Conjugation Invariants in Word Hopf algebras

Neşet Deniz Turgay

Submitted to Swansea University in fulfilment of the requirements for the Degree of Doctor of Philosophy

Swansea University

2012

Contents

1 Introduction **1**
- 1.1 A short history of Hopf algebras 1
- 1.2 What makes the Leibniz-Hopf algebra interesting? 2
- 1.3 The mod p dual Steenrod algebra and commutative ring spectrum 3
- 1.4 How does this thesis related to the topological journey above ? 4
- 1.5 Outline 5

2 Preliminaries **7**
- 2.1 Algebraic aspects 7
  - 2.1.1 Algebra 7
  - 2.1.2 Graded modules, and graded algebras 8
  - 2.1.3 Coalgebra 10
- 2.2 Bialgebra, convolution and Hopf algebra 12
  - 2.2.1 Bialgebra 12
  - 2.2.2 Convolution 13
  - 2.2.3 Hopf Algebra 18
- 2.3 Words 18
  - 2.3.1 Properties of words 19
- 2.4 Leibniz-Hopf Algebra 24
- 2.5 Mod p Leibniz-Hopf algebra 29
- 2.6 Dual Leibniz-Hopf Algebra 30
- 2.7 Mod p dual Leibniz-Hopf algebra 35

3 Conjugation Invariants in the mod 2 Dual Leibniz-Hopf Algebra **36**
- 3.1 Linear independence 37
- 3.2 Spanning set for $(\chi_{\mathcal{F}_2^*} - 1)$ 41

4 Conjugation Invariants in the mod p Dual Leibniz-Hopf Algebra **53**
4.1 Linear Independence 53
4.2 Spanning set for $\mathrm{Im}(\chi_{\mathcal{F}_p^*} + 1)$ 55

5 Conjugation Invariants in the Dual Leibniz-Hopf Algebra **63**
5.1 Linear Independence 63
5.2 Spanning set for $\mathrm{Im}(\chi_{\mathcal{F}^*} + 1)$ 64

6 Conjugation Invariants in the Leibniz-Hopf Algebra **68**
6.1 Linear independence 69
6.2 Spanning set for $\mathrm{Im}(\chi_{\mathcal{F}} + 1)$ 72

7 Conjugation Invariants in the mod p Leibniz-Hopf Algebra **85**
7.1 Linear Independence 86
7.2 Spanning set for $\mathrm{Im}(\chi_{\mathcal{F}_p} + 1)$ 86

8 Correspondence between the matrices of $\chi_{\mathcal{F}_2^*} - 1$ and $\chi_{\mathcal{F}_2} - 1$ **88**

9 Conjugation Invariants in the mod 2 Leibniz-Hopf Algebra **97**

A Calculations in the dual Leibniz-Hopf Algebra **102**
A.1 Table list in degree 4 102
A.2 Table list in degree 5 104

B Calculations in the mod 2 dual Leibniz-Hopf Algebra **106**
B.1 Table list in degree 4 106
B.2 Table list in degree 5 107
B.3 Table list in degree 5 108

C Calculations in the Leibniz-Hopf Algebra **109**
C.1 Table list in degree 4 109
C.2 Table list in degree 5 111

D Calculations in the mod 2 Leibniz-Hopf Algebra **114**
D.1 Table list in degree 4 114
D.2 Table list in degree 5 115

Bibliography **117**

Abstract

We calculate a basis for the free submodule formed by the invariants in the Leibniz-Hopf algebra under the Hopf algebra conjugation operation. We also give bases for the submodules of conjugation invariants in the dual Leibniz-Hopf algebra and in the mod p reductions of both the Leibniz-Hopf algebra and its dual.

Acknowledgements

I would like to thank my supervisor Martin Crossley for encouraging me to use my imagination and creativity both prior to and during the completion of this work. I enjoyed doing mathematics with his beautiful mathematical thinking guidance.

Secondly I would like to thank my co-supervisor Thomas Brzezinski for his support and motivation. I would also like to thank past and present staff and at Swansea University Mathematics Department. My PhD colleagues must be thanked for making my time here enjoyable and worthwhile in our office.

I most graciously thank the College of Science Mathematics Department for kindly providing the Departmental scholarship which facilitated my post-graduate studies.

I would like to thank Nige Ray for inviting me to present my work in Manchester University. I also would like to thank Reg Wood, Grant Walker and Nige Ray for their useful comments on my work for further directions.

I would also like to thank Edinburgh and Cambridge Universities for giving me chance to present my work in the BTM2011 and BTM2012.

The final months of my thesis studies were made easier by my stay in St. Thomas where I was given emphatic support by friends and neighbours.

Finally I wish to thank my family; Faruk Turgay, Aynur Turgay and Gunes Turgay, for every kind of support imaginable.

Chapter 1

Introduction

In this chapter, we will first give the historical development of Hopf algebras and its link between with the other areas of mathematics. Secondly, we will give the main motivation of this thesis which is related to the Steenrod algebra and ring spectra. Finally we give a brief description for each chapter of this thesis.

1.1 A short history of Hopf algebras

In algebraic topology Hopf algebras are named by the work of Heinz Hopf in the 1940's. Armand Borel coined the expression *Hopf algebra* in 1953, honoring the foundational work of Heinz Hopf [2]. Pierre Cartier gave the first formal definition of Hopf algebra in connection with cocommutative bialgebra with his work *hyper-algebra* in 1956 [2].

John Milnor showed the Steenrod algebra is an example of Hopf algebra in the 1960's [26]. In 1965, J. Milnor and J.Moore gave the definition of Hopf algebra in the sense of *graded bialgebra* [27]. In 1966, Bertram Kostant introduced Hopf algebra in the modern sense, i.e., expressing antipode [24].

After that Hopf algebras have started being applied into different fields. In the 1970's Giancarlo Rota applied Hopf algebras into combinatorics. In 1986, quantum groups are introduced by Drinfeld [13], which give rises the applications of Hopf algebras to physics and invariant theory for knots and links.

1.2 What makes the Leibniz-Hopf algebra interesting?

The Hopf algebra *Symm* of symmetric functions is central to many other areas of mathematics such as[20]:

$$\begin{aligned} Symm &\simeq R_{rat}(GL_\infty), \text{ the ring of rational representations of the infinite linear group} \\ &\simeq H^*(BU), \text{ the cohomology of classfying space } BU \\ &\simeq H_*(BU), \text{ the homology of classfying space } BU \\ &\simeq R(W), \text{ the representative ring of the functor of the (big) Witt vectors} \\ &\simeq U(\wedge), \text{the universal } \lambda\text{-ring on one generator.} \end{aligned}$$

There are two important generalizations of the Hopf algebra of symmetric functions which are the Hopf algebra of noncommutative symmetric functions and its graded dual the Hopf algebra of quasisymmetric functions. The Leibniz-Hopf algebra has been studied as the 'ring of noncommutative symmetric functions' [18, 19, 22, 15], and is known to be isomorphic to the Solomon Descent algebra [30] (with the 'inner' product [16]). A topological model for this Hopf algebra is given by interpreting it as the homology of the loop space of the suspension of the infinite complex projective space, $H_*(\Omega\Sigma\mathbf{C}P^\infty)$. Moreover, the antipode in $H_*(\Omega\Sigma\mathbf{C}P^\infty)$ arises from the time-inversion of loops. As antipodes are unique for Hopf algebras, this gives a geometric interpretation for the antipode in the Leibniz Hopf algebra.[4, Section 1]

The graded dual of the Leibniz-Hopf algebra, is the ring of quasi-symmetric functions with the outer coproduct [25], which has been studied in [6, 14, 18, 17, 19, 21, 22, 25]. It is also known to topologists as the cohomology of $\Omega\Sigma\mathbf{C}P^\infty$[4, Theorem 1.1]. We now need to be more careful. This is because: we know the cohomology of a space is always graded commutative. And, by Remark 2.1.7 the reader can conclude that the graded dual of the Leibniz-Hopf algebra is commutative in the strict sense rather than in the graded sense. On the other hand, the degree n part of the graded dual of the Leibniz-Hopf algebra is isomorphic to the degree $2n$ part of the cohomology of $\Omega\Sigma\mathbf{C}P^\infty$[4, Remark 1.2].

The graded dual of the Leibniz-Hopf algebra was also the subject of the Ditters conjecture [5, 18, 22], making it relevant to a wide area of combinatorics, algebra and topology. Quasi-symmetric functions are introduced to deal with the combinatorics of P-partitions and the counting of permutations with given descent sets.[18]. Moreover, a first link between Hopf algebras and quasi-symmetric functions was found by Ehrenborg[14].

After given the importance of Leibniz-Hopf algebra and its dual, let us give more motivation related to algebraic topology.

1.3 The mod p dual Steenrod algebra and commutative ring spectrum

The reader is referred to [1, Lecture 3] fore more information about the topics covered in here. We will now give a short motivation regarding how the conjugation in the dual Steenrod algebra is related to a commutative ring spectrum.

A ring spectrum[3] is a spectrum E equipped with a homotopy-associative multiplication map

$$\mu : E \wedge E \to E,$$

($\wedge$ is smash product), which has a two-sided homotopy unit. E is said to be commutative if the following diagram:

$$\begin{array}{ccc} E \wedge E & \xrightarrow{\tau} & E \wedge E \\ & \searrow^{\mu} & \downarrow^{\mu} \\ & & E \end{array} \tag{1.1}$$

is homotopy-commutative, where τ is the usual switch map.

On the other hand, the generalised homology groups of a spectrum X with coefficients in E are given by

$$E_\star(X) = \pi_\star(E \wedge X),$$

where π is the stable homotopy group. Now we are ready to give the link between the ring spectra and the Steenrod algebra. For to do it we choose E as Eilenberg-Maclane spectrum, $K(Z_p)$, then

$$E_\star(E) = \pi_\star(E \wedge E),$$

the homology of E with coefficients in E is the the mod p dual Steenrod algebra, $\mathcal{A}_p^*$, and the conjugation map on $\pi_\star(E \wedge E)$ is precisely the map induced on $\pi * (E \wedge E)$:

$$\pi_*(E \wedge E) \xrightarrow{\tau^*} \pi_*(E \wedge E),$$

by switching the factors in the smash product. Thus, conjugation in the $\mathcal{A}_p^*$ is relevant to study in commutativity of ring spectra.

Moreover, we take the homotopy of a smash product of n copies of E, $\pi*(E\wedge\cdots\wedge E)$, and $\pi_*(E^{\wedge_n}) = E_\star(E^{\wedge_{n-1}})$, the E cohomology of an $n-1$-fold product of copies of E.[10, Section 1]

Conjugation invariants and Spectral sequences

The gamma homology theory which is introduced by Sarah Whitehouse and Alan Robinson[29] developed to study higher homotopy commutativity of ring spectra.

Let Σ_n denotes the symmetric group S_n on a finite set of n symbols. Expressions like $H^m(\Sigma_n;\pi_*(E^{\wedge_n}))$ arise in spectral sequences for gamma cohomology of an E_∞-ring spectrum E,[9, Section 1]. For E suitably nice, this is $H^m(\Sigma_n;(E_*E)^{\otimes(n-1)})$, the Σ_n action is described in [33, Section 1]

By the section 1.3 it may seen for $n = 2$ and $E = K(Z_2)$ we have: $H^*(\Sigma_2;\mathcal{A}_2^*)$ and Σ_2 acts by the conjugation in $E_*E = \mathcal{A}_2^*$. And to understand whole cohomology $H^*(\Sigma_2;\mathcal{A}_2^*)$, one can use the conjugation invariants in $\mathcal{A}_2^*$. This is because Σ_2 invariants form $H^0(\Sigma_2;\mathcal{A}_2^*)$, from which we can conclude that the conjugation invariants on $\mathcal{A}_2^*$ is relevant to study in spectral sequences.

1.4 How does this thesis related to the topological journey above ?

We now first give some more details regarding the Leibniz-Hopf algebra which we will full explain in the chapter 1 of this thesis. After that we will shortly explain how the conjugation invariants in Leibniz-Hopf algebra and its dual is related to the Steenrod algebra and its dual.

The"Leibniz-Hopf algebra" is the free associative **Z**-algebra $\mathcal{F}$ on one generator S^n in each positive degree. Let $\mathcal{F}_2$ be the mod-2 reduction of this Hopf algebra. Now, let S^n represent Steenrod operations, then $\mathcal{A}_2$, is defined as a quotient of $\mathcal{F}_2$ by the Adem Relations[31]:

$$S^aS^b = \sum_{j=0}^{[\frac{a}{2}]} \begin{pmatrix} b-1-j \\ a-2j \end{pmatrix} S^{a+b-j}S^j, \quad 0 < a < 2b.$$

Hence, we have a projection:

$$\pi : \mathcal{F}_2 \to \mathcal{A}_2,$$

then by dualizing we have an injection:

$$\pi^\star : \mathcal{A}_2^* \hookrightarrow \mathcal{F}_2^*.$$

So information about conjugation invariants in the mod 2 dual Leibniz-Hopf algebra, $\mathcal{F}_2^*$, should lead to corresponding information in the mod 2 dual Steenrod algebra, ${A_2}^*$. In more details, the intersection of $\mathrm{Im}(\pi^\star)$ with the conjugation invariants in $\mathcal{F}_2^*$ may give the related information for the conjugation invariants in $\mathcal{A}_2^*$.

Note that the same problem for the $\mathcal{A}_p^*$ is satisfactorily solved in[10, Section 1]. The results in Chapter 4 in this thesis may be thought as a different solution approach to this problem.

One may also think to use the results on Chapter 9 of this thesis for the conjugation invariants in $\mathcal{A}_2$. Unfortunately, after some calculations, the reader will see there is not a promising relation between conjugation invariants in the $\mathcal{F}_2$ and $\mathcal{A}_2$.

Now after giving the motivation let us briefly describe the content of each chapter in the following:

1.5 Outline

This thesis consists of eight chapters except for the introduction.

Chapter 2 begins by introducing necessary backgrounds and new terminologies: Palindromes and non palindromes which are explained in details. In this chapter, an alternative proof is given for the conjugation formula in the Leibniz-Hopf algebra and in the dual Leibniz-Hopf algebra.

Chapter 3 is inspired by [9], and [7] is based on this chapter. It explains an approach for the invariant problem under the conjugation in the mod 2 dual Leibniz-Hopf algebra. The ring of conjugation invariants in the mod 2 dual Steenrod algebra arises when one considers commutativity of ring spectra [1].

Motivated by this, I have studied the fixed points in the mod 2 dual Leibniz-Hopf algebra under this conjugation action. It is shown that, like in the dual Steenrod algebra, these invariants are "approximately" half of the whole algebra, although we are able to give a much more precise statement than was possible for the Steenrod algebra.

[8] is based on the rest of the chapters.

In Chapter 4, I am interested in the conjugation problem for any odd prime number in the mod p dual case. It is shown in which way the results differs from mod 2 dual case.

Chapter 5 focuses on the conjugation invariant problem in the integral case, and gives details how to deal with that problem without a vector space structure, but with an adaptation of the arguments in mod p case. In particular, we conclude that the results in the integral case coincide with the the mod p dual case.

Chapter 6 we turn attention to the Leibniz-Hopf algebra. A basis is calculated for the free submodule formed by the conjugation invariants in this Hopf algebra.

Chapter 7 deals with the fixed point problem under conjugation in mod p Leibniz-Hopf algebra. The mod p Steenrod algebra naturally occurs as a quotient of the mod p Leibniz-Hopf algebra [31]. Motivated by this it is shown that the conjugation invariants coincides with the invariants in the integral case.

Chapter 8 exploits the duality between the mod 2 dual Leibniz-Hopf algebra and the mod 2 Leibniz-Hopf algebra to get information about conjugation invariants in the latter case from the former.

Chapter 9 then builds on this to solve the conjugation invariant problem in the mod 2 reduction of the Leibniz-Hopf algebra.

Chapter 2

Preliminaries

2.1 Algebraic aspects

See [11], [32], and [27] for further details on topics in this section. In this section we give definitions of algebras; coalgebras by commutative diagrams. These definitions lead to definition of graded algebras.

2.1.1 Algebra

We will now define the simplest structure of an algebra over R. As a convention, unless otherwise stated R will denote a commutative ring with unit.

Definition 2.1.1. An R-algebra is an R-module A with an R-module morphism $\varphi : A \otimes_R A \to A$ called multiplication.

Remark 2.1.2. *We write $A \otimes A$ when we mean $A \otimes_R A$ in this chapter.*

For a given R-algebra A:

i. The algebra is said to be associative if the diagram

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\varphi \otimes 1_A} & A \otimes A \\ {\scriptstyle 1_A \otimes \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ A \otimes A & \xrightarrow{\varphi} & A \end{array} \tag{2.1}$$

is commutative, where 1_A denotes the identity morphism on A.

ii. The algebra is said to be commutative if the diagram

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\tau} & A \otimes A \\ & {\scriptstyle \varphi}\searrow & \downarrow{\scriptstyle \varphi} \\ & & A \end{array} \tag{2.2}$$

is commutative, where $\tau : A \otimes A \to A \otimes A$, is the twisting map, i.e, $\tau(a \otimes b) = (b \otimes a)$, for $a, b \in A$.

iii. The algebra is said to be unital if there exists a morphism $\mu : R \to A$ which should satisfy:

$$\begin{array}{ccccc} R \otimes A & \xrightarrow{\mu \otimes 1_A} & A \otimes A & \xleftarrow{1_A \otimes \mu} & A \otimes R \\ & \searrow^{\approx} & \downarrow \varphi & \swarrow^{\approx} & \\ & & A & & \end{array} \tag{2.3}$$

Remark 2.1.3. *We write (A, φ, μ) or simply A when we mean an R-algebra A which is associative and unital. Similarly, we write 1_A when we mean the identity morphism on A.*

Let D, E be two algebras, a homomorphism $f : D \to E$ of algebras, is an R-module homomorphism such that the diagrams

$$\begin{array}{ccc} D \otimes D & \xrightarrow{\varphi_D} & D \\ \downarrow f \otimes f & & \downarrow f \\ E \otimes E & \xrightarrow{\varphi_E} & E, \end{array} \tag{2.4}$$

$$\begin{array}{ccc} R & \xrightarrow{\mu_D} & D \\ \downarrow 1_R & & \downarrow f \\ R & \xrightarrow{\mu_E} & E, \end{array} \tag{2.5}$$

are commutative. Alternatively we say, a homomorphism $f : D \to E$ of algebras, is a R-module homomorphism, which commutes with multiplication and preserves unit.

2.1.2 Graded modules, and graded algebras

Let $A = (A_i)$ be a sequence of R-modules where $i \geq 0$, then A is called a *graded* R-module. Components of A, A_i, are then said to be in *degree* or *dimension* of i. Let F and G be graded R-modules. By a graded R-module homomorphism $h : F \to G$, we mean a sequence $h_i : F_i \to G_i$ of R-module homomorphisms. For given graded R-modules F and G, we define a graded module $F \otimes G$ by

$$F \otimes G = \left((F \otimes G)_j\right),$$

where

$$(F \otimes G)_j = \sum_{i=0}^{j} F_i \otimes G_{j-i}.$$

Definition 2.1.4. A graded R-module A is *finite type* if every component of A, i.e., A_n is finitely generated.

Definition 2.1.5. Let A be a graded R-module with multiplication

$$\varphi : A \otimes A \to A.$$

A is called a *graded algebra* if for all $p, q \geq 0$

$$\varphi(A_p \otimes A_q) \subset A_{p+q}.$$

Remark 2.1.6. *If A has the unit, then the unit is of degree zero.*

Let A be a graded R-algebra, then similarly the associativity, and unit property can be defined by using the diagrams (2.1), and (2.3).

Remark 2.1.7. *Some authors define "commutative" in the graded case so that an algebra A is commutative if, and only if, $ab = (-1)^{|a||b|}ba$ for all $a, b \in A$. We do not follow this convention; througout this thesis the word "commutative" will mean strict commutativity not graded commutativity.*

Definition 2.1.8. A unital graded algebra A over R is connected if $\mu : R \to A_0$ is an isomorphism.

Given two graded R-modules F and G we defined $F \otimes G$ to be graded module. We will see now how $F \otimes G$ becomes an algebra over R.

Remark 2.1.9. *When we have more than one algebra, to make it more clear, we write (A, φ_A, μ_A) using subscripts to emphasis which product belongs to which algebra.*

Definition 2.1.10. If we have two R graded algebra (F, φ_F, μ_F) and(G, φ_G, μ_G), then $F \otimes G$ is the algebra over R with multiplication the composition given by,

$$F \otimes G \otimes F \otimes G \xrightarrow{1_F \otimes \tau \otimes 1_G} F \otimes F \otimes G \otimes G \xrightarrow{\varphi_F \otimes \varphi_G} F \otimes G, \tag{2.6}$$

where τ is the twisting morphism and unit

$$R = R \otimes R \xrightarrow{\mu_F \otimes \mu_G} F \otimes G. \tag{2.7}$$

Remark 2.1.11. *By the composition (2.6) and diagram (2.7) we have :*

$$\varphi_{F\otimes G} = (\varphi_F \otimes \varphi_G) \circ (1_F \otimes \tau \otimes 1_G), \quad \mu_{F\otimes G} = \mu_F \otimes \mu_G.$$

2.1.3 Coalgebra

We now give the definitions and properties for an R-coalgebra by reversing all the arrows of morphisms in the definition of algebras in section 2.1.1.

Definition 2.1.12. An R-coalgebra is a R-module C with a R-module module morphism $\Delta : C \to C \otimes C$, called comultiplication.

For a given R-coalgebra C:

i. The coalgebra is said to be coassociative if the diagram

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\otimes C \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle 1_C\otimes\Delta} \\ C\otimes C & \xrightarrow{\Delta\otimes 1_C} & C\otimes C\otimes C \end{array} \tag{2.8}$$

is commutative.

For $c \in C$, let $\Delta(c) = \sum_{i=1}^{n} d_i \otimes e_i$, where $d_i, e_i \in C$.

By diagram (2.8) we have the following equations:

$$(\Delta \otimes 1_C) \circ \Delta(c) = (\Delta \otimes 1_C) \sum_{i=1}^{n} d_i \otimes e_i \quad = \sum_{i=1}^{n} \sum_{j=1}^{m_i} (r_{ij} \otimes s_{ij}) \otimes e_i, \tag{2.9}$$

where $\Delta(d_i) = \sum_{j=1}^{m_i} (r_{ij} \otimes s_{ij})$, where $r_{ij}, s_{ij} \in C$.

Similarly,

$$(1 \otimes \Delta) \circ \Delta(c) = (1 \otimes \Delta) \sum_{i=1}^{n} d_i \otimes e_i \quad = \sum_{i=1}^{n} \sum_{j=1}^{p_i} d_i \otimes (y_{ij} \otimes z_{ij}), \tag{2.10}$$

where $\Delta(e_i) = \sum_{j=1}^{p_i} (y_{ij} \otimes z_{ij})$, $y_{ij}, z_{ij} \in C$.

Alternatively, by the equation (2.9) and (2.10) coalgebra C is said to be coassociative if

$$\sum_{i=1}^{n}\sum_{j=1}^{m_i}(r_{ij}\otimes s_{ij})\otimes e_i=\sum_{i=1}^{n}\sum_{j=1}^{p_i}d_i\otimes(y_{ij}\otimes z_{ij})=\sum_{i=1}^{n}\sum_{j=1}^{p_i}d_i\otimes y_{ij}\otimes z_{ij}. \tag{2.11}$$

ii. The coalgebra is said to be cocommutative if the diagram

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\otimes C \\ & {\scriptstyle\Delta}\searrow & \downarrow{\scriptstyle\tau} \\ & & C\otimes C \end{array} \tag{2.12}$$

is commutative, where τ is the *twisting* morphism. By the diagram (2.12) for $c\in C$, we have,

$$\Delta(c)=\sum_{i=1}^{n}c_i\otimes d_i=\sum_{i=1}^{n}d_i\otimes c_i \quad \text{with} \quad c_i,d_i\in C.$$

iii. The coalgebra is said to be counital if Δ has a co-unit $\epsilon : R\to C$ which should satisfy

$$\begin{array}{ccccc} & & C & & \\ & \swarrow & \downarrow{\scriptstyle\Delta} & \searrow & \\ R\otimes C & \xleftarrow{\epsilon\otimes\ 1_C} & C\otimes C & \xrightarrow{1_C\otimes\epsilon} & C\otimes R \end{array} \tag{2.13}$$

Let M and N be R-coalgebras, A homomorphism $f : M\to N$ of coalgebras, is a R-module homomorphism such that the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\Delta_M} & M\otimes M \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\otimes f} \\ N & \xrightarrow{\Delta_N} & N\otimes N, \end{array} \tag{2.14}$$

$$\begin{array}{ccc} M & \xrightarrow{\epsilon_M} & R \\ \downarrow f & & \downarrow I_R \\ N & \xrightarrow{\epsilon_N} & R, \end{array} \tag{2.15}$$

are commutative. By the diagrams (2.14) and (2.15) we have,

$$(f \otimes f) \circ \Delta_M = \Delta_N \circ f \quad and \quad \epsilon_M = \epsilon_N(f).$$

Remark 2.1.13. *We write (C, Δ, ϵ) or simply C when we mean an R-coalgebra C which is coassociative and counital. Beside this, when we have more than one coalgebra, to make it more clear, we write $(C, \Delta_C, \epsilon_C)$ using subscripts "C" to emphasis which coproduct belongs to which coalgebra.*

Definition 2.1.14. If we have two R graded coalgebras $(M, \Delta_M, \epsilon_M)$ and $(N, \Delta_N, \epsilon_N)$, then $M \otimes N$ is the coalgebra over R with comultiplication the composition is given by

$$M \otimes N \overset{\Delta_M \otimes \Delta_N}{\longrightarrow} M \otimes M \otimes N \otimes N \overset{1_M \otimes \tau \otimes 1_N}{\longrightarrow} M \otimes N \otimes M \otimes N, \tag{2.16}$$

and counit

$$M \otimes N \overset{\epsilon_M \otimes \epsilon_N}{\longrightarrow} R \otimes R \approx R. \tag{2.17}$$

Remark 2.1.15. *By the composition (2.16), and diagram (2.17) we have :*

$$\Delta_{M \otimes N} = (1_M \otimes \tau \otimes 1_N) \circ (\Delta_M \otimes \Delta_N), \ and \quad \epsilon_{M \otimes N} = \epsilon_M \otimes \epsilon_N$$

2.2 Bialgebra, convolution and Hopf algebra

See [11], [32], [27] for further details on topics in this section.

2.2.1 Bialgebra

We defined algebra and coalgebra. To be able to give a definition for another algebra structure which is *bialgebra*, we will first introduce the following proposition:

Proposition 2.2.1. *Let (B, φ, μ) be an R-algebra and let (B, Δ, ϵ) be an R-coalgebra, then the following are equivalent:*

i. Δ *and* ϵ *are algebra morphisms.*

ii. φ *and* μ *are coalgebra morphisms.*

Proof. It is easily seen by using commutative diagrams for algebra and coalgebra morphism. □

Definition 2.2.2. A bialgebra is an R-module, endowed with an algebra structure (B, φ, μ) and a coalgebra structure (B, Δ, ϵ), where either φ and μ are coalgebra morphisms or Δ and ϵ are algebra morphisms. It is denoted by $(B, \varphi, \mu, \Delta, \epsilon)$.

We defined bialgebra structure. As a next step, we first need to define a new morphism which is called *convulution.*

2.2.2 Convolution

Let (A, φ, μ) be an algebra and (C, Δ, ϵ) be a coalgebra. Let $\mathrm{Hom}(C, A)$ denote the set of of R-module morphisms from C to A. Let $f, g \in \mathrm{Hom}(C, A)$, then we can define a morphism from C to A as follows,

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\varphi} A. \tag{2.18}$$

The new morphism we get by composition of morphisms above,

$$\varphi \circ (f \otimes g) \circ \Delta \tag{2.19}$$

is called *convolution* of f and g. It is denoted by $f \star g$. By definition, more explicitly for all $c \in C$ we have,

$$\begin{aligned}(f \star g)(c) &= \varphi \circ (f \otimes g) \circ \Delta(c). \\ &= \varphi(\sum_{i=1}^{n} f(d_i) \otimes g(e_i)) \\ &= \sum_{i=1}^{n} f(d_i) g(e_i),\end{aligned}$$

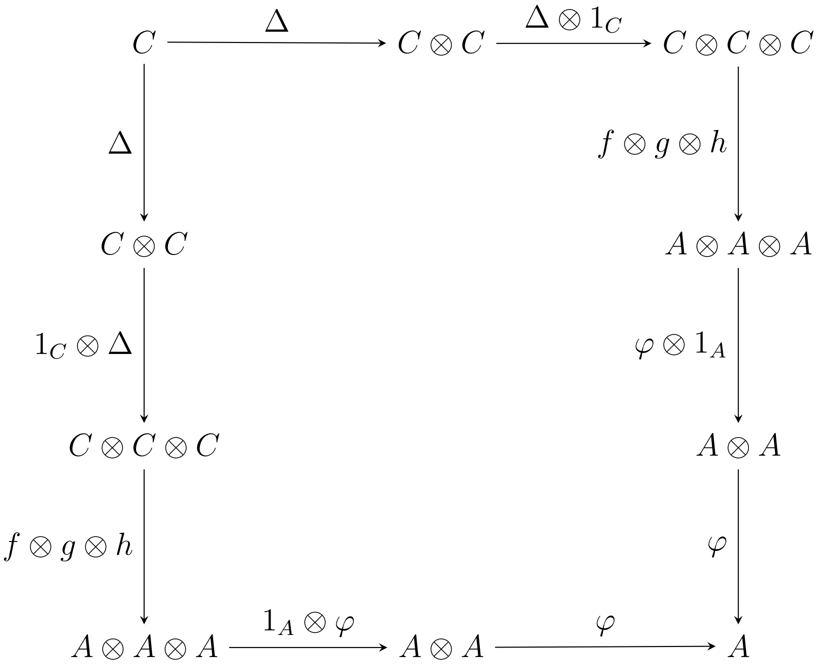
where $\Delta(c) = \sum_{i=1}^{n} d_i \otimes e_i$, for some $d_i, e_i \in C$.

Proposition 2.2.3. *Let* (A, φ, μ) *be an algebra and* (C, Δ, ϵ) *a coalgebra, then* $\mathrm{Hom}(C, A)$ *is a* monoid *under the operation* $\star$ *with unit which is given by composition:*

$$C \xrightarrow{\epsilon} R \xrightarrow{\mu} A.$$

Proof. i. Proof that $\mathrm{Hom}(C, A)$ is associative with operation $\star$.

Let $f, g, h \in \mathrm{Hom}(C, A)$. $\mathrm{Hom}(C, A)$ is associative if the following diagram commutes.



First we need to show the following equations hold.

1. $\varphi \circ (\varphi \otimes 1_A) \circ ((f \otimes g) \otimes h) \circ (\Delta \otimes 1_C) \circ \Delta = (f \star g) \star h$.
2. $\varphi \circ (1_A \otimes \varphi) \circ (f \otimes (g \otimes h)) \circ (1_C \otimes \Delta) \circ \Delta = f \star (g \star h)$.

By equation (2.9) in definition 2.1.12, for all $c \in C$ we have:

$$\varphi \circ (\varphi \otimes 1_A) \circ ((f \otimes g) \otimes h) \circ (\Delta \otimes 1_C) \circ \Delta(c) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} (f(r_{ij})g(s_{ij}))h(e_i), \tag{2.20}$$

where $r_{ij}, s_{ij}, e_i \in C$. On the other hand, for all $c \in C$ we have;

$$\begin{aligned} (f \star g) \star h(c) &= \sum_{i=1}^{n} (f \star g)(d_i)h(e_i) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m_i} (f(r_{ij})g(s_{ij}))h(e_i) \end{aligned} \tag{2.21}$$

for some $r_{ij}, s_{ij}, e_i \in C$. By equation (2.20) and equation (2.21) 1. holds. Similarly using equation (2.10) we can also show 2. holds. Finally, since A is associative and C is coassociative, for all $c \in C$ we have:

$$\begin{aligned}(f \star g) \star h(c) &= \varphi \circ (\varphi \otimes 1_A) \circ ((f \otimes g) \otimes h) \circ (\Delta \otimes 1_C) \circ \Delta(c) \\ &= \varphi \circ (\varphi \otimes 1_A) \circ ((f \otimes g) \otimes h) \circ (1_C \otimes \Delta) \circ \Delta(c) \\ &= \varphi \circ (\varphi \otimes 1_A) \circ (f \otimes (g \otimes h)) \circ (1_C \otimes \Delta) \circ \Delta(c) \\ &= \varphi \circ (1_A \otimes \varphi) \circ (f \otimes (g \otimes h)) \circ (1_C \otimes \Delta) \circ \Delta(c) \\ &= f \star (g \star h)(c).\end{aligned}$$

Therefore, $(f \star g) \star h = f \star (g \star h)$, in other words $\mathrm{Hom}(C, A)$ is associative with respect to $\star$.

Remark 2.2.4. *Unless otherwise stated we will denote identity element of an algebraic structure A by I_A,*

ii. Proof that the unit of $\mathrm{Hom}(C, A)$ is $\mu \circ \epsilon$.

Let $h \in \mathrm{Hom}(C, A)$, then for any $c \in C$

$$\begin{aligned}\big((h \star (\mu \circ \epsilon)\big)(c) &= \varphi \circ (h \otimes \mu \circ \epsilon) \circ \Delta(c) \\ &= \sum_{i=1}^{n} h(d_i)(\mu \circ \epsilon)(e_i) \\ &= \sum_{i=1}^{n} h(d_i)\mu(\epsilon(e_i)) \\ &= \sum_{i=1}^{n} h(d_i)\epsilon(e_i)\mu(I_R) \\ &= \sum_{i=1}^{n} h(d_i)\epsilon(e_i) I_A \\ &= h(c),\end{aligned} \tag{2.22}$$

where $\Delta(c) = \sum_{i=1}^{n} d_i \otimes e_i$, for some $d_i, e_i \in C$.

We used the definition of counit to show the equatliy of the last step of equation (2.4.6). Therefore $h \star (\mu\epsilon) = h$. Similarly, $(\mu\epsilon) \star h = h$. By i. and ii. $\mathrm{Hom}(C, A)$ is a *monoid.*

□

Remark 2.2.5. *The operation, $\star$ is said to be convolution product.*

We have now one more step to define *Hopf* algebra. For to do that, we introduce a new term which is called the *antipode*. Let $(H, \varphi, \mu, \Delta, \epsilon)$ be a bialgebra. Now H has both algebra and coalgebra structure, to be more precise, denote the underlying algebra of H by H^a, and underlying coalgebra of H by H^c. Then by Proposition 2.2.2 $\mathrm{Hom}(H^c, H^a)$ is also a monoid with the convolution product, $\star$. Now we can give the definition of an antipode.

Definition 2.2.6. Let $(H, \varphi, \mu, \Delta, \epsilon)$ be a bialgebra. An endomorphism $S : H \to H$ is called an antipode of a bialgebra H, if S is the two sided inverse element of the identity morphism $1_H : H \to H$ with respect to the convolution product in $\mathrm{Hom}(H^c, H^a)$.

Therefore S is an antipode if and only if S satisfies,

$$\varphi \circ (1_H \otimes S) \circ \Delta = \mu \circ \epsilon = \varphi \circ (S \otimes 1_H) \circ \Delta. \tag{2.23}$$

Corollary 2.2.7. *If an antipode exists, then it is unique.*

Proof. An antipode S of H is a two sided inverse in $\mathrm{Hom}(H^c, H^a)$. Beside this by Proposition 2.2.2 $\mathrm{Hom}(H^c, H^a)$ is associative, therefore S is unique. □

Proposition 2.2.8. *Let $(H, \varphi, \mu, \Delta, \epsilon)$, be a bialgebra, then the antipode S has following properties.*

i. S is an anti-automorphism, i.e., $S(h_1h_2) = S(h_2)S(h_1)$ where h_1 and $h_2 \in H$.

ii. S preserves the identity element.

iii. If H is commutative or cocommutative, then $S \circ S = S^2 = 1_H$.

Proof. **Proof of i.** See [32, Proposition 4.0.1] for the proof of i.

Proof that ii. By Definition 2.2.6 we have:

$$S \star 1_H(I_H) = \varphi \circ (S \otimes 1_H) \circ \Delta(I_H) = \mu \circ \epsilon(I_H). \tag{2.24}$$

On the other hand, H is a bialgebra, hence by Proposition 2.2.1 ϵ is an algebra morphism which means $\epsilon(I_H) = I_R$. We also know $\mu(I_R) = I_H$, hence $\mu \circ \epsilon(I_H) = \mu(I_R) = I_H$. Beside this $\Delta(I_H) = I_H \otimes I_H$. Therefore the equation (2.24) turns out

$$S \star 1(I_H) = S(I_H)1_H = S(I_H) = \mu \circ \epsilon(I_H) = I_H.$$

This completes the proof.

Proof of iii. By Definiton 2.2.6 we can easily observe that 1_H is the inverse of S with respect to $\star$. To make the proof we only need to show that $S^2 = S \circ S$ is also right or left inverse of S, therefore S^2 must equal identity homomorphism, 1_H. Let H be commutative algebra, and let $\Delta(c) = \sum_{i=1}^{n} d_i \otimes e_i$, where $d_i, e_i \in H$. For all $c \in H$ we have:

$$\begin{aligned}
(S^2 \star S)(c) &= \varphi \circ (S^2 \otimes S) \circ \Delta(c). \\
&= \varphi \circ (S^2 \otimes S)(\sum_{i=1}^{n} d_i \otimes e_i) \\
&= \varphi(\sum_{i=1}^{n} S^2(d_i) \otimes S(e_i)) \\
&= \sum_{i=1}^{n} S^2(d_i) S(e_i) \\
&= S(\sum_{i=1}^{n} e_i S(d_i)) \quad \text{S is anti-automorphism.} \qquad (2.25) \\
&= S(\sum_{i=1}^{n} S(d_i) e_i) \quad \text{H is commutative.} \\
&= S(\mu \circ \epsilon(c)) \quad \text{definition of } S. \\
&= S(\epsilon(c) I_H)) \\
&= \epsilon(c) S(I_H)) \quad \text{S is R module homomorphism.} \\
&= \epsilon(c) I_H \quad \text{S preserves unit.} \\
&= (\mu \circ \epsilon)(c)
\end{aligned}$$

We showed S^2 is left inverse of S, hence $S^2 = 1_H$. If H is cocommutative then we have:

$$\sum_{i=1}^{n} c_i \otimes d_i = \sum_{i=1}^{n} d_i \otimes c_i \quad \text{with} \quad c_i, d_i \in C.$$

Hence it is easily seen that equation (2.25) turns into:

$$
\begin{aligned}
(S^2 \star S)(c) &= \varphi \circ (S^2 \otimes S) \circ \Delta(c). \\
&= \varphi \circ (S^2 \otimes S)(\sum_{i=1}^{n} d_i \otimes e_i) \\
&= \varphi \circ (S^2 \otimes S)(\sum_{i=1}^{n} e_i \otimes d_i) \quad \text{H is cocomutative.} \\
&= \varphi(\sum_{i=1}^{n} S^2(e_i) \otimes S(d_i)) \\
&= \sum_{i=1}^{n} S^2(e_i) S(d_i) \\
&= S(\sum_{i=1}^{n} d_i S(e_i)) \quad \text{S is anti-automorphism.} \\
&= S(\mu \circ \epsilon(c)) \quad \text{definition of } S. \\
&= \mu \circ \epsilon(c).
\end{aligned} \tag{2.26}
$$

Similarly, we showed S^2 is left inverse of S, hence $S^2 = 1_H$. Note that we used the same Δ for equation (2.26) which we used for (2.25). This finishes the proof.

□

2.2.3 Hopf Algebra

Definition 2.2.9. A Hopf algebra is a bialgebra with an antipode.

Let K be a commutative ring with unit. We give one of the important properties of Dual Hopf algebras.

Proposition 2.2.10. *If H is graded projective K-module of finite type, then $(H, \varphi, \mu, \Delta, \epsilon)$, is a Hopf algebra with multiplication φ,comultiplication Δ,unit μ, and counit ϵ if and only if $(H^*, \Delta^*, \epsilon^*, \varphi^*, \mu^*,)$ is a Hopf algebra with multiplication Δ^*, comultiplication φ^*, unit ϵ^*, and counit μ^*. [27, Proposition 4.8]*

2.3 Words

See[28, Chapter 1] for more detailed information on topics given in this section. We now give basic concepts about words.

Definition 2.3.1. Let E be an *alphabet*, that is a non-empty set of symbols, i.e., $E = \{a, b, c\}$. Its elements will be called *letters.* A word over the alphabet E is a finite sequence of elements of E:

$$(a_1, a_2, a_3, \ldots, a_n), \quad a_i \in E.$$

The set of all words over E is denoted by $\mathcal{W}$. A product on $\mathcal{W}$ is defined by *concatenation*:

$$(a_1, a_2, a_3, \ldots, a_m)(b_1, b_2, b_3, \ldots, b_k) = (a_1, a_2, a_3, \ldots, a_m, b_1, b_2, b_3, \ldots, b_k).$$

The product is associative, which allows writing a word $(a_1, a_2, a_3, \ldots, a_n)$ as $a_1, a_2, a_3, \ldots, a_n$ by identifying a letter $a_i \in E$ with sequence (a_i).

Remark 2.3.2. *In the rest of this thesis a word* $(a_1, a_2, a_3, \ldots, a_n)$ *is denoted by* $a_1, a_2, a_3, \ldots, a_n$.

The sequence without any letter is called the *empty* word which is the neutral element for multiplication. $\mathcal{W}$ has a product which is associative and it is also combined with the unit, so $\mathcal{W}$ becomes a *monoid*.

Remark 2.3.3. *The alphabet E does not need to be finite, whereas a word is finite.*

As a convention, in the rest of this thesis we will use the alphabet $E = \mathbb{N}$. Since $\mathbb{N}$ is our alphabet, we can also add the letters.

Definition 2.3.4. Let $w = w_1, \ldots, w_p$ be a word, then the total number of letters, p, is the length of w. The degree of w is $w_1 + \ldots + w_p$. It will be denoted by $|w|$.

Of course the empty word has the length of zero.

2.3.1 Properties of words

We first give the following proposition which will be an important tool for the following section.

Proposition 2.3.5. *Let $\mathcal{R}$ be the set of all words of degree n, where $n \geq 1$. Then*

$$\mathcal{R} = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n,$$

where $A_i = \{i, l_1, \ldots, l_s : \ l_1, \ldots, l_s \text{ is a word of degree } \ n - i\}, i = 1, \ldots, n.$

Proof. Let $\mathcal{R}$ be the set of all words of degree n, where $n \geq 1$, in other words, we have:

$$\mathcal{R} = \sqcup_i \{\text{words of degree } n, \text{first letter is } i\}.$$

Let $A_i = \{i, l_1, \ldots, l_s : \ l_1, \ldots, l_s \text{ is a word of degree } \ n-i\}, i = 1, \ldots, n$. One can observe that if $w \in \mathcal{R}$, then by definition 2.3.4 the first letter of w must be in $\{1, \ldots, n\}$, and the remaining must form a word of degree $n-i$, hence $w \in A_i$. This completes the proof.

