NILPOTENCE IN THE MOD-P STEENROD ALGEBRA

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Contents

Abstract

This thesis explores nilpotence in the Steenrod Algebra $\mathcal{A}$ for an odd prime p. Three major topic areas are discussed here. In Chapter 3, we show that the Milnor element P_t^s has nilpotence $p[\frac{s}{t}] + p$ in the mod-p Steenrod Algebra $\mathcal{A}$ if p is an odd prime number. A similar result was proved by K. Monks when $p = 2$. In Chapter 4, we find formulas for the nilpotence of elements $P(n)$ for certain values of n, and generalize other results proved by Monks when $p = 2$. In chapter 5, we adapt to odd primes results of Monks [19] regarding hit elements and antiautomorphism.

Chapter 1

Introduction

The study of Steenrod algebra began with Steenrod's work constructing stable cohomology operations acting on cohomology theory with $\mathbf{Z}_2$-coefficients. In [24] and [25] he defined operations, the Steenrod square operations

$$\mathrm{Sq}^i : H^k(X; \mathbf{Z}_2) \longrightarrow H^{k+i}(X; \mathbf{Z}_2)$$

which help us to solve some questions in algebraic topology. By composition these operations give rise to an algebra of operations, the mod-2 Steenrod algebra, acting on the $\mathbf{Z}_2$-cohomology groups of spaces. The structure of this algebra was elucidated by Adem [3], Cartan [7], and Serre [20]. In particular, they showed that the mod-2 Steenrod algebra is the tensor algebra on the Sq^i's modulo the Adem relations: For $0 < i < 2j$,

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{k=0}^{[\frac{i}{2}]} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k.$$

In [20] Serre also observed that the representability of the cohomology groups implies that this algebra is the complete algebra of stable cohomology operations. In [15] Milnor observed that the Steenrod algebra has the structure of a co-commutative Hopf algebra. Therefore the dual is a commutative algebra.

The problem of determining the nilpotence height of elements of the Steenrod algebra has been of recent interest. For prime 2, the nilpotence height of Sq^{2^n} was conjectured to be $2n + 2$ by Steve Wilson in 1975. In [9] Donald M. Davis showed

that $(\mathrm{Sq}^{2^n})^{2n+1} \neq 0$. At the same time, Davis verified by computer calculation that $(\mathrm{Sq}^{2^n})^{2n+2} = 0$ for $n \leq 5$. Kenneth Monks also verified this for $n = 6, 7$. In 1994 this conjecture was proved by Walker and Wood (see [28]). Later they generalized this result for odd primes p (see [29]).

The nilpotence height of Milnor elements P_t^s was determined by Kenneth Monks [18] when $p = 2$. We adapt Monks' results about the nilpotence height of Milnor elements P_t^s to odd primes in Theorem 3.1.1 which says that the nilpotence height of Milnor elements P_t^s is exactly $p([\frac{s}{t}] + 1)$. In [17] Monks found the nilpotence of certain families of elements Sq^n. In Theorem 4.1.1 and Theorem 4.1.2, we adapt these results to odd primes. For example, we prove that the nilpotence heights of $P(\frac{p^k((p-1)p^m-1)+1}{p-1})$ and $P(p^m - 1)$ are $m + 2$, $m + 1$ respectively. In Theorem 4.1.3, we find formulas for the nilpotence of elements $P(n)$ for the certain values of n. In Theorem 5.1.1 and Theorem 5.1.2, we generalize results of Monks [19] about hit polynomials and the antiautomorphism for odd primes.

Chapter 2

The Steenrod Algebra and Its Anti-automorphism

2.1 The Steenrod Algebra and Its Dual

In this chapter, we will review much of background knowledge needed for the rest of this dissertation. Let p be an odd prime number and $\mathbf{Z}_p = \mathrm{Z}/p\mathrm{Z}$. Let

$$\beta : H^q(X; \mathbf{Z}_p) \longrightarrow H^{q+1}(X; \mathbf{Z}_p)$$

be the Bockstein coboundary operator associated to the exact sequence

$$0 \to \mathbf{Z}_p \to \mathbf{Z}_{p^2} \to \mathbf{Z}_p \to 0.$$

It is known that β is natural for mapping of spaces, that $\beta^2 = 0$, and that

$$\beta(xy) = \beta(x)y + (-1)^q x\beta(y)$$

where $\dim x = q$. Let $P^i : H^q(X; \mathbf{Z}_p) \longrightarrow H^{q+2i(p-1)}(X; \mathbf{Z}_p)$ be a natural homomorphism which has the following axioms:

1)$P^0 : H^q(X; \mathbf{Z}_p) \longrightarrow H^q(X; \mathbf{Z}_p)$ is the identity homomorphism.

2)If $\dim x = 2k$, then $P^k(x) = x^p$.

3)If $\dim x > 2k$, then $P^k(x) = 0$.

4)**Cartan Formula:** If $x, y \in H^*(X; Z_p)$, then

$$P^k(xy) = \sum_i P^i(x) P^{k-i}(y)$$

5)**Adem Relations:** If $a < pb$, then

$$P^a P^b = \sum_{t=0}^{[\frac{a}{p}]} (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} P^{a+b-t} P^t$$

If $a \leq pb$ then

$$\begin{aligned} P^a \beta P^b &= \textstyle\sum_{t=0}^{[\frac{a}{p}]} (-1)^{a+t} \binom{(p-1)(b-t)}{a-pt} \beta P^{a+b-t} P^t \\ &+ \textstyle\sum_{t=0}^{[\frac{a-1}{p}]} (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt-1} \beta P^{a+b-t} \beta P^t. \end{aligned}$$

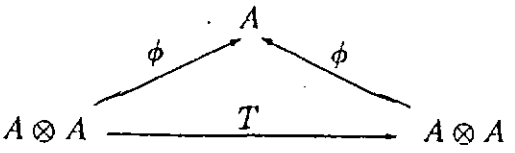
Let $M = (M_i)$ be a sequence of R-modules where R is a commutative ring and $i \geq 0$. Then M is called a **graded module.** We say the elements of M_i have degree i or dimension i. A homomorphism $f : A \to B$ of graded modules is a sequence of homomorphisms $f_i : A_i \to B_i$. If M and N are graded modules, we define the graded module $M \otimes N = \sum_i M_i \otimes N_{r-i}$. A graded R-module A is called a graded algebra if there is a homomorphism $\phi : A \otimes A \to A$ and a unit element 1. The algebra is said be associative if the following diagram is commutative;

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\phi \otimes 1} & A \otimes A \\ \downarrow{\scriptstyle 1 \otimes \phi} & & \downarrow{\scriptstyle \phi} \\ A \otimes A & \xrightarrow{\phi} & A \end{array}$$

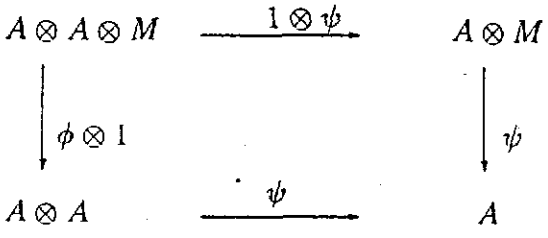
Let B be a graded algebra. Let $T : A \otimes B \longrightarrow B \otimes A$ be the map defined by

$$T(a \otimes b) = (-1)^{pq} b \otimes a$$

where $\dim a = p$ and $\dim b = q$. We say the algebra is commutative if the following diagram commutes;



A homomorphism $f : A \to B$ of algebras is a homomorphism of modules which commutes with multiplication, i.e. $f\phi_A = \phi_B(f \otimes f)$ and such that $f(1) = 1$. Let M be a graded module and A be a graded algebra. M is called an A-module if there is a map $\psi : A \otimes M \longrightarrow M$ which respects the unit of A and such that the following diagram commutes;



If B is a graded algebra, then $A \otimes B$ is given a graded algebra structure by the multiplication

$$A \otimes B \otimes A \otimes B \to A \otimes A \otimes B \otimes B \to A \otimes B.$$

If N is a B-module, then $M \otimes N$ is an $A \otimes B$-module by the mapping

$$A \otimes B \otimes M \otimes N \to A \otimes M \otimes B \otimes N \to M \otimes N.$$

The ground ring R may be regarded as a graded module such that $R_i = 0$ if $i > 0$. We say a graded algebra is **augmented** if there is an algebra homomorphism $\epsilon : A \longrightarrow R$.

Now we can define **the mod-p Steenrod algebra** $\mathcal{A}$ to be a graded associative algebra generated by the element P^i of degree $2i(p-1)$ and the element β of degree

1. A monomial in $\mathcal{A}$ can be written in the form

$$\beta^{\epsilon_0} P^{r_1} \beta^{\epsilon_1} P^{r_2} \beta^{\epsilon_2} \dots P^{r_k} \beta^{\epsilon_k}$$

where $\epsilon_i = 0$ or 1 and $r_i \geq 0$. We denote this monomial by P^I where $I = (\epsilon_0, r_1, \epsilon_1, \dots, r_k, \epsilon_k)$. A sequence I is called **admissible** if $r_i \geq pr_{i+1} + \epsilon_i$ for $i \geq 1$. Hence P^I will be called **admissible monomial** if I is admissible.

Proposition 2.1.1 *The admissible monomials P^I form a basis for the mod-p Steenrod algebra $\mathcal{A}$.*

This can be proved by using Adem relation (see [26]).

Let A be an augmented graded algebra over a commutative ring R with unit. We say A is a **Hopf algebra** if

1) There is a "diagonal map" of algebras $\psi : A \longrightarrow A \otimes A$.

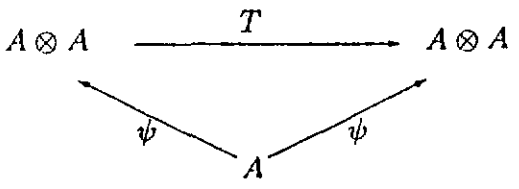
2) The following compositions are both the identity;

$$\begin{array}{ccccc} A & \longrightarrow & A \otimes A & \xrightarrow{\epsilon \otimes 1} & R \otimes A \\ & & \downarrow{\scriptstyle 1 \otimes \epsilon} & & \downarrow{\scriptstyle \simeq} \\ & & A \otimes R & \xrightarrow{\simeq} & A \end{array}$$

We say ψ is **associative** if the following diagram commutes;

$$\begin{array}{ccc} A & \xrightarrow{\psi} & A \otimes A \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi \otimes 1} \\ A \otimes A & \xrightarrow{1 \otimes \psi} & A \otimes A \otimes A \end{array}$$

We say ψ is **commutative** if the following diagram commutes;



Theorem 2.1.2 *The mod-p Steenrod algebra $\mathcal{A}$ is a Hopf algebra with commutative and associative diagonal map.*

We recall the structure of dual algebra $\mathcal{A}^*$ of $\mathcal{A}$. It was computed by Milnor [15].

