'}$.

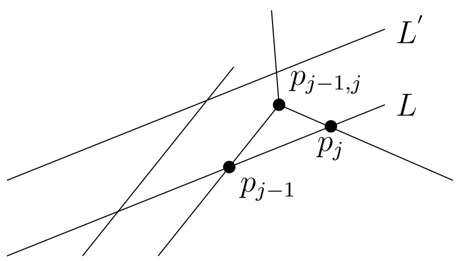

Figure 6.7. Choosing $p_{j-1,j}$ as a 3-valent vertex.

Basically we are applying the dual (recall the duality between a tropical curve C and the corresponding Subdiv$_C$) of the algorithm (e.g. for a left turn in the algorithm we create a vertex above L). We proceed inductively until all the lines crossing $L^{'}$ are ends at infinity. So we have showed that all the counted combinatorial types are realizable.

Example 6.5. *$N(0, \triangle_3)$ and $N(-1, \triangle_3)$ are the first two non-trivial numbers. $N(0, \triangle_3)$ is calculated in [1]. We will calculate $N(-1, \triangle_3)$.*

We choose λ as in Example 6.3. We need to find paths of length 7 connecting $(3, 0)$ and $(0, 3)$, subject to moves in Figure 6.3. Figure 6.8 shows the first possible moves. We will not find paths starting with b or c, since any path starting with these moves has zero multiplicity. Figure 6.9 (on the next page) shows all the possible paths starting with move a, with multiplicity. So we have, $N(-1, \triangle_3) = 21$.

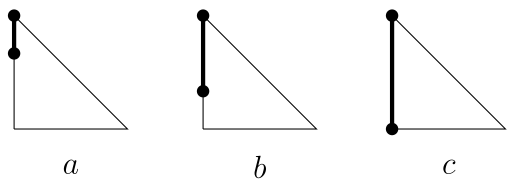

Figure 6.8. First possible moves.