□

Corollary 2.3.6. *Let $\mathcal{R}$ be the set of all words of degree n, where $n \geq 1$, then the cardinality of $\mathcal{R}$ is 2^{n-1}.*

Proof. We proceed by induction on the degree n. Let $\mathcal{R}$ be the set of all words of degree n, where $n \geq 1$, in other words we have:

$$\mathcal{R} = \{b_1, b_2, \ldots, b_p : \ b_1 + b_2 + \ldots + b_p = n\} \quad b_1, \ldots, b_p > 0.$$

When $n = 1$, there is only one word in degree one, which is the word 1, then the cardinality of $\mathcal{R}$, namely $|\mathcal{R}| = 2^{1-1} = 1$. Hence, the first step of induction is satisfied.

On the other hand, by Proposition 2.3.5, in degree n, we can find the cardinality of $\mathcal{R}$ by the following equation:

$$|\mathcal{R}| = |A_1| + |A_2| + \cdots + |A_{n-1}| + |A_n|, \tag{2.27}$$

where A_i is defined as follow:

$$A_i = \{i, l_1, \ldots, l_s : \ l_1, \ldots, l_s \text{ is a word of degree } \ n-i\}, i = 1, \ldots, n.$$

Note that when $i = n$, $A_n = \{n\}$, hence, $|A_n| = 1$. Beside this, by definition of A_i, for $i = 1, \ldots, n-1$, it is seen that, the cardinality of A_i, namely $|A_i|$ is equal to the cardinality of the set of all words of degree $n-i$. Hence, by the inductive hypothesis $|A_i| = 2^{(n-i)-1}$. Therefore, equation 2.27 turns out:

$$|\mathcal{R}| = 2^{n-2} + 2^{n-3} + \cdots + 2^0 + 1, \tag{2.28}$$

from which we can conclude that $|\mathcal{R}| = 2^{n-1}$. This completes the proof. □

As we said $\mathbb{N}$ is our alphabet, so we have a totally ordered property. Let $w = w_1, \ldots, w_p$ be a word, we now define new terminologies in the following:

Definition 2.3.7. If $w_1, \ldots, w_p = w_p, \ldots, w_1$, then w is called a palindrome. If the length of w is odd, then w is called an odd-length palindrome which will be denoted by OLP. If the length of w is even, then w is called an even-length palindrome which will be denoted by ELP.

Definition 2.3.8. If $w_1, \ldots, w_p \neq w_p, \ldots, w_1$, then w is called a non-palindrome.

Definition 2.3.9. If $w_1, \ldots, w_p > w_p, \ldots, w_1$ in dictionary order, then w is called a higher non-palindrome. It will be denoted by HNP.

Definition 2.3.10. If $w_1, \ldots, w_p < w_p, \ldots, w_1$ in dictionary order, then w is called a lower non-palindrome. It will be denoted by LNP.

HNPs, LNPs, ELPs and OLPs will play an important role in the following chapters. We first introduce interesting observations regarding these words as follows.

Proposition 2.3.11. *Even-length palindromes have even degree, so there are no even-length palindromes in odd degrees.*

Proof. Assume that $w = i_1, \ldots, i_k, i_{k+1}, \ldots, i_{2k}$ is an even-length palindrome. We can easily observe that the left part of w is $i_1, \ldots, i_k$ which is the reverse of the right part of w, namely $i_{k+1}, \ldots, i_{2k}$. Thus, the degree of w is $i_1 + \ldots + i_{2k} = i_1 + \cdots + i_k + i_k + \ldots + i_1 = 2(i_1 + \cdots + i_k)$ which is even. □

Proposition 2.3.12. *There is a one-to-one correspondence between higher non-palindromes and lower non-palindromes of any fixed degree.*

Proof. Let $b_p, \ldots, b_1$ be a higher non-palindrome. Then $b_p, \ldots, b_1 > b_1, \ldots, b_p$ in dictionary order. So, $b_1, \ldots, b_p < b_p, \ldots, b_1$, then $b_1, \ldots, b_p$ is a lower non-palindrome. This means the reverse of a higher non-palindrome is a lower non-palindrome and every reverse of a lower non-palindrome is a higher non-palindrome. □

Corollary 2.3.13. *For any fixed degree, the number of higher non-palindromes and lower non-palindromes are equal for all degrees.*

Proof. The proof is straightforward by Proposition 2.3.12. □

Now using the observations above we give the following results.

Proposition 2.3.14. *In degree n, n positive integer:*

i. The number of even-length palindromes is $2^{\frac{n}{2}-1}$ if n is even, and 0 if n is odd;

ii. The number of odd-length palindromes is $2^{\frac{n}{2}-1}$ if n is even, and $2^{\frac{n-1}{2}}$ if n is odd;

iii. The number of higher non-palindromes is $2^{n-2} - 2^{\frac{n}{2}-1}$ if n is even, and $2^{n-2} - 2^{\frac{n-1}{2}-1}$ if n is odd; and

iv. *The number of lower non-palindromes is* $2^{n-2} - 2^{\frac{n}{2}-1}$ *if* n *is even, and* $2^{n-2} - 2^{\frac{n-1}{2}-1}$ *if* n *is odd.*

Proof. i. By proposition 2.3.11 it is clear that even-length palindromes can only occur in even degrees which means the number of them is zero in odd degrees. Let n be the degree, where it is a positive even integer, and let $i_1, \ldots, i_k, i_{k+1}, \ldots, i_{2k}$ be an even-length palindrome. Then its degree is $2(i_1 + \ldots + i_k)$, so $i_1 + \ldots + i_k = \frac{n}{2}$, therefore $i_1, \ldots, i_k$ has degree $\frac{n}{2}$, but it can be any word in degree $\frac{n}{2}$. By Corollary 2.3.6 the number of all words in degree $\frac{n}{2}$ is $2^{\frac{n}{2}-1}$. And for any word, $i_1, \ldots, i_k$, of degree $\frac{n}{2}$ we get a degree n, $S^{i_1,\ldots,i_k,\ldots,i_{2k}}$ palindrome. So, the number of even-length palindromes is $2^{\frac{n}{2}-1}$.

ii. Let n be an even integer, then there is a one-to-one correspondence from the set of all even-length palindromes in degree n to the set of all odd-length palindromes in degree n given by,

$$i_1, \ldots, i_k, i_{k+1}, \ldots, i_{2k} \longmapsto i_1, \ldots, i_k + i_{k+1}, \ldots, i_{2k}$$

with inverse given by,

$$i_1, \ldots, i_k, \ldots, i_{2k-1} \longmapsto i_1, \ldots, \frac{i_k}{2}, \frac{i_k}{2}, \ldots, i_{2k-1}.$$

(Note that i_k must be even because n is an even degree.) Therefore, the number of odd-length palindromes is equal to the number of even-length palindromes in even degrees, which is $2^{\frac{n}{2}-1}$. Now let's consider odd degrees

Let n be an odd integer and let $i_1, \ldots, i_{k+1}, \ldots, i_{2k+1}$ be an odd-length palindrome. Then $i_1, \ldots, i_{k+1}, \ldots, i_{2k+1}$ has a middle term, namely i_{k+1}, where $i_{k+1} \geq 1$ and a left part word, namely $i_1, \ldots, i_k$ which is the reverse of its right part word, i.e, $i_{k+2}, \ldots, i_{2k+1}$.

The degree of $i_1, \ldots, i_k, i_{k+1}, \ldots, i_{2k+1}$ is $2(i_1 + \ldots + i_k) + i_{k+1}$. Beside this, since $i_{k+1} \geq 1$, $(i_1 + \cdots + i_k) \leq \frac{n-1}{2}$,therefore $i_1, \ldots, i_k$ has degree which is less than or equal to $\frac{n-1}{2}$. But the left part word $i_1, \ldots, i_k$ can be any word of degree which is less than or equal to $\frac{n-1}{2}$.

On the other hand, the middle term, i_{k+1}, is determined by n and the degree of $i_1, \ldots, i_k$; we have $i_{k+1} = n - 2(i_1 + \cdots + i_k)$. Hence, for any word of degree less than or equal to $\frac{n-1}{2}$ we get a degree n OLP, $S^{i_1,\ldots,i_{k+1},\ldots,i_{2k+1}}$, where $i_{k+1} = n - 2(i_1 + \cdots + i_k)$.

Therefore, the number of all odd-length palindromes is equal to the number of words with degree $0, 1, \ldots, \frac{n-1}{2}$, i.e.,

$$1 + 1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{\frac{n-1}{2}-1} = 2^{\frac{n-1}{2}}.$$

Therefore, the number of all odd-palindromes is $2^{\frac{n-1}{2}}$.

iii. By Corollary 2.3.6 the number of all words in degree n is 2^{n-1}. Let n be n positive odd integer. By (i) and (ii), the total number of palindromes is $2^{\frac{n-1}{2}}$. So by Corollary 2.3.13 the number of higher non-palindromes is

$$\frac{2^{n-1} - 2^{\frac{n-1}{2}}}{2} = 2^{n-2} - 2^{\frac{n-1}{2}-1}.$$

Let n be an even integer. By (i) and (ii), the total number of palindromes is

$$2^{\frac{n}{2}-1} + 2^{\frac{n}{2}-1} = 2^{\frac{n}{2}}.$$

So by Corollary 2.3.13 the number of higher non-palindromes is

$$\frac{2^{n-1} - 2^{\frac{n}{2}}}{2} = 2^{n-2} - 2^{\frac{n}{2}-1}.$$

This completes the proof.

iv. By (iii) and corollary 2.3.13 the number of lower non-palindromes and lower non-palindromes is equal to the number of higher non-palindromes. This completes the proof.

□

We now give two important terminologies: coarsening and refinement. These will be one of the important tools for the following chapters.

Definition 2.3.15. Let $e_1, \ldots, e_m$ be a word. $r_1, \ldots, r_n$ is a coarsening of $e_1, \ldots, e_m$ if there exist $k_1, \ldots, k_{n-1}, k_n$ with $r_1 = e_1 + \ldots + e_{k_1}$, $r_2 = e_{k_1+1} + \ldots + e_{k_2}$, $r_n = e_{k_{n-1}+1} + \ldots + e_m$, and $1 \leq k_1 < k_2 < \ldots < k_{n-1} < k_n = m$.

Remark 2.3.16. *Alternatively, we can give the following definiton for coarsening. Let $b_1, \ldots, b_p$ be a word. A coarsening of $b_1, \ldots, b_p$ is a word which can be obtained from $b_1, \ldots, b_p$ by turning some of the commas ',' $b_1, \ldots, b_p$ into "+"s.*

Example 2.3.17. *Coarsenings of the word* $2, 2, 1$ *are the words* $2, 2, 1$ *,* 2+2, 1 *,* $2, 2+1$ *,and* $2+2+1$ *i.e.,* $2, 2, 1$ *,* $4, 1$ *,* $2, 3$ *,and* 5.

According to this Definition 2.3.15 we can define refinement of a word as in the following.

Definition 2.3.18. Let $b_1, \ldots, b_n$ be a word. $c_1, \ldots, c_m$ is a refinement of $b_1, \ldots, b_n$ if there exists $k_1, \ldots, k_{n-1}, k_n$ with $b_1 = c_1 + \ldots + c_{k_1}$, $b_2 = c_{k_1+1} + \ldots + c_{k_2}$, $b_n = c_{k_{n-1}+1} + \ldots + c_m$,and $1 \leq k_1 < k_2 < \ldots < k_{n-1} < k_n = m$.

Note that the *empty* word has only one refinement which is empty word.

Example 2.3.19. *The refinements of word* $2, 2, 1$ *are* $2, 2, 1$, $2, 1, 1, 1$, $1, 1, 2, 1$, $1, 1, 1, 1, 1$.

Remark 2.3.20. *When we say a Word Hopf algebra we mean an algebra which has a basis of words. In this context, in this thesis, we are interested in some word Hopf Algebras: the Leibniz-Hopf algebra, the dual Leibniz-Hopf algebra. And for any prime p, the mod p reduction of these algebras and its duals.*

Firstly, we will introduce the Leibniz-Hopf Algebra.

2.4 Leibniz-Hopf Algebra

Definition 2.4.1. Let $\mathcal{F}$ denote the free unital associative $\mathbf{Z}$ algebra on generators $S^1, S^2, S^3, \ldots$ including the empty word which is denoted by S^0. $\mathcal{F}$ is spanned by 'words' (of finite length) in the 'letters' $S^1, S^2, S^3, \ldots$. The unit of $\mathcal{F}$ is S^0.

We now give more details regarding a basis of this free $\mathbf{Z}$ algebra.

Definition 2.4.2. Let $b_1, \ldots, b_k$ be a "word", then $S^{b_1} S^{b_2} \cdots S^{b_k}$ is called the corresponding basis element of $\mathcal{F}$. And we will abbreviate this to $S^{b_1, b_2, \ldots, b_k}$. The number of letters of the word is called the *length* of the basis element.

We can give $\mathcal{F}$ a grading by S^i has degree i. Hence, $\mathcal{F}$ is a graded algebra, i.e., $\mathcal{F} = \oplus_{n \geq 0} \mathcal{F}_n$, where $\mathcal{F}_n$, denotes the degree n part of $\mathcal{F}$. Moreover, $\mathcal{F}$ is connected, i.e, $\mathcal{F}_0 \approx \mathbf{Z}$.

Proposition 2.4.3. *For each* $n \geq 0$, $\mathcal{F}_n$ *has a basis consisting of words whose indices sum to n, i.e,*

$$\mathcal{F}_n = \{S^{i_1, i_2, \ldots, i_k} \ : i_1 + i_2 + \cdots + i_k = n\}.$$

Example 2.4.4. *The degree* 4 *part of* $\mathcal{F}$, *namely* $\mathcal{F}_4$, *has basis elements:*

$$S^4, S^{3,1}, S^{2,2}, S^{1,3}, S^{2,1,1}, S^{1,2,1}, S^{1,1,2}, S^{1,1,1,1}.$$

Proposition 2.4.5. *In any degree of $\mathcal{F}$, the dimension of $\mathcal{F}_n$, where $n \geq 1$, is calculated by the formula 2^{n-1}.*

Proof. By corollary 2.3.6 the number of words of degree n is 2^{n-1}. This completes the proof. □

For given two basis elements, $S^{a_1,a_2,a_3,\ldots,a_n}$ and $S^{b_1,b_2,b_3\ldots,b_k}$, multiplication φ is given by *concatenation*, which is determined by

$$\varphi(S^{a_1,a_2,a_3,\ldots,a_n} \otimes S^{b_1,b_2,b_3\ldots,b_k}) = S^{a_1,a_2,a_3\ldots,a_n,b_1,b_2,b_3\ldots,b_k},$$

where the *letters* $S^{a_1}, S^{a_2}, \ldots, S^{a_n}, S^{b_1}, S^{b_2} \ldots, S^{b_k} \in \mathcal{F}$.

Furthermore, comultiplication Δ is determined on $\mathcal{F}$ by

$$\Delta(S^n) = \sum_{i=0}^{n} S^i \otimes S^{n-i},$$

and requiring that Δ be an algebra morphism.

Example 2.4.6.

$$\begin{aligned}\Delta(S^{2,1}) &= \Delta(S^2)\Delta(S^1)\\ &= (S^0 \otimes S^2 + S^1 \otimes S^1 + S^2 \otimes S^0)(S^0 \otimes S^1 + S^1 \otimes S^0)\\ &= (S^0 \otimes S^{2,1} + S^1 \otimes S^2 + S^1 \otimes S^{1,1} + S^{1,1} \otimes S^1 + S^2 \otimes S^1\\ &+ S^{2,1} \otimes S^0)\end{aligned}$$

Definition 2.4.7. The counit ε is given by

$$\varepsilon(S^n) = \begin{cases} 1, & \text{if} \quad n = 0 \\ 0, & \text{if} \quad n \geq 1, \end{cases}$$

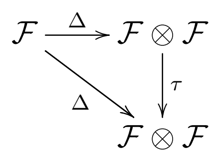
and requiring that ϵ be an algebra morphism.

$(\mathcal{F}, \Delta, \epsilon)$ has a coalgebra structure with coproduct, Δ, and counit ϵ.

Proposition 2.4.8. $(\mathcal{F}, \varphi, \mu, \Delta, \epsilon)$ *is a bialgebra.*

Proposition 2.4.9. $\mathcal{F}$ *is cocommutative with coproduct* Δ.

Proof. To show the cocommutativitiy we need to show that the following diagram is



(2.29)

commutative. Let $S^{n_1,\dots,n_q} \in \mathcal{F}$, then using the fact that Δ is an algebra morphism we have:

$$\begin{aligned}
\Delta(S^{n_1,\dots,n_q}) &= \Delta(S^{n_1})\Delta(S^{n_2})\cdots\Delta(S^{n_q}) \\
&= \sum_{i_1=0}^{n_1}\cdots\sum_{i_q=0}^{n_q}(S^{i_1}\otimes S^{n_1-i_1})\cdots(S^{i_q}\otimes S^{n_q-i_q}) \\
&= \sum_{i_1=0}^{n_1}\cdots\sum_{i_q=0}^{n_q}(S^{i_1,\dots,i_q}\otimes S^{n_1-i_1,\dots,n_q-i_q}) \\
&= \sum_{j_1=0}^{n_1}\cdots\sum_{j_q=0}^{n_q}(S^{n_1-j_1,\dots,n_q-j_q}\otimes S^{j_1,\dots,j_q}) \\
&= \tau\Delta(S^{n_1,\dots,n_q}),
\end{aligned}$$

where for $k = 1,\dots,q$, $\Delta(S^{n_k}) = \sum_{i_k=0}^{n_k} S^{i_k}\otimes S^{n_k-i_k}$. Hence, by the equation above it is easily seen that diagram (2.29) is commutative, so $\mathcal{F}$ is cocommutative. This completes the proof. □

Remark 2.4.10. *"Cocommutativity" means strict, not graded, just like commutativity in Remark 2.1.7.*

Proposition 2.4.11. *$(\mathcal{F}, \varphi, \mu, \Delta, \epsilon)$ is a Hopf algebra.*

By Proposition 2.4.8 $\mathcal{F}$ is a bialgebra. To show that $\mathcal{F}$ is Hopf algebra, what is left to show is that $\mathcal{F}$ has an antipode which will be denoted by $\chi_{\mathcal{F}}$. Before giving a proof, we need following lemma.

Lemma 2.4.12. *The antipode for $\mathcal{F}$ may be recursiveley defined by $\chi_{\mathcal{F}}(S^0) = S^0$, and for any $x \in F_n$, $n \geq 1$,*

$$\chi_{\mathcal{F}}(x) = -\sum_{i=1}^{m} y_i\chi_{\mathcal{F}}(z_i),$$

where

$$\Delta(x) = S^0\otimes x + \sum_{i=1}^{m} y_i\otimes z_i$$

and $|z_i| < n$.

Proof. Assume that for $x \in \mathcal{F}_n$, $n \geq 1$

$$\Delta(x) = S^0 \otimes x + \sum_{i=1}^{m} y_i \otimes z_i.$$

Substituting our formula for $\Delta(x)$ into the equation (2.23) in definition 2.2.6, we arrive at:

$$\varphi \circ (1_{\mathcal{F}} \otimes \chi_{\mathcal{F}}) \circ \Delta(x) = \mu \circ \epsilon(x)$$

$$\varphi \circ (1_{\mathcal{F}} \otimes \chi_{\mathcal{F}})(S^0 \otimes x + \sum_{i=1}^{m} y_i \otimes z_i) = 0 \quad \text{by definition 2.4.7 } \epsilon(x) = 0$$

$$\varphi(S^0 \otimes \chi_{\mathcal{F}}(x) + \sum_{i=1}^{m} y_i \otimes \chi_{\mathcal{F}}(z_i) = 0$$

$$\chi_{\mathcal{F}}(x) + \sum_{i=1}^{m} y_i \chi_{\mathcal{F}}(z_i) = 0$$

$$\chi_{\mathcal{F}}(x) = -\sum_{i=1}^{m} y_i \chi_{\mathcal{F}}(z_i),$$

which shows $\chi_{\mathcal{F}}$ satisfies recursive formula. □

Proposition 2.4.13. *For the special case, where $x = S^n$, the antipode,$\chi_{\mathcal{F}}$, is given by*

$$\chi_{\mathcal{F}}(S^n) = \sum (-1)^k S^{i_1,\ldots,i_k},$$

where the summation is over all refinements $S^{i_1,\ldots,i_k}$ of S^n.

Proof. The proof will proceed by induction on the degree n. By Proposition 2.2.7, $\chi_{\mathcal{F}}(S^0) = S^0$. Hence the first step of induction, $n = 0$, is satisfied.

Since $\Delta(S^n) = \sum_{i=0}^{n} S^i \otimes S^{n-i}$, Proposition 2.4.12 shows that we have a recursive formula for antipode which is given by

$$\chi_{\mathcal{F}}(S^n) = -\sum_{i=1}^{n} S^i \chi_{\mathcal{F}}(S^{n-i}). \tag{2.30}$$

Now let consider the equation (2.30). It expands in the following:

$$\chi_{\mathcal{F}}(S^n) = -\Big(S^1 \chi_{\mathcal{F}}(S^{n-1}) + S^2 \chi_{\mathcal{F}}(S^{n-2}) + \cdots + S^n \chi_{\mathcal{F}}(S^{n-n})\Big). \tag{2.31}$$

Since $\chi_{\mathcal{F}}(S^{n-n}) = \chi_{\mathcal{F}}(S^0) = S^0 = 1_{\mathcal{F}}$, the equation (2.31) turns into :

$$\chi_{\mathcal{F}}(S^n) = -\Big(S^1 \sum (-1)^{k_1} S^{r_{11},r_{22},\ldots,r_{1k_1}} + S^2 \sum (-1)^{k_2} S^{r_{21},r_{22},\ldots,r_{2k_2}} + \cdots + S^n \sum (-1)^{k_n} S^{r_{n1},r_{n2},\ldots,r_{nk_n}},\Big) \tag{2.32}$$

where the first summation is over all refinements $r_{11}, r_{12}, \ldots, r_{1k_1}$ of $n-1$, the second is over all refinements $r_{21}, r_{22}, \ldots, r_{2k_2}$ of $n-2, \ldots$, and the last one is over all refinements $r_{n1}, r_{n2}, \ldots, r_{nk_n}$ of $n-n=0$, i.e., empty word. Hence the length of $r_{n1}, r_{n2}, \ldots, r_{nk_n}$, namely k_n is zero. More precisely, by distributive property of the product in $\mathcal{F}$, the equation (2.32) turns into:

$$\chi_{\mathcal{F}}(S^n) = \sum(-1)^{k_1+1} S^{1,r_{11},r_{22},\ldots,r_{1k_1}} + \sum(-1)^{k_2+1} S^{2,r_{21},r_{22},\ldots,r_{2k_2}} + (-1)^{k_n+1} S^n, \tag{2.33}$$

In the language of Proposition 2.3.5, we observe that each summation on the right hand side of equation (2.33) is over A_i, where $i = 1, \ldots n$. and each summand in the summation is coming with coefficients $(-1)^{k_i+1}$, where $k_i + 1$ is the length of the summand. I.e., the summation $\sum(-1)^{k_1+1} S^{1,r_{11},r_{22},\ldots,r_{1k_1}}$ is over A_1, and each summand $S^{1,r_{11},r_{22},\ldots,r_{1k_1}}$ has coefficient $(-1)^{k_1+1}$, similarly, the summation $\sum(-1)^{k_2+1} S^{2,r_{21},r_{22},\ldots,r_{2k_2}}$ is over A_2, and each summand $S^{2,r_{21},r_{22},\ldots,r_{2k_2}}$ has coefficient $(-1)^{k_2+1}$, and so on. Note that, when $i = n$ the summation has only one summand which is S^n and S^n has coefficient $(-1)^{k_n+1} = (-1)^1$, since $k_n = 0$.

Moreover, by the definition 2.3.18 the set of all refinements of the length 1 word n corresponds to $\mathcal{R}$ which is the finite union of these A_i. Hence, in the language of Proposition 2.3.5 the right hand side of equation(2.33) is the sum of all refinements of S^n. This completes the proof. □

Corollary 2.4.14. *Let $S^{b_1,\ldots,b_p} \in \mathcal{F}$, then the antipode,$\chi_{\mathcal{F}}$, is given by*

$$\chi_{\mathcal{F}}(S^{b_1,\ldots,b_p}) = \sum(-1)^n S^{t_1,\ldots,t_n},$$

where the summation is over all refinements $S^{t_1,\ldots,t_n}$ of $S^{b_p,\ldots,b_1}$.

Proof. Let $S^{b_1,b_2,\ldots,b_p} \in \mathcal{F}$. By Proposition 2.2.8 $\chi_{\mathcal{F}}$ is an antiautomorphism, so

$$\chi_{\mathcal{F}}(S^{b_1,\ldots,b_p}) = \chi_{\mathcal{F}}(S^{b_p})\chi_{\mathcal{F}}(S^{b_{p-1}}) \cdots \chi_{\mathcal{F}}(S^{b_2})\chi_{\mathcal{F}}(S^{b_1}).$$

More explicitly by Proposition 2.4.13 we have:

$$\chi_{\mathcal{F}}(S^{b_1,\ldots,b_p}) = \sum(-1)^{k_1} S^{i_1,\ldots,i_{k_1}} \sum(-1)^{(k_2-k_1)} S^{i_{k_1+1},\ldots,i_{k_2}} \cdots \sum(-1)^{(k_p-k_{p-1})} S^{i_{k_{p-1}+1},\ldots,i_{k_p}}, \tag{2.34}$$

where $S^{i_1,\ldots,i_{k_1}}$ is a refinement of S^{b_p}, similarly, $S^{i_{k_1+1},\ldots,i_{k_2}}$ is a refinement of $S^{b_{p-1}}$, $\cdots$, $S^{i_{k_{p-1}+1},\ldots,i_{k_p}}$ is a refinement of S^{b_1}. We know the product of $\mathcal{F}$ is concatenation. Hence, equation (2.34) turns into:

$$\chi_{\mathcal{F}}(S^{b_1,\ldots,b_p}) = \sum(-1)^{k_p} S^{i_1,\ldots,i_{k_1},i_{k_1+1},\ldots,i_{k_2},i_{k_2+1},\ldots,k_{p-1}+1,\ldots,i_{k_p}}, \tag{2.35}$$

where $S^{i_1,\dots,i_{k_1},i_{k_1+1},\dots,i_{k_2},i_{k_2+1},\dots,k_{p-1+1},\dots,i_{k_p}}$ is a refinement of $S^{b_p,\dots,b_1}$, because $S^{i_1,\dots,i_{k_1}}$ is a refinement of S^{b_p}, similarly, $S^{i_{k_1+1},\dots,i_{k_2}}$ is a refinement of $S^{b_{p-1}}$, $\cdots$, $S^{i_{k_{p-1}+1},\dots,i_{k_p}}$ is a refinement of S^{b_1}. This completes the proof.

□

Remark 2.4.15. *By proposition 2.2.8 the antipode, χ, is an anti-endomorphism of $\mathcal{F}$, hence in the conjugation formula we first take the reverse of $S^{b_1,\dots,b_p}$, namely $S^{b_p,\dots,b_1}$, then apply refinement operation to $S^{b_p,\dots,b_1}$. We consider more details of antipode in the following chapters.*

Example 2.4.16.

$$\chi_{\mathcal{F}}(S^{3,1}) = S^{1,3} - S^{1,2,1} - S^{1,1,2} + S^{1,1,1,1}.$$

2.5 Mod p Leibniz-Hopf algebra

In this section for any prime p we consider the mod p Leibniz-Hopf algebra. We give more details while considering $p = 2$, i.e., the mod 2 Leibniz-Hopf algebra.

The free unital associative $\mathbf{Z}/p$ algebra on generators $S^1, S^2, S^3, \dots$ has the same algebraic structure as $\mathcal{F}$, but everything takes place over field $\mathbf{Z}/p$. It is denoted by $\mathcal{F}_p$, and it is a Hopf algebra, so has the antipode which is denoted by $\chi_{\mathcal{F}_p}$. Moreover, $\chi_{\mathcal{F}_p}$ is defined by the same formula as $\chi_{\mathcal{F}}$.

On the other hand, for the mod 2 Leibniz-Hopf algebra, the antipode is denoted by $\chi_{\mathcal{F}_2}$. Since we work on mod 2, the formula for antipode $\chi_{\mathcal{F}}$ in Corollary 2.4.14 is simplified into the antipode formula for $\mathcal{F}_2$ in the following:

Remark 2.5.1. *To make it more clear, the reader should keep in mind that $\mathcal{F}_n$ denotes the mod n reduction of $\mathcal{F}$, whereas $(\mathcal{F}_n)_m$ denotes the degree m part of $\mathcal{F}_n$.*

Definition 2.5.2. Let $S^{b_1,\dots,b_p} \in \mathcal{F}_2$, then the antipode, $\chi_{\mathcal{F}_2}$, is given by

$$\chi_{\mathcal{F}_2}(S^{b_1,\dots,b_p}) = \sum S^{t_1,\dots,t_n}$$

where the summation is over all refinements $S^{t_1,\dots,t_n}$ of $S^{b_p,\dots,b_1}$.

There are examples which are given for the antipode of $\mathcal{F}_3$ and $\mathcal{F}_2$ as follows.

Example 2.5.3. *The image of $S^{3,2}$ under $\chi_{\mathcal{F}_3}$ is given by*

$$\chi_{\mathcal{F}_3}(S^{3,2}) = S^{2,3} - S^{2,2,1} - S^{2,1,2} + S^{2,1,1,1} - S^{1,1,3} + S^{1,1,2,1} + S^{1,1,1,2} - S^{1,1,1,1,1}.$$

Example 2.5.4. *The image of* $S^{3,2}$ *under* $\chi_{\mathcal{F}_2}$ *is given by*

$$\chi_{\mathcal{F}_2}(S^{3,2}) = S^{2,3} + S^{2,2,1} + S^{2,1,2} + S^{2,1,1,1} + S^{1,1,3} + S^{1,1,2,1} + S^{1,1,1,2} + S^{1,1,1,1,1}.$$

In the next section we give basic algebraic details of the Dual-Leibniz Hopf algebra.

2.6 Dual Leibniz-Hopf Algebra

The Leibniz-Hopf algebra, $(\mathcal{F}, \varphi, \mu, \Delta, \epsilon)$, is graded and finite type, and since $\mathcal{F}$ is free $\mathbf{Z}$ module, it is a projective module. Hence by Proposition 2.2.10, dualising $\mathcal{F}$ we obtain a new Hopf algebra $(\mathcal{F}^*, \Delta^*, \epsilon^*, \varphi^*, \mu^*)$. This is the dual Leibniz Hopf algebra. In fact by the dual of $\mathcal{F}$ we mean the graded dual,i.e., $\mathcal{F}^* = \oplus_n \mathrm{Hom}(\mathcal{F}_n, \mathbf{Z}) = \mathcal{F}_n^*$, where $\mathcal{F}_n^*$ denotes the degree n component of $\mathcal{F}^*$. Since $\mathcal{F}$ is finite type so is $\mathcal{F}^*$. $\mathcal{F}^*$ is also connected.

Remark 2.6.1. *Note that the product,* Δ^**, and coproduct,* φ^**, in* $\mathcal{F}^*$*, are defined as dual of coproduct* Δ *and product* φ *in* $\mathcal{F}$*. And similarly unit,* ϵ^**, and counit,* μ^**, in* $\mathcal{F}^*$*, are defined as dual of counit* ϵ *and unit* μ *in* $\mathcal{F}$*.*

Definition 2.6.2. We know a basis for $\mathcal{F}$ is given by all words $S^{b_1,b_2,\ldots,b_k}$. We denote the dual basis for the free $\mathbf{Z}$-module, $\mathcal{F}^*$, by subscripts: $\{S_{b_1,b_2,\ldots,b_k}\}$. The dual basis element of $\mathcal{F}^*$ is defined with the duality given by.

$$S_{b_1,b_2,b_3,\ldots,b_k}(S^{j_1,j_2,\ldots,j_n}) = \begin{cases} 1 & \text{if } k=n, \text{ and } b_1 = j_1, b_2 = j_2, \ldots, b_k = j_k \\ 0 & \text{otherwise,} \end{cases} \tag{2.36}$$

where $S^{j_1,j_2\ldots,j_n}$ is a basis element of $\mathcal{F}$.

The length of the dual basis element $S_{b_1,b_2,b_3,\ldots,b_k}$ will be the length of the word $b_1, b_2, b_3, \ldots, b_k$.

Proposition 2.6.3. *In any degree* $n \geq 1$ *of* $\mathcal{F}^*$*, dimension of* $\mathcal{F}_n^*$*, is calculated by the formula* 2^{n-1}*.*

Proof. Follows immediately from Proposition 2.4.5 □

Recalling Remark 2.6.1, the multiplication of the dual Leibniz-Hopf algebra is defined as the dual of coproduct, Δ^*, and this product structure is given by the overlapping shuffle product, which is defined below. For the reader's convenience, let us recall Hazewinkel's notation.

In Hazewinkel's language[17], $\mathcal{F}$ is denoted by $\mathcal{Z} = \mathbf{Z} < Z_1, Z_2, \ldots >$ on generators $Z_1, Z_2, \ldots$ which corresponds to $S^1, S^2, \ldots$ in this thesis.

On the other hand the graded dual of $\mathcal{F}$ is denoted by $\mathcal{M}$[17][Section 1], and is called *overlapping shuffle algebra*. Moreover, the multiplication of $\mathcal{M}$ is defined as the dual of the coproduct which is denoted by μ and corresponds to Δ in this thesis. And this product structure, Δ^*, in the dual algebra is precisely given by the *overlapping shuffle product*[17][Section 6], which can be described in the following:

Definition 2.6.4. Let $S_{a_1,\ldots,a_k}$ and $S_{b_1,\ldots,b_m} \in \mathcal{F}^*$ so, $S_{a_1,\ldots,a_k}$ has length k, and $S_{b_1,\ldots,b_m}$ has length m. Overlapping shuffle product of $S_{a_1,\ldots,a_k}$ and $S_{b_1,\ldots,b_m}$ is defined by

$$\Delta^*(S_{a_1,\ldots,a_k} \otimes S_{b_1,\ldots,b_m}) = \sum_h h(S_{a_1,\ldots,a_k,b_1,\ldots,b_m}),$$

where h inserts a number of 0s into $a_1,\ldots,a_k$ (up to m), and inserts a number of 0s into $b_1,\ldots,b_m$ (up to k), and then adds the first indices together, then the second and so on. The sum is over all such h for which the result contains no 0.[6, Section 2]

Example 2.6.5. *Let* $S_{3,2}$ *and* $S_4 \in \mathcal{F}^*$

$$\Delta^*(S_{3,2} \otimes S_4) = S_{3,2,4} + S_{3,4,2} + S_{4,3,2} + S_{7,2} + S_{3,6}.$$

Proposition 2.6.6. *The overlapping shuffle product is commutative.*

Proof. By Proposition 2.4.9 $\mathcal{F}$ is cocomutative with coproduct Δ, therefore $\mathcal{F}^*$ is commutative with Overlapping shuffle product, Δ^*. □

Coproduct, φ^*, which is called excision or cut is given by

$$\varphi^*(S_{b_1,\ldots,b_k}) = \sum_{i=0}^{k} S_{b_1,\ldots,b_i} \otimes S_{b_{i+1},\ldots,b_k}, \quad \text{where} \quad S_{b_0} = S_0,$$

where S_0 is the identity of $\mathcal{F}^*$.

Example 2.6.7.

$$\varphi^*(S_{4,3,2}) = S_0 \otimes S_{4,3,2} + S_4 \otimes S_{3,2} + S_{4,3} \otimes S_2 + S_{4,3,2} \otimes S_0.$$

For any given $S_{j_1,\ldots,j_q} \in \mathcal{F}^*$, counit μ^* is given by

$$\mu^*(S_{j_1,\ldots,j_q}) = \begin{cases} 1, & \text{if} \quad j_1,\ldots,j_q \text{ has degree zero} \\ 0, & \text{otherwise} \end{cases}$$

Proposition 2.6.8. *The antipode,* $\chi_{\mathcal{F}^*}$, *is dual to the antipode* $\chi_{\mathcal{F}}$.

Proof. $\mathcal{F}$ is a graded algebra, so $\mathcal{F} = \oplus_n \mathcal{F}_n$, where $\mathcal{F}_n$ denotes the degree n part of $\mathcal{F}$. Moreover, the antipode, $\chi_{\mathcal{F}}$, is a graded $\mathbf{Z}$ module morphism, i.e., $\chi_{\mathcal{F}} = \oplus_n \chi_{\mathcal{F}_n}$. Similarly, product, coproduct, unit and counit are graded $\mathbf{Z}$ module morphisms, i.e, $\varphi = \oplus_n \varphi_n$, coproduct $\Delta = \oplus_n \Delta_n$, unit, $\mu = \oplus_n \mu_n$ and counit, $\epsilon = \oplus_n \epsilon_n$. By definition 2.2.6 the antipode, $\chi_{\mathcal{F}}$, on $(\mathcal{F}, \varphi, \mu, \Delta, \epsilon)$ satisfies following equation:

$$\varphi \circ (\chi_{\mathcal{F}} \otimes 1_{\mathcal{F}}) \circ \Delta = \mu \circ \epsilon = \varphi \circ (1_{\mathcal{F}} \otimes \chi_{\mathcal{F}}) \circ \Delta. \tag{2.37}$$

Applying contravariant graded functor $\mathrm{Hom}(-, \mathbf{Z})$ to the equation (2.37) we have:

$$\Delta^* \circ (\chi_{\mathcal{F}} \otimes 1_{\mathcal{F}})^* \circ \varphi^* = \epsilon^* \circ \mu^* = \Delta^* \circ (1_{\mathcal{F}} \otimes \chi_{\mathcal{F}})^* \circ \varphi^* \tag{2.38}$$

Since $(1_{\mathcal{F}} \otimes \chi_{\mathcal{F}})^* = 1_{\mathcal{F}} \otimes \chi_{\mathcal{F}}^*$, and $(\chi_{\mathcal{F}} \otimes 1_{\mathcal{F}})^* = \chi_{\mathcal{F}}^* \otimes 1_{\mathcal{F}}$, we have:

$$\Delta^* \circ (\chi_{\mathcal{F}}^* \otimes 1_{\mathcal{F}}) \circ \varphi^* = \epsilon^* \circ \mu^* = \Delta^* \circ (1 \otimes \chi_{\mathcal{F}}^*) \circ \varphi^* \tag{2.39}$$

where $\chi_{\mathcal{F}}^*$ is dual of $\chi_{\mathcal{F}}$ and satisfies the equation (2.39) which is for being antipode. By Coroallary 2.2.7 antipode is unique, hence $\chi_{\mathcal{F}}^*$ is the antipode for $(\mathcal{F}^*, \Delta^*, \mu^*, \varphi^*, \epsilon^*)$. This completes the proof.

□

Note that the Proposition 2.6.8 can be generalised for any Hopf algebra that has a dual Hopf algebra.

Remark 2.6.9. *In general for an infinite dimensional R-algebra $\mathcal{A}$ we do not have an isomorphism:*

$$\mathcal{A}^* \otimes \mathcal{A}^* \approx (\mathcal{A} \otimes \mathcal{A})^*.$$

But we have:

$$\mathcal{F}^* \otimes \mathcal{F}^* \approx (\mathcal{F} \otimes \mathcal{F})^*.$$

To be able to understand the isomorphism above, we recall the following properties of $\mathcal{F}$:

i. $\mathcal{F}$ is infinite dimensional free $\mathbf{Z}$ algebra which is graded, finite type and connected, i.e, $\mathcal{F} = \oplus_{n \geq 0} \mathcal{F}_n$, where $\mathcal{F}_n$ denotes finite rank free $\mathbf{Z}$-modules in each degree of n.

ii. By the dual of $\mathcal{F}$ we mean the *graded dual*, i.e., $\mathcal{F}^* = \oplus_{n \geq 0} \mathcal{F}_n^*$, where $\mathcal{F}_n^* = \mathrm{Hom}(\mathcal{F}_n, \mathbf{Z})$ which is the dual of $\mathcal{F}_n$, so $\mathcal{F}_n^*$ is free of rank n.

Keeping in mind the properties of $\mathcal{F}$ above, lets consider tensor product of $\mathcal{F}^*$ by itself:

$$\begin{aligned}\mathcal{F}^* \otimes \mathcal{F}^* &= \oplus_n (\mathcal{F}^* \otimes \mathcal{F}^*)_n \\ &= \oplus_n (\oplus_{i=0}^n \mathcal{F}_i^* \otimes \mathcal{F}_{n-i}^*) \\ &= \oplus_n \Big(\oplus_{i=0}^n \operatorname{Hom}(\mathcal{F}_i, \mathbf{Z}) \otimes \operatorname{Hom}(\mathcal{F}_{n-i}, \mathbf{Z}) \Big) \\ &= \oplus_n \oplus_{i=0}^n \operatorname{Hom}(\mathcal{F}_i, \mathbf{Z}) \otimes \operatorname{Hom}(\mathcal{F}_{n-i}, \mathbf{Z}).\end{aligned}$$

On the other hand, we now consider $(\mathcal{F} \otimes \mathcal{F})^*$, but before that we remind some properties of contravariant Hom functor and graded contravariant Hom functor on free $\mathbf{Z}$ modules:

iii. $\operatorname{Hom}(\mathcal{F}_i \otimes \mathcal{F}_j, \mathbf{Z}) \approx \operatorname{Hom}(\mathcal{F}_i, \mathbf{Z}) \otimes \operatorname{Hom}(, \mathbf{Z})$ because $\mathcal{F}_i, \mathcal{F}_j$ are free of finite rank.

iv. The functor $\operatorname{Hom}(-, \mathbf{Z})$ preserves finite direct sums, i.e., $\operatorname{Hom}(\oplus_{i=0}^n \mathcal{F}_n, \mathbf{Z}) \approx \oplus_{i=0}^n \operatorname{Hom}(\mathcal{F}_n, \mathbf{Z})$.

Now consider dual of $\mathcal{F} \otimes \mathcal{F}$:

$$\begin{aligned}(\mathcal{F} \otimes \mathcal{F})^* &= \oplus_n \operatorname{Hom}\Big((\mathcal{F} \otimes \mathcal{F})_n, \mathbf{Z} \Big) \quad \text{by ii.} \\ &= \oplus_n \operatorname{Hom}\Big(\oplus_{i=0}^n \mathcal{F}_i \otimes \mathcal{F}_{n-i}, \mathbf{Z} \Big) \\ &\approx \oplus_n \oplus_{i=0}^n \operatorname{Hom}(\mathcal{F}_i \otimes \mathcal{F}_{n-i}, \mathbf{Z}) \quad \text{by iv.} \\ &\approx \oplus_n \oplus_{i=0}^n \operatorname{Hom}(\mathcal{F}_i, \mathbf{Z}) \otimes \operatorname{Hom}(\mathcal{F}_{n-i}, \mathbf{Z}) \quad \text{by iii.} \\ &= \mathcal{F}^* \otimes \mathcal{F}^*.\end{aligned}$$

Note that the property iii. does not hold for infinite dimension case. And the functor $\operatorname{Hom}(-, \mathbf{Z})$ does not preserve infinite direct sums.