Proposition 2.1.3 *There is a unique algebra map $\phi : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ such that for all $i \geq 0$*

$$\phi(P^i) = \sum_{j+k=i} P^j \otimes P^k$$

and

$$\phi(\beta) = \beta \otimes 1 + 1 \otimes \beta.$$

This can be proved by using the Kunneth Theorem and Cartan formula. Since the mod-p Steenrod algebra is a co-commutative Hopf algebra, its dual $\mathcal{A}^*$ is a commutative Hopf algebra.

Theorem 2.1.4 (Milnor) *The algebra $\mathcal{A}^*$ is the tensor product of a polynomial algebra on generators ξ_i $i \geq 1$ of degree $2(p^i - 1)$ by an exterior algebra on generators τ_i $i \geq 0$ of degree $2p^i - 1$. Moreover the coproduct ϕ is given by the formula*

$$\phi(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{p^i} \otimes \xi_i$$

and

$$\phi(\tau_k) = \tau_k \otimes 1 + \sum_{i=0}^{k-1} \xi_{k-i}^{p^i} \otimes \tau_i.$$

So the monomial basis in $\mathcal{A}^*$ can be written in the form

$$\xi^I = \tau_0^{\epsilon_0}\xi_1^{r_1}\tau_1^{\epsilon_1}\cdots$$

Then dual to the the monomial basis ξ^I is a basis for the mod-p Steenrod algebra known as Milnor basis. So $Q_0^{\epsilon_0}Q_1^{\epsilon_1}\cdots P(r_1,r_2,\ldots)$ will denote the dual of ξ^I.

The mod p Steenrod algebra is given as a graded $\mathbf{Z}_p$-module on the Milnor basis with product and coproduct defined in terms of that basis. So $\mathcal{A}$ is a $\mathbf{Z}_p$-module with basis symbols $Q_0^{\epsilon_0}Q_1^{\epsilon_1}\cdots P(r_1,r_2,\ldots)$ such that $\epsilon_i = 0$ or 1, $r_j \geq 0$. The degree of

$$Q_0^{\epsilon_0}Q_1^{\epsilon_1}\cdots P(r_1,r_2,\ldots)$$

is given by $\sum_i \epsilon_i(2p^i-1)+\sum_j 2r_j(p^j-1)$. The product is given by

$$P(r_1,r_2,\ldots)\cdot P(s_1,s_2,\ldots) = \sum_X \beta(X)P(t_1,t_2,\ldots) \tag{2.1}$$

where the sum is taken over all matrices $X=(x_{ij})$ satisfying

$$\sum_i x_{ij} = s_j \tag{2.2}$$

$$\sum_j x_{ij}p^j = r_i \tag{2.3}$$

$$\sum_{i+j=k} x_{ij} = t_k \tag{2.4}$$

$$\beta(X) = \prod_h (x_{h,0}, x_{h-1,1},\ldots,x_{0,h}) \in \mathbf{Z}_p, \tag{2.5}$$

where $(n_1,n_2,\ldots,n_h)$ is the multinomial coefficient $\frac{(n_1+n_2+\cdots n_h)!}{n_1!n_2!\cdots n_h!}$ mod p. The coproduct is given by

$$\phi(Q_r) = Q_r\otimes 1 + 1\otimes Q_r \tag{2.6}$$

$$\phi(P(r_1,r_2,\ldots)) = \sum_{s_i+t_i=r_i} P(s_1,s_2,\ldots)\otimes P(t_1,t_2,\ldots). \tag{2.7}$$

We say that a matrix $X=(x_{ij})$ is (R,S)-allowable if it satisfies the conditions (2.2), (2.3).

2.2 The Canonical Anti-automorphism

Let A be a Hopf algebra. There is a homomorphism $\chi : A \longrightarrow A$ defined by the properties:

(i) $\chi(1) = 1$

(ii) If $\phi(a) = \sum a'_i \otimes a''_i$ where $\dim a > 0$, then $\sum a'_i \chi(a''_i) = 0$ (see [16]).

So in the Steenrod algebra there exists a unique homomorphism $\chi : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying (i) and (ii). It can be proved that χ is an anti-automorphism in the sense $\chi(ab) = \chi(b)\chi(a)$, χ is one-to-one and onto, and so it maps basis elements of $\mathcal{A}$ into new basis elements of $\mathcal{A}$. Similarly there is a canonical anti-automorphism for the dual Hopf algebra $\mathcal{A}^*$ with diagonal map $\phi(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{p^i} \otimes \xi_i$. Defining relation becomes

$$\sum_{i=0}^{k} \xi_{k-i}^{p^i} \chi(\xi_i) = 0.$$

Donald M. Davis [8] proved nice formulae involving the canonical anti-automorphism χ of the mod-p Steenrod algebra.

Theorem 2.2.1

$$\chi(P^{p^{n-1}+p^{n-2}+\cdots+p+1}) = (-1)^n P^{p^{n-1}} P^{p^{n-2}} \cdots P^p P^1.$$

Theorem 2.2.1 follows easily by induction on n from the following two results. Let $S(i)$ denote the sum of all Milnor basis elements of the form P^R in dimension i.

Theorem 2.2.2 *([15])*

$$\chi(P^i) = (-1)^i S(2i(p-1)).$$

Proposition 2.2.3 *([8])*

$$P^m S(l) = \sum_R \binom{\sum p^i r_i}{pm} P^R.$$

Chapter 3

Nilpotence of the element P_t^s

3.1 The Main Results

In [17] it was shown that the element P_t^s has nilpotence height $2[\frac{s}{t}] + 2$ in the mod 2 Steenrod algebra. Here the method and result are generalized to show that for an odd prime p the element P_t^s has nilpotence height $p[\frac{s}{t}] + p$ in the mod p Steenrod algebra. Recall that P_t^s is the Milnor basis element $P(r_1, r_2, \ldots, r_t)$ with $r_t = p^s$ and $r_i = 0$ for $i < t$.

Theorem 3.1.1 *For all $s \geq 0$, $t \geq 1$,*

(i) $(P_t^s)^{p[\frac{s}{t}]+p} = 0$

(ii) $(P_t^s)^{p[\frac{s}{t}]+p-1} \neq 0.$

Thus the nilpotence height of P_t^s is exactly $p\left[\frac{s}{t}\right] + p$. The upper bound (i) on the nilpotence order was proved in [17] for the case $p = 2$. The lower bound (ii) was proved in [18] for the case $p = 2$. Here we generalize these arguments.

Let $P_t(r_1, r_2, \ldots, r_m)$ be the Milnor basis element $P(s_1, s_2, \ldots, s_{tm})$ where $s_{ti} = r_i$ and $s_j = 0$ if t does not divide j. In particular $P_t(p^s) = P_t^s$ and $P_1(n) = P(n)$.

If $R = (r_1, r_2, \ldots, r_m)$ is a sequence of nonnegative integers, we will write $P_t(R)$ for the corresponding Milnor basis element. The degree of $P_t(R)$ is $2\,|R|_t$ where $|R|_t = \sum_{i=1}^{\infty} (p^{it} - 1)\, r_i$ and the excess of $P_t(R)$ is $2e(R)$ where $e(R) = \sum_{i=1}^{\infty} r_i$. For

a fixed t let B_t be the vector subspace of A with basis the set of all $P_t(R)$. For $P_t^s \in B_t$ write $\widehat{P_t^s}$ for $(-1)^s\chi(P_t^s)$ where χ denotes the canonical anti-automorphism of B_t.

We will introduce the useful notation: each natural number a has a unique expansion

$$a = \sum_{i=0}^{\infty} \alpha_i(a)p^i \tag{3.1}$$

with $0 \leq \alpha_i(a) < p$. Let $0 \leq b < p^t$, and $i < t$. Then the following are obvious,

$$\alpha_i(a) = \alpha_{i+t}(p^t a + b) \tag{3.2}$$

$$\alpha_i(b) = \alpha_i(p^t a + b). \tag{3.3}$$

Definition 3.1.2 *We say that m and n have no carries if $\alpha_i(m) + \alpha_i(n) \leq p-1$ for all i. This is equivalent to the condition that the binomial coefficient $\binom{m+n}{m}$ is nonzero* mod$p$ *. If m and n have no carries, we will write $m \asymp n$.*

Define $\gamma_t(m) = \sum_{i=0}^{m-1} p^{it}$ for any integer $m \geq 0$ and $t \geq 1$ where $\gamma_t(0) = 0$. The following fact is immediate.

Fact 3.1.3 $\gamma_t(m+1) = p^t\gamma_t(m) + 1$.

Proposition 3.1.4 *For any $t \geq 1$, $m \geq 0$, and $i < t$,*

$$P_t((p-1)\,p^i\gamma_t(m+1)) \cdot P_t^{mt+i} = 0. \tag{3.4}$$

Proof: By Milnor's product formula, $P_t(a) \cdot P_t(b) = \sum u_j \cdot P_t(a+b-(p^t+1)j, j)$ where the sum is taken over all j such that $(a-p^tj) \asymp (b-j)$, and u_j is a unit mod p. So it is sufficient to show that $(p-1)p^i\gamma_t(m+1) - p^tj$ and $p^{mt+i} - j$ have at least one carry for any $0 \leq j \leq p^i(p-1)\gamma_t(m)$. This is the content of the following lemma.

Lemma 3.1.5 *For all $t \geq 1$, $m \geq 0$, $i < t$ and $0 \leq j \leq p^i(p-1)\gamma_t(m)$, there exists a nonnegative integer k such that $\alpha_{kt+i}(p^i(p-1)\gamma_t(m+1) - p^tj) + \alpha_{kt+i}(p^{mt+i} - j) \geq p$.*

Proof: We will proceed by induction on m. Let

$$X = (p-1)p^i\gamma_t(m+1) - p^t j \tag{3.5}$$

$$Y = p^{mt+i} - j. \tag{3.6}$$

If $m = 0$, then $j = 0$, $X = (p-1)p^i$, and $Y = p^i$. So we have $\alpha_i(X) + \alpha_i(Y) = p$.

Assume that for all $t \geq 1$, $0 \leq i < t$, and $0 \leq j \leq p^i(p-1)\gamma_t(m-1)$, there exists a nonnegative integer k such that $\alpha_{kt+i}(p^i(p-1)\gamma_t(m) - p^t j) + \alpha_{kt+i}(p^{(m-1)t+i} - j) \geq p$.

Choose $t \geq 1$, $i < t$, and $0 \leq j \leq p^i(p-1)\gamma_t(m)$. If $j = 0$, then

$$\begin{aligned} X &= (p-1)p^i\gamma_t(m+1) = (p-1)\left(p^{mt+i} + p^{(m-1)t+i} + p^{(m-2)t+i} + \cdots + p^{t+i} + p^i\right) \\ Y &= p^{mt+i} \end{aligned}$$

Hence $\alpha_{mt+i}(X) + \alpha_{mt+i}(Y) = p$.

Suppose $j \neq 0$.

