Beyond Endoscopy via the Trace Formula

Salim Ali Altuğ

Ph.D. Thesis

Princeton Univeristy

Advisers: Robert P. Langlands
Peter C. Sarnak

Abstract

Motivated by the paper [18] of Langlands, we show how one can study the elliptic part of the trace formula in analytic applications. The thesis consists of several parts. First, by using an appropriate approximate functional equation we show how to rewrite the elliptic part of the trace formula which allows one to control the Artin L-functions that that appear in the formula and to smooth out the singularities of orbital integrals. In the second part we analyze the resulting formula, isolate contributions of special representations an develop asymptotic expansions of various functions that are involved in the analysis which are needed in various analytic applications (in particular for the one suggested in [18]). In the final part we apply the theory developed in the previous parts to give a new proof of the $\frac{1}{4}$-bound of Kuznetsov ([15]) towards the Ramanujan conjecture, and to carry out the analysis suggested in [18] for the standard representation. This last part in particular establishes the first case of the method as suggested in [18].

Acknowledgements

First, I would like to thank my advisors Robert Langlands and Peter Sarnak. I am indebted for their guidance and support throughout my graduate school years. Having the opportunity to see how they view and approach mathematics has been invaluable to me, both as a mathematician and as a person.

I have also benefited greatly from conversations with several other mathematicians while working on parts of this work. I would like to thank Matthew Young for pointing out the functional equation in (3.1) which made most of the material in Chapter 3 simpler. During a visit to the University of Chicago I had the chance to have many stimulating discussions with Ngô Bao Châu and Eddie Herman whom I would like to thank for their hospitality and interest. It is my pleasure to thank Arul Shankar and Jacob Tsimerman for very many inspiring conversations about the ideas in this thesis.

The role that Jill LeClair has played in every graduate students life in the mathematics department is immeasurable. Fine Hall would be in complete mayhem without her, and I really appreciate everything she has done for me and my fellow students.

I would like to express my deepest gratitude to my dear friends both inside and outside of Princeton. I will not list their names here (as this would add another chapter to this already long thesis) but rather stress that their company has made graduate school immensely better. I am privileged to have met every one of them.

To my family I would like to say thanks for their constant love and support. In particular to my parents Nahide and Uğur, and my sister Ece: While they may not know much of whats contained in this thesis it is undoubtedly true that it would not be possible without them. They shaped me into who I am, and I am forever grateful to them.

Last but certainly not the least, I am deeply thankful to Maša. She put up with hearing on too many occasions, “One more lemma and it is done!”. Yet, she consistently believed me when I said it and was always there for me afterwards. Thank you.

Tesadüflere.

Contents

Chapter 1

Introduction

1.1 Motivation and a little bit of history

1.1.1 Beyond Endoscopy

Beyond endoscopy, suggested by Langlands in [18], is an attempt to attack the general functoriality conjectures. It can be described as a two step process. The first step is to identify, for a given reductive algebraic group G, those automorphic representations π of G that are functorial transfers from other groups. The second step is to compare this data for different groups. In order to isolate those representations π one considers the behavior of the automorphic L-functions $L(s, \pi, \rho)$ at $s = 1$ for various finite dimensional representations ρ of the dual group ${}^LG$. One expects those π for which $L(s, \pi, \rho)$ has a pole at $s = 1$ for some non-trivial ρ to be functorial transfers from other groups. Thus a central problem is

Problem. *Given G, F as above and $\rho : {}^LG \to GL_n(\mathbb{C})$ for some n, isolate those cuspidal automorphic representations π of $G(\mathbb{A}_F)$ for which $L(s, \pi, \rho)$ has a pole at $s = 1$.*

In [18], Langlands proposed attacking this problem by the trace formula. For what follows, in order to simplify the notation, let us take the base fields $F = \mathbb{Q}$, the field of rational numbers. Also let us fix a finite dimensional representation, ρ, of the dual group ${}^LG$. In order to count those forms for which the automorphic L-function, $L(s, \pi, \rho)$, has a pole at $s = 1$, he starts with the logarithmic derivative of the L-function. Term by term differentiation leads to

$$-\frac{d}{ds}\log(L(s, \pi, \rho)) = \sum_p \sum_{r=1}^{\infty} \frac{\operatorname{tr}(\rho^r(A(\pi_p)))}{p^{rs}} \log(p) \tag{1.1}$$

Where $A(\pi_p)$ are the local parameters of π at prime p. In order to capture the pole of $-\frac{d}{ds}L(s, \pi, \rho)$ one considers the partial averages

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p^r<X} \operatorname{tr}(\rho(A(\pi_p))^r) \log(p) \tag{1.2}$$

Langlands' idea is to use the trace formula to express (a weighted average of) each term, $\operatorname{tr}(\rho(A(\pi_p))^r)$, explicitly and to analyze the resulting expression. More precisely, for a test function $f_\infty \in C_c^\infty(G(\mathbb{R}))$ (although the compactly supported assumption is not necessary) one can consider

$$\sum_{p^r<X} \sum_{\pi} \operatorname{tr}(\pi_\infty(f_\infty)) \operatorname{tr}(\rho(A(\pi_p))^r) \tag{1.3}$$

Where the inner sum over π runs over the cuspidal automorphic representations of G. For a suitable choice of test function $f^{p^r,\rho} \in C_c^\infty(G(\mathbb{A}))$ the inner sum over π can be expressed as the trace of the operator $R_{cusp}(f^{p^r,\rho})$ (see §8.1 for more on this), and hence one can use the trace formula to get an expression for this sum. The trace formula expresses the sum over π in (1.3) as a sum of spectral and geometric terms:

$$\sum_\pi \mathrm{tr}(\pi_\infty(f_\infty))\,\mathrm{tr}(\rho(A(\pi_p))^r) = \sum(geometric) - \sum(spectral) \tag{1.4}$$

The idea proposed in [18] is to replace the sum over π with (1.4) and to analyze the resulting expression in.

In [18] Langlands carries a preliminary analysis of each term on the right hand side of the sum in (1.4) for the group $GL(2)$, $F = \mathbb{Q}$ and $\rho = \mathrm{Sym}^k$, the k'th symmetric power representation ${}^LG = GL(2, \mathbb{C})$, for various n.

Spectral terms in (1.4) are fairly explicit (at least in the case of $GL(2)$) and can be treated in a straight forward fashion. The sum over geometric terms on the other hand runs over conjugacy classes of elements in $GL_2(\mathbb{Q})$ and the part that has the arithmetical information, and needs special attention, is the sum over elliptic conjugacy classes, i.e. the classes whose eigenvalues determine a quadratic extension of $\mathbb{Q}$. Analyzing these sums, even in the rather elementary first non-trivial case of $GL(2)$, is a major task and so far is open except for a few cases.

1.1.2 Related works with the Peterson-Kuznetsov formula

Soon after [18], Sarnak, in [24], had a rough analysis of the problem for the group $GL(2)$ and symmetric power representations. In his letter, he first points out that the averages like (1.2) that run over prime power coefficients become technically difficult and may not even be within reach. Rather than taking the logarithmic derivative, he suggested taking the L-functions themselves and considering averages of the form

$$\lim_{X\to\infty} \frac{1}{X} \sum_{n<X} a_n(\rho(\pi)) \tag{1.5}$$

Where $a_n(\rho(\pi))$ is defined by,

$$\begin{aligned} L(s,\pi,\rho) &= \prod_p \det\left(Id - \tfrac{\rho(A(\pi_p))}{p^s}\right)^{-1} \\ &= \sum_{n=1}^{\infty} \tfrac{a_n(\rho(\pi))}{n^s} \end{aligned}$$

Instead of the order of the pole, these averages are to capture the residue of the pole of $L(s, \pi, \rho)$ at $s = 1$, so in particular the limit in (1.5) is expected to be more complicated than the one in (1.2). On the other hand the sum running over integers rather than primes makes it technically friendlier. In order to analyze sums of type (1.5), in this specific case of $GL(2)$ and symmetric power representations, Sarnak proposed using the Petersson-Kuznetsov formula, a relative trace formula. In his letter he then went on to give a sketch of how this was to be carried out and pointed out to the difficulties that may arise. The relative trace formula approach was then carried out for $G = GL(2)$ and $\rho = \mathrm{Sym}^2$, the symmetric square representation, by Venkatesh in his thesis, [29]. Moreover, Herman in his thesis, [9], carried the same approach and isolated those forms that are base changes by analyzing the so-called Asai L-function again by using the Petersson-Kuznetsov formula.

1.2 This thesis

This thesis contains the first worked out case of Langlands' original suggestion in [18] to use the trace formula (as opposed to the relative trace formulae which, as mentioned before, was the subject matter of [29] and [9]) to analyze the behavior of the automorphic L-functions, $L(s, \pi, \rho)$, as $s \to 1^+$. As is already observed in §2.2 of [18], the problematic part of the analysis is to analyze the averages of the so-called "elliptic terms[1]" in the geometric side of (1.4). The main aim of the thesis is to rewrite the elliptic part of the trace formula by using the approximate functional equation and to analyze the resulting expression. This results in a form of the elliptic part (and hence the trace formula itself) which is suitable for analytic applications. We then go on and give two applications of the developed theory. In the first of these applications, in Chapter 8, we recover the classical $\frac{1}{4}$-bound of Kuznetsov's (cf. [15]) towards Ramanujan conjectures using the trace formula and prove (cf. Theorem 8.4.1):

Theorem 1.2.1. *Let[2] $f_\infty \in C_c^\infty(\mathbb{R}_+\backslash GL_2(\mathbb{R}))$ and p be a prime. Then for any $\epsilon > 0$,*

$$\sum_{\pi} \operatorname{tr} \pi_\infty(f_\infty) \operatorname{tr}(A(\pi_p)) = O_{f_\infty,\epsilon}(p^{\frac{1}{4}+\epsilon})$$

Where $A(\pi_p)$ is the local parameter at the prime p and the lo the sum is running over cuspidal automorphic representations π with trivial central character and unramified at every finite place.

The second application is to Beyond Endoscopy. In Chapter 9 we carry out the first case of Beyond Endoscopy as described in [18] by using the trace formula. We analyze the asymptotic behavior of the averages in (1.5) for $\rho =$ the standard representation and prove (cf. Corollary 9.12.1):

Theorem 1.2.2. *Let $k \geq 4$ be an even integer. Then,*

$$\lim_{X\to\infty} \frac{1}{X} \sum_{n<X} \sum_{f} a_n(f) = 0$$

Where the sum is running over cuspidal Hecke eigenforms of weight k with full level and $a_n(f)$ is the n'th coefficient of the attached L-function as defined in (1.5).

We would like to also emphasize that, although not included in here, the methods of Chapter 9 are also capable of carrying out the limit of (1.5) for $\rho = \mathrm{Sym}^2$, the symmetric square representation.

Below, we will give a more detailed description and main results of each chapter. In order to be able to describe the results we briefly need to go over the elliptic part of the trace formula. For most of the thesis (everything except Chapter 9) we will be working with the setup in [18]. That is, we will be considering the space[3] $L^2(\mathbb{R}_+\backslash GL_2(\mathbb{A}))$, and automorphic representations that are unramified at every finite place. Rather than going through the details of the choice of the test functions and

[1] Note here that the terminology "elliptic term" is used slightly different than the way we are using it here. In [18] it is used as the difference of the terms that comes from elliptic conjugacy classes with the contribution of the non-Ramanujan terms, however here we are using it for the contribution of the elliptic conjugacy classes.

[2] We note that the notation $\mathbb{R}_+\backslash GL_2(\mathbb{R})$ is not standard. We identify $\mathbb{R}_+$ with the scalar matrices with positive entries.

[3] Once again the notation is not standard, and what we mean by $\mathbb{R}_+$ is the scalar matrices with positive entries.

orbital integrals, here we simply quote the elliptic part of the trace formula[4] for (1.4). We refer the interested reader to [18] (pg. 19) for details.

Let $n \geq 1$ be an integer. The elliptic part of the trace formula for (1.4) is

$$\sum_{\substack{m\in\mathbb{Z}\\ \pm}} \theta_\infty^\pm\left(\frac{m}{2\sqrt{n}}\right) \sum_{\substack{f^2|(m^2\pm 4n)\\ \frac{m^2\pm 4n}{f^2}\equiv 0,1 \bmod 4}} L\left(1, \left(\frac{(m^2\pm 4n)/f^2}{\cdot}\right)\right) \tag{1.6}$$

Where $\left(\frac{D}{\cdot}\right)$ is the Kronecker symbol and θ_∞ is a certain function on $\mathbb{R}$ which depends on f_∞ (for precise definition and properties see Chapter 4). Then the averages (as in (1.5)) to be analyzed are

$$\sum_{n<X} \sum_{\substack{m\in\mathbb{Z}\\ \pm}} \theta_\infty^\pm\left(\frac{m}{2\sqrt{n}}\right) \sum_{\substack{f^2|(m^2\pm 4n)\\ \frac{m^2\pm 4n}{f^2}\equiv 0,1 \bmod 4}} L\left(1, \left(\frac{(m^2\pm 4n)/f^2}{\cdot}\right)\right) \tag{1.7}$$

1.2.1 Chapter by chapter contents

- **Chapter 2.**

  In this chapter we review the elliptic part of the trace formula. We start by equation (60) of [18], and then step by step rewrite the elliptic part so that it becomes more suitable for analytic applications. The main result of the chapter is Theorem 2.1.2.

- **Chapter 3.**

  This chapter starts with a review basic facts about the approximate functional equation. Then using a result of Soundararajan and Young in [28] (which they attribute to Bykovskii, [3], and Zaiger, [31]) we derive an approximate functional equation for

$$\sum_{\substack{f^2|(m^2\pm 4p)\\ \frac{m^2\pm 4p}{f^2}\equiv 0,1 \bmod 4}} L\left(1, \left(\frac{(m^2\pm 4p)/f^2}{\cdot}\right)\right) \tag{1.8}$$

  The interesting point here is that the approximate functional equation is for the whole weighted sum of L-functions instead of summing an approximate functional equations of each L-function[5]. This is the content of Corollary 3.2.4.

  We also would like to mention that the estimates of lemmas 3.2.1 and 3.3.1 are used throughout the thesis.

- **Chapter 4.**

  Chapter 4 is quite technical but at the same time fundamental for the rest of the thesis. It is concerned with the analysis of the singularities of archimedean orbital integrals. The overall idea is to apply Poisson summation to the m-sum in (1.6). This however is not straightforward

[4] We note that the sum over π in (1.4) runs over cuspidal automorphic representations that are unramified at every finite place. Although this restricts the automorphic representations we are dealing with, from an analytical point of view does not create extra difficulties, since allowing ramification brings in only finitely many congruence conditions which do not create analytical problems. Therefore the case considered here more or less is analytically the "generic case".

[5] This makes the formula much more suitable to use since otherwise one needs to deal with issues that arise because of the fact that the corresponding quadratic characters are not necessarily primitive.

since the function θ_∞ is not smooth. On the other hand we know exactly the type of the singularity (cf. (4.1)). In this chapter we first show that by choosing the parameters in the approximate functional equation of the previous chapter appropriately (cf. §4.3) one can "smooth-out" θ_∞. The idea is the following: The function $\theta_\infty(x)$ has a singularity of the form $\sqrt{|1-x^2|}$. Let Φ be Schwarz class. Since Φ decays faster than any polynomial, the product

$$\theta_\infty(x)\,\Phi\left(\tfrac{C}{|1-x^2|^\alpha}\right) \tag{1.9}$$

is smooth for any $\alpha > 0$ and $C \in \mathbb{R}\backslash\{0\}$, and one can poly Poisson summation to this product function. This is the content of Proposition 4.3.1 and resolves the problem related to singularities, however, introduces parameters (the function Φ and the parameters C and α). The importance of these parameters, of which the most important one is C, is that they control the behavior of the Fourier transform of (1.9). More precisely, we are interested in the asymptotic behavior and decay properties of

$$\int \theta_\infty(x)\Phi\left(\tfrac{C}{|1-x^2|^\alpha}\right)e(-x\eta)dx \tag{1.10}$$

We note that the asymptotic behavior of (1.10) depends on the relative sizes of C and η (The precise quantity determining the asymptotic expansions is the size of $C^{\frac{1}{\alpha}}\eta$.). The important point is that in the context we will be using (1.10), C and η will depend on other parameters (for example f and n) and these parameters ail be summed over as in (1.7). Therefore we need to know exactly how the asymptotic behavior of (1.10) depends on these parameters. The asymptotic expansion of (1.10) (and related functions) occupies the rest of the chapter and the results are summarized in theorems 4.8.1 and 4.8.2. The important point in these theorems is that when $C^{\frac{1}{\alpha}}\eta$ is large the Fourier transform decays fast. However when this is not the case, the Forier transform does not decay that fast but rather oscillates due to the appearance of the factor $e(\pm\eta)$ up front. We use this in the last chapter to estimate certain sums related to beyond endoscopy.

- **Chapter 5.**

  Throughout Chapter 5 we fix a positive integer n and consider (1.6) for this fixed n. In this chapter we apply Poisson summation to the m-sum in (1.6) and isolate the contribution of certain special representations that appears in [18].

  We only consider the " $-$ " sign in (1.6) in the first three sections. In the first two sections we rewrite (1.6) and apply Poisson summation. The dual sum is given in lemmas 5.2.1 to 5.2.4 (There are four pieces of the sum coming from the approximate functional equation and the sign of the quadratic character.) The third section is dedicated to the analysis of the dominant term in the dual sum (i.e. the term corresponding to $\xi = 0$) and isolation of the contribution of special representations.

  We take a moment to elaborate on the contribution of special representations. In [18] (cf. beginning of §2.2 and/or page 36) it is emphasized that the (possible) contribution of representations that are not of Ramanujan type is to be isolated. These are one dimensional representations and for our setting the only such representation is the trivial representation. Moreover, in equations (31), (32) of [18] the contribution of another special representation, denoted there by $\mathrm{tr}(\xi_0(f_\infty))$, is brought into attention. It is concluded in [18] that this representation in general gives a non-zero contribution to the limit in (1.5) and should, in principle, cancel with a part of the contribution of the elliptic term.

  Lemmas 5.3.6 to 5.3.9 is is concerned with the isolation of the contribution of the two special representations. Lemmas 5.3.6 and 5.3.7 isolates (a part of) the contribution of the trivial representation, whereas Lemma 5.3.8 isolates (a part of) the contribution of $\mathrm{tr}(\xi_0(f_\infty)$. All of these lemmas are based on contour shifts.

§5.4 goes through the content of the previous sections for the “ + ” sign in (1.6), and isolates the contribution of the special representations in lemmas 5.4.3 to 5.4.5.

The final result of the chapter is summarized in Theorem 5.5.2. The estimate in Proposition 5.5.3 is not very important, and will be used only in Chapter 8.

- **Chapter 6.**

Chapter 6 is a about the analysis of certain character sums that appear after applying Poisson summation to the elliptic part. and in the analysis of the limit in (1.6). The pertinent sums are denoted by

$$Kl_{t,f}(\xi,n) := \sum_{\substack{a,\ \mathrm{mod}\ 4tf^2 \\ a^2\equiv 4b\ \mathrm{mod}\ f^2 \\ \frac{a^2-4b}{f^2}\equiv 0,1\ \mathrm{mod}\ 4}} \left(\frac{(a^2-4b)/f^2}{t}\right) e\left(\frac{a\xi}{4tf^2}\right) e\left(\frac{b\nu}{4tf^2}\right)$$

and

$$\omega_{t,f}(\xi,\nu) := \sum_{\substack{a,b,\ \mathrm{mod}\ 4tf^2 \\ a^2\equiv 4n\ \mathrm{mod}\ f^2 \\ \frac{a^2-4n}{f^2}\equiv 0,1\ \mathrm{mod}\ 4}} \left(\frac{(a^2-4n)/f^2}{t}\right) e\left(\frac{a\xi}{4tf^2}\right)$$

The reason for the notation “Kl” is because these sums are analogues of classical Kloosterman sums (cf. [24] equation 70). The bounds we get for $Kl_{t,f}(\xi,n)$ are based on the Weil bound on Kloosterman sums.

The character sums $\omega_{t,f}(\xi,\nu)$, on the other hand, will appear in Chapter 9 and are related to beyond endoscopy limit for the standard representation (these appear after a second Poisson sum on the n-sum in (1.7)). These are bounded by *explicit calculation*. We note there that the fact that these sums are explicitly computable only in special cases (like the standard representation and symmetric square).

- **Chapter 7.**

Chapter 7 consists of estimates on the sums that appear after Poisson summation. We separated this chapter from the rest of the exposition because the analysis involved is tedious but straightforward. The results of this chapter are used in establishing the results of the next two chapters.

This chapter is very technical and can be skipped in a first read.

- **Chapter 8.**

In Chapter 8 we give a new proof of the classical result of Kuznetsov towards Ramanujan conjectures. Recall that the Ramanujan conjecture (for $GL(2)$, see [25] for the general statement) states that the trace of the local parameters $A(\pi_p)$ (the parameters in (1.1)) for a cuspidal automorphic representation π is bounded by 2. For special[6] π this conjecture is a celebrated theorem of Deligne ([5]), and in general is wide open. It is also well known (cf. [20]) that the Ramanujan conjecture (in its general form) follows from functoriality conjectures, however these conjectures as well (if not more) are wide open.

[6] π for which the infinity component, π_∞, is holomorphic discrete series.

It is worthwhile to note that although the Ramanujan conjecture itself seems to not be within reach of the current techniques, for applications in analytic number theory one often does not need the full conjecture and some improvement on the trivial bound, $\mathrm{tr}(A(\pi_p)) = O(p^{\frac{1}{2}})$, (For $GL(n)$ the trivial bound is $\mathrm{tr}(A(\pi_p)) = O(p^{\frac{n-1}{2}})$, cf. [25]) is good enough. For any $\theta \geq 0$, let $R_\pi(\theta)$ denote the bound

$$\mathrm{tr}(A(\pi_p)) = O_\theta(p^\theta) \qquad (R_\pi(\theta))$$

Note that the Ramanujan conjecture would imply $R_\pi(0)$ and the trivial bound is $R_\pi(\frac{1}{2})$.

This chapter is devoted (roughly) to prove the bound $R_\pi(\frac{1}{4}+\epsilon)$ for any $\epsilon > 0$ More precisely, we prove the bound

$$\sum_\pi \mathrm{tr}(\pi_\infty(f_\infty))\,\mathrm{tr}(A(\pi_p)) = O_{f_\infty,\epsilon}(p^{\frac{1}{4}+\epsilon})$$

where π runs over cuspidal automorphic representations that are unramified at every finite place and f_∞ is more or less arbitrary, cf. Theorem 8.4.1. This bound, of course, is well known[7] and was first proved by Kuznetsov (in [15]) by using Kuznetsov trace formula (a relative trace formula). The novelty here is in the method than the result. The proof we give here is entirely based on the trace formula. We also mention that the only attempt, that we know of, in the literature using the trace formula appears to be by Moreno in [22], where he recovers the trivial bound $p^{\frac{1}{2}}$.

- **Chapter 9.**

Finally in Chapter 9 we carry out the promised application to Beyond Endoscopy and show that the limit in (1.5) is 0 for $\rho = \mathrm{Sym}^1$, the standard representation. We note that this result follows form the well known holomorphy of the standard L-function (known in much greater generality, cf. [8]), and once again here the novelty is in the method rather than the result.

Since the result is not new, we do not aim for any generality and instead of using the general trace formula, use the Selberg trace formula for holomorphic forms of weight $k > 2$ (formula 9.1). This formula comes as a sum of three different terms, and correspondingly the average in 1.5 has three different contribution, denoted in Chapter 8 by A, B and C. The final result is Corollary 9.12.1 where we show that *the sum of* A, B and C is 0.

There is an interesting point here that we would like to draw attention to. Although the sum of A, B and C is 0, each one of them individually is not necessarily 0. Indeed we show that (B) is 0 (Proposition 9.2.1) and (C) $= \frac{-1}{2(k-1)}$ (Proposition 9.2.2). We then go on and show that the contribution of the elliptic part, (A), is also non-zero and equals exactly $\frac{1}{2(k-1)}$ and cancels the contribution of (C).

1.3 Final comments

First, we would like to have a few words about the contents and generality of the methods. The main idea, as mentioned before, is to use an approximate functional equation on the elliptic part of the trace formula to handle the Artin L-functions, which appear in the trace formula as volume factors, as well as to smooth-out the singularities of orbital integrals, and then to apply Poisson summation on, what is called in [7], Hitchin basis. Although we carry out this idea in the setting of [18] (i.e. $L^2(\mathbb{R}_+\backslash GL_2(\mathbb{A}))$ with representations unramified at every finite place) we would like to mention that the theory is completely general and applicable in the most general case. The

[7] In fact much better bounds are known, cf. [25].

important points are the approximate functional equation of Chapter 2 and the analysis of Chapter 3, both of which are completely general.

Second, we would like to say a few words about the trace formula and Beyond Endoscopy. Historically, the use of the trace formula in establishing functoriality has been via comparison of trace formulae of different groups (as in the case of endoscopy). The idea of Beyond Endoscopy is to use the trace formula, possibly combined with the techniques of analytic number theory, in a way to pass beyond this comparative approach. This thesis establishes a basis for such an approach and gives the first applications to Beyond Endoscopy. As mentioned before, with the methods of Chapter 9 one can also carry out the analysis for the symmetric square representation. We also would like to mention that for higher symmetric powers the sum over n in (1.5) starts running over sparse sequences and the second Poisson summation (on the n-sum) of Chapter 9 seems to be problematic. We however believe that there is a lot more structure in the singularities of orbital integrals to be uncovered and the methods of chapters 2 to 7 are open to further analysis and potential results.

Finally we would like to draw attention to the following interesting feature, appearing in chapters 4 and 8.

Let us start with Chapter 4. As we mentioned above, there are special representations that we isolate in the elliptic part after applying Poisson summation. In the context we are working there are two such representations, the trivial representation, $\mathbf{1}$, and the representation denoted by ξ_0 in [18] equation (31) and (32).

The trivial representation gives the main contribution[8] to the trace formula, and hence gives the main contribution to Poisson summation too. It is therefore expected to appear in the term corresponding to $\xi = 0$ (with the notation of Chapter 4) after applying Poisson summation. As we explained above, this indeed is the case and we show it in lemmas 5.3.6 and 5.3.7. We also mention here that in [7] this idea of applying Poisson summation and isolating the contribution of the trivial representation is carried out by different means for groups G that are semisimple, simply connected and for which[9] $G = G_{der}$.

On the other hand the second representation, ξ_0, is quite mysterious. Its contribution to the trace formula survives the limit of (1.5). Langlands, in [18] page 25, calls this "accidental" and says that this contribution should also appear in the elliptic term and cancel. In lemmas 5.3.8 and 5.4.5 we show that this is indeed the case and the contribution of ξ_0 does appear in the elliptic term with opposite sign.

Similarly in Chapter 8, the last term in 9.1 gives a non-zero contribution to the limit (Proposition 9.2.2). Since we know that the standard automorphic L-function, $L(s, \pi)$, is holomorphic we expect this contribution to cancel with contribution of the rest of (9.1). Once again this turns out to be the case and this contribution indeed cancels with the *non-zero* contribution of the elliptic term (cf. Theroem 9.6.3).

In order to explain the phenomenon we first recall the approximate functional equation (cf. Corollary

[8] Its contribution to 1.4 is $p^{\frac{1}{2}}$ whereas the Ramanujan conjecture tells us that 1.4 should be bounded by 2.

[9] Where G_{der} denotes the derived group. A typical example for such a group is $G = SL(2)$. We also mention that $G = GL(2)$ is not such a group however with a slight modification the results of [7] still applies to $GL(2)$.

3.2.4). It expresses (1.8) as,

$$\sum_{\substack{f^2|(m^2-4n) \\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\left(\frac{(m^2-4n)/f^2}{t}\right) = \sum_{\substack{f^2|(m^2-4n) \\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\left(\frac{(m^2-4n)/f^2}{t}\right) F\left(\frac{tf^2}{A}\right) \qquad (\bullet)$$

$$+ \sum_{\substack{f^2|(m^2-4n) \\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \left(\frac{\pi f^2}{|4n-m^2|}\right)^{\frac{1}{2}} \sum_{t=1}^{\infty}\left(\frac{(m^2-4n)/f^2}{t}\right) H_{\pm}\left(\frac{tf^2 A}{|m^2-4n|}\right) \qquad (\bullet\bullet)$$

Where $A > 0$ is a parameter that we are free to choose, F is a test function and $H_{\pm}$ is a certain transform of F. One should think of the functions F and $H_{\pm}$ as smooth approximations to characteristic functions of $(0, 1)$ since their role is to truncate the sums essentially to $tf^2 \ll A$ and $tf^2 \ll \frac{|m^2-4n|}{A}$.

The sum in ($\bullet$) essentially the Dirichlet series in (1.6) truncated to the range $tf^2 \ll A$, whereas the dual sum in ($\bullet\bullet$) appears because of the functional equation (cf. (3.1)) and is not apparent in (1.6). Accordingly the elliptic part of the trace formula breaks up into a sum of two parts (cf. Lemma 5.1.1).

The point is that after using the approximate functional equation, the contribution of ξ_∞ and that of the last term of (9.1) both appear in the parts corresponding to the dual sums, in particular in the part which is does not originally appear in (1.6). This seems to be quite interesting and mysterious feature of the approach.

1.3.1 Notation

The notation (hopefully) is consistent throughout the thesis. Below are a few symbols that we use without defining.

- $e(x)$.

  Unless otherwise stated $e(x) := e^{2\pi i x}$.

- Fourier transforms.

  All Fourier transforms are with respect to the character $e(x)$ and normalized so that we have $\hat{\hat{f}}(x) = f(-x)$.

- $\ll$.

  We use this symbol and the big-O notation more or less interchangeably. We hope that the way we use it will be clear from context. We use the notation $f(x) \ll_\alpha g(x)$ to mean that there is a constant, $K_\alpha > 0$, depending only on α (note that α can be a multi-index) such that $|f(x)| < K_\alpha |g(x)|$.

Chapter 2

Elliptic Part of the Trace Formula

In this chapter we will quickly review the elliptic part of the trace formula as given in equation (60) of [18]. We will be working in the setting of [18] as explained at the end of page 13 and the top of page 14 as well as at the beginning of §2.1 of that reference. We will be considering automorphic representations on $L^2(\mathbb{R}_+GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}))$ that are unramified at every finite place.

Let p be a prime, $k \geq 1$ and $n \geq 1$ be integers. Let[1] f^{p^k} be as in equation (11) of [18]. Then equation (60) of [18] gives the following expression for the elliptic part of the trace formula for the trace of the operator $R(f^{p^k})$

$$\sum_{\substack{m\in\mathbb{Z} \\ \pm}} \frac{\psi(m,\pm p^k)}{p^{\frac{k}{2}}} \mu_D \sum_{f|s} f \prod_{q|f} \left(1 - \left(\frac{D}{q}\right) q^{-1}\right) \tag{2.1}$$

Where[2],

- $\psi(m, \pm p^k)$ is the archimedean orbital integral of f_∞ as given in equation (26) of [18], and computed with respect to the normalizations of §2.1 of [18].
- $$m^2 - 4p^k = s^2 D$$
  Where D is the fundamental discriminant.
- $$\mu_D = vol(\mathbb{R}_+G_\gamma(\mathbb{Q})\backslash G_\gamma)(\mathbb{A})$$
  Where γ is an element in $GL_2(\mathbb{Q})$ with trace m and determinant $\pm p^k$, and the volume is computed with respect to the normalizations of §2.1 of [18].

Since everything is multiplicative, It is trivial to see that the same formula carries to the case when we change our function f^{p^k} to $f^n := f_\infty \prod_{q|n} T_q^{v_q(n)} \prod_{q\nmid n} T_q^0$ for an arbitrary integer n, where for any integer $k \geq 0$ the notation T_q^k is as on page (19) of [18]. i.e. The elliptic part of the trace formula for $R(f^n)$ is

$$\sum_{\substack{m\in\mathbb{Z} \\ \pm}} \frac{\psi(m,\pm n)}{\sqrt{n}} \mu_D \sum_{f|s} f \prod_{q|f} \left(1 - \left(\frac{D}{q}\right) q^{-1}\right) \tag{2.2}$$

[1]Note the slight difference in notation. In equation (11) of [18] there appears the representation, ρ, of the dual group and the function f depends on this. For our purposes we take $\rho = \mathrm{Sym}^k$, and in order to emphasize the dependence of f^p (with the notation of [18]) on k we write it as f^{p^k}. We remark that this indeed corresponds to considering the p^n'th Hecke operator instead of p'th in classical language.

[2]Note that the dependence of the variables s, D adnd γ on m and p^k are suppressed.

Where D and s are defined by the equation $m^2 \pm 4n = s^2 D$.

2.1 Rewriting the elliptic part

In this section by using Dirichlet's class number formula we will rewrite (2.2) so that it will become more suitable for analysis. We first quote some more results on the functions that appear in (2.2).

By the paragraph at the end of page 21 of [18] we know that for any $r > 0$, $\psi(rm, \pm r^2 n) = \psi(m, \pm n)$. Taking $r = \frac{1}{2\sqrt{n}}$ we see that there exist functions $\tilde{\theta}^{\pm}_{\infty}(x) \in C_c(\mathbb{R})$ such that

$$\psi(m, \pm n) = \tilde{\theta}^{\pm}_{\infty}\left(\frac{m}{2\sqrt{n}}\right) \tag{2.3}$$

By the computations[3] of §2.1 of [18] we have,

$$\mu_D = \begin{cases} 2h(D)R(D) & \text{if } D > 0 \\ \frac{2\pi h(D)}{\omega_D} & \text{if } D < 0 \end{cases} \tag{2.4}$$

Where $h(D)$ denotes the class number of the field $\mathbb{Q}(\sqrt{D})$, $R(D)$ is the regulator of $\mathbb{Q}(\sqrt{D})$ (which is given by $\ln|\epsilon|$ for a fundamental unit ϵ) and ω_D denotes the number of roots of unity in $\mathbb{Q}(\sqrt{D})$.

Recall Dirichlet's class number formula (cf. Corollary following Theorem 5 of Chapter 8 in [17]),

$$L\left(1, \left(\tfrac{D}{\cdot}\right)\right) = \begin{cases} \frac{2h(D)R(D)}{\sqrt{|D|}} & \text{if } D > 0 \\ \frac{2\pi h(D)}{\omega_D\sqrt{|D|}} & \text{if } D < 0 \end{cases} \tag{2.5}$$

Using (2.4) and (2.5) we then see that

$$\mu_D = \sqrt{|D|}L\left(1, \left(\tfrac{D}{\cdot}\right)\right) \tag{2.6}$$

Combining (2.3), (2.6) and (2.2) we get that the elliptic part can be written as

$$\sum_{\substack{m\in\mathbb{Z} \\ \pm}} \frac{\sqrt{|D|}}{\sqrt{n}}\tilde{\theta}^{\pm}_{\infty}\left(\frac{m}{2\sqrt{n}}\right) L\left(1, \left(\tfrac{D}{\cdot}\right)\right) \sum_{f|s} f \prod_{q|f}\left(1 - \left(\frac{D}{q}\right)q^{-1}\right) \tag{2.7}$$

Note that $D = \frac{m^2 \mp 4n}{s^2}$. Therefore $\sqrt{|D|} = \frac{\sqrt{m^2 \mp 4n}}{s}$. Using this we get that (2.7) is

$$\sum_{\substack{m\in\mathbb{Z} \\ \pm}} \frac{\sqrt{|m^2 \mp 4n|}}{\sqrt{n}}\tilde{\theta}^{\pm}_{\infty}\left(\frac{m}{2\sqrt{n}}\right) L\left(1, \left(\tfrac{D}{\cdot}\right)\right) \sum_{f|s} \frac{f}{s} \prod_{q|f}\left(1 - \left(\frac{D}{q}\right)q^{-1}\right) \tag{2.8}$$

Lettting

$$\theta^{\pm}_{\infty}(x) := \sqrt{|x^2 - 1|}\tilde{\theta}^{\pm}_{\infty}(x)$$

we see that (2.8) is

$$2\sum_{\substack{m\in\mathbb{Z} \\ \pm}} \theta^{\pm}_{\infty}\left(\frac{m}{2\sqrt{n}}\right) L\left(1, \left(\tfrac{D}{\cdot}\right)\right) \sum_{f|s} \frac{f}{s} \prod_{q|f}\left(1 - \left(\frac{D}{q}\right)q^{-1}\right) \tag{2.9}$$

We now turn to the L-value and the sum over f.

[3]Note that in the computation at the end of page 17 of [18] it is assumed that $D \neq -2$ or -3. It can be easily seen that the formula in (2.3) is the general form of the volume.

Lemma 2.1.1.

$$L\left(1,\left(\tfrac{D}{\cdot}\right)\right)\sum_{f|s}\tfrac{f}{s}\prod_{q|f}\left(1-\left(\tfrac{D}{q}\right)q^{-1}\right)=\sum_{f|s}\tfrac{1}{f}L\left(1,\left(\tfrac{Ds^2/f^2}{\cdot}\right)\right)$$

Proof. Using the change of variables $f\mapsto\frac{s}{f}$ we get

$$L\left(1,\left(\tfrac{D}{\cdot}\right)\right)\sum_{f|s}\tfrac{f}{s}\prod_{q|f}\left(1-\left(\tfrac{D}{q}\right)q^{-1}\right)=L\left(1,\left(\tfrac{D}{\cdot}\right)\right)\sum_{f|s}\tfrac{1}{f}\prod_{q|\frac{s}{f}}\left(1-\left(\tfrac{D}{q}\right)q^{-1}\right)$$

Note that

$$\begin{aligned}L\left(1,\left(\tfrac{D}{\cdot}\right)\right)\prod_{q|\frac{s}{f}}\left(1-\left(\tfrac{D}{q}\right)q^{-1}\right)&=\prod_{p}\left(1-\left(\tfrac{D}{p}\right)p^{-1}\right)^{-1}\prod_{q|\frac{s}{f}}\left(1-\left(\tfrac{D}{q}\right)q^{-1}\right)\\&=\prod_{p}\left(1-\left(\tfrac{D}{p}\right)p^{-1}\right)^{-1}\prod_{q|\frac{Ds^2}{f^2}}\left(1-\left(\tfrac{D}{q}\right)q^{-1}\right)\\&=\prod_{p}\left(1-\left(\tfrac{Ds^2/f^2}{p}\right)p^{-1}\right)^{-1}\\&=L\left(1,\left(\tfrac{Ds^2/f^2}{\cdot}\right)\right)\end{aligned}$$

The lemma follows. □

Using Lemma 2.1.1 we can rewrite (2.9) as

$$2\sum_{\substack{m\in\mathbb{Z}\\ \pm}}\theta_{\infty}^{\pm}\left(\tfrac{m}{2\sqrt{n}}\right)\sum_{f|s}\tfrac{1}{f}L\left(1,\left(\tfrac{Ds^2/f^2}{\cdot}\right)\right)\tag{2.10}$$

Finally using Theorem 26.39 of [14] we see that the requirement that $f\mid s$ is equivalent to $f^2\mid Ds^2=m^2\mp 4n$ and $\frac{m^2\mp 4n}{f^2}\equiv 0,1 \bmod 4$. Therefore we finally get,

Theorem 2.1.2. *The elliptic part of the trace formula in the setting of this chapter can be written as*

$$2\sum_{\substack{m\in\mathbb{Z}\\ \pm}}\theta_{\infty}^{\pm}\left(\tfrac{m}{2\sqrt{n}}\right)\sum_{\substack{f^2|m^2\mp 4n\\ \frac{m^2\mp 4n}{f^2}\equiv 0,1 \bmod 4}}\tfrac{1}{f}L\left(1,\left(\tfrac{Ds^2/f^2}{\cdot}\right)\right)\tag{2.11}$$

Chapter 3

Approximate Functional Equation

In this chapter we will derive an approximate functional equation for the weighted sum of the L-values that appear in (2.11). We will first review the functional equation that the sum over f of the L-values satisfy. The point to pay attention is that the weights (i.e. the f-sum) in (2.11) are arranged so that the whole f-sum itself satisfies[1] a functional equation. Once we have the functional equation we will derive an approximate functional equation in the usual way. For most of the material on the approximate functional equation we will follow §10.4 of [11].

3.1 Functional Equation

Let $\delta \in \mathbb{Z}$ be a discriminant, i.e. $\delta \equiv 0, 1 \bmod 4$, and $D = D_\delta$ be the fundamental discriminant it defines. Let $L(z, \delta)$ be defined by

$$L(z, \delta) := \sum_{\substack{f^2 \mid \delta \\ \delta/f^2 \equiv 0,1 \bmod 4}} \frac{1}{f^{2z-1}} L\left(z, \left(\frac{\delta/f^2}{\cdot}\right)\right)$$

Let $\Lambda(z, \delta)$ be the completed L-function, i.e.

$$\Lambda(z, \delta) := \left(\frac{|\delta|}{\pi}\right)^{\frac{z}{2}} \Gamma\left(\frac{z + e_\delta}{2}\right) L(z, \delta)$$

Where $e_\delta = 0, 1$ depending on $\delta > 0$ or < 0 respectively. Then the completed L-function satisfies the following functional equation:

Proposition 3.1.1.

$$\Lambda(z, \delta) = \Lambda(1 - z, \delta) \tag{3.1}$$

In particular we have

$$L(z, \delta) = \left(\frac{|\delta|}{\pi}\right)^{\frac{1}{2} - z} \frac{\Gamma\left(\frac{1 - z + e_\delta}{2}\right)}{\Gamma\left(\frac{z + e_\delta}{2}\right)} L(1 - z, \delta) \tag{3.1'}$$

Proof. This is the content of Lemma 2.1 of [28]. We only note that in the indicated lemma it is implicit that in the sum over f the integer d is a fundamental discriminant. This is the same as replacing the f-sum which runs $df^2 = \delta$ by $f \mid l$. It turns out that this functional equation was also observed earlier by several other authors in related contexts (Bykovskii, [3], and Zagier, [31]). We

[1] We would like to thank Matthew Young for pointing this out, and referring us to his paper with Soundararajan [28] (cf. Lemma 2.1 of [28]).

refer the reader to the proof of Lemma 2.1 of [28] and the references in §2 of the same reference for more on the history.

□

3.2 Approximate Functional Equation

For any Dirichlet series that has a functional equation we can write an approximate functional equation. We will take almost all of this material from Chapter 10 §10.4 of [11] but instead of using the classical functional equation we will use (3.1′).

Let $F \in C^{\infty}(\mathbb{R}^+)$ be

$$F(x) = \tfrac{1}{2K_0(2)} \int_x^{\infty} e^{-y-\frac{1}{y}} \tfrac{dy}{y} \tag{3.2}$$

Where $K_s(z)$ denotes the s'th modified Bessel function of the second kind. Then,

Lemma 3.2.1. *For every $x > 0$ we have*

$$0 < F(x) < \tfrac{e^{-x}}{2K_0(2)} \tag{3.3}$$

and

$$0 < 1 - F(x) < \tfrac{e^{-\frac{1}{x}}}{2K_0(2)} \tag{3.4}$$

Proof. [11] pg. 257. □

Let $\tilde{F}(z)$ denote the Mellin transform of F. i.e.

$$\tilde{F}(z) = \int_0^{\infty} F(u) u^z \tfrac{du}{u}$$

We have the following lemma about the analytic behavior of $\tilde{F}$:

Lemma 3.2.2. *Explicitly; $\tilde{F}(z) = \frac{1}{z}\frac{K_z(2)}{K_0(2)}$. It is holomorphic except for a simple pole at $z = 0$ with residue* 1*. Furthermore, $\tilde{F}(z)$ is odd, and for $z = \sigma + it \in \mathbb{C}$ we have the uniform bound $\tilde{F}(z) \ll |z|^{|\sigma|-1} e^{-\frac{\pi}{2}|t|}$*

Proof. [11] pg. 257-258.

□

Proposition 3.2.3 (Approximate functional equation)**.** *The setting is as above. Then for any $z \in \mathbb{C}$ we have,*

$$\sum_{\substack{f^2|\delta \\ \delta/f^2 \equiv 0,1 \bmod 4}} \tfrac{1}{f^{2z-1}} L\left(z, \left(\tfrac{\delta/f^2}{\cdot}\right)\right) = \sum_{\substack{f^2|\delta \\ \delta/f^2 \equiv 0,1 \bmod 4}} \tfrac{1}{f^{2z-1}} \sum_{t=1}^{\infty} \tfrac{1}{t^z} \left(\tfrac{\delta/f^2}{t}\right) F\left(\tfrac{tf^2}{A}\right)$$
$$+ \left(\tfrac{|\delta|}{\pi}\right)^{\frac{1}{2}-z} \sum_{\substack{f^2|\delta \\ \delta/f^2 \equiv 0,1 \bmod 4}} \tfrac{1}{f^{1-2z}} \sum_{t=1}^{\infty} \tfrac{1}{t^{1-z}} \left(\tfrac{\delta/f^2}{t}\right) H_{\delta,z}\left(\tfrac{tf^2 A}{|\delta|}\right)$$

Where,

$$H_{\delta,z}(y) := \frac{1}{2\pi i}\int_{\Re(u)=1} \frac{\Gamma\left(\frac{1+u-z+e_\delta}{2}\right)}{\Gamma\left(\frac{z-u+e_\delta}{2}\right)}(\pi y)^{-u}\tilde{F}(u)\,du$$

Proof. Let $\tilde{F}$ denote the mellon transform of F. For $\Re(u) = \sigma \gg 0$ by Mellin inversion we have

$$\sum_{\substack{f^2|\delta \\ \delta/f^2\equiv 0,1 \bmod 4}} \frac{1}{f^{2z-1}}\sum_{t=1}^{\infty}\frac{1}{t^z}\left(\frac{\delta/f^2}{t}\right)F\left(\frac{tf^2}{A}\right) = \frac{1}{2\pi i}\int_{\Re(u)=\sigma}\sum_{\substack{f^2|\delta \\ \delta/f^2\equiv 0,1 \bmod 4}}\frac{1}{f^{2(z+u)-1}}$$
$$L\left(z+u,\left(\frac{\delta/f^2}{\cdot}\right)\right)A^u\tilde{F}(u)\,du$$
$$= \frac{1}{2\pi i}\int_{\Re\tilde{F}(u)=\sigma} L(z+u,\delta)A^u\tilde{F}(u)\,du$$

Then shifting the contour to $\Re(u) = \sigma' < 0$ picks up the pole of $\tilde{F}(u)$ at $u = 0$ and gives

$$\frac{1}{2\pi i}\int_{\Re(u)=\sigma} L(z+u,\delta)A^u\tilde{F}(u)\,du = L(z,\delta) + \frac{1}{2\pi i}\int_{\Re(u)=\sigma'} L(z+u,\delta)A^u\tilde{F}(u)\,du$$

Using the change of variables $u \mapsto -u$ and using the oddness of $\tilde{F}$ transforms the σ'-integral to

$$\frac{1}{2\pi i}\int_{\Re(u)=\sigma'} L(z+u,\delta)A^u\tilde{F}(u)\,du = -\frac{1}{2\pi i}\int_{\Re(u)=-\sigma'} L(z-u,\delta)A^{-u}\tilde{F}(u)\,du$$

Finally using the functional equation (3.1′) gives

$$\frac{1}{2\pi i}\int_{\Re(u)=\sigma'} L(z-u,\delta)A^{-u}\tilde{F}(u)\,du = \frac{1}{2\pi i}\int_{\Re(u)=-\sigma'}\left(\frac{|\delta|}{\pi}\right)^{\frac{1}{2}+u-z}\frac{\Gamma\left(\frac{1+u-z+e_\delta}{2}\right)}{\Gamma\left(\frac{z-u+e_\delta}{2}\right)}L(1-z+u,\delta)A^{-u}\tilde{F}(u)\,du$$

Therefore we get,

$$\sum_{\substack{f^2|\delta \\ \delta/f^2\equiv 0,1 \bmod 4}}\frac{1}{f^{2z-1}}L\left(z,\left(\frac{\delta/f^2}{\cdot}\right)\right) = \sum_{\substack{f^2|\delta \\ \delta/f^2\equiv 0,1 \bmod 4}}\frac{1}{f^{2z-1}}\sum_{t=1}^{\infty}\frac{1}{t^z}\left(\frac{\delta/f^2}{t}\right)F\left(\frac{tf^2}{A}\right)$$
$$+\left(\frac{|\delta|}{\pi}\right)^{\frac{1}{2}-z}\sum_{\substack{f^2|\delta \\ \delta/f^2\equiv 0,1 \bmod 4}}\frac{1}{f^{1-2z}}\sum_{t=1}^{\infty}\frac{1}{t^{1-z}}\left(\frac{\delta/f^2}{t}\right)H_{\delta,z}\left(\frac{tf^2A}{|\delta|}\right)$$

We note that in the statement of the proposition we took $\sigma' = 1$ for convenience.

□

Corollary 3.2.4. *The setting is as above.*

$$\sum_{\substack{f^2|\delta \\ \delta/f^2\equiv 0,1 \bmod 4}}\frac{1}{f}L\left(1,\left(\frac{\delta/f^2}{\cdot}\right)\right) =$$
$$\sum_{\substack{f^2|\delta \\ \delta/f^2\equiv 0,1 \bmod 4}}\frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\left(\frac{\delta/f^2}{t}\right)F\left(\frac{tf^2}{A}\right) + \sqrt{\pi}\sum_{\substack{f^2|\delta \\ \delta/f^2\equiv 0,1 \bmod 4}}\left(\frac{f^2}{|\delta|}\right)^{\frac{1}{2}}\sum_{t=1}^{\infty}\left(\frac{\delta/f^2}{t}\right)H_\delta\left(\frac{tf^2A}{|\delta|}\right)$$

Where by abuse of notation we denote $H_{\delta,1}$ by H_δ.

We note that although there seems to be a dependence of $H_{\delta,z}$ on δ, it is no way serious. $H_{\delta,z}$ depends only on the sign of δ and once we fix the sign it is independent of δ. For this reason, from now on we will call $H_{\delta,z}$ as $H_{\pm,z}$, (similarly for H_δ) where it is implicitly understood that the sign, $\pm$, is the sign of δ.

3.3 Estimates on $H_\pm$

We have the following bound on $H_\pm$:

Lemma 3.3.1. *For any $\Re(x) \geq 1$ we have*

$$H_\pm(x) \ll \tfrac{1}{x} e^{-\frac{\sqrt{x}}{2}} \tag{3.5}$$

Proof. The only difference between H_+ and H_- is the difference in the Γ-factors, so we start with bounding those. By Stirling's formula we have

$$\frac{\Gamma\left(\frac{1+u}{2}\right)}{\Gamma\left(\frac{2-u}{2}\right)} \sim \left(\tfrac{u}{2e}\right)^{u-\frac{1}{2}} \left(1+\tfrac{1}{u}\right)^{\frac{u}{2}} \left(\tfrac{2}{u}-1\right)^{\frac{u-1}{2}}$$

Since $|1+\frac{u}{2}| \leq \sqrt{e}$ and for $\Re(u) \geq 1$, $|\frac{2}{u}-1| \leq 3^{\frac{\Re(u)}{2}}$ we have

$$\frac{\Gamma\left(\frac{1+u}{2}\right)}{\Gamma\left(\frac{2-u}{2}\right)} \ll \left(\tfrac{3u}{2e}\right)^{u-\frac{1}{2}}$$

Where the implied constant is absolute. On the other hand, by equation 10.63 of [11] (pg. 259) we have

$$\frac{\Gamma\left(\frac{u}{2}\right)}{\Gamma\left(\frac{1-u}{2}\right)} \ll |\tfrac{1}{u}-1|^{\Re(u)-\frac{1}{2}} |u|^{\Re(u)-\frac{1}{2}}$$

Which, for $\Re(u) \geq 1$ gives the bound

$$\frac{\Gamma\left(\frac{u}{2}\right)}{\Gamma\left(\frac{1-u}{2}\right)} \ll |u|^{\Re(u)-\frac{1}{2}}$$

Where the implied constant is, once more, absolute. Substituting these bounds into the definitions for $H_\pm(x)$, and using the bound on $\tilde{F}(x)$ given in Lemma 3.2.2 we get

$$\begin{aligned} H_\pm(x) &\ll \int_{(1)} |\tfrac{3u}{2e}|^{\Re(u)-\frac{1}{2}} |x|^{-\Re(u)} |u|^{\Re(u)-1} e^{-\frac{\pi|\Im(u)|}{2}} du \\ &= \int_{(1)} \left(\tfrac{3}{2e}\right)^{\Re(u)-\frac{1}{2}} |u|^{2\Re(u)-\frac{3}{2}} |x|^{-\Re(u)} e^{-\frac{\pi|\Im(u)|}{2}} du \end{aligned}$$

Shifting the contour to $\Re(u) = \max\{1, \sqrt{\frac{2x}{3}}\}$ then gives

$$\begin{aligned} H_\pm(x) &\ll \tfrac{1}{x^{3/4}} e^{-\sqrt{\frac{2u}{3}}} \\ &\ll \tfrac{1}{x} e^{-\frac{\sqrt{x}}{2}} \end{aligned}$$

Where the implied constant is absolute. $\square$

Chapter 4

Analysis of the Singularities of Archimedean Orbital Integrals and Asymptotic Expansions

4.1 Singularities of orbital integrals

We start by recalling the asymptotic behavior of archimedean orbital integrals. We will be very brief and present only what we need without going into detail or proofs. For a more detailed introduction we refer to [1] and the references therein, and for the asymptotic behavior of orbital integrals see [13], [16] and [27] and references therein.

For a test function, $f \in C_c^\infty(GL_2(\mathbb{R}))$, let us denote the orbital integral of f at the (regular) element $\gamma \in GL_2(\mathbb{R})$ by $Orb(f;\gamma)$. i.e.

$$Orb(f;\gamma) := \int_{T_\gamma \backslash GL_2(\mathbb{R})} f(g\gamma g^{-1}) d\bar{g}$$

Where T_γ is the centralizer of γ in $GL_2(\mathbb{R})$ which is a maximal torus (since γ is regular). The measure normalizations are as given in [27]. We are interested in the behavior of $Orb(f;\gamma)$ as γ approaches a central element. As is given in [18] (page 21 equation (26)) and [27] (equation (1)), around $z \in \mathbb{Z}(\mathbb{R})$: There exists a Weyl group invariant neighborhood, N_z, of z and smooth functions $G_1, G_2 \in C_c^\infty(N_z)$ (depending on the point z) such that

$$Orb(f;\gamma) = G_1(\gamma) + \tfrac{|\gamma_1\gamma_2|^{\frac{1}{2}}}{|\gamma_1-\gamma_2|} G_2(\gamma)$$

Where γ_1, γ_2 are the eigenvalues of the element γ. Note that as γ approaches a central element $Orb(f;\gamma)$ has a singularity of the prescribed form form $\frac{\gamma_1\gamma_2}{(\gamma_1-\gamma_2)^2}$ (the discriminant function of $GL(2)$).

When we parametrize the element γ with its trace $= m$, and determinant $= n$, the discriminant becomes

$$\tfrac{(\gamma_1-\gamma_2)^2}{\gamma_1\gamma_2} = \left(\tfrac{m^2}{n} - 4\right)$$

Therefore, in the (m,n) coordinates, the asymptotic behavior of the orbital integral can be expressed as

$$Orb(f;\gamma) = G_1(m,n) + \left|4 - \tfrac{m^2}{n}\right|^{-1/2} G_2(m,n)$$

Furthermore, by choosing a central character (as we do when using the trace formula), we can further assume that G_1 and G_2 depend only on the ratio $\frac{m}{\sqrt{|n|}}$ and the sign of n. Also, when sign of n is negative, the element that (m,n) parametrizes is in the split torus and G_1 vanishes.

For what follows, by abuse of notation we will write the variables of our functions as $\frac{m}{\sqrt{n}}$ instead of $\frac{m}{\sqrt{|x|}}$ and a sign, and will keep the dependence on the sign of n implicit. We will be considering the functions $\theta_\infty\left(\frac{m}{\sqrt{n}}\right)$ which are the products of $Orb(\gamma, f)$ with the square root of the discriminant and the local volume factors as defined in [18]. Let $x := \frac{m}{\sqrt{n}}$. By the above decomposition (see Lemma 2.a.4 of [18]) we see that around $x = \pm 2$ (Note that this is not the same x as in Lemma 2.a.4 of [18].) we have

$$\theta_\infty(x) = |4 - x^2|^{\frac{1}{2}}\theta_{\infty,1}(x) + \theta_{\infty,2}(x) \tag{4.1}$$

Where $\theta_{\infty,1(x)}$ is smooth and supported inside $(-2,2)$ and $\theta_{\infty,2}(x)$ is smooth and compactly supported. The most important property of these functions that we will use is the following asymptotic expansions around ± 2 (see Theorem 2.4.a of [21]):

$$\theta_{\infty,2}(\pm 2 + x) = \sum_{l=l_2^\pm(\theta)}^{\infty} \alpha_l^\pm x^l \tag{4.1-a}$$

$$|4 - (\pm 2 \mp x)^2|^{\frac{1}{2}}\theta_{\infty,1}(\pm 2 \mp x) = \sum_{l=l_1^\pm(\theta)}^{\infty} \beta_l^\pm x^{l+\frac{1}{2}} \tag{4.1-b}$$

4.2 A very brief summary of the context

Since it has been some time since the last time we discussed any of these I will quickly recall the context in which the approximate functional equation arise. For convenience assume that everything is unramified outside ∞ and the central character is trivial.

Recall that the approximate functional equation allows one to express the values of an L-function as a sum of two absolutely convergent series. More precisely for a primitive Dirichlet character χ (for this discussion there is no loss of generality in assuming that the character is primitive) of conductor $\mathfrak{f}_\chi$ it reads as

$$L(z,\chi) = \sum_{t=1}^{\infty} \frac{\chi(t)}{t^z}\Phi\left(\frac{t}{A}\right) + W_{\bar{\chi}}\left(\frac{\pi}{\mathfrak{f}_\chi}\right)^{\frac{1}{2}} \sum_{t=1}^{\infty} \frac{\bar{\chi}(t)}{t^{1-z}}\tilde{\Phi}_\chi\left(\frac{tA}{\mathfrak{f}_\chi}\right)$$

Here $A > 0$ is a parameter that is to be chosen, $W_{\bar{\chi}}$ is the root number for the complex conjugate character and Φ is a smooth cut-off function (in particular it is rapidly decreasing, say in the Schwarz space), and $\tilde{\Phi}_\chi$ is a certain transform of Φ, whose dependence on χ is actually only on $\chi(-1)$, and itself is rapidly decreasing.

Fix a test function $f_\infty \in C_c^\infty(G(\mathbb{R}))$. For any n, by choosing the standard functions, $f_{v,n}$ at each place v, to give the trace of the n'th Hecke operator, form $f := f_\infty \prod f_{v,n} \in C_c^\infty(G(\mathbb{A}))$. Then (after fixing measures etc.) the elliptic part of the geometric side of the trace formula for the operator $R(f)$ (roughly) reads as

$$\sum_{m\in\mathbb{Z}} \theta_\infty\left(\frac{m}{\sqrt{n}}\right) \sum_{\substack{f^2|(m^2-4n) \\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \frac{1}{f} L\left(1, \left(\frac{(m^2-4n)/f^2}{\cdot}\right)\right)$$

We will be treating the above sum one f at a time and for our purpose of explaining the smoothing of the archimedean orbital integral we can (and do) ignore it and pretend that $f = 1$. Then we have the sum,

$$\sum_{m\in\mathbb{Z}} \theta_\infty\left(\frac{m}{\sqrt{n}}\right) L\left(1, \left(\frac{m^2-4n}{\cdot}\right)\right)$$

Our aim is to apply Poisson summation to the m-sum. We would like to expand the L-function into the Dirichlet series and then interchange the two sums. In order to do so, we replace the L-function with the approximate functional equation and get

$$\sum_{m\in\mathbb{Z}} \theta_\infty\left(\frac{m}{\sqrt{n}}\right)\left\{\sum_{t=1}^{\infty}\frac{1}{t}\left(\frac{m^2-4n}{t}\right)\Phi\left(\frac{t}{A}\right) + \frac{\sqrt{\pi}}{|m^2-4n|^{\frac{1}{2}}}\sum_{t=1}^{\infty}\left(\frac{m^2-4n}{t}\right)\tilde{\Phi}_{e_{m,n}}\left(\frac{tA}{|m^2-4n|}\right)\right\}$$

Where $e_{m,n} = (1+sign(m^2-4n))/2$. Since the $e_{m,n}$ and $\tilde{\Phi}_{e_{m,n}}$ depend on the sign of m^2-4n we first split the sum into two parts

$$\sum_{m^2-4n<0} \theta_\infty\left(\frac{m}{\sqrt{n}}\right)\left\{\sum_{t=1}^{\infty}\frac{1}{t}\left(\frac{m^2-4n}{t}\right)\Phi\left(\frac{t}{A}\right) + \frac{\sqrt{\pi}}{|m^2-4n|^{\frac{1}{2}}}\sum_{t=1}^{\infty}\left(\frac{m^2-4n}{t}\right)\tilde{\Phi}_{e_{m,n}}\left(\frac{tA}{|m^2-4n|}\right)\right\}$$
$$+\sum_{m^2-4n>0} \theta_\infty\left(\frac{m}{\sqrt{n}}\right)\left\{\sum_{t=1}^{\infty}\frac{1}{t}\left(\frac{m^2-4n}{t}\right)\Phi\left(\frac{t}{A}\right) + \frac{\sqrt{\pi}}{|m^2-4n|^{\frac{1}{2}}}\sum_{t=1}^{\infty}\left(\frac{m^2-4n}{t}\right)\tilde{\Phi}_{e_{m,n}}\left(\frac{tA}{|m^2-4n|}\right)\right\}$$

We can now interchange the inner and outer sums, break the m-sum into arithmetic progressions mod$t$ and get

$$\sum_{t=1}^{\infty}\frac{1}{t}\sum_{a \bmod t}\left(\frac{a^2-4n}{t}\right)\left\{\sum_{\substack{m^2-4n<0\\ m\equiv a \bmod t}}\theta_\infty\left(\frac{m}{\sqrt{n}}\right)\Phi\left(\frac{t}{A}\right) + \sum_{\substack{m^2-4n>0\\ m\equiv a \bmod t}}\theta_\infty\left(\frac{m}{\sqrt{n}}\right)\Phi\left(\frac{t}{A}\right)\right\}$$
$$+\sqrt{\pi}\sum_{t=1}^{\infty}\sum_{a \bmod t}\left(\frac{a^2-4n}{t}\right)\left\{\sum_{\substack{m^2-4n<0\\ m\equiv a \bmod t}}\frac{1}{|m^2-4n|^{\frac{1}{2}}}\theta_\infty\left(\frac{m}{\sqrt{n}}\right)\tilde{\Phi}_-\left(\frac{tA}{|m^2-4n|}\right)\right.$$
$$\left.+\sum_{\substack{m^2-4n>0\\ m\equiv a \bmod t}}\frac{1}{|m^2-4n|^{\frac{1}{2}}}\theta_\infty\left(\frac{m}{\sqrt{n}}\right)\tilde{\Phi}_+\left(\frac{tA}{|m^2-4n|}\right)\right\}$$

Where $\tilde{\Phi}_-$ and $\tilde{\Phi}_+$ are the transforms which we used to label by the subscript $e_{m,n}$.

4.3 Smoothing orbital integrals

We would now like to apply Poisson summation to the m-sums that appear above. There are two problems in applying Poisson summation to the above sum: The first problem is that the sums are not complete (the m-sums are running over either $|m| < 2\sqrt{n}$ or $|n| > 2\sqrt{n}$). The second is the problems about the singularities of the function $|m^2-4n|^{\frac{-1}{2}}$.

To get around these problems at once we will use the rapid decay of our test function Φ (and of $\tilde{\Phi}_{1,2}$). Recall that we still haven't chosen the parameter A. Let $1 > \alpha > 0$ be a constant (we will later choose $\alpha \sim \frac{1}{2}$) and let

$$A = |m^2-4n|^\alpha$$

With this choice, the sum above is

$$\sum_{t=1}^{\infty}\frac{1}{t}\sum_{a \bmod t}\left(\frac{a^2-4n}{t}\right)\left\{\sum_{\substack{m^2-4n<0\\ m\equiv a \bmod t}}\theta_\infty\left(\frac{m}{\sqrt{n}}\right)\Phi\left(\frac{t}{(m^2-4n)^\alpha}\right)+\sum_{\substack{m^2-4n>0\\ m\equiv a \bmod t}}\theta_\infty\left(\frac{m}{\sqrt{n}}\right)\Phi\left(\frac{t}{(4n-m^2)^\alpha}\right)\right\}$$
$$+\sqrt{\pi}\sum_{t=1}^{\infty}\sum_{a \bmod t}\left(\frac{a^2-4n}{t}\right)\left\{\sum_{\substack{m^2-4n<0\\ m\equiv a \bmod t}}\frac{1}{|m^2-4n|^{\frac12}}\theta_{\infty,-}\left(\frac{m}{\sqrt{n}}\right)\tilde{\Phi}_-\left(\frac{t}{(m^2-4n)^{1-\alpha}}\right)\right.$$
$$\left.+\sum_{\substack{m^2-4n>0\\ m\equiv a \bmod t}}\frac{1}{|m^2-4n|^{\frac12}}\theta_{\infty,+}\left(\frac{m}{\sqrt{n}}\right)\tilde{\Phi}_+\left(\frac{t}{(4n-m^2)^{1-\alpha}}\right)\right\}$$

For any function Φ and constants $C>0$ and $1>\alpha>0$, define $\Theta_{\infty,\Phi,C,\alpha,\pm}$ by

$$\Theta_{\infty,\Phi,C,\alpha,-}(x) := \begin{cases}\theta_\infty(x)\Phi\left(\frac{C}{(4-x^2)^\alpha}\right) & \text{if } |x|<2\\ 0 & \text{otherwise}\end{cases} \qquad (\#)$$
$$\Theta_{\infty,\Phi,C,\alpha,+}(x) := \begin{cases}\theta_\infty(x)\Phi\left(\frac{C}{(x^2-4)^\alpha}\right) & \text{if } |x|>2\\ 0 & \text{otherwise}\end{cases}$$

Then the above sum can be rewritten as

$$\sum_{t=1}^{\infty}\frac{1}{t}\sum_{a \bmod t}\left(\frac{a^2-4n}{t}\right)\left\{\sum_{m\equiv a \bmod t}\Theta_{\infty,\Phi,tn^{-\alpha},\alpha,-}\left(\frac{m}{\sqrt{n}}\right)+\sum_{m\equiv a \bmod t}\Theta_{\infty,\Phi,tn^{-\alpha},\alpha,+}\left(\frac{m}{\sqrt{n}}\right)\right\}$$
$$+\sqrt{\pi}\sum_{t=1}^{\infty}\sum_{a \bmod t}\left(\frac{a^2-4n}{t}\right)\left\{\sum_{m\equiv a \bmod t}\frac{1}{|m^2-4n|^{\frac12}}\Theta_{\infty,\tilde{\Phi}_-,tn^{1-\alpha},1-\alpha,-}\left(\frac{m}{\sqrt{n}}\right)\right.$$
$$\left.+\sum_{m\equiv a \bmod t}\frac{1}{|m^2-4n|^{\frac12}}\Theta_{\infty,\tilde{\Phi}_+,tn^{1-\alpha},1-\alpha,-}\left(\frac{m}{\sqrt{n}}\right)\right\}$$

Next proposition shows that as $\frac{m}{\sqrt{n}}\to\pm2$ the test functions Φ and $\tilde{\Phi}$ decay faster than any polynomial which makes the functions $\Theta_{\infty,\Phi,tn^{-\alpha},\alpha,\pm}\left(\frac{m}{\sqrt{n}}\right)$ and $\frac{1}{|4-x^2|^{\frac12}}\Theta_{\infty,\tilde{\Phi}_\pm,tn^{1-\alpha},1-\alpha,\pm}(x)$ smooth. More precisely, we have the following:

Proposition 4.3.1. *Let $C>0$ and $1>\alpha>0$ be constants. For any Schwarz function $\Phi(x)$, the functions $\Theta_{\infty,\Phi,tn^{-\alpha},\alpha,\pm}(x)$ and $\frac{1}{|4-x^2|^{\frac12}}\Theta_{\infty,\tilde{\Phi}_\pm,tn^{1-\alpha},1-\alpha,\pm}(x)$ are smooth.*

Proof. The only problem is around $x=\pm2$ and everything is symmetric so without loss of generality we will analyze the derivative only near $x=2$. We will also show the result only for $Theta_{\infty,\Phi,tn^{-\alpha},\alpha,-}(x)$ because the argument carries verbatim to the other function.

We will show that the left and right limits of the difference quotient are both 0. The right derivative is trivially 0 because the function is 0 there. Consider the left limit of the difference quotient,

$$\lim_{h\to0^+}\frac{\theta_\infty(2-h)F\left(\frac{C}{(4-(2-h)^2)^\alpha}\right)}{h}=\lim_{h\to0^+}\frac{\theta_\infty(2-h)F\left(\frac{C}{(4h-h^2)^\alpha}\right)}{h}$$

Since Φ is Schwarz class, for any $M > 0$ we have

$$\Phi(x) = O_M(x^{-M})$$

From this we conclude that as $h \to 0^+$

$$\Phi\left(\frac{C}{(4h-h^2)^\alpha}\right) = O_M\left(\left(\frac{(4h-h^2)^\alpha}{C}\right)^M\right)$$

Therefore,

$$\frac{\theta_\infty(2-h)F\left(\frac{C}{(4h-h^2)^\alpha}\right)}{h} = O_M\left(\frac{\theta_\infty(2-h)}{h}\left(\frac{(4h-h^2)^\alpha}{C}\right)^M\right)$$

Since by equation (4.1) θ_∞ is bounded we get that the limit is 0, proving the differentiability. Since M was arbitrary, this indeed proves that all the derivatives exists. The argument for the rest of the functions are exactly the same.

$\square$

Proposition 4.3.1 allows us to apply Poisson summation to the m-sums. However, in order to be able to analyze the resulting dual sum we need to understand the decay properties of the Fourier transforms of the functions $\Theta_{\infty,\Phi,tn^{-\alpha},\alpha,\pm}(x)$ and $\frac{1}{|4-x^2|^{\frac{1}{2}}}\Theta_{\infty,\tilde{\Phi}_\pm,tn^{1-\alpha},1-\alpha,\pm}(x)$.