In conclusion, being $\mathcal{F}$ is finite type and taking the graded dual of $\mathcal{F}$ lead us to have an isomorphism: $\mathcal{F}^* \otimes \mathcal{F}^* \approx (\mathcal{F} \otimes \mathcal{F})^*$.

Proposition 2.6.10. *Let $S_{b_1,\ldots,b_p} \in \mathcal{F}^*$, then the antipode, $\chi_{\mathcal{F}}^*$, is given by*

$$\chi_{\mathcal{F}}^*(S_{b_1,\ldots,b_p}) = (-1)^p \sum S_{r_1,\ldots,r_f},$$

where the summation is over all coarsenings $r_1, \ldots, r_f$ of $b_p, \ldots, b_1$ [14].

Proof. Both $\chi_{\mathcal{F}}^*$ and $\chi_{\mathcal{F}}$ are graded $\mathbf{Z}$-module homomorphisms. Moreover, by Proposition 2.6.8 we know $\chi_{\mathcal{F}}^*$ is defined as graded dual of $\chi_{\mathcal{F}}$, i.e., $\chi_{\mathcal{F}^*} = \oplus_n \chi_{F_n^*}$. So we have the following:

$$\chi_{\mathcal{F}_n^*} : (\mathcal{F}^*)_n \to \mathcal{F}_n^*, \quad \chi_{\mathcal{F}_n^*}(S_{b_1,\ldots,b_p}) = S_{b_1,\ldots,b_p} \circ \chi_{\mathcal{F}_n} : \mathcal{F}_n \to \mathbf{Z}, \qquad (2.40)$$

where for each $n \geq 0$, $\chi_{\mathcal{F}_n^*}$ is a **Z**-module homomorphism, and $S_{b_1,\dots,b_p} \in \mathcal{F}_n^*$. Beside this, since $\mathcal{F}^*$ is of finite type, so for each n, $\mathcal{F}_n^*$ is a free module of finite rank.

To have a complete description for $\chi_{\mathcal{F}_n^*}(S_{b_1,\dots,b_p})$ in equation (2.40), we need to evaluate it for all basis elements $S^{j_1,\dots,j_n}$ of $\mathcal{F}_n$. Let $S^{j_1,\dots,j_n}$ be any basis element in $\mathcal{F}_n$,then we can evaluate $S^{j_1,\dots,j_n}$ under $\chi_{\mathcal{F}_n^*}(S_{b_1,\dots,b_p})$ as follows:

$$\chi_{\mathcal{F}_n^*}(S_{b_1,\dots,b_p})(S^{j_1,\dots,j_n}) = S_{b_1,\dots,b_p}\Big(\chi_{\mathcal{F}_n}(S^{j_1,\dots,j_n})\Big) \tag{2.41}$$

Beside this, we know by Corollary 2.4.14 we have:

$$\chi_{\mathcal{F}_n^*}(S^{j_1,\dots,j_n}) = \sum(-1)^g S^{t_1,\dots,t_g}, \tag{2.42}$$

where the summation is over all refinements $S^{t_1,\dots,t_g}$ of $S^{j_n,\dots,j_1}$. Hence substituting equation (2.42) in equation (2.41) we arrive at:

$$\chi_{\mathcal{F}_n^*}(S_{b_1,\dots,b_p})(S^{j_1,\dots,j_n}) = S_{b_1,\dots,b_p}\Big(\sum(-1)^g S^{t_1,\dots,t_g}\Big), \tag{2.43}$$

where the summation is over all refinements $S^{t_1,\dots,t_g}$ of $S^{j_n,\dots,j_1}$.

On the other hand, by definition 2.6.2 $S_{b_1,\dots,b_p}$ is defined with the duality given by.

$$S_{b_1,\dots,b_p}(S^{i_1,i_2,\dots,i_y}) = \begin{cases} 1 & p = y \text{ and } b_1 = i_1, b_2 = i_2, \dots, b_p = i_p \\ 0 & \text{otherwise,} \end{cases} \tag{2.44}$$

where $S^{i_1,i_2,\dots,i_y}$ is a basis element of $\mathcal{F}_n$. According to equation (2.44), the right hand side of equation (2.43) equals:

$$S_{b_1,\dots,b_p}\Big(\sum(-1)^g S^{t_1,\dots,t_g}\Big) = \begin{cases} (-1)^g & p = g, \text{ and } b_1 = t_1, b_2 = t_2, \dots, b_p = t_p \\ 0 & \text{otherwise,} \end{cases} \tag{2.45}$$

where $S^{t_1,\dots,t_g}$ is refinement of $S^{j_n,\dots,j_1}$. Now to be more precise let's re-write equation 2.43. Since the right hand side of equation (2.43) equals the right hand side of equation 2.45, then we have:

$$\chi_{\mathcal{F}_n^*}(S_{b_1,\dots,b_p})(S^{j_1,\dots,j_n}) = \begin{cases} (-1)^p & \text{if } S^{b_1,\dots,b_p} \text{ is a refinement of } S^{j_n,\dots,j_1} \\ 0 & \text{otherwise,} \end{cases}, \tag{2.46}$$

Beside this, if $S^{b_1,\dots,b_p}$ is a refinement of $S^{j_n,\dots,j_1}$, then $S^{j_n,\dots,j_1}$ is a coarsening of $S^{b_1,\dots,b_p}$. And using the fact if $j_n,\dots,j_1$ is a coarsening of $b_1,\dots,b_p$, then $j_1,\dots,j_n$ is a coarsening of $b_p,\dots,b_1$ in fact, $S^{j_1,\dots,j_n}$ is a coarsening of $S^{b_p,\dots,b_1}$. Hence, using these facts we re-write equation (2.46) in the following:

$$\chi_{\mathcal{F}_n^*}(S_{b_1,\ldots,b_p})(S^{j_1,\ldots,j_n}) = \begin{cases} (-1)^p & \text{if } S^{j_1,\ldots,j_n} \text{ is a coarsening of } S^{b_p,\ldots,b_1} \\ 0 & \text{otherwise,} \end{cases}, \tag{2.47}$$

Hence,

$$\chi_{\mathcal{F}}^*(S_{b_1,\ldots,b_p}) = (-1)^p \sum S_{j_1,\ldots,j_n},$$

where the summation is over all coarsenings $j_1, \ldots, j_n$ of $b_p, \ldots, b_1$. This completes the proof.

□

Example 2.6.11.

$$\chi_{\mathcal{F}^*}(S_{3,2,1}) = -S_{1,2,3} - S_{3,3} - S_{1,5} - S_6.$$

2.7 Mod p dual Leibniz-Hopf algebra

In this section for any prime p we consider the mod p dual Leibniz-Hopf algebra. We give more details while considering $p = 2$, i.e., mod 2 dual Leibniz-Hopf algebra. By the mod p dual Leibniz-Hopf algebra we mean $\mathcal{F}^* \otimes \mathbf{Z}/p$ which has the same algebraic structure as $\mathcal{F}^*$, but everything takes place over field $\mathbf{Z}/p$. It is denoted by $\mathcal{F}_p^*$. Like $\mathcal{F}^*$, $\mathcal{F}_p^*$ is a Hopf algebra with the antipode which is denoted by $\chi_{\mathcal{F}_p^*}$. $\chi_{\mathcal{F}_p^*}$ is defined by the same formula as $\chi_{\mathcal{F}^*}$. For the prime two, i.e., the mod 2 dual Leibniz Hopf algebra, the antipode is denoted by $\chi_{\mathcal{F}_2^*}$ Since we work on mod 2, the formula for antipode $\chi_{\mathcal{F}}^*$ in Proposition 2.6.10 is simplified into the antipode formula for $\chi_{\mathcal{F}_2^*}$ in the following:

$$\chi_{\mathcal{F}^*}(S_{b_1,\ldots,b_p}) = \sum S_{t_1,\ldots,t_n},$$

where the summation is over all coarsenings $S_{t_1,\ldots,t_n}$ of $S_{b_p,\ldots,b_1}$.

Our main goal will be to find the conjugation invariants in $\mathcal{F}$, $\mathcal{F}_p$ and dual of these algebras.

Chapter 3

Conjugation Invariants in the mod 2 Dual Leibniz-Hopf Algebra

$\mathrm{Im}(\chi_{\mathcal{F}_2^*}-1)$ and $\mathrm{Ker}(\chi_{\mathcal{F}_2^*}-1)$ are subvector spaces of $\mathcal{F}_2^*$. An element $w \in \mathcal{F}_2^*$ is an invariant under conjugation, $\chi_{\mathcal{F}_2^*}$, if $\chi_{\mathcal{F}_2^*}(w) = w$. In other words, $(\chi_{\mathcal{F}_2^*}-1)(w) = 0$. Thus, $\mathrm{Ker}(\chi_{\mathcal{F}_2^*}-1)$ is formed by the conjugation invariants in $\mathcal{F}_2^*$. Hence, if we can determine a basis for the $\mathrm{Ker}(\chi_{\mathcal{F}_2^*}-1)$, then we can find all conjugation invariants.

In this chapter, we will determine a basis for this vector space by proving Theorem 3.0.2 which is the main theorem of this chapter.

Remark 3.0.1. *In the rest of this thesis, in mod* 2 *cases, the reader should keep in mind that the identity map* -1 *will be the same as* $+1$.

Theorem 3.0.2. *A basis for* $\mathrm{Ker}(\chi_{\mathcal{F}_2^*}-1)$ *consists of:*

i. in even degrees, the $(\chi_{\mathcal{F}_2^*}-1)$*-image of all higher non-palindromes and all even-length palindromes*

ii. in odd degrees, the $(\chi_{\mathcal{F}_2^*}-1)$*-image of all higher non-palindromes and the* $\lambda_{F_2{}^*}$*-image of all odd-length palindromes,*

Here $\lambda_{\mathcal{F}_2^*}$ denotes the sum of all "left coarsenings", which we will fully define in Section 3.2.

For the beginning of a proof for Theorem 3.0.2, we will first prove Theorem 3.0.3.

Theorem 3.0.3. *The image of* $(\chi_{\mathcal{F}_2^*}-1)$ *on* $\mathcal{F}_2^*$ *has a basis consisting of the* $(\chi_{\mathcal{F}_2^*}-1)$*-images of all higher non-palindromes and all even-length palindromes.*

Note that there are no even-length palindromes in odd degrees. Before giving a proof for Theorem 3.0.3, we first consider linearly independent elements in $\mathrm{Im}(\chi^*_{\mathcal{F}_2} - 1)$.

3.1 Linear independence

Proposition 3.1.1. *Let $S_{i_1,\ldots,i_{2k}}$ be an even-length palindrome. Among the summands of longest length in $(\chi_{\mathcal{F}_2^*} - 1)(S_{i_1,\ldots,i_{2k}})$ there is one odd-length palindrome, $S_{i_1,\ldots,i_k+i_{k+1},\ldots,i_{2k}}$, and this odd-length palindrome does not occur as a longest summand in the $(\chi_{\mathcal{F}_2^*} - 1)$-image of any other even-length palindrome.*

Proof. In the $\chi_{\mathcal{F}_2^*}$-image of a length m basis element, all summands have length less than or equal to m, and the only length m summand is the reverse of the length m basis element. Thus, in the $(\chi_{\mathcal{F}_2^*} - 1)$-image of a palindrome, say, $S_{b_1,\ldots,b_r}$, all the summands have length strictly shorter than the length of $S_{b_1,\ldots,b_r}$. Because, $S_{b_1,\ldots,b_r}$ has coefficient 1 as a summand of $\chi_{\mathcal{F}_2^*}(S_{b_1,\ldots,b_r})$ and -1 as a summand of $(-1)(S_{b_1,\ldots,b_r})$. Hence, they cancel each other, so $S_{b_1,\ldots,b_r}$ occurs having a coefficient zero as a summand of $(\chi_{\mathcal{F}_2^*} - 1)(S_{b_1,\ldots,b_r})$.

If $S_{i_1,\ldots,i_{2k}}$ is an even-length palindrome, then in $(\chi_{\mathcal{F}_2^*} - 1)(S_{i_1,\ldots,i_{2k}})$, there are $2k - 1$ summands of length $2k - 1$, namely

$$S_{i_1+i_2,i_3,\ldots,i_{2k}}, S_{i_1,i_2+i_3,i_4,\ldots,i_{2k}}, \ldots, S_{i_1,\ldots,i_{2k-2},i_{2k-1}+i_{2k}}.$$

Among these longest $2k - 1$ length summands, as noted in the proof of Proposition 2.3.14 , there is an odd-length palindrome, namely

$$S_{i_1,\ldots,i_{k-1},i_k+i_{k+1},i_{k+2},\ldots,i_{2k}}.$$

Moreover, it is the only palindrome among these summands.

Now, let $S_{j_1,\ldots,j_{2l}}$ be another even-length palindrome, then similarly the only longest length palindrome of $(\chi_{\mathcal{F}_2^*} - 1)(S_{j_1,\ldots,j_{2k}})$ is

$$S_{j_1,\ldots,j_{l-1},j_l+j_{l+1},j_{l+2},\ldots,j_{2l}},$$

which is a summand of $(\chi_{\mathcal{F}_2^*} - 1)(S_{j_1,\ldots,j_{2l}})$ with $2l - 1$ length. For this to equal $S_{i_1,\ldots,i_{k-1},i_k+i_{k+1},i_{k+2},\ldots,i_{2k}}$, we must have $l = k, j_1 = i_1, \ldots, j_{l-1} = i_{k-1}, j_{l+2} = i_{k+2}, \ldots, j_{2l} = i_{2k}$ and $j_l + j_{l+1} = i_k + i_{k+1}$. Since, $S_{i_1,\ldots,i_{2k}}$ and $S_{j_1,\ldots,j_{2l}}$ are both ELPs , so we have equality: $j_{l+1} = j_l$ and $i_{k+1} = i_k$, from which can deduce that $j_l = i_k$ and $j_{l+1} = i_{k+1}$, then it follows that $S_{j_1,\ldots,j_{2l}} = S_{i_1,\ldots,i_{2k}}$. This completes the proof. ☐

Theorem 3.1.2. *Let $w_1, \ldots, w_m$ be all the higher non-palindromes in even degrees, and let $e_1, \ldots, e_z$ be all the even-length palindromes in even degrees. Then $(\chi_{\mathcal{F}_2^*} - 1)(w_1), \ldots, (\chi_{\mathcal{F}_2^*} - 1)(w_m), (\chi_{\mathcal{F}_2^*} - 1)(e_1), \ldots, (\chi_{\mathcal{F}_2^*} - 1)(e_z)$ are linearly independent.*

Proof. Let $w_1, \ldots, w_m$ be all the higher non-palindromes in even degrees, and let $e_1, \ldots, e_z$ be all the even-length palindromes in even degrees. Assume that $v_1, \ldots, v_k$ are distinct elements of $\{w_1, \ldots, w_m, e_1, \ldots, e_z\}$ with the property that;

$$(\chi_{\mathcal{F}_2^*} - 1)(v_1) + \cdots + (\chi_{\mathcal{F}_2^*} - 1)(v_k) = 0. \tag{3.1}$$

Moreover let's order these elements according to their length as follows:

$$length(v_1) \leq length(v_2) \leq \cdots \leq length(v_k),$$

and so that even-length palindromes of any length l come before higher non-palindromes of length l.

We know $v_1, \ldots, v_k$ are distinct elements of the set, $\{w_1, \ldots, w_m, e_1, \ldots, e_z\}$. Hence, either v_k is an even-length palindrome or v_k is a higher non-palindrome. If v_k is a higher non-palindrome, say with length r, then there are exactly two length r summands in $(\chi_{\mathcal{F}_2^*} - 1)(v_k)$. One of them is an HNP, v_k itself which comes from $-1(v_k)$, and the other one is the reverse of v_k, an LNP, which is a summand of $\chi_{\mathcal{F}_2^*}(v_k)$. All other summands of $\chi_{\mathcal{F}_2^*}(v_k)$ have length strictly less than r.

Furthermore, v_k cannot occur in the $(\chi_{\mathcal{F}_2^*} - 1)$-image of any other length r HNP, say $v_k^{'}$. Because, similarly, $(\chi_{\mathcal{F}_2^*} - 1)(v_k^{'})$ has only two length r summands, namely $v_k^{'}$, and its reverse, an LNP. And v_k is neither an LNP nor equal $v_k^{'}$.

Moreover, v_k cannot occur in the $(\chi_{\mathcal{F}_2^*} - 1)$-image of any HNPs of length shorter than r, say $v_k^{''}$ with length $r^{'}$. This is because, by the same argument above the longest summands of $(\chi_{\mathcal{F}_2^*} - 1)(v_k^{''})$ have length $r^{'}$, and $r^{'} < r$.

On the other hand, v_k cannot occur as a summand of ELP of length r or of a shorter length ELP under $(\chi_{\mathcal{F}_2^*} - 1)$. Because by the argument in the proof of Proposition 3.1.1, in the $(\chi_{\mathcal{F}_2^*} - 1)$-image of a palindrome, all summands have strictly shorter length than the palindrome. Hence all the summands of the $(\chi_{\mathcal{F}_2^*} - 1)$-image of an r length ELP will be of length which is strictly shorter than r from which we can deduce v_k cannot be any of these summands. In addition by the same argument above it can be easily seen v_k cannot occur as a summand of ELP of a shorter length than r under $(\chi_{\mathcal{F}_2^*} - 1)$.

We have established that v_k cannot occur in the image of a shorter higher non-palindrome, or in any other higher non-palindrome of the same length under $(\chi_{\mathcal{F}_2^*} - 1)$, and we also have showed that v_k cannot occur in the image of an ELP of the same length or of a shorter length ELP under $(\chi_{\mathcal{F}_2^*} - 1)$. Therefore v_k cannot occur in $(\chi_{\mathcal{F}_2^*} - 1)(v_1), \cdots, (\chi_{\mathcal{F}_2^*} - 1)(v_{k-1})$.

This summand cannot be cancelled, so the left-hand side of equation (3.1) cannot be zero. This contradiction shows that v_k is not a higher non-palindrome. Thus, there are no higher non-palindromes of the same length as v_k, because of our second assumption about the order of $v_1, \ldots, v_k$. Hence, v_k must be an ELP, so has length r. In this case, by Proposition 3.1.1 there is a unique odd-length palindrome summand in $(\chi_{\mathcal{F}_2^*} - 1)(v_k)$ of length $r-1$, and this odd-length palindrome does not occur as a longest summand in the $(\chi_{\mathcal{F}_2^*} - 1)$-image of any other even-length palindrome. Hence, this odd-length palindrome summand cannot occur in the $(\chi_{\mathcal{F}_2^*} - 1)$-image of any shorter length ELP or of any other ELP of the same length, namely r.

As noted above, there are no higher non-palindromes of the same length as v_k. So for this $r-1$ length OLP to be cancelled, it must be occur in the $(\chi_{\mathcal{F}_2^*} - 1)$-image of any HNP of length less than or equal to $r-1$.

This $r-1$ length OLP cannot occur in the $(\chi_{\mathcal{F}_2^*} - 1)$-image of any HNP of length $r-1$, because the longest summands in the $(\chi_{\mathcal{F}_2^*} - 1)-$ image of any $r-1$ length HNP are HNP itself and its reverse, and neither of them are OLP.

Furthermore, by the length consideration which is noted above, it is clear that this $r-1$ length OLP cannot occur in the $(\chi_{\mathcal{F}_2^*} - 1)$-image of any HNP of length strictly less than $r-1$. Hence, this $r-1$ length OLP cannot occur in the $(\chi_{\mathcal{F}_2^*} - 1)$-image of any HNP of length less than or equal to $r-1$.

Hence if v_k is an ELP, then this $r-1$ length OLP, which is a summand of $(\chi_{\mathcal{F}_2^*} - 1)(v_k)$ cannot occur in the $(\chi_{\mathcal{F}_2^*} - 1)$-image of any shorter length ELP or of any other ELP of the same length. And it also cannot occur in the $(\chi_{\mathcal{F}_2^*} - 1)$-image of any HNP of length strictly less than the length of w_k. Therefore it cannot occur in $(\chi_{\mathcal{F}_2^*} - 1)(v_1), \ldots, (\chi_{\mathcal{F}_2^*} - 1)(v_{k-1})$. Thus, it cannot be cancelled, so the left-hand side of equation (3.1) cannot equal zero. This contradicts to our initial assumption. Hence,

$$(\chi_{\mathcal{F}_2^*} - 1)(w_1), \ldots, (\chi_{\mathcal{F}_2^*} - 1)(w_m), (\chi_{\mathcal{F}_2^*} - 1)(e_1), \ldots, (\chi_{\mathcal{F}_2^*} - 1)(e_z)$$

are linearly independent. This proves the theorem.

□

Now we consider linearly independent elements in $\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$ for odd degrees.

Theorem 3.1.3. *In odd degrees, the higher non-palindromes in $\mathcal{F}_2^*$ have linearly independent images under $(\chi_{\mathcal{F}_2^*} - 1)$.*

Proof. The same argument in the proof of Theorem 3.1.2 also applies here. □

Remark 3.1.4. *Recall that, in odd degrees there are no ELPs.*

Corollary 3.1.5. *In each even degree* $2n$*,*

$$\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1) = \mathrm{Ker}(\chi_{F_2^*} - 1),$$

and

$$\dim \mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1) = \frac{1}{2} \dim(\mathcal{F}_2^*)_{2n}.$$

Proof. By Proposition 2.3.14 in each even degree $2n$, there are $2^{2n-2} - 2^{n-1}$ HNPs and 2^{n-1} ELPs, so there are 2^{2n-2} elements in the linearly independent subset of $\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$ given by Theorem 3.1.2. Hence,

$$\dim \mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1) \geq 2^{2n-2} = \frac{1}{2} \dim(\mathcal{F}_2^*)_{2n}.$$

In $\mathcal{F}_2^*$, the multiplication is overlapping shuffle, so it is commutative. Hence, by Proposition 2.2.8 we have:

$$\chi^2_{\mathcal{F}_2^*} = 1.$$

Therefore, we arrive at:

$$(\chi_{\mathcal{F}_2^*} - 1)(\chi_{F_2^*} - 1) = \chi^2_{\mathcal{F}_2^*} - 2\chi_{\mathcal{F}_2^*} + 1 = 0,$$

from which we can deduce:

$$\mathrm{Im}(\chi_{F_2^*} - 1) \subset \mathrm{Ker}(\chi_{F_2^*} - 1),$$

hence,

$$\dim \mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1) \geq \dim \mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1).$$

Furthermore, by the Rank-Nullity Theorem in each even degree $2n$ we have:

$$\dim \mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1) + \dim \mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1) = (\dim \mathcal{F}_2^*)_{2n},$$

therefore

$$\dim \mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1) = \dim \mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1) = \frac{1}{2} \dim(\mathcal{F}_2^*)_{2n}.$$

□

Theorem 3.1.6. *In even degrees, the image of* $(\chi_{\mathcal{F}_2^*} - 1)$ *has a basis consisting of the* $(\chi_{\mathcal{F}_2^*} - 1)$*-images of all higher non-palindromes and all even-length palindromes.*

Proof. By Theorem 3.1.2 the $(\chi_{\mathcal{F}_2^*} - 1)$-image of all higher non-palindromes and all even-length palindromes are linearly independent. On the other hand, by Corollary 3.1.5 for in any fixed degree, dim $\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1) = \frac{1}{2}\dim F_2^*$. Beside this, by Proposition 2.3.14 the number of all higher non-palindromes and even-length palindromes is equal to dim $\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$. Thus, $(\chi_{\mathcal{F}_2^*} - 1)$ images of all higher non-palindromes and all even-length palindromes also span $\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$. Therefore, they form a basis for $\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$ in even degrees of ${F_2}^*$. □

Corollary 3.1.7. *In even degrees,* $\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1) = \mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$.

3.2 Spanning set for $(\chi_{\mathcal{F}_2^*} - 1)$

We will first show that, in odd degrees, the $(\chi_{\mathcal{F}_2^*} - 1)$-images of all OLPs can be expressed in terms of the $(\chi_{\mathcal{F}_2^*} - 1)$-images of HNPs by the following proposition:

Proposition 3.2.1. *Let* $w_0 = S_{i_1,\ldots,i_{k+1},i_{k+2},\ldots,i_{2k+1}}$ *be an odd-length palindrome. Then*

$$(\chi_{\mathcal{F}_2^*} - 1)(w_0) = \sum (\chi_{\mathcal{F}_2^*} - 1)(S_{l_1,\ldots,l_m,i_{k+2},\ldots,i_{2k+1}}), \tag{3.2}$$

where the summation is over all proper coarsenings $l_1, \ldots, l_m$ *of* $i_1, \ldots, i_{k+1}$.

Note that proper condition ensures that $S_{l_1,\ldots,l_m,i_{k+2}\cdots,i_{2k+1}}$ is a higher non-palindrome. To prove this proposition, we need some technical results.

Example 3.2.2. *By Proposition 3.2.1, for OLP,* $S_{1,1,2,1,1}$*, we have:*

$$(\chi_{\mathcal{F}_2^*} - 1)(S_{1,1,2,1,1}) = (\chi_{\mathcal{F}_2^*} - 1)(S_{4,1,1}) + (\chi_{\mathcal{F}_2^*} - 1)(S_{2,2,1,1}) + (\chi_{\mathcal{F}_2^*} - 1)(S_{1,3,1,1})$$

Lemma 3.2.3. *Let* $S_{i_1,\ldots,i_{k+1},i_{k+2},\ldots,i_{2k+1}}$ *be an odd-length palindrome, and let* $l_1, \ldots, l_m$ *be a proper coarsening of* $i_1, \ldots, i_{k+1}$*, and let* v *be any summand of* $(\chi_{\mathcal{F}_2^*} - 1)(S_{l_1,\ldots,l_m,i_{k+2}\cdots,i_{2k+1}})$*. Then the number of proper coarsenings* $q_1, \ldots, q_r$ *of* $i_1, \ldots, i_{k+1}$ *for which* v *is a summand of* $(\chi_{\mathcal{F}_2^*} - 1)(S_{q_1,\ldots,q_r,i_{k+2}\cdots,i_{2k+1}})$ *is odd.*

Proof. Let $S_{i_1,\dots,i_{k+1},i_{k+2},\dots,i_{2k+1}}$ be an odd-length palindrome, and let v be a summand of $(\chi_{\mathcal{F}_2^*}-1)(S_{l_1,\dots,l_m,i_{k+2},\dots,i_{2k+1}})$, where $l_1,\dots,l_m$ is a proper coarsening of $i_1,\dots,i_{k+1}$. Then, either $v=S_{l_1,\dots,l_m,i_{k+2},\dots,i_{2k+1}}$ or v is a summand of $\chi_{\mathcal{F}_2^*}(S_{l_1,\dots,l_m,i_{k+2},\dots,i_{2k+1}})$. In other words, $v=S_{l_1,\dots,l_m,i_{k+2}\dots,i_{2k+1}}$ or v is a coarsening of $S_{i_{2k+1},\dots,i_{k+2},l_m,\dots,l_1}$, where $l_m,\dots,l_1$ is a proper coarsening of $i_{k+1},\dots,i_1$.($l_m,\dots,l_1$ is reverse of the word $l_1,\dots,l_m$.)

If v is a summand of $(\chi_{\mathcal{F}_2^*}-1)(S_{q_1,\dots,q_r,i_{k+2}\dots,i_{2k+1}})$, where $q_1,\dots,q_r$ is a proper coarsening of $i_1,\dots,i_{k+1}$, then similarly, $v=S_{q_1,\dots,q_r,i_{k+2},\dots,i_{2k+1}}$ or v is a coarsening of $S_{i_{2k+1},\dots,i_{k+2},q_r,\dots,q_1}$.

Of course, if $v=S_{l_1,\dots,l_m,i_{k+2}\dots,i_{2k+1}}$, then there is only one proper coarsening $q_1,\dots,q_r$ of $i_1,\dots,i_{k+1}$ for which $v=S_{q_1,\dots,q_r,i_{k+2},\dots,i_{2k+1}}$ or v is a coarsening of $S_{i_{2k+1},\dots,i_{k+2},q_r,\dots,q_1}$, namely $q_1,\dots,q_r=l_1,\dots,l_m$.

If $v=S_{l_1,\dots,l_m,i_{k+2},\dots,i_{2k+1}}$ and v is a coarsening of $S_{i_{2k+1},\dots,i_{k+2},q_r,\dots,q_1}$,then $S_{l_1,\dots,l_m,i_{k+2},\dots,i_{2k+1}}$ is a coarsening of $S_{i_{2k+1}\dots i_{k+2},q_r\dots,q_1}$. There are no proper coarsenings $q_1,\dots,q_r$ for which this holds. This is because if $S_{l_1,\dots,l_m,i_{k+2},\dots,i_{2k+1}}$ is a coarsening of $S_{i_{2k+1},\dots i_{k+2},q_r,\dots,\dots,q_1}$, then by the definition of coarsening $i_{2k+1}\geq q_1$. We also know $q_1\geq i_1$, since $q_1,\dots,q_r$ is a coarsening of $i_1,\dots,i_{k+1}$. So, $i_{2k+1}\geq q_1\geq i_1$. Beside this, $i_1=i_{2k+1}$, since $S_{i_1,\dots,i_{k+1},i_{k+2},\dots,i_{2k+1}}$ is a palindrome. Hence, we have equality: $i_{2k+1}=q_1=i_1$. Thus, we see that $l_1,\dots,l_m,i_{k+2}\dots,i_{2k}$ is a coarsening of $i_{2k+1},\dots,i_{k+2},q_r,\dots,q_2$, where $q_2,\dots,q_r$ is a coarsening of $i_2,\dots,i_{k+1}$. Consequently, we can apply the same argument to preceding term to see that $i_{2k}=q_2=i_2$, and so on until we see that $q_1=i_1,q_2=i_2,\dots,q_k=i_k$.

On the other hand, we know $q_1,\dots,q_r$ is a coarsening of $i_1,\dots,i_{k+1}$, and we have determined the first k part of $q_1,\dots,q_r$. So, $i_1,i_2,\dots,i_k,q_{k+1},\dots,q_r$ is a coarsening of $i_1,\dots,i_{k+1}$. Hence we must now have that $q_{k+1},\dots,q_r$ is a coarsening of i_{k+1}. And by the definition of coarsening, it is clear that this can only happen if $r=k+1$ which means $q_r=i_{k+1}$. Hence, $q_1,\dots,q_r=i_1,\dots,i_{k+1}$ is completely determined. Therefore $q_1,\dots,q_r$ is not a proper coarsening of $i_1,\dots,i_{k+1}$.

Thus, we see that if $v=S_{l_1,\dots,l_m,i_{k+2}\dots,i_{2k+1}}$, then there is only one proper coarsening which gives this summand namely, $q_1,\dots,q_r=l_1,\dots,l_m$.

If v is a coarsening of $S_{i_{2k+1},\dots,i_{k+2},l_m,\dots,l_1}$ and $v=S_{q_1,\dots,q_r,i_{k+2},\dots,i_{2k+1}}$, then there are no proper coarsenings $q_1,\dots,q_r$ for which this can happen. This can be seen by the same argument as above with $q_1,\dots,q_r$ and $l_1,\dots,l_m$ interchanged.

Finally, suppose that v is a coarsening of $S_{i_{2k+1},\dots,i_{k+2},l_m,\dots,l_1}$, and also a coarsening of $S_{i_{2k+1},\dots,i_{k+2},q_r,\dots,q_1}$. It is easily seen that, in this case, v is a coarsening of $S_{i_{2k+1},\dots,i_{k+2},i_{k+1},\dots,i_1}$.

Moreover, each proper coarsening of $S_{i_{2k+1},\dots,i_{k+2},i_{k+1},\dots,i_1}$ is obtained by turning at least one of the $2k$ commas of this palindrome into pluses. Thus, we can go from $S_{i_{2k+1},\dots,i_{k+2},i_{k+1},\dots,i_1}$ to v by turning a number of these $2k$ commas into pluses.

Remembering that we assumed v is a coarsening of $S_{i_{2k+1},\dots,i_{k+2},l_m,\dots,l_1}$, and also a coarsening of $S_{i_{2k+1},\dots,i_{k+2},q_r,\dots,q_1}$ so specifically, v is obtained by turning some (or none) of the k commas of $i_{2k+1},\dots,i_{k+1}$ into pluses, and turning some (at least one) of the k commas of $i_{k+1},\dots,i_1$ into pluses since $l_m,\dots,l_1$ is a proper coarsening of $i_{k+1},\dots,i_1$.

Let t be the number of commas in $i_{k+1},\dots,i_1$ that are turned into pluses in v, then $l_m,\dots,l_1$ corresponds to choosing a subset of these t commas. There are 2^t such subsets, and hence, there are 2^t coarsenings $q_1,\dots,q_r$ of $i_1,\dots,i_{k+1}$ with the property that v is a coarsening of $S_{i_{2k+1},\dots,i_{k+2},q_r,\dots,q_1}$. However this includes the empty set, which must be excluded for $q_1,\dots,q_r$ to be proper. Thus, there are 2^t-1 proper coarsenings $q_1,\dots,q_r$ of $i_1,\dots,i_{k+1}$ such that $S_{i_{2k+1},\dots,i_{k+2},q_r\dots,q_1}$ has v as a coarsening. And 2^t-1 is odd, this completes the proof.

□

Lemma 3.2.4. *Let $i_1,\dots,i_{k+1},i_{k+2},\dots,i_{2k+1}$ be an odd-length palindrome and let*

$$A=\{S_{j_1,\dots,j_m}: j_1,\dots,j_m \text{ is a proper coarsening of } i_1,\dots,i_{k+1},i_{k+2},\dots,i_{2k+1}\}$$

$$B=\{S_{l_1,\dots,l_n,i_{k+2},\dots,i_{2k+1}}: l_1,\dots,l_n \text{ is a proper coarsening of } i_1,\dots,i_{k+1}\}$$

$$C=\{S_{c_1,\dots,c_s}: c_1,\dots,c_s \text{ is a coarsening of } i_{2k+1},\dots,i_{k+2},l_n\dots,l_1 \text{ where } l_n,\dots,l_1 \text{ is a proper coarsening of } i_{k+1},\dots,i_1\}$$

then $B\cap C=\emptyset$, and $A=B\cup C$.

Proof. i. Proof of $B\cap C=\emptyset$.

If $x\in B$, then $x=S_{l_1,\dots,l_n,i_{k+2},\dots,i_{2k+1}}$, where $l_1,\dots,l_n$ is a proper coarsening of $i_1,\dots,i_{k+1}$. So the last k terms in x are $i_{k+2},\dots,i_{2k+1}=i_k,\dots,i_1$, since $S_{i_1,\dots,i_{k+1},\dots,i_{2k+1}}$ is a palindrome.

On the other hand, if $x\in C$, then $x=S_{c_1,\dots,c_s}$, where $c_1,\dots,c_s$ is a coarsening of $i_{2k+1},\dots,i_{k+2},m_n,\dots,m_1$, and $m_n,\dots,m_1$ is a proper coarsening of $i_{k+1},\dots,i_1$.

Hence, if $x\in B\cap C$, then the last term in x is $i_1=c_s$, and $c_s\geq m_1$ since $c_1,\dots,c_s$ is a coarsening of $i_{2k+1},\dots,i_{k+2},m_n,\dots,m_1$. On the other hand, $m_1 \geq i_1$ because, $m_n, \ldots, m_1$ is a proper coarsening of $i_{k+1}, \ldots, i_1$. Hence, $i_1 = c_s \geq m_1 \geq i_1$, from which we can conclude that we have the equality: $i_1 = c_s = m_1 = i_1$. Thus the penultimate term in x is $i_2 = c_{s-1} \geq m_2 \geq i_2$, continuing this, we find that $i_1 = c_s$, $i_2 = c_{s-1}, \ldots, i_k = c_{s-(k-1)}$. Thus the last k terms of $m_n, \ldots, m_1$ are $i_k \ldots, i_1$. However $m_n \ldots, m_1$ is a proper coarsening of $i_{k+1}, \ldots, i_1$ so, this cannot happen. Hence, there is no $x \in B \cap C$, i.e, $B \cap C = \emptyset$.

ii. Proof of $B \cup C \subset A$.

If $x \in B$, then $x = S_{l_1,\ldots,l_n,i_{k+2},\ldots,i_{2k+1}}$, where $l_1, \ldots, l_n$ is a proper coarsening of $i_1, \ldots, i_{k+1}$. Hence, by definition 2.3.15, it is clear that $S_{l_1,\ldots,l_n,i_{k+2}\ldots,i_{2k+1}}$ is also a proper coarsening of $S_{i_1,\ldots,i_{k+1},i_{k+2},\ldots,i_{2k+1}}$. Hence,

$$B \subset A.$$

On the other hand, if $x \in C$, then $x = S_{c_1,\ldots,c_s}$, where $c_1, \ldots, c_s$ is a coarsening of $i_{2k+1}, \ldots, i_{k+2}, m_n, \ldots, m_1$, and $m_n, \ldots, m_1$ is a proper coarsening of $i_{k+1}, \ldots, i_1$. Again it is clear that x is also a proper coarsening of $S_{i_1,\ldots,i_{k+1},i_{k+2},\ldots,i_{2k+1}}$. Thus

$$C \subset A.$$

Hence, we arrive at:

$$B \cup C \subset A.$$

iii. Proof of $A \subset B \cup C$.

Let $S_{j_1,\ldots,j_m}$ be any element of A. We need to show that either $S_{j_1,\ldots,j_m} \in B$ or $S_{j_1,\ldots,j_m} \in C$.

Since $S_{j_1,\ldots,j_m}$ is an element of A, then $j_1, \ldots, j_m$ is a proper coarsening of $i_1, \ldots, i_{k+1}, i_{k+2}, \ldots, i_{2k+1}$. Thus, $j_1, \ldots, j_m$ is obtained by changing some (at least one) $2k$ commas of $i_1, \ldots, i_{k+1}, i_{k+2}, \ldots, i_{2k+1}$ into pluses.

Particularly, if the last k indices of $j_1, \ldots, j_m$ match with the last k indices of $i_1, \ldots, i_{k+1}, i_{k+2}, \ldots, i_{2k+1}$, i.e., $j_{m-(k-1)} = i_{k+2}, \ldots, j_{m-1} = i_{2k}, j_m = i_{2k+1}$, then $j_1, \ldots, j_m$ is $j_1, \ldots, j_{m-k}, i_{k+2}, \ldots, i_{2k+1}$, where $j_1, \ldots, j_{m-k}$ is a proper coarsening of $i_1, \ldots, i_{k+1}$, because $j_1, \ldots, j_m$ is a proper coarsening of $i_1, \ldots, i_{k+1}, i_{k+2}, \ldots, i_{2k+1}$.

Now, suppose that this is not the case. So, $S_{j_1,\ldots,j_m}$ is obtained by

a. Turning at least one of the last k commas of $i_1, \ldots, i_{k+1}, \ldots, i_{2k+1}$ into pluses and

b. Turning some (or none) of the first k commas of $S_{i_1,\ldots,i_{k+1},\ldots,i_{2k+1}}$ into pluses.

Consider the result of only doing case (a), i.e. changing some of the last k commas of $S_{i_1,\ldots,i_{k+1},i_{k+2},\ldots,i_{2k+1}}$: we get $S_{i_1,\ldots,i_k,l_n,\ldots,l_1}$ where $l_n,\ldots,l_1$ is a proper coarsening of $i_{k+1},\ldots,i_{2k+1}$. Since $S_{i_1,\ldots,i_k,i_{k+1},\ldots,i_{2k+1}}$ is a palindrome so $i_{k+1},\ldots,i_{2k+1} = i_{k+1},\ldots,i_1$. Hence, $l_n,\ldots,l_1$ is then a proper coarsening of $i_{k+1},\ldots,i_1$, and we have equality: $S_{i_1,\ldots,i_k,l_n,\ldots,l_1} = S_{i_{2k+1},\ldots,i_{k+2},l_n,\ldots,l_1}$.

As a next step, applying (b) to $S_{i_{2k+1},\ldots,i_{k+2},l_n,\ldots,l_1}$ we can easily see that $S_{j_1,\ldots,j_m}$ is obtained from $S_{i_{2k+1},\ldots,i_{k+2},l_n,\ldots,l_1}$ by a coarsening. So $S_{j_1,\ldots,j_m} \in C$.

By i and ii we arrive: $A = B \cup C$. This completes the proof.

□

Example 3.2.5. *By Lemma 3.2.4, for OLP,* $S_{1,3,1}$*, we have:* $A = \{S_{4,1}, S_{1,4}, S_5\}$*,* $B = \{S_{4,1}\}$*, and* $C = \{S_{1,4}, S_5\}$*.*

Proof of Proposition 3.2.1. Let $w_0 = S_{i_1,\ldots,i_{k+1},i_{k+2},\ldots,i_{2k+1}}$ be an OLP, then by the definition of $\chi_{\mathcal{F}_2^*}$, and using the fact that w_0 is a palindrome, $(\chi_{\mathcal{F}_2^*}-1)(w_0)$ is the sum of all the proper coarsenings of $S_{i_1,\ldots,i_{k+1},i_{k+2},\ldots,i_{2k+1}}$. In other words, in the language of Lemma 3.2.4 we have:

$$(\chi_{\mathcal{F}_2^*} - 1)(w_0) = \sum_{a\in A} a. \tag{3.3}$$

Now consider the sum:

$$\sum (\chi_{\mathcal{F}_2^*} - 1)(S_{l_1,\ldots,l_m,i_{k+2},\ldots,i_{2k+1}}), \tag{3.4}$$

where the summation is over all proper coarsenings $l_1,\ldots,l_m$ of $i_1,\ldots,i_{k+1}$. It is clear that $S_{l_1,\ldots,l_m,i_{k+2}\ldots,i_{2k+1}}$ is an HNP, because $l_1,\ldots,l_m$ is strictly greater than $i_{2k+1},\ldots,i_{k+2}$ in the dictionary order.

Moreover, let v be a word, if v is in the sum in (3.4), then it must be a summand of $(\chi_{\mathcal{F}_2^*} - 1)(S_{l_1,\ldots,l_m,i_{k+2}\ldots,i_{2k+1}})$ for a proper coarsening $l_1,\ldots,l_m$ of $i_1,\ldots,i_{k+1}$. Then $v = S_{l_1,\ldots,l_m,i_{k+2}\ldots,i_{2k+1}}$ or v is a proper coarsening of $S_{i_{2k+1},\ldots,i_{k+2},l_m,\ldots,l_1}$. Hence, in the language of Lemma 3.2.4, $v \in B$ or $v \in C$, so $v \in A$, which means v is a summand in $\sum_{a\in A} a$.