Case 1: $\alpha_i(j-1) \neq p-1$.

$$X = (p-1)p^i(p^t\gamma_t(m) + 1) - p^t j = p^t[(p-1)p^i\gamma_t(m) - j] + (p-1)p^i.$$

Since $\alpha_i(j-1) \neq p-1$, $j-1 = p^{i+1}q + p^i h + r$ where $0 \leq r < p^i$ and $0 \leq h < p-1$.

$$\begin{aligned} Y &= p^{mt+i} - j = p^{mt+i} - (p^{i+1}q + p^i h + r + 1) \\ &= p^{i+1}[p^{mt-1} - (q+1)] + (p-1-h)p^i + (p^i - (r+1)). \end{aligned}$$

Since $r < p^i$, the term $(p^i - (r+1))$ is nonnegative and less than p^i. So $\alpha_i(X) + \alpha_i(Y) = 2p - h - 2 \geq p$.

Case 2: $\alpha_i(j-1) = p-1$.

Let $j - 1 = p^t q + r$ where $0 \leq r < p^t$. Since $\alpha_i(j-1) = p-1$, $p^i(p-1) \leq r < p^t$ by (3.3). In the assumption we have $j \leq p^i(p-1)\gamma_t(m)$. Then $p^t q + r + 1 \leq p^i(p-1)(p^t\gamma_t(m-1) + 1)$. Solving for $q+1$ shows

$$q + 1 \leq p^i(p-1)\gamma_t(m-1) + \frac{p^i(p-1) - r - 1 + p^t}{p^t}.$$

Since $p^i(p-1) \leq r$, $\frac{p^i(p-1)-r-1+p^t}{p^t} < 1$. Thus $q + 1 \leq p^i(p-1)\gamma_t(m-1)$ and so by the inductive hypothesis there exists a nonnegative integer k such that

$$\alpha_{kt+i}((p-1)p^i\gamma_t(m) - p^t(q+1)) + \alpha_{kt+i}(p^{(m-1)t+i} - (q+1)) \geq p.$$

Let k be any such value. We will show that $\alpha_{(k+1)t+i}(X) + \alpha_{(k+1)t+i}(Y) \geq p$ which will complete the induction and hence the proof.

Now since $\alpha_i(j-1) = p-1$, $\alpha_i(r) = p-1$. Hence $\alpha_i(p^t - 1 - r) = 0$. Therefore $p^t - 1 - r + (p-1)p^i < p^t$. Then we have

$$\alpha_{kt+i}((p-1)p^i\gamma_t(m) - p^t(q+1)) = \alpha_{kt+i}((p-1)p^i\gamma_t(m) - p^t(q+1) + p^t - 1 - r)$$

$$= \alpha_{(k+1)t+i}(p^t\left[(p-1)p^i\gamma_t(m) - p^t(q+1) + p^t - 1 - r\right] + (p-1)p^i) = \alpha_{(k+1)t+i}(X)$$

by (3.2) and (3.3).

$$Y = p^{mt+i} - (p^tq + r + 1) = p^t(p^{(m-1)t+i} - (q+1)) + (p^t - (1+r)).$$

Thus

$$\alpha_{kt+i}(p^{(m-1)t+i} - (q+1)) = \alpha_{(k+1)t+i}(p^t(p^{(m-1)t+i} - (q+1)) + (p^t - (1+r))) = \alpha_{(k+1)t+i}(Y)$$

by (3.2). By the induction hypothesis, $\alpha_{(k+1)t+i}(X) + \alpha_{(k+1)t+i}(Y) \geq p$. ■

Using Davis' method [8] we can derive the formulae,

Proposition 3.1.6

$$P_t(u) \cdot \hat{P}_t(v) = \sum_R \binom{|R|_t + e(R)}{p^t u} P_t(R) \tag{3.7}$$

$$\hat{P}_t(u) \cdot P_t(v) = \sum_R \binom{e(R)}{v} P_t(R) \tag{3.8}$$

where the sum is taken over all R for which $|R|_t = (p^t - 1)(u + v)$.

Proof: The formula (3.7) is exactly Corollary 1*a* in [10]. So we will prove formula (3.8). Using Gallant's Proposition 1 that $\chi(P_t^s)$ is $(-1)^s$ times the sum of all $P_t(R)$ in the appropriate dimension, we see that the product $\hat{P}_t(u) \cdot P_t(v)$ contains a term

$$\prod_i \binom{r_i}{s_i} P_t(r_1, r_2, \ldots)$$

for each Milnor matrix,

	0	...	0	v
*	0	...	0	s_1
0	0	...	0	0
⋮	⋮		⋮	⋮
0	0	...	0	0
$r_1 - s_1$	0	...	0	s_2
0	0	...	0	0
⋮	⋮		⋮	⋮
0	0	...	0	0
$r_2 - s_2$	0	...	0	s_3
0	0	...	0	0
⋮	⋮		⋮	⋮

such that $v = \sum_i s_i$. Thus

$$\hat{P}_t(u) \cdot P_t(v) = \sum_R \sum_S \prod_i \binom{r_i}{s_i} P_t(R)$$

where S ranges over sequences $(s_1, s_2, \ldots)$ having $\sum_i s_i = v$. The equation

$$\sum_S \prod_i \binom{r_i}{s_i} = \binom{\sum_i r_i}{v}$$

follows immediately from considering the coefficient of x^v in the expansion of

$$(1+x)^{r_1} \cdot (1+x)^{r_2} \cdots (1+x)^{r_n}.$$

Hence we have

$$\hat{P}_t(u) \cdot P_t(v) = \sum_R \binom{\sum_i r_i}{v} P_t(R) = \sum_R \binom{e(R)}{v} P_t(R). \quad \blacksquare$$

Using these formulae, we can prove the following proposition.

Proposition 3.1.7 *If a, b and t are positive integers with $a \geq t$, then*

$$\hat{P}_t(p^a(p-1)) \cdot P_t(p^{a+b} - p^a(p-1)) = P_t(p^{a+b} - p^{a-t}(p-1)) \cdot \hat{P}_t(p^{a-t}(p-1)). \quad (3.9)$$

Proof: Using (3.7) and (3.8), we have

$$P_t(p^{a+b} - p^{a-t}(p-1)) \cdot \hat{P}_t(p^{a-t}(p-1)) = \sum_R \binom{|R|_t + e(R)}{p^{a+b+t} - p^a(p-1)} P_t(R)$$

and

$$\hat{P}_t(p^a(p-1)) \cdot P_t(p^{a+b} - p^a(p-1)) = \sum_R \binom{e(R)}{p^{a+b} - p^a(p-1)} P_t(R)$$

where $1 \leq e(R) \leq p^{a+b}$ and $|R|_t = (p^t - 1)p^{a+b}$.

In order to prove these sums are equal, we need to show that their binomial coefficients are equivalent mod p. Note that

$$\binom{e(R)}{p^{a+b} - p^a(p-1)} = \binom{e(R)}{e(R) - p^{a+b} + p^a(p-1)} \tag{3.10}$$

$$\binom{|R|_t + e(R)}{p^{a+b+t} - p^a(p-1)} = \binom{|R|_t + e(R)}{e(R) - p^{a+b} + p^a(p-1)}. \tag{3.11}$$

Case 1: If $1 \leq e(R) < p^{a+b} - p^a(p-1)$, then both binomial coefficients are zero because $e(R) - p^{a+b} + p^a(p-1)$ is a negative integer.

Case 2: If $p^{a+b} - p^a(p-1) \leq e(R) < p^{a+b}$, then

$$\binom{|R|_t + e(R)}{e(R) - p^{a+b} + p^a(p-1)} = \binom{\sum_{i=a+b}^{a+b+t-1} \alpha_i(|R|_t)p^i + \sum_{i=0}^{a+b-1} \alpha_i(e(R))p^i}{\sum_{i=0}^{a+b-1} \alpha_i(e(R) - p^{a+b} + p^a(p-1))p^i}$$

$$\equiv \prod_{i=a+b}^{a+b+t-1} \binom{\alpha_i(|R|_t)}{0} \prod_{i=0}^{a+b-1} \binom{\alpha_i(e(R))}{\alpha_i(e(R) - p^{a+b} + p^a(p-1))} \bmod p$$

$$\equiv \binom{e(R)}{e(R) - p^{a+b} + p^a(p-1)} \bmod p.$$

Hence

$$\binom{|R|_t + e(R)}{p^{a+b+t} - p^a(p-1)} \equiv \binom{e(R)}{p^{a+b} - p^a(p-1)} \bmod p.$$

Case 3: If $e(R) = p^{a+b}$, then

$$\binom{e(R)}{p^{a+b} - p^a(p-1)} = \binom{p^{a+b}}{p^a(p-1)} \equiv 0 \bmod p$$

and

$$\binom{|R|_t + e(R)}{p^{a+b+t} - p^a(p-1)} = \binom{p^{a+b+t}}{p^a(p-1)} \equiv 0 \bmod p.$$

So the result holds. ■

If n, k and t are positive integers with $n = k + mt$ for some m, we define

$$(X_t)_k^n = P_t\left(p^n(p-1)\right) \cdot P_t\left(p^{n-t}(p-1)\right) \cdots P_t\left(p^k(p-1)\right) \tag{3.12}$$

The following corollary is immediate from Proposition 3.0.10.

Corollary 3.1.8 *Let $m \geq 1$, $c \geq 0$, and $a = c + mt$. Then*

$$\chi((X_t)_{c+t}^a) \cdot P_t\left(p^{a+b} - p^a(p-1)\right) = P_t\left(p^{a+b} - p^c(p-1)\right) \cdot \chi((X_t)_c^{a-t}).$$

For any Milnor basis element θ, define $\kappa_\theta : A \longrightarrow A$ by

$$\phi(x) = \sum \kappa_\theta(x) \otimes \theta \tag{3.13}$$

where ϕ is the diagonal map in A. These operations were first defined by Kristensen [12]. We will call it "stripping by θ". It follows from (3.13) that stripping a Milnor basis element by $\theta = P(t_1, t_2 \ldots)$ is given by

$$\kappa_\theta\left(P(r_1, r_2, \ldots)\right) = P(r_1 - t_1, r_2 - t_2, \ldots) \tag{3.14}$$

where the right hand side is taken to be 0 if $r_i < t_i$ for any i (see [28] and [29] for more details).

For any $x, y \in A$,

$$\sum \kappa_{P(R)}(x \cdot y) \otimes P(R) = (\sum \kappa_{P(I)}(x) \otimes P(I))(\sum \kappa_{P(J)}(y) \otimes P(J))$$

$$= \sum \kappa_{P(I)}(x)\kappa_{P(J)}(y) \otimes P(I)P(J) = \sum \lambda_R^{I,J} \kappa_{P(I)}(x)\kappa_{P(J)}(y) \otimes P(R)$$

where $\lambda_R^{I,J} = \langle P(I)P(J), \xi^R \rangle$ and the sums are taken over all I, J, R such that $R = I + J$. So

$$\kappa_{P(R)}(x \cdot y) = \sum_{I,J,R} \lambda_R^{I,J} \kappa_{P(I)}(x)\kappa_{P(J)}(y).$$

In particular, if $\kappa_t^s = \kappa_{P_t^s}$, then $\kappa_t^s(x \cdot y) = \sum_{j=0}^t \kappa_{t-j}^{s+j}(x)\kappa_j^s(y)$ follows from the formula $\phi(\xi_k) = \sum_{i=0}^k \xi_{k-i}^{p^i} \otimes \xi_i$.