Recall that the functions $\Theta_{\infty,\Phi,tn^{-\alpha},\alpha,\pm}(x)$ are defined by (#), and are the product of $\theta_\infty(x)$ with the appropriate cut-off function. On the other hand the function $\theta_\infty(x)$, by (4.1) is the sum of $|1-x^2|^{\frac{1}{2}}\theta_{\infty,1}(x)$ and $\theta_{\infty,2}(x)$, where $\theta_{\infty,1}(x)$ is supported in $(-2,2)$ and is smooth there, and $\theta_{\infty,2}(x)$ is smooth of compact support.

Therefore, in order to analyze the Fourier transform of $\Theta_{\infty,\Phi,tn^{-\alpha},\alpha,\pm}(x)$ and $\frac{1}{|4-x^2|^{\frac{1}{2}}}\Theta_{\infty,\tilde{\Phi}_\pm,tn^{1-\alpha},1-\alpha,\pm}(x)$ we need to understand the Fourier transforms of functions of type

$$|1-x^2|^{\frac{\rho}{2}}g(x)\Phi\left(\frac{C}{|1-x^2|^\alpha}\right)$$

Where $\rho = 0, \pm 1$ and $|1-x^2|^{\frac{\rho}{2}}g(x)$ has an asymptotic expansion around ± 1 of the form given in (4.1-a) or (4.1-b) depending on ρ. We will do this in the following sections.

Important note. It might not be clear at this point as to why we need to analyze the asymptotic expansion of Fourier transforms of the above functions since by Proposition 4.3.1 they are all smooth and therefore decay faster than any polynomial. This is indeed correct, that they are smooth and they do decay faster than any polynomial. However the constants involved depend on the parameters α and C. For us $C = \frac{t}{n^\alpha}$ and since in the applications we have in mind we will be summing on t and n, we need the explicit dependence of the Fourier transforms on C. This is subtler than the crude decay estimate that comes for free from the smoothness of the involved functions.

4.4 Asymptotic expansions Fourier transforms - 1

In this section we will derive the asymptotic behavior of the following type of Fourier transforms:

$$\int_{-1}^{1} g(x)\Phi\left(\frac{C}{(1-x^2)^\alpha}\right)e(-x\eta)dx \tag{4.2}$$

Where $\Phi(x)$ is a Schwarz class functions, $g(x)$ is smooth of compact support and the integral is taken over $(-1,1)$. We want the explicit dependence on the variables η and C, when $0<\alpha<1$ is a fixed constant. (For future reference, α will be around $\frac{1}{2}$.) The asymptotic behavior will depend on the support of $g(x)$, size of C and the relative sizes of $C^{\frac{1}{\alpha}}$ and η, and we will treat each one of these cases separately.

Throughout the proofs we will be using integration by parts on composition of functions multiple times, so it is convenient to recall the so called Faà di Bruno's formula: Let $k\in\mathbb{N}$ and f and g be k-times differentiable functions. Then the k'th derivative of the composition $f\circ g(x)$ is given by the explicit formula (cf. [23])

$$\frac{d^k}{dx^k}f(g(x)) = \sum_{\substack{a_1+2a_2+3a_3+\cdots+ka_k=k\\ a:=a_1+a_2+\cdots+a_k\\ a_i\in\mathbb{N}}} \frac{k!}{a_1!a_2!\cdots a_k!} f^{(a)}(g(x)) \left(\frac{g'(x)}{1!}\right)^{a_1}\left(\frac{g''(x)}{2!}\right)^{a_2}\cdots\left(\frac{g^{(k)}(x)}{k!}\right)^{a_k} \tag{4.3}$$

Proposition 4.4.1. *Let $1>\lambda>0$. Then for any integer $k>0$ there exists a constant $\tilde{\kappa}(\alpha,k)>0$ such that for $-1+\lambda<x<1-\lambda$ we have*

$$\frac{d^k}{dx^k}\Phi\left(\frac{C}{(1-x^2)^\alpha}\right) \le \lambda^{-k} \sum_{\substack{a_1+2a_2+3a_3+\cdots+ka_k=k\\ a:=a_1+a_2+\cdots+a_k\\ a_i\in\mathbb{N}}} \tilde{\kappa}(\alpha,k)C^a\lambda^{-a\alpha}\Phi^{(a)}\left(\frac{C}{(1-x^2)^\alpha}\right) \tag{4.4}$$

Proof. This is a direct consequence of equation (4.3) with $g(x)=\frac{C}{(1-x^2)^\alpha}$ and $f(x)=\Phi(x)$. Applying the formula we get

$$\frac{d^k}{dx^k}\Phi\left(\frac{C}{((1-x^2)^\alpha)}\right) = \sum_{\substack{a_1+2a_2+3a_3+\cdots+ka_k=k\\ a:=a_1+a_2+\cdots+a_k\\ a_i\in\mathbb{N}}} \frac{k!}{a_1!a_2!\cdots a_k!}\Phi^{(a)}\left(\frac{C}{(1-x^2)^\alpha}\right)$$
$$\left(\frac{1}{1!}\frac{d}{dx}\left(\frac{C}{(1-x^2)^\alpha}\right)\right)^{a_1}\left(\frac{1}{2!}\frac{d^2}{dx^2}\left(\frac{C}{(1-x^2)^\alpha}\right)\right)^{a_2}\cdots\left(\frac{1}{k!}\frac{d^k}{dx^k}\left(\frac{C}{(1-x^2)^\alpha}\right)\right)^{a_k}$$

$$= \sum_{\substack{a_1+2a_2+3a_3+\cdots+ka_k=k\\ a:=a_1+a_2+\cdots+a_k\\ a_i\in\mathbb{N}}} \frac{k!C^a}{a_1!a_2!\cdots a_k!}\Phi^{(a)}\left(\frac{C}{(1-x^2)^\alpha}\right)$$
$$\left(\frac{1}{1!}\frac{d}{dx}\left(\frac{1}{(1-x^2)^\alpha}\right)\right)^{a_1}\left(\frac{1}{2!}\frac{d^2}{dx^2}\left(\frac{1}{(1-x^2)^\alpha}\right)\right)^{a_2}\cdots\left(\frac{1}{k!}\frac{d^k}{dx^k}\left(\frac{1}{(1-x^2)^\alpha}\right)\right)^{a_k} \tag{4.5}$$

For any $b\in\mathbb{Z}_{>0}$, using the binomial theorem and the fact that $-1+\lambda<x<1-\lambda$, we have

$$\begin{aligned}\frac{d^b}{dx^b}\left(\frac{1}{(1-x^2)^\alpha}\right) &= \sum_{l=0}^{b}\binom{b}{l}\frac{d^l}{dx^l}\frac{1}{(1-x)^\alpha}\frac{d^{1-l}}{dx^{1-l}}\frac{1}{(1+x)^\alpha}\\ &= \frac{\alpha(\alpha+1)\cdots(\alpha+ab1)}{(1-x^2)^\alpha}\left[\frac{1}{(1-x)^b}+\frac{(-1)^b}{(1+x)^b}\right]+\sum_{l=1}^{b-1}\binom{a}{l}\frac{(-1)^{a-l}\alpha(\alpha+1)\cdots(\alpha+l-1)\alpha(\alpha+1)\cdots(\alpha+b-l-1)}{(1-x^2)^\alpha(1-x)^l(1+x)^{b-l}}\\ &\le \kappa(\alpha,b)\lambda^{-\alpha-b}\end{aligned}$$

where $\kappa(\alpha,a)$ is a constant depending only on a and α. Substituting this bound into (4.5) we then get

$$\frac{d^k}{dx^k}\Phi\left(\frac{C}{((1-x^2)^\alpha)}\right) \le \sum_{\substack{a_1+2a_2+3a_3+\cdots+ka_k=k\\ a:=a_1+a_2+\cdots+a_k\\ a_i\in\mathbb{N}}} \frac{k!C^a}{a_1!a_2!\cdots a_k!}\Phi^{(a)}\left(\frac{C}{(1-x^2)^\alpha}\right)$$
$$\left(\frac{1}{1!}\kappa(\alpha,1)\lambda^{-\alpha-1}\right)^{a_1}\left(\frac{1}{2!}\kappa(\alpha,2)\lambda^{-\alpha-2}\right)^{a_2}\cdots\left(\frac{1}{k!}\kappa(\alpha,k)\lambda^{-\alpha-k}\right)^{a_k}$$

$$\leq \lambda^{-k} \sum_{\substack{a_1+2a_2+3a_3+\cdots+ka_k=k \\ a:=a_1+a_2+\cdots+a_k \\ a_i\in\mathbb{N}}} \tilde{\kappa}(\alpha,k) C^a \lambda^{-a\alpha} \Phi^{(a)}\left(\tfrac{C}{(1-x^2)^\alpha}\right)$$

Where $\tilde{\kappa}(\alpha, k) > 0$ is a constant that only depends on α and k. □

For the rest of this chapter we will analyze the Fourier transform (4.2). We will see that depending on the support of g and the relative sizes of C and η, the Fourier transform behaves differently. The main case is when the support of g is $(-1, 1)$, and $C \to 0$ and $C^{\frac{1}{\alpha}}\eta < C^\delta$ for any $\delta > 0$, in which case we will get an asymptotic expansion which will be used in later chapters.

For the rest of the section we will work our way to this final case step by step starting from simple cases where the Fourier transform decays very fast independent of any parameter.

4.4.1 Warm-up I: $C > K$ for some constant K.

In this case integration by parts implies that the Fourier transform decays faster than any polynomial, and this decay is independent of C. More precisely,
Proposition 4.4.2. *Let $C > K$ for some constant $K > 0$. Then for any $M > 0$*

$$\int_{-1}^{1} g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right) e(-x\eta)dx = O_{\Phi,g,\alpha,M,K}(\eta^{-M})$$

Proof. Integrating by parts M-times we get

$$\int_{-1}^{1} g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right) e(-x\eta)dx = \tfrac{1}{(-2\pi i\eta)^M} \int_{-1}^{1} \tfrac{d^M}{dx^M}\left\{g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)\right\} e(-x\eta)dx \tag{4.6}$$

By the binomial theorem we have

$$\tfrac{d^M}{dx^M}\left\{g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)\right\} = \sum_{l=0}^{M} \binom{M}{l} \tfrac{d^{M-l}}{dx^{M-l}} g(x) \tfrac{d^l}{dx^l}\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right) \tag{4.7}$$

Substituting equation (4.7) into equation (4.6) gives

$$\int_{-1}^{1} g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right) e(-x\eta)dx = \tfrac{1}{(-2\pi i\eta)^M} \sum_{l=0}^{M} \binom{M}{l} \int_{-1}^{1} \tfrac{d^{M-l}}{dx^{M-l}} g(x) \tfrac{d^l}{dx^l}\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right) e(-x\eta)dx \tag{4.8}$$

Since $g(x)$ is smooth for any $0 \leq l \leq k$ we have

$$\tfrac{d^{M-l}}{dx^{M-l}} g(x) = O_{g,M}(1) \tag{4.9}$$

On the other hand, for any $0 \leq l \leq M$, by (4.3)

$$\tfrac{d^l}{dx^l}\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right) = \sum_{\substack{a_1+2a_2+3a_3+\cdots+la_l=l \\ a:=a_1+a_2+\cdots+a_l \\ a_i\in\mathbb{N}}} \tfrac{l!C^a}{a_1!a_2!\cdots a_l!}\Phi^{(a)}\left(\tfrac{C}{(1-x^2)^\alpha}\right)$$
$$\left(\tfrac{1}{1!}\tfrac{d}{dx}\left(\tfrac{1}{(1-x^2)^\alpha}\right)\right)^{a_1}\left(\tfrac{1}{2!}\tfrac{d^2}{dx^2}\left(\tfrac{1}{(1-x^2)^\alpha}\right)\right)^{a_2}\cdots\left(\tfrac{1}{l!}\tfrac{d^l}{dx^l}\left(\tfrac{1}{(1-x^2)^\alpha}\right)\right)^{a_l} \tag{4.10}$$

Furthermore by the binomial theorem we have

$$\frac{d^b}{dx^b}\left(\frac{1}{(1-x^2)^\alpha}\right)$$
$$=\frac{\alpha(\alpha+1)\cdots(\alpha+b-1)}{(1-x^2)^\alpha}\left[\frac{1}{(1-x)^b}+\frac{(-1)^b}{(1+x)^b}\right]+\sum_{m=1}^{b-1}\binom{b}{m}\frac{(-1)^{b-m}\alpha(\alpha+1)\cdots(\alpha+m-1)\alpha(\alpha+1)\cdots(\alpha+b-m-1)}{(1-x^2)^\alpha(1-x)^m(1+x)^{b-m}} \tag{4.11}$$

Substituting (4.11) into (4.10) we see that if we can bound generic term

$$\frac{1}{(1-x^2)^\alpha(1-x)^{ma_j}(1+x)^{(j-m)a_j}}\Phi^{(a)}\left(\frac{C}{(1-x^2)^\alpha}\right) \tag{4.12}$$

where $a_j \leq \frac{l}{j}$ and $0 \leq m \leq j$. Since Φ is of Schwarz class and $a < M$, for any $A > 0$ we have

$$\Phi^{(a)}(x) = O_{\Phi,M,A}(x^{-A})$$

Using this we get

$$\Phi^{(a)}\left(\frac{C}{(1-x^2)^\alpha}\right) = O_{\Phi,M,A}\left(\frac{(1-x^2)^{\alpha A}}{C^A}\right) \tag{4.13}$$

Substituting (4.13) into (4.12) we get

$$\frac{1}{(1-x^2)^\alpha(1-x)^{ma_j}(1+x)^{(b-m)a_j}}\Phi^{(a)}\left(\frac{C}{(1-x^2)^\alpha}\right) = O_{\Phi,M,A}\left(\frac{(1-x^2)^{\alpha A}}{(1-x^2)^\alpha(1-x)^{ma_j}(1+x)^{(j-m)a_j}C^A}\right) \tag{4.14}$$

Since $jm \leq ja_j \leq l \leq M$ Choose (and fix) A such that $\lfloor \alpha A\rfloor = \lfloor M+\alpha+5\rfloor$, and note that $0 \leq 1-x^2 \leq 1$. We then get

$$\frac{(1-x^2)^{\alpha A}}{(1-x^2)^\alpha(1-x)^{ma_j}(1+x)^{(b-m)a_j}C^A} = O_{\Phi,\alpha,M}(C^{-A}) \tag{4.15}$$

Substituting (4.15) back into (4.14), (4.11) and then into (4.10) we get

$$\frac{d^l}{dx^l}\Phi\left(\frac{C}{(1-x^2)^\alpha}\right) = O_{\Phi,\alpha,M}\left(\frac{C^a}{C^A}\right) \tag{4.16}$$

Recalling $\alpha < 1$ we get $A - M > 0$. Then noting that $a \leq l \leq M$ and substituting (4.16) into (4.8) gives

$$\int_{-1}^{1} g(x)\Phi\left(\frac{C}{(1-x^2)^\alpha}\right)e(-x\eta)dx = O_{\Phi,g,\alpha,M}(\eta^{-M}C^{M-A})$$

Finally using $C > K$, we get the proposition. □

4.4.2 Warm-up II: $C < K$ for some constant K, and the support of $g(x)$ does not intersect $\{\pm 1\}$.

If the support of $g(x)$ does not intersect $\{\pm 1\}$, then the integrand is smooth and we can again use integration by parts to get the decay of the Fourier transform.

Lemma 4.4.3. *Let $1 > \lambda > 0$. Then for any integer $k > 0$ and for $-1+\lambda < x < 1-\lambda$ we have*

$$\frac{d^k}{dx^k}\Phi\left(\frac{C}{(1-x^2)^\alpha}\right) = O_{\Phi,\alpha,k,K,\lambda}(1) \tag{4.17}$$

Proof. Since $C < K$,

$$C^a \lambda^{-a\alpha} \leq K^a \lambda^{-a\alpha} \tag{4.18}$$

Substituting (4.18) into (4.4) we get

$$\tfrac{d^k}{dx^k}\Phi\left(\tfrac{C}{((1-x^2)^\alpha)}\right) \leq \lambda^{-k} \sum_{\substack{a_1+2a_2+3a_3+\cdots+ka_k=k\\ a:=a_1+a_2+\cdots+a_k\\ a_i\in\mathbb{N}}} \tilde{\kappa}(\alpha,k)K^a\lambda^{-a\alpha}\Phi^{(a)}\left(\tfrac{C}{(1-x^2)^\alpha}\right) \tag{4.19}$$

Since $C < K$, for $-1+\lambda < x < 1-\lambda$,

$$\tfrac{C}{(1-x^2)^\alpha} < K\lambda^{-\alpha}$$

Therefore for $-1+\lambda < x < 1-\lambda$ and for any $0 < a < k$ we have

$$\Phi^{(a)}\left(\tfrac{C}{(1-x^2)^\alpha}\right) = O_{\Phi,\alpha,k,K,\lambda}(1)$$

Substituting this bound into (4.19) we then get

$$\begin{aligned}\tfrac{d^k}{dx^k}\Phi\left(\tfrac{C}{((1-x^2)^\alpha)}\right) &= O_{\Phi,\alpha,k,\kappa}\left(\lambda^{-k}\tilde{\kappa}(\alpha,k)\sum_{\substack{a_1+2a_2+3a_3+\cdots+ka_k=k\\ a:=a_1+a_2+\cdots+a_k\\ a_i\in\mathbb{N}}} K^a\lambda^{-a\alpha}\right)\\ &= O_{\Phi,\alpha,k,K,\lambda}(1)\end{aligned}$$

□

Corollary 4.4.4. *Assume that $C < K$ for some constant K, and the support of $g(x)$ does not intersect $\{\pm 1\}$. Then for every $M > 0$ we have*

$$\int_{-1}^{1} g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)e(-x\eta)dx = O_{\Phi,g,\alpha,M,K}(\eta^{-M})$$

Proof. Integrating by parts M-times gives

$$\int_{-1}^{1} g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)e(-x\eta)dx = \tfrac{1}{(-2\pi i\eta)^M}\int_{-1}^{1}\tfrac{d^M}{dx^M}\left\{g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)\right\}e(-x\eta)dx \tag{4.20}$$

By the binomial theorem we have

$$\tfrac{d^M}{dx^M}\left\{g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)\right\} = \sum_{l=0}^{M}\binom{M}{l}\tfrac{d^{M-l}}{dx^{M-l}}g(x)\tfrac{d^l}{dx^l}\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)$$

Let $\lambda > 0$ be a constant such that the support of $g(x)$ be contained in $(-1+\lambda, 1-\lambda)$. Then by Lemma 4.4.3 we have

$$\tfrac{d^M}{dx^M}\left\{g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)\right\} = O_{\Phi,\alpha,M,\lambda,K}\left(\sum_{l=0}^{M}\binom{M}{l}\tfrac{d^{M-l}}{dx^{M-l}}g(x)\right)$$

Since $g(x)$ is smooth and its support is in $(-1,1)$, $g^{(M-l)}(x) = O_{g,M-l}(1)$. Combining these, we get

$$\tfrac{d^M}{dx^M}\left\{g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)\right\}O_{\Phi,g,\alpha,M,K}(1)$$

Substituting this back into (4.20) finishes the proof.

□

4.4.3 Support of $g(x)$ intersects $\{\pm 1\}$ non-trivially and $C \to 0$.

There are two sub cases of this case depending on the size of η relative to $C^{-\frac{1}{\alpha}}$.

There exists $\delta > 0$ such that $\eta > C^{-\frac{1}{\alpha}-\delta}$.

In this case, once again integration by parts resolves the problem.

Proposition 4.4.5. *Assume that there exists $\delta > 0$ such that $\eta > C^{-\frac{1}{\alpha}-\delta}$. Then for any $M > 0$ we have*

$$\int_{-1}^{1} g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right) e(-x\eta)dx = O_{\Phi,g,\alpha,M,\delta}(\eta^{-M})$$

Proof. We proceed as in the proof of Proposition 4.4.2. For any integer $M_0 > 0$, integration by parts M_0-times and going through steps (4.6) to (4.12) reduces the problem to bounding terms of the form

$$\frac{1}{(1-x^2)^\alpha(1-x)^{ma_j}(1+x)^{(j-m)a_j}}\Phi^{(a)}\left(\tfrac{C}{(1-x^2)^\alpha}\right) \tag{4.21}$$

where $l \le M_0$, $1 \le a \le l$, $a_j \le \frac{l}{j}$, $0 \le m \le j$ and $-1 < x < 1$. We split $(-1,1)$ into three parts,

$$I = I_1 \cup I_2 \cup I_3$$

Where,

$$\begin{aligned} I_1 &:= (-1, -1 + C^{\frac{1}{\alpha}+\frac{\delta}{2}}) \\ I_2 &:= (-1 + C^{\frac{1}{\alpha}+\frac{\delta}{2}}, 1 - C^{\frac{1}{\alpha}+\frac{\delta}{2}}) \\ I_3 &:= (1 - C^{\frac{1}{\alpha}+\frac{\delta}{2}}, 1) \end{aligned}$$

For x belonging to each of the I_1, I_2, I_3 we will bound(4.21) separately.

- $x \in I_3$. Since $x \in I_3$, $(1-x^2)^{-\alpha} \ge 2^{-\alpha}C^{-1-\delta\alpha}$. Hence $\frac{C}{(1-x^2)^\alpha} \ge 2^{-\alpha}C^{-\delta\alpha}$, and in particular $\frac{C}{(1-x^2)^\alpha} \to \infty$. Therefore, since Φ is of Schwarz class for any $A > 0$ we have

  $$\Phi^{(a)}(x) = O_{\Phi,M_0,A}(x^{-A})$$

  Therefore,

  $$\Phi^{(a)}\left(\tfrac{C}{(1-x^2)^\alpha}\right) = O_{\Phi,M_0,A}\left(\tfrac{(1-x^2)^{\alpha A}}{C^A}\right) \tag{4.22}$$

  Substituting this back into (4.21) gives

  $$\frac{1}{(1-x^2)^\alpha(1-x)^{ma_j}(1+x)^{(j-m)a_j}}\Phi^{(a)}\left(\tfrac{C}{(1-x^2)^\alpha}\right) = O_{\Phi,M_0,A}\left(\frac{(1-x^2)^{\alpha A}}{C^A(1-x^2)^\alpha(1-x)^{ma_j}(1+x)^{(j-m)a_j}}\right)$$

  Since $x \in I_3$, $(1-x^2)^{\alpha A} \le C^{A+\alpha\delta A}$. Therefore

  $$\frac{1}{(1-x^2)^\alpha(1-x)^{ma_j}(1+x)^{(j-m)a_j}}\Phi^{(a)}\left(\tfrac{C}{(1-x^2)^\alpha}\right) = O_{\Phi,M_0,A}\left(\frac{C^{\alpha\delta A}}{(1-x^2)^\alpha(1-x)^{ma_j}(1+x)^{(j-m)a_j}}\right) \tag{4.23}$$

  Substituting (4.23) into equation (4.10) gives

  $$\frac{d^l}{dx^l}\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right) = O_{\Phi,M_0,A,\delta}\left(C^{\alpha\delta A}C^{-1-\delta\alpha}\right)$$

Choosing $\lfloor A \rfloor = \lfloor \frac{2}{\alpha\delta} \rfloor$ and noting that $C < 1$ then gives

$$\frac{d^l}{dx^l}\Phi\left(\frac{C}{(1-x^2)^\alpha}\right) = O_{\Phi,\alpha,\delta}(1) \tag{4.24}$$

Finally, using (4.9) and substituting (4.24) into (4.6) we get

$$\frac{1}{(-2\pi i\eta)^{M_0}}\int_{I_3}\frac{d^{M_0}}{dx^{M_0}}\left\{g(x)\Phi\left(\frac{C}{(1-x^2)^\alpha}\right)\right\}e(-x\eta)dx = O_{\Phi,g,\alpha,M_0,\delta}(\eta^{-M_0}) \tag{4.25}$$

This finishes the treatment of I_3.

- $x \in I_1$. This case is the exact symmetric of $x \in I_3$ and the same bound in (4.25) applies.
- $x \in I_2$. In this case since $(1-x^2)^\alpha \geq C^{1+\delta\alpha}$, we have $\frac{C}{(1-x^2)^\alpha} \leq C^{-\delta\alpha}$. Hence, for any $a > 0$ we have

$$\Phi^{(a)}\left(\frac{C}{(1-x^2)^\alpha}\right) = O_{\Phi,M_0,a}(1)$$

Substituting this back into (4.12) gives

$$\frac{1}{(1-x^2)^\alpha(1-x)^{ma_j}(1+x)^{(j-m)a_j}}\Phi^{(a)}\left(\frac{C}{(1-x^2)^\alpha}\right) = O_{\Phi,M_0,a}\left(\frac{1}{(1-x^2)^\alpha(1-x)^{ma_j}(1+x)^{(j-m)a_j}}\right)$$

Substituting this back into (4.10) then gives

$$\frac{d^l}{dx^l}\Phi\left(\frac{C}{(1-x^2)^\alpha}\right) = O_{\Phi,\alpha,M_0}\left(\frac{C}{(1-x^2)^\alpha(1-x^2)^l}\right) \tag{4.26}$$

Substituting (4.26) into (4.7), and using (4.9) we get

$$\begin{aligned}\int_{I_2} g(x)\Phi\left(\frac{C}{(1-x^2)^\alpha}\right)e(-x\eta)dx &= O_{\Phi,g,\alpha,M_0}\left(\eta^{-M_0}C\max_{x\in I_2}\left\{\frac{1}{(1-x^2)^{\alpha+M_0}}\right\}\right)\\ &= O_{\Phi,g,\alpha,M_0}\left(\eta^{-k_0}\frac{C}{C^{\left(\frac{1}{\alpha}+\frac{\delta}{2}\right)(\alpha+M_0)}}\right)\\ &= O_{\Phi,g,\alpha,M}\left(\eta^{-M_0}C^{-\left(\frac{\delta\alpha}{2}+\frac{\delta M_0}{2}+\frac{M_0}{\alpha}\right)}\right) \qquad (\star)\end{aligned}$$

Where we used $(1-x^2)^{-1} \leq C^{\frac{1}{\alpha}+\delta}$ when passing from the first to second equality. Note that since $\eta > C^{-\frac{1}{\alpha}-\delta}$ we have

$$C^{-1} < \eta^{\frac{\alpha}{1+\delta\alpha}}$$

Finally, substituting this in $(\star)$ gives

$$\begin{aligned}\int_{I_2} g(x)\Phi\left(\frac{C}{(1-x^2)^\alpha}\right)e(-x\eta)dx &= O_{\Phi,g,\alpha,M_0}\left(\eta^{-M_0}\eta^{\frac{\alpha}{1+\delta\alpha}\left(\frac{\delta\alpha}{2}+\frac{\delta M_0}{2}+\frac{M_0}{\alpha}\right)}\right)\\ &= O_{\Phi,g,\alpha,M_0}\left(\eta^{-\frac{\delta\alpha M_0}{1+\delta\alpha}+\frac{\delta\alpha M_0}{2(1+\delta\alpha)}+\frac{\delta\alpha^2}{2(1+\delta\alpha)}}\right)\\ &= O_{\Phi,g,\alpha,M_0}\left(\eta^{-\frac{\delta\alpha M_0}{2(1+\delta\alpha)}+\frac{\delta\alpha^2}{2(1+\delta\alpha)}}\right) \qquad (\star\star)\end{aligned}$$

Now choosing $M_0 > \frac{2(1+\delta\alpha)}{\delta\alpha}(M+1)$ and combining (4.25) and $(\star\star)$ finishes the proof.

$\square$

$\eta C^{\frac{1}{\alpha}} < C^{-\delta}$ **for every** $\delta > 0$**.**

In this case the integral behaves quite differently. It does not decay arbitrarily fast in η. The first term in the asymptotic expansion is η^{-1} times an oscillating factor, and the rest of the asymptotic expansion comes as a polynomial in η and $C^{\frac{1}{\alpha}}\eta$.

Proposition 4.4.6. *For any $M > 0$, there exists constants $a_m(l), d_r(s)$ which depend only on the indicated parameters such that the following asymptotic expansions hold:*

$$\int_{-1}^{1} g(x)\Phi\left(\frac{C}{(1-x^2)^{\alpha}}\right) e(-x\eta)dx$$
$$= e(-\eta)\sum_{l=l_g^+}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM} \frac{\alpha_l^+ a_m(l) d_r(s)}{s!}\left(\frac{i}{\pi\eta}\right)^{m+r+s+1} \mathcal{T}_{\alpha,m+r+2s}(\Phi)\left(C^{\frac{1}{\alpha}}\eta\right)$$
$$+ e(\eta)\sum_{l=l_g^-}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM} \frac{(-1)^l\alpha_l^- a_m(l) d_r(s)}{s!}\left(\frac{-i}{\pi\eta}\right)^{m+r+s+1} \mathcal{T}_{\alpha,m+r+2s}(\Phi)\left(-C^{\frac{1}{\alpha}}\eta\right)$$
$$+ O_{\Phi,g,M}(\eta^{-M})$$

Where,

$$g(x) = \sum_{l=l_g^{\pm}}^{\infty} \alpha_l^{\pm}(x \mp 1)^l$$

is the asymptotic expansion of $g(x)$ around ± 1, and for $1 > \alpha > 0$, $\beta > 0$ and $\kappa \in \mathbb{Z}_{\geq 0}$, $\mathcal{T}(\Phi)(z)$ is defined by

$$\mathcal{T}_{\alpha,\kappa}(\Phi)(y) := \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \mathcal{M}(\Phi)(z)\left(\frac{i}{\pi y}\right)^{\alpha z}\Gamma(\kappa+\alpha z+1)dz$$

where $\mathcal{M}$ denotes the Mellin transform.

Proof. Let $\frac{1}{2} > \epsilon > 0$ be fixed. We break the integral into three parts according to the size of x. Let,

$$(-1,1) = (-1, -\sqrt{1-|\eta|^{\epsilon-1}}) \cup (-\sqrt{1-|\eta|^{\epsilon-1}}, \sqrt{1-|\eta|^{\epsilon-1}}) \cup (\sqrt{1-|\eta|^{\epsilon-1}}, 1)$$

We will analyze the integral on each of these regions separately. The middle region will be small compared to the first and last (this is because x stays away from the singularities in the middle region) and the main asymptotic contribution will come from the first and the last.

In order to formally carry the above argument, it is more convenient (due to the integration by parts argument in the middle region) to introduce a smooth cut-off function that separates the middle region from the rest.

Smoothing the cut-off. The general idea is to introduce an even function $\tilde{\phi}$ (which will depend on η and ϵ) that is 1 up to a neighborhood of $\sqrt{1-|\eta|^{\epsilon-1}}$ and smoothly decays to 0 again in a small neighborhood of $\sqrt{1-|\eta|^{\epsilon-1}}$.

Let ϕ be a function such that:

- $supp(\phi) = (0,1)$ and $\int \phi = 1$,
- $\lim_{x\to 0^+}\phi(x) = 0$, $\lim_{x\to 1^-}\phi(x) = 1$,
- ϕ is smooth, positive and monotonically increasing,

- ϕ decays faster than any power of x around 0,
- All the right derivatives of ϕ at 0 and the left derivatives at 1 vanishes,

(Note that for example $\phi(x) = \kappa \int_0^x e^{-\frac{1}{1-y}-\frac{1}{y}} dy$, where κ is normalized so that the value at $x = 1$ is 1, is such a function.) Using ϕ we construct $\tilde{\phi}$ as follows:

$$\tilde{\phi}(x) = \begin{cases} 0 & \text{if } |x| \leq \sqrt{1-2|\eta|^{\epsilon-1}} \\ \phi\left(\frac{|x|-\sqrt{1-2|\eta|^{\epsilon-1}}}{\sqrt{1-|\eta|^{\epsilon-1}}-\sqrt{1-2|\eta|^{\epsilon-1}}}\right) & \text{if } \sqrt{1-2|\eta|^{\epsilon-1}} < |x| < \sqrt{1-|\eta|^{\epsilon-1}} \\ 1 & \text{if } \sqrt{1-|\eta|^{\epsilon-1}} \leq |x| \end{cases} \tag{4.27}$$

Using $\tilde{\phi}$ we rewrite the integral as follows

$$\begin{aligned} \int_{-1}^{1} g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right) e(-x\eta)dx &= \int_{-1}^{1} g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)\left((1-\tilde{\phi}(x)) + \tilde{\phi}(x)\right) e(-x\eta)dx \\ &= A + B \end{aligned}$$

Where,

$$\begin{aligned} A &:= \int_{-1}^{1} (1-\tilde{\phi}(x))g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right) e(-x\eta)dx \\ B &:= \int_{-1}^{1} \tilde{\phi}(x)g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right) e(-x\eta)dx \end{aligned}$$

We will analyze A and B separately. Note that the integration in the A integral is over $|x| < \sqrt{1-|\eta|^{\epsilon-1}}$ and the integration in B is over $1 > |x| > \sqrt{1-2|\eta|^{\epsilon-1}}$. Furthermore the integrands in A and B are the same as the original integrand except for $\sqrt{1-2|\eta|^{\epsilon-1}} < |x| < \sqrt{1-|\eta|^{\epsilon-1}}$.

Lemma 4.4.7. *For any $M > 0$,*

$$A = O_{\Phi,g,\alpha,\phi,M}\left(C|\eta|^{\alpha(1-\epsilon)}|\eta|^{-M}\right)$$

Proof. Without loss of generality we can assume $\eta > 0$ (if not, we can replace η with $-\eta$ without changing the argument). For any $M_0 > 0$, repeated integration by parts M_0-times yields,

$$A = \tfrac{1}{(-2\pi i\eta)^{M_0}} \int_{-1}^{1} \tfrac{d^{M_0}}{dx^{M_0}} \left\{(1-\tilde{\phi}(x))g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)\right\} e(-x\eta)dx \tag{4.28}$$

By the multinomial theorem we have

$$\begin{aligned} &\tfrac{d^{M_0}}{dx^{M_0}} \left\{(1-\tilde{\phi}(x))g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)\right\} \\ &\qquad = \sum_{l_1+l_2+l_3=M_0} \binom{M_0}{l_1\, l_2\, l_3} \tfrac{d^{l_1}}{dx^{l_1}}(1-\tilde{\phi}(x))\tfrac{d^{l_2}}{dx^{l_2}}g(x)\tfrac{d^{l_3}}{dx^{l_3}}\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right) \end{aligned} \tag{4.29}$$

Since $g(x)$ is smooth we have,

$$\tfrac{d^l}{dx^l} g(x) = O_{g,M_0}(1) \tag{4.30}$$

for all $0 \leq l \leq M_0$. On the other hand by definition (4.27), and noting that

$$\sqrt{1-\eta^{\epsilon-1}} - \sqrt{1-2\eta^{\epsilon-1}} = \tfrac{1}{2}\eta^{\epsilon-1} + O(\eta^{2(\epsilon-1)})$$

for any $0 \le l \le M_0$ we have

$$\begin{aligned}
\frac{d^l}{dx^l}(1-\tilde{\phi}(x)) &\le \frac{d^l}{dx^l}\phi\left(\frac{|x|-\sqrt{1-2\eta^{\epsilon-1}}}{\sqrt{1-\eta^{\epsilon-1}}-\sqrt{1-2\eta^{\epsilon-1}}}\right)+1 \\
&= O\left(\eta^{l(1-\epsilon)}\phi^{(l)}\left(\frac{|x|-\sqrt{1-2\eta^{\epsilon-1}}}{\sqrt{1-\eta^{\epsilon-1}}-\sqrt{1-2\eta^{\epsilon-1}}}\right)+1\right) \\
&= O_{\phi,M_0}\left(\eta^{l(1-\epsilon)}\right) \qquad (\dagger)
\end{aligned}$$

Finally, once again, let $0 \le l \le M_0$. Then[1] by Proposition 4.4.1 there are constants $\tilde{\kappa}(\alpha, k_0)$ such that

$$\begin{aligned}
\frac{d^l}{dx^l}\Phi\left(\frac{C}{(1-x^2)^\alpha}\right) &\le \sum_{\substack{a_1+2a_2+3a_3+\cdots+la_l=l \\ a:=a_1+a_2+\cdots+a_l \\ a_i\in\mathbb{N}}} \tilde{\kappa}(\alpha, M_0)C^a\eta^{(l+\alpha a)(1-\epsilon)}\Phi^{(a)}\left(\frac{C}{(1-x^2)^\alpha}\right) \\
&= \sum_{\substack{a_1+2a_2+3a_3+\cdots+la_l=l \\ a:=a_1+a_2+\cdots+a_l \\ a_i\in\mathbb{N}}} \bar{\kappa}(\Phi, \alpha, M_0)C^a\eta^{(l+\alpha a)(1-\epsilon)} \qquad (\dagger\dagger)
\end{aligned}$$

for some constant $\bar{\kappa}(\Phi, \alpha, M_0)$. Substituting (4.30), (†) and (††) into (4.29) gives

$$\begin{aligned}
&\frac{d^{M_0}}{dx^{M_0}}\left\{\tilde{\phi}(x)g(x)\Phi\left(\frac{C}{(1-x^2)^\alpha}\right)\right\} \\
&= O_{\Phi,g,\alpha,\phi,M_0}\left(\sum_{l_1+l_2+l_3=M_0}\eta^{l_1(1-\epsilon)}\sum_{\substack{a_1+2a_2+\cdots+l_3a_{l_3}=l_3 \\ a:=a_1+a_2+\cdots+a_{l_3} \\ a_i\in\mathbb{N}}}C^a\eta^{(l_3+\alpha a)(1-\epsilon)}\right) \\
&= O_{\Phi,g,\alpha,\phi,M_0}\left(\sum_{\substack{l_1+l_2+l_3=M_0 \\ a_1+2a_2+\cdots+l_3a_{l_3}=l_3 \\ a:=a_1+a_2+\cdots+a_{l_3} \\ a_i\in\mathbb{N}}}\left(C^{\frac{1}{\alpha}}\eta^{1-\epsilon}\right)^{\alpha a}\eta^{(l_1+l_3)(1-\epsilon)}\right) \qquad (4.31)
\end{aligned}$$

Since $\eta < C^{\frac{-1}{\alpha}+\delta}$ for every $\delta > 0$, $\eta^{1-\epsilon} < C^{\frac{-1}{\alpha}}$. Therefore the maximum of (4.31) occurs when $l_1 = l_2 = 0$ and $l_3 = M_0$. Hence we get,

$$\frac{d^{M_0}}{dx^{M_0}}\left\{(1-\tilde{\phi}(x))g(x)\Phi\left(\frac{C}{(1-x^2)^\alpha}\right)\right\} = O_{\Phi,g,\alpha,\phi,M_0}\left(C\eta^{\alpha(1-\epsilon)}\eta^{M_0(1-\epsilon)}\right) \qquad (4.32)$$

Substituting (4.32) into (4.28) then gives

$$A = O_{\Phi,g,\alpha,\phi,M_0}\left(C\eta^{\alpha(1-\epsilon)}\eta^{-M_0\epsilon}\right) \qquad (4.33)$$

Choosing $M_0 = \frac{M}{\epsilon}$ gives the result.

□

[1] Note that since we have smoothed our integrand, $\tilde{\phi}(x)g(x)\Phi\left(\frac{C}{(1-x^2)^\alpha}\right)$, it and all its derivatives are differentiable around the boundary and furthermore all of these derivatives vanish because $\tilde{\phi}$ and all its derivatives vanish to infinite order at the boundary. This is what allows us to use Proposition (4.4.1).

We will now analyze the integral in B. By definition,

$$\begin{aligned} B &= \int_{-1}^{1} \tilde{\phi}(x)g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)e(-x\eta)dx \\ &= \int_{-1}^{0} \tilde{\phi}(x)g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)e(-x\eta)dx + \int_{0}^{1} \tilde{\phi}(x)g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)e(-x\eta)dx \\ &= B_1 + B_2 \end{aligned}$$

The analysis of B_1 and B_2 are symmetric. So without loss of generality we analyze B_2 in detail and sketch the analysis for B_1.

Lemma 4.4.8. *For any $M > 0$, there exists constants $a_m(l), d_r(s)$ which depend only on the indicated parameters such that the following asymptotic expansions hold:*

$$B_2 = e(-\eta)\sum_{l=l_g^+}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM} \frac{\alpha_l^+ a_m(l) d_r(s)}{s!}\left(\tfrac{i}{\pi\eta}\right)^{m+r+s+1}\mathcal{T}_{\alpha,m+r+2s}\left(C^{\frac{1}{\alpha}}\eta\right) + O_{\Phi,g,\phi,M}(\eta^{-M})$$

Where,

$$g(x) = \sum_{l=l_g^+}^{\infty} \alpha_l^+(x-1)^l$$

is the asymptotic expansion of $g(x)$ around $x=1$, and for $1>\alpha>0$, $\beta>0$ and $\kappa\in\mathbb{Z}_{\geq 0}$, $\mathcal{T}(\Phi)(z)$ is defined by

$$\mathcal{T}_{\alpha,\kappa}(\Phi)(y) := \tfrac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\mathcal{M}(\Phi)(z)\left(\tfrac{i}{\pi y}\right)^{\alpha z}\Gamma(\kappa+\alpha z+1)dz$$

where $\mathcal{M}$ denotes the Mellin transform.

Proof. The analysis will be tedious but the idea is simple. We will use the asymptotic expansion of $g(x)$ around $x=1$, and then use the Taylor expansion of the exponential together with the binomial theorem to control $e(-\eta\sqrt{1-u^{-1}})$.

Substituting the asymptotic expansion of $g(x)$ into B_2 we get

$$\sum_{l=l_g^+}^{\infty}\alpha_l^+\int_0^1 (x-1)^l\tilde{\phi}(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)e(-x\eta)dx \tag{4.34}$$

Since the support of $\tilde{\phi}(x)$ is in $(\sqrt{1-2\eta^{\epsilon-1}},1)$, for any $A_0>0$ we have

$$\sum_{l=A_0+1}^{\infty}\alpha_l^+\int_0^1 (x-1)^l\tilde{\phi}(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)e(-x\eta)dx = O_{g,A_0}(\eta^{(\epsilon-1)(A_0+1)}) \tag{4.35}$$

Let $u^{-1}=1-x^2$. Then the integral becomes,

$$\sum_{l=l_g^+}^{A_0}\alpha_l^+\int_1^\infty \frac{\left(\sqrt{1-u^{-1}}-1\right)^l}{2u^2\sqrt{1-u^{-1}}}\tilde{\phi}\left(\sqrt{1-u^{-1}}\right)\Phi\left(Cu^\alpha\right)e\left(-\eta\sqrt{1-u^{-1}}\right)du \tag{4.36}$$

Next, note that for any $l\geq 0$,

$$\frac{(\sqrt{1-u^{-1}}-1)^l}{2u^2\sqrt{1-u^{-1}}} = \tfrac{1}{l+1}\tfrac{d}{du}(\sqrt{1-u^{-1}}-1)^{l+1}$$

and the Taylor expansion of $\sqrt{1-x}-1$ around $x=0$ is

$$\sqrt{1-x}-1=\sum_{m=1}^{\infty}\binom{1/2}{m}(-x)^m$$

Using these we get

$$\begin{aligned}
\tfrac{(\sqrt{1-u^{-1}}-1)^l}{2u^2\sqrt{1-u^{-1}}} &= \tfrac{1}{l+1}\tfrac{d}{du}\left[\sum_{m=1}^{\infty}\binom{1/2}{m}(-u)^{-m}\right]^{l+1} \\
&= \left[\sum_{m=1}^{\infty}\binom{1/2}{m}(-u)^{-m}\right]^{l}\left[\sum_{m=1}^{\infty}\binom{1/2}{m}m(-u)^{-m-1}\right] \\
&= \tfrac{1}{u^2}\sum_{m=l}^{\infty}a_m(l)u^{-m} \qquad (\dagger)
\end{aligned}$$

Where the coefficients $a_m(l)$ depend only on m and l. Substituting ($\dagger$) into (4.36) gives

$$\sum_{l=l_g^+}^{A_0}\sum_{m=l}^{\infty}\alpha_l^+a_m(l)\int_1^{\infty}u^{-m-2}\tilde{\phi}\left(\sqrt{1-u^{-1}}\right)\Phi\left(Cu^{\alpha}\right)e\left(-\eta\sqrt{1-u^{-1}}\right)du$$

Once again, since the support of $\tilde{\phi}(x)$ is in $(\sqrt{1-2|\eta|^{\epsilon-1}},1)$, the support of $\tilde{\phi}\left(\sqrt{1-u^{-1}}\right)$ is in $(\frac{|\eta|^{1-\epsilon}}{2},\infty)$. Therefore for any $A_1\geq A_0$ we have,

$$\sum_{l=l_g^+}^{A_0}\sum_{m=A_1+1}^{\infty}\alpha_l^+a_m(l)\int_1^{\infty}u^{-m-2}\tilde{\phi}\left(\sqrt{1-u^{-1}}\right)\Phi\left(Cu^{\alpha}\right)e\left(-\eta\sqrt{1-u^{-1}}\right)du$$
$$=O_{g,A_0,A_1}(|\eta|^{(\epsilon-1)(A_1+2)}) \qquad (4.37)$$

Next, we analyze the exponential. By the binomial theorem we have

$$\sqrt{1-u^{-1}}=\sum_{r=0}^{\infty}\binom{1/2}{r}(-u)^{-r}$$

Substituting this into the exponential we get

$$\begin{aligned}
e\left(-\eta\sqrt{1-u^{-1}}\right) &= e\left(-\eta\sum_{r=0}^{\infty}\binom{1/2}{r}(-u)^{-2r}\right) \\
&= e\left(-\eta+\tfrac{\eta u^{-1}}{2}\right)e\left(-\eta u^{-2}\sum_{r=2}^{\infty}\binom{1/2}{r}(-u)^{-r+2}\right)
\end{aligned}$$

For notational convenience, let $b_r=\binom{1/2}{r}(-1)^r$. Then,

$$e\left(-\eta\sqrt{1-u^{-1}}\right)=e\left(-\eta+\tfrac{\eta u^{-1}}{2}\right)e\left(-\eta u^{-2}\sum_{r=2}^{\infty}b_ru^{-r+2}\right)$$

By the Taylor expansion of the exponential we get

$$\begin{aligned}
e\left(-\eta u^{-2}\sum_{r=2}^{\infty}b_ru^{-r+2}\right) &= \sum_{s=0}^{\infty}\tfrac{1}{s!}\left[-2\pi i\eta u^{-2}\sum_{r=2}^{\infty}b_ru^{-r+2}\right]^s \\
&= \sum_{s=0}^{\infty}\tfrac{(-2\pi i\eta u^{-2})^s}{s!}\left[\sum_{r=2}^{\infty}b_ru^{-r+2}\right]^s
\end{aligned}$$

Substituting this back into the integral in (4.37) we get

$$\sum_{l=l_g^+}^{A_0}\sum_{m=l}^{A_1}\alpha_l^+ a_m(l)\int_1^\infty u^{-m-2}\tilde{\phi}\left(\sqrt{1-u^{-1}}\right)\Phi\left(Cu^\alpha\right) e\left(-\eta+\tfrac{\eta u^{-1}}{2}\right)\sum_{s=0}^{\infty}\tfrac{(-2\pi i\eta u^{-2})^s}{s!}\left[\sum_{r=2}^{\infty}b_r u^{-r+2}\right]^s du \quad (4.38)$$

Since the support of $\tilde{\phi}$ limits the lower limit of integration to $u>\frac{|\eta|^{1-\epsilon}}{2}$, for any $A_2>0$ we have

$$\sum_{l=l_g^+}^{A_0}\sum_{m=l}^{A_1}\alpha_l^+ a_m(l)\int_1^\infty u^{-m-2}\tilde{\phi}\left(\sqrt{1-u^{-1}}\right)\Phi\left(Cu^\alpha\right) e\left(-\eta+\tfrac{\eta u^{-1}}{2}\right)\sum_{s=A_2+1}^{\infty}\tfrac{(-2\pi i\eta u^{-2})^s}{s!}\left[\sum_{r=2}^{\infty}b_r u^{-r+2}\right]^s du = O_{g,A_0,A_1}(|\eta|^{-A_2-2-l_{g,1}}|\eta|^{\epsilon(l_{g,1}+2A_2+3)}) \quad (4.39)$$

Similarly for any $A_3\geq 2$ we have

$$e(-\eta)\sum_{l=l_g^+}^{A_0}\sum_{m=l}^{A_1}\sum_{s=1}^{A_2}\frac{\alpha_l^+ a_m(l)(-2\pi i\eta)^s}{s!}\int_1^\infty u^{-m-2s-2}\tilde{\phi}\left(\sqrt{1-u^{-1}}\right)\Phi\left(Cu^\alpha\right) e\left(\tfrac{\eta u^{-1}}{2}\right)\left[\sum_{r=A_3+1}^{\infty}b_r u^{-r+2}\right]^s du = O_{g,A_0,A_1,A_2}(|\eta|^{-A_3-l_{g,1}-1}|\eta|^{\epsilon(A_3+l_{g,1}+2)}) \quad (4.40)$$

Therefore we are left with

$$e(-\eta)\sum_{l=l_g^+}^{A_0}\sum_{m=l}^{A_1}\sum_{s=0}^{A_2}\sum_{r=0}^{sA_3}\frac{\alpha_l^+ a_m(l)d_r'(s)(-2\pi i\eta)^s}{s!}\int_1^\infty u^{-m-r-2s-2}\tilde{\phi}\left(\sqrt{1-u^{-1}}\right)\Phi\left(Cu^\alpha\right) e\left(\tfrac{\eta u^{-1}}{2}\right) du \quad (4.41)$$

Where $d_r'(s)$ depend only on r and s. Hence we need to analyze the asymptotic behavior of integrals of the form

$$\int_1^\infty u^{-\kappa-2}\tilde{\phi}\left(\sqrt{1-u^{-1}}\right)\Phi\left(Cu^\alpha\right) e\left(\tfrac{\eta u^{-1}}{2}\right) du$$

for integers $\kappa\geq l_{g,1}$. Let $u\mapsto u^{-1}$. Then we have

$$\int_0^1 u^\kappa\tilde{\phi}\left(\sqrt{1-u}\right)\Phi\left(\tfrac{C}{u^\alpha}\right) e\left(\tfrac{\eta u}{2}\right) du \quad (4.42)$$

Recall that the support of $\tilde{\phi}\left(\sqrt{1-u}\right)$ is $u<2\eta^{\epsilon-1}$ and for $u<\eta^{\epsilon-1}$, $\tilde{\phi}\left(\sqrt{1-u}\right)=1$. Therefore,

$$(4.42)=\int_0^{\eta^{\epsilon-1}} u^\kappa\Phi\left(\tfrac{C}{u^\alpha}\right) e\left(\tfrac{\eta u}{2}\right) du+\int_{\eta^{\epsilon-1}}^{2\eta^{\epsilon-1}} u^\kappa\tilde{\phi}\left(\sqrt{1-u}\right)\Phi\left(\tfrac{C}{u^\alpha}\right) e\left(\tfrac{\eta u}{2}\right) du$$

We will analyze these two integrals one by one. The analysis will depend on the sign of η. We will give the detailed analysis for $\eta>0$ and then sketch the necessary modifications for $\eta<0$.

- $\eta>0$.

– $\int_0^{\eta^{\epsilon-1}} u^\kappa \Phi\left(Cu^{-\alpha}\right) e\left(\frac{\eta u}{2}\right) du.$

Let $\mathcal{M}(\Phi)(z)$ denote the Mellin transform of Φ. Then by the Mellin inversion formula for some $\beta > 0$, we have

$$\int_0^{\eta^{\epsilon-1}} u^\kappa \Phi\left(Cu^{-\alpha}\right) e\left(\tfrac{\eta u}{2}\right) du = \int_0^{\eta^{\epsilon-1}} u^\kappa \left\{\tfrac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \mathcal{M}(\Phi)(z)\left(Cu^{-\alpha}\right)^{-z} dz\right\} e\left(\tfrac{\eta u}{2}\right) du$$

Interchanging the integrals we get

$$\tfrac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \mathcal{M}(\Phi)(z)C^{-z}\left\{\int_0^{\eta^{\epsilon-1}} u^{\kappa+\alpha z} e\left(\tfrac{\eta u}{2}\right) du\right\} dz \tag{4.43}$$

We treat the inner integral separately by treating it as a complex function. By the Cauchy integral formula, we can write the integral as the sum of two integrals on the vertical lines $0 \to i\infty$ and $\eta^{\epsilon-1} + i\infty \to \eta^{\epsilon-1}$.

$$\begin{aligned}\int_0^{\eta^{\epsilon-1}} u^{\kappa+\alpha z} e\left(\tfrac{\eta u}{2}\right) du &= \int_0^{i\infty} u^{\kappa+\alpha z} e\left(\tfrac{\eta u}{2}\right) du + \int_{\eta^{\epsilon-1}+i\infty}^{\eta^{\epsilon-1}} u^{\kappa+\alpha z} e\left(\tfrac{\eta u}{2}\right) du \\ &= \int_0^{i\infty} u^{\kappa+\alpha s} e\left(\tfrac{\eta u}{2}\right) du + e\left(\tfrac{\eta^\epsilon}{2}\right)\int_{i\infty}^0 (\eta^{\epsilon-1}+u)^{\kappa+\alpha s} e\left(\tfrac{\eta u}{2}\right) du \\ &= \left(\tfrac{i}{\pi\eta}\right)^{\kappa+\alpha z+1}\int_0^\infty u^{\kappa+\alpha z} e^{-u} du \\ &\quad + ie\left(\tfrac{\eta^\epsilon}{2}\right)\int_\infty^0 (\eta^{\epsilon-1}+iu)^{\kappa+\alpha z} e^{-\pi\eta u} du \end{aligned} \tag{$\circ$}$$

In the next paragraph we will show that ($\circ$) cancels ($\circ\circ$). Ignoring that for now, we have

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \mathcal{M}(\Phi)(z)C^{-z}\left(\tfrac{i}{\pi\eta}\right)^{\kappa+\alpha z+1}\int_0^\infty u^{\kappa+\alpha z} e^{-u} du dz \\ &= \left(\tfrac{i}{\pi\eta}\right)^{\kappa+1}\tfrac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \mathcal{M}(\Phi)(z)\left(\tfrac{i}{\pi C^{\frac{1}{\alpha}}\eta}\right)^{\alpha z}\Gamma(\kappa+\alpha z+1) dz \\ &= \left(\tfrac{i}{\pi\eta}\right)^{\kappa+1}\mathcal{T}_{\alpha,\kappa}(\Phi)(C^{\frac{1}{\alpha}}\eta) \end{aligned} \tag{4.44}$$

This finishes this part of the analysis.

– $\int_{\eta^{\epsilon-1}}^{2\eta^{\epsilon-1}} u^\kappa \tilde{\phi}\left(\sqrt{1-u}\right)\Phi\left(\frac{C}{u^\alpha}\right) e\left(\frac{\eta u}{2}\right) du.$

As before, using the Mellin transform of F we get that the integral is equal to

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{\beta_1-i\infty}^{\beta_1+i\infty} \mathcal{M}(\Phi)(z_1)C^{-z_1}\left\{\int_{\eta^{\epsilon-1}}^{2\eta^{\epsilon-1}} u^{\kappa+\alpha z_1}\tilde{\phi}\left(\sqrt{1-u}\right) e\left(\tfrac{\eta u}{2}\right) du\right\} dz_1 \\ &= \tfrac{1}{2\pi i}\int_{\beta_1-i\infty}^{\beta_1+i\infty} \mathcal{M}(\Phi)(z_1)C^{-z_1}\left\{\int_{\eta^{\epsilon-1}}^{2\eta^{\epsilon-1}} u^{\kappa+\alpha z_1}\phi\left(\tfrac{\sqrt{1-u}-\sqrt{1-2\eta^{\epsilon-1}}}{\sqrt{1-\eta^{\epsilon-1}}-\sqrt{1-2\eta^{\epsilon-1}}}\right) e\left(\tfrac{\eta u}{2}\right) du\right\} dz_1\end{aligned}$$

Consider the inner integral,

$$\int_{\eta^{\epsilon-1}}^{2\eta^{\epsilon-1}} u^{\kappa+\alpha z_1}\phi\left(\tfrac{\sqrt{1-u}-\sqrt{1-2\eta^{\epsilon-1}}}{\sqrt{1-\eta^{\epsilon-1}}-\sqrt{1-2\eta^{\epsilon-1}}}\right) e\left(\tfrac{\eta u}{2}\right) du$$

Using Mellin inversion once again, this time on ϕ,

$$\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_{\eta^{\epsilon-1}}^{\eta^{2\epsilon-1}}u^{\kappa+\alpha z_1}\left(\frac{\sqrt{1-u}-\sqrt{1-2\eta^{\epsilon-1}}}{\sqrt{1-\eta^{\epsilon-1}}-\sqrt{1-2\eta^{\epsilon-1}}}\right)^{-z_2}e\left(\frac{\eta u}{2}\right)du dz_2$$

Note that,

$$\frac{\sqrt{1-u}-\sqrt{1-2\eta^{\epsilon-1}}}{\sqrt{1-\eta^{\epsilon-1}}-\sqrt{1-2\eta^{\epsilon-1}}}=2-\frac{u}{\eta^{\epsilon-1}}+O\left(\frac{u^2+\eta^{2(\epsilon-1)}}{\eta^{\epsilon-1}}\right)$$

Also, since $\phi(x)$ decays faster than any polynomial at 0 it's Mellin transform is holomorphic. Therefore, shifting the contour to $\Re(z_2)=-A_5$ we see that

$$\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\left\{\int_{\eta^{\epsilon-1}}^{2\eta^{\epsilon-1}}u^{\kappa+\alpha z_1}\left(\frac{u^2+\eta^{2(\epsilon-1)}}{\eta^{\epsilon-1}}\right)^{-z_2}e\left(\frac{\eta u}{2}\right)du\right\}dz_2$$
$$=O_{\phi,A_5}(\eta^{A_5(\epsilon-1)}) \quad (4.45)$$

Using this, we are left with analyzing

$$\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_{\eta^{\epsilon-1}}^{2\eta^{\epsilon-1}}u^{\kappa+\alpha z_1}\left(2-\frac{u}{\eta^{\epsilon-1}}\right)^{-z_2}e\left(\frac{\eta u}{2}\right)du dz_2$$

Letting $u=\eta^{\epsilon-1}+v$ gives

$$\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_{0}^{\eta^{\epsilon-1}}(\eta^{\epsilon-1}+v)^{\kappa+\alpha z_1}\left(1-\frac{v}{\eta^{\epsilon-1}}\right)^{-z_2}e\left(\frac{\eta(\eta^{\epsilon-1}+v)}{2}\right)dv dz_2$$

We again shift the contours in the integral so that it becomes an integral over the vertical lines $0\to i\infty$ and $\eta^{\epsilon-1}+i\infty\to\eta^{\epsilon-1}$.

$$\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_{0}^{i\infty}(\eta^{\epsilon-1}+v)^{\kappa+\alpha z_1}\left(1-\frac{v}{\eta^{\epsilon-1}}\right)^{-z_2}e\left(\frac{\eta(\eta^{\epsilon-1}+v)}{2}\right)dv dz_2$$
$$+\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_{\eta^{\epsilon-1}}^{\eta^{\epsilon-1}+i\infty}(\eta^{\epsilon-1}+v)^{\kappa+\alpha z_1}\left(1-\frac{v}{\eta^{\epsilon-1}}\right)^{-z_2}e\left(\frac{\eta(\eta^{\epsilon-1}+v)}{2}\right)dv dz_2$$

By the binomial theorem we have

$$\left(1-\frac{v}{\eta^{\epsilon-1}}\right)^{-z_2}=1+\sum_{f=1}^{\infty}\binom{-z_2}{f}\left(\frac{v}{\eta^{\epsilon-1}}\right)^{f}$$

Substituting this back into the first integral (and using the normalization that $\int\phi=1$) we get

$$\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_{0}^{i\infty}(\eta^{\epsilon-1}+v)^{\kappa+\alpha z_1}e\left(\frac{\eta(\eta^{\epsilon-1}+v)}{2}\right)dv dz_2$$
$$=ie\left(\frac{\eta^{\epsilon}}{2}\right)\int_{0}^{\infty}(\eta^{\epsilon-1}+iu)^{\kappa+\alpha z}e^{-\pi\eta u}du \quad (\circ\circ)$$

Hence this integral cancels the second integral in ($\circ$). Therefore we are left with

$$\begin{aligned}
&\tfrac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\sum_{f=1}^{\infty}\binom{-z_2}{f}\int_0^{i\infty}(\eta^{\epsilon-1}+v)^{\kappa+\alpha z_1}\left(\tfrac{-v}{\eta^{\epsilon-1}}\right)^f e\left(\tfrac{\eta(\eta^{\epsilon-1}+v)}{2}\right)dvdz_2\\
&+\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_{\eta^{\epsilon-1}}^{\eta^{\epsilon-1}+i\infty}(\eta^{\epsilon-1}+v)^{\kappa+\alpha z_1}\left(1-\tfrac{v}{\eta^{\epsilon-1}}\right)^{-z_2}e\left(\tfrac{\eta(\eta^{\epsilon-1}+v)}{2}\right)dvdz_2\\
&=\tfrac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\sum_{f=1}^{\infty}\binom{-z_2}{f}\int_0^{i\infty}(\eta^{\epsilon-1}+v)^{\kappa+\alpha z_1}\left(\tfrac{-v}{\eta^{\epsilon-1}}\right)^f e\left(\tfrac{\eta(\eta^{\epsilon-1}+v)}{2}\right)dvdz_2\\
&+\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_0^{i\infty}(2\eta^{\epsilon-1}+v)^{\kappa+\alpha z_1}\left(\tfrac{-v}{\eta^{\epsilon-1}}\right)^{-z_2}e\left(\tfrac{\eta(2\eta^{\epsilon-1}+v)}{2}\right)dvdz_2\\
&=ie\left(\tfrac{\eta^{\epsilon}}{2}\right)\tfrac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\sum_{f=1}^{\infty}\binom{-z_2}{f}\int_0^{\infty}(\eta^{\epsilon-1}+iv)^{\kappa+\alpha z_1}\left(\tfrac{-iv}{\eta^{\epsilon-1}}\right)^f e^{-\pi\eta v}dvdz_2\\
&+ie\left(\tfrac{\eta^{\epsilon}}{2}\right)\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_0^{\infty}(2\eta^{\epsilon-1}+iv)^{\kappa+\alpha z_1}\left(\tfrac{-iv}{\eta^{\epsilon-1}}\right)^{-z_2}e^{-\pi\eta v}dvdz_2
\end{aligned}$$

Note that if $v>\eta^{-1+\frac{\epsilon}{2}}$, then $e^{-\pi\eta v}=e^{-\pi\eta(v-\eta^{-1+\frac{\epsilon}{2}}+\eta^{-1+\frac{\epsilon}{2}})}=e^{-\pi\eta(v-\eta^{-1+\frac{\epsilon}{2}})}e^{-\pi\eta^{-\frac{\epsilon}{2}}}$. Therefore, we have

$$\begin{aligned}
&\int_0^{\infty}(2\eta^{\epsilon-1}+iv)^{\kappa+\alpha z_1}\left(\tfrac{-iv}{\eta^{\epsilon-1}}\right)^{-z_2}e^{-\pi\eta v}dv\\
&=\int_{\eta^{-1+\frac{\epsilon}{2}}}^{\infty}(2\eta^{\epsilon-1}+iv)^{\kappa+\alpha z_1}\left(\tfrac{-iv}{\eta^{\epsilon-1}}\right)^{-z_2}e^{-\pi\eta v}dv\\
&+\int_0^{\eta^{-1+\frac{\epsilon}{2}}}(2\eta^{\epsilon-1}+iv)^{\kappa+\alpha z_1}\left(\tfrac{-iv}{\eta^{\epsilon-1}}\right)^{-z_2}e^{-\pi\eta v}dv\\
&=O(e^{-\eta^{\frac{\epsilon}{2}}})+\int_0^{\eta^{-1+\frac{\epsilon}{2}}}(2\eta^{\epsilon-1}+iv)^{\kappa+\alpha z_1}\left(\tfrac{-iv}{\eta^{\epsilon-1}}\right)^{-z_2}e^{-\pi\eta v}dv \qquad (\dagger\dagger)
\end{aligned}$$

Taking out the exponentially small error in ($\dagger\dagger$) we have,

$$\begin{aligned}
&e\left(\tfrac{\eta^{\epsilon}}{2}\right)\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_0^{\infty}(2\eta^{\epsilon-1}+iv)^{\kappa+\alpha z_1}\left(\tfrac{-iv}{\eta^{\epsilon-1}}\right)^{-z_2}e^{-\pi\eta v}dvdz_2\\
&=e\left(\tfrac{\eta^{\epsilon}}{2}\right)\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_0^{\eta^{-1+\frac{1}{2}}}(2\eta^{\epsilon-1}+iv)^{\kappa+\alpha z_1}\left(\tfrac{-iv}{\eta^{\epsilon-1}}\right)^{-z_2}e^{-\pi\eta v}dvdz_2
\end{aligned}$$

Finally, once again, since ϕ vanishes to infinite order at 0, it's Mellin transform is holomorphic, and hence for any $M>0$, shifting the contour to $\Re(z_2)=-A_6$, we get

$$\begin{aligned}
&e\left(\tfrac{\eta^{\epsilon}}{2}\right)\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_0^{\eta^{-1+\frac{1}{2}}}(2\eta^{\epsilon-1}+iv)^{\kappa+\alpha z_1}\left(\tfrac{-iv}{\eta^{\epsilon-1}}\right)^{-z_2}e^{-\pi\eta v}dvdz_2\\
&\qquad=O_{\phi,A_6}(\eta^{\frac{-A_6\epsilon}{2}}) \qquad (4.46)
\end{aligned}$$

Now for any $M>0$, choosing $A_1=\cdots=A_4=2M$ and $A_5=A_6=\frac{2M}{\epsilon}$, and combining (4.35), (4.37), (4.39), (4.40), (4.44), (4.45) and (4.46) gives the asymptotic of B_2 as

$$B_2=e(-\eta)\sum_{l=l_g^+}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM}\frac{\alpha_l^+a_m(l)d_r(s)}{s!}\left(\frac{i}{\pi\eta}\right)^{m+r+s+1}\mathcal{T}_{\alpha,m+r+2s}(\Phi)\left(C^{\frac{1}{\alpha}}\eta\right)+O_{\Phi,M}(\eta^{-M})$$

Where $d_r(s) = 2^s d'_r(s)$. This finishes the proof of Lemma 4.4.8 when $\eta > 0$.

- $\eta < 0$.

  In this case we again use Mellin inversion, but this time instead of using the Cauchy integral formula on the upper half-plane, we use it on the lower half plane. Then, for example, the inner integral in (4.43) becomes

$$\begin{aligned}\int_0^{|\eta|^{\epsilon-1}} u^{\kappa+\alpha z} e\left(\tfrac{\eta u}{2}\right) du &= \int_0^{-i\infty} u^{\kappa+\alpha z} e\left(\tfrac{\eta u}{2}\right) du + \int_{|\eta|^{\epsilon-1}-i\infty}^{|\eta|^{\epsilon-1}} u^{\kappa+\alpha z} e\left(\tfrac{\eta u}{2}\right) du \\ &= \left(\tfrac{i}{\pi\eta}\right)^{\kappa+\alpha z+1} \Gamma(\kappa+\alpha z+1) - ie\left(\tfrac{-|\eta|^{\epsilon}}{2}\right) \int_{\infty}^{0} (|\eta|^{\epsilon-1} - iu)^{\kappa+\alpha z} e^{\pi\eta u} du\end{aligned}$$

  We then carry the rest of the analysis exactly the same as the case for $\eta > 0$. The result stays the same.

□

We will now sketch the analysis for B_1.

Lemma 4.4.9. *For any $M > 0$, there exists constants $a_m(l), d_r(s)$ which depend only on the indicated parameters such that the following asymptotic expansions hold:*

$$B_1 = e(\eta) \sum_{l=l_g^-}^{2M} \sum_{m=l}^{2M} \sum_{s=0}^{2M} \sum_{r=0}^{2sM} \frac{(-1)^l \alpha_l^- a_m(l) d_r(s)}{s!} \left(\frac{-i}{\pi\eta}\right)^{m+r+s+1} \mathcal{T}_{\alpha,m+r+2s}(\Phi)\left(-C^{\frac{1}{\alpha}}\eta\right) + O_{\Phi,g,\phi,M}(\eta^{-M})$$

Where,

$$g(x) = \sum_{l_g^-}^{\infty} \alpha_l^- (x+1)^l$$

is the asymptotic expansion of $g(x)$ around $x = -1$, and for $1 > \alpha > 0$, $\beta > 0$ and $\kappa \in \mathbb{Z}_{\geq 0}$, $\mathcal{T}(\Phi)(z)$ is defined by

$$\mathcal{T}_{\alpha,\kappa}(\Phi)(y) := \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \mathcal{M}(\Phi)(z) \left(\frac{i}{\pi y}\right)^{\alpha z} \Gamma(\kappa+\alpha z+1) dz$$

where $\mathcal{M}$ denotes the Mellin transform

Proof. The proof goes through the same steps as in the proof of Lemma 4.4.8, and we will only indicate the necessary modifications. All the error bounds apply verbatim so we will ignore those throughout the proof.

Since the integral is now around -1, we use the asymptotic expansion around $x = -1$ given above. The substitution $u^{-1} = 1 - x^2$ stays the same with the modification that since $x < 0$ we now have $x = -\sqrt{1-u^{-1}}$ in the formulas starting with (4.36). With this modification (and noting that the limits of integration in (4.36) is reversed, which brings in an extra factor of -1) $a_m(l)$ gets changed to $(-1)^l a_m(l)$. For the rest of the analysis the only changes are that η becomes $-\eta$ and the contour shifts take place in the lower half plane rather than the upper one. Therefore we get,

$$B_1 = e(\eta) \sum_{l=l_g^-}^{2M} \sum_{m=l}^{2M} \sum_{s=0}^{2M} \sum_{r=0}^{2sM} \frac{(-1)^l \alpha_l^- a_m(l) d_r(s)}{s!} \left(\frac{-i}{\pi\eta}\right)^{m+r+s+1} \mathcal{T}_{\alpha,m+r+2s}(\Phi)\left(-C^{\frac{1}{\alpha}}\eta\right) \tag{4.47}$$

□

Combining lemmas 4.4.7, 4.4.8 and 4.4.9 finishes the proof of Proposition 4.4.6.