On the other hand, the coefficient of v in (3.4) is the number of proper coarsenings $q_1,\ldots,q_r$ of $i_1,\ldots,i_{k+1}$ for which $(\chi_{\mathcal{F}_2^*}-1)(S_{q_1,\ldots,q_r,i_{k+2}\ldots,i_{2k+1}})$ has v as a summand. By Lemma 3.2.3 this number is odd. Hence it is one in mod 2. Therefore, we have :

$$\sum (\chi_{\mathcal{F}_2^*} - 1)(S_{l_1,\ldots,l_m,i_{k+2},\ldots,i_{2k+1}}) = \sum_{v\in A} v. \tag{3.5}$$

Consequently, by equation (3.3) and equation (3.5) we arrive at:

$$(\chi_{\mathcal{F}_2^*} - 1)(w_o) = \sum (\chi_{\mathcal{F}_2^*} - 1)(S_{l_1,\ldots,l_m,i_{k+2},\ldots,i_{2k+1}}),$$

where the summation is over all proper coarsenings $l_1, \ldots, l_m$ of $i_1, \ldots, i_{k+1}$. This completes the proof. □

We will show that the $(\chi_{\mathcal{F}_2^*} - 1)$-image of any LNP can be expressed in terms of the $(\chi_{\mathcal{F}_2^*} - 1)$-image of HNPs. Before that, we need the following technical result.

Proposition 3.2.6. *Let $S_{i_1,\ldots,i_n}$ be a lower non-palindrome. Then*

$$(\chi_{\mathcal{F}_2^*} - 1)(S_{i_1,\ldots,i_n}) = (\chi_{\mathcal{F}_2^*} - 1)(S_{i_n,\ldots,i_1}) + \sum (\chi_{\mathcal{F}_2^*} - 1)(S_{j_1,\ldots,j_k}),$$

where the summation is over all proper coarsenings $j_1, \ldots, j_k$ of $i_1, \ldots, i_n$.

Proof. Let $S_{i_1,\ldots,i_n}$ be a lower non-palindrome, then we have a corresponding HNP which is $S_{i_n,\ldots,i_1}$. Applying $(\chi_{\mathcal{F}_2^*} - 1)$ to this HNP we get:

$$(\chi_{\mathcal{F}_2^*} - 1)(S_{i_n,\ldots,i_1}) = S_{i_n,\ldots,i_1} + S_{i_1,\ldots,i_n} + \sum S_{j_1,\ldots,j_k}, \tag{3.6}$$

where $j_1, \ldots, j_k$ is a proper coarsenings of $i_1, \ldots, i_n$.

In $\mathcal{F}_2^*$, we know

$$(\chi_{\mathcal{F}_2^*} - 1) \circ (\chi_{\mathcal{F}_2^*} - 1) = 0.$$

Therefore, applying $(\chi_{\mathcal{F}_2^*} - 1)$ to both sides of equation (3.6) yields:

$$0 = (\chi_{\mathcal{F}_2^*} - 1)(S_{i_n,\ldots,i_1}) + (\chi_{\mathcal{F}_2^*} - 1)(S_{i_1,\ldots,i_n}) + \sum (\chi_{\mathcal{F}_2^*} - 1)(S_{j_1,\ldots,j_k}), \tag{3.7}$$

where the summation is over all proper coarsenings $j_1, \ldots, j_k$ of $i_1, \ldots, i_n$. Re-writing equation (3.7) we have:

$$(\chi_{\mathcal{F}_2^*} - 1)(S_{i_1,\ldots,i_n}) = (\chi_{\mathcal{F}_2^*} - 1)(S_{i_n,\ldots,i_1}) + \sum (\chi_{\mathcal{F}_2^*} - 1)(S_{j_1,\ldots,j_k}),$$

where $j_1, \ldots, j_k$ is a proper coarsening of $i_1, \ldots, i_n$. This completes the proof. □

Theorem 3.2.7. *Let w_0 be a lower non-palindrome, then $(\chi_{\mathcal{F}_2^*} - 1)(w_0)$ can be written as a linear combination of $(\chi_{\mathcal{F}_2^*} - 1)$-images of higher non-palindromes.*

Proof. Let w_0 be a lower non-palindrome in odd degrees, the proof is by induction on length of w_0. A lower non-palindrome must have length greater than or equal to two, because otherwise it is a palindrome. If length of w_0 is two, then take LNP, $w_0 = S_{a,b}$. Its image under $(\chi_{\mathcal{F}_2^*} - 1)$:

$$(\chi_{\mathcal{F}_2^*} - 1)(w_0) = S_{a,b} + S_{b,a} + S_{a+b} = (\chi_{\mathcal{F}_2^*} - 1)(S_{b,a}),$$

where $S_{b,a}$ is a higher non-palindrome.

Now assume that all lower non-palindromes of length strictly less than y ($y > 2$) have the $(\chi_{\mathcal{F}_2^*} - 1)$-images that can be written as linear combinations of the $(\chi_{\mathcal{F}_2^*} - 1)$-images of higher non-palindromes. Let $w_0 = S_{b_1,\ldots,b_y}$ be a length y LNP, then by Proposition 3.2.6 we have:

$$(\chi_{\mathcal{F}_2^*} - 1)(w_0) = (\chi_{\mathcal{F}_2^*} - 1)(S_{b_y,\ldots,b_1}) + \sum(\chi_{\mathcal{F}_2^*} - 1)(S_{g_1,\ldots,g_p}), \tag{3.8}$$

where $S_{b_y,\ldots,b_1}$ is a higher non-palindrome and each $S_{g_1,\ldots,g_p}$ is either

i. a higher non-palindrome,

ii. an odd-length palindrome, in which case $(\chi_{\mathcal{F}_2^*} - 1)(S_{g_1,\ldots,g_p})$ is a linear combination of $(\chi_{\mathcal{F}_2^*} - 1)$-images of higher non-palindromes by Proposition 3.2.1, or

iii. a lower non-palindrome. In this case the inductive hypothesis applies, because $g_1, \ldots, g_p$ is a proper coarsening of the reverse of w_0, namely $S_{b_y,\ldots,b_1}$, so has length strictly less than y. Hence, $(\chi_{\mathcal{F}_2^*} - 1)(S_{g_1,\ldots,g_p})$ is a linear combination of the $(\chi_{\mathcal{F}_2^*} - 1)$-images of higher non-palindromes.

   Thus, in each case, $(\chi_{\mathcal{F}_2^*} - 1)(S_{g_1,\ldots,g_p})$ can be written as a linear combination of the $(\chi_{\mathcal{F}_2^*} - 1)$-images of higher non-palindromes. This completes the proof.

□

Theorem 3.2.8. *In odd degrees, the image of* $(\chi_{\mathcal{F}_2^*} - 1)$ *has a basis consisting of the* $(\chi_{\mathcal{F}_2^*} - 1)$*-images of all higher non-palindromes.*

Proof. In odd degrees, HNPs, LNPs, and OLPs form a basis for $\mathcal{F}_2^*$. Hence, $\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$ is spanned by the $(\chi_{\mathcal{F}_2^*} - 1)$-image of HNPs, LNPs, and OLPs. We can reduce this spanning set: by Proposition 3.2.1 the $(\chi_{\mathcal{F}_2^*} - 1)$-image of an OLPs can be written as a linear combination of the $(\chi_{\mathcal{F}_2^*} - 1)$-images of HNPs, and by Theorem 3.2.7 the $(\chi_{\mathcal{F}_2^*} - 1)$-image of LNPs also can be written as a linear combination of the $(\chi_{\mathcal{F}_2^*} - 1)$-images of HNPs. Hence the

$(\chi_{\mathcal{F}_2^*}-1)$-image of OLPs and LNPs are linearly dependent with the $(\chi_{\mathcal{F}_2^*}-1)$-image of HNPs. Therefore, in odd degrees, the $(\chi_{\mathcal{F}_2^*}-1)$-image of all HNPs also spans $\operatorname{Im}(\chi_{\mathcal{F}_2^*}-1)$.

On the other hand, by Theorem 3.1.3, in odd degrees, the $(\chi_{\mathcal{F}_2^*}-1)$-images of higher non-palindromes are also linearly independent. Hence they form a basis for $\operatorname{Im}(\chi_{\mathcal{F}_2^*}-1)$. This proves the theorem. □

Proof of Theorem 3.0.3. By Theorem 3.1.6 in even degrees, $(\chi_{\mathcal{F}_2^*}-1)$ has a basis consisting of the $(\chi_{\mathcal{F}_2^*}-1)$-images of all higher non-palindromes and even-length palindromes. On the other hand, by Theorem 3.2.8 in odd degrees, $(\chi_{\mathcal{F}_2^*}-1)$ has a basis consisting of the $(\chi_{\mathcal{F}_2^*}-1)$ images of all higher non-palindromes. This proves the theorem. □

Corollary 3.2.9. *In the mod-2 dual Leibniz-Hopf algebra, $\mathcal{F}_2^*$, the dimension of the* $\operatorname{Im}(\chi_{F_2^*}-1)$ *in degree m is:*

$$\dim \operatorname{Im}(\chi_{\mathcal{F}_2^*}-1)_m = \begin{cases} 2^{2n-2}, & \text{if } m=2n, \\ 2^{2n-3}-2^{n-2}, & \text{if } m=2n-1. \end{cases}$$

Proof. By Corollary 3.1.5, in $2n$ degrees, the dimension of $\operatorname{Im}(\chi_{\mathcal{F}_2^*}-1)$ is 2^{2n-2}. On the other hand, by Proposition 2.3.14, in degree $2n-1$ there are $2^{2n-3}-2^{n-2}$ HNPs, so there are $2^{2n-3}-2^{n-2}$ elements in basis which is given by Theorem 3.2.8. Hence the dimension of $\operatorname{Im}(\chi_{\mathcal{F}_2^*}-1)$ is $2^{2n-3}-2^{n-2}$. This completes the proof. □

Corollary 3.2.10. *In the mod-2 dual Leibniz-Hopf algebra, $\mathcal{F}_2^*$, the dimension of the* $\operatorname{Ker}(\chi_{F_2^*}-1)$ *in degree m is:*

$$\dim \operatorname{Ker}(\chi_{\mathcal{F}_2^*}-1)_m = \begin{cases} 2^{2n-2}, & \text{if } m=2n, \\ 2^{2n-3}+2^{n-2}, & \text{if } m=2n-1. \end{cases}$$

Proof. In degree $2n$, by the Rank-Nullity Theorem we have:

$$\dim \operatorname{Im}(\chi_{\mathcal{F}_2^*}-1) + \dim \operatorname{Ker}(\chi_{\mathcal{F}_2^*}-1) = 2^{2n-1}.$$

Hence, by Corollary 3.2.9 it is clear that we have:

$$\dim \operatorname{Ker}(\chi_{\mathcal{F}_2^*}-1) = 2^{2n-1}-2^{2n-2} = 2^{2n-2}.$$

On the other hand, by using the same argument above, in degree $2n-1$, we have:

$$\dim \operatorname{Ker}(\chi_{\mathcal{F}_2^*}-1) = 2^{2n-1}-(2^{2n-3}-2^{n-2}) = 2^{2n-3}+2^{n-2}.$$

□

We have established the dimension for $\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)$. We will now give a proof for the main theorem of this chapter. Firstly, we need to give technical results and introduce a new terminology, which is the semi-image "$\lambda_{F_2^*}$."

Definition 3.2.11. Let $S_{b_1,\ldots,b_p}$ be a basis element of $\mathcal{F}_2^*$. The semi-image "$\lambda_{\mathcal{F}_2^*}$" is defined as $\lambda_{\mathcal{F}_2^*}(S_{b_1,\ldots,b_p}) = \sum(S_{l_1,\ldots,l_n})$, summed over all coarsenings $S_{l_1,\ldots,l_n}$ of $S_{b_1,\ldots,b_p}$ for which the last $\lfloor \frac{p}{2} \rfloor$ terms of $l_1,\ldots,l_n$ are the same as the last $\lfloor \frac{p}{2} \rfloor$ terms of $b_1,\ldots,b_p$.

Example 3.2.12. *The $\lambda_{\mathcal{F}_2^*}$-image of the odd-length palindrome $S_{1,1,2,1,1}$ is:*

$$\lambda_{\mathcal{F}_2^*}(S_{1,1,2,1,1}) = S_{1,1,2,1,1} + S_{2,2,1,1} + S_{1,3,1,1} + S_{4,1,1}.$$

Theorem 3.2.13. *In odd degrees, let $p_1,\ldots,p_r$ be all the odd-length palindromes, and let $h_1,\ldots,h_s$ be all the higher non-palindromes. Then*

$$\lambda_{\mathcal{F}_2^*}(p_1),\ldots,\lambda_{\mathcal{F}_2^*}(p_r),(\chi_{\mathcal{F}_2^*}-1)(h_1),\ldots,(\chi_{\mathcal{F}_2^*}-1)(h_s)$$

are linearly independent.

Proof. Let $p_1,\ldots,p_r$ are all the odd-length palindromes in odd degrees, and let $h_1,\ldots,h_s$ be all the higher non-palindromes in odd degrees. Suppose $p_1,\ldots,p_k$ are some distinct elements of $\{p_1,\ldots,p_r\}$ and $h_1,\ldots,h_l$ are some distinct elements of $\{h_1,\ldots,h_s\}$ with the property that:

$$\lambda_{\mathcal{F}_2^*}(p_1)+\cdots+\lambda_{\mathcal{F}_2^*}(p_k) = (\chi_{\mathcal{F}_2^*}-1)(h_1)+\cdots+(\chi_{\mathcal{F}_2^*}-1)(h_l). \tag{3.9}$$

Moreover, let's order these elements according to their lengths in a non-decreasing order, i.e,

$$length(p_k) \geq length(p_{k-1}) \geq \cdots \geq length(p_1), \tag{3.10}$$

and

$$length(h_l) \geq length(h_{l-1}) \geq \cdots \geq length(h_1). \tag{3.11}$$

Let m be the length of p_k, then by definition 3.2.11, the only length m summand in $\lambda_{\mathcal{F}_2^*}(p_k)$ is p_k, namely p_k itself. On the other hand, by the ordering assumption (3.10), there can be other OLPs that have length m on the left hand side of equation 3.9. To be more precise, let i be the smallest index such that p_i has length m, then similarly, in $\lambda_{\mathcal{F}_2^*}(p_i)$, there is only one summand of the same length as p_i, namely p_i itself. Consequently, the only length m summands in $\lambda_{\mathcal{F}_2^*}(p_1)+\cdots+\lambda_{\mathcal{F}_2^*}(p_k)$ will be those p_i that have length m, i.e., $p_i, p_{i+1},\ldots,p_{k-1},p_k$. And $p_1,\ldots,p_{i-1}$ will have length strictly less than m.

Beside this, since $p_1, \ldots, p_k$ are all distinct, $p_i, p_{i+1}, \ldots, p_{k-1}, p_k$ cannot cancel, so the maximal-length summands on the left hand side of equation (3.9) have length m and are palindromes.

Now, let's consider the right hand side of equation 3.9. Let n be the length of h_l, then the only length n summands in $(\chi_{\mathcal{F}_2^*} - 1)(h_l)$ are h_l and its reverse, which is an LNP. Again, by the assumption of ordering, (3.11), there can be other HNPs that have length n on the right hand side of equation 3.9. Let j be the smallest index such that h_j has length n, then in the same manner, the only length n summands in $(\chi_{\mathcal{F}_2^*} - 1)(h_j)$ are h_j and its reverse. Following this, the only length n summands in $(\chi_{\mathcal{F}_2^*} - 1)(h_1) + \cdots + (\chi_{\mathcal{F}_2^*} - 1)(h_l)$ are $h_j, h_{j+1}, \ldots, h_l$ and the reverse of those HNPs. And $h_1, \ldots, h_{j-1}$ will have length which is strictly less than n.

Furthermore, since $h_1, \ldots, h_l$ are all distinct, $h_j, h_{j+1}, \ldots, h_l$ and the reverse of those HNPs cannot cancel, so the maximal-length summand on the right hand side of equation (3.9) have length n and are HNPs and LNPs. In other words these n length summands are non palindromes.

Finally, we see that, the maximal-length summands on the left hand side of equation (3.9) are palindromes, whereas the maximal-length summands on the right hand side of equation (3.9) are non-palindromes. This leads to a contradiction which shows that equation (3.9) cannot hold unless both sides are zero. Therefore,

$$\lambda_{\mathcal{F}_2^*}(p_1), \ldots, \lambda_{\mathcal{F}_2^*}(p_r), (\chi_{\mathcal{F}_2^*} - 1)(h_1), \ldots, (\chi_{\mathcal{F}_2^*} - 1)(h_s)$$

are linearly independent. This completes the proof.

□

We can now give the proof of the main theorem.

Proof of Theorem 3.0.2. In even degrees, by Corollary 3.1.7 we have:

$$\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1) = \mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1).$$

Therefore a basis for $\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$ is also a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)$, and by Theorem 3.1.6 the image of $(\chi_{\mathcal{F}_2^*} - 1)$ has a basis consisting of the $(\chi_{\mathcal{F}_2^*} - 1)$-images of all higher non-palindromes and all even-length palindromes. Hence in even degrees, this basis is also a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)$.

Now, let us consider odd degrees. In $\mathcal{F}_2^*$, we have:

$$(\chi_{\mathcal{F}_2^*} - 1) \circ (\chi_{\mathcal{F}_2^*} - 1) = 0,$$

so the $(\chi_{\mathcal{F}_2^*} - 1)$-image of all HNPs are in $\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)$.

On the other hand, in odd degrees, by Proposition 3.2.1 $\lambda_{\mathcal{F}_2^*}$-image of an odd-length palindrome is also in $\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)$. Moreover, by Theorem 3.2.13 $(\chi_{\mathcal{F}_2^*} - 1)$-image of all HNPs and $\lambda_{\mathcal{F}_2^*}$-image of all OLPs are linearly independent.

Beside this, by Proposition 2.3.14 the number of all HNPs and OLPs is:

$$(2^{2n-3} - 2^{n-2}) + 2^{n-1} = 2^{2n-3} + 2^{n-2},$$

which is exactly $\dim \mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)$. Hence, $(\chi_{\mathcal{F}_2^*} - 1)$-image of all HNPs and $\lambda_{\mathcal{F}_2^*}$-image of all OLPs also span $\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)$. Therefore, $(\chi_{\mathcal{F}_2^*} - 1)$-image of all HNPs and $\lambda_{\mathcal{F}_2^*}$-image of all OLPs form a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)$ in odd degrees of $\mathcal{F}_2^*$. □

Corollary 3.2.14. *In odd degrees, $\lambda_{\mathcal{F}_2^*}$-images of all odd-length palindromes form a basis for* $\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)/\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$.

Proof. Suppose that there are some odd-length palindromes $p_1, \ldots, p_k$ such that;

$$\lambda_{\mathcal{F}_2^*}(p_1), \ldots, \lambda_{\mathcal{F}_2^*}(p_k) \in \mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)/\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$$

with the property that:

$$\lambda_{\mathcal{F}_2^*}(p_1) + \cdots + \lambda_{\mathcal{F}_2^*}(p_k) \equiv 0 \quad mod\ \mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1),$$

which means $\lambda_{\mathcal{F}_2^*}(p_1) + \cdots + \lambda_{\mathcal{F}_2^*}(p_k) \in \mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$. And by Theorem 3.0.3 we know, in odd degrees $(\chi_{\mathcal{F}_2^*} - 1)$-image of higher non-palindromes form a basis for $\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$ which implies that there are higher non-palindromes $h_1, \ldots, h_k$ with the property that:

$$\lambda_{\mathcal{F}_2^*}(p_1) + \cdots + \lambda_{\mathcal{F}_2^*}(p_k) = (\chi_{\mathcal{F}_2^*} - 1)(h_1) + \cdots + (\chi_{\mathcal{F}_2^*} - 1)(h_l) \qquad (3.12)$$

But by the same argument in the proof of Theorem 3.2.13, equation (3.12) cannot hold unless both sides are zero, so it is a contradiction. Therefore, the $\lambda_{\mathcal{F}_2^*}$-image of all OLPs are linearly independent mod $\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$.

On the other hand, since $\mathcal{F}_2^*$ is a finite type,

$$\dim(\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)/\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)) = \dim \mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1) - \dim \mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1),$$

in each degree. Therefore, by Corollary 3.2.9 and Corollary 3.2.10 in each degree n we have:

$$\dim(\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)/\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)) = 2^{n-1}.$$

Beside this, by the Proposition 2.3.14 the number of OLPs in $2n-1$ degrees is 2^{n-1}. Hence, the $\lambda_{\mathcal{F}_2}$-images: $\lambda_{\mathcal{F}_2^*}(p_1), \ldots, \lambda_{\mathcal{F}_2^*}(p_k)$ also span $\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)/\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$, so the $\lambda_{\mathcal{F}_2^*}$-image of all odd-length palindromes form a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)/\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$.

□

Corollary 3.2.15. *[7] In degree m, the quotient* $\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)/\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$*, i.e., the Tate cohomology of* $\mathbf{Z}/2$ *acting on* $\mathcal{F}_2^*$ *by conjugation, has dimension*

$$\dim\left(\frac{\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)_m}{\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)_m}\right) = \begin{cases} 0, & \text{if } m = 2n, \\ 2^{n-1}, & \text{if } m = 2n-1 \end{cases}$$

Proof. It can be seen by Corollary 3.1.7 and by Corollary 3.2.14. □

Chapter 4

Conjugation Invariants in the mod p Dual Leibniz-Hopf Algebra

For any odd prime p, both $\mathrm{Im}(\chi_{\mathcal{F}_p^*}+1)$ and $\mathrm{Ker}(\chi_{\mathcal{F}_p^*}-1)$ are subvector spaces of $\mathcal{F}_p^*$. In particular, $\mathrm{Ker}(\chi_{\mathcal{F}_p^*}-1)$ is formed by the conjugation invariants in $\mathcal{F}_p^*$. In this chapter, we determine a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_p^*}-1)$ by proving Theorem 4.0.1 which is the main theorem of this chapter.

Theorem 4.0.1. *For any odd prime p, $\mathrm{Ker}(\chi_{\mathcal{F}_p^*}-1)$ has a basis consisting of the $(\chi_{\mathcal{F}_p^*}+1)$-images of all higher non-palindromes and all even-length palindromes.*

As Theorem 4.0.1 suggests, $\mathrm{Ker}(\chi_{\mathcal{F}_p^*}-1)$ coincides with $\mathrm{Im}(\chi_{\mathcal{F}_p^*}+1)$, we will consider a basis for $\mathrm{Im}(\chi_{\mathcal{F}_p^*}+1)$ to determine a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_p^*}-1)$. We will prove Theorem 4.0.1 by showing the following.

Theorem 4.0.2. *For any odd prime p, the image of $(\chi_{\mathcal{F}_p^*}+1)$ has a basis consisting of the $(\chi_{\mathcal{F}_p^*}+1)$-images of all higher non-palindromes and all even-length palindromes.*

To prove this theorem, we first consider linearly independent elements in $\mathrm{Im}(\chi_{\mathcal{F}_p^*}+1)$.

4.1 Linear Independence

Theorem 4.1.1. *Let $w_1,\ldots,w_m$ be all the higher non-palindromes in even degrees, and let $e_1,\ldots,e_z$ be all the even-length palindromes in even degrees.*

Then

$$(\chi_{\mathcal{F}_p^*}+1)(w_1),\ldots,(\chi_{\mathcal{F}_p^*}+1)(w_m),(\chi_{\mathcal{F}_p^*}+1)(e_1),\ldots,(\chi_{\mathcal{F}_p^*}+1)(e_z)$$

are linearly independent.

Proof. Let $w_1,\ldots,w_m$ be all the higher non-palindromes in even degrees, and let $e_1,\ldots,e_z$ be all the even-length palindromes in even degrees. Assume that $v_1,\ldots,v_k$ are distinct elements of $\{w_1,\ldots,w_m,e_1,\ldots,e_z\}$ with the property that;

$$b_1(\chi_{\mathcal{F}_p^*}+1)(v_1)+b_2(\chi_{\mathcal{F}_p^*}+1)(v_2)+\cdots+b_k(\chi_{\mathcal{F}_p^*}+1)(v_k)=0, \qquad (4.1)$$

for some non-zero coefficients $b_1,\ldots,b_k\in\mathbf{Z}/p$.

Moreover, let's order these elements according to their lengths in the following:

$$length(v_1)\leq length(v_2)\leq\cdots\leq length(v_k). \qquad (4.2)$$

Since $\{v_1,\ldots,v_k\}$ is a subset of $\{w_1,\ldots,w_m,e_1,\ldots,e_z\}$, either v_k is an ELP or an HNP.

If v_k is a higher non-palindrome, then by the same argument as in the proof of Theorem 3.1.2, v_k cannot occur in the image of a shorter higher non-palindrome, or in any other higher non-palindrome of the same length under $(\chi_{\mathcal{F}_p^*}+1)$. This is because, $\chi_{\mathcal{F}_2^*}$ is defined as mod 2 reduction of the formula, $\chi_{\mathcal{F}^*}$. Beside this, v_k, itself is one of the longest summands in $(\chi_{F_p^*}+1)(v_k)$ which comes from $(+1)(v_k)$ and is an HNP.

Note that again, we use the fact that v_k is one of the longest summands of $(\chi_{\mathcal{F}_p^*}+1)(v_k)$. Moreover the presence of v_k implies to be the right hand side of equation (4.1) is not zero, so we have contradiction. The key point is the presence of v_k with non zero coefficient as a summand of $(\chi_{\mathcal{F}_p^*}+1)(v_k)$.

In addition, v_k cannot occur as a summand of an even-length palindrome of the same length under $(\chi_{\mathcal{F}_p^*}+1)$. This is because, the longest summand of this ELP under $(\chi_{\mathcal{F}_p^*}+1)$ is itself, an ELP with coefficient 2, whereas, v_k is an HNP.

Moreover, by length consideration, the longest-length summands of shorter ELPs under $(\chi_{\mathcal{F}_p^*}+1)$ cannot also include v_k. Therefore, v_k cannot occur in the image of an ELP of the same length or of a shorter length ELP under $(\chi_{\mathcal{F}_p^*}+1)$.

We have established that v_k cannot occur in the image of a shorter higher non-palindrome, or in any other higher non-palindrome of the same length under $(\chi_{\mathcal{F}_p^*}+1)$, and we also have shown that v_k cannot occur in the image of an ELP of the same length or of a shorter length ELP under $(\chi_{\mathcal{F}_p^*}+1)$. Therefore, v_k cannot occur in $b_1(\chi_{\mathcal{F}_p^*}+1)(v_1), b_2(\chi_{\mathcal{F}_p^*}+1)(v_2),\ldots,b_{k-1}(\chi_{\mathcal{F}_p^*}+$

$1)(v_{k-1})$. Hence, v_k cannot be cancelled, so v_k occurs with non-zero coefficient b_k on the left-hand side of the equation (4.1). Therefore the left-hand side of equation (4.1) cannot equal zero.

This contradiction shows that v_k is not an HNP. Thus, there are no HNPs of the same length as v_k, because of our second assumption about the order, (4.2). Hence, v_k must be an ELP, and as we stated above, the longest summand of v_k under $(\chi_{\mathcal{F}_p^*}+1)$ is v_k, itself, an ELP. Thus, it is clear that, this ELP, v_k , cannot occur in the $(\chi_{\mathcal{F}_p^*}+1)$-image of any other ELP of the same length. And by length considerations, it is clear that v_k cannot occur in the $(\chi_{\mathcal{F}_p^*}+1)$-image of any shorter length LNP or of any shorter length of HNP. Hence, it cannot occur in $b_1(\chi_{\mathcal{F}_p^*}+1)(v_1), b_2(\chi_{\mathcal{F}_p^*}+1)(v_2), \ldots, b_{k-1}(\chi_{\mathcal{F}_p^*}+1)(v_{k-1})$. Thus v_k cannot be cancelled, so v_k occurs with non-zero coefficient b_k on the left-hand side of the equation (4.1), therefore the left-hand side of equation (4.1) cannot equal zero. This contradicts our initial assumption, so

$$(\chi_{\mathcal{F}_p^*}+1)(w_1), \ldots, (\chi_{\mathcal{F}_p^*}+1)(w_m), (\chi_{\mathcal{F}_p^*}+1)(e_1), \ldots, (\chi_{\mathcal{F}_p^*}+1)(e_z)$$

are linearly independent. This proves the theorem. □

Theorem 4.1.2. *In odd degrees, the higher non-palindromes have linearly independent images under* $(\chi_{\mathcal{F}_p^*}+1)$.

Proof. The same argument as in the proof of Theorem 4.1.1 applies here. □

For the remainder proof of Theorem 4.0.2, we need to determine a spanning set for $\mathrm{Im}(\chi_{\mathcal{F}_p^*}+1)$.

4.2 Spanning set for $\mathrm{Im}(\chi_{\mathcal{F}_p^*}+1)$

We will first show that, in odd degrees, the $(\chi_{\mathcal{F}_p^*}+1)$-images of all OLPs can be expressed in terms of the $(\chi_{\mathcal{F}_p^*}+1)$-images HNPs by the following relation:

Proposition 4.2.1. *Let* $S_{i_1,\ldots,i_k,i_{k+1},\ldots,i_{2k+1}}$ *be an odd-length palindrome. Then*

$$(\chi_{\mathcal{F}_p^*}+1)(-S_{i_1,\ldots,i_k,i_{k+1},\ldots,i_{2k+1}}) = \sum(\chi_{\mathcal{F}_p^*}+1)(S_{l_1,\ldots,l_m,i_{k+2},\ldots,i_{2k+1}}), \quad (4.3)$$

where the summation is over all proper coarsenings $l_1,\ldots,l_m$ *of* $i_1,\ldots,i_{k+1}$.

Note that the proper condition implies that $S_{l_1,\ldots,l_m,i_{k+2},\ldots,i_{2k+1}}$ is an HNP. To make a proof more manageable, Proposition 4.2.1 is stated in an equivalent form in Proposition 4.2.2.

Proposition 4.2.2. *Let $S_{i_1,\ldots,i_{k+1},i_{k+2},\ldots,i_{2k+1}}$ be an odd-length palindrome. Then*

$$\sum(\chi_{\mathcal{F}_p^*}+1)(S_{l_1,\ldots,l_m,i_{k+2},\ldots,i_{2k+1}})=0, \tag{4.4}$$

where the summation is over all coarsenings $l_1,\ldots,l_m$ of $i_1,\ldots,i_{k+1}$.

To give a proof for Proposition 4.2.2, we need following technical results:

Lemma 4.2.3. $\sum_{j=0}^{n}\binom{n}{j}(-1)^j=0$*, where n is a non-negative integer.*

Proof. Let n be a non-negative integer, substituting $a=-1$ and $b=1$ in the the binomial theorem

$$(a+b)^n=\sum_{j=0}^{n}\binom{n}{j}a^jb^{n-j},$$

it is easily seen that:

$$\sum_{j=0}^{n}\binom{n}{j}(-1)^j= \quad 0.$$

□

Corollary 4.2.4. *Let X be a finite set, then the number of odd-cardinality subsets of X is equal to the number of even-cardinality subsets of X.*

Proof. Let X be a finite n-element set, where n is an positive integer, then by Lemma 4.2.3, more explicitly we have:

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\binom{n}{4}-\binom{n}{5}+\binom{n}{6}-\binom{n}{7}+\cdots+(-1)^n\binom{n}{n}=0. \tag{4.5}$$

There are two cases to consider for n, either

Case1. n is even, or

Case2. n is odd.

In case 1, n is even, then by equation 4.5 we have:

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\binom{n}{6}+\cdots+\binom{n}{n}=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\binom{n}{7}+\cdots+\binom{n}{n-1}. \tag{4.6}$$

In case 2, n is odd, then by equation 4.5 we have:

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\binom{n}{6}+\cdots+\binom{n}{n-1} = \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\binom{n}{7}+\cdots+\binom{n}{n} \tag{4.7}$$

Considering the two cases above, it can be easily seen that the right hand side of equation (4.6) and equation (4.7) are the the sum of odd-cardinality subsets of X, and the left hand side of equation (4.6) and equation (4.7) are the sum of even-cardinality subsets of X. By the equality in equation (4.6) and equation (4.7), the number of odd-cardinality subsets of X is equal to the number of even-cardinality subsets of X. This completes the proof. □

Now we can introduce a proof of Proposition 4.2.2.

Proof of Proposition 4.2.2. Let $S_{i_1,\dots,i_{k+1},i_{k+2},\dots,i_{2k+1}}$ be an odd-length palindrome. Let $S_{d_1,\dots,d_n}$ be any word. If $S_{d_1,\dots,d_n}$ occurs in the sum (4.4) then, we shall show that it occurs with coefficient zero.

Before giving a proof we can observe that $S_{i_1,\dots,i_{k+1},i_{k+2},\dots,i_{2k+1}}$ cannot be in the $(\chi_{\mathcal{F}_p^*}+1)(S_{i_1,\dots,i_{k+1},i_{k+2},\dots,i_{2k+1}})$, so cannot be in the sum (4.4). This is because $S_{i_1,\dots,i_{k+1},i_{k+2},\dots,i_{2k+1}}$ has coefficient plus one as a summand of $1(S_{i_1,\dots,i_{k+1},i_{k+2},\dots,i_{2k+1}})$, and has coefficient $(-1)^{2k+1}$, i.e., minus one as a summand of $\chi^*_{\mathcal{F}_p}(S_{i_1,\dots,i_{k+1},i_{k+2},\dots,i_{2k+1}})$, hence they cancel each other.

If $S_{d_1,\dots,d_n}$ occurs in the sum (4.4) at all, it must be a summand of $(\chi^*_{\mathcal{F}_p}+1)(S_{l_1,\dots,l_m,i_{k+2},\dots,i_{2k+1}})$ for some coarsening $l_1,\dots,l_m$ of $i_1,\dots,i_{k+1}$, then, either

A. $S_{d_1,\dots,d_n} = S_{l_1,\dots,l_m,i_{k+2},\dots,i_{2k+1}}$ or

B. $S_{d_1,\dots,d_n}$ is a summand of $\chi_{\mathcal{F}_p^*}(S_{l_1,\dots,l_m,i_{k+2},\dots,i_{2k+1}})$.

In case A, if $S_{d_1,\dots,d_n} = S_{l_1,\dots,l_m,i_{k+2},\dots,i_{2k+1}}$, then $S_{d_1,\dots,d_n}$ occurs having coefficient one as a summand of $1(S_{l_1,\dots,l_m,i_{k+2},\dots,i_{2k+1}})$. Now, to find the coefficient of $S_{d_1,\dots,d_n}$ in the sum, we also need the number of other coarsenings $c_1,\dots,c_q$ of $i_1,\dots,i_{k+1}$ for which $S_{d_1,\dots,d_n}$ is a summand of $(\chi_{\mathcal{F}_p^*}+1)(S_{c_1,\dots,c_q,i_{k+2},\dots,i_{2k+1}})$.

If $S_{d_1,\dots,d_n} = S_{l_1,\dots,l_m,i_{k+2},\dots,i_{2k+1}}$, and $S_{d_1,\dots,d_n}$ is a summand of $(\chi_{\mathcal{F}_p^*}+1)(S_{c_1,\dots,c_q,i_{k+2},\dots,i_{2k+1}})$, then either

i. $S_{d_1,\dots,d_n} = S_{c_1,\dots,c_q,i_{k+2},\dots,i_{2k+1}}$ or

ii. $S_{d_1,\dots,d_n}$ is a summand of $\chi_{\mathcal{F}_p^*}(S_{c_1,\dots,c_q,i_{k+2},\dots,i_{2k+1}})$.

If $S_{d_1,\dots,d_n} = S_{c_1,\dots,c_q,i_{k+2},\dots,i_{2k+1}}$, since we also have equality above: $S_{d_1,\dots,d_n} = S_{l_1,\dots,l_m,i_{k+2},\dots,i_{2k+1}}$, then $q=m$, $c_1=l_1,\dots,c_q=l_m$, i.e, $c_1,\dots,c_q = l_1,\dots,l_m$, so there are no "other" coarsenings for which $S_{d_1,\ldots,d_n}$ occurs as a summand of $1(S_{l_1,\ldots,l_m,i_{k+2},\ldots,i_{2k+1}})$.

If $S_{d_1,\ldots,d_n}$ is a summand of $\chi_{\mathcal{F}_p^*}(S_{c_1,\ldots,c_q,i_{k+2},\ldots,i_{2k+1}})$, then $S_{d_1,\ldots,d_n}$ is a coarsening of $S_{i_{2k+1},\ldots,i_{k+2},c_q,\ldots,c_1}$. Hence $d_n \geq c_1$. On the other hand, since $c_1,\ldots,c_q$ is a coarsening of $i_1,\ldots,i_{k+1}$, we also have $c_1 \geq i_1$. Beside this, $i_1 = i_{2k+1}$, since $S_{i_1,\ldots,i_{k+1},i_{k+2},\ldots,i_{2k+1}}$ is an OLP. In addition, using the fact that $S_{d_1,\ldots,d_n} = S_{l_1,\ldots,l_m,i_{k+2},\ldots,i_{2k+1}}$, then $i_{2k+1} = d_n$, from which we can deduce $i_1 = i_{2k+1} = d_n$. Hence $d_n \geq c_1 \geq i_1 = d_n$, so we have equality: $d_n = c_1 = i_1$. It then follows that $d_{n-1} \geq c_2 \geq i_2 = d_{n-1}$, so $d_{n-2} = c_2 = i_2$. Repeating the same argument, we find that $c_3 = i_3, c_4 = i_4, \ldots, c_k = i_k$, so c_{k+1} must equal i_{k+1} and $q = k+1$, so $c_1,\ldots,c_q$ is not a proper coarsening. So, if $S_{d_1,\ldots,d_n}$ is a summand of $\chi_{\mathcal{F}_p^*}(S_{c_1,\ldots,c_q,i_{k+2},\ldots,i_{2k+1}})$, then there is only one other coarsening $c_1,\ldots,c_q$ for which $S_{d_1,\ldots,d_n}$ is a summand of $\chi_{\mathcal{F}_p^*}(S_{c_1,\ldots,c_q,i_{k+2},\ldots,i_{2k+1}})$, namely the improper one $c_1,\ldots,c_q = i_1,\ldots,i_{k+1}$. And by definition of $\chi_{\mathcal{F}_p^*}$ $S_{d_1,\ldots,d_n}$ occurs having a coefficient $(-1)^{2k+1}$ as a summand of $\chi_{\mathcal{F}_p^*}(S_{i_1,\ldots,i_{k+1},i_{k+2}\ldots,i_{2k+1}})$.

Hence, if $S_{d_1,\ldots,d_n} = S_{l_1,\ldots,l_m,i_{k+2},\ldots,i_{2k+1}}$, then the coefficient of $S_{d_1,\ldots,d_n}$ in sum (4.4) is zero, because $S_{d_1,\ldots,d_n}$ occurs having a coefficient 1 as a summand of $1(S_{l_1,\ldots,l_m,i_{k+2},\ldots,i_{2k+1}})$, and has a coefficient $(-1)^{2k+1}$ as a summand in $\chi^*_{\mathcal{F}_p}(S_{i_1,\ldots,i_{k+1},i_{k+2}\ldots,i_{2k+1}})$, and in no other terms.

In case B, if $S_{d_1,\ldots,d_n}$ is a summand of $\chi_{\mathcal{F}_p^*}(S_{l_1,\ldots,l_m,i_{k+2},\ldots,i_{2k+1}})$, then $S_{d_1,\ldots,d_n}$ is a coarsening of $S_{i_{2k+1},\ldots,i_{k+2},l_m,\ldots,l_1}$.

If $S_{d_1,\ldots,d_n}$ also occurs as $S_{c_1,\ldots,c_q,i_{k+2},\ldots,i_{2k+1}}$ for some coarsening $c_1,\ldots,c_q$ of $i_{k+1},\ldots,i_{2k+1}$, then we are back in case Aii, the argument in that case shows that $S_{d_1,\ldots,d_n}$ has coefficient zero. So we may assume that $S_{d_1,\ldots,d_n}$ does not occur as $S_{c_1,\ldots,c_q,i_{k+2},\ldots,i_{2k+1}}$ for any coarsening $c_1,\ldots,c_q$.

To find the coefficient of $S_{d_1,\ldots,d_n}$ in the sum, we need to find the number of all coarsenings $c_1,\ldots,c_q$ of $i_1,\ldots,i_{k+1}$ for which $S_{d_1,\ldots,d_n}$ is a summand of $\chi_{\mathcal{F}_p^*}(S_{c_1,\ldots,c_q,i_{k+2},\ldots,i_{2k+1}})$. Each such coarsening $c_1,\ldots,c_q$ contributes $(-1)^{k+q}$ to the coefficient of $S_{d_1,\ldots,d_n}$ in the sum (4.4). This is because, for each such coarsening $c_1,\ldots,c_q$ $S_{d_1,\ldots,d_n}$ occurs as a summand of $\chi_{\mathcal{F}_p^*}(S_{c_1,\ldots,c_q,i_{k+2},\ldots,i_{2k+1}})$, and by definition 2.4.13 any summand of $\chi_{\mathcal{F}_p^*}(S_{c_1,\ldots,c_q,i_{k+2},\ldots,i_{2k+1}})$ occurs having a coefficient $(-1)^{k+q}$. Hence $\chi_{\mathcal{F}_p^*}(S_{c_1,\ldots,c_q,i_{k+2},\ldots,i_{2k+1}})$ has $S_{d_1,\ldots,d_n}$ as a summand with a coefficient $(-1)^{k+q}$.

If $S_{d_1,\ldots,d_n}$ is a summand of $\chi_{\mathcal{F}_p^*}(S_{l_1,\ldots,l_m,i_{k+2},\ldots,i_{2k+1}})$, then $S_{d_1,\ldots,d_n}$ is a coarsening of $S_{i_{2k+1},\ldots,i_{k+2},l_m,\ldots,l_1}$. Since $S_{i_{2k+1},\ldots,i_{k+2},l_m,\ldots,l_1}$, is a coarsening of $S_{i_1,\ldots,i_{k+1},i_{k+2},\ldots,i_{2k+1}} = S_{i_{2k+1},\ldots,i_{k+2},i_{k+1},\ldots,i_1}$, then it follows that $S_{d_1,\ldots,d_n}$ is also a coarsening of $S_{i_{2k+1},\ldots,i_{k+2},i_{k+1},\ldots,i_1}$.

Moreover, each coarsening is obtained by turning some of the $2k$ commas of $i_{2k+1},\ldots,i_{k+2},i_{k+1},\ldots,i_1$ into pluses. Hence $d_1,\ldots,d_n$ is. Concretely,

$d_1, \ldots, d_n$ is obtained by turning $2k - (n-1)$ commas into pluses . Some of these (possibly none) will be from the k commas of $i_{2k+1}, \ldots, i_{k+1}$, the rest (including at least one because of our assumption above in the case B) will be from the k commas in $i_{k+1}, \ldots, i_1$. Let z be the number of commas taken from $i_{k+1}, \ldots, i_1$ so $1 \leq z \leq (2k - (n-1))$.

If $c_q, \ldots, c_1$ is a coarsening of $i_{k+1}, \ldots, i_1$ such that $d_1, \ldots, d_n$ is a coarsening of $i_{2k+1}, \ldots, i_{k+2}, c_q, \ldots, c_1$, then $c_q, \ldots, c_1$ is obtained from $i_{k+1}, \ldots, i_1$ by turning a subset of those z commas into pluses, a subset of cardinality $k - (q-1)$. Moreover, each such coarsening arises from exactly one such subset, and there are 2^z such subsets (2^z is even, since $z \geq 1$).