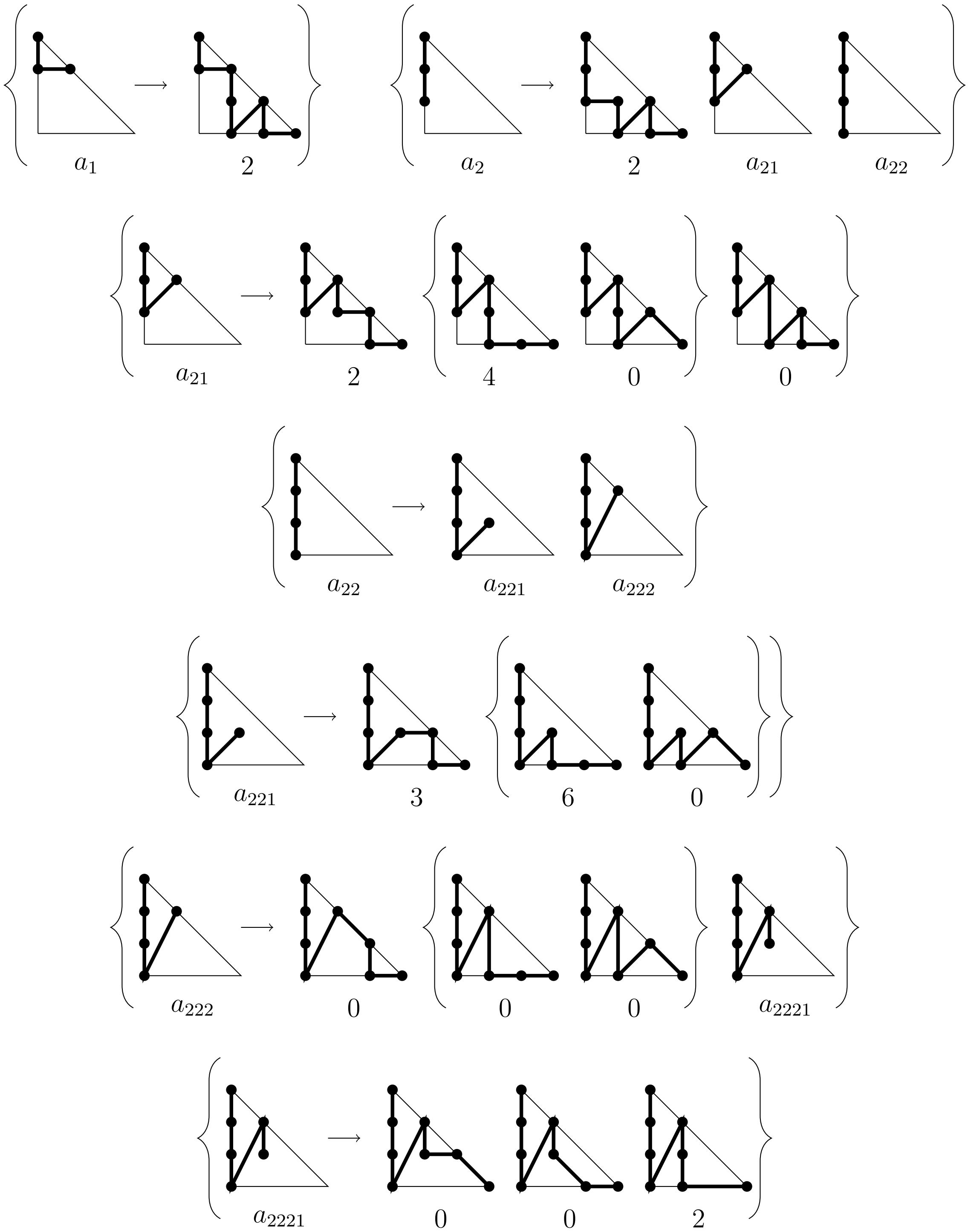

Figure 6.9. Calculating $N(-1, \triangle_3)$.

7. CONCLUSION

In Chapters 2 and 3, we defined tropical curves by using two different approaches. In the former we parametrized tropical curves in $\mathbb{R}^2$ with abstract graphs, and in the latter we saw them as purely algebraic objects with respect to the tropical algebra.

In Chapter 4, we defined the tropical enumerative problem analogous to the classical enumerative problem in $(\mathbb{C}^*)^2$. The aim was to show the correspondence between the two problems. We partially succeed in this in Chapter 5. Finally in Chapter 6, we gave a combinatorial solution to the tropical enumerative problem, which also solves the classical problem by the correspondence theorem in tropical geometry.

Throughout this work, we have followed " Enumerative tropical algebraic geometry in $\mathbb{R}^2$" by Grigory Mikhalkin. We do not claim originality of any idea discussed here. We apologize for any possible mistake made in referencing.

REFERENCES

1. Mikhalkin, G., "Enumerative Tropical Algebraic Geometry in $\mathbb{R}^2$", *Journal of the American Mathematical Society*, Vol. 18, No. 2, pp. 313–377, 2005.

2. Brugallé, E. and K. Shaw, "A Bit of Tropical Geometry", *American Mathematical Monthly*, Vol. 121, No. 7, pp. 563–589, 2014.

3. Gathmann, A., "Tropical Algebraic Geometry", *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Vol. 108, No. 1, pp. 3–32, 2006.

4. Maclagan, D. and B. Sturmfels, *Introduction to Tropical Geometry*, Vol. 161 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, 2015.

5. Itenberg, I., G. Mikhalkin and E. Shustin, *Tropical Algebraic Geometry*, Vol. 35 of *Oberwolfach Seminars*, Birkhäuser Verlag, Basel, second edn., 2009.

6. Mikhalkin, G., "Decomposition into Pairs-of-Pants for Complex Algebraic Hypersurfaces", *Topology. An International Journal of Mathematics*, Vol. 43, No. 5, pp. 1035–1065, 2004.

7. Viro, O. Y., "Real Plane Algebraic Curves: Constructions with Controlled Topology", *Algebra i Analiz*, Vol. 1, No. 5, pp. 1–73, 1989.

8. Risler, J.-J., "Construction D'Hypersurfaces Réelles (D'Après Viro)", *Astérisque*, Vol. 1992/93, No. 216, pp. 69–86, 1993.

9. Kontsevich, M. and Y. Manin, "Gromov-Witten Classes, Quantum Cohomology, and Enumerative Geometry", *Mirror symmetry II*, Vol. 1, pp. 607–653, American Mathematical Society, Providence, RI, 1997.

10. Caporaso, L. and J. Harris, “Counting Plane Curves of Any Genus”, *Inventiones Mathematicae*, Vol. 131, No. 2, pp. 345–392, 1998.

11. Passare, M. and H. Rullgård, “Amoebas, Monge-Ampère Measures, and Triangulations of the Newton Polytope”, *Duke Mathematical Journal*, Vol. 121, No. 3, pp. 481–507, 2004.

12. Brugallé, E. and H. Markwig, “Deformation of Tropical Hirzebruch Surfaces and Enumerative Geometry”, 2013, http://arxiv.org/pdf/1303.1340v4.pdf, [June 2015].