□

4.5 Asymptotic expansions of Fourier transforms - 2

This section is complementary to the previous section. We are concerned with the asymptotic behavior of the following type of Fourier transforms:

$$\int_{\mathbb{R}\setminus(-1,1)} h(x)\Phi\left(\tfrac{C}{(x^2-1)^\alpha}\right) e(-x\eta)dx \tag{4.48}$$

Where $\Phi(x)$ is again Schwarz class and $h(x)$ is smooth of compact support, but this time the integral is over $\mathbb{R}\setminus(-1,1)$. We again want the decay properties of (4.48) explicitly in terms of η and C. We will be very brief in this section as the methods and results are exactly the same as the ones of the previous section.

4.5.1 Warm-up I: $C > K$ for some constant K.

Proposition 4.5.1. *Let $C > K$ for some constant $K > 0$. Then for any $M > 0$*

$$\int_{\mathbb{R}\setminus(-1,1)} h(x)\Phi\left(\tfrac{C}{(x^2-1)^\alpha}\right) e(-x\eta)dx = O_{\Phi,h,\alpha,M,K}(\eta^{-M})$$

Proof. See the proof of Proposition 4.4.2. □

4.5.2 Warm-up II: $C < K$ for some constant K, and the support of $h(x)$ does not intersect $\{\pm 1\}$.

Proposition 4.5.2. *Assume that $C < K$ for some constant K, and the support of $h(x)$ does not intersect $\{\pm 1\}$. Then for every $M > 0$ we have*

$$\int_{\mathbb{R}\setminus(-1,1)} h(x)\Phi\left(\tfrac{C}{(x^2-1)^\alpha}\right) e(-x\eta)dx = O_{\Phi,h,\alpha,M,K}(\eta^{-M})$$

Proof. See the proof of Corollary 4.4.4. □

4.5.3 Support of $h(x)$ intersects $\{\pm 1\}$ non-trivially and $C \to 0$.

There exists $\delta > 0$ such that $\eta > C^{-\frac{1}{\alpha}-\delta}$.

Proposition 4.5.3. *Assume that there exists $\delta > 0$ such that $\eta > C^{-\frac{1}{\alpha}-\delta}$. Then for any $M > 0$ we have*

$$\int_{\mathbb{R}\setminus(-1,1)} h(x)\Phi\left(\tfrac{C}{(x^2-1)^\alpha}\right) e(-x\eta)dx = O_{\Phi,h,\alpha,M,\delta}(\eta^{-M})$$

Proof. See the proof of Proposition 4.4.5 □

$\eta C^{\frac{1}{\alpha}} < C^{-\delta}$ **for every** $\delta > 0$**.**

Proposition 4.5.4. *For any $M > 0$, there exists constants $a_m(l), d_r(s)$ (note that these are the same constants that appear in Proposition 4.4.6) which depend only on the indicated parameters such that the following asymptotic expansions hold:*

$$\begin{aligned}
&\int_{\mathbb{R}\setminus(-1,1)} h(x)\Phi\left(\tfrac{C}{(x^2-1)^\alpha}\right) e(-x\eta)dx \\
&= -e(-\eta)\sum_{l=l_h^+}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM} \tfrac{\alpha_l^+ a_m(l) d_r(s)}{s!}\left(\tfrac{i}{\pi\eta}\right)^{m+r+s+1}\mathcal{T}_{\alpha,m+r+2s}(\Phi)\left(-C^{\frac{1}{\alpha}}\eta\right) \\
&- e(\eta)\sum_{l=l_h^-}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM} \tfrac{(-1)^l\alpha_l^- a_m(l) d_r(s)}{s!}\left(\tfrac{-i}{\pi\eta}\right)^{m+r+s+1}\mathcal{T}_{\alpha,m+r+2s}(\Phi)\left(C^{\frac{1}{\alpha}}\eta\right) \\
&+ O_{\Phi,g,M}(\eta^{-M})
\end{aligned}$$

Where,

$$h(x) = \sum_{l=l_h^\pm}^{\infty} \alpha_l^\pm (x\mp 1)^l$$

is the asymptotic expansion of $h(x)$ around ± 2, and for $1 > \alpha > 0$, $\beta > 0$ and $\kappa \in \mathbb{Z}_{\geq 0}$, $\mathcal{T}(\Phi)(z)$ is defined by

$$\mathcal{T}_{\alpha,\kappa}(\Phi)(y) := \tfrac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \mathcal{M}(\Phi)(z)\left(\tfrac{i}{\pi y}\right)^{\alpha z}\Gamma(\kappa+\alpha z+1)dz$$

where $\mathcal{M}$ denotes the Mellin transform.

Proof. The proof of Proposition 4.4.6 goes through and gives the desired asymptotic expansion. For completeness we present the necessary changes here, this is essentially a repetition of the proof of Proposition 4.4.6.

Note. Since essentially the same argument goes through for the analysis of $\eta > 0$ and $\eta < 0$ (The only change is at the very last part of analysis where we us the opposite half-plane while analyzing the integrals is (4.58), see the last paragraph of Lemma 4.4.8.) throughout the proof we will, without loss of generality, assume that $\eta > 0$.

We introduce the cut-off function as follows:

$$\tilde{\phi}(x) = \begin{cases} 1 & \text{if } 1 < |x| \leq \sqrt{1+\eta^{\epsilon-1}} \\ \phi\left(\frac{\sqrt{1+2\eta^{\epsilon-1}}-|x|}{\sqrt{1+2\eta^{\epsilon-1}}-\sqrt{1+\eta^{\epsilon-1}}}\right) & \text{if } \sqrt{1+\eta^{\epsilon-1}} < |x| < \sqrt{1+2\eta^{\epsilon-1}} \\ 0 & \text{if } \sqrt{1+2\eta^{\epsilon-2}} \leq |x| \end{cases}$$

Where $\phi(x)$ is the same function introduced in the proof of Proposition 4.4.6. Using this, we decompose the integral as

$$\begin{aligned}
&\int_{\mathbb{R}\setminus(-1,1)} h(x)\Phi\left(\tfrac{C}{(x^2-1)^\alpha}\right) e(-x\eta)dx \\
&= \int_{\mathbb{R}\setminus(-1,1)} h(x)(1-\phi(x))\Phi\left(\tfrac{C}{(x^2-1)^\alpha}\right) e(-x\eta)dx + \int_{\mathbb{R}\setminus(-1,1)} h(x)\phi(x)\Phi\left(\tfrac{C}{(x^2-1)^\alpha}\right) e(-x\eta)dx \\
&= A + B
\end{aligned}$$

A is bounded exactly in Lemma 4.4.7 and gives

$$A = O_{\Phi,h,\alpha,\phi,M}\left(C\eta^{\alpha(1-\epsilon)}\eta^{-M}\right) \tag{4.49}$$

Moving on to B, we break it into two as

$$\begin{aligned} B &= \int_{\mathbb{R}\setminus(-1,1)} h(x)\phi(x)\Phi\left(\tfrac{C}{(x^2-1)^\alpha}\right)e(-x\eta)dx \\ &= \int_{-\infty}^{-1} h(x)\phi(x)\Phi\left(\tfrac{C}{(x^2-1)^\alpha}\right)e(-x\eta)dx + \int_1^\infty h(x)\phi(x)\Phi\left(\tfrac{C}{(x^2-1)^\alpha}\right)e(-x\eta)dx \\ &= B_1 + B_2 \end{aligned}$$

We analyze B_2 exactly as in the proof of Proposition 4.4.6. Expand $h(x)$ around $x = 1$,

$$\sum_{l=l_h^+}^{\infty} \alpha_l^+(x-1)^l$$

Substituting this into B_2 we get

$$\sum_{l=l_h^+}^{\infty} \alpha_l^+ \int_1^\infty (x-1)^l\tilde{\phi}(x)\Phi\left(\tfrac{C}{(x^2-1)^\alpha}\right)e(-x\eta)dx \tag{4.50}$$

Since the support of $\tilde{\phi}(x)$ is in $(1, \sqrt{1+2\eta^{\epsilon-1}})$, for any $A_0 > 0$ we have

$$\sum_{l=A_0+1}^{\infty} \alpha_l^+ \int_1^\infty (x-1)^l\tilde{\phi}(x)\Phi\left(\tfrac{C}{(x^2-1)^\alpha}\right)e(-x\eta)dx = O_{h,A_0}(\eta^{(\epsilon-1)(A_0+1)}) \tag{4.51}$$

Let $u^{-1} = x^2 - 1$. Then the integral becomes,

$$\sum_{l=l_h^+}^{A_0} \alpha_l^+ \int_0^\infty \tfrac{\left(\sqrt{u^{-1}+1}-1\right)^l}{2u^2\sqrt{u^{-1}+1}}\tilde{\phi}\left(\sqrt{u^{-1}+1}\right)\Phi\left(Cu^\alpha\right)e\left(-\eta\sqrt{u^{-1}+1}\right)du \tag{4.52}$$

Next, note that for any $l \geq 0$,

$$\tfrac{(\sqrt{u^{-1}+1}-1)^l}{2u^2\sqrt{u^{-1}+1}} = \tfrac{-1}{l+1}\tfrac{d}{du}(\sqrt{u^{-1}+1}-1)^{l+1}$$

and the Taylor expansion of $\sqrt{x+1}-1$ around $x = 0$ is

$$\sqrt{x+1}-1 = \sum_{m=1}^{\infty}\binom{1/2}{m}x^m$$

Using these we get

$$\begin{aligned} \tfrac{(\sqrt{u^{-1}+1}-1)^l}{2u^2\sqrt{u^{-1}+1}} &= \tfrac{-1}{l+1}\tfrac{d}{du}\left[\sum_{m=1}^{\infty}\binom{1/2}{m}u^{-m}\right]^{l+1} \\ &= -\left[\sum_{m=1}^{\infty}\binom{1/2}{m}u^{-m}\right]^l\left[\sum_{m=1}^{\infty}\binom{1/2}{m}(-m)u^{-m-1}\right] \\ &= \tfrac{1}{u^2}\sum_{m=l}^{\infty}(-1)^m a_m(l)u^{-m} \end{aligned} \tag{†}$$

Where the coefficients $a_m(l)$ are the same coefficients that appear in Lemma 4.4.8. Substituting (†) into (4.52) gives

$$\sum_{l=l_h^+}^{A_0}\sum_{m=l}^{\infty}\alpha_l^+(-1)^m a_m(l)\int_0^\infty u^{-m-2}\tilde{\phi}\left(\sqrt{u^{-1}+1}\right)\Phi\left(Cu^\alpha\right)e\left(-\eta\sqrt{u^{-1}+1}\right)du$$

Once again, since the support of $\tilde{\phi}(x)$ is in $(1,\sqrt{1+2\eta^{\epsilon-1}})$, the support of $\tilde{\phi}\left(\sqrt{u^{-1}+1}\right)$ is in $(\frac{\eta^{1-\epsilon}}{2},\infty)$. Therefore for any $A_1 \geq A_0$ we have,

$$\sum_{l=l_h^+}^{A_0}\sum_{m=A_1+1}^{\infty}\alpha_l^+(-1)^m a_m(l)\int_0^\infty u^{-m-2}\tilde{\phi}\left(\sqrt{u^{-1}+1}\right)\Phi\left(Cu^\alpha\right)e\left(-\eta\sqrt{u^{-1}+1}\right)du$$
$$= O_{h,A_0,A_1}(\eta^{(\epsilon-1)(A_1+2)}) \quad (4.53)$$

Next, we analyze the exponential. By the binomial theorem we have

$$\sqrt{u^{-1}+1} = \sum_{r=0}^{\infty}\binom{1/2}{r}u^{-r}$$

Substituting this into the exponential we get

$$e\left(-\eta\sqrt{u^{-1}+1}\right) = e\left(-\eta\sum_{r=0}^{\infty}\binom{1/2}{r}u^{-2r}\right)$$
$$= e\left(-\eta-\tfrac{\eta u^{-1}}{2}\right)e\left(-\eta u^{-2}\sum_{r=2}^{\infty}\binom{1/2}{r}u^{-r+2}\right)$$

For notational convenience and consistence with Proposition (4.4.6), let $b_r = \binom{1/2}{r}(-1)^r$. Then,

$$e\left(-\eta\sqrt{u^{-1}+1}\right) = e\left(-\eta-\tfrac{\eta u^{-1}}{2}\right)e\left(-\eta u^{-2}\sum_{r=2}^{\infty}b_r(-u)^{-r+2}\right)$$

By the Taylor expansion of the exponential we get

$$e\left(-\eta u^{-2}\sum_{r=2}^{\infty}b_r(-u)^{-r+2}\right) = \sum_{s=0}^{\infty}\tfrac{1}{s!}\left[-2\pi i\eta u^{-2}\sum_{r=2}^{\infty}b_r(-u)^{-r+2}\right]^s$$
$$= \sum_{s=0}^{\infty}\tfrac{(-2\pi i\eta u^{-2})^s}{s!}\left[\sum_{r=2}^{\infty}b_r(-u)^{-r+2}\right]^s$$

Substituting this back into the integral in (4.52) we get

$$\sum_{l=l_h^+}^{A_0}\sum_{m=l}^{A_1}\alpha_l^+(-1)^m a_m(l)\int_0^\infty u^{-m-2}\tilde{\phi}\left(\sqrt{u^{-1}+1}\right)\Phi\left(Cu^\alpha\right)$$
$$e\left(-\eta-\tfrac{\eta u^{-1}}{2}\right)\sum_{s=0}^{\infty}\tfrac{(-2\pi i\eta u^{-2})^s}{s!}\left[\sum_{r=2}^{\infty}b_r(-u)^{-r+2}\right]^s du \quad (4.54)$$

Since the support of $\tilde{\phi}$ limits the lower limit of integration to $u > \frac{\eta^{1-\epsilon}}{2}$, for any $A_2 > 0$ we have

$$\sum_{l=l_h^+}^{A_0}\sum_{m=l}^{A_1}\alpha_l^+(-1)^m a_m(l)\int_0^\infty u^{-m-2}\tilde{\phi}\left(\sqrt{u^{-1}+1}\right)\Phi\left(Cu^\alpha\right)e\left(-\eta-\tfrac{\eta u^{-1}}{2}\right)$$
$$\sum_{s=A_2+1}^{\infty}\tfrac{(-2\pi i\eta u^{-2})^s}{s!}\left[\sum_{r=2}^{\infty}b_r(-u)^{-r+2}\right]^s du = O_{h,A_0,A_1}(\eta^{-A_2-2-l_{g,1}}\eta^{\epsilon(l_{g,1}+2A_2+3)}) \quad (4.55)$$

Similarly for any $A_3 \geq 2$ we have

$$e(-\eta)\sum_{l=l_h}^{A_0}\sum_{m=l}^{A_1}\sum_{s=1}^{A_2}\tfrac{\alpha_l^+(-1)^m a_m(l)(-2\pi i\eta)^s}{s!}\int_0^\infty u^{-m-2s-2}\tilde{\phi}\left(\sqrt{u^{-1}+1}\right)\Phi\left(Cu^\alpha\right)e\left(\tfrac{-\eta u^{-1}}{2}\right)$$
$$\left[\sum_{r=A_3+1}^{\infty}b_r(-u)^{-r+2}\right]^s du = O_{h,A_0,A_1,A_2}(\eta^{-A_3-l_{g,1}-1}\eta^{\epsilon(A_3+l_{g,1}+2)}) \quad (4.56)$$

Therefore we are left with

$$e(-\eta)\sum_{l=l_h}^{A_0}\sum_{m=l}^{A_1}\sum_{s=0}^{A_2}\sum_{r=0}^{sA_3}\tfrac{\alpha_l^+(-1)^{m+r}a_m(l)d_r'(s)(-2\pi i\eta)^s}{s!}$$
$$\int_0^\infty u^{-m-r-2s-2}\tilde{\phi}\left(\sqrt{u^{-1}+1}\right)\Phi\left(Cu^\alpha\right)e\left(\tfrac{-\eta u^{-1}}{2}\right)du \quad (4.57)$$

Where $d_r'(s)$ are the same constants that appear in Lemma 4.4.8. We now need to analyze the asymptotic behavior of integrals of the form

$$\int_0^\infty u^{-\kappa-2}\tilde{\phi}\left(\sqrt{u^{-1}+1}\right)\Phi\left(Cu^\alpha\right)e\left(\tfrac{-\eta u^{-1}}{2}\right)du$$

for integers $\kappa \geq l_{h,1}$. Let $u \mapsto u^{-1}$. Then we have

$$\int_0^\infty u^\kappa\tilde{\phi}\left(\sqrt{1+u}\right)\Phi\left(\tfrac{C}{u^\alpha}\right)e\left(\tfrac{-\eta u}{2}\right)du \quad (4.58)$$

Recall that the support of $\tilde{\phi}\left(\sqrt{1+u}\right)$ is $u < 2\eta^{\epsilon-1}$ and for $u < \eta^{\epsilon-1}$, $\tilde{\phi}\left(\sqrt{1+u}\right) = 1$. Therefore,

$$(4.57) = \int_0^{\eta^{\epsilon-1}}u^\kappa\Phi\left(\tfrac{C}{u^\alpha}\right)e\left(\tfrac{-\eta u}{2}\right)du + \int_{\eta^{\epsilon-1}}^{2\eta^{\epsilon-1}}u^\kappa\tilde{\phi}\left(\sqrt{1+u}\right)\Phi\left(\tfrac{C}{u^\alpha}\right)e\left(\tfrac{-\eta u}{2}\right)du$$

We will analyze these two integrals one by one.

- $\int_0^{\eta^{\epsilon-1}}u^\kappa\Phi\left(Cu^{-\alpha}\right)e\left(\frac{-\eta u}{2}\right)du$.

  Let $\mathcal{M}(\Phi)(z)$ denote the Mellin transform of Φ. Then by the Mellin inversion formula for some $\beta > 0$, we have

  $$\int_0^{\eta^{\epsilon-1}}u^\kappa\Phi\left(\tfrac{C}{u^\alpha}\right)e\left(\tfrac{-\eta u}{2}\right)du = \int_0^{\eta^{\epsilon-1}}u^\kappa\left\{\tfrac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\mathcal{M}(\Phi)(z)\left(Cu^{-\alpha}\right)^{-z}dz\right\}e\left(\tfrac{xiu}{2}\right)du$$

  Interchanging the integrals we get

  $$\tfrac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\mathcal{M}(\Phi)(z)C^{-z}\left\{\int_0^{\eta^{\epsilon-1}}u^{\kappa+\alpha z}e\left(\tfrac{-\eta u}{2}\right)du\right\}dz \quad (4.59)$$

We treat the inner integral separately by treating it as a complex function. By the Cauchy integral formula, we can write the integral as the sum of two integrals on the vertical lines $0 \to -i\infty$ and $\eta^{\epsilon-1} - i\infty \to \eta^{\epsilon-1}$.

$$\begin{aligned}\int_0^{\eta^{\epsilon-1}} u^{\kappa+\alpha z} e\left(\tfrac{-\eta u}{2}\right) du &= \int_0^{-i\infty} u^{\kappa+\alpha z} e\left(\tfrac{-\eta u}{2}\right) du + \int_{\eta^{\epsilon-1}-i\infty}^{\eta^{\epsilon-1}} u^{\kappa+\alpha z} e\left(\tfrac{-\eta u}{2}\right) du \\ &= \int_0^{-i\infty} u^{\kappa+\alpha s} e\left(\tfrac{-\eta u}{2}\right) du + e\left(\tfrac{-\eta^{\epsilon}}{2}\right) \int_{-i\infty}^{0} (\eta^{\epsilon-1}+u)^{\kappa+\alpha s} e\left(\tfrac{-\eta u}{2}\right) du \\ &= \left(\tfrac{-i}{\pi\eta}\right)^{\kappa+\alpha z+1} \int_0^{\infty} u^{\kappa+\alpha z} e^{-u} du \\ &\quad - ie\left(\tfrac{\eta^{\epsilon}}{2}\right) \int_{\infty}^{0} (\eta^{\epsilon-1} - iu)^{\kappa+\alpha z} e^{-\pi\eta u} du \qquad (\circ)\end{aligned}$$

In the next paragraph we will show that $(\circ)$ cancels $(\circ\circ)$. Ignoring that for now, we have

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \mathcal{M}(\Phi)(z) C^{-z} \left(\tfrac{-i}{\pi\eta}\right)^{\kappa+\alpha z+1} \int_0^{\infty} u^{\kappa+\alpha z} e^{-u} du dz \\ &= \left(\tfrac{-i}{\pi\eta}\right)^{\kappa+1} \tfrac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \mathcal{M}(\Phi)(z) \left(\frac{-i}{\pi C^{\frac{1}{\alpha}}\eta}\right)^{\alpha z} \Gamma(\kappa+\alpha z+1) dz \\ &= \left(\tfrac{-i}{\pi\eta}\right)^{\kappa+1} \mathcal{T}_{\alpha,\kappa}(\Phi)(-C^{\frac{1}{\alpha}}\eta) \qquad (4.60)\end{aligned}$$

This finishes this part of the analysis.

- $\int_{\eta^{\epsilon-1}}^{2\eta^{\epsilon-1}} u^{\kappa} \tilde{\phi}\left(\sqrt{1+u}\right) \Phi\left(\frac{C}{u^{\alpha}}\right) e\left(\frac{-\eta u}{2}\right) du.$

As before, using the Mellin transform of F we get that the integral is equal to

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{\beta_1-i\infty}^{\beta_1+i\infty} \mathcal{M}(\Phi)(z_1) C^{-z_1} \left\{ \int_{\eta^{\epsilon-1}}^{2\eta^{\epsilon-1}} u^{\kappa+\alpha z_1} \tilde{\phi}\left(\sqrt{1+u}\right) e\left(\tfrac{-\eta u}{2}\right) du \right\} dz_1 \\ &= \tfrac{1}{2\pi i}\int_{\beta_1-i\infty}^{\beta_1+i\infty} \mathcal{M}(\Phi)(z_1) C^{-z_1} \left\{ \int_{\eta^{\epsilon-1}}^{2\eta^{\epsilon-1}} u^{\kappa+\alpha z_1} \phi\left(\frac{\sqrt{1+2\eta^{\epsilon-1}}-\sqrt{1+u}}{\sqrt{1+2\eta^{\epsilon-1}}-\sqrt{1+\eta^{\epsilon-1}}}\right) e\left(\tfrac{-\eta u}{2}\right) du \right\} dz_1\end{aligned}$$

Consider the inner integral,

$$\int_{\eta^{\epsilon-1}}^{2\eta^{\epsilon-1}} u^{\kappa+\alpha z_1} \phi\left(\frac{\sqrt{1+2\eta^{\epsilon-1}}-\sqrt{1+u}}{\sqrt{1+2\eta^{\epsilon-1}}-\sqrt{1+\eta^{\epsilon-1}}}\right) e\left(\tfrac{-\eta u}{2}\right) du$$

Using Mellin inversion once again, this time on ϕ,

$$\tfrac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty} \mathcal{M}(\phi)(z_2) \int_{\eta^{\epsilon-1}}^{\eta^{2\epsilon-1}} u^{\kappa+\alpha z_1} \left(\frac{\sqrt{1+2\eta^{\epsilon-1}}-\sqrt{1+u}}{\sqrt{1+2\eta^{\epsilon-1}}-\sqrt{1+\eta^{\epsilon-1}}}\right)^{-z_2} e\left(\tfrac{-\eta u}{2}\right) du dz_2$$

Note that,

$$\frac{\sqrt{1+2\eta^{\epsilon-1}}-\sqrt{1+u}}{\sqrt{1+2\eta^{\epsilon-1}}-\sqrt{1+\eta^{\epsilon-1}}} = 2 - \tfrac{u}{\eta^{\epsilon-1}} + O\left(\tfrac{u^2+\eta^{2(\epsilon-1)}}{\eta^{\epsilon-1}}\right)$$

Also, since $\phi(x)$ decays faster than any polynomial at 0 it's Mellin transform is holomorphic. Therefore, shifting the contour to $\Re(z_2) = -A_5$ we see that

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty} \mathcal{M}(\phi)(z_2) \left\{ \int_{\eta^{\epsilon-1}}^{2\eta^{\epsilon-1}} u^{\kappa+\alpha z_1} \left(\tfrac{u^2+\eta^{2(\epsilon-1)}}{\eta^{\epsilon-1}}\right)^{-z_2} e\left(\tfrac{-\eta u}{2}\right) du \right\} dz_2 \\ &= O_{\phi,A_5}(\eta^{A_5(\epsilon-1)}) \qquad (4.61)\end{aligned}$$

Using this, we are left with analyzing

$$\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_{\eta^{\epsilon-1}}^{2\eta^{\epsilon-1}}u^{\kappa+\alpha z_1}\left(2-\frac{u}{\eta^{\epsilon-1}}\right)^{-z_2}e\left(\frac{-\eta u}{2}\right)dudz_2$$

Letting $u=\eta^{\epsilon-1}+v$ gives

$$\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_{0}^{\eta^{\epsilon-1}}(\eta^{\epsilon-1}+v)^{\kappa+\alpha z_1}\left(1-\frac{v}{\eta^{\epsilon-1}}\right)^{-z_2}e\left(\frac{-\eta(\eta^{\epsilon-1}+v)}{2}\right)dvdz_2$$

We again shift the contours in the integral so that it becomes an integral over the vertical lines $0\to -i\infty$ and $\eta^{\epsilon-1}-i\infty\to\eta^{\epsilon-1}$.

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_{0}^{-i\infty}(\eta^{\epsilon-1}+v)^{\kappa+\alpha z_1}\left(1-\frac{v}{\eta^{\epsilon-1}}\right)^{-z_2}e\left(\frac{-\eta(\eta^{\epsilon-1}+v)}{2}\right)dvdz_2\\&+\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_{\eta^{\epsilon-1}}^{\eta^{\epsilon-1}-i\infty}(\eta^{\epsilon-1}+v)^{\kappa+\alpha z_1}\left(1-\frac{v}{\eta^{\epsilon-1}}\right)^{-z_2}e\left(\frac{-\eta(\eta^{\epsilon-1}+v)}{2}\right)dvdz_2\end{aligned}$$

By the binomial theorem we have

$$\left(1-\frac{v}{\eta^{\epsilon-1}}\right)^{-z_2}=1+\sum_{f=1}^{\infty}\binom{-z_2}{f}\left(\frac{v}{\eta^{\epsilon-1}}\right)^{f}$$

Substituting this back into the first integral (and using the normalization that $\int\phi=1$) we get

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_{0}^{-i\infty}(\eta^{\epsilon-1}+v)^{\kappa+\alpha z_1}e\left(\frac{-\eta(\eta^{\epsilon-1}+v)}{2}\right)dvdz_2\\&=-ie\left(\frac{-\eta^{\epsilon}}{2}\right)\int_{0}^{\infty}(\eta^{\epsilon-1}-iu)^{\kappa+\alpha z}e^{-\pi\eta u}du\end{aligned}\qquad(\circ\circ)$$

Hence this integral cancels the second integral in $(\circ)$. Therefore we are left with

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\sum_{f=1}^{\infty}\binom{-z_2}{f}\int_{0}^{-i\infty}(\eta^{\epsilon-1}+v)^{\kappa+\alpha z_1}\left(\frac{-v}{\eta^{\epsilon-1}}\right)^{f}e\left(\frac{-\eta(\eta^{\epsilon-1}+v)}{2}\right)dvdz_2\\&+\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_{\eta^{\epsilon-1}}^{\eta^{\epsilon-1}-i\infty}(\eta^{\epsilon-1}+v)^{\kappa+\alpha z_1}\left(1-\frac{v}{\eta^{\epsilon-1}}\right)^{-z_2}e\left(\frac{-\eta(\eta^{\epsilon-1}+v)}{2}\right)dvdz_2\\&=\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\sum_{f=1}^{\infty}\binom{-z_2}{f}\int_{0}^{-i\infty}(\eta^{\epsilon-1}+v)^{\kappa+\alpha z_1}\left(\frac{-v}{\eta^{\epsilon-1}}\right)^{f}e\left(\frac{-\eta(\eta^{\epsilon-1}+v)}{2}\right)dvdz_2\\&+\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_{0}^{-i\infty}(2\eta^{\epsilon-1}+v)^{\kappa+\alpha z_1}\left(\frac{-v}{\eta^{\epsilon-1}}\right)^{-z_2}e\left(\frac{-\eta(2\eta^{\epsilon-1}+v)}{2}\right)dvdz_2\\&=-ie\left(\frac{\eta^{\epsilon}}{2}\right)\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\sum_{f=1}^{\infty}\binom{-z_2}{f}\int_{0}^{\infty}(\eta^{\epsilon-1}-iv)^{\kappa+\alpha z_1}\left(\frac{iv}{\eta^{\epsilon-1}}\right)^{f}e^{-\pi\eta v}dvdz_2\\&-ie\left(\frac{\eta^{\epsilon}}{2}\right)\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_{0}^{\infty}(2\eta^{\epsilon-1}-iv)^{\kappa+\alpha z_1}\left(\frac{iv}{\eta^{\epsilon-1}}\right)^{-z_2}e^{-\pi\eta v}dvdz_2\end{aligned}$$

Note that if $v > \eta^{-1+\frac{\epsilon}{2}}$, then $e^{-\pi\eta v} = e^{-\pi\eta(v-\eta^{-1+\frac{\epsilon}{2}}+\eta^{-1+\frac{\epsilon}{2}})} = e^{-\pi\eta(v-\eta^{-1+\frac{\epsilon}{2}})}e^{-\pi\eta^{-\frac{\epsilon}{2}}}$. Therefore, we have

$$\begin{aligned}
&\int_0^\infty (2\eta^{\epsilon-1} - iv)^{\kappa+\alpha z_1}\left(\tfrac{iv}{\eta^{\epsilon-1}}\right)^{-z_2} e^{-\pi\eta v}dv \\
&= \int_{\eta^{-1+\frac{\epsilon}{2}}}^\infty (2\eta^{\epsilon-1} - iv)^{\kappa+\alpha z_1}\left(\tfrac{iv}{\eta^{\epsilon-1}}\right)^{-z_2} e^{-\pi\eta v}dv \\
&+ \int_0^{\eta^{-1+\frac{\epsilon}{2}}} (2\eta^{\epsilon-1} - iv)^{\kappa+\alpha z_1}\left(\tfrac{iv}{\eta^{\epsilon-1}}\right)^{-z_2} e^{-\pi\eta v}dv \\
&= O(e^{-\eta^{\frac{\epsilon}{2}}}) + \int_0^{\eta^{-1+\frac{\epsilon}{2}}} (2\eta^{\epsilon-1} - iv)^{\kappa+\alpha z_1}\left(\tfrac{iv}{\eta^{\epsilon-1}}\right)^{-z_2} e^{-\pi\eta v}dv \qquad (\dagger\dagger)
\end{aligned}$$

Taking out the exponentially small error in (††) we have,

$$\begin{aligned}
&e\left(\tfrac{\eta^\epsilon}{2}\right)\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_0^\infty (2\eta^{\epsilon-1} - iv)^{\kappa+\alpha z_1}\left(\tfrac{iv}{\eta^{\epsilon-1}}\right)^{-z_2} e^{-\pi\eta v}dvdz_2 \\
&= e\left(\tfrac{\eta^\epsilon}{2}\right)\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_0^{\eta^{-1+\frac{1}{2}}} (2\eta^{\epsilon-1} - iv)^{\kappa+\alpha z_1}\left(\tfrac{iv}{\eta^{\epsilon-1}}\right)^{-z_2} e^{-\pi\eta v}dvdz_2
\end{aligned}$$

Finally, once again, since ϕ vanishes to infinite order at 0, it's Mellin transform is holomorphic, and hence for any $M > 0$, shifting the contour to $\Re(z_2) = -A_6$, we get

$$\begin{aligned}
&e\left(\tfrac{\eta^\epsilon}{2}\right)\frac{1}{2\pi i}\int_{\beta_2-i\infty}^{\beta_2+i\infty}\mathcal{M}(\phi)(z_2)\int_0^{\eta^{-1+\frac{1}{2}}} (2\eta^{\epsilon-1} - iv)^{\kappa+\alpha z_1}\left(\tfrac{iv}{\eta^{\epsilon-1}}\right)^{-z_2} e^{-\pi\eta v}dvdz_2 \\
&= O_{\phi,A_6}(\eta^{\frac{-A_6\epsilon}{2}}) \qquad (4.62)
\end{aligned}$$

Now for any $M > 0$, choosing $A_1 = \cdots = A_4 = 2M$ and $A_5 = A_6 = \frac{2M}{\epsilon}$, and combining (4.51), (4.53), (4.55), (4.56), (4.60), (4.61) and (4.62) gives the asymptotic of B_2 as

$$B_2 = -e(-\eta)\sum_{l=l_h^+}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM}\frac{\alpha_l^+ a_m(l)d_r(s)}{s!}\left(\tfrac{i}{\pi\eta}\right)^{m+r+s+1}\mathcal{T}_{\alpha,m+r+2s}(\Phi)\left(-C^{\frac{1}{\alpha}}\eta\right) + O_{\Phi,M}(\eta^{-M})$$

Where $d_r(s) = 2^s d_r'(s)$ (note that the power of $(-1)^{m+r}$ cancels, it is not a typo that it is $\left(\frac{i}{\pi\eta}\right)$). Finally, the asymptotic expansion of B_1 is obtained by following the same steps as

$$B_1 = -e(\eta)\sum_{l=l_h^-}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM}\frac{(-1)^l\alpha_l^- a_m(l)d_r(s)}{s!}\left(\tfrac{-i}{\pi\eta}\right)^{m+r+s+1}\mathcal{T}_{\alpha,m+r+2s}(\Phi)\left(C^{\frac{1}{\alpha}}\eta\right) + O_{\Phi,M}(\eta^{-M})$$

The proposition follows.

□

4.6 Asymptotic expansion of Fourier transforms - 3

In this section we will state an asymptotic expansion for the integrals of the form

$$\int_{-1}^{1}(1-x^2)^{\frac{\varrho}{2}}g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)e(-x\eta)dx \qquad (4.63)$$

Where $g(x)$ is a smooth function and ρ is an integer. The methods are the same as the previous sections. To carry the integration by parts arguments, one just needs to note that the function $(1-x^2)^{\frac{\rho}{2}}\Phi\left(\frac{C}{(1-x^2)^\alpha}\right)$ is smooth and its derivatives are dominated by the derivatives of $\Phi\left(\frac{C}{(1-x^2)^\alpha}\right)$.

4.6.1 Warm-up I: $C > K$ for some constant K.

Proposition 4.6.1. *Let $C > K$ for some constant $K > 0$. Then for any $M > 0$*

$$\int_{-1}^{1}(1-x^2)^{\frac{\rho}{2}}g(x)\Phi\left(\frac{C}{(1-x^2)^\alpha}\right)e(-x\eta)dx = O_{\Phi,g,\alpha,M,K,\rho}(\eta^{-M})$$

Proof. See the proof of Proposition 4.4.2. □

4.6.2 Warm-up II: $C < K$ for some constant K, and the support of $g(x)$ does not intersect $\{\pm 1\}$.

Proposition 4.6.2. *Assume that $C < K$ for some constant K, and the support of $g(x)$ does not intersect $\{\pm 1\}$. Then for every $M > 0$ we have*

$$\int_{-1}^{1}(1-x^2)^{\frac{\rho}{2}}g(x)\Phi\left(\frac{C}{(1-x^2)^\alpha}\right)e(-x\eta)dx = O_{\Phi,g,\alpha,M,K,\rho}(\eta^{-M})$$

Proof. See the proof of Corollary 4.4.4. □

4.6.3 Support of $g(x)$ intersects $\{\pm 1\}$ non-trivially and $C \to 0$.

There exists $\delta > 0$ such that $\eta > C^{-\frac{1}{\alpha}-\delta}$.

Proposition 4.6.3. *Assume that there exists $\delta > 0$ such that $\eta > C^{-\frac{1}{\alpha}-\delta}$. Then for any $M > 0$ we have*

$$\int_{-1}^{1}(1-x^2)^{\frac{\rho}{2}}g(x)\Phi\left(\frac{C}{(1-x^2)^\alpha}\right)e(-x\eta)dx = O_{\Phi,g,\alpha,M,\delta,\rho}(\eta^{-M})$$

Proof. See the proof of Proposition 4.4.5 □

$\eta C^{\frac{1}{\alpha}} < C^{-\delta}$ for every $\delta > 0$.

Proposition 4.6.4. *For any $M > 0$, there exists constants $a_m(l), d_r(s)$ (note that these are the same constants that appear in Proposition 4.4.6) which depend only on the indicated parameters* *such that the following asymptotic expansions hold:*

$$\begin{aligned}
&\int_{-1}^{1}(1-x^2)^{\frac{\rho}{2}}g(x)\Phi\left(\tfrac{C}{(1-x^2)^\alpha}\right)e(-x\eta)dx\\
&= e(-\eta)\sum_{l=l_g^+}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM}\frac{\beta_l^+ a_m(l)d_r(s)}{s!}\left(\tfrac{i}{\pi\eta}\right)^{m+r+s+\frac{\rho+2}{2}}\mathcal{T}_{\alpha,m+r+2s+\frac{\rho}{2}}(\Phi)\left(C^{\frac{1}{\alpha}}\eta\right)\\
&+ e(\eta)\sum_{l=l_g^-}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM}\frac{(-1)^l\beta_l^- a_m(l)d_r(s)}{s!}\left(\tfrac{-i}{\pi\eta}\right)^{m+r+s+\frac{\rho+2}{2}}\mathcal{T}_{\alpha,m+r+2s+\frac{\rho}{2}}(\Phi)\left(-C^{\frac{1}{\alpha}}\eta\right)\\
&+ O_{\Phi,g,M,\rho}(\eta^{-M})
\end{aligned}$$

Where,

$$(1-x^2)^{\frac{\rho}{2}}g(x) = (1-x^2)^{\frac{\rho}{2}}\sum_{l=l_g^\pm}^{\infty}\beta_l^\pm(x\mp 1)^l$$

is the asymptotic expansion of $(1-x^2)^{\frac{\rho}{2}}g(x)$ *around* ± 1*, and for* $1>\alpha>0$*,* $\beta>0$ *and* $\kappa\in\mathbb{Z}_{\geq 0}$*,* $\mathcal{T}(\Phi)(z)$ *is defined by*

$$\mathcal{T}_{\alpha,\kappa}(\Phi)(y) := \tfrac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\mathcal{M}(\Phi)(z)\left(\tfrac{i}{\pi y}\right)^{\alpha z}\Gamma(\kappa+\alpha z+1)dz$$

where $\mathcal{M}$ *denotes the Mellin transform.*

Proof. The proof follows the same lines as the proof of Proposition 4.4.6. With the difference that since we have an extra factor of $(x^2-1)^{\frac{\rho}{2}}$ in the numerator, (†) becomes

$$\tfrac{1}{u^{2+\frac{\rho}{2}}}\sum_{m=l}^{\infty}a_m(l)u^{-m}$$

which brings the shift by $\frac{\rho}{2}$.

□

4.7 Asymptotic expansion of Fourier transforms - 4

In this section we will state an asymptotic expansion for the integrals of the form

$$\int_{\mathbb{R}\setminus(-1,1)}(x^2-1)^{\frac{\rho}{2}}h(x)\Phi\left(\tfrac{C}{(x^2-1)^\alpha}\right)e(-x\eta)dx \tag{4.64}$$

Where $h(x)$ is a smooth function of compact support and the integral is taken outside of $(-1,1)$. The methods are the same as the previous sections. To carry the integration by parts arguments, one just needs to note that the function $(x^2-1)^{\frac{\rho}{2}}\Phi\left(\frac{C}{(x^2-1)^\alpha}\right)$ is smooth and its derivatives decay faster than the derivatives of $\Phi\left(\frac{C}{(x^2-1)^\alpha}\right)$.

4.7.1 Warm-up I: $C > K$ for some constant K.

Proposition 4.7.1. *Let $C > K$ for some constant $K > 0$. Then for any $M > 0$*

$$\int_{\mathbb{R}\setminus(-1,1)} (x^2-1)^{\frac{\rho}{2}} h(x)\Phi\left(\tfrac{C}{(x^2-1)^{\alpha}}\right) e(-x\eta)dx = O_{\Phi,h,\alpha,M,K,\rho}(\eta^{-M})$$

Proof. See the proof of Proposition 4.4.2. □

4.7.2 Warm-up II: $C < K$ for some constant K, and the support of $h(x)$ does not intersect $\{\pm 1\}$.

Proposition 4.7.2. *Assume that $C < K$ for some constant K, and the support of $h(x)$ does not intersect $\{\pm 1\}$. Then for every $M > 0$ we have*

$$\int_{\mathbb{R}\setminus(-1,1)} (x^2-1)^{\frac{\rho}{2}} h(x)\Phi\left(\tfrac{C}{(x^2-1)^{\alpha}}\right) e(-x\eta)dx = O_{\Phi,h,\alpha,M,K,\rho}(\eta^{-M})$$

Proof. See the proof of Corollary 4.4.4. □

4.7.3 Support of $h(x)$ intersects $\{\pm 1\}$ non-trivially and $C \to 0$.

There exists $\delta > 0$ such that $\eta > C^{-\frac{1}{\alpha}-\delta}$.

Proposition 4.7.3. *Assume that there exists $\delta > 0$ such that $\eta > C^{-\frac{1}{\alpha}-\delta}$. Then for any $M > 0$ we have*

$$\int_{\mathbb{R}\setminus(-1,1)} (x^2-1)^{\frac{\rho}{2}} h(x)\Phi\left(\tfrac{C}{(x^2-1)^{\alpha}}\right) e(-x\eta)dx = O_{\Phi,h,\alpha,M,\delta,\rho}(\eta^{-M})$$

Proof. See the proof of Proposition 4.4.5 □

$\eta C^{\frac{1}{\alpha}} < C^{-\delta}$ for every $\delta > 0$.

Proposition 4.7.4. *For any $M > 0$, there exists constants $a_m(l), d_r(s)$ (note that these are the same constants that appear in Proposition 4.4.6) which depend only on the indicated parameters such that the following asymptotic expansions hold:*

$$\begin{aligned}
&\int_{\mathbb{R}\setminus(-1,1)} (x^2-1)^{\frac{\rho}{2}} h(x)\Phi\left(\tfrac{C}{(x^2-1)^{\alpha}}\right) e(-x\eta)dx \\
&= -e(-\eta)\sum_{l=l_h^+}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM} \frac{\beta_l^+ a_m(l) d_r(s)}{s!}\left(\tfrac{i}{\pi\eta}\right)^{m+r+s+\frac{\rho+2}{2}} \mathcal{T}_{\alpha,m+r+2s+\frac{\rho}{2}}(\Phi)\left(-C^{\frac{1}{\alpha}}\eta\right) \\
&- e(\eta)\sum_{l=l_h^-}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM} \frac{\beta_l^-(-1)^l a_m(l) d_r(s)}{s!}\left(\tfrac{-i}{\pi\eta}\right)^{m+r+s+\frac{\rho+2}{2}} \mathcal{T}_{\alpha,m+r+2s+\frac{\rho}{2}}(\Phi)\left(C^{\frac{1}{\alpha}}\eta\right) \\
&+ O_{\Phi,h,M,\rho}(\eta^{-M})
\end{aligned}$$

Where,

$$(x^2 \pm 1)^{\frac{\varrho}{2}} h(x) = (x^2 \pm 1)^{\frac{\varrho}{2}} \sum_{l=l_h^\pm} \beta_l^\pm (x \mp 1)^l$$

is the asymptotic expansion of $(x^2 \pm 1)^{\frac{\varrho}{2}} g(x)$ *around* ± 1*, and for* $1 > \alpha > 0$*,* $\beta > 0$ *and* $\kappa \in \mathbb{Z}_{\geq 0}$*,* $\mathcal{T}(\Phi)(z)$ *is defined by*

$$\mathcal{T}_{\alpha,\kappa}(\Phi)(y) := \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \mathcal{M}(\Phi)(z) \left(\frac{i}{\pi y}\right)^{\alpha z} \Gamma(\kappa + \alpha z + 1) dz$$

where $\mathcal{M}$ *denotes the Mellin transform.*

Proof. The proof follows the same lines as the proof of Proposition 4.5.4 with the change of u^{-2} becoming $u^{-2-\frac{\varrho}{2}}$ in (†).

□

4.8 Asymptotic expansions of Fourier transforms of smoothed orbital integrals

In this section we collect the results of the previous sections and give the promised asymptotic expansion of the smoothed orbital integrals depending on the relevant parameters.

Recall the definitions,

$$\Theta_{\infty,\Phi,C,\alpha,-}(x) := \begin{cases} \theta_\infty(x)\Phi\left(\frac{C}{(4-x^2)^\alpha}\right) & \text{if } |x|<2 \\ 0 & \text{otherwise} \end{cases}$$
$$\Theta_{\infty,\Phi,C,\alpha,+}(x) := \begin{cases} \theta_\infty(x)\Phi\left(\frac{C}{(x^2-4)^\alpha}\right) & \text{if } |x|>2 \\ 0 & \text{otherwise} \end{cases}$$

Also recall that by (4.1) we have

$$\theta_\infty(x) = |4-x^2|^{\frac{1}{2}}\theta_{\infty,1}(x) + \theta_{\infty,2}(x)$$

Substituting this in the definition above and noting that $\theta_{\infty,1}$ is supported only in $|x|<2$ we have,

$$\Theta_{\infty,\Phi,C,\alpha,-}(x) := \begin{cases} \left\{(4-x^2)^{\frac{1}{2}}\theta_{\infty,1}(x) + \theta_{\infty,2}(x)\right\}\Phi\left(\frac{C}{(4-x^2)^\alpha}\right) & \text{if } |x|<2 \\ 0 & \text{otherwise} \end{cases}$$
$$\Theta_{\infty,\Phi,C,\alpha,+}(x) := \begin{cases} \theta_{\infty,2}(x)\Phi\left(\frac{C}{(x^2-4)^\alpha}\right) & \text{if } |x|>2 \\ 0 & \text{otherwise} \end{cases}$$

Using this we get,

Theorem 4.8.1. *For any $\Phi_1,\Phi_2 \in \mathcal{S}(\mathbb{R})$, and any $M>0$, there exists constants $a_m(l), d_r(s)$ which depend only on the indicated parameters such that the following asymptotic expansions hold:*

$$\begin{aligned}
&\widehat{\Theta}_{\infty,\Phi_1,C,\alpha,+}(\eta) + \widehat{\Theta}_{\infty,\Phi_2,C,\alpha,-}(\eta) \\
&= e(-\eta)\sum_{l=l_2^+(\theta)}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM}\frac{\alpha_l^+ a_m(l)d_r(s)}{s!}\left(\frac{i}{\pi\eta}\right)^{m+r+s+1}\left\{\mathcal{T}_{\alpha,m+r+2s}(\Phi_2)\left(C^{\frac{1}{\alpha}}\eta\right) - \mathcal{T}_{\alpha,m+r+2s}(\Phi_1)\left(-C^{\frac{1}{\alpha}}\eta\right)\right\} \\
&+ e(\eta)\sum_{l=l_2^-(\theta)}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM}\frac{\alpha_l^-(-1)^l a_m(l)d_r(s)}{s!}\left(\frac{-i}{\pi\eta}\right)^{m+r+s+1}\left\{\mathcal{T}_{\alpha,m+r+2s}(\Phi_2)\left(-C^{\frac{1}{\alpha}}\eta\right) - \mathcal{T}_{\alpha,m+r+2s}(\Phi_1)\left(C^{\frac{1}{\alpha}}\eta\right)\right\} \\
&+ e(-\eta)\sum_{l=l_1^+(\theta)}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM}\frac{\beta_l^+ a_m(l)d_r(s)}{s!}\left(\frac{i}{\pi\eta}\right)^{m+r+s+\frac{3}{2}}\mathcal{T}_{\alpha,m+r+2s+\frac{1}{2}}(\Phi_2)\left(C^{\frac{1}{\alpha}}\eta\right) \\
&+ e(\eta)\sum_{l=l_1^-(\theta)}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM}\frac{\beta_l^-(-1)^l a_m(l)d_r(s)}{s!}\left(\frac{-i}{\pi\eta}\right)^{m+r+s+\frac{3}{2}}\mathcal{T}_{\alpha,m+r+2s+\frac{1}{2}}(\Phi_2)\left(-C^{\frac{1}{\alpha}}\eta\right) \\
&+ O_{\Phi_1,\Phi_2,\theta_\infty,M}(\eta^{-M})
\end{aligned}$$

Where,

$$\theta_{\infty,2}(\pm 1 + x) = \sum_{l=l_2^{\pm}(\theta)}^{\infty} \alpha_l^{\pm} x^l$$

$$|1-(\pm 1 \mp x)^2|^{\frac{1}{2}} \theta_{\infty,1}(\pm 1 \mp x) = |1-(\pm 1 \mp x)^2|^{\frac{1}{2}} \sum_{l=l_1^{\pm}(\theta)}^{\infty} \beta_l^{\pm} x^l$$

are the asymptotic expansions around ± 1*, and for* $1 > \alpha > 0$*,* $\beta > 0$ *and* $\kappa \in \mathbb{Z}_{\geq 0}$*,* $\mathcal{T}(\Phi)(z)$ *is defined by*

$$\mathcal{T}_{\alpha,\kappa}(\Phi)(y) := \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \mathcal{M}(\Phi)(z) \left(\frac{i}{\pi y}\right)^{\alpha z} \Gamma(\kappa + \alpha z + 1) dz$$

where $\mathcal{M}$ *denotes the Mellin transform.*

Theorem 4.8.2. *For any* $\Phi_1, \Phi_2 \in \mathcal{S}(\mathbb{R})$*, and any* $M > 0$*, there exists constants* $a_m(l), d_r(s)$ *which depend only on the indicated parameters such that the following asymptotic expansions hold:*

$$\int_{-1}^{1} \left(\theta_{\infty,1}(x) + \frac{\theta_{\infty,2}(x)}{\sqrt{1-x^2}}\right) \Phi_1\left(\frac{C}{(1-x^2)^{\alpha}}\right) e(-x\eta) dx + \int_{\mathbb{R}\setminus(-1,1)} \frac{\theta_{\infty,2}(x)}{\sqrt{x^2-1}} \Phi_2\left(\frac{C}{(x^2-1)^{\alpha}}\right) e(-x\eta) dx$$

$$= e(-\eta) \sum_{l=l_1^{+}(\theta)}^{2M} \sum_{m=l}^{2M} \sum_{s=0}^{2M} \sum_{r=0}^{2sM} \frac{\beta_l^{+} a_m(l) d_r(s)}{s!} \left(\frac{i}{\pi\eta}\right)^{m+r+s+1} \mathcal{T}_{\alpha,m+r+2s}(\Phi_1)\left(C^{\frac{1}{\alpha}}\eta\right)$$

$$+ e(\eta) \sum_{l=l_1^{-}(\theta)}^{2M} \sum_{m=l}^{2M} \sum_{s=0}^{2M} \sum_{r=0}^{2sM} \frac{\beta_l^{-}(-1)^l a_m(l) d_r(s)}{s!} \left(\frac{-i}{\pi\eta}\right)^{m+r+s+1} \mathcal{T}_{\alpha,m+r+2s}(\Phi_1)\left(-C^{\frac{1}{\alpha}}\eta\right)$$

$$+ e(-\eta) \sum_{l=l_2^{+}(\theta)}^{2M} \sum_{m=l}^{2M} \sum_{s=0}^{2M} \sum_{r=0}^{2sM} \frac{\alpha_l^{+} a_m(l) d_r(s)}{s!} \left(\frac{i}{\pi\eta}\right)^{m+r+s+\frac{1}{2}} \left\{\mathcal{T}_{\alpha,m+r+2s-\frac{1}{2}}(\Phi_1)\left(C^{\frac{1}{\alpha}}\eta\right) - \mathcal{T}_{\alpha,m+r+2s-\frac{1}{2}}(\Phi_2)\left(-C^{\frac{1}{\alpha}}\eta\right)\right\}$$

$$+ e(\eta) \sum_{l=l_2^{-}(\theta)}^{2M} \sum_{m=l}^{2M} \sum_{s=0}^{2M} \sum_{r=0}^{2sM} \frac{\alpha_l^{-}(-1)^l a_m(l) d_r(s)}{s!} \left(\frac{-i}{\pi\eta}\right)^{m+r+s+\frac{1}{2}} \left\{\mathcal{T}_{\alpha,m+r+2s-\frac{1}{2}}(\Phi_1)\left(-C^{\frac{1}{\alpha}}\eta\right) - \mathcal{T}_{\alpha,m+r+2s-\frac{1}{2}}(\Phi_2)\left(C^{\frac{1}{\alpha}}\eta\right)\right\}$$

$$+ O_{\Phi_1,\Phi_2,\theta_\infty,M}(\eta^{-M})$$

Where,

$$\theta_{\infty,2}(\pm 1 + x) = \sum_{l=l_2^{\pm}(\theta)}^{\infty} \alpha_l^{\pm} x^l$$

$$|1-(\pm 1 \mp x)^2|^{\frac{1}{2}} \theta_{\infty,1}(\pm 1 \mp x) = |1-(\pm 1 \mp x)^2|^{\frac{1}{2}} \sum_{l=l_1^{\pm}(\theta)}^{\infty} \beta_l^{\pm} x^l$$

‘ are the asymptotic expansions around ± 1*, and for* $1 > \alpha > 0$*,* $\beta > 0$ *and* $\kappa \in \mathbb{Z}_{\geq 0}$*,* $\mathcal{T}(\Phi)(z)$ *is defined by*

$$\mathcal{T}_{\alpha,\kappa}(\Phi)(y) := \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \mathcal{M}(\Phi)(z) \left(\frac{i}{\pi y}\right)^{\alpha z} \Gamma(\kappa + \alpha z + 1) dz$$

where $\mathcal{M}$ *denotes the Mellin transform.*

Chapter 5

Poisson Summation

By Theorem 2.1.2 the elliptic part of the trace formula for the n'th Hecke operator is given by

$$2 \sum_{\substack{\pm \\ N:=\pm n}} \sum_{\substack{m\in\mathbb{Z} \\ m^2-4N\neq\square}} \theta_\infty^{\pm}\left(\frac{m}{2\sqrt{|N|}}\right) \sum_{\substack{f^2|(m^2-4N) \\ \frac{m^2-4N}{f^2}\equiv 0,1 \bmod 4}} \frac{1}{f} L\left(1, \left(\frac{(m^2-4N)/f^2}{\cdot}\right)\right) \tag{5.1}$$

Where $\theta_\infty^{\pm}$ is given by

$$\theta_\infty^{\mp}(x) := \mathcal{O}(f_\infty; (x, \mp 1))\sqrt{|x^2 \pm 1|}$$

and by $(x, \mp 1)$ we denote the element $\gamma \in GL_2(\mathbb{R})$ that has trace x and determinant ∓ 1.

The sums for $N = n$ and $N = -n$ will be treated separately, however the methods are general and carries verbatim to both cases. Keeping this in mind we will only go into detail for $N = n$ in what follows, and state the results for $N = -n$. To avoid extra notation we will denote θ_∞^{+} simply by θ_∞.

5.1 Rewriting the formula ($N > 0$)

Our aim in this section will be to rewrite (5.1) and isolate the contribution of the special representations as indicated in [18] (i.e. the trivial representation and the contribution of formula (31) of [18], denoted by $\xi_0(f_\infty)$ in that reference) while keeping track of the rest of the terms. We note that the isolation of the trivial representation in a far more general context is carried out in an adelic setting in the recent paper of Frenkel, Langlands and Ngo, [7]. They isolate the contribution of the trivial representation as the dominant term in the dual sum of the elliptic part, after performing Poisson summation on the Hitchin base. We note however that in [7] no attempt is made on the analysis of the remaining terms. This is for a good reason that the approach of [7] is based on a multiplicative truncation of the Artin L-functions that appear in the elliptic part, and the resulting expression, after applying Poisson summation, does not seem to be suitable for analysis.

To the extent of using Poisson summation on the Hitchin base (more or less the Poisson sum on the m-sum in our context, see the lemma below) our approach will be the same as the one in [7]. The main difference, which allows us to go further than [7], is that instead of the truncation used in [7] we will use the approximate functional equation (cf. Corollary 3.2.4). This not only allows us to isolate the special representations, but also lets us to control the remaining terms for further analysis.

Lemma 5.1.1. *For any $A > 0$ we have*

$$(5.1)_{N>0} = 2 \sum_{\substack{m\in\mathbb{Z} \\ m^2-4n\neq\square}} \theta_\infty\left(\frac{m}{2\sqrt{n}}\right) \left\{ \sum_{\substack{f^2|(m^2-4n) \\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \frac{1}{f} \sum_{t=1}^{\infty} \frac{1}{t} \left(\frac{(m^2-4n)/f^2}{t}\right) F\left(\frac{tf^2}{A}\right) \right. \tag{5.2}$$

$$\left. + \sum_{\substack{f^2|(m^2-4n) \\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \left(\frac{\pi f^2}{|4n-m^2|}\right)^{\frac{1}{2}} \sum_{t=1}^{\infty} \left(\frac{(m^2-4n)/f^2}{t}\right) H_\pm\left(\frac{tf^2A}{|m^2-4n|}\right) \right\} \tag{5.3}$$

Where,

$$H_\pm(x) := \frac{1}{2\pi i} \int_{(1)} \frac{\Gamma\left(\frac{e_\pm+u}{2}\right)}{\Gamma\left(\frac{1+e_\pm-u}{2}\right)} (\pi x)^{-u} \tilde{F}(u) du,$$

the sign in $H_\pm$ is the sign of m^2-4n, $e_- = 1$ and $e_+ = 0$.

Proof. Follows from Corollary 3.2.4. $\square$

Note that both $F(x)$ and $H_\pm(x)$ are rapidly decreasing which means for large values of t the corresponding terms in (5.4) and (5.5) are very small, hence the t-sums in (5.4) and (5.5) essentially run up to A and $\frac{|4n-m^2|}{f^2A}$ respectively.

We would like to apply Poisson summation to the m-sums in (5.4) and (5.5) however in order to do so we need to have a sum over complete lattice $\mathbb{Z}$. The sums in (5.4) and (5.5) are missing the m

for which $m^2 - 4n$ is a perfect square. We add and subtract[1] these values and rewrite (5.1) as

$$(5.1)_{N>0} = 2\sum_{m\in\mathbb{Z}} \theta_\infty\left(\frac{m}{2\sqrt{n}}\right)\left\{ \sum_{\substack{f^2|(m^2-4n)\\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\left(\frac{(m^2-4n)/f^2}{t}\right)F\left(\frac{tf^2}{A}\right) \right. \tag{5.2$'$}$$

$$\left. + \sum_{\substack{f^2|(m^2-4n)\\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \left(\frac{\pi f^2}{|4n-m^2|}\right)^{\frac{1}{2}}\sum_{t=1}^{\infty}\left(\frac{(m^2-4n)/f^2}{t}\right)H_{\pm}\left(\frac{tf^2A}{|m^2-4n|}\right)\right\} \tag{5.3$'$}$$

$$-2\sum_{\substack{m\in\mathbb{Z}\\ m^2-4n=\square}} \theta_\infty\left(\frac{m}{2\sqrt{n}}\right)\left\{ \sum_{\substack{f^2|(m^2-4n)\\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\left(\frac{(m^2-4n)/f^2}{t}\right)F\left(\frac{tf^2}{A}\right) \right. \tag{$\dagger$}$$

$$\left. + \sum_{\substack{f^2|(m^2-4n)\\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \left(\frac{\pi f^2}{|4n-m^2|}\right)^{\frac{1}{2}}\sum_{t=1}^{\infty}\left(\frac{(m^2-4n)/f^2}{t}\right)H_{\pm}\left(\frac{tf^2A}{|m^2-4n|}\right)\right\} \tag{$\dagger\dagger$}$$

For the rest of this chapter we will be only concerned with (5.2$'$) and (5.3$'$) and leave (†) and (††) as the way they are. We also note that if the support of θ_∞ is in $[-2, 2]$ then since there are no (m, n) with $m^2 - 4n = \square$ and $m^2 - 4n < 0$, (†) and (††) are both empty.

We take a moment to explain the manipulations to come. We introduced the approximate functional equation in order to be able to be able to approximate the L-values with relatively short sums (i.e. truncating the t-sum) while keeping the error small. How short we can truncate, and the error, depends on the conductor of the quadratic character. In particular by the symmetry of the approximate functional equation we can truncate both (5.2$'$) and (5.3$'$) at the square root of the conductor of the quadratic characters that appear in (5.1).

Furthermore note that since the support of θ_∞ is compact the m-sum is finite and goes only up to $Cn^{1/2}$ for some constant $C > 0$ that depends on the support of θ_∞. We will denote this by $m \ll n^{1/2}$. Also since $m \ll n^{1/2}$ and $f^2 \mid m^2 - 4n$, we necessarily have $f \ll n^{1/2}$. We will be using these observation repeatedly without reference in the text.

We now interchange the f sum with the m sum (note that the f-sum is a finite sum because of the compact support of θ_∞) and rewrite (5.2$'$) and (5.3$'$) as

$$(5.2') = 2\sum_{f=1}^{\infty} \sum_{\substack{m\in\mathbb{Z}\\ f^2|(m^2-4n)\\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \theta_\infty\left(\frac{m}{2\sqrt{n}}\right)\frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\left(\frac{(m^2-4n)/f^2}{t}\right)F\left(\frac{tf^2}{A}\right) \tag{5.4}$$

[1] Note that we cannot do this directly to (5.1) since the L-functions corresponding to $m^2 - 4n = \square$ are modified ζ-functions which are not defined at 1. Therefore we need to introduce some type of truncation to the t-sums before adding and subtracting these values. After introducing the approximate functional equation, as explained in the paragraph above the t-sums essentially become finite sums and we can add and subtract the missing terms.

$$(5.3') = 2\sqrt{\pi}\sum_{f=1}^{\infty} f \sum_{\substack{m\in\mathbb{Z} \\ f^2|(m^2-4n) \\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \theta_\infty\left(\tfrac{m}{2\sqrt{n}}\right)\frac{1}{\sqrt{|m^2-4n|}}\sum_{t=1}^{\infty}\left(\tfrac{(m^2-4n)/f^2}{t}\right)H_\pm\left(\tfrac{tf^2A}{|m^2-4n|}\right) \tag{5.5}$$

In what follows, we will apply Poisson summation to the m-sums in (5.4) and (5.5). We remark that the functions $\theta_\infty\left(\frac{m}{2\sqrt{n}}\right)F\left(\frac{tf^2}{A}\right)$ and $\frac{1}{\sqrt{|m^2-4n|}}\theta_\infty\left(\frac{m}{2\sqrt{n}}\right)H_\pm\left(\frac{tf^2A}{|4n-m^2|}\right)$ are in general singular, therefore care must be taken in applying Poisson summation.

We resolve this problem by choosing[2] $A=|m^2-4n|^\alpha$, for some $1>\alpha>0$, as in Chapter 4. With this choice (5.4) and (5.5) respectively become

$$(5.4) = 2\sum_{f=1}^{\infty}\sum_{\substack{m\in\mathbb{Z} \\ f^2|(m^2-4n) \\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \theta_\infty\left(\tfrac{m}{2\sqrt{n}}\right)\frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\left(\tfrac{(m^2-4n)/f^2}{t}\right)F\left(\tfrac{tf^2}{|m^2-4n|^\alpha}\right) \tag{5.4$'$}$$

$$(5.5) = 2\sqrt{\pi}\sum_{f=1}^{\infty} f \sum_{\substack{m\in\mathbb{Z} \\ f^2|(m^2-4n) \\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \theta_\infty\left(\tfrac{m}{2\sqrt{n}}\right)\frac{1}{\sqrt{|m^2-4n|}}\sum_{t=1}^{\infty}\left(\tfrac{(m^2-4n)/f^2}{t}\right)H_\pm\left(\tfrac{tf^2}{|m^2-4n|^{1-\alpha}}\right) \tag{5.5$'$}$$

Furthermore define $\Theta_{\infty,F,\alpha,\pm}(m,n,t)$ and $\Theta_{\infty,H,\alpha,\pm}(m,n,t)$ by

$$\Theta_{\infty,F,\alpha,+}(m,n,t) := \begin{cases} \theta_\infty\left(\frac{m}{2\sqrt{n}}\right)F\left(\frac{t}{(m^2-4n)^\alpha}\right) & \text{if } m^2-4n>0 \\ 0 & \text{if } m^2-4n\le 0\end{cases}$$

$$\Theta_{\infty,H,\alpha,+}(m,n,t) := \begin{cases} \frac{1}{\sqrt{m^2-4n}}\theta_\infty\left(\frac{m}{2\sqrt{n}}\right)H_+\left(\frac{t}{(m^2-4n)^{1-\alpha}}\right) & \text{if } m^2-4n>0 \\ 0 & \text{if } m^2-4n\le 0\end{cases}$$

$$\Theta_{\infty,F,\alpha,-}(m,n,t) := \begin{cases} 0 & \text{if } m^2-4n\ge 0 \\ \theta_\infty\left(\frac{m}{2\sqrt{n}}\right)F\left(\frac{t}{(4n-m^2)^\alpha}\right) & \text{if } m^2-4n<0\end{cases}$$

$$\Theta_{\infty,H,\alpha,-}(m,n,t) := \begin{cases} 0 & \text{if } m^2-4n\ge 0 \\ \frac{1}{\sqrt{4n-m^2}}\theta_\infty\left(\frac{m}{2\sqrt{n}}\right)H_-\left(\frac{t}{(4n-m^2)^{1-\alpha}}\right) & \text{if } m^2-4n<0\end{cases}$$

By Proposition 4.3.1 the functions $\Theta_{\infty,F,\alpha,\pm}(m,n,t)$ and $\Theta_{\infty,H,\alpha,\pm}(m,n,t)$ are smooth and we have

$$(5.4') = 2\sum_{f=1}^{\infty}\sum_{\substack{m\in\mathbb{Z} \\ f^2|(m^2-4n) \\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\left(\tfrac{(m^2-4n)/f^2}{t}\right)\Theta_{\infty,F,\alpha,+}(m,n,tf^2) \tag{5.4$'$-a}$$

$$+\,2\sum_{f=1}^{\infty}\sum_{\substack{m\in\mathbb{Z} \\ f^2|(m^2-4n) \\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\left(\tfrac{(m^2-4n)/f^2}{t}\right)\Theta_{\infty,F,\alpha,-}(m,n,tf^2) \tag{5.4$'$-b}$$

[2] For Beyond Endoscopy purposes we will be taking $\alpha=\frac{1}{2}$. We are leaving α as a variable here with applications to bounds towards the Ramanujan conjecture in mind.

$$(5.5') = 2\sqrt{\pi}\sum_{f=1}^{\infty} f \sum_{\substack{m\in\mathbb{Z}\\ f^2|(m^2-4n)\\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \sum_{t=1}^{\infty} \left(\frac{(m^2-4n)/f^2}{t}\right)\Theta_{\infty,H,\alpha,+}(m,n,tf^2) \tag{5.5$'$-a}$$

$$+ 2\sqrt{\pi}\sum_{f=1}^{\infty} f \sum_{\substack{m\in\mathbb{Z}\\ f^2|(m^2-4n)\\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \sum_{t=1}^{\infty} \left(\frac{(m^2-4n)/f^2}{t}\right)\Theta_{\infty,H,\alpha,-}(m,n,tf^2) \tag{5.5$'$-b}$$

In the next section we will apply Poisson summation to all of the sums above. However before applying Poisson summation we would like to note that the t-sums in (5.4$'$-a), (5.4$'$-b), (5.5$'$-a) and (5.5$'$-b) are effectively limited to a short range with respect to n. More precisely we have the following:

Proposition 5.1.2. *For any $\delta_0 > 0$ we have,*

$$\sum_{f=1}^{\infty}\frac{1}{f} \sum_{\substack{m\in\mathbb{Z}\\ f^2|(m^2-4n)\\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \sum_{\substack{t=1\\ tf^2\gg n^{\alpha+\delta_0}}}^{\infty} \frac{1}{t}\left(\frac{(m^2-4n)/f^2}{t}\right)\Theta_{\infty,F,\alpha,\pm}(m,n,tf^2) = O_{\theta_\infty}\left(ne^{-n^{\delta_0}}\right)$$

$$\sum_{f=1}^{\infty} f \sum_{\substack{m\in\mathbb{Z}\\ f^2|(m^2-4n)\\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \sum_{\substack{t=1\\ tf^2\gg n^{1-\alpha+\delta_0}}}^{\infty} \left(\frac{(m^2-4n)/f^2}{t}\right)\Theta_{\infty,H,\alpha,\pm}(m,n,tf^2) = O_{\theta_\infty}\left(n^{\frac{3}{2}-\alpha}e^{\frac{-n^{\frac{\delta_0}{2}}}{2}}\right)$$

Proof. First of all note that since θ_∞ is of compact support there exists a constant K such that the m-sum runs only up to $K\sqrt{n}$. Furthermore, since $f^2 \mid m^2 - 4n$ we necessarily have $f < K\sqrt{n}$ too.