The coarsening $c_q, \ldots, c_1$ contributes $(-1)^{k+q}$ to the coefficient of $S_{d_1,\ldots,d_n}$, so this contribution is ± 1 according to parity of $k+q$, i.e., $+1$ if $k-(q-1)$ is odd and -1 if $k-(q-1)$ is even. Since $z \geq 1$, by Corollary 4.2.4 the number of subsets of odd cardinality is equal to the number of subsets of even cardinality, and hence net contribution to the coefficient of $S_{d_1,\ldots,d_n}$ is zero. (Note that counting the number of such coarsenings $c_q, \ldots, c_1$ means the counting the number of such sequences $c_q, \ldots, c_1$.) Hence, in both case A and case B, $S_{d_1,\ldots,d_n}$ occurs with coefficient zero in the sum (4.4). This completes the proof □

We will now show that the $(\chi_{\mathcal{F}_p^*}+1)$-images of all LNPs can be expressed in terms of the $(\chi_{\mathcal{F}_p^*}+1)$-images of all HNPs by the following proposition:

Proposition 4.2.5. *Let $S_{i_1,\ldots,i_n}$ be a lower non-palindrome. Then*

$$(\chi_{\mathcal{F}_p^*}+1)(S_{i_1,\ldots,i_n}) = (-1)^n\Big((\chi_{\mathcal{F}_p^*}+1)(S_{i_n,\ldots,i_1}) + \sum(\chi_{\mathcal{F}_p^*}+1)(S_{j_1,\ldots,j_k})\Big),$$

where the summation is over all proper coarsenings $S_{j_1,\ldots,j_k}$ of $S_{i_n,\ldots,i_1}$.

Proof. Let $S_{i_1,\ldots,i_n}$ be a LNP, then applying $(\chi_{\mathcal{F}_p^*}-1)$ to this LNP we have:

$$(\chi_{\mathcal{F}_p^*}-1)(S_{i_1,\ldots,i_n}) = -S_{i_i,\ldots,i_n} + (-1)^n S_{i_n,\ldots,i_1} + \sum(-1)^n S_{j_1,\ldots,j_k}, \qquad (4.8)$$

where $S_{j_1,\ldots,j_k}$ are all proper coarsenings of $S_{i_n,\ldots,i_1}$. On the other hand, in ${\mathcal{F}_p}^*$, multiplication is overlapping shuffle which is commutative, so by Proposition 2.2.8 we have:

$$\chi^2_{\mathcal{F}_p^*} = 1.$$

Therefore, we have:

$$(\chi_{\mathcal{F}_p^*}-1)(\chi_{F_p^*}+1) = \chi^2_{\mathcal{F}_p^*} + \chi_{\mathcal{F}_p^*} - \chi_{\mathcal{F}_p^*} - 1 = 0.$$

Therefore, applying $(\chi_{\mathcal{F}_p^*} + 1)$ to both side of equation (4.8) we arrive at;

$$0 = (\chi_{\mathcal{F}_p^*}+1)(-S_{i_1,\ldots,i_n})+(\chi_{\mathcal{F}_p^*}+1)\Big((-1)^n S_{i_n,\ldots,i_1})+\sum(-1)^n(\chi_{\mathcal{F}_p^*}+1)(S_{j_1,\ldots,j_k})\Big).$$

Thus, we have:

$$(\chi_{\mathcal{F}_p^*} + 1)(S_{i_1,\ldots,i_n}) = (-1)^n \Big((\chi_{\mathcal{F}_p^*} + 1)(S_{i_n,\ldots,i_1}) + \sum(\chi_{\mathcal{F}_p^*} + 1)(S_{j_1,\ldots,j_k})\Big),$$

where the summation is over all proper coarsenings $S_{j_1,\ldots,j_k}$ of $S_{i_n,\ldots,i_1}$. This completes the proof. □

Theorem 4.2.6. *If w_0 is a low non palindrome in degree $2n-1$, then $(\chi_{\mathcal{F}_p^*} + 1)(w_0)$ can be written as a linear combination of $(\chi_{\mathcal{F}_p^*} + 1)$-images of higher non palindromes.*

Proof. By induction on length of w_0. A low non palindrome must have length greater than or equal to two, because, otherwise it is a palindrome. If length of w_0 is two, then $(w_0) = S_{a,b}$, and we have:

$$(\chi_{\mathcal{F}_p^*} + 1)(w_0) = S_{a,b} + (-1)^2 S_{b,a} + (-1)^2 S_{a+b} = (\chi_{\mathcal{F}_p^*} + 1)(S_{b,a}),$$

where $S_{b,a}$ is a high non palindrome.

Now assume that all LNPs which have strictly shorter length than y have $(\chi_{\mathcal{F}_p^*}+1)$-images that can be written as linear combinations of the $(\chi_{\mathcal{F}_p^*}+1)$-images of HNPs. Let $w_0 = S_{b_1,\ldots,b_y}$ be an LNP with length y. By proposition 4.2.5,

$$(\chi_{\mathcal{F}_p^*} + 1)(w_0) = (-1)^n((\chi_{\mathcal{F}_p^*} + 1)(S_{b_y,\ldots,b_1}) + \sum(\chi_{\mathcal{F}_p^*} + 1)(S_{g_1,\ldots,g_p})). \quad (4.9)$$

where $S_{b_y,\ldots,b_1}$ is a high non palindrome and each $S_{g_1,\ldots,g_p}$ is either

i. a higher non-palindrome,

ii. an odd-length palindrome, in which case $(\chi_{\mathcal{F}_p^*} + 1)(S_{g_1,\ldots,g_p})$ is a linear combination of the $(\chi_{\mathcal{F}_p^*} + 1)$-images of HNPs by Proposition 4.2.1, or

iii. a lower non-palindrome. In this case the inductive hypothesis applies because $g_1, \ldots, g_p$ is a proper coarsening of the reverse of w_0, namely $S_{b_y,\ldots,b_1}$, so has length strictly less than y. Hence, the $(\chi_{\mathcal{F}_p^*}+1)(S_{g_1,\ldots,g_p})$ is a linear combination of the $(\chi_{\mathcal{F}_p^*} + 1)$-images of HNPs.

   Thus, in each case, the $(\chi_{\mathcal{F}_p^*} + 1)(S_{g_1,\ldots,g_p})$ can be written as a linear combination of the $(\chi_{\mathcal{F}_p^*} + 1)$-images of HNPs.

$\square$

Now we can introduce a proof of Theorem 4.0.2.

Proof of Proposition 4.0.2. In even degrees, by Theorem 4.1.1 the $(\chi_{F_p^*}+1)$-images of all HNPs and ELPs are linearly independent in $\mathrm{Im}(\chi_{F_p^*}+1)$.

Furthermore, HNPs, LNPs, ELPs, and OLPs form a basis for F_p^*. Hence, $\mathrm{Im}(\chi_{F_p^*}+1)$ is spanned by $(\chi_{F_p^*}+1)$-image of these basis elements. By Proposition 4.2.1 and and by Theorem 4.2.6 we can reduce this spanning set to $(\chi_{F_p^*}+1)$-images of all HNPs and ELPs. Hence, they form a basis for $\mathrm{Im}(\chi_{\mathcal{F}_p^*}+1)$.

On the other hand, in odd degrees, recalling Remark 3.1.4 it can easily seen that $(\chi_{F_p^*}+1)$-image of all HNPs span $\mathrm{Im}(\chi_{F_p^*}+1)$, since the same argument in even degrees also applies this case. Moreover, by Theorem 4.1.2 the $(\chi_{F_p^*}+1)$-image of all HNPs are linearly independent in $\mathrm{Im}(\chi_{F_p^*}+1)$. Therefore they form a basis for $\mathrm{Im}(\chi_{F_p^*}+1)$. This proves the theorem. $\square$

We know a basis for vector space $\mathrm{Im}(\chi_{\mathcal{F}_p^*}+1)$. Now showing that:

$$\mathrm{Ker}(\chi_{\mathcal{F}_p^*}-1)=\mathrm{Im}(\chi_{\mathcal{F}_p^*}+1),$$

we will deduce a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_p^*}-1)$.

Theorem 4.2.7. *On $\mathcal{F}_p^*$, we have:*

$$\mathrm{Ker}(\chi_{\mathcal{F}_p^*}-1)=\mathrm{Im}(\chi_{\mathcal{F}_p^*}+1).$$

Proof.

i. Proof of $\mathrm{Im}(\chi_{\mathcal{F}_p^*}+1)\subset\mathrm{Ker}(\chi_{\mathcal{F}_p^*}-1)$.

By the proof of Proposition 4.2.5 we have:

$$(\chi_{\mathcal{F}_p^*}-1)(\chi_{F_p^*}+1)=0,$$

from which we can deduce:

$$\mathrm{Im}(\chi_{F_p^*}+1)\subset\mathrm{Ker}(\chi_{F_p^*}-1).$$

ii. Proof of $\mathrm{Ker}(\chi_{\mathcal{F}_p^*}-1)\subset\mathrm{Im}(\chi_{\mathcal{F}_p^*}+1)$:

In $\mathcal{F}_p^*$, if $x\in\mathrm{Ker}(\chi_{\mathcal{F}_p^*}-1)$, then $\chi_{\mathcal{F}_p^*}(x)=x$, hence $(\chi_{\mathcal{F}_p^*}+1)(x)=2x$, so there is an element $x\in\mathcal{F}_p^*$, such that $(\chi_{\mathcal{F}_p^*}+1)(x)=2x$, hence $2x\in\mathrm{Im}(\chi_{\mathcal{F}_p^*}+1)$. For the remainder of the proof, in that case, we need to show that $x\in\mathrm{Im}(\chi_{\mathcal{F}_p^*}+1)$. On the other hand, if $(\chi_{\mathcal{F}_p^*}+1)(x)=2x$, using the fact that the characteristic of $\mathcal{F}_p^*$ is not equal two, we arrive at:

$$(\chi_{\mathcal{F}_p^*}+1)(\frac{x}{2})=x,$$

from which we conclude that $x \in \mathrm{Im}(\chi_{\mathcal{F}_p^*} + 1)$. Since for any $x \in \mathrm{Ker}(\chi_{\mathcal{F}_p^*} - 1)$ we show that $x \in \mathrm{Im}(\chi_{\mathcal{F}_p^*} + 1)$, hence $\mathrm{Ker}(\chi_{\mathcal{F}_p^*} - 1) \subset \mathrm{Im}(\chi_{\mathcal{F}_p^*} + 1)$.

By i. and ii the proof is complete.

□

We now give the proof of the main theorem of this chapter:

Proof of Theorem 4.0.1. By Proposition 4.2.7 we have :

$$\mathrm{Im}(\chi_{\mathcal{F}_p^*} + 1) = \mathrm{Ker}(\chi_{\mathcal{F}_p^*} - 1).$$

Therefore a basis for $\mathrm{Im}(\chi_{\mathcal{F}_p^*} + 1)$ is also a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_p^*} - 1)$. By Theorem 4.0.2 $(\chi_{F_p^*} + 1)$-image of all HNPs and ELPs form a basis for $\mathrm{Im}(\chi_{F_p^*} + 1)$, hence they also form a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_p^*} - 1)$.This proves the theorem. □

Now we can state the dimension for $\mathrm{Ker}(\chi_{\mathcal{F}_p^*} - 1)$.

Corollary 4.2.8. *In the mod-p dual Leibniz-Hopf algebra, $\mathcal{F}_p^*$, the dimension of the conjugation invariants in degree m is:*

$$\dim \mathrm{Ker}(\chi_{\mathcal{F}} - 1)_m = \begin{cases} 2^{2n-2}, & \text{if } m = 2n, \\ 2^{2n-3} - 2^{n-2}, & \text{if } m = 2n - 1. \end{cases}$$

Proof. By Proposition 2.3.14, in degree $2n$, there are $2^{2n-2} - 2^{n-1}$ HNPs and 2^{n-1} ELPs, so there are 2^{2n-2} elements in basis given by Theorem 4.0.1. Similarly, in degree $2n-1$ there are $2^{2n-3} - 2^{n-2}$ HNPs, so there are $2^{2n-3} - 2^{n-2}$ elements in basis given by Theorem 4.0.1. This completes the proof. □

Chapter 5

Conjugation Invariants in the Dual Leibniz-Hopf Algebra

A submodule of a free R-module need not be a free R-module. However, we know $\mathcal{F}^*$ is free over $\mathbf{Z}$ and using the fact that $\mathbf{Z}$ is a principal ideal domain[23, Theorem 6.1], the submodules: $\mathrm{Im}(\chi_{\mathcal{F}^*} + 1)$ and $\mathrm{Ker}(\chi_{\mathcal{F}^*} - 1)$ are also free over $\mathbf{Z}$. Similar to $\mathrm{Ker}(\chi_{\mathcal{F}_p^*} - 1)$, $\mathrm{Ker}(\chi_{\mathcal{F}^*} - 1)$ is also formed by the conjugation invariants.

In this chapter, using the previous results in the mod p Leibniz-Hopf algebra, $\mathcal{F}_p^*$, we will show that how we can take an easy approach to find a basis for $\mathrm{Ker}(\chi_{\mathcal{F}^*} - 1)$ by proving Theorem 5.0.1.

Theorem 5.0.1. *A basis for* $\mathrm{Ker}(\chi_{\mathcal{F}^*} - 1)$ *consists of the* $(\chi_{\mathcal{F}^*} + 1)$*-image of all higher non-palindromes and all even-length palindromes.*

As Theorem 5.0.1 implies $\mathrm{Ker}(\chi_{\mathcal{F}^*} - 1)$ coincides with $\mathrm{Im}(\chi_{\mathcal{F}^*} + 1)$. Firstly, we will consider a basis for $\mathrm{Im}(\chi_{\mathcal{F}^*} + 1)$ by the following theorem.

Theorem 5.0.2. *A basis for* $\mathrm{Im}(\chi_{\mathcal{F}^*} + 1)$ *consists of the* $(\chi_{\mathcal{F}^*} + 1)$*-images of all higher non-palindromes and all even-length palindromes.*

For the proof of Theorem 5.0.2, we first consider linearly independent elements in $\mathrm{Im}(\chi_{\mathcal{F}^*} + 1)$.

5.1 Linear Independence

Theorem 5.1.1. *In even degrees, let* $w_1, \ldots, w_m$ *be all the higher non-palindromes , and let* $e_1, \ldots, e_z$ *be all the even-length palindromes. Then* $(\chi_{\mathcal{F}^*}+1)(w_1), \ldots, (\chi_{\mathcal{F}^*}+1)(w_m), (\chi_{\mathcal{F}^*}+1)(e_1), \ldots, (\chi_{\mathcal{F}^*}+1)(e_z)$ *are linearly independent.*

Proof. Our proof starts with the observation that the definition of the conjugation in $\mathcal{F}^*$, $\chi_{\mathcal{F}^*}$, is same as the definition of conjugation in $\mathcal{F}_p^*$, $\chi_{\mathcal{F}_p^*}$. Furthermore, in the proof of Theorem 4.1.1 we did not refer to coefficients of the summands of ELPs and HNPs under $(\chi_{\mathcal{F}_p^*}+1)$, hence the same proof of Theorem 4.1.1 works also here. □

Theorem 5.1.2. *In odd degrees, the higher non-palindromes in $\mathcal{F}^*$ have linearly independent images under $(\chi_{\mathcal{F}^*}+1)$.*

Proof. Again, in the proof of Theorem 4.1.2 we did not refer to coefficients of summands of HNPs under $\chi_{\mathcal{F}^*}$. Hence the same argument as in the proof of Theorem 4.1.2 also applies here. □

For the proof of Theorem 5.0.2, we are left with the task of ascertaining a spanning set for $\mathrm{Im}(\chi_{F^*}+1)$

5.2 Spanning set for $\mathrm{Im}(\chi_{\mathcal{F}^*}+1)$

We will now show $(\chi_{\mathcal{F}^*}+1)$ is spanned by $(\chi_{F^*}+1)$-images of all HNPs and all ELPs. Let's first have a look the relation between the $(\chi_{\mathcal{F}^*}+1)$-image of OLPs and the $(\chi_{\mathcal{F}^*}+1)$-image of HNPs.

Proposition 5.2.1. *Let $S_{i_1,\ldots,i_k,i_{k+1},\ldots,i_{2k+1}}$ be an odd-length palindrome. Then*

$$(\chi_{\mathcal{F}^*}+1)(-S_{i_1,\ldots,i_k,i_{k+1},\ldots,i_{2k+1}}) = \sum(\chi_{\mathcal{F}^*}+1)(S_{l_1,\ldots,l_m,i_{k+2},\ldots,i_{2k+1}}), \quad (5.1)$$

where the summation is over all proper coarsenings $l_1,\ldots,l_m$ of $i_1,\ldots,i_{k+1}$.

Note that the proper condition implies that $S_{l_1,\ldots,l_m,i_{k+2},\ldots,i_{2k+1}}$ is an HNP.

Proof. The proof is same as the proof of Proposition 4.2.1. □

The relation between the $(\chi_{\mathcal{F}^*}+1)$-images of all LNPs and the $(\chi_{\mathcal{F}^*}+1)$-images of all HNPs is given by the following Theorem:

Theorem 5.2.2. *Let w_0 be a lower non-palindrome in $2n-1$ degrees, then $(\chi_{\mathcal{F}^*}+1)(w_0)$ can be written as a linear combination of $(\chi_{\mathcal{F}^*}+1)$-images of higher non-palindromes.*

Proof. The proof for Theorem 5.2.2 is similar to proof of Theorem 4.2.6. □

We can now give a proof for Theorem 5.0.2.

Proof of Theorem 5.0.2. Recalling Remark 3.1.4, by Theorem 4.0.2, in all degrees of $\mathcal{F}^*$, the $(\chi_{\mathcal{F}_p^*}+1)$-images of HNPs and ELPs span $\mathrm{Im}(\chi_{\mathcal{F}_p^*}+1)$. On the other hand, we know $\chi_{\mathcal{F}}$ is same as $\chi_{\mathcal{F}_p}$. Therefore, the $(\chi_{\mathcal{F}^*}+1)$-image of HNPs and ELPs also span $\mathrm{Im}(\chi_{\mathcal{F}^*}+1)$. Moreover, by Theorem 5.1.1 and by Theorem 5.1.2 in all degrees of $\mathcal{F}^*$, $(\chi_{\mathcal{F}^*}+1)$-image of all HNPs and ELPs are linearly independent in $\mathrm{Im}(\chi_{\mathcal{F}^*}+1)$. Hence they form a basis for $\mathrm{Im}(\chi_{\mathcal{F}^*}+1)$. This completes the proof. □

We have determined a basis for free submodule $\mathrm{Im}(\chi_{\mathcal{F}^*}+1)$. Now showing that:

$$\mathrm{Ker}(\chi_{\mathcal{F}^*}-1)=\mathrm{Im}(\chi_{\mathcal{F}^*}+1),$$

we will give a basis for $\mathrm{Ker}(\chi_{\mathcal{F}^*}-1)$.

Theorem 5.2.3. *In $\mathcal{F}^*$, we have:*

$$\mathrm{Ker}(\chi_{\mathcal{F}^*}-1)=\mathrm{Im}(\chi_{\mathcal{F}^*}+1).$$

Proof.

i. Proof that $\mathrm{Im}(\chi_{\mathcal{F}^*}+1)\subset\mathrm{Ker}(\chi_{\mathcal{F}^*}-1)$.

   $\mathcal{F}^*$ has the same multiplication as $\mathcal{F}_p^*$. Thus, remainder of the proof is same as proof of the i.part of Theorem 4.2.7.

ii. Proof that $\mathrm{Ker}(\chi_{\mathcal{F}^*}-1)\subset\mathrm{Im}(\chi_{\mathcal{F}^*}+1)$.

   If $x\in\mathrm{Ker}(\chi_{\mathcal{F}^*}-1)$, then $\chi_{\mathcal{F}^*}(x)=x$, hence $(\chi_{\mathcal{F}^*}+1)(x)=2x$, so $2x\in\mathrm{Im}(\chi_{\mathcal{F}^*}+1)$. For the remainder of the proof, we will show that if $2x\in\mathrm{Im}(\chi_{\mathcal{F}^*}+1)$, then $x\in\mathrm{Im}(\chi_{\mathcal{F}^*}+1)$. We first deal with the even degrees of $\mathcal{F}^*$.

   Let $w_1,\dots,w_m$ be all the higher non-palindromes in even degrees, and let $e_1,\dots,e_z$ be all the even-length palindromes in even degrees. Assume that $v_1,\dots,v_k$ are distinct elements in $\{w_1,\dots,w_m,e_1,\dots,e_z\}$, then by Theorem 5.0.2, there are distinct elements, $v_1,\dots,v_k$ in the set,$\{w_1,\dots,w_m,e_1,\dots,e_z\}$, such that:

$$2x=(\chi_{\mathcal{F}^*}+1)(b_1v_1+b_2v_2+\cdots+b_kv_k),\tag{5.2}$$

   for some coefficients $b_1,b_2,\dots,b_k\in\mathbf{Z}$. Since $\chi_{\mathcal{F}^*}+1$ is a $\mathbf{Z}$-module homomorphism, moreover, equation (5.2) has form:

$$2x=b_1(\chi_{\mathcal{F}^*}+1)(v_1)+b_2(\chi_{\mathcal{F}^*}+1)(v_2)+\cdots+b_k(\chi_{\mathcal{F}^*}+1)(v_k).\tag{5.3}$$

   Moreover, let's order $v_1,\dots,v_k$ according to their lengths in the following:

$$length(v_1)\le length(v_2)\le\cdots\le length(v_k).$$

Since $\{v_1, \ldots, v_k\}$ is a subset of $\{w_1, \ldots, w_m, e_1, \ldots, e_z\}$, either v_k is an ELP or an HNP. If v_k is an HNP, then v_k cannot occur in $b_1(\chi_{\mathcal{F}^*} + 1)(v_1) + b_2(\chi_{\mathcal{F}^*} + 1)(v_2) + \cdots + b_{k-1}(\chi_{\mathcal{F}^*} + 1)(v_{k-1})$. This is because, the definition of $\chi_{\mathcal{F}^*}$ is same as the definition of $\chi_{\mathcal{F}_p^*}$, and $(\chi_{\mathcal{F}^*} + 1)(v_k)$ has a summand which has the same length as v_k, which is v_k itself, an HNP, and comes from $(+1)(v_k)$. Hence, the same argument as in the proof of Theorem 4.1.1 applies here. Consequently, v_k cannot be cancelled, so v_k occurs with a coefficient b_k on the right-hand side of equation (5.3).

On the other hand, $2x \in \mathcal{F}^*$, and $\mathcal{F}^*$ has a basis, therefore $2x$ can be written uniquely as a linear combination of basis elements, so the coefficients of these basis elements are even. Expressing the right hand side of equation (5.3) with these basis elements, one basis element has coefficient b_k, so it is even. We have established that b_k is even.

If v_k is an ELP say with length r, then there can be no HNPs of the same length because of our second assumption about the order of $v_1, \ldots, v_k$. Moreover, Proposition 3.1.1 can be easily adapted to this case to see there is a unique odd-length palindrome summand in $(\chi_{\mathcal{F}^*} + 1)(v_k)$ of length $r-1$ with coefficient $(-1)^r = 1$, since r is even. In addition by the same argument in the proof of Theorem 3.1.2 this $r-1$ length OLP cannot occur in $b_1(\chi_{\mathcal{F}^*} + 1)(v_1) + b_2(\chi_{\mathcal{F}^*} + 1)(v_2) + \cdots + b_{k-1}(\chi_{\mathcal{F}^*} + 1)(v_{k-1})$. Thus, it cannot be cancelled, so this $r-1$ length OLP occurs with non-zero coefficient b_k on the right-hand side of the equation (5.3).

Since, $2x \in \mathcal{F}^*$, so in the same manner above b_k is even. Now we have established that whether v_k is an ELP or v_k is an HNP, it occurs with an even coefficient b_k, say $b_k = 2\bar{b_k}$, where $\bar{b_k} \in \mathbf{Z}$.

Now we define $\dot{x} = x - \bar{b_k}(\chi_{\mathcal{F}^*} + 1)(v_k)$. In particular, $\dot{x} \in \text{Ker}(\chi_{\mathcal{F}^*} - 1)$, because $x \in \text{Ker}(\chi_{\mathcal{F}^*} - 1)$, and by the proof of i above, $\bar{b_k}(\chi_{\mathcal{F}^*} + 1)(v_k) \in \text{Ker}(\chi_{\mathcal{F}^*} - 1)$. If we re-write equation (5.3) with respect to $\dot{x}$, we have:

$$2\dot{x} = b_1(\chi_{\mathcal{F}^*} + 1)(v_1) + b_2(\chi_{\mathcal{F}^*} + 1)(v_2) + \cdots + b_{k-1}(\chi_{\mathcal{F}^*} + 1)(v_{k-1}). \quad (5.4)$$

Now by the same argument above, b_{k-1} is even, say $b_{k-1} = 2\bar{b}_{k-1}$, where $\bar{b}_{k-1} \in \mathbf{Z}$. Thus, now we define $\ddot{x} = \dot{x} - \bar{b}_{k-1}(\chi_{\mathcal{F}^*} + 1)(v_{k-1})$. In the same manner above, this is in $\text{Ker}(\chi_{\mathcal{F}^*} - 1)$.

Repeating this argument we see that coefficients $b_{k-2}, \ldots, b_1$ occur as even, say $b_{k-2} = 2\bar{b}_{k-2}, \ldots, b_1 = 2\bar{b_1}$, where $\bar{b}_{k-2}, \ldots, \bar{b_1} \in \mathbf{Z}$. Hence we can re-write equation (5.3) as follows:

$$2x = 2\bar{b_1}\chi_{\mathcal{F}^*+1}(v_1) + 2\bar{b_2}\chi_{\mathcal{F}^*+1}(v_2) + \cdots + 2\bar{b_k}\chi_{\mathcal{F}^*+1}(v_k). \quad (5.5)$$

Dividing each side of equation (5.5) by 2 and re-writing, we obtain:

$$x = \bar{b_1}\chi_{F+1}(v_1) + \bar{b_2}\chi_{\mathcal{F}^*+1}(v_2) + \cdots + \bar{b_k}\chi_{\mathcal{F}^*+1}(v_k), \tag{5.6}$$

from which we can deduce that $x \in \text{Im}(\chi_{\mathcal{F}^*} + 1)$. Since we can do the same argument for all $x \in \text{Ker}(\chi_{\mathcal{F}^*} - 1)$, then $\text{Ker}(\chi_{\mathcal{F}^*} - 1) \subset \text{Im}(\chi_{\mathcal{F}^*} + 1)$. In addition by i we have $\text{Im}(\chi_{\mathcal{F}^*} + 1) \subset \text{Ker}(\chi_{\mathcal{F}^*} - 1)$. Hence $\text{Im}(\chi_{\mathcal{F}^*} + 1) = \text{Ker}(\chi_{\mathcal{F}^*} - 1)$ in even degrees of $\mathcal{F}^*$.

It is easily seen that the same proof for the even case above works for odd degree case, since there is no ELP in odd degrees. Hence, $\text{Im}(\chi_{\mathcal{F}^*} + 1) = \text{Ker}(\chi_{\mathcal{F}^*} - 1)$ in odd degrees of $\mathcal{F}^*$.

Finally by i. and ii. on $\mathcal{F}^*$ we see that:

$$\text{Im}(\chi_{\mathcal{F}^*} + 1) = \text{Ker}(\chi_{\mathcal{F}^*} - 1).$$

This completes the proof.

□

We now give the proof of the main theorem of this chapter:
Proof of Theorem 5.0.1 By Theorem 5.2.3 we have :

$$\text{Im}(\chi_{\mathcal{F}^*} + 1) = \text{Ker}(\chi_{\mathcal{F}^*} - 1).$$

Therefore a basis for $\text{Im}(\chi_{\mathcal{F}^*}+1)$ is also a basis for $\text{Ker}(\chi_{\mathcal{F}^*}-1)$. By Theorem 5.0.2 $(\chi_{F^*} + 1)$-image of all HNPs and ELPs form a basis for $\text{Im}(\chi_{F^*} + 1)$, hence they also form a basis for $\text{Ker}(\chi_{\mathcal{F}^*} - 1)$. This proves the theorem.

Now we can state the rank for $\text{Ker}(\chi_{\mathcal{F}^*} - 1)$.

Corollary 5.2.4. *In the dual Leibniz-Hopf algebra, $\mathcal{F}^*$, the rank of the conjugation invariants is:*

$$\text{rank Ker}(\chi_{\mathcal{F}^*} - 1)_m = \begin{cases} 2^{2n-2}, & \text{if } m = 2n, \\ 2^{2n-3} - 2^{n-2}, & \text{if } m = 2n-1. \end{cases}$$

Proof. The same proof for Corollary 4.2.8 also works for here. □

Chapter 6

Conjugation Invariants in the Leibniz-Hopf Algebra

Like $\mathcal{F}^*$, $\mathcal{F}$ is also free over $\mathbf{Z}$. Bearing in the mind that $\mathbf{Z}$ is principal ideal domain, both of the submodules $\mathrm{Im}(\chi_{\mathcal{F}} + 1)$ and $\mathrm{Ker}(\chi_{\mathcal{F}} - 1)$ are also free over $\mathbf{Z}$[23, Theorem 6.1], and $\mathrm{Ker}(\chi_{\mathcal{F}^*} - 1)$ is also formed by the conjugation invariants.

In this chapter, we will determine a basis for this submodule $\mathrm{Ker}(\chi_{\mathcal{F}^*} - 1)$ by proving Theorem 6.0.1 which is the main theorem of this chapter:

Theorem 6.0.1. *A basis for* $\mathrm{Ker}(\chi_{\mathcal{F}} - 1)$ *consists of:*

i. the $(\chi_{\mathcal{F}} + 1)$*-image of all higher non-palindromes and all odd-length palindromes in even degrees.*

ii. the $(\chi_{\mathcal{F}} + 1)$*-image of all higher non-palindromes in odd degrees.*

Before proving this theorem, similar to $\mathcal{F}^*$, we also first consider a basis for $\mathrm{Im}(\chi_{\mathcal{F}} + 1)$ to determine a basis for $\mathrm{Ker}(\chi_{\mathcal{F}} - 1)$ in the following theorem:

Theorem 6.0.2. *In the Leibniz-Hopf algebra,* $\mathcal{F}$*, in degree* n*, the submodule* $\mathrm{Im}(\chi_{\mathcal{F}} + 1)$ *has a basis consisting of:*

i. the $(\chi_{\mathcal{F}} + 1)$*-images of all odd-length palindromes and higher non-palindromes, if* n *is even, or*

ii. the $(\chi_{\mathcal{F}} + 1)$*-images of all higher non-palindromes, if* n *is odd.*

To give a proof, we first consider linearly independent elements in $\mathrm{Im}(\chi_{\mathcal{F}} + 1)$.

6.1 Linear independence

Proposition 6.1.1. *Let $S^{i_1,\dots,i_k,\dots,i_{2k-1}}$ be an odd-length palindrome in even degrees. Among the summands of shortest length in $(\chi_{\mathcal{F}}+1)(S^{i_1,\dots,i_k,\dots,i_{2k-1}})$ there is one even-length palindrome, $S^{i_1,\dots,i_{k-1},\frac{i_k}{2},\frac{i_k}{2},i_{k+2},\dots,i_{2k-1}}$, and this even-length palindrome does not occur as a shortest-length-summand in the (χ_F+1)-image of any other odd-length palindrome.*

Proof. We consider even degrees. In the $(\chi_{\mathcal{F}}+1)$-image of an OLP, say $S^{i_1,\dots,i_k,\dots,i_{2k-1}}$, all summands have length strictly bigger than the length of $S^{i_1,\dots,i_k,\dots,i_{2k-1}}$. This is because $S^{i_1,\dots,i_k,\dots,i_{2k-1}}$ has coefficient 1 as a summand of $1(S^{i_1,\dots,i_k,\dots,i_{2k-1}})$, and $-1^{(2k-1)}$ as a summand of $\chi_{\mathcal{F}}(S^{i_1,\dots,i_k,\dots,i_{2k-1}})$. Hence these coefficients cancel each other, so $S^{i_1,\dots,i_k,\dots,i_{2k-1}}$ occurs having coefficient zero as a summand of $(\chi_{\mathcal{F}}+1)(S^{i_1,\dots,i_k,\dots,i_{2k-1}})$. And the other summands of $\chi_{\mathcal{F}}(S^{i_1,\dots,i_k,\dots,i_{2k-1}})$ are proper refinements of $S^{i_1,\dots,i_k,\dots,i_{2k-1}}$, so have length strictly bigger than the length of $S^{i_1,\dots,i_k,\dots,i_{2k-1}}$.

Hence it is clear that the summands of $(\chi_{\mathcal{F}}+1)(S^{i_1,\dots,i_k,\dots,i_{2k-1}})$, are all proper refinements of $S^{i_{2k-1},\dots,i_k,\dots,i_1} = S^{i_1,\dots,i_k,\dots,i_{2k-1}}$, and as noted in the proof of Proposition 2.3.14, there is an ELP of length $2k$, namely

$$S^{i_1,\dots,i_{k-1},\frac{i_k}{2},\frac{i_k}{2},i_{k+2},\dots,i_{2k-1}}$$

as a refinement of $S^{i_1,\dots,i_k,\dots,i_{2k-1}}$.(Note that i_k must be even because we work in even degrees). Moreover, $S^{i_1,\dots,i_{k-1},\frac{i_k}{2},\frac{i_k}{2},i_{k+2},\dots,i_{2k-1}}$ is the only shortest length palindrome among the summands of $(\chi_{\mathcal{F}}+1)(S^{i_1,\dots,i_k,\dots,i_{2k-1}})$.

Let $S^{j_1,\dots,j_l,\dots,j_{2l-1}}$ be another OLP. Similarly, the only shortest length palindrome as a summand of $(\chi_{\mathcal{F}}+1)(S^{j_1,\dots,j_l,\dots,j_{2l-1}})$ is

$$S^{j_1,\dots,j_{l-1},\frac{j_l}{2},\frac{j_l}{2},j_{l+2},\dots,j_{2l-1}}.$$

For this to equal $S^{i_1,\dots,i_{k-1},\frac{i_k}{2},\frac{i_k}{2},i_{k+2},\dots,i_{2k-1}}$, we must have:

$$l=k, j_1=i_1,\dots,j_{l-1}=i_{k-1}, j_{l+2}=i_{k+2},\dots,j_{2l-1}=i_{2k-1}$$

and $\frac{i_k}{2}=\frac{j_l}{2}$, so we have equality: $j_l=i_k$ from which we can deduce that:

$$S^{j_1,\dots,j_l,\dots,j_{2l-1}} = S^{i_1,\dots,i_k,\dots,i_{2k-1}}.$$

This completes the proof. □

Theorem 6.1.2. *Let $w_1,\dots,w_m$ be all the higher non-palindromes in even degrees, and let $o_1,\dots,o_z$ be all the odd-length palindromes in even degrees, then*

$$(\chi_{\mathcal{F}}+1)(v_1),\dots,(\chi_{\mathcal{F}}+1)(v_m),(\chi_{\mathcal{F}}+1)(o_1),\dots,(\chi_{\mathcal{F}}+1)(o_z)$$

are linearly independent.

Proof. Let $w_1, \ldots, w_m$ be all the higher non-palindromes in even degrees, and let $o_1, \ldots, o_z$ be all the odd-length palindromes in even degrees. Assume that $v_1, \ldots, v_k$ are distinct elements of $\{w_1, \ldots, w_m, o_1, \ldots, o_z\}$, with the property that;

$$a_1(\chi_{\mathcal{F}} + 1)(v_1) + \ldots + a_{k-1}(\chi_{\mathcal{F}} + 1)(v_{k-1}) + a_k(\chi_{\mathcal{F}} + 1)(v_k) = 0, \quad (6.1)$$

for some non-zero integer coefficients $a_1, \ldots, a_k$.

Moreover, let's order these elements according to their lengths as follows:

$$length(v_k) \leq length(v_{k-1}) \leq \cdots \leq length(v_1),$$

and so that OLPs of any length l come after HNPs of length l. Since $\{v_1, \ldots, v_k\} \subset \{v_1, \ldots, v_m, o_1, \ldots, o_z\}$, either v_k is an odd-length palindrome or v_k is a higher non-palindrome. Now let consider the case where v_k is an higher non-palindrome.

If v_k is a higher non-palindrome, then $(\chi_{\mathcal{F}} + 1)(v_k)$ has exactly two summands which has the same length as v_k. One of them is an HNP, v_k, itself, which comes from $(+1)(v_k)$, and the other one is the reverse of v_k, an LNP, which is a summand of $\chi_{\mathcal{F}}(v_k)$. All the other summands in the $(\chi_{\mathcal{F}} + 1)(v_k)$ have length strictly greater than the length of v_k. This can be deduced easily by considering definition of $\chi_{\mathcal{F}}$ and identity morphism.

Furthermore, v_k cannot occur in the $(\chi_{\mathcal{F}} + 1)$-image of any other HNP of the same length, because, if there is another HNP of the same length, say $v_k^{'}$, then similarly, $(\chi_{\mathcal{F}} - 1)(v_k^{'})$ has exactly two summands which have the same length as $v_k^{'}$, which are $v_k^{'}$ itself, and its reverse which is an LNP, and the other summands have length strictly greater than $v_k^{'}$. It is obvious that v_k is different than $v_k^{'}$, v_k is not an LNP and v_k cannot equal a word that has length strictly greater than its length.

Moreover, by length considerations and the same argument above, it can easily seen that v_k cannot occur in the $(\chi_{\mathcal{F}} + 1)$-image of any longer length HNP.

Now, we have established v_k cannot occur in the $(\chi_{\mathcal{F}} + 1)$-image of any longer length HNP, or in any other HNP of the same length. Therefore, v_k cannot occur in $a_1(\chi_{\mathcal{F}} + 1)(v_1), \ldots, a_{k-1}(\chi_{\mathcal{F}} + 1)(v_{k-1})$. Hence, v_k cannot be cancelled, so, v_k occurs with a coefficient b_k on the right hand side of equation (6.1). Hence the left-hand side of equation (6.1) cannot equal zero. This contradiction shows that v_k is not a higher non-palindrome. Thus, there are no higher non-palindromes of the same length as v_k, because of our second assumption about the order of $v_1, \ldots, v_k$. Hence, v_k must be an odd-length palindrome, say with length r. Now lets consider this case. If v_k is an OLP, then by Proposition 6.1.1 there is an even-length palindrome summand in

$(\chi_{\mathcal{F}}+1)(v_k)$ of length $r+1$. In addition, this even-length palindrome does not occur as a shortest-length-summand in the $(\chi_{\mathcal{F}}+1)$-image of any other odd-length palindrome. Hence, this even-length palindrome cannot occur in the $(\chi_{\mathcal{F}}+1)$-image of any other OLP of length greater than or equal to the length of v_k.

As noted above, there are no higher non-palindromes of the same length as v_k,(we assumed w_k has length r in the preceding paragraph). For the remainder of the proof, we will now show that this $r+1$ length ELP cannot occur in the $(\chi_{\mathcal{F}}+1)$-image of any HNP which has length greater than $r+1$, or equal to $r+1$.

Now let's recall the beginning of our proof. We know $(\chi_{\mathcal{F}}+1)$-image of an $r+1$ length HNP have exactly two summands with length $r+1$, which are HNP itself, and its reverse, an LNP. And all the other summands in the $(\chi_{\mathcal{F}}+1)$-image of $r+1$ length HNP have length strictly greater than $r+1$. It is obvious that this $r+1$ length ELP cannot equal an $r+1$ length HNP, it cannot equal $r+1$ length LNP and it cannot equal any word of length strictly greater than $r+1$.

Furthermore, by the same argument as in the preceding paragraph, it is clear that an $r+1$ length ELP cannot occur in the $(\chi_{\mathcal{F}}+1)$-image of any higher non-palindrome of length strictly bigger than $r+1$.

Hence we established that, if v_k is an OLP there is an ELP as a summand of $(\chi_{\mathcal{F}}+1)(v_k)$ which cannot occur in the $(\chi_{\mathcal{F}}+1)$-image of any other OLP of length greater than or equal to the length of v_k. And, this ELP cannot occur in the $(\chi_{\mathcal{F}}+1)$-image of any HNP of length strictly greater than v_k. Hence, it cannot occur in $a_1(\chi_{\mathcal{F}}+1)(v_1)+\ldots+a_{k-1}(\chi_{\mathcal{F}}+1)(v_{k-1})$. Thus this ELP cannot be cancelled, so occurs with non-zero coefficient a_k on the left hand side of (6.1) from which we can deduce the left hand side of equation (6.1) cannot be zero. This contradicts our initial assumption. Hence,

$$(\chi_{\mathcal{F}}+1)(w_1),\ldots,(\chi_{\mathcal{F}}+1)(w_m),(\chi_{\mathcal{F}}+1)(o_1),\ldots,(\chi_{\mathcal{F}}+1)(o_z)$$

are linearly independent. This proves the theorem. □

Theorem 6.1.3. *In odd degrees, the higher non-palindromes have linearly independent images under* $(\chi_{\mathcal{F}}+1)$.

Proof. The proof of Theorem 6.1.2 also works for proof of Theorem 6.1.3 □

Remark 6.1.4. *In odd degrees, The argument in the proof of Theorem 6.1 doesn't show that OLPs have linearly independent image under* $(\chi_{\mathcal{F}}+1)$*, because Proposition 6.1.1 doesn't apply in* $2n-1$ *degrees since there is no ELP in odd degree, and so there is not a unique ELP among the summands of shortest length* $(\chi_{\mathcal{F}}+1)$*-image of an OLP.*

To complete the proof of Theorem 6.0.2, in odd and even degrees, we will now determine a spanning set for $\mathrm{Im}(\chi_{\mathcal{F}}+1)$.

6.2 Spanning set for $\mathrm{Im}(\chi_{\mathcal{F}}+1)$

In odd degrees, firstly, we will show that the $(\chi_{\mathcal{F}}+1)$-images of all OLPs can be expressed in terms of the $(\chi_{\mathcal{F}}+1)$-images HNPs by the following proposition:

Proposition 6.2.1. *Let $S^{i_1,\dots,i_k,i_{k+1},\dots,i_{2k+1}}$ be an odd-length palindrome with odd degree. Then*

$$(\chi_{\mathcal{F}}+1)((-1)^{2k+1}S^{i_1,\dots,i_k,i_{k+1},\dots,i_{2k+1}}) = \sum(\chi_{\mathcal{F}}+1)((-1)^{k+l}S^{i_1,\dots,i_k,j_1\dots,j_l}), \tag{6.2}$$

summed over all proper refinements $j_1,\dots j_l$ of $i_{k+1},\dots,i_{2k+1}$ where $j_1 \geq \frac{(i_{k+1})+1}{2}$.

Note that the proper condition implies that $S^{i_1,\dots,i_k,j_1\dots,j_l}$ is an HNP. To make a proof more manageable, Proposition 6.2.1 is stated in an equivalent form in Proposition 6.2.2.