Lemma 3.1.9 *If x and y belong to B_t then stripping by P_t^s is a derivation, i.e.*

$$\kappa_t^s(x \cdot y) = \kappa_t^s(x) \cdot y + x \cdot \kappa_t^s(y). \tag{3.15}$$

Proof: The proof follows from the formula above since all of the other terms in the sum correspond to stripping by Milnor basis elements which are not in B_t and therefore give zero. ■

Definition 3.1.10 *A sequence $R = (r_1, r_2, \ldots)$ is called a t-representation of a positive integer m if $m = \sum_i \gamma_t(i) r_i$. A function $\mu_t(m) = \min\{\sum_i r_i : (r_1, r_2, \ldots)$ is a t-representation of $m\}$ is called a minimum excess function.*

Lemma 3.1.11 *For $m \geq 0$,*

$$\mu_t(m) = \min\{e(P_t(R)) : |R|_t = (p^t - 1)m\}.$$

Proof: There is a $1-1$ correspondence between Milnor basis elements $P_t(R)$ satisfying $|R|_t = (p^t - 1)m$ and t-representations of m given by

$$P_t(R) \longleftrightarrow m = \sum_i r_i \gamma_t(i).$$

Under this correspondence, $e(P_t(R))$ corresponds to the number $\sum_i r_i$ which is used in determining $\mu_t(m)$. The lemma follows immediately from this observation. ■

Thus we have following lemma which is analogous to Davis' formula (Theorem 2.2.1).

Lemma 3.1.12 *For all $m \geq 0$ and $t \geq 1$,*

$$\widehat{P}_t((p-1)p^{t-1}\gamma_t(m)) = (X_t)_{t-1}^{mt-1}.$$

Proof: We will prove it by induction on m. If $m = 1$ then the result holds since $P_t(p^{t-1}(p-1))$ is the only $P_t(R)$ in this dimension. Assume that the result holds for $m-1$. By (3.7),

$$(X_t)_{t-1}^{mt-1} = P_t((p-1)p^{mt-1}) \cdot (X_t)_{t-1}^{(m-1)t-1} = P_t((p-1)p^{mt-1}) \cdot \widehat{P}_t((p-1)p^{t-1}\gamma_t(m-1))$$

$$= \sum_R \binom{|R|_t + e(R)}{(p-1)p^{(m+1)t-1}} P_t(R)$$

where the sum is taken over all R such that $|R|_t = \sum_i (p^{it}-1)r_i = (p-1)p^{t-1}(p^{mt}-1)$. By Lemma 3.1.11,

$$\mu_t((p-1)p^{t-1}\gamma_t(m)) \leq \sum_i r_i \leq (p-1)p^{t-1}\gamma_t(m).$$

By [10], the t-representation of $(p-1)p^{t-1}\gamma_t(m)$ with $r_m = (p-1)p^{t-1}$ and all other $r_i = 0$ has minimal $\sum_i r_i$. Hence $\mu_t((p-1)p^{t-1}\gamma_t(m)) = (p-1)p^{t-1}$. So we have

$$(p-1)p^{t-1}(p^{mt}-1) + (p-1)p^{t-1} \leq \sum_i p^{it} r_i \leq (p-1)p^{t-1}\gamma_t(m+1);$$

that is,

$$(p-1)p^{(m+1)t-1} \leq \sum_i p^{it} r_i \leq (p-1)p^{t-1}\gamma_t(m+1).$$

Hence $\left(\frac{\sum_i p^{it} r_i}{(p-1)p^{(m+1)t-1}}\right) \equiv 1 \bmod p$. By Proposition 1 in [10],

$$\hat{P}_t((p-1)p^{t-1}\gamma_t(m)) = \sum_R P_t(R) \tag{3.16}$$

where the sum is taken over all R such that $|R|_t = (p-1)p^{t-1}(p^{mt}-1)$. Therefore we have $(X_t)_{t-1}^{mt-1} = \hat{P}_t((p-1)p^{t-1}\gamma_t(m))$. ■

The following lemma is analogous to Lemma 1.4 in [29] .

Lemma 3.1.13 *(i) For $s \geq 0$ and $t \geq 1$ let κ_t^s denote the operation of stripping by P_t^s. Then*

$$\kappa_t^s(\hat{P}_t(k)) = \hat{P}_t(k - p^s)$$

where $P_t(k-p^s) = 0$ for $k < p^s$.

(ii) Let Θ be an element in B_t which gives zero when stripped by P_t^s, and assume $\Theta \cdot \hat{P}_t(ip^s) \cdot (P_t^s)^m = 0$. Then $\Theta \cdot \hat{P}_t((i-1)p^s) \cdot (P_t^s)^{m+1} = 0$.

Proof: (i) We will prove it by induction on k. Note that

$$\sum_{n \in Z} P_t(n) \cdot \chi(P_t(k-n)) = 0.$$

Hence we have

$$\sum_{n \in Z} (-1)^{k-n} P_t(n) \cdot \hat{P}_t(k-n) = 0 \tag{3.17}$$

with the convention that $P_t(i) = 0$ and $\hat{P}_t(i) = 0$ if $i < 0$. Applying the derivation κ_t^s to (3.17), we have

$$\kappa_t^s(\hat{P}_t(k)) + \sum_{n>0}(-1)^n\{\kappa_t^s(P_t(n)) \cdot \hat{P}_t(k-n) + P_t(n) \cdot \kappa_t^s(\hat{P}_t(k-n))\} = 0.$$

By the induction hypothesis, we obtain

$$\kappa_t^s(\hat{P}_t(k)) + \sum_{n\neq 0}(-1)^n P_t(n-p^s) \cdot \hat{P}_t(k-n) + \sum_{m\neq p^s}(-1)^{m-p^s} P_t(m-p^s) \cdot \hat{P}_t(k-m) = 0.$$

All terms cancel except the term with $n = p^s$ in the first sum. Hence

$$\kappa_t^s(\hat{P}_t(k)) = \hat{P}_t(k-p^s).$$

This completes the induction.

(ii) If $\Theta \cdot \hat{P}_t(ip^s) \cdot (P_t^s)^m = 0$ then

$$\kappa_t^s(\Theta) \cdot \hat{P}_t(ip^s) \cdot (P_t^s)^m + \Theta \cdot \kappa_t^s(\hat{P}_t(ip^s)) \cdot (P_t^s)^m + \Theta \cdot \hat{P}_t(ip^s) \cdot \kappa_t^s\,(P_t^s)^m = 0.$$

By part (i),

$$\Theta \cdot \hat{P}_t((i-1)p^s) \cdot (P_t^s)^m + m\Theta \cdot \hat{P}_t(ip^s) \cdot (P_t^s)^{m-1} = 0.$$

So

$$\Theta \cdot \hat{P}_t((i-1)p^s) \cdot (P_t^s)^{m+1} = -m\Theta \cdot \hat{P}_t(ip^s) \cdot (P_t^s)^m = 0. \quad \blacksquare$$

Let E be the exterior sub-algebra of A generated by $\{P_t^0 | t \geq 1\}$. There is an algebra isomorphism between A and $A//E$. This isomorphism $F : A \longrightarrow A//E$ is given by

$$F\,(P(t_1, t_2, t_3 \ldots)) = [P(pt_1, pt_2, pt_3 \ldots)]$$

where $[x]$ is equivalence class in $A//E$ of $x \in A$ (see [14]). Suppose $(P_t^s)^n = 0$. Then $[0] = [P_t^s]^n = F((P_t^{s-1})^n)$. Since F is an algebra isomorphism, $(P_t^{s-1})^n = 0$. So by iterating we see that if $(P_t^s)^n = 0$ then $(P_t^i)^n = 0$ for all $i \leq s$. This proves the following lemma.

Lemma 3.1.14 *Let $t > 1$.*

(i) If Theorem 3.1.1(i) holds for all $s \equiv -1 \bmod t$, then it holds for all s.

(ii) If Theorem 3.1.1(ii) holds for all $s \equiv 0 \bmod t$, then it holds for all s.

3.2 The Proof of The Main Result

The proof the Theorem 3.1.1(i) now follows by imitating the proof of [17] for the case $p = 2$. We proceed by proving the following two equations by induction on k for $1 \leq k \leq m$.

$$\chi((X_t)_{kt-1}^{mt-1}) \cdot \left(P_t^{mt-1}\right)^{p(k-1)+1} = 0 \tag{3.18}$$

$$\chi((X_t)_{kt-1}^{(m-1)t-1}) \cdot \left(P_t^{mt-1}\right)^{pk} = 0. \tag{3.19}$$

If $k = 1$ then

$$\chi((X_t)_{t-1}^{mt-1}) \cdot P_t^{mt-1} = P_t\left((p-1)p^{t-1}\gamma_t(m)\right) \cdot P_t^{mt-1} = 0$$

by Lemma 3.1.12 and Proposition 3.1.4.

Next we show that equation (3.18) for k implies equation (3.19) for k. The assumption says that

$$\chi((X_t)_{kt-1}^{(m-1)t-1})\chi(P_t(P^{mt-1}(p-1)))\left(P_t^{mt-1}\right)^{p(k-1)+1} = 0.$$

On applying Lemma 3.1.13 $(p-1)$ times with $\Theta = \chi((X_t)_{kt-1}^{(m-1)t-1})$, we obtain

$$\chi((X_t)_{kt-1}^{(m-1)t-1}) \cdot \left(P_t^{mt-1}\right)^{pk} = 0,$$

as desired.

Now we show that equation (3.19) for k implies equation (3.18) for $k+1$. Indeed we have the following equations with $z = p^{mt} - p^{kt-1}(p-1)$. At the second step we use Corollary 3.1.8 with $a = mt-1$, $b = 1$, and $c = kt-1$. The induction hypothesis is used at the third step.

$$\begin{aligned}
\chi((X_t)_{(k+1)t-1}^{mt-1}) \cdot \left(P_t^{mt-1}\right)^{pk+1} &= \chi((X_t)_{kt-1+t}^{mt-1}) \cdot (P_t^{mt-1}) \cdot \left(P_t^{mt-1}\right)^{pk} \\
&= P_t(z) \cdot \chi((X_t)_{kt-1}^{mt-1-t}) \cdot \left(P_t^{mt-1}\right)^{pk} \\
&= P_t(z) \cdot 0 = 0.
\end{aligned}$$

Thus by induction on k we see that both equation (3.18) and (3.19) hold for $1 \leq k \leq m$. Equation (3.19) for $k = m$ proves the theorem for all $s \equiv -1 \bmod t$. By Lemma 3.1.14(i) this proves Theorem 3.1.1(i).