Recall that by (3.3) we know that $F(x) = O(e^{-x})$, therefore we have

$$\Theta_{\infty,F,\alpha,\pm}(m,n,tf^2) = O_{\theta_\infty}(e^{-\frac{tf^2}{n^\alpha}})$$

Therefore the first sum is bounded by,

$$\begin{aligned}
&\ll_{\theta_\infty} \sum_{f\ll\sqrt{n}}\frac{1}{f}\sum_{\substack{m\in\mathbb{Z}\\ f^2|(m^2-4n)}}\sum_{\substack{t=1\\ tf^2\gg n^{\alpha+\delta_0}}}^{\infty}\frac{1}{t}e^{\frac{-tf^2}{n^\alpha}}\\
&\le n\sum_{f\ll\sqrt{n}}\frac{1}{f^3}\sum_{\substack{t=1\\ tf^2\gg n^{\alpha+\delta_0}}}^{\infty}\frac{1}{t}e^{\frac{-tf^2}{n^\alpha}}\\
&\ll ne^{-n^{\delta_0}}\sum_{f\ll\sqrt{n}}\frac{1}{f^3}\\
&\le ne^{-n^{\delta_0}}
\end{aligned}$$

To bound the second sum we use the bound in (3.5) which gives

$$\Theta_{\infty,H,\alpha,\pm}(m,n,tf^2) = O_{\theta_\infty}\left(\frac{n^{1-\alpha}}{n^{\frac{1}{2}}tf^2}e^{-\frac{\left(\frac{tf^2}{n^{1-\alpha}}\right)^{\frac{1}{2}}}{2}}\right)$$

Then the second sum is bounded by,

$$\begin{aligned}
&\ll_{\theta_\infty} \sum_{f \ll \sqrt{n}} f \sum_{\substack{m \in \mathbb{Z} \\ f^2|(m^2-4n)}} \sum_{\substack{t=1 \\ tf^2 \gg n^{1-\alpha+\delta_0}}}^{\infty} \frac{n^{1-\alpha}}{n^{\frac{1}{2}} tf^2} e^{-\frac{\left(\frac{tf^2}{n^{1-\alpha}}\right)^{\frac{1}{2}}}{2}} \\
&\leq n^{\frac{3}{2}-\alpha} \sum_{f \ll \sqrt{n}} \frac{1}{f^3} \sum_{\substack{t=1 \\ tf^2 \gg n^{1-\alpha+\delta_0}}}^{\infty} \frac{1}{t} e^{-\frac{\left(\frac{tf^2}{n^{1-\alpha}}\right)^{\frac{1}{2}}}{2}} \\
&\ll n^{\frac{3}{2}-\alpha} e^{\frac{-n^{\frac{\delta_0}{2}}}{2}} \sum_{f > n^{\frac{1-\alpha+\delta_0}{2}+}} \frac{1}{f^3} \\
&\leq n^{\frac{3}{2}-\alpha} e^{\frac{-n^{\frac{\delta_0}{2}}}{2}}
\end{aligned}$$

The proposition follows. □

By Proposition 5.1.2 we immediately get,
Corollary 5.1.3. *For any $\delta_0 > 0$ we have,*

$$(5.4'\text{-a}) = 2 \sum_{f \ll n^{\frac{\alpha+\delta_0}{2}}} \sum_{\substack{m \in \mathbb{Z} \\ f^2|(m^2-4n) \\ \frac{m^2-4n}{f^2} \equiv 0,1 \bmod 4}} \frac{1}{f} \sum_{\substack{t \\ tf^2 \ll n^{\alpha+\delta_0}}} \frac{1}{t} \left(\frac{(m^2-4n)/f^2}{t}\right) \Theta_{\infty,F,\alpha,+}(m,n,tf^2) + O_{\theta_\infty}(ne^{-n^{\delta_0}})$$

$$(5.4'\text{-b}) = 2 \sum_{f \ll n^{\frac{\alpha+\delta_0}{2}}} \sum_{\substack{m \in \mathbb{Z} \\ f^2|(m^2-4n) \\ \frac{m^2-4n}{f^2} \equiv 0,1 \bmod 4}} \frac{1}{f} \sum_{\substack{t \\ tf^2 \ll n^{\alpha+\delta_0}}} \frac{1}{t} \left(\frac{(m^2-4n)/f^2}{t}\right) \Theta_{\infty,F,\alpha,-}(m,n,tf^2) + O_{\theta_\infty}(ne^{-n^{\delta_0}})$$

$$(5.5'\text{-a}) = 2\sqrt{\pi} \sum_{f \ll n^{\frac{1-\alpha+\delta_0}{2}}} f \sum_{\substack{m \in \mathbb{Z} \\ f^2|(m^2-4n) \\ \frac{m^2-4n}{f^2} \equiv 0,1 \bmod 4}} \sum_{\substack{t \\ tf^2 \ll n^{1-\alpha+\delta_0}}} \left(\frac{(m^2-4n)/f^2}{t}\right) \Theta_{\infty,H,\alpha,+}(m,n,tf^2) + O_{\theta_\infty}(n^{\frac{3}{2}-\alpha} e^{\frac{-n^{\frac{\delta_0}{2}}}{2}})$$

$$(5.5'\text{-b}) = 2\sqrt{\pi} \sum_{f \ll n^{\frac{1-\alpha+\delta_0}{2}}} f \sum_{\substack{m \in \mathbb{Z} \\ f^2|(m^2-4n) \\ \frac{m^2-4n}{f^2} \equiv 0,1 \bmod 4}} \sum_{\substack{t \\ tf^2 \ll n^{1-\alpha+\delta_0}}} \left(\frac{(m^2-4n)/f^2}{t}\right) \Theta_{\infty,H,\alpha,-}(m,n,tf^2) + O_{\theta_\infty}(n^{\frac{3}{2}-\alpha} e^{\frac{-n^{\frac{\delta_0}{2}}}{2}})$$

Proof. Obvious.

□

5.2 Poisson summation ($N > 0$)

5.2.1 Poisson summation on (5.4′-a)

Lemma 5.2.1.

$$(5.4'\text{-a}) = \sqrt{n}\sum_{f=1}^{\infty}\tfrac{1}{f^3}\sum_{t=1}^{\infty}\tfrac{1}{t^2}\sum_{\xi\in\mathbb{Z}}\left\{\int_{|x|>1}\theta_\infty(x)\,F\left(\tfrac{tf^2(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)e\left(\tfrac{-x\xi\sqrt{n}}{2tf^2}\right)dx\right\}Kl_{t,f}(\xi,n) \tag{5.6}$$

Where,

$$Kl_{t,f}(\xi,n) := \sum_{\substack{a \bmod 4tf^2\\ a^2-4n\equiv 0 \bmod f^2\\ \frac{a^2-4n}{f^2}\equiv 0,1 \bmod 4}}\left(\tfrac{(a^2-4n)/f^2}{t}\right)e\left(\tfrac{a\xi}{4tf^2}\right)$$

Proof. Everything converges absolutely so we can interchange the sums freely. Interchange the t and m-sums:

$$(5.4'\text{-a}) = 2\sum_{f=1}^{\infty}\sum_{t=1}^{\infty}\tfrac{1}{t}\sum_{\substack{m\in\mathbb{Z}\\ f^2|(m^2-4n)\\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}}\tfrac{1}{f}\left(\tfrac{(m^2-4n)/f^2}{t}\right)\Theta_{\infty,F,\alpha,+}(m,n,tf^2)$$

Observe that the Legendre symbol and the divisibility condition are periodic mod$4f^2$ so once we fix the value of m modulo $4tf^2$ all the arithmetic dependence is fixed and we get

$$(5.4'\text{-a}) = 2\sum_{f=1}^{\infty}\sum_{t=1}^{\infty}\tfrac{1}{t}\sum_{\substack{a \bmod 4tf^2\\ a^2-4n\equiv 0 \bmod f^2\\ \frac{a^2-4n}{f^2}\equiv 0,1 \bmod 4}}\tfrac{1}{f}\left(\tfrac{(a^2-4n)/f^2}{t}\right)\sum_{m\equiv a \bmod 4tf^2}\Theta_{\infty,F,\alpha,+}(m,n,tf^2)$$

By Poisson summation on the m-sum we get

$$\sum_{m\equiv a \bmod 4tf^2}\Theta_{\infty,F,\alpha,+}(m,n,t) = \tfrac{1}{4tf^2}\sum_{\xi\in\mathbb{Z}}\widehat{\Theta}_{\infty,F,\alpha,+}\left(\tfrac{\xi}{4tf^2},n,tf^2\right)e\left(\tfrac{a\xi}{4tf^2}\right)$$

Where,

$$\begin{aligned}\widehat{\Theta}_{\infty,F,\alpha,+}\left(\tfrac{\xi}{4tf^2},n,tf^2\right) &= \int_{\mathbb{R}}\Theta_{\infty,F,\alpha,+}(x,n,t)e\left(\tfrac{-x\xi}{4tf^2}\right)dx\\ &= \int_{|x|>2\sqrt{n}}\theta_\infty\left(\tfrac{x}{2\sqrt{n}}\right)F\left(\tfrac{tf^2}{(x^2-4n)^\alpha}\right)e\left(\tfrac{-x\xi}{4tf^2}\right)dx\\ &= 2\sqrt{n}\int_{|x|>1}\theta_\infty(x)\,F\left(\tfrac{tf^2(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)e\left(\tfrac{-x\xi\sqrt{n}}{2tf^2}\right)dx\end{aligned}$$

The lemma follows. □

5.2.2 Poisson summation on (5.4′-b)

We handle (5.4′-b) in the same way as (5.4′-a).

Lemma 5.2.2.

$$(5.4'\text{-b}) = \sqrt{n}\sum_{f=1}^{\infty}\tfrac{1}{f^3}\sum_{t=1}^{\infty}\tfrac{1}{t^2}\sum_{\xi\in\mathbb{Z}}\left\{\int_{|x|<1}\theta_\infty(x)\,F\left(\tfrac{tf^2(4n)^{-\alpha}}{(1-x^2)^{\alpha}}\right)e\left(\tfrac{-x\xi\sqrt{n}}{2tf^2}\right)dx\right\}Kl_{t,f}(\xi,n) \tag{5.7}$$

Where,

$$Kl_{t,f}(\xi,n) := \sum_{\substack{a \bmod 4tf^2\\ a^2-4n\equiv 0 \bmod f^2\\ \frac{a^2-4n}{f^2}\equiv 0,1 \bmod 4}}\left(\tfrac{(a^2-4n)/f^2}{t}\right)e\left(\tfrac{a\xi}{4tf^2}\right)$$

Proof. Identical to the proof of Lemma 5.2.1.

□

5.2.3 Poisson summation on (5.5′-a)

Lemma 5.2.3.

$$(5.5'\text{-a}) = \tfrac{\sqrt{\pi}}{2}\sum_{f=1}^{\infty}\tfrac{1}{f}\sum_{t=1}^{\infty}\tfrac{1}{t}\sum_{\xi\in\mathbb{Z}}\left\{\int_{|x|>1}\tfrac{\theta_\infty(x)}{\sqrt{x^2-1}}H_+\left(\tfrac{tf^2(4n)^{\alpha-1}}{(x^2-1)^{1-\alpha}}\right)e\left(\tfrac{-x\sqrt{n}\xi}{2tf^2}\right)dx\right\}Kl_{t,f}(\xi,n) \tag{5.8}$$

Where,

$$Kl_{t,f}(\xi,n) := \sum_{\substack{a \bmod 4tf^2\\ a^2-4n\equiv 0 \bmod f^2\\ \frac{a^2-4n}{f^2}\equiv 0,1 \bmod 4}}\left(\tfrac{(a^2-4n)/f^2}{t}\right)e\left(\tfrac{a\xi}{4tf^2}\right)$$

Proof. Interchanging t and m sums and breaking the m-sum into arithmetic progressions $\bmod 4tf^2$ (as in Lemma 5.2.1) gives

$$(5.5'\text{-a}) = 2\sqrt{\pi}\sum_{f=1}^{\infty}f\sum_{t=1}^{\infty}\sum_{\substack{a \bmod 4tf^2\\ f^2|(a^2-4n)\\ \frac{a^2-4n}{f^2}\equiv 0,1 \bmod 4}}\left(\tfrac{(a^2-4n)/f^2}{t}\right)\sum_{m\equiv 1 \bmod 4tf^2}\Theta_{\infty,H,\alpha,+}(m,n,tf^2)$$

Applying Poisson summation to the m-sum gives,

$$\sum_{m\equiv a \bmod 4tf^2}\Theta_{\infty,H,\alpha,+}(m,n,tf^2) = \tfrac{1}{4tf^2}\sum_{\xi\in\mathbb{Z}}\widehat{\Theta}_{\infty,H,\alpha,+}\left(\tfrac{\xi}{4tf^2},n,tf^2\right)e\left(\tfrac{a\xi}{4tf^2}\right)$$

Where,

$$\begin{aligned}
\widehat{\Theta}_{\infty,H,\alpha,+}\left(\tfrac{\xi}{4tf^2},n,tf^2\right) &= \int_{\mathbb{R}}\Theta_{\infty,H,\alpha,+}(x,n,tf^2)e\left(\tfrac{-x\xi}{4tf^2}\right)dx\\
&= \int_{|x|>2\sqrt{n}}\tfrac{1}{\sqrt{x^2-4n}}\theta_\infty\left(\tfrac{x}{2\sqrt{n}}\right)H_+\left(\tfrac{tf^2}{(x^2-4n)^{1-\alpha}}\right)e\left(\tfrac{-x\xi}{4tf^2}\right)dx\\
&= \int_{|x|>1}\tfrac{1}{\sqrt{x^2-1}}\theta_\infty(x)\,H_+\left(\tfrac{tf^2(4n)^{\alpha-1}}{(x^2-1)^{1-\alpha}}\right)e\left(\tfrac{-x\sqrt{n}\xi}{2tf^2}\right)dx
\end{aligned}$$

The lemma follows. □

5.2.4 Poisson summation on (5.5′-b)

Lemma 5.2.4.

$$(5.5'\text{-b}) = \frac{\sqrt{\pi}}{2}\sum_{f=1}^{\infty}\frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\sum_{\xi\in\mathbb{Z}}\left\{\int_{|x|<1}\frac{\theta_\infty(x)}{\sqrt{1-x^2}}H_-\left(\frac{tf^2(4n)^{\alpha-1}}{(1-x^2)^{1-\alpha}}\right)e\left(\frac{-x\sqrt{n}\xi}{2tf^2}\right)dx\right\}Kl_{t,f}(\xi,n) \tag{5.9}$$

Where,

$$Kl_{t,f}(\xi,n) := \sum_{\substack{a \bmod 4tf^2 \\ a^2-4n\equiv 0 \bmod f^2 \\ \frac{a^2-4n}{f^2}\equiv 0,1 \bmod 4}}\left(\frac{(a^2-4n)/f^2}{t}\right)e\left(\frac{a\xi}{4tf^2}\right)$$

Proof. Identical to the proof of Lemma 5.2.3. □

5.3 The terms corresponding to $\xi = 0$ $(N > 0)$

In this section we will analyze the terms corresponding to $\xi = 0$ in (5.6), (5.7), (5.8) and (5.9). We will show that the sum of these terms gives the contribution of the trivial representation plus an explicit error term (which, in order of of magnitude in a precise sense, is small).

5.3.1 An auxiliary Dirichlet series

In this subsection we will explicitly calculate a Dirichlet series that will occur frequently in the upcoming analysis of the dominant term. Let $D(z;n)$ be defined by,

$$D(z;n) := \sum_{f=1}^{\infty}\frac{1}{f^{2z+1}}\sum_{t=1}^{\infty}\frac{Kl_{t,f}(0,n)}{t^{z+1}} \tag{5.10}$$

Lemma 5.3.1.

$$D(z;n) = \prod_p D_p(z;n)$$

Where for each prime p $D_p(z;n)$ is defined by

$$D_p(z;n) := \sum_{k=0}^{\infty}\frac{1}{p^{k(2z+1)}}\sum_{l=0}^{\infty}\frac{Kl_{p^l,p^k}(0,n)}{p^{l(z+1)}}$$

Proof. Chinese remainder theorem. □

Lemma 5.3.2. *Let $p\nmid 2n$ be an odd prime. Then,*

$$D_p(z;n) = \frac{\left(1-\frac{1}{p^{z+1}}\right)}{\left(1-\frac{1}{p^{2z}}\right)}$$

Proof. In order to compute $D_p(z;n)$ we need to compute $Kl_{p^l,p^k}(0,n)$ for various values of k,l.

- $l = k = 0$.

  In this case the $Kl_{1,1}(0,n)$ is obviously 1.

- $l > 0$, $k = 0$.

$$\begin{aligned} Kl_{p^l,1}(0,n) &= \sum_{a \bmod p^l} \left(\frac{a^2-4n}{p^l}\right) \\ &= \sum_{a_0 \bmod p} \left(\frac{a_0^2-4n}{p^l}\right) \sum_{a_1 \bmod p^{l-1}} 1 \\ &= p^{l-1} \sum_{a_0 \bmod p} \left(\frac{a_0^2-4n}{p^l}\right) \qquad (i) \end{aligned}$$

  This last sum depends on the parity of l.

  - $l \equiv 0 \bmod 2$.

    In this case

$$\begin{aligned} (i) &= p^{l-1} \sum_{\substack{a_0 \bmod p \\ a_0^2 \neq 4n}} 1 \\ &= p^l - p^{l-1}\left(1 + \left(\frac{n}{p}\right)\right) \qquad (ii) \end{aligned}$$

  - $l \equiv 1 \bmod 2$.

    In this case

$$(i) = -p^{l-1} \qquad (iii)$$

    Where we used Lemma 2 of Appendix A of [18].

- $l = 0$, $k > 0$.

  Since $p \neq 2$,

$$\begin{aligned} Kl_{1,p^k}(0,n) &= \sum_{\substack{a \bmod p^{2k} \\ a^2 \equiv 4n \bmod p^{2k}}} 1 \\ &= 1 + \left(\frac{n}{p}\right) \qquad (iv) \end{aligned}$$

- $l, k > 0$. First of all the sum is clearly 0 unless n is a square mod$p$. If n is a square mod$p$, since $p \neq 2$, then there are two exactly square-roots of $4n$ modulo p^{2k}. Let us denote them by

n_1, n_2. Then,

$$\begin{aligned}
Kl_{p^l,p^k}(0,n) &= \sum_{\substack{a \bmod p^{l+2k} \\ a^2 \equiv 4n \bmod p^{2k}}} \left(\frac{(a^2-4n)/p^{2k}}{p^l}\right) \\
&= p^{l-1} \sum_{\substack{a_0 \bmod p^{1+2k} \\ a_0^2 \equiv 4n \bmod p^{2k}}} \left(\frac{(a_0^2-4n)/p^{2k}}{p^l}\right) \\
&= p^{l-1} \sum_{\substack{a_0 \bmod p^{1+2k} \\ a_0 \equiv n_j \bmod p^{2k}}} \left(\frac{(a_0^2-4n)/p^{2k}}{p^l}\right) \\
&= p^{l-1} \sum_{\substack{a_2 \bmod p \\ j=1,2}} \left(\frac{a_2 n_j}{p^l}\right) \\
&= p^{l-1} \begin{cases} 0 & \text{if } l \equiv 1 \bmod 2 \\ (p-1)\left(1+\left(\frac{n}{p}\right)\right) & \text{if } l \equiv 0 \bmod 2 \end{cases}
\end{aligned} \tag{v}$$

We can now compute $D_p(z;n)$.

$$\begin{aligned}
D_p(z;n) &= \sum_{k=0}^{\infty} \frac{1}{p^{k(2z+1)}} \sum_{l=0}^{\infty} \frac{Kl_{p^l,p^k}(0,n)}{p^{l(z+1)}} \\
&= 1 + \sum_{l=1}^{\infty} \frac{Kl_{p^l,1}(0,n)}{p^{l(z+1)}} + \sum_{k=1}^{\infty} \frac{Kl_{1,p^k}(0,n)}{p^{k(2z+1)}} + \sum_{k=1}^{\infty} \frac{1}{p^{k(2z+1)}} \sum_{l=1}^{\infty} \frac{Kl_{p^l,p^k}(0,n)}{p^{l(z+1)}}
\end{aligned}$$

Using (i) to (v), we then get:

$$\begin{aligned}
D_p(z;n) &= 1 + \sum_{l=1}^{\infty} \frac{Kl_{p^l,1}(0,n)}{p^{l(z+1)}} + \sum_{k=1}^{\infty} \frac{Kl_{1,p^k}(0,n)}{p^{k(2z+1)}} + \sum_{k=1}^{\infty} \frac{1}{p^{k(2z+1)}} \sum_{l=1}^{\infty} \frac{Kl_{p^l,p^k}(0,n)}{p^{l(z+1)}} \\
&= 1 + \sum_{l=1}^{\infty} \frac{Kl_{p^{2l},1}(0,n)}{p^{2l(z+1)}} + \sum_{l=0}^{\infty} \frac{Kl_{p^{2l+1},1}(0,n)}{p^{(2l+1)(z+1)}} + \sum_{k=1}^{\infty} \frac{Kl_{1,p^k}(0,n)}{p^{k(2z+1)}} + \sum_{k=1}^{\infty} \frac{1}{p^{k(2z+1)}} \sum_{l=1}^{\infty} \frac{Kl_{p^{2l},p^k}(0,n)}{p^{2l(z+1)}} \\
&= 1 + \left(1 - \frac{1}{p}\left(1+\left(\frac{n}{p}\right)\right)\right) \sum_{l=1}^{\infty} \frac{p^{2l}}{p^{2l(z+1)}} - \frac{1}{p} \sum_{l=0}^{\infty} \frac{p^{2l+1}}{p^{(2l+1)(z+1)}} + \left(1+\left(\frac{n}{p}\right)\right) \sum_{k=1}^{\infty} \frac{1}{p^{k(2z+1)}} \\
&\quad + \left(1-\frac{1}{p}\right)\left(1+\left(\frac{n}{p}\right)\right) \sum_{k=1}^{\infty} \frac{1}{p^{k(2z+1)}} \sum_{l=1}^{\infty} \frac{p^{2l}}{p^{2l(z+1)}} \\
&= 1 + \left(1 - \frac{1}{p}\left(1+\left(\frac{n}{p}\right)\right)\right) \sum_{l=1}^{\infty} \frac{1}{p^{2lz}} - \frac{1}{p} \sum_{l=0}^{\infty} \frac{1}{p^{(2l+1)z}} + \left(1+\left(\frac{n}{p}\right)\right) \sum_{k=1}^{\infty} \frac{1}{p^{k(2z+1)}} \\
&\quad + \left(1-\frac{1}{p}\right)\left(1+\left(\frac{n}{p}\right)\right) \sum_{k=1}^{\infty} \frac{1}{p^{k(2z+1)}} \sum_{l=1}^{\infty} \frac{1}{p^{2lz}} \\
&= \frac{1-\frac{1}{p^{z+1}}}{1-\frac{1}{p^{2z}}} + \left(1+\left(\frac{n}{p}\right)\right) \sum_{k=1}^{\infty} \frac{1}{p^{k(2z+1)}} \sum_{l=0}^{\infty} \frac{1}{p^{2lz}} - \frac{1}{p}\left(1+\left(\frac{n}{p}\right)\right) \sum_{k=0}^{\infty} \frac{1}{p^{k(2z+1)}} \sum_{l=1}^{\infty} \frac{1}{p^{2lz}} \\
&= \frac{1-\frac{1}{p^{z+1}}}{1-\frac{1}{p^{2z}}}
\end{aligned}$$

□

Lemma 5.3.3. *Let $p \mid n$ be an odd prime, and let $v_p(n)$ denote the p-adic valuation of n. Then,*

$$D_p(z;n) = \frac{\left(1-\frac{1}{p^{z(v_p(n)+1)}}\right)\left(1-\frac{1}{p^{z+1}}\right)}{\left(1-\frac{1}{p^{2z}}\right)\left(1-\frac{1}{p^{z}}\right)}$$

Proof. The computation depends on the parity of the p-adic valuation of n. Let $r = v_p(n)$ throughout the proof. As in the proof of Lemma 5.3.2 we need to compute the values of $Kl_{p^l,p^k}(0,n)$ first.

- $r \equiv 1 \bmod 2$. We divide the computation into cases depending on the values of l and k.
  - $l = k = 0$.

    In this case $Kl_{p^l,p^k}(0,n)$ is obviously 1.
  - $l > 0$, $k = 0$.

    Since $p \mid n$,

    $$\begin{aligned} Kl_{p^l,1}(0,n) &= \sum_{a \bmod p^l} \left(\frac{a^2-4n}{p^l}\right) \\ &= \sum_{a \bmod p^l} \left(\frac{a^2}{p^l}\right) \\ &= p^l - p^{l-1} \end{aligned} \tag{i}$$

  - $2k > r$.

    In this case we use the assumption that $r \equiv 1 \bmod 2$ and that $p \neq 2$. Note that in this case the sum runs over $a \bmod p^{2k}$ such that $a^2 \equiv 4n \bmod p^{2k}$. If $a \bmod p^{2k}$ is such that $a^2 \equiv 4n \bmod p^{2k}$ then we need to have $a = p^{\frac{r+1}{2}} a_0$ for some $a_0 \bmod p^{2k-\frac{r+1}{2}}$. But in this case $a^2 = p^{r+1}a_0^2 \equiv 0 \bmod p^{r+1}$, therefore we cannot have $a^2 \equiv 4n \bmod p^{2k}$. Hence in this case the sum is 0.
  - $r > 2k > 0$, $l > 0$.

    In this case $n \equiv 0 \bmod p^{2k}$ and hence in order to have $a^2 \equiv a \bmod p^{2k}$ we necessarily have $a \equiv 0 \bmod p^k$. Then the sum is,

    $$\begin{aligned} Kl_{p^l,p^k}(0,n) &= \sum_{\substack{a \bmod p^{l+2k} \\ a^2 \equiv 4n \bmod p^{2k}}} \left(\frac{(a^2-4n)/p^{2k}}{p^l}\right) \\ &= \sum_{a_0 \bmod p^{l+k}} \left(\frac{a_0^2}{p^l}\right) && (\star) \\ &= p^{k+l} - p^{k+l-1} && (ii) \end{aligned}$$

    Where in passing to $(\star)$ we used the assumption that $r \equiv 1 \bmod 2$ so that $a/p^{2k} \equiv 0 \bmod p$.
  - $r > 2k > 0$, $l = 0$.

    In this case,

$$\begin{aligned} Kl_{1,p^k}(0,n) &= \sum_{\substack{a \bmod p^{2k} \\ a^2 \equiv 4n \bmod p^{2k}}} 1 \\ &= p^k \end{aligned} \qquad (iii)$$

Using (i), (ii), (iii) and the argument above we see that,

$$\begin{aligned} D_p(z;n) &= 1 + \sum_{l=1}^{\infty} \frac{Kl_{p^l,1}(0,n)}{p^{l(z+1)}} + \sum_{k=1}^{\infty} \frac{Kl_{1,p^k}(0,n)}{p^{k(2z+1)}} + \sum_{k=1}^{\infty} \frac{1}{p^{k(2z+1)}} \sum_{l=1}^{\infty} \frac{Kl_{p^l,p^k}(0,n)}{p^{l(z+1)}} \\ &= 1 + \left(1 - \tfrac{1}{p}\right) \sum_{l=1}^{\infty} \frac{p^l}{p^{l(z+1)}} + \sum_{k=1}^{\frac{r-1}{2}} \frac{p^k}{p^{k(2z+1)}} + \left(1 - \tfrac{1}{p}\right) \sum_{k=1}^{\frac{r-1}{2}} \frac{p^k}{p^{k(2z+1)}} \sum_{l=1}^{\infty} \frac{p^l}{p^{l(z+1)}} \\ &= \left(1 - \tfrac{1}{p}\right) \sum_{k=0}^{\frac{r-1}{2}} \frac{1}{p^{2kz}} \sum_{l=1}^{\infty} \frac{1}{p^{lz}} + \sum_{k=0}^{\frac{r-1}{2}} \frac{1}{p^{2kz}} \\ &= \frac{1 - \frac{1}{p^{z+1}}}{1 - \frac{1}{p^z}} \sum_{k=0}^{\frac{r-1}{2}} \frac{1}{p^{2kz}} \\ &= \frac{\left(1 - \frac{1}{p^{z+1}}\right)\left(1 - \frac{1}{p^{z(r+1)}}\right)}{\left(1 - \frac{1}{p^z}\right)\left(1 - \frac{1}{p^{2z}}\right)} \end{aligned}$$

Which finishes the proof in the case of $r \equiv 1 \bmod 2$.

- $r \equiv 0 \bmod 2$.

  Let $r = 2r_0$. We proceed as before and first compute $Kl_{p^l,p^k}(0,n)$. The computation once again depends on the values of l and k.

  - $l = k = 0$.

    Once again $Kl_{1,1}(0,n) = 1$.

  - $l > 0$, $k = 0$.

    In this case the result of (i) is still valid.

  - $r_0 \geq k > 0$, $l = 0$.

    In this case (iii) is still valid.

  - $k > r_0$, $l = 0$.

    In this case we need to compute,

$$Kl_{1,p^k}(0,n) = \sum_{\substack{a \bmod p^{2k} \\ a^2 \equiv 4n \bmod p^{2k}}} 1$$

Let $n = p^{2r_0}n_0$. Then the sum vanishes unless $\left(\frac{n_0}{p}\right) = 1$. If this is the case then we have,

$$\begin{aligned} Kl_{1,p^k}(0,n) &= \sum_{\substack{a \bmod p^{2k} \\ a^2 \equiv 4n \bmod p^{2k}}} 1 \\ &= \sum_{\substack{a_0 \bmod p^{2k-r_0} \\ a_0^2 \equiv 4n_0 \bmod p^{2k-2r_0}}} 1 \\ &= 2p^{r_0} \end{aligned}$$

Therefore in this case,

$$Kl_{1,p^k}(0,n) = \left(1 + \left(\frac{n_0}{p}\right)\right) p^{r_0} \tag{iv}$$

– $r_0 > k > 0$, $l > 0$.

  In this case (ii) is still valid.

– $k = r_0$, $l > 0$.

  In this case we have,

$$\begin{aligned} Kl_{p^l,p^k}(0,n) &= \sum_{\substack{a \bmod p^{l+2k} \\ a^2 \equiv 4n \bmod p^{2k}}} \left(\frac{(a^2-4n)/p^{2k}}{p^l}\right) \\ &= \sum_{a_0 \bmod p^{l+k}} \left(\frac{a_0^2-4n_0}{p^l}\right) \\ &= \begin{cases} -p^{l+k-1} & \text{if } l \equiv 1 \bmod 2 \\ p^{l+k} - p^{l+k-1}\left(1 + \left(\frac{n_0}{p}\right)\right) & \text{if } l \equiv 0 \bmod 2 \end{cases} \end{aligned}$$

  Where we used Lemma 2 of Appendix A of [18] in the first line.

– $k > r_0$, $l > 0$.

  In this case we need to compute,

$$Kl_{p^l,p^k}(0,n) = \sum_{\substack{a \bmod p^{l+2k} \\ a^2 \equiv 4n \bmod p^{2k}}} \left(\frac{(a^2-4n)/p^{2k}}{p^l}\right)$$

  Let $n = p^{2r_0}n_0$, where $v_p(n_0) = 0$. Then, since $p \neq 2$, in order to have $a^2 \equiv 4n \bmod p^{2k}$ we need to have $\left(\frac{n_0}{p}\right) = 1$, i.e. n_0 is a square modulo p. This, by Hensel's lemma, implies that n_0 is a square modulo p^{2k-2r_0+1}. Let us assume that this is the case and denote the square-roots (which there are exactly two since $p \neq 2$) of n_0 modulo p^{2k-2r_0+1} by u_1, u_2, i.e. $u_j^2 \equiv n_0 \bmod p^{2k-2r_0+1}$. Then $Kl_{p^l,p^k}(0,n)$ can be written as,

$$\begin{aligned}
Kl_{p^l,p^k}(0,n) &= \sum_{\substack{a \bmod p^{l+2k} \\ a^2 \equiv 4n \bmod p^{2k}}} \left(\frac{(a^2-4n)/p^{2k}}{p^l}\right) \\
&= p^{l-1} \sum_{\substack{a_0 \bmod p^{1+2k} \\ a_0^2 \equiv 4n \bmod p^{2k}}} \left(\frac{(a_0^2-4n)/p^{2k}}{p^l}\right) \\
&= p^{l-1} \sum_{\substack{a_1 \bmod p^{1+2k-r_0} \\ a_1^2 \equiv 4n_0 \bmod p^{2(k-r_0)}}} \left(\frac{(a_1^2-4n_0)/p^{2(k-r_0)}}{p^l}\right) \\
&= p^{l+r_0-1} \sum_{j=1,2} \sum_{a_3 \bmod p} \left(\frac{a_3 u_j}{p^l}\right) \\
&= p^{l+r_0-1} 2 \begin{cases} 0 & \text{if } l \equiv 1 \bmod 2 \\ p-1 & \text{if } l \equiv 0 \bmod 2 \end{cases}
\end{aligned}$$

We therefore get,

$$Kl_{p^l,p^k}(0,n) = p^{l+r_0}\left(1-\tfrac{1}{p}\right)\left(1+\left(\tfrac{n_0}{p}\right)\right) \begin{cases} 0 & \text{if } l \equiv 1 \bmod 2 \\ 1 & \text{if } l \equiv 0 \bmod 2 \end{cases} \qquad (v)$$

We can now compute $D_p(z;n)$. By using (i) to (v) we get,

$$\begin{aligned}
D_p(z;n) &= 1 + \sum_{l=1}^{\infty} \frac{Kl_{p^l,1}(0,n)}{p^{l(z+1)}} + \sum_{k=1}^{r_0} \frac{Kl_{1,p^k}(0,n)}{p^{k(2z+1)}} + \sum_{k=r_0+1}^{\infty} \frac{Kl_{1,p^k}(0,n)}{p^{k(2z+1)}} \\
&\quad + \sum_{k=1}^{r_0-1} \frac{1}{p^{k(2z+1)}} \sum_{l=1}^{\infty} \frac{Kl_{p^l,p^k}(0,n)}{p^{l(z+1)}} + \frac{1}{p^{r_0(2z+1)}} \sum_{l=1}^{\infty} \frac{Kl_{p^l,p^{r_0}}(0,n)}{p^{l(z+1)}} \\
&\quad + \sum_{k=r_0+1}^{\infty} \frac{1}{p^{k(2z+1)}} \sum_{l=1}^{\infty} \frac{Kl_{p^l,p^k}(0,n)}{p^{l(z+1)}} \\
&= 1 + \left(1-\tfrac{1}{p}\right) \sum_{l=1}^{\infty} \frac{1}{p^{lz}} + \sum_{k=1}^{r_0} \frac{1}{p^{2kz}} + p^{r_0}\left(1+\left(\tfrac{n_0}{p}\right)\right) \sum_{k=r_0+1}^{\infty} \frac{1}{p^{k(2z+1)}} \\
&\quad + \left(1-\tfrac{1}{p}\right) \sum_{k=1}^{r_0-1} \frac{1}{p^{2kz}} \sum_{l=1}^{\infty} \frac{1}{p^{lz}} + \frac{p^{r_0}}{p^{r_0(2z+1)}} \left(1-\tfrac{1}{p}\left(1+\left(\tfrac{n_0}{p}\right)\right)\right) \sum_{l=1}^{\infty} \frac{1}{p^{2lz}} \\
&\quad - \frac{p^{r_0-1}}{p^{r_0(2z+1)}} \sum_{l=0}^{\infty} \frac{1}{p^{(2l+1)z}} + p^{r_0}\left(1-\tfrac{1}{p}\right)\left(1+\left(\tfrac{n_0}{p}\right)\right) \sum_{k=r_0+1}^{\infty} \frac{1}{p^{k(2z+1)}} \sum_{l=1}^{\infty} \frac{1}{p^{2lz}}
\end{aligned}$$

$$
\begin{aligned}
&= 1 + \left(1 - \tfrac{1}{p}\right) \sum_{l=1}^{\infty} \tfrac{1}{p^{lz}} + \sum_{k=1}^{r_0} \tfrac{1}{p^{2kz}} + \left(1 - \tfrac{1}{p}\right) \sum_{k=1}^{r_0-1} \tfrac{1}{p^{2kz}} \sum_{l=1}^{\infty} \tfrac{1}{p^{lz}} \\
&\qquad + \tfrac{p^{r_0}}{p^{r_0(2z+1)}} \sum_{l=1}^{\infty} \tfrac{1}{p^{2lz}} - \tfrac{p^{r_0-1}}{p^{r_0(2z+1)}} \sum_{l=0}^{\infty} \tfrac{1}{p^{(2l+1)z}} \\
&= 1 + \left(1 - \tfrac{1}{p}\right) \sum_{k=0}^{r_0-1} \tfrac{1}{p^{2kz}} \sum_{l=1}^{\infty} \tfrac{1}{p^{lz}} + \sum_{k=1}^{r_0} \tfrac{1}{p^{2kz}} + \frac{\left(\frac{1}{p^z} - \frac{1}{p}\right)}{p^{(2r_0+1)z}\left(1 - \frac{1}{p^{2z}}\right)} \\
&= \frac{\left(1 - \frac{1}{p^{2(r_0+1)z}}\right)}{\left(1 - \frac{1}{p^{2z}}\right)} + \frac{\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p^{2r_0 z}}\right)}{p^z\left(1 - \frac{1}{p^{2z}}\right)\left(1 - \frac{1}{p^z}\right)} + \frac{\left(\frac{1}{p^z} - \frac{1}{p}\right)}{p^{(2r_0+1)z}\left(1 - \frac{1}{p^{2z}}\right)} \\
&= 1 + \sum_{k=0}^{r_0-1} \tfrac{1}{p^{2kz}} \sum_{l=1}^{\infty} \tfrac{1}{p^{lz}} + \sum_{k=1}^{r_0} \tfrac{1}{p^{2kz}} - \tfrac{1}{p} \sum_{k=0}^{r_0-1} \tfrac{1}{p^{2kz}} \sum_{l=1}^{\infty} \tfrac{1}{p^{lz}} - \frac{1}{p^{(2r_0+1)z+1}\left(1 - \frac{1}{p^{2z}}\right)} \\
&= 1 + \frac{\left(1 - \frac{1}{p^{2r_0 z}}\right)}{p^{2z}\left(1 - \frac{1}{p^z}\right)\left(1 - \frac{1}{p^{2z}}\right)} - \frac{\left(1 - \frac{1}{p^{(2r_0+1)z}}\right)}{p^{z+1}\left(1 - \frac{1}{p^z}\right)\left(1 - \frac{1}{p^{2z}}\right)} \\
&= \frac{\left(1 - \frac{1}{p^{(2r_0+1)z}}\right)\left(1 - \frac{1}{p^{z+1}}\right)}{\left(1 - \frac{1}{p^z}\right)\left(1 - \frac{1}{p^{2z}}\right)}
\end{aligned}
$$

This finishes the proof of the lemma.

□

Lemma 5.3.4.

$$
D_2(z;n) = 4 \begin{cases} \dfrac{\left(1 - \frac{1}{2^{z+1}}\right)}{\left(1 - \frac{1}{2^{2z}}\right)} & \textit{if } 2 \nmid n \\[2ex] \dfrac{\left(1 - \frac{1}{2^{z(v_2(n)+1)}}\right)\left(1 - \frac{1}{2^{z+1}}\right)}{\left(1 - \frac{1}{2^{2z}}\right)\left(1 - \frac{1}{2^z}\right)} & \textit{if } 2 \mid n \end{cases}
$$

Proof. The proof follows the same lines of the proofs of lemmas 5.3.2 and 5.3.3. The the only difference is we use properties of the Kronecker symbol modulo 2. We skip the details.

□

Corollary 5.3.5.

$$
D(z;n) = 4 \tfrac{\zeta(2z)}{\zeta(z+1)} \prod_{p|n} \left(1 - \tfrac{1}{p^{z(v_p(n)+1)}}\right) \left(1 - \tfrac{1}{p^z}\right)^{-1}
$$

Proof. Follows from lemmas 5.3.2, 5.3.3 and 5.3.4.

□

5.3.2 Analysis of (5.6) for $\xi = 0$

In this subsection we will analyze the term corresponding to $\xi = 0$ in (5.6). Let

$$
(5.6)_{\xi=0} := \sqrt{n} \sum_{f=1}^{\infty} \tfrac{1}{f^3} \sum_{t=1}^{\infty} \tfrac{1}{t^2} \left\{ \int_{|x|>1} \theta_\infty(x)\, F\left(\tfrac{t f^2 (4n)^{-\alpha}}{(x^2-1)^\alpha}\right) dx \right\} Kl_{t,f}(0,n)
$$

Lemma 5.3.6.

$$(5.6)_{\xi=0} = D(1;n)\sqrt{n}\int_{|x|>1}\theta_\infty(x)\,dx$$
$$+\frac{2\sqrt{n}\tilde{F}(\frac{-1}{2})}{\zeta(\frac{3}{2})}\int_{|x|>1}\theta_\infty(x)\left(\frac{(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)^{\frac{1}{2}}dx\prod_{p|n}\frac{1-p^{-(v_p(n)+1)/2}}{1-p^{1/2}}$$
$$+\sqrt{n}\int_{|x|>1}\theta_\infty(x)\left[\frac{1}{2\pi i}\int_{(-1)}\tilde{F}(u)D(u+1;n)\left(\frac{(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)^{-u}du\right]dx$$

Proof. By Mellin inversion we have,

$$F(y) = \frac{1}{2\pi i}\int_{(1)}\tilde{F}(u)y^{-u}du$$

Substituting this into the expression for $(5.6)_{\xi=0}$ leaves,

$$\sqrt{n}\sum_{f=1}^{\infty}\frac{1}{f^3}\sum_{t=1}^{\infty}\frac{1}{t^2}\left\{\int_{|x|>1}\theta_\infty(x)\left[\frac{1}{2\pi i}\int_{(1)}\tilde{F}(u)\left(\frac{tf^2(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)^{-u}du\right]dx\right\}Kl_{t,f}(0,n)$$

Interchanging the t and f-sums with the integral (everything converges a bsolutely) we have

$$\sqrt{n}\int_{|x|>1}\theta_\infty(x)\left[\frac{1}{2\pi i}\int_{(1)}\tilde{F}(u)\left\{\sum_{f=1}^{\infty}\frac{1}{f^{2u+3}}\sum_{t=1}^{\infty}\frac{Kl_{t,f}(0,n)}{t^{u+2}}\right\}\left(\frac{(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)^{-u}du\right]dx$$
$$=\sqrt{n}\int_{|x|>1}\theta_\infty(x)\left[\frac{1}{2\pi i}\int_{(1)}\tilde{F}(u)D(u+1;n)\left(\frac{(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)^{-u}du\right]dx \qquad (\bullet)$$

By Corollary 5.3.5 we know that $D(u+1;n) = 4\frac{\zeta(2(u+1))}{\zeta(u+2)}\prod_{p|n}\left(1-\frac{1}{p^{(u+1)(v_p(n)+1)}}\right)\left(1-\frac{1}{p^{(u+1)}}\right)^{-1}$. In particular this shows that $D(u+1;n)$ is holomorphic for $\Re(u) > \frac{-1}{2}$, has a simple pole at $u = \frac{-1}{2}$ with residue[3] $\frac{2}{\zeta(\frac{3}{2})}\prod_{p|n}\left(1-\frac{1}{p^{\frac{(v_p(n)+1)}{2}}}\right)\left(1-\frac{1}{p^{\frac{1}{2}}}\right)^{-1}$, and is holomorphic for $-1 \le \Re(u) < \frac{-1}{2}$.

Also recall that by Lemma 3.2.2 we know that $\tilde{F}(u)$ is holomorphic everywhere except for a simple pole at $u = 0$ with residue 1.

Therefore, shifting the count our of integration in the u-integral to $\frac{-1}{2} < \Re(u) = \sigma < 0$ only picks up the pole of $\tilde{F}(u)$ at $u = 0$ and gives the equality,

$$(\bullet) = D(1;n)\sqrt{n}\int_{|x|>1}\theta_\infty(x)\,dx$$
$$+\sqrt{n}\int_{|x|>1}\theta_\infty(x)\left[\frac{1}{2\pi i}\int_{(\sigma)}\tilde{F}(u)D(u+1;n)\left(\frac{(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)^{-u}du\right]dx$$

[3] 2 is not a typo

Then moving the σ-contour to $-1 < \Re(u) = \sigma' < \frac{-1}{2}$ picks up the pole of $D(1+u;n)$ at $u = \frac{-1}{2}$ and leaves,

$$(\bullet) = D(1;n)\sqrt{n}\int_{|x|>1}\theta_\infty(x)\,dx + \frac{2\sqrt{n}\tilde{F}\left(\frac{-1}{2}\right)}{\zeta\left(\frac{3}{2}\right)}\int_{|x|>1}\theta_\infty(x)\left(\frac{(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)^{\frac{1}{2}}dx\prod_{p|n}\frac{1-p^{-(v_p(n)+1)/2}}{1-p^{1/2}}$$
$$+\sqrt{n}\int_{|x|>1}\theta_\infty(x)\left[\frac{1}{2\pi i}\int_{(\sigma')}\tilde{F}(u)D(u+1;n)\left(\frac{(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)^{-u}du\right]dx$$

Finally note that $D(u+1;n)$ is holomorphic all the way to $\Re(u) = -1$ ($D(u+1;n)$ is easily shown to be 0 at $u = -1$ (L'Hospital's rule)), and by moving the σ'-contour to $\Re(u) = -1$ we finally get

$$(\bullet) = D(1;n)\sqrt{n}\int_{|x|>1}\theta_\infty(x)\,dx + \frac{2\sqrt{n}\tilde{F}\left(\frac{-1}{2}\right)}{\zeta\left(\frac{3}{2}\right)}\int_{|x|>1}\theta_\infty(x)\left(\frac{(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)^{\frac{1}{2}}dx\prod_{p|n}\frac{1-p^{-(v_p(n)+1)/2}}{1-p^{1/2}}$$
$$+\sqrt{n}\int_{|x|>1}\theta_\infty(x)\left[\frac{1}{2\pi i}\int_{(-1)}\tilde{F}(u)D(u+1;n)\left(\frac{(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)^{-u}du\right]dx$$

□

5.3.3 Analysis of (5.7) for $\xi = 0$

Let,

$$(5.7)_{\xi=0} := \sqrt{n}\sum_{f=1}^{\infty}\frac{1}{f^3}\sum_{t=1}^{\infty}\frac{1}{t^2}\left\{\int_{|x|<1}\theta_\infty(x)\,F\left(\frac{tf^2(4n)^{-\alpha}}{(1-x^2)^\alpha}\right)dx\right\}Kl_{t,f}(0,n)$$

Lemma 5.3.7.

$$\begin{aligned}(5.7)_{\xi=0} &= D(1;n)\sqrt{n}\int_{|x|<1}\theta_\infty(x)\,dx\\ &+ \frac{2\sqrt{n}\tilde{F}\left(\frac{-1}{2}\right)}{\zeta\left(\frac{3}{2}\right)}\int_{|x|<1}\theta_\infty(x)\left(\frac{(4n)^{-\alpha}}{(1-x^2)^\alpha}\right)^{\frac{1}{2}}dx\prod_{p|n}\frac{1-p^{-(v_p(n)+1)/2}}{1-p^{1/2}}\\ &+\sqrt{n}\int_{|x|<1}\theta_\infty(x)\left[\frac{1}{2\pi i}\int_{(-1)}\tilde{F}(u)D(u+1;n)\left(\frac{(4n)^{-\alpha}}{(1-x^2)^\alpha}\right)^{-u}du\right]dx\end{aligned}$$

Proof. Identical to the proof of Lemma 5.3.6.

□

5.3.4 Analysis of (5.8) for $\xi = 0$

Let,

$$(5.8)_{\xi=0} := \frac{\sqrt{\pi}}{2}\sum_{f=1}^{\infty}\frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\left\{\int_{|x|>1}\frac{\theta_\infty(x)}{\sqrt{x^2-1}}H_+\left(\frac{tf^2(4n)^{\alpha-1}}{(x^2-1)^{1-\alpha}}\right)dx\right\}Kl_{t,f}(0,n)$$

Lemma 5.3.8.

$$(5.8)_{\xi=0} = \frac{2\sqrt{n}\tilde{F}\left(\frac{1}{2}\right)}{\zeta\left(\frac{3}{2}\right)} \int_{|x|>1} \frac{\theta_\infty(x)}{\sqrt{x^2-1}} \left(\frac{(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)^{\frac{1}{2}} dx \prod_{p|n} \frac{1-p^{-(v_p(n)+1)/2}}{1-p^{1/2}} - \frac{\prod_{p|n}(v_p(n)+1)}{2} \int_{|x|>1} \frac{\theta_\infty(x)}{\sqrt{x^2-1}} dx$$
$$+ \frac{\sqrt{\pi}}{2} \int_{|x|>1} \frac{\theta_\infty(x)}{\sqrt{x^2-1}} \left[\frac{1}{2\pi i} \int_{\mathcal{C}_\epsilon} \frac{\Gamma\left(\frac{u}{2}\right)\tilde{F}(u)}{\Gamma\left(\frac{1-u}{2}\right)} D(u;n) \left(\frac{\pi(4n)^{\alpha-1}}{(x^2-1)^{1-\alpha}}\right)^{-u} du\right] dx$$

Where $\epsilon > 0$ be such that $\zeta(u+1)$ does not have any zeros for $|u| < \epsilon$ (Such an epsilon exists since $\zeta(u+1)$ is non-zero at $u = 0$ and the zeta function is meropmorphic.),
$\mathcal{C}_\epsilon = \{(0, it) \mid t \in (-\infty, -\epsilon) \cup (\epsilon, \infty)\} \cup C_\epsilon$, and C_ϵ denotes the left-half of the circle of radius ϵ around 0.

Proof. Recall the definition of $H_+(x)$,

$$H_+(x) = \frac{1}{2\pi i} \int_{(1)} \frac{\Gamma\left(\frac{u}{2}\right)}{\Gamma\left(\frac{1-u}{2}\right)} (\pi x)^{-u} \tilde{F}(u) du$$

Substituting this into $(5.8)_{\xi=0}$ gives

$$(5.8)_{\xi=0} = \frac{\sqrt{\pi}}{2} \sum_{f=1}^{\infty} \frac{1}{f} \sum_{t=1}^{\infty} \frac{1}{t} \left\{ \int_{|x|>1} \frac{\theta_\infty(x)}{\sqrt{x^2-1}} \left[\frac{1}{2\pi i} \int_{(1)} \frac{\Gamma\left(\frac{u}{2}\right)\tilde{F}(u)}{\Gamma\left(\frac{1-u}{2}\right)} \left(\frac{\pi t f^2 (4n)^{\alpha-1}}{(x^2-1)^{1-\alpha}}\right)^{-u} du\right] dx \right\} Kl_{t,f}(0,n)$$
$$= \frac{\sqrt{\pi}}{2} \int_{|x|>1} \frac{\theta_\infty(x)}{\sqrt{x^2-1}} \left[\frac{1}{2\pi i} \int_{(1)} \frac{\Gamma\left(\frac{u}{2}\right)\tilde{F}(u)}{\Gamma\left(\frac{1-u}{2}\right)} \left\{ \sum_{f=1}^{\infty} \frac{1}{f^{2u+1}} \sum_{t=1}^{\infty} \frac{Kl_{t,f}(0,n)}{t^{u+1}} \right\} \left(\frac{\pi(4n)^{\alpha-1}}{(x^2-1)^{1-\alpha}}\right)^{-u} du\right] dx$$
$$= \frac{\sqrt{\pi}}{2} \int_{|x|>1} \frac{\theta_\infty(x)}{\sqrt{x^2-1}} \left[\frac{1}{2\pi i} \int_{(1)} \frac{\Gamma\left(\frac{u}{2}\right)\tilde{F}(u)}{\Gamma\left(\frac{1-u}{2}\right)} D(u;n) \left(\frac{\pi(4n)^{\alpha-1}}{(x^2-1)^{1-\alpha}}\right)^{-u} du\right] dx \qquad (\bullet)$$

Where the interchange of sums and the integral is justified by the absolute convergence for $\Re(u)$ large.

Recall that $D(u;n) = 4\frac{\zeta(2u)}{\zeta(u+1)} \prod_{p|n} \left(1 - \frac{1}{p^{u(v_p(n)+1)}}\right) \left(1 - \frac{1}{p^u}\right)^{-1}$. We see that $D(u;n)$ has a simple pole at $u = \frac{1}{2}$ with residue $\frac{2}{\zeta\left(\frac{3}{2}\right)} \prod_{p|n} \left(1 - \frac{1}{p^{(v_p(n)+1)/2}}\right) \left(1 - \frac{1}{p^{1/2}}\right)^{-1}$. Hence shifting the contour to $0 < \Re(u) = \sigma < \frac{1}{2}$ picks up the pole of $D(u;n)$ at $u = \frac{1}{2}$ and gives,

$$(\bullet) = \frac{2\sqrt{n}\tilde{F}\left(\frac{1}{2}\right)}{\zeta\left(\frac{3}{2}\right)} \int_{|x|>1} \theta_\infty(x) \left(\frac{(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)^{\frac{1}{2}} dx \prod_{p|n} \frac{1-p^{-(v_p(n)+1)/2}}{1-p^{1/2}}$$
$$+ \frac{\sqrt{\pi}}{2} \int_{|x|>1} \frac{\theta_\infty(x)}{\sqrt{x^2-1}} \left[\frac{1}{2\pi i} \int_{(\sigma)} \frac{\Gamma\left(\frac{u}{2}\right)\tilde{F}(u)}{\Gamma\left(\frac{1-u}{2}\right)} D(u;n) \left(\frac{\pi(4n)^{\alpha-1}}{(x^2-1)^{1-\alpha}}\right)^{-u} du\right] dx$$

Now note the following: $\Gamma\left(\frac{u}{2}\right)$ has a simple pole at $u = 0$ (with residue 2),
$D(u;n) = u4\zeta(2u) \prod_{p|n} (v_p(n)+1) + O(u^2)$ around $u = 0$ (by L'Hospital), and finally $\tilde{F}(u)$ has a simple pole at $u = 0$ (with residue 1). Therefore we see that the product $\frac{\Gamma\left(\frac{u}{2}\right)\tilde{F}(u)}{\Gamma\left(\frac{1-u}{2}\right)} D(u;n)$ has a simple pole at $u = 0$ with residue $\frac{-4\prod_{p|n}(v_p(n)+1)}{\sqrt{\pi}}$.

By the above paragraph, moving the line of integration to $\mathcal{C}_\epsilon$ picks up the simple pole of the product $\frac{\Gamma(\frac{u}{2})\tilde{F}(u)}{\Gamma(\frac{1-u}{2})}D(u;n)$ with residue $\frac{-4\prod_{p|n}(v_p(n)+1)}{\sqrt{\pi}}$. Therefore we have,

$$(\bullet) = \frac{2\sqrt{n}\tilde{F}(\frac{1}{2})}{\zeta(\frac{3}{2})}\int_{|x|>1}\theta_\infty(x)\left(\frac{(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)^{\frac{1}{2}}dx\prod_{p|n}\frac{1-p^{-(v_p(n)+1)/2}}{1-p^{1/2}} - 2\prod_{p|n}(v_p(n)+1)\int_{|x|>1}\frac{\theta_\infty(x)}{\sqrt{x^2-1}}dx$$
$$+\frac{\sqrt{\pi}}{2}\int_{|x|>1}\frac{\theta_\infty(x)}{\sqrt{x^2-1}}\left[\frac{1}{2\pi i}\int_{\mathcal{C}_\epsilon}\frac{\Gamma(\frac{u}{2})\tilde{F}(u)}{\Gamma(\frac{1-u}{2})}D(u;n)\left(\frac{\pi(4n)^{\alpha-1}}{(x^2-1)^{1-\alpha}}\right)^{-u}du\right]dx$$

The lemma follows.

□

5.3.5 Analysis of (5.9) for $\xi = 0$

Let,

$$(5.9)_{\xi=0} := \frac{\sqrt{\pi}}{2}\sum_{f=1}^{\infty}\frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\left\{\int_{|x|<1}\frac{\theta_\infty(x)}{\sqrt{1-x^2}}H_-\left(\frac{tf^2(4n)^{\alpha-1}}{(1-x^2)^{1-\alpha}}\right)dx\right\}Kl_{t,f}(0,n)$$

Lemma 5.3.9.

$$(5.9)_{\xi=0} = \frac{2\sqrt{n}\tilde{F}(\frac{1}{2})}{\zeta(\frac{3}{2})}\int_{|x|<1}\theta_\infty(x)\left(\frac{(4n)^{-\alpha}}{(1-x^2)^\alpha}\right)^{\frac{1}{2}}dx\prod_{p|n}\frac{1-p^{-(v_p(n)+1)/2}}{1-p^{1/2}}$$
$$+\frac{\sqrt{\pi}}{2}\int_{|x|<1}\frac{\theta_\infty(x)}{\sqrt{1-x^2}}\left[\frac{1}{2\pi i}\int_{\mathcal{C}_\epsilon}\frac{\Gamma(\frac{1+u}{2})\tilde{F}(u)}{\Gamma(\frac{2-u}{2})}D(u;n)\left(\frac{\pi(4n)^{\alpha-1}}{(1-x^2)^{1-\alpha}}\right)^{-u}du\right]dx$$

Where $\epsilon > 0$ be such that $\zeta(u+1)$ does not have any zeros for $|u| < \epsilon$ (Such an epsilon exists since $\zeta(u+1)$ is non-zero at $u = 0$ and the zeta function is meropmorphic.),
$\mathcal{C}_\epsilon = \{(0, it) \mid t \in (-\infty, -\epsilon) \cup (\epsilon, \infty)\} \cup C_\epsilon$, and C_ϵ denotes the left-half of the circle of radius ϵ around 0.

Proof. The proof goes exactly as in the proof of Lemma 5.3.8. We shift the contour picking up the pole of $D(u;n)$ at $u = \frac{1}{2}$. The only difference is that this time the Γ-quotient, $\frac{\Gamma(\frac{1+u}{2})}{\Gamma(\frac{2-u}{2})}$ is holomorphic at $u = 0$ therefore we do not pick up the second term in the first line of Lemma 5.3.8.

□

5.4 $N < 0$.

We essentially just change n to $-n$ and everything in the previous sections go through without a change. We just note that when $N < 0$, we necessarily have $m^2 - 4N = m^2 + 4n > 0$, hence we do not have to break the summation range for m into two cases as in the equations (5.4′-a) to (5.5′-b). Let us start by the analogue of Lemma 5.1.1.

Lemma 5.4.1. *For any $A > 0$ we have*

$$(5.1)_{N<0} = 2 \sum_{\substack{m\in\mathbb{Z} \\ m^2+4n\neq\square}} \theta_\infty^-\left(\tfrac{m}{2\sqrt{n}}\right) \left\{ \sum_{\substack{f^2|(m^2+4n) \\ \frac{m^2+4n}{f^2}\equiv 0,1 \bmod 4}} \tfrac{1}{f} \sum_{t=1}^{\infty} \tfrac{1}{t} \left(\tfrac{(m^2+4n)/f^2}{t}\right) F\left(\tfrac{tf^2}{A}\right) \right. \tag{5.11}$$

$$\left. + \sum_{\substack{f^2|(m^2+4n) \\ \frac{m^2+4n}{f^2}\equiv 0,1 \bmod 4}} \left(\tfrac{\pi f^2}{m^2+4n}\right)^{\frac{1}{2}} \sum_{t=1}^{\infty} \left(\tfrac{(m^2+4n)/f^2}{t}\right) H_+\left(\tfrac{tf^2A}{m^2+4n}\right) \right\} \tag{5.12}$$

Where,

$$H_+(x) := \tfrac{1}{2\pi i} \int_{(1)} \frac{\Gamma\left(\frac{u}{2}\right)}{\Gamma\left(\frac{1-u}{2}\right)} (\pi x)^{-u} \tilde{F}(u)du$$

Proof. Follows from Corollary 3.2.4. □

Next we add and subtract the split elements.

$$(5.1)_{N<0} = 2 \sum_{m\in\mathbb{Z}} \theta_\infty^-\left(\tfrac{m}{2\sqrt{n}}\right) \left\{ \sum_{\substack{f^2|(m^2+4n) \\ \frac{m^2+4n}{f^2}\equiv 0,1 \bmod 4}} \tfrac{1}{f} \sum_{t=1}^{\infty} \tfrac{1}{t} \left(\tfrac{(m^2+4n)/f^2}{t}\right) F\left(\tfrac{tf^2}{A}\right) \right. \tag{5.11$'$}$$

$$\left. + \sum_{\substack{f^2|(m^2+4n) \\ \frac{m^2+4n}{f^2}\equiv 0,1 \bmod 4}} \left(\tfrac{\pi f^2}{m^2+4n}\right)^{\frac{1}{2}} \sum_{t=1}^{\infty} \left(\tfrac{(m^2+4n)/f^2}{t}\right) H_+\left(\tfrac{tf^2A}{m^2+4n}\right) \right\} \tag{5.12$'$}$$

$$-2 \sum_{\substack{m\in\mathbb{Z} \\ m^2+4n=\square}} \theta_\infty^-\left(\tfrac{m}{2\sqrt{n}}\right) \left\{ \sum_{\substack{f^2|(m^2+4n) \\ \frac{m^2+4n}{f^2}\equiv 0,1 \bmod 4}} \tfrac{1}{f} \sum_{t=1}^{\infty} \tfrac{1}{t} \left(\tfrac{(m^2+4n)/f^2}{t}\right) F\left(\tfrac{tf^2}{A}\right) \right. \tag{$\dagger\dagger\dagger$}$$

$$\left. + \sum_{\substack{f^2|(m^2+4n) \\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \left(\tfrac{\pi f^2}{m^2+4n}\right)^{\frac{1}{2}} \sum_{t=1}^{\infty} \left(\tfrac{(m^2+4n)/f^2}{t}\right) H_+\left(\tfrac{tf^2A}{m^2+4n}\right) \right\} \tag{$\dagger\dagger\dagger\dagger$}$$

Once again, we will leave ($\dagger\dagger$) and ($\dagger\dagger\dagger\dagger$) to be handled later.

As in equations (5.4) and (5.5) we interchange the f-sum with the m-sum, rewrite (5.11′) and (5.12′), and choose $A = (m^2+4n)^\alpha$. This results in,

$$(5.11') = 2\sum_{f=1}^{\infty} \sum_{\substack{m\in\mathbb{Z} \\ f^2|(m^2+4n) \\ \frac{m^2+4n}{f^2}\equiv 0,1 \bmod 4}} \theta_\infty^-\left(\frac{m}{2\sqrt{n}}\right)\frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\left(\frac{(m^2+4n)/f^2}{t}\right)F\left(\frac{tf^2}{(m^2+4n)^\alpha}\right) \tag{5.13}$$

$$(5.12') = 2\sqrt{\pi}\sum_{f=1}^{\infty} f \sum_{\substack{m\in\mathbb{Z} \\ f^2|(m^2+4n) \\ \frac{m^2+4n}{f^2}\equiv 0,1 \bmod 4}} \theta_\infty^-\left(\frac{m}{2\sqrt{n}}\right)\frac{1}{\sqrt{m^2+4n}}\sum_{t=1}^{\infty}\left(\frac{(m^2+4n)/f^2}{t}\right)H_+\left(\frac{tf^2}{m^2+4n^{1-\alpha}}\right) \tag{5.14}$$

We note that since m^2+4n do not need to go through the definitions of $\Theta_{\infty,F,\alpha,+}$ or $\Theta_{\infty,H,\alpha,+}$ since m^2+4n is always positive.

Next step are the truncations in Proposition 5.1.2. The analogue of Corollary 5.1.3 is:
Corollary 5.4.2. *For any $\delta_0 > 0$ we have,*

$$(5.13) = 2\sum_{f<n^{\frac{\alpha+\delta_0}{2}}} \sum_{\substack{m\in\mathbb{Z} \\ f^2|(m^2+4n) \\ \frac{m^2+4n}{f^2}\equiv 0,1 \bmod 4}} \frac{1}{f}\sum_{t<\frac{n^{\alpha+\delta_1}}{f^2}}\frac{1}{t}\left(\frac{(m^2+4n)/f^2}{t}\right)\theta_\infty^-\left(\frac{m}{2\sqrt{n}}\right)F\left(\frac{tf^2}{(m^2+4n)^\alpha}\right)$$
$$+ O_{\theta_\infty^-}(\sqrt{n}\log^2(n)e^{-n^{2\delta_0}} + \sqrt{n}\log(n)e^{-n^{\delta_1}})$$

$$(5.14) = 2\sqrt{\pi}\sum_{f<n^{\frac{1-\alpha+\delta_0}{2}}} f \sum_{\substack{m\in\mathbb{Z} \\ f^2|(m^2+4n) \\ \frac{m^2+4n}{f^2}\equiv 0,1 \bmod 4}} \sum_{t<\frac{n^{1-\alpha+\delta_1}}{f^2}}\left(\frac{(m^2+4n)/f^2}{t}\right)\frac{\theta_\infty^-\left(\frac{m}{2\sqrt{n}}\right)}{\sqrt{m^2+4n}}H_+\left(\frac{tf^2}{(m^2+4n)^{1-\alpha}}\right)$$
$$+ O_{\theta_\infty^-}(n^{1-\alpha}\log^2(n)e^{-\frac{n^{\delta_0}}{2}} + n^{1-\alpha}\log(n)e^{\frac{-n^{\frac{\delta_1}{2}}}{2}})$$

Next is the application of Poisson summation. The following lemma is the analogue of lemmas 5.2.1 and 5.2.3.
Lemma 5.4.3.

$$(5.13) = \sqrt{n}\sum_{f=1}^{\infty}\frac{1}{f^3}\sum_{t=1}^{\infty}\frac{1}{t^2}\sum_{\xi\in\mathbb{Z}}\left\{\int_{\mathbb{R}}\theta_\infty^-(x)\,F\left(\frac{tf^2(4n)^{-\alpha}}{(x^2+1)^\alpha}\right)e\left(\frac{-x\xi\sqrt{n}}{2tf^2}\right)dx\right\}Kl_{t,f}(\xi,-n) \tag{5.15}$$

$$(5.14) = \frac{\sqrt{\pi}}{2}\sum_{f=1}^{\infty}\frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\sum_{\xi\in\mathbb{Z}}\left\{\int_{\mathbb{R}}\frac{\theta_\infty^-(x)}{\sqrt{x^2+1}}H_+\left(\frac{tf^2(4n)^{\alpha-1}}{(x^2+1)^{1-\alpha}}\right)e\left(\frac{-x\sqrt{n}\xi}{2tf^2}\right)dx\right\}Kl_{t,f}(\xi,-n) \tag{5.16}$$

Where,

$$Kl_{t,f}(\xi,n) := \sum_{\substack{a \bmod 4tf^2 \\ a^2-4n\equiv 0 \bmod f^2 \\ \frac{a^2-4n}{f^2}\equiv 0,1 \bmod 4}}\left(\frac{(a^2-4n)/f^2}{t}\right)e\left(\frac{a\xi}{4tf^2}\right)$$

We are only left with analyzing the terms corresponding to $\xi = 0$ in (5.15) and (5.16). For that we simply note that the Dirichlet series $D(z;n)$ does not depend on the sign of n, therefore $D(z;n) = D(z;-n)$.

The analogue of Lemma 5.3.6 is
Lemma 5.4.4.

$$(5.15)_{\xi=0} = D(1;n)\sqrt{n}\int_{\mathbb{R}} \theta^{-}_{\infty}(x)\,dx + \frac{2\sqrt{n}\tilde{F}\left(\frac{-1}{2}\right)}{\zeta\left(\frac{3}{2}\right)}\int_{\mathbb{R}} \theta^{-}_{\infty}(x)\left(\frac{(4n)^{-\alpha}}{(x^2+1)^{\alpha}}\right)^{\frac{1}{2}} dx \prod_{p|n} \frac{1-p^{-(v_p(n)+1)/2}}{1-p^{1/2}} + \sqrt{n}\int_{|x|>1} \theta^{-}_{\infty}(x)\left[\frac{1}{2\pi i}\int_{(-1)} \tilde{F}(u)D(u+1;n)\left(\frac{(4n)^{-\alpha}}{(x^2+1)^{\alpha}}\right)^{-u} du\right] dx$$

The analogue of Lemma 5.3.8 is
Lemma 5.4.5.

$$(5.16)_{\xi=0} = \frac{2\sqrt{n}\tilde{F}\left(\frac{1}{2}\right)}{\zeta\left(\frac{3}{2}\right)}\int_{\mathbb{R}} \frac{\theta^{-}_{\infty}(x)}{\sqrt{x^2+1}}\left(\frac{(4n)^{-\alpha}}{(x^2+1)^{\alpha}}\right)^{\frac{1}{2}} dx \prod_{p|n}\frac{1-p^{-(v_p(n)+1)/2}}{1-p^{1/2}} - 2\prod_{p|n}(v_p(n)+1)\int_{\mathbb{R}} \frac{\theta^{-}_{\infty}(x)}{\sqrt{x^2+1}}dx + \frac{\sqrt{\pi}}{2}\int_{\mathbb{R}} \frac{\theta^{-}_{\infty}(x)}{\sqrt{x^2+1}}\left[\frac{1}{2\pi i}\int_{\mathcal{C}_{\epsilon}} \frac{\Gamma\left(\frac{u}{2}\right)\tilde{F}(u)}{\Gamma\left(\frac{1-u}{2}\right)} D(u;n)\left(\frac{\pi(4n)^{\alpha-1}}{(x^2+1)^{1-\alpha}}\right)^{-u} du\right] dx$$

Where $\epsilon > 0$ be such that $\zeta(u+1)$ does not have any zeros for $|u| < \epsilon$ (Such an epsilon exists since $\zeta(u+1)$ is non-zero at $u = 0$ and the zeta function is meropmorphic.),
$\mathcal{C}_{\epsilon} = \{(0, it) \mid t \in (-\infty, -\epsilon) \cup (\epsilon, \infty)\} \cup C_{\epsilon}$, and C_{ϵ} denotes the left-half of the circle of radius ϵ around 0.

5.5 Identification of the contribution of special representations and a summary of results so far

In this section we will combine the results of the previous sections, identify the contribution of the trivial representation to the trace formula, as given in equation (65) of [18], and the contribution given in equation (32) in [18][4].

We start by rewriting the elliptic part of the trace formula after Poisson summation.

Lemma 5.5.1. *For any $0 < \alpha < 1$ we have,*

$$\begin{aligned}
(5.1) = {} & \sqrt{n}\sum_{f=1}^{\infty}\tfrac{1}{f^3}\sum_{t=1}^{\infty}\tfrac{1}{t^2}\sum_{\xi\in\mathbb{Z}}\left\{\int_{|x|>1}\theta_\infty^+(x)\,F\left(\tfrac{tf^2(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)e\left(\tfrac{-x\xi\sqrt{n}}{2tf^2}\right)dx\right\}Kl_{t,f}(\xi,n)\\
&+\sqrt{n}\sum_{f=1}^{\infty}\tfrac{1}{f^3}\sum_{t=1}^{\infty}\tfrac{1}{t^2}\sum_{\xi\in\mathbb{Z}}\left\{\int_{|x|<1}\theta_\infty^+(x)\,F\left(\tfrac{tf^2(4n)^{-\alpha}}{(1-x^2)^\alpha}\right)e\left(\tfrac{-x\xi\sqrt{n}}{2tf^2}\right)dx\right\}Kl_{t,f}(\xi,n)\\
&+\tfrac{\sqrt{\pi}}{2}\sum_{f=1}^{\infty}\tfrac{1}{f}\sum_{t=1}^{\infty}\tfrac{1}{t}\sum_{\xi\in\mathbb{Z}}\left\{\int_{|x|>1}\tfrac{\theta_\infty^+(x)}{\sqrt{x^2-1}}H_+\left(\tfrac{tf^2(4n)^{\alpha-1}}{(x^2-1)^{1-\alpha}}\right)e\left(\tfrac{-x\sqrt{n}\xi}{2tf^2}\right)dx\right\}Kl_{t,f}(\xi,n)\\
&+\tfrac{\sqrt{\pi}}{2}\sum_{f=1}^{\infty}\tfrac{1}{f}\sum_{t=1}^{\infty}\tfrac{1}{t}\sum_{\xi\in\mathbb{Z}}\left\{\int_{|x|<1}\tfrac{\theta_\infty^+(x)}{\sqrt{1-x^2}}H_-\left(\tfrac{tf^2(4n)^{\alpha-1}}{(1-x^2)^{1-\alpha}}\right)e\left(\tfrac{-x\sqrt{n}\xi}{2tf^2}\right)dx\right\}Kl_{t,f}(\xi,n)\\
&+2\sum_{f=1}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ f^2|(m^2+4n)\\ \frac{m^2+4n}{f^2}\equiv 0,1 \bmod 4}}\theta_\infty^-\left(\tfrac{m}{2\sqrt{n}}\right)\tfrac{1}{f}\sum_{t=1}^{\infty}\tfrac{1}{t}\left(\tfrac{(m^2+4n)/f^2}{t}\right)F\left(\tfrac{tf^2}{(m^2+4n)^\alpha}\right)\\
&+2\sqrt{\pi}\sum_{f=1}^{\infty}f\sum_{\substack{m\in\mathbb{Z}\\ f^2|(m^2+4n)\\ \frac{m^2+4n}{f^2}\equiv 0,1 \bmod 4}}\theta_\infty^-\left(\tfrac{m}{2\sqrt{n}}\right)\tfrac{1}{\sqrt{m^2+4n}}\sum_{t=1}^{\infty}\left(\tfrac{(m^2+4n)/f^2}{t}\right)H_+\left(\tfrac{tf^2}{m^2+4n^{1-\alpha}}\right)\\
&-\left((\dagger)+(\dagger\dagger)+(\dagger\dagger\dagger)+(\dagger\dagger\dagger\dagger)\right)
\end{aligned}$$

Proof. Just collected the results of previous sections. ∎

Next, we isolate the contribution of the trivial representation in the sum of the $\xi = 0$ terms.