Proposition 6.2.2. *Let $S^{i_1,\dots,i_k,i_{k+1},\dots,i_{2k+1}}$ be an odd-length palindrome with odd degree. Then*

$$\sum(\chi_{\mathcal{F}}+1)((-1)^{k+l}S^{i_1,\dots,i_k,j_1\dots,j_l}) = 0, \tag{6.3}$$

summed over all refinements $j_1,\dots j_l$ of $i_{k+1},\dots,i_{2k+1}$, where $j_1 \geq \frac{(i_{k+1})+1}{2}$.

To give a proof for Proposition 6.2.2, we need following lemmas:

Lemma 6.2.3. *Let $s_1,\dots,s_m$ be any word, where $s_1,\dots,s_m$ sum to positive odd integer p. Let k be the largest number for which $s_k+\dots+s_m \geq \frac{p+1}{2}$, where $1 \leq k$. Then*

$$k = 1 \Leftrightarrow s_1 \geq \frac{p+1}{2}.$$

Proof. Let $s_1,\dots,s_m$ be any word. Let k be the largest number such that $s_k+\dots+s_m \geq \frac{p+1}{2}$, where $s_1+\dots+s_m = p$ for a positive odd integer p.

i. If $k=1$, then $s_1+\dots+s_m \geq \frac{p+1}{2}$ which means $s_2+\dots+s_m < \frac{p+1}{2}$. Hence, $p-(s_2+\dots+s_m) > p-(\frac{p+1}{2})$. Since $p-(s_2+\dots+s_m) = s_1$, then we have $s_1 > \frac{p-1}{2}$. As before, we note that p is an odd integer, the inequality $s_1 > \frac{p-1}{2}$ is equivalent to saying $s_1 \geq \frac{p+1}{2}$.

ii. If $k \neq 1$, then $k \geq 2$, so $s_2 + \cdots + s_m \geq s_k + \cdots + s_m \geq \frac{p+1}{2}$, since $s_1 + s_2 + \cdots + s_m = p$, and $s_2 + \cdots + s_m \geq \frac{p+1}{2}$, then $s_1 < \frac{p+1}{2}$ by the same argument in i case.

□

Lemma 6.2.4. *Let $S^{r_1,\ldots,r_m}$ be any word and $S^{i_1,\ldots,i_b}$ be any proper coarsening of $S^{r_1,\ldots,r_m}$ then, the number of even-length sequences $S^{q_1,\ldots,q_n}$ that are coarsenings of $S^{r_1,\ldots,r_m}$ and refinements of $S^{i_1,\ldots,i_b}$ is equal to the number of odd length sequences $S^{q_1,\ldots,q_n}$ that are coarsenings of $S^{r_1,\ldots,r_m}$ and refinements of $S^{i_1,\ldots,i_b}$.*

Proof. Let $S^{r_1,\ldots,r_m}$ be any word and $S^{i_1,\ldots,i_b}$ be any proper coarsening of $S^{r_1,\ldots,r_m}$ then, $S^{r_1,\ldots,r_m}$ has $m-1$ commas and $S^{i_1,\ldots,i_b}$ has $b-1$ commas. Since $S^{i_1,\ldots,i_b}$ is a coarsening of $S^{r_1,\ldots,r_m}$, these $b-1$ commas are a subset of the $m-1$ commas in $S^{r_1,\ldots,r_m}$,and the complementary subset has $(m-1)-(b-1) = m-b$ commas which have been turned into pluses.

For $S^{q_1,\ldots,q_n}$ to be a coarsening of $S^{r_1,\ldots,r_m}$, it must also be obtained by turning some of the $m-1$ commas into pluses. And for $S^{q_1,\ldots,q_n}$ to be a refinement of $S^{i_1,\ldots,i_b}$, the selection of commas must be chosen from the $m-b$ commas that were turned into pluses in $S^{i_1,\ldots,i_b}$. In other words such sequences $S^{q_1,\ldots,q_n}$ corresponds the $m-n$ element subsets of a set $m-b$ elements. So the number of such sequences $S^{q_1,\ldots,q_n}$ is given by $\binom{m-b}{m-n}$. The parity of the sequences correspond to the parity of the subset. There are two cases to consider for length of m, either

Case1. m is even,

or

Case2. m is odd.

In case 1, m is even, then even-length sequences $S^{q_1,\ldots,q_n}$ correspondence to even order subsets,and odd-length sequences $S^{q_1,\ldots,q_n}$ correspond to odd-cardinality subsets.

In case 2, m is odd, then even-length sequences $S^{q_1,\ldots,q_n}$ correspondence to odd order subsets, and odd-length sequences $S^{q_1,\ldots,q_n}$ correspondence to even-cardinality subsets.

By Corollary 4.2.4, the number of odd-cardinality subsets of $m-b$ element set is equal to the number of even-cardinality subsets of $m-b$ elements set . Therefore, in both case 1 and 2, the number of odd length such subsets $S^{q_1,\ldots,q_n}$ is equal to number of even-length such $S^{q_1,\ldots,q_n}$ subsets. And we know each such sequences $S^{q_1,\ldots,q_n}$ is a coarsening of $S^{r_1,\ldots,r_m}$ and a refinement of $S^{i_1,\ldots,i_b}$. This completes the proof. □

Now we can introduce a proof of Proposition 6.2.2.

Proof of Proposition 6.2.2. Let $S^{i_1,\dots,i_k,i_{k+1},\dots,i_{2k+1}}$ be an odd-length palindrome with odd degree, let $j_1 \dots, j_l$ be a refinement of $i_{k+1},\dots,i_{2k+1}$, and let $S^{d_1,\dots,d_n}$ be any word in odd degrees, we will show that the coefficient of $S^{d_1,\dots,d_n}$ in

$$\sum(\chi_{\mathcal{F}}+1)((-1)^{k+l}S^{i_1,\dots,i_k,j_1\dots,j_l}) \tag{6.4}$$

is zero.

If $S^{d_1,\dots,d_n}$ occurs in sum (6.4) at all, it must be a summand of $(\chi_{\mathcal{F}}+1)((-1)^{k+l}S^{i_1,\dots,i_k,j_1,\dots,j_l})$ for some refinement $j_1,\dots,j_l$ of $i_{k+1},\dots,i_{2k+1}$ with $j_1 \geq \frac{(i_{k+1})+1}{2}$, then either

A. $S^{d_1,\dots,d_n} = S^{i_1,\dots,i_k,j_1,\dots,j_l}$ or

B. $S^{d_1,\dots,d_n}$ is a summand of $\chi_{\mathcal{F}}((-1)^{k+l}S^{i_1,\dots,i_k,j_1,\dots,j_l})$.

In case A, if $S^{d_1,\dots,d_n} = S^{i_1,\dots,i_k,j_1,\dots,j_l}$ then, $S^{d_1,\dots,d_n}$ occurs having a coefficient $(-1)^{k+l}$ as a summand of $1((-1)^{k+l}S^{i_1,\dots,i_k,j_1,\dots,j_l})$, where 1 is identity homomorphism. Therefore, $S^{d_1,\dots,d_n}$ has length $k+l$, so we have equality: $k+l=n$. However, to find the coefficient of $S^{d_1,\dots,d_n}$ in sum (6.4) , we also need the number of other refinements $c_1,\dots,c_q$ of $i_{k+1},\dots,i_{2k+1}$ with $c_1 \geq \frac{(i_{k+1})+1}{2}$ for which $S^{d_1,\dots,d_n}$ is a summand of $(\chi_F+1)((-1)^{k+q}S^{i_1,\dots,i_k,c_1,\dots,c_q})$.

If $S^{d_1,\dots,d_n} = S^{i_1,\dots,i_k,j_1,\dots,j_l}$, where $j_1,\dots,j_l$ is a refinement of $i_{k+1},\dots,i_{2k+1}$, and $S^{d_1,\dots,d_n}$ is a summand of $(\chi_{\mathcal{F}}+1)((-1)^{k+q}S^{i_1,\dots,i_k,c_1,\dots,c_q})$, where $c_1,\dots,c_q$ is a refinement of $i_{k+1},\dots,i_{2k+1}$, then either

i. $S^{d_1,\dots,d_n} = S^{i_1,\dots,i_k,c_1,\dots,c_q}$

or

ii. $S^{d_1,\dots,d_n}$ is a summand of $\chi_{\mathcal{F}}((-1)^{k+q}S^{i_1,\dots,i_k,c_1,\dots,c_q})$.

If $S^{d_1,\dots,d_n} = S^{i_1,\dots,i_k,c_1,\dots,c_q}$, then $c_1,\dots,c_q = j_1,\dots,j_l$, which means we have equality: $q=l$, so there are no "other" refinements for which $S^{d_1,\dots,d_n}$ occurs as a summand of $1((-1)^{k+q}S^{i_1,\dots,i_k,c_1,\dots,c_q})$.

If $S^{d_1,\dots,d_n}$ is a summand of $\chi_{\mathcal{F}}((-1)^{k+q}S^{i_1,\dots,i_k,c_1,\dots,c_q})$, then $S^{d_1,\dots,d_n}$ is a refinement of $S^{c_q,\dots,c_1,i_k,\dots,i_1}$, so $d_1 \leq c_q$. In addition, $c_1,\dots,c_q$ is a refinement of $i_{k+1},\dots,i_{2k+1}$, similarly $c_q \leq i_{2k+1}$, so it is easily seen that $d_1 \leq c_q \leq i_{2k+1}$. Beside this, $i_1,\dots,i_k,i_{k+1},i_{k+2}\dots,i_{2k+1}$ is an OLP, hence $d_1 \leq c_q \leq i_{2k+1} = i_1$. Nevertheless, we also have that $S^{d_1,\dots,d_n} = S^{i_1,\dots,i_k,j_1,\dots,j_l}$, so $i_1 = d_1$, hence, $i_1 = d_1 \leq c_q \leq i_{2k+1} = i_1$, from which we can deduce that $i_1 = d_1 = c_q$. Having established that $i_1 = d_1 = c_q$, we can now see $S^{d_2,\dots,d_n} = S^{i_2,\dots,i_k,j_1,\dots,j_l}$, and $S^{d_2,\dots,d_n}$ is a refinement of $S^{c_{q-1},\dots,c_1,i_k,\dots,i_1}$. Consequently and similarly,

$i_2 = d_2 \leq c_{q-1} \leq i_{2k} = i_2$, therefore $i_2 = d_2 = c_{q-1}$, and so on up to $i_k = d_k \leq c_{q-(k-1)} \leq i_{k+2} = i_k$, i.e., $i_k = d_k = c_{q-(k-1)}$, and $d_{k+1} \leq c_{q-k} \leq i_{k+1}$. We established that $i_1 = c_q, i_2 = c_{q-1}, \ldots, i_k = c_{q-(k-1)}$. Hence, we can now see $S^{d_{k+1},\ldots,d_n} = S^{j_1,\ldots,j_l}$, so $d_{k+1} = j_1$ which means we have $j_1 = d_{k+1} \leq c_{q-k}$, i.e., $c_{q-k} \geq d_{k+1} = j_1$. We also know $j_1 \geq \frac{(i_{k+1})+1}{2}$ from which we can deduce $c_{q-k} \geq \frac{(i_{k+1})+1}{2}$.

On the other hand, $c_1, \ldots, c_q$ is a refinement of $i_{k+1}, \ldots, i_{2k+1} = i_{k+1}, \ldots, i_1$ and we have determined the last k part of $c_1, \ldots, c_q$. So, $c_1, c_2, \ldots, c_{q-k}, i_k, \ldots, i_1$ is a refinement of $i_{k+1}, i_k, \ldots, i_1$. Hence, we must now have that $c_1, c_2, \ldots, c_{q-k}$ is a refinement of i_{k+1}. Hence $c_1 + c_2 + \cdots + c_{q-k} = i_{k+1}$, and we know $c_1 \geq \frac{(i_{k+1})+1}{2}$. In addition from the preceding paragraph we also know $c_{q-k} \geq \frac{(i_{k+1})+1}{2}$. If $q - k > 1$, then we have:

$$c_1 + c_2 + \cdots + c_{q-k} \geq c_1 + c_{q-k} \geq \frac{(i_{k+1})+1}{2} + \frac{(i_{k+1})+1}{2} = (i_{k+1}) + 1 > i_{k+1}.$$

We established $c_1 + c_2 + \cdots + c_{q-k} = i_{k+1}$, hence this can only happen if $q - k = 1$, and $c_1 = i_{k+1}$. Moreover, in the preceding paragraph we have already established $i_1 = c_q, i_2 = c_{q-1}, \ldots i_k = c_{q-(k-1)}$. Thus, $c_1 \ldots, c_q = i_{k+1}, i_k, \ldots, i_1$ is completely determined. Therefore, there is only one other refinement $c_1, \ldots, c_q$ of $i_{k+1}, \ldots, i_{2k+1}$ for which $S^{d_1,\ldots,d_n}$ is a summand of $(\chi_{F_2} + 1)((-1)^{k+q} S^{i_1,\ldots,i_k,c_1,\ldots,c_q})$ which is the improper one $c_1, \ldots, c_q = i_{k+1}, \ldots, i_{2k+1}$.

On the other hand, we know $S^{d_1,\ldots,d_n}$ has length $k + l = n$ and by definition 2.4.13 $S^{d_1,\ldots,d_n}$ occurs having a coefficient $(-1)^{2k+1}(-1)^{k+l}$as a summand in $\chi_F((-1)^{2k+1} S^{i_1,\ldots,i_k,i_{k+1},\ldots,i_{2k+1}})$. Hence, if $S^{d_1,\ldots,d_n} = S^{i_1,\ldots,i_k,j_1,\ldots,j_l}$, then the coefficient of $S^{d_1,\ldots,d_n}$ in the sum is zero, because $S^{d_1,\ldots,d_n}$ occurs having a coefficient $(-1)^{k+l}$ as a summand of $1((-1)^{k+l} S^{i_1,\ldots,i_k,j_1,\ldots,j_l})$ and has a coefficient $(-1)^{2k+1}(-1)^{k+l}$ as a summand in $\chi_{\mathcal{F}}((-1)^{2k+1} S^{i_1,\ldots,i_k,i_{k+1},\ldots,i_{2k+1}})$ and in no other terms.

In case B, if $S^{d_1,\ldots,d_n}$ is a summand of $\chi_{\mathcal{F}}((-1)^{k+l} S^{i_1,\ldots,i_k,j_1,\ldots,j_l})$, then $S^{d_1,\ldots,d_n}$ is a refinement of $(-1)^{k+l} S^{j_l,\ldots,j_1,i_k,\ldots,i_1}$ where $j_l, \ldots, j_1$ is a refinement of $i_{2k+1}, \ldots, i_{k+1}$ with $j_1 \geq \frac{(i_{k+1})+1}{2}$.

If $S^{d_1,\ldots,d_n}$ also occurs as $S^{i_1,\ldots,i_k,c_1,\ldots,c_q}$ for some refinement $c_1, \ldots, c_q$ of $i_{k+1}, \ldots, i_{2k+1}$, then we are back in case A, the argument in that case shows that its coefficient is zero. So we may assume that $S^{d_1,\ldots,d_n}$ does not occur as $S^{i_1,\ldots,i_k,c_1,\ldots,c_q}$ for any refinement $c_1, \ldots, c_q$ with $c_1 \geq \frac{(i_{k+1})+1}{2}$.

To find the coefficient of $S^{d_1,\ldots,d_n}$ in the sum, we need to find the number of all refinements $c_1, \ldots, c_q$ of $i_{k+1}, \ldots, i_{2k+1}$ with $c_1 \geq \frac{(i_{k+1})+1}{2}$ for which $S^{d_1,\ldots,d_n}$ is a summand of $(\chi_{\mathcal{F}})((-1)^{k+q} S^{i_1,\ldots,i_k,c_1,\ldots,c_q})$. Each such refinement $c_1, \ldots, c_q$ contributes $(-1)^{n+k+q}$ to the coefficient of $S^{d_1,\ldots,d_n}$ in the sum. This is because, for each such refinement $c_1, \ldots, c_q$, $S^{d_1,\ldots,d_n}$ occurs as a summand of $(\chi_F)((-1)^{k+q}S^{i_1,\ldots,i_k,c_1,\ldots,c_q})$, and by definition 2.4.13 and since $\chi_{\mathcal{F}}$ is a module homomorphism, $(\chi_{\mathcal{F}})((-1)^{k+q}S^{i_1,\ldots,i_k,c_1,\ldots,c_q})$ has $S^{d_1,\ldots,d_n}$ as a summand with a coefficient $(-1)^n(-1)^{k+q} = (-1)^{n+k+q}$.

Since n, k are fixed, the coefficient of $S^{d_1,\ldots,d_n}$ is determined by the number of such refinements $c_1, \ldots, c_q$ with q odd and the number with q even. We shall show that the number of odd-length refinements and the number of even-length refinements is equal. Therefore, since each such refinement $c_1, \ldots, c_q$ contributes -1 or $+1$, according to the parity of q, then they cancel each other. Hence, $S^{d_1,\ldots,d_n}$ occurs with coefficient zero in the sum. In other words if the the number of odd length such refinements $c_1, \ldots, c_q$ matches the number of even-length such refinements $c_1, \ldots, c_q$, then $S^{d_1,\ldots,d_n}$ occurs with coefficient 0 in the sum.

Since $S^{d_1,\ldots,d_n}$ is a refinement of $S^{j_l,\ldots,j_1,i_k,\ldots,i_1}$ and $S^{j_l,\ldots,j_1,i_k,\ldots,i_1}$ is a refinement of $S^{i_1,\ldots,i_k,i_{k+1},\ldots,i_{2k+1}} = S^{i_{2k+1},\ldots,i_{k+1},i_k,\ldots,i_1}$, it follows that $S^{d_1,\ldots,d_n}$ is a refinement of $S^{i_{2k+1},\ldots,i_{k+1},i_k,\ldots,i_1}$. Hence, there is an index g with $1 \leq g \leq n$ such that $S^{d_1,\ldots,d_g}$ is a refinement of $S^{i_{2k+1},\ldots,i_{k+1}}$ and $S^{d_{g+1},\ldots,d_n}$ is a refinement of $S^{i_k,\ldots,i_1}$.

So more explicitly, to find the coefficient of $S^{d_1,\ldots,d_g,d_{g+1},\ldots,d_n}$, we need to find the number of refinements $c_1, \ldots, c_q$ of $i_{k+1}, \ldots, i_{2k+1} = i_{k+1}, \ldots, i_1$ with $c_1 \geq \frac{(i_{k+1})+1}{2}$ for which $S^{d_1,\ldots,d_g,d_{g+1},\ldots,d_n}$ is a refinement of $(-1)^{k+q}S^{c_q,\ldots,c_1,i_k\ldots,i_1}$.

Thus, in other words, the sequences $c_1, \ldots, c_q$ with $c_1 \geq \frac{(i_{k+1})+1}{2}$ that we want to count are refinements of $i_{k+1}, \ldots, i_{2k+1} = i_{k+1}, \ldots, i_1$ that admit $d_1, \ldots, d_g$ as a refinement of $c_q, \ldots, c_1$ with $c_1 \geq \frac{(i_{k+1})+1}{2}$. And it is clear that counting the number of such sequences $c_q, \ldots, c_1$ means counting the number of such sequences $c_1, \ldots, c_q$.

If $d_1, \ldots, d_g$ is a refinement of $c_q, \ldots, c_1$ then, $c_q, \ldots, c_1$ is a coarsening of $d_1 \ldots, d_g$. In particular, $c_1 = d_e + \cdots + d_g$ for some e in the range $1 \leq e \leq g$. For c_1 to be greater than or equal to $\frac{(i_{k+1})+1}{2}$ we need e to be small enough. Precisely, let f be the largest index such that $d_f + \cdots + d_g \geq \frac{(i_{k+1})+1}{2}$ where $1 \leq f \leq g$. Then,

$$d_e + \cdots + d_{f-1} + d_f + \cdots + d_g \geq \frac{(i_{k+1}) + 1}{2} \Leftrightarrow e \leq f. \tag{6.5}$$

This corresponds to $c_q, \ldots, c_1$ being a coarsening of $d_1, \ldots, d_{f-1}, d_f + \cdots + d_g$. And more precisely, we also want the coarsening $c_q, \ldots, c_1$ of $d_1, \ldots, d_{f-1}, d_f + \cdots + d_g$ to be a refinement of $i_1, \ldots, i_{k+1}$.

Hence, counting the number of coarsenings $c_q, \ldots, c_1$ of $d_1, \ldots, d_g$ for which $c_1 \geq \frac{(i_{k+1})+1}{2}$ means counting the number of coarsenings $c_q, \ldots, c_1$ of $d_1, \ldots, d_{f-1}, d_f + \cdots + d_g$.

If $c_q, \ldots, c_1$ is a coarsening of $d_1, \ldots, d_{f-1}, d_f + \cdots + d_g$ and is a refinement of $i_1, \ldots, i_{k+1}$ then, by Lemma 6.2.4 the number of odd length coarsenings $c_q, \ldots, c_1$ of $d_1, \ldots, d_g$ with $c_1 \geq \frac{(i_{k+1})+1}{2}$ is equal to the number of even-length coarsenings $c_q, \ldots, c_1$ of $d_1, \ldots, d_g$ with $c_1 \geq \frac{(i_{k+1})+1}{2}$ as long as $i_1, \ldots, i_{k+1}$ is a proper coarsening of $d_1, \ldots, d_{f-1}, d_f + \cdots + d_g$. This completes the proof in the case where "$d_1, \ldots, d_{f-1}, d_f + \cdots + d_g$" is a proper refinement of $i_1, \ldots, i_{k+1}$.

If $i_1, \ldots, i_{k+1}$ is not a proper coarsening of $d_1, \ldots, d_{f-1}, d_f + \cdots + d_g$ then, $f = k+1$, $d_1, d_2, \ldots, d_k = i_1, i_2, \ldots, i_k$ and $d_f + \cdots + d_g = i_{k+1}$ where f is the largest number for which $d_f + \ldots + d_g \geq \frac{(i_{k+1})+1}{2}$. Then by Lemma 6.2.3 $d_f \geq \frac{(i_{k+1})+1}{2}$. Therefore, if we set $c_1 = d_f, c_2 = d_{f+1}, \ldots, c_{n+1-f} = d_n$ and $q = n+1-f$, then $c_1, \ldots, c_q$ is a refinement of $i_{k+1}, \ldots, i_1 = i_{k+1}, \ldots, i_{2k+1}$ with $c_1 \geq \frac{(i_{k+1})+1}{2}$ which ensures $S^{d_1,\ldots,d_n}$ occurs as $S^{i_1,\ldots,i_k,c_1,\ldots,c_q}$ for some refinement $c_1, \ldots, c_q$ of $i_{k+1}, \ldots, i_{2k+1}$, thus this only occurs when we are in case A, for which we have already proved that the coefficient is zero.

Hence, in both case A and case B, $S^{d_1,\ldots,d_n}$ occurs with coefficient zero in the sum over all refinements. This completes the proof. □

Secondly, in odd degrees, we will show that $(\chi_F + 1)$-images of all LNPs can be expressed in terms of the $(\chi_F + 1)$-images HNPs. Before that we need following technical result:

Proposition 6.2.5. *Let $S^{i_1,\ldots,i_n}$ be a lower non-palindrome with odd degree. Then*

$$(\chi_{\mathcal{F}} + 1)(S^{i_1,\ldots,i_n}) = (-1)^n(\chi_{\mathcal{F}} + 1)(S^{i_n,\ldots,i_1}) + \sum(-1)^k(\chi_{\mathcal{F}} + 1)(S^{j_1,\ldots,j_k}),$$

where the summation is over all proper refinements $S^{j_1,\ldots,j_k}$ of $S^{i_n,\ldots,i_1}$.

Proof. Let $S^{i_1,\ldots,i_n}$ be a lower non-palindrome with odd degree. Applying $(\chi_{\mathcal{F}} - 1)$ to $S^{i_1,\ldots,i_n}$ we arrive at:

$$(\chi_{\mathcal{F}} - 1)(S^{i_1,\ldots,i_n}) = -S^{i_i,\ldots,i_n} + (-1)^n S^{i_n,\ldots,i_1} + \sum(-1)^k S^{j_1,\ldots,j_k}, \quad (6.6)$$

where the summation is over all proper refinements $S^{j_1,\ldots,j_k}$ of $S^{i_n,\ldots,i_1}$. By Proposition 2.4.9, $\mathcal{F}$ is cocommutative, so by Proposition 2.2.8 we have:

$$\chi_{\mathcal{F}}^2 = 1.$$

Hence, $(\chi_{\mathcal{F}} + 1)(\chi_{\mathcal{F}} - 1) = 0$. Therefore, applying module homomorphism $(\chi_{\mathcal{F}} + 1)$ to both sides of equation (6.6) we get;

$$0 = (\chi_{\mathcal{F}}+1)(-S^{i_1,\ldots,i_n}) + (\chi_{\mathcal{F}}+1)((-1)^n S^{i_n,\ldots,i_1}) + \sum(-1)^k(\chi_{\mathcal{F}}+1)(S^{j_1,\ldots,j_k}).$$

Thus,

$$(\chi_{\mathcal{F}} + 1)(S^{i_1,\dots,i_n}) = (-1)^n(\chi_{\mathcal{F}} + 1)(S^{i_n,\dots,i_1}) + \sum(-1)^k(\chi_F + 1)(S^{j_1,\dots,j_k}),$$

where the summation is over all proper refinements $S^{j_1,\dots,j_k}$ of $S^{i_n,\dots,i_1}$. This completes the proof. □

Theorem 6.2.6. *If v_0 is a lower non-palindrome in degree $2n-1$, then $(\chi_{\mathcal{F}} + 1)(v_0)$ can be written as a linear combination of $(\chi_{\mathcal{F}} + 1)$-images of higher non-palindromes.*

Proof. The proof is proved by decreasing induction on length of v_0. In degree $2n-1$, the longest possible length is $2n-1$, and there is only one element of length $2n-1$, namely $S^{1,1,1,\dots,1,1}$. It is unique and is an odd length palindrome, so the hypothesis is true for all LNPs of length greater than or equal to $2n-1$, since there are none.

Now assume that all lower non-palindromes of length strictly bigger than p have $(\chi_{\mathcal{F}}+1)$-images that can be written as linear combinations of $(\chi_{\mathcal{F}}+1)$-images of higher non-palindromes. Let $v_0 = S^{b_1,\dots,b_p}$ a lower non-palindrome. By Proposition 6.2.5 we have:

$$(\chi_{\mathcal{F}} + 1)(v_0) = (\chi_{\mathcal{F}} + 1)(-1)^p(S^{b_p,\dots,b_1}) + \sum(-1)^y(\chi_{\mathcal{F}} + 1)(S^{g_1,\dots,g_y}), \quad (6.7)$$

where $S^{b_p,\dots,b_1}$ is a higher non-palindrome and each $S^{g_1,\dots,g_y}$ is either

i. a higher non-palindrome,

ii. an odd-length palindrome, in which case by Proposition 6.2.1 $(\chi_{\mathcal{F}} + 1)(S^{g_1,\dots,g_y})$ is a linear combination of $(\chi_{\mathcal{F}} + 1)$-images of higher non-palindromes, or

iii. a lower non-palindrome. In this case the inductive hypothesis applies because $S^{g_1,\dots,g_y}$ is a proper coarsening of the reverse of v_0 so has length strictly bigger than p. Hence, $(\chi_{\mathcal{F}}+1)(S^{g_1,\dots,g_y})$ is a linear combination of $(\chi_{\mathcal{F}} + 1)$-images of higher non-palindromes.

Thus, in each case, $(\chi_{\mathcal{F}} + 1)(S^{g_1,\dots,g_y})$ can be written as a linear combination of $(\chi_F + 1)$-images of higher non-palindromes. This completes the proof.

□

Now in even degrees, we will show that $(\chi_F + 1)$-images of an LNP or ELP can be expressed in terms of the $(\chi_F + 1)$-images of HNPs and OLPs . Before that we need following technical results:

Proposition 6.2.7. *Let $S^{i_1,\dots,i_{2n}}$ be an even-length palindrome in degree $2n$ of $\mathcal{F}$, then there is an odd-length palindrome $S^{i_1,\dots,i_n+i_{n+1},\dots,i_{2n}}$ such that,*

$$(\chi_{\mathcal{F}}+1)(S^{i_1,\dots,i_{2n}}) = 2(\chi_{\mathcal{F}}+1)(S^{i_1,\dots,i_n+i_{n+1},\dots,i_{2n}})+\sum(-1)^{k+1}(\chi_{\mathcal{F}}+1)(S^{j_1,\dots,j_k}),$$

where the summation is over all proper refinements $S^{j_1,\dots,j_k}$ of $S^{i_i,\dots,i_n+i_{n+1},\dots,i_{2n}}$ with $j_1,\dots,j_k \neq i_i,\dots,i_{2n}$.

Proof. Let $S^{i_1,\dots,i_{2n}}$ be an even-length palindrome in degree $2n$ of $\mathcal{F}$, then there is an odd length palindrome namely, $S^{i_i,\dots,i_n+i_{n+1},\dots,i_{2n}}$ among the proper coarsenings of $S^{i_1,\dots,i_{2n}}$. And applying $(\chi_{\mathcal{F}}-1)$ to $S^{i_i,\dots,i_n+i_{n+1},\dots,i_{2n}}$ we have;

$$\begin{aligned}(\chi_{\mathcal{F}}-1)(S^{i_i,\dots,i_n+i_{n+1},\dots,i_{2n}}) &= -S^{i_i,\dots,i_n+i_{n+1},\dots,i_{2n}} + (-1)^{2n-1}S^{i_{2n},\dots,i_{n+1}+i_n,\dots,i_1}\\ &\quad+\sum(-1)^k S^{j_1,\dots,j_k}, \end{aligned} \tag{6.8}$$

where the summation is over all proper refinements $S^{j_1,\dots,j_k}$ of $S^{i_{2n},\dots,i_{n+1}+i_n,\dots,i_1}$ $= S^{i_1,\dots,i_n+i_{n+1},\dots,i_{2n}}$. $S^{i_i,\dots,i_n+i_{n+1},\dots,i_{2n}}$ is a proper coarsening of $S^{i_1,\dots,i_{2n}}$, in other words, $S^{i_i,\dots,i_{2n}}$ is a proper refinement of $S^{i_i,\dots,i_n+i_{n+1},\dots,i_{2n}}$. Hence there is one $S^{j_1,\dots,j_k}$ which equals $S^{i_i,\dots,i_{2n}}$. According to this refinement, if we re-write the equation (6.8), then more explicitly we get:

$$(\chi_{\mathcal{F}}-1)(S^{i_i,\dots,i_n+i_{n+1},\dots,i_{2n}}) = -2S^{i_i,\dots,i_n+i_{n+1},\dots,i_{2n}}+S^{i_1,\dots,i_{2n}}+\sum(-1)^k S^{j_1,\dots,j_k}, \tag{6.9}$$

where the summation is over all proper refinements $S^{j_1,\dots,j_k}$ of $S^{i_i,\dots,i_n+i_{n+1},\dots,i_{2n}}$ with $j_1,\dots,j_k \neq i_1,\dots,i_{2n}$. On $\mathcal{F}$ we know $(\chi_{\mathcal{F}}+1)(\chi_{\mathcal{F}}-1)=0$, therefore, if we apply module homomorphism (χ_F+1) to both sides of equation (6.9) and rewrite it, we have;

$$0 = (\chi_{\mathcal{F}}+1)(-2S^{i_i,\dots,i_n+i_{n+1},\dots,i_{2n}} + S^{i_1,\dots,i_{2n}}) + \sum(-1)^k(\chi_{\mathcal{F}}+1)(S^{j_1,\dots,j_k}). \tag{6.10}$$

Consequently, by equation (6.10) we have;

$$(\chi_{\mathcal{F}}+1)(S^{i_1,\dots,i_{2n}}) = 2(\chi_{\mathcal{F}}+1)(S^{i_i,\dots,i_n+i_{n+1},\dots,i_{2n}})+\sum(-1)^{k+1}(\chi_{\mathcal{F}}+1)(S^{j_1,\dots,j_k}), \tag{6.11}$$

where the summation is over all proper refinements $S^{j_1,\dots,j_k}$ of $S^{i_i,\dots,i_n+i_{n+1},\dots,i_{2n}}$ with $j_1,\dots,j_k \neq i_1,\dots,i_{2n}$. This completes the proof. □

Theorem 6.2.8. *If e_0 is an even-length palindrome or lower non-palindrome in degree $2n$, then $(\chi_{\mathcal{F}}+1)(e_0)$ can be written as a linear combination of $(\chi_{\mathcal{F}}+1)$-images of higher non-palindromes and odd-length palindromes.*

Proof. The proof is by decreasing induction on length of e_0. In degree $2n$, the longest possible length is $2n$, and there is only one element of length $2n$, namely $S^{1,1,1,\ldots,1,1,\ldots,1,1,1}$, beside this, $(\chi_{\mathcal{F}}+1)(S^{1,1,1,\ldots,1,1,\ldots,1,1,1}) = 2(\chi_{\mathcal{F}}+1)(S^{1,1,1,\ldots,(1+1),\ldots,1,1,1})$, and $S^{1,1,1,\ldots,(1+1),\ldots,1,1,1}$ is an OLP. Thus every word of length greater than or equal to $2n$ has $(\chi_{\mathcal{F}}+1)$-image that can be written as a a linear combination of $(\chi_{\mathcal{F}}+1)$-images of higher non-palindromes and of odd-length palindromes.

Now assume that all basis elements of length strictly greater than r have $(\chi_{\mathcal{F}}+1)$-images that can be written as linear combinations of $(\chi_{\mathcal{F}}+1)$-images of HNPs and of OLPs. Let $e_0 = S^{b_1,\ldots,b_r}$ be an LNP. By Proposition 6.2.5 we have;

$$(\chi_{\mathcal{F}}+1)(S^{b_1,\ldots,b_r}) = (-1)^r(\chi_{\mathcal{F}}+1)(S^{b_r,\ldots,b_1}) + \sum(-1)^s(\chi_{\mathcal{F}}+1)(S^{h_1,\ldots,h_s}), \tag{6.12}$$

where $S^{b_r,\ldots,b_1}$ is an HNP and $S^{h_1,\ldots,h_s}$ is a proper refinement of $S^{b_r,\ldots,b_1}$. Every term on the right hand side of equation (6.12) has length greater than or equal to r. Any term of length strictly greater than r is dealt with by the inductive hypothesis. And the only term of length r is $S^{b_r,\ldots,b_1}$ which is an HNP. Thus, every term on the right hand side of equation (6.12) can be written as a linear combination of $(\chi_{\mathcal{F}}+1)$-images of HNPs and of OLPs.

For the remainder of the proof we need to consider where e_0 is an ELP. Let $e_0 = S^{c_1,\ldots,c_r}$ be an ELP, by Proposition 6.2.7 we have an OLP $S^{c_1,\ldots,c_{\frac{r}{2}}+c_{\frac{r}{2}+1},\ldots,c_r}$ such that ;

$$(\chi_{\mathcal{F}}+1)(S^{c_1,\ldots,c_r}) = (\chi_{\mathcal{F}}+1)(2S^{c_1,\ldots,c_{\frac{r}{2}}+c_{\frac{r}{2}+1},\ldots,c_r}) + \sum(-1)^{k+1}(\chi_{\mathcal{F}}+1)(S^{j_1,\ldots,j_k}), \tag{6.13}$$

where the summation is over all proper refinements $S^{j_1,\ldots,j_k}$ of $S^{c_1,\ldots,c_{\frac{r}{2}}+c_{\frac{r}{2}+1},\ldots,c_r}$ with $j_1,\ldots,j_k \neq c_1,\ldots,c_r$, and $S^{c_1,\ldots,c_{\frac{r}{2}}+c_{\frac{r}{2}+1},\ldots,c_r}$ is an OLP. Every term on the right hand side of equation (6.13) has length greater than or equal to $r-1$. All terms with length greater than r are dealt with by the inductive hypothesis. This leaves only the terms with length $r-1$ or r to deal with. There is only one term with length $r-1$, namely $S^{c_1,\ldots,c_{\frac{r}{2}}+c_{\frac{r}{2}+1},\ldots,c_r}$ and it is an OLP. The length r terms can be either HNPs or LNPs. It is clear that inductive hypothesis applies to HNPs, and we have already shown that $(\chi_{\mathcal{F}}+1)$-image of length r LNPs can be written as a linear combination of $(\chi_{\mathcal{F}}+1)$-images of HNPs and of OLPs. Thus, every term on the right hand side of equation (6.13) can be written as a linear combination of $(\chi_{\mathcal{F}}+1)$-images of HNPs and of OLPs.

In conclusion, whether e_0 is an ELP or an LNP in degree $2n$, $(\chi_{\mathcal{F}}+1)(e_0)$ can be written as a linear combination of $(\chi_{\mathcal{F}}+1)$-images of HNPs and of OLPs. $\square$

We now will introduce a basis for $\text{Im}(\chi_{\mathcal{F}} + 1)$.

Theorem 6.2.9. *In even degrees, the image of* $(\chi_{\mathcal{F}} + 1)$ *has a basis consisting of the* $(\chi_{\mathcal{F}} + 1)$ *images of all higher non-palindromes and all odd-length palindromes.*

Proof. In even degrees, by Theorem 6.1.2 the image of $(\chi_{\mathcal{F}} + 1)$ all higher non-palindromes and of odd-length palindromes are linearly independent.

On the other hand, HNPs,LNPs, ELPs and OLPs form a basis for $\mathcal{F}$. Hence, $\text{Im}(\chi_{\mathcal{F}} + 1)$ is spanned by $(\chi_{\mathcal{F}} + 1)$-image of HNPs, LNPs, ELPs and OLPs. On the other hand, by Theorem 6.2.6 and Theorem 6.2.8 and we can reduce this to $(\chi_{\mathcal{F}} + 1)$-image of all HNPs and OLPs . Hence, $(\chi_{\mathcal{F}} + 1)$ all HNPs and of all OLPs form a basis for $\text{Im}(\chi_{\mathcal{F}}+1)$ in even degrees of $\mathcal{F}$. This proves the theorem. □

Theorem 6.2.10. *In odd degrees , the image of* $(\chi_{\mathcal{F}}+1)$ *has a basis consisting of the* $(\chi_{\mathcal{F}} + 1)$ *images of all higher non-palindromes.*

Proof. In odd degrees, by Theorem 6.1.3 the $(\chi_{\mathcal{F}} + 1)$ image of the all higher non-palindromes are linearly independent. On the other hand, we know HNPs, LNPs and OLPs form a basis for $\mathcal{F}$. Hence, $\text{Im}(\chi_{\mathcal{F}} + 1)$ is spanned by the $(\chi_{\mathcal{F}} + 1)$-image of HNPs,LNPs, and OLPs. Moreover, by Proposition 6.2.1 and Theorem 6.2.6 we can reduce this to the $(\chi_{\mathcal{F}} + 1)$-images of HNPs. Hence, the $(\chi_{\mathcal{F}} + 1)$-images of all HNPs form a basis for $\text{Im}(\chi_{\mathcal{F}} + 1)$ in odd degrees of F. This proves the theorem. □

Proof of Proposition 6.0.2. The proof is easily seen by Theorem 6.2.10 and Theorem 6.2.9. □

We know a basis for free submodule $\text{Im}(\chi_{\mathcal{F}} + 1)$. Now showing that:

$$\text{Ker}(\chi_{\mathcal{F}} - 1) = \text{Im}(\chi_{\mathcal{F}} + 1),$$

we will give a basis for $\text{Ker}(\chi_{\mathcal{F}} - 1)$.

Theorem 6.2.11. *In* $\mathcal{F}$, *we have:*

$$\text{Im}(\chi_{\mathcal{F}} + 1) = \text{Ker}(\chi_{\mathcal{F}} - 1).$$

Proof. We first show that $\text{Im}(\chi_{\mathcal{F}} + 1) \subset \text{Ker}(\chi_{\mathcal{F}} - 1)$ in all degrees of $\mathcal{F}$ and then we will consider the $2n$ and $2n - 1$ degrees separately in the proof of $\text{Ker}(\chi_{\mathcal{F}} - 1) \subset \text{Im}(\chi_{\mathcal{F}} + 1)$.

i. Proof that $\text{Im}(\chi_{\mathcal{F}}+1) \subset \text{Ker}(\chi_{\mathcal{F}}-1)$. By the argument in the proof Proposition 6.2.5, we know

$$(\chi_{\mathcal{F}}-1)(\chi_{\mathcal{F}}+1)=0,$$

which implies

$$\text{Im}(\chi_{\mathcal{F}}+1) \subset \text{Ker}(\chi_{\mathcal{F}}-1).$$

ii. Proof that $\text{Ker}(\chi_{\mathcal{F}}-1) \subset \text{Im}(\chi_{\mathcal{F}}+1)$. If $x \in \text{Ker}(\chi_{\mathcal{F}}-1)$, then $\chi_{\mathcal{F}}(x)=x$, hence $(\chi_{\mathcal{F}}+1)(x)=2x$, so $2x \in \text{Im}(\chi_{\mathcal{F}}+1)$. To complete the proof, we will show that if $2x \in \text{Im}(\chi_{\mathcal{F}}+1)$, then $x \in \text{Im}(\chi_{\mathcal{F}}+1)$. We first deal with the even degrees of $\mathcal{F}$.

Let $w_1,\dots,w_m$ be all the higher non-palindromes and $o_1,\dots,o_z$ be all the odd length palindromes in degree $2n$, then by Theorem 6.2.9, there are $v_1,\dots,v_k$ distinct elements of $\{w_1,\dots,w_m,o_1,\dots,o_z\}$ such that;

$$2x=(\chi_{\mathcal{F}}+1)(a_1v_1+a_2v_2+\cdots+a_kv_k), \tag{6.14}$$

for some coefficients $a_1,a_2,\dots,a_k \in \mathbf{Z}$. Since $\chi_{\mathcal{F}}+1$ is a $\mathbf{Z}$ module homomorphism, more explicitly equation (6.14) has in the following form:

$$2x=a_1\chi_{\mathcal{F}+1}(v_1)+a_2\chi_{\mathcal{F}+1}(v_2)+\cdots+a_k\chi_{\mathcal{F}+1}(a_k). \tag{6.15}$$

Moreover, let's order $v_1,\dots,v_k$ as follows

$$length(v_k) \leq length(v_{k-1}) \leq \cdots \leq length(v_1),$$

and so that odd-length palindromes of any length l come after HNPs of length l. Since $\{v_1,\dots,v_k\}$ is chosen from the set $\{w_1,\dots,w_m,o_1,\dots,o_z\}$ then either v_k is an odd-length palindrome or v_k is a higher non-palindrome. If v_k is an HNP, then the same argument in proof of Theorem 6.1 applies here so, v_k cannot occur in $a_1(\chi_{\mathcal{F}}+1)(v_1),\dots,a_{k-1}(\chi_{\mathcal{F}}+1)(v_{k-1})$, and in addition v_k occurs with coefficient a_k on the right hand side of equation 6.15.

On the other hand $2x \in \mathcal{F}$, and $\mathcal{F}$ has a basis, therefore $2x$ can be written uniquely as a linear combination of basis elements, so the coefficients of these basis elements are even. In equation (6.15) expressing the right hand side with these basis, one basis element has coefficient a_k, so it is even. We have established that a_k is even.