Proof of Theorem 3.1.1(*ii*): We follow the method of Davis, using properties of the coproduct ϕ in A to find elements $\xi^{R_{m,t}(i)}$ in the dual algebra A^* for $i < pm$ such that

$$\langle \left(P_t^{(m-1)t}\right)^i, \xi^{R_{m,t}(i)} \rangle \neq 0. \tag{3.20}$$

If $i = kp + w$ with $0 \leq k \leq m-1$ and $0 \leq w \leq p-1$, then let

$$\xi^{R_{m,t}(i)} = \xi_t^{r_1(i)} \xi_{2t}^{r_2(i)} \cdots \xi_{(k+1)t}^{r_{k+1}(i)} \tag{3.21}$$

where $r_l(i) = a_l(i) + b_l(i)$ such that

$$a_l(i) = \begin{cases} (w-p+1)p^{(m-1)t} & \text{if } l = 1 \\ 0 & \text{if } l > 1 \end{cases}$$

and

$$b_l(i) = \begin{cases} (p-1)p^{(m-k-1)t} & \text{if } l = k+1 \\ (p^{t+1}-p+1)p^{(m-k-1)t} & \text{if } l = k \\ (p^t-1)p^{(m-l-1)t+1} & \text{if } 1 \leq l < k \end{cases}$$

For example,

1. for $k = 0$, $R_{m,t}(i) = (wp^{(m-1)t})$
2. for $k = 1$, $R_{m,t}(i) = ((w+1)p^{(m-1)t} - (p-1)p^{(m-2)t}, (p-1)p^{(m-2)t})$
3. for $k = 2$, $R_{m,t}(i) = ((w+1)p^{(m-1)t} - p^{(m-2)t+1}, p^{(m-2)t+1} - (p-1)p^{(m-3)t}, (p-1)p^{(m-3)t})$
4. for $k = 3$, $R_{m,t}(i) = ((w+1)p^{(m-1)t} - p^{(m-2)t+1}, (p^t-1)p^{(m-3)t+1}, p^{(m-3)t+1} - (p-1)p^{(m-4)t}, (p-1)p^{(m-4)t})$

The proof of (3.20) depends on the following calculation.

Lemma 3.2.1

$$\phi(\xi^{R_{m,t}(i)}) = (z\xi^{R_{m,t}(i-1)} + \xi') \otimes \xi_t^{p^{(m-1)t}} + \sum_j a_j \otimes b_j$$

where $b_j \neq \xi_t^{p^{(m-1)t}}$ *for all* j, ξ' *is divisible by* $\xi_t^{p^{(m-1)t+1}}$, *and* $z = w$ *if* $w > 0$, $z = 1$ *if* $w = 0$.

This implies (3.20), since it follows by induction on i that

$$\langle \left(P_t^{(m-1)t}\right)^i, \xi^{R_{m,t}(i)}\rangle = \langle \left(P_t^{(m-1)t}\right)^{i-1}, w\xi^{R_{m,t}(i-1)}\rangle \langle P_t^{(m-1)t}, \xi_t^{p^{(m-1)t}}\rangle \neq 0,$$

since the evaluation of ξ' on all elements of the subalgebra $A(mt-1)$ is zero. By Lemma 3.1.14(ii), this implies Theorem 3.1.1(ii).

Proof of Lemma 3.2.1 Let $R_{m,t}(i)$ be as in (3.21) above. Since we are only interested in terms of the form $A \otimes \xi_t^{p^{(m-1)t}}$, we will work mod terms involving ξ_{jt} for $j > 1$ in the second factor. So we have

$$\phi(\xi^{R_{m,t}(i)}) = \phi(\prod_{l=1}^{k+1} \xi_{lt}^{r_l(i)}) = \prod_{l=1}^{k+1} \phi(\xi_{lt})^{r_l(i)} \equiv \prod_{l=1}^{k+1} (\xi_{lt} \otimes 1 + \xi_{(l-1)t}^{p^t} \otimes \xi_t)^{r_l(i)}. \quad (3.22)$$

Note that

$$\sum_{l=1}^{k+1} r_l(i) = (w+1)p^{(m-1)t}.$$

We only want terms of the form $A \otimes \xi_t^{p^{(m-1)t}}$. When $w = 0$, the only such term is obtained from the product $\prod_{l=1}^{k+1} (\xi_{(l-1)t}^{p^t \cdot} \otimes \xi_t)^{r_l(i)}$ and hence

$$A = \prod_{l=1}^{k+1} \xi_{(l-1)t}^{p^t r_l(i)} = \prod_{l=1}^{k} \xi_{lt}^{p^t b_{l+1}(i)} = \prod_{l=1}^{k} \xi_{lt}^{b_l(i-1)} = \xi^{R_{m,t}(i-1)}$$

since $i - 1 = (k-1)p + p - 1$.

Suppose $w > 0$. The l-th factor of (3.22) can be written as

$$\begin{cases} (\xi_{(k+1)t}^{p^{(m-k-1)t}} \otimes 1 + \xi_{kt}^{p^{(m-k)t}} \otimes \xi_t^{p^{(m-k-1)t}})^{p-1} & \text{if } l = k+1 \\ (\xi_{kt}^{p^{(m-k-1)t}} \otimes 1 + \xi_{(k-1)t}^{p^{(m-k)t}} \otimes \xi_t^{p^{(m-k-1)t}})^{p^{t+1}-p+1} & \text{if } l = k \\ (\xi_{lt}^{p^{(m-l-1)t+1}} \otimes 1 + \xi_{(l-1)t}^{p^{(m-l)t+1}} \otimes \xi_t^{p^{(m-l-1)t+1}})^{p^t-1} & \text{if } 1 < l < k \\ (\xi_t^{p^{(m-2)t+1}} \otimes 1 + 1 \otimes \xi_t^{p^{(m-2)t+1}})^{(w+1)p^{t-1}-1} & \text{if } l = 1,\ 1 < k \\ (\xi_t^{p^{(m-2)t}} \otimes 1 + 1 \otimes \xi_t^{p^{(m-2)t}})^{(w+1)p^t-p+1} & \text{if } l = 1,\ k = 1 \end{cases}$$

We expand each of these factors. If we choose for each l the term with the j_l-th power of the second term, then we must have ($k \geq 2$)

$$j_1 p^{(m-2)t+1} + \sum_{l=2}^{k-1} j_l p^{(m-l-1)t+1} + j_k p^{(m-k-1)t} + j_{k+1} p^{(m-k-1)t} = p^{(m-1)t} \quad (3.23)$$

and $\binom{(m+1)p^{t-1}-1}{j_1}$ is a unit, $0 \le j_l \le p^t - 1$ for $2 \le l \le k-1$, $\binom{p^{t+1}-p+1}{j_k}$ is a unit, and $0 \le j_{k+1} \le p-1$.

To satisfy (3.23) we must have $j_{k+1} + j_k \equiv 0 \bmod p^{t+1}$. Since $\binom{p^{t+1}-p+1}{j_k}$ is a unit, either $j_k \equiv 0 \bmod p$ or $j_k \equiv 1 \bmod p$. Therefore either $j_{k+1} = 0$ or $p-1$ since $j_{k+1} \le p-1$.

If $j_{k+1} = 0$, then $j_k = 0$. So we have following equation

$$j_1 p^{(m-2)t+1} + \sum_{l=2}^{k-1} j_l p^{(m-l-1)t+1} = p^{(m-1)t}. \tag{3.24}$$

Since $j_{k-1}p^{(m-k)t+1} \equiv 0 \bmod p^{(m-k+1)t+1}$, j_{k-1} is multiple of p^t. Therefore $j_{k-1} = 0$ since $j_{k-1} \le p^t - 1$. By a similar argument, $j_{k-2} = j_{k-3} = \cdots = j_2 = 0$. So we have $j_1 p^{(m-2)t+1} = p^{(m-1)t}$ which implies $j_1 = p^{t-1}$. Therefore we obtain a term of the form $A \otimes \xi_t^{p^{(m-1)t}}$ with $A = A_1 A_2 \cdots A_k A_{k+1}$ where

$$\begin{aligned}
A_1 &= w\xi_t^{wp^{(m-1)t} - p^{(m-2)t+1}} \\
A_2 &= \xi_{2t}^{(p^t-1)p^{(m-3)t+1}} \\
&\cdots \\
A_{k-1} &= \xi_{(k-1)t}^{(p^t-1)p^{(m-k)t+1}} \\
A_k &= \xi_{kt}^{(p^{t+1}-p+1)p^{(m-k-1)t}} \\
A_{k+1} &= \xi_{(k+1)t}^{(p-1)p^{(m-k-1)t}}.
\end{aligned}$$

Hence $A = \xi^{R_{m,t}(i-1)}$.

If $j_{k+1} = p-1$, then we have $j_k = p^{t+1} - p + 1$. From (3.23),

$$j_1 p^{(m-2)t+1} + \sum_{l=2}^{k-2} j_l p^{(m-l-1)t+1} + j_{k-1}p^{(m-k)t+1} + p^{(m-k)t+1} = p^{(m-1)t}.$$

So $j_{k-1} \equiv -1 \bmod p^t$. Hence $j_{k-1} = p^t - 1$ since $j_{k-1} \le p^t - 1$. By a similar argument, $j_{k-2} = j_{k-3} = \cdots = j_2 = p^t - 1$. If we put these j_l in the equation (3.23) for $2 \le l \le k-2$, we will have $(j_1 + 1)p^{(m-2)t+1} = p^{(m-1)t}$. This implies that $j_1 = p^{t-1} - 1$. Therefore we obtain the other term of the form $B \otimes \xi_t^{p^{(m-1)t}}$ with

Chapter 4

Nilpotence Of Certain $P(n)$

4.1 The Main Results

In this chapter we shall adapt to odd primes results proved by Monks [17] and find formulas for nilpotence of certain $P(n)$.

Theorem 4.1.1 *Suppose* $r_i \equiv p-1 \bmod p$ *for* $1 \leq i \leq m$. *Then*

(*i*) $Nil(P(r_1,\ldots,r_m) \leq \min\{k | r_m < p^{(k-1)m+1}-1\}$.

(*ii*) $Nil(P(r_1,\ldots,r_m)) = Nil(P(p^k r_1 + p^k - 1,\ldots,p^k r_{m-1} + p^k - 1, p^k r_m - \frac{p^k-1}{p-1}))$ *for all* $k \in N$.