Theorem 5.5.2. *Let $(5.1)^{\pm}_{\xi=0}$ denote the contribution of the $\xi = 0$ term to the Poisson sum where $\pm$ denotes the sign of N in (5.1). Then,*

[4] We note that there is a slight error in equation (65) of the [18]. The integer m in the exponent of p should be $m + 1$.

$$
\begin{aligned}
(5.1)^{-}_{\xi=0} + (5.1)^{+}_{\xi=0} &= \operatorname{tr}(\mathbf{1}(f)) - \operatorname{tr}(\xi_0(f)) \\
&+ \sqrt{n}\int_{|x|>1} \theta^{+}_{\infty}(x) \left[\frac{1}{2\pi i}\int_{(-1)} \tilde{F}(u) D(u+1;n) \left(\frac{(4n)^{-\alpha}}{(x^2-1)^{\alpha}}\right)^{-u} du\right] dx \\
&+ \sqrt{n}\int_{|x|<1} \theta^{+}_{\infty}(x) \left[\frac{1}{2\pi i}\int_{(-1)} \tilde{F}(u) D(u+1;n) \left(\frac{(4n)^{-\alpha}}{(1-x^2)^{\alpha}}\right)^{-u} du\right] dx \\
&+ \frac{\sqrt{\pi}}{2}\int_{|x|>1} \frac{\theta^{+}_{\infty}(x)}{\sqrt{x^2-1}} \left[\frac{1}{2\pi i}\int_{\mathcal{C}_\epsilon} \frac{\Gamma\left(\frac{u}{2}\right)\tilde{F}(u)}{\Gamma\left(\frac{1-u}{2}\right)} D(u;n) \left(\frac{\pi(4n)^{\alpha-1}}{(x^2-1)^{1-\alpha}}\right)^{-u} du\right] dx \\
&+ \frac{\sqrt{\pi}}{2}\int_{|x|<1} \frac{\theta^{+}_{\infty}(x)}{\sqrt{1-x^2}} \left[\frac{1}{2\pi i}\int_{\mathcal{C}_\epsilon} \frac{\Gamma\left(\frac{1+u}{2}\right)\tilde{F}(u)}{\Gamma\left(\frac{2-u}{2}\right)} D(u;n) \left(\frac{\pi(4n)^{\alpha-1}}{(1-x^2)^{1-\alpha}}\right)^{-u} du\right] dx \\
&+ \sqrt{n}\int_{|x|>1} \theta^{-}_{\infty}(x) \left[\frac{1}{2\pi i}\int_{(-1)} \tilde{F}(u) D(u+1;n) \left(\frac{(4n)^{-\alpha}}{(x^2+1)^{\alpha}}\right)^{-u} du\right] dx \\
&+ \frac{\sqrt{\pi}}{2}\int_{\mathbb{R}} \frac{\theta^{-}_{\infty}(x)}{\sqrt{x^2+1}} \left[\frac{1}{2\pi i}\int_{\mathcal{C}_\epsilon} \frac{\Gamma\left(\frac{u}{2}\right)\tilde{F}(u)}{\Gamma\left(\frac{1-u}{2}\right)} D(u;n) \left(\frac{\pi(4n)^{\alpha-1}}{(x^2+1)^{1-\alpha}}\right)^{-u} du\right] dx
\end{aligned}
$$

Where the notation ξ_0 is as in equation (32) of [18].

Proof. By lemmas 5.3.6, 5.3.7, 5.3.8 and 5.3.9 we have,

$$
\begin{aligned}
(5.1)^{-}_{\xi=0} &= D(1;n)\sqrt{n}\int_{\mathbb{R}} \theta_{\infty}(x)\, dx && ((5.1)^{-}_{\xi=0}\text{-a}) \\
&+ \frac{2\sqrt{n}\tilde{F}\left(\frac{-1}{2}\right)}{\zeta\left(\frac{3}{2}\right)}\int_{|x|>1} \theta_{\infty}(x) \left(\frac{(4n)^{-\alpha}}{(x^2-1)^{\alpha}}\right)^{\frac{1}{2}} dx \prod_{p|n} \frac{1-p^{-(v_p(n)+1)/2}}{1-p^{1/2}} && ((5.1)^{-}_{\xi=0}\text{-b}) \\
&+ \frac{2\sqrt{n}\tilde{F}\left(\frac{1}{2}\right)}{\zeta\left(\frac{3}{2}\right)}\int_{|x|>1} \theta_{\infty}(x) \left(\frac{(4n)^{-\alpha}}{(x^2-1)^{\alpha}}\right)^{\frac{1}{2}} dx \prod_{p|n} \frac{1-p^{-(v_p(n)+1)/2}}{1-p^{1/2}} && ((5.1)^{-}_{\xi=0}\text{-c}) \\
&+ \frac{2\sqrt{n}\tilde{F}\left(\frac{-1}{2}\right)}{\zeta\left(\frac{3}{2}\right)}\int_{|x|<1} \theta_{\infty}(x) \left(\frac{(4n)^{-\alpha}}{(1-x^2)^{\alpha}}\right)^{\frac{1}{2}} dx \prod_{p|n} \frac{1-p^{-(v_p(n)+1)/2}}{1-p^{1/2}} && ((5.1)^{-}_{\xi=0}\text{-d}) \\
&+ \frac{2\sqrt{n}\tilde{F}\left(\frac{1}{2}\right)}{\zeta\left(\frac{3}{2}\right)}\int_{|x|<1} \theta_{\infty}(x) \left(\frac{(4n)^{-\alpha}}{(1-x^2)^{\alpha}}\right)^{\frac{1}{2}} dx \prod_{p|n} \frac{1-p^{-(v_p(n)+1)/2}}{1-p^{1/2}} && ((5.1)^{-}_{\xi=0}\text{-e}) \\
&- 2\prod_{p|n}(v_p(n)+1)\int_{|x|>1} \frac{\theta_{\infty}(x)}{\sqrt{x^2-1}} dx && ((5.1)^{-}_{\xi=0}\text{-f}) \\
&+ \sqrt{n}\int_{|x|>1} \theta_{\infty}(x) \left[\frac{1}{2\pi i}\int_{(-1)} \tilde{F}(u) D(u+1;n) \left(\frac{(4n)^{-\alpha}}{(x^2-1)^{\alpha}}\right)^{-u} du\right] dx && ((5.1)^{-}_{\xi=0}\text{-g}) \\
&+ \sqrt{n}\int_{|x|<1} \theta_{\infty}(x) \left[\frac{1}{2\pi i}\int_{(-1)} \tilde{F}(u) D(u+1;n) \left(\frac{(4n)^{-\alpha}}{(1-x^2)^{\alpha}}\right)^{-u} du\right] dx && ((5.1)^{-}_{\xi=0}\text{-h}) \\
&+ \frac{\sqrt{\pi}}{2}\int_{|x|>1} \frac{\theta_{\infty}(x)}{\sqrt{x^2-1}} \left[\frac{1}{2\pi i}\int_{\mathcal{C}_\epsilon} \frac{\Gamma\left(\frac{u}{2}\right)\tilde{F}(u)}{\Gamma\left(\frac{1-u}{2}\right)} D(u;n) \left(\frac{\pi(4n)^{\alpha-1}}{(x^2-1)^{1-\alpha}}\right)^{-u} du\right] dx && ((5.1)^{-}_{\xi=0}\text{-i}) \\
&+ \frac{\sqrt{\pi}}{2}\int_{|x|<1} \frac{\theta_{\infty}(x)}{\sqrt{1-x^2}} \left[\frac{1}{2\pi i}\int_{\mathcal{C}_\epsilon} \frac{\Gamma\left(\frac{1+u}{2}\right)\tilde{F}(u)}{\Gamma\left(\frac{2-u}{2}\right)} D(u;n) \left(\frac{\pi(4n)^{\alpha-1}}{(1-x^2)^{1-\alpha}}\right)^{-u} du\right] dx && ((5.1)^{-}_{\xi=0}\text{-j})
\end{aligned}
$$

On the other hand lemmas 5.4.4 and 5.4.5 gives,

$$(5.1)^{+}_{\xi=0} = D(1;n)\sqrt{n}\int_{\mathbb{R}} \theta^{-}_{\infty}(x)\,dx \tag{$(5.1)^+_{\xi=0}$-a}$$

$$+ \frac{2\sqrt{n}\tilde{F}\left(\frac{-1}{2}\right)}{\zeta\left(\frac{3}{2}\right)}\int_{\mathbb{R}} \theta^{-}_{\infty}(x)\left(\frac{(4n)^{-\alpha}}{(x^2-1)^{\alpha}}\right)^{\frac{1}{2}} dx \prod_{p|n}\frac{1-p^{-(v_p(n)+1)/2}}{1-p^{1/2}} \tag{$(5.1)^+_{\xi=0}$-b}$$

$$+ \frac{2\sqrt{n}\tilde{F}\left(\frac{1}{2}\right)}{\zeta\left(\frac{3}{2}\right)}\int_{\mathbb{R}} \frac{\theta^{-}_{\infty}(x)}{\sqrt{x^2+1}}\left(\frac{(4n)^{-\alpha}}{(x^2+1)^{\alpha}}\right)^{\frac{1}{2}} dx \prod_{p|n}\frac{1-p^{-(v_p(n)+1)/2}}{1-p^{1/2}} \tag{$(5.1)^+_{\xi=0}$-c}$$

$$- 2\prod_{p|n}(v_p(n)+1)\int_{\mathbb{R}} \frac{\theta^{-}_{\infty}(x)}{\sqrt{x^2+1}}dx \tag{$(5.1)^+_{\xi=0}$-d}$$

$$+ \sqrt{n}\int_{|x|>1} \theta^{-}_{\infty}(x)\left[\frac{1}{2\pi i}\int_{(-1)} \tilde{F}(u)D(u+1;n)\left(\frac{(4n)^{-\alpha}}{(x^2+1)^{\alpha}}\right)^{-u} du\right] dx \tag{$(5.1)^+_{\xi=0}$-e}$$

$$+ \frac{\sqrt{\pi}}{2}\int_{\mathbb{R}} \frac{\theta^{-}_{\infty}(x)}{\sqrt{x^2+1}}\left[\frac{1}{2\pi i}\int_{\mathcal{C}_\epsilon} \frac{\Gamma\left(\frac{u}{2}\right)\tilde{F}(u)}{\Gamma\left(\frac{1-u}{2}\right)}D(u;n)\left(\frac{\pi(4n)^{\alpha-1}}{(x^2+1)^{1-\alpha}}\right)^{-u} du\right] dx \tag{$(5.1)^+_{\xi=0}$-f}$$

Recall that $D(1;n) = 4\frac{1-\frac{1}{p^{v_p(n)+1}}}{1-\frac{1}{p}}$, therefore by comparing the formulas for $((5.1)^{-}_{\xi=0}\text{-a})+((5.1)^{+}_{\xi=0}\text{-a})$ with equation[5] (65) of [18] we first get:

$$\operatorname{tr}(\mathbf{1}(f)) = ((5.1)^{-}_{\xi=0}\text{-a}) + ((5.1)^{+}_{\xi=0}\text{-a}) \tag{$\star$}$$

Then by comparing the sum $((5.1)^{-}_{\xi=0}\text{-f})+((5.1)^{+}_{\xi=0}\text{-d})$ with equation (32) of [18] we see that (with the notation of [18])

$$-\operatorname{tr}(\xi_0(f)) = ((5.1)^{-}_{\xi=0}\text{-f}) + ((5.1)^{+}_{\xi=0}\text{-d}) \tag{$\star\star$}$$

Now recall that by Lemma 3.2.2 we know that $\tilde{F}$ is odd[6]. Therefore we get

$$((5.1)^{-}_{\xi=0}\text{-b}) + ((5.1)^{-}_{\xi=0}\text{-c}) = 0 \tag{$\bullet$}$$

$$((5.1)^{-}_{\xi=0}\text{-d}) + ((5.1)^{-}_{\xi=0}\text{-e}) = 0 \tag{$\bullet\bullet$}$$

$$((5.1)^{+}_{\xi=0}\text{-b}) + ((5.1)^{+}_{\xi=0}\text{-c}) = 0 \tag{$\bullet\bullet\bullet$}$$

Finally

Combining $(\star)$, $(\star\star)$, $(\bullet)$, $(\bullet\bullet)$, $(\bullet\bullet\bullet)$ finishes the proof of the proposition. □

We will now show that the extra terms in Theorem 5.5.2 are small, which will be useful in §8.3. We note that the estimate in Proposition 5.5.3 is very crude and one can actually do better with slightly more effort (One can for example use the zero-free region of the ζ-function together with an integration by parts argument.), however since we will only be using this proposition in §8.3 we do not aim for the best bound. We also note that once we further average the sums in Proposition 5.5.3 over n, then one gets even better bounds (cf. §9.4 and §9.5).

[5] The integer m of equation (65) of [18] (which should be $m+1$) in our notation is $v_p(n)$.

[6] We note that the oddness of $\tilde{F}$ is completely inessential for the argument. The whole argument is valid for an arbitrary $\tilde{F}$. If $\tilde{F}$ is not odd, then we would get $-\tilde{F}(-v)$ in the dual part of the approximate functional equation.

Proposition 5.5.3. *For any $\kappa > 0$,*

$$
\begin{aligned}
&+ \sqrt{n} \int_{|x|>1} \theta_\infty^+(x) \left[\frac{1}{2\pi i} \int_{(-1)} \tilde{F}(u) D(u+1;n) \left(\frac{(4n)^{-\alpha}}{(x^2-1)^\alpha} \right)^{-u} du \right] dx \\
&+ \sqrt{n} \int_{|x|<1} \theta_\infty^+(x) \left[\frac{1}{2\pi i} \int_{(-1)} \tilde{F}(u) D(u+1;n) \left(\frac{(4n)^{-\alpha}}{(1-x^2)^\alpha} \right)^{-u} du \right] dx \\
&+ \frac{\sqrt{\pi}}{2} \int_{|x|>1} \frac{\theta_\infty^+(x)}{\sqrt{x^2-1}} \left[\frac{1}{2\pi i} \int_{\mathcal{C}_\epsilon} \frac{\Gamma\left(\frac{u}{2}\right)\tilde{F}(u)}{\Gamma\left(\frac{1-u}{2}\right)} D(u;n) \left(\frac{\pi(4n)^{\alpha-1}}{(x^2-1)^{1-\alpha}} \right)^{-u} du \right] dx \\
&+ \frac{\sqrt{\pi}}{2} \int_{|x|<1} \frac{\theta_\infty^+(x)}{\sqrt{1-x^2}} \left[\frac{1}{2\pi i} \int_{\mathcal{C}_\epsilon} \frac{\Gamma\left(\frac{1+u}{2}\right)\tilde{F}(u)}{\Gamma\left(\frac{2-u}{2}\right)} D(u;n) \left(\frac{\pi(4n)^{\alpha-1}}{(1-x^2)^{1-\alpha}} \right)^{-u} du \right] dx \\
&+ \sqrt{n} \int_{|x|>1} \theta_\infty^-(x) \left[\frac{1}{2\pi i} \int_{(-1)} \tilde{F}(u) D(u+1;n) \left(\frac{(4n)^{-\alpha}}{(x^2+1)^\alpha} \right)^{-u} du \right] dx \\
&+ \frac{\sqrt{\pi}}{2} \int_{\mathbb{R}} \frac{\theta_\infty^-(x)}{\sqrt{x^2+1}} \left[\frac{1}{2\pi i} \int_{\mathcal{C}_\epsilon} \frac{\Gamma\left(\frac{u}{2}\right)\tilde{F}(u)}{\Gamma\left(\frac{1-u}{2}\right)} D(u;n) \left(\frac{\pi(4n)^{\alpha-1}}{(x^2+1)^{1-\alpha}} \right)^{-u} du \right] dx \\
&= O_{\theta_\infty, F, \alpha, \epsilon}\left(n^{\frac{1}{2}-\alpha+\kappa} + 1 \right)
\end{aligned}
$$

Proof. Observing that for $\Re(u) = 0$

$$
\begin{aligned}
D(u;n) &= 4 \frac{\zeta(2u)}{\zeta(u+1)} \prod_{p|n} \left(1 - \frac{1}{p^{u(v_p(n)+1)}}\right) \left(1 - \frac{1}{p^u}\right)^{-1} \\
&= 4 \frac{\zeta(2u)}{\zeta(u+1)} \sum_{d|n} \frac{1}{d^u} \\
&= O(\sigma_0(n)) \\
&= O_\kappa(n^\kappa)
\end{aligned}
$$

for any $\kappa > 0$. The lemma follows.

$\square$

5.6 Final form of the formula

After applying Poisson summation to the elliptic term we have the following expression for any $\delta_0, \epsilon, \kappa > 0$:

$$
\begin{aligned}
&+\sqrt{n}\sum_{f\ll n^{\frac{\alpha+\delta_0}{2}}}\frac{1}{f^3}\sum_{\substack{t\\ tf^2\ll n^{\alpha+\delta_0}}}\frac{1}{t^2}\sum_{\xi\in\mathbb{Z}\setminus\{0\}}Kl_{t,f}(\xi,n)\\
&\qquad\left\{\int_{|x|>1}\theta_\infty^+(x)\,F\left(\frac{tf^2(4n)^{-\alpha}}{(x^2-1)^\alpha}\right)e\left(\frac{-x\xi\sqrt{n}}{2tf^2}\right)dx+\int_{|x|<1}\theta_\infty^+(x)\,F\left(\frac{tf^2(4n)^{-\alpha}}{(1-x^2)^\alpha}\right)e\left(\frac{-x\xi\sqrt{n}}{2tf^2}\right)dx\right\}\\
&+\frac{\sqrt{\pi}}{2}\sum_{f<n^{\frac{1-\alpha+\delta_0}{2}}}\frac{1}{f}\sum_{\substack{t\\ tf^2\ll n^{1-\alpha+\delta_0}}}\frac{1}{t}\sum_{\xi\in\mathbb{Z}\setminus\{0\}}Kl_{t,f}(\xi,n)\\
&\qquad\left\{\int_{|x|>1}\frac{\theta_\infty^+(x)}{\sqrt{x^2-1}}H_+\left(\frac{tf^2(4n)^{\alpha-1}}{(x^2-1)^{1-\alpha}}\right)e\left(\frac{-x\sqrt{n}\xi}{2tf^2}\right)dx+\int_{|x|<1}\frac{\theta_\infty^+(x)}{\sqrt{1-x^2}}H_-\left(\frac{tf^2(4n)^{\alpha-1}}{(1-x^2)^{1-\alpha}}\right)e\left(\frac{-x\sqrt{n}\xi}{2tf^2}\right)dx\right\}\\
&+\sqrt{n}\sum_{f<n^{\frac{\alpha+\delta_0}{2}}}\frac{1}{f^3}\sum_{\substack{t\\ tf^2\ll n^{\alpha+\delta_0}}}\frac{1}{t^2}\sum_{\xi\in\mathbb{Z}\setminus\{0\}}Kl_{t,f}(\xi,n)\left\{\int_{\mathbb{R}}\theta_\infty^-(x)\,F\left(\frac{tf^2(4n)^{-\alpha}}{(x^2+1)^\alpha}\right)e\left(\frac{-x\xi\sqrt{n}}{2tf^2}\right)dx\right\}\\
&+\frac{\sqrt{\pi}}{2}\sum_{f<n^{\frac{1-\alpha+\delta_0}{2}}}\frac{1}{f}\sum_{\substack{t\\ tf^2\ll n^{1-\alpha+\delta_0}}}\frac{1}{t}\sum_{\xi\in\mathbb{Z}\setminus\{0\}}Kl_{t,f}(\xi,n)\left\{\int_{\mathbb{R}}\frac{\theta_\infty^-(x)}{\sqrt{x^2+1}}H_+\left(\frac{tf^2(4n)^{\alpha-1}}{(x^2+1)^{1-\alpha}}\right)e\left(\frac{-x\sqrt{n}\xi}{2tf^2}\right)dx\right\}\\
&+\mathrm{tr}(\mathbf{1}(f))-\mathrm{tr}(\xi_0(f))+O(n^{\frac{1}{2}-\alpha+\kappa}+1)
\end{aligned}
$$

Chapter 6

Analysis of Certain Character Sums

In this chapter we will analyze the character sums $Kl_{t,f}(\xi,n)$ and certain relatives of them, denoted by $\omega^{\{1\}}_{t,f}(\xi,\nu)$ (see (6.4)), and give bounds on them. The bounds on $Kl_{t,f}(\xi,\nu)$ we will derive will essentially be the Weil bound on the Kloosterman sums [30]. The important factor in the bounds will be the dependence on the t and f-parameters, since those will be summed over later. The bound on $\omega^{\{1\}}_{t,f}(\xi,\nu)$ is obtained by explicit computation of the sum.

6.1 $Kl_{t,f}(\xi,n)$

We start by $Kl_{t,f}(\xi,\nu)$. Note the following twisted multiplicativity of the sums in question which will allow us to reduce bounding the sums for general tuples (t,f) to prime powers.

Before stating the lemma let us introduce some notation. For any integer $a\in\mathbb{Z}\backslash\{0\}$ and any prime p, let $v_p(a)$ denote the p-adic valuation of a as usual. Also, for any two non-zero integer $A,B\in\mathbb{Z}\backslash\{0\}$ let A_B denote the B-part of A. i.e.

$$A_B := \left\{ C \mid \mathbb{Z}_{\geq 0} \mid \; p \mid C \Rightarrow p \mid B \;,\; \gcd\left(\tfrac{A}{C}, B\right) = 1 \right\} \tag{$\star$}$$

Note that with this notation we have $p^{v_p(A)} = A_p$ for any prime p.

Lemma 6.1.1. *Let $t,f\in\mathbb{Z}_{>0}$ and $\xi\in\mathbb{Z}$. Then,*

$$Kl_{t,f}(\xi,n) = \prod_p Kl_{t_p,f_p}\left(\left(\tfrac{4tf^2}{(4tf^2)_p}\right)_p^{-1}\xi, n\right)$$

Where for any $a,b\in\mathbb{Z}$ such that $\gcd(a,b)=1$, $(a)_b^{-1}$ denotes the inverse of a mod b.

Proof. By the Chinese remainder theorem we know that the map

$$\begin{aligned}(a_{p_1},a_{p_2},\cdots,a_{p_l}) &\xrightarrow{\varphi} \sum_{j=1}^{l} a_{p_j} \tfrac{4tf^2}{(4tf^2)_{p_j}} \left(\tfrac{4tf^2}{(4tf^2)_{p_j}}\right)_{p_j}^{-1} \\ &:= \varphi(a_{p_1},\cdots,a_{p_l})\end{aligned}$$

gives an isomorphism of $\prod_{j=1}^{l}\mathbb{Z}/(4tf^2)_{p_j}\mathbb{Z}$ onto $\mathbb{Z}/4tf^2\mathbb{Z}$. Furthermore note that $\varphi(a_1,\cdots,a_l)\equiv a_{p_j} \bmod (4tf^2)_{p_j}$ Therefore we have,

$$
\begin{aligned}
Kl_{t,f}(\xi,n) &= \sum_{\substack{a \bmod 4tf^2 \\ a^2-4n\equiv 0 \bmod f^2 \\ \frac{a^2-4n}{f^2_{p_j}}\equiv 0,1 \bmod 4}} \left(\frac{(a^2-4n)/f^2}{t}\right) e\left(\frac{a\xi}{4tf^2}\right) \\
&= \sum_{j=1}^{l} \sum_{\substack{a_{p_j} \bmod (4tf^2)_{p_j} \\ a_{p_j}^2-4n\equiv 0 \bmod f^2_{p_j} \\ \frac{a_{p_j}^2-4n}{f^2_{p_j}}\equiv 0,1 \bmod 4_{p_j}}} e\left(\frac{\varphi(a_{p_1},\cdots,a_{p_l})\xi}{4tf^2}\right) \prod_{j=1}^{l} \left(\frac{(\varphi(a_{p_1},\cdots,a_{p_l})^2-4n)/f^2}{t_{p_j}}\right) \\
&= \sum_{j=1}^{l} \sum_{\substack{a_{p_j} \bmod (4tf^2)_{p_j} \\ a_{p_j}^2-4n\equiv 0 \bmod f^2_{p_j} \\ \frac{a_{p_j}^2-4n}{f^2_{p_j}}\equiv 0,1 \bmod 4_{p_j}}} e\left(\frac{\left(\sum_{j=1}^{l} a_{p_j}\frac{4tf^2}{(4tf^2)_{p_j}}\left(\frac{4tf^2}{(4tf^2)_{p_j}}\right)^{-1}_{p_j}\right)\xi}{4tf^2}\right) \prod_{j=1}^{l} \left(\frac{(a_{p_j}^2-4n)/f^2_{p_j}}{t_{p_j}}\right) \\
&= \sum_{j=1}^{l} \sum_{\substack{a_{p_j} \bmod (4tf^2)_{p_j} \\ a_{p_j}^2-4n\equiv 0 \bmod f^2_{p_j} \\ \frac{a_{p_j}^2-4n}{f^2_{p_j}}\equiv 0,1 \bmod 4_{p_j}}} \prod_{j=1}^{l} \left(\frac{(a_{p_j}^2-4n)/f^2_{p_j}}{t_{p_j}}\right) e\left(\frac{a_{p_j}\left(\frac{4tf^2}{(4tf^2)_{p_j}}\right)^{-1}_{p_j}\xi}{(4tf^2)_{p_j}}\right) \\
&= \prod_{j=1}^{l} \sum_{\substack{a_{p_j} \bmod (4tf^2)_{p_j} \\ a_{p_j}^2-4n\equiv 0 \bmod f^2_{p_j} \\ \frac{a_{p_j}^2-4n}{f^2_{p_j}}\equiv 0,1 \bmod 4_{p_j}}} \left(\frac{(a_{p_j}^2-4n)/f^2_{p_j}}{t_{p_j}}\right) e\left(\frac{a_{p_j}\left(\frac{4tf^2}{(4tf^2)_{p_j}}\right)^{-1}_{p_j}\xi}{(4tf^2)_{p_j}}\right) \\
&= \prod_{p} Kl_{t_p,f_p}\left(\left(\frac{4tf^2}{(4tf^2)_p}\right)^{-1}_{p}\xi, n\right)
\end{aligned}
$$

□

Lemma 6.1.1 reduces bounding $Kl_{t,f}(\xi,n)$ to bounding[1]

$$
Kl_{p^{k_1},p^{k_2}}(\xi,n) = \sum_{\substack{a \bmod p^{k_1+2k_2} \\ a^2-4n\equiv 0 \bmod p^{2k_2} \\ \frac{a^2-4n}{p^{2k_2}}\equiv 0,1 \bmod 4}} \left(\frac{(a^2-4n)/p^{2k_2}}{p^{k_1}}\right) e\left(\frac{a\xi}{p^{k_1+2k_2}}\right)
$$

where p is a prime and k_1, k_2 and k_3 are non-negative integers. We start by bounding the sum when $k_2 = 0$ and $p \neq 2$.

[1]By abuse of notation we absorb the factor of $\left(\frac{4tf^2}{(4tf^2)_p}\right)^{-1}_p$ is ξ. Note that this does not change the bounds to follow since $\left(\frac{4tf^2}{(4tf^2)_p}\right)^{-1}_p$ is relatively prime to p.

Lemma 6.1.2. *Let $p \neq 2$ be a prime. Then,*

$$Kl_{p,1}(\xi, n) = \begin{cases} p-1 & \text{if } p \mid \xi \text{ and } p \mid 4n \\ O(\sqrt{p}) & \text{otherwise} \end{cases}$$

Proof. By definition,

$$Kl_{p,1}(\xi, n) = \sum_{a \bmod p} \left(\frac{a^2-4n}{p}\right) e\left(\frac{a\xi}{p}\right)$$

We divide the proof into two cases according to $p \mid \xi$ or not.

- $\gcd(\xi, p) = p$.

  In this case we have $e\left(\frac{a\xi}{p}\right) = 1$ so the sum reduces to

  $$\sum_{a \bmod p} \left(\frac{a^2-4n}{p}\right) = \begin{cases} p-1 & \text{if } p \mid 4n \\ -1 & \text{if } p \nmid 4n \end{cases}$$

- $\gcd(\xi, p) = 1$.

  In this case $Kl_{p,1}(\xi, n)$ is a Kloosterman sum (see [**?**] equation (70)). Therefore by the Weil bound (cf. [30]) we then have

  $$Kl_{p,1}(\xi, n) \leq 2\sqrt{p}$$

The lemma follows. □

Corollary 6.1.3.

$$Kl_{p^{k_1},1}(\xi, n) = \begin{cases} O(p^{k_1}) & \text{if } v_p(\xi) \geq k_1 \\ O(p^{k_1-\frac{1}{2}}) & \text{if } v_p(\xi) = k_1 - 1 \\ 0 & \text{if } v_p(\xi) < k_1 - 1 \end{cases}$$

Proof. There are two cases depending on the parity of k_1.

- $k_1 \equiv 0 \bmod 2$. In this case the sum reduces to

$$\begin{aligned}
Kl_{p^{k_1},1} &= \sum_{a \bmod p^{k_1}} \left(\frac{a^2-4n}{p^{k_1}}\right) e\left(\frac{a\xi}{p^{k_1}}\right) \\
&= \sum_{\substack{a \bmod p^{k_1} \\ a^2-4n \neq 0 \bmod p}} e\left(\frac{a\xi}{p^{k_1}}\right) \\
&= \sum_{\substack{a \bmod p \\ a_0^2-4n \neq 0 \bmod p}} e\left(\frac{a_0\xi}{p^{k_1}}\right) \sum_{a_1 \bmod p^{k_1-1}} e\left(\frac{a_1\xi}{p^{k_1-1}}\right) \\
&= \sum_{\substack{a_0 \bmod p \\ a_0^2-4n \neq 0 \bmod p}} e\left(\frac{a_0\xi}{p^{k_1}}\right) \begin{cases} 0 & \text{if } p^{k_1-1} \nmid \xi \\ p^{k_1-1} & \text{if } p^{k_1-1} \mid \xi \end{cases} \\
&= \left\{\begin{matrix} p-\left(1+\left(\frac{4n}{p}\right)\right) & \text{if } p^{k_1} \mid \xi \\ 0 & \text{if } p^{k_1} \nmid \xi \text{ and } \left(\frac{4n}{p}\right) = -1 \\ -1 & \text{if } p^{k_1} \nmid \xi \text{ and } p \mid 4n \\ -2\cos\left(\frac{4\pi\xi\sqrt{n}}{p^{k_1}}\right) & \text{if } p^{k_1} \nmid \xi \text{ and } \left(\frac{4n}{p}\right) = 1 \end{matrix}\right\} \left\{\begin{matrix} 0 & \text{if } p^{k_1-1} \nmid \xi \\ p^{k_1-1} & \text{if } p^{k_1-1} \mid \xi \end{matrix}\right\} \\
&= p^{k_1-1} \begin{cases} 0 & \text{if } v_p(\xi) < k_1 - 1 \\ p-\left(1+\left(\frac{4n}{p}\right)\right) & \text{if } v_p(\xi) \geq k_1 \\ 0 & \text{if } v_p(\xi) = k_1 - 1 \text{ and } \left(\frac{4n}{p}\right) = -1 \\ -1 & \text{if } v_p(\xi) = k_1 - 1 \text{ and } p \mid 4n \\ -2\cos\left(\frac{4\pi\xi\sqrt{n}}{p^{k_1}}\right) & \text{if } v_p(\xi) = k_1 - 1 \text{ and } \left(\frac{4n}{p}\right) = 1 \end{cases} \qquad (\dagger)
\end{aligned}$$

Where $\sqrt{n}$ denotes the square root of n modulo p.

- $k_1 \equiv 1 \bmod 2$. Proceeding as in the previous case we get

$$\begin{aligned}
Kl_{p^{k_1},1} &= \sum_{a \bmod p^{k_1}} \left(\frac{a^2-4n}{p^{k_1}}\right) e\left(\frac{a\xi}{p^{k_1}}\right) \\
&= \sum_{a_0 \bmod p} \left(\frac{a_0^2-4n}{p}\right) e\left(\frac{a_0\xi}{p^{k_1}}\right) \sum_{a_1 \bmod p^{k_1-1}} e\left(\frac{a_1\xi}{p^{k_1-1}}\right) \\
&= \sum_{a_0 \bmod p} \left(\frac{a_0^2-4n}{p}\right) e\left(\frac{a_0\xi}{p^{k_1}}\right) \begin{cases} 0 & \text{if } p^{k_1-1} \nmid \xi \\ p^{k_1-1} & \text{if } p^{k_1-1} \mid \xi \end{cases} \\
&= p^{k_1-1} \begin{cases} 0 & \text{if } v_p(\xi) < k_1 - 1 \\ p-1 & \text{if } v_p(\xi) \geq k_1 \text{ and } p \mid 4n \\ -1 & \text{if } v_p(\xi) \geq k_1 \text{ and } p \nmid 4n \end{cases} \qquad (\dagger\dagger) \\
&\leq p^{k_1-1} \begin{cases} 2\sqrt{p} & \text{if } v_p(\xi) = k_1 - 1 \end{cases}
\end{aligned}$$

Where we used Lemma 6.1.1 to obtain the last inequality.

The lemma now follows from (†) and (††).

□

Next, we will bound the sum when $p \neq 2$ and $k_3 = 0$, but $k_2 \neq 0$.

Proposition 6.1.4. *Let $p \neq 2$ be a prime. Then,*

$$Kl_{p^{k_1},p^{k_2}} = \begin{cases} O\left(p^{k_1+\min\{k_2,\frac{v_p(n)}{2}\}}\right) & \text{if } n \equiv \square \bmod p^{2k_2} \text{ and } v_p(\xi) \geq k_1 + \min\{k_2 + \frac{v_p(n)}{2}\} \\ O\left(p^{k_1+\min\{k_2+\frac{v_p(n)}{2}\}-\frac{1}{2}}\right) & \text{if } n \equiv \square \bmod p^{2k_2} \text{ and } v_p(\xi) = k_1 + \min\{k_2, \frac{v_p(n)}{2}\} - 1 \\ 0 & \text{if } 4n \neq \square \bmod p^{2k_2} \text{ or } v_p(\xi) < k_1 + \min\{k_2, \frac{v_p(n)}{2}\} - 1 \end{cases}$$

Proof. Note that since $p \equiv 1 \bmod 2$ the condition that $\frac{a^2-4n}{f^2} \equiv 0, 1 \bmod 4$ is vacuous. We will once again divide the computations into cases, this time depending on $k_1 = 0$ and $k_1 > 0$.

- $k_1 = 0$.

  In this case we have,

$$Kl_{1,p^{k_2}}(\xi, n) = \sum_{\substack{a \bmod p^{2k_2} \\ a^2-4n\equiv 0 \bmod p^{2k_2}}} e\left(\frac{a\xi}{p^{2k_2}}\right)$$

  The sum is 0 unless n is a square modp^{2k_2}. If n is a square we have two sub-cases depending on $\frac{v_p(n)}{2} \geq k_2$ or not.

  - $\frac{v_p(n)}{2} \geq k_2$.

    In this case $a^2 \equiv 4n \equiv 0 \bmod p^{2k_2}$ which implies (since $p \neq 2$) $a \equiv 0 \bmod p^{k_2}$. Therefore the sum is,

$$\begin{aligned} Kl_{1,p^{k_2}} &= \sum_{\substack{a \bmod p^{2k_2} \\ a\equiv 0 \bmod p^{k_2}}} e\left(\frac{a\xi}{p^{2k_2}}\right) \\ &= \sum_{a_0 \bmod p^{k_2}} e\left(\frac{a_0\xi}{p^{k_2}}\right) \\ &= \begin{cases} 0 & \text{if } v_p(\xi) < k_2 \\ p^{k_2} & \text{if } v_p(\xi) \geq k_2 \end{cases} \end{aligned} \tag{i}$$

  - $\frac{v_p(n)}{2} < k_2$.

    Let $v_p(n) = 2r$ (Note that $v_p(n)$ is even since otherwise the sum is necessarily 0), and $n_0 \bmod {}^\times p^{2k_2-2r}$ be such that $n = n_0^2 p^{2r}$. Then, $a^2 \equiv 2n \bmod p^{2k_2}$ if and only if $a = p^r a_0$ for some $a_0 \bmod p^{2k_2-r}$ where $a_0^2 \equiv 4n_0 \bmod p^{2k_2-2r}$. Finally $a_0^2 \equiv 4n_0 \bmod p^{2k-2r}$ if and only if (note that $p \neq 2$) $a_0 = \pm 2\sqrt{n_0} + a_1 p^{2k-2r}$ for some $a_1 \bmod p^r$, where by abuse of notation we use $\sqrt{n_0}$ to denote a square root of n_0 modulo p^{2k_2-r}. Hence,

$$\begin{aligned} Kl_{1,p^{k_2}} &= \sum_{\substack{a_0 \bmod p^{2k_2-r} \\ a_0^2\equiv 4n_0^2 \bmod p^{2k_2-2r}}} e\left(\frac{a_0\xi}{p^{2k_2-r}}\right) \\ &= 2\cos\left(\frac{2\sqrt{n}\xi}{p^{2k_2}}\right) \sum_{a_1 \bmod p^r} e\left(\frac{a_1\xi}{p^r}\right) \\ &= 2\cos\left(\frac{2\sqrt{n}\xi}{p^{2k_2}}\right) \begin{cases} 0 & \text{if } n \neq \square \bmod p^{2k_2} \text{ or } v_p(\xi) < \frac{v_p(n)}{2} \\ p^{\frac{v_p(n)}{2}} & \text{if } n \equiv \square \bmod p^{2k_2} \text{ and } v_p(\xi) \geq \frac{v_p(n)}{2} \end{cases} \end{aligned} \tag{ii}$$

    Where we use $\sqrt{n}$ to denote the square root of n modulo p^{2k_2}.

Combining (i) and (ii) gives

$$Kl_{1,p^{k_2}}(\xi,n)=\begin{cases}0 & \text{if } n\neq\square \bmod p^{2k_2} \text{ or } v_p(\xi)<\min\{k_2,\frac{v_p(n)}{2}\}\\ p^{k_2} & \text{if } n\equiv\square \bmod p^{2k_2},\ \frac{v_p(n)}{2}\geq k_2 \text{ and } v_p(\xi)\geq k_2\\ 2p^{\frac{v_p(n)}{2}}\cos\left(\frac{2\sqrt{n}\xi}{p^{2k_2}}\right) & \text{if } n\equiv\square \bmod p^{2k_2},\ \frac{v_p(n)}{2}<k_2 \text{ and } v_p(\xi)\geq\frac{v_p(n)}{2}\end{cases}$$

Where $\sqrt{n}$ denotes a square root of n modulo p^{2k_2}. This in particular implies

$$Kl_{1,p^{k_2}}(\xi,n)=\begin{cases}0 & \text{if } 4n\neq\square \bmod p^{2k_2} \text{ or } v_p(\xi)<\min\{k_2,\frac{v_p(n)}{2}\}\\ O\left(p^{\min\{k_2,\frac{v_p(n)}{2}\}}\right) & \text{if } 4n\equiv\square \bmod p^{2k_2} \text{ and } v_p(\xi)\geq\min\{k_2,\frac{v_p(n)}{2}\}\end{cases}\tag{6.1}$$

- $k_1>0$.

We once again note that the sum vanishes unless $4n\equiv\square \bmod p^{2k_2}$. Assuming this let $a=a_0+a_1p^{2k_2+1}$. Then,

$$\begin{aligned}Kl_{p^{k_1},p^{k_2},1}(\xi,n)&=\sum_{\substack{a_0 \bmod p^{2k_2+1}\\ a_0^2\equiv 4n \bmod p^{2k_2}}}\left(\frac{(a_0^2-4n)/p^{2k_2}}{p^{k_1}}\right)e\left(\frac{a_0\xi}{p^{k_1+2k_2}}\right)\sum_{a_1 \bmod p^{k_1-1}}e\left(\frac{a_1\xi}{p^{k_1-1}}\right)\\ &=\sum_{\substack{a_0 \bmod p^{2k_2+1}\\ a_0^2\equiv 4n \bmod p^{2k_2}}}\left(\frac{(a_0^2-4n)/p^{2k_2}}{p^{k_1}}\right)e\left(\frac{a_0\xi}{p^{k_1+2k_2}}\right)\begin{cases}0 & \text{if } v_p(\xi)<k_1-1\\ p^{k_1-1} & \text{if } v_p(\xi)\geq k_1-1\end{cases}\end{aligned}\tag{iii}$$

Therefore the sum vanishes unless $v_p(\xi)\geq k_1-1$. We move on assuming that the sum does not vanish. Let $v_p(n)=2r$ (Note that $v_p(n)$ is even otherwise the sum is 0.) We have two cases according to $r\geq k_2$ or $r<k_2$.

  - $\frac{v_p(n)}{2}\geq k_2$. In this case we need to have $a_0^2\equiv 0 \bmod p^{2k_2}$ therefore $a_0\equiv 0 \bmod p^{k_2}$. Hence the sum becomes,

$$p^{k_1-1}\sum_{\substack{a_0 \bmod p^{2k_2+1}\\ a_0\equiv 0 \bmod p^{k_2}}}\left(\frac{(a_0^2-4n)/p^{2k_2}}{p^{k_1}}\right)e\left(\frac{a_0\xi}{p^{k_1+2k_2}}\right)=p^{k_1-1}\sum_{a_2 \bmod p^{k_2+1}}\left(\frac{a_2^2-4n/p^{2k_2}}{p^{k_1}}\right)e\left(\frac{a_2\xi}{p^{k_1+k_2}}\right)$$

  This last sum was computed in the proof of Corollary 6.1.3. Using this we get,

$$Kl_{p^{k_1},p^{k_2}}(\xi,n)=$$

$$\begin{cases}0 & \text{if } v_p(\xi)<k_1+k_2-1\\ 0 & \text{if } k_1\equiv 0 \bmod 2,\ v_p(\xi)=k_1-1 \text{ and } \left(\frac{4n/p^{2k_2}}{p}\right)=-1\\ -p^{k_1+k_2-1} & \text{if } k_1\equiv 0 \bmod 2,\ v_p(\xi)=k_1-1 \text{ and } p\mid 4n/p^{2k_2}\\ -2p^{k_1+k_2-1}\cos\left(\frac{4\pi\xi\sqrt{n}}{p^{k_1+2k_2}}\right) & \text{if } k_1\equiv 0 \bmod 2,\ v_p(\xi)=k_1-1 \text{ and } \left(\frac{4n/p^{2k_2}}{p}\right)=1\\ p^{k_1+k_2}\left(1-\frac{1+\left(\frac{4n/p^{2k_2}}{p}\right)}{p}\right) & \text{if } k_1\equiv 0 \bmod 2 \text{ and } v_p(\xi)\geq k_1+k_2\\ \varphi\left(p^{k_1+k_2}\right) & \text{if } k_1\equiv 1 \bmod 2,\ v_p(\xi)\geq k_1+k_2 \text{ and } p\mid 4n/p^{2k_2}\\ -p^{k_1+k_2-1} & \text{if } k_1\equiv 1 \bmod 2,\ v_p(\xi)\geq k_1+k_2 \text{ and } p\nmid 4n/p^{2k_2}\\ O\left(p^{k_1+k_2-\frac{1}{2}}\right) & \text{if } k_1\equiv 1 \bmod 2 \text{ and } v_p(\xi)=k_1+k_2-1\end{cases}$$

Which in particular implies,

$$Kl_{p^{k_1},p^{k_2}}(\xi,n) = \begin{cases} O\left(p^{k_1+k_2}\right) & \text{if } v_p(\xi) \geq k_1+k_2 \\ O\left(p^{k_1+k_2-\frac{1}{2}}\right) & \text{if } v_p(\xi) = k_1+k_2-1 \\ 0 & \text{if } v_p(\xi) < k_1+k_2-1 \end{cases}$$

– $\frac{v_p(n)}{2} < k_2$.

We are summing over $a_0 \in \mathbb{Z}/p^{2k_2+1}\mathbb{Z}$ that satisfy $a_0^2 - 4n \equiv 0 \bmod p^{2k_2}$. Let $n = p^{2r}n_0$. Then $a_0^2 \equiv 4n \bmod p^{2k_2}$ if and only if $a_0 = 2p^r a_1$ for some $a_1 \in \mathbb{Z}/p^{2k_2-r+1}\mathbb{Z}$ which satisfies $a_1^2 \equiv n_0 \bmod p^{2k_2-2r}$. Therefore the sum can be written as

$$\sum_{\substack{a_0 \bmod p^{2k_2+1} \\ a_0^2 \equiv 4n \bmod p^{2k_2}}} \left(\frac{(a_0^2-4n)/p^{2k_2}}{p^{k_1}}\right) e\left(\frac{a_0\xi}{p^{k_1+2k_2}}\right)$$
$$= \sum_{\substack{a_1 \bmod p^{2k_2-r+1} \\ a_1^2 \equiv 4n_0 \bmod p^{2k_2-2r}}} \left(\frac{(a_1^2-4n_0)/p^{2k_2-2r}}{p^{k_1}}\right) e\left(\frac{a_1\xi}{p^{k_1+2k_2-r}}\right)$$

Let $a_1 = a_2 + a_3 p^{2k_2-2r+1}$, where $a_2 \in \mathbb{Z}/p^{2k_2-2r+1}\mathbb{Z}$ and $a_3 \in \mathbb{Z}/p^r\mathbb{Z}$. Substituting this into the sum we get,

$$\begin{aligned} &\sum_{\substack{a_1 \bmod p^{2k_2-r+1} \\ a_1^2 \equiv 4n_0 \bmod p^{2k_2-2r}}} \left(\frac{(a_1^2-4n_0)/p^{2k_2-2r}}{p^{k_1}}\right) e\left(\frac{a_1\xi}{p^{k_1+2k_2-r}}\right) \\ &= \sum_{\substack{a_3 \bmod p^r \\ a_2 \bmod p^{2k_2-2r+1} \\ a_2^2 \equiv 4n_0 \bmod p^{2k_2-2r}}} \left(\frac{(a_2^2-4n_0)/p^{2k_2-2r}}{p^{k_1}}\right) e\left(\frac{(a_2+a_3p^{2k_2-2r+1})\xi}{p^{k_1+2k_2-r}}\right) \\ &= \sum_{\substack{a_2 \bmod p^{2k_2-2r+1} \\ a_2^2 \equiv 4n_0 \bmod p^{2k_2-2r}}} \left(\frac{(a_2^2-4n_0)/p^{2k_2-2r}}{p^{k_1}}\right) e\left(\frac{a_2\xi}{p^{k_1+2k_2-r}}\right) \sum_{a_3 \bmod p^r} e\left(\frac{a_3\xi}{p^{k_1+r-1}}\right) \\ &= \sum_{\substack{a_2 \bmod p^{2k_2-2r+1} \\ a_2^2 \equiv 4n_0 \bmod p^{2k_2-2r}}} \left(\frac{(a_2^2-4n_0)/p^{2k_2-2r}}{p^{k_1}}\right) e\left(\frac{a_2\xi}{p^{k_1+2k_2-r}}\right) \begin{cases} 0 & \text{if } v_p(\xi) < k_1+r-1 \\ p^r & \text{if } v_p(\xi) \geq k_1+r-1 \end{cases} \end{aligned} \tag{iv}$$

The only remaining part is to calculate the character sum above. To do so, note that the solutions to $x^2 - 4n + 0 \bmod p^{2k_2-2r}$ where $x \in \mathbb{Z}/p^{2k_2-2r+1}\mathbb{Z}$ are obtained by $\pm 2\sqrt{n_0} + a_4 p^{2k_2-2r}$, where by abuse of notation we denote[2] a solution to $x^2 - 4n_0 \equiv 0 \bmod p^{2k_2-2r+1}$

[2] To keep uniformity, one can pick a p-adic solution (which exists by Hensel's lemma) at once and consider its reductions. Note that the sum is independent of all the choices that are made.

by $\sqrt{n_0}$, and $a_4 \in \mathbb{Z}/p\mathbb{Z}$. Therefore we have,

$$\sum_{\substack{a_2 \bmod p^{2k_2-2r+1} \\ a_2^2 \equiv 4n_0 \bmod p^{2k_2-2r}}} \left(\frac{(a_2^2-4n_0)/p^{2k_2-2r}}{p^{k_1}}\right) e\left(\frac{a_2\xi}{p^{k_1+2k_2-r}}\right)$$

$$= \sum_{\substack{a_4 \bmod p \\ \pm}} \left(\frac{\pm 4\sqrt{n_0}a_4}{p^{k_1}}\right) e\left(\frac{\pm 2\sqrt{n}\xi}{p^{k_1+2k_2}}\right) e\left(\frac{a_4\xi}{p^{k_1+r}}\right)$$

$$= \begin{cases} 2(p-1)\cos\left(\frac{2\sqrt{n}\xi}{p^{k_1+2k_2}}\right) & \text{if } k_1 \equiv 0 \bmod 2 \text{ and } v_p(\xi) \geq k_1 + r \\ -2\cos\left(\frac{2\sqrt{n}\xi}{p^{k_1+2k_2}}\right) & \text{if } k_1 \equiv 0 \bmod 2 \text{ and } v_p(\xi) = k_1 + r - 1 \\ 0 & \text{if } k_1 \equiv 1 \bmod 2 \text{ and } v_p(\xi) \geq k_1 + r \\ \begin{array}{l} G_p\left(\left(\frac{\sqrt{n_0}\xi/\xi_p}{p}\right) e\left(\frac{2\sqrt{n}\xi}{p^{k_1+2k_2}}\right)\right) \\ +G_p\left(\left(\frac{-\sqrt{n_0}\xi/\xi_p}{p}\right) e\left(\frac{-2\sqrt{n}\xi}{p^{k_1+2k_2}}\right)\right) \end{array} & \text{if } k_1 \equiv 1 \bmod 2 \text{ and } v_p(\xi) = k_1 + r - 1 \end{cases}$$

Where G_p denotes the Gauss sum modulo p, so in particular $|G_p| = \sqrt{p}$.

Combining this last calculation with (iii) and (iv) gives

$$Kl_{p^{k_1},p^{k_2}} = \begin{cases} O\left(p^{k_1+\frac{v_p(n)}{2}}\right) & \text{if } n \equiv \square \bmod p^{2k_2} \text{ and } v_p(\xi) \geq k_1 + \frac{v_p(n)}{2} \\ O\left(p^{k_1+\frac{v_p(n)}{2}-\frac{1}{2}}\right) & \text{if } n \equiv \square \bmod p^{2k_2} \text{ and } v_p(\xi) = k_1 + \frac{v_p(n)}{2} - 1 \\ 0 & \text{if } 4n \not\equiv \square \bmod p^{2k_2} \text{ or } v_p(\xi) < k_1 + \frac{v_p(n)}{2} - 1 \end{cases}$$

Combining the result for the cases $\frac{v_p(n)}{2} \geq k_2$ and $\frac{v_p(n)}{2} < k_2$ then gives the following bounds when $k_1 > 0$.

$$Kl_{p^{k_1},p^{k_2}} =$$

$$\begin{cases} O\left(p^{k_1+\min\{k_2,\frac{v_p(n)}{2}\}}\right) & \text{if } n \equiv \square \bmod p^{2k_2} \text{ and } v_p(\xi) \geq k_1 + \min\{k_2 + \frac{v_p(n)}{2}\} \\ O\left(p^{k_1+\min\{k_2+\frac{v_p(n)}{2}\}-\frac{1}{2}}\right) & \text{if } n \equiv \square \bmod p^{2k_2} \text{ and } v_p(\xi) = k_1 + \min\{k_2, \frac{v_p(n)}{2}\} - 1 \\ 0 & \text{if } 4n \not\equiv \square \bmod p^{2k_2} \text{ or } v_p(\xi) < k_1 + \min\{k_2, \frac{v_p(n)}{2}\} - 1 \end{cases} \tag{6.2}$$

Finally combining (6.1) and (6.2) finishes the proof of the proposition. □

The same bound essentially holds for the prime $p = 2$. We state this as the next proposition.

Proposition 6.1.5.

$$Kl_{2^{k_1},2^{k_2}} = \begin{cases} O\left(2^{k_1+\min\{k_2,\frac{v_2(n)}{2}\}}\right) & \textit{if } n \equiv \square \bmod 2^{2k_2} \textit{ and } v_2(\xi) \geq k_! + \min\{k_2 + \frac{v_2(n)}{2}\} \\ O\left(2^{k_1+\min\{k_2+\frac{v_2(n)}{2}\}-\frac{1}{2}}\right) & \textit{if } n \equiv \square \bmod 2^{2k_2} \textit{ and } v_2(\xi) = k_1 + \min\{k_2, \frac{v_2(n)}{2}\} - 1 \\ 0 & \textit{if } 4n \not\equiv \square \bmod 2^{2k_2} \textit{ or } v_2(\xi) < k_1 + \min\{k_2, \frac{v_2(n)}{2}\} - 1 \end{cases}$$

Proof. o

□

Combining propositions 6.1.4 and 6.1.5 then gives,

Corollary 6.1.6. *For any prime p we have the following bound,*

$$Kl_{p^{k_1},p^{k_2}} = \begin{cases} O\left(p^{k_1+\min\{k_2,\frac{v_p(n)}{2}\}}\right) & \text{if } n\equiv\square \bmod p^{2k_2} \text{ and } v_p(\xi)\geq k_1+\min\{k_2+\frac{v_p(n)}{2}\} \\ O\left(p^{k_1+\min\{k_2+\frac{v_p(n)}{2}\}-\frac{1}{2}}\right) & \text{if } n\equiv\square \bmod p^{2k_2} \text{ and } v_p(\xi)= k_1+\min\{k_2,\frac{v_p(n)}{2}\}-1 \\ 0 & \text{if } 4n\not\equiv\square \bmod p^{2k_2} \text{ or } v_p(\xi)< k_1+\min\{k_2,\frac{v_p(n)}{2}\}-1 \end{cases}$$

Proof. Obvious.

□

By using Lemma 6.1.1 and corollaries 6.1.3 and 6.1.6 we get the following bound.
Corollary 6.1.7. *Let $\xi,t,f,n\in\mathbb{Z}$. For a non-zero integer A let $rad(A)=\prod_{p|A}p$ be the radical of A and $A_\square$ denote the greatest square divisor of A. Then,*

$$Kl_{t,f}(\xi,n) = \begin{cases} 0 & \text{if } 4n\not\equiv\square \bmod f^2 \text{ or } \frac{t\gcd(f,\sqrt{n_\square})}{rad(t)}\nmid\xi \\ O\left(\frac{t\gcd(f,\sqrt{n_\square})}{\sqrt{rad(t/\gcd(t,\xi))}}\right) & \text{if } 4n\equiv\square \bmod f^2 \text{ and } \frac{t\gcd(f,\sqrt{n_\square})}{rad(t)}\mid\xi \end{cases}$$

This corollary finishes the treatment of $Kl_{t,f}(\xi,n)$.

6.2 $\omega^{\{1\}}_{t,f}(\xi,\nu)$

We begin by defining in general for each $k\in\mathbb{Z}_{\geq1}$,

$$\omega^{\{k\}}_{t,f}(\xi,\nu) := \sum_{b \bmod 4tf^2} Kl_{t,f,N}(\xi,b^k)e\left(\frac{b\nu}{4tf^2}\right) \tag{6.3}$$

$$= \sum_{\substack{a,b \bmod 4tf^2\\ a^2-4b^k\equiv0 \bmod f^2\\ \frac{a^2-4b^k}{f^2}\equiv0,1 \bmod 4}} \left(\frac{(a^2-4b^k)/f^2}{t}\right)e\left(\frac{a\xi}{4tf^2}\right)e\left(\frac{b\nu}{4tf^2}\right) \tag{6.4}$$

Lemma 6.2.1.

$$\omega^{\{k\}}_{t,f}(\xi,\nu) = \prod_p \omega^{\{k\}}_{p^{v_p(t)},p^{v_p(f)}}\left(\left(\frac{4tf^2}{(4tf^2)_p}\right)^{-1}\xi,\left(\frac{4tf^2}{(4tf^2)_p}\right)^{-1}\nu\right)$$

Proof. Follows from the Chinese Remainder Theorem as in Lemma 6.1.1.

□

These character sums will be important for applications beyond endoscopy for the symmetric k'th power representation. Since the only case of interest for this thesis is $k=1$, below we fix $k=1$ and give a detailed analysis of $\omega^{\{1\}}_{t,f}(\xi,\nu)$ that will be used in Chapter 8. We would like to note that for $k=1,2$ the exact values of $\omega^{\{k\}}_{p^r,p^s,N}(\xi,\nu)$ can be explicitly calculated and this plays a crucial role in the determination of the limits that we consider. In general, we do not expect an answer as explicit as the cases for $k=1,2$.
Lemma 6.2.2. *Let $p\nmid 2\nu$. Then*

$$\omega^{\{1\}}_{p^r,p^s}(\xi,\nu) = \begin{cases} 0 & \text{if } r>1 \\ p^{r+s}e\left(\frac{-\xi^2\bar{\nu}}{p^{r+2s}}\right) & \text{if } r\leq1 \end{cases}$$

Proof. For the proof let ϵ_p denote the sign of the Gauss sum modulo p, and recall that $\epsilon_p^2 = \left(\frac{-1}{p}\right)$.

- $r = s = 0$. Then the sum is trivially 1.
- $r > 0$, $s = 0$. Then,

$$\begin{aligned}\omega^{\{1\}}_{p^r,1,1}(\xi,\nu) &= \sum_{a,b \bmod p^r} \left(\frac{a^2-4b}{p^r}\right) e\left(\frac{a\xi}{p^r}\right) e\left(\frac{b\nu}{p^r}\right)\\ &= \sum_{a_0,b_0 \bmod p} \left(\frac{a_0^2-4b_0}{p^r}\right) e\left(\frac{a_0\xi}{p^r}\right) e\left(\frac{b_0\nu}{p^r}\right) \sum_{a_1,b_1 \bmod p^{r-1}} e\left(\frac{a_1\xi}{p^{r-1}}\right) e\left(\frac{b_1\nu}{p^{r-1}}\right)\end{aligned}$$

Not that the inner sum vanishes unless $p^{r-1} \mid \xi$ and $p^{r-1} \mid \nu$. Since by the assumption $\gcd(p,\nu) = 1$, this means that the sum vanishes unless $r = 1$. For $r = 1$, we have

$$\begin{aligned}\omega^{\{1\}}_{p,1}(\xi,\nu) &= \sum_{a,b \bmod p} \left(\frac{a^2-4b}{p}\right) e\left(\frac{a\xi}{p}\right) e\left(\frac{b\nu}{p}\right)\\ &= \sum_{a \bmod p} e\left(\frac{2a\xi+a^2\nu}{p}\right) \sum_{b \bmod p} \left(\frac{-b}{p}\right) e\left(\frac{b\nu}{p}\right)\\ &= pe\left(\frac{-\xi^2\bar{\nu}}{p}\right)\end{aligned}$$

- $r = 0$, $s > 0$. Then,

$$\begin{aligned}\omega^{\{1\}}_{1,p^s}(\xi,\nu) &= \sum_{\substack{a,b \bmod p^{2s}\\ a^2\equiv 4b \bmod p^{2s}}} e\left(\frac{a\xi}{p^{2s}}\right) e\left(\frac{b\nu}{p^{2s}}\right)\\ &= \sum_{a \bmod p^{2s}} e\left(\frac{2a\xi+a^2\nu}{p^{2s}}\right)\\ &= p^s e\left(\frac{-\xi^2\bar{\nu}}{p^{2s}}\right)\end{aligned}$$

- $r > 0, s > 0$. Then, as in the case of $\omega^{\{1\}}_{p^r,1}(\xi,\nu)$, the sum vanishes unless $r = 1$. For $r = 1$ we get,

$$\begin{aligned}\omega^{\{1\}}_{p,p^s}(\xi,\nu) &= \sum_{\substack{a,b \bmod p^{1+2s}\\ a^2\equiv b \bmod p^{2s}}} \left(\frac{(a^2-4b)/p^{2s}}{p}\right) e\left(\frac{a\xi}{p^{1+2s}}\right) e\left(\frac{b\nu}{p^{1+2s}}\right)\\ &= \sum_{a \bmod p^{1+2s}} e\left(\frac{a\xi}{p^{1+2s}}\right) \sum_{\substack{b \bmod p^{1+2s}\\ a^2\equiv 4b \bmod p^{2s}}} \left(\frac{(a^2-4b)/p^{2s}}{p}\right) e\left(\frac{b\nu}{p^{1+2s}}\right)\\ &= \sum_{a \bmod p^{1+2s}} e\left(\frac{a\xi}{p^{1+2s}}\right) \sum_{b_0 \bmod p} \left(\frac{-4b_0}{p}\right) e\left(\frac{\bar{4}a^2\nu}{p^{1+2s}}\right) e\left(\frac{b_0\nu}{p}\right)\\ &= \sqrt{p}\left(\frac{-\nu}{p}\right)\epsilon_p \sum_{a \bmod p^{1+2s}} e\left(\frac{2a\xi+a^2\nu}{p^{1+2s}}\right)\\ &= p^{1+s} e\left(\frac{-\xi^2\bar{\nu}}{p^{1+2s}}\right)\end{aligned}$$

$\square$

Lemma 6.2.3. *Let $p\nmid 2$ and $p\mid\nu$, $p\nmid\gcd(\xi,\nu)$. Let $\nu=p^k\nu_0$ where $p\nmid\nu_0$. Then,*

$$\omega^{\{1\}}_{p^r,p^s}(\xi,\nu)=\begin{cases}1 & \text{if } (r,s)=(0,0)\\ 0 & \text{if } (r,s)\neq(0,0)\end{cases}$$

Proof.

- $r=s=0$. The sum is trivially 1.
- $r>0$, $s=0$. Then,

$$\begin{aligned}\omega^{\{1\}}_{p^r,1}(\xi,\nu)&=\sum_{a,b \bmod p^r}\left(\frac{a^2-4b}{p^r}\right)e\left(\frac{a\xi}{p^r}\right)e\left(\frac{b\nu}{p^r}\right)\\&=\sum_{a_0,b_0 \bmod p}\left(\frac{a_0^2-4b_0}{p^r}\right)e\left(\frac{a_0\xi}{p^r}\right)e\left(\frac{b_0\nu}{p^r}\right)\sum_{a_1,b_1 \bmod p^{r-1}}e\left(\frac{a_1\xi}{p^{r-1}}\right)e\left(\frac{b_1\nu}{p^{r-1}}\right)\end{aligned}$$

  The sum vanishes unless $p^{r-1}\mid\gcd(\xi,\nu)$. Therefore by the assumption we need to have $r=1$. For $r=1$ the sum is,

$$\begin{aligned}\omega^{\{1\}}_{p,1}(\xi,\nu)&=\sum_{a,b \bmod p}\left(\frac{a^2-4b}{p}\right)e\left(\frac{a\xi}{p}\right)\\&=\sum_{a \bmod p}e\left(\frac{a\xi}{p}\right)\sum_{b \bmod p}\left(\frac{b}{p}\right)\\&=0\end{aligned}$$

- $r=0$, $s>0$. Then,

$$\begin{aligned}\omega^{\{1\}}_{1,p^s}(\xi,\nu)&=\sum_{\substack{a,b \bmod p^{2s}\\ a^2\equiv 4b \bmod p^{2s}}}e\left(\frac{a\xi}{p^{2s}}\right)e\left(\frac{b\nu}{p^{2s}}\right)\\&=\sum_{a \bmod p^{2s}}e\left(\frac{2a\xi+a^2\nu}{p^{2s}}\right)\end{aligned}$$

  This sum vanishes if $k\geq 2s$. If not,

$$\begin{aligned}&=\sum_{a_0 \bmod p^{2s-k}}e\left(\frac{2a_0\xi+a_0^2\nu}{p^{2s}}\right)\sum_{a_1 \bmod p^k}e\left(\frac{2a_1\xi}{p^k}\right)\\&=0\end{aligned}$$

- $r>0$, $s>0$. Then once again $r=1$ otherwise the sum vanishes. If $r=1$, then

$$\begin{aligned}\omega^{\{1\}}_{p,p^s}(\xi,\nu)&=\sum_{\substack{a,b \bmod p^{1+2s}\\ a^2\equiv 4b \bmod p^{2s}}}\left(\frac{(a^2-4b)/p^{2s}}{p}\right)e\left(\frac{a\xi}{p^{1+2s}}\right)e\left(\frac{b\nu}{p^{1+2s}}\right)\\&=\sum_{a \bmod p^{1+2s}}e\left(\frac{a\xi}{p^{1+2s}}\right)\sum_{\substack{b \bmod p^{1+2s}\\ a^2\equiv 4b \bmod p^{2s}}}\left(\frac{(a^2-4b)/p^{2s}}{p}\right)e\left(\frac{b\nu}{p^{1+2s}}\right)\\&=\sum_{a \bmod p^{1+2s}}e\left(\frac{a\xi}{p^{1+2s}}\right)e\left(\frac{\bar{4}a^2\nu}{p^{1+2s}}\right)\sum_{b_0 \bmod p}\left(\frac{-b_0}{p}\right)e\left(\frac{b_0\nu}{p}\right)\\&=0\end{aligned}$$

$\square$

Lemma 6.2.4. *Let $p \nmid 2$ and $p \mid \gcd(\xi,\nu)$. Let $v_p(\gcd(\xi,\nu)) = p^k$. Then,*

- *For $r = 0$,*

$$\omega^{\{1\}}_{1,p^s}(\xi,\nu) = \begin{cases} 0 & \text{if } k < 2s \text{ and } p^{k+1} \mid \nu \\ p^{s+\frac{k}{2}}\epsilon_{p^k}\left(\frac{\nu_0}{p^k}\right) e\left(\frac{-\xi_0^2\bar{\nu}_0}{p^{2s-k}}\right) & \text{if } k < 2s \text{ and } p^{k+1} \nmid \nu \\ p^{2s} & \text{if } k \geq 2s \end{cases}$$

- *For $r > 0$,*

$$\omega^{\{1\}}_{p^r,p^s}(\xi,\nu) =$$

$$\begin{cases} 0 & \text{if } k < r-1 \\ 0 & \text{if } k < r+2s \text{ and } p^{k+1} \mid \nu \\ -p^{2(r-1)+s+\frac{1}{2}}\epsilon_p\left(\frac{\nu_0}{p}\right) e\left(\frac{-\xi_0^2\bar{\nu}_0}{p}\right) & \text{if } k = r-1,\ p^{k+1} \nmid \nu \text{ and } r \equiv 0 \bmod 2 \\ p^{2r+s-1} e\left(\frac{-\xi_0^2\bar{\nu}_0}{p}\right) & \text{if } k = r-1,\ p^{k+1} \nmid \nu \text{ and } r \equiv 1 \bmod 2 \\ 0 & \text{if } k \geq r \text{ and } r \equiv 1 \bmod 2 \\ \phi(p^{r+s}) p^{\frac{r+k}{2}} \epsilon_{k-r}\left(\frac{\nu_0}{p^{k-r}}\right) e\left(\frac{-\xi_0^2\bar{\nu}_0}{p^{r+2s-k}}\right) & \text{if } r \leq k < r+2s,\ p^{k+1} \nmid \nu \text{ and } r \equiv 0 \bmod 2 \\ \phi(p^{2(r+s)}) & \text{if } k \geq r+2s \text{ and } r \equiv 0 \bmod 2 \end{cases}$$

Where $\frac{\xi}{p^k} = \xi_0$ and $\frac{\nu}{p^k} = \nu_0$

Proof.