If v_k is an OLP, then there can be no HNPs of the same length because of our assumption about ordering lengths of $v_1,\dots,v_k$. Furthermore, by the same argument as in the proof of Theorem 6.1.2 there is an ELP summand in $(\chi_{\mathcal{F}}+1)(v_k)$ of length 1 more than the length of v_k with coefficient $-1^{(k+1)} = 1$, since k is odd. And it cannot occur in $a_1(\chi_{\mathcal{F}}+1)(v_1), \ldots, a_{k-1}(\chi_{\mathcal{F}}+1)(v_{k-1})$. In addition v_k occurs with coefficient a_k on the right hand side of equation (6.15).

Since, $2x \in \mathcal{F}$, so in the same manner above a_k is even. Once we established that whether v_k is an OLP or v_k is an HNP, it occurs with an even coefficient a_k, say $a_k = 2\bar{a_k}$, where $\bar{a_k} \in \mathbf{Z}$. By the same argument in the proof of Theorem 5.2.3 it is easily seen that x can be written as follows

$$x = (\chi_{\mathcal{F}}+1)(\bar{a_1}v_1 + \bar{a_2}v_2 + \cdots + \bar{a_k}v_k), \tag{6.16}$$

where $\bar{a_j} = 2a_j \in \mathbf{Z}$ for $j = 1, \ldots, k$. By equation (6.16) $x \in \mathrm{Im}(\chi_{\mathcal{F}}+1)$. Since we can do the same argument for all $x \in \mathrm{Ker}(\chi_{\mathcal{F}}-1)$, then $\mathrm{Ker}(\chi_{\mathcal{F}}-1) \subset \mathrm{Im}(\chi_{\mathcal{F}}+1)$. By i. we know $\mathrm{Im}(\chi_{\mathcal{F}}+1) \subset \mathrm{Ker}(\chi_{\mathcal{F}}-1)$. Therefore $\mathrm{Im}(\chi_{\mathcal{F}}+1) = \mathrm{Ker}(\chi_{\mathcal{F}}-1)$ in even degrees of $\mathcal{F}$.

Now let consider odd degrees of $\mathcal{F}$. If $x \in \mathrm{Ker}(\chi_{\mathcal{F}}-1)$, in the same manner as in even degrees, $2x \in \mathrm{Im}(\chi_{\mathcal{F}}+1)$. Then, by Theorem 6.2.10 there are distinct HNPs $h_1, h_2, \ldots, h_y$ such that;

$$2x = (\chi_{\mathcal{F}}+1)(a_1h_1 + a_2h_2 + \cdots + a_yh_y). \tag{6.17}$$

for some coefficients $a_1, a_2, \ldots, a_y \in \mathbf{Z}$.

And in the same manner as in the proof for even degrees, we can see that coefficients $a_1, a_2, \ldots, a_y \in \mathbf{Z}$ occurs even in the right hand side of equation (6.17), and we can easily see that $x \in \mathrm{Ker}(\chi_{\mathcal{F}}-1)$. By i. we know $\mathrm{Im}(\chi_{\mathcal{F}}+1) \subset \mathrm{Ker}(\chi_{\mathcal{F}}-1)$. Therefore $\mathrm{Im}(\chi_{\mathcal{F}}+1) = \mathrm{Ker}(\chi_{\mathcal{F}}-1)$ in even degrees of $\mathcal{F}$.

Hence, we show that $\mathrm{Im}(\chi_{\mathcal{F}}+1) \subset \mathrm{Ker}(\chi_{\mathcal{F}}-1)$ in all degrees of $\mathcal{F}$. □

Finally we give the proof of main theorem of this chapter:

Proof of Theorem 6.0.1. By Proposition 6.2.11 we have :

$$\mathrm{Im}(\chi_{\mathcal{F}}+1) = \mathrm{Ker}(\chi_{\mathcal{F}}-1).$$

It is clear that a basis for $\mathrm{Im}(\chi_{\mathcal{F}}+1)$ gives also a basis for $\mathrm{Ker}(\chi_{\mathcal{F}}-1)$. Hence, by Theorem 6.2.10 $\mathrm{Ker}(\chi_{\mathcal{F}}-1)$ has a basis consisting of the $(\chi_{\mathcal{F}}+1)$ images of all higher non-palindromes in odd degrees of $\mathcal{F}$.

On the other hand, by Theorem 6.2.9 $\mathrm{Ker}(\chi_{\mathcal{F}}-1)$ has a basis consisting of the $(\chi_{\mathcal{F}}+1)$ images of all higher non-palindromes and all odd-length palindromes in even degrees of $\mathcal{F}$. This proves the theorem. □

Now we can state the dimension for $\mathrm{Ker}(\chi_{\mathcal{F}} - 1)$.

Corollary 6.2.12. *In the Leibniz-Hopf algebra, $\mathcal{F}$, the rank of the conjugation invariants in degree m is:*

$$\text{rank } \mathrm{Ker}(\chi_{\mathcal{F}} - 1)_m = \begin{cases} 2^{2n-2}, & \text{if } m = 2n, \\ 2^{2n-3} - 2^{n-2}, & \text{if } m = 2n-1. \end{cases}$$

Proof. By Proposition 2.3.14, in degree $2n-1$ there are $2^{2n-3} - 2^{n-2}$ HNPs, so there are $2^{2n-3} - 2^{n-2}$ elements in basis given by Theorem 6.2.10.

Similarly, in degree $2n$, there are $2^{2n-2} - 2^{n-1}$ HNPs and 2^{n-1} OLPs, so there are 2^{2n-2} elements in basis given by Theorem 6.2.9. This completes the proof. □

Chapter 7

Conjugation Invariants in the mod p Leibniz-Hopf Algebra

For any odd prime p, similar to $\mathcal{F}_p^*$, we have subvector spaces: $\mathrm{Im}(\chi_{\mathcal{F}_p}+1)$ and $\mathrm{Ker}(\chi_{\mathcal{F}_p}-1)$. Furthermore, $\mathrm{Ker}(\chi_{\mathcal{F}_p}-1)$ is formed by the conjugation invariants in $\mathcal{F}_p$.

In this chapter, using the results in the Leibniz-Hopf algebra, $\mathcal{F}$, we will show how we can take an easy approach to find a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_p}-1)$. We now introduce the main theorem of this chapter:

Theorem 7.0.1. *A basis for* $\mathrm{Ker}(\chi_{\mathcal{F}_p}-1)$ *consists of:*

i. the $(\chi_{\mathcal{F}_p}+1)$*-image of all higher non-palindromes and all odd-length palindromes in even degrees.*

ii. the $(\chi_{\mathcal{F}_p}+1)$*-image of all higher non-palindromes in odd degrees.*

As theorem 7.0.1 implies, $\mathrm{Ker}(\chi_{\mathcal{F}_p}-1)$ coincides with $\mathrm{Im}(\chi_F+1)$, like in $\mathcal{F}$, we will now consider a basis for $\mathrm{Im}(\chi_{\mathcal{F}_p}+1)$ to determine a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_p}-1)$. To prove Theorem 7.0.1, we will first give a proof for the following Theorem 7.0.2.

Theorem 7.0.2. *For any odd prime p, in the degree n part of the mod p Leibniz-Hopf algebra, $\mathcal{F}_p$, the image of* $(\chi_{\mathcal{F}_p}+1)$ *has a basis consisting of:*

i. the $(\chi_{\mathcal{F}_p}+1)$*-images of all higher non-palindromes and odd-length palindromes, if n is even, or*

ii. the $(\chi_{\mathcal{F}_p}+1)$*-images of all higher non-palindromes, if n is odd.*

To give a proof for Theorem 7.0.2 , we first consider a linearly independent set in $\mathrm{Im}(\chi_{F_p}+1)$.

7.1 Linear Independence

Proposition 7.1.1. *In even degrees, let* $v_1, \ldots, v_m$ *be all the higher non-palindromes, and let* $o_1, \ldots, o_z$ *be all the odd-length palindromes, then* $(\chi_{\mathcal{F}_p} + 1)(v_1), \ldots, (\chi_{\mathcal{F}_p} + 1)(v_m), (\chi_{\mathcal{F}_p} + 1)(o_1), \ldots, (\chi_{\mathcal{F}_p} + 1)(o_z)$ *are linearly independent.*

Proof. We use the fact that the conjugation in $\mathcal{F}_p$, namely, $\chi_{\mathcal{F}_p}$ is defined as same as $\chi_{\mathcal{F}}$. Beside this, in the proof of Theorem 6.1 we did not refer to the coefficients of summands of OLPs and HNPs under $\chi_{\mathcal{F}} + 1$, hence, the same argument as in the proof of Theorem 6.1 also applies here. □

Proposition 7.1.2. *In odd degrees, the higher non-palindromes have linearly independent images under* $(\chi_{\mathcal{F}_p} + 1)$.

Proof. Again, in the proof of Theorem 6.1.3 we did not refer to the coefficients of summands of HNPs under $\chi_{\mathcal{F}} + 1$. Hence the same argument as in the proof of Theorem 6.1.3 also applies here. □

To complete the proof of Theorem 7.0.2, we will now determine a spanning set for $\mathrm{Im}(\chi_{\mathcal{F}_p} + 1)$.

7.2 Spanning set for $\mathrm{Im}(\chi_{\mathcal{F}_p} + 1)$

Proof of Theorem 7.0.2

Proof of i.

By Theorem 6.0.2, in even degrees, the $(\chi_{\mathcal{F}} + 1)$-images of HNPs and OLPs span $\mathrm{Im}(\chi_{\mathcal{F}} + 1)$. On the other hand, we know $\chi_{\mathcal{F}_p}$ is same as $\chi_{\mathcal{F}}$. Therefore, the $(\chi_{\mathcal{F}_p} + 1)$-image of HNPs and the $(\chi_{\mathcal{F}_p} + 1)$-image of OLPs also span $\mathrm{Im}(\chi_{\mathcal{F}_p} + 1)$. Moreover, by Proposition 7.1.1 $(\chi_{\mathcal{F}_p} + 1)$-image of HNPs and $(\chi_{\mathcal{F}_p} + 1)$-image of OLPs are linearly independent, hence they form a basis for $\mathrm{Im}(\chi_{\mathcal{F}_p} + 1)$.

Proof of ii.

By Theorem 6.0.2, in odd degrees, the $(\chi_{\mathcal{F}_p} + 1)$-image of HNPs span $\mathrm{Im}(\chi_{\mathcal{F}_p} + 1)$. On the other hand, by Proposition 7.1.2 the $(\chi_{\mathcal{F}_p} + 1)$-images of HNPs are linearly independent. Hence the $(\chi_{\mathcal{F}_p} + 1)$-images of HNPs form a basis for $\mathrm{Im}(\chi_{\mathcal{F}_p} + 1)$.

Theorem 7.2.1. *In the mod p dual Hopf-Leibniz algebra, we have:*

$$\mathrm{Im}(\chi_{\mathcal{F}_p} + 1) = \mathrm{Ker}(\chi_{\mathcal{F}_p} - 1).$$

Proof. i. Proof that $\mathrm{Im}(\chi_{\mathcal{F}_p}+1) \subset \mathrm{Ker}(\chi_{\mathcal{F}_p}+1)$.

Like $\mathcal{F}$, $\mathcal{F}_p$ is also cocommutative, so by Proposition 2.2.8 we have:

$$\chi^2_{\mathcal{F}_p} = 1.$$

Therefore we arrive at:

$$(\chi_{\mathcal{F}_p}-1)(\chi_{\mathcal{F}_p}+1) = 0,$$

from which we can deduce:

$$\mathrm{Im}(\chi_{\mathcal{F}_p}+1) \subset \mathrm{Ker}(\chi_{\mathcal{F}_p}-1).$$

ii. Proof that $\mathrm{Ker}(\chi_{\mathcal{F}_p}-1) \subset \mathrm{Im}(\chi_{\mathcal{F}_p}+1)$.

In F_p, if $x \in \mathrm{Ker}(\chi_{\mathcal{F}_p}-1)$, then $\chi_{\mathcal{F}_p}(x) = x$, hence $(\chi_{\mathcal{F}_p}+1)(x) = 2x$, so $2x \in \mathrm{Im}(\chi_{\mathcal{F}_p}+1)$. In that case, we need to show that if $x \in \mathrm{Im}(\chi_{\mathcal{F}_p}+1)$. On the other hand, if $(\chi_{\mathcal{F}_p}+1)(x) = 2x$, bearing in mind that the characteristic of $\mathcal{F}_p$ is not equal two, we have:

$$(\chi_{\mathcal{F}_p}+1)(\frac{x}{2}) = x,$$

hence $x \in \mathrm{Im}(\chi_{\mathcal{F}_p}+1)$.

By i. and ii. the proof is complete.

□

We now give the proof of the main theorem of this chapter:

Proof of Theorem 7.0.1. By Proposition 7.2.1 in n degrees we have :

$$\mathrm{Im}(\chi_{\mathcal{F}_p}+1) = \mathrm{Ker}(\chi_{\mathcal{F}_p}-1).$$

Therefore, Theorem 7.0.2 gives the basis for the relevant degrees. This completes the proof. □

Now we can state the dimension for $\mathrm{Ker}(\chi_{\mathcal{F}_p}-1)$.

Corollary 7.2.2. *In the mod p Leibniz-Hopf algebra, $\mathcal{F}_p$, the dimension of the conjugation invariants in degree m is:*

$$\dim \mathrm{Ker}(\chi_{\mathcal{F}_p}-1)_m = \begin{cases} 2^{2n-2}, & \text{if } m = 2n, \\ 2^{2n-3} - 2^{n-2}, & \text{if } m = 2n-1. \end{cases}$$

Proof. The proof is same as in the proof of Corollary 6.2.12. □

Chapter 8

Correspondence between the matrices of $\chi_{\mathcal{F}_2^*} - 1$ and $\chi_{\mathcal{F}_2} - 1$

We first give details regarding dual basis of a given vector space, and define dual of a given linear transformation which will be also a linear transformation. Secondly, we will introduce Theorem 8.0.3 which tells the correspondence between matrices of these two linear transformations.

Definition 8.0.1. Let W be a finite dimensional vector space over the field $\mathbb{F}$ with basis $B = \{k_1, k_2 \ldots, k_n\}$, then we can define a dual basis of B, which is denoted by B^*, and given by $B^* = \{k_1^*, \ldots, k_n^*\}$, where $k_i^* : W \to \mathbb{F}$ is defined for each k_j by the following relation :

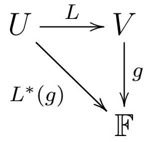
$$k_i^*(k_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

When we say dual basis, we understand in the sense of Definition 8.0.1 in the following theorem of this chapter.

Definition 8.0.2. Let $L : U \to V$ be a linear transformation, where U and V are finite dimensional vector spaces over $\mathbb{F}$. We define the dual of L, denoted by L^*, as the map given by:

$$L^* : V^* \to U^*, \quad L^*(g) = g \circ L : V \to \mathbb{F},$$

for each $g \in V^*$. This can be expressed as a commutative diagram in the following way:



It is clear from the set up that L^* is a linear transformation between V^* and U^*.

Theorem 8.0.3. *Let $L : U \to V$ be a linear transformation, where U and V are finite dimensional vector spaces over $\mathbb{F}$. Let $\mathcal{U} = \{u_1, u_2 \dots, u_n\}$ and $\mathcal{V} = \{k_1, k_2 \dots, k_m\}$ be bases of U and V respectively. Suppose that the matrix of L with respect to these two bases is C. Then the transpose C^t is the matrix of L^* with respect to the dual bases $\mathcal{V}^* = \{k_1^*, k_2^* \dots, k_m^*\}$ of V^* and $\mathcal{U}^* = \{u_1^*, u_2^* \dots, u_n^*\}$ of U^*.*

Proof. Let $L : U \to V$ be a linear transformation where, U, and V are vector spaces with bases $\mathcal{U} = \{u_1, \dots, u_n\}$ and $\mathcal{V} = \{k_1, \dots, k_m\}$ respectively, which are finite dimensional over $\mathbb{F}$, then for a basis element $u_j \in \mathcal{U}$, we have $L(u_j) = c_{1,j}k_1 + c_{2,j}k_2 + \dots + c_{m,j}k_m$, where $c_{i,j} \in \mathbb{F}$. Hence the matrix of L with respect to bases $\mathcal{U}$ and V:

$$[C]_{\mathcal{U},\mathcal{V}} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & \vdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}.$$

We know that the dual of L, namely $L^* : V^* \to U^*$ is also a linear transformation, where V^* and U^* are dual vector spaces with dual bases $\mathcal{V}^* = \{k_1^*, \dots, k_m^*\}$ and $\mathcal{U}^* = \{u_1^*, \dots, u_n^*\}$ respectively. We now determine the matrix of L^* with respect to these two dual bases. For to do that, we need to write $L^*(k_i^*)$ in terms of basis elements of U^*:

$$L^*(k_i^*) = \sum_{j=1}^{n} d_{j,i}u_j^*, \quad \text{where} \quad d_{j,j} \in \mathbb{F},$$

and we need to determine the scalars $d_{j,i}$. By Definition 8.0.1 and 8.0.2 we have:

$$L^*(k_i^*)(u_j) = k_i^*(L(u_j)) = k_i^*(c_{1,j}k_1 + c_{2,j}k_2 + \dots + c_{m,j}k_m) = 0 + k_i^*(c_{i,j}k_i) = c_{i,j}$$

from which we can deduce that the coefficient of u_j^* in $L^*(k_i^*)$ is $c_{i,j}$, hence,

$$L^*(k_i^*) = \sum_{j=1}^{n} c_{i,j} u_j^*,$$

and the matrix of L^* with respect to $\mathcal{V}^*$ and $\mathcal{U}^*$ is given by,

$$[C]_{\mathcal{U},\mathcal{V}}^T = [C]_{\mathcal{V}^*,\mathcal{U}^*} = \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{m1} \\ c_{12} & c_{22} & \cdots & c_{m2} \\ \vdots & & \vdots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{mn} \end{bmatrix}.$$

□

Now in the same sense let's consider conjugation. Both conjugations $\chi_{\mathcal{F}_2^*}$ and $\chi_{\mathcal{F}_2}$ are linear transformations, so $\chi_{\mathcal{F}_2^*}-1$ and $\chi_{\mathcal{F}_2}-1$ are. By Proposition 2.6.8 $\chi_{\mathcal{F}_2^*} = (\chi_{\mathcal{F}_2})^*$ i.e., the antipode on $\mathcal{F}_2^*$ is dual to the antipode on $\mathcal{F}_2$. Consequently, $\chi_{\mathcal{F}_2^*} - 1 = (\chi_{\mathcal{F}_2} - 1)^*$. In the light of this duality, lets first give an example in even degrees and consider what the matrix of $\chi_{\mathcal{F}_2^*} - 1$ tells us about the linearly independent elements in $\mathrm{Im}(\chi_{\mathcal{F}_2} - 1)$.

Example 8.0.4. *In degree* 4, *let* $\chi_{\mathcal{F}_2^*} - 1 : (\mathcal{F}_2^*)_4 \to (\mathcal{F}_2^*)_4$ *be the linear transformation, and take bases* Y^* *in the domain and* S^* *in the range, where* Y^* *is equal* S^**, with these bases ordered in the lexicographical order, that is:*

$$Y^* = S^* = \{S_4, S_{3,1}, S_{2,2}, S_{2,1,1}, S_{1,3}, S_{1,2,1}, S_{1,1,2}, S_{1,1,1,1}\}.$$

Then, there is a matrix for $\chi_{\mathcal{F}_2^*} - 1$ *with respect to* Y^* *and* S^* *which is denoted by* $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{Y^*,S^*}$*, and is given by:*

$$\left[\chi_{\mathcal{F}_2^*} - 1\right]_{Y^*,S^*} = \begin{array}{c} \\ S_4 \\ S_{3,1} \\ S_{2,2} \\ S_{2,1,1} \\ S_{1,3} \\ S_{1,2,1} \\ S_{1,1,2} \\ S_{1,1,1,1} \end{array} \begin{array}{c} \begin{array}{cccccccc} S_4 & S_{3,1} & S_{2,2} & S_{2,1,1} & S_{1,3} & S_{1,2,1} & S_{1,1,2} & S_{1,1,1,1} \end{array} \\ \begin{pmatrix} 0 & 1 & \mathbf{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & \mathbf{1} & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

In that case, by Theorem 8.0.3 we can conclude that for the linear transformation, $(\chi_{\mathcal{F}_2} - 1) : (\mathcal{F}_2)_4 \to (\mathcal{F}_2)_4$*, with respect to the bases:* S *in the* *domain and Y in the range, where $S = Y$, with these bases ordered in the lexicographical order, namely:*

$$S = Y = \{S^4, S^{3,1}, S^{2,2}, S^{2,1,1}, S^{1,3}, S^{1,2,1}, S^{1,1,2}, S^{1,1,1,1}\},$$

the matrix for $\chi_{\mathcal{F}_2} - 1$ with respect to the bases S and Y, denoted by $[\chi_{\mathcal{F}_2} - 1]_{S,Y}$ and given by:

$$[\chi_{\mathcal{F}_2} - 1]_{S,Y} = \begin{array}{c} \\ S^4 \\ S^{3,1} \\ S^{2,2} \\ S^{2,1,1} \\ S^{1,3} \\ S^{1,2,1} \\ S^{1,1,2} \\ S^{1,1,1,1} \end{array} \begin{array}{c} \begin{array}{cccccccc} S^4 & S^{3,1} & S^{2,2} & S^{2,1,1} & S^{1,3} & S^{1,2,1} & S^{1,1,2} & S^{1,1,1,1} \end{array} \\ \left(\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \mathbf{1} & 0 & 0 & 1 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & \mathbf{1} & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & \mathbf{1} & 1 & 0 \end{array} \right) \end{array}.$$

And $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{Y^*,S^*} = ([(\chi_{\mathcal{F}_2} - 1)]_{S,Y})^T$ from which we can conclude that $[\chi_{\mathcal{F}_2} - 1]_{S,Y} = (\left[(\chi_{\mathcal{F}_2^*} - 1)\right]_{Y^*,S^*})^T$.

Remark 8.0.5. *In matrix $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{Y^*,S^*}$, each column indicates $(\chi_{\mathcal{F}_2^*} - 1)$-image of a basis element in $(\mathcal{F}_2^*)_4$, and in matrix $[\chi_{\mathcal{F}_2} - 1]_{S,Y}$ each column indicates $(\chi_{\mathcal{F}_2} - 1)$-image of a basis element in $(\mathcal{F}_2)_4$.*

By Theorem 3.1.6, in even degrees of $\mathcal{F}_2^*$, the image of $(\chi_{\mathcal{F}_2^*} - 1)$ has a basis consisting of the $(\chi_{\mathcal{F}_2^*} - 1)$-images of all HNPs and all ELPs. In matrix $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{Y^*,S^*}$, the columns which have the highlighted coefficients refer to these basis elements. Since they are basis elements, they are linearly independent. Furthermore, these highlighted coefficients are *witness* elements to all HNPs and all ELPs to have linearly independent elements under $(\chi_{\mathcal{F}_2^*} - 1)$.

As we know the matrix $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{Y^*,S^*}$ is the transpose of $[\chi_{\mathcal{F}_2} - 1]_{S,Y}$, hence the highlighted coefficients in $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{Y^*,S^*}$ are obtained by transposing the columns which have the highlighted coefficients in matrix $[\chi_{\mathcal{F}_2} - 1]_{S,Y}$. By Example 8.0.4, considering the correspondence between the columns which have the highlighted coefficients in matrix $[\chi_{\mathcal{F}_2} - 1]_{S,Y}$ and its transpose. Unfortunately, we see that the matrix $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{Y^*,S^*}$ does not lead the matrix $[\chi_{\mathcal{F}_2} - 1]_{S,Y}$ to give a clear pattern regarding linearly independent elements in $\mathrm{Im}(\chi_{\mathcal{F}_2} - 1)$.

To have a clear pattern, for the linear transformation, $\chi_{\mathcal{F}_2^*} - 1 : (\mathcal{F}_2^*)_4 \to (\mathcal{F}_2^*)_4$, let's take different bases which we will introduce in the following example.

Example 8.0.6. *In degree 4, let $(\chi_{\mathcal{F}_2^*} - 1) : ({F_2}^*)_4 \to ({F_2}^*)_4$, be the linear transformation, and take bases C^* in the domain and B^* in the range which are ordered in the following: $C^* = \{S_4, S_{1,3}, S_{1,2,1}, S_{1,1,2}, S_{1,1,1,1}, S_{2,1,1}, S_{3,1}, S_{2,2}\}$, and $B^* = \{S_{2,2}, S_{1,3}, S_{1,1,2}, S_{1,1,1,1}, S_4, S_{3,1}, S_{2,1,1}, S_{1,2,1}\}$, then there is a matrix for $(\chi_{\mathcal{F}_2^*} - 1)$ with respect to C^* in the domain and, B^* in the range, denoted by $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{C^*,B^*}$, is given by,*

$$\left[\chi_{\mathcal{F}_2^*} - 1\right]_{C^*,B^*} = \begin{array}{c} \\ S_{2,2} \\ S_{1,3} \\ S_{1,1,2} \\ S_{1,1,1,1} \\ S_4 \\ S_{3,1} \\ S_{2,1,1} \\ S_{1,2,1} \end{array} \begin{array}{c} \begin{array}{cccccccc} S_4 & S_{1,3} & S_{1,2,1} & S_{1,1,2} & S_{1,1,1,1} & S_{2,1,1} & S_{3,1} & S_{2,2} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & \mathbf{1} \\ 0 & 1 & 1 & 1 & 1 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 1 & 1 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

By Theorem 8.0.3 we can see that for the linear transformation, $(\chi_{\mathcal{F}_2} - 1) : (F_2)_4 \to (\mathcal{F}_2)_4$, with respect to bases B in the domain and C in the range, with these bases ordered: $B = \{S^{2,2}, S^{1,3}, S^{1,1,2}, S^{1,1,1,1}, S^4, S^{3,1}, S^{2,1,1}, S^{1,2,1}\}$, $C = \{S^4, S^{1,3}, S^{1,2,1}, S^{1,1,2}, S^{1,1,1,1}, S^{2,1,1}, S^{3,1}, S^{2,2}\}$, the matrix for $\chi_{\mathcal{F}_2} - 1 : (\mathcal{F}_2)_4 \to (\mathcal{F}_2)_4$, with respect to bases B in the domain and C in the range, denoted by $[\chi_{\mathcal{F}_2} - 1]_{B,C}$, and given by

$$[\chi_{\mathcal{F}_2} - 1]_{B,C} = \begin{array}{c} \\ S^4 \\ S^{1,3} \\ S^{1,2,1} \\ S^{1,1,2} \\ S^{1,1,1,1} \\ S^{2,1,1} \\ S^{3,1} \\ S^{2,2} \end{array} \begin{array}{c} \begin{array}{cccccccc} S^{2,2} & S^{1,3} & S^{1,1,2} & S^{1,1,1,1} & S^4 & S^{3,1} & S^{2,1,1} & S^{1,2,1} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & \mathbf{1} \\ 1 & 1 & 1 & 0 & 1 & 0 & \mathbf{1} & 0 \\ 0 & 1 & 0 & 0 & 1 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

And $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{C^*,B^*} = ([\chi_{\mathcal{F}_2} - 1]_{B,C})^T$. Hence, $(\left[\chi_{\mathcal{F}_2^*} - 1\right]_{C^*,B^*})^T = [\chi_{\mathcal{F}_2} - 1]_{B,C}$

Remark 8.0.7. *In example 8.0.6, we did a different choice of bases.*

More specifically, for any choice of basis ordering on the left hand side of C^*, we set the right hand side of C^*, namely $\{S_{1,1,1,1}, S_{2,1,1}, S_{3,1}, S_{2,2}\}$ to include all ELPs and HNPs according to their lengths in non-increasing order *so that same length of basis elements ordered in reverse lexicographical order, where ELPs come after HNPs.*

On the other hand, for any choice of basis ordering on the left hand side of B^**, we set the right hand side of* B^**, namely* $\{S_4, S_{3,1}, S_{2,1,1}, S_{1,2,1}\}$ *to include all OLPs and HNPs according to their lengths in non-decreasing order so that same length of basis elements ordered in reverse lexicographical order, where OLPs come after HNPs.*

By example 8.0.6, we can now see the matrix $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{C^*,B^*}$ leads the matrix $[\chi_{\mathcal{F}_2} - 1]_{B,C}$ to give a clear pattern regarding linearly independent elements in $\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$. To understand this pattern, let us consider the correspondence between the highlighted coefficients in matrices $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{C^*,B^*}$ and $[\chi_{\mathcal{F}_2} - 1]_{B,C}$.

Like in example 8.0.4, the columns in the matrix $(\left[\chi_{\mathcal{F}_2^*} - 1\right]_{C^*,B^*})$ which have the highlighted coefficients refer to linearly independent elements of $(\chi_{\mathcal{F}_2^*} - 1)$-image, and these highlighted coefficients, the witness elements to ELPs and HNPs being linearly independent under $(\chi_{\mathcal{F}_2^*} - 1)$-image, form a diagonal in $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{C^*,B^*}$. Beside this, these witness elements are obtained by transposing the columns which have the highlighted coefficients in $[\chi_{\mathcal{F}_2} - 1]_{B,C}$. Moreover, in example 8.0.6 we see that, these highlighted coefficients in $[\chi_{\mathcal{F}_2} - 1]_{B,C}$ also form a diagonal, and are also witness elements to OLPs and HNPs to have linearly independent elements under $(\chi_{\mathcal{F}_2} - 1)$.

Precisely, let us consider the correspondence between the witness elements in $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{C^*,B^*}$ and $[\chi_{\mathcal{F}_2} - 1]_{B,C}$. As in $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{C^*,B^*}$ HNPs: $S_{3,1}, S_{2,1,1}$ are witness elements to HNPs: $S_{3,1}, S_{2,1,1}$ having linearly independent images under $(\chi_{\mathcal{F}_2^*} - 1)$, and OLPs: $S_4, S_{1,2,1}$ are witness elements to ELPs: $S_{2,2}, S_{1,1,1,1}$ having linearly independent images under $(\chi_{\mathcal{F}_2^*} - 1)$.

On the other hand, likewise in $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{C^*,B^*}$, in $[\chi_{\mathcal{F}_2} - 1]_{B,C}$, HNPs: $S^{3,1}, S^{2,1,1}$ are witness elements to HNPs: $S^{3,1}, S^{2,1,1}$ having linearly independent images under $(\chi_{\mathcal{F}_2} - 1)$, and ELPs: $S^{2,2}, S^{1,1,1,1}$ are witness elements to OLPs: $S^4, S^{1,2,1}$ having linearly independent images under $(\chi_{\mathcal{F}_2} - 1)$.

Remark 8.0.8. *The OLPs which are witness elements in* $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{C^*,B^*}$ *are interchanging with ELPs being witness elements in* $[\chi_{\mathcal{F}_2} - 1]_{B,C}$*.*

We can now introduce the following generalization in all even degrees.

In degree $2n$, let

$$(\chi_{\mathcal{F}_2^*} - 1) : (\mathcal{F}_2^*)_{2n} \to (\mathcal{F}_2^*)_{2n}$$

be a linear transformation, and if we take bases E^* in the domain and D^* in the range, where D^* is formed taking LNPs and OLPs in the given degree, in any order followed by taking ELPs and HNPs according to their lengths in non-increasing order so that the same length of basis elements are ordered in reverse lexicographical order, where ELPs come after HNPs, and D^* is formed taking LNPs and ELPs in the given degree, in any order followed by taking OLPs and HNPs according to their lengths in non-decreasing order so that same length of basis elements are ordered in reverse lexicographical order, where OLPs come after HNPs. Then we have a matrix for $\chi_{\mathcal{F}_2^*} - 1$ with respect to E^* in the domain and D^* in the range denoted by $\left[\chi_{\mathcal{F}_2^*} - 1\right]_{E^*,D^*}$, and given by:

$$[\chi_{F_2{}^*} - 1]_{E^*,D^*} = \left[\begin{array}{ccccc|cccccccc}
* & . & . & . & * & * & . & . & . & . & . & . & * \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
* & . & . & . & * & * & . & . & . & . & . & . & * \\
\hline
* & . & . & . & * & * & . & . & . & . & . & * & \mathbf{1} \\
. & . & . & . & . & . & . & . & . & . & * & \mathbf{1} & 0 \\
. & . & . & . & . & . & . & . & * & . & . & . & . \\
. & . & . & . & . & . & . & * & . & . & . & . & . \\
. & . & . & . & . & . & * & . & . & . & . & . & . \\
. & . & . & . & . & * & \mathbf{1} & 0 & . & . & . & . & . \\
* & . & . & . & * & \mathbf{1} & 0 & . & . & . & . & . & 0
\end{array}\right]$$

By Theorem 8.0.3 we have a matrix for linear transformation

$$(\chi_{\mathcal{F}_2^*} - 1) : (\mathcal{F}_2^*)_{2n} \to (\mathcal{F}_2^*)_{2n}$$

with basis D in the domain and E in the range which is denoted $[\chi_{\mathcal{F}_2} - 1]_{D,E}$, and is given by

$$[\chi_{\mathcal{F}_2} - 1]_{D,E} = \left[\begin{array}{ccccc|cccccccc}
* & . & . & . & * & * & . & . & . & . & . & . & * \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
* & . & . & . & * & * & . & . & . & . & . & . & * \\
\hline
* & . & . & . & * & * & . & . & . & . & . & * & \mathbf{1} \\
. & . & . & . & . & . & . & . & . & . & * & \mathbf{1} & 0 \\
. & . & . & . & . & . & . & . & * & . & . & . & . \\
. & . & . & . & . & . & . & * & . & . & . & . & . \\
. & . & . & . & . & . & * & . & . & . & . & . & . \\
. & . & . & . & . & * & \mathbf{1} & 0 & . & . & . & . & . \\
* & . & . & . & * & \mathbf{1} & 0 & . & . & . & . & . & 0
\end{array}\right].$$

And

$$\left[\chi_{\mathcal{F}_2^*} - 1\right]_{E^*,D^*} = ([\chi_{F_2} - 1]_{D,E})^T.$$

Hence for choice of bases E^* and D^* above, in all even degrees, we can always get suitable bases D and, E giving the linearly independent elements under $\chi_{\mathcal{F}_2} - 1$. Therefore we have proved the following Theorem.

Theorem 8.0.9. *Let $v_1, \ldots, v_m$ be all the higher non-palindromes with even degree, and let $o_1, \ldots, o_z$ be all the odd length palindromes with even degree. Then $(\chi_{\mathcal{F}_2} - 1)(v_1), \ldots, (\chi_{F_2} - 1)(v_m), (\chi_{F_2} - 1)(o_1), \ldots, (\chi_{F_2} - 1)(o_z)$ are linearly independent.*

Bearing in the mind that there is no ELP in odd degrees, one can adapt the generalization in even degrees for odd degrees taking E^* and D^* in the same order with considering only HNPs. Therefore we have proved the following Theorem.

Theorem 8.0.10. *In odd degrees, the higher non-palindromes in $\mathcal{F}_2$ have linearly independent images under $(\chi_{\mathcal{F}_2} - 1)$.*

Now let's introduce the dimension of $\operatorname{Im}(\chi_{\mathcal{F}_2} - 1)$.

Theorem 8.0.11. *In the mod-2 Leibniz-Hopf algebra, $\mathcal{F}_2$, the dimension of the* $\operatorname{Im}(\chi_{\mathcal{F}_2} - 1)$ *in degree m is equal to the dimension of* $\operatorname{Im}(\chi_{F_2^*} - 1)$*, i.e,*

$$\dim \operatorname{Im}(\chi_{\mathcal{F}_2} - 1)_m = \begin{cases} 2^{2n-2}, & \text{if } m = 2n, \\ 2^{2n-3} - 2^{n-2}, & \text{if } m = 2n - 1. \end{cases}$$

Proof. Using the fact that for each positive integer n,

$$(\chi_{\mathcal{F}_2} - 1) : (\mathcal{F}_2)_n \to (\mathcal{F}_2)_n,$$

is a linear transformation on finite vector space, namely $(\mathcal{F}_2)_n$, we know:

$$\dim \operatorname{Im}(\chi_{\mathcal{F}_2} - 1) =: \text{rank of } (\chi_{\mathcal{F}_2} - 1).$$

Furthermore,

$$\text{rank of } (\chi_{\mathcal{F}_2} - 1) = \text{rank of } (\chi_{\mathcal{F}_2} - 1)^T$$

Beside this, since $(\mathcal{F}_2)_n$ is a finite dimensional vector space, $(\mathcal{F}_2^*)_n$ is also a finite dimensional vector space. On the other hand, by Theorem 8.0.3

$$\text{rank of } (\chi_{\mathcal{F}_2} - 1)^T = \text{rank of } (\chi_{\mathcal{F}_2^*} - 1),$$

where

$$(\chi_{\mathcal{F}_2*} - 1) : (\mathcal{F}_2^*)_n \to (\mathcal{F}_2^*)_n,$$

is a linear transformation on finite dimensional vector space, namely $(\mathcal{F}_2^*)_n$. So we have:

$$\text{rank of } (\chi_{\mathcal{F}_2} - 1) = \text{rank of } (\chi_{\mathcal{F}_2^*} - 1).$$

Similarly,

$$\dim \operatorname{Im}(\chi_{\mathcal{F}_2^*} - 1) =: \text{rank of } (\chi_{\mathcal{F}_2^*} - 1).$$

Hence,

$$\dim \operatorname{Im}(\chi_{\mathcal{F}_2} - 1) = \dim \operatorname{Im}(\chi_{\mathcal{F}_2^*} - 1),$$

And, the remainder of the proof is easily seen by Corollary 3.2.9. □

Theorem 8.0.12. *In even degrees, the image of* $(\chi_{\mathcal{F}_2} - 1)$ *is spanned by the* $(\chi_{\mathcal{F}_2} - 1)$*-images of of all higher non-palindromes and all odd-length palindromes.*

Proof. In even degrees, by Theorem 8.0.9 $(\chi_{\mathcal{F}_2} - 1)$-images of all HNPs and OLPs are linearly independent. Beside by Proposition 2.3.14 the number of all HNPs and OLPs exactly matches $\dim \operatorname{Im}(\chi_{\mathcal{F}_2} - 1)$ which is given by Theorem 8.0.11. Hence, $(\chi_{\mathcal{F}_2} - 1)$-image of all HNPs and OLPs also span $\operatorname{Im}(\chi_{\mathcal{F}_2} - 1)$. □

Theorem 8.0.13. *In odd degrees, the image of* $(\chi_{\mathcal{F}_2} - 1)$ *is spanned by the* $(\chi_{\mathcal{F}_2} - 1)$*-images of all higher non-palindromes*

By the theorems above we established a linearly independent set and a spanning set for $\operatorname{Im}(\chi_{\mathcal{F}_2} - 1)$ in all degrees. Hence we proved the following theorem.

Theorem 8.0.14. *In the mod 2 Leibniz-Hopf algebra,* $\mathcal{F}_2$*, in degree* n*,* $\operatorname{Im}(\chi_{\mathcal{F}_2} - 1)$ *has a basis consisting of:*

i. the $(\chi_{\mathcal{F}_2} - 1)$*-images of all higher non-palindromes and, odd-length palindromes if* n *is even, or*

ii. the $(\chi_{\mathcal{F}_2} - 1)$*-images of all higher non-palindromes, if* n *is odd.*

Chapter 9

Conjugation Invariants in the mod 2 Leibniz-Hopf Algebra

In this chapter, for prime two, we have also subvector spaces: $\mathrm{Im}(\chi_{\mathcal{F}_2}-1)$ and $\mathrm{Ker}(\chi_{\mathcal{F}_2}-1)$ of $\mathcal{F}_2$. Moreover, $\mathrm{Ker}(\chi_{\mathcal{F}_2}-1)$ is also formed by the conjugation invariants in $\mathcal{F}_2$. Using the results in the previous chapter, we will show that how we can take an easy approach to find a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_2}-1)$. We now introduce the main theorem of this chapter:

Theorem 9.0.1. *A basis for* $\mathrm{Ker}(\chi_{\mathcal{F}_2}-1)$ *consists of:*

i. in even degrees, $(\chi_{\mathcal{F}_2}-1)$*-image of all higher non-palindromes and all odd-length palindromes*

ii. in odd degrees, $(\chi_{\mathcal{F}_2}-1)$*-image of all higher non-palindromes and* ρ*-image of all odd-length palindromes.*

Here ρ denotes the sum of " right refinement", which we will fully define in the following section.

Before giving a proof we will first introduce the dimension of $\mathrm{Ker}(\chi_{\mathcal{F}_2}-1)$ in the following theorem.

Theorem 9.0.2. *In the mod 2 Leibniz-Hopf algebra, the dimension of the* $\mathrm{Ker}(\chi_{\mathcal{F}_2}-1)$ *in degree m is equal to the dimension of* $\mathrm{Ker}(\chi_{F_2^*}-1)$*, i.e,*

$$\dim \mathrm{Ker}(\chi_{\mathcal{F}_2}-1)_m = \begin{cases} 2^{2n-2}, & \text{if } m = 2n, \\ 2^{2n-3}+2^{n-2}, & \text{if } m = 2n-1. \end{cases}$$

Proof. By Rank and nullity Theorem, and Theorem 8.0.11, one can easily see that

$$\dim \mathrm{Ker}(\chi_{\mathcal{F}_2}-1) = \dim \mathrm{Ker}(\chi_{\mathcal{F}_2^*}-1).$$

The remainder of the proof can be seen by Corollary 3.2.10. □

Now we will first give a basis for $\mathrm{Ker}(\chi_{F_2} - 1)$ in even degrees.

Corollary 9.0.3. *In even degrees,* $\mathrm{Ker}(\chi_{F_2} - 1) = \mathrm{Im}(\chi_{F_2} - 1)$.

Proof. The proof of Corollary 3.1.7 can easily adapt to this case without difficulty. □

Now we will give a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)$. Firstly, we need to give technical results and introduce a new terminology," ρ."

Definition 9.0.4. Let $S^{i_1,\ldots,i_{2k+1}}$ be an odd-length palindrome. Define the ρ-image to be

$$\rho(S^{i_1,\ldots,i_{2k+1}}) = \sum S^{i_1,\ldots,i_k,j_1,\ldots,j_l},$$

where summation is over all refinements $j_1, \ldots, j_l$ of $i_{k+1}, \ldots, i_{2k+1}$ that have $j_1 \geq \frac{i_{k+1}}{2}$.

Example 9.0.5.

$$\rho(S^{2,3,2}) = S^{2,3,2} + S^{2,3,1,1} + S^{2,2,1,2} + S^{2,2,1,1,1}.$$

Remark 9.0.6. *By mod 2 reduction of the equation 6.2 in the Proposition 6.2.1 in odd degrees, we can easily see that ρ-image of all OLPs in* $\mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)$.