Theorem 4.1.2 (*i*) *For* $m \geq 0$, $k \geq 1$,

$$Nil(P(\frac{p^k((p-1)p^m-1)+1}{p-1})) = m+2.$$

(*ii*) *For* $m \geq 0$,

$$Nil(P(p^m-1)) = m+1.$$

Theorem 4.1.3 *For* $n \geq 2$,

$$Nil(P(n)) = \begin{cases} [\frac{p}{n}]+1 & \text{if } n < p \\ 2p & \text{if } n = p. \end{cases}$$

For any integer l, let

$$\nu_p(l) = \min\{m | l \equiv 0 \bmod p^m\}.$$

Proposition 4.1.4 *If $l \equiv -1 \bmod p$, then $Nil(P(l)) > \nu_p(l+1)$.*

Our method of proving this involves introducing sub-algebras O_k of $\mathcal{A}$ defined as follows:

Definition 4.1.5 *O_k is the Z_p-subspace of $\mathcal{A}$ whose basis is the set of Milnor basis elements*

$$B_k = \{P(r_1, r_2, \ldots, r_m) \ : \ r_i \equiv -1 \bmod p^{k+1} \text{ for } i < m \text{ and } (p-1)r_m \equiv 1 \bmod p^{k+1}\}.$$

We will denote $\mathcal{O} = O_0$.

The following two results will be proved later.

Proposition 4.1.6 *O_k is a subalgebra of A for all $k \in N$.*

Proposition 4.1.7 *There is an algebra monomorphism $\lambda : \mathcal{O} \to \mathcal{O}$ defined by*

$$\lambda(P(r_1, r_2, \ldots, r_m)) = P(pr_1 + p - 1, pr_2 + p - 1, \ldots, pr_{m-1} + p - 1, pr_m - 1).$$

If $\lambda^{(0)}$ is the identity map on $\mathcal{O}$, and $\lambda^{(k)} = \lambda \circ \lambda^{(k-1)}$ for $k > 1$, then $\lambda^{(k)}$ is also a monomorphism for every k. It is easy to check that

$$\lambda^{(k)}(P(r_1, r_2, \ldots, r_m)) = P(p^k r_1 + p^k - 1, \ldots, p^k r_{m-1} + p^k - 1, p^k r_m - \frac{p^k - 1}{p - 1}).$$

From the definition of λ, we have $\lambda(O_k) = O_{k+1}$, and hence the restriction of λ to O_k is an isomorphism between O_k and O_{k+1}. Therefore for any $a \in \mathcal{O}$ the nilpotence of a is equal to the nilpotence of $\lambda^{(k)}(a)$ for $k \in N$. This explanation proves part (ii) of Theorem 4.1.1.

Let $n = \sum_{i=0}^{\infty} \alpha_i(n) p^i$ be the p-adic expansion of an integer n where $0 \leq \alpha_i(n) < p$.

Lemma 4.1.8 *Let $m = \sum_i \alpha_i(m)p^i$ and $n = \sum_i \alpha_i(n)p^i$ be p-adic expansions. Then*

$$\binom{m}{n} \equiv \prod_i \binom{\alpha_i(m)}{\alpha_i(n)} \pmod{p}.$$

Proof: Note that

$$(1+x)^m = (1+x)^{\sum_i \alpha_i(m)p^i} \equiv (1+x^{p^i})^{\sum_i \alpha_i(m)} = \prod_i (1+x^{p^i})^{\alpha_i(m)} = \prod_i \sum_j \binom{\alpha_i(m)}{j} x^{j \cdot p^i}$$

From this observation, the coefficient $\binom{m}{n}$ of x^n in the expansion $(1+x)^m$ is equivalent to

$$\prod_i \binom{\alpha_i(m)}{\alpha_i(n)}. \quad \blacksquare$$

Corollary 4.1.9 $(n_1, n_2, \ldots, n_k) \equiv (pn_1 + p - 1, pn_2, \ldots, pn_j, \ldots, pn_k) \bmod p.$

Proof: For $l \leq k$, let $S_l = n_l + n_{l+1} + \cdots + n_k$. Then using Lemma 4.1.8, we have

$$\begin{aligned}(n_1, n_2, \ldots, n_k) &= \binom{S_1}{n_1} \cdot \binom{S_2}{n_2} \cdots \binom{S_{k-1}}{n_{k-1}} \\ &\equiv \binom{pS_1+p-1}{pn_1+p-1} \cdot \binom{pS_2}{pn_2} \cdots \binom{pS_{k-1}}{pn_{k-1}} \bmod p \\ &= (pn_1 + p - 1, pn_2, \ldots, pn_k). \quad \blacksquare\end{aligned}$$

Lemma 4.1.10 *Let $P(r_1, r_2, \ldots, r_m), P(s_1, s_2, \ldots, s_n) \in \mathcal{O}$. If $\beta(X)P(t_1, t_2, \ldots, t_l)$ is a nonzero term in $P(r_1, r_2, \ldots, r_m) \cdot P(s_1, s_2, \ldots, s_n)$, then*

$$P(t_1, t_2, \ldots, t_l) \in \mathcal{O}, \tag{4.1}$$

$$l = m + n, \tag{4.2}$$

and

$$x_{ij} \equiv \begin{cases} 0 \bmod p & \text{if } j > 0 \text{ and } i < m \\ p - 1 \bmod p & \text{if } j = 0,\ i \leq m \text{ or } i = m,\ j \leq n. \end{cases} \tag{4.3}$$

Proof: By (2.3), we have

$$r_i = \sum_j p^j x_{ij}.$$

Since $P(r_1, r_2, \ldots, r_m) \in \mathcal{O}$, $r_i \equiv p-1 \bmod p$ for $i \leq m$. Hence $x_{i0} \equiv p-1 \bmod p$ for $i \leq m$. When we combine this result with (2.5) and Corollary 4.1.9, we have $x_{ij} \equiv 0 \bmod p$ for $i+j \leq m$ and $j > 0$. Let $d < n$ and assume that $x_{mj} \equiv p-1 \bmod p$ for $0 < j \leq d$, and $x_{ij} \equiv 0 \bmod p$ for $i+j \leq m+d$, $j > 0$, and $i < m$. Since $P(s_1, s_2, \ldots, s_n) \in \mathcal{O}$, $s_{d+1} \equiv p-1 \bmod p$. Hence $x_{m,d+1} \equiv p-1 \bmod p$ by (2.2). Using (2.5) again, $x_{ij} \equiv 0 \bmod p$ for $i+j = m+d+1$ and $j > d+1$. Thus by finite induction on d, we have shown that $x_{ij} \equiv p-1 \bmod p$ if and only if $j = 0$, $i \leq m$ or $i = m$, $j \leq n$.

By (2.3), $P(t_1, t_2, \ldots, t_l) \in \mathcal{O}$ and $t_l = x_{mn} \equiv p-1 \bmod p$, $m+n = l$. ■

4.2 The Proof of The Main Results

Proof of Proposition 4.1.7: It is convenient to define $\lambda(R)$ by

$$\lambda(r_1, r_2, \ldots, r_m) = (pr_1 + p - 1, pr_2 + p - 1, \ldots, pr_{m-1} + p - 1, pr_m - 1).$$

It is easy to check that λ is injective. It remains to show that λ is a homomorphism. Let $R = (r_1, r_2, \ldots, r_m)$, $S = (s_1, s_2, \ldots, s_n)$, and $T = (t_1, t_2, \ldots, t_{m+n})$. Let $\hat{X} = (\hat{x}_{ij})$ be a $(\lambda(R), \lambda(S))$-allowable matrix. Using the proof of Lemma 4.0.11, we have $\hat{x}_{ij} \equiv p-1 \bmod p$ if and only if $j = 0$, $i \leq m$ or $i = m$, $j \leq n$. So we can define

$$\hat{x}_{ij} = \begin{cases} px_{ij} + p - 1 & \text{if } j = 0,\ i \leq m \text{ or } i = m\ ,\ j < n \\ px_{ij} - 1 & \text{if } j = n,\ i = m \\ px_{ij} & \text{otherwise.} \end{cases}$$

So we would like to show that $\hat{X} = (\hat{x}_{ij})$ is $(\lambda(R), \lambda(S))$-allowable if and only if $X = (x_{ij})$ is (R, S)-allowable, and that $\beta(X) = \beta(\hat{X})$.

Define

$$\epsilon_j = \begin{cases} p-1 & \text{if } j < n \\ -1 & \text{if } j = n \end{cases}$$

Let $\lambda(P(S)) = P(\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n)$ where $\hat{s}_j = p \cdot s_j + \epsilon_j$. We will check condition (2.2).

$$\sum_{i=0}^{m} \hat{x}_{ij} = \hat{s}_j \iff \sum_{i=0}^{m-1} p \cdot \hat{x}_{ij} + p \cdot x_{mj} + \epsilon_j = p \cdot s_j + \epsilon_j \iff \sum_{i=0}^{m} x_{ij} = s_j.$$

Let $\lambda(P(R)) = P(\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_m)$ where $\hat{r}_i = p \cdot r_i + p - 1$ for $1 \leq i < m$ and $\hat{r}_m = p \cdot r_m - 1$. If $1 \leq i < m$, then

$$\sum_{j=0}^{n} p^j \cdot \hat{x}_{ij} = \hat{r}_i \iff \sum_{j=1}^{n} p^{j+1} \cdot x_{ij} + p \cdot x_{i0} + p - 1 = p \cdot r_i + p - 1 \iff \sum_{j=0}^{n} p^j \cdot x_{ij} = r_i.$$

If $i = m$, then

$$\sum_{j=0}^{n} p^j \cdot \hat{x}_{mj} = \hat{r}_m \iff \sum_{j=0}^{n-1} p^j \cdot (p \cdot x_{mj} + p - 1) + p^n(p \cdot x_{mn} - 1) = pr_m - 1$$

$$\iff \sum_{j=0}^{n-1} p^{j+1} \cdot x_{mj} + (p-1) \sum_{j=0}^{n-1} p^j + p^{n+1} x_{mn} - p^n = pr_m - 1$$

$$\iff \sum_{j=0}^{n} p^j x_{mj} = r_m.$$

So condition (2.3) holds for both $\hat{X}$ and X.