- $r = s = 0$. The sum is trivially 1.
- $r > 0$, $s = 0$. Then,

$$\begin{aligned} \omega^{\{1\}}_{p^r,1}(\xi,\nu) &= \sum_{a,b \bmod p^r} \left(\frac{a^2-4b}{p^r}\right) e\left(\frac{a\xi}{p^r}\right) e\left(\frac{b\nu}{p^r}\right) \\ &= \sum_{a_0,b_0 \bmod p} \left(\frac{a_0^2-4b_0}{p^r}\right) e\left(\frac{a_0\xi}{p^r}\right) e\left(\frac{b_0\nu}{p^r}\right) \sum_{a_1,b_1 \bmod p^{r-1}} e\left(\frac{a_1\xi}{p^{r-1}}\right) e\left(\frac{b_1\nu}{p^{r-1}}\right) \end{aligned}$$

This last sum vanishes unless $k \geq r-1$. In which case we get,

$$\begin{aligned} & p^{2(r-1)} \sum_{a_0,b_0 \bmod p} \left(\frac{a_0^2-4b_0}{p^r}\right) e\left(\frac{a_0\xi}{p^r}\right) e\left(\frac{b_0\nu}{p^r}\right) \\ &= p^{2(r-1)} \sum_{a_0 \bmod p} e\left(\frac{2a_0\xi+a_0^2\nu}{p^r}\right) \sum_{b_0 \bmod p} \left(\frac{-b_0}{p^r}\right) e\left(\frac{b_0\nu}{p^r}\right) \\ &= p^{2(r-1)} \begin{cases} 0 & \text{if } k = r-1 \text{ and } p^{k+1} \mid \nu \\ -\sqrt{p}\epsilon_p\left(\frac{\nu_0}{p}\right) e\left(\frac{-\xi_0^2\bar{\nu}_0}{p}\right) & \text{if } k = r-1,\ p^{k+1} \nmid \nu \text{ and } r \equiv 0 \bmod 2 \\ pe\left(\frac{-\xi_0^2\bar{\nu}_0}{p}\right) & \text{if } k = r-1,\ p^{k+1} \nmid \nu \text{ and } r \equiv 1 \bmod 2 \\ \phi(p^2) & \text{if } k \geq r \text{ and } r \equiv 0 \bmod 2 \\ 0 & \text{if } k \geq r \text{ and } r \equiv 1 \bmod 2 \end{cases} \end{aligned}$$

- $r=0$, $s>0$. Then,

$$\omega^{\{1\}}_{1,p^s}(\xi,\nu) = \sum_{\substack{a,b \bmod p^{2s} \\ a^2\equiv 2b \bmod p^{2s}}} e\left(\frac{a\xi}{p^{2s}}\right) e\left(\frac{b\nu}{p^{2s}}\right)$$
$$= \sum_{a \bmod p^{2s}} e\left(\frac{2a\xi+a^2\nu}{p^{2s}}\right)$$

This sum is p^{2s} if $k\geq 2s$. If $k<2s$,

$$\sum_{a \bmod p^{2s}} e\left(\frac{2a\xi+a^2\nu}{p^{2s}}\right) = p^k \sum_{a_0 \bmod p^{2s-k}} e\left(\frac{2a_0\xi+a_0^2\nu}{p^{2s}}\right)$$
$$= \begin{cases} 0 & \text{if } p^{k+1}\mid \nu \\ p^{s+\frac{k}{2}}\epsilon_{p^k}\left(\frac{\nu_0}{p^k}\right) e\left(\frac{-\xi_0^2\bar{\nu}_0}{p^{2s-k}}\right) & \text{if } p^{k+1}\nmid \nu \end{cases}$$

- $r>0$, $s>0$. Then once again the sum vanishes unless $k\geq r-1$.

  - $k=r-1$.

$$\omega^{\{1\}}_{p^r,p^s}(\xi,\nu)$$
$$= \sum_{\substack{a,b \bmod p^{r+2s} \\ a^2\equiv 4b \bmod p^{2s}}} \left(\frac{(a^2-4b)/p^{2s}}{p^r}\right) e\left(\frac{a\xi}{p^{r+2s}}\right) e\left(\frac{b\nu}{p^{r+2s}}\right)$$
$$= p^{2(r-1)} \sum_{\substack{a_0,b_0 \bmod p^{1+2s} \\ a_0^2\equiv 4b_0 \bmod p^{2s}}} \left(\frac{(a_0^2-4b_0)/p^{2s}}{p^r}\right) e\left(\frac{a_0\xi_0}{p^{1+2s}}\right) e\left(\frac{b_0\nu_0}{p^{1+2s}}\right)$$
$$= p^{2(r-1)} \sum_{a_0 \bmod p^{1+2s}} e\left(\frac{a_0\xi_0}{p^{1+2s}}\right) \sum_{\substack{b_0 \bmod p^{1+2s} \\ a_0^2\equiv 4b_0 \bmod p^{2s}}} \left(\frac{(a_0^2-4b_0)/p^{2s}}{p^r}\right) e\left(\frac{b_0\nu_0}{p^{1+2s}}\right)$$
$$= p^{2(r-1)} \sum_{a_0 \bmod p^{1+2s}} e\left(\frac{2a_0\xi_0+a_0^2\nu_0}{p^{1+2s}}\right) \sum_{b_1 \bmod p} \left(\frac{-b_0}{p^r}\right) e\left(\frac{b_1\nu_0}{p}\right)$$
$$= p^{2(r-1)} \begin{cases} 0 & \text{if } k=r-1 \text{ and } p^{k+1}\mid\nu \\ -p^{s+\frac{1}{2}}\epsilon_p\left(\frac{\nu_0}{p}\right) e\left(\frac{-\xi_0^2\bar{\nu}_0}{p^{1+2s}}\right) & \text{if } k=r-1,\ p^{k+1}\nmid\nu \text{ and } r\equiv 0 \bmod 2 \\ p^{s+1} e\left(\frac{-\xi_0^2\bar{\nu}_0}{p^{1+2s}}\right) & \text{if } k=r-1,\ p^{k+1}\nmid\nu \text{ and } r\equiv 1 \bmod 2 \end{cases}$$

  - $r\leq k<r+2s$.

Let $\frac{\xi}{p^{r-1}} = \xi_1$ and $\frac{\nu}{p^{r-1}} = \nu_1$. Then,

$$
\begin{aligned}
&\omega^{\{1\}}_{p^r,p^s}(\xi,\nu) \\
&= \sum_{\substack{a,b \bmod p^{r+2s} \\ a^2 \equiv 4b \bmod p^{2s}}} \left(\frac{(a^2-4b)/p^{2s}}{p^r}\right) e\left(\frac{a\xi}{p^{r+2s}}\right) e\left(\frac{b\nu}{p^{r+2s}}\right) \\
&= p^{2(r-1)} \sum_{\substack{a_0,b_0 \bmod p^{1+2s} \\ a_0^2 \equiv 4b_0 \bmod p^{2s}}} \left(\frac{(a_0^2-4b_0)/p^{2s}}{p^r}\right) e\left(\frac{a_0\xi_1}{p^{1+2s}}\right) e\left(\frac{b_0\nu_1}{p^{1+2s}}\right) \\
&= p^{2(r-1)} \sum_{a_0 \bmod p^{1+2s}} e\left(\frac{a_0\xi_1}{p^{1+2s}}\right) \sum_{\substack{b_0 \bmod p^{1+2s} \\ a_0^2 \equiv 4b_0 \bmod p^{2s}}} \left(\frac{(a_0^2-4b_0)/p^{2s}}{p^r}\right) e\left(\frac{b_0\nu_1}{p^{1+2s}}\right) \\
&= p^{2(r-1)} \sum_{a_0 \bmod p^{1+2s}} e\left(\frac{2a_0\xi_1 + a_0^2\nu_1}{p^{1+2s}}\right) \sum_{b_1 \bmod p} \left(\frac{-b_0}{p^r}\right) \\
&= p^{k+r-1} \sum_{a_1 \bmod p^{r+2s-k}} e\left(\frac{2a_0\xi_0 + a_0^2\nu_0}{p^{r+2s-k}}\right) \sum_{b_1 \bmod p} \left(\frac{-b_0}{p^r}\right) \\
&= \begin{cases} 0 & \text{if } r \le k < r+2s,\ p^{k+1} \mid \nu \\ & \quad \text{and } r \equiv 0 \bmod 2 \\ \phi(p^{r+s}) p^{\frac{r+k}{2}} \epsilon_{p^{k-r}} \left(\frac{\nu_0}{p^{k-r}}\right) e\left(\frac{-\xi_0^2\bar{\nu}_0}{p^{r+2s-k}}\right) & \text{if } r \le k < r+2s,\ p^{k+1} \nmid \nu \\ & \quad \text{and } r \equiv 0 \bmod 2 \\ 0 & \text{if } r \le k < r+2s \\ & \text{and } r \equiv 1 \bmod 2 \end{cases}
\end{aligned}
$$

– $k \geq r + 2s$.

$$
\begin{aligned}
\omega^{\{1\}}_{p^r,p^s}(\xi,\nu) &= \sum_{\substack{a,b \bmod p^{r+2s} \\ a^2 \equiv 4b \bmod p^{2s}}} \left(\frac{(a^2-4b)/p^{2s}}{p^r}\right) \\
&= p^{2(r-1)} \sum_{\substack{a_0,b_0 \bmod p^{1+2s} \\ a_0^2 \equiv 4b_0 \bmod p^{2s}}} \left(\frac{(a_0^2-4b_0)/p^{2s}}{p^r}\right) \\
&= p^{2(r-1)} \sum_{a_0 \bmod p^{1+2s}} \sum_{\substack{b_0 \bmod p^{1+2s} \\ a_0^2 \equiv 4b_0 \bmod p^{2s}}} \left(\frac{(a_0^2-4b_0)/p^{2s}}{p^r}\right) \\
&= p^{2(r-1)} \sum_{a_0 \bmod p^{1+2s}} \sum_{b_1 \bmod p} \left(\frac{-b_0}{p^r}\right) \\
&= \begin{cases} \phi(p^{2r+2s}) & \text{if } k \ge 2s + r \text{ and } r \equiv 0 \bmod 2 \\ 0 & \text{if } k \ge r + 2s \text{ and } r \equiv 1 \bmod 2 \end{cases}
\end{aligned}
$$

□

Chapter 7

Some Auxiliary Estimates

Recall the formula in §5.6 in Cahpter 5 that, for any $\kappa > 0$, after applying Poisson summation to the m-sum we are left with

$$
\begin{aligned}
&+\sqrt{n}\sum_{f<n^{\frac{\alpha+\delta_0}{2}}}\frac{1}{f^3}\sum_{\substack{t\\ tf^2\ll n^{\alpha+\delta_0}}}\frac{1}{t^2}\sum_{\xi\in\mathbb{Z}\setminus\{0\}}Kl_{t,f}(\xi,n)\left\{\mathcal{J}^{+}_{F,\alpha,+}\left(\tfrac{tf^2}{(4n)^{\alpha}},\tfrac{\xi\sqrt{n}}{2tf^2}\right)+\mathcal{J}^{+}_{F,\alpha,-}\left(\tfrac{tf^2}{(4n)^{\alpha}},\tfrac{\xi\sqrt{n}}{2tf^2}\right)\right\}\\
&+\tfrac{\sqrt{\pi}}{2}\sum_{f<n^{\frac{1-\alpha+\delta_0}{2}}}\frac{1}{f}\sum_{\substack{t\\ tf^2\ll n^{1-\alpha+\delta_0}}}\frac{1}{t}\sum_{\xi\in\mathbb{Z}\setminus\{0\}}Kl_{t,f}(\xi,n))\\
&\qquad\qquad\left\{\mathcal{J}^{1,+}_{H,1-\alpha,+}\left(\tfrac{tf^2}{(4n)^{1-\alpha}},\tfrac{\xi\sqrt{n}}{2tf^2}\right)+\mathcal{J}^{1,+}_{H,1-\alpha,-}\left(\tfrac{tf^2}{(4n)^{1-\alpha}},\tfrac{\xi\sqrt{n}}{2tf^2}\right)\right\}\\
&+\sqrt{n}\sum_{f<n^{\frac{\alpha+\delta_0}{2}}}\frac{1}{f^3}\sum_{\substack{t\\ tf^2\ll n^{\alpha+\delta_0}}}\frac{1}{t^2}\sum_{\xi\in\mathbb{Z}\setminus\{0\}}Kl_{t,f}(\xi,n)\mathcal{J}^{-}_{F,\alpha}\left(\tfrac{tf^2}{(4n)^{\alpha}},\tfrac{\xi\sqrt{n}}{2tf^2}\right)\\
&+\tfrac{\sqrt{\pi}}{2}\sum_{f<n^{\frac{1-\alpha+\delta_0}{2}}}\frac{1}{f}\sum_{\substack{t\\ tf^2\ll n^{1-\alpha+\delta_0}}}\frac{1}{t}\sum_{\xi\in\mathbb{Z}\setminus\{0\}}Kl_{t,f}(\xi,n)\mathcal{J}^{1,-}_{H,1-\alpha}\left(\tfrac{tf^2}{(4n)^{1-\alpha}},\tfrac{\xi\sqrt{n}}{2tf^2}\right)\\
&+\mathrm{tr}(\mathbf{1}(f))-\mathrm{tr}(\xi_0(f))+O(n^{\frac{1}{2}-\alpha+\kappa}+1)
\end{aligned}
$$

Where for any $C > 0$ and $\eta \in \mathbb{C}$,

$$
\begin{aligned}
\mathcal{J}^{+}_{F,\alpha,+}(C,\eta) &= \int_{|x|>1}\theta_\infty(x)\,F\left(\tfrac{C}{(x^2-1)^{\alpha}}\right)e(-x\eta)\,dx\\
\mathcal{J}^{+}_{F,\alpha,-}(C,\eta) &= \int_{|x|<1}\theta_\infty(x)\,F\left(\tfrac{C}{(1-x^2)^{\alpha}}\right)e(-x\eta)\,dx\\
\mathcal{J}^{1,+}_{H,1-\alpha,+}(C,\eta) &= \int_{|x|>1}\tfrac{\theta_\infty(x)}{\sqrt{x^2-1}}H_+\left(\tfrac{C}{(x^2-1)^{1-\alpha}}\right)e(-x\eta)\,dx\\
\mathcal{J}^{1,+}_{H,1-\alpha,-}(C,\eta) &= \int_{|x|<1}\tfrac{\theta_\infty(x)}{\sqrt{1-x^2}}H_-\left(\tfrac{C}{(1-x^2)^{1-\alpha}}\right)e(-x\eta)\,dx\\
\mathcal{J}^{-}_{F,\alpha}(C,\eta) &= \int_{\mathbb{R}}\theta_\infty(x)\,F\left(\tfrac{C}{(1+x^2)^{\alpha}}\right)e(-x\eta)\,dx\\
\mathcal{J}^{-}_{H,1-\alpha}(C,\eta) &= \int_{\mathbb{R}}\tfrac{\theta_\infty(x)}{\sqrt{1+x^2}}H_+\left(\tfrac{C}{(1+x^2)^{1-\alpha}}\right)e(-x\eta)\,dx
\end{aligned}
$$

and,

$$Kl_{t,f}(\xi,n) = \sum_{\substack{a \bmod 4tf^2 \\ a^2-4n\equiv 0 \bmod f^2 \\ \frac{a^2-4n}{f^2}\equiv 0,1 \bmod 4}} \left(\frac{(a^2-4n)/f^2}{t}\right) e\left(\frac{a\xi}{4tf^2}\right)$$

Note that $\mathcal{J}^-_{F,\alpha}$ and $\mathcal{J}^-_{H,1-\alpha}$ are both *smooth.*

In this chapter we will give various estimates on the ξ-sum. In particular we will show that for large values of ξ the sum is small. We will first bound the character sums by the Weil bound and then by using the asymptotic expansions derived in Chapter 4 to show that the sum exhibits cancellation for certain ranges of ξ.

7.1 Asymptotic expansions and bounds on $\mathcal{T}_{\alpha,\kappa}(\Phi)$

In this section we will derive asymptotic expansions and bounds for the transforms $\mathcal{T}_{\alpha,\kappa}(\Phi)(y)$ where $0<\alpha<1$, $\kappa\in\mathbb{R}_+$ and $\Phi\in\mathcal{S}(\mathbb{R})$. Recall that $\mathcal{T}_{\alpha,\kappa}(\Phi)(y)$ is defined by

$$\mathcal{T}_{\alpha,\kappa}(\Phi)(y) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \mathcal{M}(\Phi)(z)\left(\frac{i}{\pi y}\right)^{\alpha z}\Gamma(\kappa+\alpha z+1)dz$$

Where $\Re(\beta)>0$. The following proposition will be useful when the size of y is large.
Lemma 7.1.1. *For any $M>0$ and $y\neq 0$ we have,*

$$\mathcal{T}_{\alpha,\kappa}(\Phi)(y) = O_{M,,\Phi,\alpha,\kappa}(|y|^{-M})$$

Proof. Since $\Phi\in\mathcal{S}(\mathbb{R})$ its Mellin transform is bounded on vertical strips. Therefore shifting the contour to $\Re(z)=\frac{M}{\alpha}$ gives the result.

□

The next proposition describes the asymptotic behavior of $\mathcal{T}_{\alpha,\kappa}(\Phi)(y)$ for small values of y.
Lemma 7.1.2. *For any $M>0$ and $y\neq 0$ we have,*

$$\begin{aligned}\mathcal{T}_{\alpha,\kappa}(\Phi)(y) &= \sum_{\sigma}{}' \left(\frac{i}{\pi y}\right)^{\alpha\sigma}\Gamma(\kappa+\alpha\sigma+1)Res_{z=\sigma}(\mathcal{M}(\Phi)(z)) \\ &+ \sum_{0\le\rho\le M}{}' \frac{(-1)^\rho}{\rho!}\mathcal{M}(\Phi)\left(-\frac{1+\kappa+\rho}{\alpha}\right)\left(\frac{\pi y}{i}\right)^{1+\kappa+\rho} \\ &+ \sum_{\tau}\left(\frac{i}{\pi y}\right)^{\alpha\tau} Res_{z=\tau}(\Gamma(\kappa+\alpha z+1)\mathcal{M}(\Phi)(z)) \\ &+ O_{M,\Phi,\alpha,\kappa}(|y|^{\frac{M+\kappa+1}{\alpha}+\frac{1}{2}})\end{aligned}$$

Where the first sum is over the poles of $\mathcal{M}(\Phi)(z)$ that are in the strip $-\frac{M-\kappa-1}{\alpha}-\frac{1}{2}\le\Re(z)\le 0$ and are not poles of of $\Gamma(\kappa+\alpha z+1)$ (hence the $'$ in the summation sign), the second sum is over non-negative integers $0\le\rho\le M$ such that $\mathcal{M}(\Phi)(z)$ does not have a pole at $-\frac{1+\kappa+\rho}{\alpha}$ (this is why we have the $'$ on the summation sign), and the third is over the poles that are common to both $\mathcal{M}(\Phi)(z)$ and $\Gamma(\kappa+\alpha z+1)$ and are in the strip $-\frac{M-\kappa-1}{\alpha}-\frac{1}{2}\le\Re(z)\le 0$.

Proof. Shifting the contour to $\Re(z) = -\frac{M-\kappa-1}{\alpha} - \frac{1}{2}$ picks up the poles of $\mathcal{M}(\Phi)(z)$ and the poles of the Γ-function. Recalling that $\Gamma(x)$ has simple poles at integers $-\rho$ (for $\rho \geq 0$) with residue $\frac{(-1)^\rho}{\rho!}$ gives

$$\begin{aligned}
\mathcal{T}_{\alpha,\kappa}(\Phi)(y) = & \sum_{\sigma}{}' \left(\frac{i}{\pi y}\right)^{\alpha\sigma} \Gamma(\kappa+\alpha\sigma+1) Res_{z=\sigma}(\mathcal{M}(\Phi)(z)) \\
& + \sum_{0\leq\rho\leq M}{}' \frac{(-1)^\rho}{\rho!} \left(\frac{\pi y}{i}\right)^{1+\kappa+\rho} \mathcal{M}(\Phi)\left(-\frac{1+\kappa+\rho}{\alpha}\right) \\
& + \sum_{\tau} \left(\frac{i}{\pi y}\right)^{\alpha\tau} Res_{z=\tau}(\Gamma(\kappa+\alpha z+1)\mathcal{M}(\Phi)(z)) \\
& + \frac{1}{2\pi i}\int_{\Re(z)=\frac{-M-\kappa-1}{\alpha}-\frac{1}{2}} \mathcal{M}(\Phi)(z)\left(\frac{i}{\pi y}\right)^{\alpha z}\Gamma(\kappa+\alpha z+1)dz
\end{aligned} \tag{$\star$}$$

Where the sums are as in the statement of the proposition. Since $\Phi \in \mathcal{S}(\mathbb{R})$ it is bounded on vertical strips, and we have

$$\frac{1}{2\pi i}\int_{\Re(z)=\frac{-M-\kappa-1}{\alpha}-\frac{1}{2}} \mathcal{M}(\Phi)(z)\left(\frac{i}{\pi y}\right)^{\alpha z}\Gamma(\kappa+\alpha z+1)dz = O_{M,\Phi,\alpha,\kappa}(|y|^{M+\kappa+1+\frac{\alpha}{2}})$$

Combining this with $(\star)$ gives the result.

$\square$

7.2 Asymptotic expansions and bounds on $\mathcal{J}^{+}_{F,\alpha,+}(C,\eta)+\mathcal{J}^{+}_{F,\alpha,-}(C,\eta)$ and $\mathcal{J}^{1,+}_{H,\alpha,+}(C,\eta)+\mathcal{J}^{1,+}_{H,\alpha,-}(C,\eta)$

Recall that by Theorem 5.5.2 after applying Poisson summation to the ξ-sum we get certain sums of the functions $\mathcal{J}^{+}_{F,\alpha,+}(C,\eta)+\mathcal{J}^{+}_{F,\alpha,-}(C,\eta)$ and $\mathcal{J}^{1,+}_{H,\alpha,+}(C,\eta)+\mathcal{J}^{1,+}_{H,\alpha,-}(C,\eta)$. In this section we will give certain estimates and asymptotics on these functions which will be useful respectively for large and small values of the parameter $|C^{\frac{1}{\alpha}}\eta|$. The results of this section will be simply translations of the results obtained in Chapter 4 on the analysis of singularities.

Proposition 7.2.1. *Fo any $M>0$,*

$$\mathcal{J}^{+}_{F,\alpha,+}(C,\eta)+\mathcal{J}^{+}_{F,\alpha,-}(C,\eta)=O_{F,\theta_\infty,\alpha,M}(|\eta|^{-M-1}+\eta^{-1}|C^{\frac{1}{\alpha}}\eta|^{-M})$$
$$\mathcal{J}^{1,+}_{H,\alpha,+}(C,\eta)+\mathcal{J}^{1,+}_{H,\alpha,-}(C,\eta)=O_{F,\theta_\infty,\alpha,M}(|\eta|^{-M-1}+\eta^{-\frac{1}{2}}|C^{\frac{1}{\alpha}}\eta|^{-M})$$

Proof. By theorems 4.8.1 and 4.8.2 we know that $\mathcal{J}^{+}_{F,\alpha,+}(C,\eta)+\mathcal{J}^{+}_{F,\alpha,-}(C,\eta)$ and $\mathcal{J}^{1,+}_{H,\alpha,+}(C,\eta)+\mathcal{J}^{1,+}_{H,\alpha,-}(C,\eta)$ are both bounded by sums of $M+1$ terms each of which is bounded by $\eta^{-1}\mathcal{T}_{\alpha,\kappa}(F)(C^{\frac{1}{\alpha}}\eta)$ and $\eta^{-\frac{1}{2}}\mathcal{T}_{\alpha,\kappa}(F)(C^{\frac{1}{\alpha}}\eta)$ respectively. The proposition now follows from Lemma 7.1.1. □

Proposition 7.2.2. *There exist constants*[1] $A^{\pm,j}_{l,m,s,r,\rho}\in\mathbb{C}$, *for $j=1,2,3$, such that, for any $M>0$ we have,*

$$\begin{aligned}
&\mathcal{J}^{+}_{F,\alpha,+}(C,\eta)+\mathcal{J}^{+}_{F,\alpha,-}(C,\eta)\\
&=e(-\eta)\sum_{l=l_2^+(\theta)}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM}\sum_{\rho=0}^{M}A^{+,1}_{l,m,s,r,\rho}((-1)^\rho+(-1)^{m+r})C^{\frac{1+m+r+s}{\alpha}}(C^{\frac{1}{\alpha}}\eta)^{s+\rho}\\
&+e(\eta)\sum_{l=l_2^-(\theta)}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM}\sum_{\rho=0}^{M}A^{-,1}_{l,m,s,r,\rho}((-1)^\rho+(-1)^{m+r})C^{\frac{1+m+r+s}{\alpha}}(C^{\frac{1}{\alpha}}\eta)^{s+\rho}\\
&+e(-\eta)\sum_{l=l_1^+(\theta)}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM}A^{+,2}_{l,m,s,r,\rho}\left(\frac{1}{\eta}\right)^{m+r+s+\frac{3}{2}}\\
&+e(-\eta)\sum_{l=l_2^+(\theta)}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM}\sum_{\rho=0}^{M}A^{+,3}_{l,m,s,r,\rho}C^{\frac{\frac{3}{2}+m+r+s}{\alpha}}(C^{\frac{1}{\alpha}}\eta)^{s+\rho}\\
&+e(\eta)\sum_{l=l_1^-(\theta)}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM}A^{-,2}_{l,m,s,r,\rho}\left(\frac{1}{\eta}\right)^{m+r+s+\frac{3}{2}}\\
&+e(-\eta)\sum_{l=l_2^+(\theta)}^{2M}\sum_{m=l}^{2M}\sum_{s=0}^{2M}\sum_{r=0}^{2sM}\sum_{\rho=0}^{M}A^{-,3}_{l,m,s,r,\rho}C^{\frac{\frac{3}{2}+m+r+s}{\alpha}}(C^{\frac{1}{\alpha}}\eta)^{s+\rho}\\
&+O_{F,\theta_\infty,\alpha,M}(\eta^{-M}+|\eta|^{-1}|C^{\frac{1}{\alpha}}\eta|^{M+1})
\end{aligned}$$

Where,

$$\theta_\infty(x)=|4-x^2|^{\frac{1}{2}}\theta_{\infty,1}(x)+\theta_{\infty,2}(x)$$

[1] The superscripts $\pm$ on the constants $A^{\pm,j}$ indicates the torus, and have nothing to do with the superscripts $\pm$ in $\mathcal{J}^{\pm}$ which indicates the sign of the determinant.

and $l_{1,2}^{\pm}(\theta)$ *is defined by,*

$$\theta_{\infty,2}(\pm 1 + x) = \sum_{l=l_2^{\pm}(\theta)}^{\infty} \alpha_l^{\pm} x^l$$
$$|1-(\pm 1 \mp x)^2|^{\frac{1}{2}}\theta_{\infty,1}(\pm 1 \mp x) = |1-(\pm 1 \mp x)^2|^{\frac{1}{2}} \sum_{l=l_1^{\pm}(\theta)}^{\infty} \beta_l^{\pm} x^l$$

for some coefficients $\alpha_l^{\pm}$ *and* $\beta_l^{\pm}$ *in* $\mathbb{C}$.

Proof. We will be using Theorem 4.8.1. With the notation of Theorem 4.8.1 we have $\Phi_1 = \Phi_2 = F$. To get the desired result we only need to apply Lemma 7.1.2 to $\mathcal{T}_{\alpha,\kappa}(F)(C^{\frac{1}{\alpha}}\eta)$ fro $\kappa = m+r+2s$ and $\kappa = m+r+2s+\frac{1}{2}$.

In order to apply Lemma 7.1.2 we need to locate the poles of $\mathcal{M}(F)(z)$.By Lemma 3.2.2 we know that the only pole of $\mathcal{M}(F)(z)$ is at $z=0$, which is simple and has residue 1. Therefore Lemma 7.1.2 implies that

$$\mathcal{T}_{\alpha,\kappa}(F)(C^{\frac{1}{\alpha}}\eta) = \kappa! + \sum_{\rho=0}^{M} \frac{(-1)^{\rho}}{\rho!}\left(\frac{\pi C^{\frac{1}{\alpha}}\eta}{i}\right)^{1+\kappa+\rho} \mathcal{M}(F)\left(-\frac{1+\kappa+\rho}{\alpha}\right) + O_{M,F,\alpha,\kappa}(|C^{\frac{1}{\alpha}}\eta|^{M+\kappa+1\alpha+\frac{\alpha}{2}})$$

and

$$\mathcal{T}_{\alpha,\kappa}(F)(-C^{\frac{1}{\alpha}}\eta) = \kappa! + (-1)^{\kappa+1}\sum_{\rho=0}^{M} \frac{1}{\rho!}\left(\frac{\pi C^{\frac{1}{\alpha}}\eta}{i}\right)^{1+\kappa+\rho} \mathcal{M}(F)\left(-\frac{1+\kappa+\rho}{\alpha}\right) + O_{M,F,\alpha,\kappa}(|C^{\frac{1}{\alpha}}\eta|^{M+\kappa+1+\frac{\alpha}{2}}) \quad (\circ)$$

Hence we get,

$$\mathcal{T}_{\alpha,\kappa}(F)\left(C^{\frac{1}{\alpha}}\eta\right) - \mathcal{T}_{\alpha,\kappa}(F)\left(-C^{\frac{1}{\alpha}}\eta\right) = \sum_{\rho=0}^{M} \frac{(-1)^{\rho}+(-1)^{\kappa}}{\rho!}\left(\frac{\pi C^{\frac{1}{\alpha}}\eta}{i}\right)^{1+\kappa+\rho} \mathcal{M}(F)\left(-\frac{1+\kappa+\rho}{\alpha}\right) + O_{M,F,\alpha,\kappa}(|C^{\frac{1}{\alpha}}\eta|^{M+\kappa+1+\frac{\alpha}{2}})$$

In particular this implies,

$$\mathcal{T}_{\alpha,m+r+2s}(F)\left(C^{\frac{1}{\alpha}}\eta\right) - \mathcal{T}_{\alpha,m+r+2s}(F)\left(-C^{\frac{1}{\alpha}}\eta\right) = \sum_{\rho=0}^{M} \frac{(-1)^{\rho}+(-1)^{m+r}}{\rho!}\left(\frac{\pi C^{\frac{1}{\alpha}}\eta}{i}\right)^{1+m+r+2s+\rho} \mathcal{M}(F)\left(-\frac{1+m+r+2s+\rho}{\alpha}\right) + O_{M,F,\alpha,m,r,s}(|C^{\frac{1}{\alpha}}\eta|^{M+m+r+2s+1+\frac{\alpha}{2}}) \quad (\circ\circ)$$

The proposition now follows from combining Theorem 4.8.1 with ($\circ$) and ($\circ\circ$). □

Proposition 7.2.3. *There exist constants*[2] $U_{m,r,s,\sigma,H_{\pm}}^{j,\pm}$, $V_{m,r,s,\sigma,H_{\pm}}^{l,\pm}$, *for* $j=1,2,3$, $l=1,2$, *depend-*

[2]The superscripts $\pm$ on the constants $U^{j,\pm}$ and $V^{l,\pm}$ indicates the torus, and have nothing to do with the superscripts $\pm$ in $\mathcal{J}^{\pm}$ which indicates the sign of the determinant.

ing only on the indicated parameters such that for any $M > 0$ we have,

$$
\begin{aligned}
&\mathcal{J}^{1,+}_{H,\alpha,+}(C,\eta) + \mathcal{J}^{1,+}_{H,\alpha,-}(C,\eta) \\
&= e(-\eta) \sum_{l=l_1^+(\theta)}^{2M} \sum_{m=l}^{2M} \sum_{s=0}^{2M} \sum_{r=0}^{2sM} \Bigg\{ U^{1,+}_{m,r,s,H_-} \eta^{-(1+m+r+s)} + \sum_{\sigma=0}^{M} U^{2,+}_{m,r,s,\sigma,H_-} (C\eta^{1-\alpha})^{2\sigma+1} \eta^{-(m+r+s+1)} \\
&\qquad\qquad + \sum_{\rho=0}^{M} U^{3,+}_{m,r,s,\rho,H_-} C^{\frac{1+m+r+s}{1-\alpha}} (C^{\frac{1}{1-\alpha}}\eta)^{s+\rho} \Bigg\} \\
&+ e(\eta) \sum_{l=l_1^-(\theta)}^{2M} \sum_{m=l}^{2M} \sum_{s=0}^{2M} \sum_{r=0}^{2sM} \Bigg\{ U^{1,-}_{m,r,s,H_-} \eta^{-(1+m+r+s)} + \sum_{\sigma=0}^{M} U^{1,-}_{m,r,s,\sigma,H_-} (C\eta^{1-\alpha})^{2\sigma+1} \eta^{-(m+r+s+1)} \\
&\qquad\qquad + \sum_{\rho=0}^{M} U^{1,-}_{m,r,s,\rho,H_-} C^{\frac{1+m+r+s}{1-\alpha}} (C^{\frac{1}{1-\alpha}}\eta)^{s+\rho} \Bigg\} \\
&+ e(-\eta) \sum_{l=l_2^+(\theta)}^{2M} \sum_{m=l}^{2M} \sum_{s=0}^{2M} \sum_{r=0}^{2sM} \Bigg\{ \eta^{-(m+r+s+\frac{1}{2})} \sum_{\sigma=0}^{M} V^{1,+}_{m,r,s,\sigma,H_-,H_+} (C\eta^{1-\alpha})^{2\sigma} \\
&\qquad\qquad + C^{\frac{\frac{1}{2}+m+r+s}{1-\alpha}} \sum_{\rho=0}^{M} V^{2,+}_{m,r,s,\sigma,H_-,H_+} (C^{\frac{1}{1-\alpha}}\eta)^{s+\rho} \Bigg\} \\
&+ e(\eta) \sum_{l=l_2^-(\theta)}^{2M} \sum_{m=l}^{2M} \sum_{s=0}^{2M} \sum_{r=0}^{2sM} \Bigg\{ \eta^{-(m+r+s+\frac{1}{2})} \sum_{\sigma=0}^{M} V^{1,-}_{m,r,s,\sigma,H_-,H_+} (C\eta^{1-\alpha})^{2\sigma} \\
&\qquad\qquad + C^{\frac{\frac{1}{2}+m+r+s}{1-\alpha}} \sum_{\rho=0}^{M} V^{2,-}_{m,r,s,\sigma,H_-,H_+} (C^{\frac{1}{1-\alpha}}\eta)^{s+\rho} \Bigg\} \\
&+ O_{H_+,H_-,\theta_\infty,M}(|\eta|^{-M} + |\eta|^{\frac{-1}{2}} |C^{\frac{1}{1-\alpha}}\eta|^{M+1})
\end{aligned}
$$

Proof. We will be using Theorem 4.8.2. With the notation Theorem 4.8.2 we have $\Phi_1 = H_-$ and $\Phi_2 = H_+$. Therefore we need to understand $\mathcal{T}_{1-\alpha,m+r+2s}(H_-)(C^{\frac{1}{1-\alpha}}\eta)$ and $\mathcal{T}_{1-\alpha,m+r+2s-\frac{1}{2}}(H_\pm)(C^{\frac{1}{1-\alpha}}\eta)$.

In order to do so we first need to understand the poles of the Mellin transforms $\mathcal{M}(H_\pm)(z)$. By definition $H_\pm(z)$ is given as a Mellin inverse, hence we have

$$\mathcal{M}(H_\pm)(z) = \pi^{-z} \frac{\Gamma\left(\frac{e_\pm+z}{2}\right)}{\Gamma\left(\frac{1+e_\pm-z}{2}\right)} \mathcal{M}(F)(z)$$

We also recall that $e_- = 1$ and $e_+ = 0$.

By Lemma 3.2.2 we know that $\mathcal{M}(F)(z)$ has a single simple pole at $z = 0$ with residue 1 and holomorphic elsewhere. Therefore all the other poles of $\mathcal{M}(H_\pm)(z)$ are governed by the poles of $\Gamma\left(\frac{e_\pm+z}{2}\right)$ and $\Gamma\left(\frac{1+e_\pm-z}{2}\right)$. The function $\Gamma\left(\frac{e_\pm+z}{2}\right)$ has simple poles at $z = -2\sigma - e_\pm$ for non-negative integers $\sigma \in \mathbb{Z}_{\geq 0}$. On the other hand the Γ-function does not have zeros, so that $\Gamma\left(\frac{1+e_\pm-z}{2}\right)$ does not give a pole, and since it does not have a pole at $1 + 2e_\pm + 2\sigma$ for $\sigma \in \mathbb{Z}_{\geq 0}$ it does not cancel any of the previous poles. Therefore we conclude;

- $\mathcal{M}(H_+)(z)$ has poles at $z = -2\sigma$ for $\sigma \in \mathbb{Z}_{\geq 0}$. All of them except the one at $\sigma = 0$ are simple and the one at $\sigma = 0$ is a double pole.

- $\mathcal{M}(H_-)(z)$ has poles at $z=-2\sigma-1$ and at $z=0$. All of these poles are simple.

Using the above observations we also see that none of the poles of $\mathcal{M}(H_\pm)(z)$ are poles of $\Gamma(m+r+2s+1+(1-\alpha)z)$ or $\Gamma(m+r+2s+\frac{1}{2}+(1-\alpha)z)$ as long as $\alpha\neq\frac{1}{2}$, and in this case we get double poles for $\mathcal{T}_{1-\alpha,m+r+2s-\frac{1}{2}}(H_-)(C^{\frac{1}{1-\alpha}}\eta)$. Therefore by Lemma 7.1.2 we get

$$\begin{aligned}
\mathcal{T}_{1-\alpha,m+r+2s}(H_-)(C^{\frac{1}{1-\alpha}}\eta) &= \sqrt{\pi}(m+r+2s)! \\
&+\sum_{\sigma=0}^{M}\left(\frac{\pi C^{\frac{1}{1-\alpha}}\eta}{i}\right)^{(1-\alpha)(2\sigma+1)} \\
&\qquad \Gamma(1+m+r+2s-(1-\alpha)(2\sigma+1))Res_{z=-2\sigma-1}(\mathcal{M}(H_-)(z)) \\
&+\sum_{\rho=0}^{M}\frac{(-1)^\rho}{\rho!}\mathcal{M}(H_-)\left(-\frac{1+m+r+2s+\rho}{1-\alpha}\right)\left(\frac{\pi C^{\frac{1}{1-\alpha}}\eta}{i}\right)^{1+m+r+2s+\rho} \\
&+O_{M,H_-,\alpha,\kappa}(|C^{\frac{1}{1-\alpha}}\eta|^{M+m+r+2s+1+\frac{1-\alpha}{2}}) \\
\mathcal{T}_{1-\alpha,m+r+2s-\frac{1}{2}}(H_-)(C^{\frac{1}{1-\alpha}}\eta) &= \sqrt{\pi}(m+r+2s-\tfrac{1}{2})! \\
&+\sum_{\sigma=0}^{M}\left(\frac{\pi C^{\frac{1}{1-\alpha}}\eta}{i}\right)^{(1-\alpha)(2\sigma+1)} \\
&\qquad \Gamma(\frac{1}{2}+m+r+2s-(1-\alpha)(2\sigma+1))Res_{z=-2\sigma-1}(\mathcal{M}(H_-)(z)) \\
&+\sum_{\rho=0}^{M}\frac{(-1)^\rho}{\rho!}\mathcal{M}(H_-)\left(-\frac{\frac{1}{2}+m+r+2s+\rho}{1-\alpha}\right)\left(\frac{\pi C^{\frac{1}{1-\alpha}}\eta}{i}\right)^{\frac{1}{2}+m+r+2s+\rho} \\
&+O_{M,H_-,\alpha,\kappa}(|C^{\frac{1}{1-\alpha}}\eta|^{M+m+r+2s+\frac{1}{2}+\frac{1-\alpha}{2}}) \\
\mathcal{T}_{1-\alpha,m+r+2s-\frac{1}{2}}(H_+)(C^{\frac{1}{\alpha}}\eta) &= \sum_{\sigma=0}^{M}\left(\frac{\pi C^{\frac{1}{1-\alpha}}\eta}{i}\right)^{2(1-\alpha)\sigma} \\
&\qquad \Gamma(\frac{1}{2}+m+r+2s-2(1-\alpha)\sigma)Res_{z=-2\sigma}(\mathcal{M}(H_+)(z)) \\
&+\sum_{\rho=0}^{M}\frac{(-1)^\rho}{\rho!}\mathcal{M}(H_+)\left(-\frac{\frac{1}{2}+m+r+2s+\rho}{1-\alpha}\right)\left(\frac{\pi C^{\frac{1}{1-\alpha}}\eta}{i}\right)^{\frac{1}{2}+m+r+2s+\rho} \\
&+O_{M,\Phi,\alpha,\kappa}(|C^{\frac{1}{1-\alpha}}\eta|^{M+m+r+2s+\frac{1}{2}+\frac{1-\alpha}{2}})
\end{aligned}$$

Combining the coefficients of the σ-sums (including the constatnt $\sqrt{\pi}(m+r+2s-\frac{1}{2})$) finishes the proof of the lemma.

□

7.3 Rewriting the sums

In the upcoming sections we will be estimating our sums for certain ranges of the parameters. However because of the dependence of $Kl_{t,f}(\xi,n)$ on various primes dividing n we would like to first rewrite the sums so that the gcd becomes easily tractable.

Lemma 7.3.1. *For any non-zero integer $n \in \mathbb{Z}$ let $n_\square$ denote the largest square divisor of n. Then,*

$$\sum_{f \ll n^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^3} \sum_{t \ll \frac{n^{\alpha+\delta_0}}{f^2}} \frac{1}{t^2} \sum_{\xi \neq 0} Kl_{t,f}(\xi, n) \left\{ \mathcal{J}_{F,\alpha,+}\left(\frac{tf^{2\alpha}}{(4n)^\alpha}, \frac{2\xi\sqrt{n}}{tf^2}\right) + \mathcal{J}_{F,\alpha,-}\left(\frac{tf^{2\alpha}}{(4n)^\alpha}, \frac{2\xi\sqrt{n}}{tf^2}\right)\right\}$$

$$= \sum_{d_n^2 | n} \frac{1}{d_n^3} \sum_{\substack{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n} \\ \gcd(f, \frac{n}{d_n^2})=1}} \frac{1}{f^3} \sum_{\substack{t_0=1 \\ t_0 : \square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} \frac{1}{t_0^2} \sum_{\substack{t_1 \ll \frac{n^{\alpha+\delta_0}}{t_0 (fd_n)^2} \\ rad(t_1)|t_0}} \frac{1}{t_1^2} \sum_{d_{t_0} | t_0} \sum_{\substack{\xi \neq 0 \\ \gcd(\frac{t_0}{d_{t_0}}, \xi)=1}} Kl_{t_0 t_1, fd_n}(t_1 d_n d_{t_0} \xi, n)$$

$$\left\{ \mathcal{J}_{F,\alpha,+}\left(\frac{t_0 t_1 (fd_n)^2}{(4n)^\alpha}, \frac{2d_{t_0}\xi\sqrt{n}}{t_0 d_n f^2}\right) + \mathcal{J}_{F,\alpha,-}\left(\frac{t_0 t_1 (fd_n)^2}{(4n)^\alpha}, \frac{2d_{t_0}\xi\sqrt{n}}{t_0 d_n f^2}\right)\right\}$$

Proof. The proof is a straightforward calculation. Starting from the original sum the steps that lead to the equality are,

$$= \sum_{d_n^2 | n} \frac{1}{d_n^3} \sum_{\substack{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n} \\ \gcd(f, \frac{n}{d_n^2})=1}} \frac{1}{f^3} \sum_{\substack{t_0=1 \\ t_0 : \square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} \frac{1}{t_0^2} \sum_{\substack{t_1 \ll \frac{n^{\alpha+\delta_0}}{t_0 (fd_n)^2} \\ rad(t_1)|t_0}} \frac{1}{t_1^2} \sum_{\xi \neq 0} Kl_{t_0 t_1, fd_n}(\xi, n)$$

$$\left\{ \mathcal{J}_{F,\alpha,+}\left(\frac{t_0 t_1 (fd_n)^2}{(4n)^\alpha}, \frac{2\xi\sqrt{n}}{t_0 t_1 (fd_n)^2}\right) + \mathcal{J}_{F,\alpha,-}\left(\frac{t_0 t_1 (fd_n)^2}{(4n)^\alpha}, \frac{2\xi\sqrt{n}}{t_0 t_1 (fd_n)^2}\right)\right\}$$

$$= \sum_{d_n^2 | n} \frac{1}{d_n^3} \sum_{\substack{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n} \\ \gcd(f, \frac{n}{d_n^2})=1}} \frac{1}{f^3} \sum_{\substack{t_0=1 \\ t_0 : \square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} \frac{1}{t_0^2} \sum_{\substack{t_1 \ll \frac{n^{\alpha+\delta_0}}{t_0 (fd_n)^2} \\ rad(t_1)|t_0}} \frac{1}{t_1^2} \sum_{\xi \neq 0} Kl_{t_0 t_1, fd_n}(t_1 d_n \xi, n)$$

$$\left\{ \mathcal{J}_{F,\alpha,+}\left(\frac{t_0 t_1 (fd_n)^2}{(4n)^\alpha}, \frac{2\xi\sqrt{n}}{t_0 d_n f^2}\right) + \mathcal{J}_{F,\alpha,-}\left(\frac{t_0 t_1 (fd_n)^2}{(4n)^\alpha}, \frac{2\xi\sqrt{n}}{t_0 d_n f^2}\right)\right\}$$

$$= \sum_{d_n^2 | n} \frac{1}{d_n^3} \sum_{\substack{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n} \\ \gcd(f, \frac{n}{d_n^2})=1}} \frac{1}{f^3} \sum_{\substack{t_0=1 \\ t_0 : \square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} \frac{1}{t_0^2} \sum_{\substack{t_1 \ll \frac{n^{\alpha+\delta_0}}{t_0 (fd_n)^2} \\ rad(t_1)|t_0}} \frac{1}{t_1^2} \sum_{d_{t_0} | t_0} \sum_{\substack{\xi \neq 0 \\ \gcd(t_0, \xi)=d_{t_0}}} Kl_{t_0 t_1, fd_n}(t_1 d_n \xi, n)$$

$$\left\{ \mathcal{J}_{F,\alpha,+}\left(\frac{t_0 t_1 (fd_n)^2}{(4n)^\alpha}, \frac{2\xi\sqrt{n}}{t_0 d_n f^2}\right) + \mathcal{J}_{F,\alpha,-}\left(\frac{t_0 t_1 (fd_n)^2}{(4n)^\alpha}, \frac{2\xi\sqrt{n}}{t_0 d_n f^2}\right)\right\}$$

$$= \sum_{d_n^2 | n} \frac{1}{d_n^3} \sum_{\substack{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n} \\ \gcd(f, \frac{n}{d_n^2})=1}} \frac{1}{f^3} \sum_{\substack{t_0=1 \\ t_0 : \square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} \frac{1}{t_0^2} \sum_{\substack{t_1 \ll \frac{n^{\alpha+\delta_0}}{t_0 (fd_n)^2} \\ rad(t_1)|t_0}} \frac{1}{t_1^2} \sum_{d_{t_0} | t_0} \sum_{\substack{\xi \neq 0 \\ \gcd(\frac{t_0}{d_{t_0}}, \xi)=1}} Kl_{t_0 t_1, fd_n}(t_1 d_n d_{t_0} \xi, n)$$

$$\left\{ \mathcal{J}_{F,\alpha,+}\left(\frac{t_0 t_1 (fd_n)^2}{(4n)^\alpha}, \frac{2d_{t_0}\xi\sqrt{n}}{t_0 d_n f^2}\right) + \mathcal{J}_{F,\alpha,-}\left(\frac{t_0 t_1 (fd_n)^2}{(4n)^\alpha}, \frac{2d_{t_0}\xi\sqrt{n}}{t_0 d_n f^2}\right)\right\}$$

Where, in the first line we used the fact that the vacation of n at a place dividing n has to be even for the sum to not to vanish, therefore the condition that $\gcd(f, \frac{n_\square}{d_n^2}) = 1$ ($n_\square$ is the largest square divisor of n) can be written as $\gcd(f, \frac{n}{d_n^2}) = 1$. Moreover in passing from the second equality to the third we used Corollary 6.1.7 which tells us that the sum vanishes unless $d_n t_1 \mid \xi$ and in passing to the last line we used the fact that t_1 is square-free. $\square$

Although the formulas look complicated Lemma 7.3.1 is useful because of its explicitness. The interesting part of the sum is the sum over ξ and t_0. Since everything is explicit we will be able to use Corollary 6.1.7 to bound the inner character sum in terms of the rest of the parameters. Explicitly, Corollary 6.1.7 gives us

$$Kl_{t_0t_1,d_nf}(t_1d_nd_{t_0}\xi,n)=O(d_nt_1\sqrt{d_{t_0}t_0}) \tag{7.1}$$

Where the notation is as in Lemma 7.3.1.

7.4 Auxiliary estimates

In this section we will prove several lemmas that will be used in the upcoming sections to prove the relevant bounds.

Lemma 7.4.1. *For any $\epsilon>0$, and any integer $M\in\mathbb{Z}_{>0}$ we have,*

$$\sqrt{n}\sum_{f\ll n^{\frac{\alpha+\delta_0}{2}}}\frac{1}{f^3}\sum_{t\ll\frac{n^{\alpha+\delta_0}}{f^2}}\frac{1}{t^2}\sum_{\frac{\xi\sqrt{n}}{tf^2}\gg n^{\epsilon}}|Kl_{t,f}(\xi,n)|\left(\frac{\xi\sqrt{n}}{tf^2}\right)^{-M-1}$$
$$=O_{M,\alpha,\delta_0,\epsilon}\left(\frac{n^{\frac{1}{4}}}{n^{(M+\frac{1}{2})\epsilon)}}+\frac{n^{\frac{\alpha+\delta_0}{2}}}{n^{M\epsilon}}\right) \tag{7.2}$$

In particular, for any $\tilde{\delta}>0$ if one can chooses $M=\max\{1,\frac{1}{4\epsilon}-\frac{1}{2},\frac{\alpha+\delta_0}{2\epsilon}\}+\tilde{\delta}$, then (7.2) is $O_M(n^{-\tilde{\delta}})$.

Proof. First note that if $tf^2\ll n^{\frac{1}{2}-\epsilon}$ then we automatically have the condition $\frac{\xi\sqrt{n}}{tf^2}\gg n^{\epsilon}$. We will therefore separate the treatment into two cases depending on $tf^2\ll n^{\frac{1}{2}-\epsilon}$ and $tf^2\gg n^{\frac{1}{2}-\epsilon}$.

- $tf^2\ll n^{\frac{1}{2}-\epsilon}$.

  By Lemma 7.3.1 we can rewrite (7.2) as

$$\sqrt{n}\sum_{d_n^2|n}\frac{1}{d_n^3}\sum_{\substack{f\ll\frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}\\ \gcd(f,\frac{n}{d_n^2})=1}}\frac{1}{f^3}\sum_{\substack{t_0=1\\ t_0:\square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}}\frac{1}{t_0^2}\sum_{\substack{t_1\ll\frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2}\\ rad(t_1)|t_0\\ t_0t_1(d_nf)^2\ll n^{\frac{1}{2}-\epsilon}}}\frac{1}{t_1^2}\sum_{d_{t_0}|t_0}$$
$$\sum_{\substack{\xi\in\mathbb{Z}\setminus\{0\}\\ \gcd(\frac{t_0}{d_{t_0}},\xi)=1}}|Kl_{t_0t_1,fd_n}(t_1d_nd_{t_0}\xi,n)|\left(\frac{d_{t_0}\xi\sqrt{n}}{t_0d_nf^2}\right)^{-M-1}$$

  By (7.1) we have $|Kl_{t_0t_1,d_nf}(t_1d_nd_{t_0}\xi,n)|=O(d_nt_1\sqrt{d_{t_0}t_0})$. Substituting this into the equation gives the following bound

$$\sqrt{n}\sum_{d_n^2|n}\frac{1}{d_n^2}\sum_{f\ll\frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}}\frac{1}{f^3}\sum_{\substack{t_0=1\\ t_0:\square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}}\frac{1}{t_0^{\frac{3}{2}}}\sum_{\substack{t_1\ll\frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2}\\ rad(t_1)|t_0\\ t_0t_1(d_nf)^2\ll n^{\frac{1}{2}-\epsilon}}}\frac{1}{t_1}\sum_{d_{t_0}|t_0}\sqrt{d_{t_0}}\sum_{\xi\neq0}\left(\frac{d_{t_0}\xi\sqrt{n}}{t_0d_nf^2}\right)^{-M-1}$$

Since $tf^2 \ll n^{\frac{1}{2}-\epsilon}$ we bound the ξ-sum trivially and are left with,

$$\frac{1}{n^{\frac{M}{2}}} \sum_{d_n^2|n} d_n^{M-1} \sum_{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}} f^{2M-1} \sum_{\substack{t_0=1 \\ t_0:\square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} t_0^{M-\frac{1}{2}} \sum_{\substack{t_1 \ll \frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2} \\ rad(t_1)|t_0 \\ t_0t_1(d_nf)^2 \ll n^{\frac{1}{2}-\epsilon}}} \frac{1}{t_1} \sum_{d_{t_0}|t_0} \frac{1}{d_{t_0}^{M+\frac{1}{2}}}$$

$$= O_{M,\alpha,\epsilon}\left(\frac{n^{\frac{1}{4}}}{n^{(M+\frac{1}{2})\epsilon}}\right) \quad (\star)$$

Note that in the last line we used the fact that $\alpha \geq \frac{1}{2}$.

- $tf^2 \gg n^{\frac{1}{2}-\epsilon}$.

Again by Lemma 7.3.1 we can rewrite (7.2) as

$$\sqrt{n} \sum_{d_n^2|n} \frac{1}{d_n^3} \sum_{\substack{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n} \\ \gcd(f, \frac{n}{d_n^2})=1}} \frac{1}{f^3} \sum_{\substack{t_0=1 \\ t_0:\square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} \frac{1}{t_0^2} \sum_{\substack{t_1 \ll \frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2} \\ rad(t_1)|t_0 \\ t_0t_1(d_nf)^2 \gg n^{\frac{1}{2}-\epsilon}}} \frac{1}{t_1^2} \sum_{d_{t_0}|t_0}$$

$$\sum_{\substack{\frac{d_{t_0}\xi\sqrt{n}}{t_0d_nf^2} \gg n^{\epsilon} \\ \gcd(\frac{t_0}{d_{t_0}},\xi)=1}} |Kl_{t_0t_1,fd_n}(t_1d_nd_{t_0}\xi, n)| \left(\frac{d_{t_0}\xi\sqrt{n}}{t_0d_nf^2}\right)^{-M-1}$$

By (7.1) we have $|Kl_{t_0t_1,d_nf}(t_1d_nd_{t_0}\xi, n)| = O(d_nt_1\sqrt{d_{t_0}t_0})$, and also since $d_nf < n^{\frac{\alpha+\delta_0}{2}}$ we necessarily have $d_n < n^{\frac{\alpha+\delta_0}{2}}$. Substituting these in the equation above we get

$$(7.2) \leq$$

$$\frac{1}{n^{\frac{M}{2}}} \sum_{\substack{d_n^2|n \\ d_n < n^{\frac{\alpha+\delta_0}{2}}}} \frac{1}{d_n^2} \sum_{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}} \frac{1}{f^3} \sum_{\substack{t_0=1 \\ t_0:\square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} \frac{1}{t_0^{\frac{3}{2}}} \sum_{\substack{t_1 \ll \frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2} \\ rad(t_1)|t_0 \\ t_0t_1(d_nf)^2 \gg n^{\frac{1}{2}-\epsilon}}} \frac{1}{t_1} \sum_{d_{t_0}|t_0} \sqrt{d_{t_0}} \sum_{\frac{d_{t_0}\xi\sqrt{n}}{t_0d_nf^2} \gg n^{\epsilon}} \left(\frac{t_0d_nf^2}{d_{t_0}\xi}\right)^{M+1}$$

The ξ-sum is bounded trivially if $\frac{d_{t_0}\sqrt{n}}{t_0d_nf^2} \gg n^{\epsilon}$ otherwise it is bounded by $\left(\frac{d_{t_0}n^{\frac{1}{2}-\epsilon}}{t_0d_nf^2}\right)^M$. Then we get the following bound for (7.2):

$$(7.2) \ll$$

$$\frac{1}{n^{\frac{M}{2}}} \sum_{\substack{d_n^2|n \\ d_n<n^{\frac{\alpha+\delta_0}{2}}}} d_n^{M-1} \sum_{f\ll\frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}} f^{2M-1} \sum_{\substack{t_0=1 \\ t_0:\square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} t_0^{M-\frac{1}{2}} \sum_{\substack{t_1\ll\frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2} \\ rad(t_1)|t_0 \\ t_0t_1(d_nf)^2\gg n^{\frac{1}{2}-\epsilon}}} \frac{1}{t_1} \sum_{\substack{d_{t_0}|t_0 \\ \frac{t_0d_nf^2}{d_{t_0}}\ll n^{\frac{1}{2}-\epsilon}}} \frac{1}{d_{t_0}^{M+\frac{1}{2}}}$$

$$+ \frac{1}{n^{M\epsilon}} \sum_{\substack{d_n^2|n \\ d_n<n^{\frac{\alpha+\delta_0}{2}}}} \frac{1}{d_n} \sum_{f\ll\frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}} \frac{1}{f} \sum_{\substack{t_0=1 \\ t_0:\square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} \frac{1}{t_0^{\frac{1}{2}}} \sum_{\substack{t_1\ll\frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2} \\ rad(t_1)|t_0 \\ t_0t_1(d_nf)^2\gg n^{\frac{1}{2}-\epsilon}}} \frac{1}{t_1} \sum_{\substack{d_{t_0}|t_0 \\ \frac{t_0d_nf^2}{d_{t_0}}\gg n^{\frac{1}{2}-\epsilon}}} \frac{1}{\sqrt{d_{t_0}}}$$

Using the change of variables $d_{t_0} \mapsto \frac{t_0}{d_{t_0}}$, the above sum can be written as

$$(7.2) \ll$$

$$\frac{1}{n^{\frac{M}{2}}} \sum_{\substack{d_n^2|n \\ d_n<n^{\frac{\alpha+\delta_0}{2}}}} d_n^{M-1} \sum_{f\ll\frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}} f^{2M-1} \sum_{\substack{t_0=1 \\ t_0:\square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} \frac{1}{t_0} \sum_{\substack{t_1\ll\frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2} \\ rad(t_1)|t_0 \\ t_0t_1(d_nf)^2\gg n^{\frac{1}{2}-\epsilon}}} \frac{1}{t_1} \sum_{\substack{d_{t_0}|t_0 \\ d_{t_0}d_nf^2\ll n^{\frac{1}{2}-\epsilon}}} d_{t_0}^{M+\frac{1}{2}}$$

$$+ \frac{1}{n^{M\epsilon}} \sum_{\substack{d_n^2|n \\ d_n<n^{\frac{\alpha+\delta_0}{2}}}} \frac{1}{d_n} \sum_{f\ll\frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}} \frac{1}{f} \sum_{\substack{t_0=1 \\ t_0:\square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} \frac{1}{t_0} \sum_{\substack{t_1\ll\frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2} \\ rad(t_1)|t_0 \\ t_0t_1(d_nf)^2\gg n^{\frac{1}{2}-\epsilon}}} \frac{1}{t_1} \sum_{\substack{d_{t_0}|t_0 \\ d_{t_0}d_nf^2\gg n^{\frac{1}{2}-\epsilon}}} \sqrt{d_{t_0}}$$

By rearranging the d_{t_0}, t_1 and t_0 sums the above is bounded by

$$(7.2) \ll \frac{1}{n^{\frac{M}{2}}} \sum_{\substack{d_n^2|n \\ d_n<n^{\frac{\alpha+\delta_0}{2}}}} d_n^{M-1} \sum_{f\ll\frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}} f^{2M-1} \sum_{\substack{d\ll\frac{n^{\alpha+\delta_0}}{(d_nf)^2} \\ dd_nf^2\ll n^{\frac{1}{2}-\epsilon}}} d^{M-\frac{1}{2}} \sum_{t=1}^{\frac{n^{\alpha+\delta_0}}{d(fd_n)^2}} \frac{1}{t}$$

$$+ \frac{1}{n^{M\epsilon}} \sum_{\substack{d_n^2|n \\ d_n<n^{\frac{\alpha+\delta_0}{2}}}} \frac{1}{d_n} \sum_{f\ll\frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}} \frac{1}{f} \sum_{\substack{d\ll\frac{n^{\alpha+\delta_0}}{(d_nf)^2} \\ dd_nf^2\gg n^{\frac{1}{2}-\epsilon}}} \frac{1}{\sqrt{d}} \sum_{t=1}^{\frac{n^{\alpha+\delta_0}}{d(fd_n)^2}} \frac{1}{t}$$

$$\ll_{M,\alpha,\delta_0,\epsilon} \frac{n^{\frac{1}{4}}}{n^{(M+\frac{1}{2})\epsilon)}} + \frac{n^{\frac{\alpha+\delta_0}{2}}}{n^{M\epsilon}} \qquad (\star\star)$$

Where in the last line we used that $\alpha \geq \frac{1}{2}$.

The lemma follows from combining $(\star)$ and $(\star\star)$

$\square$

Lemma 7.4.2. *For any $\epsilon > 0$, and any integer $M \in \mathbb{Z}_{>0}$ we have,*

$$\sqrt{n}\sum_{f\ll n^{\frac{\alpha+\delta_0}{2}}}\frac{1}{f^3}\sum_{t\ll\frac{n^{\alpha+\delta_0}}{f^2}}\frac{1}{t^2}\sum_{\xi\gg\frac{n^{\frac{1}{2}+\epsilon}}{(tf^2)^{\frac{1}{\alpha}-1}}}|Kl_{t,f}(\xi,n)|\left(\frac{\xi\sqrt{n}}{tf^2}\right)^{-1}\left(\frac{\xi(tf^2)^{\frac{1}{\alpha}-1}}{\sqrt{n}}\right)^{-M}$$

$$=O_{M,\alpha,\delta_0,\epsilon}\left(\frac{n^{\frac{1}{2}(\alpha+\delta_0)}}{n^{M\epsilon}}+\frac{n^{\frac{1/2+\epsilon}{2(\frac{1}{\alpha}-1)}}}{n^{M\epsilon}}\right)\tag{7.3}$$

In particular for any $\tilde{\delta} > 0$, we can choose $M = \max\{1, \frac{\alpha(\frac{1}{2}+\epsilon)}{2\epsilon(1-\alpha)}\} + \tilde{\delta}$ so that (7.3) is $O_{M,\tilde{\delta}}(n^{-\tilde{\delta}})$.

Proof. We proceed as in Lemma 7.4.1. Using Lemma 7.3.1 equation (7.3) can be rewritten as

$$\sqrt{n}\sum_{d_n^2|n}\frac{1}{d_n^3}\sum_{\substack{f\ll\frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}\\ \gcd(f,\frac{n}{d_n^2})=1}}\frac{1}{f^3}\sum_{\substack{t_0=1\\ t_0:\square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}}\frac{1}{t_0^2}\sum_{\substack{t_1\ll\frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2}\\ rad(t_1)|t_0}}\frac{1}{t_1^2}\sum_{d_{t_0}|t_0}$$
$$\sum_{d_nd_{t_0}t_1\xi\gg\frac{n^{\frac{1}{2}+\epsilon}}{(t_0t_1d_n^2f^2)^{\frac{1}{\alpha}-1}}}|Kl_{t_0t_1,fd_n}(t_1d_nd_{t_0}\xi,n)|\frac{n^{\frac{M-1}{2}}}{(t_0t_1d_n^2f^2)^{M(\frac{1}{\alpha}-1)-1}(d_nd_{t_0}t_1\xi)^{M+1}}$$

Substituting the bound $Kl_{t_0t_1,d_nf}(t_1d_nd_{t_0}\xi,n) = O(d_nt_1\sqrt{d_{t_0}t_0})$ and recalling that $d_nf < n^{\frac{\alpha+\delta_0}{2}}$ shows that (7.3) is bounded by

$$n^{\frac{M}{2}}\sum_{\substack{d_n^2|n\\ d_n\ll n^{\frac{\alpha+\delta_0}{2}}}}\frac{1}{d_n^2}\sum_{f\ll\frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}}\frac{1}{f^3}\sum_{\substack{t_0=1\\ t_0:\square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}}\frac{1}{t_0^{\frac{3}{2}}}\sum_{\substack{t_1\ll\frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2}\\ rad(t_1)|t_0}}\frac{1}{t_1}\sum_{d_{t_0}|t_0}d_{t_0}$$
$$\sum_{d_nd_{t_0}t_1\xi\gg\frac{n^{\frac{1}{2}+\epsilon}}{(t_0t_1d_n^2f^2)^{\frac{1}{\alpha}-1}}}\frac{1}{(t_0t_1d_n^2f^2)^{M(\frac{1}{\alpha}-1)-1}(d_nd_{t_0}t_1\xi)^{M+1}}$$

$$\ll_M n^{\frac{M}{2}}\sum_{\substack{d_n^2|n\\ d_n\ll n^{\frac{\alpha+\delta_0}{2}}}}\frac{1}{d_n^2}\sum_{f\ll\frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}}\frac{1}{f^3}\sum_{\substack{t_0=1\\ t_0:\square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}}\frac{1}{t_0^{\frac{3}{2}}}\sum_{\substack{t_1\ll\frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2}\\ rad(t_1)|t_0}}\frac{1}{t_1}\sum_{d_{t_0}|t_0}d_{t_0}$$
$$\left(\frac{(d_nd_{t_0}t_1)^M(t_0t_1d_n^2f^2)^{M(\frac{1}{\alpha}-1)}}{(t_0t_1d_n^2f^2)^{M(\frac{1}{\alpha}-1)-1}(d_nd_{t_0}t_1)^{M+1}n^{M(\frac{1}{2}+\epsilon)}}+\frac{1_{d_nd_{t_0}t_1\gg\frac{n^{\frac{1}{2}+\epsilon}}{(t_0t_1d_n^2f^2)^{\frac{1}{\alpha}-1}}}}{(t_0t_1d_n^2f^2)^{M(\frac{1}{\alpha}-1)-1}(d_nd_{t_0}t_1)^{M+1}}\right)$$

$$= \frac{1}{n^{M\epsilon}} \sum_{\substack{d_n^2 | n \\ d_n \ll n^{\frac{\alpha+\delta_0}{2}}}} \frac{1}{d_n} \sum_{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}} \frac{1}{f} \sum_{\substack{t_0=1 \\ t_0 : \square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} \frac{1}{t_0^{\frac{1}{2}}} \sum_{\substack{t_1 \ll \frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2} \\ rad(t_1)|t_0}} \frac{1}{t_1} \sum_{\substack{d_{t_0}|t_0 \\ d_n d_{t_0} t_1 \ll \frac{n^{\frac{1}{2}+\epsilon}}{(t_0 t_1 (d_n f)^2)^{\frac{1}{\alpha}-1}}}} 1 \qquad (\star)$$

$$+ n^{\frac{M}{2}} \sum_{\substack{d_n^2 | n \\ d_n \ll n^{\frac{\alpha+\delta_0}{2}}}} \frac{1}{d_n^{M(\frac{2}{\alpha}-1)+1}} \sum_{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}} \frac{1}{f^{2M(\frac{1}{\alpha}-1)+1}}$$

$$\sum_{\substack{t_0=1 \\ t_0 : \square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} \frac{1}{t_0^{M(\frac{1}{\alpha}-1)+\frac{1}{2}}} \sum_{\substack{t_1 \ll \frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2} \\ rad(t_1)|t_0}} \frac{1}{t_1^{\frac{M}{\alpha}+1}} \sum_{\substack{d_{t_0}|t_0 \\ d_n d_{t_0} t_1 \gg \frac{n^{\frac{1}{2}+\epsilon}}{(t_0 t_1 d_n^2 f^2)^{\frac{1}{\alpha}-1}}}} \frac{1}{d_{t_0}^{M+1}} \qquad (\star\star)$$

By interchanging the d_{t_0}-sum with the t_0-sum and using the fact that $\alpha \geq \frac{1}{2}$ (which in particular implies that $\frac{1-\alpha}{\alpha} \leq 1$), $(\star)$ is easily seen to be bounded by a constant (depending only on M, α and δ_0) times $\frac{n^{\frac{\alpha+\delta_0}{2}}}{n^{M\epsilon}} \log(n)\}$. Next we bound $(\star\star)$. Since $d_n d_{t_0} t_1 \gg \frac{n^{\frac{1}{2}+\epsilon}}{(t_0 t_1 d_n^2 f^2)^{\frac{1}{\alpha}-1}}$ we either have $d_n t_1 \gg \frac{n^{\frac{1}{2}+\epsilon}}{(t_0 t_1 d_n^2 f^2)^{\frac{1}{\alpha}-1}}$ or $d_{t_0} \gg \frac{n^{\frac{1}{2}+\epsilon}}{d_n t_1 (t_0 t_1 d_n^2 f^2)^{\frac{1}{\alpha}-1}}$. Therefore $(\star\star)$ is bounded by

$$\ll_M n^{\frac{M}{2}} \sum_{\substack{d_n^2 | n \\ d_n \ll n^{\frac{\alpha+\delta_0}{2}}}} \frac{1}{d_n^{M(\frac{2}{\alpha}-1)+1}} \sum_{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}} \frac{1}{f^{2M(\frac{1}{\alpha}-1)+1}} \sum_{\substack{t_0=1 \\ t_0 : \square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} \frac{1}{t_0^{M(\frac{1}{\alpha}-1)+\frac{1}{2}}} \sum_{\substack{t_1 \ll \frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2} \\ rad(t_1)|t_0}} \frac{1}{t_1^{\frac{M}{\alpha}+1}}$$

$$\left(\left(\frac{n^{\frac{1}{2}+\epsilon}}{d_n t_1 (t_0 t_1 d_n^2 f^2)^{\frac{1}{\alpha}-1}} \right)^{-M} + 1_{d_n t_1 \gg \frac{n^{\frac{1}{2}+\epsilon}}{(t_0 t_1 d_n^2 f^2)^{\frac{1}{\alpha}-1}}} \right)$$

$$\ll_{M,\alpha,\delta_0,\epsilon} \frac{n^{\frac{1}{2}(\alpha+\delta_0)}}{n^{M\epsilon}} + n^{\frac{M}{2}} \sum_{\substack{d_n^2 | n \\ d_n \ll n^{\frac{\alpha+\delta_0}{2}}}} \frac{1}{d_n^{M(\frac{2}{\alpha}-1)+1}} \sum_{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}} \frac{1}{f^{2M(\frac{1}{\alpha}-1)+1}} \sum_{\substack{t_0=1 \\ t_0 : \square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} \frac{1}{t_0^{M(\frac{1}{\alpha}-1)+\frac{1}{2}}}$$

$$\left(\left(\frac{n^{\frac{1}{2}+\epsilon}}{d_n^{\frac{2}{\alpha}-1} (t_0 f^2)^{\frac{1}{\alpha}-1}} \right)^{-M} + 1_{d_n^{\frac{2}{\alpha}-1} \gg \frac{n^{\frac{1}{2}+\epsilon}}{(t_0 f^2)^{(\frac{1}{\alpha}-1)}}} \right)$$

$$\ll_{M,\alpha,\delta_0,\epsilon} \frac{n^{\frac{1}{2}(\alpha+\delta_0)}}{n^{M\epsilon}} + n^{\frac{M}{2}} \sum_{\substack{d_n^2|n \\ d_n \ll n^{\frac{\alpha+\delta_0}{2}}}} \frac{1}{d_n^{M(\frac{2}{\alpha}-1)+1}} \sum_{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}} \frac{1}{f^{2M(\frac{1}{\alpha}-1)+1}}$$
$$\left(\left(\frac{n^{\frac{1}{2}+\epsilon}}{d_n^{\frac{2}{\alpha}-1} f^{2(\frac{1}{\alpha}-1)}} \right)^{-M+\frac{1}{2(\frac{1}{\alpha}-1)}} + 1_{d_n^{\frac{2}{\alpha}-1} f^{2(\frac{1}{\alpha}-1)} \gg n^{\frac{1}{2}+\epsilon}} \right)$$
$$\ll_{M,\alpha,\delta_0,\epsilon} \frac{n^{\frac{1}{2}(\alpha+\delta_0)}}{n^{M\epsilon}} + \frac{n^{\frac{1/2+\epsilon}{2(\frac{1}{\alpha}-1)}}}{n^{M\epsilon}} + n^{\frac{M}{2}} \sum_{\substack{d_n^2|n \\ d_n \ll n^{\frac{\alpha+\delta_0}{2}}}} \frac{1}{d_n^{M(\frac{2}{\alpha}-1)+1}} \left(\frac{d_n^{M(\frac{2}{\alpha}-1)}}{n^{M(\frac{1}{2}+\epsilon)}} + 1_{d_n^{\frac{2}{\alpha}-1} \gg n^{\frac{1}{2}+\epsilon}} \right)$$
$$\ll_{M,\alpha,\delta_0,\epsilon} \frac{n^{\frac{1}{2}(\alpha+\delta_0)}}{n^{M\epsilon}} + \frac{n^{\frac{1/2+\epsilon}{2(\frac{1}{\alpha}-1)}}}{n^{M\epsilon}}$$

The lemma follows.