Theorem 9.0.7. *In odd degrees, let $p_1, \ldots, p_r$ be all the odd-length palindromes, and let $h_1, \ldots, h_s$ be all the higher non-palindromes. Then* $\rho(p_1), \ldots, \rho(p_r), (\chi_{F_2} - 1)(h_1), \ldots, (\chi_{F_2} - 1)(h_s)$ *are linearly independent.*

Proof. Let $p_1, \ldots, p_r$ are all the odd-length palindromes in odd degrees, and let $h_1, \ldots, h_s$ be all the higher non-palindromes in odd degrees. Suppose $p_1, \ldots, p_k$ are some distinct elements of $\{p_1, \ldots, p_r\}$ and $h_1, \ldots, h_l$ are some distinct elements of $\{h_1, \ldots, h_s\}$ with the property that:

$$\rho(p_1) + \cdots + \rho(p_k) = (\chi_{\mathcal{F}_2} - 1)(h_1) + \cdots + (\chi_{\mathcal{F}_2} - 1)(h_l) \qquad (9.1)$$

Moreover, let's order these elements according to their lengths in a non-increasing order, i.e,

$$length(p_k) \leq length(p_{k-1}) \leq \cdots \leq length(p_1), \qquad (9.2)$$

and

$$length(h_l) \leq length(h_{l-1}) \leq \cdots \leq length(h_1). \qquad (9.3)$$

Let m be the length of p_k, then by definition 9.0.4, the only length m summand in $\rho(p_k)$ is p_k, namely p_k itself. On the other hand, by the ordering assumption (9.2), there can be other OLPs that have length m on the left hand side of equation 9.1. To be more precise, let i be the smallest index such that p_i has length m, then similarly, in $\rho(p_i)$, there is only summand of the same length as p_i, namely p_i itself. Consequently, the only length m summands in $\rho(p_1) + \cdots + \rho(p_k)$ will be those p_i that have length m, i.e., $p_i, p_{i+1}, \ldots, p_{k-1}, p_k$. And $p_1, \ldots, p_{i-1}$ will have length strictly greater than m.

Beside this, since $p_1, \ldots, p_k$ are all distinct, $p_i, p_{i+1}, \ldots, p_{k-1}, p_k$ cannot cancel, so the minimal-length summands on the left hand side of equation (9.1) have length m and are palindromes.

Now, let's consider the right hand side of equation (9.1). Let n be the length of h_l, then the only length n summands in $(\chi_{\mathcal{F}_2} - 1)(h_l)$ are h_l and its reverse, which is an LNP. Again, by the assumption of ordering (9.3), there can be other HNPs that have length n on the right hand side of equation 9.1. Let j be the smallest index such that h_j has length n, then in the same manner, the only length n summands in $(\chi_{\mathcal{F}_2} - 1)(h_j)$ are h_j and its reverse. Following this, the only length n summands in $(\chi_{\mathcal{F}_2} - 1)(h_1) + \ldots + (\chi_{\mathcal{F}_2} - 1)(h_l)$ are $h_j, h_{j+1}, \ldots, h_l$ and the reverse of those HNPs. And $h_1, \ldots, h_{j-1}$ will have length which is strictly greater than n.

Furthermore, since $h_1, \ldots, h_l$ are all distinct, $h_j, h_{j+1}, \ldots, h_l$ and the reverse of those HNPs cannot cancel, so the minimal-length summand on the right hand side of equation (9.1) have length n and are HNPs and LNPs. In other words these n length summands are non palindromes.

Finally, we see that, the minimal-length of summands on the left hand side of equation (9.1) are palindromes, whereas the minimall-length of summands on the right hand side of equation (9.1) are non-palindromes. This leads to a contradiction which shows that equation (9.1) cannot hold unless both sides are zero. Therefore,

$$\rho(p_1), \ldots, \rho(p_k), (\chi_{\mathcal{F}_2} - 1)(h_1), \ldots, (\chi_{\mathcal{F}_2} - 1)(h_l)$$

are linearly independent. This completes the proof.

□

Theorem 9.0.8. *In odd degrees, $(\chi_{\mathcal{F}_2}-1)$-images of all higher non-palindromes and ρ-images of all odd-length palindromes form a basis for* $\mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)$.

Proof. In $\mathcal{F}_2$ we have:

$$(\chi_{\mathcal{F}_2} - 1) \circ (\chi_{\mathcal{F}_2} - 1) = 0,$$

so the $(\chi_{\mathcal{F}_2} - 1)$-image of all HNPs are in $\mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)$. On the other hand, in odd degrees, by remark 9.0.6 the ρ-image of an odd-length palindrome is also in $\mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)$. Moreover, by Theorem 9.0.7 the $(\chi_{\mathcal{F}_2} - 1)$-image of all HNPs and ρ-image of all OLPs are linearly independent. Beside this, by Proposition 2.3.14 the number of all HNPs and OLPs is:

$$(2^{2n-3} - 2^{n-2}) + 2^{n-1} = 2^{2n-3} + 2^{n-2},$$

which is exactly $\dim \mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)$. Hence, $(\chi_{\mathcal{F}_2} - 1)$-image of all HNPs and ρ-image of all OLPs also span $\mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)$. Therefore, $(\chi_{\mathcal{F}_2} - 1)$-image of all HNPs and ρ-image of all OLPs form a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)$ in odd degrees of $\mathcal{F}_2$. □

We can give a proof for the main theorem of this chapter.

Proof of Theorem 9.0.1. In even degrees, by Corollary 9.0.3 we have:

$$\mathrm{Ker}(\chi_{F_2} - 1) = \mathrm{Im}(\chi_{\mathcal{F}_2} - 1).$$

Therefore a basis for $\mathrm{Im}(\chi_{\mathcal{F}_2} - 1)$ is also a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)$, and by Theorem 8.0.14 the image of $(\chi_{\mathcal{F}_2} - 1)$ has a basis consisting of the $(\chi_{\mathcal{F}_2} - 1)$-images of all higher non-palindromes and all odd-length palindromes. Hence this basis is also a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)$. The remainder of the proof can be easily seen by Theorem 9.0.8. □

Corollary 9.0.9. *In odd degrees, ρ-images of all the odd-length palindromes form a basis for* $\mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)/\mathrm{Im}(\chi_{\mathcal{F}_2} - 1)$.

Proof. Suppose that there are some odd-length palindromes $p_1, \ldots, p_k$ such that;

$$\rho(p_1), \ldots, \rho(p_k) \in \mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)/\mathrm{Im}(\chi_{\mathcal{F}_2} - 1)$$

with the property that:

$$\rho(p_1) + \cdots + \rho(p_k) \equiv 0 \quad mod\ \mathrm{Im}(\chi_{\mathcal{F}_2} - 1),$$

which means $\rho(p_1) + \cdots + \rho(p_k) \in \mathrm{Im}(\chi_{\mathcal{F}_2} - 1)$. And by Theorem 8.0.14 we know that, in odd degrees, $(\chi_{\mathcal{F}_2} - 1)$-image of higher non-palindromes form a basis for $\mathrm{Im}(\chi_{\mathcal{F}_2} - 1)$ which implies that there are higher non-palindromes $h_1, \ldots, h_k$ with the property that:

$$\rho_(p_1) + \cdots + \rho_(p_k) = (\chi_{\mathcal{F}_2} - 1)(h_1) + \cdots + (\chi_{\mathcal{F}_2} - 1)(h_l). \qquad (9.4)$$

But by the same argument in the proof of Theorem 9.0.7, equation (9.4) cannot hold unless both sides are zero, so it is a contradiction. Therefore, the ρ-image of all OLPs are linearly independent mod $\mathrm{Im}(\chi_{\mathcal{F}_2} - 1)$.

On the other hand, since $\mathcal{F}_2$ is of finite type, so we have:

$$\dim(\mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)/\mathrm{Im}(\chi_{\mathcal{F}_2} - 1)) = \dim \mathrm{Ker}(\chi_{\mathcal{F}_2} - 1) - \dim \mathrm{Im}(\chi_{\mathcal{F}_2} - 1),$$

in each degree. Therefore, by Theorem 8.0.11 and Theorem 9.0.2 we have:

$$\dim(\mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)/\mathrm{Im}(\chi_{\mathcal{F}_2} - 1)) = 2^{n-1}.$$

Beside this, by the Proposition 2.3.14 the number of OLPs in $2n-1$ degrees is 2^{n-1}. Hence,the ρ-images: $\rho(p_1), \ldots, \rho(p_k)$ also span $\mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)/\mathrm{Im}(\chi_{\mathcal{F}_2} - 1)$, so the ρ-image of all odd-length palindromes form a basis for $\mathrm{Ker}(\chi_{\mathcal{F}_2^*} - 1)/\mathrm{Im}(\chi_{\mathcal{F}_2^*} - 1)$.

□

Corollary 9.0.10. *In degree m, the quotient $\mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)/\mathrm{Im}(\chi_{\mathcal{F}_2} - 1)$, i.e., the Tate cohomology of $\mathbf{Z}/2$ acting on $\mathcal{F}_2$ by conjugation, has dimension*

$$\dim\left(\frac{\mathrm{Ker}(\chi_{\mathcal{F}_2} - 1)_m}{\mathrm{Im}(\chi_{\mathcal{F}_2} - 1)_m}\right) = \begin{cases} 0, & \text{if } m = 2n, \\ 2^{n-1}, & \text{if } m = 2n - 1 \end{cases}$$

Proof. It can maybe seen by Corollary 9.0.3 and by Corollary 9.0.9. □

Appendix A

Calculations in the dual Leibniz-Hopf Algebra

In this Appendix the $(\chi_{\mathcal{F}^*} \pm 1)$-image of all degree 4 and 5 basis elements are listed. In particular the summands of these basis elements under $\mathrm{Im}(\chi_{\mathcal{F}^*} \pm 1)$ are listed according to non-increasing length order. The given tables can also be used for the mod p dual Leibniz algebra.

A.1 Table list in degree 4

$(\chi_{\mathcal{F}^*} - 1)$(**basis for** $\mathcal{F}^*$)	**RESULT**
$(\chi_{\mathcal{F}^*} - 1)(S_4)$	$-2S_4$
$(\chi_{\mathcal{F}^*} - 1)(S_{3,1})$	$-S_{3,1} + S_{1,3} + S_4$
$(\chi_{\mathcal{F}^*} - 1)(S_{2,2})$	S_4
$(\chi_{\mathcal{F}^*} - 1)(S_{1,3})$	$-S_{1,3} + S_{3,1} + S_4$
$(\chi_{\mathcal{F}^*} - 1)(S_{2,1,1})$	$-S_{2,1,1} - S_{1,1,2} - S_{2,2} - S_{1,3} - S_4$
$(\chi_{\mathcal{F}^*} - 1)(S_{1,2,1})$	$-2S_{1,2,1} - S_{3,1} - S_{1,3} - S_4$
$(\chi_{\mathcal{F}^*} - 1)(S_{1,1,2})$	$-S_{1,1,2} - S_{2,1,1} - S_{3,1} - S_{2,2} - S_4$
$(\chi_{\mathcal{F}^*} - 1)(S_{1,1,1,1})$	$S_{2,1,1} + S_{1,2,1} + S_{1,1,2} + S_{1,3} + S_{3,1} + S_{2,2} + S_4$

$(\chi_{\mathcal{F}^*}+1)$(**basis for** $\mathcal{F}^*$)	**RESULT**
$(\chi_{\mathcal{F}^*}+1)(S_4)$	0
$(\chi_{\mathcal{F}^*}+1)(S_{3,1})$	$S_{3,1}+S_{1,3}+S_4$
$(\chi_{\mathcal{F}^*}+1)(S_{2,2})$	$2S_{2,2}+S_4$
$(\chi_{\mathcal{F}^*}+1)(S_{1,3})$	$S_{1,3}+S_{3,1}+S_4$
$(\chi_{\mathcal{F}^*}+1)(S_{2,1,1})$	$S_{2,1,1}-S_{1,1,2}-S_{2,2}-S_{1,3}-S_4$
$(\chi_{\mathcal{F}^*}+1)(S_{1,2,1})$	$-S_{3,1}-S_{1,3}-S_4$
$(\chi_{\mathcal{F}^*}+1)(S_{1,1,2})$	$S_{1,1,2}-S_{2,1,1}-S_{3,1}-S_{2,2}-S_4$
$(\chi_{\mathcal{F}^*}+1)(S_{1,1,1,1})$	$2S_{1,1,1,1}+S_{2,1,1}+S_{1,2,1}+S_{1,1,2}+S_{1,3}+S_{3,1}$ $+S_{2,2}+S_4$

A.2 Table list in degree 5

$(\chi_{\mathcal{F}^*}-1)$(**basis for** $\mathcal{F}^*$)	**RESULT**
$(\chi_{\mathcal{F}^*}-1)(S_5)$	$-2S_5$
$(\chi_{\mathcal{F}^*}-1)(S_{4,1})$	$-S_{4,1}+S_{1,4}+S_5$
$(\chi_{\mathcal{F}^*}-1)(S_{1,4})$	$-S_{1,4}+S_{4,1}+S_5$
$(\chi_{\mathcal{F}^*}-1)(S_{3,2})$	$-S_{3,2}+S_{2,3}+S_5$
$(\chi_{\mathcal{F}^*}-1)(S_{2,3})$	$-S_{2,3}+S_{3,2}+S_5$
$(\chi_{\mathcal{F}^*}-1)(S_{3,1,1})$	$-S_{3,1,1}-S_{1,1,3}-S_{1,4}-S_{2,3}-S_5$
$(\chi_{\mathcal{F}^*}-1)(S_{2,2,1})$	$-S_{2,2,1}-S_{1,2,2}-S_{3,2}-S_{1,4}-S_5$
$(\chi_{\mathcal{F}^*}-1)(S_{2,1,2})$	$-2S_{2,1,2}-S_{2,3}-S_{3,2}-S_5$
$(\chi_{\mathcal{F}^*}-1)(S_{1,3,1})$	$-2S_{1,3,1}-S_{4,1}-S_{1,4}-S_5$
$(\chi_{\mathcal{F}^*}-1)(S_{1,2,2})$	$-S_{1,2,2}-S_{2,2,1}-S_{4,1}-S_{2,3}-S_5$
$(\chi_{\mathcal{F}^*}-1)(S_{1,1,3})$	$-S_{1,1,3}-S_{3,1,1}-S_{4,1}-S_{3,2}-S_5$
$(\chi_{\mathcal{F}^*}-1)(S_{2,1,1,1})$	$-S_{2,1,1,1}+S_{1,1,1,2}+S_{2,1,2}+S_{1,2,2}S_{1,1,3}+S_{2,3}$ $+S_{1,4}+S_{3,2}+S_5$
$(\chi_{\mathcal{F}^*}-1)(S_{1,2,1,1})$	$-S_{1,2,1,1}+S_{1,1,2,1}+S_{2,2,1}+S_{1,3,1}+S_{1,1,3}+$ $S_{2,3}+S_{1,4}+S_{4,1}+S_5$
$(\chi_{\mathcal{F}^*}-1)(S_{1,1,2,1})$	$-S_{1,1,2,1}+S_{1,2,1,1}+S_{3,1,1}+S_{1,3,1}+S_{1,2,2}+S_{3,2}$ $+S_{1,4}+S_{4,1}+S_5$
$(\chi_{\mathcal{F}^*}-1)(S_{1,1,1,2})$	$-S_{1,1,1,2}+S_{2,1,1,1}+S_{3,1,1}+S_{2,2,1}+S_{2,1,2}+S_{3,2}$ $+S_{2,3}+S_{4,1}+S_5$
$(\chi_{\mathcal{F}^*}-1)(S_{1,1,1,1,1})$	$-2S_{1,1,1,1,1}-S_{2,1,1,1}-S_{1,2,1,1}-S_{1,1,2,1}-S_{1,1,1,2}$ $-S_{2,2,1}-S_{2,1,2}-S_{1,2,2}-S_{3,1,1}-S_{1,3,1}-S_{1,1,3}$ $-S_{3,2}-S_{2,3}-S_{1,4}-S_{4,1}-S_5$

$(\chi_{\mathcal{F}^*}+1)$(**basis for** $\mathcal{F}^*$)	**RESULT**
$(\chi_{\mathcal{F}^*}+1)(S_5)$	0
$(\chi_{\mathcal{F}^*}+1)(S_{4,1})$	$S_{4,1}+S_{1,4}+S_5$
$(\chi_{\mathcal{F}^*}+1)(S_{1,4})$	$S_{1,4}+S_{4,1}+S_5$
$(\chi_{\mathcal{F}^*}+1)(S_{3,2})$	$S_{3,2}+S_{2,3}+S_5$
$(\chi_{\mathcal{F}^*}+1)(S_{2,3})$	$S_{2,3}+S_{3,2}+S_5$
$(\chi_{\mathcal{F}^*}+1)(S_{3,1,1})$	$S_{3,1,1}-S_{1,1,3}-S_{1,4}-S_{2,3}-S_5$
$(\chi_{\mathcal{F}^*}+1)(S_{2,2,1})$	$S_{2,2,1}-S_{1,2,2}-S_{3,2}-S_{1,4}-S_5$
$(\chi_{\mathcal{F}^*}+1)(S_{2,1,2})$	$-S_{2,3}-S_{3,2}-S_5$
$(\chi_{\mathcal{F}^*}+1)(S_{1,3,1})$	$-S_{4,1}-S_{1,4}-S_5$
$(\chi_{\mathcal{F}^*}+1)(S_{1,2,2})$	$S_{1,2,2}-S_{2,2,1}-S_{4,1}-S_{2,3}-S_5$
$(\chi_{\mathcal{F}^*}+1)(S_{1,1,3})$	$S_{1,1,3}-S_{3,1,1}-S_{4,1}-S_{3,2}-S_5$
$(\chi_{\mathcal{F}^*}+1)(S_{2,1,1,1})$	$S_{2,1,1,1}+S_{1,1,1,2}+S_{2,1,2}+S_{1,2,2}S_{1,1,3}+S_{2,3}$ $+S_{1,4}+S_{3,2}+S_5$
$(\chi_{\mathcal{F}^*}+1)(S_{1,2,1,1})$	$S_{1,2,1,1}+S_{1,1,2,1}+S_{2,2,1}+S_{1,3,1}+S_{1,1,3}+$ $S_{2,3}+S_{1,4}+S_{4,1}+S_5$
$(\chi_{\mathcal{F}^*}+1)(S_{1,1,2,1})$	$S_{1,1,2,1}+S_{1,2,1,1}+S_{3,1,1}+S_{1,3,1}+S_{1,2,2}+S_{3,2}$ $+S_{1,4}+S_{4,1}+S_5$
$(\chi_{\mathcal{F}^*}+1)(S_{1,1,1,2})$	$S_{1,1,1,2}+S_{2,1,1,1}+S_{3,1,1}+S_{2,2,1}+S_{2,1,2}+S_{3,2}$ $+S_{2,3}+S_{4,1}+S_5$
$(\chi_{\mathcal{F}^*}+1)(S_{1,1,1,1,1})$	$-S_{2,1,1,1}-S_{1,2,1,1}-S_{1,1,2,1}-S_{1,1,1,2}$ $-S_{2,2,1}-S_{2,1,2}-S_{1,2,2}-S_{3,1,1}-S_{1,3,1}-S_{1,1,3}$ $-S_{3,2}-S_{2,3}-S_{1,4}-S_{4,1}-S_5$

Appendix B

Calculations in the mod 2 dual Leibniz-Hopf Algebra

In this Appendix the $(\chi_{\mathcal{F}_2^*} - 1)$-image of all degree 4 and 5 basis elements are listed.

B.1 Table list in degree 4

$(\chi_{\mathcal{F}_2^*} - 1)$(**basis for** $\mathcal{F}_2^*$)	**RESULT**
$(\chi_{\mathcal{F}_2{}^*} - 1)(S_4)$	0
$(\chi_{\mathcal{F}_2^*} - 1)(S_{3,1})$	$S_{3,1} + S_{1,3} + S_4$
$(\chi_{\mathcal{F}_2^*} - 1)(S_{2,2})$	S_4
$(\chi_{\mathcal{F}_2^*} - 1)(S_{1,3})$	$S_{1,3} + S_{3,1} + S_4$
$(\chi_{\mathcal{F}_2^*} - 1)(S_{2,1,1})$	$S_{2,1,1} + S_{1,1,2} + S_{2,2} + S_{1,3} + S_4$
$(\chi_{\mathcal{F}_2^*} - 1)(S_{1,2,1})$	$S_{3,1} + S_{1,3} + S_4$
$(\chi_{\mathcal{F}_2^*} - 1)(S_{1,1,2})$	$S_{1,1,2} + S_{2,1,1} + S_{3,1} + S_{2,2} + S_4$
$(\chi_{\mathcal{F}_2^*} - 1)(S_{1,1,1,1})$	$S_{2,1,1} + S_{1,2,1} + S_{1,1,2} + S_{1,3} + S_{3,1} + S_{2,2} + S_4$

B.2 Table list in degree 5

$(\chi_{\mathcal{F}_2^*} - 1)$(**basis for** $\mathcal{F}_2^*$)	**RESULT**
$(\chi_{\mathcal{F}_2^*} - 1)(S_5)$	0
$(\chi_{\mathcal{F}_2^*} - 1)(S_{4,1})$	$S_{4,1} + S_{1,4} + S_5$
$(\chi_{\mathcal{F}_2^*} - 1)(S_{1,4})$	$S_{1,4} + S_{4,1} + S_5$
$(\chi_{\mathcal{F}_2^*} - 1)(S_{3,2})$	$S_{3,2} + S_{2,3} + S_5$
$(\chi_{\mathcal{F}_2^*} - 1)(S_{2,3})$	$S_{2,3} + S_{3,2} + S_5$
$(\chi_{\mathcal{F}_2^*} - 1)(S_{3,1,1})$	$S_{3,1,1} + S_{1,1,3} + S_{1,4} + S_{2,3} + S_5$
$(\chi_{\mathcal{F}_2^*} - 1)(S_{2,2,1})$	$S_{2,2,1} + S_{1,2,2} + S_{3,2} + S_{1,4} + S_5$
$(\chi_{\mathcal{F}_2^*} - 1)(S_{2,1,2})$	$S_{2,3} + S_{3,2} + S_5$
$(\chi_{\mathcal{F}_2^*} - 1)(S_{1,3,1})$	$S_{4,1} + S_{1,4} + S_5$
$(\chi_{\mathcal{F}_2^*} - 1)(S_{1,2,2})$	$S_{1,2,2} + S_{2,2,1} + S_{4,1} + S_{2,3} + S_5$
$(\chi_{\mathcal{F}_2^*} - 1)(S_{1,1,3})$	$S_{1,1,3} + S_{3,1,1} + S_{4,1} + S_{3,2} + S_5$
$(\chi_{\mathcal{F}_2^*} - 1)(S_{2,1,1,1})$	$S_{2,1,1,1} + S_{1,1,1,2} + S_{2,1,2} + S_{1,2,2} + S_{1,1,3} + S_{2,3}$ $+ S_{1,4} + S_{3,2} + S_5$

B.3 Table list in degree 5

$(\chi_{\mathcal{F}_2^*}-1)$(**basis for** $\mathcal{F}_2^*$)	**RESULT**
$(\chi_{\mathcal{F}_2^*}-1)(S_{2,1,1,1})$	$S_{2,1,1,1}+S_{1,1,1,2}+S_{2,1,2}+S_{1,2,2}+S_{1,1,3}+S_{2,3}$ $+S_{1,4}+S_{3,2}+S_5$
$(\chi_{\mathcal{F}_2^*}-1)(S_{1,2,1,1})$	$S_{1,2,1,1}+S_{1,1,2,1}+S_{2,2,1}+S_{1,3,1}+S_{1,1,3}+$ $S_{2,3}+S_{1,4}+S_{4,1}+S_5$
$(\chi_{\mathcal{F}_2^*}-1)(S_{1,1,2,1})$	$S_{1,1,2,1}+S_{1,2,1,1}+S_{3,1,1}+S_{1,3,1}+S_{1,2,2}+S_{3,2}$ $+S_{1,4}+S_{4,1}+S_5$
$(\chi_{\mathcal{F}_2^*}-1)(S_{1,1,1,2})$	$S_{1,1,1,2}+S_{2,1,1,1}+S_{3,1,1}+S_{2,2,1}+S_{2,1,2}+S_{3,2}$ $+S_{2,3}+S_{4,1}+S_5$
$(\chi_{\mathcal{F}_2^*}-1)(S_{1,1,1,1,1})$	$S_{2,1,1,1}+S_{1,2,1,1}+S_{1,1,2,1}+S_{1,1,1,2}+S_{2,2,1}+S_{2,1,2}$ $+S_{1,2,2}+S_{3,1,1}+S_{1,3,1}+S_{1,1,3}+S_{3,2}+S_{2,3}$ $+S_{1,4}+S_{4,1}+S_5$

basis for $\mathrm{Ker}(\chi_{\mathcal{F}_2^*}-1)/\mathrm{Im}(\chi_{\mathcal{F}_2^*}-1)$	
$\lambda_{\mathcal{F}_2^*}(S^{1,1,1,1,1})$	$S^{1,1,1,1,1}+S^{2,1,1,1}+S^{1,2,1,1}$ $+S^{3,1,1}$

Appendix C

Calculations in the Leibniz-Hopf Algebra

In this Appendix the $(\chi_{\mathcal{F}} \pm 1)$-image of all degree 4 and 5 basis elements are listed. In particular the summands of these basis elements under $\mathrm{Im}(\chi \pm 1)$ are listed according to non-decreasing length order. The given tables can also be used for the mod p Leibniz algebra.

C.1 Table list in degree 4

$(\chi_{\mathcal{F}} - 1)$(**basis for** $\mathcal{F}$)	**RESULT**
$(\chi_{\mathcal{F}} - 1)(S^4)$	$-2S^4 + S^{3,1} + S^{2,2} - S^{2,1,1} + S^{1,3} - S^{1,2,1}$ $- S^{1,1,2} + S^{1,1,1,1}$
$(\chi_{\mathcal{F}} - 1)(S^{3,1})$	$-S^{3,1} + S^{1,3} - S^{1,2,1} - S^{1,1,2} + S^{1,1,1,1}$
$(\chi_{\mathcal{F}} - 1)(S^{2,2})$	$-S^{2,1,1} - S^{1,1,2} + S^{1,1,1,1}$
$(\chi_{\mathcal{F}} - 1)(S^{2,1,1})$	$-S^{2,1,1} - S^{1,1,2} + S^{1,1,1,1}$
$(\chi_{\mathcal{F}} - 1)(S^{1,3})$	$-S^{1,3} + S^{3,1} - S^{2,1,1} - S^{1,2,1} + S^{1,1,1,1}$
$(\chi_{\mathcal{F}} - 1)(S^{1,2,1})$	$-2S^{1,2,1} + S^{1,1,1,1}$
$(\chi_{\mathcal{F}} - 1)(S^{1,1,2})$	$-S^{1,1,2} - S^{2,1,1} + S^{1,1,1,1}$
$(\chi_{\mathcal{F}} - 1)(S^{1,1,1,1})$	0

$(\chi_{\mathcal{F}}+1)$(**basis for** $\mathcal{F}$)	**RESULT**
$(\chi_{\mathcal{F}}+1)(S^{4})$	$S^{3,1}+S^{2,2}-S^{2,1,1}+S^{1,3}-S^{1,2,1}$ $-S^{1,1,2}+S^{1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{3,1})$	$S^{3,1}+S^{1,3}-S^{1,2,1}-S^{1,1,2}+S^{1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{2,2})$	$2S^{2,2}-S^{2,1,1}-S^{1,1,2}+S^{1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{2,1,1})$	$S^{2,1,1}-S^{1,1,2}+S^{1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{1,3})$	$S^{1,3}+S^{3,1}-S^{2,1,1}-S^{1,2,1}+S^{1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{1,2,1})$	$S^{1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{1,1,2})$	$S^{1,1,2}-S^{2,1,1}+S^{1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{1,1,1,1})$	$2S^{1,1,1,1}$

C.2 Table list in degree 5

$(\chi_{\mathcal{F}}-1)$(**basis for** $\mathcal{F}$)	**RESULT**
$(\chi_{\mathcal{F}}-1)(S^{5})$	$-2S^{5}+S^{4,1}+S^{3,2}-S^{3,1,1}+S^{2,3}-S^{2,2,1}$ $-S^{2,1,2}+S^{2,1,1,1}+S^{1,4}-S^{1,3,1}-S^{1,2,2}+$ $S^{1,2,1,1}-S^{1,1,3}+S^{1,1,2,1}+S^{1,1,1,2}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}-1)(S^{4,1})$	$-S^{4,1}+S^{1,4}-S^{1,3,1}-S^{1,2,2}+S^{1,2,1,1}$ $-S^{1,1,3}+S^{1,1,2,1}+S^{1,1,1,2}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}-1)(S^{3,2})$	$-S^{3,2}+S^{2,3}-S^{2,2,1}-S^{2,1,2}+S^{2,1,1,1}$ $-S^{1,1,3}+S^{1,1,2,1}+S^{1,1,1,2}S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}-1)(S^{1,4})$	$-S^{1,4}+S^{4,1}-S^{3,1,1}-S^{2,2,1}+S^{2,1,1,1}$ $-S^{1,3,1}+S^{1,2,1,1}+S^{1,1,2,1}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}-1)(S^{2,3})$	$-S^{2,3}+S^{3,2}-S^{3,1,1}-S^{2,1,2}+S^{2,1,1,1}$ $-S^{1,2,2}+S^{1,2,1,1}+S^{1,1,1,2}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}-1)(S^{3,1,1})$	$-S^{3,1,1}-S^{1,1,3}+S^{1,1,2,1}+S^{1,1,1,2}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}-1)(S^{2,2,1})$	$-S^{2,2,1}+-S^{1,2,2}+S^{1,2,1,1}+S^{1,1,1,2}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}-1)(S^{2,1,2})$	$-2S^{2,1,2}+S^{2,1,1,1}+S^{1,1,1,2}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}-1)(S^{1,3,1})$	$-2S^{1,3,1}+S^{1,2,1,1}+S^{1,1,2,1}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}-1)(S^{1,2,2})$	$-S^{1,2,2}-S^{2,2,1}+S^{2,1,1,1}+S^{1,1,2,1}-S^{1,1,1,1,1}$

$(\chi_{\mathcal{F}}-1)$(**basis for** $\mathcal{F}$)	**RESULT**
$(\chi_{\mathcal{F}}-1)(S^{1,1,3})$	$-S^{1,1,3}-S^{3,1,1}+S^{2,1,1,1}+S^{1,2,1,1}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}-1)(S^{2,1,1,1})$	$-S^{2,1,1,1}+S^{1,1,1,2}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}-1)(S^{1,2,1,1})$	$-S^{1,2,1,1}+S^{1,1,2,1}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}-1)(S^{1,1,2,1})$	$-S^{1,1,2,1}+S^{1,2,1,1}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}-1)(S^{1,1,1,2})$	$-S^{1,1,1,2}+S^{2,1,1,1}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}-1)(S^{1,1,1,1,1})$	$-2S^{1,1,1,1,1}$

$(\chi_{\mathcal{F}}+1)$(**basis for** $\mathcal{F}$)	**RESULT**
$(\chi_{\mathcal{F}}+1)(S^{5})$	$S^{4,1}+S^{3,2}-S^{3,1,1}+S^{2,3}-S^{2,2,1}-S^{2,1,2}$ $+S^{2,1,1,1}+S^{1,4}-S^{1,3,1}-S^{1,2,2}+S^{1,2,1,1}$ $-S^{1,1,3}+S^{1,1,2,1}+S^{1,1,1,2}+-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{4,1})$	$S^{4,1}+S^{1,4}-S^{1,3,1}-S^{1,2,2}+S^{1,2,1,1}-S^{1,1,3}$ $+S^{1,1,2,1}+S^{1,1,1,2}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{3,2})$	$S^{3,2}+S^{2,3}-S^{2,2,1}-S^{2,1,2}+S^{2,1,1,1}-S^{1,1,3}$ $+S^{1,1,2,1}+S^{1,1,1,2}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{1,4})$	$S^{1,4}+S^{4,1}-S^{3,1,1}-S^{2,2,1}+S^{2,1,1,1}+S^{1,3,1}$ $+S^{1,2,1,1}+S^{1,1,2,1}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{2,3})$	$S^{2,3}+S^{3,2}-S^{3,1,1}-S^{2,1,2}+S^{2,1,1,1}-S^{1,2,2}$ $+S^{1,2,1,1}+S^{1,1,1,2}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{3,1,1})$	$S^{3,1,1}-S^{1,1,3}+S^{1,1,2,1}+S^{1,1,1,2}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{2,2,1})$	$S^{2,2,1}-S^{1,2,2}+S^{1,2,1,1}+S^{1,1,1,2}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{2,1,2})$	$S^{2,1,1,1}+S^{1,1,1,2}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{1,3,1})$	$S^{1,2,1,1}+S^{1,1,2,1}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{1,2,2})$	$S^{1,2,2}-S^{2,2,1}+S^{2,1,1,1}+S^{1,1,2,1}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{1,1,3})$	$S^{1,1,3}-S^{3,1,1}+S^{2,1,1,1}+S^{1,2,1,1}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{2,1,1,1})$	$S^{2,1,1,1}+S^{1,1,1,2}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{1,2,1,1})$	$S^{1,2,1,1}+S^{1,1,2,1}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{1,1,2,1})$	$S^{1,1,2,1}+S^{1,2,1,1}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{1,1,1,2})$	$S^{1,1,1,2}+S^{2,1,1,1}-S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}}+1)(S^{1,1,1,1,1})$	0

Appendix D

Calculations in the mod 2 Leibniz-Hopf Algebra

In this Appendix the $(\chi_{\mathcal{F}_2} - 1)$-image of all degree 4 and 5 basis elements are listed.

D.1 Table list in degree 4

$(\chi_{\mathcal{F}_2} - 1)$(**basis for** $\mathcal{F}_2$)	**RESULT**
$(\chi_{\mathcal{F}_2} - 1)(S^4)$	$S^{3,1} + S^{2,2} + S^{2,1,1} + S^{1,3} + S^{1,2,1} + S^{1,1,2} + S^{1,1,1,1}$
$(\chi_{\mathcal{F}_2} - 1)(S^{3,1})$	$S^{3,1} + S^{1,3} + S^{1,2,1} + S^{1,1,2} + S^{1,1,1,1}$
$(\chi_{\mathcal{F}_2} - 1)(S^{2,2})$	$S^{2,1,1} + S^{1,1,2} + S^{1,1,1,1}$
$(\chi_{\mathcal{F}_2} - 1)(S^{2,1,1})$	$S^{2,1,1} + S^{1,1,2} + S^{1,1,1,1}$
$(\chi_{\mathcal{F}_2} - 1)(S^{1,3})$	$S^{1,3} + S^{3,1} + S^{2,1,1} + S^{1,2,1} + S^{1,1,1,1}$
$(\chi_{\mathcal{F}_2} - 1)(S^{1,2,1})$	$S^{1,1,1,1}$
$(\chi_{\mathcal{F}_2} - 1)(S^{1,1,2})$	$S^{1,1,2} + S^{2,1,1} + S^{1,1,1,1}$
$(\chi_{\mathcal{F}_2} - 1)(S^{1,1,1,1})$	0

D.2 Table list in degree 5

$(\chi_{\mathcal{F}_2}-1)$(**basis for** $\mathcal{F}_2$)	**RESULT**
$(\chi_{\mathcal{F}_2}-1)(S^{5})$	$S^{4,1}+S^{3,2}+S^{3,1,1}+S^{2,3}+S^{2,2,1}$ $+S^{2,1,2}+S^{2,1,1,1}+S^{1,4}+S^{1,3,1}+S^{1,2,2}+$ $S^{1,2,1,1}+S^{1,1,3}+S^{1,1,2,1}+S^{1,1,1,2}+S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}_2}-1)(S^{4,1})$	$S^{4,1}+S^{1,4}+S^{1,3,1}+S^{1,2,2}+S^{1,2,1,1}$ $+S^{1,1,3}+S^{1,1,2,1}+S^{1,1,1,2}+S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}_2}-1)(S^{3,2})$	$S^{3,2}+S^{2,3}+S^{2,2,1}+S^{2,1,2}+S^{2,1,1,1}$ $+S^{1,1,3}+S^{1,1,2,1}+S^{1,1,1,2}+S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}_2}-1)(S^{1,4})$	$S^{1,4}+S^{4,1}+S^{3,1,1}+S^{2,2,1}+S^{2,1,1,1}$ $+S^{1,3,1}+S^{1,2,1,1}+S^{1,1,2,1}+S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}_2}-1)(S^{2,3})$	$S^{2,3}+S^{3,2}+S^{3,1,1}+S^{2,1,2}+S^{2,1,1,1}$ $+S^{1,2,2}+S^{1,2,1,1}+S^{1,1,1,2}+S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}_2}-1)(S^{3,1,1})$	$S^{3,1,1}+S^{1,1,3}+S^{1,1,2,1}+S^{1,1,1,2}+S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}_2}-1)(S^{2,2,1})$	$S^{2,2,1}+S^{1,2,2}+S^{1,2,1,1}+S^{1,1,1,2}+S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}_2}-1)(S^{2,1,2})$	$S^{2,1,1,1}+S^{1,1,1,2}+S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}_2}-1)(S^{1,3,1})$	$S^{1,2,1,1}+S^{1,1,2,1}+S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}_2}-1)(S^{1,2,2})$	$+S^{1,2,2}+S^{2,2,1}+S^{2,1,1,1}+S^{1,1,2,1}+S^{1,1,1,1,1}$

$(\chi_{\mathcal{F}_2}-1)$(**basis for** $\mathcal{F}_2$)	**RESULT**
$(\chi_{\mathcal{F}_2}-1)(S^{1,1,3})$	$S^{1,1,3}+S^{3,1,1}+S^{2,1,1,1}+S^{1,2,1,1}+S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}_2}-1)(S^{2,1,1,1})$	$S^{2,1,1,1}+S^{1,1,1,2}+S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}_2}-1)(S^{1,2,1,1})$	$S^{1,2,1,1}+S^{1,1,2,1}+S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}_2}-1)(S^{1,1,2,1})$	$S^{1,1,2,1}+S^{1,2,1,1}+S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}_2}-1)(S^{1,1,1,2})$	$S^{1,1,1,2}+S^{2,1,1,1}+S^{1,1,1,1,1}$
$(\chi_{\mathcal{F}_2}-1)(S^{1,1,1,1,1})$	0

basis for $\mathrm{Ker}(\chi_{\mathcal{F}_2}-1)/\mathrm{Im}(\chi_{\mathcal{F}_2}-1)$	
$\rho(S^{1,1,1,1,1})$	$S^{1,1,1,1,1}$

Bibliography

[1] J. F. Adams, *Lectures on generalised cohomology*, Springer Lecture Notes in Mathematics, vol. 99, pp. 1-138 (Springer, 1969).

[2] N. Andruskiewitsch, W. F. Santos, *The beginnings of the theory of Hopf algebras*, arXiv:0901.2460.

[3] A. Baker and B. Richter, *Classical obstructions and S-algebras*, Structured ring spectra, eds. LMS Lecture Note Series 315 (2004) 133 - 149.

[4] A. Baker, B. Richter, *Quasisymmetric functions from a topological point of view*, Math. Scand. 103 (2008) 208–242.

[5] M. D. Crossley, *The Steenrod Algebra and Other Copolynomial Hopf Algebras*, Bull. London Math. Soc. 32 (2000) 609–614.

[6] M. D. Crossley, *Some Hopf Algebras of Words*, Glasg. Math. J. 48 (2006) 575–582.

[7] M. D. Crossley, N.D. Turgay, *Conjugation Invariants in the mod* 2 *dual Leibniz-Hopf algebra*, To appear in Comm. Algebra.

[8] M. D. Crossley, N. D. Turgay, *Conjugation Invariants in the Leibniz Hopf Algebra* Preprint.

[9] M. D. Crossley, S. Whitehouse, *On conjugation invariants in the dual Steenrod algebra*, Proc. Amer. Math. Soc. 128 (2000) 2809–2818.

[10] M. D. Crossley, S. Whitehouse, *Higher conjugation cohomology in commutative Hopf algebras*, Proc. Edinb. Math. Soc. (2) 44 (2001) 19–26.

[11] S. Dascalescu, C. Nastasescu, and, S. Raianu, *Hopf Algebras An Introduction*, Chapman & Hall/CRC Pure and Applied Mathematics Series., Volume 235, 2000.

[12] E.J. Ditters, *Curves and formal (co)groups*, Invent. Math. 17 (1972) 1–20.

[13] V. Drinfeld, *Quantum groups*, Proc. Int. Congr. Math., Berkeley 1986, Vol. 1, 798-820, 1987.

[14] R. Ehrenborg, *On posets and Hopf algebras*, Adv. Math. 119 (1996) 1–25.

[15] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh, J.-Y. Thibon, *Noncommutative symmetric functions*, Adv. Math. 112 (1995) 218–348.

[16] I. Gessel, *Multipartite P-partitions and inner products of skew Schur functions*, Contemp. Math., 34, Amer. Math. Soc., Providence, RI, 1984.

[17] M. Hazewinkel, *Generalized overlapping shuffle algebras*, J. Math. Sci. (New York) 106 (2001) 3168–3186.

[18] M. Hazewinkel, *The algebra of quasi-symmetric functions is free over the integers*, Adv. Math. 164 (2001) 283–300.

[19] M. Hazewinkel, *Symmetric functions, noncommutative symmetric functions, and quasisymmetric functions. Monodromy and differential equations*, Acta Appl. Math. 75 (2003) 55–83.

[20] Hazewinkel *Symmetric functions, noncommutative symmetric functions, and quasisymmetric functions*,(2004), arXiv:math/0410468v1 [math.QA].

[21] M. Hazewinkel, *Symmetric functions, noncommutative symmetric functions, and quasisymmetric functions. Monodromy and differential equations. II*, Acta Appl. Math. 85 (2005) 319–340.

[22] M. Hazewinkel, *Explicit polynomial generators for the ring of quasisymmetric functions over the integers.* Acta Appl. Math. 109 (2010) 39–44.

[23] T.W. Hungerford, *Algebra*, Grad. Texts in Math. Springer, Volume 73, 1974.

[24] Bertram Kostant, *Groups over Z*, Proc. Sympos. Pure Math. 9, Amer. Math. Soc.,(1966) 90–98.

[25] C. Malvenuto, C. Reutenauer, *Duality between Quasi-Symmetric Functions and the Solomon Descent Algebra*, J. Algebra 177 (1995) 967–982.

[26] J. Milnor, *The Steenrod algebra and its Dual*, Ann. of Math., Volume 67,(1958), 150-171.

[27] J. Milnor, J.Moore. *On the structure of Hopf algebras*, Ann. of Math. 81, (1965), 211-264.

[28] M. Lothaire, *Combinatoric on Words*, Encyclopedia of mathematics and its applications., Volume 17, 1983.

[29] A.Robinson, S. Whitehouse, *Operads and* Γ*-homology of commutative rings*, Mathematical Proceedings of the Cambridge Philosophical Society, 132, (2002), 197-234.

[30] L. Solomon, *A Mackey formula in the group ring of a Coxeter group*, J. Algebra 41 (1976) 255–264.

[31] N. Steenrod, D.B.A. Epstein, *Cohomology Operations*, Princeton Univ. Press, Princeton, 1962.

[32] M.E. Sweedler, *Hopf algebras*, Benjamin, 1969.

[33] S. Whitehouse, *Symmetric group actions on tensor products of Hopf algebroids*, Communications in Algebra, Volume 29,(2001), 3351-3363.