By Corollary 4.1.9, $(a_1, a_2, \ldots, a_h) \equiv (p \cdot a_1 + \gamma_1, p \cdot a_2 + \gamma_2, \ldots, p \cdot a_h + \gamma_h) \bmod p$ where $\gamma_i = p - 1$ for at most one $1 \leq i \leq h$ and is zero otherwise. Hence

$$\prod_h (x_{h,0}, x_{h-1,1}, \ldots, x_{0,h}) \equiv \prod_h (\hat{x}_{h,0}, \hat{x}_{h-1,1}, \ldots, \hat{x}_{0,h}) \pmod{p}.$$

Let $P(t_1, t_2, \ldots t_{m+n})$ be the summand of $P(R) \cdot P(S)$ associated with X and let $P(\hat{T}) = P(\hat{t}_1, \hat{t}_2, \ldots \hat{t}_{m+n})$ be the summand of $\lambda(P(R)) \cdot \lambda(P(S))$ associated with $\hat{X}$. Then for $h < m + n$,

$$\hat{t}_h = \sum_{i+j=h} \hat{x}_{ij} = \sum_{i+j=h} p \cdot x_{ij} + p - 1 = p \cdot t_h - 1$$

and

$$\hat{t}_{mn} = \hat{x}_{mn} = p \cdot x_{mn} - 1 = p \cdot t_{mn} - 1.$$

Therefore $P(\hat{T}) = \lambda(P(T))$. ■

Proof of Proposition 4.1.6:

By lemma 4.1.10 , $x \cdot y \in \mathcal{O}$ for all $x, y \in B_0$. Since $\lambda^{(k)}(\mathcal{O}) = O_k$ and λ is an algebra homomorphism, the result holds. ■

Proof of Theorem 4.1.1(ii):

Let $P(T) = P(t_1, \ldots, t_l)$ be any summand of $P(r_1, \ldots, r_m)^{k-1} = P(R)^{k-1}$ with nonzero coefficient. By Lemma 4.1.10, we have $l = (k-1)m$. Let $X = (x_{ij})$ be any (R, S)-allowable matrix with nonzero coefficient. As shown in the proof of Lemma 4.1.10 , $x_{ij} \equiv p-1 \bmod p$ if $i = m$ and $j \leq (k-1)m$. From this and (2.5), we have

$$r_m = \sum_{j=0}^{(k-1)m} p^j \cdot x_{m,j} \geq \sum_{j=0}^{(k-1)m} (p-1)p^j = p^{(k-1)m+1} - 1.$$

Hence when $r_m < p^{(k-1)m+1} - 1$, an (R, S)-allowable matrix with nonzero coefficient does not exist. Therefore $P(R)^k = 0$. ■

Proof of Proposition 4.0.4: Let $l = p^m a - 1$ where a is not divisible by p. For each $1 \leq h \leq m$, we can define $S_{l,h} = (s_{l,h,1}, s_{l,h,2}, \ldots, s_{l,h,h})$ by

$$s_{l,h,i} = \begin{cases} (p-1)(p^{m-i}a + 1) & \text{if } 1 \leq i < h \\ p^{m-h+1}a - 1 & \text{if } i = h. \end{cases}$$

We shall prove that $\beta(Y)P(S_{l,h})$ is a nonzero term in $P(l)^h$ for $1 \leq h \leq m$ by induction on h, where $Y = (y_{ij})$ is the matrix associated with $P(l)P(l)^{h-1}$, and hence $P(l)^m \neq 0$.

If $h = 1$, then $P(S_{l,1})$ is a nonzero term in $P(l)$. By induction hypothesis, assume that $P(S_{l,h})$ has a nonzero coefficient in $P(l)^h$ for $h < m$. Suppose that $P(S_{l,h+1})$ is a nonzero term in $P(l)P(R)$ for some $P(R) = P(r_1, r_2, \ldots, r_h)$ such that $P(R)$ is a nonzero term in $P(l)^h$. Let $X = (x_{ij})$ be the associated matrix. By (2.4), we have $x_{1h} = s_{l,h+1,h+1} = p^{m-h}a - 1$ and from (2.3)

$$l = \sum_{j=0}^{h} x_{1j}p^j = \sum_{j=0}^{h-1} x_{1j}p^j + p^m a - p^h$$

and hence $\sum_{j=0}^{h-1} x_{1j}p^j = p^h - 1$. By (4.3), we have $x_{1j} \equiv p-1 \bmod p$. Since $\sum_{j=0}^{h-1} x_{1j}p^j = p^h - 1$, (2.3) holds if and only if $x_{1j} = p-1$ for $1 \leq j < h$. However by (2.2) and (2.4), for $1 \leq j < h$ we have

$$r_j = x_{0j} + x_{1j} = (s_{l,h+1,j} - p + 1) + p - 1 = (p-1)p^{m-j}a + p - 1 = s_{l,h,j}$$

and

$$\begin{aligned} r_h &= x_{0h} + x_{1h} = (s_{l,h+1,h} - p + 1) + s_{l,h+1,h+1} \\ &= (p-1)(p^{m-h}a + 1) - p + 1 + p^{m-(h+1)+1}a - 1 \\ &= p^{m-h+1}a - 1 = s_{l,h,h}. \end{aligned}$$

So we have proved that if $\beta(X)P(S_{l,h+1})$ is a nonzero term in $P(l)P(R)$, then $P(R) = P(S_{l,h})$. Moreover by our construction, the conditions (2.2) and (2.3) hold for the matrix X associated with $P(l)P(R)$. Since $s_{l,h,j} \equiv p-1 \bmod p$, the multinomial coefficient $(p-1, s_{l,h,j} - p + 1)$ is a unit. Therefore $P(S_{l,h+1})$ has nonzero coefficient in $P(l)P(S_{l,h})$ extending the induction. ■

Proof of Theorem 4.1.2(ii): We know that $Nil(P(p^m-1)) \leq m+1$ by Theorem 4.1.1(i) and $Nil(P(p^m - 1)) > m$ by Proposition 4.1.3. Hence $Nil(P(p^m - 1)) = m + 1$. ■

Proof of Theorem 4.1.2(i): For $k \geq 1$, $m \geq 0$, we have

$$\lambda^{(k-1)}(P(p^{m+1} - 1)) = P(p^{k-1}(p^{m+1} - 1) - \frac{p^{k-1} - 1}{p-1}) = P(\frac{p^k((p-1)p^m - 1) + 1}{p-1}).$$

Thus using Theorem 4.1.1(ii) and Theorem 4.1.2(ii),

$$Nil(P(\frac{p^k((p-1)p^m - 1) + 1}{p-1})) = Nil(P(p^{m+1} - 1)) = m + 2. \quad ■$$

Proof of Theorem 4.1.3: When $n = p$, it is proved by Walker and Wood (see [29]).

Let $n < p$. Claim $P(n)^j = uP(nj)$ if $nj < p$ where u is nonzero. We will prove this by induction on j.

If $j = 1$, it is true. Suppose the statement is true for $j - 1$. By induction hypothesis,

$$P(n)^j = P(n)P(n)^{j-1} = uP(n)P(n(j-1)).$$

Since $n < p$, $P(n)^j = u\binom{jn}{n}P(jn)$.

Suppose $j = [\frac{p}{n}] + 1$. Then $nj \geq p$. By our claim, $P(n)^{j-1} = uP((j-1)n)$. Then $P(n)^j = P(n)P(n)^{j-1} = uP(n)P((j-1)n)$. Since $nj \geq p$,

$$P(n)^j = u\binom{jn}{n}P(jn) = 0 \bmod p.$$

This completes the proof. ■

Chapter 5

On The Action of Steenrod Operations In Polynomial Algebras

5.1 The Main Results

Let $P_s = Z_p[x_1, x_2, \ldots, x_s]$. A polynomial $N \in P_s$ is said to be hit if it is in the image of the action $\bar{A} \otimes P_s \longrightarrow P_s$, i.e. $N \in \bar{A}P_s$ where $\bar{A}$ is the augmentation ideal of $\mathcal{A}$, i.e. $N = \sum_i P^i M_i$ for some $M_i \in P_s$.

We are interested in determining the image of the action $\bar{A} \otimes P_s \longrightarrow P_s$: the space of elements in P_s that are hit by positive dimensional Steenrod operations. In [30], when $p = 2$ Wood showed that if $\alpha(d+s) > s$ then every polynomial of degree d in P_s is hit where $\alpha(d+s)$ denotes the number of ones in the binary expansion of $d+s$. In [23] Singer generalized Wood's result conjectured by Peterson and identified a larg er class of hit polynomials. In [22] Silverman generalized a result of Wood and proved a conjecture of Singer. In [19] Monks extended a result of Wood to determine a new family of hit polynomials in P_s.

The following results are odd-primary analogues of results of Monks [19].

Theorem 5.1.1 *Let H and K be polynomials of degree $2h$, $2k$ respectively. If $h <$*

$\mu_t(k)$, *then* HK^{p^t} *is hit.*

Theorem 5.1.2 *For* $s, t \geq 1$, $0 \leq k < s$, *and* $k < t$,

$$\widehat{P_t}(p^s - p^k) = P_t((p-1)p^{s-1})P_t((p-1)p^{s-2}) \cdots P_t((p-1)p^k).$$

We list some lemmas we need to prove Theorem 5.1.1 and Theorem 5.1.2. Most of these lemmas involve the function μ_t defined in Definition 3.1.10. The following lemma is analogous to Lemma 2.1 in [19].

Lemma 5.1.3 *For all* $t, m \geq 1$, $\mu_t(m) \leq \frac{p^t-1}{p-1}\mu_1(m)$.

Proof: There exist positive integers $l_1, l_2, \ldots, l_{\mu_1(m)}$ such that

$$m = \sum_{i=1}^{\mu_1(m)} \gamma_1(l_i).$$

For each l_i, let $l_i = tq_i + r_i$ where q_i and r_i are non-negative integers and $0 \leq r_i < t$.

$$\begin{aligned} m &= \textstyle\sum_{i=1}^{\mu_1(m)} \gamma_1(l_i) = \sum_{i=1}^{\mu_1(m)} \frac{p^{tq_i+r_i}-1}{p-1} \\ &= \textstyle\sum_{i=1}^{\mu_1(m)} [\frac{p^t-p^{r_i}}{p-1}\frac{p^{tq_i}-1}{p^t-1} + \frac{p^{r_i}-1}{p-1}\frac{p^{t(q_i+1)}-1}{p^t-1}] \\ &= \textstyle\sum_{j=1}^{\frac{p^t-p^{r_i}}{p-1}} \sum_{i=1}^{\mu_1(m)} \gamma_t(q_i) + \sum_{j=1}^{\frac{p^{r_i}-1}{p-1}} \sum_{i=1}^{\mu_1(m)} \gamma_t(q_i+1) \end{aligned}$$

This yields a t-decomposition of m with $\frac{p^t-1}{p-1}\mu_1(m)$ terms. This completes the proof. ■

Lemma 5.1.4 *If* $m \leq p^t$ *then* $\mu_t(m) = m$.