$\square$

Lemma 7.4.3. *Let $u, s, \rho \in \mathbb{Z}_{\geq 0}$ and $\kappa \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Then, for any $\epsilon > 0$ and $\delta_0 > 0$ we have,*

$$\sqrt{n} \sum_{f \ll n^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^3} \sum_{t \ll \frac{n^{\alpha+\delta_0}}{f^2}} \frac{1}{t^2} \sum_{\frac{n^{\epsilon}\sqrt{n}}{(tf^2)^{\frac{1}{\alpha}-1}} \gg \xi} |Kl_{t,f}(\xi,n)| \left(\frac{(tf^2)^{\frac{1}{\alpha}}}{n} \right)^{\kappa+u+s} \left(\frac{(tf^2)^{\frac{1}{\alpha}-1}\xi}{\sqrt{n}} \right)^{s+\rho}$$
$$= O_{s,\rho,\alpha,\delta_0,\epsilon}\left(n^{\frac{\alpha}{2}+\epsilon(s+\rho+1)+\delta_0(\frac{\kappa+u+s-1}{\alpha}+\frac{1}{2})} \right) \quad (7.4)$$

Proof.

$$(7.4) = \frac{1}{n^{\kappa+u+s+\frac{s+\rho-1}{2}}} \sum_{f \ll n^{\frac{\alpha+\delta_0}{2}}} f^{\frac{2(\kappa+u+2s+\rho)}{\alpha}-(2(s+\rho)+3)} \sum_{t \ll \frac{n^{\alpha+\delta_0}}{f^2}} t^{\frac{\kappa+u+2s+\rho}{\alpha}-(s+\rho)-2} \sum_{\frac{n^{\epsilon}\sqrt{n}}{t^{\frac{1}{\alpha}-1}} \gg \xi} Kl_{t,f}(\xi,n)\xi^{s+\rho}$$

In order to bound this last sum we proceed exactly as in the proof of lemmas 7.4.1 and 7.4.2 and first rewrite the sum and then use (7.1).

$$(7.4) = \frac{1}{n^{\kappa+u+s+\frac{s+\rho-1}{2}}} \sum_{\substack{d_n^2|n \\ d_n \ll n^{\frac{\alpha+\delta_0}{2}}}} d_n^{\frac{2(\kappa+u+2s+\rho)}{\alpha}-(2(s+\rho)+3)} \sum_{\substack{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n} \\ \gcd(f,\frac{n}{d_n^2})=1}} f^{\frac{2(\kappa+u+2s+\rho)}{\alpha}-(2(s+\rho)+3)}$$
$$\sum_{\substack{t_0=1 \\ t_0:\square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} t_0^{\frac{\kappa+u+2s+\rho}{\alpha}-(s+\rho)-2} \sum_{\substack{t_1 \ll \frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2} \\ rad(t_1)|t_0}} t_1^{\frac{\kappa+u+2s+\rho}{\alpha}-(s+\rho)-2}$$
$$\sum_{d_{t_0}|t_0} \sum_{\substack{\frac{n^{\epsilon}\sqrt{n}}{(t_1t_0(d_nf)^2)^{\frac{1}{\alpha}-1}} \gg t_1d_nd_{t_0}\xi \\ \gcd(\frac{t_0}{d_{t_0}},\xi)=1}} |Kl_{t_0t_1,fd_n}(t_1d_nd_{t_0}\xi,n)|(t_1d_nd_{t_0}\xi)^{s+\rho}$$

Note that the ξ-sum is empty unless $t_1 d_n d_{t_0}(t_1 t_0 (d_n f)^2)^{\frac{1}{\alpha}-1} \ll n^{\frac{1}{2}+\epsilon}$, hence by using (7.1) we can bound the ξ-sum by

$$\sum_{\substack{\frac{n^{\epsilon}\sqrt{n}}{(t_1 t_0 (d_n f)^2)^{\frac{1}{\alpha}-1}} \gg t_1 d_n d_{t_0}\xi \\ \gcd(\frac{t_0}{d_{t_0}},\xi)=1}} |Kl_{t_0 t_1, f d_n}(t_1 d_n d_{t_0}\xi, n)|(t_1 d_n d_{t_0}\xi)^{s+\rho} \ll_{s,\rho} \left(\frac{n^{\frac{1}{2}+\epsilon}}{(t_1 t_0 (d_n f)^2)^{\frac{1}{\alpha}-1}}\right)^{s+\rho+1} \frac{\sqrt{t_0}}{\sqrt{d_{t_0}}}$$

Substituting this into the expression above for (7.4) gives,

$$\begin{aligned}
(7.4) \ll_{s,\rho} & \frac{n^{1+\epsilon(s+\rho+1)}}{n^{\kappa+u+s}} \sum_{\substack{d_n^2 | n \\ d_n \ll n^{\frac{\alpha+\delta_0}{2}}}} d_n^{\frac{2(\kappa+u+s)}{\alpha}-(1+\frac{2}{\alpha})} \sum_{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}} f^{\frac{2(\kappa+u+s)}{\alpha}-(1+\frac{2}{\alpha})} \\
& \sum_{\substack{t_0=1 \\ t_0:\square-free}}^{\frac{n^{\alpha+\delta_0}}{(f d_n)^2}} t_0^{\frac{\kappa+u+s}{\alpha}-(\frac{1}{2}+\frac{1}{\alpha})} \sum_{\substack{t_1 \ll \frac{n^{\alpha+\delta_0}}{t_0 (f d_n)^2} \\ rad(t_1)|t_0}} t_1^{\frac{\kappa+u+s}{\alpha}-(1+\frac{1}{\alpha})} \sum_{d_{t_0}|t_0} \frac{1}{\sqrt{d_{t_0}}} \\
\ll_{s,\rho,\alpha,\delta_0,\epsilon} & \frac{n^{1+\epsilon(s+\rho+1)}}{n^{\kappa+u+s}} \sum_{\substack{d_n^2 | n \\ d_n \ll n^{\frac{\alpha+\delta_0}{2}}}} d_n^{\frac{2(\kappa+u+s)}{\alpha}-(1+\frac{2}{\alpha})} \sum_{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}} f^{\frac{2(\kappa+u+s)}{\alpha}-(1+\frac{2}{\alpha})} \\
& \left(n^{\frac{\alpha}{2}+\epsilon(s+\rho+1)+\delta_0(\frac{\kappa+u+s-1}{\alpha}+\frac{1}{2})}+1\right) \\
\ll_{s,\rho,\alpha,\delta_0,\epsilon} & n^{\frac{\alpha}{2}+\epsilon(s+\rho+1)+\delta_0(\frac{\kappa+u+s-1}{\alpha}+\frac{1}{2})}
\end{aligned}$$

$\square$

Lemma 7.4.4. *Let $u, s \in \mathbb{Z}_{\geq 0}$ and $\kappa \in \frac{1}{2}\mathbb{Z}_{\geq \frac{3}{2}}$. Then, for any $\epsilon_1 > 0$, $\frac{1}{2} > \epsilon_2 > 0$ and $\delta_0 > 0$ we have,*

$$\sqrt{n} \sum_{f \ll n^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^3} \sum_{t \ll \frac{n^{\alpha+\delta_0}}{f^2}} \frac{1}{t^2} \sum_{\frac{n^{\epsilon_1}\sqrt{n}}{(tf^2)^{\frac{1}{\alpha}-1}} \gg \xi \gg n^{\epsilon_2}} |Kl_{t,f}(\xi, n)| \left(\frac{tf^2}{\xi\sqrt{n}}\right)^{\kappa+u+s}$$
$$= O_{\kappa,u,s,\epsilon_1,\epsilon_2}\left(\left(\frac{n^{\alpha+\delta_0}}{n^{\frac{1}{2}+\epsilon_2}}\right)^{\kappa+u+s-\frac{1}{2}} n^{\frac{1}{4}+\frac{\epsilon_2}{2}}\right) \quad (7.5)$$

Proof. Once again, as in the lemmas above, we will rewrite (7.5) and use (7.1).

$$(7.5) = \frac{1}{n^{\frac{\kappa+u+s-1}{2}}} \sum_{f \ll n^{\frac{\alpha+\delta_0}{2}}} f^{2(\kappa+u+s)-3} \sum_{t \ll \frac{n^{\alpha+\delta_0}}{f^2}} t^{\kappa+u+s-2} \sum_{\frac{n^{\epsilon_1}\sqrt{n}}{(tf^2)^{\frac{1}{\alpha}-1}} \gg \xi \gg n^{\epsilon_2}} Kl_{t,f}(\xi,n) \frac{1}{\xi^{\kappa+u+s}}$$

$$= \frac{1}{n^{\frac{\kappa+u+s-1}{2}}} \sum_{\substack{d_n^2 | n \\ d_n \ll n^{\frac{\alpha+\delta_0}{2}}}} d_n^{2(\kappa+u+s)-3} \sum_{\substack{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n} \\ \gcd(f, \frac{n}{d_n^2})=1}} f^{2(\kappa+u+s)-3} \sum_{\substack{t_0=1 \\ t_0:\square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} t_0^{\kappa+u+s-2}$$

$$\sum_{\substack{t_1 \ll \frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2} \\ rad(t_1)|t_0}} t_1^{\kappa+u+s-2} \sum_{d_{t_0}|t_0} \sum_{\substack{\frac{n^{\epsilon_1}\sqrt{n}}{(t_1 t_0 (d_n f)^2)^{\frac{1}{\alpha}-1}} \gg t_1 d_n d_{t_0} \xi \gg n^{\epsilon_2} \\ \gcd(\frac{t_0}{d_{t_0}}, \xi)=1}} |Kl_{t_0 t_1, f d_n}(t_1 d_n d_{t_0} \xi, n)| \frac{1}{(t_1 d_n d_{t_0} \xi)^{\kappa+u+s}}$$

$$\ll_{\kappa,u,s,\epsilon_2} \frac{1}{n^{(\kappa+u+s-1)(\frac{1}{2}+\epsilon_2)}} \sum_{\substack{d_n^2 | n \\ d_n \ll n^{\frac{\alpha+\delta_0}{2}}}} d_n^{\kappa+u+s-2} \sum_{f \ll \frac{n^{\frac{\alpha+\delta_0}{2}}}{d_n}} f^{2(\kappa+u+s)-3} \sum_{\substack{t_0=1 \\ t_0:\square-free}}^{\frac{n^{\alpha+\delta_0}}{(fd_n)^2}} t_0^{\kappa+u+s-\frac{3}{2}}$$

$$\sum_{\substack{t_1 \ll \frac{n^{\alpha+\delta_0}}{t_0(fd_n)^2} \\ rad(t_1)|t_0}} \frac{1}{t_1} \sum_{d_{t_0}|t_0} \frac{1}{d_{t_0}^{\kappa+u+s-\frac{3}{2}}}$$

$$\ll_{\kappa,u,s,\epsilon_2} \frac{1}{n^{(\kappa+u+s-1)(\frac{1}{2}+\epsilon_2)}} \sum_{\substack{d_n^2 | n \\ d_n \ll n^{\frac{\alpha+\delta_0}{2}}}} d_n^{\kappa+u+s-2} \sum_{f \ll \frac{n^{\frac{1}{2}+\epsilon_1-\epsilon_2}}{d_n^{2+\frac{\alpha}{1-\alpha}}}} f^{2(\kappa+u+s)-3}$$

$$\left(\left(\frac{n^{\alpha+\delta_0}}{(d_n f)^2} \right)^{\kappa+u+s-\frac{1}{2}} + 1 \right)$$

$$\ll_{\kappa,u,s,\epsilon_2} \frac{n^{(\alpha+\delta_0)(\kappa+u+s-\frac{1}{2})}}{n^{(\kappa+u+s-1)(\frac{1}{2}+\epsilon_2)}}$$

The lemma follows.

□

Lemma 7.4.5. *For any $\epsilon > 0$, and any integer $M \in \mathbb{Z}_{>0}$ we have,*

$$\sum_{f \ll n^{\frac{1-\alpha+\delta_0}{2}}} \frac{1}{f} \sum_{t \ll \frac{n^{1-\alpha+\delta_0}}{f^2}} \frac{1}{t} \sum_{\frac{\xi\sqrt{n}}{tf^2} \gg n^{\epsilon}} |Kl_{t,f}(\xi,n)| \left(\frac{\xi\sqrt{n}}{tf^2} \right)^{-M-1}$$

$$= O_{M,\alpha,\delta_0,\epsilon}\left(\frac{n^{\frac{1}{4}}}{n^{(M+\frac{3}{2})\epsilon}} + \frac{n^{1-\frac{3\alpha}{2}+\delta_0}}{n^{M\epsilon}} \right) \quad (7.6)$$

In particular, for any $\tilde{\delta} > 0$ $M = \max\{1, \frac{1}{4\epsilon} - \frac{3}{2}, \frac{2-3\alpha}{2\epsilon}\} + \tilde{\delta}$, then (7.6) is $O_M(n^{-\tilde{\delta}})$.

Proof. Same as the proof of Lemma 7.4.1

□

Lemma 7.4.6. *For any $\epsilon > 0$, and any integer $M \in \mathbb{Z}_{>0}$ we have,*

$$\sum_{f \ll n^{\frac{1-\alpha+\delta_0}{2}}} \frac{1}{f} \sum_{t \ll \frac{n^{1-\alpha+\delta_0}}{f^2}} \frac{1}{t} \sum_{\xi \gg \frac{n^{\frac{1}{2}+\epsilon}}{(tf^2)^{\frac{1}{1-\alpha}-1}}} |Kl_{t,f}(\xi, n)| \left(\frac{\xi\sqrt{n}}{tf^2}\right)^{-1} \left(\frac{\xi(tf^2)^{\frac{1}{1-\alpha}-1}}{\sqrt{n}}\right)^{-M}$$

$$= O_{M,\alpha,\delta_0,\epsilon}\left(\frac{n^{\frac{3(1-\alpha)}{2\alpha}(\frac{1}{2}+\epsilon)}\log(n)}{n^{\frac{1}{2}+M\epsilon}}\right) \quad (7.7)$$

In particular for any $\tilde{\delta} > 0$, we can choose $M = \max\{1, \frac{1}{\epsilon}\left(\frac{3(1-\alpha)(\frac{1}{2}+\epsilon)}{2\alpha} - \frac{1}{2}\right)\} + \tilde{\delta}$ so that (7.7) is $O_{M,\tilde{\delta}}(n^{-\tilde{\delta}})$.

Proof. Same as the proof of Lemma 7.4.2

□

Lemma 7.4.7. *Let $u, s, \rho \in \mathbb{Z}_{\geq 0}$ and $\kappa \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Then, for any $\epsilon > 0$ and $\delta_0 > 0$ we have,*

$$\sum_{f \ll n^{\frac{1-\alpha+\delta_0}{2}}} \frac{1}{f} \sum_{t \ll \frac{n^{1-\alpha+\delta_0}}{f^2}} \frac{1}{t} \sum_{\frac{n^{\epsilon}\sqrt{n}}{(tf^2)^{\frac{1}{1-\alpha}-1}} \gg \xi} Kl_{t,f}(\xi, n) \left(\frac{(tf^2)^{\frac{1}{1-\alpha}}}{n}\right)^{\kappa+u+s} \left(\frac{(tf^2)^{\frac{1}{1-\alpha}-1}\xi}{\sqrt{n}}\right)^{s+\rho}$$

$$= O_{s,\rho,\alpha,\delta_0}\left(n^{1-\frac{3\alpha}{2}+\epsilon(s+\rho+1)+\delta_0(\frac{\kappa+u+s-1}{1-\alpha}+\frac{3}{2})}\right) \quad (7.8)$$

Proof. Same as the proof of Lemma 7.4.3.

□

Lemma 7.4.8. *Let $u, s, \tau \in \mathbb{Z}_{\geq 0}$ and $\kappa \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Then, for any $\epsilon > 0$ and $\delta_0 > 0$ we have,*

$$\sum_{f \ll n^{\frac{1-\alpha+\delta_0}{2}}} \frac{1}{f} \sum_{t \ll \frac{n^{1-\alpha+\delta_0}}{f^2}} \frac{1}{t} \sum_{\frac{n^{\epsilon}\sqrt{n}}{(tf^2)^{\frac{1}{1-\alpha}-1}} \gg \xi} Kl_{t,f}(\xi, n) \left(\frac{(tf^2)^{\alpha}\xi^{1-\alpha}}{n^{\frac{1-\alpha}{2}}}\right)^{\tau} \left(\frac{\xi\sqrt{n}}{tf^2}\right)^{-(u+s+\kappa)}$$

$$= O_{s,\rho,\alpha,\delta_0}\left(n^{1-\frac{3\alpha}{2}+\epsilon(\tau+1)+\delta_0(\frac{\kappa+u+s-1}{1-\alpha}+\frac{3}{2})}\right) \quad (7.9)$$

Proof. Same as the proof of Lemma 7.4.3.

□

Lemma 7.4.9. *Let $u, s \in \mathbb{Z}_{\geq 0}$ and $\kappa \in \frac{1}{2}\mathbb{Z}_{\geq \frac{3}{2}}$. Then, for any $\epsilon_1 > \epsilon_2 > 0$ and $\delta_0 > 0$ we have,*

$$\sum_{f \ll n^{\frac{1-\alpha+\delta_0}{2}}} \frac{1}{f} \sum_{t \ll \frac{n^{1-\alpha+\delta_0}}{f^2}} \frac{1}{t} \sum_{\frac{n^{\epsilon_1}\sqrt{n}}{(tf^2)^{\frac{1}{1-\alpha}-1}} \gg \xi \gg n^{\epsilon_2}} |Kl_{t,f}(\xi, n)| \left(\frac{tf^2}{\xi\sqrt{n}}\right)^{\kappa+u+s}$$

$$= O_{\kappa,u,s,\delta_0,\epsilon_1,\epsilon_2}\left(\left(\frac{n^{1-\alpha+\delta_0}}{n^{\frac{1}{2}+\epsilon_2}}\right)^{\kappa+u+s} n^{\frac{1-\alpha+\delta_0}{2}+\epsilon_2}\right) \quad (7.10)$$

Proof. Same as the proof of Lemma 7.4.4

□

Chapter 8

$\frac{1}{4}$-Bound

In this chapter we will give the first application of the theory developed in the previous chapters and give a new proof of the so-called $\frac{1}{4}$-bound of Kuznetsov towards the Ramanujan conjectures.

We will prove the bound for forms that are of full level (This is because we have developed the theory only for representations that are unramified at every finite place and have trivial central character.). However we would like to note that nothing in the previous chapters is special to this ramification restriction and can be generalized to ambitrary representations and forms almost verbatim.

8.1 Setup

In this section we will give a brief description of the setup in which we will be using the trace formula. Since this material is well-known and the general definitions of various concepts (like of an automorphic representation or the decomposition of the L^2-space to come) are lengthy (and not really needed for our purposes here) we will not be precise with the definitions and instead refer the interested reader to [2].

Let $G = GL(2)$. We will be considering the decomposition of the right regular representation, R, of $G(\mathbb{A})$ on the space $L^2(\mathbb{R}_+G(\mathbb{Q})\backslash G(\mathbb{A}))$. By the theory of Eisenstein series (see [10] for the case of $GL(2)$ and [19] in general) we know that this L^2 space decompose into an invariant direct sum (in a precise sense which we will not define here) of a continuous integral, denoted here by $L^2(\mathbb{R}_+G(\mathbb{Q})\backslash G(\mathbb{A}))_{cont}$, with a discrete part, denoted by $L^2_0(\mathbb{R}_+G(\mathbb{Q})\backslash G(\mathbb{A}))$. i.e.

$$L^2(\mathbb{R}_+G(\mathbb{Q})\backslash G(\mathbb{A})) = L^2_{cont}(\mathbb{R}_+G(\mathbb{Q})\backslash G(\mathbb{A})) \oplus L^2_0(\mathbb{R}_+G(\mathbb{Q})\backslash G(\mathbb{A}))$$

Furthermore the discrete part, L^2_0, decomposes further into a direct sum

$$L^2_0(\mathbb{R}_+G(\mathbb{Q})\backslash G(\mathbb{A})) = L^2_{disc}(\mathbb{R}_+G(\mathbb{Q})\backslash G(\mathbb{A})) \oplus L^2_{cusp}(\mathbb{R}_+G(\mathbb{Q})\backslash G(\mathbb{A}))$$

Where L^2_{res} denotes those functions in L^2_0 that appear as the residue of an Eisenstein series and L^2_{cusp} denotes the space of cuspidal functions, the orthogonal complement of L^2_{res} in L^2_0.

Corresponding to the above decomposition we get a decomposition of the right regular representation R as

$$R = R_{cont} \oplus R_{res} \oplus R_{cusp}$$

and the classical spaces of automoprhic forms (as functions on the upper half plane $\mathbb{H}$) embeds as sub representation of R_{cusp}.

It is well known (cf. [6]) that each automorphic representation π is unramified (i.e. has a vector fixed under the hyperspecial maximal compact subgroup) at almost every place, and at each such place there is an associated semi simple conjugacy class in the dual group, ${}^{L}G$, the so-called Satake parameter of π. We will denote this conjugacy class at the place p by $A(\pi_p) \subset GL(2, \mathbb{C})$ and will be concerned with its trace when π is cuspidal.

By a suitable choice of test functions we will compute this trace by the trace formula. For each prime p and integer m let $f_p^{(m)}$ denote the characteristic function of the set

$$\left\{ X \in Mat_{2\times 2}(\mathbb{Z}_p) \mid |\det(X)|_p = p^{-m} \right\}$$

As is shown in page 19 of [18], for an unramified representation π_p of the local group $GL_2(\mathbb{Q}_p)$ trace of the operator $p^{-\frac{1}{2}}\pi_p(f_p^{(1)})$ is the trace of $A(\pi_p)$. Let $f_\infty \in C_c^\infty(\mathbb{R}_+ \backslash GL_2(\mathbb{R}))$ arbitrary and set

$$f := \tfrac{1}{\sqrt{p}} f_\infty f_p^{(1)} \prod_{v \neq p} f_v^{(0)}$$

With this choice of test functions the trace of the operator $R_{cusp}(f)$ will be

$$\operatorname{tr}(R_{cusp}(f)) = \sum_{\pi} \operatorname{tr}(\pi_\infty(f_\infty)) \operatorname{tr}(A(\pi_p))$$

where the sum runs over those π that have trivial central character and are unramified at every finite place v.

We will be interested in the trace of $R_{cusp}(f)$. Following §2.2 of [18] in order to calculate this trace we will use the formula on page 271-272 of [12]. At this point we remark that the formula in [12] gives the trace of the operator $R_0(f)$, therefore in order to get the trace of $R_{cusp}(f)$ we need to subtract those representations which occur discretely but are not cuspidal. For $G = GL(2)$ these are all one dimensional, and in our case (trivial central character and unramified at every finite place) only the trivial representation is present. We have already isolated this contribution in Theorem 5.5.2 and in what follows will subtract this contribution and then analyze the difference.

As explained in page 23-24 of [18] the formula in [12] has to be modified slightly. For time and space reasons we will neither give a detailed explanation of the formula in [12] nor talk about the necessary modifications in our case. Instead we refer the interested reader to [18] and [12].

In the next section we will cite the bounds that are already established in [18]. This will leave aside only the elliptic term (term (ii) in [12]). Then we will use the theory developed in the previous chapters to handle the elliptic term and establish the desired bound.

8.2 Everything but the elliptic term

In this section we will cite the relevant bounds from [18]. As in page 19 of [18], for a place v and an integer $k \geq 0$, let $f_v^{(k)}$ denote the characteristic function of the set $\left\{ X \in Mat_{2\times 2}(\mathbb{Z}_v) \mid |\det(X)|_v = q_v^{-k} \right\}$ divided by $q_v^{k/2}$. The function Φ of [12] is then the function

$$f^{(p,1)} = f_\infty f_p^{(1)} \prod_{v \neq p} f_v^{(0)}$$

In the following lemmas the number on the left of the inequality refers to the corresponding term on page 271-272 of [12].

Lemma 8.2.1. *The contribution of (i) of [12] is,*

$$(i) = O_{f_\infty}(1)$$

Proof. By equation $(TF.1)$ of [18] we have

$$(i) = \sum_{z \in Z(\mathbb{Q})} \mu(\mathbb{R}_+ G(\mathbb{Q}) \backslash G(\mathbb{A})) f^{(p,1)}(z)$$

The lemma is now obvious. □

Lemma 8.2.2. *The contribution of (iv) of [12] is,*

$$(iv) = O_{f_\infty}(1)$$

Proof. By the first paragraph of §2.3 of [18] we know that once we fix f_∞ (which we did) this term only contributes for finitely many p. The lemma follows

□

Lemma 8.2.3. *The contribution of* (v) *of [12] is,*

$$(v) = 0$$

Proof. Follows from the second paragraph of §2.3 of [18].

□

Lemma 8.2.4. *The contribution of (vi) of [12] is,*

$$(vi) = \int_{|x|>1} \frac{\theta_\infty(x)}{\sqrt{x^2-1}} dx + \int_{\mathbb{R}} \frac{\theta_\infty^-(x)}{\sqrt{x^2+1}} dx$$

Proof. This is already mentioned several times in chapter 5. The lemma is a consequence of equation (32) of [18] (taking into account that the factor of 2 is a mistake in [18] which comes from a mistake in equation (30), the 4 up front in the latter equation should be a 2) where $m = 1$.

□

Lemma 8.2.5. *The contribution of (vii) of [12] is,*

$$(vii) = O_{f_\infty}(1)$$

Proof. By (33), (34) and (35) of [18] we have,

$$(vii) = \frac{1}{4\pi} \int_{-i\infty}^{i\infty} \mathrm{tr}(\xi_s(f_\infty))(p^{\frac{is}{2}} + p^{\frac{-is}{2}}) \left\{ -\frac{1}{2} \frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} - \frac{1}{2} \frac{\Gamma'\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{1+s}{2}\right)} - \frac{\zeta'(1-s)}{\zeta(1-s)} - \frac{\zeta'(1+s)}{\zeta(1+s)} \right\} d|s|$$

The lemma now follows from the fact that the only dependence on p is on $p^{\frac{is}{2}} + p^{-\frac{is}{2}}$, which is bounded by 2.

□

Lemma 8.2.6. *The contribution of* $(viii)$ *of* [12] *is,*

$$(viii) = O_{f_\infty}(1)$$

Proof. From the last paragraph of §2.6 of [18] we see that the contribution of $(viii)$ is given by

$$\frac{1}{4\pi}\int_{-i\infty}^{i\infty} \operatorname{tr}(R^{-1}(s)R'(s)\xi_s(f_\infty))(p^{is}+p^{-is})d|s|$$

Again by the same paragraph we also know that,

$$|\operatorname{tr}(R^{-1}(s)R'(s)\xi_s(f_\infty))| = O_{f_\infty}(s^{-2})$$

Therefore we get

$$\begin{aligned}(viii) &= \frac{1}{4\pi}\int_{-i\infty}^{i\infty} \operatorname{tr}(R^{-1}(s)R'(s)\xi_s(f_\infty))(p^{is}+p^{-is})d|s| \\ &= O_{f_\infty}(\int_{-\infty}^{\infty} s^{-2}ds) \\ &= O_{f_\infty}(1)\end{aligned}$$

□

8.3 The elliptic term

Recall from Chapter 2 Theorem 2.1.2 that the elliptic term (term (ii) of [12]) of the trace formula is given by

$$2\sum_{\substack{\pm \\ N:=\pm p}} \sum_{\substack{m\in\mathbb{Z} \\ m^2-4N\neq\square}} \theta_\infty^\mp\left(\frac{m}{2\sqrt{|N|}}\right) \sum_{\substack{f^2|(m^2-4N) \\ \frac{m^2-4N}{f^2}\equiv 0,1 \bmod 4}} \frac{1}{f} L\left(1, \left(\frac{(m^2-4N)/f^2}{\cdot}\right)\right) \tag{8.1}$$

Where $\theta_\infty^\pm$ is given by

$$\theta_\infty^\mp(x) := \mathcal{O}(f_\infty; (x,\mp 1))\sqrt{|x^2\pm 1|}$$

and by $(x,\mp 1)$ we denote the element $\gamma\in GL_2(\mathbb{R})$ that has trace x and determinant ∓ 1.

We proceed as in that section and replace the L-functions with the corresponding approximate functional equation and get

$$\begin{aligned}(8.1) = 2\sum_{\substack{m\in\mathbb{Z} \\ \pm \\ m^2\mp 4p\neq\square}} \theta_\infty^\pm\left(\frac{m}{2\sqrt{p}}\right) \Bigg\{ &\sum_{\substack{f^2|(m^2\mp 4p) \\ \frac{m^2\mp 4p}{f^2}\equiv 0,1 \bmod 4}} \frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\left(\frac{(m^2\mp 4p)/f^2}{t}\right) F\left(\frac{tf^2}{|m^2\mp 4p|^\alpha}\right) \\ &+ \sum_{\substack{f^2|(m^2\mp 4p) \\ \frac{m^2\mp 4p}{f^2}\equiv 0,1 \bmod 4}} \left(\frac{\pi f^2}{|m^2\mp 4p|}\right)^{\frac{1}{2}} \sum_{t=1}^{\infty}\left(\frac{(m^2\mp 4p)/f^2}{t}\right) H_\pm\left(\frac{tf^2}{|m^2\mp 4p|^{1-\alpha}}\right)\Bigg\}\end{aligned}$$

Where $0<\alpha<1$ and

$$H_\pm(x) := \frac{1}{2\pi i}\int_{(1)} \frac{\Gamma\left(\frac{e_\pm+u}{2}\right)}{\Gamma\left(\frac{1+e_\pm-u}{2}\right)} (\pi x)^{-u}\tilde{F}(u)du,$$

the sign in $H_{\pm}$ is the sign of m^2-4p, $e_-=1$ and $e_+=0$. For our purposes here we will choose

$$\alpha = \tfrac{1}{2}$$

We then add and subtract the missing elements in the sum above and get

$$(8.1) = 2\sum_{\substack{m\in\mathbb{Z}\\ \pm}} \theta_\infty^{\pm}\left(\frac{m}{2\sqrt{p}}\right)\left\{ \sum_{\substack{f^2|(m^2\mp 4p)\\ \frac{m^2\mp 4p}{f^2}\equiv 0,1 \bmod 4}} \frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\left(\frac{(m^2\mp 4p)/f^2}{t}\right) F\left(\frac{tf^2}{\sqrt{|m^2\mp 4p|}}\right)\right. \tag{8.2}$$

$$\left. + \sum_{\substack{f^2|(m^2\mp 4p)\\ \frac{m^2\mp 4p}{f^2}\equiv 0,1 \bmod 4}} \left(\frac{\pi f^2}{|m^2\mp 4p|}\right)^{\frac{1}{2}}\sum_{t=1}^{\infty}\left(\frac{(m^2\mp 4p)/f^2}{t}\right) H_{\pm}\left(\frac{tf^2}{\sqrt{|m^2\mp 4p|}}\right)\right\} \tag{8.3}$$

$$-2\sum_{\substack{m\in\mathbb{Z}\\ \pm\\ m^2\mp 4p=\square}} \theta_\infty^{\pm}\left(\frac{m}{2\sqrt{p}}\right)\left\{ \sum_{\substack{f^2|(m^2\mp 4p)\\ \frac{m^2\mp 4n}{f^2}\equiv 0,1 \bmod 4}} \frac{1}{f}\sum_{t=1}^{\infty}\frac{1}{t}\left(\frac{(m^2\mp 4p)/f^2}{t}\right) F\left(\frac{tf^2}{\sqrt{|m^2\mp 4p|}}\right)\right. \tag{8.4}$$

$$\left. + \sum_{\substack{f^2|(m^2\mp 4p)\\ \frac{m^2\mp 4p}{f^2}\equiv 0,1 \bmod 4}} \left(\frac{\pi f^2}{|m^2\mp 4p|}\right)^{\frac{1}{2}}\sum_{t=1}^{\infty}\left(\frac{(m^2\mp 4p)/f^2}{t}\right) H_{\pm}\left(\frac{tf^2}{\sqrt{|m^2\mp 4p|}}\right)\right\} \tag{8.5}$$

Let us first bound (8.4) and (8.5)

Lemma 8.3.1. *There exists a constant $K_{f_\infty}>0$ depending only on f_∞ such that for any $\epsilon>0$ we have*

$$(8.4)+(8.5) = \begin{cases} O_\epsilon(p^\epsilon) & \textit{if } p<K_{f_\infty}\\ 0 & \textit{if } p>K_{f_\infty}\end{cases}$$

Proof. The m-sum is running over integers m such that $m^2\mp 4p=a^2$ for some $a\in\mathbb{Z}$. Letting $u=(m+a)$ and $v=\pm(m-a)$(the sign depending on the sign of $m^2\mp 4p$), using the fact that p is odd we see that the tuples (m,a) satisfying $m^2\mp 4p=a^2$ are given by

$$(m,a)\in\{(2,2p),(2p,2)\}$$

Therefore there are at most two terms in the sum. Moreover for each one of these we have $m^2\mp 4p=(p-1)^2$ or $(p+1)^2$ depending on the sign and because of the compact support of θ_∞ we observe that for large p (8.4) and (8.5) are 0. Using these and the rapid decay of F and $H_\pm$ which limits both t-sums to $t<p^{\frac{1}{2}+\epsilon}$ for any $\epsilon>0$ (cf. Proposition 5.1.2) we get that there is a constant K depending only on f_∞ such that,

$$(8.4)+(8.5) = \begin{cases} O_\epsilon(p^\epsilon) & \text{if } p<K\\ 0 & \text{if } p>K\end{cases}$$

for any $\epsilon>0$. The lemma follows.

$\square$

By Lemma 8.3.1 we get

$$(8.1) = 2\sum_{\substack{m\in\mathbb{Z}\\ \pm}} \theta_\infty^\pm\left(\tfrac{m}{2\sqrt{p}}\right)\left\{\sum_{\substack{f^2|(m^2\mp 4p)\\ \frac{m^2\mp 4p}{f^2}\equiv 0,1 \bmod 4}} \tfrac{1}{f}\sum_{t=1}^\infty \tfrac{1}{t}\left(\tfrac{(m^2\mp 4p)/f^2}{t}\right) F\left(\tfrac{tf^2}{\sqrt{|m^2\mp 4p|}}\right)\right.$$

$$\left. + \sum_{\substack{f^2|(m^2\mp 4p)\\ \frac{m^2\mp 4p}{f^2}\equiv 0,1 \bmod 4}} \left(\tfrac{\pi f^2}{|m^2\mp 4p|}\right)^{\frac{1}{2}}\sum_{t=1}^\infty \left(\tfrac{(m^2\mp 4p)/f^2}{t}\right) H_\pm\left(\tfrac{tf^2}{\sqrt{|m^2\mp 4p|}}\right)\right\}$$

$$+\begin{cases} O_\epsilon(p^\epsilon) & \text{if } p < K_{f_\infty}\\ 0 & \text{if } p > K_{f_\infty}\end{cases} \qquad (\bullet)$$

Next, we apply Poisson summation to the m-sum. By the formula in §5.6, for any $\delta_0, \epsilon > 0$ we have

$$\begin{aligned}
&(8.2) + (8.3)\\
&= \mathrm{tr}(\mathbf{1}(f)) - \mathrm{tr}(\xi_0(f))\\
&+ \sqrt{p}\sum_{f\ll p^{\frac{1+\delta_0}{4}}} \tfrac{1}{f^3}\sum_{\substack{t\\ tf^2\ll p^{\frac{1+\delta_0}{2}}}} \tfrac{1}{t^2}\sum_{\xi\in\mathbb{Z}\setminus\{0\}} Kl_{t,f}(\xi,p)\\
&\quad\left\{\int_{|x|>1}\theta_\infty^+(x)\,F\left(\tfrac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{x^2-1}}\right)e\left(\tfrac{-x\xi\sqrt{p}}{2tf^2}\right)dx + \int_{|x|<1}\theta_\infty^+(x)\,F\left(\tfrac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{1-x^2}}\right)e\left(\tfrac{-x\xi\sqrt{p}}{2tf^2}\right)dx\right\}\\
&+ \tfrac{\sqrt{\pi}}{2}\sum_{f<p^{\frac{1+\delta_0}{4}}} \tfrac{1}{f}\sum_{\substack{t\\ tf^2\ll p^{\frac{1+\delta_0}{2}}}} \tfrac{1}{t}\sum_{\xi\in\mathbb{Z}\setminus\{0\}} Kl_{t,f}(\xi,p)\\
&\quad\left\{\int_{|x|>1}\tfrac{\theta_\infty^+(x)}{\sqrt{x^2-1}}H_+\left(\tfrac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{x^2-1}}\right)e\left(\tfrac{-x\sqrt{p}\xi}{2tf^2}\right)dx + \int_{|x|<1}\tfrac{\theta_\infty^+(x)}{\sqrt{1-x^2}}H_-\left(\tfrac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{1-x^2}}\right)e\left(\tfrac{-x\sqrt{p}\xi}{2tf^2}\right)dx\right\}\\
&+ \sqrt{p}\sum_{f<p^{\frac{1+\delta_0}{4}}} \tfrac{1}{f^3}\sum_{\substack{t\\ tf^2\ll p^{\frac{1+\delta_0}{2}}}} \tfrac{1}{t^2}\sum_{\xi\in\mathbb{Z}\setminus\{0\}} Kl_{t,f}(\xi,p)\left\{\int_{\mathbb{R}}\theta_\infty^-(x)\,F\left(\tfrac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{x^2+1}}\right)e\left(\tfrac{-x\xi\sqrt{p}}{2tf^2}\right)dx\right\}\\
&+ \tfrac{\sqrt{\pi}}{2}\sum_{f<p^{\frac{1+\delta_0}{4}}} \tfrac{1}{f}\sum_{\substack{t\\ tf^2\ll p^{\frac{1+\delta_0}{2}}}} \tfrac{1}{t}\sum_{\xi\in\mathbb{Z}\setminus\{0\}} Kl_{t,f}(\xi,n)\left\{\int_{\mathbb{R}}\tfrac{\theta_\infty^-(x)}{\sqrt{x^2+1}}H_+\left(\tfrac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{x^2+1}}\right)e\left(\tfrac{-x\sqrt{p}\xi}{2tf^2}\right)dx\right\}\\
&+ O(1 + p^{-\epsilon} + p^{\frac{3}{2}-\alpha}e^{-\delta_0})
\end{aligned}$$

Where,

$$\mathrm{tr}(\xi_0(f)) = \int_{|x|>1}\frac{\theta_\infty(x)}{\sqrt{x^2-1}}dx + \int_{\mathbb{R}}\frac{\theta_\infty^-(x)}{\sqrt{x^2+1}}dx$$

We also note that the exponent κ that appears in the formula in §5.6 disappears since it comes from bounding the number of divisors, $\sigma_0(n)$ (cf. the proof of Proposition 5.5.3) and since in our case $n = p$, a prime, $\sigma_0(p) = 2$, and hence $\kappa = 0$. Also since $\alpha = \frac{1}{2}$ we have $\frac{1}{2} - \alpha + \kappa = 0$.

We will now give estimates on the sums above.

Lemma 8.3.2. *For any $\epsilon > 0$ and $M \in \mathbb{Z}_{\geq 1}$ we have,*

$$\sqrt{p} \sum_{f<p^{\frac{1+\delta_0}{4}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll p^{\frac{1+\delta_0}{2}}}} \frac{1}{t^2} \sum_{\frac{\xi\sqrt{p}}{tf^2} \gg p^\epsilon} Kl_{t,f}(\xi,p) \left\{ \int_{\mathbb{R}} \theta_\infty^-(x)\, F\left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{x^2+1}}\right) e\left(\frac{-x\xi\sqrt{p}}{2tf^2}\right) dx \right\}$$
$$= O_{M,\delta_0,\epsilon}\left(\frac{p^{\frac{1}{4}}}{p^{(M+\frac{1}{2})\epsilon)}} + \frac{p^{\frac{1+\delta_0}{4}}}{p^{M\epsilon}}\right)$$

Proof. Note that $\frac{tf^2(4p)^{-\alpha}}{(x^2+1)^\alpha}$ is a smooth function, therefore the integral, being the Fourier transform of a smooth function, decays faster than any polynomial. (We note that this case is different than the first two cases where the integrand is still smooth however the so to speak"smoothness" of the function depends on the parameters which are reflected on the decay of the Fourier transform.) Therefore by Lemma 7.4.1 for any $M > 0$ we get

$$\sqrt{p} \sum_{f<p^{\frac{1+\delta_0}{4}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll p^{\frac{1+\delta_0}{2}}}} \frac{1}{t^2} \sum_{\frac{\xi\sqrt{p}}{tf^2} \gg p^\epsilon} Kl_{t,f}(\xi,p) \left\{ \int_{\mathbb{R}} \theta_\infty^-(x)\, F\left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{x^2+1}}\right) e\left(\frac{-x\xi\sqrt{p}}{2tf^2}\right) dx \right\}$$
$$= O_{M,\delta_0,\epsilon}\left(\frac{p^{\frac{1}{4}}}{p^{(M+\frac{1}{2})\epsilon)}} + \frac{p^{\frac{1+\delta_0}{4}}}{p^{M\epsilon}}\right)$$

□

Lemma 8.3.3. *For any $\epsilon > 0$,*

$$\sqrt{p} \sum_{f<p^{\frac{1+\delta_0}{4}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll p^{\frac{1+\delta_0}{2}}}} \frac{1}{t^2} \sum_{\frac{\xi\sqrt{p}}{tf^2} \ll p^\epsilon} Kl_{t,f}(\xi,p) \left\{ \int_{\mathbb{R}} \theta_\infty^-(x)\, F\left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{x^2+1}}\right) e\left(\frac{-x\xi\sqrt{p}}{2tf^2}\right) dx \right\}$$
$$= O_{\theta_\infty,F,\epsilon}\left(p^{\frac{1}{4}+\frac{\epsilon}{2}}\right)$$

Proof. Note that if $tf^2 \ll p^{\frac{1}{2}-\epsilon}$ then $\frac{\sqrt{p}}{tf^2} \gg p^\epsilon$ therefore we necessarily have $tf^2 \gg p^{\frac{1}{2}-\epsilon}$. Therefore we need to bound,

$$\sqrt{p} \sum_{f<p^{\frac{1+\delta_0}{4}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll p^{\frac{1+\delta_0}{2}} \\ tf^2 \gg p^{\frac{1}{2}-\epsilon}}} \frac{1}{t^2} \sum_{\frac{\xi\sqrt{p}}{tf^2} \ll p^\epsilon} Kl_{t,f}(\xi,p) \left\{ \int_{\mathbb{R}} \theta_\infty^-(x)\, F\left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{x^2+1}}\right) e\left(\frac{-x\xi\sqrt{p}}{2tf^2}\right) dx \right\} \qquad (\star)$$

We proceed as in Lemma 7.4.1, first rewrite the sum and then use (6.2).

$$(\star) \ll_{\theta_\infty,F} \sqrt{p} \sum_{f \ll p^{\frac{1+\delta_0}{4}}} \frac{1}{f^3} \sum_{\substack{t_0=1 \\ t_0:\square-free}}^{\frac{p^{\frac{1+\delta_0}{2}}}{f^2}} \frac{1}{t_0^2} \sum_{\substack{t_1 \ll \frac{p^{\frac{1+\delta_0}{2}}}{t_0 f^2} \\ rad(t_1)|t_0 \\ t_0 t_1 f^2 \gg p^{\frac{1}{2}-\epsilon}}} \frac{1}{t_1^2} \sum_{d_{t_0}|t_0} \sum_{\substack{d_{t_0} t_1 \xi \ll p^\epsilon \\ \xi \neq 0 \\ \gcd(\frac{t_0}{d_{t_0}},\xi)=1}} |Kl_{t_0 t_1,f}(t_1 d_{t_0}\xi,p)|$$

Using (6.2) we bound this sum by

$$(\star) \le p^{\frac{1}{2}+\epsilon} \sum_{f<p^{\frac{1+\delta_0}{4}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll p^{\frac{1+\delta_0}{2}} \\ tf^2 \gg p^{\frac{1}{2}-\epsilon}}} \frac{1}{t^{\frac{3}{2}}}$$
$$\le p^{\frac{1}{4}+\frac{\epsilon}{2}}$$

The lemma follows. □

Lemma 8.3.4. *For any $\epsilon > 0$ and $M \in \mathbb{Z}_{\ge 2}$,*

$$\frac{\sqrt{\pi}}{2} \sum_{f<p^{\frac{1+\delta_0}{4}}} \frac{1}{f} \sum_{\substack{t \\ tf^2 \ll p^{\frac{1+\delta_0}{2}}}} \frac{1}{t} \sum_{\frac{\xi\sqrt{p}}{tf^2} \gg p^{\epsilon}} Kl_{t,f}(\xi, n) \left\{ \int_{\mathbb{R}} \frac{\theta_\infty^-(x)}{\sqrt{x^2+1}} H_+ \left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{x^2+1}} \right) e\left(\frac{-x\sqrt{p}\xi}{2tf^2} \right) dx \right\}$$
$$= O_{M,\alpha,\delta_0,\epsilon}\left(\frac{n^{\frac{1}{4}}}{n^{(M+\frac{3}{2})\epsilon}} + \frac{n^{1-\frac{3\alpha}{2}+\delta_0}}{n^{M\epsilon}} \right)$$

Proof. The proof of Lemma 8.3.2 goes through, we only need to change the reference to Lemma 7.4.1 to Lemma 7.4.5. □

Lemma 8.3.5. *For any $\epsilon > 0$,*

$$\frac{\sqrt{\pi}}{2} \sum_{f<p^{\frac{1+\delta_0}{4}}} \frac{1}{f} \sum_{\substack{t \\ tf^2 \ll p^{\frac{1+\delta_0}{2}}}} \frac{1}{t} \sum_{\frac{\xi\sqrt{p}}{tf^2} \ll p^{\epsilon}} Kl_{t,f}(\xi, n) \left\{ \int_{\mathbb{R}} \frac{\theta_\infty^-(x)}{\sqrt{x^2+1}} H_+ \left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{x^2+1}} \right) e\left(\frac{-x\sqrt{p}\xi}{2tf^2} \right) dx \right\}$$
$$= O_{\theta_\infty,H_+,\delta_0,\epsilon}\left(p^{\frac{1}{4}+\frac{3\delta_0}{4}+\epsilon} \right)$$

Proof. Same as the proof of Lemma 8.3.3. □

Lemma 8.3.6. *For any $\epsilon > 0$ such that $\epsilon > \delta_0$ and $M \in \mathbb{Z}_{\ge 2}$,*

$$\sqrt{p} \sum_{f \ll p^{\frac{1+\delta_0}{4}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll p^{\frac{1+\delta_0}{2}}}} \frac{1}{t^2} \sum_{\xi \gg \frac{p^{\frac{1}{2}+\epsilon}}{tf^2}} Kl_{t,f}(\xi, p)$$
$$\left\{ \int_{|x|>1} \theta_\infty^+(x) F\left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{x^2-1}} \right) e\left(\frac{-x\xi\sqrt{p}}{2tf^2} \right) dx + \int_{|x|<1} \theta_\infty^+(x) F\left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{1-x^2}} \right) e\left(\frac{-x\xi\sqrt{p}}{2tf^2} \right) dx \right\}$$
$$= O_{M,\delta_0,\epsilon}\left(\frac{p^{\frac{1}{4}+\frac{\delta_0}{2}}}{p^{M\epsilon}} + \frac{p^{\frac{1}{4}+\frac{\epsilon}{2}}}{p^{M\epsilon}} + \frac{p^{\frac{1}{4}}}{p^{(M+\frac{1}{2})(\epsilon-2\delta_0)}} + \frac{p^{\frac{1}{4}+\frac{\delta_0}{2}}}{p^{M(\epsilon-2\delta_0)}} \right)$$

Proof. By Proposition 7.2.1, when $\xi \gg \frac{n^{\frac{1}{2}+\epsilon}}{tf^2}$ we have

$$\left\{ \int_{|x|>1} \theta_\infty^+(x) F\left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{x^2-1}} \right) e\left(\frac{-x\xi\sqrt{p}}{2tf^2} \right) dx + \int_{|x|<1} \theta_\infty^+(x) F\left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{1-x^2}} \right) e\left(\frac{-x\xi\sqrt{p}}{2tf^2} \right) dx \right\}$$
$$= O_{F,\theta_\infty,M}\left(\left(\frac{\sqrt{p}\xi}{tf^2} \right)^{-M-1} + \left(\frac{\sqrt{p}\xi}{tf^2} \right)^{-1} \left(\frac{tf^2\xi}{\sqrt{p}} \right)^{-M} \right)$$

Therefore we need to only bound,

$$\bullet \sqrt{p} \sum_{f \ll p^{\frac{1+\delta_0}{4}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll p^{\frac{1+\delta_0}{2}}}} \frac{1}{t^2} \sum_{\xi \gg \frac{p^{\frac{1}{2}+\epsilon}}{tf^2}} |Kl_{t,f}(\xi,p)| \left(\frac{\sqrt{p}\xi}{tf^2}\right)^{-M-1} \tag{8.6}$$

$$\bullet \sqrt{p} \sum_{f \ll p^{\frac{1+\delta_0}{4}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll p^{\frac{1+\delta_0}{2}}}} \frac{1}{t^2} \sum_{\xi \gg \frac{p^{\frac{1}{2}+\epsilon}}{tf^2}} |Kl_{t,f}(\xi,p)| \left(\frac{\sqrt{p}\xi}{tf^2}\right)^{-1} \left(\frac{tf^2\xi}{\sqrt{p}}\right)^{-M} \tag{8.7}$$

For (8.6) note that since $tf^2 \ll p^{\frac{1+\delta_0}{2}}$ we necessarily have $\frac{\xi\sqrt{p}}{tf^2} \gg n^{\epsilon-\delta_0}$. Since $\epsilon > \delta_0$, $\epsilon - \delta_0 > 0$ and we can apply Lemma 7.4.1 to conclude that

$$(8.6) = O_{M,\delta_0,\epsilon}\left(\frac{p^{\frac{1}{4}}}{p^{(M+\frac{1}{2})(\epsilon-2\delta_0)}} + \frac{p^{\frac{1}{4}+\frac{\delta_0}{2}}}{p^{M(\epsilon-2\delta_0)}}\right)$$

For (8.7) we use Lemma 7.4.2 to conclude

$$(8.7) = O_{M,\delta_0,\epsilon}\left(\frac{p^{\frac{1}{4}+\frac{\delta_0}{2}}}{p^{M\epsilon}} + \frac{p^{\frac{1}{4}+\frac{\epsilon}{2}}}{p^{M\epsilon}}\right)$$

The lemma follows. $\square$

Lemma 8.3.7. *For any $\epsilon > 0$ such that $\epsilon > \delta_0$, and $M \in \mathbb{Z}_{\geq 2}$, and $m, s, \rho \leq 2M$, $r \leq sM$,*

$$\sqrt{p} \sum_{f \ll p^{\frac{1+\delta_0}{4}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll p^{\frac{1+\delta_0}{2}}}} \frac{1}{t^2} \sum_{\xi \ll \frac{p^{\frac{1}{2}+\epsilon}}{tf^2}} Kl_{t,f}(\xi,p)$$
$$\left\{\int_{|x|>1} \theta_\infty^+(x) F\left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{x^2-1}}\right) e\left(\frac{-x\xi\sqrt{p}}{2tf^2}\right) dx + \int_{|x|<1} \theta_\infty^+(x) F\left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{1-x^2}}\right) e\left(\frac{-x\xi\sqrt{p}}{2tf^2}\right) dx\right\}$$
$$= O_{M,\delta_0,\epsilon}\left(p^{\frac{1}{4}+\epsilon(M+2)-\delta_0} + (p^{\delta_0(1+m+r+s)} + p^{\delta_0(M+\frac{1}{2})})p^{\frac{1}{4}+\epsilon} + p^{\frac{1}{4}+\epsilon(s+\rho+1)+2\delta_0(\frac{3}{4}+m+r+s)}\right)$$

Proof. By Proposition 7.2.2 we know that for any $M \in \mathbb{Z}_{\geq 2}$ the function

$$\left\{\int_{|x|>1} \theta_\infty^+(x) F\left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{x^2-1}}\right) e\left(\frac{-x\xi\sqrt{p}}{2tf^2}\right) dx + \int_{|x|<1} \theta_\infty^+(x) F\left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{1-x^2}}\right) e\left(\frac{-x\xi\sqrt{p}}{2tf^2}\right) dx\right\}$$

has an asymptotic expansion which is a is sum of terms of the form

- $\left(\frac{t^2f^4}{4p}\right)^{1+m+r+s} \left(\frac{tf^2\xi}{8\sqrt{p}}\right)^{s+\rho}$
- $\left(\frac{2tf^2}{\sqrt{p}\xi}\right)^{m+r+s+\frac{3}{2}}$
- $\left(\frac{t^2f^4}{4p}\right)^{\frac{3}{2}+m+r+s} \left(\frac{tf^2\xi}{8\sqrt{p}}\right)^{s+\rho}$

where $m, s, \rho \leq 2M$ and $r \leq sM$, with an error of size

$$\left(\frac{2tf^2}{\sqrt{p}\xi}\right)^{M+1} + \left(\frac{2tf^2}{\sqrt{p}\xi}\right)\left(\frac{tf^2\xi}{8\sqrt{p}}\right)^{M+1}$$

We will bound each one of those separately. Therefore we need to bound:

- $$\sqrt{p}\sum_{f\ll p^{\frac{1+\delta_0}{4}}}\frac{1}{f^3}\sum_{\substack{t\\ tf^2\ll p^{\frac{1+\delta_0}{2}}}}\frac{1}{t^2}\sum_{\xi\ll\frac{p^{\frac{1}{2}+\epsilon}}{tf^2}}|Kl_{t,f}(\xi,p)|\left(\frac{t^2f^4}{4p}\right)^{1+m+r+s}\left(\frac{tf^2\xi}{8\sqrt{p}}\right)^{s+\rho} \tag{8.8}$$
- $$\sqrt{p}\sum_{f\ll p^{\frac{1+\delta_0}{4}}}\frac{1}{f^3}\sum_{\substack{t\\ tf^2\ll p^{\frac{1+\delta_0}{2}}}}\frac{1}{t^2}\sum_{\xi\ll\frac{p^{\frac{1}{2}+\epsilon}}{tf^2}}|Kl_{t,f}(\xi,p)|\left(\frac{2tf^2}{\sqrt{p}\xi}\right)^{m+r+s+\frac{3}{2}} \tag{8.9}$$
- $$\sqrt{p}\sum_{f\ll p^{\frac{1+\delta_0}{4}}}\frac{1}{f^3}\sum_{\substack{t\\ tf^2\ll p^{\frac{1+\delta_0}{2}}}}\frac{1}{t^2}\sum_{\xi\ll\frac{p^{\frac{1}{2}+\epsilon}}{tf^2}}|Kl_{t,f}(\xi,p)|\left(\frac{t^2f^4}{4p}\right)^{\frac{3}{2}+m+r+s}\left(\frac{tf^2\xi}{8\sqrt{p}}\right)^{s+\rho} \tag{8.10}$$
- $$\sqrt{p}\sum_{f\ll p^{\frac{1+\delta_0}{4}}}\frac{1}{f^3}\sum_{\substack{t\\ tf^2\ll p^{\frac{1+\delta_0}{2}}}}\frac{1}{t^2}\sum_{\xi\ll\frac{p^{\frac{1}{2}+\epsilon}}{tf^2}}|Kl_{t,f}(\xi,p)|\left(\frac{2tf^2}{\sqrt{p}\xi}\right)\left(\frac{tf^2\xi}{8\sqrt{p}}\right)^{M+1} \tag{8.11}$$
- $$\sqrt{p}\sum_{f\ll p^{\frac{1+\delta_0}{4}}}\frac{1}{f^3}\sum_{\substack{t\\ tf^2\ll p^{\frac{1+\delta_0}{2}}}}\frac{1}{t^2}\sum_{\xi\ll\frac{p^{\frac{1}{2}+\epsilon}}{tf^2}}|Kl_{t,f}(\xi,p)|\left(\frac{2tf^2}{\sqrt{p}\xi}\right)^{M+1} \tag{8.12}$$

- (8.8), (8.10)

  By Lemma 7.4.3 (where we take $\kappa=1$ for (8.8) and $\kappa=\frac{3}{2}$ for (8.10) $u=m+r$ and $\alpha=\frac{1}{2}$) we immediately get the bound

  $$(8.8)+(8.10)=O_{s,\rho,\delta_0,\epsilon}(p^{\frac{1}{4}+\epsilon(s+\rho+1)+2\delta_0(\frac{3}{4}+m+r+s)})$$

- (8.9), (8.12)

  This time we use Lemma 7.4.4 (where for (8.9) we take $\kappa=\frac{3}{2}$, $u=m+r$, and for (8.12) we take $\kappa+u+s=M+1$, and for both $\alpha=\frac{1}{2}$ and $\epsilon_1=\epsilon_2=\epsilon$) and get

  $$(8.9)+(8.12)=O_{M,m,r,s,\epsilon}\left((p^{\delta_0(1+m+r+s)}+p^{\delta_0(M+\frac{1}{2})})p^{\frac{1}{4}+\epsilon}\right)$$

- (8.11)

  Since $\xi\neq 0$ and $tf^2\ll p^{\frac{1+\delta_0}{2}}$ we have $\frac{tf^2}{\sqrt{p}\xi}\ll p^{\frac{\delta_0}{2}}$. Therefore,

  $$(8.11)\leq p^{\frac{1+\delta_0}{2}}\sum_{f\ll p^{\frac{1+\delta_0}{4}}}\frac{1}{f^3}\sum_{\substack{t\\ tf^2\ll p^{\frac{1+\delta_0}{2}}}}\frac{1}{t^2}\sum_{p^\epsilon\ll\xi\ll\frac{p^{\frac{1}{2}+\epsilon}}{tf^2}}|Kl_{t,f}(\xi,p)|\left(\frac{tf^2\xi}{8\sqrt{p}}\right)^{M+1}$$

  Now using Lemma 7.4.3 (where we take $\kappa=u=s=0$, $\rho=M+1$ and $\alpha=\frac{1}{2}$) we get

  $$(8.11)=O_{\delta_0,\epsilon,M}\left(p^{\frac{1}{4}+\epsilon(M+2)-\delta_0}\right)$$

$\square$

Lemma 8.3.8. *For any $\epsilon > 0$ such that $\epsilon > \delta_0$ and $M \in \mathbb{Z}_{\geq 2}$,*

$$\frac{\sqrt{\pi}}{2}\sum_{f\ll p^{\frac{1+\delta_0}{4}}}\frac{1}{f}\sum_{\substack{t\\ tf^2\ll p^{\frac{1+\delta_0}{2}}}}\frac{1}{t}\sum_{\xi\gg\frac{p^{\frac{1}{2}+\epsilon}}{tf^2}} Kl_{t,f}(\xi,p)$$
$$\left\{\int_{|x|>1}\frac{\theta_\infty^+(x)}{\sqrt{x^2-1}}H_+\left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{x^2-1}}\right)e\left(\frac{-x\sqrt{p}\xi}{2tf^2}\right)dx+\int_{|x|<1}\frac{\theta_\infty^+(x)}{\sqrt{1-x^2}}H_-\left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{1-x^2}}\right)e\left(\frac{-x\sqrt{p}\xi}{2tf^2}\right)dx\right\}$$
$$=O_{M,\delta_0,\epsilon}\left(\frac{p^{\frac{1}{4}}}{p^{(M+\frac{3}{2})\epsilon}}+\frac{p^{\frac{1}{4}+\delta_0}}{p^{M\epsilon}}+\frac{p^{\frac{1}{4}}\log(p)}{p^{(M-\frac{3}{2})\epsilon}}\right)$$

Proof. Same as the proof of Lemma 8.3.6 except that we refer to lemmas 7.4.5 and 7.4.6 instead of lemmas 7.4.1 and 7.4.2 respectively. □

Lemma 8.3.9. *For any $\epsilon > 0$ such that $\epsilon > \delta_0$, and $M \in \mathbb{Z}_{\geq 2}$, and $m, s, \rho, \sigma \leq 2M$, $r \leq sM$,*

$$\frac{\sqrt{\pi}}{2}\sum_{f\ll p^{\frac{1+\delta_0}{4}}}\frac{1}{f}\sum_{\substack{t\\ tf^2\ll p^{\frac{1+\delta_0}{2}}}}\frac{1}{t}\sum_{\xi\ll\frac{p^{\frac{1}{2}+\epsilon}}{tf^2}} Kl_{t,f}(\xi,p)$$
$$\left\{\int_{|x|>1}\frac{\theta_\infty^+(x)}{\sqrt{x^2-1}}H_+\left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{x^2-1}}\right)e\left(\frac{-x\sqrt{p}\xi}{2tf^2}\right)dx+\int_{|x|<1}\frac{\theta_\infty^+(x)}{\sqrt{1-x^2}}H_-\left(\frac{tf^2(4p)^{-\frac{1}{2}}}{\sqrt{1-x^2}}\right)e\left(\frac{-x\sqrt{p}\xi}{2tf^2}\right)dx\right\}$$
$$=O_{M,\delta_0,\epsilon}\Big(p^{\frac{1}{4}+\delta_0(\frac{3}{2}+m+r+s)}+p^{\frac{1}{4}+2\epsilon(\sigma+1)+2\delta_0(m+r+s+\frac{3}{4})}+p^{\frac{1}{4}+\epsilon(s+\rho+1)+2\delta_0(m+r+s+\frac{3}{4})}$$
$$+p^{\frac{1}{4}+2\epsilon(\sigma+\frac{1}{2})+2\delta_0(m+r+s+\frac{1}{2})}+p^{\frac{1}{4}+\epsilon(s+\rho+1)+2\delta_0(m+r+s+\frac{1}{4})}+p^{\frac{1}{4}+2\epsilon(M+\frac{3}{2})+\frac{\delta_0}{2}}+p^{\frac{1}{4}+\delta_0(M+1)}\Big)$$

Proof. Same as the proof of Lemma 8.3.7, the only modification is that instead of referring to lemmas 7.4.3 and 7.4.4 we refer to lemmas 7.4.6 and 7.4.8 respectively and there is an extra term coming from the additional term involving the sum in Lemma 7.4.7 in the asymptotic expansion of Proposition 7.2.3. □

Corollary 8.3.10. *For any $\tilde{\epsilon} > 0$ we have*

$$(8.2)+(8.3)=\mathrm{tr}(\boldsymbol{1}(f))-\mathrm{tr}(\xi_0(f))+O_{f_\infty,F,H_\pm,\tilde{\epsilon}}(p^{\frac{1}{4}+\tilde{\epsilon}})$$

Where

$$\mathrm{tr}(\xi_0(f))=\int_{|x|>1}\frac{\theta_\infty(x)}{\sqrt{x^2-1}}dx+\int_{\mathbb{R}}\frac{\theta_\infty^-(x)}{\sqrt{x^2+1}}dx$$

Proof. We choose $M = 2$, $\epsilon = 2\delta_0 = \frac{\tilde{\epsilon}}{10}$. The rest follows from lemmas 8.3.2 to 8.3.9. □

8.4 The $\frac{1}{4}$-Bound

Theorem 8.4.1 ($\frac{1}{4}$-Bound). *Let $f_\infty \in C_c^\infty(\mathbb{R}_+\backslash GL_2(\mathbb{R}))$ and p be a prime. Then for any $\epsilon > 0$,*

$$\sum_{\pi} \operatorname{tr} \pi_\infty(f_\infty) A(\pi_p) = O_{f_\infty,\epsilon}(p^{\frac{1}{4}+\epsilon})$$

Where the sum is running over cuspidal automorphic representations π with trivial central character and unramified at every finite place.

Proof. As already observed in §8.1 we use the trace formula on pages 271-272 of [12] to calculate the sum on the left. There are seven terms of the formula, labeled by (i) to $(viii)$ ((iii) is not present since we are in characteristic zero). By lemmas 8.2.1 to 8.2.6 we have

$$(i) + (iv) + (v) + (vi) + (vii) + (viii) = \int_{|x|>1} \frac{\theta_\infty(x)}{\sqrt{x^2-1}} dx + \int_{\mathbb{R}} \frac{\theta_\infty^-(x)}{\sqrt{x^2+1}} dx + O_{f_\infty}(1)$$

Then the elliptic term, (ii), by Corollary 8.3.10 is

$$(ii) = \operatorname{tr}(\mathbf{1}(f)) - \int_{|x|>1} \frac{\theta_\infty(x)}{\sqrt{x^2-1}} dx + \int_{\mathbb{R}} \frac{\theta_\infty^-(x)}{\sqrt{x^2+1}} dx + O_{f_\infty,\epsilon}(p^{\frac{1}{4}+\epsilon})$$

Therefore we have

$$(i) + (ii) + (iv) + (v) + (vi) + (vii) + (viii) - \operatorname{tr}(\mathbf{1}(f)) = O_{f_\infty,\epsilon}(p^{\frac{1}{4}+\epsilon})$$

$\square$

Chapter 9

Standard Representation

In this last chapter we will give another application of the theory developed above, which actually was the motivation for all the thesis. We will calculate the average of the trace of the n'th Hecke operator on the space of cuspidal automorphic representations as prescribed in [18] by the trace formula.

9.1 The Result

Because of time and space concerns we will not be aiming for any generality and instead of using the general trace formula, as we have been doing previously, we will instead use the Selberg trace formula (the formula at the end of [26], for a recent account see Proposition 28.5 of [14]). This choice is inessential since the analysis of previous chapters are completely general and is not the main point we would like to emphasize here.

Let $k > 2$ be an *even* integer. Recall (cf. [26] or [14]) that the Selberg trace formula expresses the trace of the n'th Hecke operator acting on modular forms of weight k and full level as the sum

$$\tfrac{-1}{2} \sum_{m<\sqrt{n}} \tfrac{\bar{\rho}^{k-1}-\rho^{k-1}}{\rho-\bar{\rho}} \sum_{\substack{f^2|(4n-m^2) \\ \frac{m^2-4n}{f^2} \equiv 0,1 \bmod 4}} h_w\left(\tfrac{m^2-4n}{f^2}\right) + \tfrac{k-1}{12\sqrt{n}} \Box_n - \tfrac{1}{2} \sum_{d|n} \min\left(\tfrac{d}{\sqrt{n}}, \tfrac{\sqrt{n}}{d}\right)^{k-1} \tag{9.1}$$

Where,

$$\rho = m + \sqrt{m^2-4n}$$

$$\Box_n = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise} \end{cases}$$

$$h_w\left(\tfrac{m^2-4n}{f^2}\right) = \frac{\text{class number of the order with discriminant } \frac{m^2-4n}{f^2}}{\text{number of roots of unity in } \mathbb{Q}(\sqrt{m^2-4n})}$$

Note. Our normalization for the Hecke operators is such that the Ramanujan conjecture takes the form $|a_p| \leq 2$. In particular this means we are taking the formula for the trace of T_n in [26] and dividing it by $n^{\frac{k-1}{2}}$.

As in Chapter 2 by using Dirichlet's class number formula, we can rewrite 9.1 as

$$\sum_{m\in\mathbb{Z}} \theta_\infty\left(\frac{m}{2\sqrt{n}}\right) \sum_{\substack{f^2|(m^2-4n)\\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \frac{1}{f} L\left(1, \left(\frac{(m^2-4n)/f^2}{\cdot}\right)\right) \tag{9.2}$$

$$+\frac{k-1}{12\sqrt{n}}\square_n - \frac{1}{2}\sum_{d|n} \min\left(\frac{d}{\sqrt{n}}, \frac{\sqrt{n}}{d}\right)^{k-1} \tag{9.3}$$

Where θ_∞ is given by

$$\theta_\infty(x) := \begin{cases} \frac{i}{8\pi}[(x+\sqrt{x^2-1})^{k-1} - (x-\sqrt{x^2-1})^{k-1}] & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$$

and $\square_n = 1, 0$ if n is a perfect square or not respectively. We will be analyzing the averages of both (9.2) and (9.3) over n. More precisely, we will be interested in

$$\lim_{X\to\infty} \frac{2}{X} \sum_{n<X} \sum_{m\in\mathbb{Z}} \theta_\infty\left(\frac{m}{2\sqrt{n}}\right) \sum_{\substack{f^2|(m^2-4n)\\ \frac{m^2-4n}{f^2}\equiv 0,1 \bmod 4}} \frac{1}{f} L\left(1, \left(\frac{(m^2-4n)/f^2}{\cdot}\right)\right) \tag{A}$$

$$\lim_{X\to\infty} \frac{1}{X} \sum_{n<X} \frac{k-1}{12\sqrt{n}}\square_n \tag{B}$$

$$-\frac{1}{2}\lim_{X\to\infty} \frac{1}{X} \sum_{n<X} \sum_{d|n} \min\left(\frac{d}{\sqrt{n}}, \frac{\sqrt{n}}{d}\right)^{k-1} \tag{C}$$

The rest of the chapter will be devoted to showing that

$$B = 0$$

and

$$A = -C$$

The second equality is coincidental and will not be the main point. The main point is to control what is left over once we isolate C in A. In order to do this we will use the theory developed in chapters 3 to 7.