Proof: Let $m \leq p^t$. Then $m \leq p^t < p^t + 1 = \gamma_t(2)$. The only possible t-decomposition of m is a sequence of m ones because γ_t is strictly increasing. ■

Let $L = (l_1, l_2, \ldots, l_n)$ be any sequence of nonnegative integers. Define

$$|L| = \sum_{i=1}^{n} l_i \tag{5.1}$$

$$\nu(L) = \max_i l_i \tag{5.2}$$

and

$$Y_t(L) = \sum_{i=1}^{n} \gamma_t(l_i). \tag{5.3}$$

Suppose that $l_1 \geq l_2 \geq \cdots \geq l_n$ and that $|L| \geq 1$. For this sequence, we can define

$$\delta(L) = (l_1', l_2', \ldots, l_n') \tag{5.4}$$

where

$$l_i' = \begin{cases} l_i - 1 & \text{if } l_i = l_1 \text{ and } (l_{i+1} \neq l_1 \text{ or } i = n) \\ l_i & \text{if otherwise.} \end{cases} \tag{5.5}$$

It is easy to verify that

$$l_1' \geq l_2' \geq \cdots \geq l_n' \tag{5.6}$$

$$|\delta(L)| = |L| - 1 \tag{5.7}$$

$$\nu(\delta(L)) \leq \nu(L) \tag{5.8}$$

and

$$Y_t(\delta(L)) = Y_t(L) - p^{t(\nu(L)-1)}. \tag{5.9}$$

We can define δ^r to be the r-fold composition of δ with itself (δ^0 is the identity function) for $0 \leq r \leq |L|$. Let $F_L = (f_1, f_2, \ldots, f_{|L|})$ be the sequence given by

$$f_i = Y_t(\delta^{i-1}(L)) - Y_t(\delta^i(L)). \tag{5.10}$$

Since $\delta^{|L|}(L) = (0, 0, \ldots, 0)$ and $Y_t(\delta^{|L|}(L)) = 0$,

$$\begin{aligned} |F_L| &= \textstyle\sum_{i=1}^{|L|} [Y_t(\delta^{i-1}(L)) - Y_t(\delta^i(L))] = Y_t(\delta^0(L)) - Y_t(\delta^{|L|}(L)) \\ &= Y_t(L). \end{aligned} \tag{5.11}$$

Lemma 5.1.5 *If $m < (p-1)p^s$, then $\mu_t(m) \leq \mu_t(m + (p-1)p^s)$.*

Proof: Assume that $L = (l_1, l_2, \ldots, l_n)$ is a t-decomposition of $m + (p-1)p^s$. Without loss of generality we can also assume that $l_1 \geq l_2 \geq \cdots \geq l_n$. By definition we have $Y_t(L) = m + (p-1)p^s$ and so by (5.11)

$$\sum_{i=0}^{|L|} f_i = m + (p-1)p^s.$$

So F_K is a non-increasing sequence whose power is $m+(p-1)p^s$. We need following lemma

Lemma 5.1.6 *If* $(p-1)p^b \leq a < p^{b+1}$, $\sum_{i=1}^{r} p^{x_i} = a$, *and* $p^{x_1} \geq p^{x_2} \geq \cdots \geq p^{x_r}$ *then there is a* $q \in \{1, \ldots, r\}$ *such that* $\sum_{i=1}^{q} p^{x_i} = (p-1)p^b$.

Proof: If $a = (p-1)p^b$ then we can take $q = r$ and we are done. Assume that $(p-1)p^b < a$. Since $p^{b+1} > a$, we have $p^b \geq p^{x_1} \geq p^{x_2} \geq \cdots \geq p^{x_{q+1}}$. Let q be the largest integer such that $\sum_{i=1}^{q} p^{x_i} \leq (p-1)p^b$. Then $(p-1)p^b - \sum_{i=1}^{q} p^{x_i} \equiv 0 \bmod p^{x_{q+1}}$ and $(p-1)p^b < \sum_{i=1}^{q+1} p^{x_i}$ and hence $\sum_{i=1}^{q} p^{x_i} = (p-1)p^b$. ■

From Lemma 5.1.6 there exists $q \in \{1, \ldots, |L|\}$ such that $\sum_{i=1}^{q} f_i = (p-1)p^s$. Thus

$$\sum_{i=1}^{q} [Y_t(\delta^{i-1}(L)) - Y_t(\delta^i(L))] = Y_t(L) - Y_t(\delta^q(L)) = m + (p-1)p^s - Y_t(\delta^q(L)) = (p-1)p^s$$

and hence $Y_t(\delta^q(L)) = m$. Therefore $\mu_t(m) \leq \mu_t(m + (p-1)p^s)$. ■

Using this result we can prove

Lemma 5.1.7 $\mu_t(p^s - p^k) \geq (p-1)p^k$ *where* s, t, *and* k *are any integers such that* $s, t \geq 1$, $0 \leq k < s$, *and* $k < t$.

Proof: We will prove it by induction on s. If $s = k+1$ then $\mu_t(p^s - p^k) = \mu_t((p-1)p^k) = (p-1)p^k$ by Lemma 5.1.4. Assume that it is true for $s-1$. Then by Lemma 5.1.5, $\mu_t(p^s - p^k) = \mu_t((p-1)p^{s-1} + p^{s-1} - p^k) \geq \mu_t(p^{s-1} - p^k)$. By inductive hypothesis, $\mu_t(p^{s-1} - p^k) \geq (p-1)p^k$. Hence $\mu_t(p^s - p^k) \geq (p-1)p^k$. ■

5.2 The Proof of The Main Results

The key idea in Wood's argument which is that for any $u, w \in P_s$ and any $\theta \in \mathcal{A}$, we have $u \cdot \theta w \equiv \widehat{\theta} u \cdot w$ modulo hit elements. In particular, if $e(\widehat{\theta}) > \deg(u)$, then $u \cdot \theta w$ is hit. Using this we will prove Theorem 5.1.1. We accomplish this with the aid of the following lemma.

Lemma 5.2.1 *If $N \in P_s$ is any element of degree $2k$, then for any $t \geq 1$,*

$$P_t(k) \cdot N = N^{p^t}. \tag{5.12}$$

Proof: We will prove this by induction on the number of variables in N. Suppose

$$N = x_{i_1}^{h_1} x_{i_2}^{h_2} \cdots x_{i_n}^{h_n}.$$

Let $n = 1$. Then

$$P_t(k)x^k = (x^k)^{p^t}. \tag{5.13}$$

So the result holds for $n = 1$. Assume that the result holds for all monomials comprised of less than n variables. Let $N_1 = x_{i_1}^{h_1} x_{i_2}^{h_2} \cdots x_{i_{n-1}}^{h_{n-1}}$ so that $N = N_1 x_{i_n}^{h_n}$. Let $\psi : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ be the diagonal map of $\mathcal{A}$. Then $\psi(P_t(k)) = \sum_{i=0}^{k} P_t(k-i) \otimes P_t(i)$. So

$$\begin{aligned} P_t(k) \cdot N &= \textstyle\sum_{i=0}^{k} P_t(k-i)N_1 \cdot P_t(i)x_{i_n}^{h_n} \\ &= \textstyle\sum_{i=0}^{h_n-1} P_t(k-i)N_1 \cdot P_t(i)x_{i_n}^{h_n} + P_t(k-h_n)N_1 \cdot P_t(h_n)x_{i_n}^{h_n} \\ &\quad + \textstyle\sum_{i=h_n+1}^{k} P_t(k-i)N_1 \cdot P_t(i)x_{i_n}^{h_n}. \end{aligned}$$

Since $e(P_t(k-i)) > \frac{1}{2}\deg(N_1)$, $\sum_{i=0}^{h_n-1} P_t(k-i)N_1 \cdot P_t(i)x_{i_n}^{h_n} = 0$. Similarly $\sum_{i=h_n+1}^{k} P_t(k-i)N_1 \cdot P_t(i)x_{i_n}^{h_n} = 0$ because $e(P_t(i)) > \frac{1}{2}\deg(x_{i_n}^{h_n})$. By induction, we have

$$P_t(k) \cdot N = P_t(h_n - i)N_1 \cdot P_t(i)x_{i_n}^{h_n} = N_1^{p^t}(x_{i_n}^{h_n})^{p^t}.$$

Hence we obtain $P_t(k) \cdot N = N^{p^t}$. ■

Wood's argument shows that $HK^{p^t} \equiv \widehat{P}_t(k)H \cdot K$ modulo hit elements. Hence if $e(\widehat{P}_t(k)) > h$, then $\widehat{P}_t(k)H = 0$ and hence HK^{p^t} is hit. Therefore it remains to show that $e(\widehat{P}_t(k)) = \mu_t(k)$. The following lemma was proved by Gallant [10].

Lemma 5.2.2

$$\widehat{P}_t(k) = \sum_{R} P_t(R)$$

where $|R|_t = (p^t - 1)k$.

By Lemma 3.1.11, $\mu_t(k)$ is exactly the minimum excess of the element $P_t(R)$ where $|R|_t = (p^t-1)k$. On the other hand, $\widehat{P}_t(k)$ is the summand of all $P_t(R)$ where $|R|_t = (p^t-1)k$ by Lemma 5.2.2. Hence $e(\widehat{P}_t(k)) = \mu_t(k)$. This completes the proof of Theorem 5.1.1.

Proof of Theorem 5.1.2 : We will prove this by induction on s. Suppose that $s = k+1$. Since for $k < t$ the only nonzero element $P_t(R)$ of B_t with $|R|_t = (p^t-1)(p-1)p^k$ is $P_t((p-1)p^k)$, $\widehat{P}_t(p^s-p^k) = \widehat{P}_t((p-1)p^k) = P_t((p-1)p^k)$. This proves the theorem for $s = k+1$.

Assume that it is true for $s-1$. Using the induction hypothesis and Corollary 1.a in [10], we have

$$\begin{aligned} P_t((p-1)p^{s-1})P_t((p-1)p^{s-2})\cdots P_t((p-1)p^k) &= P_t((p-1)p^{s-1})\widehat{P}_t(p^{s-1}-p^k) \\ &= \textstyle\sum_R \binom{\sum_i p^{it}r_i}{(p-1)p^{t+s-1}} P_t(R). \end{aligned}$$

where the sum is taken over all R such that $|R|_t = (p^t-1)(p^s-p^k)$. Since $\widehat{P}_t(p^s-p^k)$ is the sum of all $P_t(R)$ where $|R|_t = (p^t-1)(p^s-p^k)$, it is sufficient to show that

$$\binom{\sum_i p^{it}r_i}{(p-1)p^{t+s-1}} \equiv 1 \bmod p.$$

By Lemma 5.1.3 and Lemma 5.1.7, $\sum_i r_i \geq \mu_t(p^s-p^k) \geq (p-1)p^k$. For $s > k$ and $t \geq 1$ we have

$$(p-1)(p^k-p^{s+t-1}) + (p^t-1)(p^s-p^k) = (p^s-p^{k+1})(p^{t-1}-1) \geq 0.$$

Hence

$$\sum_i p^{it}r_i = \sum_i (p^{it}-1)r_i + \sum_i r_i \geq (p^t-1)(p^s-p^k) + (p-1)p^k \geq (p-1)p^{s+t-1}$$

On the other hand,

$$(p^t-1)\sum_i r_i \leq \sum_i (p^{it}-1)r_i = (p^t-1)(p^s-p^k).$$

So $\sum_i r_i \leq p^s - p^k$. Using this inequality, we have

$$\begin{aligned} \textstyle\sum_i p^{it}r_i &= \textstyle\sum_i (p^{it}-1)r_i + \sum_i r_i \\ &\leq (p^t-1)(p^s-p^k) + p^s - p^k \\ &\leq p^{s+t}. \end{aligned}$$

Hence $\binom{\sum_i p^{l_i} r_i}{(p-1)p^{t+s-1}} \equiv 1 \bmod p$ by Lucas's theorem [13] . This completes the proof.
■

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