9.2 (B) and (C)

Proposition 9.2.1.

$$\lim_{X\to\infty} \frac{1}{X} \sum_{n<X} \frac{k-1}{12\sqrt{n}}\square_n = 0$$

Proof. The first is obvious since $\sum_{n<X} n^{-\frac{1}{2}} = O(\sqrt{X})$. □

Proposition 9.2.2.

$$-\frac{1}{2}\lim_{X\to\infty} \frac{1}{X} \sum_{n<X} \sum_{\substack{d|n\\ d\le\sqrt{n}}} \left(\frac{d}{\sqrt{n}}\right)^{k-1} = \frac{-1}{2(k-1)}$$

Proof. We can rewrite the sum as

$$\sum_{\substack{ab<X \\ a\leq b}} \left(\tfrac{a}{b}\right)^{\frac{k-1}{2}}$$

Comparing this sum with the double integral

$$\iint_{\substack{a,b\geq 1 \\ ab<X \\ a\leq b}} \left(\tfrac{a}{b}\right)^{\frac{k-1}{2}} dadb = \tfrac{X}{k-1}$$

finishes the proof. □

9.3 Analysis of (A) - 1: Poisson Summation

As in Chapter 5 in order to analyze the elliptic term, we will apply Poisson summation to the m-sum. Note that we are in a simpler situation now that the m-sum actually runs over all integers and we do not have to add the split elements as we had to in Chapter 5. We note that although this is convenient it is completely inessential to the general argument.

Fix n for the time being. For $0 < \alpha < 1$ (to be chosen later, it will be slightly larger than $\frac{1}{2}$), using the approximate functional equation (cf. Lemma 6.1.1) and then applying Poisson summation to the m-sum (cf. §5.6) we get

$$\sqrt{n} \sum_{f \ll n^{\frac{\alpha+\delta_0}{2}}} \tfrac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll n^{\alpha+\delta_0}}} \tfrac{1}{t^2} \sum_{\xi \in \mathbb{Z}\setminus\{0\}} Kl_{t,f}(\xi, n) \left\{ \int_{|x|<1} \theta_\infty(x)\, F\left(\tfrac{tf^2(4n)^{-\alpha}}{(1-x^2)^\alpha}\right) e\left(\tfrac{-x\xi\sqrt{n}}{2tf^2}\right) dx \right\} \tag{9.2-a}$$

$$+ \tfrac{\sqrt{\pi}}{2} \sum_{f < n^{\frac{1-\alpha+\delta_0}{2}}} \tfrac{1}{f} \sum_{\substack{t \\ tf^2 \ll n^{1-\alpha+\delta_0}}} \tfrac{1}{t} \sum_{\xi \in \mathbb{Z}\setminus\{0\}} Kl_{t,f}(\xi, n) \left\{ \int_{|x|<1} \tfrac{\theta_\infty(x)}{\sqrt{1-x^2}} H_-\left(\tfrac{tf^2(4n)^{\alpha-1}}{(1-x^2)^{1-\alpha}}\right) e\left(\tfrac{-x\sqrt{n}\xi}{2tf^2}\right) dx \right\} \tag{9.2-b}$$

$$+ \sqrt{n} \sum_{f=1}^{\infty} \tfrac{1}{f^3} \sum_{t=1}^{\infty} \tfrac{Kl_{t,f}(0,n)}{t^2} \left\{ \int_{|x|<1} \theta_\infty(x)\, F\left(\tfrac{tf^2(4n)^{-\alpha}}{(1-x^2)^\alpha}\right) dx \right\} \tag{9.2-c}$$

$$+ \tfrac{\sqrt{\pi}}{2} \sum_{f=1}^{\infty} \tfrac{1}{f} \sum_{t=1}^{\infty} \tfrac{Kl_{t,f}(0,n)}{t} \left\{ \int_{|x|<1} \tfrac{\theta_\infty(x)}{\sqrt{1-x^2}} H_-\left(\tfrac{tf^2(4n)^{\alpha-1}}{(1-x^2)^{1-\alpha}}\right) dx \right\} \tag{9.2-d}$$

Note that the extra terms in §5.6 are not present since $\theta_\infty(x)$ is not supported outside of $|x| < 1$. We will analyze the average over n of (9.2-a) to (9.2-d).

9.4 Analysis of (A) - 1: Average of (9.2-c)

In this section we will analyze

$$\sum_{n<X} \sqrt{n} \sum_{f=1}^{\infty} \tfrac{1}{f^3} \sum_{t=1}^{\infty} \tfrac{Kl_{t,f}(0,n)}{t^2} \left\{ \int_{|x|<1} \theta_\infty(x)\, F\left(\tfrac{tf^2(4n)^{-\alpha}}{(1-x^2)^\alpha}\right) dx \right\} \tag{9.3$_X$}$$

Lemma 9.4.1. *For $\tau > 0$,*

$$(9.3_X) = 4\int \theta_\infty(x)\left[\frac{1}{2\pi i}\int_{(\tau)} \zeta(2(u+1))\tilde{F}(u)\frac{X^{\frac{3}{2}+\alpha u}}{\frac{3}{2}+\alpha u}4^{\alpha u}(1-x^2)^{\alpha u}du\right]dx + o_{\theta_\infty,F}(X)$$

Proof. We will follow the analysis of Chapter 5. By Perron's formula we have

$$(9.3_X) = \frac{1}{2\pi i}\int_{(\sigma)}\left[\sum_{n=1}^{\infty}\frac{1}{n^{z-\frac{1}{2}}}\sum_{f=1}^{\infty}\frac{1}{f^3}\sum_{t=1}^{\infty}\frac{Kl_{t,f}(0,n)}{t^2}\left\{\int_{|x|<1}\theta_\infty(x)F\left(\frac{tf^2(4n)^{-\alpha}}{(1-x^2)^\alpha}\right)dx\right\}\right]\frac{X^z}{z}dz$$

Then (as in Lemma 7.4.4) we use Mellin inversion on F and get,

$$(9.3_X) =$$
$$\int\theta_\infty(x)\left[\frac{1}{(2\pi i)^2}\int_{(\sigma)}\int_{(\tau)}\left\{\sum_{n=1}^{\infty}\frac{1}{n^{z-\alpha u-\frac{1}{2}}}\sum_{f=1}^{\infty}\frac{1}{f^{2u+3}}\sum_{t=1}^{\infty}\frac{Kl_{t,f}(0,n)}{t^{u+2}}\right\}\frac{X^z}{z}dz\tilde{F}(u)4^{\alpha u}(1-x^2)^{\alpha u}du\right]dx$$

Where $\tau > 0$ and $\sigma > \alpha u + \frac{3}{2}$ (The interchange of integrals and sums is justified by the trivial bound, $Kl_{t,f}(0,n) < t$, $\tau > 0$ and $\sigma > \alpha u + \frac{3}{2}$.). By Corollary 5.3.5 we have

$$\begin{aligned}\sum_{f=1}^{\infty}\frac{1}{f^{2u+3}}\sum_{t=1}^{\infty}\frac{Kl_{t,f}(0,n)}{t^{u+2}} &= 4\frac{\zeta(2(u+1))}{\zeta(u+2)}\prod_{p|n}\left(1-\frac{1}{p^{(u+1)(v_p(n)+1)}}\right)\left(1-\frac{1}{p^{u+1}}\right)^{-1}\\ &= 4\frac{\zeta(2(u+1))}{\zeta(u+2)}\left(\sum_{d|n}\frac{1}{d^{u+1}}\right) \qquad (\star)\end{aligned}$$

Note also that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^{z-\alpha u-\frac{1}{2}}}\sum_{d|n}\frac{1}{d^{u+1}} &= \sum_{a=1}^{\infty}\frac{1}{a^{z-\alpha u-\frac{1}{2}}}\sum_{b=1}^{\infty}\frac{1}{b^{z+(1-\alpha)u+\frac{1}{2}}}\\ &= \zeta(z-\alpha u-\tfrac{1}{2})\zeta(z+(1-\alpha)u+\tfrac{1}{2}) \qquad (\star\star)\end{aligned}$$

Using $(\star)$ and $(\star\star)$ we get

$$(9.3_X) =$$
$$4\int\theta_\infty(x)\left[\frac{1}{(2\pi i)^2}\int_{(\sigma)}\int_{(\tau)}\left\{\frac{\zeta(z-\alpha u-\frac{1}{2})\zeta(z+(1-\alpha)u+\frac{1}{2})\zeta(2(u+1))\tilde{F}(u)}{\zeta(u+2)}\right\}\frac{X^z}{z}4^{\alpha u}(1-x^2)^{\alpha u}dzdu\right]dx$$

We now shift the z-contour. Note that if we can shift the contour to a line $\Re(z) < \sigma' < 1$ then the resulting expression does not contribute to the limit. Let $\sigma' < 1$. Then moving the z-contour to $\Re(z) = \sigma'$ picks up the poles of product $\zeta(z-\alpha u-\frac{1}{2})\zeta(z+(1-\alpha)u+\frac{1}{2})$ at $z = \alpha u + \frac{3}{2}$ and (possibly) at $z = \frac{1}{2} - (1-\alpha)u$.

Therefore we get,

$$(9.3_X) = 4\int \theta_\infty(x)\left[\frac{1}{2\pi i}\int_{(\tau)}\left\{\frac{\zeta(u+2)\zeta(2(u+1))\tilde{F}(u)}{\zeta(u+2)}\right\}\frac{X^{\frac{3}{2}+\alpha u}}{\frac{3}{2}+\alpha u}4^{\alpha u}(1-x^2)^{\alpha u}du\right]dx$$
$$+4\int \theta_\infty(x)\left[\frac{1}{2\pi i}\int_{(\tau)}\left\{\frac{\zeta(-u)\zeta(2(u+1))\tilde{F}(u)}{\zeta(u+2)}\right\}\frac{X^{\frac{1}{2}-(1-\alpha)u}}{\frac{1}{2}-(1-\alpha)u}4^{\alpha u}(1-x^2)^{\alpha u}du\right]dx + o_{\theta_\infty,\alpha}(X)$$

The lemma follows since the second term is $O_{\theta_\infty,F,\alpha}(X^{\frac{1}{2}})$.

□

Lemma 9.4.2.

$$(9.3_X) = \zeta(2)\frac{4X^{\frac{3}{2}}}{3}\int \theta_\infty(x)dx + 2\frac{4^{\frac{-\alpha}{2}}\tilde{F}(\frac{-1}{2})X^{\frac{3}{2}-\frac{\alpha}{2}}}{\frac{3}{2}-\frac{\alpha}{2}}\int \theta_\infty(x)(1-x^2)^{\frac{-\alpha}{2}}dx + o_{\theta_\infty,\alpha}(X)$$

Proof. We use Lemma 9.4.1. Shifting the u-contour to $\Re(u) = \tau' < \frac{-1}{\alpha}$ picks up the pole of $\tilde{F}(u)$ at $u = 0$ and the pole of the ζ-function at $u = \frac{-1}{2}$. The first has residue 1 an the second has residue $\frac{1}{2}$. Therefore we get

$$(9.3_X) = \zeta(2)\frac{4X^{\frac{3}{2}}}{3}\int \theta_\infty(x)dx + 2\frac{4^{\frac{1-\alpha}{2}}\tilde{F}(\frac{-1}{2})}{\frac{3}{2}-\frac{\alpha}{2}}\int \theta_\infty(x)(1-x^2)^{\frac{-\alpha}{2}}dx$$
$$+4\int \theta_\infty(x)\left[\frac{1}{2\pi i}\int_{(\tau')}\zeta(2(u+1))\tilde{F}(u)\frac{X^{\frac{3}{2}+\alpha u}}{\frac{3}{2}+\alpha u}4^{\alpha u}(1-x^2)^{\alpha u}du\right]dx + o_{\theta_\infty,\alpha}(X)$$

The lemma follows.

□

9.5 Analysis of (A) - 2: Average of (9.2-d)

In this section we will analyze

$$\sum_{n<X}\frac{\sqrt{\pi}}{2}\sum_{f=1}^{\infty}\frac{1}{f}\sum_{t=1}^{\infty}\frac{Kl_{t,f}(0,n)}{t}\left\{\int_{|x|<1}\frac{\theta_\infty(x)}{\sqrt{1-x^2}}H_-\left(\frac{tf^2(4n)^{\alpha-1}}{(1-x^2)^{1-\alpha}}\right)dx\right\} \qquad (9.4_X)$$

Lemma 9.5.1.

$$(9.4_X) = 2\sqrt{\pi}\int\frac{\theta_\infty(x)}{\sqrt{1-x^2}}\left[\frac{1}{2\pi i}\int_{(1)}\frac{\Gamma\left(\frac{1+u}{2}\right)}{\Gamma\left(\frac{2-u}{2}\right)}\left\{\zeta(2u)\tilde{F}(u)\right\}\frac{X^{1+(1-\alpha)u}}{1+(1-\alpha)u}\left(\frac{\pi 4^{(1-\alpha)}}{(1-x^2)^{(\alpha-1)}}\right)^u du\right]dx + o_{\theta_\infty,\alpha}(X)$$

Proof. As in the proof of Lemma 9.4.1 we use Perron's formal and the definition

$$H_-(x) = \frac{1}{2\pi i}\int_{(1)}\frac{\Gamma\left(\frac{1+u}{2}\right)}{\Gamma\left(\frac{2-u}{2}\right)}(\pi x)^{-u}\tilde{F}(u)du$$

to rewrite (9.4_X) as

$$(9.4_X) = \frac{\sqrt{\pi}}{2}\int\frac{\theta_\infty(x)}{\sqrt{1-x^2}}$$
$$\left[\frac{1}{(2\pi i)^2}\int_{(\sigma)}\int_{(1)}\frac{\Gamma\left(\frac{1+u}{2}\right)}{\Gamma\left(\frac{2-u}{2}\right)}\left\{\sum_{n=1}^{\infty}\frac{1}{n^{z-(1-\alpha)u}}\sum_{f=1}^{\infty}\frac{1}{f^{1+2u}}\sum_{t=1}^{\infty}\frac{Kl_{t,f}(0,n)}{t^{1+u}}\right\}\frac{X^z}{z}\left(\frac{4^{(1-\alpha)}}{\pi(1-x^2)^{(\alpha-1)}}\right)^u\tilde{F}(u)dudz\right]dx$$

Once again by Corollary 5.3.5

$$\sum_{f=1}^{\infty}\frac{1}{f^{1+2u}}\sum_{t=1}^{\infty}\frac{Kl_{t,f}(0,n)}{t^{1+u}}=4\frac{\zeta(2u)}{\zeta(u+1)}\left(\sum_{d|n}\frac{1}{d^u}\right)$$

Hence,

$$\sum_{n=1}^{\infty}\frac{1}{n^{z-(1-\alpha)u}}\sum_{f=1}^{\infty}\frac{1}{f^{1+2u}}\sum_{t=1}^{\infty}\frac{Kl_{t,f}(0,n)}{t^{1+u}}=4\frac{\zeta(z+\alpha u)\zeta(z-(1-\alpha)u)\zeta(2u)}{\zeta(u+1)}$$

Therefore,

$$\begin{aligned}&(9.4_X)=\\ &2\sqrt{\pi}\int\frac{\theta_\infty(x)}{\sqrt{1-x^2}}\left[\frac{1}{(2\pi i)^2}\int_{(\sigma)}\int_{(1)}\frac{\Gamma\left(\frac{1+u}{2}\right)}{\Gamma\left(\frac{2-u}{2}\right)}\left\{\frac{\zeta(z+\alpha u)\zeta(z-(1-\alpha)u)\zeta(2u)\tilde{F}(u)}{\zeta(u+1)}\right\}\frac{X^z}{z}\left(\frac{4^{(1-\alpha)}}{\pi(1-x^2)^{(\alpha-1)}}\right)^u dudz\right]dx\end{aligned}$$

We move the σ-contour to $\Re(z)=\sigma'<1$, picking up the poles at $z=1+(1-\alpha)u$ and (possibly) at $z=1-\alpha u$. This gives,

$$\begin{aligned}&(9.4_X)\\ &=2\sqrt{\pi}\int\frac{\theta_\infty(x)}{\sqrt{1-x^2}}\left[\frac{1}{2\pi i}\int_{(1)}\frac{\Gamma\left(\frac{1+u}{2}\right)}{\Gamma\left(\frac{2-u}{2}\right)}\left\{\zeta(2u)\tilde{F}(u)\right\}\frac{X^{1+(1-\alpha)u}}{1+(1-\alpha)u}\left(\frac{4^{(1-\alpha)}}{\pi(1-x^2)^{(\alpha-1)}}\right)^u du\right]dx\\ &+2\sqrt{\pi}\int\frac{\theta_\infty(x)}{\sqrt{1-x^2}}\left[\frac{1}{(2\pi i)^2}\int_{(1)}\frac{\Gamma\left(\frac{1+u}{2}\right)}{\Gamma\left(\frac{2-u}{2}\right)}\left\{\frac{\zeta(1-u)\zeta(2u)\tilde{F}(u)}{\zeta(u+1)}\right\}\frac{X^{1-(1-\alpha)u}}{1-(1-\alpha u)}\left(\frac{4^{(1-\alpha)}}{\pi(1-x^2)^{(\alpha-1)}}\right)^u du\right]dx\\ &+2\sqrt{\pi}\int\frac{\theta_\infty(x)}{\sqrt{1-x^2}}\left[\frac{1}{(2\pi i)^2}\int_{(\sigma')}\int_{(1)}\frac{\Gamma\left(\frac{1+u}{2}\right)}{\Gamma\left(\frac{2-u}{2}\right)}\left\{\frac{\zeta(z+\alpha u)\zeta(z-(1-\alpha)u)\zeta(2u)\tilde{F}(u)}{\zeta(u+1)}\right\}\frac{X^z}{z}\left(\frac{4^{(1-\alpha)}}{\pi(1-x^2)^{(\alpha-1)}}\right)^u dudz\right]dx\end{aligned}$$

The exponent of X is the second term and the last is smaller than 1. The lemma follows.

$\square$

Lemma 9.5.2.

$$(9.4_X)=-\pi X\int\frac{\theta_\infty(x)}{\sqrt{1-x^2}}dx+2\frac{4^{\frac{-\alpha}{2}}\tilde{F}\left(\frac{1}{2}\right)X^{\frac{3}{2}-\frac{\alpha}{2}}}{\frac{3}{2}-\frac{\alpha}{2}}\int\theta_\infty(x)(1-x^2)^{-\alpha}dx+o_{\theta_\infty,\alpha}(X)$$

Proof. We use Lemma 9.5.1 and shift the u-contour to $\Re(u)=\tau'<0$. This picks up the poles of the ζ-function at $u=\frac{1}{2}$ with residue $\frac{1}{2}$ and the pole of $\tilde{F}(u)$ at $u=0$ with residue 1. Using the identities $\zeta(0)=\frac{-1}{2}$ and $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

$$\begin{aligned}(9.4_X)&=-\pi X\int\frac{\theta_\infty(x)}{\sqrt{1-x^2}}dx+2\frac{4^{\frac{-\alpha}{2}}\tilde{F}\left(\frac{1}{2}\right)X^{\frac{3}{2}-\frac{\alpha}{2}}}{\frac{3}{2}-\frac{\alpha}{2}}\int\theta_\infty(x)(1-x^2)^{-\alpha}dx\\ &+2\sqrt{\pi}\int\frac{\theta_\infty(x)}{\sqrt{1-x^2}}\left[\frac{1}{2\pi i}\int_{(\tau')}\frac{\Gamma\left(\frac{1+u}{2}\right)}{\Gamma\left(\frac{2-u}{2}\right)}\left\{\zeta(2u)\tilde{F}(u)\right\}\frac{X^{1+(1-\alpha)u}}{1+(1-\alpha)u}\left(\frac{4^{(1-\alpha)}}{\pi(1-x^2)^{(\alpha-1)}}\right)^u du\right]dx\\ &+o_{\theta_\infty,\alpha}(X)\end{aligned}$$

The lemma now follows from $\Re(\tau')<0$. $\square$

9.6 Analysis of (A) - 3: Average of (9.2-c)+(9.2-d)

Lemma 9.6.1.

$$\int \theta_\infty(x)dx = 0$$

Proof. Recall that $\theta_\infty(x) = \frac{i}{8\pi}[(x+i\sqrt{1-x^2})^{k-1} - (x-i\sqrt{x^2-1})^{k-1}$. Therefore,

$$\int \theta_\infty(x)dx = \frac{i}{8\pi}\int_{-1}^{1}[(x+i\sqrt{1-x^2})^{k-1} - (x-i\sqrt{x^2-1})^{k-1}]dx$$

Changing the variables to $x = \cos(\theta)$ gives

$$\begin{aligned}\frac{i}{8\pi}\int_{-1}^{1}[(x+i\sqrt{1-x^2})^{k-1} - (x-i\sqrt{x^2-1})^{k-1}]dx &= \frac{-1}{4\pi}\int_{-\pi}^{0}\cos(\theta)\sin((k-1)\theta)d\theta \\ &= 0\end{aligned}$$

Where, in the first equality we used the fact that the second equality follows from integration by parts twice and using the fact that $k \equiv 0 \bmod 2$.

□

Lemma 9.6.2.

$$\int \frac{\theta_\infty(x)}{\sqrt{1-x^2}}dx = \frac{-1}{2\pi(k-1)}$$

Proof. Once again recall that $\theta_\infty(x) = \frac{i}{8\pi}[(x+i\sqrt{1-x^2})^{k-1} - (x-i\sqrt{x^2-1})^{k-1}]$. Then by using the change of variables $x = \cos(\theta)$ we get,

$$\begin{aligned}\int \frac{\theta_\infty(x)}{\sqrt{1-x^2}}dx &= \frac{-1}{4\pi}\int_{-\pi}^{0}\cos(\theta)\frac{\sin((k-1)\theta)}{-\cos(\theta)}d\theta \\ &= \frac{-1}{2\pi(k-1)}\end{aligned}$$

Where, we used the fact that $\cos(\theta) < 0$ for the range of integration and $\sqrt{1-x^2} > 0$ so in the change of variables we have $\sqrt{1-x^2} = -\cos(\theta)$, and in the second equality we used $k \equiv 0 \bmod 2$ in the last equality. □

Theorem 9.6.3.

$$\text{(9.2-c)} + \text{(9.2-d)} = \frac{1}{2(k-1)} + o_{\theta_\infty,\alpha}(X)$$

Proof. Using lemmas 9.4.2 and 9.5.2 we have

$$\begin{aligned}\text{(9.2-c)} + \text{(9.2-d)} = \zeta(2)\frac{4X^{\frac{3}{2}}}{3}\int \theta_\infty(x)dx + 2\frac{4^{\frac{-\alpha}{2}}\tilde{F}(\frac{-1}{2})X^{\frac{3}{2}-\frac{\alpha}{2}}}{\frac{3}{2}-\frac{\alpha}{2}}\int \theta_\infty(x)(1-x^2)^{\frac{-\alpha}{2}}dx \\ -\pi\int \frac{\theta_\infty(x)}{\sqrt{1-x^2}}dx + 2\frac{4^{\frac{-\alpha}{2}}\tilde{F}(\frac{1}{2})X^{\frac{3}{2}-\frac{\alpha}{2}}}{\frac{3}{2}-\frac{\alpha}{2}}\int \theta_\infty(x)(1-x^2)^{-\alpha}dx + o_{\theta_\infty,\alpha}(X)\end{aligned}$$

By using lemmas 9.6.1 and 9.6.2 we then get the statement of the theorem. □

9.7 Analysis of (A) - 4: The Strategy for the averages of (9.2-a) and (9.2-b)

We will be using the theory developed in chapters 4 to 7. Let us start with a few preliminary observations.

Lemma 9.7.1. *For any $k > 0$ and $\epsilon > 0$*

$$\lim_{X\to\infty} \tfrac{1}{X} \sum_{n<X} n^{-\epsilon} = 0$$
$$\lim_{X\to\infty} \tfrac{1}{X} \sum_{n<X} n^{k} e^{-n^{\epsilon}} = 0$$

Proof. Obvious. □

By Lemma 9.7.1 we see that the error terms in Corollary 5.1.3 do not contribute to the limits.

Next, we consider the ξ-sums. We have an auxiliary lemma which will be useful later in the sequel. Let Φ be a smooth approximation to the characteristic function of $(\frac{1}{2}, 2)$.

Lemma 9.7.2.

$$\sum_{n\equiv b \bmod 4tf^2} e\left(\tfrac{\xi\sqrt{n}}{2tf^2}\right)\Phi\left(\tfrac{n}{X}\right) = \tfrac{1}{4tf^2}\sum_{\nu} e\left(\tfrac{b\nu}{4tf^2}\right)\int e\left(\tfrac{\xi\sqrt{u}}{2tf^2}\right)\Phi\left(\tfrac{u}{X}\right)e\left(\tfrac{-u\nu}{4tf^2}\right)du$$

Moreover for any $r > 0$,

$$\int e\left(\tfrac{\xi\sqrt{u}}{2tf^2}\right)\Phi\left(\tfrac{u}{X}\right)e\left(\tfrac{-u\nu}{4tf^2}\right)du = \begin{cases} O_{r,\Phi}\left(X^{1-r}\left(\tfrac{tf^2}{\nu}\right)^{r} + \tfrac{\xi(tf^2)^r}{X^r\nu^{r+1}}\right) & \text{if } \nu\neq 0 \\ O_{r,\Phi}(X) & \text{if } \nu = 0 \end{cases}$$

Proof. Applying Poisson summation gives the first statement.

$$\tfrac{1}{4tf^2}\sum_{\nu} e\left(\tfrac{b\nu}{4tf^2}\right)\int e\left(\tfrac{\xi\sqrt{u}}{2tf^2}\right)\Phi\left(\tfrac{u}{X}\right)e\left(\tfrac{-u\nu}{4tf^2}\right)du$$

On the other hand for $\nu \neq 0$ by completing to square we get

$$\begin{aligned}\int e\left(\tfrac{\xi\sqrt{u}}{2tf^2}\right)\Phi\left(\tfrac{u}{X}\right)e\left(\tfrac{-u\nu}{4tf^2}\right)du &= X\int \Phi(u)\, e\left(\tfrac{-X\nu(u-2\xi(X\nu)^{-1}\sqrt{u})}{4tf^2}\right)du \\ &= X\int \Phi(u)e\left(\tfrac{-X\nu[(\sqrt{u}-\xi(X\nu)^{-1})^2-\xi^2(X\nu)^{-2}]}{4tf^2}\right)du \\ &= Xe\left(\tfrac{\xi^2(X\nu)^{-1}}{4tf^2}\right)\int \Phi(u)e\left(\tfrac{-X\nu(\sqrt{u}-\xi(X\nu)^{-1})^2}{4tf^2}\right)du \\ &= Xe\left(\tfrac{\xi^2(X\nu)^{-1}}{4tf^2}\right)\int \Phi(u^2)e\left(\tfrac{-X\nu(u-\xi(X\nu)^{-1})^2}{4tf^2}\right)2udu \\ &= Xe\left(\tfrac{\xi^2(X\nu)^{-1}}{4tf^2}\right)\int \Phi(u^2)e\left(\tfrac{-X\nu u^2}{4tf^2}\right)2\left(u+\tfrac{\xi}{X\nu}\right)du\end{aligned}$$

$$= Xe\left(\tfrac{\xi^2(X\nu)^{-1}}{4tf^2}\right)\int \Phi(u)e\left(\tfrac{-X\nu u}{4tf^2}\right)du + \tfrac{2\xi}{\nu}e\left(\tfrac{\xi^2(X\nu)^{-1}}{4tf^2}\right)\int \Phi(u)e\left(\tfrac{-X\nu u}{4tf^2}\right)\tfrac{du}{\sqrt{u}}$$

Integrating by parts r-times we get that

$$\int e\left(\frac{\xi\sqrt{u}}{2tf^2}\right)\Phi\left(\frac{u}{X}\right)e\left(\frac{-u\nu}{4tf^2}\right)du = O_{r,\Phi}\left(X^{1-k}\left(\frac{tf^2}{\nu}\right)^r + \frac{\xi(tf^2)^r}{X^r\nu^{r+1}}\right)$$

For $\nu = 0$ we have

$$\begin{aligned}\int e\left(\frac{\xi\sqrt{u}}{2tf^2}\right)\Phi\left(\frac{u}{X}\right)du &= X\int e\left(\frac{\xi\sqrt{X}\sqrt{u}}{2tf^2}\right)\Phi(u)\,du \\ &= 2X\int e\left(\frac{\xi\sqrt{X}u}{2tf^2}\right)\Phi\left(u^2\right)udu \\ &= O_{r,\Phi}(X)\end{aligned}$$

The lemma follows. □

There will be various ranges of the ξ-sum we will be treating differently. Roughly speaking the idea is the following:

- (9.2-a)
  - $\frac{(tf^2)^{\frac{1-\alpha}{\alpha}}\xi}{\sqrt{n}} \gg n^\epsilon$.

    For this range we will use Proposition 7.2.1 which tells us that the integrals decay fast.
  - $\frac{(tf^2)^{\frac{1-\alpha}{\alpha}}\xi}{\sqrt{n}} \ll n^\epsilon$ and $\frac{\xi\sqrt{n}}{tf^2} \gg n^\epsilon$.

    In this case we will use the asymptotic expansion in Proposition 7.2.2 which tells us that the functions oscillate and there will be significant cancellation in the n-sum.
  - $\frac{\xi\sqrt{n}}{tf^2} \ll n^\epsilon$.

    In this case we will use the cancellation in the character sum to get the desired result.
- (9.2-b).

  Th analogous ranges are:
  - $\frac{(tf^2)^{\frac{\alpha}{1-\alpha}}\xi}{\sqrt{n}} \gg n^\epsilon$.
  - $\frac{(tf^2)^{\frac{\alpha}{1-\alpha}}\xi}{\sqrt{n}} \ll n^\epsilon$ and $\frac{\xi\sqrt{n}}{tf^2} \gg n^\epsilon$.
  - $\frac{\xi\sqrt{n}}{tf^2} \ll n^\epsilon$.

9.8 Analysis of (A) - 5: Estimates for the first range

In this section we will estimate (9.2-a) for the range $\frac{(tf^2)^{\frac{1-\alpha}{\alpha}}\xi}{\sqrt{n}} \gg n^\epsilon$ and (9.2-b) for the range $\frac{(tf^2)^{\frac{\alpha}{1-\alpha}}\xi}{\sqrt{n}} \gg n^\epsilon$.

Lemma 9.8.1. *For any $\epsilon > 0$ such that $\epsilon > \frac{\delta_0}{\alpha}$ and any $M > 0$ we have,*

$$\sqrt{n} \sum_{f \ll n^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll n^{\alpha+\delta_0}}} \frac{1}{t^2} \sum_{\frac{\xi (tf^2)^{\frac{1-\alpha}{\alpha}}}{\sqrt{n}} \gg n^{\epsilon}} Kl_{t,f}(\xi, n) \left\{ \int_{|x|<1} \theta_\infty(x) F\left(\tfrac{tf^2(4n)^{-\alpha}}{(1-x^2)^\alpha}\right) e\left(\tfrac{-x\xi\sqrt{n}}{2tf^2}\right) dx \right\}$$

$$= O_{M,\alpha,\delta_0,\epsilon}\left(\frac{n^{\frac{1}{2}(\alpha+\delta_0)}}{n^{M(\epsilon-\frac{\delta_0}{\alpha})}} + \frac{n^{\frac{1/2+\epsilon-\frac{\delta_0}{\alpha}}{2(\frac{1}{\alpha}-1)}}}{n^{M(\epsilon-\frac{\delta_0}{\alpha})}} + \frac{n^{\frac{1}{4}}}{n^{(M+\frac{1}{2})\epsilon)}} + \frac{n^{\frac{\alpha+\delta_0}{2}}}{n^{M\epsilon}} \right)$$

Therefore, by choosing $M > \frac{1}{\epsilon - \frac{\delta_0}{\alpha}}$ we see that these terms do not contribute to the limit.

Proof. By Proposition 7.2.1 we know that for any $M > 0$ the integral satisfy the bound

$$\left(\frac{\xi\sqrt{n}}{tf^2}\right)^{-M} + \left(\frac{\xi\sqrt{n}}{tf^2}\right)^{-1} \left(\frac{\xi (tf^2)^{\frac{1-\alpha}{\alpha}}}{\sqrt{n}}\right)^{-M}$$

Note that since $tf^2 \ll n^{\alpha+\delta_0}$ and $\frac{\xi(tf^2)^{\frac{1-\alpha}{\alpha}}}{\sqrt{n}} \gg n^\epsilon$ we necessarily have $\frac{\xi\sqrt{n}}{tf^2} \gg n^{\epsilon-\delta_0/\alpha}$. The lemma now follows from 7.4.1 and 7.4.2.

□

Lemma 9.8.2. *For any $\epsilon > 0$ such that $\epsilon > \frac{\delta_0}{1-\alpha}$ and any $M > 0$ we have,*

$$\frac{\sqrt{\pi}}{2} \sum_{f < n^{\frac{1-\alpha+\delta_0}{2}}} \frac{1}{f} \sum_{\substack{t \\ tf^2 \ll n^{1-\alpha+\delta_0}}} \frac{1}{t} \sum_{\frac{\xi (tf^2)^{\frac{\alpha}{1-\alpha}}}{\sqrt{n}} \gg n^{\epsilon}} Kl_{t,f}(\xi, n) \left\{ \int_{|x|<1} \frac{\theta_\infty(x)}{\sqrt{1-x^2}} H_-\left(\tfrac{tf^2(4n)^{\alpha-1}}{(1-x^2)^{1-\alpha}}\right) e\left(\tfrac{-x\sqrt{n}\xi}{2tf^2}\right) dx \right\}$$

$$= O_{M,\alpha,\delta_0,\epsilon}\left(\frac{n^{\frac{1}{4}}}{n^{(M+\frac{3}{2})(\epsilon-\frac{\delta_0}{1-\alpha})}} + \frac{n^{1-\frac{3\alpha}{2}+\delta_0}}{n^{M(\epsilon-\frac{\delta_0}{1-\alpha})}} + \frac{n^{\frac{3(1-\alpha)}{2\alpha}(\frac{1}{2}+\epsilon)} \log(n)}{n^{\frac{1}{2}+M\epsilon}} \right)$$

In particular if we choose $M > \frac{1}{\epsilon - \frac{\delta_0}{1-\alpha}}$ we see that these terms do not contribute to the limit.

Proof. The same as the proof of Lemma 9.8.1 but this time we refer to lemmas 7.4.5 and 7.4.6 instead of lemmas 7.4.1 and 7.4.2.

□

9.9 Analysis of (A) - 6: Estimates for the second range

In this section we will be estimating the average over n of (9.2-a) for the range $\frac{(tf^2)^{\frac{1-\alpha}{\alpha}}\xi}{\sqrt{n}} \ll n^\epsilon$ and $\frac{\xi\sqrt{n}}{tf^2} \gg n^\epsilon$ and (9.2-b) for the range $\frac{(tf^2)^{\frac{\alpha}{1-\alpha}}\xi}{\sqrt{n}} \ll n^\epsilon$ and $\frac{\xi\sqrt{n}}{tf^2} \gg n^\epsilon$.

For notational convenience let us denote the two ranges by $R(\xi, t, f, n, \alpha, \epsilon)$ and $R(\xi, t, f, n, 1-\alpha, \epsilon)$. i.e.

$$R(\xi, t, f, n, \alpha) = \left\{ (\xi, t, f, n) \mid \frac{(tf^2)^{\frac{1-\alpha}{\alpha}}\xi}{\sqrt{n}} \ll n^\epsilon \text{ and } \frac{\xi\sqrt{n}}{tf^2} \gg n^\epsilon \right\}$$

We will be summing over n. Since all the ranges of summation depend on n it is convenient to sum n over dayadic ranges, i.e. ranges of the form $\frac{X}{2} < n < X$. For that purpose let Φ be a smooth approximation to the characteristic function of $(\frac{1}{2}, 1)$ and consider

$$\sum_{l=0}^{\log(X)-1} \sum_{n\in\mathbb{Z}} \sqrt{n}\Phi\left(\frac{n2^l}{X}\right) \sum_{f \ll \left(\frac{X}{2^l}\right)^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll \left(\frac{X}{2^l}\right)^{\alpha+\delta_0}}} \frac{1}{t^2} \sum_{\xi \in R(\xi,t,f,\frac{X}{2^l},\alpha,\epsilon)} Kl_{t,f}(\xi,n)$$
$$\left\{ \int_{|x|<1} \theta_\infty(x) F\left(\frac{tf^2(4n)^{-\alpha}}{(1-x^2)^\alpha}\right) e\left(\frac{-x\xi\sqrt{n}}{2tf^2}\right) dx \right\}$$

and

$$\sum_{l=0}^{\log(X)-1} \sum_{n\in\mathbb{Z}} \frac{\sqrt{\pi}}{2}\Phi\left(\frac{n2^l}{X}\right) \sum_{f \ll \left(\frac{X}{2^l}\right)^{\frac{1-\alpha+\delta_0}{2}}} \frac{1}{f} \sum_{\substack{t \\ tf^2 \ll \left(\frac{X}{2^l}\right)^{1-\alpha+\delta_0}}} \frac{1}{t} \sum_{\xi \in R(\xi,t,f,\frac{X}{2^l},1-\alpha,\epsilon)} Kl_{t,f}(\xi,n)$$
$$\left\{ \int_{|x|<1} \frac{\theta_\infty(x)}{\sqrt{1-x^2}} H_-\left(\frac{tf^2(4n)^{\alpha-1}}{(1-x^2)^{1-\alpha}}\right) e\left(\frac{-x\sqrt{n}\xi}{2tf^2}\right) dx \right\}$$

Lemma 9.9.1. *For $\epsilon < \frac{1}{2}$,*

$$\lim_{X\to\infty} \frac{1}{X} \sum_{l=0}^{\log(X)-1} \sum_{f \ll X^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll X^{\alpha+\delta_0}}} \frac{1}{t^2} \sum_{\xi \in R(\xi,t,f,\frac{X}{2^l},\alpha,\epsilon)}$$
$$\sum_{n\in\mathbb{Z}} \sqrt{n}\Phi\left(\frac{n}{X}\right) Kl_{t,f}(\xi,n) \left\{ \int_{|x|<1} \theta_\infty(x) F\left(\frac{tf^2(4n)^{-\alpha}}{(1-x^2)^\alpha}\right) e\left(\frac{-x\xi\sqrt{n}}{2tf^2}\right) dx \right\} = 0$$

Proof. Let $M \geq k$ be an integer. We will consider the M'th order asymptotic expansion of the integral derived in Chapter 4. By Proposition 7.2.3 we know that the integral is equal to a sum of terms of the form

$$\bullet e\left(\frac{-\xi\sqrt{n}}{2tf^2}\right)\left(\frac{\xi\sqrt{n}}{tf^2}\right)^{-(\frac{3}{2}+m+r+s)}$$
$$\bullet e\left(\frac{-\xi\sqrt{n}}{2tf^2}\right)\left(\frac{(tf^2)^{\frac{1}{\alpha}}}{n}\right)^{1+m+r+s}\left(\frac{\xi(tf^2)^{\frac{1-\alpha}{\alpha}}}{\sqrt{n}}\right)^{s+\rho}$$

with an error of

$$\left(\frac{\xi\sqrt{n}}{tf^2}\right)^{-M-1} + \left(\frac{\xi\sqrt{n}}{tf^2}\right)^{-1}\left(\frac{\xi(tf^2)^{\frac{1-\alpha}{\alpha}}}{\sqrt{n}}\right)^{M+1}$$

We first start with the error terms. By lemmas 7.4.1 and 7.4.2 we see that the error terms contribute

$$O_{M,\alpha,\delta_0,\epsilon}\left(\frac{n^{\frac{1}{4}}}{n^{(M+\frac{1}{2})\epsilon)}} + \frac{n^{\frac{\alpha+\delta_0}{2}}}{n^{M\epsilon}} + \frac{n^{\frac{1}{2}(\alpha+\delta_0)}}{n^{M\epsilon}} + \frac{n^{\frac{1/2+\epsilon}{2(\frac{1}{\alpha}-1)}}}{n^{M\epsilon}}\right)$$

which, when averaged over $n \sim X$ gives a contribution of

$$O_{M,\alpha,\delta_0,\epsilon}\left(\frac{X^{\frac{1}{4}}}{X^{(M+\frac{1}{2})\epsilon)}} + \frac{X^{\frac{\alpha+\delta_0}{2}}}{X^{M\epsilon}} + \frac{X^{\frac{1}{2}(\alpha+\delta_0)}}{X^{M\epsilon}} + \frac{X^{\frac{1/2+\epsilon}{2(\frac{1}{\alpha}-1)}}}{X^{M\epsilon}}\right)$$

As long as we choose $M > \max\{\frac{1}{4}, \frac{\alpha+\delta_0}{2}, \frac{1/2+\epsilon}{2(\frac{1}{\alpha}-1)}\}\epsilon^{-1}$, the limit of this expression as $X \to \infty$ is 0 and does not contribute to the limit.

Next, we consider the oscillating terms. For notational convenience we will bound the term corresponding to $l = 0$. The bound we will obtain will be a power saving in X, and since there are at most $\log(X)$ terms in the l-sum this will imply the result.

- $e\left(\frac{-\xi\sqrt{n}}{2tf^2}\right)\left(\frac{\xi\sqrt{n}}{tf^2}\right)^{-(\frac{3}{2}+m+r+s)}$.

  We focus on
$$\sum_{n\in\mathbb{Z}} \sqrt{n}\Phi\left(\frac{n}{X}\right) Kl_{t,f}(\xi,n) e\left(\frac{-\xi\sqrt{n}}{2tf^2}\right)\left(\frac{\xi\sqrt{n}}{tf^2}\right)^{-(\frac{3}{2}+m+r+s)}$$

  Since $n \sim X$ by partial summation bounding the above sum is implied by bounding
$$\left(\frac{\xi}{tf^2}\right)^{-(\frac{1}{2}+m+r+s)} X^{\frac{1}{4}-\frac{m+r+s}{2}} \sum_n \Phi\left(\frac{n}{X}\right) e\left(\frac{\xi\sqrt{n}}{tf^2}\right) Kl_{t,f}(\xi,n) \qquad (\star)$$

  Breaking the n-sum into arithmetic progressions mod$4tf^2$ and applying Poisson summation then gives
$$\sum_n \Phi\left(\frac{n}{X}\right) e\left(\frac{\xi\sqrt{n}}{tf^2}\right) Kl_{t,f}(\xi,n)$$
$$= \frac{1}{4tf^2}\sum_{\nu}\left\{\int e\left(\frac{\xi\sqrt{u}}{2tf^2}\right)\Phi\left(\frac{u}{X}\right) e\left(\frac{-u\nu}{4tf^2}\right) du\right\} \sum_{b \bmod 4tf^2} Kl_{tf^2}(\xi,n) e\left(\frac{b\nu}{4tf^2}\right)$$

  Recall the notation of (6.4),
$$\omega^{\{1\}}_{t,f}(\xi,n) = \sum_{b \bmod 4tf^2} Kl_{tf^2}(\xi,n) e\left(\frac{b\nu}{4tf^2}\right)$$

  By Lemma 9.7.2, for any $e > 0$, the sum above is
$$O_{e,\Phi}\left(X|\omega^{\{1\}}_{t,f}(\xi,0)| + X^{1-e}(tf^2)^{e-1}\sum_{\nu}|\omega^{\{1\}}_{t,f}(\xi,\nu)|\nu^{-e}\right.$$
$$\left. + X^{-e}\xi(tf^2)^{e-1}\sum_{\nu}|\omega^{\{1\}}_{t,f}(\xi,\nu)|\nu^{-e+1}\right)$$

  For $\nu \neq 0$ by Lemma 6.2.4 we have $|\omega^{\{1\}}_{t,f}(\xi,\nu)| \leq f^2t^2$. Using this, we get
$$\sum_{f \ll X^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll X^{\alpha+\delta_0}}} \frac{1}{t^2} \sum_{\xi \in R(\xi,t,f,X,\alpha,\epsilon)} \left(\frac{\xi\sqrt{n}}{tf^2}\right)^{-(\frac{3}{2}+m+r+s)} \frac{1}{4tf^2}$$
$$\sum_{\nu\neq 0}\left\{\int e\left(\frac{\xi\sqrt{u}}{2tf^2}\right)\Phi\left(\frac{u}{X}\right) e\left(\frac{-u\nu}{4tf^2}\right) du\right\} \sum_{b \bmod 4tf^2} Kl_{tf^2}(\xi,n) e\left(\frac{b\nu}{4tf^2}\right)$$
$$\ll_{e,\Phi} X^{\frac{1}{4}-\frac{m+r+s}{2}} \sum_{f \ll X^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll X^{\alpha+\delta_0}}} \frac{1}{t} \sum_{\xi \in R(\xi,t,f,\frac{X}{2^l},\alpha,\epsilon)} \left(\frac{\xi}{tf^2}\right)^{-(\frac{3}{2}+m+r+s)}$$
$$\left(X^{1-e}(tf^2)^{e-1} + X^{-e}\xi(tf^2)^{e-1}\right)$$
$$\ll_{e,\Phi,\delta_0,\epsilon,m,r,s} X^{\frac{5+2\alpha+2\delta_0}{4}-\frac{m+r+s}{2}(1-\alpha-\delta_0)-e(\alpha-1+\delta_0)}$$

Since $m+r+s \geq 0$ if we choose e such that $\frac{5+2\alpha+2\delta_0}{4} - e(\alpha-1+\delta_0) > 0$ then the sum is bounded by 1. Therefore this sum does not contribute to the limit.

Now we pass to the $\nu = 0$-term. This time we use Lemma 6.2.4 and see that $\omega_{t,f}^{\{1\}}(\xi,0)$ vanishes unless $tf^2 \mid \xi$. And if $tf^2 \mid \xi$ then it is bounded by t^2f^2. Then using the bound X on the integral we get,

$$\sum_{f \ll X^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll X^{\alpha+\delta_0}}} \frac{1}{t^2} \sum_{\xi \in R(\xi,t,f,\frac{X}{2^l},\alpha,\epsilon)} \left(\frac{\xi\sqrt{n}}{tf^2}\right)^{-(\frac{3}{2}+m+r+s)} \frac{\omega_{t,f}(\xi,0)}{4tf^2} \int e\left(\frac{\xi\sqrt{u}}{2tf^2}\right) \Phi\left(\frac{u}{X}\right) du$$
$$X \sum_{f \ll X^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll X^{\alpha+\delta_0}}} \frac{1}{t} \sum_{tf^2\xi \in R(tf^2\xi,t,f,\frac{X}{2^l},\alpha,\epsilon)} \left(\xi\sqrt{n}\right)^{-(\frac{3}{2}+m+r+s)}$$
$$\ll_{\alpha,\delta_0} X^{\frac{1}{4}-\frac{m+r+s}{2}}$$

Therefore this sum does not contribute to the limit.

- $e\left(\frac{-\xi\sqrt{n}}{2tf^2}\right)\left(\frac{(tf^2)^{\frac{1}{\alpha}}}{n}\right)^{1+m+r+s} \left(\frac{\xi(tf^2)^{\frac{1-\alpha}{\alpha}}}{\sqrt{n}}\right)^{s+\rho}$

We proceed as above. Consider the n-sum:

$$\sum_{n\in\mathbb{Z}} \sqrt{n}\Phi\left(\frac{n}{X}\right) Kl_{t,f}(\xi,n) e\left(\frac{-\xi\sqrt{n}}{2tf^2}\right)\left(\frac{(tf^2)^{\frac{1}{\alpha}}}{n}\right)^{1+m+r+s} \left(\frac{\xi(tf^2)^{\frac{1-\alpha}{\alpha}}}{\sqrt{n}}\right)^{s+\rho}$$

Since $n \sim X$ it is enough to estimate

$$\frac{(tf^2)^{\frac{1+m+r+s+(1-\alpha)(s+\rho)}{\alpha}-1}\xi^{s+\rho+1}}{X^{m+r+s+\frac{s+\rho}{2}}} \sum_{n\in\mathbb{Z}} \Phi\left(\frac{n}{X}\right) Kl_{t,f}(\xi,n) e\left(\frac{-\xi\sqrt{n}}{2tf^2}\right) \tag{$\star\star$}$$

Once again we apply Poisson summation to the n-sum, use lemmas 9.7.2 and 6.2.4. This gives for any $e > 0$

$$\sum_{n\in\mathbb{Z}} \Phi\left(\frac{n}{X}\right) Kl_{t,f}(\xi,n) e\left(\frac{-\xi\sqrt{n}}{2tf^2}\right) = \frac{1}{4tf^2}\sum_{\nu}\left\{\int \Phi\left(\frac{u}{X}\right) e\left(\frac{\xi\sqrt{u}}{2tf^2}\right) e\left(\frac{-\nu u}{4tf^2}\right) du\right\} \omega_{t,f}^{\{1\}}(\xi,\nu)$$
$$\ll_{\Phi,e} \frac{X}{tf^2}|\omega_{t,f}^{\{1\}}(\xi,0)| + X^{1-e}t(tf^2)^{e-1} + X^{-e}\xi t(tf^2)^{e-1}$$

We substitute each one of the terms above into ($\star\star$) and bound the sum over f, t and ξ.

–

$$\sum_{f \ll X^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll X^{\alpha+\delta_0}}} \frac{1}{t^2} \sum_{\xi \in R(\xi,t,f,X,\alpha,\epsilon)} \frac{(tf^2)^{\frac{1+m+r+s+(1-\alpha)(s+\rho)}{\alpha}-1}\xi^{s+\rho+1}}{X^{m+r+s+\frac{s+\rho}{2}}} X^{1-e}t(tf^2)^{e-1}$$
$$= O_{\Phi,e,\epsilon,\delta_0,\alpha}\left(X^{1+\alpha(e-2)+\epsilon(s+\rho+2)+\frac{\delta_0}{\alpha}(1+m+r+s-2+e\alpha)-e}\right)$$

Therefore as long as $e > 1+\alpha(e-2)+\epsilon(s+\rho+2)+\frac{\delta_0}{\alpha}(1+m+r+s-2+e\alpha)$ this part does not contribute to the limit (recall that $0<\alpha<1$).

–

$$\sum_{f\ll X^{\frac{\alpha+\delta_0}{2}}}\frac{1}{f^3}\sum_{\substack{t\\ tf^2\ll X^{\alpha+\delta_0}}}\frac{1}{t^2}\sum_{\xi\in R(\xi,t,f,X,\alpha,\epsilon)}\frac{(tf^2)^{\frac{1+m+r+s+(1-\alpha)(s+\rho)}{\alpha}-1}\xi^{s+\rho+2}}{X^{m+r+s+\frac{s+\rho}{2}}}X^{-e}t(tf^2)^{e-1}$$
$$= O_{\Phi,e,\epsilon,\delta_0,\alpha}\left(X^{\alpha-\frac{1}{2}+e\alpha+\epsilon(s+\rho+3)+\delta_0(\frac{1+m+r+s-3(1-\alpha)}{\alpha}+e-2)-e}\right)$$

Therefore as long as $e>\alpha-\frac{1}{2}+e\alpha+\epsilon(s+\rho+3)+\delta_0(\frac{1+m+r+s-3(1-\alpha)}{\alpha}+e-2)$ this does not contribute to the limit.

–

$$\sum_{f\ll X^{\frac{\alpha+\delta_0}{2}}}\frac{1}{f^3}\sum_{\substack{t\\ tf^2\ll X^{\alpha+\delta_0}}}\frac{1}{t^2}\sum_{\xi\in R(\xi,t,f,X,\alpha,\epsilon)}\frac{(tf^2)^{\frac{1+m+r+s+(1-\alpha)(s+\rho)}{\alpha}-1}\xi^{s+\rho+1}}{X^{m+r+s+\frac{s+\rho}{2}}}\frac{X}{tf^2}|\omega^{\{1\}}_{t,f}(\xi,0)|$$

Once again by Lemma 6.2.4 and see that $\omega^{\{1\}}_{t,f}(\xi,0)$ vanishes unless $tf^2\mid\xi$ and is bounded by t^2f^2 if $tf^2\mid\xi$. Since we are in the range $\frac{\xi(tf^2)^{\frac{1-\alpha}{\alpha}}}{\sqrt{X}}\ll X^{\epsilon}$, if we let $\xi=tf^2\xi_0$ we see that $\xi_0(tf^2)^{\frac{1}{\alpha}}\ll X^{\frac{1}{2}+\epsilon}$. Therefore we need to have $tf^2\ll X^{\alpha(\frac{1}{2}+\epsilon)}$. Using this we get

$$\sum_{f\ll X^{\frac{\alpha+\delta_0}{2}}}\frac{1}{f^3}\sum_{\substack{t\\ tf^2\ll X^{\alpha+\delta_0}}}\frac{1}{t}\sum_{tf^2\xi_0\in R(tf^2\xi_0,t,f,X,\alpha,\epsilon)}\frac{(tf^2)^{\frac{1+m+r+s+(1-\alpha)(s+\rho)}{\alpha}-1}(tf^2\xi_0)^{s+\rho+1}}{X^{m+r+s+\frac{s+\rho}{2}}}X$$
$$= O_{\Phi,\epsilon,\alpha}\left(X^{1+(\frac{1}{2}+\epsilon)(m+r+s-1)-(m+r+s+\frac{s+\rho}{2})}\right)$$

Which does not contribute to the limit as long as $\epsilon<\frac{1}{2}$.

This finishes the proof of the lemma.

□

Lemma 9.9.2. *For $\epsilon<\frac{1}{2}$,*

$$\lim_{X\to\infty}\frac{1}{X}\sum_{l=0}^{\log(X)-1}\sum_{f\ll X^{\frac{\alpha+\delta_0}{2}}}\frac{1}{f}\sum_{\substack{t\\ tf^2\ll X^{1-\alpha+\delta_0}}}\frac{1}{t}\sum_{\xi\in R(\xi,t,f,\frac{X}{2^l},1-\alpha,\epsilon)}$$
$$\sum_{n\in\mathbb{Z}}\Phi\left(\frac{n}{X}\right)Kl_{t,f}(\xi,n)\left\{\int_{|x|<1}\frac{\theta_\infty(x)}{\sqrt{1-x^2}}H_-\left(\frac{tf^2(4n)^{\alpha-1}}{(1-x^2)^{1-\alpha}}\right)e\left(\frac{-x\sqrt{n}\xi}{2tf^2}\right)dx\right\}=0$$

Proof. The proof follows the same line of argument as the proof of Lemma 9.9.1 the only difference is now one refers to the asymptotic expansion in Proposition 7.2.3 rather than 7.2.2.

□

9.10 Analysis of (A) - 7: Estimates for the third range and the choice of the parameter α

In this section we will estimate the average over n of (9.2-a) and (9.2-b) for the ranges $\frac{\xi\sqrt{n}}{tf^2}\ll n^{\epsilon}$.

We first start with (9.2-d). Recall that we have $tf^2 \ll n^{1-\alpha+\delta_0}$. Here $\delta_0 > 0$ is arbitrary, and hence can be chosen arbitrarily small. Since $tf^2 \ll n^{1-\alpha+\delta_0}$, we have $\frac{\xi\sqrt{n}}{tf^2} \gg n^{\alpha-\frac{1}{2}-\delta_0}$ Then, by choosing $\alpha = \frac{1}{2}+\delta_0+\epsilon$ we see that $\frac{\xi\sqrt{n}}{tf^2} \gg n^{\epsilon}$. But on the other hand since the third range requires $\frac{\xi\sqrt{n}}{tf^2} \ll n^{\epsilon}$, we conclude that (9.2-b) is empty for the third range with this choice of α.

We turn back to (9.2-a). We would like to estimate

$$\sum_{n<X} \sqrt{n} \sum_{f \ll n^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll n^{\alpha+\delta_0}}} \frac{1}{t^2} \sum_{\frac{\xi\sqrt{n}}{tf^2} \ll n^{\epsilon}} Kl_{t,f}(\xi,n) \left\{ \int_{|x|<1} \theta_\infty(x) F\left(\frac{tf^2(4n)^{-\alpha}}{(1-x^2)^\alpha}\right) e\left(\frac{-x\xi\sqrt{n}}{2tf^2}\right) dx \right\}$$

Once again we break the n-sum into dyadic ranges by introducing, Φ, a smooth approximation to the characteristic function of $(\frac{1}{2},1)$ and instead of the above sum, consider

$$\sum_{l=0}^{\log(X)-1} \sum_{n} \Phi\left(\frac{n2^l}{X}\right)\sqrt{n}$$
$$\sum_{f \ll \left(\frac{X}{2^l}\right)^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll \left(\frac{X}{2^l}\right)^{\alpha+\delta_0}}} \frac{1}{t^2} \sum_{\frac{\xi\sqrt{X}}{2^{\frac{l}{2}} tf^2} \ll X^{\epsilon}} Kl_{t,f}(\xi,n) \left\{ \int_{|x|<1} \theta_\infty(x) F\left(\frac{tf^2(4n)^{-\alpha}}{(1-x^2)^\alpha}\right) e\left(\frac{-x\xi\sqrt{n}}{2tf^2}\right) dx \right\}$$

We will consider each dyadic piece separately and focus on

$$\sum_{n} \Phi\left(\frac{n}{X}\right)\sqrt{n}$$
$$\sum_{f \ll X^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll X^{\alpha+\delta_0}}} \frac{1}{t^2} \sum_{\frac{\xi\sqrt{X}}{tf^2} \ll X^{\epsilon}} Kl_{t,f}(\xi,n) \left\{ \int_{|x|<1} \theta_\infty(x) F\left(\frac{tf^2(4n)^{-\alpha}}{(1-x^2)^\alpha}\right) e\left(\frac{-x\xi\sqrt{n}}{2tf^2}\right) dx \right\} \quad (\bullet)$$

Applying Poisson summation to the n-sum then gives

$$\sum_{n} \Phi\left(\frac{n}{X}\right)\sqrt{n} Kl_{t,f}(\xi,n) \left\{ \int_{|x|<1} \theta_\infty(x) F\left(\frac{tf^2(4n)^{-\alpha}}{(1-x^2)^\alpha}\right) e\left(\frac{-x\xi\sqrt{n}}{2tf^2}\right) dx \right\}$$
$$= \frac{1}{4tf^2} \sum_{\nu} \omega_{t,f}^{\{1\}}(\xi,\nu) \left[\int \Phi\left(\frac{u}{X}\right)\sqrt{u} \left\{ \int_{|x|<1} \theta_\infty(x) F\left(\frac{tf^2(4u)^{-\alpha}}{(1-x^2)^\alpha}\right) e\left(\frac{-x\xi\sqrt{u}}{2tf^2}\right) dx \right\} e\left(\frac{-u\nu}{4tf^2}\right) du \right] \quad (\Sigma)$$

The next lemma shows that the terms besides $\nu = 0$ are small in size.

Lemma 9.10.1. *Let $\nu \neq 0$. Then for any $M > 0$,*

$$\int_{|x|<1} \theta_\infty(x) \left\{ \int \Phi\left(\frac{u}{X}\right)\sqrt{u} F\left(\frac{tf^2(4u)^{-\alpha}}{(1-x^2)^\alpha}\right) e\left(\frac{-x\xi\sqrt{u}}{2tf^2}\right) e\left(\frac{-u\nu}{4tf^2}\right) du \right\} dx$$
$$= O_{\theta_\infty,F,\Phi,\alpha,M}\left(\frac{(tf^2)^M}{\nu^M X^{M-\frac{3}{2}}} \left(\left(\frac{tf^2}{X^\alpha}\right)^M + \left(\frac{\xi\sqrt{X}}{tf^2}\right)^M \right) \right)$$

Proof. Changing the variables to $u \mapsto uX$, we can rewrite the integral as

$$X^{\frac{3}{2}} \int \Phi(u)\sqrt{u} \left\{ \int_{|x|<1} \theta_\infty(x) F\left(\frac{tf^2(4Xu)^{-\alpha}}{(1-x^2)^\alpha}\right) e\left(\frac{-x\xi\sqrt{Xu}}{2tf^2}\right) dx \right\} e\left(\frac{-Xu\nu}{4tf^2}\right) du$$

We then change the order of integration (justified since the functions are continuous and the range of integration is compact),

$$X^{\frac{3}{2}} \int_{|x|<1} \theta_{\infty}(x) \left\{ \int \Phi(u) \sqrt{u} F\left(\frac{tf^2(4Xu)^{-\alpha}}{(1-x^2)^{\alpha}}\right) e\left(\frac{-x\xi\sqrt{Xu}}{2tf^2}\right) e\left(\frac{-Xu\nu}{4tf^2}\right) du \right\} dx$$

Integration by parts M-times (integrating the last exponential and differentiating the rest) then gives

$$X^{\frac{3}{2}} \int_{|x|<1} \theta_{\infty}(x) \left\{ \int \Phi(u) \sqrt{u} F\left(\frac{tf^2(4Xu)^{-\alpha}}{(1-x^2)^{\alpha}}\right) e\left(\frac{-x\xi\sqrt{Xu}}{2tf^2}\right) e\left(\frac{-Xu\nu}{4tf^2}\right) du \right\} dx$$
$$= O_{\theta_{\infty},F,\Phi,\Phi,\alpha,M}\left(\frac{(tf^2)^M}{\nu^M X^{M-\frac{3}{2}}}\left(\left(\frac{tf^2}{X^{\alpha}}\right)^M + \left(\frac{\xi\sqrt{X}}{tf^2}\right)^M\right)\right)$$

□

Lemma 9.10.2.

$$\sum_{f \ll X^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^5} \sum_{\substack{t \\ tf^2 \ll X^{\alpha+\delta_0}}} \frac{1}{t^3} \sum_{\frac{\xi\sqrt{X}}{tf^2} \ll X^{\epsilon}}$$
$$\sum_{\nu \neq 0} \omega_{t,f}^{\{1\}}(\xi,\nu) \left[\int \Phi\left(\frac{u}{X}\right) \sqrt{u} \left\{ \int_{|x|<1} \theta_{\infty}(x) F\left(\frac{tf^2(4u)^{-\alpha}}{(1-x^2)^{\alpha}}\right) e\left(\frac{-x\xi\sqrt{u}}{2tf^2}\right) dx \right\} e\left(\frac{-u\nu}{4tf^2}\right) du \right]$$
$$\ll_{\theta_{\infty},F,\Phi,M,\alpha,\delta_0\epsilon} X^{(\alpha-1)M+2\delta_0(M+\frac{1}{2})+1+\epsilon+\alpha} + X^{M(\alpha-1+\delta_0)+1+\alpha+\delta_0}$$

Proof. By Lemma 9.10.1, for any $M > 0$ we have,

$$\sum_{f \ll X^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^5} \sum_{\substack{t \\ tf^2 \ll X^{\alpha+\delta_0}}} \frac{1}{t^3} \sum_{\frac{\xi\sqrt{X}}{tf^2} \ll X^{\epsilon}}$$
$$\sum_{\nu \neq 0} \omega_{t,f}^{\{1\}}(\xi,\nu) \left[\int \Phi\left(\frac{u}{X}\right) \sqrt{u} \left\{ \int_{|x|<1} \theta_{\infty}(x) F\left(\frac{tf^2(4u)^{-\alpha}}{(1-x^2)^{\alpha}}\right) e\left(\frac{-x\xi\sqrt{u}}{2tf^2}\right) dx \right\} e\left(\frac{-u\nu}{4tf^2}\right) du \right]$$

$$\ll_{\theta_{\infty},F,\Phi,\alpha,M} \sum_{f \ll X^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^5} \sum_{\substack{t \\ tf^2 \ll X^{\alpha+\delta_0}}} \frac{1}{t^3} \sum_{\frac{\xi\sqrt{X}}{tf^2} \ll X^{\epsilon}}$$
$$\sum_{\nu \neq 0} \omega_{t,f}^{\{1\}}(\xi,\nu) \left(\frac{(tf^2)^M}{\nu^M X^{M-\frac{3}{2}}}\left(\left(\frac{tf^2}{X^{\alpha}}\right)^M + \left(\frac{\xi\sqrt{X}}{tf^2}\right)^M\right)\right)$$

By Lemma 6.2.4 we have the rough bound $|\omega_{\xi,\nu}^{\{1\}}(\xi,\nu)| \leq t^2 f^2$ we get that the above is bounded by,

$$\leq \sum_{f \ll X^{\frac{\alpha+\delta_0}{2}}} \frac{1}{f^3} \sum_{\substack{t \\ tf^2 \ll X^{\alpha+\delta_0}}} \frac{1}{t} \sum_{\frac{\xi\sqrt{X}}{tf^2} \ll X^{\epsilon}} \left(\frac{(tf^2)^M}{X^{M-\frac{3}{2}}}\left(\left(\frac{tf^2}{X^{\alpha}}\right)^M + \left(\frac{\xi\sqrt{X}}{tf^2}\right)^M\right)\right)$$
$$\ll_{M,\alpha,\delta_0\epsilon} X^{(\alpha-1)M+2\delta_0(M+\frac{1}{2})+1+\epsilon+\alpha} + X^{M(\alpha-1+\delta_0)+1+\alpha+\delta_0}$$

□

Lemma 9.10.3. *For $\epsilon < \frac{1}{2}$, the term for $\nu = 0$ in (Σ) does not contribute to the limit.*

Proof. Recall that the term for $\nu = 0$ in (Σ) is

$$\tfrac{1}{4tf^2}\omega_{t,f}^{\{1\}}(\xi,0)\left[\int \Phi\left(\tfrac{u}{X}\right)\sqrt{u}\left\{\int_{|x|<1}\theta_\infty(x)\,F\left(\tfrac{tf^2(4u)^{-\alpha}}{(1-x^2)^\alpha}\right)e\left(\tfrac{-x\xi\sqrt{u}}{2tf^2}\right)dx\right\}du\right]$$

By Lemma 6.2.4 we see that $\omega_{t,f}^{\{1\}}(\xi,0) = 0$ unless $tf^2 \mid \xi$. However the sum over ξ in $(\bullet)$ is over $\xi \neq 0$ such that $\frac{\xi\sqrt{X}}{tf^2} \ll X^\epsilon$. Since $tf^2 \mid \xi$ and $\xi \neq 0$ otherwise the sum vanishes, let $\xi = tf^2\xi_0$ for some non-zero integer ξ_0. Then ξ_0 needs to satisfy $\xi_0 \ll X^{\epsilon-\frac{1}{2}}$. Since $\epsilon < \frac{1}{2}$, such ξ_0 is not present in the sum for large X and the lemma follows.

$\square$

9.11 Analysis of (A) - 8: Choosing the parameters δ_0, ϵ and M, and the final contribution

In this section we will finally choose the parameters δ_0 and ϵ and show that the contribution of (A) to the limit is $\frac{1}{k-1}$. We recall from the previous section that we had already chosen $\alpha = \frac{1}{2}+\delta_0+\epsilon$

Theorem 9.11.1. *Let* $\delta_0 = \frac{1}{64}$, $\epsilon = \frac{1}{4}$, $M = 6$. *For this choice,*

$$(\mathrm{A}) = \frac{1}{2(k-1)}$$

Proof. Recall from §9.3 that

$$(\mathrm{A}) = \lim_{X\to\infty} \frac{1}{X}((9.2\text{-a}) + (9.2\text{-b}) + (9.2\text{-c}) + (9.2\text{-d}))$$

Theorem 9.6.3 then shows that

$$\lim_{X\to\infty} \frac{1}{X}((9.2\text{-c}) + (9.2\text{-d})) = \frac{1}{2(k-1)}$$

Note that for the choice in the statement of the theorem

$$\begin{aligned} \alpha &= \tfrac{1}{2} + \delta_0 + \epsilon \\ &= \tfrac{1}{2} + \tfrac{1}{64} + \tfrac{1}{4} \\ &= \tfrac{49}{64} \\ 1-\alpha &= \tfrac{1}{2} - \tfrac{1}{16} - \tfrac{1}{4} \\ &= \tfrac{15}{64} \end{aligned}$$

Therefore we have,

$$\begin{aligned} \epsilon &= \tfrac{1}{4} < \tfrac{1}{2} && (i) \\ \epsilon &= \tfrac{1}{4} > \tfrac{1}{49} = \tfrac{\delta_0}{\alpha} && (ii) \\ \epsilon &= \tfrac{1}{4} > \tfrac{1}{15} = \tfrac{\delta_0}{1-\alpha} && (iii) \\ M &= 6 > \tfrac{196}{45} = \tfrac{1}{\epsilon - \frac{\delta_0}{\alpha}} && (iv) \\ M &= 6 > \tfrac{60}{11} = \tfrac{1}{\epsilon - \frac{\delta_0}{1-\alpha}} && (v) \\ (\alpha-1)M + 2\delta_0(M+\tfrac{1}{2}) + 1 + \alpha + \epsilon &= \tfrac{52}{64} < 1 && (vi) \\ (\alpha - 1 + \delta_0)M + 1 + \alpha + \delta_0 &= \tfrac{15}{32} < 1 && (vii) \end{aligned}$$

Recall that at the end of §9.7 we have divided the averages of (9.2-a) and (9.2-b) into three ranges and estimated them in lemmas 9.8.1 to 9.10.3. Using these estimates we will now show that with the choices we do not get any extra contribution to the limit.

- By the estimates of Lemma 9.8.1 and *ii* and *iv* there is no contribution from the range in Lemma 9.8.1.
- By the estimates of Lemma 9.8.2 and *iii* and *v* there is no contribution from the range in Lemma 9.8.2.
- By the estimates of lemmas 9.9.1 and 9.9.2 and *i* there is no contribution from the ranges in lemmas 9.9.1 and 9.9.2.

- By the estimates of Lemma 9.10.2 and v and vi there is no contribution from the range in Lemma 9.10.2. (Note that for this range we will divide the estimates in Lemma 9.10.2 by X before taking the limit, and the extra sum over l does not contribute to the limit.)

Since by the paragraph at the beginning of §9.10 and the choice of $\alpha = \frac{1}{2} + \epsilon + \delta_0$ and Lemma 9.10.3 the rest of the terms also do not contribute to the limit, the statement of the theorem follows.

□

9.12 The final answer

Corollary 9.12.1.

$$(\mathrm{A}) + (\mathrm{B}) + (\mathrm{C}) = 0$$

Proof. By propositions 9.2.1 and 9.2.2 we have

$$(\mathrm{B}) + C = \tfrac{-1}{2(k-1)}$$

By theorems 9.6.3 and 9.11.1

$$(\mathrm{A}) = \tfrac{1}{2(k-1)}$$

The corollary follows. □

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